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Fuzzy Stationary Schrödinger Equation with Correlated Fuzzy Boundaries

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Abstract

This article introduces the space of A-linearly correlated fuzzy complex numbers. Using this space, we study the stationary Schrödinger equation with boundary conditions are given by fuzzy complex numbers. This equation plays an special role in Quantum Mechanics describing the state of the system. We apply the formalism to the step potential, generating quantum results consistent with traditional quantum results.

Keywords: Fuzzy complex number A-linearly correlated fuzzy complex number fuzzy complex process fuzzy stationary Schrödinger equation.

1 Introduction

In Quantum Mechanics, the state of the system is described by the Schrödinger equation

$$\frac{d^2}{dx^2}\Psi(x) + \frac{2m}{\hbar^2}(E - V(x))\Psi(x) = 0, \tag{1}$$

where $\Psi : \mathbb{R} \longrightarrow \mathbb{C}$ is the wave function at the position coordinates x, $x \in \mathbb{R}$. Furthermore, the function $V : \mathbb{R} \longrightarrow \mathbb{R}$ is the potential in position coordinate space to which the particle of mass m is subject and $\hbar$ is the reduced Planck's constant. The quantity E, represents the energy of the particle. The first term of (1) is related to the square of the moment operator, which in the representation of positions is given by $\mathbf{p} = -i\hbar\frac{d}{dx}$ [1].

The Schrödinger equation can be solved for particular cases of potentials as analyzed in [1]. Among these potentials, we can highlight the barrier that is widely used in Engineering and Physics problems [2], especially in the study of periodic systems. In this case, the solution of the Schrödinger equation can become very laborious for an arbitrary number of barriers. In [3], the authors presented a systematic way of studying systems with an arbitrary number of barriers. In the classical formulation of quantum mechanics, the complete specification of the initial conditions of the particle are incorporated in the initial conditions for the function $\Psi(x)$, that is, $\Psi(x_{i,0})$ and $\Psi'(x_{i,0}) = \frac{d}{dx_i}\Psi(x_{i,0})$ where $x_{i,0}$ is related to the positions on which the initial conditions were imposed. A step potential may represent, to a first approximation, a p-n junction in semiconductor devices [4, 5]. This fact justifies that the initial conditions $\Psi(x_{i,0})$ and $\Psi'(x_{i,0})$ are not precisely known, since the step is a first approximation to the real potential.

A natural way to deal with quantities that are not precisely known is through the fuzzy set theory [6]. In particular, fuzzy differential equations emerged as a way of modeling uncertainty present in certain phenomena in a dynamical environment [7]. Furthermore, when studying evolutionary systems from initial conditions, it may be necessary to consider the possibility of interactivity among all uncertain quantities (see [8–13]). The interactivity between uncertain values given by fuzzy numbers is provided by joint possibility distributions [14]. Intuitively, interactivity models how the possible values of two or more uncertain quantities can be jointly assumed. Fuzzy processes that consider interactivity are called autocorrelated processes [15]. They are similar to the autocorrelated processes that occur in statistical analysis of time series.

Physics is a Science in which uncertainties are naturally present either due to experimental or fundamental characteristics, such as in Quantum Mechanics. Several problems in Physics can be described through initial value and boundary problems. In this sense, recent techniques have been observed that increasingly allow us to study fuzzy differential equations associated with physical problems. Recently, in [16] the authors studied the linear drag problem in the small Reynolds number regime with initial conditions given by fuzzy sets. In [17], authors studied the Schrödinger equation considering the initial conditions modeled by fuzzy numbers and using the derivative in the Kandel-Fridman Ming sense. Physically, as the wave function assumes complex values, the initial conditions could be interpreted as fuzzy complex numbers. Furthermore, considering that the initial conditions are fuzzy complex numbers, we may have the presence of interactivity in the phenomenon studied and, in this case, the solution of (1) can be viewed as an autocorrelated fuzzy process [15]. Note that such an assumption was not taken into account by the authors in [17]. In [18] the authors presented a solution of the fuzzy Schrödinger equation using colony programming (ACP). In [19] the authors studied fuzzy initial value problems that describe mechanical vibrations. In [20] the authors studied a type of harmonic oscillator with initial conditions given by fuzzy sets. In [21], authors investigated the existence and singularity of solutions for the fractional fuzzy Schrodinger equation.

This work presents the space of A-Linearly correlated complex fuzzy numbers that is composed by the set of fuzzy complex numbers that satisfy a special type of interactivity relationship called linear correlation with a fixed fuzzy number A. Under certain conditions on A, we show that this space can be equipped with the structure of a Banach space. This space corresponds to an appropriate environment to study the stationary Schrödinger equation whose initial conditions are given by fuzzy complex numbers. In addition, apply the formalism to a potential step and analyze the physical results obtained.

2 Preliminaries

In this section, we review some fundamentals concepts about fuzzy sets.

Let U be a topological space. A fuzzy set A of U is a mapping $A : U \to [0,1]$, where $A(x)$ denotes the degree that the element x belongs to the fuzzy set A [22]. With a fuzzy subset A of U, we can associate with a family of subsets of U called α-levels of A. For every $\alpha \in [0,1]$, the α-level of the fuzzy set A is defined by

$$[A]_\alpha = \begin{cases} \{x \in U\,;\;\; A(x) \geq \alpha\} & \text{if } \; 0 < \alpha \leq 1 \\ \overline{\{x \in U\,;\;\; A(x) > 0\}} & \text{if } \; \alpha = 0, \end{cases} \tag{2}$$

where the set $\{x \in U\,;\;\; A(x) > 0\}$ is the support of the A, denoted by $supp(A)$ and $\overline{X}$ denotes the closure of the subset X of U [22].

A fuzzy subset A of $\mathbb{R}$ is a fuzzy number if every α-level of A is a bounded, closed, and nonempty interval of $\mathbb{R}$ [22]. We denote the α-level of a fuzzy number A by

$$[A]_\alpha = [a_-(\alpha), a_+(\alpha)] = \left[a_\alpha^-, a_\alpha^+\right],$$

for all $\alpha \in [0,1]$.

We use the symbol $\mathbb{R}_\mathcal{F}$ to denote the set of all fuzzy numbers. A fuzzy number A is said to be positive (negative) if $a_1^- \geq 0$ ($a_1^+ \leq 0$). The set of fuzzy numbers whose support does not contain $0 \in \mathbb{R}$ is denoted by $\mathbb{R}_\mathcal{F}^* = \{A \in \mathbb{R}_\mathcal{F};\;\; 0 \notin supp(A)\}$ (cf. [23]). An example of fuzzy number is the triangular fuzzy number, whose α-levels are given by $[A]_\alpha = [(m - a_0^-)\alpha + a_0^-, (m - a_0^+)\alpha + a_0^+]$, for all $\alpha \in [0,1]$, where $[A]_0 = [a_0^-, a_0^+]$ and $\{m\} = [A]_1$. A triangular fuzzy number is also denoted by the triple $(a_0^-;\; m;\; a_0^+)$ [22].

A fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ is said to be symmetric with respect to $x \in \mathbb{R}$ if $A(x-y) = A(x+y)$, for all $y \in \mathbb{R}$. We say that A is non-symmetric or asymmetric if there exists no x such that A is symmetric [24].

Proposition 1 [24] *A fuzzy number A is symmetric with respect to $x \in \mathbb{R}$, if and only if, $a_\alpha^+ = 2x - a_\alpha^-$, for all $\alpha \in [0,1]$.*

The standard addition and the scalar product on fuzzy number can be defined as follows. Let $A, B \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$. The standard sum of A and B and the product of the scalar λ and A are respectively the fuzzy numbers $A+B$ and λA that are defined levelwise by

$$[A+B]_\alpha = \left[a_\alpha^- + b_\alpha^-,\ a_\alpha^+ + b_\alpha^+\right]$$

and

$$[\lambda A]_\alpha = \begin{cases} [\lambda a_\alpha^-,\ \lambda a_\alpha^+] \ if \ \ \lambda \geq 0 \\ [\lambda a_\alpha^+,\ \lambda a_\alpha^-] \ if \ \ \lambda < 0, \end{cases}$$

for all $\alpha \in [0,1]$.

A fuzzy complex number Z can be expressed in the form $C + Di$, where C and D are fuzzy numbers and i is the imaginary unit (which satisfies the equation $i^2 = -1$) [25]. We denote the set of all fuzzy complex number by $\mathbb{C}_{\mathcal{F}}$. Similar to the classical case, standard arithmetic operations on the the class of fuzzy complex numbers can be defined in terms of arithmetic operations on $\mathbb{R}_{\mathcal{F}}$. More precisely, let $Z = C + Di$, $W = A + Bi$, and $\omega = \lambda + i\theta$, we define

$$Z + W = (C + A) + i(D + B) \tag{3}$$

and

$$\omega Z = (\lambda C + (-\theta)D) + i\,(\theta C + \lambda D)\,. \tag{4}$$

One can easily verify that these operations satisfies the following interesting distributive law:

$$\omega(Z + W) = \omega Z + \omega W. \tag{5}$$

3 The Space of A-Linearly Correlated Fuzzy Complex Numbers - $\mathbb{C}_{\mathcal{F}(A)}$

This section we introduce the space A-linearly correlated fuzzy complex numbers and study differentiability of functions of the form $F(t) = q(t)A + r(t)$, where $A \in \mathbb{R}_{\mathcal{F}}$ and $q, r : \mathbb{R} \longrightarrow \mathbb{C}$.

In [24], authors introduced the operator $\psi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}_{\mathcal{F}}$ associating each vector $(q, r) \in \mathbb{R}^2$ with the the fuzzy number

$$[\psi_1(q,r)]_\alpha = q[A]_\alpha + r, \tag{6}$$

for all $\alpha \in [0,1]$ and $q, r \in \mathbb{R}$. The range of the operator ψ_1, denoted by $\mathbb{R}_{\mathcal{F}(A)}$, is the space of A-linearly correlated fuzzy numbers, that is, $\mathbb{R}_{\mathcal{F}(A)} = \{\psi_1(q,r) = qA + r;\ (q,r) \in \mathbb{R}^2\}$.

In this article, we propose an extension of the space $\mathbb{R}_{\mathcal{F}(A)}$ for the complex case as follows. For each $A \in \mathbb{R}_{\mathcal{F}}$, we can define the operator $\psi_A : \mathbb{C}^2 \longrightarrow \mathbb{C}_{\mathcal{F}}$ associating each vector $(q,r) \in \mathbb{C}^2$ with the complex the fuzzy number

$$[\psi_A(q,r)]_\alpha = q[A]_\alpha + r, \tag{7}$$

for all $\alpha \in [0,1]$ and $q, r \in \mathbb{C}$.

Note that ψ is well defined. In fact, we can easily show that the α-levels of $\psi_A(q,r)$ given by $[\psi_A(q,r)]_\alpha = \{qx + r \in \mathbb{C};\ x \in [A]_\alpha\}$ according to definition of fuzzy complex number. We denote the operator ψ_A by $\psi_A(q,r) = qA + r$. The range of the operator ψ_A, denoted by $\mathbb{C}_{\mathcal{F}(A)}$, is the space of A-linearly correlated fuzzy complex numbers, that is,

$$\mathbb{C}_{\mathcal{F}(A)} = \left\{\psi_A(q,r);\ (q,r) \in \mathbb{C}^2\right\}.$$

Remark 1 The set of complex number $\mathbb{C}$ can be embedded in $\mathbb{C}_{\mathcal{F}(A)}$ since every complex number r can be identified with the complex fuzzy number $\psi_A(0,r) \in \mathbb{C}_{\mathcal{F}(A)}$, that is, $\mathbb{C} \subset \mathbb{C}_{\mathcal{F}(A)}$.

Theorem 2 [24] *Given $A \in \mathbb{R}_{\mathcal{F}}$, the operator $\psi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}_{\mathcal{F}}$ given by $\psi_1(p,q) = pA + q$ is injective if and only if A is non-symmetric.*

The next theorem is the extension of the Theorem 2 for the case $\psi_A : \mathbb{C}^2 \longrightarrow \mathbb{C}_{\mathcal{F}}$.

Theorem 3 *Let $A \in \mathbb{R}_{\mathcal{F}}$. The operator $\psi_A : \mathbb{C}^2 \longrightarrow \mathbb{C}_{\mathcal{F}}$ given by*

$$\psi_A(p + ir, q + is) = (p + ir)A + (q + is) = (pA + q) + i(rA + s)$$

is injective if and only if A is non-symmetric.

Proof Note that, by definition of the arithmetic operations on $\mathbb{C}_{\mathcal{F}}$ and the definition of ψ_1, we obtain the function ψ_A can be rewritten in terms of the function ψ_1 as follows:

$$\begin{aligned}\psi_A(p + ir, q + is) &= (p + ir)A + (q + is) \\ &= (pA + q) + i(rA + s) \\ &= \psi_1(p,q) + i\psi_1(r,s).\end{aligned} \tag{8}$$

Suppose that ψ_A is injective. Let $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}$. If $(p,q) \neq (\tilde{p},\tilde{q})$, then $\psi_A(p + i0, q + i0) = \psi_1(p,q) \neq \psi_A(\tilde{p} + i0, \tilde{q} + i0) = \psi_1(\tilde{p},\tilde{q})$. This implies that ψ_1 is injective. Applying Theorem 2, we conclude that A is non-symmetric.

Now, suppose that A is non-symmetric. Let $p + ir, q + ir \in \mathbb{C}$ such that $\psi_A(p + ir, q + ir) = \psi_1(p,q) + i\psi_1(r,s) = \psi_1(\tilde{p},\tilde{q}) + i\psi_1(\tilde{r},\tilde{s}) = \psi_A(\tilde{p} + i\tilde{r}, \tilde{q} + i\tilde{s})$. Thus, we have $\psi_1(p,q) = \psi_1(\tilde{p},\tilde{q})$ and $\psi_1(r,s) = \psi(\tilde{r},\tilde{s})$. From Theorem 2, we have ψ_1 is injective which implies that $p = \tilde{p}$, $q = \tilde{q}$, $r = \tilde{r}$ and $s = \tilde{s}$. Therefore, ψ_A is injective. □

Remark 2 Equation 8 reveal an existence of a bijection between the spaces $\mathbb{R}^2_{\mathcal{F}(A)}$ and $\mathbb{C}_{\mathcal{F}(A)}$. More precisely, with every $Z \in \mathbb{C}_{\mathcal{F}(A)}$, we can associate a unique pair $(C, D) \in \mathbb{R}^2_{\mathcal{F}(A)}$ such that $Z = C + iD$. In this case, we say that C is real part of Z and D is the imaginary part of Z.

From Theorem 3, if the fuzzy number A is non-symmetric, then the operator $\psi_A : \mathbb{C}^2 \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ is a bijection. This bijection can be used to induce an algebraic structure of vector space over $\mathbb{C}_{\mathcal{F}(A)}$ as follows.

Let $A \in \mathbb{R}_{\mathcal{F}}$ be non-symmetric. For all $B, C \in \mathbb{C}_{\mathcal{F}(A)}$ and $\omega \in \mathbb{C}$, the set $\mathbb{C}_{\mathcal{F}(A)}$ is a vector space associated to the scalar field $\mathbb{C}$ with the vector addition and the scalar product defined respectively by

i) $B \oplus C = \psi_A \left(\psi_A^{-1}(B) + \psi_A^{-1}(C) \right)$,

ii) $\omega \cdot_\psi B = \psi_A \left(\beta\, \psi_A^{-1}(B) \right)$.

Additionally, $\mathbb{C}_{\mathcal{F}(A)}$ is isomorphic to $\mathbb{C}^2$ via the linear isomorphism ψ_A.

Proof This result follows immediately from Theorem 3 (which ensures that $\psi_A : \mathbb{C}^2 \to \mathbb{C}_{\mathcal{F}(A)}$ is bijective) and from definitions of the operations $\oplus$ and $\cdot_\psi$. □

The following proposition ensures that the operation $\cdot_\psi$ in Corollary 3 coincides to the standard scalar product on fuzzy complex numbers given in Equation (4).

Proposition 4 *Let $A \in \mathbb{R}_{\mathcal{F}}$ be non-symmetric. For every $Z \in \mathbb{C}_{\mathcal{F}(A)}$ and $\omega \in \mathbb{C}$, we have that $\omega B = \omega \cdot_\psi B$.*

Proof Since $Z \in \mathbb{C}_{\mathcal{F}(A)}$, there exist $p, q \in \mathbb{C}$ such that $Z = pA + q = \psi_A(p,q)$. From Equation 5, we have that

$$\begin{aligned} \omega Z &= \omega(pA + q) = \omega pA + \omega q \\ &= \psi_A(\omega p, \omega q) = \omega \cdot_\psi Z. \end{aligned}$$

□

In view of Proposition 4 and for the sake of simplicity, we simply denote the scalar product in Corollary 3 by ωB instead of $\omega \cdot_\psi B$. Moreover, let $Z, W \in \mathbb{C}_{\mathcal{F}(A)}$, the additive inverse of Z is given by $-Z = (-1)Z$, and the difference of Z and W is defined by

$$Z \ominus W = Z \oplus -W.$$

The next corollary states that, for every non-symmetric fuzzy number, we can define a Banach space that is isomorphic to the complex Banach space $\left(\mathbb{C}^2, \|\cdot\|\right)$.

Let $\|\cdot\|$ be a norm of $\mathbb{C}^2$ and $A \in \mathbb{R}_{\mathcal{F}}$ be a non-symmetric. The vector space $\left(\mathbb{C}_{\mathcal{F}(A)}, \oplus, \cdot_{\psi}\right)$ forms a Banach space with norm $\|\cdot\|_{\psi_A}$ give by $\|B\|_{\psi_A} = \left\|\psi_A^{-1}(B)\right\|$ for all $B \in \mathbb{C}_{\mathcal{F}(A)}$.

Proof This result follows from Theorem 3 (which ensures that $\psi_A : \mathbb{C}^2 \to \mathbb{C}_{\mathcal{F}(A)}$ is bijective) and the definition of $\|\cdot\|_{\psi_A}$. □

Remark 3 Similar to the classical case, one can note that the real Banach space $\mathbb{R}_{\mathcal{F}(A)}$ can be embedded into complex Banach space $\mathbb{C}_{\mathcal{F}(A)}$ by considering the mappings $B \mapsto B + i0$ and $\lambda \mapsto \lambda + i0$ for all $B \in \mathbb{R}_{\mathcal{F}(A)}$ and for all $\lambda \in \mathbb{R}$. Thus, the vector addition and scalar product on $\mathbb{R}_{\mathcal{F}(A)}$ can be obtained from the operations $\oplus$ and $\cdot_{\psi}$ on $\mathbb{C}_{\mathcal{F}(A)}$ in terms of the above mappings, that is,

$$B \oplus C = (B + i0) \oplus (C + i0) \quad \textit{and} \quad \lambda B = (\lambda + i0)(B + i0). \tag{9}$$

Thus, using the connection between the addition on real numbers and the addition on complex numbers, we can easily verify that the operations given in Equation 9 coincide with the ones induced by the bijection ψ_1.

Given $B, C \in \mathbb{C}_{\mathcal{F}(A)}$, an important question in this article is how to define the product of B and C. In [23], Ban and Bede introduces the concept of cross-product on the class $\mathbb{R}_{\mathcal{F}}^{\wedge}$ composed by fuzzy numbers such that their 1-levels are unit sets of $\mathbb{R}$, that is,

$$\mathbb{R}_{\mathcal{F}}^{\wedge} = \{A \in \mathbb{R}_{\mathcal{F}};\ \ \exists!\ x_0 \in \mathbb{R} \text{ such that } A(x_0) = 1\}.$$

Note that $\mathbb{R}_{\mathcal{F}}^{\wedge}$ contains the subset of triangular fuzzy numbers and, therefore, the set of real numbers. In fact, this binary operation can be viewed as multiplication operation on $\mathbb{R}_{\mathcal{F}}^{\wedge}$, since it extends the the multiplication on real numbers [23].

For a given asymmetric fuzzy number A, Longo *et al.* defined the cross-product on $\mathbb{R}_{\mathcal{F}(A)}$ by replacing the standard operations by those from the real Banach space $\mathbb{R}_{\mathcal{F}(A)}$ with addition and scalar product induced by the bijection ψ_1 [26]. As we mentioned in Remark 3, these operations can be rewritten in terms of the operations on $\mathbb{C}_{\mathcal{F}(A)}$ as in Equation 9.

Definition 1 [26] Let $A \in \mathbb{R}_{\mathcal{F}}^{\wedge}$ be non-symmetric and $B, C \in \mathbb{R}_{\mathcal{F}(A)}$. The A-cross product between B and C is defined as the fuzzy number W given by

$$W = B \odot C = cB \oplus bC \ominus bc. \tag{10}$$

where $[B]_1 = \{b\}$ and $[C]_1 = \{c\}$.

According to [26], $B \odot C$ is a fuzzy number of the space $\mathbb{R}_{\mathcal{F}(A)}$ and is defined for all $B, C \in \mathbb{R}_{\mathcal{F}(A)}$. In addition, the A-cross product $B \odot C$ can be written in terms of the coefficients $(p, r), (q, s) \in \mathbb{R}^2$ such that $B = pA + r$ and $C = qA + s$ as follows:

$$B \odot C = \psi_1(bq + cp, bs + cr - cb)$$

$$= (bq + cp)A + (bs + cr - cb) \in \mathbb{R}_{\mathcal{F}(A)}.$$

Let $[A]_1 = \{a\}$, we have $b = pa + r$ and $c = qa + s$. These imply that $bq + cp = (pa+r)q+(qa+s)p = 2pqa+rq+sp$ and $bs+cr-cb = (pa+r)s+(qa+s)r-(pa+r)(qa+s) = -a^2pq+rs$. Therefore, we obtain the following alternative expression:

$$\begin{aligned} B \odot C &= \psi_1 \left(2pqa + rq + sp, -a^2pq + rs\right) \\ &= (2pqa + rq + sp)\, A + \left(-a^2pq + rs\right). \end{aligned} \tag{11}$$

Based on Remark 3, Definition 1 can be easily extended to $\mathbb{C}_{\mathcal{F}(A)}$ as follows.

Definition 2 Let $A \in \mathbb{R}^{\wedge}_{\mathcal{F}}$ be non-symmetric. For $Z, W \in \mathbb{C}_{\mathcal{F}(A)}$, we define the A-cross product of Z and W by

$$Z \odot W = wZ \oplus zW \ominus zw \tag{12}$$

where $[Z]_1 = \{z\}$ and $[W]_1 = \{w\}$.

The following theorem states that the A-cross product on $\mathbb{C}_{\mathcal{F}(A)}$ can be terms of the real and imaginary parts of the operands such as in the crisp case.

Theorem 5 *Let $A \in \mathbb{R}^{\wedge}_{\mathcal{F}}$ be non-symmetric. For $Z, W \in \mathbb{C}_{\mathcal{F}(A)}$ such that $Z = B+iC$ and $W = D + iE$, with $B, C, D, E \in \mathbb{R}_{\mathcal{F}(A)}$, we have*

$$Z \odot W = [(B \odot D)\ \ominus\ (C \odot E)]\ \oplus\ i\,[(B \odot E)\ \oplus\ (C \odot D)]. \tag{13}$$

Proof Let A be a non-symmetric fuzzy number such that $[A]_1 = \{a\}$. For every $Z, W \in \mathbb{C}_{\mathcal{F}(A)}$, there exist unique complex numbers $u = p + ir, v = q + is, \tilde{u} = \tilde{p} + i\tilde{r}, \tilde{v} = \tilde{q} + i\tilde{s}$, with $p, \tilde{p}, q, \tilde{q}, r, \tilde{r}, s, \tilde{s} \in \mathbb{R}$, such that $Z = uA + v$ and $W = \tilde{u}A + \tilde{v}$.

On the one hand, since $[Z]_1 = \{z = ua + v\}$ and $[W]_1 = \{w = \tilde{u}a + \tilde{v}\}$. From Definition 2, we have

$$Z \odot W = wZ \oplus zW \ominus zw \tag{14}$$

$$= \psi_A \left(wu + z\tilde{u}, wv + z\tilde{v} - zw\right). \tag{15}$$

Moreover, we have

$$\begin{aligned} wu + z\tilde{u} &= (\tilde{u}a + \tilde{v})u + (ua + v)\tilde{u} \\ &= 2au\tilde{u} + u\tilde{v} + v\tilde{u} \\ &= (2a(p\tilde{p} - r\tilde{r}) + p\tilde{q} - s\tilde{r} + q\tilde{p} - r\tilde{s}) \\ &\quad\ i(2a(p\tilde{r} + r\tilde{p}) + p\tilde{s} + s\tilde{p} + q\tilde{r} + r\tilde{q}), \end{aligned} \tag{16}$$

and

$$\begin{aligned} wv + z\tilde{v} - zw &= (\tilde{u}a + \tilde{v})v + (ua + v)\tilde{v} - (ua + v)(\tilde{u}a + \tilde{v}) \\ &= -a^2u\tilde{u} + vv \\ &= (-a^2(p\tilde{p} - r\tilde{r}) + q\tilde{q} - s\tilde{s}) \end{aligned} \tag{17}$$

$$i(-a^2(p\tilde{r}+\tilde{p}r)+q\tilde{s}+\tilde{q}s).$$

On the other hand, from Equation 8, we have that $Z=\psi_1(p,q)+i\psi_1(r,s)$ and $W=\psi_1(\tilde{p},\tilde{q})+i\psi_1(\tilde{r},\tilde{s})$, where $B=\psi_1(p,q)$, $C=\psi_1(r,s)$, $D=\psi_1(\tilde{p},\tilde{q})$, and $E=\psi_1(\tilde{r},\tilde{s})$. Using Equation 11, we obtain

$$\begin{aligned}
B\odot D &= \psi_1(p,q)\odot\psi_1(\tilde{p},\tilde{q}) = \psi_1(2ap\tilde{p}+q\tilde{p}+\tilde{q}p,-a^2p\tilde{p}+q\tilde{q}),\\
C\odot E &= \psi_1(r,s)\odot\psi_1(\tilde{r},\tilde{s}) = \psi_1(2ar\tilde{r}+s\tilde{r}+\tilde{s}r,-a^2r\tilde{r}+s\tilde{s}),\\
B\odot E &= \psi_1(p,q)\odot\psi_1(\tilde{r},\tilde{s}) = \psi_1(2ap\tilde{r}+q\tilde{r}+\tilde{s}p,-a^2p\tilde{r}+q\tilde{s}),\\
C\odot D &= \psi_1(r,s)\odot\psi_1(\tilde{p},\tilde{q}) = \psi_1(2a\tilde{p}r+\tilde{q}r+s\tilde{p},-a^2\tilde{p}r+\tilde{q}s).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(B\odot D)\ominus(C\odot E) &= \psi_1(2a(p\tilde{p}-r\tilde{r})+\tilde{q}p-s\tilde{r}+q\tilde{p}-\tilde{s}r,\\
&\qquad -a^2(p\tilde{p}-r\tilde{r})+q\tilde{q}-s\tilde{s})
\end{aligned}$$

and

$$\begin{aligned}
(B\odot E)\oplus(C\odot D) &= \psi_1(2a(p\tilde{r}+r\tilde{p})+\tilde{s}p+s\tilde{p}+q\tilde{r}+\tilde{q}r,\\
&\qquad -a^2(p\tilde{r}+r\tilde{p})+q\tilde{s}+s\tilde{q}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&[(B\odot D)\ \ominus\ (C\odot E)]\ \oplus\ i\,[(B\odot E)\ \oplus\ (C\odot D)] =\\
&= \psi_A(2a(p\tilde{p}-r\tilde{r})+p\tilde{q}-s\tilde{r}+q\tilde{p}-r\tilde{s})+\\
&\quad i(2a(p\tilde{r}+r\tilde{p})+p\tilde{s}+s\tilde{p}+q\tilde{r}+r\tilde{q}),\\
&\quad (-a^2(p\tilde{p}-r\tilde{r})+q\tilde{q}-s\tilde{s})+\\
&\quad i(-a^2(p\tilde{r}+\tilde{p}r)+q\tilde{s}+\tilde{q}s))\\
&= \psi_A(wu+z\tilde{u},wv+z\tilde{v}-zw)\\
&= Z\odot W.
\end{aligned}$$

□

The notion of an A-linearly interactive complex fuzzy process is fundamental in this work and is stated as follows.

Definition 3 Given $A\in\mathbb{R}_{\mathcal{F}}$ an A-linearly interactive fuzzy complex process is a function $F:[a,b]\longrightarrow\mathbb{C}_{\mathcal{F}(A)}$ defined by $F(t)=q(t)A+r(t)$, where $q,r:[a,b]\longrightarrow\mathbb{C}$.

If $q(t)=q_1(t)+iq_2(t)$ and $r(t)=r_1(t)+ir_2(t)$, the conjugate of an A-linearly interactive fuzzy complex process F is the function

$$\begin{aligned}
F^*(t) &= q(t)^*A+q(t)^*\\
&= (q_1(t)-iq_2(t))A+(r_1(t)-ir_2(t)).
\end{aligned}$$

In what follows, unless otherwise stated, we assume that A is a non-symmetric fuzzy number. Thus, in this case, from Corollary 3, we have that the space $(\mathbb{C}_{\mathcal{F}(A)},\oplus,\cdot_\psi,\|\cdot\|_{\psi_A})$ is a Banach space over the field of complex numbers. This fact allows us to use the concept of the Fréchet derivative for functions $F:[a,b]\longrightarrow\mathbb{C}_{\mathcal{F}(A)}$.

Definition 4 An A-linearly interactive complex fuzzy process $F : [a, b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ is said to be Fréchet differentiable at $t \in (a, b)$ if there exists a continuous linear operator $F'[t] : \mathbb{R} \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ such that

$$F(t+h) = F(t) \oplus F'[t](h) \oplus \omega(h),$$

with $\lim_{h\to 0} \frac{\|\omega(h)\|_{\psi_A}}{|\, h\,|} = 0.$

The next theorem provides a practical way to compute the Fréchet derivative of an A-linearly interactive complex fuzzy process.

Theorem 6 *Let $A \in \mathbb{R}_{\mathcal{F}(A)}$ be non-symmetric and $F : [a, b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ such that $F(t) = q(t)A + r(t)$ for all $t \in [a, b]$. The function F is Fréchet differentiable at $t \in (a, b)$ if, and only if, q and r are differentiable at t. Additionally, $F'[t](h) = q'(t)hA + r'(t)h$ for all $h \in \mathbb{R}$.*

Proof The proof follows from the isomorphism between $\mathbb{C}_{\mathcal{F}(A)}$ and $\mathbb{C}^2$ (see Corollary 3). More precisely, if $q'(t)$ and $r'(t)$ exist, then $F'[t](h) = q'(t)hA + r'(t)h$, $h \in \mathbb{R}$, is a continuous linear operator that satisfies the hypotheses of Definition 4.

On the other hand, let $F'[t] : \mathbb{R} \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ be the Fréchet derivative of F at t. Since $F'[t]$ is a continuous linear operator, $F'[t]$ is of the form $F'[t](h) = hF'[t](1) = hq_0A + hr_0$, for some fixed $q_0, r_0 \in \mathbb{C}$. From Definition 4, we have that

$$\begin{aligned}
0 &= \lim_{h\to 0} \frac{\|\omega(h)\|_{\psi_A}}{|\, h\,|} \\
&= \lim_{h\to 0} \frac{\left\|F(t+h) \ominus F(t) \ominus F'[t](h)\right\|_{\psi_A}}{|\, h\,|} \\
&= \lim_{h\to 0} \frac{\|\psi_A\left(q(t+h) - q(t) - hq_0, r(t+h) - r(t) - hr_0\right)\|_{\psi_A}}{|\, h\,|} \\
&= \lim_{h\to 0} \left\| \left(\frac{q(t+h) - q(t)}{h} - q_0,\ \frac{r(t+h) - r(t)}{h} - r_0 \right) \right\|.
\end{aligned}$$

The last equality implies that $q'(t) = q_0$ and $r'(t) = r_0$. □

A well-known result is that the Fréchet derivative is a linear operator in the space of the functions from a Banach space to another Banach space over the same field [27]. This result can be stated to the class of A-linearly interactive fuzzy complex process as follows.

Proposition 7 *Let $A \in \mathbb{R}_{\mathcal{F}(A)}$ be non-symmetric. If $F, G : [a, b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ are Fréchet differentiable on $[a, b]$ then $(F \oplus G)'[t] = F'[t] \oplus G'[t]$ and $(\lambda F)'[t] = \lambda F'[t]$ for all $t \in [a, b]$ and $\lambda \in \mathbb{C}$.*

Since $\left(\mathbb{C}_{\mathcal{F}(A)}, \oplus, \cdot_\psi, \|.\|_{\psi_A}\right)$ is a Banach space isomorphic to $\mathbb{C}^2$, one can also define a notion of derivative of $F : [a,b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ at some point $t \in [a,b]$ by means of a limit of quotients such as in calculus for real-valued functions.

Definition 5 Let $A \in \mathbb{R}_{\mathcal{F}(A)}$ be non-symmetric and $F : [a,b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ such that $F(t) = q(t)A + r(t)$. We say that F is differentiable at $t \in (a,b)$ if there exists $F'(t) \in \mathbb{C}_{\mathcal{F}(A)}$ such that

$$\lim_{h \to 0} \frac{1}{h} \left(F(t+h) \ominus F(t)\right) = F'(t). \tag{18}$$

Theorem 8 *The function $F : [a,b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ is Fréchet differentiable at $t \in (a,b)$ if, and only if, F is differentiable (in the sense of the (18)) at t. Moreover,*

$$F'[t](1) = F'(t) = \psi_A\left(q'(t), r'(t)\right) = q'(t)A + r'(t), \tag{19}$$

where $F'[t]$ is the Fréchet derivative of F at t.

Proof On the one hand, if $F'[t]$ exists, then, from its linearity, we have $F'[t](h) = hF'[t](1)$ for all h. Using this observation, we have that

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{\left\|F(t+h) \ominus F(t) \ominus hF'[t](1)\right\|_{\psi_A}}{\mathsf{h}} \\ &= \lim_{h \to 0} \left\| \frac{1}{h}(F(t+h) \ominus F(t)) \ominus F'[t](1) \right\|. \end{aligned}$$

Thus, by Definition 5, $F'[t](1)$ is the derivative of F at t.

On the other hand, if F is differentiable at $t \in (a,b)$, then

$$\begin{aligned} 0 &= \lim_{h \to 0} \left\| \frac{1}{h}(F(t+h) \ominus F(t)) \ominus F'(t) \right\| \\ &= \lim_{h \to 0} \left\| \frac{1}{h}(F(t+h) \ominus F(t) \ominus hF'(t)) \right\| \\ &= \lim_{h \to 0} \frac{\left\|F(t+h) \ominus F(t) \ominus hF'(t)\right\|}{\mathsf{h}}. \end{aligned}$$

From Definition 4, the continuous linear operator $g(h) = hF'(t)$ is the Fréchet derivative of F at t.

Therefore, we have that $F'[t](1) = F'(t)$ and, from Theorem 6, we conclude that

$$F'(t) = \psi_A\left(q'(t), r'(t)\right) = q'(t)A + r'(t).$$

□

The connection revealed by Theorem 8 implies that the notion of differentiability in Definition 5 inherits all interesting properties regarding Fréchet differentiability, such as the linearity of the derivative operator and the implication of the continuity of a function at the points where it is differentiable. On the other hand, in contrast to Fréchet derivative, Definition 5 provides a natural way to define and deal with derivative of higher order of A-linearly interactive fuzzy complex processes. From Theorem 8, the function $F : [a, b] \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ has derivative of order $n \geq 1$ at $t \in (a, b)$ if, and only if, the functions $q, r : [a, b] \longrightarrow \mathbb{C}$ have derivatives of order n at t and is given by

$$F^{(n)}(t) = \psi_A\left(q^{(n)}(t), r^{(n)}(t)\right) = q^{(n)}(t)A + r^{(n)}(t). \tag{20}$$

4 Fuzzy stationary Schrödinger equation

The physics of a traditional quantum system is mainly obtained from the Hamiltonian operator, denoted by $\mathbf{H}$, and the wave function $\Psi(x)$. Usually, the Hamiltonian is given by

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \mathbf{V}(x), \tag{21}$$

where $\mathbf{p}$ is the momentum operator, which in the representation of positions is given by $\mathbf{p} = -i\hbar\frac{d}{dx}$ and $\mathbf{V}(x)$ is the potential energy that in the representation of positions is given by the function $V(x)$. Thus, in the steady state, we have

$$\mathbf{H}\Psi(x) = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi(x) = E\Psi(x). \tag{22}$$

Equation (22), with contorn conditions $\Psi(x_{1,0})$ and $\Psi'(x_{1,0}) \in \mathbb{C}_{\mathcal{F}(A)}$, is given by

$$\begin{cases} \dfrac{d^2}{dx^2}\Psi(x) \oplus \dfrac{2m}{\hbar^2}(E - V(x))\Psi(x) = 0 \\ \\ \Psi(x_{1,0}) = q_1A + r_1 \in \mathbb{C}_{\mathcal{F}(A)} \\ \\ \Psi^{'}(x_{1,0}) = q_2A + r_2 \in \mathbb{C}_{\mathcal{F}(A)}, \end{cases} \tag{23}$$

is known as stationary Schrödinger equation with boundary conditions given by linearly correlated complex fuzzy numbers. If A is a non-symmetric fuzzy number, the function $\Psi : \mathbb{R} \longrightarrow \mathbb{C}_{\mathcal{F}(A)}$ represents the wave function for each $x \in \mathbb{R}$. Moreover, the function $V : \mathbb{R} \longrightarrow \mathbb{R}$ is a potential where the particle of the mass m is subject and $\hbar$ is the reduced Planck constant. The quantity E, means the energy of the particle. A very important and fundamental physical situation is related to the situation in which the potential $V(x)$ can be considered constant. It is important to emphasize that this is a fundamental step for the study of quantum systems with varying potentials, as we can always approximate an arbitrary potential curve by a set of step potentials [2–5]. So let us consider the case that the potential is a step function:

$$V(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ V_0 \text{ if } x > 0. \end{cases} \tag{24}$$

Figure 1, graphically illustrates the potential given by (24) in which we can verify two sectors, one with null potential defined in $(-\infty, 0]$, called sector I and another sector with potential V_0 defined in $(0, \infty)$ called sector II. Furthermore, in sector II we have two possible situations, one where $E > V_0$ and the other where $E < V_0$ which leads us to the need to obtain the solution of the Shcrödinger equation in only two situations, one for $E > V_0$ and another $E < V_0$.

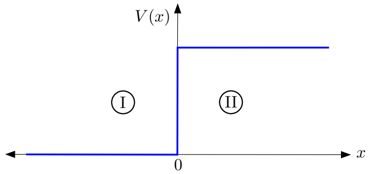

Fig. 1 Graphical representation of the potential, in which we can highlight two regions, one with domain $(-\infty, 0]$, called sector I, and another with domain $(0, \infty)$, identified as sector II.

Considering the potential given by (24), we have that:

$$\begin{cases} \dfrac{d^2}{dx^2}\Psi(x) \oplus \dfrac{2m}{\hbar^2}(E - V_0)\Psi(x) = 0 \\ \\ \Psi(0) = q_1 A + r_1 \in \mathbb{C}_{\mathcal{F}(A)} \\ \\ \Psi'(0) = q_2 A + r_2 \in \mathbb{C}_{\mathcal{F}(A)}. \end{cases} \tag{25}$$

Thus, the Problem (25) implies a fuzzy wave function given by

$$\Psi(x) = \begin{cases} \Psi_I(x) = q_I(x)A + r_I(x) & \text{if } x \leq 0 \\ \Psi_{II}(x) = q_{II}(x)A + r_{II}(x) & \text{if } x > 0, \end{cases} \tag{26}$$

and consequently, it is necessary to obtain the set $\{q_I(x), r_I(x), q_{II}(x), r_{II}(x)\}$ to physically study the system. In this sense, it is important to distinguish two fundamental physical situations, one in which the energy of the particle E is greater than the energy of the barrier V_0, and the other in which the energy of the particle is less than the barrier.

4.1 The case where $E > V_0$

For the case $E > V_0$, let us introduce the positive constants $\{k_I, k_{II}\}$ defined by

$$E = \frac{\hbar^2 k_I^2}{2m} \quad \text{and} \quad E - V_0 = \frac{\hbar^2 k_{II}^2}{2m}. \tag{27}$$

Now, consider that in Equation (25), $\Psi(x) \in \mathbb{C}_{\mathcal{F}(A)}$ and $q, r : \mathbb{R} \longrightarrow \mathbb{C}$ are continuous and Riemann integrable functions. By Theorem 8, to obtain the fuzzy wave function, it is only necessary to find the elements $\{q, r\}$ from the following classic initial value problems:

$$\begin{cases} q_\tau^{''}(x) + k_\tau^2 q_\tau(x) = 0 \\ q_\tau(0) = q_{\tau,1} \\ q_\tau^{'}(0) = q_{\tau,2} \end{cases} \tag{28}$$

and

$$\begin{cases} r_\tau^{''}(x) + k_\tau^2 r_\tau(x) = 0 \\ r_\tau(0) = r_{\tau,1} \\ r_\tau^{'}(0) = r_{\tau,2}, \end{cases} \tag{29}$$

where τ can take values I and II, thus identifying the different sectors. Let's start by analyzing the sector $\tau = I$, but only for the part $q_I(x)$, because the solution of the part in $r_I(x)$ is obtained similarly. The function $q_I(x)$ is associated with two plane waves, one evolving from left to right, characterized by $e^{Ik_I x}$, and the other evolving from right to left, characterized by $e^{-Ik_I x}$. It is important to emphasize that the wave function traveling from right to left is associated with the reflection generated by the barrier (step).

For sector II the function $q_{II}(x)$ is also associated with two plane waves functions, one traveling from left to right, characterized by $e^{Ik_{II}x}$, and another traveling from right to left, characterized by $e^{-Ik_{II}x}$. However, the wave traveling from right to left is discarded from the physical point of view, as there is no physical agent capable of generating this term, normally related to the process of reflection. Therefore, all multiplicative constants of $e^{-Ik_{II}x}$ of sector II, can be considered null. Therefore, we have that $q_I(x) = A_1 e^{ik_I x} + A_2 e^{-ik_I x}$, $q_{II}(x) = A_5 e^{ik_{II}x}$, $r_I(x) = A_3 e^{ik_I x} + A_4 e^{-ik_I x}$ and $r_{II}(x) = A_6 e^{ik_{II}x}$, where A_1, A_2, A_3, A_4, A_5 and A_6 are constants. Using the fact that $\Psi_I(0) = \Psi_{II}(0)$ we have that $q_{1,I}(0) = q_{1,II}(0)$ and $r_{1,I}(0) = r_{1,II}(0)$. On the other hand, as $\Psi_I^{'}(0) = \Psi_{II}^{'}(0)$ it follows that $q_{2,I}(0) = q_{2,II}(0)$ and $r_{2,I}(0) = r_{2,II}(0)$. Thus,

$$A_1 = \frac{1}{2}\frac{(k_I + k_{II})}{k_I} A_5, \;\; A_2 = \frac{1}{2}\frac{(k_I - k_{II})}{k_I} A_5, \tag{30}$$

and

$$A_3 = \frac{1}{2}\frac{(k_I + k_{II})}{k_I} A_6, \;\; A_4 = \frac{1}{2}\frac{(k_I - k_{II})}{k_I} A_6. \tag{31}$$

Moreover,

$$\begin{aligned} q_I(x) &= \tfrac{A_5}{2k_I}\left[(k_I + k_{II})e^{ik_I x} + (k_I - k_{II})e^{-ik_I x}\right] \\ &= \tfrac{A_5}{k_I}\left[i\sin(k_I x)k_{II} + k_I\cos(k_I x)\right] \end{aligned} \tag{32}$$

and

$$q_{II}(x) = A_5 e^{ik_{II}x} = A_5\left[\cos(k_{II}x) + i\sin(k_{II}x)\right]. \tag{33}$$

To get $r_I(x)$ and $r_{II}(x)$ just replace A_5 by A_6, or $A_5 = \beta A_6$:

$$\begin{aligned} r_I(x) &= \tfrac{A_6}{2k_I}\left[(k_I + k_{II})e^{ik_I x} + (k_I - k_{II})e^{-ik_I x}\right] \\ &= \tfrac{A_6}{k_I}\left[i\sin(k_I x)k_{II} + k_I\cos(k_I x)\right] \end{aligned} \tag{34}$$

and

$$r_{II}(x) = A_6 e^{ik_{II}x} = A_6 \left[\cos(k_{II}x) + i\sin(k_{II}x)\right]. \tag{35}$$

Furthermore, if A is an non-symmetrical fuzzy number, we have all quantities to obtain $\Psi_I(x)$ and $\Psi_{II}(x)$ and, consequently, the fuzzy wave function given by (26) which can be written as follow:

$$\Psi(x)_{E>V_0} = \begin{cases} \left[i\sin(k_I x)k_{II} + k_I\cos(k_I x)\right]\frac{A_6}{k_I}(1+\beta A) & \text{if } x \leq 0 \\ \left[\cos(k_{II}x) + i\sin(k_{II}x)\right]A_6(1+\beta A) & \text{if } x > 0, \end{cases} \tag{36}$$

where $A_5 = \beta A_6$. Given $\beta = 1$, the fuzzy wave function (36) can be written in the form

$$\Psi(x)_{E>V_0} = \begin{cases} \frac{A_6}{k_I}\left[ik_{II}\sin(k_I x) + k_I\cos(k_I x)\right](A+1) & \text{if } x \leq 0 \\ \left(A_6 e^{ik_{II}x}\right)A + A_6 e^{ik_{II}x} & \text{if } x > 0. \end{cases} \tag{37}$$

Given the different possible approaches to the physical interpretation of (37), one that deserves special analysis is related to considering the traditional quantum part associated with the function $r(x)$. Although not necessary, in this case, if we consider the fuzzy number A with a sufficiently small diameter, we can associate the fuzzy part as a correction to the traditional quantum result, which can open many perspectives of applications for this formalism. Another important point to be highlighted is that the set of functions $\{q, r\}$ can even be a set of orthogonal functions, which opens up the possibility of applications and interpretations beyond the traditional ones.

Remark 4 Note that

$$\begin{aligned} \Psi_I(0) &= \Psi_{II}(0) = A_6(1+\beta A) = r_1 + q_1 A, \\ \Psi'_I(0) &= \Psi'_{II}(0) = ik_{II}A_6(1+\beta A) = ik_{II}\Psi_{II}(0). \end{aligned} \tag{38}$$

With that, we can impose boundary condition for $\Psi_I(0) = \Psi_{II}(0)$ with A_6 imaginary and consequently $\Psi'_I(0) = \Psi'_{II}(0)$ will be real, assuming β real.

Figure 2 show the graphical representation of the real part of the fuzzy wave function given by (37) for two differents energy values, one fuzzy number A and $A_6 = 1$ and $\beta = 1$. In the first graph, we have $E = 1.5$ and $A = (-0.3;\ 0;\ 0.1)$. In the second graph, we consider the energy $E = 1.05$ and the same uncertainty to observe what happens in the energy region close to $V_0 = 1$. In both cases, the wave functions have the same diameter. Another important point that we can highlight is that using the points where the real part of the wavefunction is zero, we obtain the classical results for the wavelength. Furthermore, we can observe in the case where $E = 1.05$, as the energy is very close to the potential of the barrier V_0, the wavelength of the sector II increases, showing a tendency of the oscillation to transform into an exponential decay . With this, we see that the fuzzification process for this quantic case provides us with physics consistent with classical physics.

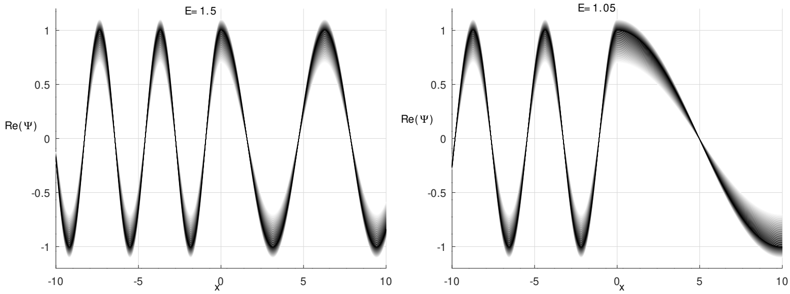

Fig. 2 Graphical representation of the real part of the fuzzy wave functions for the case where $E > V_0$ with $A = (-0.3;\ 0;\ 0.1)$, $A_6 = 1$, $V_0 = 1$ for energy values: $E = 1.5$ and $E = 1.05$.

Figure 3 show the graphical representation of the real part of the fuzzy wave function given by (37) for one energy value, two differents fuzzy numbers A, $A_6 = 1$ and $\beta = 1$. We chose the fuzzy number $A = (-0.05;\ 0;\ 0.005)$, very close to zero, with fixed energy $E = 1.5$ to illustrate the physical effect of reducing uncertainty in the system. In both cases, the fuzzy wave functions have the same wavelength. As the uncertainty decreased, we recovered the traditional classical result.

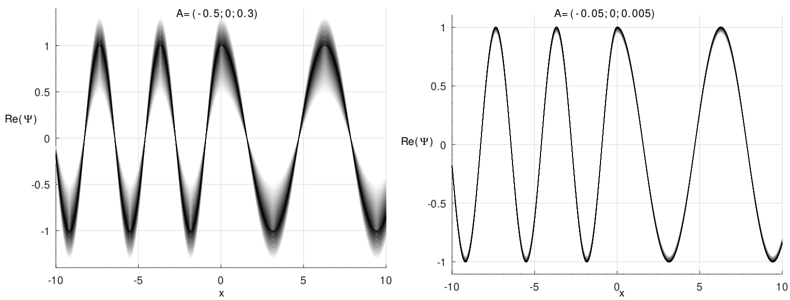

Fig. 3 Graphical representation of the real part of the fuzzy wave functions for for fixed energy $E = 1.5, A_6 = 1, V_0 = 1$ for the fuzzy numbers: $A = (-0.5;\ 0;\ 0.3)$ and $A = (-0.05;\ 0;\ 0.005)$

Remark 5 The real function $\|\Psi\|^2$ has a probabilistic interpretation. For A non-symmetric, the norm $\|\Psi(x)\|$ is given by $\|q(x), r(x)\|_{\mathbb{C}^2}$. We define the probability density $P(x)$, as the norm-squared of the wave function

$$P(x) = \|\Psi(x)\|^2 . \tag{39}$$

This probability density so defined as positive. The physical interpretation of the wave function arises because we assume that $P(x)dx$ is the probability to find the particle in the interval $[x, x+dx]$.

4.2 The case where $E < V_0$

To find the fuzzy wave function in the case $E < V_0$, we will proceed in an analogous way to the case $E \geq V_0$. For the sector I, the physical properties of the wave function are the same as the case $E > V_0$. For sector II, we have

$$q''_{II}(x) - \rho^2 q_{II}(x) = 0, \quad \text{where} \quad \rho^2 = \frac{2\,m\,(V_0 - E)}{\hbar^2}, \tag{40}$$

whose solution is given by

$$q_{II}(x) = A_5 e^{-\rho x} + A_7 e^{\rho x}. \tag{41}$$

For since, $\rho > 0$ e therefore $e^{\rho x}$ is unrealistica. We will consider the $A_7 = 0$, and consequently we have

$$q_{II}(x) = A_5 e^{-\rho x}. \tag{42}$$

Applying the boundary conditions at $x = 0$, we have

$$\begin{array}{l} A_1 + A_2 = A_5, \quad ik_I(A_1 - A_2) = -\rho A_5 \\ A_1 = \frac{1}{2}\frac{(k_I + i\rho)}{k_I} A_5, \quad A_2 = -\frac{I}{2}\frac{(ik_I + \rho)}{k_I} A_5. \end{array} \tag{43}$$

So we have

$$q_I(x) = \frac{A_5}{k_I}\big(k_I \cos(k_I x) - \sin(k_I x)\rho\big) \tag{44}$$

and

$$r_I(x) = \frac{A_6}{k_I}\big(k_I \cos(k_I x) - \sin(k_I x)\rho\big). \tag{45}$$

For sector II, implies that

$$r_{II}(x) = A_6 e^{-\rho x}. \tag{46}$$

Let's consider $A_5 = \beta A_6$, with constant β. Thus

$$\begin{array}{l} \Psi_I(x) = [k_I \cos(k_I x) - \sin(k_I x)\rho]\, \frac{A_6}{k_I}(1 + \beta A), \\ \Psi_{II}(x) = e^{-\rho x} A_6 (1 + \beta A). \end{array} \tag{47}$$

Furthemore,

$$\Psi_I(0) = \Psi_{II}(0) \;=\; A_6(1 + \beta A) = r_1 + q_1 A \tag{48}$$

$$\Psi'_I(0) = \Psi'_{II}(0) \;=\; -\rho A_6(1 + \beta A) = -\rho \Psi_I(0) \tag{49}$$

and hence $r_1 = A_6$, $q_1 = A_6\beta$, $r_2 = -\rho r_1$ and $q_2 = -\rho q_1$. Therefore, an boundary condition given by a real number implies a real boundary condition for $\Psi^{'}_{II}(0) = \Psi^{'}_{II}(0)$, with a proportionality factor equal to $-\rho$.

Therefore, the fuzzy wave function (26) with $\Psi_I(x)$ and $\Psi_{II}(x)$ given by (47) can be written in the form

$$\Psi(x)_{E<V_0} = \begin{cases} \frac{A_6}{k_I}\left[k_I\cos(k_I x) - \rho\sin(k_I x)\right](A+\beta) & \text{if } x \leq 0 \\ (A_6 e^{-\rho x})A + A_6 e^{-\rho x} & \text{if } x > 0. \end{cases} \tag{50}$$

Figure 4 illustrates the graphical representation of the fuzzy wave function given by (50) for $A = (-0.3;\ 0;\ 0.1)$, $\beta = 1$, $A_6 = 1$, $E = 0.1$ and $E = 0.95$. By choosing $E = 0.95$ we observe classical quantum physics for $E \cong V_0$. It is observed that the diameter of the fuzzy wave function does not depend on energy, as expected. For the case where $E = 0.1$ we can use the points that the fuzzy wave function vanishes to obtain the classic result for the wavelength. Furthermore, the graph of the fuzzy wave function for $E = 0.1$ decays faster than the case $E = 0.95$, as physically expected. With this, we observed that the fuzzification process is well structured, as it shows results consistent with those expected classically.

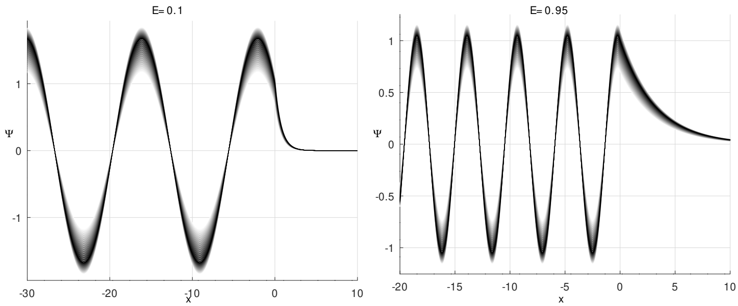

Fig. 4 Graphical representation of the fuzzy wave functions for the case where $E > V_0$ with $A = (-0.3;\ 0;\ 0.1)$, $\beta = 1$, $A_6 = 1$, $V_0 = 1$ for energy values: $E = 0.1$ and $E = 0.95$.

Figure 5 shows the graphical representation of the fuzzy wave function given by (50) for energy $E = 0.5$, $\beta = 1$, $A_6 = 1$ and two differents fuzzy numbers: $A = (-0.5;\ 0;\ 0.3)$ and $A = (-0.05;\ 0;\ 0.005)$. The choice of this last fuzzy number shows us the physical effects of the reduction of uncertainty in the studied phenomenon. We can verify that the fuzzy wavelength in the I sector is the same for both chosen fuzzy numbers.

The current density is an important quantity in quantum mechanics

$$J(x) = \frac{i\hbar}{2m}\left[\Psi(x)\Psi^{*\prime}(x) - \Psi^*(x)\Psi'(x)\right], \tag{51}$$

where $\Psi^*(x)$ denotes the conjugate of the function $\Psi(x)$.

The fuzzy current density is defined by

$$J_{\text{fuzzy}}(x) = \frac{i\hbar}{2m}\left[(\Psi(x)\odot\Psi^{*\prime}(x)) \ominus (\Psi^*(x)\odot\Psi'(x))\right]. \tag{52}$$

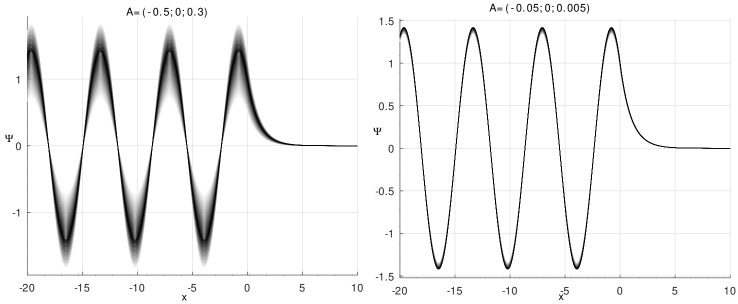

Fig. 5 Graphical representation of the fuzzy wave functions for fixed energy $E = 0.5$, $\beta = 1$, $A_6 = 1$, $V_0 = 1$ for the fuzzy numbers: $A = (-0.5;\ 0;\ 0.3)$ and $A = (-0.05;\ 0;\ 0.005)$.

In the classic case, the current density for the case $E < V_0$ is zero. Considering $x > 0$ and the initial conditions:

$$\Psi(0)_{E<V_0} = q_1 A = A_5 A \quad and \quad \Psi'(0)_{E<V_0} = -\rho q_1 A = -\rho A_5 A, \tag{53}$$

where A is the non-symmetric fuzzy number, then, the fuzzy wave function is given by

$$\Psi(x) = \left(A_5 e^{-\rho x}\right) A, \quad x > 0. \tag{54}$$

Using (12), since the imaginary parts of the corresponding fuzzy complex numbers are zero, we have

$$\Psi(x) \odot \Psi^{*\prime}(x) = \Psi^{*}(x) \odot \Psi'(x). \tag{55}$$

Therefore,

$$\begin{aligned} J_{\text{fuzzy}}(x) &= \frac{i\hbar}{2m} \left((\Psi(x) \odot \Psi^{*\prime}(x)) \ \ominus \ (\Psi^{*}(x) \odot \Psi'(x)) \right) \\ &= 0. \end{aligned} \tag{56}$$

Thus, we verify that for $E < V_0$, the fuzzy current density is null, which allow us to make an interpretation in analogy to the classic one, in which the reflection coefficient is null. Therefore, we can conclude that for $x < 0$, we have the superposition of fuzzy waves of the same amplitude propagating in opposite directions, generating a stationary fuzzy wave function.

Statements and Declarations

Authorship contributions

These authors contributed equally to this work.

Ethical approval

Not applicable.

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Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

Informed Consent

Not applicable.

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