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Sums of graded algebras, images of graded polynomials and f-zpd algebras

Somas de álgebras graduadas, imagens de polinômios graduados e álgebras f-zpd

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Resumo

Esta tese tem como objetivo apresentar resultados na direção de três problemas distintos.

Mostramos que álgebras graduadas que são soma de duas subálgebras homogêneas satisfazendo identidades graduadas nem sempre são gr-PI álgebras. Além disso, apresentamos condições suficientes para uma tal soma satisfazer alguma identidade polinomial graduada.

Consideramos imagens de polinômios multilineares graduados sobre a álgebra graduada de matrizes triangulares superiores e classificamos tais imagens para certas graduações. Obtemos uma descrição completa no caso de dimensão baixa nos ambientes ordinário e de Jordan. Também estudamos o caso de matrizes triangulares superiores com involução graduada e de dimensão baixa, onde classificamos imagens de polinômios multilineares nestas álgebras bem como mostramos que tais imagens nem sempre são subespaços.

Uma generalização de álgebras zpd é apresentada (as chamadas álgebras f-zpd) e mostramos que nem sempre a álgebra das matrizes é f-zpd. Fornecemos vários exemplos de polinômios f em que a álgebra das matrizes é f-zpd, e consideramos um problema do tipo Nullstellensatz que está relacionado com a classe de álgebras introduzida.

Palavras-chave: imagens de polinômios, conjectura de L'vov-Kaplansky, identidades polinomiais, polinômios centrais, somas de álgebras, matrizes triangulares superiores, álgebras de Jordan, álgebras de Lie, involuções, álgebras graduadas, involuções graduadas, álgebras zpd, álgebras f-zpd, Nullstellensatz.

Abstract

The main goal of this thesis is to present results in the direction of three distinct problems.

We show that graded algebras which are sum of two homogeneous subalgebras satisfying graded identities are not always gr-PI algebras. Moreover, we give sufficient conditions for the sum to satisfy some graded polynomial identity.

We consider images of multilinear graded polynomials on the graded algebra of upper triangular matrices and we classify such images for certain gradings. We obtain a full description in the case of small dimension for the ordinary and Jordan settings. We also study the case of upper triangular matrices with graded involution and of small dimension, where we classify the images of multilinear polynomials on these algebras, moreover we show that such images are not always vector subspaces.

A generalization of zpd algebras is presented (the so called f-zpd algebras) and we prove that the full matrix algebra is not always f-zpd. We give several examples of polynomials f where the full matrix algebra is f-zpd, and we also consider a problem of Nullstellensatz type which is related to the class of algebras introduced.

Keywords: images of polynomials, L'vov-Kaplansky conjecture, polynomial identities, central polynomials, sums of algebras, upper triangular matrices, Jordan algebras, Lie algebras, involutions, graded algebras, graded involutions, zpd algebras, f-zpd algebras, Nullstellensatz.

Contents

Introduction

Images of polynomials on algebras appear in several results from Linear Algebra and Ring Theory. For instance the well-known Cayley-Hamilton Theorem states that the image of the characteristic polynomial of a matrix $A \in M_n(F)$ on A itself is the null matrix. Another example comes when the image of a polynomial on some algebra is always zero, or even is always contained in the center of the algebra. These polynomials are known as polynomial identity and central polynomial for the algebra, respectively.

Since polynomial identities for algebras are a particular example of images of polynomials on algebras, we may say that the first results concerning the latter appeared with the works of Dehn [18] and Wagner [65] in the twenties and thirties of the 20th century. However, a different first result concerning images of polynomials on algebras is addressed in the literature. In 1937, Shoda [56] showed that the image of the commutator $[x, y] := xy - yx$ on the full matrix algebra $M_n(F)$ with entries in a field F of characteristic zero is exactly the subspace $sl_n(F)$ of traceless matrices. Shoda's result was generalized in the fifties by Albert and Muckenhoupt [2] for matrices over arbitrary fields.

The result obtained by Shoda, Albert and Muckenhoupt can be equivalently read as follows: the image of the polynomial $xy - yx$ on the full matrix algebra is a vector space. Now a question that appears is whether the image of an arbitrary polynomial on the full matrix algebra is always a vector space. As one might already expect, this is not the case in general. For instance, for an integer $n \geqslant 2$, the image of the polynomial $f(x) = x^n$ on $M_n(F)$ is not a vector space (see [52, Example 4]). It is not clear why the image of the commutator on $M_n(F)$ is a vector space while the image of $f(x) = x^n$ on the same algebra is not. However, it is worth noticing that the commutator satisfies an additional property that the latter does not. The commutator is what we call a multilinear polynomial. We recall here that a polynomial in the free associative algebra is said multilinear if each one of its variables appears exactly once in every monomial of this polynomial. So, what to expect if we consider images of multilinear polynomials on the full matrix algebra instead? This question goes in the direction of the following one posed by L'vov in [30, Problem 1.98].

Question. (L'vov) Is the image of a multilinear polynomial on the full matrix algebra a vector space?

The difficulty of studying the question above lies in the lack of structure of the image of a multilinear polynomial on some algebra. On the other hand the linear span of such images have a nice behaviour. In fact, they are Lie ideals of the algebra. In the particular case of the full matrix algebra $M_n(F)$, its Lie ideals are well known (up to some mild conditions on the ground field F). They are precisely $\{0\}$, F (scalar matrices), $sl_n(F)$ and $M_n(F)$ itself, according to an old result due to Herstein [36]. Thus, in case L'vov's Question has a positive answer, then a full description of the images of multilinear polynomials on $M_n(F)$ will be also obtained. The problem of obtaining a full description of multilinear polynomials on matrices is attributed to Kaplansky (see [38]). As a resumé of all these discussion we have the following conjecture.

Conjecture. (L'vov-Kaplansky) The image of a multilinear polynomial on $M_n(F)$ is $\{0\}, F, sl_n(F)$ or $M_n(F)$. Equivalently, the image of a multilinear polynomial on $M_n(F)$ is a vector space.

Little is known concerning the above conjecture. The image of a multilinear polynomial of degree 2 on $M_n(F)$ is an easy consequence from the results of Shoda, Albert and Muckenhoupt. In fact, the image must be $\{0\}, sl_n(F)$ or $M_n(F)$. In 2012, Kanel-Belov, Malev and Rowen [38] showed that the L'vov-Kaplasnky conjecture is true for 2×2 matrices with entries in a quadratically closed field. The conjecture remains open besides these two results. However partial results were obtained in the case of polynomials of degree 3 [22] and also in the case of 3×3 matrices [39].

The L'vov-Kaplansky conjecture has shown to be a truly challenging problem, and in attempting to approach it several variations of it have appeared. We firstly comment about a weaker version of the L'vov-Kaplansky conjecture, the so called Mesyan conjecture [52]. The latter states that the image of a multilinear polynomial of a fixed degree m on $M_n(F)$ always contains $sl_n(F)$, provided n is large enough. Mesyan's conjecture is true for poylnomials of degree 3 [52] and also for polynomials of degree 4 [16, 26], and it remains open for polynomials of degree 5 or more. Subalgebras of $M_n(F)$ were also considered. In [23], Fagundes considered the subalgebra of strictly upper triangular matrices J, and proved that the image of a multilinear polynomial of degree m on J^k is J^{km}. A similar problem was also posed for upper triangular matrices $UT_n(F)$, where partial results were obtained in [25] by Fagundes and de Mello, and simultaneously fully solved in [31] by Gargate and de Mello in case the ground field is infinite, and in [48] by Luo and Wang for fields with at least $n(n-1)/2$ elements. The infinite-dimensional case was settled by Vitas in [63], where the author proved that non-zero multilinear polynomials are surjective on algebras with surjective inner derivations. In particular, every non-zero multilinear polynomial is surjective on the algebra $End(V)$ of endomorphisms of an infinite-dimensional vector space V.

The nonassociative cases were also explored. Here we point out the work [51] where Malev, Yavich and Shayer obtained a full description of the images of multilinear

Jordan polynomials on the Jordan algebra given by a particular nondegenerate symmetric bilinear form. Their result covers the cases of images of multilinear Jordan polynomials on the 2×2 self-adjoint matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions). In the Lie case, we have the paper [40] where Kanel-Belov, Malev and Rowen investigated the images of arbitrary Lie polynomials (not necessarily multilinear) on the Lie algebra $sl_2(F)$, while in [64] Špenko showed that non-zero multilinear Lie polynomials of degree up to four have images equal to $sl_n(F)$ on the Lie algebra $gl_n(F)$ (excluded the trivial case of polynomial of degree 1, of course).

In the same way that gradings on algebras showed to be an important tool for the theory of algebras with polynomial identities (PI-algebras), one might also hope for the same in the case of images of polynomials on algebras. This motivates the study of such images on algebras with additional structure, such as group gradings or involutions, for instance. Besides a first result concerning images of graded polynomials on algebras that was given in 2000 by Kulyamin [47], just recently this topic has been more explored. We draw the readers' attention to the recent paper [17] from 2023 where Centrone and de Mello considered images of multilinear graded polynomials on the full matrix algebra $M_n(F)$ endowed with the natural $\mathbb{Z}_n$-grading (Vasilovsky's grading). The authors classified the linear span of the image of a multilinear graded polynomial on $M_n(F)$ with the aforementioned grading, and conjectured that the same subspaces obtained are actually the possible images that a multilinear graded polynomial can take on this algebra. They also proved that their conjecture is true in the small (but not trivial) cases of polynomials of degree 2 and also for 2×2 matrices. Also in 2023, Gargate and de Mello [32] gave an equivalent form of the L'vov-Kaplansky conjecture in terms of images of multilinear (nongraded) polynomials on the algebra $M_n(F)$ endowed with the Vasilovsky's grading. In particular, their result shows us the importance of studying images of graded polynomials on graded algebras. The involutive case was explored by Franca and Urure in the papers [60] and [61]. They considered the upper triangular matrix algebra $UT_n(F)$ with the reflexive involution, and classified the images of multi-homogeneous Lie polynomials on the skew-symmetric part of $UT_n(F)$ for $n \leqslant 4$, and of multilinear Jordan polynomials of degree up to 3 on the symmetric part of $UT_n(F)$.

We now turn out attention to the particular case where the image of a polynomial on some algebra $\mathcal{A}$ is $\{0\}$. As we already mentioned such polynomials are called polynomial identities for the algebra $\mathcal{A}$. The study of PI theory is commonly divided in three parts that we stress in the following. The first part concerns the description of all polynomial identities of a given algebra, the second one is about studying the variety of algebras defined by a given set of identities, while the last one is concerned in deciding if a given algebra satisfies some polynomial identity or not. Recently most of the work done in PI theory is in the direction of the two first parts. Concerning the third part, it is well known that finite-dimensional algebras are PI. Moreover, we are also able to construct new PI algebras from old ones. Indeed, homomorphic images of PI algebras are also PI, direct product of PI algebras satisfying a common identity is again PI, and subalgebras of PI algebras are PI algebras. This is actually part of the contents of Birkhoff's theorem which states that any variety can be described as a class of algebras closed under taking subalgebras, homomorphic images and direct products (for instance, see [21]). Besides those results, it is not an easy task to construct PI algebras, or find sufficient conditions on algebras so that these become PI algebras. A celebrated result in this direction is the one given by Regev [53] where he uses combinatorial tools to establish that the tensor product of PI algebras is also PI.

As we have commented, direct product of PI algebras satisfying a common identity is again a PI algebra. This direct product coincides with the external direct sum of PI algebras in case we have finitly many algebras. However, as one might expect, a quite different situation is settled when we consider internal (direct) sums, that is, algebras $\mathcal{A} = \mathcal{B} + \mathcal{C}$ such that both $\mathcal{B}$ and $\mathcal{C}$ are PI subalgebras of $\mathcal{A}$. Then the natural question that arises is whether $\mathcal{A}$ satisfies some polynomial identity or not. This question was posed by Beidar and Mikhalev [9] in 1995. In this same paper the authors proved that sums of rings satisfying product of commutators as identities is also a PI ring. Their proof is far from being trivial, using structure results from the theory of associative rings and also the so called Amitsur's method (see the book [55]), which is another result in the direction of existence of PI algebras. Besides the partial result given by Beidar and Mikhalev on their own question, we can not consider this as a first result in the direction of the aforementioned problem. In fact, in 1963 Kegel [44] showed that the sum of two nilpotent rings is still nilpotent. Actually, Beidar and Mikhalev's question is a particular case of a problem posed by Szép [58, 59] in the late fifties concerning which properties one may obtain on a ring $\mathcal{A} = \mathcal{B} + \mathcal{C}$ provided one has some information about the subrings $\mathcal{B}$ and $\mathcal{C}$.

In 1996, Kępczyk and Puczylowski initiated a sequence of papers concerning the study of radicals of rings which are sums of two subrings using ideas that go back to [3]. However some of their results were also related to Beidar and Mikhalev's question. We mention here the paper [43] where Kępczyk and Puczylowski obtained important results concerning the problem of sums of PI rings. For instance, an important theorem obtained by the authors in [43] was that a class of rings which is closed under homomorphic images, direct powers and such that every prime ring in the class is GPI (satisfies some generalized polynomial identity) is actually a class of PI rings. This result is of fundamental importance in the positive solution of Beidar and Mikhalev's question given by Kępczyk [41] in 2016. Besides Kępczyk's theorem being interesting by itself, it is worth noticing that it also has connections with major problems in Ring Theory. For instance, we mention here the paper [42] where Kępczyk used his positive answer to Beidar and Mikhalev's problem to obtain a new equivalent condition to Koethe's conjecture (which states, in one among several of its equivalent forms, that the sum of left nil ideals is still nil in any ring).

The third and last topic of this thesis, but not less important, is concerning a class of algebras called zero product determined algebras (which we briefly call zpd algebras). These algebras first appeared in the works of Brešar and Šemrl [15] and of Alaminos, Brešar, Extremera and Villena [1]. In the first paper the authors described commutativity preserving linear maps on finite-dimensional central simple algebras, while in the second one it was studied derivations and some related maps on C^*-algebras. By the way how these algebras appeared one might expect their usefulness for applications. In fact, several results on linear preserving maps and characterization of derivations on some algebras have being obtained by using zpd algebras (see Chapter 3 of [13] for instance).

We recall that an algebra $\mathcal{A}$ is said zero product determined if for every bilinear functional φ on $\mathcal{A}$ such that $\varphi(x, y) = 0$ whenever $xy = 0$, then $\varphi(x, y) = \tau(xy)$ for all $x, y \in \mathcal{A}$, where τ is a linear functional on $\mathcal{A}$ which depends on φ. Algebras generated by idempotents are one of the examples of such algebras. In particular, the full matrix algebra is zpd. On the other hand unital domains are not zpd algebras, except the case where the domain is one-dimensional. A surprising result in the theory of zpd algebras is the main theorem from [14] where Brešar showed that in the finite-dimensional unital associative setting, zpd algebras are precisely those generated by idempotents. However, it is still open the problem of existence of an infinite-dimensional unital zpd algebra which is not generated by idempotents. We recommend the book [13] for further details concerning this class of algebras.

In this thesis, we present results on the three topics previously discussed: description of images of polynomials on algebras, algebras with polynomial identities and zpd algebras. Let us briefly discuss these results and how they are organized in the present text.

In the first chapter of this thesis we set the main definitions and results that will be used in the whole text. The reader familiar with the basic notions of images of polynomials on graded algebras might skip the first chapter, perhaps using it just for consulting notations.

The second chapter is devoted to studying graded algebras which are sums of two homogeneous subalgebras. In Section 2.1 we show that an analogue of Kępczyk's theorem can not be expected in the group-graded setting. In the rest of the chapter we give sufficient conditions on a graded algebra $\mathcal{A} = \mathcal{B} + \mathcal{C}$ such that $\mathcal{A}$ satisfies some graded polynomial identity. Hence in Section 2.2 we show that if $\mathcal{B}$ and $\mathcal{C}$ are gr-PI (satisfy some graded identities) and $\mathcal{B}$ is also a two-sided ideal, then $\mathcal{A}$ is also gr-PI. In the last section of this chapter (Section 2.3), we introduce the notion of graded semi-identities for $\mathcal{A} = \mathcal{B} + \mathcal{C}$ and we give sufficient conditions on such graded semi-identities in order to obtain graded identities for $\mathcal{A}$. The results from Chapter 2 are original and were published in [28].

In the third chapter we discuss images of multilinear polynomials on upper triangular matrices with several additional structures. The Section 3.1 is only devoted to present a quick review on gradings on $UT_n(F)$ (the upper triangular matrix algebra with entries in a field F). Section 3.2 shows the non existence of graded central polynomials for $UT_n(F)$ with an arbitrary grading. In Section 3.3, we prove that the image of a multilinear graded polynomial in neutral variables on the neutral component of $UT_n(F)$ is always a homogeneous vector space, regardless the grading. In Section 3.4 we study the images of multilinear graded polynomials on $UT_n(F)$ with certain gradings given by the group $\mathbb{Z}_q$. Under some mild conditions on the ground field F we obtain a precise description of such images, in particular they are always homogeneous vector spaces. In Section 3.5 we apply some ideas from the previous section to obtain a sufficient condition for the image of an ordinary multilinear polynomial on the full matrix algebra to contain the subspace of traceless matrices. Such condition relies on graded polynomials obtained from the ordinary one. Section 3.6 is concerned with the study of images of multilinear graded polynomials on upper triangular matrix algebras of small dimension. In Subsection 3.6.1 we show that such images on $UT_2(F)$ and $UT_3(F)$ are always homogeneous subspaces. An analogous result is proved in Subsection 3.6.2 where we considered the graded Jordan algebra $UJ_2(F)$ of 2×2 upper triangular matrices. In Subsection 3.6.3 we have obtained a classification of multilinear graded Jordan polynomials on the Jordan algebra $UJ_3(F)$ endowed with the elementary natural $\mathbb{Z}_3$-grading. The last subsection (Section 3.6.4) deals with the graded involutive case on $UT_2(F)$ and $UT_3(F)$. During this subsection we prove that the images of multilinear graded polynomials with involution on $UT_2(F)$ are always a homogeneous subspace, and the same result was obtained for $UT_3(F)$ provided the grading on this algebra is not trivial. Besides those results, we show that an analogue of L'vov-Kaplansky conjecture can not be expected in the graded involutive case. We prove that there exists a multilinear (graded) polynomial with involution so that its image on $UT_n(F), n \geqslant 3$, $(UT_n(F), n \geqslant 4)$ is not a vector space. The results from this chapter are new. The ones from Subsection 3.2 to Subsection 3.6.3 are published in [27], while the results from Subsection 3.6.4 are submitted for publication [24].

In the last chapter we present a generalization of zpd algebras, that we call f-zpd algebras (f an associative multilinear polynomial). We introduce this concept in Section 4.1. In Section 4.2 we are mostly interested in investigating whether the full matrix algebra $M_n(F)$ is f-zpd. While in Subsection 4.2.1 we prove that this is not always the case, in Subsections 4.2.2, 4.2.3 we present some multilinear polynomials f so that $M_n(F)$ is f-zpd. In Subsection 4.2.4 we show how one can construct an f-zpd algebra from given ones using composition of polynomials. The last section (Section 4.3) is devoted to present a related problem to f-zpd algebras. The problem goes in the direction of a multilinear version of an old result due to Amitsur [4]. We show that if a multilinear polynomial g of degree m preserves the zeros of a multilinear polynomial f of degree m on the full matrix algebra $M_n(F)$, then g and f are linearly dependent, provided $m < 2n - 3$. The results from this last chapter are new and are submitted for publication [8].

1 Preliminaries

The main goal of this chapter is to establish the main definitions, notations and basic results from the theory of algebras used during all this thesis. The reader already familiar with the notions presented in this chapter may skip it without further problems in following the thesis.

1.1 Graded algebras

A nonassociative algebra will mean a not necessarily associative algebra. The word "algebra" in this thesis will stand for an associative algebra.

Definition 1.1. *Let $\mathcal{V}$ be a vector space over a field F and let G be a group. We define a G-grading on $\mathcal{V}$ as a decomposition*

$$\Gamma\colon \mathcal{V} = \bigoplus_{g\in G} \mathcal{V}_g$$

into a direct sum of subspaces $\mathcal{V}_g$. In case $\mathcal{A}$ is a nonassociative algebra over F, a G-grading on $\mathcal{A}$ is defined as a G-grading on the vector space $\mathcal{A}$ over F satisfying the additional condition

$$\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{gh},$$

for all $g, h \in G$.

Definition 1.2. *The subspaces $\mathcal{V}_g$ are called homogeneous components of the grading* Γ *on $\mathcal{V}$. A non-zero element $v \in \mathcal{V}_g$ is said to have homogeneous degree g, and we denote it by* $\deg(v) = g$.

Definition 1.3. *A subspace $\mathcal{U}$ of a G-graded space $\mathcal{V}$ is called homogeneous if*

$$\mathcal{U} = \bigoplus_{g\in G}(\mathcal{U} \cap \mathcal{V}_g).$$

Definition 1.4. *We define the support of a G-grading* Γ *on the vector space $\mathcal{V}$ as the subset*

$$\operatorname{supp}(\Gamma) = \{g \in G | \mathcal{V}_g \neq 0\}.$$

Definition 1.5. *Given a G-graded algebra $\mathcal{A}$ and a homogeneous ideal $\mathcal{I}$ of $\mathcal{A}$, we define a G-grading on the quotient $\mathcal{A}/\mathcal{I}$ by setting $(\mathcal{A}/\mathcal{I})_g = \{a + \mathcal{I} | a \in \mathcal{A}_g\}$.*

We will use the multiplicative notation for the neutral element 1 of a group. Obviously the homogeneous component $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}$, and it is also called the *neutral component* of the graded algebra $\mathcal{A}$. The following result is folklore.

Proposition 1.6. *Let $\mathcal{A}$ be a G-graded unital nonassociative algebra. Then its unit $1 \in \mathcal{A}$ lies in the neutral component.*

Definition 1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two G-graded nonassociative algebras. A homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ is called a graded homomorphism if $\varphi(\mathcal{A}_g) \subset \mathcal{B}_g$ for all $g \in G$. In case φ is an isomorphism one can see that we actually have $\varphi(\mathcal{A}_g) = \varphi(\mathcal{B}_g)$, and in this case we say that $\mathcal{A}$ and $\mathcal{B}$ are graded isomorphic algebras.*

Besides the associative algebras, other two classes of algebras which will be also explored on this thesis are the Jordan and Lie algebras. For that reason, let us recall their definitions and a few examples.

Definition 1.8. *Let $\mathcal{J}$ be a nonassociative commutative algebra. We say that $\mathcal{J}$ is a Jordan algebra if $(a^2b)a = a^2(ba)$ for all $a, b \in \mathcal{J}$.*

We can always obtain a Jordan structure from an associative algebra. Indeed, given an algebra $\mathcal{A}$ over a field of characteristic different from 2, one just need to consider the product $a \circ b = (ab + ba)/2$ for all $a, b \in \mathcal{A}$. The Jordan algebra obtained in this way is denoted by $\mathcal{A}^{(+)}$.

Remark 1.9. *Given $a, b, c \in \mathcal{J}$ (a Jordan algebra), we denote $(a, b, c) = (ab)c - a(bc)$ (the associator of the elements a, b, c in this order).*

Definition 1.10. *Let $\mathcal{L}$ be a nonassociative algebra. We say that $\mathcal{L}$ is a Lie algebra if $a^2 = 0$ (which implies anti-commutativity in case $\mathcal{L}$ is an algebra over a field of characteristic different from 2) and $(ab)c + (ca)b + (bc)a = 0$ (Jacobi identity) for all $a, b, c \in \mathcal{L}$.*

As in the Jordan case, one can also obtain Lie structures from associative ones. Let $\mathcal{A}$ be an algebra and consider the product $[a, b] = ab - ba$ for all $a, b \in \mathcal{A}$. This defines a Lie algebra which will be denoted by $\mathcal{A}^{(-)}$. Every Lie algebra can be realized as a subalgebra of some $\mathcal{A}^{(-)}$, where $\mathcal{A}$ is an associative algebra; this is the content of the well known Poincaré-Birkhoff-Witt theorem. The analogous statement for Jordan algebras fails, there exist Jordan algebras that are not isomorphic to subalgebras of any $\mathcal{A}^{(+)}$ given from an associative algebra $\mathcal{A}$. Such Jordan algebras are called exceptional; on the contrary $\mathcal{J}$ is special. We recall here that the notion of a Jordan algebra can be extended to the case of base field of characteristic 2, these are the so-called quadratic Jordan algebras. Since we shall not need this notion we will not discuss it further.

Remark 1.11. *In both Jordan and Lie cases, we use the left-normed orientation for product of three or more elements if there are no parentheses (or brackets) in the product.*

An example of group graded algebra that will be exhaustively used in this thesis is the free G-graded algebra (in different classes of algebras). For that reason let us introduce this algebra in the following examples.

Example 1.12. *Let G be a group and let $X_G = \{x_i^{(g)} | g \in G, i = 1, 2, \dots\}$ be a set of noncommutative and nonassociative variables. Let $F\{X_G\}$ be the free G-graded nonassociative algebra. This means it is the vector space with basis consisting of all monomials (with all possible valid dispositions of parentheses) on X_G. The degree of a monomial is the product (in G) of the degrees of its variables. The free associative G-graded algebra is defined as $F\langle X_G\rangle = F\{X_G\}/K$ where K is the T-ideal generated by $(x_i^{(g_i)} x_j^{(g_j)}) x_k^{(g_k)} - x_i^{(g_i)} (x_j^{(g_j)} x_k^{(g_k)})$. Notice that $F\langle X_G\rangle$ is G-graded since K is a homogeneous ideal of $F\{X_G\}$. The elements from $F\langle X_G\rangle$ are called G-graded polynomials.*

Example 1.13. *Let G be a group and let $X_G = \{x_i^{(g)} | g \in G, i = 1, 2, \dots\}$ be a set of noncommutative and nonassociative variables. Let $F\{X_G\}$ be the free G-graded nonassociative algebra.*

- *We denote by I the intersection of all homogeneous ideals of $F\{X_G\}$ containing the set $\{f_1^2, (f_1 f_2) f_3 + (f_2 f_3) f_1 + (f_3 f_1) f_2 | f_1, f_2, f_3 \in F\{X_G\}\}$;*
- *We denote by J the intersection of all homogeneous ideals of $F\{X_G\}$ containing the set $\{f_1 f_2 - f_2 f_1, (f_1 f_1)(f_2 f_1) - ((f_1 f_1) f_2) f_1 | f_1, f_2 \in F\{X_G\}\}$.*

Hence we have that

- *$\mathcal{L}(X_G) = F\{X_G\}/I$ is the free G-graded Lie algebra (and its elements are called G-graded Lie polynomials);*
- *$\mathcal{J}(X_G) = F\{X_G\}/J$ is the free G-graded Jordan algebra (and its elements are called G-graded Jordan polynomials).*

Remark 1.14. *Notice that we recover the ordinary setting in all definitions above by considering the trivial G-grading on $\mathcal{A}$, which is the one given by the trivial group $G = \{e\}$.*

Definition 1.15. *A polynomial $f = f(x_1^{(g_1)}, \dots, x_m^{(g_m)}) \in F\langle X_G\rangle$ (resp. $\mathcal{L}(X_G)$ / $\mathcal{J}(X_G)$) is multilinear if each variable $x_i^{(g_i)}$ appears in every monomial of f exactly once.*

Remark 1.16. *In particular one can notice that multilinear polynomials have the following form in the associative setting*

$$f = \sum_{\sigma \in S_m} \alpha_\sigma x_{\sigma(1)}^{(g_{\sigma(1)})} \cdots x_{\sigma(m)}^{(g_{\sigma(m)})}, \quad \alpha_\sigma \in F,$$

where S_m denotes the symmetric group of degree m.

Remark 1.17. *Given an m-tuple $(g_1, \dots, g_m) \in G^m$, the vector space of multilinear G-graded polynomials in m variables $x_1^{(g_1)}, \dots, x_m^{(g_m)}$ will be denoted by $P^{(g_1, \dots, g_m)}$. More generally, given a decomposition $m = m_1 + \cdots + m_k$ where $m_i \in \mathbb{N}$, the space of multilinear polynomials in m_i homogeneous variables of degree g_i is denoted by $P_{m_1, \dots, m_k}$.*

1.2 Images of graded polynomials on graded algebras

We start this subsection with the definition of image of a graded polynomial on some graded algebra.

Definition 1.18. *Let $f(x_1^{(g_1)}, \ldots, x_m^{(g_m)}) \in F\langle X_G \rangle$ (resp. $\mathcal{L}(X_G)/\mathcal{J}(X_G)$) and let $\mathcal{A}$ be a G-graded algebra (resp. Lie/Jordan algebra). We define the image of f on $\mathcal{A}$ (denoted by $f(\mathcal{A})$) as the image of the function*

$$\begin{array}{rccc} \tilde{f}\colon & \mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_m} & \to & \mathcal{A} \\ & (a_1, \ldots, a_m) & \mapsto & f(a_1, \ldots, a_m). \end{array}$$

Equivalently

$$f(\mathcal{A}) = \{f(a_1, \ldots, a_m) | a_i \in \mathcal{A}_{g_i}\}.$$

We observe that $\tilde{f}$ is a multilinear function whenever f is a multilinear polynomial.

In the next proposition we present some basic properties of images of graded polynomials on algebras. The proposition is written in the associative setting, however the same properties hold in the Lie and Jordan cases.

Proposition 1.19 ([27])**.** *Let $f \in F\langle X_G \rangle$ and let Γ be a G-grading on the algebra $\mathcal{A}$.*

1. *$f(\mathcal{A})$ is invariant under graded endomorphisms of $F\langle X_G \rangle$;*
2. *If $1 \in \mathcal{A}$ and $f \in F\langle X_G \rangle$ is multilinear in neutral variables and the sum of its coefficients is non-zero, then $f(\mathcal{A}) = \mathcal{A}_1$, the neutral component in the grading on $\mathcal{A}$;*
3. *Assume that there exists an one-dimensional subspace $\mathcal{V}$ of $\mathcal{A}$ such that $f(\mathcal{A}) \subset \mathcal{V}$ and assume that $\lambda f(\mathcal{A}) \subset f(\mathcal{A})$ for every $\lambda \in F$. Then either $f(\mathcal{A}) = \{0\}$ or $f(\mathcal{A}) = \mathcal{V}$;*
4. *If $\operatorname{supp}(\Gamma)$ is abelian and $f \in F\langle X_G \rangle$ is multilinear, then $f(\mathcal{A})$ is entirely contained in some homogeneous component.*

Proof. The proof of items 1, 3 and 4 are immediate. For the second item, we just need to recall that $1 \in \mathcal{A}_1$ (Proposition 1.6), and that

$$f(a, 1, \ldots, 1) = f(1, \ldots, 1)a \quad \text{for all} \quad a \in \mathcal{A}_1,$$

where $f(1, \ldots, 1)$ is non-zero by hypothesis. □

The following lemma is a useful tool when one is dealing with images of polynomials on associative algebras of small dimension.

Lemma 1.20 ([50]). *Let F be a field, and let $\mathcal{V}_1, \ldots, \mathcal{V}_m, \mathcal{V}$ be vector spaces over F. Assume that the image of a multilinear map $f : \prod_{i=1}^{m} \mathcal{V}_i \to \mathcal{V}$ contains two linearly independent vectors. Then the image of f contains a 2-dimensional vector subspace.*

As a direct corollary, we have the following

Corollary 1.21. *Assume that the image of a multilinear polynomial f on some algebra $\mathcal{A}$ is contained in some 2-dimensional vector space. Then $f(\mathcal{A})$ is a vector subspace.*

Proof. Let $f(\mathcal{A}) \subset \mathcal{V}$, where $\dim(\mathcal{V}) = 2$. If $f(\mathcal{A})$ contains two linearly independent vectors, by the previous lemma $f(\mathcal{A}) = \mathcal{V}$. Now assume that any two vectors in $f(\mathcal{A})$ are linearly dependent. Then $f(\mathcal{A}) \subset \mathcal{U}$, where $\mathcal{U}$ is an 1-dimensional subspace. Hence, $f(\mathcal{A}) = \{0\}$ or $f(\mathcal{A}) = \mathcal{U}$ by Proposition 1.19 (3). □

Another important property concerning images of graded polynomials on algebras is that the homogeneous subspace structure of the image is invariant under graded homomorphic images.

Proposition 1.22 ([27]). *Let G be a group and let $\mathcal{A}$ and $\mathcal{B}$ be two G-graded algebras such that $\mathcal{B}$ is a graded homomorphic image of $\mathcal{A}$. Let $f \in F\langle X_G \rangle$ be a graded polynomial and assume that $f(\mathcal{A})$ is a homogeneous subspace of $\mathcal{A}$. Then $f(\mathcal{B})$ is also a homogeneous subspace of $\mathcal{B}$.*

Proof. Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be a graded epimorphism. We start by noticing that

$$\phi(f(\mathcal{A})) = f(\phi(\mathcal{A})).$$

Then taking $b_i^{(1)}, b_i^{(2)} \in \mathcal{B}_{g_i}$, $i = 1, \ldots, m$, we have $b_i^{(j)} = \phi(a_i^{(j)})$ for some $a_i^{(j)} \in \mathcal{A}_{g_i}$, since ϕ is surjective. This leads us to

$$\begin{aligned}
&\alpha f(b_1^{(1)}, \ldots, b_m^{(1)}) + f(b_1^{(2)}, \ldots, b_m^{(2)}) \\
&= \alpha f(\phi(a_1^{(1)}), \ldots, \phi(a_m^{(1)})) + f(\phi(a_1^{(2)}), \ldots, \phi(a_m^{(2)})) \\
&= \phi(\alpha f(a_1^{(1)}, \ldots, a_m^{(1)}) + f(a_1^{(2)}, \ldots, a_m^{(2)}))
\end{aligned}$$

which is an element from $\phi(f(\mathcal{A})) = f(\phi(\mathcal{A}))$ since we are assuming that $f(\mathcal{A})$ is a subspace. Then $f(\mathcal{B})$ is also a subspace.

Now we assume that $f(\mathcal{A})$ is a homogeneous subspace and let $b = \bar{b}_{h_1} + \cdots + \bar{b}_{h_k}$ be an element in $f(\mathcal{B})$ written as the sum of its homogeneous components. Since $b \in f(\mathcal{B})$ let

$$b = f(b_1, \ldots, b_m) = f(\phi(a_1), \ldots, \phi(a_m)) = \phi(f(a_1, \ldots, a_m))$$

for some $b_i = \phi(a_i)$, and let

$$f(a_1, \ldots, a_m) = \overline{a}_{g_1} + \cdots + \overline{a}_{g_l}$$

be the sum of its homogeneous components. It follows that

$$b = \phi(\overline{a}_{g_1}) + \cdots + \phi(\overline{a}_{g_l})$$

and since ϕ is a graded homomorphism we must have $k = l$ and every g_t must be equal to some h_s. Without loss of generality, we assume $\overline{b}_{g_t} = \phi(\overline{a}_{g_t})$. Now it is enough to use that $f(\mathcal{A})$ is homogeneous and $\phi(f(\mathcal{A})) = f(\phi(\mathcal{A}))$. $\square$

Remark 1.23. *In the proof of Proposition 1.22 we have not used the associativity of $\mathcal{A}$. Therefore it also holds for arbitrary algebras, in particular an analogous proposition holds for graded Jordan algebras.*

An important problem concerning images of polynomials on algebras is the well-known L'vov-Kaplansky conjecture.

Conjecture 1.24 ([30])**.** *The image of $f \in P_m$ on the full matrix algebra $M_n(F)$ is a vector space.*

Equivalently, the image of a multilinear polynomial on $M_n(F)$ is one of the four subspaces: $\{0\}$, F (viewed as the space of scalar matrices), $sl_n(F)$ (the space of traceless matrices) or $M_n(F)$.

An important example of image of a graded polynomial on some graded algebra $\mathcal{A}$ is the one whose image is identically zero. Such polynomial is called a graded polynomial identity for the algebra $\mathcal{A}$.

Definition 1.25. *Let $f \in F\langle X_G\rangle$ (resp. $\mathcal{L}(X_G)/\mathcal{J}(X_G)$) and let $\mathcal{A}$ be a G-graded algebra (resp. Lie/Jordan algebra). We say that $f = 0$ is a graded polynomial identity for $\mathcal{A}$ if $f(\mathcal{A}) = 0$. Equivalently, $f = 0$ is a graded polynomial identity for $\mathcal{A}$ if*

$$f \in \bigcap \ker(\varphi),$$

where the intersection runs over all graded homomorphisms $\varphi : F\langle X_G\rangle \to \mathcal{A}$ (resp. φ with domain in $\mathcal{L}(X_G)/\mathcal{J}(X_G)$).

We say that a nonassociative algebra $\mathcal{A}$ is gr-PI if $\mathcal{A}$ satisfies some non-zero graded polynomial identity.

We denote by $Id_G(\mathcal{A})$ the two-sided ideal of all graded identities for $\mathcal{A}$. It turns out that $Id_G(\mathcal{A})$ is actually a T_G-ideal, that is, an ideal which is invariant under graded endomorphisms of the free G-graded associative (resp. Lie/Jordan) algebra.

Given a subset $S \subset F\langle X_G\rangle$ (resp. $\mathcal{L}(X_G)/\mathcal{J}(X_G)$), we denote by $\langle S\rangle^{T_G}$ the T_G-ideal generated by S, that is, the intersection of all T_G-ideals that contain S.

Definition 1.26. *Let $f \in F\langle X_G \rangle$ be a polynomial with zero constant term and let $\mathcal{A}$ be a G-graded algebra. We say that f is a graded central polynomial for $\mathcal{A}$ if $f = 0$ is not a graded identity for $\mathcal{A}$ and $f(\mathcal{A}) \subset Z(\mathcal{A})$, where $Z(\mathcal{A})$ denotes the center of $\mathcal{A}$.*

2 Sums of gr-PI algebras

In this chapter we will deal with the problem of when a graded algebra $\mathcal{A} = \mathcal{B}+\mathcal{C}$, graded by a group G, which is a sum of two homogeneous subalgebras $\mathcal{B}$ and $\mathcal{C}$ both satisfying graded polynomial identities, also does satisfy some graded polynomial identity. Whereas the problem has a negative answer in general (as we will see in the first section), in the second and third sections we present sufficient conditions on the algebra $\mathcal{A} = \mathcal{B} + \mathcal{C}$ in order to turn it into a gr-PI algebra. During this chapter, all algebras are over an arbitrary field F. The results from this chapter are new and were published in the journal Linear Algebra and its Applications [28]. This is a joint work with Plamen Koshlukov.

Before going into the main topic of this chapter, let us recall that when the algebra $\mathcal{A}$ is endowed with the trivial grading then sums of PI algebras is again PI. This result is due to M. Kępczyk.

Theorem 2.1 ([41]). *Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be an algebra that is a sum of two PI subalgebras. Then $\mathcal{A}$ is also a PI algebra.*

In an attempt to approach a graded version of Theorem 2.1, one may try to reduce the problem from the graded setting to the ordinary one. This can be done by looking for sufficient conditions on the graded identities on $\mathcal{B}$ and $\mathcal{C}$ which imply the existence of ordinary identities on these subalgebras. As an example of such sufficient condition we recall the following result from [7, 11].

Theorem 2.2. *Let G be a finite group and let $\mathcal{A}$ be a G-graded algebra. If the neutral component $\mathcal{A}_1$ is a PI-algebra, then $\mathcal{A}$ is a PI-algebra.*

As a direct consequence we have the following corollary.

Corollary 2.3. *Let G be a finite group and let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be an algebra that is the sum of two homogeneous subalgebras. If $\mathcal{B}$ and $\mathcal{C}$ satisfy graded polynomial identities in neutral variables, then $\mathcal{A}$ is a PI algebra.*

We give below a slight improvement of the corollary above. This improvement is concerning gradings on algebras by monoids, whose definition remains the same as in the group graded case. Let us start with the following lemma.

Lemma 2.4. *Let M be a monoid and let $\mathcal{V}$ be an M-graded vector space. Let $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ be a sum of two homogeneous subspaces. Then $\mathcal{V}_m = \mathcal{V}_{1,m} + \mathcal{V}_{2,m}$ for each $m \in M$, where $\mathcal{V}_{i,m}$ is the homogeneous component of degree m of $\mathcal{V}_i$, $i = 1, 2$.*

Proof. Let $m \in M$. Since $\mathcal{V}_{1,m} = \mathcal{V}_m \cap \mathcal{V}_1$ and $\mathcal{V}_{2,m} = \mathcal{V}_m \cap \mathcal{V}_2$ it follows that $\mathcal{V}_{1,m} + \mathcal{V}_{2,m} \subset \mathcal{V}_m$. Reciprocally let $v \in \mathcal{V}_m$. Hence $v = v_1 + v_2$ for some $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$. Writing

$$v_1 = v_{1,m_1} + \cdots + v_{1,m} + \cdots + v_{1,m_n} \quad \text{and} \quad v_2 = v_{2,m_1} + \cdots + v_{2,m} + \cdots + v_{2,m_n}$$

such that $v_{1,m_i} \in \mathcal{V}_{1,m_i}$, $v_{1,m} \in \mathcal{V}_{1,m}$, $v_{2,m_i} \in \mathcal{V}_{2,m_i}$, $v_{2,m} \in \mathcal{V}_{2,m}$. We can assume without loss of generality that $m_1, \ldots, m, \ldots, m_n$ are pairwise distinct. Therefore

$$0 = v - v_1 - v_2 = (v - v_{1,m} - v_{2,m}) - (v_{1,m_1} + v_{2,m_1}) - \cdots - (v_{1,m_n} + v_{2,m_n}).$$

This implies $v = v_{1,m} + v_{2,m}$ and hence we have $\mathcal{V}_m = \mathcal{V}_{1,m} + \mathcal{V}_{2,m}$. □

Let $\mathcal{A}$ be an algebra graded by a monoid M. Assume that $\mathcal{A} = \mathcal{B} + \mathcal{C}$ is a sum of two homogeneous subalgebras. Then Lemma 4.38 gives us that $\mathcal{A}_1 = \mathcal{B}_1 + \mathcal{C}_1$. Since $\mathcal{B}_1$ and $\mathcal{C}_1$ are subalgebras of $\mathcal{A}_1$, Theorems 2.1 and 2.2 lead us to the following result.

Theorem 2.5. *Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be an algebra graded by a monoid M such that $\mathcal{B}$ and $\mathcal{C}$ are two homogeneous subalgebras. If $\mathcal{B}$ and $\mathcal{C}$ satisfy polynomial identities in neutral variables, then the same holds for the algebra $\mathcal{A}$.*

2.1 Sums of gr-PI subalgebras do not always satisfy graded identities

The main goal of this section is to show that there exist a group G and a G-graded algebra $\mathcal{A} = \mathcal{B} + \mathcal{C}$ sum of two homogeneous subalgebras such that both $\mathcal{B}$ and $\mathcal{C}$ are gr-PI algebras but $\mathcal{A}$ is not.

For simplicity let us denote by $\mathcal{F}$ the free associative algebra $F\langle X \rangle$.

Consider the F-algebra $\mathcal{A} = M_2(\mathcal{F})$ and a $\mathbb{Z}_2$-grading on it given by $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ where

$$\mathcal{A}_{\bar{0}} = \begin{pmatrix} \mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix} \quad \text{and} \quad \mathcal{A}_{\bar{1}} = \begin{pmatrix} 0 & \mathcal{F} \\ \mathcal{F} & 0 \end{pmatrix}.$$

We define now two subalgebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ by setting

$$\mathcal{B} = \begin{pmatrix} \mathcal{F} & \mathcal{F} \\ 0 & \mathcal{F} \end{pmatrix} \quad \text{and} \quad \mathcal{C} = \begin{pmatrix} \mathcal{F} & 0 \\ \mathcal{F} & \mathcal{F} \end{pmatrix}.$$

Lemma 2.6. *The subalgebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ are homogeneous in the grading, $\mathcal{A} = \mathcal{B} + \mathcal{C}$, and both $\mathcal{B}$ and $\mathcal{C}$ satisfy the graded identity $x_1^{(\bar{1})} x_2^{(\bar{1})} = 0$.*

Proof. All statements are immediate. □

Now it is enough to prove that $\mathcal{A}$ satisfies no $\mathbb{Z}_2$-graded identities.

Proposition 2.7. *The $\mathbb{Z}_2$-graded algebra $\mathcal{A}$ satisfies no graded identities.*

Proof. Suppose, on the contrary, that $\mathcal{A}$ does satisfy a non-zero graded identity f. Without loss of generality we can assume f multilinear

$$f = f(x_1^{(\bar{0})}, \ldots, x_m^{(\bar{0})}, x_1^{(\bar{1})}, \ldots, x_k^{(\bar{1})}).$$

We shall use an argument with generic matrices. We take the elements

$$a_i = \begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix} \quad \text{and} \quad b_i = \begin{pmatrix} 0 & w_i \\ t_i & 0 \end{pmatrix}, \qquad i \geqslant 1$$

in $\mathcal{A}$ where u_i, v_i, w_i, t_i are distinct variables in the set X. These generic elements multiply as follows:

$$a_i a_j = \begin{pmatrix} u_i u_j & 0 \\ 0 & v_i v_j \end{pmatrix}, \qquad a_i b_j = \begin{pmatrix} 0 & u_i w_j \\ v_i t_j & 0 \end{pmatrix},$$
$$b_i b_j = \begin{pmatrix} w_i t_j & 0 \\ 0 & t_i w_j \end{pmatrix}, \qquad b_j a_i = \begin{pmatrix} 0 & w_j v_i \\ t_j u_i & 0 \end{pmatrix}.$$

Take a monomial $\boldsymbol{m} = \boldsymbol{m}_1 x_{j_1}^{(\bar{1})} \boldsymbol{m}_2 x_{j_2}^{(\bar{1})} \cdots x_{j_k}^{(\bar{1})} \boldsymbol{m}_{k+1}$ where $(j_1, \ldots, j_k)$ is a permutation of $(1, \ldots, k)$, and the $\boldsymbol{m}_i$ are monomials that do not contain variables of homogeneous degree $\bar{1}$ (some of the $\boldsymbol{m}_i$ may be empty). Suppose that $\boldsymbol{m}$ participates in f with non-zero coefficient. As f is a graded identity for $\mathcal{A}$ then $f(a_1, \ldots, a_m, b_1, \ldots, b_k) = 0$ in $\mathcal{A}$.

Let us evaluate $\boldsymbol{m}$ on a_1, $\ldots$, a_m, and b_1, $\ldots$, b_k. Depending on the parity of k we get a matrix in either $\mathcal{A}_0$ or $\mathcal{A}_1$. Assume k even. Then at position $(1,1)$ of the resulting matrix we will have an entry of the type

$$\alpha_1 w_{j_1} \beta_2 t_{j_2} \alpha_3 w_{j_3} \beta_4 t_{j_4} \cdots$$

where α_1 is the product of the entries u_i at position $(1,1)$ of the matrices a_i that appear in the substitution for $\boldsymbol{m}_1$, in their respective order. Similarly β_2 is the product of the entries v_i that come from the matrices in the second block, $\boldsymbol{m}_2$, and so on, alternating the $(1,1)$ and $(2,2)$ entries of these blocks consecutively.

Since we obtain a monomial in the free associative algebra $F\langle X\rangle$, it must cancel out with the monomial coming from some other term of f. But our monomial comes from only one term of f, namely from $\boldsymbol{m}$. Thus the resulting monomial in $F\langle X\rangle$ cannot cancel out, and this proves that the coefficient of $\boldsymbol{m}$ must be 0. Hence if k is even we are done. When k is odd the argument is analogous, and we omit it. □

Remark 2.8. *One could expect a positive answer by requiring $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$ instead of $\mathcal{A} = \mathcal{B} + \mathcal{C}$. However, even in the direct sum case we have the following example: we consider the $\mathbb{Z}_2$-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ as in the previous example, and we take*

$$\mathcal{B} = \begin{pmatrix} \mathcal{F} & \mathcal{F} \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \mathcal{C} = \begin{pmatrix} 0 & 0 \\ \mathcal{F} & \mathcal{F} \end{pmatrix}.$$

Clearly $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$*, and both* $\mathcal{B}$ *and* $\mathcal{C}$ *are homogeneous subalgebras satisfying* $x_1^{(\bar{1})} x_2^{(\bar{1})} = 0$*.*

Remark 2.9. *The same two examples given above transfer to the graded Lie case. Indeed, let* $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ *as before, and consider the Lie algebra* $\mathcal{A}^{(-)}$*. Concerning the first example, we write* $\mathcal{A}^{(-)} = \mathcal{B}^{(-)} + \mathcal{C}^{(-)}$*, and one can easily see that* $\mathcal{B}^{(-)}$ *and* $\mathcal{C}^{(-)}$ *are homogeneous Lie subalgebras of* $\mathcal{A}^{(-)}$*, both satisfying* $[x_1^{(1)}, x_2^{(1)}] = 0$*. It remains to prove that* $\mathcal{A}^{(-)}$ *does not satisfy* $\mathbb{Z}_2$*-graded Lie identities. Assume the contrary, and consider the embedding of the free* $\mathbb{Z}_2$*-graded Lie algebra into* $F\langle X_{\mathbb{Z}_2}\rangle^{(-)}$*. Hence, we would have a* $\mathbb{Z}_2$*-graded (associative) identity for* $\mathcal{A}$*, which can not occur in light of Proposition 2.7. The example from Remark 2.8 is treated analogously.*

2.2 The sum of a gr-PI ideal and a gr-PI subalgebra

In this section we study graded algebras which are a sum of a homogeneous ideal and a homogeneous subalgebra, both of them satisfying graded polynomial identities. Our goal in this section is to show that the sum itself is also gr-PI.

Throughout this section G will denote a finite group.

We recall that the operations on some direct power $\prod \mathcal{A}$ of $\mathcal{A}$ are given by the operations on $\mathcal{A}$ component-wise. We can extend the G-grading of $\mathcal{A}$ to the algebra $\prod \mathcal{A}$ as follows:

$$\prod \mathcal{A} = \bigoplus_{g \in G} \mathbf{A}_g \text{ such that } \mathbf{A}_g = \prod \mathcal{A}_g$$

where $\mathcal{A}_g$ is the homogeneous component of degree g of $\mathcal{A}$.

Let us start with the following easy lemma.

Lemma 2.10. *If* $\mathcal{A}$ *is a gr-PI algebra, then* $\prod \mathcal{A}$ *is also a gr-PI algebra.*

Proof. It is enough to notice that if $\mathcal{A}$ satisfies $f = 0$, then $\prod \mathcal{A}$ also does. □

Lemma 2.11. *Let* $\mathcal{A}$ *be a* G*-graded algebra which is not gr-PI. Then some direct power of* $\mathcal{A}$ *contains a homogeneous subalgebra that is graded isomorphic to some free* G*-graded associative algebra.*

Proof. Take $\mathcal{H} = F\langle X_G\rangle \backslash \{0\}$. By hypothesis, for each $h \in \mathcal{H}$ we have $h \notin Id_G(\mathcal{A})$. Hence there exists some graded homomorphism $\phi_h \colon F\langle X_G\rangle \to \mathcal{A}$ such that $\phi_h(h) \neq 0$. We define $\psi \colon F\langle X_G\rangle \to \prod_{h \in \mathcal{H}} \mathcal{A}$ by $\psi(f) = (\phi_h(f))_h$. It is easy to check that ψ is a graded homomorphism. Moreover, if $h \in \mathcal{H}$ then $\phi_h(h) \neq 0$ and then $\psi(h) \neq 0$. We conclude that ψ is a graded embedding of $F\langle X_G\rangle$ into $\prod_{h \in \mathcal{H}} \mathcal{A}$. □

Lemma 2.12. *Let $F\langle X_G\rangle$ be the free graded associative algebra and let $\mathcal{L}$ be a homogeneous left ideal of $F\langle X_G\rangle$. Then $\mathcal{L}$ does not satisfy any graded polynomial identity.*

Proof. Assume that $\mathcal{L}$ satisfies some graded polynomial identity $f(x_1^{(g_1)}, \ldots, x_n^{(g_n)})$. Without loss of generality we can assume that f is multilinear of degree n. Let $w \in \mathcal{L}\backslash\{0\}$ be a homogeneous element, and let us say $\deg(w) = g$. For $i = 1, \ldots, n$, we consider the variables $x_i^{(h_i)}$ where $h_i = g_i g^{-1}$. Hence $\deg(x_i^{(h_i)} w) = g_i$ and then $f(x_1^{(h_1)} w, \ldots, x_n^{(h_n)} w) = 0$.

On the other hand, we write f and w as follows:

$$f = m_1 + \cdots + m_k \quad \text{and} \quad w = w_1 + \cdots + w_l$$

as sums of their distinct non-zero monomials, respectively. Then

$$f(x_1^{(h_1)} w, \ldots, x_n^{(h_n)} w) = \sum_{i=1}^{k} \sum_{j_1,\ldots,j_n=1}^{l} m_i(x_1^{(h_1)} w_{j_1}, \ldots, x_n^{(h_n)} w_{j_n}). \tag{2.1}$$

We claim that $f(x_1^{(h_1)} w, \ldots, x_n^{(h_n)} w)$ is a non-zero polynomial and this will lead us to a contradiction. To this end, it is enough to prove that the monomials in (2.1) are pairwise distinct.

For a fixed index i, we must have

$$\boldsymbol{m_p} = m_i(x_1^{(h_1)} w_{p_1}, \ldots, x_n^{(h_n)} w_{p_n}) \neq m_i(x_1^{(h_1)} w_{q_1}, \ldots, x_n^{(h_n)} w_{q_n}) = \boldsymbol{m_q}$$

provided that the n-tuples $(p_1, \ldots, p_n) \neq (q_1, \ldots, q_n)$. Indeed, without loss of generality we may assume $p_1 \neq q_1$ and $m_1 = \alpha x_1^{(g_1)} \cdots x_n^{(g_n)}$ for some non-zero scalar $\alpha \in F$. Then we write

$$\boldsymbol{m_p} = \alpha x_1^{(g_1)} w_{p_1} w' \text{ and } \boldsymbol{m_q} = \alpha x_1^{(g_1)} w_{q_1} w''$$

and since $w_{p_1} \neq w_{q_1}$, then $\boldsymbol{m_p} \neq \boldsymbol{m_q}$.

It remains to analyse the monomials

$$\boldsymbol{m_p} = m_i(x_1^{(h_1)} w_{p_1}, \ldots, x_n^{(h_n)} w_{p_n}) \text{ and } \boldsymbol{m_q} = m_j(x_1^{(h_1)} w_{q_1}, \ldots, x_n^{(h_n)} w_{q_n})$$

for $i \neq j$. We start by rewriting the monomials $m_i(x_1^{(g_1)}, \ldots, x_n^{(g_n)}) = m' x_r^{(g_r)} m''$ and $m_j(x_1^{(g_1)}, \ldots, x_n^{(g_n)}) = m''' x_s^{(g_s)} m''''$, where $r \neq s$ and m', m'', m''', m'''' are suitable monomials. We notice that we may have $m' = m'''$. Therefore we suppose that the monomial $\boldsymbol{m_p}$ starts with $m'(x_1^{(h_1)} w_{p_1}, \ldots, x_n^{(h_n)} w_{p_n}) x_r^{(g_r)} w_{p_r}$ and $\boldsymbol{m_q}$ starts with $m'''(x_1^{(h_1)} w_{q_1}, \ldots, x_n^{(h_n)} w_{q_n}) x_s^{(g_s)} w_{q_s}$. Since $x_r^{(g_r)} \neq x_s^{(g_s)}$, then we must have $\boldsymbol{m_p} \neq \boldsymbol{m_q}$, even if $m'(x_1^{(h_1)} w_{p_1}, \ldots, x_n^{(h_n)} w_{p_n}) = m'''(x_1^{(h_1)} w_{q_1}, \ldots, x_n^{(h_n)} w_{q_n})$. □

Before stating the main theorem of this section, we recall that a graded algebra $\mathcal{A}$ is called gr-prime if the product of two non-zero homogeneous ideals of $\mathcal{A}$ is still non-zero.

Theorem 2.13. *Let $\mathfrak{A}$ be a class of G-graded algebras closed under graded homomorphic images and direct powers. Assume that every gr-prime algebra in $\mathfrak{A}$ has a non-zero homogeneous ideal satisfying some graded polynomial identity. Then $\mathfrak{A}$ is a class of gr-PI algebras.*

Proof. Assume that some algebra $\mathcal{A} \in \mathfrak{A}$ is not gr-PI. Then Lemma 2.11 gives us the existence of some homogeneous subalgebra $\mathcal{T}$ of $\prod \mathcal{A}$ which is graded isomorphic to $F\langle X_G \rangle$. By the Zorn Lemma, there exists a homogeneous ideal $\mathcal{I}$ of $\prod \mathcal{A}$ that is maximal with respect to the property $\mathcal{T} \cap \mathcal{I} = \{0\}$. Since $\mathfrak{A}$ is closed under homomorphic images and direct powers then $\overline{\mathcal{A}} = (\prod \mathcal{A})/\mathcal{I} \in \mathfrak{A}$. We claim that $\overline{\mathcal{A}}$ is a gr-prime algebra. In fact, let $\mathcal{J}_1/\mathcal{I}$ and $\mathcal{J}_2/\mathcal{I}$ be non-zero homogeneous ideals of $\overline{\mathcal{A}}$. Hence there exist $x \in \mathcal{T} \cap \mathcal{J}_1 \backslash \{0\}$ and $y \in \mathcal{T} \cap \mathcal{J}_2 \backslash \{0\}$, and then $0 \neq (x + \mathcal{I})(y + \mathcal{I}) \in (\mathcal{J}_1/\mathcal{I})(\mathcal{J}_2/\mathcal{I})$, which proves our claim. Therefore $\overline{\mathcal{A}}$ has a non-zero homogeneous ideal $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{I}$ satisfying a graded polynomial identity. Since $\overline{\mathcal{L}} \neq 0$, the maximality of $\mathcal{I}$ implies the existence of a non-zero homogeneous element $x \in \mathcal{L} \cap \mathcal{T}$, that is, $x \notin \mathcal{I}$. Recalling that

$$\overline{\mathcal{T}} = (\mathcal{T} + \mathcal{I})/\mathcal{I} \cong \mathcal{T}/(\mathcal{T} \cap \mathcal{I}) \cong \mathcal{T} \cong F\langle X_G \rangle,$$

we have that $\overline{\mathcal{T}}\overline{x} \subset \overline{\mathcal{L}}$. In other words, we have a non-zero homogeneous left ideal of $\overline{\mathcal{T}}$ which satisfies a graded polynomial identity. But this is an absurd by Lemma 2.12. □

Corollary 2.14. *Let $\mathcal{A}$ be a G-graded algebra such that $\mathcal{A} = \mathcal{B} + \mathcal{C}$ where $\mathcal{B}$ is a homogeneous ideal of $\mathcal{A}$ and $\mathcal{C}$ is a homogeneous subalgebra of $\mathcal{A}$. Moreover assume that $\mathcal{B}$ and $\mathcal{C}$ both satisfy graded polynomial identities. Then $\mathcal{A}$ is a gr-PI algebra.*

Proof. Consider the class $\mathfrak{A}$ of all G-graded algebras $\mathcal{A} = \mathcal{B} + \mathcal{C}$ where $\mathcal{B}$ is a homogeneous gr-PI ideal and $\mathcal{C}$ is a homogeneous gr-PI subalgebra. Notice that every non-zero algebra in $\mathfrak{A}$ contains some non-zero homogeneous gr-PI ideal, since in case $\mathcal{B} = \{0\}$, we have $\mathcal{C} = \mathcal{A}$ which is an ideal of $\mathcal{A}$. Moreover, one can see that $\mathfrak{A}$ contains the class of all G-graded algebras satisfying some graded polynomial identity and actually we aim to show that these two classes are the same.

Given $\mathcal{A} \in \mathfrak{A}$, notice that $\prod \mathcal{A} = \prod \mathcal{B} + \prod \mathcal{C}$, $\prod \mathcal{B}$ is a homogeneous ideal of $\prod \mathcal{A}$ and $\prod \mathcal{C}$ is a homogeneous subalgebra of $\prod \mathcal{A}$. Moreover, Lemma 2.10 gives us that both $\prod \mathcal{B}$ and $\prod \mathcal{C}$ satisfy graded polynomial identities provided $\mathcal{B}$ and $\mathcal{C}$ also do. Hence $\prod \mathcal{A} \in \mathfrak{A}$. Now let $\varphi \colon \mathcal{A} \to \mathcal{A}_2$ be a graded epimorphism where $\mathcal{A} = \mathcal{B} + \mathcal{C} \in \mathfrak{A}$. Then $\mathcal{A}_2 = \varphi(\mathcal{B}) + \varphi(\mathcal{C})$ where $\varphi(\mathcal{B})$ is a homogeneous gr-PI ideal of $\mathcal{A}$ and $\varphi(\mathcal{C})$ is a homogeneous gr-PI subalgebra of $\mathcal{A}$. Therefore we also have $\mathcal{A}_2 \in \mathfrak{A}$.

Since every non-zero algebra in $\mathfrak{A}$ contains some non-zero homogeneous gr-PI ideal, then we can apply Theorem 2.13 to get that $\mathfrak{A}$ is a class of gr-PI algebras. In particular, $\mathcal{A} \in \mathfrak{A}$ is a gr-PI algebra. □

We note that no additional information is given on the identity satisfied by $\mathcal{A}$. In the following we will prove that in case the group G is abelian, then we can obtain a concrete identity for the sum $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

Corollary 2.15. *Let $\mathcal{A}$ be a G-graded algebra such that G is an abelian group and $\mathcal{A} = \mathcal{B} + \mathcal{C}$, where $\mathcal{B}$ is a homogeneous ideal of $\mathcal{A}$ and $\mathcal{C}$ is a homogeneous subalgebra of $\mathcal{A}$. Moreover assume that $\mathcal{B}$ satisfies an identity $f(x_1^{(h_1)}, \dots, x_m^{(h_m)})$ and $\mathcal{C}$ satisfies and identity $g(x_1^{(g_1)}, \dots, x_n^{(g_n)})$. Then $\mathcal{A}$ is a gr-PI algebra and we can explicitly compute an identity for $\mathcal{A}$.*

Proof. We can assume without loss of generality that g is multilinear. Notice that $\mathcal{A}/\mathcal{B}$ is a homomorphic image of $\mathcal{C}/(\mathcal{B} \cap \mathcal{C})$. Since g is an identity for $\mathcal{C}$, then the same holds for $\mathcal{A}/\mathcal{B}$. Thus, $g(\mathcal{A}) \subset \mathcal{B}$. Since G is abelian and g is multilinear then we have that g is actually an homogeneous element in $F\langle X_G \rangle$, say of homogeneous degree h. Therefore the polynomial p_i given by the product of g with a variable of homogeneous degree $h^{-1}h_i$ is a polynomial of homogeneous degree h_i, for each $i = 1, \dots, m$. We can also assume that the variables occuring in all polynomials p_i are pairwise distinct. We conclude that $f(p_1, \dots, p_m)$ is a non-zero polynomial which is an identity for $\mathcal{A} = \mathcal{B} + \mathcal{C}$. $\square$

Corollary 2.16. *Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be a G-graded algebra, where $\mathcal{B}$ is a homogeneous ideal satisfying some ordinary polynomial identity $f(x_1, \dots, x_m)$ and $\mathcal{C}$ is a homogeneous subalgebra satisfying some graded identity $g(x_1^{(g_1)}, \dots, x_n^{(g_n)})$. Then the algebra $\mathcal{A}$ satisfies the following graded identity*

$$f(g(x_{11}^{(g_1)}, \dots, x_{1n}^{(g_n)}), \dots, g(x_{m1}^{(g_1)}, \dots, x_{mn}^{(g_n)})).$$

Proof. We evaluate each homogeneous variable of degree h by some element $a_h \in \mathcal{A}_h$. By Lemma 2.4, we can write $a_h = b_h + c_h$ where $b_h \in \mathcal{B}_h$ and $c_h \in \mathcal{C}_h$. Since $\mathcal{B}$ is an ideal of $\mathcal{A}$ we obtain that each evaluation of g on homogeneous elements of $\mathcal{A}$ is a sum of homogeneous elements of $\mathcal{B}$ plus an evaluation of g on $\mathcal{C}$. This last evaluation of g on $\mathcal{C}$ is zero actually zero since g is a graded polynomial identity for $\mathcal{C}$. We therefore get an evaluation of f on elements of $\mathcal{B}$, and now it is enough to use that f is an ordinary polynomial identity for $\mathcal{B}$ to get the desired conclusion. $\square$

Corollary 2.17. *Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be a G-graded algebra, where $\mathcal{B}$ is a homogeneous ideal of $\mathcal{A}$ satisfying a graded polynomial identity $f(x_1^{(g_1)}, \dots, x_m^{(g_m)})$ and $\mathcal{C}$ is a homogeneous subalgebra satisfying some ordinary polynomial identity $g(x_1, \dots, x_n)$. Then $\mathcal{A}$ satisfies the following graded identity*

$$f(g(x_{11}^{(g_1)}, x_{12}^{(1)}, \dots, x_{1n}^{(1)}), \dots, g(x_{m1}^{(g_n)}, x_{m2}^{(1)}, \dots, x_{mn}^{(1)})).$$

Proof. The proof follows the argument from the previous corollary. $\square$

2.3 Graded semi-identities

Throughout this section G will denote a finite group.

We will define the so-called graded semi-identities for the sum $\mathcal{A} = \mathcal{B} + \mathcal{C}$ and we will show how some particular graded semi-identities can ensure the existence of graded polynomial identities for $\mathcal{A}$.

As a motivation for the following definition we recall that Lemma 2.4 gives us that $\mathcal{A}_g = \mathcal{B}_g + \mathcal{C}_g$ for each $g \in G$, hence every $a_g \in \mathcal{A}_g$ is a sum $a_g = b_g + c_g$, $b_g \in \mathcal{B}_g$, $c_g \in \mathcal{C}_g$. Thus it is convenient to consider free variables corresponding to the homogeneous components $\mathcal{B}_g$ and $\mathcal{C}_g$ and look for their interplay with the variables corresponding to $\mathcal{A}_g$.

We introduce the following sets of variables

$$Y_G = \{y_i^{(g)} | g \in G, i = 1, 2, \dots\} \quad \text{and} \quad Z_G = \{z_i^{(g)} | g \in G, i = 1, 2, \dots\},$$

and we consider the free G-graded algebra $F\langle Y_G \cup Z_G\rangle$. We set $x_i^{(g)} = y_i^{(g)} + z_i^{(g)}$ for each $g \in G$, and then we may consider $F\langle X_G\rangle \subset F\langle Y_G \cup Z_G\rangle$. We are now ready to introduce the definition of graded semi-identities for a sum $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

Definition 2.18. *Let*

$$f = f(y_1^{(h_1)}, \dots, y_m^{(h_m)}, z_1^{(\tilde{h}_1)}, \dots, z_n^{(\tilde{h}_n)}) \in F\langle Y_G \cup Z_G\rangle,$$

such that $h_1, \dots, h_m, \tilde{h}_1, \dots, \tilde{h}_n \in G$. *We say that* $f = 0$ *is a graded semi-identity for* $\mathcal{A} = \mathcal{B} + \mathcal{C}$ *if*

$$f(b_1^{(h_1)}, \dots, b_m^{(h_m)}, c_1^{(\tilde{h}_1)}, \dots, c_n^{(\tilde{h}_n)}) = 0$$

for all $b_i^{(h_i)} \in \mathcal{B}_{h_i}$, $c_j^{(\tilde{h}_j)} \in \mathcal{C}_{\tilde{h}_j}$. *We say that a graded semi-identity* $f = 0$ *is trivial if* $f \in Id_G(\mathcal{A})$.

We notice here that the notion of a graded semi-identity depends on the decomposition $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

Example 2.19. *Let* $\mathcal{F}$ *be an* F*-algebra without* 1 *such that* $\mathcal{F} = \mathcal{L}_1 + \mathcal{L}_2$, *where* $\mathcal{L}_1$ *is a subalgebra of* $\mathcal{F}$ *and* $\mathcal{L}_2$ *is an ideal of* $\mathcal{F}$ *(for instance, take* $\mathcal{F} = F\langle X\rangle$, *the free associative algebra without* 1, $\mathcal{L}_1$ *the subalgebra generated by all monomials which do not contain the variable* x_1, *and* $\mathcal{L}_2$ *the subalgebra generated by the monomials that contain* x_1*).*

We now set

$$\mathcal{A} = \begin{pmatrix} \mathcal{F} & \mathcal{L}_2 \\ \mathcal{L}_2 & \mathcal{F} \end{pmatrix}.$$

Consider the $\mathbb{Z}_2$*-grading on* $\mathcal{A}$ *given by* $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ *where*

$$\mathcal{A}_0 = \begin{pmatrix} \mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix} \quad \text{and} \quad \mathcal{A}_1 = \begin{pmatrix} 0 & \mathcal{L}_2 \\ \mathcal{L}_2 & 0 \end{pmatrix}.$$

Notice then that $\mathcal{A} = \mathcal{B} + \mathcal{C}$, where

$$\mathcal{B} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_1 \end{pmatrix} \quad \text{and} \quad \mathcal{C} = \begin{pmatrix} \mathcal{L}_2 & \mathcal{L}_2 \\ \mathcal{L}_2 & \mathcal{L}_2 \end{pmatrix},$$

and both $\mathcal{B}$ and $\mathcal{C}$ are homogeneous subalgebras of $\mathcal{A}$. Now, it is clear that $y_1^{(1)} = 0$ is a nontrivial graded semi-identity for $\mathcal{A}$.

In this section we will present some nontrivial graded semi-identities satisfied by $\mathcal{A}$ that imply the existence of graded polynomial identities for $\mathcal{A}$. We now define a type of multilinear polynomial in $F\langle Y_G \cup Z_G\rangle$ which we will consider as a graded semi-identity for $\mathcal{A}$.

Definition 2.20. *Let $y_1, \ldots, y_d \in Y_G$ and $x_{d+1}, \ldots, x_{2d-1} \in X_G$ be homogeneous variables. We define the following polynomial*

$$Sp_d(y_1, \ldots, y_d; x_{d+1}, \ldots, x_{2d-1}) = \sum_{\sigma \in S_d} \alpha_\sigma y_{\sigma(1)} x_{d+1} y_{\sigma(2)} x_{d+2} \cdots x_{2d-1} y_{\sigma(d)}$$

where $\alpha_\sigma \in F$, and S_d stands for the symmetric group permuting $\{1, \ldots, d\}$.

The polynomial defined above was influenced by a generalization of the *Capelli identity*, the so-called *sparse identity* (see for instance [10]). We will be interested in the case where the variables y's are of the same homogeneous degree g and the x's are all of homogeneous degree g^{-1}. We denote the latter polynomial as $Sp_d[Y^{(g)}, X^{(g^{-1})}]$.

2.3.1 Codimensions modulo graded semi-identities

From now on we assume that $G = \{g_1, \ldots, g_k\}$.

Given $n \in \mathbb{N}$ we write $n = n_1 + \cdots + n_k$, where $n_1, \ldots, n_k$ are non-negative integers. We recall the vector space of multilinear graded polynomials $P_{n_1,\ldots,n_k}$ in n_i homogeneous variables of degrees g_i, respectively. Precisely

$$\begin{aligned} P_{n_1,\ldots,n_k} = \operatorname{span}\{u_{\sigma(1)} \cdots u_{\sigma(n)} \mid \sigma \in S_n, u_{i_1} = x_{i_1}^{(g_1)} \text{ for } i_1 \in \{1, \ldots, n_1\}, \\ u_{n_1+i_2} = x_{n_1+i_2}^{(g_2)} \text{ for } i_2 \in \{1, \ldots, n_2\}, \ldots, \\ u_{n_1+\cdots n_{k-1}+i_k} = x_{n_1+\cdots n_{k-1}+i_k}^{(g_k)} \text{ for } i_k \in \{1, \ldots, n_k\}\}. \end{aligned}$$

Note that $\dim(P_{n_1,\ldots,n_k}) = n!$ and therefore we have the following straightforward lemma.

Lemma 2.21. *If there exists a positive integer $n = n_1 + \cdots + n_k$ such that*

$$\dim\left(\frac{P_{n_1,\ldots,n_k}}{P_{n_1,\ldots,n_k} \cap Id_G(A)}\right) < n!,$$

then $\mathcal{A}$ satisfies a multilinear graded polynomial identity of degree n in n_i variables of homogeneous degree g_i, respectively.

Denoting by $Id^s_G(\mathcal{A})$ the set of all graded semi-identities of $\mathcal{A}$ (including the identically zero polynomial), it follows immediately that $Id_G(\mathcal{A})$, $Id_G(\mathcal{B})$, $Id_G(\mathcal{C}) \subset Id^s_G(\mathcal{A})$. Moreover $Id^s_G(\mathcal{A})$ is an ideal of $F\langle Y_G \cup Z_G\rangle$ which is invariant under all graded endomorphisms that preserve both $F\langle Y_G\rangle$ and $F\langle Z_G\rangle$.

The vector space of multilinear polynomials in $F\langle Y_G \cup Z_G\rangle$ is defined as

$$\begin{aligned} V_{n_1,\dots,n_k} = \operatorname{span}\{v_{\sigma(1)}\cdots v_{\sigma(n)} \mid \sigma \in S_n, v_{i_1} \in \{y^{(g_1)}_{i_1}, z^{(g_1)}_{i_1}\} \text{ for } i_1 \in \{1,\dots,n_1\},\dots,\\ v_{n_1+\cdots+n_{k-1}+i_k} \in \{y^{(g_k)}_{n_1+\cdots+n_{k-1}+i_k}, z^{(g_k)}_{n_1+\cdots+n_{k-1}+i_k}\} \text{ for } i_k \in \{1,\dots,n_k\}\}. \end{aligned}$$

One can easily see that $P_{n_1,\dots,n_k} \subset V_{n_1,\dots,n_k}$. Moreover, if

$$f \in P_{n_1,\dots,n_k} \cap (V_{n_1,\dots,n_k} \cap Id^s_G(\mathcal{A})),$$

then f is a graded semi-identity in the homogeneous variables y and z, and f can be written as a polynomial in the variables $x^{(g)}_i = y^{(g)}_i + z^{(g)}_i$. Now given any $a_g \in \mathcal{A}_g$, Lemma 2.4 yields the existence of $b_g \in \mathcal{B}_g$ and $c_g \in \mathcal{C}_g$ such that $a_g = b_g + c_g$. Hence an evaluation of the variable $x^{(g)}_i$ on some element a_g implies an evaluation of the variables $y^{(g)}_i$ on some b_g and $z^{(g)}_i$ on some c_g, respectively. Since f is a graded semi-identity, then such evaluation must be zero on $\mathcal{A}$. This shows that f is a graded identity. Therefore we conclude that

$$P_{n_1,\dots,n_k} \cap Id_G(\mathcal{A}) = P_{n_1,\dots,n_k} \cap (V_{n_1,\dots,n_k} \cap Id^s_G(\mathcal{A})).$$

As a consequence of the discussion above we have the following lemma.

Lemma 2.22. *If there exists a positive integer $n = n_1 + \cdots + n_k$ such that*

$$\dim\left(\frac{V_{n_1,\dots,n_k}}{V_{n_1,\dots,n_k} \cap Id^s_G(\mathcal{A})}\right) < n!$$

then $\mathcal{A}$ satisfies some multilinear graded polynomial identity of degree n in n_i variables of homogeneous degree g_i, for $i = 1, \dots, k$.

Proof. Notice that

$$\frac{P_{n_1,\dots,n_k}}{P_{n_1,\dots,n_k} \cap Id_G(\mathcal{A})} = \frac{P_{n_1,\dots,n_k} \cap V_{n_1,\dots,n_k}}{P_{n_1,\dots,n_k} \cap (V_{n_1,\dots,n_k} \cap Id^s_G(\mathcal{A}))} \hookrightarrow \frac{V_{n_1,\dots,n_k}}{V_{n_1,\dots,n_k} \cap Id^s_G(\mathcal{A})}.$$

and finally apply Lemma 2.21. □

We consider one further decomposition on the space $V_{n_1,\dots,n_k}$. In this decomposition we will determine precisely when v_i is given by either $y^{(g_j)}_i$ or $z^{(g_j)}_i$. For each $j = 1, \dots, k$ we consider integers $0 \leqslant r_j \leqslant n_j$, and $1 \leqslant t_{j_1} < \cdots < t_{j_{r_j}} \leqslant n_j$. Denoting $\mathbf{r} = (r_1,\dots,r_k)$ and $\mathbf{t} = (t_{1_1},\dots,t_{1_{r_1}},\dots,t_{k_1},\dots,t_{k_{r_k}})$, we define

$$\begin{aligned} V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}} = \operatorname{span}\{v_{\sigma(1)}\cdots v_{\sigma(n)} \mid \sigma \in S_n, v_{i_1} &= y^{(g_1)}_{i_1} \text{ for } i_1 \in \{t_{1_1},\dots,t_{1_{r_1}}\}, \text{ and}\\ v_{i_1} &= z^{(g_1)}_{i_1} \text{ for } i_1 \in \{1,\dots,n_1\}\backslash\{t_{1_1},\dots,t_{1_{r_1}}\},\dots,\\ v_{n_1+\cdots+n_{k-1}+i_k} &= y^{(g_k)}_{n_1+\cdots+n_{k-1}+i_k} \text{ for } i_k \in \{t_{k_1},\dots,t_{k_{r_k}}\} \text{ and}\\ v_{n_1+\cdots+n_{k-1}+i_k} &= z^{(g_k)}_{n_1+\cdots+n_{k-1}+i_k} \text{ for } i_k \in \{1,\dots,n_k\}\backslash\{t_{k_1},\dots,t_{k_{r_k}}\}\}. \end{aligned}$$

In other words, in $V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$ we have k groups of distinct variables, with $n_1, \dots, n_k$ variables in each one of them, respectively. In the i-th group of variables we take r_i among them: the ones with indices $\{t_{i_1}, \dots, t_{i_{r_i}}\}$, $0 \leqslant r_i \leqslant n_i$. These variables are the $y^{(g_i)}$, and the remaining variables from this group are $z^{(g_i)}$.

One can notice that

$$V_{n_1,\dots,n_k} = \bigoplus_{\substack{r_1=0 \\ 1\leqslant t_{1_1}<\cdots<t_{1_{r_1}}\leqslant n_1}}^{n_1} \cdots \bigoplus_{\substack{r_k=0 \\ 1\leqslant t_{k_1}<\cdots<t_{k_{r_k}}\leqslant n_k}}^{n_k} V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}.$$

In particular, a graded semi-identity in $V_{n_1,\dots,n_k}$ can be written as a sum of polynomials in $V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$. The next lemma shows that each one of these polynomials in $V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$ are also graded semi-identities.

Lemma 2.23. *The following decomposition of $V_{n_1,\dots,n_k} \cap Id_G^s(\mathcal{A})$ holds*

$$V_{n_1,\dots,n_k} \cap Id_G^s(\mathcal{A}) = \bigoplus_{\substack{r_1=0 \\ 1\leqslant t_{1_1}<\cdots<t_{1_{r_1}}\leqslant n_1}}^{n_1} \cdots \bigoplus_{\substack{r_k=0 \\ 1\leqslant t_{k_1}<\cdots<t_{k_{r_k}}\leqslant n_k}}^{n_k} (V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}} \cap Id_G^s(\mathcal{A})).$$

Proof. Let $f \in V_{n_1,\dots,n_k} \cap Id_G^s(\mathcal{A})$ and write

$$f = \sum_{r_1=0}^{n_1} \sum_{1\leqslant t_{1_1}<\cdots<t_{1_{r_1}}\leqslant n_1} \cdots \sum_{r_k=0}^{n_k} \sum_{1\leqslant t_{k_1}<\cdots<t_{k_{r_k}}\leqslant n_k} f_{\mathbf{r},\mathbf{t}}. \tag{2.2}$$

We have to show that each $f_{\mathbf{r},\mathbf{t}} \in V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$ is also a graded semi-identity of $\mathcal{A}$. We proceed by induction on the number of terms of the sum (2.2). The base case is when $f = f_{\mathbf{r},\mathbf{t}}$, and here we already have $f_{\mathbf{r},\mathbf{t}} \in Id_G^s(\mathcal{A})$. From now on we suppose that there exist at least two non-zero terms in (2.2), and we take two distinct of them, say $f_{\mathbf{r},\mathbf{t}}$ and $\tilde{f}_{\tilde{\mathbf{r}},\tilde{\mathbf{t}}}$. Hence there exists some variable, which without loss of generality we will suppose $y_i^{(g_1)}$, such that it occurs in $f_{\mathbf{r},\mathbf{t}}$ but not in $\tilde{f}_{\tilde{\mathbf{r}},\tilde{\mathbf{t}}}$. Now we write $f = f_1 + f_2$, where f_1 is the sum of the terms from (2.2) that contain the variable $y_i^{(g_1)}$ and f_2 is the sum of terms from (2.2) which do not. Evaluating $y_i^{(g_1)}$ by 0 we obtain that f_2 is a consequence of f and therefore f_2 is a graded semi-identity. Hence the same happens to $f_1 = f - f_2$. Now it is enough to apply the induction hypothesis to both f_1 and f_2. □

We finish this section by showing how the decomposition above can be used to prove the existence of a graded polynomial identity for $\mathcal{A}$.

In order to simplify our notation we will write just $V_{n_1,\dots,n_k,\mathbf{r}}$ instead of $V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$, where $\mathbf{t} = (1, \dots, r_1, \dots, 1, \dots, r_k)$. Note that given any $\mathbf{t}$ there exists a graded isomorphism of vector spaces $V_{n_1,\dots,n_k,\mathbf{r}} \cong V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$ such that

$$V_{n_1,\dots,n_k,\mathbf{r}} \cap Id_G^s(\mathcal{A}) \cong V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}} \cap Id_G^s(\mathcal{A}).$$

We also note that there exist exactly $\binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$ vector spaces $V_{n_1,\dots,n_k,\mathbf{r},\mathbf{t}}$ isomorphic to $V_{n_1,\dots,n_k,\mathbf{r}}$.

Lemma 2.24. *If there exists a positive integer $n = n_1 + \cdots + n_k$ such that*

$$\dim\left(\frac{V_{n_1,\ldots,n_k,\mathbf{r}}}{V_{n_1,\ldots,n_k,\mathbf{r}} \cap Id_G^s(\mathcal{A})}\right) < \frac{n!}{2^n},$$

for all $\mathbf{r}$, then $\mathcal{A}$ satisfies some multilinear graded polynomial identity of degree n in n_i variables of homogeneous degree g_i, for $i = 1, \ldots, k$.

Proof. By Lemma 2.23 we have

$$\begin{aligned}\dim\frac{V_{n_1,\ldots,n_k}}{V_{n_1,\ldots,n_k} \cap Id_G^s(\mathcal{A})} &= \sum_{r_1=0}^{n_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k}\dim\frac{V_{n_1,\ldots,n_k,\mathbf{r}}}{V_{n_1,\ldots,n_k,\mathbf{r}} \cap Id_G^s(\mathcal{A})}\\ &< \sum_{r_1=0}^{n_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k}\frac{n!}{2^n}\\ &= 2^{n_1}\cdots 2^{n_k}\frac{n!}{2^n} = n!\end{aligned}$$

Now it suffices to apply Lemma 2.22. □

2.3.2 A generating set for $V_{n,0,\ldots,0,\mathbf{r}}$ modulo $Id_G^s(\mathcal{A})$

In this section we give a generating set for $V_{n,0,\ldots,0,\mathbf{r}}$ by using the language of good permutations of the symmetric group. This notion appeared in the proof of the well known theorem of Regev about the exponential upper bound of the codimension sequence of a PI algebra, see [53]. Good sequences and similar combinatorial notions have been extensively used in studying numerical invariants of PI algebras such as codimensions, cocharacters, and so on.

Definition 2.25. *Let $n \in \mathbb{N}$ and $1 \leqslant d \leqslant n$. We say that $\sigma \in S_n$ is a d-bad permutation if there exist $1 \leqslant i_1 < \cdots < i_d \leqslant n$ such that $\sigma(i_1) > \cdots > \sigma(i_d)$. Otherwise we call $\sigma \in S_n$ a d-good permutation.*

We recall a well known result about the number of d-good permutations (see [54]). It can be obtained by using the theorem of Dilworth in Combinatorics.

Lemma 2.26. *The number of d-good permutations in S_n is at most $(d-1)^{2n}$.*

We will adopt the following convention: given $d > n$, then every permutation in S_n is d-good. Such convention does not change the maximum number of d-good permutations. Indeed, in this case the number is exactly $n!$ which is less than $(d-1)^{2n}$.

Definition 2.27. *Let $\boldsymbol{m}$ be the monomial*

$$v_1 y_{\sigma(1)}^{(h_1)}\cdots y_{\sigma(i_1)}^{(h_{i_1})} v_2 y_{\sigma(i_1+1)}^{(h_{i_1+1})}\cdots y_{\sigma(i_2)}^{(h_{i_2})} v_3 \cdots v_l y_{\sigma(i_{l-1}+1)}^{(h_{i_{l-1}+1})}\cdots y_{\sigma(i_l)}^{(h_{i_l})} v_{l+1} \in V_{n_1,\ldots,n_k,\mathbf{r},\mathbf{t}}$$

where $v_1, \ldots, v_{l+1}$ are (eventually empty) words in the homogeneous variables of type z. We say that $\boldsymbol{m}$ is a d-y-good monomial if the permutation $\sigma \in S_l$ is a d-good one.

We recall a combinatorial fact concerning finite groups, [7, Lemma 4.1].

Lemma 2.28. *Every product of $|H|d$ words in a finite group H contains a product of d consecutive trivial subwords.*

In the next lemma we will assume that $\mathcal{A}$ satisfies a graded semi-identity of the form $Sp_{d_1}[Y^{(g)}; X^{(g^{-1})}]$, for some $g \in G$. Without loss of generality we write $g_1 = g$ and we denote $n_1 = n$ and $r_1 = r$. We also use $o(g)$ to denote the order of the element $g \in G$ (and of the cyclic subgroup generated by g).

Lemma 2.29. *The space $V_{n,0,\ldots,0,\mathbf{r}}$ is generated, modulo $Id^s_G(\mathcal{A})$, by all $(2d_1-1)o(g)$-y-good monomials for all $\mathbf{r}$.*

Proof. Following our convention we may assume $(2d_1 - 1)o(g) \leqslant r$. The proof will be done by contradiction. Thus we assume that $V_{n,0,\ldots,0,\mathbf{r}}$ is not generated by the $(2d_1 - 1)o(g)$-y-good monomials modulo $Id^s_G(\mathcal{A})$. Hence the set $\mathfrak{U}$ of all $(2d_1 - 1)o(g)$-y-bad monomials which cannot be written as a linear combination of $(2d_1 - 1)o(g)$-y-good monomials modulo $Id^s_G(\mathcal{A})$ is nonempty. We order the variables in Y_G of homogeneous degree g as $y_1^{(g)} < y_2^{(g)} < \cdots$ and we take in $\mathfrak{U}$ the partial order given lexicographically from the left to the right in the variables in Y_G of homogeneous degree g, only. Let $\boldsymbol{m}_\tau \in \mathfrak{U}$ be a minimal element, where $\tau \in S_r$ is the permutation which defines the positions of the variables from Y_G of homogeneous degree g in this minimal element. In particular, $\boldsymbol{m}_\tau$ is a $(2d_1 - 1)o(g)$-y-bad monomial and hence there exist $1 \leqslant i_1 < \cdots < i_{(2d_1-1)o(g)} \leqslant r$ such that $\tau(i_1) > \cdots > \tau(i_{(2d_1-1)o(g)})$.

We write $\boldsymbol{m}_\tau = w_0 w_1 w_2 \cdots w_{(2d_1-1)o(g)} w_{(2d_1-1)o(g)+1}$, where w_j is the word that starts with $y^{(g)}_{\tau(i_j)}$ and ends just before $y^{(g)}_{\tau(i_{j+1})}$, $j = 1, \ldots, (2d_1 - 1)o(g) - 1$, $w_{(2d_1-1)o(g)} = y^{(g)}_{\tau(i_{(2d_1-1)o(g)})}$, w_0 and $w_{(2d_1-1)o(g)+1}$ are suitable words.

Applying Lemma 2.28 to the subgroup generated by g, we obtain the existence of $2d_1 - 1$ consecutive subwords $\overline{w}_1, \ldots, \overline{w}_{2d_1-1}$ from $w_1 \cdots w_{(2d_1-1)o(g)}$ where $\deg(\overline{w}_j) = 1$, $j = 1, \ldots, 2d_1 - 1$.

Hence we rewrite $\boldsymbol{m}_\tau = \overline{w}_0 \overline{w}_1 \cdots \overline{w}_{2d_1-1} \overline{w}_{2d_1}$, where $\overline{w}_0$ and $\overline{w}_{2d_1}$ are suitable words. Note that for each $j = 1, \ldots, d_1 - 1$, we can write $\overline{w}_{2j-1}\overline{w}_{2j} = y^{(g)}_{\tau(i_l)} m_{d_1+j}$ for some l (which depends on j) and m_{d_1+j} is the subword of $\overline{w}_{2j-1}\overline{w}_{2j}$ obtained by deleting its first variable. Denote $m_j = y^{(g)}_{\tau(i_l)}$ for $j = 1, \ldots, d_1 - 1$, and take m_{d_1} as the first variable of $\overline{w}_{2d_1-1}$.

In this way we have

$$\boldsymbol{m}_\tau = m_0 m_1 m_{d_1+1} m_2 \cdots m_{d_1-1} m_{2d_1-1} m_{d_1} m_{2d_1}$$

where $m_0 = \overline{w}_0$, m_{2d_1} is a suitable word, $\deg(m_j) = g$ for $j = 1, \ldots, d_1$ and $\deg(m_{d_1+j}) = g^{-1}$ for $j = 1, \ldots, d_1 - 1$. Indeed, it follows from $\deg(\overline{w}_{2j-1}\overline{w}_{2j}) = 1$

and $\deg(y^{(g)}_{\tau(i_l)}) = g$ that $\deg(m_{d_1+j}) = g^{-1}$. We also note that each m_j is evaluated on Y_G of homogeneous degree g and each m_{d_1+j} is evaluated on X_G of homogeneous degree g^{-1}. Moreover, we have $m_1 > \cdots > m_{d_1}$.

Recalling that $Sp_{d_1}[Y^{(g)}; X^{(g^{-1})}] \in Id^s_G(\mathcal{A})$ and that the scalar related to the identity permutation can be assumed as 1, we can write

$$\boldsymbol{m}_\tau = \sum_{\sigma \in S_{d_1} \setminus \{id\}} -\alpha_\sigma m_0 m_{\sigma(1)} m_{d_1+1} m_{\sigma(2)} \cdots m_{2d_1-1} m_{\sigma(d_1)} m_{2d_1} \quad (\mathrm{mod}\ Id^s_G(\mathcal{A})).$$

Since for each $\sigma \in S_{d_1} \setminus \{id\}$ we have

$$\boldsymbol{m}_\sigma = m_0 m_{\sigma(1)} m_{d_1+1} m_{\sigma(2)} \cdots m_{2d_1-1} m_{\sigma(d_1)} m_{2d_1} < \boldsymbol{m}_\tau,$$

the minimality of $\boldsymbol{m}_\tau$ leads us to $m_\sigma \notin \mathfrak{U}$. Therefore each $\boldsymbol{m}_\sigma$ can be written as a linear combination of $(2d_1 - 1)o(g)$-y-good monomials modulo the graded semi-identities of $\mathcal{A}$, and hence the same happens for $\boldsymbol{m}_\tau$. This is a contradiction to $m_\tau \in \mathfrak{U}$. $\square$

2.3.3 Existence of a graded identity for $\mathcal{A}$

First of all let us estimate the dimension of $V_{n,0,\ldots,0,\mathbf{r}}$ modulo $Id^s_G(\mathcal{A})$. We start by recalling the following result from [7].

Theorem 2.30. *Let $\mathcal{A}$ be an algebra graded by a finite group G. If the neutral component of $\mathcal{A}$ satisfies a polynomial identity of degree d then*

$$\dim \frac{P^{(h_1,\ldots,h_n)}}{P^{(h_1,\ldots,h_n)} \cap Id_G(\mathcal{A})} \leqslant (|G|d - 1)^{2n}$$

for every $n \in \mathbb{N}$ and $(h_1, \ldots, h_n) \in G^n$.

For our main goal of this section we consider one last decomposition of the vector space $V_{n,0,\ldots,0,\mathbf{r}}$ into the subspaces

$$\begin{aligned} U_{n,0,\ldots,0,\mathbf{r},u,\mathbf{p},\mathbf{q}} = \mathrm{span}\{& y_{\sigma(1)} \ldots y_{\sigma(p_1)} z_{\tau(1)} \cdots z_{\tau(q_1)} \cdots y_{\sigma(p_1+\cdots+p_{u-1}+1)} \cdots y_{\sigma(r)} \times \\ & \times z_{\tau(q_1+\cdots+q_{u-1}+1)} \cdots z_{\tau(n-r)} \mid \sigma \in S_r, \tau \in S_{n-r}\} \end{aligned}$$

where $r = p_1 + \cdots + p_u$ and $n - r = q_1 + \cdots + q_u$ are such that $p_1 \geqslant 0$, $q_u \geqslant 0$, and $p_2, \ldots, p_u, q_1, \ldots, q_{u-1} > 0$, and the homogeneous variables were written without their homogeneous degrees for simplicity.

Hence we have

$$V_{n,0,\ldots,0,\mathbf{r}} = \bigoplus_{u,\mathbf{p},\mathbf{q}} U_{n,0,\ldots,0,\mathbf{r},u,\mathbf{p},\mathbf{q}}. \tag{2.3}$$

Definition 2.31. *A composition of a positive integer n sequence of integers $(n_1, \ldots, n_k)$ such that $n = n_1 + \cdots + n_k$.*

Remark 2.32. *The number of compositions of a positive integer n is exactly 2^{n-1}. In particular, one can see that there exist at most $2^r \cdot 2^{n-r} = 2^n$ vector spaces $U_{n,0,\ldots,0,\mathbf{r},u,\mathbf{p},\mathbf{q}}$ appearing in the decomposition of $V_{n,0,\ldots,0,\mathbf{r}}$ in (2.3).*

Lemma 2.33. *Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be a G-graded algebra sum of two homogeneous subalgebras. We assume that $\mathcal{A}$ satisfies the graded semi-identity $Sp_{d_1}[Y^{(g)}; X^{(g^{-1})}]$ and $\mathcal{C}_1$ satisfies a polynomial identity of degree d_2. Then the following inequality holds*

$$\dim \frac{V_{n,0,\ldots,0,\mathbf{r}}}{V_{n,0,\ldots,0,\mathbf{r}} \cap Id^s_G(\mathcal{A})} \leqslant 2^n((2d_1-1)o(g)-1)^{2r}(|G|d_2-1)^{2(n-r)}(r+1)^{n-r}.$$

Proof. We start the proof by recalling that Lemma 2.29 gives us a generating set for $V_{n,0,\ldots,0,\mathbf{r}}$ modulo $Id^s_G(A)$, namely the set of all $(2d_1-1)o(g)$-$y^{(g)}$-good monomials. By (2.3) it is enough to count the number of $(2d_1-1)o(g)$-$y^{(g)}$-good monomials modulo the graded semi-identities of $\mathcal{A}$ in $U_{n,0,\ldots,0,\mathbf{r},u,\mathbf{p},\mathbf{q}}$. First of all note that there exist at most $((2d_1-1)o(g)-1)^{2r}$ dispositions of the homogeneous variables y modulo $Id^s_G(\mathcal{A})$. We also have

$$\binom{n-r}{q_1,\ldots,q_u} = \frac{(n-r)!}{q_1!\cdots q_u!}$$

different manners of distributing the homogeneous variables z that occur in blocks of q_1 consecutive ones, $\ldots$, q_u consecutive ones. By Theorem 2.30, each consecutive group of homogeneous variables z can be written as a linear combination of at most $(|G|d_2-1)^{2q_i}$ monomials modulo $Id^s_G(A)$. Hence $U_{n,0,\ldots,0,\mathbf{r},u,\mathbf{p},\mathbf{q}}$ is generated by at most

$$((2d_1-1)o(g)-1)^{2r}\binom{n-r}{q_1,\ldots,q_u}(|G|d_2-1)^{2q_1}\cdots(|G|d_2-1)^{2q_u}$$

monomials.

Therefore the multinomial theorem enables us to bound the multinomial coefficient corresponding to the homogeneous variables z by u^{n-r} and since $u \leqslant r+1$ we write

$$\binom{n-r}{q_1,\ldots,q_u} \leqslant (r+1)^{n-r}.$$

We finish the proof of this lemma by observing that there exist at most 2^n vector spaces of the form $U_{n,0,\ldots,0,\mathbf{r},u,\mathbf{p},\mathbf{q}}$ (see Remark 2.32). □

We are ready to prove the main result of this section. We recall the following inequality known as Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n; \quad \text{and} \quad \left(\frac{n}{e}\right)^n < n!, \quad \text{for all} \quad n \in \mathbb{N} \tag{2.4}$$

where the meanings of e and π are the obvious ones.

Theorem 2.34. *Let $\mathcal{A} = \mathcal{B} + \mathcal{C}$ be a G-graded algebra which is a sum of two homogeneous subalgebras. If $\mathcal{A}$ satisfies the graded semi-identity $Sp_{d_1}[Y^{(g)}; X^{(g^{-1})}]$ and $\mathcal{C}_1$ satisfies some ordinary polynomial identity of degree d_2, then $\mathcal{A}$ satisfies some graded multilinear polynomial identity of degree n in homogeneous variables of degree g, where n is the least integer greater or equal to α^α such that $\alpha = 8e((2d_1-1)o(g)-1)^2(|G|d_2-1)^2$.*

Proof. Lemma 2.24 reduces the existence of the required graded polynomial identity to the existence of a positive integer $n \in \mathbb{N}$ such that

$$\dim \frac{V_{n,0,\ldots,0,\mathbf{r}}}{V_{n,0,\ldots,0,\mathbf{r}} \cap Id_G^s(\mathcal{A})} < \frac{n!}{2^n},$$

for every $\mathbf{r}$.

We begin by assuming that $r \neq 0$ and we will show that there exists $n \in \mathbb{N}$ (which does not depend on r) such that

$$8^n e^n((2d_1-1)o(g)-1)^{2n}(|G|d_2-1)^{2n} r^{n-r} \leqslant n^n.$$

Recall $\alpha = 8e((2d_1-1)o(g)-1)^2(|G|d_2-1)^2$. Take $n \in \mathbb{N}$ as the least integer satisfying $n \geqslant \alpha^\alpha$, we claim that $\alpha^n r^{n-r} \leqslant n^n$. We can consider $r < n$, since for $r = n$ the inequality $\alpha^n \leqslant n^n$ follows from $\alpha < n$. We consider two cases now.

Case 1: $r \leqslant \dfrac{n}{\alpha}$. In this case we have $\alpha r \leqslant n$ and hence $\alpha^n r^{n-r} \leqslant (\alpha r)^n \leqslant n^n$.

Case 2: $\dfrac{n}{\alpha} < r < n$. In this second case we have $n < \alpha r$, that is, there exists a positive real number v such that $\alpha r = n + v$. The minimality of n leads us to $r < \alpha^\alpha$. We also note that $\alpha^v > 1$. Therefore,

$$\begin{aligned} n^n &\geqslant (\alpha^\alpha)^n = \alpha^n(\alpha^{\alpha-1})^{n-r}(\alpha^{\alpha-1})^r = \alpha^n(\alpha^{\alpha-1})^{n-r}\alpha^{\alpha r - r} \\ &= \alpha^n(\alpha^{\alpha-1})^{n-r}\alpha^{n-r+v} = \alpha^n(\alpha^\alpha)^{n-r}\alpha^v > \alpha^n r^{n-r}, \end{aligned}$$

which finishes the second case and the proof of the claim.

Since $r \neq 0$, we have $r + 1 \leqslant 2r$ and hence

$$\begin{aligned} &4^n e^n((2d_1-1)o(g)-1)^{2r}(|G|d_2-1)^{2(n-r)}(r+1)^{n-r} \\ \leqslant &8^n e^n((2d_1-1)o(g)-1)^{2n}(|G|d_2-1)^{2n} r^{n-r} \leqslant n^n \end{aligned}$$

which implies in turn that

$$2^n((2d_1-1)o(g)-1)^{2r}(|G|d_2-1)^{2(n-r)}(r+1)^{n-r} \leqslant \frac{n^n}{e^n 2^n} < \frac{n!}{2^n}.$$

The only case left is when $r = 0$; we note that for any positive integer $n \geqslant 4e(|G|d_2-1)^2$ (in particular for the same n chosen in the case $r \neq 0$) we have $4^n e^n(|G|d_2-1)^{2n} \leqslant n^n$, and then it follows that

$$2^n(|G|d_2-1)^{2n} \leqslant \frac{n^n}{e^n 2^n} < \frac{n!}{2^n}$$

Now it is enough to apply Lemma 2.33 in order to finish the proof. □

Remark 2.35. *If we take $\mathcal{F}$, $\mathcal{L}_1$, $\mathcal{L}_2$ as defined in Example 2.19 for $\mathcal{F}$ given by the free associative algebra $F\langle X\rangle$ without 1, then $\mathcal{A}$ does not satisfy an identity in variables of homogeneous degree 1, but satisfies a graded semi-identity of the type Sp_d. However, this does not contradict our last theorem, since clearly $\mathcal{C}_1$ is not PI.*

3 Images of graded polynomials on upper triangular matrices

In this chapter we will deal with the problem of classifying images of polynomials on algebras with additional structure. We will be mostly concerned with the upper triangular matrix algebra $UT_n(F)$ with entries in a field F. When the field F is clear in the context, then we will use the simpler notation UT_n instead of the previous one. The results from Section 3.2 to Section 3.6 are new and two papers were written from them. The first one is published in the Canadian Journal of Mathematics [27] in a joint work with Plamen Koshlukov. The second one is published on arXiv and has been submitted for publication in a specialized journal [24].

3.1 A review of gradings on UT_n

In this first section we will give a short review of group-gradings on the associative upper triangular matrix algebra. Thus let us start with the following definition.

Definition 3.1. *Let G be a group. A G-grading Γ on UT_n is said elementary if all matrix units e_{ij} are homogeneous in this grading. Equivalently, we say that Γ is elementary if there exists a sequence $(g_1, \ldots, g_n) \in G^n$ such that $\deg(e_{ij}) = g_i^{-1}g_j$ for all i, j. The grading Γ is also called elementary induced by the sequence $(g_1, \ldots, g_n) \in G^n$.*

The importance of elementary gradings on UT_n is given in the next theorem due to Valenti and Zaicev in 2007.

Theorem 3.2. *[62] Let Γ be a G-grading on UT_n. Then Γ is graded isomorphic to some elementary G-grading.*

Clearly, it is true that knowing the homogeneous degree of the matrices e_{ij} in an elementary grading gives us the homogeneous components of it. The next result shows us that in order to completely determine the elementary grading it is enough to know the homogeneous degree of the matrix units $e_{i,i+1}$, $i = 1, \ldots, n-1$, only.

Proposition 3.3. *[20] Every elementary G-grading on UT_n is uniquely determined by the homogeneous degree of the elements in the first diagonal of a matrix in the Jacobson radical of UT_n.*

We finish this quick review with two results also from [20] about the neutral component of elementary gradings on UT_n.

Lemma 3.4. *Let Γ be an elementary grading on UT_n. Then the neutral component of Γ contains the subspace of all diagonal matrices.*

Proposition 3.5. *Let Γ be an elementary grading on UT_n. Then the neutral component of Γ is isomorphic (as F-algebras) to the direct sum*

$$UT_{n_1} \oplus \cdots \oplus UT_{n_k}.$$

3.2 Graded central polynomials for upper triangular matrices

The main goal of this section is to prove that there exist no graded central polynomials for upper triangular matrices. It is well known that the algebra of upper block triangular matrices has no central polynomials in case the number of blocks is greater than 1, see [33, Lemma 1]. In particular, the upper triangular matrix algebra $UT_n(F)$ has no central polynomials.

Theorem 3.6. *Let $UT_n = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G-grading on the algebra of upper triangular matrices over a field. If $n > 1$ then there exist no graded central polynomials for $\mathcal{A}$.*

Proof. By Theorem 3.2 we have that $\mathcal{A}$ is graded isomorphic to some elementary grading on UT_n. Hence we may reduce our problem to the elementary gradings. Now we assume that $f \in F\langle X_G \rangle$ is a polynomial with zero constant term, such that $f(\mathcal{A})$ is contained in $Z(\mathcal{A})$ (which can be identified with the ground field F). We write f as $f = f_1 + f_2$ where f_1 contains neutral variables only and f_2 has at least one non neutral variable in each of its monomials. Consider $a_1, \ldots, a_m \in \mathcal{A}_1$, and $b_1, \ldots, b_l$ arbitrary elements of homogeneous degree $\neq 1$ that occur in f. Hence, in light of Lemma 3.4 we have that $f(a_1, \ldots, a_m, b_1, \ldots, b_l) = f_1(\overline{a}_1, \ldots, \overline{a}_m) + j$ where $j \in J$, the Jacobson radical of $\mathcal{A}$, and $\overline{a}_i$ is the diagonal part of a_i. Since $f(\mathcal{A}) \subset F$, then $j = 0$ and hence $f(\mathcal{A}) = f_1(\mathcal{D})$, where $\mathcal{D}$ denotes the subspace of diagonal matrices. Now, notice that if $\lambda_1, \ldots, \lambda_m \in F$ are arbitrary, then

$$f_1(\lambda_1 e_{11}, \ldots, \lambda_m e_{11}) = f_1(\lambda_1, \ldots, \lambda_m) e_{11}.$$

Since $f_1(\mathcal{D}) \subset F$, we must have $f_1(\lambda_1, \ldots, \lambda_m) = 0$. Hence, for diagonal matrices $D_i = \sum_{k=1}^{n} \lambda_k^{(i)} e_{kk}$ we have

$$f_1(D_1, \ldots, D_m) = \sum_{k=1}^{n} f_1(\lambda_k^{(1)}, \ldots, \lambda_k^{(m)}) e_{kk} = 0,$$

and thus $f(\mathcal{A}) = \{0\}$. We conclude the non existence of graded central polynomials for UT_n. $\square$

As an immediate consequence we have the following corollary.

Corollary 3.7. *The space of scalar matrices can not be realized as the image of some multilinear graded polynomial on UT_n.*

3.3 The neutral component of UT_n

We start this section by recalling the following definition from [31].

Definition 3.8. *Let $f \in F\langle X\rangle$. We say that f has commutator degree r if*

$$f \in \langle [x_1, x_2] \cdots [x_{2r-1}, x_{2r}] \rangle^T \quad \textit{and} \quad f \notin \langle [x_1, x_2] \cdots [x_{2r+1}, x_{2r+2}] \rangle^T.$$

We say that f has commutator degree 0 if f is not a consequence of the commmutator.

We now recall a result due to Gargate and de Mello concerning the description of images of multilinear polynomials on upper triangular matrices.

Theorem 3.9. *[31] Let F be an infinite field and let $f \in P_m$. Then $f(UT_n) = J^r$ if and only if f has commutator degree r.*

A slight improvement of Theorem 3.9 was given shortly afterwards.

Theorem 3.10. *[48] Let F be a field with at least $n(n-1)/2$ elements. Then the image of $f \in P_m$ on UT_n is J^r, where r is the commutator degree of f.*

As a consequence of these results we have the following proposition.

Proposition 3.11. *Let F be a field with at least $n(n-1)/2$ elements. The image of a multilinear polynomial on $\mathcal{T} = UT_{n_1} \oplus \cdots \oplus UT_{n_k}$ is either $\mathcal{T}$ or some power of its Jacobson radical J.*

Proof. Let $f \in F\langle X\rangle$ be a multilinear polynomial. By [12, Proposition 5.60] we have

$$J = J_1 \oplus \cdots \oplus J_k,$$

where J_i stands for the Jacobson radical of UT_{n_i}, $i = 1, \ldots, k$. Now it is enough to apply Theorem 3.10 to see that

$$f(\mathcal{T}) = f(UT_{n_1}) \oplus \cdots \oplus f(UT_{n_k}) = J_1^r \oplus \cdots \oplus J_k^r = J^r,$$

where r is the commutator degree of f. □

In light of Proposition 1.22, Proposition 3.5 and Proposition 3.11, we have the following corollary.

Corollary 3.12. *The image of a multilinear polynomial in neutral variables on UT_n is a homogeneous vector subspace, regardless of the grading on UT_n.*

3.4 Certain $\mathbb{Z}_q$-gradings

Apart from the last section of this chapter, we will adopt the following notation: the set of neutral variables in X_G will be denoted by $Y_G = \{y_1, y_2, \dots\}$ and the non-neutral ones will be denoted by $Z_G = \{z_1, z_2, \dots\}$. Of course we will make clear what the homogeneous degrees of the variables z_i's are. We shall say that the variables of type y are even while those of type z are odd.

Throughout this section $\mathcal{A}$ will denote the algebra of upper triangular matrices UT_n endowed with the elementary $\mathbb{Z}_q$-grading given by the following sequence in $\mathbb{Z}_q^n$

$$(\overline{0}, \overline{1}, \dots, \overline{q-2}, \underbrace{\overline{q-1}, \overline{q-1}, \dots, \overline{q-1}}_{n-q+1 \text{ times}}),$$

where $q \leqslant n$ are integers. Our goal is to give a complete description of the images of multilinear graded polynomials on $\mathcal{A}$.

One can see that for $q = n$ we recover the natural $\mathbb{Z}_n$-grading on UT_n given by $\deg e_{ij} = j - i \pmod{n}$ for every $i \leqslant j$.

For $l = 0, 1, \dots, q-1$, let us describe the homogeneous component $\mathcal{A}_{\bar{l}}$. First of all we notice that the homogeneous components of $\mathcal{A}$ are all given in blocks in the form

$$\begin{pmatrix} A & C \\ & B \end{pmatrix},$$

where $A \in UT_{q-1}(F), B \in UT_{n-q+1}(F)$ and $C \in M_{q-1,n-q+1}(F)$. More precisely, the neutral component $\mathcal{A}_{\bar{0}}$ is such that A is a diagonal matrix, B is an arbitrary triangular matrix and $C = 0$. In other words, $\mathcal{A}_{\bar{0}} = F \oplus \cdots \oplus F \oplus UT_{n-q+1}(F)$. Concerning the homogeneous component $\mathcal{A}_{\bar{l}}$ where $l \neq 0$, we have A as matrix with non-zero entries in the $l+1$ diagonal only (the main diagonal is counted as the first one), $B = 0$ and C as a matrix with non-zero entries in the $q-l$ row only. Thus a general element in $\mathcal{A}_{\bar{l}}$ has the form

$$\begin{pmatrix} & F & & & & & \\ & & \ddots & & & & \\ & & & F & & & \\ & & & & F & F & \cdots & F \\ & & & & & & \end{pmatrix}.$$

In other words,

$$\mathcal{A}_{\bar{l}} = \operatorname{span}\{e_{i,i+l}, e_{q-l,j} \mid i = 1, \dots, q-l, j = q+1, \dots, n\}.$$

For $1 \leqslant r \leqslant n-q$ we also define the following homogeneous subspaces of $\mathcal{A}_{\bar{l}}$

$$\mathcal{B}_{\bar{l},r} = \operatorname{span}\{e_{q-l,j} \mid j = q+r, \dots, n\}.$$

Notice then that $\mathcal{B}_{\bar{l},r}$ is just the subspace of $\mathcal{A}_{\bar{l}}$ whose entire $l+1$ diagonal is zero and where the non-zero entries occur only in the $q-l$ row (some entries in this row might be zero from left to the right, depending on the value of r).

An easy computation shows that the following are graded identities for $\mathcal{A}$

$$[y_1, y_2]z = 0 \tag{3.1}$$

$$z_1 z_2 = 0 \tag{3.2}$$

$$[y_1, y_2] \cdots [y_{2(n-q+1)-1}, y_{2(n-q+1)}] = 0 \tag{3.3}$$

where the variables y_i are neutral ones, z, z_1, z_2 are non-neutral variables and $\deg(z_1) + \deg(z_2) = \bar{0}$.

We state several lemmas concerning the description of some graded polynomials on $\mathcal{A}$. In the upcoming lemmas, unless otherwise stated, we assume that the field F has at least $n(n-1)/2$ elements and $f \in F\langle X_G \rangle$ is a multilinear polynomial.

In the next two lemmas we will assume that

$$f = f(z_1, \ldots, z_l, y_{l+1}, \ldots, y_m)$$

where $\deg(z_i) = \bar{1}$, $1 \leqslant i \leqslant l$. It is obvious that in this case one must have $f(\mathcal{A})$ as a subset of $\mathcal{A}_{\bar{l}}$. Modulo the identity (3.1) we rewrite the polynomial f as

$$f = \sum_{\boldsymbol{i_1}, \ldots, \boldsymbol{i_l}} y_{\boldsymbol{i_1}} z_1 y_{\boldsymbol{i_2}} z_2 \cdots y_{\boldsymbol{i_l}} z_l g_{\boldsymbol{i_1}, \ldots, \boldsymbol{i_l}} + h \tag{3.4}$$

where $y_{\boldsymbol{i_j}} = y_{i_{j_1}} \cdots y_{i_{j_{k_j}}}$ is such that $i_{j_1} < \cdots < i_{j_{k_j}}$. Moreover $g_{\boldsymbol{i_1}, \ldots, \boldsymbol{i_l}}$ is the polynomial obtained by permuting the neutral variables whose indices are different from either of $\boldsymbol{i_1}$, $\ldots$, $\boldsymbol{i_l}$, and forming a linear combination of such monomials. Furthermore, h is the sum of polynomials that differ from the first summand of f by nontrivial permutations of the odd variables.

Among all polynomials $g_{\boldsymbol{i_1}, \ldots, \boldsymbol{i_l}}$ (including those defined analogously in h), we choose one, say g, of least commutator degree r. Up to permuting the odd variables, we can assume that the polynomial g occurs in the first summand of f.

Hence, in case $1 \leqslant r \leqslant n-q$, we can improve the inclusion $f(\mathcal{A}) \subset \mathcal{A}_{\bar{l}}$ to $f(\mathcal{A}) \subset \mathcal{B}_{\bar{l},r}$. Our goal is to prove that $f(\mathcal{A}) = \mathcal{A}_{\bar{l}}$ in case $r = 0$ and $f(\mathcal{A}) = \mathcal{B}_{\bar{l},r}$ otherwise.

Lemma 3.13. *Let $1 \leqslant l \leqslant q-1$. If $1 \leqslant r \leqslant n-q$, then $f(\mathcal{A}) = \mathcal{B}_{\bar{l},r}$.*

Proof. Let $\bar{g} = y_{\boldsymbol{i_1}} z_1 y_{\boldsymbol{i_2}} z_2 \cdots y_{\boldsymbol{i_l}} z_l g$ be the non-zero summand of f written as above, where the commutator degree of g is r. We consider the following evaluations: the variables in $y_{\boldsymbol{i_1}}$ by $e_{q-l,q-l}$, the ones in $y_{\boldsymbol{i_2}}$ by $e_{q-l+1,q-l+1}, \ldots$, and all variables in $y_{\boldsymbol{i_l}}$ by $e_{q-1,q-1}$. We also put $z_1 = e_{q-l,q-l+1}$, $z_2 = e_{q-l+1,q-l+2}$, $\ldots$, $z_{l-1} = e_{q-2,q-1}$, and $z_l = \sum_{k=q}^{n} w_k e_{q-1,k}$. Since

g has commutator degree r, Theorem 3.10 enables us to evaluate the even variables in g by matrices from

$$\begin{pmatrix} 0 & 0 \\ 0 & UT_{n-q+1} \end{pmatrix}$$

in order to obtain the matrix $e_{q,q+r} + e_{q+1,q+r+1} + \cdots + e_{n-r,n}$.

Note that the evaluations that we have considered allow us to reduce the study of the image of f to the polynomial $\overline{g}$. Under these evaluations we have

$$\begin{aligned} \overline{g} &= (w_q e_{q-l,q} + w_{q+1} e_{q-l,q+1} + \cdots + w_n e_{q-l,n})(e_{q,q+r} + e_{q+1,q+r+1} + \cdots + e_{n-r,n}) \\ &= w_q e_{q-l,q+r} + w_{q+1} e_{q-l,q+r+1} + \cdots + w_{n-r} e_{q-l,n}. \end{aligned}$$

Taking a matrix $B \in \mathcal{B}_{\overline{l},r}$, say $B = b_q e_{q-l,q+r} + \cdots + b_{n-r} e_{q-l,n}$ we can easily realize B as image of $\overline{g}$ by choosing $w_j = b_j$, $j = q, \ldots, n-r$. Hence $f(\mathcal{A}) = \mathcal{B}_{\overline{l},r}$. □

Before analysing the case of zero commutator degree, let us recall the following elementary and well known result.

Lemma 3.14. *Let F be a finite field with d elements and let $f = f(w_1, \ldots, w_m)$ be a non-zero commutative polynomial. If $\deg_{w_i}(f) \leqslant d-1$ for all $i = 1, \ldots, m$, then there exist $a_1, \ldots, a_m \in F$ such that $f(a_1, \ldots, a_m) \neq 0$.*

Corollary 3.15. *Let F be a finite field with d elements and let*

$$f_1(w_1 \ldots, w_m), \ldots, f_{d-1}(w_1, \ldots, w_m)$$

be non-zero commutative polynomials. If $\deg_{w_i}(f_j) \leqslant 1$ for all i, j, then there exist $a_1, \ldots, a_m \in F$ such that

$$f_1(a_1, \ldots, a_m) \neq 0, \ldots, f_{d-1}(a_1, \ldots, a_m) \neq 0.$$

We also recall the following well-known lemma.

Lemma 3.16. *Let $f \in F\langle X\rangle$ be multilinear. Then f has commutator degree 0 if and only if the sum of its coefficients is different from 0.*

Lemma 3.17. *If F is a field with at least n elements and $r = 0$, then $f(\mathcal{A}) = \mathcal{A}_{\overline{l}}$.*

Proof. Denoting by $\mathcal{D}$ the homogeneous subspace of diagonal matrices of $\mathcal{A}$, we consider the following homogeneous subalgebra of $\mathcal{A}$:

$$\mathcal{S} = \mathcal{D} \oplus \bigoplus_{1 \leqslant l \leqslant q-1} \mathcal{A}_{\overline{l}}.$$

We will show that $f(\mathcal{S})$ is exactly $\mathcal{A}_{\overline{l}}$ which is enough to conclude the lemma. Note that $\mathcal{S}$ still satisfies the identity (3.2) and it also satisfies $[y_1, y_2] = 0$.

By the identity $[y_1, y_2] = 0$, we may write the polynomial $\overline{g}$ (according to Lemma 3.13) as

$$\beta y_{\boldsymbol{i_1}} z_1 y_{\boldsymbol{i_2}} z_2 \cdots y_{\boldsymbol{i_l}} z_l y_{\boldsymbol{i_{l+1}}}$$

where β is the sum of all coefficients of the polynomial g and $y_{\boldsymbol{i_{l+1}}}$ is the product of the variables of g in increasing order of the indices. Since $r = 0$, we get from Definition 3.8 and Lemma 3.16 that $\beta \neq 0$.

Now we write $f = f(z_1, \ldots, z_l, y_{l+1}, \ldots, y_m)$ as

$$f = \sum_{j=1}^{l} f_j$$

where f_j is the sum of all monomials of f such that the variable z_l is in the j-th position in relation to the odd variables.

For each $j = 1, \ldots, l$, we write

$$f_j = \sum_{\sigma \in S_l^{(j)}} f_{j,\sigma}$$

where $S_l^{(j)} = \{\sigma \in S_l | \sigma(j) = l\}$, $f_{j,\sigma}$ is the sum of all monomials of f_j where the order of the odd variables is given by the permutation σ.

Taking $z_i = \sum_{k=1}^{q-1} w_k^{(i)} e_{k,k+1} + w_q^{(i)} e_{q-1,q+1} + \cdots + w_{n-1}^{(i)} e_{q-1,n}$ and $y_j = \sum_{k=1}^{n} w_k^{(j)} e_{kk}$ we have

$$f_{l,id}(z_1, \ldots, z_l, y_{l+1}, \ldots, y_m) = \sum_{k=1}^{q-l} p_k w_k^{(1)} w_{k+1}^{(2)} \cdots w_{k+l-1}^{(l)} e_{k,k+l}$$
$$+ p_{q-l+1} w_{q-l}^{(1)} \cdots w_{q-2}^{(l-1)} w_q^{(l)} e_{q-l,q+1} + \cdots + p_{n-l} w_{q-l}^{(1)} \cdots w_{q-2}^{(l-1)} w_{n-1}^{(l)} e_{q-l,n}$$

where p_k, $k = 1, \ldots, n-l$, are polynomials in the variables $w^{(l+1)}, \ldots, w^{(m)}$. We note that all polynomials p_k, $k = 1, \ldots, n-l$, are non-zero ones. Indeed, we just have to check that different monomials in $f_{l,id}$ give different monomials in p_k. To this end, note that if m_1 and m_2 are different monomials in $f_{l,id}$, then there exists some even variable y_j such that the quantity of preceding odd variables in relation to y_j is distinct in m_1 and m_2. This gives us variables $w^{(j)}$ with different lower indices in the two monomials in p_k given by m_1 and m_2, which proves our claim. Moreover, we note that every variable in each monomial of the polynomial p_k appears exactly once.

Since we have at most $n-1$ polynomials p_k, by Corollary 3.15 there exist evaluations of the even variables y_j by diagonal matrices D_j such that p_k take non-zero values simultaneously for all k. Hence

$$
\begin{aligned}
f_l = &\sum_{k=1}^{q-l}\left(\sum_{\sigma\in S_l^{(l)}}\alpha_\sigma w_k^{(\sigma(1))}\cdots w_{k+l-2}^{(\sigma(l-1))}\right)w_{k+l-1}^{(l)}e_{k,k+l}\\
&+\left(\sum_{\sigma\in S_l^{(l)}}\alpha_\sigma w_{q-l}^{(\sigma(1))}\cdots w_{q-2}^{(\sigma(l-1))}\right)w_q^{(l)}e_{q-l,q+1}\\
&+\cdots+\left(\sum_{\sigma\in S_l^{(l)}}\alpha_\sigma w_{q-l}^{(\sigma(1))}\cdots w_{q-2}^{(\sigma(l-1))}\right)w_{n-1}^{(l)}e_{q-l,n}
\end{aligned}
$$

with $\alpha_{id}\neq 0$ because the coefficients α are determined by the polynomials p_k. So the polynomials inside the brackets above are non-zero ones and each of their monomials have variables of degree one. Applying Corollary 3.15 once again we may evaluate the variables $z_1,\ldots,z_{l-1}$ by matrices in $C_1,\ldots,C_{l-1}\in\mathcal{A}_{\bar{1}}$ such that all these polynomials take non-zero values on F. Denote by $\alpha_{l,k}\in F\backslash\{0\}$, $k=1,\ldots,q-l$, the values of the polynomials inside the brackets after such evaluations.

Therefore

$$
\begin{aligned}
&f(C_1,\ldots,C_{l-1},z_l,D_{l+1},\ldots,D_m)\\
&=\sum_{k=1}^{q-l}\left(\alpha_{1,k}w_k^{(l)}+\cdots+\alpha_{l-1,k}w_{k+l-2}^{(l)}+\alpha_{l,k}w_{k+l-1}^{(l)}\right)e_{k,k+l}\\
&+\left(\alpha_{1,q}w_{q-l}^{(l)}+\cdots+\alpha_{l-1,q}w_{q-2}^{(l)}+\alpha_{l,q-l}w_q^{(l)}\right)e_{q-l,q+1}\\
&+\cdots+\left(\alpha_{1,n-1}w_{q-l}^{(l)}+\cdots+\alpha_{l-1,n-1}w_{q-2}^{(l)}+\alpha_{l,q-l}w_{n-1}^{(l)}\right)e_{q-l,n}
\end{aligned}
$$

where $\alpha_{l,k}\neq 0$ for every $k=1,\ldots,q-l$.

Then given a matrix $B=\sum_{k=1}^{q-l}b_ke_{k,k+l}+b_{q-l+1}e_{q-l,q+1}+\cdots+b_{n-l}e_{q-l,n}\in\mathcal{A}_{\bar{l}}$ we take

$$f(C_1,\ldots,C_{l-1},z_l,D_{l+1},\ldots,D_m)=B$$

and we obtain a linear system in the variables $w^{(l)}$ whose solution (not necessarily unique) can be found recursively. $\square$

Corollary 3.18. *Let $f\in F\langle X|\mathbb{Z}_q\rangle$ be a multilinear polynomial of non-neutral homogeneous degree. Then $f(\mathcal{A})$ is $\{0\},\mathcal{B}_{\bar{l},r}$ or $\mathcal{A}_{\bar{l}}$.*

Proof. By Lemmas 3.13 and 3.17, we already know the image of f on $\mathcal{A}$ in case the non-neutral variables on f are those of homogeneous degree $\bar{1}$ only. Let us now consider the general case. Modulo the graded identities (3.1), (3.2), (3.3), let f and g be as in the comments before Lemma 3.13 and let r be the commutator degree of g. Hence $f(\mathcal{A})\subset\mathcal{B}_{\bar{l},r}$ if $r\neq 0$ and $f(\mathcal{A})\subset\mathcal{A}_{\bar{l}}$ otherwise. Recalling that the image is invariant under endomorphisms of the free $\mathbb{Z}_q$-graded algebra (see Proposition 1.19(1)), the image of the polynomial $\tilde{f}$ obtained from f by evaluating every non-neutral variable z_i of homogeneous degree $\overline{k}$ by a product of k variables of homogeneous degree $\overline{1}$, is contained in $f(\mathcal{A})$. But the polynomial g defined for f (see the comments before Lemma 3.13) is the same as the one defined for $\tilde{f}$. This allows us to get $\mathcal{B}_{\overline{l},r} \subseteq \tilde{f}(\mathcal{A})$ in case $r \neq 0$ and $\mathcal{A}_{\overline{l}} \subseteq \tilde{f}(\mathcal{A})$ otherwise. □

The corollary above allows us to state the following definition.

Definition 3.19. *Let $f \in F\langle X|\mathbb{Z}_q\rangle$ be a multilinear polynomial. We say that f has right-commutator-degree r modulo $Id_{\mathbb{Z}_q}(\mathcal{A})$ if r is the minimal commutator degree of all those polynomials of the form $g_{\boldsymbol{i_1},\ldots,\boldsymbol{i_l}}$ (include those in h) appearing in f modulo $Id_{\mathbb{Z}_q}(\mathcal{A})$ in* (3.4).

We notice that the right-commutator-degree modulo $Id_{\mathbb{Z}_q}(\mathcal{A})$ of a multilinear polynomial is well-defined by the image of f on $\mathcal{A}$ via Corollary 3.18.

Theorem 3.20. *Let F be a field with at least $n(n-1)/2$ elements, let $UT_n = \bigoplus_{k\in\mathbb{Z}_q} \mathcal{A}_k$ be endowed with the elementary $\mathbb{Z}_q$-grading given by the sequence $(\overline{0}, \overline{1}, \ldots, \overline{q-2}, \overline{q-1}, \ldots, \overline{q-1})$ and let $f \in F\langle X|\mathbb{Z}_q\rangle$ be a multilinear polynomial. Then $f(UT_n)$ is*

- *$\{0\}$, if f is a graded polynomial identity for UT_n;*
- *J^r, if f is a polynomial in neutral variables and has commutator-degree r, where J stands for the Jacobson radical of $\mathcal{A}_{\overline{0}}$;*
- *$\mathcal{B}_{\overline{l},r}$, if f has right-commutator-degree r modulo $Id_{\mathbb{Z}_q}(UT_n)$, $r = 1, \ldots, n-q$;*
- *$\mathcal{A}_{\overline{l}}$, if f has right-commutator-degree 0 modulo $Id_{\mathbb{Z}_q}(UT_n)$.*

In particular, the image is always a homogeneous vector subspace.

Proof. The proof is clear from Proposition 3.11 and Corollary 3.18. □

Remark 3.21. *Considering similar computations one can easily see that the result is also valid for the elementary $\mathbb{Z}_q$-grading defined by the sequence $(\overline{q-1}, \ldots, \overline{q-1}, \overline{q-2}, \ldots, \overline{1}, \overline{0})$, where we have $n-q+1$ copies of $\overline{q-1}$ in the beginning of the sequence.*

In the next corollary we are assuming that F is a field of characteristic zero and $\mathcal{A} = UT_n$ is endowed with the elementary $\mathbb{Z}_q$-grading given by the sequence $(\overline{0}, \overline{1}, \ldots, \overline{q-2}, \overline{q-1}, \ldots, \overline{q-1})$.

Corollary 3.22. *The $T_{\mathbb{Z}_q}$-ideal $Id_{\mathbb{Z}_q}(\mathcal{A})$ is generated by the graded identities (3.1), (3.2), and (3.3).*

Proof. Let $f \in Id_{\mathbb{Z}_q}(\mathcal{A})$. Since the ground field has characteristic zero, we may assume that f is multilinear. Notice that in the proof of Theorem 3.20 and in the lemmas that precede it, we have shown that if f is not a consequence of the identities (3.1), (3.2), and (3.3), then $f(\mathcal{A}) \neq \{0\}$. In other words, if $f \in Id_{\mathbb{Z}_q}(\mathcal{A})$, then f is a consequence of the aforementioned identities. □

We recall that in case $q = n$ we have the natural $\mathbb{Z}_n$-grading on UT_n.

Corollary 3.23. *Let F be a field with at least n elements, let UT_n be endowed with the natural $\mathbb{Z}_n$-grading and let $f \in F\langle X|\mathbb{Z}_n\rangle$ be a multilinear polynomial. Then the image of f on UT_n is either zero or some homogeneous component.*

Proof. It follows from the proof of Lemma 3.17. □

Remark 3.24. *Notice that the same result also holds if we consider the natural $\mathbb{Z}$-grading on UT_n. Analogous results hold for the lower triangular matrix algebra LT_n as well (we will use this remark in the next section).*

3.5 Traceless matrices

As an application of the results obtained in the previous section, we now give a sufficient condition for the subspace of the traceless matrices to be contained in the image of a multilinear polynomial on the full matrix algebra.

We start by recalling the following result from [5].

Theorem 3.25. *[5] Let D be a division ring, $n \geqslant 2$ an integer, and $A \in M_n(D)$ a non-central matrix. Then A is similar (conjugate) to a matrix in $M_n(D)$ with at most one non-zero entry on the main diagonal. In particular, if A has trace zero, then it is similar to a matrix in $M_n(D)$ with only zeros on the main diagonal.*

Consider the natural $\mathbb{Z}$-grading on $M_n(F) = \bigoplus_{r\in\mathbb{Z}} M_n(F)_r$ given by

$$M_n(F)_r = \begin{cases} \operatorname{span}\{e_{k,k+r} \mid k = 1, \dots, n-r\}, & \text{if } 0 \leqslant r \leqslant n-1 \\ \operatorname{span}\{e_{k-r,k} \mid k = 1, \dots, n+r\}, & \text{if } -n+1 \leqslant r \leqslant -1 \\ \{0\}, & \text{elsewhere} \end{cases} \tag{3.5}$$

Theorem 3.26. *Let $n \geqslant 2$ be an integer, let F be a field with at least $(n-1)n+1$ elements where $char(F)$ does not divide n, and let $f \in F\langle X\rangle$ be a multilinear polynomial. If $f(y_1, \dots, y_{m-1}, z) \notin \langle [y_1, y_2]\rangle^{T_{\mathbb{Z}}}$ for every non-neutral variable z, then $f(M_n(F))$ contains $sl_n(F)$.*

Proof. Since $\mathrm{char}(F)$ does not divide n, then we have that any non-zero traceless matrix is non-central. Using further that $f(M_n(F))$ is invariant under automorphisms of $M_n(F)$, by Theorem 3.25 it is enough to show that $f(M_n(F))$ contains all matrices with zero diagonal. Let A be a zero diagonal matrix and write A as the sum of its homogeneous components (with respect to the $\mathbb{Z}$-grading on $M_n(F)$ given by (3.5))

$$A = \sum_{i=-n+1}^{-1} A_i + \sum_{i=1}^{n-1} A_i$$

where $A_i = \sum_{k=1}^{n+i} a_{k-i,k} e_{k-i,k}$ for $i = -n+1, \ldots, -1$ and $A_i = \sum_{k=1}^{n-i} a_{k,k+i} e_{k,k+i}$ for $i = 1, \ldots, n-1$.

By hypothesis and from Corollary 3.23 we have that $f(y_1, \ldots, y_{m-1}, z^{(i)})$ is not a graded polynomial identity for UT_n with the natural $\mathbb{Z}$-grading, for every variable $z^{(i)}$ of homogeneous degree i where $1 \leqslant i \leqslant n-1$.

We now consider the following evaluations on generic matrices: $y_j = \sum_{k=1}^{n} w_k^{(j)} e_{kk}$ for all $j = 1, \ldots, m-1$ and $z^{(i)} = \sum_{k=1}^{n-i} w_k^{(m,i)} e_{k,k+i}$.

Hence

$$f(y_1, \ldots, y_{m-1}, z^{(i)}) = \sum_{k=1}^{n-i} p_{k,i} w_k^{(m,i)} e_{k,k+i}$$

where $p_{k,i}$ is a polynomial in the variables $w_k^{(j)}$. Since $f \notin Id_{\mathbb{Z}}(UT_n)$, Corollary 3.23 gives us that the image of $f(y_1, \ldots, y_{m-1}, z^{(i)})$ on UT_n is exactly $(UT_n)_i$. Hence all $p_{k,i}$ are non-zero polynomials. Moreover notice that $p_{k,i}$ is such that all its monomials are multilinear ones.

Analogously, we also have that $f(y_1, \ldots, y_{m-1}, z^{(i)})$ is not a graded polynomial identity for the lower triangular matrix algebra LT_n endowed with the natural $\mathbb{Z}$-grading, for $i = -n+1, \ldots, -1$. Therefore

$$f(y_1, \ldots, y_{m-1}, z^{(i)}) = \sum_{k=1}^{n-i} q_{k,i} w_k^{(m,-i)} e_{k+i,k}$$

where $z^{(i)} = \sum_{k=1}^{n+i} w_k^{(m,-i)} e_{k-i,k}$ and $q_{k,i}$ are non-zero commutative polynomials with multilinear monomials.

The number of polynomials $p_{k,i}$ and $q_{k,i}$ is exactly $(n-1)n$. We now apply Corollary 3.15 to get an evaluation of all variables $w_k^{(j)}$ such that the polynomials $p_{k,i}$ and $q_{k,i}$ assume simultaneously non-zero values in F. Such evaluations give us diagonal matrices $D_1, \ldots, D_{m-1}$ such that

$$f(D_1, \ldots, D_{m-1}, z^{(i)}) = \sum_{k=1}^{n-i} \alpha_{i,k} w_k^{(m,i)} e_{k,k+i}$$

where $\alpha_{i,k}$ are non-zero scalars. Thus each matrix A_i can be realized as $f(D_1, \dots, D_{m-1}, B_i)$ for a suitable matrix $B_i \in (UT_n)_i$, for every $i = 1, \dots, n-1$. Similarly we also have that each matrix A_i can be realized as $f(D_1, \dots, D_{m-1}, C_i)$ for a suitable matrix $C_i \in (LT_n)_i$, for all $i = -n+1, \dots, -1$. Hence

$$A = \sum_{i=-n+1}^{-1} A_i + \sum_{i=1}^{n-1} A_i = \sum_{i=-n+1}^{-1} f(D_1, \dots, D_{m-1}, C_i) + \sum_{i=1}^{n-1} f(D_1, \dots, D_{m-1}, B_i)$$

and it is enough to use the linearity of f in one variable to get $A \in f(M_n(F))$. □

Corollary 3.27. *Let* $\mathrm{char}(F) = 0$ *and consider the multilinear polynomial*

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_{m-1}} \alpha_\sigma [x_m, x_{\sigma(1)}, \dots, x_{\sigma(m-1)}] \in F\langle X \rangle,$$

where $\sum_{\sigma \in S_{m-1}} \alpha_\sigma \neq 0$*. Then* $f(M_n(F)) = sl_n(F)$*.*

Proof. Consider the polynomial

$$f(y_1, \dots, y_{m-1}, z_m) \in F\langle X_{\mathbb{Z}} \rangle.$$

In light of the Jacobi identity, one can see that modulo the $T_{\mathbb{Z}}$-ideal $\langle [y_1, y_2] \rangle^{T_{\mathbb{Z}}}$ we have

$$[z_m, y_{\sigma(1)}, \dots, y_{\sigma(m-1)}] = [z_m, y_1, \dots, y_{m-1}]$$

for all $\sigma \in S_{m-1}$. Hence, modulo $\langle [y_1, y_2] \rangle^{T_{\mathbb{Z}}}$, we can write f as

$$f = (\sum_{\sigma \in S_{m-1}} \alpha_\sigma)[z_m, y_1, \dots, y_{m-1}].$$

Hence, for $j = 1, \dots, m-1$ we take $D_j = \sum_{i=1}^{n} i e_{ii}$ and $D_m \in M_n(F)$ a non-zero non-neutral homogeneous element concerning the grading given in (3.5), and now one can easily check that

$$f(D_1, \dots, D_{m-1}, D_m) = (\sum_{\sigma \in S_{m-1}} \alpha_\sigma) D_m \neq 0.$$

This implies in $f \notin \langle [y_1, y_2] \rangle^{T_{\mathbb{Z}}}$. Now it is enough to apply Theorem 3.26 to get $sl_n(F) \subset f(M_n(F))$. Since the opposite inclusion is trivial, we get the equality. □

We notice that the corollary above recovers the Shoda's result [56] about commutators on the full matrix algebra.

3.6 Small dimension cases

During this section we will deal with the images of multilinear graded polynomials on upper triangular matrices of small dimension and endowed with different additional structures. We will see that a way clearer picture can be obtained when we have small dimension. Precisely in the associative and Jordan settings the image is always a homogeneous vector space, regardless of the grading. However in the involution case we will see that there exist multilinear polynomials such that their images on the upper triangular matrix algebra is not a vector space.

3.6.1 The associative setting

Throughout this section F will denote an arbitrary field.

Let us start with the case of 2×2 matrices. This is actually an easy consequence of Proposition 1.22.

Theorem 3.28. *Let $UT_2 = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be some grading on $\mathcal{A}$ and let $f \in F\langle X_G \rangle$ be a multilinear graded polynomial. Then $f(\mathcal{A})$ is a homogeneous subspace of $\mathcal{A}$.*

Proof. By Theorem 3.2 and Proposition 1.22, it is enough to consider images of multilinear graded polynomials on elementary gradings only. We notice that just two elementary G-gradings can be defined on $\mathcal{A} = UT_2$. Indeed, an elementary grading on UT_2 is completely determined by the homogeneous degree of e_{12}. If $\deg(e_{12}) = 1$, then we have the trivial grading, and we apply Theorem 3.10. Hence we assume $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}\}$ and $\mathcal{A}_g = \operatorname{span}\{e_{12}\}$, where $g \neq 1$. In this grading the images of multilinear polynomials in neutral variables are handled by Lemma 3.16 and Proposition 1.19 (2). Since $\mathcal{A}_g^2 = \{0\}$, it is enough to consider multilinear polynomials in one variable of homogeneous degree g and all remaining variables of neutral degree. In this case the image is contained in $\mathcal{A}_g$ and by Proposition 1.19 (3) we are done. $\square$

Now we prove an analogous fact to Theorem 3.28 with $\mathcal{A} = UT_3$ instead of UT_2. From now on in this subsection we assume that $\mathcal{A}$ is endowed with some elementary G-grading given by a tuple $(g_1, g_2) \in G^2$. Hence $g_1 = \deg(e_{12})$, $g_2 = \deg(e_{23})$, and $g_3 := g_1 g_2 = \deg(e_{13})$.

Hence the elementary gradings on UT_3 are exactly the ones given by the following relations.

(I) $\{1\} \cap \{g_1, g_2, g_3\} \neq \varnothing$.

(a) $g_1 = g_2 = 1$, which implies $g_3 = 1$;

(b) $g_1 = 1$ and $g_2 \neq 1$, which implies $g_3 = g_2$;

(c) $g_2 = 1$ and $g_1 \neq 1$, which implies $g_3 = g_1$;

(d) $g_3 = 1$, $g_1 \neq 1$ and $g_1 = g_2$;

(e) $g_3 = 1, g_1 \neq 1, g_2 \neq 1$ and $g_1 \neq g_2$.

(II) $\{1\} \cap \{g_1, g_2, g_3\} = \varnothing$.

(a) $1, g_1, g_2, g_3$ are pairwise distinct elements;

(b) $g_1 = g_2 \neq g_3$.

In the following lemmas we discuss the grading on UT_3 determined by each relation above and we give a precise description of the respective image of a multilinear graded polynomial on such a graded algebra.

Lemma 3.29. *Let UT_3 be endowed with the grading (I)(b). Then $f(UT_3)$ is a homogeneous subspace.*

Proof. We denote $g_2 = g$, then we have $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}, e_{12}\}$ and $\mathcal{A}_g = \operatorname{span}\{e_{13}, e_{23}\}$. We notice that $\mathcal{A}_g^2 = \{0\}$ and hence we only need to analyse multilinear polynomials in at most one variable of homogeneous degree g.

The case when f is a multilinear polynomial in neutral variables is settled by Lemma 3.16 and Proposition 1.19(2).

Now we consider f as a multilinear polynomial in one non-neutral variable and $m-1$ neutral ones. Since $\mathcal{A}$ satisfies the graded identity $z[y_1, y_2] = 0$, then modulo this identity we write f as

$$\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m-1} h_{i_1,\ldots,i_k} z_m y_{i_1} \cdots y_{i_k}.$$

If all polynomials $h_{i_1,\ldots,i_k}$ have commutator degree different from 0, then $f(UT_3) \subset \operatorname{span}\{e_{13}\}$ and then we apply Proposition 1.19(3). Otherwise we may assume without loss of generality that $h_{1,\ldots,k}$ has commutator degree 0. Then we perform the following evaluations: $y_1 = \cdots = y_k = e_{33}$, $y_j = e_{11} + e_{22}$ for every $j \notin \{1, \ldots, k\}$, and $z_m = \alpha^{-1}(a_1 e_{13} + a_2 e_{23})$, where α is the sum of the coefficients of $h_{1,\ldots,k}$. Notice that under such an evaluation we have $a_1 e_{13} + a_2 e_{23} \in f(UT_3)$ which proves that $f(UT_3) = \mathcal{A}_g$. □

Lemma 3.30. *Let UT_3 be endowed with the grading (I)(c). Then $f(UT_3)$ is a homogeneous subspace.*

Proof. Notice that $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}, e_{23}\}$, $\mathcal{A}_{g_1} = \operatorname{span}\{e_{12}, e_{13}\}$ and also that $\mathcal{A}$ satisfies the identities $[y_1, y_2]z = 0$ and $z_1 z_2 = 0$. Thus, the proof is similar to the one for (I)(b). □

Lemma 3.31. *Let UT_3 be endowed with the grading (I)(e). Then $f(UT_3)$ is a homogeneous subspace.*

Proof. Here we must have $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$, $\mathcal{A}_{g_1} = \operatorname{span}\{e_{12}\}$, $\mathcal{A}_{g_2} = \operatorname{span}\{e_{23}\}$. Notice that $\mathcal{A}_{g_1}^2 = \mathcal{A}_{g_2}^2 = \{0\}$, $\mathcal{A}_{g_2}\mathcal{A}_{g_1} = \{0\}$, and $\mathcal{A}_{g_1}\mathcal{A}_{g_2} \subset \operatorname{span}\{e_{13}\}$.

The case when f is a multilinear polynomial in neutral variables can be treated as in the grading (I)(b). Hence we may consider f is a multilinear polynomial in: one variable of degree g_1 (respectively g_2) and $m-1$ neutral variables, or in one variable of degree g_1, one of degree g_2 and $m-2$ neutral ones. In each of these situations we have that $f(UT_3)$ is contained in a one-dimensional space and we apply Proposition 1.19(3). $\square$

Lemma 3.32. *Let UT_3 be endowed with the grading (II)(a). Then $f(UT_3)$ is a homogeneous subspace.*

Proof. We have $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_{g_1} = \operatorname{span}\{e_{12}\}$, $\mathcal{A}_{g_2} = \operatorname{span}\{e_{23}\}$, and $\mathcal{A}_{g_3} = \operatorname{span}\{e_{13}\}$. The only nontrivial relation among the non-neutral homogeneous components is given by $\mathcal{A}_{g_1}\mathcal{A}_{g_2} = \mathcal{A}_{g_3}$.

The case of f in neutral variables is the same as for the grading (I)(b).

Since the non-neutral components are one-dimensional, then the image of a multilinear polynomial in one non-neutral variable and $m-1$ neutral ones is always zero or the respective homogeneous component.

In case f has one variable of homogeneous degree g_1, one of degree g_2 and $m-2$ neutral ones then the image is contained in $\mathcal{A}_{g_2}$, and we are done. $\square$

Lemma 3.33. *Let UT_3 be endowed with the grading (II)(b). Then $f(UT_3)$ is a homogeneous subspace.*

Proof. Note that $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_{g_1} = \operatorname{span}\{e_{12}, e_{23}\}$ and $\mathcal{A}_{g_3} = \operatorname{span}\{e_{13}\}$. We only need to consider the case when f is a multilinear polynomial in $m-1$ neutral variables and one of homogeneous degree g_1, since the remaining cases can be treated as above. We write $f = \sum_{j=1}^{m} f_j$ where f_j is the sum of all monomials from f which contain the variable z_m in the j-th position. Hence, modulo $[y_1, y_2] = 0$ we have

$$f_j = \sum_{1 \leqslant i_1 < \cdots < i_{j-1} \leqslant m-1} \alpha_{i_1,\ldots,i_{j-1}} y_{i_1} \cdots y_{i_{j-1}} z_m y_{k_1} \cdots y_{k_{m-j}}$$

where $k_1, \ldots, k_{m-j} \in \{1, \ldots, m-1\}$ are such that $k_1 < \cdots < k_{m-j}$. We evaluate $y_i = w_1^{(i)} e_{11} + e_{22} + w_3^{(i)} e_{33}$ and $z_m = w_1^{(m)} e_{12} + w_2^{(m)} e_{23}$. Thus $f(y_1, \ldots, y_{m-1}, z_m)$ is given by

$$\begin{pmatrix} 0 & p_1(w_1^{(1)},\dots,w_1^{(m-1)})w_1^{(m)} & 0 \\ & 0 & p_2(w_3^{(1)},\dots,w_3^{(m-1)})w_2^{(m)} \\ & & 0 \end{pmatrix}$$

where $p_1(w_1^{(1)},\dots,w_1^{(m-1)}) = \sum_{j=1}^{m} \sum_{1\leqslant i_1<\cdots<i_{j-1}\leqslant m-1} \alpha_{i_1,\dots,i_{j-1}} w_1^{(i_1)}\cdots w_1^{(i_{j-1})}$ and p_2 is given analogously.

We claim that p_1 takes non-zero values on F. Indeed, assume that p_1 is a polynomial identity for F and denote $e_j = \sum_{1\leqslant i_1<\cdots<i_{j-1}\leqslant m-1} \alpha_{i_1,\dots,i_{j-1}} w_1^{(i_1)}\cdots w_1^{(i_{j-1})}$. Hence

$$p_1 = \sum_{j=1}^{m} e_j.$$

Notice that e_1 is a commutative polynomial and taking $w_1^{(1)} = \cdots = w_1^{(m-1)} = 0$ we have $e_1 = 0$. Taking $w_1^{(l)} = 1$ and zero for the remaining values w_1's we have $\alpha_l = 0$ for all $l \in \{1,\dots,m-1\}$ and hence $e_2 = 0$. Now assume $e_l = 0$ for all $l < k$, and we shall prove that $e_k = 0$. For each chosen $i_1, \dots, i_{k-1}$ we take $w_1^{(r)} = 0$ for all $r \notin \{i_1,\dots,i_{k-1}\}$, then $e_l = 0$ for all $l > k$ and $e_k = \alpha_{i_1,\dots,i_{k-1}} w_1^{(i_1)}\cdots w_1^{(i_{k-1})}$. Then we take $w_1^{(i_1)} = \cdots = w_1^{(i_{k-1})} = 1$ and we conclude that $\alpha_{i_1,\dots,i_{k-1}} = 0$. Hence $p_1 = 0$, which is a contradiction. An analogous claim holds for p_2. Therefore it is enough to use the variables $w_1^{(m)}$ and $w_2^{(m)}$ to realize any matrix in $\mathcal{A}_{g_1}$ in the image of f on UT_3. □

Lemma 3.34. *Let UT_3 be endowed with the grading (I)(d). Then $f(UT_3)$ is a homogeneous subspace.*

Proof. We denote $g = g_1$ and notice that $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$ and $\mathcal{A}_g = \operatorname{span}\{e_{12}, e_{23}\}$. Then $\mathcal{A}_g^2 \subset \operatorname{span}\{e_{13}\}$ and $\mathcal{A}$ satisfies the identities $z[y_1, y_2] = 0$ and $[y_1, y_2]z = 0$. The case when f has one variable of homogeneous degree g and $m-1$ neutral variables can be treated as in the previous lemma. The remaining cases are considered as above. □

Hence we have the following theorem.

Theorem 3.35. *Let F be an arbitrary field, let $UT_3 = \mathcal{A} = \bigoplus_{g\in G} A_g$ be some nontrivial grading on $\mathcal{A}$, and let $f \in F\langle X_G\rangle$ be a multilinear graded polynomial. Then $f(\mathcal{A})$ is a homogeneous subspace of $\mathcal{A}$. If $|F| \geqslant 3$ and $\mathcal{A}$ is equipped with the trivial grading, then the image is also a subspace.*

Proof. The proof is clear from the previous lemmas and Proposition 1.22. □

3.6.2 The Jordan setting

Throughout this subsection we assume that F is a field of characteristic different from 2 and we denote by UJ_n the Jordan algebra of the upper triangular matrices with product $a \circ b = (ab + ba)/2$. Unlike the associative setting, gradings on UJ_n are not only elementary ones. Actually, a second kind of gradings also occurs on UJ_n, the so-called mirror type gradings, and we define these below. First of all let us introduce the following notation. Let i, m be non negative integers and set

$$E^+_{i:m} = e_{i,i+m} + e_{n-i-m+1,n-i+1} \text{ and } E^-_{i:m} = e_{i,i+m} - e_{n-i-m+1,n-i+1}.$$

Definition 3.36. *A G-grading on UJ_n is called of mirror type if the matrices $E^+_{i:m}$ and $E^-_{i:m}$ are homogeneous, and $\deg(E^+_{i:m}) \neq \deg(E^-_{i:m})$.*

We recall the following theorem from [46].

Theorem 3.37. *The G-gradings on the Jordan algebra UJ_n are, up to a graded isomorphism, elementary or of mirror type. Moreover the support of a G-grading on UJ_2 is always commutative.*

In particular we have the following classification of the gradings on UJ_2.

Proposition 3.38. *Up to a graded isomorphism, the gradings on UJ_2 are given by $UJ_2 = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ where*

(I) elementary ones

(a) trivial grading;

(b) $\mathcal{A}_1 = Fe_{11} + Fe_{22}$, $\mathcal{A}_g = Fe_{12}$,

(II) mirror type ones

(a) $\mathcal{A}_1 = F(e_{11} + e_{22})$, $\mathcal{A}_g = F(e_{11} - e_{22}) + Fe_{12}$;

(b) $\mathcal{A}_1 = F(e_{11} + e_{22}) + Fe_{12}$, $\mathcal{A}_g = F(e_{11} - e_{22})$;

(c) $\mathcal{A}_1 = F(e_{11} + e_{22})$, $\mathcal{A}_g = F(e_{11} - e_{22})$, $\mathcal{A}_h = Fe_{12}$,

where $g \in G$ is an element of order 2.

Next we describe precisely the image of a multilinear graded Jordan polynomial f on some gradings considered above.

Lemma 3.39. *Let UJ_2 be endowed with the grading (I)(b) and let $f \in \mathcal{J}(X_G)$ be a multilinear polynomial. Then $f(UJ_2)$ is a homogeneous subspace.*

Proof. We start with a multilinear polynomial f in m neutral variables. We evaluate each variable y_i to an arbitrary diagonal matrix D_i. Therefore each monomial $\mathbf{m}$ in f is evaluated to $\beta D_1 \cdots D_m$, where $\beta \in F$ is the coefficient of $\mathbf{m}$. Hence

$$f(D_1, \ldots, D_m) = \alpha D_1 \cdots D_m$$

where $\alpha \in F$ is the sum of all coefficients of f. In case $\alpha = 0$, then $f = 0$ is a graded polynomial identity for UJ_2, otherwise we can take $D_2 = \cdots = D_m = I_2$ and use D_1 in order to obtain every diagonal matrix in the image of f on UJ_2.

Since UJ_2 satisfies the graded identity $z_1 z_2 = 0$ such that $\deg(z_1) = \deg(z_2) = g$, then we only need to analyse the case where f is a multilinear polynomial in $m-1$ neutral variables and one of homogeneous degree g. Obviously we must have $f(UJ_2) \subset \mathcal{A}_g$ and this homogeneous component is one-dimensional, then we are done. □

For the grading (II)(a) we recall the following lemma from [34] applied to multilinear polynomials. If no brackets are given in a product, we assume these left-normed, that is $abc = (ab)c$.

Lemma 3.40. *Let UJ_2 be endowed with the grading (II)(a) and let $f \in \mathcal{J}(X_G)_g$ be a multilinear polynomial. Then, modulo the graded identities of UJ_2, we can write f as a linear combination of monomials of the type*

$$y_1 \cdots y_l z_{i_0}(z_{i_1} z_{i_2}) \cdots (z_{i_{2m-1}} z_{i_{2m}}), 1 < \cdots < l, i_1 < i_2 < i_3 < \cdots < i_m < i_{m+1}, i_0 > 0.$$

Lemma 3.41. *Let UJ_2 be endowed with the grading (II)(a). Then $f(UJ_2)$ is a homogeneous subspace.*

Proof. Since $\dim(\mathcal{A}_1) = 1$ it follows that if the image of a multilinear polynomial on UJ_2 is contained in $\mathcal{A}_1$ then it must be either $\{0\}$ or $\mathcal{A}_1$.

Now we consider a multilinear polynomial f in homogeneous variables of degree 1 and g such that $\deg(f) = g$. Let $\boldsymbol{m} = y_1 \cdots y_l z_{i_0}(z_{i_1} z_{i_2}) \cdots (z_{i_{2m-1}} z_{i_{2m}})$ be a monomial as in Lemma 3.40. We notice that the main diagonal of a matrix in $\boldsymbol{m}(UJ_2)$ is such that the entry (k,k) is given by $(-1)^{k+1}a$, where a is the product of the entries at position $(1,1)$ of all matrices y and z. Hence every matrix in $f(UJ_2)$ is of the form

$$\begin{pmatrix} \alpha \cdot a & * \\ & -\alpha \cdot a \end{pmatrix}$$

where α is the sum of all coefficients of f.

In case $\alpha = 0$, then $f(UJ_2) \subset \mathrm{span}\{e_{12}\}$ and then the image is completely determined.

We consider now $\alpha \neq 0$. Our goal is to prove that $f(UJ_2) = \mathcal{A}_g$. Without loss of generality, we assume that the non-zero scalar occurs in the monomial $y_1 \cdots y_l z_0(z_1 z_2) \cdots (z_{2m-1} z_{2m})$. Then we take the following evaluation: $y_1 = \cdots = y_l = I_2$, $z_0 = w_1(e_{11} - e_{22}) + w_2 e_{12}$ and $z_i = e_{11} - e_{22}$ for every $i = 1, \ldots, 2m$, where w_1, w_2 are commutative variables. Therefore

$$f(y_1, \ldots, y_l, z_0, \ldots, z_{2m}) = \begin{pmatrix} \alpha w_1 & w_2 \\ & -\alpha w_1 \end{pmatrix}.$$

Since $\alpha \neq 0$, it follows that $f(UJ_2) = \mathcal{A}_g$. $\square$

Now we consider the grading (II)(b) and we recall another lemma from [34] that we state in the following.

Lemma 3.42. *Let $f \in \mathcal{J}(X)_1$ be a multilinear polynomial. Then, modulo the graded identities of UJ_2, f can be written as a linear combination of monomials of the form*

1. $(y_{i_1} \cdots y_{i_r})(z_{j_1} \cdots z_{j_l})$;
2. $(((y_i z_{j_1}) z_{j_2}) y_{i_1} \cdots y_{i_r}) z_{j_3} \cdots z_{j_l}$,

where $l \geqslant 0$ is even, $r \geqslant 0$, $i_1 < \cdots < i_r$, and $z_{j_1} < z_{j_2} < z_{j_3} < \cdots < z_{j_l}$.

Lemma 3.43. *Let UJ_2 be endowed with the grading (II)(b). Then $f(UJ_2)$ is a homogeneous subspace.*

Proof. We start with a multilinear polynomial f in m neutral variables. Notice that UJ_2 satisfies the graded identity $(y_1, y_2, y_3) = 0$. Hence, modulo the graded identities of UJ_2 we can write f as

$$f = \alpha y_1 \cdots y_m,$$

where α is the sum of the coefficients in f. If $\alpha = 0$, then f is a graded identity of UJ_2 and we are done. In case $\alpha \neq 0$ it is enough to take $y_2 = \cdots = y_m = I_2$ and use y_1 to realize an arbitrary element from $\mathcal{A}_1$ in the image of f. This implies in $f(UJ_2) = \mathcal{A}_1$.

Now we consider a multilinear polynomial f which has at least one variable of homogeneous degree g. In case $\deg(f) = g$, then $f(UJ_2)$ is completely determined, since $\dim(\mathcal{A}_g) = 1$. So we assume $\deg(f) = 1$. In case f is a multilinear polynomial in variables of homogeneous degree g, then $f(UJ_2)$ is contained in the vector space of the scalar matrices, and therefore the image is completely determined. Hence we assume further that f has at least one variable of neutral degree and let f be a multilinear polynomial in l neutral variables $y_1, \ldots, y_l$, and $m - l$ variables $z_{l+1}, \ldots, z_m$ of homogeneous degree g. Then by Lemma 3.42, we write f as

$$f = \alpha_1(y_1 \cdots y_l)(z_{l+1} \cdots z_m) + \sum_{i=1}^{l} \alpha_{i+1}(((y_i z_{l+1}) z_{l+2}) y_1 \cdots \widehat{y_i} \cdots y_l) z_{l+3} \cdots z_m.$$

Here $\widehat{y}_i$ means that the variable y_i does not appear in the product $y_1 \cdots \widehat{y}_i \cdots y_l$.

We replace $y_i = w_1^{(i)}(e_{11} + e_{22}) + w_2^{(i)} e_{12}$ and $z_j = w_1^{(j)}(e_{11} - e_{22})$, where the w's are commuting variables. Notice that the Jordan product of two matrices y_1 and y_2 is given by $y_1 \cdot y_2$ where the dot $\cdot$ stands for the usual product of matrices. On the other hand, the usual product of n matrices $y_1, \ldots, y_n$ is given by

$$\begin{pmatrix} w_1^{(1)} \cdots w_n^{(n)} & w \\ & w_1^{(1)} \cdots w_n^{(n)} \end{pmatrix},$$

where

$$w = \sum_{\substack{1 \leqslant i_1 < \cdots < i_{n-1} \leqslant n \\ i_n \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_{n-1}\}}} w_1^{(i_1)} \cdots w_1^{(i_{n-1})} w_2^{(i_n)},$$

as one can see by induction on n. Hence the image of the monomial

$$\alpha_1 (y_1 \cdots y_l)(z_{l+1} \cdots z_m)$$

on UJ_2 is equal to

$$\alpha_1 \begin{pmatrix} w_1^{(1)} \cdots w_1^{(m)} & \sum_{\substack{1 \leqslant i_1 < \cdots < i_{l-1} \leqslant l \\ i_l \in \{1,\ldots,l\} \setminus \{i_1,\ldots,i_{l-1}\}}} w_1^{(i_1)} \cdots w_1^{(i_{l-1})} w_2^{(i_l)} w_1^{(l+1)} \cdots w_1^{(m)} \\ & w_1^{(1)} \cdots w_1^{(m)} \end{pmatrix},$$

while the image of

$$\alpha_{i+1}(((y_i z_{l+1}) z_{l+2}) y_1 \cdots \widehat{y}_i \cdots y_l) z_{l+3} \cdots z_m$$

on UJ_2 is equal to

$$\alpha_{i+1} \begin{pmatrix} w_1^{(1)} \cdots w_1^{(m)} & \\ & w_1^{(1)} \cdots w_1^{(m)} \end{pmatrix}.$$

Therefore the main diagonal of $f(y_1, \ldots, y_l, z_{l+1}, \ldots, z_m)$ is given by

$$\alpha w_1^{(1)} \cdots w_1^{(m)} (e_{11} + e_{22})$$

where α is the sum of all coefficients in f. The entry at position $(1, 2)$ is given by

$$\sum_{\substack{1 \leqslant i_1 < \cdots < i_{l-1} \leqslant l \\ i_l \in \{1,\ldots,l\} \setminus \{i_1,\ldots,i_{l-1}\}}} w_1^{(i_1)} \cdots w_1^{(i_{l-1})} w_2^{(i_l)} w_1^{(l+1)} \cdots w_1^{(m)} e_{12}.$$

If $\alpha = 0$, then $f(UJ_2) \subset \operatorname{span}\{e_{12}\}$ and we are done. We thus assume $\alpha \neq 0$ and we notice that we may also assume $\alpha_1 \neq 0$, otherwise $f(UJ_2) \subset \operatorname{span}\{e_{11} + e_{22}\}$ which is an one-dimensional subspace.

We therefore take

- $w_1^{(i)} = 1$ for $i = 2, \ldots, m$;

- $w_2^{(i)} = 0$ for $i = 2, \ldots, l$.

These evaluations imply that

$$f(y_1, e_{11} + e_{22}, \ldots, e_{11} + e_{22}, e_{11} - e_{22}, \ldots, e_{11} - e_{22}) = \begin{pmatrix} \alpha w_1^{(1)} & \alpha_1 w_2^{(1)} \\ & \alpha w_1^{(1)} \end{pmatrix}.$$

Since both α and α_1 are non-zero scalars, then we can realize any matrix from $\mathcal{A}_1$ in the image of f on UJ_2, that is, $f(UJ_2) = \mathcal{A}_1$. □

Theorem 3.44. *Let $UJ_2 = \bigoplus_{g \in G} \mathcal{A}_g$ be a G-grading and let $f \in \mathcal{J}(X_G)$ be a multilinear graded Jordan polynomial. Then $f(UJ_2)$ is a homogeneous subspace.*

Proof. We first consider a nontrivial grading on UJ_2. By Remark 1.23 we may reduce the defined grading on UJ_2 to one of those described above. We note that the case of the grading (II)(c) follows from the fact that $f(UJ_2)$ is entirely contained in some homogeneous component and all of them are one-dimensional. We use Lemmas 3.39, 3.41 and 3.43 for the remaining nontrivial gradings. Now we consider the trivial grading on UJ_2. Let $f \in J(X)$ be a multilinear polynomial. We may assume that $f \notin Id(UJ_2)$. By [57], the algebra $J(X)/Id(UJ_2)$ is a special Jordan algebra, and hence we may assume f as an element in the free special Jordan algebra. Therefore, the image $f(UJ_2)$ is equal to the image of some associative polynomial on UT_2. Hence $f(UJ_2) \in \{J, UJ_2\}$ where $J = Jac(UT_2)$. □

Remark 3.45. *Consider the Lie algebra $UT_n^{(-)}$ with product given by the Lie bracket. Given a grading on $UT_n^{(-)}$, we notice that $J = [UT_n^{(-)}, UT_n^{(-)}]$ is always a homogeneous ideal. We also notice that if $f \in \mathcal{L}(X_G)$ is a multilinear polynomial of degree $\geqslant 2$, then $f(UT_n^{(-)})$ is contained in J. In particular, for $n = 2$ we must have that $f(UT_2^{(-)})$ is contained in* $\operatorname{span}\{e_{12}\}$ *which is a homogeneous subspace. Since the image of multilinear polynomials of degree* 1 *is trivial, we have that $f(UT_2^{(-)})$ is always a homogeneous subspace, regardless of the grading defined on $UT_2^{(-)}$.*

3.6.3 The natural elementary $\mathbb{Z}_3$-grading in the Jordan algebra UJ_3

In this section we study images of multilinear polynomials on the Jordan algebra $\mathcal{A} = UJ_3$ endowed with the elementary $\mathbb{Z}_3$-grading given by the sequence $(\overline{0}, \overline{1}, \overline{2})$, that is, $\mathcal{A}_{\overline{0}} = \operatorname{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_{\overline{1}} = \operatorname{span}\{e_{12}, e_{23}\}$, and $\mathcal{A}_{\overline{2}} = \operatorname{span}\{e_{13}\}$. We assume the base field is of characteristic different from 2.

We recall the following identity which holds in any Jordan algebra.

Lemma 3.46. *Let $\mathcal{J}$ be a Jordan algebra. Then*

$$abcd + adcb + bdca = (ab)(cd) + (ac)(bd) + (ad)(bc)$$

for all a, b, c, $d \in \mathcal{J}$.

Proof. See for example [37, Page 34]. □

As an easy consequence of Lemma 3.46 we have

$$abcd + adcb + bdca = abdc + acdb + bcda \tag{3.6}$$

for every a, b, c, $d \in \mathcal{J}$.

The next lemma points out some graded identities for the algebra UJ_3.

Lemma 3.47. *The identities*

$$(y_1, y_2, y_3) = 0, (y_1, z, y_2) = 0 \text{ and } z_1 z_2 = 0$$

hold for UJ_3*, where* z, z_1, z_2 *are odd variables and either* $\deg(z_1) + \deg(z_2) = \overline{0}$ *or* $\deg(z_1) = \deg(z_2) = \overline{2}$.

Proof. A straightforward computation, hence it is omitted. □

The next lemma has the same proof as [34, Lemma 5.3]. However we will consider its proof here for the sake of completeness.

Lemma 3.48. *The polynomial*

$$g = y_1(y_2(y_3z)) - \frac{1}{2}\Big(y_1(z(y_2y_3)) + y_2(z(y_1y_3)) + y_3(z(y_1y_2)) - z(y_1(y_2y_3))\Big)$$

is a consequence of (y_1, z, y_2)*, where* $\deg(z) \in \{\overline{1}, \overline{2}\}$.

Proof. Taking $a = y_2, b = y_3, c = z$ and $d = y_1$ in the identity (3.6) we have

$$-((y_2y_3)z)y_1 - ((y_1y_3)z)y_2 - ((y_1y_2)z)y_3 + ((y_2y_3)y_1)z = -((y_2z)y_1)y_3 - ((y_3z)y_1)y_2.$$

Hence, we can write $h = 2g$ as

$$\begin{aligned} h &= 2(y_1(y_2(y_3z)) - ((y_2z)y_1)y_3 - ((y_3z)y_1)y_2 \\ &= (y_3, z, y_2)y_1 + (y_2, zy_3, y_1) + (y_3, zy_2, y_1) \end{aligned}$$

which implies that g is a consequence of (y_1, z, y_2). □

Given two even variables y_i and y_j we set $y_i < y_j$ if $i < j$. Hence we define an order on words in even variables $Y_1 < Y_2$ considering the left lexicographic order in case Y_1 and Y_2 have the same length, and $Y_1 < Y_2$ in case Y_2 is longer than Y_1. For the next lemma we use ideas from [34, Lemma 5.6]. We denote by T the T_G-ideal generated by the identities from Lemma 3.47.

Lemma 3.49. *Let $f = f(y_1, \ldots, y_{m-1}, z_m) \in J(X_{\mathbb{Z}_3})$ be a multilinear polynomial, where $\deg(z_m) \in \{\overline{1}, \overline{2}\}$. Then modulo T, f is a linear combination of monomials of the form $Y_1(zY_2)$, where each Y_i is an increasingly ordered product of even variables and $Y_1 < Y_2$.*

Proof. It is enough to consider $f = f(y_1, \ldots, y_{m-1}, z_m)$ as a monomial. We apply induction on m. If $m = 1$ or $m = 2$, then the conclusion is obvious. So we assume $m \geqslant 3$ and we write $f = gh$ where g, $h \in J(X_G)$. Without loss of generality we may assume that the odd variable z_m occurs in g. Hence $h = Y_1$ and by the induction hypothesis we must have g as a linear combination of monomials of the form $Y_2(zY_3)$. On the other hand the Lemma 3.48 gives us that

$$(Y_2(zY_3))Y_1 = \frac{1}{2}\bigg(Y_1(z(Y_2Y_3)) + Y_2(z(Y_1Y_3)) + Y_3(z(Y_1Y_2)) - z(Y_1(Y_2Y_3))\bigg).$$

Now it is enough to use the identities $(y_1, y_2, y_3) = 0$, $(y_1, z, y_2) = 0$ and the commutativity of the Jordan product to get that each monomial inside the bracket on the right side of the equation above is actually in the desired form. □

Theorem 3.50. *Let F be an infinite field of characteristic different from 2 and let $f \in J(X_{\mathbb{Z}_3})$ be a multilinear graded polynomial. Then the image of f on the graded Jordan algebra UJ_3 endowed with the natural elementary $\mathbb{Z}_3$-grading is either $\{0\}$ or some homogeneous component.*

Proof. Since f is a homogeneous element in the graded algebra $J(X_{\mathbb{Z}_3})$ and $z_1z_2 = 0$ holds on UJ_3, for either $\deg z_1 + \deg z_2 = \overline{0}$ or $\deg(z_1) = \deg(z_2) = \overline{2}$, we will consider the following three cases in our proof.

Case 1: $\deg f = \overline{0}$. Here we must have $f = f(y_1, \ldots, y_m)$ and the proof is the same as the first paragraph of the proof of Lemma 3.39.

Case 2: $\deg f = \overline{1}$. Let $f = f(y_1, \ldots, y_{m-1}, z_m)$ be such that $\deg z_m = \overline{1}$. By Lemma 3.49, modulo T, we may write f as a linear combination of monomials of the form $Y_1(z_mY_2)$, where $Y_1 < Y_2$. On the other hand, given

$$y_i = \sum_{k=1}^{3} w_k^{(i)} e_{kk} \text{ and } z_m = w_1^{(m)} e_{12} + w_2^{(m)} e_{23}, \tag{3.7}$$

note that $y_iz_m = \frac{1}{2}\begin{pmatrix} 0 & (w_1^{(i)} + w_2^{(i)})w_1^{(m)} & 0 \\ & 0 & (w_2^{(i)} + w_3^{(i)})w_2^{(m)} \\ & & 0 \end{pmatrix}$ and then

$$f(y_1, \ldots, y_{m-1}, z_m) = \begin{pmatrix} 0 & p_1w_1^{(m)} & 0 \\ & 0 & p_2w_2^{(m)} \\ & & 0 \end{pmatrix}$$

where p_1 and p_2 are polynomials in the variables $w^{(i)}$, $i = 1, \ldots, m-1$. We claim that if $f \neq 0$ modulo T, then $p_1 \neq 0$ and $p_2 \neq 0$. Indeed, consider the monomial $\mathbf{m} = \alpha Y_1(z_m Y_2)$, where $Y_1 = y_{j_1} \cdots y_{j_r}$, $Y_2 = y_{l_1} \cdots y_{l_s}$ and $Y_1 < Y_2$. Note that the $(1,2)$ entry of the image of $\mathbf{m}$ under the evaluation (3.7) is given by

$$\frac{1}{4}\alpha(w_1^{(j_1)} \cdots w_1^{(j_r)} + w_2^{(j_1)} \cdots w_2^{(j_r)})w_1^{(m)}(w_1^{(l_1)} \cdots w_1^{(l_s)} + w_2^{(l_1)} \cdots w_2^{(l_s)}).$$

Hence p_1 contains the following monomials

$$\frac{1}{4}\alpha w_1^{(j_1)} \cdots w_1^{(j_r)} w_2^{(l_1)} \cdots w_2^{(l_s)} \text{ and } \frac{1}{4}\alpha w_2^{(j_1)} \cdots w_2^{(j_r)} w_1^{(l_1)} \cdots w_1^{(l_s)}.$$

Since $Y_1 < Y_2$, the two monomials above can only be obtained from the monomial $\mathbf{m}$. Hence, if $f \neq 0$ modulo T, then f contains some monomial $\mathbf{m}$ as above for some non-zero α, which will imply in non-zero monomials in p_1 that are not scalar multiple of any other one that comes from the remaining monomials of f. The same ideas also prove that $p_2 \neq 0$.

Now we use the fact that F is infinite to get evaluations of the even variables for diagonal matrices such that p_1 and p_2 assume non-zero values on F, simultaneously. We finally use the variables $w_1^{(m)}$ and $w_2^{(m)}$ to get arbitrary odd matrices in $f(UJ_3)$, that is, $f(UJ_3) = (UJ_3)_{\overline{1}}$.

Case 3: $\deg f = \overline{2}$. This last case follows from the fact that the homogeneous component of degree $\overline{2}$ is one dimensional. □

3.6.4 The (graded) involution setting

In this section F is a field of characteristic different from 2. Recall that an involution $*$ on an algebra $\mathcal{A}$ defines two subspaces $\mathcal{S}$ and $\mathcal{K}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{S} \oplus \mathcal{K}$, and where $\mathcal{S}$ consists of the symmetric elements, that is, $a \in \mathcal{A}$ such that $a^* = a$, while $\mathcal{K}$ consistis of the skew-symmetric ones, that is, $a \in \mathcal{A}$ such that $a^* = -a$.

Two algebras with involution $(\mathcal{A}, *_1)$ and $(\mathcal{B}, *_2)$ are isomorphic as algebras with involution if there exists an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(a^{*_1}) = \varphi(a)^{*_2}$, for all $a \in \mathcal{A}$. We say that two involutions $*_1$ and $*_2$ on $\mathcal{A}$ are equivalent if $(\mathcal{A}, *_1)$ and $(\mathcal{A}, *_2)$ are isomorphic as algebras with involution.

An involution $*$ on a G-graded algebra $\mathcal{A}$ is called a G-graded involution if the homogeneous components of $\mathcal{A}$ are invariant under $*$, that is, if

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

is the G-grading on $\mathcal{A}$, then $\mathcal{A}_g^* \subset \mathcal{A}_g$, for all $g \in G$. We notice that the subspaces $\mathcal{S}$ and $\mathcal{K}$ are homogeneous in the grading.

Setting $X_G = \{x_{i,g} | g \in G, i = 1, 2, \ldots\}$ and $X_G^* = \{x_{i,g}^* | g \in G, i = 1, 2, \ldots\}$, we denote by $F\langle X_G, X_G^* \rangle$ the free G-graded associative algebra with involution. By writting

$y_{i,g} = (x_{i,g}+x^*_{i,g})/2$ and $z_{i,g} = (x_{i,g}-x^*_{i,g})/2$ and denoting $Y_G = \{y_{i,g}|g \in G, i = 1, 2, \dots\}$ and $Z_G = \{z_{i,g}|g \in G, i = 1, 2, \dots\}$, we may consider elements from $F\langle X_G, X_G^*\rangle$ as polynomials in homogeneous symmetric/skew-symmetric variables. We also will use $F\langle Y_G \cup Z_G\rangle$ to denote the free G-graded associative algebra with involution.

The space of multilinear graded polynomials with involution is given by

$$P_m^{(G,*)} = \operatorname{span}\{\xi_{\sigma(1),g_1} \cdots \xi_{\sigma(m),g_m} | \xi_i \in \{y_i, z_i\}, i = 1, \dots, m, g_1, \dots, g_m \in G\}.$$

We notice here that polynomial functions given by the polynomials in P_m are not multilinear functions. For instance, the polynomial $f = y_{1,g}z_{2,h} + z_{1,g}z_{2,h}$ is a multilinear graded $*$-polynomial however the function given by it on some graded algebra with involution is not bilinear. In order to obtain multilinear functions we will need to consider the following multilinear graded $*$-polynomials from P_m.

For a fixed m-tuple $(g_1, \dots, g_m) \in G^m$, let us consider the subspace $P_{m,l}^{(G,*)}$ of $P_m^{(G,*)}$ given by

$$P_{m,l}^{(G,*)} = \operatorname{span}\{\xi_{\sigma(1)} \cdots \xi_{\sigma(m)} | \xi_i = y_{i,g_i}, i = 1, \dots, l, \xi_i = z_{i,g_i}, i = l + 1, \dots, m\}.$$

In case the G-grading is trivial, that is, G defined as the trivial group, then we simply denote $P_{m,l}^{(G,*)}$ just by $P_{m,l}^*$.

Definition 3.51. *Let $f = f(y_{1,g_1}, \dots, y_{l,g_l}, z_{l+1,g_{l+1}}, \dots, z_{m,g_m}) \in P_{m,l}^{(G,*)}$ and let $\mathcal{A}$ be an algebra with involution. We define the image of f on $\mathcal{A}$ (denoted by $f(\mathcal{A})$) as the image of the function*

$$\begin{array}{rccc} \tilde{f}\colon & \mathcal{S}_{g_1} \times \cdots \mathcal{S}_{g_l} \times \mathcal{K}_{g_{l+1}} \times \cdots \times \mathcal{K}_{g_m} & \to & \mathcal{A} \\ & (a_1, \dots, a_l, b_{l+1}, \dots, b_m) & \mapsto & f(a_1, \dots, a_l, b_{l+1}, \dots, b_m). \end{array},$$

where $a_i \in \mathcal{S}_{g_i}$ and $b_j \in \mathcal{K}_{g_j}$.

We now turn our attention to recall the following description of (graded) involutions on the upper triangular matrix algebra UT_n.

Definition 3.52. *The reflexive involution r on UT_n is defined as $A^r = QA^tQ$, where $A \in UT_n$, A^t denotes the usual transposition of matrices and Q is the permutation matrix*

$$Q = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

In case $n = 2k$ is even, we define a second involution on UT_n named the sympletic involution s given by $A^s = DA^rD^{-1}$ where $A \in UT_n$ and

$$D = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}.$$

The next result from [19] shows that involutions on upper triangular matrices are essentially the reflexive and the sympletic one (the later occurring only in matrices of even order).

Theorem 3.53 ([19]). *Every involution on UT_n is equivalent to the reflexive or the sympletic one.*

Graded involutions on UT_n are also well known. We recall them in the following.

Theorem 3.54 ([29]). *Let F be an algebraically closed field of characteristic different from 2, and let G be a group. Let Γ_1 be a G-grading on UT_n such that supp(Γ_1) generates G. Let φ_1 be a graded involution on Γ_1. Then (Γ_1, φ_1) is isomorphic to (Γ_2, φ_2) where Γ_2 is the elementary grading on UT_n induced by a sequence $(g_1, \dots, g_n) \in G^n$ such that $g_1 g_n = g_2 g_{n-1} = \cdots = g_n g_1$ and φ_2 is either r or s.*

In the following sections we describe the images of polynomials from $P_{m,l}^{(G,*)}$ on UT_2 and UT_3 and we show the difficulties of extending such results to higher dimensions.

3.6.4.1 2×2 matrices

First of all we recall that both reflexive and sympletic involutions acts on UT_2 as follows:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^r = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^s = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}.$$

In light of Theorem 3.54, one can easily see that a graded involution on UT_2 is isomorphic to either the reflexive or sympletic one, where UT_2 is endowed with either the

- $\Gamma_{2,1}$ - trivial grading;
- $\Gamma_{2,2}$ - $UT_2 = \mathcal{A}_1 \oplus \mathcal{A}_g$, where $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}\}$ and $\mathcal{A}_g = \text{span}\{e_{12}\}$.

The next three subsections are devoted to classify the images of polynomials from $P_{m,l}^{(G,*)}$ on UT_2. In particular we show that the image is always a vector space.

3.6.4.1.1 Grading $\Gamma_{2,1}$: the reflexive case

We notice that $UT_2 = \mathcal{S} \oplus \mathcal{K}$, where $\mathcal{S} = span\{e_{11} + e_{22}, e_{12}\}$ and $\mathcal{K} = span\{e_{11} - e_{22}\}$. In the following we recall a result from [19] (see also [35]).

Proposition 3.55. *Let $f \in F\langle Y \cup Z\rangle$. Then, modulo the identities with involution of (UT_2, r), we have that f is a linear combination of polynomials of the form*

$$y_1^{p_1} \cdots y_n^{p_n} z_1^{q_1} \cdots z_m^{q_m} [z_m, y_k] \text{ and } y_1^{p_1} \cdots y_n^{p_n} z_1^{q_1} \cdots z_m^{q_m}$$

where $n \geqslant 1, m \geqslant 1, p_1, \dots, p_n, q_1, \dots, q_m \geqslant 0, k \geqslant 1$.

We consider a set of commuting variables $W = \{w_j^{(i)} | i, j = 1, 2, \dots\}$ and the commutative polynomial algebra $F[W]$. Let us also set the following evaluations of the symmetric and skew-symmetric variables by matrices with entries in $F[W]$:

$$y_i = \begin{pmatrix} w_1^{(i)} & w_2^{(i)} \\ & w_1^{(i)} \end{pmatrix} \text{ and } z_j = \begin{pmatrix} w_1^{(j)} & 0 \\ & -w_1^{(j)} \end{pmatrix}. \tag{3.8}$$

Remark 3.56. *In a product $x_1 \cdots \widehat{x_i} \cdots x_m$, the hat $\widehat{}$ means that the variable x_i is missing.*

Lemma 3.57. *In light of (3.8), the entry $(1, 2)$ of $y_1 \cdots y_l$ is given by*

$$\sum_{i=1}^{l} w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(l)} w_2^{(i)}$$

Proof. It is enough to apply induction on l. □

In the following result we identify the space of scalar matrices with F and we also denote $J = \operatorname{span}\{e_{12}\}$.

Theorem 3.58. *Let $f \in P_{m,l}^*$. Then the image of f on (UT_2, r) is $\{0\}, J, F, \mathcal{K}, \mathcal{S}$, or $\mathcal{K} + J$.*

Proof. Let $f = f(y_1, \dots, y_l, z_{l+1}, \dots, z_m)$. By Proposition 3.55, we may write f modulo $Id(UT_2, r)$ as

$$f = \alpha y_1 \cdots y_l z_{l+1} \cdots z_m + \sum_{i=1}^{l} \alpha_i y_1 \cdots \widehat{y_i} \cdots y_l z_{l+1} \cdots z_{m-1} [z_m, y_i]$$

and additionally assume f non-zero modulo $Id(UT_2, r)$. Notice that if $\alpha = 0$, then $f(UT_2) = J$. Indeed, this follows from the fact that the image of $[z_m, y_i]$ is contained in J along with the later being an one-dimensional ideal of UT_2.

We may assume from now on that $\alpha \neq 0$ and let us denote $\eta = m - l$. Note that under the evaluation (3.8) and by Lemma 3.57 we have that the entry $(1, 2)$ of $y_1 \cdots y_l z_{l+1} \cdots z_m$ is given by

$$(y_1 \cdots y_l z_{l+1} \cdots z_m)_{12} = (-1)^\eta w_1^{(l+1)} \cdots w_1^{(m)} \sum_{i=1}^{l} w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(l)} w_2^{(i)}.$$

Moreover, for each $i \in \{1, \dots, l\}$, we have $[z_m, y_i] = 2 w_1^{(m)} w_2^{(i)} e_{12}$ and also

$$(y_1 \cdots \widehat{y_i} \cdots y_l z_{l+1} \cdots z_{m-1})_{11} = w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(l)} w_1^{(l+1)} \cdots w_1^{(m-1)}.$$

Thus

$$y_1 \cdots \widehat{y_1} \cdots y_l z_{l+1} \cdots z_{m-1} [z_m, y_i] = 2 w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)} w_2^{(i)} e_{12}.$$

Therefore we conclude that $f(y_1, \dots, y_l, z_{l+1}, \dots, z_m)$ is given by the following sum:

$$\alpha w_1^{(1)} \cdots w_1^{(m)} (e_{11} + (-1)^\eta e_{22}) + \sum_{i=1}^{l} ((-1)^\eta \alpha + 2\alpha_i) w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)} w_2^{(i)} e_{12}.$$

If $(-1)^\eta\alpha+2\alpha_i=0$ for all $i\in\{1,\dots,l\}$, then one can see that $f(UT_2)=F$ or $f(UT_2)=\mathcal{K}$, according to whether η is even or odd, respectively.

We assume now that $(-1)^\eta\alpha+2\alpha_k\neq 0$ for some $k\in\{1,\dots,l\}$. We thus have that $f(UT_2)\subset\mathcal{S}$, in case η is even, or $f(UT_2)\subset\mathcal{K}+J$ otherwise. We further perform the following evaluation on the commutative variables $w_j^{(i)}$:

- $w_1^{(i)}=1$ for $i=l+1,\dots,m$;
- $w_2^{(i)}=0$ for all $i\neq k$;
- $w_1^{(i)}=1$ for all $i\in\{1,\dots,l\}\backslash\{k\}$.

Therefore,

$$f(y_1,\dots,y_l,z_{l+1},\dots,z_m)=\alpha w_1^{(k)}(e_{11}+(-1)^\eta e_{22})+((-1)^\eta\alpha+2\alpha_k)w_2^{(k)}e_{12}$$

which implies in $f(UT_2)=\mathcal{S}$ or $f(UT_2)=\mathcal{K}+J$. □

3.6.4.1.2 Grading $\Gamma_{2,1}$: the symplectic case

Once again, we start recalling the following proposition from [19] (see also [35]).

Proposition 3.59. *Let $f\in F\langle Y\cup Z\rangle$. Then, modulo the identities with involution of (UT_2,s), we have that f is a linear combination of polynomials of the form*

$$y_1^{p_1}\cdots y_n^{p_n}[z_j,z_i]z_i^{q_i}z_{i+1}^{q_{i+1}}\cdots z_m^{q_m}\ \text{and}\ y_1^{p_1}\cdots y_n^{p_n}z_1^{q_1}\cdots z_m^{q_m}$$

where $j>i$.

We consider evaluations of the variables z_i's by matrices with entries in $F[W]$:

$$z_i=\begin{pmatrix} w_1^{(i)} & w_2^{(i)} \\ & -w_1^{(i)}\end{pmatrix} \tag{3.9}$$

for $i\in\{1,\dots,m\}$.

Lemma 3.60. *In light of (3.9), the matrix $z_1\cdots z_m$ is given by*

$$w_1^{(1)}\cdots w_1^{(m)}(e_{11}+(-1)^m e_{22})+\sum_{i=1}^{m}(-1)^{i+m}w_1^{(1)}\cdots\widehat{w_1^{(i)}}\cdots w_1^{(m)}w_2^{(i)}e_{12}$$

Proof. Induction on m. □

The proof of the next theorem follows the same ideas from the reflexive case.

Theorem 3.61. *Let $f\in P_{m,l}^*$. Then the image of f on (UT_2,s) is $\{0\},J,\mathcal{S},\mathcal{K},\mathcal{K}\cap\mathcal{D},\mathcal{S}+J$, where $\mathcal{D}$ denotes the space of diagonal matrices.*

Proof. By Proposition 3.59 and since $\mathcal{S} = F$, we may consider

$$f(z_1, \dots, z_m) = \alpha_0 z_1 \cdots z_m + \sum_{i=2}^{m} \alpha_i [z_i, z_1] z_2 \cdots \widehat{z_i} \cdots z_m.$$

We notice that

$$[z_i, z_1] = 2(w_1^{(i)} w_2^{(1)} - w_2^{(i)} w_1^{(1)}) e_{12}$$

and

$$(z_2 \cdots \widehat{z_i} \cdots z_m)_{22} = (-1)^m w_1^{(2)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)}.$$

Hence,

$$[z_i, z_1] z_2 \cdots \widehat{z_i} \cdots z_m = (-1)^m 2(w_1^{(2)} \cdots w_1^{(m)} w_2^{(1)} - w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)} w_2^{(i)}) e_{12}$$

and therefore the evaluation of f on the matrices $z_1, \dots, z_m$ is given by the sum of the following two matrices

$$\alpha_0 w_1^{(1)} \cdots w_1^{(m)} (e_{11} + (-1)^m e_{22})$$

and

$$\Big((-1)^m(-\alpha_0 + \sum_{i=2}^{m} 2\alpha_i) w_1^{(2)} \cdots w_1^{(m)} w_2^{(1)} + \sum_{i=2}^{m} (-1)^m((-1)^i \alpha_0 + 2\alpha_i) w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)} w_2^{(i)}\Big) e_{12}.$$

We may assume $\alpha_0 \neq 0$, otherwise the image is already determined since $f(UT_2) \subset J$.

If $-\alpha_0 + \sum_{i=2}^{m} 2\alpha_i = 0$ and $(-1)^i \alpha_0 + 2\alpha_i = 0$ for all $i \in \{2, \dots, m\}$, then it is easy to see that $f(UT_2) = \operatorname{span}\{e_{11} + (-1)^n e_{22}\} \in \{\mathcal{S}, \mathcal{K} \cap \mathcal{D}\}$. Otherwise let us first assume that $-\alpha_0 + \sum_{i=2}^{m} 2\alpha_i \neq 0$. Then we perform the following evaluation of the commutative variables $w_j^{(i)}$:

- $w_2^{(i)} = 0$ for all $i \in \{2, \dots, m\}$;
- $w_1^{(i)} = 1$ for all $i \in \{2, \dots, m\}$.

We therefore have

$$f(z_1, \dots, z_m) = \alpha_0 w_1^{(1)} (e_{11} + (-1)^m e_{22}) + (-1)^m(-\alpha_0 + \sum_{i=1}^{m} 2\alpha_i) w_2^{(1)} e_{12}$$

which implies in $f(UT_2) = \operatorname{span}\{e_{11} + (-1)^m e_{22}, e_{12}\} \in \{\mathcal{K}, \mathcal{S} + J\}$.

Assume now that $(-1)^i \alpha_0 + 2\alpha_i \neq 0$ for some $i \in \{2, \dots, m\}$. We thus set the evaluation

- $w_2^{(j)} = 0$ for all $j \in \{1, \dots, m\} \backslash \{i\}$;
- $w_1^{(j)} = 1$ for all $j \in \{1, \dots, m\} \backslash \{i\}$.

We obtain then

$$f(z_1, \ldots, z_m) = \alpha_0 w_1^{(i)}(e_{11} + (-1)^m e_{22}) + (-1)^m((-1)^i + 2\alpha_i)w_2^{(i)} e_{12},$$

which clearly leads us to $f(UT_2) = \text{span}\{e_{11} + (-1)^m e_{22}, e_{12}\} \in \{\mathcal{K}, \mathcal{S} + J\}$. □

3.6.4.1.3 Grading $\Gamma_{2,2}$

Let us first write

$$UT_2 = \mathcal{A}_1 \oplus \mathcal{A}_g,$$

where $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}\}$ and $\mathcal{A}_g = \text{span}\{e_{12}\}$. We further write

$$\mathcal{A}_1 = \mathcal{S}_1 \oplus \mathcal{K}_1 \text{ and } \mathcal{A}_g = \mathcal{S}_g \oplus \mathcal{K}_g \tag{3.10}$$

with respect to the involution $* \in \{r, s\}$.

Applying Corollary 1.21 gives us that the image of a multilinear graded polynomial with involution f on UT_2 is a vector space (the subspaces V_i are defined as the ones appearing in the decompositions in (3.10), accordingly to the homogeneous degree and symmetry of the variables occuring in f). Indeed, since $Supp(\Gamma_{2,2})$ is abelian, then $f(UT_2)$ is contained in some homogeneous component, and now one just need to notice that both $\mathcal{A}_1$ and $\mathcal{A}_g$ have dimension $\leqslant 2$.

It is also straightforward to obtain a precise classification of the images of multilinear graded $*$-polynomials on UT_2 with the grading $\Gamma_{2,2}$.

Proposition 3.62. *Let $f \in F\langle Y_G \cup Z_G\rangle$ be multilinear and consider the G-graded involution $* \in \{r, s\}$ on UT_2 with respect to the grading $\Gamma_{2,2}$. Then $f(UT_2)$ is either $\{0\}$ or some (skew-)symmetric part from some homogeneous component.*

Proof. Let us consider $* = r$, since the sympletic case is quite similar. First we note that

$$\mathcal{S}_1 = \text{span}\{e_{11} + e_{22}\}, \mathcal{K}_1 = \text{span}\{e_{11} - e_{22}\}, \mathcal{S}_g = \mathcal{A}_g \text{ and } \mathcal{K}_g = \{0\}.$$

Since $Supp(\Gamma_{2,2}) = \{1, g\}$, we may consider that only variables of homogeneous degree 1 and g occur in f. Now note that $\mathcal{S}_g$ is a nilpotent ideal of UT_2 of index 2, and it is one-dimensional as a vector space. Hence if f has at least one variable of homogeneous degree g, then $f(UT_2)$ is either $\{0\}$ or $\mathcal{S}_g$.

So we may assume now that f has only neutral variables. Since UT_2 satisfies the graded $*$-identities $[z_{1,1}, z_{2,1}] = 0$ and $[y_{1,1}, x] = 0$, where $x \in Y_G \cup Z_G$, then modulo these identities we may write f as

$$f = \alpha y_{1,1} \cdots y_{l,1} z_{l+1,1} \cdots z_{m,1},$$

for some $\alpha \in F$. This will therefore imply in $f(UT_2)$ equals to $\{0\}, \mathcal{S}_1$ or $\mathcal{K}_1$. □

We therefore conclude the following theorem.

Theorem 3.63. *Let F be an algebraically closed field of characteristic different from 2. Let $f \in P_{m,l}^{(G,*)}$. Assume that UT_2 is endowed with some G-graded involution $*$ on a grading Γ such that supp(Γ) generates G. Then the image of f on UT_2 is a homogeneous vector space.*

In the next section we show that a result like Theorem 3.63 can not be obtained if we consider matrices of order greater than 2.

3.6.4.2 The image is not always a vector space

Next we present an example which shows that an analogue of Theorem 3.63 can not be expected for upper triangular matrices of order $n \geqslant 3$. In other words, we prove that images of multilinear $*$-polynomials on UT_n are not always vector spaces. The polynomial from our example will be in skew-symmetric variables. For that reason, we recall that the skew-symmetric part of UT_n with the reflexive involution is given by

$$\mathcal{K} = \operatorname{span}\{e_{ij} - e_{n+1-j,n+1-i} | i \leqslant j, i, j = 1, \ldots, n\}.$$

Proposition 3.64. *Let $n \geqslant 3$ and let UT_n be endowed with the reflexive involution. Then the image of the multilinear polynomial $f(z_1, z_2) = z_1 z_2$ on UT_n is not a vector space.*

Proof. Let n be an odd integer (the even case can be treated analogously, up to minor adjustments), and let us assume that $f(UT_n)$ is a vector space.

Denoting $n_0 = \dfrac{n+1}{2}$ we have that

$$\begin{aligned} e_{11} + e_{nn} &= f(e_{11} - e_{nn}, e_{11} - e_{nn}) \\ e_{1n} &= f(e_{1,n_0} - e_{n_0,n}, -e_{1,n_0} + e_{n_0,n}). \end{aligned}$$

Hence we must have $e_{11} + e_{nn} + e_{1n} \in f(UT_n)$, that is, there exist $A, B \in \mathcal{K}$ such that

$$e_{11} + e_{nn} + e_{1n} = AB. \tag{3.11}$$

Let us write

$$A = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} e_{ij} \text{ and } B = \sum_{1 \leqslant i \leqslant j \leqslant n} b_{ij} e_{ij}.$$

and since $A, B \in \mathcal{K}$, we additionally have that $a_{ij} = -a_{n+1-j,n+1-i}$ and $b_{ij} = -b_{n+1-j,n+1-i}$, for all i, j.

We claim that $(AB)_{1n} = 0$, which clearly leads us to a contradiction. To prove the claim, let us compute the following entries of the product AB:

1. "Half first row": for $j = 2, \ldots, n_0$, the entry $(1, j)$ is given by

$$\sum_{i=1}^{j} a_{1i} b_{ij}$$

2. "Half last column": for $i = n_0, \ldots, n-1$, the entry (i, n) is given by

$$\sum_{j=i}^{n} a_{ij} b_{jn}.$$

We rewrite the entry (i, n) above as

$$\sum_{j=i}^{n} a_{ij} b_{jn} = \sum_{j=i}^{n} a_{n+1-j,n+1-i} b_{1,n+1-j} = \sum_{k=1}^{l} a_{kl} b_{1k},$$

for $l = 2, \ldots, n_0$.

We now prove that $a_{1j} = b_{1j} = 0$ for $j = 2, \ldots, n_0$. To this end we proceed by induction on n_0. For the base of the induction, we note that the entries $(1, 2)$ and $(n-1, n)$ along with Equation (3.11) give us

$$a_{11} b_{12} + a_{12} b_{22} = 0 \text{ and } a_{22} b_{12} + a_{12} b_{11} = 0.$$

Since $a_{11} b_{11} \neq 0$ and $a_{22} b_{22} = 0$, we therefore get $a_{12} = b_{12} = 0$. We assume now $a_{1j} = b_{1j} = 0$ for $j < n_0$. Considering the entries $(1, n_0)$ and (n_0, n) along with Equation (3.11) we have

$$\sum_{i=1}^{n_0} a_{1i} b_{i,n_0} = 0 \text{ and } \sum_{k=1}^{n_0} a_{k,n_0} b_{1k} = 0.$$

By our induction hypothesis, the two equations above reduce to

$$a_{11} b_{1,n_0} + a_{1,n_0} b_{n_0,n_0} = 0 \text{ and } a_{1,n_0} b_{11} + a_{n_0,n_0} b_{1,n_0} = 0.$$

Now it is enough to use that $a_{11} b_{11} \neq 0$ and $a_{n_0,n_0} b_{n_0,n_0} = 0$ to get $a_{1,n_0} = b_{1,n_0} = 0$.

Finally, to obtain our claim we just need to notice that

$$(AB)_{1n} = \sum_{i=1}^{n} a_{1i} b_{in} = -\sum_{i=1}^{n} a_{1i} b_{1,n+1-i}$$

and that $a_{1n} = b_{1n} = a_{1i} = b_{1i} = 0$ for $i = 2, \ldots, n_0$. □

3.6.4.3 The 3×3 case and one more example

In the last section we proved in particular that images of multilinear $*$-polynomials on UT_3 are not always vector spaces. In that situation we had UT_3 endowed with the trivial grading. In this section we will show that the picture changes when we allow UT_3 be endowed with some nontrivial grading. Moreover, we also show that the nontrivial grading setting fails if we consider upper triangular matrices of order greater than 3.

Let us recall that, up to equivalence, there is only one involution on UT_3, the reflexive one, given by:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix}^r = \begin{pmatrix} a_{33} & a_{23} & a_{13} \\ & a_{22} & a_{12} \\ & & a_{11} \end{pmatrix}.$$

Therefore, the only graded involutions on UT_3 are the ones given by the reflexive involution r and gradings:

- $\Gamma_{1,3}$ - trivial grading;
- $\Gamma_{2,3}$ - $UT_3 = \mathcal{A}_1 \oplus \mathcal{A}_g$, where $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$ and $\mathcal{A}_g = \text{span}\{e_{12}, e_{23}\}$;
- $\Gamma_{3,3}$ - $UT_3 = \mathcal{A}_1 \oplus \mathcal{A}_g \oplus \mathcal{A}_h$, where $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_g = \text{span}\{e_{12}, e_{23}\}$ and $\mathcal{A}_h = \text{span}\{e_{13}\}$.

In the next subsections we describe the images of polynomials from $P_{m,l}^{(G,*)}$ on UT_3. In particular, we show that the image is always a vector space in case UT_3 is endowed with a nontrivial grading. We also give an example of a multilinear graded $*$-polynomial whose image on UT_n $(n \geqslant 4)$ is not a vector space where UT_4 is endowed with the natural $\mathbb{Z}_n$-grading.

3.6.4.3.1 The grading $\Gamma_{2,3}$

We recall that $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$ and $\mathcal{A}_g = \text{span}\{e_{12}, e_{23}\}$. Hence,

$$\mathcal{S}_1 = \text{span}\{e_{11} + e_{33}, e_{22}, e_{13}\}, \mathcal{S}_g = \text{span}\{e_{12} + e_{23}\},$$

$$\mathcal{K}_1 = \text{span}\{e_{11} - e_{33}\} \text{ and } \mathcal{K}_g = \text{span}\{e_{12} + e_{23}\}.$$

Since the homogeneous components are invariant under the involution, we may regard the neutral component $\mathcal{A}_1$ as an algebra with involution as well. In the next lemma we notice some similarities between the reflexive case on UT_2 and $\mathcal{A}_1$.

Lemma 3.65. *The neutral component of UT_3 with reflexive involution and grading $\Gamma_{2,3}$ satisfies the following identities:*

$$(i)\ [y_1, y_2]; \quad (ii)\ [z_1, z_2]; \quad (iii)\ [y_1, z_1][y_2, z_2]; \quad (iv)\ z_1 y_1 z_2 - z_2 y_1 z_1.$$

Proof. It is immediate, so omitted. □

One can see that the identities from the lemma above are exactly the ones satisfied by UT_2 with the reflexive involution (see [19, Theorem 3.1]). So it is expected that we can use the same approach from Proposition 3.58. Let us thus consider the following evaluation of the variables y's and z's by matrices over $F[W]$:

$$y_i = \begin{pmatrix} w_1^{(i)} & 0 & w_3^{(i)} \\ & w_2^{(i)} & 0 \\ & & w_1^{(i)} \end{pmatrix} \text{ and } z_j = \begin{pmatrix} w_1^{(j)} & 0 & 0 \\ & 0 & 0 \\ & & -w_1^{(j)} \end{pmatrix} \tag{3.12}$$

Lemma 3.66. *Let $y_i, i = 1, \dots, m$, as in equation (3.12). Then the entry $(1,3)$ of $y_1 \cdots y_m$ is given by*

$$\sum_{i=1}^{m} w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)} w_3^{(i)}.$$

Proof. Induction on m. □

The proof of the next lemma follows the same ideas from Proposition 3.58.

Lemma 3.67. *Let $f \in F\langle Y \cup Z \rangle$ be multilinear. Then $f(\mathcal{A}_1)$ is $\{0\}, J, \mathcal{S}_1 \cap \mathcal{D}$, $(\mathcal{S}_1 \cap \mathcal{D}) + J, \mathcal{S}_1, \mathcal{K}_1$ or $\mathcal{K}_1 + J$, where J denotes the subspace spanned by $\{e_{13}\}$, and $\mathcal{D}$ stands for the linear span of $\{e_{11}, e_{33}\}$.*

Proof. By Proposition 3.55 we may write f as

$$f = \alpha y_1 \cdots y_l z_{l+1} \cdots z_m + \sum_{i=1}^{l} \alpha_i y_1 \cdots \widehat{y_i} \cdots y_l z_{l+1} \cdots z_{m-1} [z_m, y_i].$$

We will assume $\alpha \neq 0$, since otherwise we clearly have $f(UT_3) \in \{\{0\}, J\}$.

Let $\eta = m - l$. Then notice that the main diagonal of f is given by

$$w_1^{(1)} \cdots w_1^{(m)} (e_{11} + (-1)^{\eta} e_{33}) + \lambda w_2^{(1)} \cdots w_2^{(m)} e_{22},$$

where $\lambda = 0$ if there is no skew-symmetric variable in f, and $\lambda = 1$ otherwise. The case where f has only symmetric variables is obvious, indeed one can esaily obtain $f(UT_3) = \mathcal{S}_1$. Hence we may assume that f has skew-symmetric variables and also $\lambda = 0$. Hence

$$f(UT_3) \subset (S_1 \cap \mathcal{D}) + J \text{ or } f(UT_3) \subset \mathcal{K}_1 + J,$$

accordingly η is even or η is odd, respectively. Now let us turn our attention to the nilpotent part f. We start to noticing that $[z_m, y_i] = 2 w_3^{(i)} w_1^{(m)} e_{13}$, and therefore

$$y_1 \cdots \widehat{y_i} \cdots y_l z_{l+1} \cdots z_{m-1} [z_m, y_i] = 2 w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m-1)} w_3^{(i)} w_1^{(m)} e_{13}.$$

Hence, in light of Lemma 3.66 we must have $f(y_1, \dots, y_l, z_{l+1}, \dots, z_m)$ equals to

$$\alpha w_1^{(1)} \cdots w_1^{(m)} (e_{11} + (-1)^{\eta} e_{33}) + \sum_{i=1}^{l} ((-1)^{\eta} \alpha + 2\alpha_i) w_1^{(1)} \cdots \widehat{w_1^{(i)}} \cdots w_1^{(m)} w_3^{(i)} e_{13}.$$

Proceding in a silimar fashion as in Proposition 3.58, one can see that

$$f(UT_3) \in \{\mathcal{K}_1, \mathcal{K}_1 + J, \mathcal{S}_1 \cap \mathcal{D}, (\mathcal{S}_1 \cap \mathcal{D}) + J\},$$

which finishes the proof. $\square$

Proposition 3.68. *Let $f \in P_{m,l}^{(G,*)}$. Then $f(UT_3)$ is $\{0\}, J, \mathcal{S}_1 \cap \mathcal{D}, (\mathcal{S}_1 \cap \mathcal{D}) + J, \mathcal{S}_1, \mathcal{K}_1, \mathcal{K}_1 + J$, some one-dimensional subspace of $\mathcal{A}_1$ or $\mathcal{A}_1$. Moreover, every one-dimensional subspace of $\mathcal{A}_1$ can be realized as the image of some multilinear graded polynomial with involution on UT_3, as well as the homogeneous component $\mathcal{A}_1$.*

Proof. The proof of the first part is clear from Lemma 3.67 and Corollary 1.21. Now, fix $\alpha, \beta \in F$ and consider the one-dimensional subspace

$$V = \operatorname{span}\{\alpha e_{12} + \beta e_{23}\} \subset \mathcal{A}_1.$$

It is straightforward to check that the image of $\alpha z_{1,0} z_{1,1} + \beta z_{1,1} z_{1,0}$ on UT_3 is exactly V, and that the image of $y_{1,0} y_{1,1}$ on UT_3 is exactly $\mathcal{A}_1$. $\square$

3.6.4.3.2 The grading $\Gamma_{3,3}$

We recall that $\mathcal{A}_1 = \operatorname{span}\{e_{11}, e_{22}, e_{33}\}, \mathcal{A}_g = \operatorname{span}\{e_{12}, e_{23}\}$ and $\mathcal{A}_h = \operatorname{span}\{e_{13}\}$. We therefore write

$$UT_3 = \mathcal{S}_1 \oplus \mathcal{K}_1 \oplus \mathcal{S}_g \oplus \mathcal{K}_g \oplus \mathcal{S}_h \oplus \mathcal{K}_h$$

where $\mathcal{S}_1 = \operatorname{span}\{e_{11} + e_{33}, e_{22}\}, \mathcal{K}_1 = \operatorname{span}\{e_{11} - e_{33}\}, \mathcal{S}_g = \operatorname{span}\{e_{12} + e_{23}\}$, $\mathcal{K}_g = \operatorname{span}\{e_{12} - e_{23}\}, \mathcal{S}_h = \operatorname{span}\{e_{13}\}$ and $\mathcal{K}_h = \{0\}$. Hence we have the following result.

Proposition 3.69. *Let F be a field of characteristic different from 2 and let $f \in P_{m,l}^{(G,*)}$. Then $f(UT_3)$ is $\{0\}, \mathcal{S}_1, \mathcal{K}_1, (\mathcal{K}_1)^2$, some one-dimensional subspace of $\mathcal{A}_g$, $\mathcal{A}_g$ or $\mathcal{S}_h$. Moreover, any subspace of $\mathcal{A}_g$ can be realized as the image of some multilinear polynomial on UT_3.*

Proof. Assume first that f has (skew-)symmetric neutral variables only. Since UT_3 satisfies $[x_1, x_2]$ where x_1, x_2 are any (skew-)symmetric neutral variables, then the image of f on UT_3 is either zero or the image of a word in the form $y_1 \cdots y_l z_{l+1} \cdots z_m$ on UT_3. Clearly the former gives us $f(UT_3) \in \{\mathcal{S}_1, \mathcal{K}_1, (\mathcal{K}_1)^2\}$.

Let us assume now that f has non neutral variables. Since $x_1 x_2$ is an identity for UT_3 when x_1 has homogeneous degree g and x_2 has homogeneous degree h, then $f(UT_3)$ is always contained in some non-neutral homogeneous component. In this case, we apply Corollary 1.21 to conclude that $f(UT_3)$ is a vector subspace of $\mathcal{A}_g$ or $\mathcal{A}_h$. Moreover, we notice that $\mathcal{A}_g = f(UT_3)$ for $f = y_{1,0}y_{1,1}$, and for arbitrarily fixed $\alpha, \beta \in F$ and

$$V = \operatorname{span}\{\alpha e_{12} + \beta e_{23}\}$$

we have $V = f(UT_3)$ for $f = \alpha z_{1,0}y_{1,1} - \beta y_{1,1}z_{1,0}$. $\square$

We therefore conclude the following theorem.

Theorem 3.70. *Let F be an algebraically closed field of characteristic different from 2. Let $f \in P_{m,l}^{(G,*)}$. Assume that UT_3 is endowed with some G-graded involution $*$ on a nontrivial grading Γ such that supp(Γ) generates G. Then, the image of f on UT_3 is a homogeneous vector space.*

We finish this section by showing that, in general, Theorem 3.70 can not be expected to hold for UT_n, $n \geqslant 4$, not even for the canonical $\mathbb{Z}_n$-grading.

Proposition 3.71. *Let $n \geqslant 4$ and let UT_n be endowed with the canonical $\mathbb{Z}_n$-grading and reflexive involution. Then the image of the multilinear polynomial $f(y_{1,0}, y_{2,1}) = y_{1,0}y_{2,1}$ on UT_n is not a vector space.*

Proof. Let us suppose n even (the odd case is analogous). We recall that the symmetric parts of homogeneous degree $\overline{0}$ and $\overline{1}$ are given by

$$\mathcal{S}_0 = \operatorname{span}\{e_{ii} + e_{n+1-i,n+1-i}; i = 1, \ldots, n/2\}, \text{and}$$
$$\mathcal{S}_1 = \operatorname{span}\{e_{n/2,(n+2)/2}, e_{i,i+1} + e_{n-i,n+1-i}; i = 1, \ldots, -1 + n/2\}.$$

Assuming that $f(UT_n)$ is a vector space, and noticing that

$$f(e_{11} + e_{nn}, e_{12} + e_{n-1,n}) = e_{12}$$
$$f(e_{22} + e_{n-1,n-1}, e_{23} + e_{n-2,n-1}) = e_{23}$$

we therefore have $e_{12} + e_{23} \in f(UT_n)$ (we take $e_{23} = f(e_{22} + e_{33}, e_{23})$ in case $n = 4$). Hence there exist $A \in \mathcal{S}_0$ and $B \in \mathcal{S}_1$ such that $e_{12} + e_{23} = AB$.

Writing $A = \sum_{i=1}^{n} a_{ii}e_{ii}$ and $B = \sum_{i=1}^{n-1} b_{i,i+1}e_{i,i+1}$, we therefore have

$$(AB)_{12} = a_{11}b_{12}$$
$$(AB)_{23} = a_{22}b_{23}$$
$$(AB)_{n-1,n} = a_{n-1,n-1}b_{n-1,n} = a_{22}b_{12}$$

Since the entry $(n-1, n)$ from AB is zero, we conclude that either $(AB)_{12} = 0$ or $(AB)_{23} = 0$, a contradiction. $\square$

4 f-zpd algebras

In this chapter we introduce the class of zpd algebras as well as the main examples (non-examples) and results concerning this class of algebras. Our main reference for the basics of zpd algebras is the book [13] (see also the paper [14]). The results from Section 4.2 to Section 4.3 are submitted for publication in a specialized journal [8]. This is a joint work with Žan Bajuk, Matej Brešar and Antonio Ioppolo.

4.1 (f-)zpd algebras

We start this section by defining the class of zpd algebras.

Definition 4.1. *Let $\mathcal{A}$ be a nonassociative algebra over a field F. We say that $\mathcal{A}$ is a zero product determined algebra (zpd algebra for short) if for every bilinear functional $\varphi : \mathcal{A} \times \mathcal{A} \to F$ satisfying*

$$xy = 0 \Rightarrow \varphi(x, y) = 0 \quad (x, y \in \mathcal{A})$$

then there exists a linear functional $\tau : \mathcal{A} \to F$ such that

$$\varphi(x, y) = \tau(xy) \quad \text{for all} \quad x, y \in \mathcal{A}.$$

It might be a tough task to find nontrivial examples of zpd algebras by just having in mind the definition given above. For this reason we state the next theorem.

Theorem 4.2 ([13]). *Let $\mathcal{A}$ be an associative algebra. If $\mathcal{A}$ is generated by idempotents, then $\mathcal{A}$ is zpd.*

Remark 4.3. *The converse of Theorem 4.2 is valid when $\mathcal{A}$ is finite-dimensional.*

Example 4.4. *The field F is a zpd algebra over itself.*

Example 4.5. *The full matrix algebra $M_n(F)$ is zpd.*

Example 4.6. *Given $\mathcal{A}$ and $\mathcal{B}$ F-algebras generated by idempotents and $\mathcal{M}$ an $(\mathcal{A}, \mathcal{B})$-bimodule, then the triangular algebra*

$$\begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$$

is zpd. In particular, the algebra of upper triangular matrices $UT_n(F)$ is zpd.

Of course not every algebra is zpd. Let us see some examples.

Proposition 4.7. *[13] Let $\mathcal{A}$ be a unital nonassociative algebra of dimension greater than* 1. *If $\mathcal{A}$ has no zero-divisors, then $\mathcal{A}$ is not zpd.*

Example 4.8. *A division algebra of dimension greater than* 1 *is not zpd. In particular, the field F is the only division zpd algebra over F.*

A natural question now is how to construct examples of zpd algebras from the ones that we already know. Fortunately, the class of zpd algebras has a nice behaviour under some classic constructions. For instance, homomorphic image of unital zpd nonassociative algebra is also zpd. The same stability happens for direct sums and tensor products of zpd nonassociative algebras (see [13]).

Definition 4.9. *Let $\mathcal{A}$ be an algebra. We say that $\mathcal{A}$ is a zero Lie product determined algebra (zLpd algebra for short) if $\mathcal{A}^{(-)}$ is zpd.*

Clearly commutative algebras are zLpd.

Theorem 4.10. *If $\mathcal{A}$ is a zLpd unital algebra, then so is $M_n(\mathcal{A})$ for every $n \geqslant 1$.*

In particular, the full matrix algebra $M_n(F)$ is zLpd.

Definition 4.11. *Let $\mathcal{A}$ be an algebra. We say that $\mathcal{A}$ is zero Jordan product determined algebra (zJpd algebra for short) if $\mathcal{A}^{(+)}$ is zpd.*

Theorem 4.12. *Let $\mathcal{A}$ be an unital algebra over a field of characteristic different from* 2. *If $\mathcal{A}$ is generated by idempotents, then $\mathcal{A}$ is zJpd.*

In particular, the full matrix algebra $M_n(F)$ $(char(F) \neq 2)$ is zJpd.

Let us now summarize the results presented so far concerning the algebra $M_n(F)$. We have seem that $M_n(F)$ is zpd, zLpd and also zJpd. This means the following: for $i = 1, 2, 3$, let $\varphi_i \colon M_n(F) \times M_n(F) \to F$ be a bilinear functional such that

$$\begin{aligned} xy = 0 &\Rightarrow \varphi_1(x, y) = 0 \\ xy - yx = 0 &\Rightarrow \varphi_2(x, y) = 0 \\ xy + yx = 0 &\Rightarrow \varphi_3(x, y) = 0 \end{aligned}$$

Then there exist linear functionals $\tau_i \colon M_n(F) \to F$ such that

$$\begin{aligned} \varphi_1(x, y) &= \tau_1(xy) \\ \varphi_2(x, y) &= \tau_2(xy - yx) \\ \varphi_3(x, y) &= \tau_3(xy + yx) \end{aligned}$$

for all $x, y \in M_n(F)$.

In other words, bilinear functionals on the full matrix algebra preserving zeros of the polynomials $xy, xy - yx$ and $xy + yx$ can be described in terms of a linear functional on the algebra and the polynomial. This motivates the following definition.

Definition 4.13. *Let $\mathcal{A}$ be an algebra and let $f \in P_m$ be a multilinear polynomial of degree m. We say that $\mathcal{A}$ is an f-zero product determined algebra (f-zpd algebra for short) if for every multilinear functional $\varphi\colon \mathcal{A}^m \to F$ satisfying*

$$f(a_1, \dots, a_m) = 0 \Rightarrow \varphi(a_1, \dots, a_m) = 0 \quad (a_1, \dots, a_m \in \mathcal{A})$$

then there exists a linear functional $\tau\colon \mathcal{A} \to F$ such that

$$\varphi(a_1, \dots, a_m) = \tau(f(a_1, \dots, a_m))$$

for all $a_1, \dots, a_m \in \mathcal{A}$.

In light of Theorems 4.2, 4.10, 4.12, we can now pose the following question.

Question 4.14. *Let $f \in P_m$. Is the full matrix algebra $M_n(F)$ an f-zpd algebra?*

This chapter is concerned with dealing with Question 4.14 and related problems. We now finish this first section with some basic properties of f-zpd algebras.

Remark 4.15. *During the rest of this chapter f will always denote a multilinear polynomial of degree m.*

Proposition 4.16. *Let $\mathcal{A}$ be an algebra.*

(i) $\mathcal{A}$ is f-zpd for any polynomial identity f of $\mathcal{A}$;

(ii) Let $\alpha \in F$ be non-zero. Then $\mathcal{A}$ is f-zpd if and only if $\mathcal{A}$ is αf-zpd.

Proof. The proof is straightforward. □

Proposition 4.17. *Let $\alpha = f(1, \dots, 1) \neq 0$. Then $\mathcal{A}$ is f-zpd if and only if any multilinear functional $\varphi\colon \mathcal{A}^m \to F$ preserving zeros of f satisfies*

$$\varphi(a_1, \dots, a_m) = \alpha^{-1}\varphi(f(a_1, \dots, a_m), 1, \dots, 1)$$

for all $a_1, \dots, a_m \in \mathcal{A}$.

Proof. Assume that $\mathcal{A}$ is f-zpd. Then there exists a linear functional $\tau\colon \mathcal{A} \to F$ such that $\varphi(a_1, \dots, a_m) = \tau(f(a_1, \dots, a_m))$, for all $a_1, \dots, a_m \in \mathcal{A}$. Hence

$$\begin{aligned}\varphi(f(a_1, \dots, a_m), 1, \dots, 1) &= \tau(f(f(a_1, \dots, a_m), 1, \dots, 1)) \\ &= \alpha\tau(f(a_1, \dots, a_m)) \\ &= \alpha\varphi(a_1, \dots, a_m),\end{aligned}$$

and the desired conclusion follows. Reciprocally, it is enough to define the linear functional $\tau\colon \mathcal{A} \to F$ given by

$$\tau(a) = \alpha^{-1}\varphi(a, 1, \dots, 1) \quad \text{for all} \quad a \in \mathcal{A}. \qquad \square$$

Lemma 4.18. *An F-algebra $\mathcal{A}$ is f-zpd if and only if every multilinear functional $\varphi\colon \mathcal{A}^m \to F$ that preserves zeros of f satisfies the following condition: for all $N \geqslant 1$ and all $a_1^{(t)}, \dots, a_m^{(t)} \in \mathcal{A}$, $t = 1, \dots, N$,*

$$\sum_{t=1}^{N} f\big(a_1^{(t)}, \dots, a_m^{(t)}\big) = 0 \implies \sum_{t=1}^{N} \varphi\big(a_1^{(t)}, \dots, a_m^{(t)}\big) = 0 \tag{4.1}$$

Proof. The "only if" part is clear. To prove the "if" part, denote by $\mathcal{A}_0$ the linear span of $f(\mathcal{A})$, and observe that (4.1) implies that $\tau_0\colon \mathcal{A}_0 \to F$,

$$\tau_0\left(\sum_{t=1}^{N} f\left(a_1^{(t)}, \dots, a_m^{(t)}\right)\right) = \sum_{t=1}^{N} \varphi\left(a_1^{(t)}, \dots, a_m^{(t)}\right)$$

is a well defined linear functional on $\mathcal{A}_0$. Letting $\tau\colon \mathcal{A} \to F$ to be any linear extension of τ_0, we thus have $\varphi(a_1, \dots, a_m) = \tau\left(f(a_1, \dots, a_m)\right)$ for all $a_1, \dots, a_m \in \mathcal{A}$. $\square$

Lemma 4.19. *Let $\mathcal{A}$ be an f-zpd algebra and let $\mathcal{V}$ be a vector space over F. If a multilinear map $\Phi\colon \mathcal{A}^m \to \mathcal{V}$ preserves zeros of f, then there exists a linear map $T\colon \mathcal{A} \to \mathcal{V}$ such that*

$$\Phi(a_1, \dots, a_m) = T\left(f(a_1, \dots, a_m)\right)$$

for all $a_1, \dots, a_m \in \mathcal{A}$.

Proof. If $\mathcal{V} = F$ then this is true by the definition of an f-zpd algebra. The general case can be easily reduced to this one. Indeed, take a linear functional ω on $\mathcal{V}$ and observe that the composition $\omega \circ \Phi$ is a multilinear functional preserving zeros of f. We may therefore use Lemma 4.18 to conclude that for all $a_1^{(t)}, \dots, a_m^{(t)} \in A$,

$$\sum_{t=1}^{N} f\big(a_1^{(t)}, \dots, a_m^{(t)}\big) = 0 \implies$$

$$\omega\left(\sum_{t=1}^{N} \Phi\big(a_1^{(t)}, \dots, a_m^{(t)}\big)\right) = \sum_{t=1}^{N} (\omega \circ \Phi)\big(a_1^{(t)}, \dots, a_m^{(t)}\big) = 0.$$

Since ω is an arbitrary linear functional on $\mathcal{V}$, it follows that Φ satisfies

$$\sum_{t=1}^{N} f\big(a_1^{(t)}, \dots, a_m^{(t)}\big) = 0 \implies \sum_{t=1}^{N} \Phi\big(a_1^{(t)}, \dots, a_m^{(t)}\big) = 0.$$

We can now repeat the argument from the proof of Lemma 4.18, that is, we define the linear map $T_0\colon \mathcal{A}_0 \to \mathcal{V}$ by

$$T_0\left(\sum_{t=1}^{N} f\left(a_1^{(t)}, \dots, a_m^{(t)}\right)\right) = \sum_{t=1}^{N} \Phi\big(a_1^{(t)}, \dots, a_m^{(t)}\big)$$

(where $\mathcal{A}_0$ is the linear span of the image of f on $\mathcal{A}$) and extend it to a linear map $T\colon \mathcal{A} \to \mathcal{V}$. $\square$

4.2 Examples and non-examples of f-zpd algebras

In this section we will be focused in dealing with Question 4.14. While the first subsection is devoted to show that Question 4.14 has a negative answer in general, the following subsections give examples of multilinear polynomials f so that $M_n(F)$ is f-zpd.

4.2.1 The full matrix algebra is not always f-zpd

Until the rest of this section, assume that the field F has more than 3 elements. Pick $\alpha, \beta \in F\backslash\{0, -1\}$ with $\alpha \neq \beta$. Fix $n \geqslant 2$ and a multilinear central polynomial $c = c(x_1, \ldots, x_l)$ of $M_n(F)$, and define multilinear polynomials h_1, h_2, f, g of degree $m = l + 1$ by

$$\begin{aligned} h_1 &= c(x_1, \ldots, x_{m-2}, x_{m-1})x_m, \\ h_2 &= c(x_1, \ldots, x_{m-2}, x_m)x_{m-1}, \\ f &= h_1 + \alpha h_2, \\ g &= h_1 + \beta h_2. \end{aligned}$$

Example 4.20. *If $n = 2$, then we can take*

$$c = [x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2]$$

which is a central polynomial of minimal degree. Then h_1, h_2, f, g are of degree $m = 5$. For example,

$$\begin{aligned} f = [x_1, x_2][x_3, x_4]x_5 &+ [x_3, x_4][x_1, x_2]x_5 \\ &+ \alpha[x_1, x_2][x_3, x_5]x_4 + \alpha[x_3, x_5][x_1, x_2]x_4. \end{aligned}$$

The next proposition is related to the multilinear Nullstellensatz from the last section of this chapter.

Proposition 4.21. *Let $A_1, \ldots, A_m \in M_n(F)$. The following conditions are equivalent:*

(i) $f(A_1, \ldots, A_m) = 0$.

(ii) $g(A_1, \ldots, A_m) = 0$.

(iii) $h_1(A_1, \ldots, A_m) = h_2(A_1, \ldots, A_m) = 0$.

In particular, f and g have the same zero sets. However, g is not the sum of a scalar multiple of f and a polynomial identity.

Proof. Suppose that $f(A_1,\dots,A_m) = 0$ but $h_1(A_1,\dots,A_m) \neq 0$. The latter implies $c(A_1,\dots,A_{m-1}) \neq 0$ which together with $f(A_1,\dots,A_m) = 0$ shows that $A_m = \lambda A_{m-1}$ for some $\lambda \in F$. Hence,

$$\begin{aligned}
&(1+\alpha)h_1(A_1,\dots,A_m)\\
=&\lambda(1+\alpha)c(A_1,\dots,A_{m-1})A_{m-1}\\
=&c(A_1,\dots,A_{m-2},A_{m-1})A_m + \alpha c(A_1,\dots,A_{m-2},A_m)A_{m-1}\\
=&f(A_1,\dots,A_m) = 0.
\end{aligned}$$

As $\alpha \neq -1$, this contradicts our assumption. We have thereby shown that (i) implies (iii). Since (iii) trivially implies (i), these two conditions are equivalent. Similarly we see that (ii) and (iii) are equivalent.

Let us now prove that g is not a sum of a scalar multiple of f and polynomial identity of $M_n(F)$. Indeed, assume that $g = \lambda f + h$, where $h \in Id(M_n(F))$. This means that $g - \lambda f \in Id(M_n(F))$, that is,

$$(1-\lambda)c(x_1,\dots,x_{m-1})x_m + (\beta - \lambda\alpha)c(x_1,\dots,x_m)x_{m-1} \in Id(M_n(F)).$$

If $\lambda = 1$, then we conclude that $c(x_1,\dots,x_m)x_{m-1} \in Id(M_n(F))$ which is clearly an absurd. We assume then $\lambda \neq 1$. Taking $A_1,\dots,A_{m-1} \in M_n(F)$ such that $\gamma = c(A_1,\dots,A_{m-1}) \neq 0$ and $A_m \in M_n(F)$ linearly independent with A_{m-1}, we see that

$$(1-\lambda)\gamma A_m + (\beta - \lambda\alpha)\delta A_{m-1} = 0,$$

where $\delta \in F$. This is an absurd, and we conclude the proof of the proposition. □

The second proposition provides an evidence for the nontriviality of the results of the following subsections.

Proposition 4.22. *$M_n(F)$ is not f-zpd.*

Proof. Pick $B_1,\dots,B_{m-1} \in M_n(F)$ such that $c(B_1,\dots,B_{m-1}) \neq 0$. Hence,

$$\rho(A) = c(B_1,\dots,B_{m-2},A)$$

is a non-zero linear functional on $M_n(F)$ (here we identified scalars with scalar multiples of the identity). Let ω be any linear functional on $M_n(F)$ that is linearly independent with ρ. Define $\varphi\colon M_n(F)^m \to F$ by

$$\varphi(A_1,\dots,A_m) = c(A_1,\dots,A_{m-1})\omega(A_m).$$

Observe that the implication (i) $\implies$ (iii) from Proposition 4.21 shows that for all $A_1,\dots,A_m \in M_n(F)$,

$$f(A_1,\dots,A_m) = 0 \implies \varphi(A_1,\dots,A_m) = 0.$$

Suppose $M_n(F)$ were f-zpd. Then there would exist a linear functional τ on $M_n(F)$ such that

$$\varphi(A_1, \dots, A_m) = \tau\left(f(A_1, \dots, A_m)\right)$$

for all $A_1, \dots, A_m \in M_n(F)$. That is,

$$\begin{aligned} &c(A_1, \dots, A_{m-2}, A_{m-1})\omega(A_m) \\ =&c(A_1, \dots, A_{m-2}, A_{m-1})\tau(A_m) + \alpha c(A_1, \dots, A_{m-2}, A_m)\tau(A_{m-1}). \end{aligned}$$

Taking B_i for A_i, $i = 1, \dots, m-2$, and writing A for A_{m-1} and B for A_m, we thus have

$$\rho(A)\omega(B) = \rho(A)\tau(B) + \alpha\tau(A)\rho(B)$$

for all $A, B \in M_n(F)$. Picking any $A \notin \ker \rho$ we see that $\tau = \omega + \lambda\rho$ for some $\lambda \in F$. Consequently,

$$((1+\alpha)\lambda\rho(A) + \alpha\omega(A))\rho(B) = 0,$$

which contradicts the linear independence of ρ and ω. □

4.2.2 The generalized commutator

This subsection is devoted to the *generalized commutator*

$$f(x_1, x_2, x_3) = x_1x_2x_3 - x_3x_2x_1.$$

This is one of the polynomials that deserve special attention (see, e.g., [45]), so the question of whether the algebra $M_n(F)$ is f-zpd occurs naturally. We will show that the answer is affirmative.

Throughout this subsection, we assume that $\varphi\colon M_n(F)^3 \to F$ is a 3-linear functional such that for all $A, B, C \in M_n(F)$,

$$ABC - CBA = 0 \implies \varphi(A, B, C) = 0. \tag{4.2}$$

Our goal is to prove that φ satisfies the condition presented in Lemma 4.18. Thus, assume that $N \geqslant 1$ and that the matrices

$$A^{(t)} = \sum_{i,j=1}^{n} a_{ij}^t e_{ij}, \qquad B^{(t)} = \sum_{i,j=1}^{n} b_{ij}^t e_{ij}, \qquad C^{(t)} = \sum_{i,j=1}^{n} c_{ij}^t e_{ij},$$

$t = 1, \dots, N$, where e_{ij} are standard matrix units, satisfy

$$\sum_{t=1}^{N} A^{(t)}B^{(t)}C^{(t)} - C^{(t)}B^{(t)}A^{(t)} = 0. \tag{4.3}$$

We have to show that

$$\sum_{t=1}^{N} \varphi\left(A^{(t)}, B^{(t)}, C^{(t)}\right) = 0. \tag{4.4}$$

We proceed with a series of lemmas.

Lemma 4.23. *We have*

$$\sum_{t=1}^{N}\left(\sum_{l=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_{ik}^t b_{kl}^t c_{lj}^t - c_{ik}^t b_{kl}^t a_{lj}^t + \sum_{\substack{l=1\\l\neq i}}^{n} a_{ij}^t b_{jl}^t c_{lj}^t - c_{ij}^t b_{jl}^t a_{lj}^t\right) = 0.$$

Proof. Note that for each pair $(i,j) \in \{1,\ldots,n\}^2$,

$$\left(A^{(t)}B^{(t)}C^{(t)} - C^{(t)}B^{(t)}A^{(t)}\right)_{ij} = \sum_{k,l=1}^{n} a_{ik}^t b_{kl}^t c_{lj}^t - c_{ik}^t b_{kl}^t a_{lj}^t,$$

and hence, by (4.3),

$$\sum_{t=1}^{N}\sum_{k,l=1}^{n} a_{ik}^t b_{kl}^t c_{lj}^t - c_{ik}^t b_{kl}^t a_{lj}^t = 0.$$

Clearly $a_{ij}^t b_{ji}^t c_{ij}^t - c_{ij}^t b_{ji}^t a_{ij}^t = 0$ for all t. Hence this sum reduces to the one from the statement of the lemma. □

It is obvious that $f(A,B,A) = 0$ and so

$$\varphi(A,B,A) = 0,$$

yielding

$$\varphi(A,B,C) = -\varphi(C,B,A)$$

for all $A,B,C \in M_n(F)$. In what follows, we will use these two identities without comment.

Lemma 4.24. *We have*

$$\begin{aligned}
&\sum_{t=1}^{N}\varphi\left(A^{(t)},B^{(t)},C^{(t)}\right)\\
=&\sum_{t=1}^{N}\sum_{i,j=1}^{n}\left(\sum_{l=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_{ik}^t b_{kl}^t c_{lj}^t \varphi(e_{ik},e_{kl},e_{lj}) + \sum_{\substack{l=1\\l\neq i}}^{n} a_{ij}^t b_{jl}^t c_{lj}^t \varphi(e_{ij},e_{jl},e_{lj})\right.\\
&\left.-\sum_{l=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} c_{ik}^t b_{kl}^t a_{lj}^t \varphi(e_{ik},e_{kl},e_{lj}) - \sum_{\substack{l=1\\l\neq i}}^{n} c_{ij}^t b_{jl}^t a_{lj}^t \varphi(e_{ij},e_{jl},e_{lj})\right).
\end{aligned}$$

Proof. Clearly,

$$\begin{aligned}
\sum_{t=1}^{N}\varphi\left(A^{(t)},B^{(t)},C^{(t)}\right) &= \sum_{t=1}^{N}\varphi\left(\sum_{i,j=1}^{n} a_{ij}^t e_{ij}, \sum_{k,l=1}^{n} b_{kl}^t e_{kl}, \sum_{p,q=1}^{n} c_{pq}^t e_{pq}\right)\\
&= \sum_{t=1}^{N}\sum_{i,j,k,l,p,q=1}^{n} a_{ij}^t b_{kl}^t c_{pq}^t \varphi(e_{ij},e_{kl},e_{pq}).
\end{aligned} \tag{4.5}$$

It is easy to check that if i, j, k, l, p, q satisfy one of the following conditions

$$j \neq k \text{ and } q \neq k,$$
$$j \neq k \text{ and } i \neq l,$$
$$l \neq p \text{ and } q \neq k,$$
$$l \neq p \text{ and } i \neq l,$$

then $f(e_{ij}, e_{kl}, e_{pq}) = 0$ and so $\varphi(e_{ij}, e_{kl}, e_{pq}) = 0$. Hence we may assume that the following relations hold

$$j = k \text{ or } q = k,$$
$$j = k \text{ or } i = l,$$
$$l = p \text{ or } q = k,$$
$$l = p \text{ or } i = l.$$

We can rewrite (4.5) as

$$\begin{aligned}\sum_{t=1}^{N} \varphi\left(A^{(t)}, B^{(t)}, C^{(t)}\right) &= \sum_{t=1}^{N} \sum_{i,j,l,q=1}^{n} a_{ij}^t b_{jl}^t c_{lq}^t \varphi(e_{ij}, e_{jl}, e_{lq}) \\ &+ \sum_{t=1}^{N} \sum_{i,j,p=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{ij}^t b_{ki}^t c_{pk}^t \varphi(e_{ij}, e_{ki}, e_{pk}) \\ &+ \sum_{t=1}^{N} \sum_{i,j=1}^{n} \sum_{\substack{p=1 \\ p \neq i}}^{n} a_{ij}^t b_{ji}^t c_{pj}^t \varphi(e_{ij}, e_{ji}, e_{pj}).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{t=1}^{N} \varphi\left(A^{(t)}, B^{(t)}, C^{(t)}\right) = \sum_{t=1}^{N} \sum_{i,j=1}^{n} \Bigg(&\sum_{k,l=1}^{n} a_{ik}^t b_{kl}^t c_{lj}^t \varphi(e_{ik}, e_{kl}, e_{lj}) \\ &- \sum_{l=1}^{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} c_{ik}^t b_{kl}^t a_{lj}^t \varphi(e_{ik}, e_{kl}, e_{lj}) \\ &- \sum_{\substack{l=1 \\ l \neq i}}^{n} c_{ij}^t b_{jl}^t a_{lj}^t \varphi(e_{ij}, e_{jl}, e_{lj}) \Bigg).\end{aligned}$$

Finally, using $\varphi(e_{ij}, e_{ji}, e_{ij}) = 0$ we obtain the statement of the lemma. □

Lemma 4.25. *If $u \neq l, i$; $l \neq i$, and $j \neq k$, then*

$$\varphi(e_{ik}, e_{kl}, e_{lj}) = \varphi(e_{ik}, e_{ku}, e_{uj}).$$

Proof. Note that

$$f(e_{ik} + e_{uj}, e_{kl} + e_{ku}, e_{lj} + e_{ik}) = 0.$$

Therefore,

$$\begin{aligned}
0 &= \varphi(e_{ik} + e_{uj}, e_{kl} + e_{ku}, e_{lj} + e_{ik}) \\
&= \varphi(e_{ik}, e_{kl}, e_{lj}) + \varphi(e_{ik}, e_{kl}, e_{ik}) + \varphi(e_{ik}, e_{ku}, e_{lj}) \\
&\quad + \varphi(e_{ik}, e_{ku}, e_{ik}) + \varphi(e_{uj}, e_{kl}, e_{lj}) + \varphi(e_{uj}, e_{kl}, e_{ik}) \\
&\quad + \varphi(e_{uj}, e_{ku}, e_{lj}) + \varphi(e_{uj}, e_{ku}, e_{ik}) \\
&= \varphi(e_{ik}, e_{kl}, e_{lj}) + \varphi(e_{uj}, e_{ku}, e_{ik}).
\end{aligned}$$

□

Lemma 4.26. *If $l \neq i$ and $k \neq j$, then*

$$\varphi(e_{ij}, e_{jl}, e_{lj}) = \varphi(e_{ik}, e_{kl}, e_{lj}).$$

Proof. Note that

$$f(e_{ij} + e_{lj}, e_{jl} + e_{kl}, e_{lj} + e_{ik}) = 0.$$

Therefore,

$$\begin{aligned}
0 &= \varphi(e_{ij} + e_{lj}, e_{jl} + e_{kl}, e_{lj} + e_{ik}) \\
&= \varphi(e_{ij}, e_{jl}, e_{lj}) + \varphi(e_{ij}, e_{jl}, e_{ik}) + \varphi(e_{ij}, e_{kl}, e_{lj}) \\
&\quad + \varphi(e_{ij}, e_{kl}, e_{ik}) + \varphi(e_{lj}, e_{jl}, e_{lj}) + \varphi(e_{lj}, e_{jl}, e_{ik}) \\
&\quad + \varphi(e_{lj}, e_{kl}, e_{lj}) + \varphi(e_{lj}, e_{kl}, e_{ik}) \\
&= \varphi(e_{ij}, e_{jl}, e_{lj}) + \varphi(e_{lj}, e_{kl}, e_{ik}).
\end{aligned}$$

□

Lemma 4.27. *If $k \neq i$ and $k \neq j$, then*

$$\varphi(e_{ik}, e_{ki}, e_{ij}) = \varphi(e_{ik}, e_{kk}, e_{kj}).$$

Proof. Note that

$$f(e_{ik} + e_{kj}, e_{ki} + e_{kk}, e_{ij} + e_{ik}) = 0.$$

Therefore,

$$\begin{aligned}
0 &= \varphi(e_{ik} + e_{kj}, e_{ki} + e_{kk}, e_{ij} + e_{ik}) \\
&= \varphi(e_{ik}, e_{ki}, e_{ij}) + \varphi(e_{ik}, e_{ki}, e_{ik}) + \varphi(e_{ik}, e_{kk}, e_{ij}) \\
&\quad + \varphi(e_{ik}, e_{kk}, e_{ik}) + \varphi(e_{kj}, e_{ki}, e_{ij}) + \varphi(e_{kj}, e_{ki}, e_{ik}) \\
&\quad + \varphi(e_{kj}, e_{kk}, e_{ij}) + \varphi(e_{kj}, e_{kk}, e_{ik}) \\
&= \varphi(e_{ik}, e_{ki}, e_{ij}) + \varphi(e_{kj}, e_{kk}, e_{ik}).
\end{aligned}$$

□

Lemma 4.28. *If $k \neq i$, then*

$$\varphi(e_{ii}, e_{ik}, e_{ki}) = \varphi(e_{ik}, e_{kk}, e_{ki}).$$

Proof. Note that

$$f(e_{ii} + e_{ki}, e_{ik} + e_{kk}, e_{ki} + e_{ik}) = 0$$

and therefore

$$\begin{aligned} 0 &= \varphi(e_{ii} + e_{ki}, e_{ik} + e_{kk}, e_{ki} + e_{ik}) \\ &= \varphi(e_{ii}, e_{ik}, e_{ki}) + \varphi(e_{ii}, e_{ik}, e_{ik}) + \varphi(e_{ii}, e_{kk}, e_{ki}) \\ &\quad + \varphi(e_{ii}, e_{kk}, e_{ik}) + \varphi(e_{ki}, e_{ik}, e_{ki}) + \varphi(e_{ki}, e_{ik}, e_{ik}) \\ &\quad + \varphi(e_{ki}, e_{kk}, e_{ki}) + \varphi(e_{ki}, e_{kk}, e_{ik}) \\ &= \varphi(e_{ii}, e_{ik}, e_{ki}) + \varphi(e_{ki}, e_{kk}, e_{ik}). \end{aligned}$$

□

Lemma 4.29. *We have*

$$\varphi(e_{ii}, e_{ij}, e_{jj}) = \varphi(e_{ii}, e_{ii}, e_{ij}) = \varphi(e_{ij}, e_{jj}, e_{jj}).$$

Proof. Note that

$$f(e_{ii} + e_{jj}, e_{ii} + e_{ij}, e_{ij} + e_{ii}) = 0.$$

Therefore,

$$\begin{aligned} 0 &= \varphi(e_{ii} + e_{jj}, e_{ii} + e_{ij}, e_{ij} + e_{ii}) \\ &= \varphi(e_{ii}, e_{ii}, e_{ij}) + \varphi(e_{ii}, e_{ii}, e_{ii}) + \varphi(e_{ii}, e_{ij}, e_{ij}) \\ &\quad + \varphi(e_{ii}, e_{ij}, e_{ii}) + \varphi(e_{jj}, e_{ii}, e_{ij}) + \varphi(e_{jj}, e_{ii}, e_{ii}) \\ &\quad + \varphi(e_{jj}, e_{ij}, e_{ij}) + \varphi(e_{jj}, e_{ij}, e_{ii}) \\ &= \varphi(e_{ii}, e_{ii}, e_{ij}) + \varphi(e_{jj}, e_{ij}, e_{ii}). \end{aligned}$$

The second equality can be obtained analogously. □

The next result contains all information that we need from the previous five lemmas.

Lemma 4.30. *Given $i, j \in \{1, \ldots, n\}$, the set*

$$\Phi_{ij} = \{\varphi(e_{ik}, e_{kl}, e_{lj}) | k, l = 1 \ldots, n\} \backslash \{\varphi(e_{ij}, e_{ji}, e_{ij})\}$$

is a singleton set.

Proof. Assume first that $i \neq j$. We claim that

$$\Phi_{ij} = \{\varphi(e_{ij}, e_{jj}, e_{jj})\}.$$

Consider $\varphi(e_{ik}, e_{kl}, e_{lj})$ where $k \neq j$ and $l \neq i$. Since $j \neq i$, by Lemma 4.25 we have $\varphi(e_{ik}, e_{kl}, e_{lj}) = \varphi(e_{ik}, e_{kj}, e_{jj})$. We now apply Lemma 4.26 to get $\varphi(e_{ik}, e_{kj}, e_{jj}) = \varphi(e_{ij}, e_{jj}, e_{jj})$.

We now consider the case where $i \neq l$ and $j = k$. In this case take $u \neq j$ and by Lemma 4.26 we have $\varphi(e_{ij}, e_{jl}, e_{lj}) = \varphi(e_{iu}, e_{ul}, e_{lj})$. However $\varphi(e_{iu}, e_{ul}, e_{lj}) = \varphi(e_{ij}, e_{jj}, e_{jj})$ since $u \neq j$, $l \neq i$ and the previous case.

We now consider the case where $j \neq k$ and $i = l$. If $k = i$, then by Lemma 4.29 we have $\varphi(e_{ii}, e_{ii}, e_{ij}) = \varphi(e_{ij}, e_{jj}, e_{jj})$. In case $k \neq i$, then $\varphi(e_{ik}, e_{ki}, e_{ij}) = \varphi(e_{ik}, e_{kk}, e_{kj})$. We now apply Lemma 4.25 to obtain $\varphi(e_{ik}, e_{kk}, e_{kj}) = \varphi(e_{ik}, e_{kj}, e_{jj})$ and finally Lemma 4.26 to get $\varphi(e_{ik}, e_{kj}, e_{jj}) = \varphi(e_{ij}, e_{jj}, e_{jj})$ as desired.

We may now consider the case where $i = j$. Fix $u \neq i$. We claim that

$$\Phi_{ij} = \{\varphi(e_{ii}, e_{iu}, e_{ui})\}.$$

Initially consider $k \neq i$ and $l \neq i$. Then by Lemma 4.25 $\varphi(e_{ik}, e_{kl}, e_{li}) = \varphi(e_{ik}, e_{ku}, e_{ui})$ and by Lemma 4.26 we obtain $\varphi(e_{ik}, e_{ku}, e_{ui}) = \varphi(e_{ii}, e_{iu}, e_{ui})$.

If $k \neq i$ and $l = i$, then Lemma 4.27 $\varphi(e_{ik}, e_{ki}, e_{ii}) = \varphi(e_{ik}, e_{kk}, e_{ki})$. By Lemma 4.25 $\varphi(e_{ik}, e_{kk}, e_{ki}) = \varphi(e_{ik}, e_{ku}, e_{ui})$ and finally Lemma 4.26 implies $\varphi(e_{ik}, e_{ku}, e_{ui}) = \varphi(e_{ii}, e_{iu}, e_{ui})$.

We now consider $k = i$ and $l \neq i$. By Lemma 4.28 we have $\varphi(e_{ii}, e_{il}, e_{li}) = \varphi(e_{il}, e_{ll}, e_{li})$. By Lemma 4.25 $\varphi(e_{il}, e_{ll}, e_{li}) = \varphi(e_{il}, e_{lu}, e_{ui})$ and then Lemma 4.26 implies $\varphi(e_{il}, e_{lu}, e_{ui}) = \varphi(e_{ii}, e_{iu}, e_{ui})$. □

Theorem 4.31. *Let $f = x_1x_2x_3 - x_3x_2x_1$. Then the algebra $M_n(F)$ is f-zpd.*

Proof. As already mentioned, in light of Lemma 4.18 it is enough to prove (4.4). Considering the right-hand side of the identity given in Lemma 4.24, and denoting $\Phi_{ij} = \{\varphi_{ij}\}$ by Lemma 4.30, we therefore have

$$\begin{aligned}\sum_{t=1}^{N} \varphi\left(A^{(t)}, B^{(t)}, C^{(t)}\right) &= \sum_{t=1}^{N} \sum_{i,j=1}^{n} \left(\sum_{l=1}^{n} \sum_{\substack{k=1\\k\neq j}}^{n} a_{ik}^t b_{kl}^t c_{lj}^t - c_{ik}^t b_{kl}^t a_{lj}^t + \sum_{\substack{l=1\\l\neq i}}^{n} a_{ij}^t b_{jl}^t c_{lj}^t - c_{ij}^t b_{jl}^t a_{lj}^t \right) \varphi_{ij} \\ &= \sum_{i,j=1}^{n} \sum_{t=1}^{N} \left(\sum_{l=1}^{n} \sum_{\substack{k=1\\k\neq j}}^{n} a_{ik}^t b_{kl}^t c_{lj}^t - c_{ik}^t b_{kl}^t a_{lj}^t + \sum_{\substack{l=1\\l\neq i}}^{n} a_{ij}^t b_{jl}^t c_{lj}^t - c_{ij}^t b_{jl}^t a_{lj}^t \right) \varphi_{ij}.\end{aligned}$$

Invoking Lemma 4.23 we now obtain the desired conclusion that

$$\sum_{t=1}^{N} \varphi\left(A^{(t)}, B^{(t)}, C^{(t)}\right) = 0.$$

□

4.2.3 Polynomials given by cyclic permutations

In this subsection we deal with a multilinear polynomial f whose monomials correspond to a cyclic permutation and satisfies the condition that the sum of its coefficients is non-zero. The only assumption that we will require on our algebra is that it is generated by idempotents. In our proof we will use ideas from the proof that an algebra generated by idempotents is zJpd (see [6] or [13, Theorem 3.15]).

Let us now state the theorem.

Theorem 4.32. *Let* $\mathrm{char}(F) \neq 2$*, let* $\alpha_1, \ldots, \alpha_m \in F$ *be such that* $\sum_{i=1}^m \alpha_i \neq 0$*, and let*

$$f(x_1, \ldots, x_m) = \alpha_1 x_1 \cdots x_m + \alpha_2 x_2 \cdots x_m x_1 + \cdots + \alpha_m x_m x_1 \cdots x_{m-1}.$$

If an F*-algebra* $\mathcal{A}$ *is generated by idempotents, then* $\mathcal{A}$ *is* f*-zpd.*

Proof. In light of Proposition 4.16(ii), we may assume without loss of generality that $\sum_{i=1}^{m} \alpha_i = 1$. Let φ be an m-linear functional preserving zeros of f. By Proposition 4.17, it suffices to prove that φ satisfies

$$\varphi(a_1, \ldots, a_m) = \varphi(f(a_1, \ldots, a_m), 1, \ldots, 1) \tag{4.6}$$

for all $a_1, \ldots, a_m \in \mathcal{A}$.

Set $\alpha_{m+1} = \alpha_1$. We claim that $\alpha_i + \alpha_{i+1} \neq 0$ for some i. Indeed, if m is even then this is immediate from

$$1 = (\alpha_1 + \alpha_2) + \cdots + (\alpha_{m-1} + \alpha_m),$$

and if m is odd this follows from the observation that

$$\alpha_1 = -\alpha_2 = \cdots = \alpha_m = -\alpha_1$$

implies $2\alpha_1 = 0$ and hence $\alpha_1 = 0$, so we can again use the assumption that the sum of all α_i is 1. After relabeling, if necessary, we may assume that $i = m$, i.e.,

$$\alpha_1 + \alpha_m \neq 0.$$

Let $\mathcal{S}$ denote the set of all $s \in \mathcal{A}$ such that

$$\varphi(a_1, \ldots, a_{m-1}, s) = \varphi(f(a_1, \ldots, a_{m-1}, s), 1, \ldots, 1)$$

for all $a_1, \ldots, a_{m-1} \in \mathcal{A}$. To prove (4.6), we have to show that $\mathcal{S} = \mathcal{A}$. We will establish this by induction on m.

In the base case where $m = 2$ we have

$$f(x_1, x_2) = \alpha_1 x_1 x_2 + \alpha_2 x_2 x_1$$

with $\alpha_1 + \alpha_2 = 1$. The proof that we will give is just a minor modification of the proof that $\mathcal{A}$ is zJpd (i.e., of the case where $\alpha_1 = \alpha_2$).

Since $\mathcal{S}$ is a vector subspace of $\mathcal{A}$, it is enough to show that $\mathcal{S}$ contains every product of the form $e_1 e_2 \cdots e_n$ where e_i are idempotents in $\mathcal{A}$. We proceed by induction on n. Let $e \in \mathcal{A}$ be an idempotent, and let us first prove that $e \in \mathcal{S}$. Considering an arbitrary $a \in \mathcal{A}$ and writing $h = 1 - e$, we have

$$a = eae + hae + eah + hah$$

and hence

$$\varphi(a, e) = \varphi(eae, e) + \varphi(hae, e) + \varphi(eah, e) + \varphi(hah, e).$$

Recalling that $\alpha_1 + \alpha_2 = 1$, one can easily check that

$$f(hae, e - \alpha_1) = f(eah, e - \alpha_2) = f(eae, h) = f(hah, e) = 0,$$

which gives us the following relations

$$\begin{aligned}
\varphi(hae, e) &= \alpha_1 \varphi(hae, 1),\\
\varphi(eah, e) &= \alpha_2 \varphi(eah, 1),\\
\varphi(eae, e) &= \varphi(eae, 1),\\
\varphi(hah, e) &= 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\varphi(a, e) &= \varphi(eae, e) + \varphi(eah, e) + \varphi(hae, e) + \varphi(hah, e)\\
&= \varphi(eae, 1) + \alpha_1 \varphi(hae, 1) + \alpha_2 \varphi(eah, 1)\\
&= \varphi(eae + \alpha_1 hae + \alpha_2 eah, 1)\\
&= \varphi(f(a, e), 1).
\end{aligned}$$

We have thus shown that $e \in \mathcal{S}$.

Next, assuming that $\mathcal{S}$ contains products of n idempotents, let us prove that $\mathcal{S}$ also contains $e_1 \cdots e_n e_{n+1}$, where each $e_i \in \mathcal{A}$ is an idempotent. Write

$$h_1 = 1 - e_1, h_{n+1} = 1 - e_{n+1}, t = e_2 \cdots e_n,$$

($t = 1$ if $n = 1$), so we want to prove that $e_1 t e_{n+1} \in \mathcal{S}$. For any $a \in \mathcal{A}$, we have

$$a = e_{n+1} a e_1 + h_{n+1} a e_1 + e_{n+1} a h_1 + h_{n+1} a h_1.$$

Therefore,

$$\begin{aligned}
\varphi(a, e_1 t e_{n+1}) &= \varphi(e_{n+1} a e_1, e_1 t e_{n+1}) + \varphi(h_{n+1} a e_1, e_1 t e_{n+1})\\
&\quad + \varphi(e_{n+1} a h_1, e_1 t e_{n+1}) + \varphi(h_{n+1} a h_1, e_1 t e_{n+1})\\
&= \varphi(e_{n+1} a e_1, t e_{n+1} - h_1 t + h_1 t h_{n+1})\\
&\quad + \varphi(h_{n+1} a e_1, t e_{n+1} - h_1 t e_{n+1})\\
&\quad + \varphi(e_{n+1} a h_1, e_1 t - e_1 t h_{n+1})\\
&\quad + \varphi(h_{n+1} a h_1, e_1 t e_{n+1}).
\end{aligned}$$

Since

$$f(e_{n+1}ae_1, h_1th_{n+1}) = 0,$$
$$f(h_{n+1}ae_1, h_1te_{n+1}) = 0,$$
$$f(e_{n+1}ah_1, e_1th_{n+1}) = 0,$$
$$f(h_{n+1}ah_1, e_1te_{n+1}) = 0$$

and hence

$$\varphi(e_{n+1}ae_1, h_1th_{n+1}) = 0,$$
$$\varphi(h_{n+1}ae_1, h_1te_{n+1}) = 0,$$
$$\varphi(e_{n+1}ah_1, e_1th_{n+1}) = 0,$$
$$\varphi(h_{n+1}ah_1, e_1te_{n+1}) = 0,$$

we can conclude that

$$\varphi(a, e_1te_{n+1}) = \varphi(e_{n+1}ae_1, te_{n+1} - h_1t) + \varphi(h_{n+1}ae_1, te_{n+1}) + \varphi(e_{n+1}ah_1, e_1t).$$

Since $te_{n+1}, h_1t, te_{n+1}, e_1t$ lie in S by the induction hypothesis, it follows that

$$\begin{aligned}
\varphi(e_{n+1}ae_1, te_{n+1}) &= \varphi(\alpha_1 e_{n+1}ae_1te_{n+1} + \alpha_2 te_{n+1}ae_1, 1),\\
\varphi(e_{n+1}ae_1, h_1t) &= \varphi(\alpha_2 h_1te_{n+1}ae_1, 1),\\
\varphi(h_{n+1}ae_1, te_{n+1}) &= \varphi(\alpha_1 h_{n+1}ae_1te_{n+1}, 1),\\
\varphi(e_{n+1}ah_1, e_1t) &= \varphi(\alpha_2 e_1te_{n+1}ah_1, 1).
\end{aligned}$$

One easily checks that this implies that

$$\varphi(a, e_1te_{n+1}) = \varphi(f(a, e_1te_{n+1}), 1).$$

Hence $e_1te_{n+1} \in \mathcal{S}$, which concludes the proof for the base case where $m = 2$.

Assume now $m > 2$ and that the result holds for $m-1$, and so in particular for the polynomial $f(x_1, \ldots, x_{m-1}, 1)$ (which is also of the required form). As the $(m-1)$-linear functional $\varphi(a_1, \ldots, a_{m-1}, 1)$ preserves its zeros, we have

$$\varphi(a_1, \ldots, a_{m-1}, 1) = \varphi(f(a_1, \ldots, a_{m-1}, 1), 1, \ldots, 1) \tag{4.7}$$

for all $a_1, \ldots, a_{m-1} \in \mathcal{A}$.

As in the $m = 2$ case, it is enough to show that $\mathcal{S}$ contains every element of the form $e_1e_2 \cdots e_n$ where e_i are idempotents in $\mathcal{A}$. The proof that we will give is conceptually similar to the one just given, but the necessary changes are not obvious. We proceed by induction on n.

To handle the base case, take an idempotent $e = e_1 \in \mathcal{A}$. We must prove that $e \in \mathcal{S}$. Denote $1 - e$ by h and write $a_1 = ea_1 + ha_1$ and $a_{m-1} = a_{m-1}e + a_{m-1}h$. Thus,

$$\begin{aligned}
\varphi(a_1, \ldots, a_{m-1}, e) =&\varphi(ea_1, \ldots, a_{m-1}e, e) + \varphi(ea_1, \ldots, a_{m-1}h, e)\\
&+ \varphi(ha_1, \ldots, a_{m-1}e, e) + \varphi(ha_1, \ldots, a_{m-1}h, e).
\end{aligned} \tag{4.8}$$

It is easy to see that

$$\begin{aligned} &f(ea_1,\dots,a_{m-1}e,e-1)=0,\\ &f(ea_1,\dots,a_{m-1}h,(\alpha_1+\alpha_m)e-\alpha_m)=0,\\ &f(ha_1,\dots,a_{m-1}e,(\alpha_1+\alpha_m)e-\alpha_1)=0,\\ &f(ha_1,\dots,a_{m-1}h,e)=0. \end{aligned} \tag{4.9}$$

Of course, φ then satisfies the same identities, which can be written as

$$\begin{aligned} &\varphi(ea_1,\dots,a_{m-1}e,e)=\varphi(ea_1,\dots,a_{m-1}e,1),\\ &(\alpha_1+\alpha_m)\varphi(ea_1,\dots,a_{m-1}h,e)=\alpha_m\varphi(ea_1,\dots,a_{m-1}h,1),\\ &(\alpha_1+\alpha_m)\varphi(ha_1,\dots,a_{m-1}e,e)=\alpha_1\varphi(ha_1,\dots,a_{m-1}e,1),\\ &\varphi(ha_1,\dots,a_{m-1}h,e)=0. \end{aligned}$$

Consequently, (4.8) becomes

$$\begin{aligned} \varphi(a_1,\dots,a_{m-1},e)=\varphi(ea_1,\dots,a_{m-1}e,1)&\\ +\alpha_m(\alpha_1+\alpha_m)^{-1}\varphi(ea_1,\dots,a_{m-1}h,1)&\\ +\alpha_1(\alpha_1+\alpha_m)^{-1}\varphi(ha_1,\dots,a_{m-1}e,1).& \end{aligned}$$

Applying (4.7) it follows that

$$\begin{aligned} \varphi(a_1,\dots,a_{m-1},e)=\varphi(f(ea_1,\dots,a_{m-1}e,1),1,\dots,1)&\\ +\alpha_m(\alpha_1+\alpha_m)^{-1}\varphi(f(ea_1,\dots,a_{m-1}h,1),1,\dots,1)&\\ +\alpha_1(\alpha_1+\alpha_m)^{-1}\varphi(f(ha_1,\dots,a_{m-1}e,1),1,\dots,1).& \end{aligned}$$

Using (4.9) we obtain

$$\begin{aligned} \varphi(a_1,\dots,a_{m-1},e)=\varphi(f(ea_1,\dots,a_{m-1}e,e),1,\dots,1)&\\ +\varphi(f(ea_1,\dots,a_{m-1}h,e),1,\dots,1)&\\ +\varphi(f(ha_1,\dots,a_{m-1}e,e),1,\dots,1).& \end{aligned}$$

Since $f(ha_1,\dots,a_{m-1}h,e)=0$ (see (4.9)), it follows that

$$\varphi(a_1,\dots,a_{m-1},e)=\varphi(f(a_1,\dots,a_{m-1},e),1,\dots,1).$$

This means that $e\in\mathcal{S}$, as desired.

We may now assume that any product of n idempotents is contained in $\mathcal{S}$. Take idempotents $e_1,\dots,e_{n+1}$ and let us prove that $\mathcal{S}$ contains $e_1\cdots e_ne_{n+1}$. Write

$$h_1=1-e_1,\quad h_{n+1}=1-e_{n+1},\quad t=e_2\cdots e_n$$

($t=1$ if $n=1$). We have to show that $e_1te_{n+1}\in\mathcal{S}$. Take $a_1,a_2,\dots,a_{m-1}\in\mathcal{A}$ and write

$$a_1=e_{n+1}a_1+h_{n+1}a_1\quad\text{and}\quad a_{m-1}=a_{m-1}e_1+a_{m-1}h_1.$$

We have

$$\begin{aligned}\varphi(a_1,\dots,a_{m-1},e_1te_{n+1}) &= \varphi(e_{n+1}a_1,\dots,a_{m-1}e_1,e_1te_{n+1})\\
&\quad+\varphi(e_{n+1}a_1,\dots,a_{m-1}h_1,e_1te_{n+1})\\
&\quad+\varphi(h_{n+1}a_1,\dots,a_{m-1}e_1,e_1te_{n+1})\\
&\quad+\varphi(h_{n+1}a_1,\dots,a_{m-1}h_1,e_1te_{n+1})\\
&= \varphi(e_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}-h_1t+h_1th_{n+1})\\
&\quad+\varphi(e_{n+1}a_1,\dots,a_{m-1}h_1,e_1t-e_1th_{n+1})\\
&\quad+\varphi(h_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}-h_1te_{n+1})\\
&\quad+\varphi(h_{n+1}a_1,\dots,a_{m-1}h_1,e_1te_{n+1}).\end{aligned}$$

One easily checks that

$$\begin{aligned}f(e_{n+1}a_1,\dots,a_{m-1}e_1,h_1th_{n+1})&=0,\\
f(e_{n+1}a_1,\dots,a_{m-1}h_1,e_1th_{n+1})&=0,\\
f(h_{n+1}a_1,\dots,a_{m-1}e_1,h_1te_{n+1})&=0,\\
f(h_{n+1}a_1,\dots,a_{m-1}h_1,e_1te_{n+1})&=0.\end{aligned}\tag{4.10}$$

As φ then satisfies the same identities, it follows that

$$\begin{aligned}\varphi(a_1,\dots,a_{m-1},e_1te_{n+1}) &= \varphi(e_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}-h_1t)\\
&\quad+\varphi(e_{n+1}a_1,\dots,a_{m-1}h_1,e_1t)\\
&\quad+\varphi(h_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}).\end{aligned}$$

Since $te_{n+1},h_1t,e_1t,te_{n+1}\in S$ by the induction assumption, it follows that

$$\begin{aligned}\varphi(a_1,\dots,a_{m-1},e_1te_{n+1}) &= \varphi(f(e_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}-h_1t),1,\dots,1)\\
&\quad+\varphi(f(e_{n+1}a_1,\dots,a_{m-1}h_1,e_1t),1,\dots,1)\\
&\quad+\varphi(f(h_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}),1,\dots,1).\end{aligned}$$

Applying (4.10) we finally obtain

$$\begin{aligned}&\varphi(a_1,\dots,a_{m-1},e_1te_{n+1})\\
&\quad=\varphi(f(e_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}-h_1t+h_1th_{n+1}),1,\dots,1)\\
&\qquad+\varphi(f(e_{n+1}a_1,\dots,a_{m-1}h_1,e_1t-e_1th_{n+1}),1,\dots,1)\\
&\qquad+\varphi(f(h_{n+1}a_1,\dots,a_{m-1}e_1,te_{n+1}-h_1te_{n+1}),1,\dots,1)\\
&\qquad+\varphi(f(h_{n+1}a_1,\dots,a_{m-1}h_1,e_1te_{n+1}),1,\dots,1)\\
&\quad=\varphi(f(a_1,\dots,a_{m-1},e_1te_{n+1}),1,\dots,1).\end{aligned}$$

This means that $e_1te_{n+1}\in S$ and the proof is complete. □

4.2.4 Constructing new examples from old ones

To state our first result in this section, we need the following generalization of the definition of a f-zpd algebra.

Definition 4.33. *Let $f = f(x_1, \dots, x_m) \in F\langle x_1, x_2, \dots\rangle$ be a multilinear polynomial, let $\mathcal{A}$ be an F-algebra, and let $\mathcal{V}_1, \dots, \mathcal{V}_m$ be vector subspaces of $\mathcal{A}$. We say that the set $\mathcal{V}_1 \times \cdots \times \mathcal{V}_m$ is f-zpd if for every m-linear functional $\varphi\colon \mathcal{V}_1 \times \cdots \times \mathcal{V}_m \to F$ with the property that for all $a_i \in \mathcal{V}_i$, $i = 1, \dots, m$,*

$$f(a_1, \dots, a_m) = 0 \implies \varphi(a_1, \dots, a_m) = 0,$$

there exists a linear functional τ on $\mathcal{A}$ such that

$$\varphi(a_1, \dots, a_m) = \tau\left(f(a_1, \dots, a_m)\right)$$

for all $a_i \in \mathcal{V}_i$, $i = 1, \dots, m$.

Clearly $\mathcal{A}^m = \mathcal{A} \times \cdots \times \mathcal{A}$ being f-zpd is the same thing as $\mathcal{A}$ being f-zpd.

We draw the readers' attention here that the L'vov-Kaplansky conjecture indicates that the assumption (a) from the following theorem is not artificial. The theorem actually concerns an arbitrary algebra $\mathcal{A}$, but of course we are primarily interested in the case where $\mathcal{A} = M_n(F)$.

Theorem 4.34. *Let $\mathcal{A}$ be an F-algebra and let $k \geqslant 1$. For each $i = 1, \dots, k$, let $f_i\left(x_{i_1}, \dots, x_{i_{m_i}}\right) \in F\langle x_1, x_2, \dots\rangle$ be a multilinear polynomial in m_i variables and let $\mathcal{V}_1^{(i)}, \dots, \mathcal{V}_{m_i}^{(i)}$ be vector subspaces of $\mathcal{A}$. Set*

$$\mathcal{U}_i = \mathcal{V}_1^{(i)} \times \cdots \times \mathcal{V}_{m_i}^{(i)}.$$

Further, let $f_0 \in F\langle x_1, x_2, \dots\rangle$ be a multilinear polynomial in k variables, and let

$$f = f_0\left(f_1, \dots, f_k\right).$$

Suppose that the following three conditions are satisfied:

(a) For each $i = 1, \dots, k$, $f_i(\mathcal{U}_i)$ is a vector subspace of $\mathcal{A}$.

(b) For each $i = 1, \dots, k$, the set $\mathcal{U}_i$ is f_i-zpd.

(c) The set $f_1(\mathcal{U}_1) \times \cdots \times f_k(\mathcal{U}_k)$ is f_0-zpd.

Then the set $\mathcal{U}_1 \times \cdots \times \mathcal{U}_k$ is f-zpd.

Proof. For each $i = 1, \ldots, k$, we set

$$\mathcal{W}_i = \mathcal{U}_i \times \mathcal{U}_{i+1} \times \cdots \times \mathcal{U}_k.$$

Our goal is to prove that $\mathcal{W}_1$ is f-zpd. Thus, take a multilinear functional $\varphi\colon \mathcal{W}_1 \to F$ that preserves zeros of f, that is, for all $a_j^{(i)} \in \mathcal{V}_j^{(i)}$,

$$f\left(a_1^{(1)}, \ldots, a_{m_k}^{(k)}\right) = 0 \implies \varphi\left(a_1^{(1)}, \ldots, a_{m_k}^{(k)}\right) = 0.$$

Define the multilinear functional $\varphi_{a_1^{(2)},\ldots,a_{m_k}^{(k)}} : \mathcal{U}_1 \to F$ by

$$\varphi_{a_1^{(2)},\ldots,a_{m_k}^{(k)}}\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right) = \varphi\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}, a_1^{(2)}, \ldots, a_{m_k}^{(k)}\right).$$

Clearly $\varphi_{a_1^{(2)},\ldots,a_{m_k}^{(k)}}$ preserves zeros of f_1 on $\mathcal{U}_1$. Since $\mathcal{U}_1$ is f_1-zpd, there is a linear functional $\tau_{a_1^{(2)},\ldots,a_{m_k}^{(k)}} : \mathcal{A} \to F$ such that

$$\varphi_{a_1^{(2)},\ldots,a_{m_k}^{(k)}}\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right) = \tau_{a_1^{(2)},\ldots,a_{m_k}^{(k)}}\left(f_1\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right)\right).$$

Now define $\Phi_1\colon f_1(\mathcal{U}_1) \times \mathcal{W}_2 \to F$ by

$$\Phi_1\left(a_1, a_1^{(2)}, \ldots, a_{m_2}^{(2)}, \ldots, a_1^{(k)}, \ldots, a_{m_k}^{(k)}\right) = \tau_{a_1^{(2)},\ldots,a_{m_k}^{(k)}}(a_1).$$

The linearity of Φ_1 in the first argument is obvious. Let us prove the linearity of Φ_1 in the second argument. It is enough to show that

$$\tau_{a_1^{(2)}+b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}} = \tau_{a_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}} + \tau_{b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}} \tag{4.11}$$

and

$$\tau_{\lambda a_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}} = \lambda \tau_{a_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}} \tag{4.12}$$

where $\lambda \in F$. Take $a_1 \in f(\mathcal{U}_1)$ and write $a_1 = f_1\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right)$. Then,

$$\begin{aligned}
\tau_{a_1^{(2)}+b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}(a_1) &= \tau_{a_1^{(2)}+b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\left(f_1\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right)\right) \\
&= \varphi_{a_1^{(2)}+b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right) \\
&= \varphi\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}, a_1^{(2)} + b_1^{(2)}, a_2^{(2)}, \ldots, a_{m_k}^{(k)}\right) \\
&= \varphi\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_{m_k}^{(k)}\right) \\
&\quad + \varphi\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}, b_1^{(2)}, a_2^{(2)}, \ldots, a_{m_k}^{(k)}\right) \\
&= \varphi_{a_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right) \\
&\quad + \varphi_{b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right) \\
&= \tau_{a_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\left(f_1\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right)\right) \\
&\quad + \tau_{b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\left(f_1\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right)\right) \\
&= \left(\tau_{a_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}} + \tau_{b_1^{(2)},a_2^{(2)},\ldots,a_{m_k}^{(k)}}\right)(a_1),
\end{aligned}$$

which proves (4.11). The proof of (4.12) is similar. Analogously we see that Φ_1 is linear in other arguments, so Φ_1 is a multilinear functional. Moreover,

$$\begin{aligned}\varphi&\left(a_1^{(1)},\dots,a_{m_1}^{(1)},a_1^{(2)},\dots,a_{m_2}^{(2)},\dots,a_1^{(k)},\dots,a_{m_k}^{(k)}\right)\\&=\Phi_1\left(f_1\left(a_1^{(1)},\dots,a_{m_1}^{(1)}\right),a_1^{(2)},\dots,a_{m_2}^{(2)},\dots,a_1^{(k)},\dots,a_{m_k}^{(k)}\right)\end{aligned}$$

for all $a_j^{(i)} \in \mathcal{V}_j^{(i)}$. From this we see that the multilinear functional

$$\varphi_{a_1,a_1^{(3)},\dots,a_{m_k}^{(k)}}\colon \mathcal{U}_2 \to F,$$

given by

$$\varphi_{a_1,a_1^{(3)},\dots,a_{m_k}^{(k)}}\left(a_1^{(2)},\dots,a_{m_2}^{(2)}\right)=\Phi_1\left(a_1,a_1^{(2)},\dots,a_{m_2}^{(2)},a_1^{(3)},\dots,a_{m_k}^{(k)}\right),$$

preserves zeros of f_2 on $\mathcal{U}_2$. As $\mathcal{U}_2$ is f_2-zpd, there exists a linear functional

$$\tau_{a_1,a_1^{(3)},\dots,a_{m_k}^{(k)}}\colon \mathcal{A} \to F$$

such that

$$\varphi_{a_1,a_1^{(3)},\dots,a_{m_k}^{(k)}}\left(a_1^{(2)},\dots,a_{m_2}^{(2)}\right)=\tau_{a_1,a_1^{(3)},\dots,a_{m_k}^{(k)}}\left(f_2\left(a_1^{(2)},\dots,a_{m_2}^{(2)}\right)\right).$$

Next we define

$$\Phi_2\colon f_1(\mathcal{U}_1)\times f_2(\mathcal{U}_2)\times \mathcal{W}_3 \to F$$

by

$$\Phi_2\left(a_1,a_2,a_1^{(3)},\dots,a_{m_3}^{(3)},\dots,a_1^{(k)},\dots,a_{m_k}^{(k)}\right)=\tau_{a_1,a_1^{(3)},\dots,a_{m_k}^{(k)}}(a_2).$$

We show that Φ_2 is multilinear in a similar fashion as we showed that Φ_1 is multilinear. Moreover, we have

$$\begin{aligned}&\varphi\left(a_1^{(1)},\dots,a_{m_k}^{(k)}\right)\\=&\Phi_1\left(f_1\left(a_1^{(1)},\dots,a_{m_1}^{(1)}\right),a_1^{(2)},\dots,a_{m_2}^{(2)},\dots,a_1^{(k)},\dots,a_{m_k}^{(k)}\right)\\=&\varphi_{f_1(a_1^{(1)},\dots,a_{m_1}^{(1)}),a_1^{(3)},\dots,a_{m_k}^{(k)}}\left(a_1^{(2)},\dots,a_{m_2}^{(2)}\right)\\=&\tau_{f_1(a_1^{(1)},\dots,a_{m_1}^{(1)}),a_1^{(3)},\dots,a_{m_k}^{(k)}}\left(f_2\left(a_1^{(2)},\dots,a_{m_2}^{(2)}\right)\right)\\=&\Phi_2\left(f_1\left(a_1^{(1)},\dots,a_{m_1}^{(1)}\right),f_2\left(a_1^{(2)},\dots,a_{m_2}^{(2)}\right),a_1^{(3)},\dots,a_{m_k}^{(k)}\right).\end{aligned}$$

Repeating this process, we obtain the existence of a multilinear functional

$$\Phi\colon f_1(\mathcal{U}_1)\times f_2(\mathcal{U}_2)\times\cdots\times f_k(\mathcal{U}_k)\to F$$

satisfying

$$\varphi\left(a_1^{(1)},\dots,a_{m_k}^{(k)}\right)=\Phi\left(f_1\left(a_1^{(1)},\dots,a_{m_1}^{(1)}\right),\dots,f_k\left(a_1^{(k)},\dots,a_{m_k}^{(k)}\right)\right)$$

for $a_j^{(i)} \in \mathcal{V}_j^{(i)}$. Suppose $f_0(a_1, \ldots, a_k) = 0$ for some $a_i \in f(\mathcal{U}_i)$. Write $a_i = f_i\left(a_1^{(i)}, \ldots, a_{m_i}^{(i)}\right)$, we then have

$$f\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}, \ldots, a_1^{(k)}, \ldots, a_{m_k}^{(k)}\right) = 0.$$

Hence

$$\varphi\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}, \ldots, a_1^{(k)}, \ldots, a_{m_k}^{(k)}\right) = 0,$$

that is,

$$\Phi\left(a_1, \ldots, a_k\right) = \Phi\left(f_1\left(a_1^{(1)}, \ldots, a_{m_1}^{(1)}\right), \ldots, f_k\left(a_1^{(k)}, \ldots, a_{m_k}^{(k)}\right)\right) = 0.$$

Therefore, Φ preserves zeros of f_0. Since $f_1(\mathcal{U}_1) \times \cdots \times f_k(\mathcal{U}_k)$ is f_0-zpd, there exists a linear functional $\tau \colon \mathcal{A} \to F$ such that, for all $a_i \in f_i\left(\mathcal{U}_i\right)$,

$$\Phi\left(a_1, \ldots, a_k\right) = \tau\left(f_0\left(a_1, \ldots, a_k\right)\right).$$

The proof is complete since, for all $a_j^{(i)} \in \mathcal{A}$,

$$\varphi\left(a_1^{(1)}, \ldots, a_{m_k}^{(k)}\right) = \tau\left(f\left(a_1^{(1)}, \ldots, a_{m_k}^{(k)}\right)\right). \qquad \square$$

The following corollary is immediate.

Corollary 4.35. *Let $f_1, \ldots, f_k$ be multilinear polynomials in distinct variables and let f_0 be a multilinear polynomial in k variables. Let $\mathcal{A}$ be an F-algebra satisfying the following two conditions:*

(a) $\mathcal{A}$ is f_i-zpd for $i = 0, 1, \ldots, k$.

(b) $f_i(\mathcal{A}) = \mathcal{A}$ for $i = 1, \ldots, k$.

Then the algebra $\mathcal{A}$ is f-zpd where $f = f_0\left(f_1, \ldots, f_k\right)$.

The applicability of Theorem 4.34 and Corollary 4.35 of course depends on the validity of the L'vov-Kaplansky conjecture and its variants.

We continue with a lemma needed for another corollary to Theorem 4.34. Recall that $sl_n(F)$ stands for the Lie algebra of traceless matrices in $M_n(F)$.

Lemma 4.36. *Let $f_0 = [x_1, x_2]$ and let $\mathcal{V}_1, \mathcal{V}_2 \in \{sl_n(F), M_n(F)\}$. Then the set $\mathcal{V}_1 \times \mathcal{V}_2$ is f_0-zpd, provided that char(F) is 0 or does not divide n.*

Proof. We know that $M_n(F)$ is zLpd (see Theorem 4.10), which means that $M_n(F) \times M_n(F)$ is f_0-zpd.

Let $\varphi \colon sl_n(F) \times M_n(F) \to F$ be a bilinear functional preserving zeros of f_0. Our assumption on *char*(F) implies that $M_n(F) = sl_n(F) \oplus F \cdot 1$. Therefore, we extend

φ to $M_n(F)^2$ by setting $\varphi(1, a) = 0$ for all $a \in M_n(F)$. Let now $a, b \in M_n(F)$ be such that $[a, b] = 0$. Writing $a = [a_1, a_2] + \lambda 1$ with $a_1, a_2 \in M_n(F)$ and $\lambda \in F$, we thus have $[[a_1, a_2], b] = 0$. Since $[a_1, a_2] \in sl_n(F)$, it follows that

$$\varphi(a, b) = \varphi([a_1, a_2], b) = 0.$$

As $M_n(F)$ is zLpd, there exists a linear functional $\tau\colon M_n(F) \to F$ such that $\varphi(c, d) = \tau([c, d])$ for all $c, d \in M_n(F)$, and so in particular for all $c \in sl_n(F)$ and $d \in M_n(F)$. This proves that the set $sl_n(F) \times M_n(F)$ is f_0-zpd.

The $M_n(F) \times sl_n(F)$ case can be handled similarly, and so can be the $sl_n(F) \times sl_n(F)$ case. Indeed, one extends a bilinear functional φ defined on $sl_n(F) \times sl_n(F)$ to $M_n(F)^2$ by setting $\varphi(1, a) = \varphi(a, 1) = 0$ for all $a \in M_n(F)$. □

Corollary 4.37. *If f is a multilinear Lie monomial, then the algebra $M_n(F)$ is f-zpd, provided that* $\mathrm{char}(F)$ *is* 0 *or does not divide n.*

Proof. First let us show that $f(M_n(F))$ is a vector space. In fact, we claim that $f(M_n(F)) = sl_n(F)$, unless the degree m of f is 1, in which case $f(M_n(F))$ is obviously equal to $M_n(F)$. We may therefore assume that $m > 1$ and that our claim is true for Lie monomials of degree less than m. Write $f = [f_1, f_2]$, where f_1 and f_2 are multilinear Lie monomials in distinct variables of degree at most $m - 1$. By [2], every matrix in $sl_n(F)$ is a commutator of two matrices from $M_n(F)$. However, since $M_n(F) = sl_n(F) \oplus F \cdot 1$ by the characteristic assumption, it is actually a commutator of two matrices from $sl_n(F)$. Since, by our assumption, $f_1(M_n(F))$ and $f_2(M_n(F))$ contain $sl_n(F)$, it follows that $f(M_n(F)) = sl_n(F)$.

Let us now prove that $M_n(F)$ is f-zpd. There is nothing to prove if $m = 1$, so we may assume that $m > 1$ and that by writing $f = [f_1, f_2]$ as above we have that $M_n(F)$ is f_i-zpd, $i = 1, 2$. Since $f_i(M_n(F)) \in \{sl_n(F), M_n(F)\}$, taking into account Lemma 4.36 we can apply Theorem 4.34 to conclude that $M_n(F)$ is f-zpd. □

It is clear that the method of proof can be used for some other polynomials. For example, using the fact that the algebra $M_n(F)$ is zJpd (Theorem 4.12) and that the polynomial $f_0 = x_1x_2 + x_2x_1$ obviously satisfies $f_0(M_n(F)) = M_n(F)$ provided that $\mathrm{char}(F) \neq 2$, we see that Corollary 4.35 yields the following result (by a Jordan monomial we mean a monomial in the free special Jordan algebra).

Corollary 4.38. *Let* $\mathrm{char}(F) \neq 2$. *If f is a multilinear Jordan monomial, then the algebra $M_n(F)$ is f-zpd.*

4.3 A multilinear Nullstellensatz

In this last section we consider the situation where f and g are multilinear polynomials of the same degree m such that every zero of f in $\mathcal{A}^m$, where $\mathcal{A}$ is an algebra, is also a zero of g, that is, for all $a_1, \ldots, a_m \in \mathcal{A}$,

$$f(a_1, \ldots, a_m) = 0 \implies g(a_1, \ldots, a_m) = 0.$$

From Lemma 4.19 it is evident that the situation above it is a special case of the condition from the definition of an f-zpd algebra. It is therefore natural to ask whether the problem of describing the relation between f and g can be solved in any f-zpd algebra. In the next proposition, we give a positive answer under the assumption that $f(1, \ldots, 1) \neq 0$.

Proposition 4.39. *Let $\mathcal{A}$ be an F-algebra and let $f, g \in F\langle x_1, x_2, \ldots \rangle$ be multilinear polynomials of degree m such that every zero of f in $\mathcal{A}^m$ is a zero of g. If $\mathcal{A}$ is f-zpd and $f(1, \ldots, 1) \neq 0$, then there exist an scalar $\lambda \in F$ and a polynomial identity h of $\mathcal{A}$ such that $g = \lambda f + h$.*

Proof. It is clear from our assumptions that

$$(a_1, \ldots, a_m) \mapsto g(a_1, \ldots, a_m)$$

is an m-linear map that preserves zeros of f. Lemma 4.19 therefore shows that there exists a linear map $T\colon \mathcal{A} \to \mathcal{A}$ satisfying

$$g(a_1, \ldots, a_m) = T(f(a_1, \ldots, a_m)).$$

Thus, for every $a \in \mathcal{A}$ we have

$$g(1, \ldots, 1)a = g(a, 1, \ldots, 1) = T(f(a, 1, \ldots, 1)) = f(1, \ldots, 1)T(a).$$

Setting $\lambda = g(1, \ldots, 1)f(1, \ldots, 1)^{-1}$ we thus have $T(a) = \lambda a$ for all $a \in \mathcal{A}$, and hence

$$g(a_1, \ldots, a_m) = \lambda f(a_1, \ldots, a_m)$$

for all $a_1, \ldots, a_m \in \mathcal{A}$. This means that $h = g - \lambda f$ is a polynomial identity of $\mathcal{A}$. $\square$

Corollary 4.40. *Let $\mathrm{char}(F) \neq 2$, let $\alpha_1, \ldots, \alpha_m \in F$ be such that $\sum_{i=1}^{m} \alpha_i \neq 0$, and let*

$$f(x_1, \ldots, x_m) = \alpha_1 x_1 \cdots x_m + \alpha_2 x_2 \cdots x_m x_1 + \cdots + \alpha_m x_m x_1 \cdots x_{m-1}.$$

If an F-algebra $\mathcal{A}$ is generated by idempotents and g is a multilinear polynomial of degree m such that every zero of f in $\mathcal{A}^m$ is a zero of g, then there exist an scalar $\lambda \in F$ and a polynomial identity h of $\mathcal{A}$ such that $g = \lambda f + h$.

Proof. This is immediate from Theorem 4.32 and Proposition 4.39. □

From now on we consider the case where $\mathcal{A} = M_n(F)$. We recall from Proposition 4.21 that the conclusion of Proposition 4.39 then does not always hold. Our goal is to show that it does hold provided that $m < 2n - 3$. In fact, since, as is well known, $M_n(F)$ has no polynomial identities of degree less than $2n$, we will actually prove that f and g are linearly dependent. To this end, we start by introducing the necessary notation.

In what follows, let $\alpha_\sigma, \beta_\sigma \in F$ be such that

$$f = \sum_{\sigma \in S_m} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)},$$

$$g = \sum_{\sigma \in S_m} \beta_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}.$$

We set

$$\mathrm{Supp}(f) = \{\sigma \in S_m \mid \alpha_\sigma \neq 0\}$$

and similarly we define $\mathrm{Supp}(g)$. Further, for each $\sigma \in S_m$ we write

$$(x_1, \dots, x_m)_\sigma = x_{\sigma(1)} \cdots x_{\sigma(m)}.$$

Thus, for example,

$$f = \sum_{\sigma \in S_m} \alpha_\sigma (x_1, \dots, x_m)_\sigma.$$

We will consider evaluations $(a_1, \dots, a_m)_\sigma$ with $a_i \in M_n(F)$.

The next two definitions are standard in group theory.

Definition 4.41. *We define a metric d on S_m by letting $d(\sigma_1, \sigma_2)$ to be the least nonnegative integer k for which there exists a sequence of transpositions $\tau_1, \tau_2, \dots, \tau_k \in S_m$ such that $\tau_k \cdots \tau_1 \sigma_1 = \sigma_2$.*

The next definition concerns any subset T of S_m, but we will be actually interested in the case where $T = \mathrm{Supp}(f)$.

Definition 4.42. *Let T be a subset of S_m. We define an equivalence relation on T as follows: $\sigma_1 \sim \sigma_2$ if and only if there exists a (possibly empty) sequence of transpositions $\tau_1, \tau_2, \dots, \tau_k$ such that*

(a) $\tau_i \cdots \tau_1 \sigma_1 \in T$, $i = 1, \dots, k-1$,

(b) $\tau_k \cdots \tau_1 \sigma_1 = \sigma_2$.

The following theorem is the main result of this section.

Theorem 4.43. *Let $f, g \in F\langle x_1, x_2, \dots\rangle$ be multilinear polynomials of degree m such that every zero of f in $M_n(F)^m$ is a zero of g. If $m < 2n-3$, then there exists an scalar $\lambda \in F$ such that $f = \lambda g$.*

Proof. Set $m_0 = \dfrac{m}{2} + 1$ if m is even and $m_0 = \dfrac{m+1}{2}$ if m is odd. Note that $m_0 + 1 \leqslant n$ since $m + 3 < 2n$. Define a sequence $\boldsymbol{e} = (e_1, \dots, e_m)$ by setting

$$e_1 = e_{11}, \quad e_2 = e_{12}, \quad e_3 = e_{22}, \quad \dots, \quad e_m = \begin{cases} e_{m_0-1,m_0} & m \text{ even} \\ e_{m_0,m_0} & m \text{ odd} \end{cases}.$$

For any $\sigma, \pi \in S_m$, we have

$$(e_{\sigma(1)}, \dots, e_{\sigma(m)})_\pi = e_{\pi(\sigma(1))} \cdots e_{\pi(\sigma(m))} = \begin{cases} e_{1,m_0} & \text{if } \sigma = \pi^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, for every $\sigma \in S_m$ we have

$$\begin{aligned} \alpha_\sigma = 0 &\implies f\left(e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(m)}\right) = \alpha_\sigma e_{1,m_0} = 0 \\ &\implies g\left(e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(m)}\right) = \beta_\sigma e_{1,m_0} = 0 \\ &\implies \beta_\sigma = 0. \end{aligned}$$

We have thereby proved that

$$\operatorname{Supp}(g) \subseteq \operatorname{Supp}(f). \tag{4.13}$$

Take $\sigma \in \operatorname{Supp}(f)$ and write $\lambda = \alpha_\sigma^{-1}\beta_\sigma$, so that $\beta_\sigma = \lambda\alpha_\sigma$. We claim that

$$\beta_{\tau\sigma} = \lambda\alpha_{\tau\sigma} \tag{4.14}$$

for every transposition τ. Indeed, without loss of generality we may assume that $\sigma = (1)$ and we write $\tau = (p\,q)$ with $p < q$. We consider four cases.

Case 1: p and q are both even. In this case e_p and e_q are square-zero matrices. Hence, considering the matrices

$$\begin{aligned} a_i &= e_i, \ i \in \{1, \dots, m\} \backslash \{p, q\}, \\ a_p &= e_p + e_q, \\ a_q &= \alpha_{(1)} e_p - \alpha_\tau e_q, \end{aligned}$$

we have

$$f(a_1, \dots, a_m) = (\alpha_\tau\alpha_{(1)} - \alpha_{(1)}\alpha_\tau)e_{1,m_0} = 0.$$

This implies that

$$0 = g(a_1, \dots, a_m) = (\beta_\tau\alpha_{(1)} - \beta_{(1)}\alpha_\tau)e_{1,m_0} = (\beta_\tau - \lambda\alpha_\tau)\alpha_{(1)}e_{1,m_0},$$

which yields (4.14).

Case 2: p and q are both odd. In this case both e_p and e_q are idempotents. We reduce this case to the previous one by considering a shift on the sequence $\boldsymbol{e}$, that is,

$$\tilde{e}_1 = e_{12}, \quad \tilde{e}_2 = e_{22}, \quad \tilde{e}_3 = e_{23}, \quad \dots, \quad \tilde{e}_m = \begin{cases} e_{m_0,m_0}, & m \text{ even} \\ e_{m_0,m_0+1}, & m \text{ odd} \end{cases}.$$

Now it is enough to perform the evaluation at

$$\begin{aligned} a_i &= \tilde{e}_i, \ i \in \{1, \dots, m\} \backslash \{p, q\}, \\ a_p &= \tilde{e}_p + \tilde{e}_q, \\ a_q &= \alpha_{(1)} \tilde{e}_p - \alpha_\tau \tilde{e}_q \end{aligned}$$

and proceed as at the end of Case 1.

Case 3: p is odd and q is even. In this case e_p is an idempotent but e_q is not. The idea now is to consider a shift on the sequence from e_p on. This shift will turn the element in the q-th position into an idempotent. So an additional change will be needed in this element as well. Precisely we take

$$\begin{gathered} \tilde{e}_1 = e_{11}, \quad \dots, \quad \tilde{e}_{p-1} = e_{\frac{p-1}{2}, \frac{p-1}{2}+1}, \quad \tilde{e}_p = e_{\frac{p+1}{2}, \frac{p+1}{2}+1}, \\ \tilde{e}_{p+1} = e_{\frac{p+1}{2}+1, \frac{p+1}{2}+1}, \quad \dots, \tilde{e}_{q-1} = e_{\frac{q}{2}, \frac{q}{2}+1}, \quad \tilde{e}_q = e_{\frac{q}{2}+1, \frac{q}{2}+2}, \\ \tilde{e}_{q+1} = e_{\frac{q}{2}+2, \frac{q}{2}+2}, \quad \dots, \quad \tilde{e}_m = \begin{cases} e_{m_0,m_0+1}, & m \text{ even} \\ e_{m_0+1,m_0+1}, & m \text{ odd}\ . \end{cases} \end{gathered}$$

We consider the evaluation at

$$\begin{aligned} a_i &= \tilde{e}_i, \ i \in \{1, \dots, m\} \backslash \{p, q\}, \\ a_p &= \tilde{e}_p + \tilde{e}_q, \\ a_q &= \alpha_{(1)} \tilde{e}_p - \alpha_\tau \tilde{e}_q \end{aligned}$$

and once again we proceed as in the end of Case 1.

Case 4: p is even and q is odd. Here we have that e_p as a square-zero matrix and e_q is idempotent. We proceed similarly as in the previous case. The difference, however, is that no shift is needed at the beginning, just the change in the elements at the q-th position.

This completes the proof of our claim.

Let $[\sigma]$ denote the equivalence class of σ in $\mathrm{Supp}(f)/\sim$. Write $\lambda_{[\sigma]}$ for λ. Observe that (4.14) implies that

$$\beta_{\sigma'} = \lambda_{[\sigma]} \alpha_{\sigma'} \tag{4.15}$$

for all $\sigma' \in [\sigma]$.

In view of (4.13) and (4.15), we are left to prove that $\lambda_A = \lambda_B$ for all equivalence classes A, B in $\mathrm{Supp}(f)/\sim$. Assume this is not true and consider a pair of permutations σ_1 and σ_2 such that

$$d(\sigma_1, \sigma_2) = \min_{\substack{\pi_1 \in A, \pi_2 \in B \\ A, B \in \mathrm{Supp}(f)/\sim \\ \lambda_A \neq \lambda_B}} d(\pi_1, \pi_2) =: \ell. \tag{4.16}$$

Let f_ψ denote the reindexing of f through the permutation ψ, i.e.,

$$f_\psi = \sum_{\sigma \in S_m} \alpha_\sigma x_{\psi\sigma(1)} \cdots x_{\psi\sigma(m)}.$$

We claim that we may assume $\sigma_2\sigma_1^{-1}$ is the product of disjoint cycles

$$\sigma_2\sigma_1^{-1} = (s_1 \ s_1 - 1 \ \cdots \ 1) \cdots (s_h \ s_h - 1 \ \cdots \ s_{h-1} + 1).$$

Indeed, let us prove that the minimum ℓ in (4.16) is invariant under reindexing of the variables in both f and g, that is,

$$\ell = \min_{\substack{\pi_1 \in A, \pi_2 \in B \\ A, B \in \mathrm{Supp}(f_\psi)/\sim \\ \lambda_A \neq \lambda_B}} d(\pi_1, \pi_2). \tag{4.17}$$

We first notice that $\mathrm{Supp}(f_\psi) = \psi\mathrm{Supp}(f)$. Denoting the equivalent classes in $\mathrm{Supp}(f_\psi)/\sim$ by $[[\sigma]], \sigma \in \mathrm{Supp}(f_\psi)$, one then can see that for each $\sigma \in \mathrm{Supp}(f_\psi)$, there exists $\sigma' \in \mathrm{Supp}(f)$ such that

$$[[\sigma]] = \psi[\sigma'].$$

We also notice that for $\sigma = \psi\sigma'$ where $\sigma' \in \mathrm{Supp}(f)$, we must have $\lambda_{[[\sigma]]} = \lambda_{[\sigma']}$. In fact,

$$\lambda_{[[\sigma]]} = \alpha^{-1}_{\psi^{-1}\sigma}\beta_{\psi^{-1}\sigma} = \alpha^{-1}_{\sigma'}\beta_{\sigma'} = \lambda_{[\sigma']}.$$

Hence we conclude that the sets in the equations (4.16) and (4.17) for which we take the minimum are actually the same, and we therefore conclude the equation (4.17).

Writing $\sigma_2\sigma_1^{-1}$ as a product of h disjoint cycles, and recalling that two permutations are conjugated if and only if they have the same cycle type, we therefore have the existence of $\psi \in S_m$ such that

$$\psi\sigma_2\sigma_1^{-1}\psi^{-1} = (s_1, s_1 - 1, \ldots, 1) \cdots (s_h, s_h - 1, \ldots, s_{h-1} + 1).$$

Hence, up to reindexing both f and g by ψ, we may assume

$$\sigma_2\sigma_1^{-1} = (s_1, s_1 - 1, \ldots, 1) \cdots (s_h, s_h - 1, \ldots, s_{h-1} + 1),$$

proving our claim.

Since disjoint cycles commute, we may also assume that the first p cycles are of even length and the remaining $h-p$ are of odd length, where $0 \leqslant p \leqslant h$.

Setting $s_0 = 0$ and taking into account for instance [49], we have

$$\ell = s_h - h = \sum_{i=1}^{h} s_i - s_{i-1} - 1.$$

Finally, we also assume that $\sigma_1 = (1)$. Only minor adjustments in the proof are needed if σ_1 is an arbitrary permutation, which, however, makes the reading more difficult. Thus, from now one we will be dealing with the permutations (1) and

$$\sigma_2 = (s_1 \; s_1 - 1 \; \cdots \; 1) \cdots (s_h \; s_h - 1 \; \cdots \; s_{h-1} + 1).$$

We have $\lambda_{[(1)]} \neq \lambda_{[\sigma_2]}$.

In order to obtain a contradiction, our goal will be to construct a sequence $E \in M_n(F)^m$ such that

$$E_{(1)} = e_{1,m_0}, \quad E_{\sigma_2} = -\frac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0}, \quad E_\sigma = 0 \text{ for all } \sigma \in \mathrm{Supp}(f) \backslash \{(1), \sigma_2\}.$$

This will imply

$$f(E) = \alpha_{(1)} e_{1,m_0} - \alpha_{\sigma_2} \frac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0} = 0,$$

hence

$$g(E) = \lambda_{[(1)]} \alpha_{(1)} e_{1,m_0} - \lambda_{[\sigma_2]} \alpha_{\sigma_2} \frac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0} = (\lambda_{[(1)]} - \lambda_{[\sigma_2]}) \alpha_{(1)} e_{1,m_0} = 0,$$

ans so $\lambda_{[(1)]} = \lambda_{[\sigma_2]}$, contrary to the assumption.

As the first step, we shall construct the sequence E in three particular cases. We will see at the end that the general case will follow from these three ones.

Case 1: $p = h$ (all cycles are of even length).

By assumption, $s_i = 2q_i$, $i = 1, \dots, h$. We introduce the sequence E in blocks as follows:

$$E = (E_1, \dots, E_h, E_{h+1}),$$

- $E_1 = e_{11}, e_{12}, e_{22}, \dots, e_{q_1-1,q_1}, e_{q_1,q_1} + e_{q_1,q_1+1}, e_{q_1,q_1+1} + e_{11}$,
- $E_2 = e_{q_1+1,q_1+1}, e_{q_1+1,q_1+2}, \dots, e_{q_2-1,q_2}, e_{q_2,q_2} + e_{q_2,q_2+1}$,
  $e_{q_2,q_2+1} + e_{q_1+1,q_1+1}$,

  $\vdots$
- $E_h = e_{q_{h-1}+1,q_{h-1}+1}, \dots, e_{q_h-1,q_h}, e_{q_h,q_h} + e_{q_h,q_h+1}$,
  $e_{q_h,q_h+1} - \dfrac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{q_{h-1}+1,q_{h-1}+1}$,

- $E_{h+1} = e_{q_h+1,q_h+1}, e_{q_h+1,q_h+2}, \ldots, e_{m',m_0}$.

Here $m' = m_0$ if m is odd and $m' = m_0 - 1$ otherwise (recall that m_0 is defined at the beginning of the proof).

One can easily see that the non-zero products of matrices in E are only obtained by joining the non-zero evaluations of each block, in the increasing order of the blocks.

For $i = 1, 2, \ldots, h$, define the following sets of permutations:

$$R_i = \{(1), (s_i \ \cdots \ s_{i-1}+3 \ \ s_{i-1}+2), (s_i \ \cdots \ s_{i-1}+2 \ \ s_{i-1}+1)\}.$$

Then we get

$$E_\sigma = \begin{cases} e_{1,m_0} & \text{if } \sigma = (1) \\ -\dfrac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0} & \text{if } \sigma = \sigma_2 \\ \mu_\sigma e_{1,m_0} & \text{if } \sigma = \pi_1 \cdots \pi_h, \ \pi_i \in R_i, \ \sigma \notin \{(1), \sigma_2\} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu_\sigma \in F$.

In order to complete the proof of this case we are left to show that actually the permutations η of the third item of E_σ giving a non-zero evaluation are not elements of $\mathrm{Supp}(f)$. To this end let us prove the following facts.

- $d(\eta, (1)) < \ell$.

  This follows from a direct comparison between η and σ_2. Indeed, let $\eta = \pi_1 \cdots \pi_h$, $\pi_i \in R_i$. Since $\eta \neq \sigma_2$, we have that at least one of the π_i's is not equal to $(s_i \ \cdots \ s_{i-1}+2 \ \ s_{i-1}+1)$. Hence we have a fewer number of transpositions in the decomposition of η than in that of σ_2. As a consequence we obtain $d(\eta, (1)) < \ell$, as desired.

- $d(\eta, \sigma_2) < \ell$.

  As before let $\eta = \pi_1 \cdots \pi_h$, $\pi_i \in R_i$. Since $\eta \neq (1)$, we have that at least one of the π_i's is not equal to (1). Now consider $\sigma_2 \eta^{-1}$. If η involves a cycle of the form $(s_i \ \ldots \ s_{i-1}+2 \ \ s_{i-1}+1)$, then the elements from the set $\{s_i, \ldots, s_{i-1}+2, s_{i-1}+1\}$ are fixed in $\sigma_2\eta^{-1}$. The outcome of this is that, with respect to the i-th block, we have fewer transpositions in $\sigma_2\eta^{-1}$ than in σ_2 and we are done in this case. The other possibility is that a cycle of the form $(s_i \ \cdots \ s_{i-1}+3 \ \ s_{i-1}+2)$ occurs in η. In this case, in the i-th block of $\sigma_2\eta^{-1}$ only the transposition $(s_{i-1}+1 \ \ s_i)$ appears. Since $1 < 3 \leqslant s_i - s_{i-1} - 1$, we reach the desired conclusion $d(\eta, \sigma_2) < \ell$.

According to the above two facts we can complete the proof of this case. If $\lambda_{[\eta]} \neq \lambda_{[(1)]}$, we immediately get a contradiction to the minimality of ℓ since we have proved that $d(\eta,(1)) < \ell$. So assume that $\lambda_{[\eta]} = \lambda_{[(1)]}$. Since by hypothesis $\lambda_{[\sigma_2]} \neq \lambda_{[(1)]}$, we obtain that $\lambda_{[\eta]} \neq \lambda_{[\sigma_2]}$. Again we get a contradiction to the minimality of ℓ since we have proved that $d(\eta,\sigma_2) < \ell$.

Case 2: $h = 1$ and $s_1 = 2q+1$ is odd.

Consider the sequence $E = (E_1, E_2)$ given in two blocks as follows:

- $E_1 = e_{11}, e_{12}, e_{22}, \ldots, e_{q,q+1}, e_{q+1,q+1} - \dfrac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{11}$,
- $E_2 = e_{q+1,q+2}, e_{q+2,q+3}, \ldots, e_{m',m_0}$.

It is not difficult to see that

$$E_\sigma = \begin{cases} e_{1,m_0}, & \text{if } \sigma = (1) \\ -\dfrac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0}, & \text{if } \sigma \in \{\sigma_2, (s_1 \ \cdots \ 3 \ 2)\} \\ 0, & \text{otherwise} \end{cases}.$$

The proof of this case is complete since it is sufficient to observe that $(s_1 \ \cdots \ 3 \ 2) \notin \mathrm{Supp}(f)$.

Case 3: $h = 2$, $p = 0$ (2 odd cycles).

In this case we have that $s_1 = 2q_1 + 1$ and $s_2 = 2q_2$.

Consider the sequence $E = (E_1, E_2)$ given in two blocks as follows:

- $E_1 = e_{11}, e_{12}, e_{22}, \ldots, e_{q_1,q_1+1}, e_{q_1+1,q_1+1} + e_{11}, e_{q_1+1,q_1+2}$, $e_{q_1+2,q_1+2}, \ldots, e_{q_2-1,q_2}, e_{q_2,q_2} + e_{q_2,q_2+1}, e_{q_2,q_2+1} - \dfrac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{q_1+1,q_1+1}$,
- $E_2 = e_{q_2+1,q_2+1}, e_{q_2+1,q_2+2}, \ldots, e_{m',m_0}$.

Define the following two sets of permutations:

$$R_1 = \{(1), (s_1 \ \cdots \ 3 \ 2), (s_1 \ \cdots \ 2 \ 1)\},$$
$$R_2 = \{(1), (s_2 \ \cdots \ s_1+2 \ s_1+1)\}.$$

One can directly see that

$$E_\sigma = \begin{cases} e_{1,m_0}, & \text{if } \sigma = (1) \\ -\dfrac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0}, & \text{if } \sigma = \sigma_2 \\ \mu_\sigma e_{1,m_0}, & \text{if } \sigma = \pi_1\pi_2, \ \pi_i \in R_i, \ \sigma \notin \{(1), \sigma_2\} \text{ or } \sigma = (s_2 \ \cdots \ s_1) \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu_\sigma \in F$.

Let η be a permutation of the third item of E_σ giving a non-zero evaluation. In order to complete the proof of this case we need to show that $\eta \notin \mathrm{Supp}(f)$. Assume first that $\eta = \pi_1\pi_2$, where π_i's are permutations from R_i. In this case we are in the situation of Case 1 and, proceeding in a similar manner, we arrive at the desired conclusion. Now suppose that $\eta = (s_2 \cdots s_1 + 1\ s_1)$. In this case we obtain that

$$\sigma_2\eta^{-1} = (s_1\ s_2\ s_1 - 1\ s_1 - 2 \cdots s_0 + 2\ s_0 + 1),$$

which decomposes into $s_1 - s_0$ transpositions. Since $s_2 - s_1 - 1 \geqslant 2$ (otherwise we would have a cycle of length 1 in σ_2 and we could just ignore it), we get

$$s_1 - s_0 < s_1 - s_0 + s_2 - s_1 - 2 = (s_1 - s_0 - 1) + (s_2 - s_1 - 1) = \ell.$$

This shows that $d(\sigma_2, \eta) < \ell$. Analogously we have that $d((1), \eta) < \ell$. In fact $(s_2 \cdots s_1 + 1\ s_1)$ decomposes into $s_2 - s_1$ transpositions, which is less than $(s_1 - s_0 - 1) + (s_2 - s_1 - 1)$. Using the same approach at the end of Case 1, we get the desired conclusion also in this case.

In order to complete the proof of the theorem we are left to analyze the general situation. Recall that

$$\sigma_2 = (s_1\, s_1 - 1 \cdots 1) \cdots (s_h\, s_h - 1 \cdots s_{h-1} + 1),$$

where the first p cycles are of even length ($s_i = 2q_i$) and the remaining $h - p$ are of odd length, $p \in \{0, 1, \ldots, h\}$. Note that $h - p > 0$, otherwise we are in Case 1.

Now distinguish two situations: $h - p$ odd or $h - p$ even.

Suppose first that $h - p = 2k + 1$ is odd. In this case we construct the sequence $E = (G_1, \ldots, G_p, G'_1, \ldots, G'_k, G'_{k+1})$ in blocks as follows:

- for the blocks G_i, $i = 1, \ldots, p$, we use the idea of Case 1. More precisely we put

$$\begin{aligned} G_i = {} & e_{q_{i-1}+1,q_{i-1}+1}, e_{q_{i-1}+1,q_{i-1}+2}, e_{q_{i-1}+2,q_{i-1}+2}, \ldots, e_{q_i-1,q_i}, \\ & e_{q_i,q_i} + e_{q_i,q_i+1}, e_{q_i,q_i+1} + e_{q_{i-1}+1,q_{i-1}+1}, \end{aligned}$$

  where we assume that $q_0 = 0$;

- for the blocks G'_j, $j = 1, \ldots, k$, we mimic the first block of matrices (called E_1) given in Case 3, more precisely

$$\begin{aligned} G'_j = {} & e_{q_{j-1+p}+1,q_{j-1+p}+1}, e_{q_{j-1+p}+1,q_{j-1+p}+2}, \ldots, e_{q_{j+p},q_{j+p}+1}, \\ & e_{q_{j+p}+1,q_{j+p}+1} + e_{q_{j-1+p}+1,q_{j-1+p}+1}, e_{q_{j+p}+1,q_{j+p}+2}, \\ & e_{q_{j+p}+2,q_{j+p}+2}, \ldots, e_{q_{j+p+1}-1,q_{j+p+1}}, \\ & e_{q_{j+p+1},q_{j+p+1}} + e_{q_{j+p+1},q_{j+p+1}+1}, e_{q_{j+p+1},q_{j+p+1}+1} + e_{q_{j+p}+1,q_{j+p}+1}; \end{aligned}$$

- the block G'_{k+1} is constructed as in Case 2:

$$\begin{aligned} G'_{k+1} = {} & e_{q_{p+k+1}+1,q_{p+k+1}+1}, e_{q_{p+k+1}+1,q_{p+k+1}+2}, \ldots, e_{q_{p+k+2},q_{p+k+2}+1}, \\ & e_{q_{p+k+2}+1,q_{p+k+2}+1} - \frac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{q_{p+k+1}+1,q_{p+k+1}+1}, \\ & e_{q_{p+k+2}+1,q_{p+k+2}+2}, e_{q_{p+k+2}+2,q_{p+k+2}+2}, \ldots, e_{m',m_0}. \end{aligned}$$

Now assume that $h - p = 2k$ is even. In this case we do not have the last block G'_{k+1}. All the other ones are constructed as in the previous case, except for the last one G'_k, which becomes:

$$\begin{aligned} G'_k = {} & e_{q_{k-1+p}+1,q_{k-1+p}+1}, e_{q_{k-1+p}+1,q_{k-1+p}+2}, \ldots, e_{q_{k+p},q_{k+p}+1}, \\ & e_{q_{k+p}+1,q_{k+p}+1} + e_{q_{k-1+p}+1,q_{k-1+p}+1}, e_{q_{k+p}+1,q_{k+p}+2}, \\ & e_{q_{k+p}+2,q_{k+p}+2}, \ldots, e_{q_{k+p+1}-1,q_{k+p+1}}, \\ & e_{q_{k+p+1},q_{k+p+1}} + e_{q_{k+p+1},q_{k+p+1}+1}, e_{q_{k+p+1},q_{k+p+1}+1} - \frac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{q_{k+p}+1,q_{k+p}+1}, \\ & e_{q_{k+p+1}+1,q_{k+p+1}+1}, e_{q_{k+p+1}+1,q_{k+p+1}+2}, \ldots, e_{m',m_0}. \end{aligned}$$

In both cases, we get that the sequence E is such that

$$E_{(1)} = e_{1,m_0}, \quad E_{\sigma_2} = -\frac{\alpha_{(1)}}{\alpha_{\sigma_2}} e_{1,m_0}, \quad E_\sigma = 0 \text{ for all } \sigma \in \mathrm{Supp}(f) \backslash \{(1), \sigma_2\}.$$

In fact, each permutation $\eta \notin \{(1), \sigma_2\}$, giving a non-zero evaluation of the matrices in the sequence E, does not belong to $\mathrm{Supp}(f)$. Indeed, in computing $d(\eta, (1))$ and $d(\eta, \sigma_2)$, we always have a sum

$$l_1 + \cdots + l_h$$

where, for each i, $l_i \leqslant s_i - s_{i-1} - 1$ or $l_{i+1} + l_i \leqslant (s_{i+1} - s_i - 1) + (s_i - s_{i-1} - 1)$, and for at least one i we have that the inequality is strict. $\square$

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