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Pullback Attractors for Non-Newtonian Fluids with Shear Dependent Viscosity

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Abstract. The so-called Ladyzhenskaya model is analyzed from a non-autonomous dynamical system point of view, for weak and strong solutions. Existence of attractors when forces are time-dependent is ensured in several universes with different tempered parameter conditions, and also for fixed bounded sets. Attraction is proved in L^2 and $W^{1,p}$ norms and relationships between these families are also established.

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1. Introduction

Some fluids including polymer melts and solutions, as well as liquids in which small particles are in suspension, in mathematical physics, are called non-Newtonian fluids. This kind of fluids, unlike the Newtonian fluids, cannot be adequately described assuming a constitutive relation for the Cauchy stress linear in the symmetric part of the velocity gradient. Non-Newtonian fluids cover a broad variety of materials of widely differing material structure, leading to diverse constitutive relations for the stress tensor, what implies the existence of many different non-Newtonian models (see, for instance, [7]).

The model we are interested in was introduced and treated by O. A. Ladyzhenskaya [12–14], sometimes called the *modified Navier–Stokes system* or the *Ladyzhenskaya model* (see also Lions [16, pp. 207–221] or the seminal monograph by Malek, Nečas, Rokyta and Ružička [18, Section 5] or in the more recent by Feireisl and Pražák [8, Ch. 7 & 8] and the references therein).

The so-called Ladyzhenskaya model deals with an homogeneous and incompressible fluid without thermal effects which reads

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}\mathbb{S}(Du) + \mathrm{div}(u \otimes u) + \nabla p = f \quad \text{in } \Omega \times (\tau, \infty), \\ \mathrm{div} u = 0 \quad \text{in } \Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) \quad \text{in } \Omega, \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) is a bounded domain with Lipschitz boundary, $\tau \in \mathbb{R}$, as usual Du denotes the symmetrized gradient of the velocity field u, i.e. $Du = \frac{1}{2}(\nabla u + \nabla u^\top)$, p is the pressure, f the external force and the stress-tensor $\mathbb{S} : \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^{n\times n}_{sym}$ is assumed to satisfy the following conditions (cf. [8])

$$\begin{cases} \mathbb{S}(0) = 0, \\ (\mathbb{S}(D_1) - \mathbb{S}(D_2)) : (D_1 - D_2) \geq \nu_1(1 + \mu(|D_1| + |D_2|))^{p-2}|D_1 - D_2|^2, \\ |\mathbb{S}(D_1) - \mathbb{S}(D_2)| \leq c_1\nu_1(1 + \mu(|D_1| + |D_2|))^{p-2}|D_1 - D_2|, \end{cases} \tag{2}$$

where the positive constants ν_1 and ν_2 are the so-called generalized viscosities, and $\mu = (\nu_2/\nu_1)^{1/(p-2)}$ (with the convention that $\mu = 0$ for $p = 2$). [By the way c_1 is another positive constant; the same $c_i > 0$

will be supposed in the sequel, constants depending on the parameters of the model.] Examples of non-Newtonian tensors are for instance those of power-law type $\mathbb{S} = (\tilde{\nu}_1 + \tilde{\nu}_2|Du|^{p-2})Du$. These assumptions above imply coercivity and control of growth for the tensor $\mathbb{S}$

$$\mathbb{S}(D) : D \geq c_2(\nu_1|D|^2 + \nu_2|D|^p), \quad |\mathbb{S}(D)| \leq c_3(\nu_1|D| + \nu_2|D|^{p-1}) \text{ for } p \geq 2. \tag{3}$$

There are some difficulties in analyzing the dynamics of non-Newtonian fluids in comparison with Newtonian fluids. Even though it would be possible to consider the three-dimensional case with the uniqueness of solutions, with p large enough, there would be some additional obstacles in proving the asymptotic compactness of the process. Roughly speaking this happens because higher regularity results (say H^2) are not available. In this way, we have to explore the p-integrability of the solutions as well as the regularity of the time partial derivative.

The study of the asymptotic behavior of solutions is crucial not only for the long-term understanding of the behavior of solutions and their internal structure, but for each complete model itself, as regularity and uniqueness issues, minimal dissipative conditions and so on. Concerning the asymptotics associated to (1), in the autonomous case the analysis is rather satisfactory (stability, attractors, exponential attractors, fractal dimension, Lyapunov exponents, perturbed models, etcetera). We may mention [2,10,11,17,19,23], and [8, Sec. 7] among many others.

Under the presence of time-dependent forces (e.g. due to fluctuations or device controls), the framework becomes non-autonomous. This means that the dynamical analysis can be performed in several senses: uniform attractors seek *a fixed photo* in the phase-space; skew-product flows do similarly but under a driving system; random dynamical systems and (deterministic) pullback attractor theories look for time-dependent families whose sections attract in suitable senses. These objects point out dynamical properties of the evolution in time of the trajectories, that help to go deeper in the understanding of the model, for instance, after invariance, of regularity and uniqueness issues.

Regarding this non-autonomous case, very recently, Yang et al. [25] proved the existence of finite-dimensional pullback attractors for a simplified model where $\text{div}\mathbb{S}(Du)$ was replaced by $(\nu+\nu_0\|\nabla u\|^2_{L^2})\Delta u$. The advantage of this change is that the modified model enjoys nice properties of uniqueness and regularity. Although a completely different non-Newtonian model (fourth order), also worth to mention several contributions to the existence of pullback attractors from Zhao and collaborators [27–30].

The goal of this paper is to investigate the existence of dynamical systems and pullback attractors associated to (1), both without or with uniqueness and in different norms. The structure of the paper is as follows. Section 2 is devoted to recall briefly the abstract functional setting of the problem, focusing on weak and strong solutions, existence, regularity and uniqueness issues. Since uniqueness is unknown when p is not large enough, we summarize the main ingredients on pullback attractors for processes in a (possibly) multi-valued framework in Sect. 3. Time-dependent universes and relationships between attractors are also presented here. Section 4 deals with the asymptotic of weak solutions in L^2-norm. The assumptions on forces are minimized, and an energy method is developed. Worth to emphasize here that the uniform estimates allow arbitrarily large choices of the tempered parameter for the case $p > 2$, in contrast to the Newtonian case where the range of values of the parameter is (upper) constrained by the first eigenvalue of the Stokes operator. The parameter in a tempered universe is a measure of the growth of the perturbed initial data when a *very long* phenomena takes place, but such that this tracking property of the sections of a pullback attractor still holds. This will lead to infinite attractors of respective tempered universes, both suitably ordered (cf. Theorem 23 and Remark 25.) Finally in Sect. 5 strong solutions to (1) are considered. We pay the price of uniqueness to ensure the results proving them through the Galerkin approximations. Under additional regularity of the force term, deriving the equation, attraction in $W^{1,p}$-norm is obtained, again with an infinite family of objects, related with those of the previous paragraph. Actually, we conclude proving that they coincide under a suitable stronger assumption on the force.

2. Abstract Setting of the Problem and Known Results

In order to fit the problem in an abstract functional setting let us recall the usual functional spaces involved. Denote $\mathcal{V} := \{\varphi \in C_c^\infty(\Omega)^n : \text{div}\varphi = 0\}$, H is the closure of $\mathcal{V}$ in the $L^2(\Omega)^n$-norm and (for a general $q \in [1,\infty)$) V_q the closure of $\mathcal{V}$ in the $W^{1,q}(\Omega)^n$-norm. In H we will denote by $(\cdot,\cdot)$ the standard scalar product in $L^2(\Omega)^n$ and the corresponding norm by $|\cdot|$. The norm on V_q will be the L^q-norm of the gradient of an element (thanks to the Poincaré inequality). V_q^* denotes the topological dual of V_q, $\langle\cdot,\cdot\rangle$ the action among these spaces, and $\|\cdot\|_*$ the norm in V_q^* (without referring to q if no confusion arises).

The amount of results concerning different notions of solutions in the modern theory for PDE problems is wide. Indeed it is even more involved for non-Newtonian fluid models depending on the parameters and the regularity of the data (e.g. cf. the above cited monographs [8,18] and the references therein). Roughly speaking, while assuming more regular data and functions in the system (and larger values of the parameters) strong solutions can be obtained; on other hand, weakened assumptions or smaller values of parameters only allow to gain weak solutions, or by last (the most general case) measure-valued solutions. Here we will be mainly concerned with weak solutions (of course, under suitable values of the parameter p related to the dimension n). We recall the basic notion and existence result for our purposes.

In what follows we assume that $f \in L^{p'}_{loc}(\mathbb{R}; V_p^*)$.

Definition 1. A weak solution to (1) is an element $u \in L^\infty(\tau,T;H) \cap L^p(\tau,T;V_p)$ for any $T > \tau$ such that

$$\int_\tau^T \int_\Omega (-u\partial_t\varphi - (u \otimes u) : D\varphi + \mathbb{S}(Du) : D\varphi) dxdt = \int_\Omega u_\tau \varphi(\tau,\cdot)dx + \int_\tau^T \int_\Omega f\varphi dxdt \tag{4}$$

for any $\varphi \in C_c^\infty([\tau,T) \times \Omega)^n$ with $\text{div}\varphi = 0$.

If larger enough values of the parameter p (related to the dimension n) are assumed, we may consider the following functional operators to reformulate the problem. Namely, we borrow [17, Lemma 2.8, p. 504] to define the operator $\mathbb{T}$ by

$$\int_\tau^T \langle \mathbb{T}(u)(t), v(t)\rangle dt := \int_\tau^T \int_\Omega \mathbb{S}(Du) : Dvdxdt.$$

Then it satisfies $\mathbb{T} : L^p(\tau,T;V_p) \to L^{p'}(\tau,T;V_p^*)$ if $p > 1$.

On other hand the operator B defined by

$$\int_\tau^T \langle B(u)(t), v(t)\rangle dt := \sum_{i,j=1}^n \int_\tau^T \int_\Omega u_j(x,t)\frac{\partial}{\partial x_j}u_i(x,t)v_i(x,t)dxdt$$

satisfies $B : L^p(\tau,T;V_p) \cap L^\infty(\tau,T;H) \to L^{p'}(\tau,T;V_p^*)$ if $p \geq 1 + 2n/(n+2)$.

In this way taking $p \geq 1 + 2n/(n+2)$ the abstract formulation of the problem (1) in a free-divergence space arises

$$\frac{du}{dt} + \mathbb{T}u + B(u) = f,$$

whence a solution u satisfies $\frac{du}{dt} \in L^{p'}_{loc}(\tau,\infty;V_p^*)$. Actually any element of the class of weak solutions can be taken as a test function in the weak formulation. This leads to uniform estimates, and by last (through compactness and monotonicity arguments) to an existence result, continuity properties and energy equality (cf. Remark 3 below), which will be essential in order to study the long-time behavior).

Theorem 2. (Existence; e.g. cf. [12–14], [16, Théorème 5.1], [17, Theorem 2.14], [8,18])

(i) Suppose $p \geq 1 + 2n/(n+2)$, $f \in L^{p'}_{loc}(\mathbb{R}; V_p^*)$ and $u_\tau \in H$. Then there exists at least one weak solution to problem (1).

(ii) Besides the above, if $p \geq (n+2)/2$ then the weak solution to (1) is unique.

Remark 3. From above and the regularity of f, any weak solution (ensured by (i) provided that $p \geq 1 + 2n/(n+2)$) satisfies $u' \in L^{p'}_{loc}(\tau, \infty; V_p^*)$. Therefore there exists a representative in the class with $u \in C([\tau,\infty); H)$ and the following energy equality holds

$$|u(t)|^2 + 2\int_s^t \int_\Omega \mathbb{S}(Du) : Dudxdr = |u(s)|^2 + 2\int_s^t \langle f(r), u(r)\rangle dr \quad \forall \tau \leq s \leq t.$$

It will be convenient at some stage of the paper to assume that $\mathbb{S}$ has a potential, i.e. there exists $\Phi \in C^2(\mathbb{R}^{n\times n}; \mathbb{R}_+)$ with

$$\begin{aligned} &\partial_D \Phi(D) = \mathbb{S}(D), \\ &\partial_D^2 \Phi(D) : (B \otimes B) \geq \nu_1 (1 + \mu|D|)^{p-2}|B|^2, \\ &|\partial_D^2 \Phi(D)| \leq c_4 \nu_1 (1+\mu|D|)^{p-2}. \end{aligned} \tag{5}$$

Observe that this means control from above and below for $\Phi(D)$ for any $D \in \mathbb{R}^{n\times n}$ namely

$$c_5\nu_1(1+\mu|D|)^{p-2}|D|^2 \leq \Phi(D) \leq c_6\nu_1(1+\mu|D|)^{p-2}|D|^2. \tag{6}$$

Assuming that $\mathbb{S}$ has a potential, it can be shown a regularizing effect in the problem for a more regular external force. Actually, we have the following result.

Proposition 4. ([2, Theorem 3.3], [8, Theorem 7.32]) *Consider T, $\tau \in \mathbb{R}$ with $T > \tau$, $u_\tau \in H$ and $f \in L^2_{loc}(\mathbb{R}; L^2(\Omega)^n)$. Assume that $p > 2$ if $n = 2$ and $p \geq 12/5$ if $n = 3$. Then, any weak solution associated to the initial condition u_τ satisfies*

$$u \in L^\infty(\tau+\varepsilon, T; V_p) \text{ and } \frac{\partial u}{\partial t} \in L^2(\tau+\varepsilon, T; H)$$

for all $\varepsilon > 0$ such that $\tau + \varepsilon < T$. If $u_\tau \in V_p$ then we can take $\varepsilon = 0$.

Recall the Korn inequality, which relates the norms of the gradient and of the symmetrized gradient. Namely, for any $\varphi \in W_0^{1,q}(\Omega)^n$, $1 < q < \infty$, there exists a constant $c(q) > 0$ such that

$$\|\nabla\varphi\|_q \leq c(q)\|D\varphi\|_q.$$

For short we denote $c_0 = c(2)$ and $\tilde{c}_0 = c(p)$. We also recall the Poincaré inequality

$$\lambda_1|v|^2 \leq |\nabla v|^2 \quad \forall v \in V_2,$$

where λ_1 is the first eigenvalue of the Stokes operator with homogeneous Dirichlet boundary conditions.

3. Non-autonomous Multi-valued Dynamical Systems and Pullback Attractors

In this section we recall briefly some well known concepts and results concerning multi-valued non-autonomous dynamical systems and about the existence and relationships of minimal pullback attractors. They are just included for the sake of completeness and for the convenience of the readers, and for their proofs we refer for instance to [22] (for the autonomous case) or to [4, Section 2] or [3,6,21] among many others. Nevertheless, it is worth to mention that when uniqueness holds for the considered problem, the theory exposed below reduces to the standard non-autonomous dynamical system theory of pullback attractors for single-valued closed processes (e.g., cf. [9]).

Given a metric space (X, d_X), and denoting $\mathbb{R}^2_d = \{(t,s) \in \mathbb{R}^2 : t \geq s\}$ and $\mathcal{P}(X)$ the family of nonempty subsets of X, we recall the notion of multi-valued (semi-)process.

Definition 5. A multi-valued map $U : \mathbb{R}^2_d \times X \to \mathcal{P}(X)$ is a multi-valued process on X if

(i) $U(\tau,\tau)x = \{x\}$ for any $\tau \in \mathbb{R}$ and all $x \in X$.
(ii) $U(t,\tau)x \subset U(t,s)(U(s,\tau)x)$ for any $\tau \leq s \leq t$ and all $x \in X$, where $U(t,\tau)B := \bigcup_{z\in B} U(t,\tau)z$ for any $B \subset X$.

The process U is said to be strict if the inclusion in (ii) is an equality.

One of natural extensions of notion of continuity is upper-semicontinuity, and reads as follows. A multi-valued process U on X is upper-semicontinuous if for any pair $(t,\tau) \in \mathbb{R}^2_d$, the mapping $U(t,\tau) : X \to \mathcal{P}(X)$ satisfies for each $x \in X$ and neighborhood $N(U(t,\tau)x)$ of $U(t,\tau)x$ that there exists a neighborhood M of x such that $U(t,\tau)y \subset N(U(t,\tau)x)$ for any $y \in M$.

In the context of non-autonomous dynamical systems it seems more suitable to consider not only fixed (bounded) sets for applications but a universe $\mathcal{D}$, that is, a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. This is due to the fact that a perturbation in data does not affect only the spatial position but also a possible change in time as we will see below (this is originally motivated by the random dynamical system theory). A universe $\mathcal{D}$ is said inclusion-closed if given two families $\widehat{D}$ and $\widehat{D}'$ with $\widehat{D} \in \mathcal{D}$ and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, then it holds that $\widehat{D}' \in \mathcal{D}$.

The key ingredients for the establishment of an attraction object are absorption and asymptotic compactness.

Definition 6. A family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ is pullback $\mathcal{D}$-absorbing for a multi-valued process U if for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$ there exists $\tau(\widehat{D},t) \le t$ such that $U(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \le \tau(\widehat{D},t)$.

Observe that we do not require that the absorbing family above belongs to the universe.

Definition 7. Given a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$, a multi-valued process $U : \mathbb{R}^2_d \times X \to \mathcal{P}(X)$ is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, it holds that any sequence $\{y_n\}$ with $y_n \in U(t,\tau_n)x_n$ is relatively compact in X.

A multi-valued process U is said pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

The minimal components (in the sense of attraction) that we aim to collect are the omega-limit families. Namely, given a family $\widehat{D} = \{D(s) : s \in \mathbb{R}\} \subset \mathcal{P}(X)$, the omega limit set of $\widehat{D}$ by U at time t (when it make sense) is defined by

$$\Lambda(\widehat{D},t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau)D(\tau)}^X .$$

Now the above combine to give the following result (cf. [4, Theorem 3]).

Theorem 8. *Consider an upper-semicontinuous multi-valued process U on a metric space X with closed values, a universe $\mathcal{D}$, a pullback $\mathcal{D}$-absorbing family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ and assume that U is pullback $\widehat{D}_0$-asymptotically compact. Then the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ given by*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D},t)}^X \quad \forall t \in \mathbb{R}$$

is the minimal pullback $\mathcal{D}$-attractor, i.e. it satisfies the following four properties: (i) [compact sections] $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X for each $t \in \mathbb{R}$; (ii) [attraction] $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e. $\lim_{\tau \to -\infty} dist_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for any $\widehat{D} \in \mathcal{D}$ and all $t \in \mathbb{R}$, where $dist_X(\cdot,\cdot)$ denotes the Hausdorff semi-distance in X between two subsets of X; (iii) [negatively invariance] $\mathcal{A}_{\mathcal{D}}$ is negatively invariant under the process U, i.e. $\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$ for any $t \ge \tau$; (iv) [minimality] if $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ is a family of closed sets which pullback $\mathcal{D}$-attracts under U, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Moreover, it holds that $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for any $t \in \mathbb{R}$. Besides this, if $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ and U is a strict process, then $\mathcal{A}_{\mathcal{D}}$ is invariant under U, i.e. $\mathcal{A}_{\mathcal{D}}(t) = U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$ for any $t \ge \tau$. If $\widehat{D}_0 \in \mathcal{D}$ has closed sections and $\mathcal{D}$ is inclusion-closed then $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$.

Observe that a pullback attractor fulfilling only conditions (i)–(iii) above does not need to be unique (cf. [20]). Condition (iv) of minimality gives uniqueness. On the other hand, when the attractor $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is also the unique family of closed subsets in $\mathcal{D}$ satisfying (ii)–(iii).

Next result is useful when we need to compare several attractors, for instance in regularity framework. It is inspired in the single-valued case [9, Theorem 3.15] but valid for a multi-valued process (cf. [4, Theorem 4]).

Theorem 9. *Consider two metric spaces $\{(X_i, d_{X_i})\}_{i=1,2}$ with continuous embedding $X_1 \subset X_2$, respective universes $\mathcal{D}_i$ in $\mathcal{P}(X_i)$ for $i = 1, 2$ and $\mathcal{D}_1 \subset \mathcal{D}_2$. Assume that U is a multi-valued process in both spaces, i.e. $U : \mathbb{R}^2_d \times X_i \to \mathcal{P}(X_i)$ for $i = 1, 2$. For each $t \in \mathbb{R}$ denote*

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i}, \qquad i = 1, 2,$$

(the subscript i in the omega-limit symbol Λ_i means w.r.t. the respective topology). Then $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Moreover, $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$ if the following two conditions hold: (i) $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$; (ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ and $t \in \mathbb{R}$ there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and a $t^*_{\widehat{D}_1}$ such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $s \leq t^*_{\widehat{D}_1}$ there exists a $\tau_s < s$ such that $U(s, \tau)D_2(\tau) \subset D_1(s)$ for all $\tau \leq \tau_s$.

An immediate consequence of the above results—but worth to recall for applications—are relating attractors for a general universe (usually given by tempered condition as will be seen below) with that of the universe of fixed bounded sets, denoted by $\mathcal{D}^X_F$, i.e. $\widehat{D} \in \mathcal{D}^X_F$ if $\widehat{D} = \{D(t) = B : t \in \mathbb{R}\}$ with B a nonempty bounded subset of X. Then we have the following

Corollary 10. *Under the assumptions of Theorem 8, if $\mathcal{D}^X_F \subset \mathcal{D}$ then $\mathcal{A}_{\mathcal{D}^X_F}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$. Moreover if there exists $T \in \mathbb{R}$ such that $\bigcup_{t \leq T} D_0(t)$ is bounded in X, then $\mathcal{A}_{\mathcal{D}^X_F}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.*

Remark 11. If the problem considered guarantees uniqueness, then the associated process through the solution operator becomes single-valued. Then the above results in this section can also be applied. Actually they reduce in such case to standard results in autonomous (e.g. cf. [24]) and non-autonomous (e.g. cf. [9, Section 3]) dynamical system theories.

4. Uniform Estimates and Attractors in H

In this section we study the asymptotic behavior of solutions to (1) analyzing the existence of the minimal pullback attractors in the H-norm for various universes.

In what follows, we assume that $p \geq 1 + 2n/(n+2)$ and that $f \in L^{p'}_{loc}(\mathbb{R}, V^*_p)$.

We denote by $\Psi(\tau, u_\tau)$ the set of weak solutions to (1) in $[\tau, +\infty)$ with initial datum $u_\tau \in H$. Theorem 2 on the existence of weak solutions to (1) guarantees that $\Psi(\tau, u_\tau)$ is not empty. Moreover, we can define a multi-valued map $U : \mathbb{R}^2_d \times H \to \mathcal{P}(H)$ given by

$$U(t, \tau)u_\tau = \{u(t) : u \in \Psi(\tau, u_\tau)\}, \quad u_\tau \in H, \quad \tau \leq t.$$

Next result establishes that the multi-valued map is a strict multi-valued process which is a direct consequence of the translation and concatenation properties of weak solutions.

Lemma 12. *The multi-valued map U is a strict multi-valued process in H.*

Compactness arguments show that the multi-valued process U is upper-semicontinuous with closed values.

Lemma 13. *Let $\{u_\tau^m\} \subset H$ be a strongly convergent sequence to u_τ in H. Then, for any sequence $\{u_m\} \subset \Psi(\tau, u_\tau^m)$ there exist a subsequence of $\{u_m\}$ (relabeled the same) and $u \in \Psi(\tau, u_\tau)$, such that*

$$u_m(s) \to u(s) \quad \text{strongly in } H \text{ for any } s \geq \tau. \tag{7}$$

Proof. Consider any $T > \tau$. Note that from the energy equality for (1), the coercivity of the stress tensor $\mathbb{S}$ (3), and the Korn and Young inequalities, it follows that the sequence $\{u_m\}$ is bounded in $L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$. Therefore, the sequence $\{\frac{\partial u_m}{\partial t}\}$ is bounded in $L^{p'}(\tau, T; V_p^*)$. By the Aubin-Lions compactness Lemma, there exist a subsequence of $\{u_m\}$ (relabeled the same) and $u \in L^\infty(\tau, T; H) \cap L^p(\tau, T; V_p)$ with $\frac{\partial u}{\partial t} \in L^{p'}(\tau, T; V_p^*)$ such that

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau, T; H), \\ u_m &\rightharpoonup u \quad \text{weakly in } L^p(\tau, T; V_p), \\ \frac{\partial u_m}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } L^{p'}(\tau, T; V_p^*), \\ u_m &\to u \quad \text{strongly in } L^2(\tau, T; H). \end{aligned}$$

These convergences allow to pass to the limit in the weak formulation (4) by using the monotonicity of $\mathbb{S}$ (this can be proved exactly as in [8, Theorem 7.4, p. 176]). Moreover, it is not difficult to check that $u(\tau) = u_\tau$. Consequently, $u \in \Psi(\tau, u_\tau)$.

It remains to prove (7). Observe that $\{u_m\}$ is equicontinuous in V_p^* on $[\tau, T]$ and that $\{u_m\}$ is bounded in $C([\tau, T]; H)$. Therefore, by the Arzelà-Ascoli Theorem, up to a subsequence, there follows that

$$u_m \to u \quad \text{strongly in } C([\tau, T]; V_p^*).$$

Hence, by the boundedness of $\{u_m\}$ in $C([\tau, T]; H)$ we conclude that

$$u_m(s) \rightharpoonup u(s) \quad \text{weakly in } H \text{ for any } \tau \leq s \leq T. \tag{8}$$

Now, since the estimate

$$|z(r)|^2 \leq |z(s)|^2 + 2\int_s^r \langle f(\theta), z(\theta)\rangle d\theta, \quad \tau \leq s \leq r \leq T$$

holds for $z = u_m$ and $z = u$, it follows that the functions $J_m, J : [\tau, T] \to \mathbb{R}$ defined by

$$\begin{aligned} J_m(r) &= |u_m(r)|^2 - 2\int_\tau^r \langle f(\theta), u_m(\theta)\rangle d\theta, \\ J(r) &= |u(r)|^2 - 2\int_\tau^r \langle f(\theta), u(\theta)\rangle d\theta, \end{aligned}$$

are non-increasing and continuous (in the class of weak solutions we are choosing the continuous representatives in $C([\tau, \infty); H)$, *cf.* Remark 3), and satisfy

$$J_m(r) \to J(r) \quad \text{a.e. } r \in (\tau, T).$$

To prove that $J_m(r) \to J(r)$ for any $r \in [\tau, T]$ consider a fixed $t^* \in (\tau, T]$ and an increasing sequence $t_k \uparrow t^*$ such that $J_m(t_k) \to J(t_k)$ for all $k \geq 1$. Thus, for any $\epsilon > 0$ there exist M, $K > 0$ such that

$$\begin{aligned} |J(t_k) - J(t^*)| &\leq \frac{\epsilon}{2} \quad \text{for } k \geq K, \\ |J_m(t_K) - J(t_K)| &\leq \frac{\epsilon}{2} \quad \text{for } m \geq M. \end{aligned}$$

Since J_m is a non-increasing function, we have that

$$J_m(t^*) - J(t^*) \leq |J_m(t_K) - J(t_K)| + |J(t_K) - J(t^*)| \leq \epsilon$$

for all $m \geq M$, and consequently $\limsup_{m\to\infty} J_m(t^*) \leq J(t^*)$. Taking into account that

$$\int_\tau^{t^*} \langle f(\theta), u_m(\theta)\rangle d\theta \to \int_\tau^{t^*} \langle f(\theta), u(\theta)\rangle d\theta,$$

we deduce that $\limsup_{m\to\infty} |u_m(t^*)| \leq |u(t^*)|$. Hence, by the weak convergence (8) we conclude that (7) holds for all $s \in [\tau, T]$. By taking increasing intervals and a diagonal argument we see that, for a suitable subsequence, (7) holds true for any $s \geq \tau$. The proof is complete. □

As a direct consequence of the previous result we have that

Corollary 14. *The multi-valued process U is upper-semicontinuous with closed values.*

In the following we obtain the uniform estimates leading to the existence of absorbing families and to the property of asymptotic compactness. Just for clarity in the exposition we split the case $p > 2$ from the case $p = 2$.

Lemma 15. *Consider $p > 2$. Then for any $\eta > 0$, there exist positive constants $c_{\nu_2,p}$ and $\widehat{C}_1$ such that any weak solution to (1) satisfies*

$$|u(t)|^2 \leq e^{-\eta(t-\tau)}|u(\tau)|^2 + c_{\nu_2,p}\int_\tau^t e^{-\eta(t-s)}\|f(s)\|_*^{p'}ds + \frac{\widehat{C}_1}{\eta} \quad \forall\, t \geq \tau. \tag{9}$$

Proof. From the energy equality for (1), the coercivity of the tensor $\mathbb{S}$ (3), and the Korn and Young inequalities, there follows

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{c_2\nu_1}{c_0^2}|\nabla u|^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla u\|_p^p \leq \frac{1}{p'\epsilon^{p'}}\|f\|_*^{p'} + \frac{\epsilon^p}{p}\|\nabla u\|_p^p \quad a.e.\ t > \tau.$$

Choosing $\frac{\epsilon^p}{p} = \frac{c_2\nu_2}{2\tilde{c}_0^p}$ and denoting $c_{\nu_2,p} = \frac{2}{p'\epsilon^{p'}}$, after the Poincaré inequality

$$\frac{d}{dt}|u|^2 + \frac{2c_2\nu_1\lambda_1}{c_0^2}|u|^2 + \frac{c_2\nu_2}{\tilde{c}_0^p}\|\nabla u\|_p^p \leq c_{\nu_2,p}\|f\|_*^{p'} \quad a.e.\ t > \tau. \tag{10}$$

Since we aim to arrive at a similar expression in the LHS with $\eta|u|^2$, we make the most of the Young inequality and the remaining term $\|\nabla u\|_p^p$.

Namely we split in cases depending the value of η. Actually the only interesting case is $\eta > 2c_2\nu_1\lambda_1 c_0^{-2}$.

Denote $0 < \beta = \eta - \frac{2c_2\nu_1\lambda_1}{c_0^2}$. Consider also C_I a constant of the embedding $W_0^{1,p}(\Omega)^n \subset L^2(\Omega)^n$, i.e. $|u| \leq C_I\|\nabla u\|_p$. Then the Young inequality yields

$$|u|^2 \leq \frac{\gamma^{p/2}}{p/2}\|\nabla u\|_p^p + \frac{(p-2)C_I^{2p/(p-2)}}{p\gamma^{p/(p-2)}}.$$

Putting $\frac{\gamma^{p/2}}{p/2} = \frac{c_2\nu_2}{2\tilde{c}_0^p\beta}$ we gain

$$\beta|u|^2 \leq \frac{c_2\nu_2}{2\tilde{c}_0^p}\|\nabla u\|_p^p + \widehat{C}_1$$

where $\widehat{C}_1 = \frac{(p-2)C_I^{2p/(p-2)}\beta}{p\gamma^{p/(p-2)}}$. Then (10) reduces to

$$\frac{d}{dt}|u|^2 + \eta|u|^2 + \frac{c_2\nu_2}{2\tilde{c}_0^p}\|\nabla u\|_p^p \leq c_{\nu_2,p}\|f\|_*^{p'} + \widehat{C}_1.$$

The case $\eta \leq 2c_2\nu_1\lambda_1 c_0^{-2}$ is simpler, and in any case the above inequality is valid.

Now it is standard multiplying by $e^{\eta s}$ and integrating in $[\tau, t]$ to arrive at (9). □

The case $p = 2$ is simpler, nevertheless we include it for the sake of completeness. Observe that from (2), for $p = 2$ it holds that $c_2 = 1$ and $\nu_2 = 0$ in (3).

Lemma 16. *If $p = 2$, for any $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$ there exists a positive constant β such that any weak solution to (1) satisfies*

$$|u(t)|^2 \leq e^{-\eta(t-\tau)}|u(\tau)|^2 + \frac{\lambda_1}{\beta}\int_\tau^t e^{-\eta(t-s)}\|f(s)\|_*^2 ds \quad \forall t \geq \tau. \tag{11}$$

Proof. The energy equality and the Korn inequality yield

$$\frac{d}{dt}|u|^2 + \frac{2\nu_1}{c_0^2}|\nabla u|^2 \le 2\langle f, u\rangle \quad a.e.\ t > \tau.$$

Fix a value $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$. Arranging coefficients and using the Cauchy inequality in the RHS we arrive at

$$\frac{d}{dt}|u|^2 + \frac{\eta}{\lambda_1}|\nabla u|^2 \le \frac{\lambda_1}{\beta}\|f\|_*^2 \quad a.e.\ t > \tau, \tag{12}$$

where $\beta := 2\nu_1\lambda_1 c_0^{-2} - \eta > 0$. By the Poincaré inequality, multiplying by $e^{\eta s}$, and integrating in $[\tau, t]$, (11) holds true. □

The above uniform estimates indicate what type of tempered condition should be involved in the definition of suitable universes.

Definition 17. (Universes in H) For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma^H$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that

$$\lim_{\tau\to-\infty} e^{\sigma\tau} \sup_{v\in D(\tau)} |v|^2 = 0.$$

Observe that any $\mathcal{D}_\sigma^H$ is inclusion-closed. From the tempered condition it holds that the universe of fixed bounded sets satisfies $\mathcal{D}_F^H \subset \mathcal{D}_\sigma^H$.

To prove the existence of a pullback absorbing family, we introduce the class

$$\mathcal{I}_\sigma^{p'} = \{f \in L_{loc}^{p'}(\mathbb{R}; V_p^*) : \int_{-\infty}^0 e^{\sigma s}\|f(s)\|_*^{p'} ds < \infty\}.$$

From Lemmas 15 and 16 one deduces

Corollary 18. *If there exists $\eta > 0$ (if $p = 2$, $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$) such that $f \in \mathcal{I}_\eta^{p'}$, then the dynamical system U in H associated to the weak solutions of (1) has a pullback $\mathcal{D}_\eta^H$-absorbing family $\widehat{B}_0 = \{B_0(t) : t \in \mathbb{R}\} \subset \mathcal{D}_\eta^H$ with $B_0(t) = \overline{B}_H(0, \mathcal{R}(t))$ where*

$$\mathcal{R}^2(t) = 1 + \frac{\widehat{C}_1}{\eta} + c_{\nu_2,p}e^{-\eta t}\int_{-\infty}^t e^{\eta s}\|f(s)\|_*^{p'} ds \quad if\ p > 2,$$
$$\mathcal{R}^2(t) = 1 + \frac{\lambda_1}{\beta}e^{-\eta t}\int_{-\infty}^t e^{\eta s}\|f(s)\|_*^2 ds \quad if\ p = 2, \tag{13}$$

being $\beta = 2\nu_1\lambda_1 c_0^{-2} - \eta$.

Remark 19. Since $\mathcal{I}_\eta^{p'} \subset \mathcal{I}_\sigma^{p'}$ for any $\sigma \ge \eta$, the estimates leading to the previous result means that for $p > 2$ any universe $\mathcal{D}_\sigma^H$ (with $\sigma \ge \eta$) will have an absorbing family through U. This will be completed later in terms of existence of attractors (cf. Theorem 23 and Remarks 24 and 25). Actually, this is not only valid for a bounded domain Ω but also for domains with finite measure (nevertheless the unbounded case requires more technicalities). On the other hand, the case $p = 2$ is shown to require a smallness condition.

To accomplish the existence of minimal pullback attractors, it remains to prove the pullback asymptotic compactness. To this end, we establish some additional estimates. Again, we consider the cases $p > 2$ and $p = 2$ separately.

Lemma 20. *Consider $p > 2$ and suppose that there exists $\eta > 0$ such that $f \in \mathcal{I}_\eta^{p'}$. Then for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta^H$, there exists $\tau_1(\widehat{D}, t) < t - 3$, such that for any $\tau \le \tau_1(\widehat{D}, t)$, $u_\tau \in D(\tau)$, and $u \in \Psi(\tau, u_\tau)$, it holds*

$$\begin{cases} |u(r;\tau,u_\tau)| \le \varrho_1(t) \quad \forall\, r\in[t-3,t], \\ \displaystyle\int_{r-1}^{r} \|\nabla u(s;\tau,u_\tau)\|_p^p ds \le \varrho_2(t) \quad \forall\, r\in[t-2,t], \end{cases}$$

where

$$\varrho_1^2(t) = 1+\frac{\widehat{C}_1}{\eta} + c_{\nu_2,p}e^{\eta(3-t)}\int_{-\infty}^{t} e^{\eta s}\|f(s)\|_*^{p'}ds$$

and

$$\varrho_2(t) = \frac{\tilde{c}_0^p}{c_2\nu_2}\varrho_1^2(t) + \frac{\tilde{c}_0^p c_{\nu_2,p}}{c_2\nu_2}\int_{t-3}^{t}\|f(s)\|_*^{p'}ds.$$

Proof. Fix $t\in\mathbb{R}$. The first inequality can be proved analogously to Corollary 18. Indeed, there exists $\tau_1(\widehat{D},t) < t-3$ such that

$$e^{-\eta(t-\tau)}|u_\tau|^2 \le 1 \quad \forall \tau \le \tau_1(\widehat{D},t).$$

Therefore, it follows from (9) that

$$|u(r;\tau,u_\tau)|^2 \le \varrho_1^2(t) \quad \forall\, r\in[t-3,t],\ \tau\le\tau_1(\widehat{D},t),\ u_\tau\in D(\tau),$$

where ϱ_1 is given above.

Integrating (10) from $r-1$ to r we arrive at

$$\frac{c_2\nu_2}{\tilde{c}_0^p}\int_{r-1}^{r}\|\nabla u(s)\|_p^p ds \le |u(r-1)|^2 + c_{\nu_2,p}\int_{r-1}^{r}\|f(s)\|_*^{p'}ds$$

which finishes the proof. □

Lemma 21. *Consider $p=2$ and suppose that there exists $\eta\in(0,2\nu_1\lambda_1c_0^{-2})$ such that $f\in\mathcal{I}_\eta^2$. Then for any $t\in\mathbb{R}$ and $\widehat{D}\in\mathcal{D}_\eta^H$, there exists $\tau_1(\widehat{D},t)<t-3$, such that for any $\tau\le\tau_1(\widehat{D},t)$, $u_\tau\in D(\tau)$, and $u\in\Psi(\tau,u_\tau)$, it holds*

$$\begin{cases} |u(r;\tau,u_\tau)| \le \varrho_1(t) \quad \forall\, r\in[t-3,t], \\ \displaystyle\int_{r-1}^{r} |\nabla u(s;\tau,u_\tau)|^2 ds \le \varrho_2(t) \quad \forall\, r\in[t-2,t], \end{cases}$$

where

$$\varrho_1^2(t) = 1+\frac{\lambda_1}{\beta}e^{\eta(3-t)}\int_{-\infty}^{t} e^{\eta s}\|f(s)\|_*^2 ds$$

and

$$\varrho_2(t) = \frac{\lambda_1}{\eta}\varrho_1^2(t) + \frac{\lambda_1^2}{\beta\eta}\int_{t-3}^{t}\|f(s)\|_*^2 ds.$$

Proof. The first inequality follows from (11) similarly to the proof of Corollary 18. To obtain the second inequality, we integrate (12) from $r-1$ to r

$$\frac{\eta}{\lambda_1}\int_{r-1}^{r}|\nabla u(s)|^2 ds \le |u(r-1)|^2 + \frac{\lambda_1}{\beta}\int_{r-1}^{r}\|f(s)\|_*^2 ds,$$

which furnishes the desired inequality. □

Now, we are able to prove the asymptotic compactness of the process. We proceed as in the proof of Lemma 13 by using the continuous and non-increasing functions J_m and J.

Proposition 22. *Consider $p\ge 2$ and suppose that there exists $\eta>0$ (for $p=2$, $\eta\in(0,2\nu_1\lambda_1c_0^{-2})$) such that $f\in\mathcal{I}_\eta^{p'}$. Then, the process U is pullback $\widehat{B}_0$-asymptotically compact, where $\widehat{B}_0$ is the pullback $\mathcal{D}_\eta^H$-absorbing family given in Corollary 18.*

Proof. We prove the result for $p > 2$, the case $p = 2$ is analogous. Fix $t \in \mathbb{R}$ and consider a sequence $\tau_m \subset (-\infty, t-2]$ such that $\tau_m \to -\infty$ and $u_{\tau_m} \in B_0(\tau_m)$. We will prove that any sequence $\{v_m\}$, where $v_m \in U(t, \tau_m)u_{\tau_m}$ for all m, is relatively compact in H. We denote by $u_m \in \Psi(\tau_m, u_{\tau_m})$ a solution such that $u_m(t) = v_m$.

From Lemma 20 there exists $\tau_1(\widehat{B}_0, t) < t-3$ such that if $\tau_m < \tau_1(\widehat{B}_0, t)$ for $m \geq m_0(t)$, the sequence $\{u_m\}$ is bounded in $L^\infty(t-2, t; H) \cap L^p(t-2, t; V_p)$ for $m \geq m_0(t)$. Hence, $\{\frac{\partial u_m}{\partial t}\}$ is bounded in $L^{p'}(t-2, t; V_p^*)$. The Aubin-Lions compactness Theorem allows to conclude that there exists $u \in L^\infty(t-2, t; H) \cap L^p(t-2, t; V_p)$ with $\frac{\partial u}{\partial t} \in L^{p'}(t-2, t; V_p^*)$, such that, up to subsequences, the following convergences hold

$$
\begin{aligned}
u_m &\overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(t-2, t; H),\\
u_m &\rightharpoonup u \quad \text{weakly in } L^p(t-2, t; V_p),\\
\frac{\partial u_m}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } L^{p'}(t-2, t; V_p^*),\\
u_m &\to u \quad \text{strongly in } L^2(t-2, t; H),\\
u_m(s) &\to u(s) \quad \text{strongly in } H \text{ a.e. } s \in (t-2, t).
\end{aligned}
$$

Moreover, notice that $u \in C([t-2, t]; H)$ and that u is a weak solution to (1) on $(t-2, t)$.

Let $\{t_m\} \subset [t-2, t]$ be a sequence such that $t_m \to t_*$. Since $\{u_m(t_m)\}$ is bounded in H, there exists $v \in H$ such that $u_m(t_m) \rightharpoonup v$ in H (up to subsequences). As in the proof of Lemma 13, by means of the Arzelà-Ascoli Theorem, we can see, again up to subsequences, that

$$u_m \to u \quad \text{strongly in } C([t-2, t]; V_p^*).$$

Thus, $v = u(t_*)$ and

$$u_m(t_m) \rightharpoonup u(t_*) \quad \text{in } H. \tag{14}$$

To finish it suffices to show that $u_m \to u$ strongly in $C([t-1, t]; H)$. We argue by contradiction. Assume that there exist $\epsilon > 0$ and a sequence $\{t_m\} \subset [t-1, t]$ with $t_m \to t^*$ for some t^* such that

$$|u_m(t_m) - u(t^*))| \geq \epsilon \quad \forall m \geq m_0(t). \tag{15}$$

In analogous way as in the proof in Lemma 13, we introduce the functions J_m, $J : [t-2, t] \to \mathbb{R}$ defined by

$$
\begin{aligned}
J_m(r) &= |u_m(r)|^2 - 2\int_{t-2}^{r} \langle f(\theta), u_m(\theta)\rangle d\theta,\\
J(r) &= |u(r)|^2 - 2\int_{t-2}^{r} \langle f(\theta), u(\theta)\rangle d\theta,
\end{aligned}
$$

which are non-increasing and continuous, and satisfy

$$J_m(r) \to J(r) \quad \text{a.e. } r \in (t-2, t).$$

Let us fix $\varepsilon > 0$. Observe that $t^* \in [t-1, t]$ and therefore, from above and the continuity and non-increasing character of J, there exists $t-2 < \hat{t}_\varepsilon < t^*$ such that

$$\lim_{m\to\infty} J_m(\hat{t}_\varepsilon) = J(\hat{t}_\varepsilon), \tag{16}$$

and

$$0 \leq J(\hat{t}_\varepsilon) - J(t^*) \leq \varepsilon.$$

Since $t_m \to t^*$, there exists m_ε such that $\hat{t}_\varepsilon < t_m$ for all $m \geq m_\varepsilon$. Then,

$$\begin{aligned} J_m(t_m) - J(t^*) &\leq J_m(\hat{t}_\varepsilon) - J(t^*) \\ &\leq |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + |J(\hat{t}_\varepsilon) - J(t^*)| \\ &\leq |J_m(\hat{t}_\varepsilon) - J(\hat{t}_\varepsilon)| + \varepsilon \end{aligned}$$

for all $m \geq m_\varepsilon$, and consequently, by (16), $\limsup_{m\to\infty} J_m(t_m) \leq J(t^*) + \varepsilon$. Thus, as $\varepsilon > 0$ is arbitrary, we deduce that

$$\limsup_{m\to\infty} J_m(t_m) \leq J(t^*).$$

Taking into account that $t_m \to t^*$ and

$$\int_{t-2}^{t_m} \langle f(\theta), u_m(\theta)\rangle d\theta \to \int_{t-2}^{t^*} \langle f(\theta), u(\theta)\rangle d\theta,$$

from above we deduce that $\limsup_{m\to\infty} |u_m(t_m)| \leq |u(t^*)|$. This last inequality and (14) imply that $u_m(t_m) \to u(t^*)$ strongly in H, which contradicts (15). Hence, $u_m \to u$ in $C([t-1,t];H)$ which implies that the multi-valued process U is pullback $\widehat{B}_0$-asymptotically compact. □

We are now in position to establish the main result of this section.

Theorem 23. *Consider $p \geq 2$ and suppose that there exists $\eta > 0$ (for $p = 2$, $\eta \in (0, 2\nu_1\lambda_1 c_0^{-2})$) such that $f \in \mathcal{I}_\eta^{p'}$. Then, there exist the minimal pullback $\mathcal{D}_F^H$-attractor*

$$\mathcal{A}_{\mathcal{D}_F^H} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\},$$

and the minimal pullback $\mathcal{D}_\eta^H$-attractor

$$\mathcal{A}_{\mathcal{D}_\eta^H} = \{\mathcal{A}_{\mathcal{D}_\eta^H}(t) : t \in \mathbb{R}\},$$

for the process $U : \mathbb{R}_d \times H \to \mathcal{P}(H)$, and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\eta^H}(t) \subset B_0(t) \quad \forall\, t \in \mathbb{R}. \tag{17}$$

Moreover, if $f \in L^{p'}_{loc}(\mathbb{R}; V_p^*)$ satisfies

$$\sup_{s\leq 0}\Big(e^{-\eta s}\int_{-\infty}^{s} e^{\eta r}\|f(r)\|_*^{p'}dr\Big) < \infty, \tag{18}$$

then

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^H}(t) \quad \forall t \in \mathbb{R}. \tag{19}$$

Proof. Corollaries 14 and 18, and Proposition 22 guarantee that we can apply Theorem 8 to conclude the existence of $\mathcal{A}_{\mathcal{D}_\eta^H}$ and $\mathcal{A}_{\mathcal{D}_F^H}$.

The relationships established in (17) follow from Theorem 9 and Corollary 10.

Finally, the last statement is a consequence of the fact that for all $T \in \mathbb{R}$, $\bigcup_{t\leq T} B_0(t)$ is a bounded subset of H. Therefore, it follows from Corollary 10. □

Remark 24. It is not difficult to see that for any $\mu \geq \eta > 0$ the following two inclusions hold: $\mathcal{I}_\eta^{p'} \subset \mathcal{I}_\mu^{p'}$ and $\mathcal{D}_\eta^H \subset \mathcal{D}_\mu^H$. Hence, if $f \in L^{p'}_{loc}(\mathbb{R}; V_p^*)$ satisfies $f \in \mathcal{I}_\eta^{p'}$ for some $\eta > 0$, as a consequence of Theorem 23, the corresponding minimal pullback $\mathcal{D}_\mu^H-$attractor $\mathcal{A}_{\mathcal{D}_\mu^H}$ exists for any $\mu \in [\eta, +\infty)$ ($\mu \in [\eta, 2\nu_1\lambda_1 c_0^{-2})$ if $p = 2$) and $\mathcal{A}_{\mathcal{D}_\eta^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t)$ for any $t \in \mathbb{R}$.

Moreover, if f satisfies (18), then

$$\sup_{s\leq 0}\Big(e^{-\mu s}\int_{-\infty}^{s} e^{\mu r}\|f(r)\|_*^{p'}dr\Big) < \infty \quad \forall \mu \in [\eta, +\infty).$$

Therefore, by (19), we conclude that

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\eta^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) \quad \forall \mu \in [\eta, +\infty)\big(\mu \in [\eta, 2\nu_1\lambda_1 c_0^{-2}) \text{ if } p = 2\big).$$

Remark 25. The existence of absorbing family for the case $p > 2$ requires the existence of $\eta > 0$ such that $f \in \mathcal{I}_\eta^{p'}$. Observe that this gives more generality than the case $p = 2$. Indeed, if such assumption is satisfied for $p > 2$, then the associated process U on each universe $\mathcal{D}_\mu^H$ $(\mu \geq \eta)$ possesses an attractor $\mathcal{A}_{\mathcal{D}_\mu^H}$. According to Theorem 9, $\mathcal{A}_{\mathcal{D}_{\mu_1}^H}(t) \subset \mathcal{A}_{\mathcal{D}_{\mu_2}^H}(t)$ for any $t \in \mathbb{R}$ for $\eta \leq \mu_1 \leq \mu_2$. This also improves the corresponding results for the existence of pullback attractor for a p-Laplacian parabolic problem recently established in [5].

5. Pullback Attractors in V_p

In this section, we will strengthen the main conclusion of the previous section, Theorem 23, in the sense that we will show the existence of minimal pullback attractors in the V_p-norm. In addition, we will verify that all these families of attractors are the same object under reasonable assumptions.

From now on, we assume that the tensor stress $\mathbb{S}$ has a potential, i.e., there exists a function Φ satisfying (5). This assumption will allow to obtain some estimates for the time partial derivative of the solution u. Moreover, in order to perform our analysis we consider

$$p \geq 12/5 \quad \text{if } n = 3 \text{ and } p > 2 \quad \text{if } n = 2.$$

This range of p guarantees that problem (1) has a unique weak solution with initial data in V_p, cf. [15, Theorem 7.2] and [2, Theorems 3.2 and 3.3]. Hence we can define the process U on V_p, for each $(t, \tau) \in \mathbb{R}_d^2$.

Proposition 26. *Let $f \in L_{loc}^2(\mathbb{R}; L^2(\Omega)^n)$. Then, the map $U : \mathbb{R}_d^2 \times V_p \to V_p$ defined by $U(t, \tau)u_\tau = u(t; \tau, u_\tau)$, where u is the unique weak solution to (1), is a closed process on V_p.*

Proof. $u_\tau \in V_p$ leads to the solution $u \in C([\tau, T]; H) \cap L^\infty(\tau, T; V_p)$. By [1, Theorem 1.6] it follows that $u \in C_w((\tau, T]; V_p)$. Hence, $u(t)$ is defined on V_p, for each $t \in (\tau, T]$. Then the process U is well-defined on V_p.

To see that the process is closed, let $t \in \mathbb{R}$ with $\tau < t$ be given and suppose that $\{u_\tau^k\}$ is a sequence in V_p with $u_\tau^k \to u_\tau$ in V_p, as $k \to \infty$. Assume also that $U(t, \tau)u_\tau^k = u(t; \tau; u_\tau^k) \to v$ in V_p, as $k \to \infty$. We will show that, $v = U(t, \tau)u_\tau$. Indeed, we know by (3.6) in [2] that $U(t, \tau)u_\tau^k \to U(t, \tau)u_\tau$ in H, as $k \to \infty$. Since $V_p \hookrightarrow H$, by the uniqueness of limit, there follows that $v = U(t, \tau)u_\tau$. □

We next introduce the universes in V_p.

Definition 27. (Universes in V_p) For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma^{H,V_p}$ the class of all families $\widehat{D}_{V_p} \subset \mathcal{P}(V_p)$ of the form $\widehat{D}_{V_p} = \{D(t) \cap V_p : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\sigma^H$.

Accordingly, we will denote by $\mathcal{D}_F^{V_p}$ the class of all families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of V_p. Observe that, for any $\sigma > 0$, $\mathcal{D}_F^{V_p} \subset \mathcal{D}_\sigma^{H,V_p}$ and that D_σ^{H,V_p} is inclusion-closed.

Notice that Lemma 15 entails the existence of a $\mathcal{D}_\eta^{H,V_p}$-absorbing family.

Corollary 28. *Assume that $f \in L_{loc}^2(\mathbb{R}; L^2(\Omega)^n)$ satisfies $f \in \mathcal{I}_\eta^{p'}$ for some $\eta > 0$. Then,*

$$\widehat{B}_{0,V_p} = \{B_{0,V_p}(t) = \overline{B}_H(0, \mathcal{R}(t)) \cap V_p : t \in \mathbb{R}\},$$

where $\mathcal{R}(t)$ is given in (13), belongs to $\mathcal{D}_\eta^{H,V_p}$ and satisfies that for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\eta^H$, there exists $\tau(\widehat{D}, t) < t$ such that

$$U(t, \tau)D(\tau) \subset B_{0,V_p}(t) \quad \textit{for all } \tau \leq \tau(\widehat{D}, t).$$

In particular, the family $\widehat{B}_{0,V_p}$ is pullback $\mathcal{D}_\eta^{H,V_p}$-absorbing for the process $U : \mathbb{R}_d^2 \times V_p \to V_p$.

We next establish some uniform estimates in finite time-intervals when the initial time is shifted pullback appropriately.

Lemma 29. *Assume that $f \in L^2_{loc}(\mathbb{R}; L^2(\Omega)^n)$ satisfies $f \in \mathcal{I}^{p'}_\eta$ for some $\eta > 0$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\eta$, there exists $\tau_1(\widehat{D}, t) < t - 2$ such that, for all $\tau \le \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds*

$$\begin{aligned} \|u(r;\tau,u_\tau)\|_{1,p} &\le \varrho_3(t) \quad \forall\, r \in [t-2,t], \\ \int_{r-1}^r \Big|\frac{\partial u}{\partial t}(\theta;\tau,u_\tau)\Big|^2 d\theta &\le \varrho_4(t) \quad \forall\, r \in [t-1,t], \end{aligned} \tag{20}$$

where

$$\varrho_3(t) = \max\{\varsigma_1(t), \varsigma_2(t), \varsigma_3(t)\} \quad \text{if } n = 3, \qquad \varrho_3(t) = \varsigma_4(t) \quad \text{if } n = 2,$$

with $\varsigma_i(t)$, $i = 1, \dots, 4$ are given in (23), (25), (26), and (27) respectively, and

$$\varrho_4(t) = 2c_8(1 + (\varrho_3(t))^p) + 2(\varrho_3(t))^2 \varsigma_5(t) + 2\int_{t-2}^t |f(\theta)|^2 d\theta$$

where $\varsigma_5(t)$ is given in (28) if $n = 3$ or in (29) if $n = 2$.

Proof. The proof will be performed at level of the Galerkin approximations of the solution u (for short we omit the subscript of the Galerkin approximation). In this way, $\frac{\partial u}{\partial t}$ can be taken as test function in the weak formulation

$$\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{d}{dt}\int_\Omega \Phi(Du)dx + \Big(u \cdot \nabla u, \frac{\partial u}{\partial t}\Big) = \Big(f, \frac{\partial u}{\partial t}\Big),$$

where we have used that $\mathbb{S}$ has a potential.

By using Hölder and Young inequalities we have that

$$\frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{d}{dt}\|\Phi(Du)\|_1 \le |f|^2 + \|u\|^2_{1,p}\|u\|^2_{\frac{2p}{p-2}}. \tag{21}$$

We notice that, from (6), there exist positive constants c_7 and c_8 such that

$$c_7\|u\|^p_{1,p} \le \|\Phi(Du)\|_1 \le c_8(1 + \|u\|^p_{1,p}). \tag{22}$$

To estimate the RHS of (21), we split the proof into three cases: $n = 3$ with $12/5 \le p < 3$, $n = 3$ with $p \ge 3$, and $n = 2$ with $p > 2$.

Case (i): $n = 3$ and $12/5 \le p < 3$. By the interpolation inequality, it holds

$$\|u\|^2_{\frac{2p}{p-2}} \le \widehat{C}\|u\|^{\frac{12}{5p-6}}_{1,p}|u|^{\frac{2(5p-12)}{5p-6}},$$

where $\widehat{C}$ is a constant that depends on Ω and p. Hence, by using (22), we can estimate

$$\begin{aligned} \|u\|^2_{1,p}\|u\|^2_{\frac{2p}{p-2}} &\le \widehat{C}\|u\|^p_{1,p}\|u\|^{\frac{p(16-5p)}{(5p-6)}}_{1,p}|u|^{\frac{2(5p-12)}{(5p-6)}} \\ &\le \frac{\widehat{C}}{c_7}\|\Phi(Du)\|_1\|u\|^{\frac{p(16-5p)}{(5p-6)}}_{1,p}|u|^{\frac{2(5p-12)}{(5p-6)}}. \end{aligned}$$

By plugging this inequality into (21), we arrive at

$$\frac{d}{dt}\|\Phi(Du)\|_1 \le |f|^2 + \frac{\widehat{C}}{c_7}\|\Phi(Du)\|_1\|u\|^{\frac{p(16-5p)}{(5p-6)}}_{1,p}|u|^{\frac{2(5p-12)}{(5p-6)}}.$$

By integrating in time from s to r, where $r \in [t-2,t]$ and $s \in [r-1,r]$, and applying the Gronwall Lemma one has

$$\|\Phi(Du(r))\|_1 \le \Big(\|\Phi(Du(s))\|_1 + \int_{r-1}^r |f(\theta)|^2 d\theta\Big)\exp\Big(\frac{\widehat{C}}{c_7}\int_{r-1}^r \|u(\theta)\|^{\frac{p(16-5p)}{(5p-6)}}_{1,p}|u(\theta)|^{\frac{2(5p-12)}{(5p-6)}} d\theta\Big).$$

Now, integrating in s from $r-1$ to r yields

$$\|\Phi(Du(r))\|_1 \leq \Big(\int_{r-1}^{r} \|\Phi(Du(s))\|_1 ds + \int_{r-1}^{r} |f(\theta)|^2 d\theta\Big) \exp\Big(\frac{\widehat{C}}{c_7}\int_{r-1}^{r} \|u(\theta)\|_{1,p}^{\frac{p(16-5p)}{5p-6}} |u(\theta)|^{\frac{2(5p-12)}{5p-6}} d\theta\Big).$$

By (22) and Lemma 20 we deduce that

$$\|u(r;\tau,u_\tau)\|_{1,p} \leq \varsigma_1(t),$$

for all $r\in[t-2,t]$, $\tau\leq\tau_1(\widehat{D},t)$, and $u_\tau\in D(\tau)$, where

$$\varsigma_1^p(t) = \frac{1}{c_7}\Big[c_8\big(1+\varrho_2(t)\big) + \int_{t-3}^{t} |f(\theta)|^2 d\theta\Big] \exp\Big(\frac{\widehat{C}}{c_7}\big(\varrho_1(t)\big)^{\frac{2(5p-12)}{(5p-6)}}\big(\varrho_2(t)\big)^{\frac{16-5p}{5p-6}}\Big). \tag{23}$$

Case (ii): $n=3$ and $p\geq 3$. Let $\widetilde{C}$ be the constant of the embedding $W^{1,p}(\Omega)^n \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)^n$. From (21) we have that

$$\frac{d}{dt}\|\Phi(Du)\|_1 \leq |f|^2 + \widetilde{C}^2\|u\|_{1,p}^4.$$

Let us denote $\mathcal{U} = 1+\|\Phi(Du)\|_1$. Therefore, from (22) one has

$$\frac{d}{dt}\mathcal{U} \leq |f|^2 + \frac{\widetilde{C}^2}{c_7^{4/p}}\mathcal{U}^{4/p}. \tag{24}$$

Now, if $p\geq 4$, since $\mathcal{U}\geq 1$, then $\mathcal{U}^{4/p}\leq\mathcal{U}$. Hence,

$$\frac{d}{dt}\mathcal{U} \leq |f|^2 + \frac{\widetilde{C}^2}{c_7^{4/p}}\mathcal{U}.$$

Integrating from s to r, with $r\in[t-2,t]$ and $s\in[r-1,r]$, there follows

$$\mathcal{U}(r) \leq \mathcal{U}(s) + \int_{r-1}^{r}|f(\theta)|^2 d\theta + \frac{\widetilde{C}^2}{c_7^{4/p}}\int_{r-1}^{r}\mathcal{U}(\theta)d\theta.$$

Integrating again in s from $r-1$ to r, we have that

$$\mathcal{U}(r) \leq \int_{r-1}^{r}\mathcal{U}(s)ds + \int_{r-1}^{r}|f(\theta)|^2 d\theta + \frac{\widetilde{C}^2}{c_7^{4/p}}\int_{r-1}^{r}\mathcal{U}(\theta)d\theta.$$

By (22) and Lemma 20 we find

$$\|u(r;\tau,u_\tau)\|_{1,p} \leq \varsigma_2(t),$$

for all $r\in[t-2,t]$, $\tau\leq\tau_1(\widehat{D},t)$ and $u_\tau\in D(\tau)$, where

$$\varsigma_2^p(t) = \frac{1}{c_7}\Big[\Big(1+\frac{\widetilde{C}^2}{c_7^{4/p}}\Big)\Big(1+c_8\big(1+\varrho_2(t)\big)\Big) + \int_{t-3}^{t}|f(\theta)|^2 d\theta\Big]. \tag{25}$$

If $3\leq p<4$, let us consider $\gamma=\frac{2p-4}{p}$. Observe that $\gamma\in[\frac{2}{3},1)$. Multiplying (24) by $\mathcal{U}^{\gamma-1}$ we obtain that

$$\frac{1}{\gamma}\frac{d}{dt}(\mathcal{U}^\gamma) \leq |f|^2\mathcal{U}^{\gamma-1} + \frac{\widetilde{C}^2}{c_7^{4/p}}\mathcal{U} \leq |f|^2 + \frac{\widetilde{C}^2}{c_7^{4/p}}\mathcal{U},$$

since $\mathcal{U}^{\gamma-1}\leq 1$. Similarly as before, integrating twice, we arrive at

$$\frac{1}{\gamma}\mathcal{U}^\gamma(r) \leq \Big(\frac{1}{\gamma}+\frac{\widetilde{C}^2}{c_7^{4/p}}\Big)\int_{r-1}^{r}\mathcal{U}(s)ds + \int_{t-3}^{t}|f(\theta)|^2 d\theta,$$

for all $r \in [t-2, t]$. We conclude from Lemma 20 using (22) that

$$\|u(r;\tau,u_\tau)\|_{1,p} \le \varsigma_3(t),$$

for all $r \in [t-2,t]$, $\tau \le \tau_1(\widehat{D},t)$, and $u_\tau \in D(\tau)$, where

$$\varsigma_3^p(t) = \frac{1}{c_7}\Big(\gamma\Big[\Big(\frac{1}{\gamma} + \frac{\widetilde{C}^2}{c_7^{4/p}}\Big)\Big(1 + c_8\big(1+\varrho_2(t)\big)\Big) + \int_{t-3}^t |f(\theta)|^2 d\theta\Big]\Big)^{\frac{1}{\gamma}}. \tag{26}$$

Case (iii): $n = 2$ and $p > 2$. Observe that $\frac{2p}{p-2} > 2$. Then, by the interpolation inequality,

$$\|u\|^2_{\frac{2p}{p-2}} \le \overline{C}_p |\nabla u|^{\frac{4}{p}} |u|^{\frac{2(p-2)}{p}} \le \overline{C}\|u\|_{1,p}^{\frac{4}{p}} |u|^{\frac{2(p-2)}{p}}.$$

Integrating (21) from s to r, where $r \in [t-2,t]$ and $s \in [r-1, r]$,

$$\|\Phi(Du(r))\|_1 \le \|\Phi(Du(s))\|_1 + \int_{r-1}^r |f(\theta)|^2 d\theta + \overline{C}\int_{r-1}^r \|u(\theta)\|_{1,p}^{\frac{4+2p}{p}} |u(\theta)|^{\frac{2(p-2)}{p}} d\theta.$$

Integrating in s from $r-1$ to r and, similarly as before, by using Lemma 20 and inequality (22) we arrive at

$$\|u(r;\tau,u_\tau)\|_{1,p} \le \varsigma_4(t),$$

for all $r \in [t-2,t]$, $\tau \le \tau_1(\widehat{D},t)$ and $u_\tau \in D(\tau)$, where

$$\varsigma_4^p(t) = \frac{1}{c_7}\Big(c_8(1+\varrho_2(t)) + \overline{C}(\varrho_1(t))^{\frac{2(p-2)}{p}}(\varrho_2(t))^{\frac{4+2p}{p^2}} + \int_{t-3}^t |f(\theta)|^2 d\theta\Big). \tag{27}$$

Hence, (20) is proved.

Finally, integrating (21) from $r-1$ to r with $r \in [t-1,t]$, and using (20) and (22),

$$\begin{aligned}\int_{r-1}^r \Big|\frac{\partial u}{\partial t}(\theta)\Big|^2 d\theta &\le 2\|\Phi(Du(r-1))\|_1 + 2\int_{r-1}^r |f(\theta)|^2 d\theta + 2\int_{r-1}^r \|u(\theta)\|_{1,p}^2 \|u(\theta)\|^2_{\frac{2p}{p-2}} d\theta \\ &\le 2c_8\big(1 + (\varrho_3(t))^p\big) + 2\int_{t-2}^t |f(\theta)|^2 d\theta + 2(\varrho_3(t))^2 \int_{r-1}^r \|u(\theta)\|^2_{\frac{2p}{p-2}} d\theta.\end{aligned}$$

By using interpolation as in the previous cases, we can estimate

$$\|u(\theta)\|^2_{\frac{2p}{p-2}} \le \varsigma_5(t) = \max\{\widehat{C}(\varrho_3(t))^{\frac{12}{5p-6}}(\varrho_1(t))^{\frac{2(5p-12)}{5p-6}}, \widetilde{C}^2(\varrho_3(t))^2\} \text{ if } n = 3 \tag{28}$$

and

$$\|u(\theta)\|^2_{\frac{2p}{p-2}} \le \varsigma_5(t) = \overline{C}(\varrho_3(t))^{\frac{4}{p}}(\varrho_1(t))^{\frac{2(p-2)}{2}} \text{ if } n = 2. \tag{29}$$

In this way, we deduce that

$$\int_{r-1}^r \Big|\frac{\partial u}{\partial t}(\theta;\tau,u_\tau)\Big|^2 d\theta \le \varrho_4(t)$$

for all $r \in [t-1,t]$, $\tau \le \tau_1(\widehat{D},t)$, and $u_\tau \in D(\tau)$, where ϱ_4 is given in the statement. This finishes the proof. □

We make a further estimate for the time partial derivative $\frac{\partial u}{\partial t}$ of the solution. To this end we require more regularity on the external force.

Lemma 30. *Assume that $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega)^n)$ satisfies $f \in \mathcal{I}_\eta^{p'}$ for some $\eta > 0$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\eta^H$, there exists $\tau_1(\widehat{D},t) < t-2$, such that for all $\tau \le \tau_1(\widehat{D},t)$ and any $u_\tau \in D(\tau)$, it holds*

$$\Big|\frac{\partial u}{\partial t}(r;\tau,u_\tau)\Big| \le \varrho_5(t) \ \forall\ r \in [t-1,t],$$

where

$$\varrho_5^2(t) = \varrho_4(t)\Big[2 + C\big(\varrho_3(t)\big)^{\frac{2p}{2p-3}}\Big] + \int_{t-2}^{t}\Big|\frac{\partial f}{\partial t}(\theta)\Big|^2 d\theta \ \textit{if } n = 3,$$

and

$$\varrho_5^2(t) = \varrho_4(t)\Big[2 + C\big(\varrho_3(t)\big)^{\frac{p}{p-1}}\Big] + \int_{t-2}^{t}\Big|\frac{\partial f}{\partial t}(\theta)\Big|^2 d\theta \ \textit{if } n = 2,$$

being $C = C(p, \nu_1, c_0, \widehat{C})$.

Proof. We will make formal computations that can be justified by using the Galerkin approximations of the solution u. By differentiating in time the first equation in (1), we have

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\Big(\partial_D^2\Phi(Du)D\Big(\frac{\partial u}{\partial t}\Big)\Big) + \operatorname{div}\Big(\frac{\partial u}{\partial t}\otimes u + u\otimes\frac{\partial u}{\partial t}\Big) + \nabla\Big(\frac{\partial p}{\partial t}\Big) = \frac{\partial f}{\partial t}.$$

Multiplying by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain that

$$\frac{1}{2}\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|^2 + \int_\Omega \partial_D^2\Phi(Du)D\Big(\frac{\partial u}{\partial t}\Big) : D\Big(\frac{\partial u}{\partial t}\Big)dx + \int_\Omega\Big(\frac{\partial u}{\partial t}\otimes\frac{\partial u}{\partial t}\Big) : \nabla u dx = \Big(\frac{\partial f}{\partial t}, \frac{\partial u}{\partial t}\Big),$$

where we have used that $\operatorname{div}\big(\frac{\partial u}{\partial t}\big) = 0$.

Bu using the properties of Φ (5) and the Hölder inequality, there follows

$$\frac{1}{2}\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|^2 + \nu_1\int_\Omega(1+\mu|Du|)^{p-2}\Big|D\Big(\frac{\partial u}{\partial t}\Big)\Big|^2 dx \le \frac{1}{2}\Big|\frac{\partial f}{\partial t}\Big|^2 + \frac{1}{2}\Big|\frac{\partial u}{\partial t}\Big|^2 + \|u\|_{1,p}\Big\|\frac{\partial u}{\partial t}\Big\|_{2p'}^2. \tag{30}$$

Observe that, by the Korn inequality,

$$\frac{1}{c_0^2}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^2 \le \int_\Omega\Big|D\Big(\frac{\partial u}{\partial t}\Big)\Big|^2 dx \le \int_\Omega(1+\mu|Du|)^{p-2}\Big|D\Big(\frac{\partial u}{\partial t}\Big)\Big|^2 dx$$

and that, by the interpolation inequality, we have that

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{2p'}^2 \le \widehat{C}\Big|\frac{\partial u}{\partial t}\Big|^{\frac{2p-3}{p}}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^{\frac{3}{p}} \quad \text{if } n = 3$$

and

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{2p'}^2 \le \widehat{C}\Big|\frac{\partial u}{\partial t}\Big|^{\frac{2(p-1)}{p}}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^{\frac{2}{p}} \quad \text{if } n = 2.$$

By using the Hölder and Young inequalities in (30) we arrive at

$$\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{\nu_1}{c_0^2}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^2 \le \Big|\frac{\partial f}{\partial t}\Big|^2 + \Big|\frac{\partial u}{\partial t}\Big|^2 + C\|u\|_{1,p}^{\frac{2p}{2p-3}}\Big|\frac{\partial u}{\partial t}\Big|^2 \quad \text{if } n = 3$$

and

$$\frac{d}{dt}\Big|\frac{\partial u}{\partial t}\Big|^2 + \frac{\nu_1}{c_0^2}\Big\|\frac{\partial u}{\partial t}\Big\|_{1,2}^2 \le \Big|\frac{\partial f}{\partial t}\Big|^2 + \Big|\frac{\partial u}{\partial t}\Big|^2 + C\|u\|_{1,p}^{\frac{p}{p-1}}\Big|\frac{\partial u}{\partial t}\Big|^2 \quad \text{if } n = 2,$$

where $C = C(p, \nu_1, c_0, \widehat{C})$.

Integrating first from s to r, with $s \in [r-1, r]$ and then integrating in s from $r-1$ to r,

$$\Big|\frac{\partial u}{\partial t}(r)\Big|^2 \le 2\int_{r-1}^{r}\Big|\frac{\partial u}{\partial t}(s)\Big|^2 ds + \int_{r-1}^{r}\Big|\frac{\partial f}{\partial t}(\theta)\Big|^2 d\theta + C\int_{r-1}^{r}\|u(\theta)\|_{1,p}^{\frac{2p}{2p-3}}\Big|\frac{\partial u}{\partial t}(\theta)\Big|^2 d\theta,$$

for $n = 3$ and similarly for $n = 2$. The conclusion follows from Lemma 29. □

To prove the pullback asymptotic compactness in V_p we follow some ideas used in [26] for the semigroup of the p-Laplacian equation.

Proposition 31. *Assume that* $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega)^n)$ *satisfies* $f \in \mathcal{I}_\eta^{p'}$ *for some* $\eta > 0$. *Then, the process* U *is pullback* $\mathcal{D}_\eta^{H,V_p}$*-asymptotically compact.*

Proof. Let $\widehat{D} \in \mathcal{D}_\eta^{H,V_p}$ and $t \in \mathbb{R}$ be given. Let $u_m(t) = U(t,\tau_m)u_{\tau_m}$ be the unique weak solution to (1), where $u_{\tau_m} \in D(\tau_m)$ for each $m \in \mathbb{N}$ and $\tau_m \to -\infty$. From Lemma 29, we know that $\{u_m(t)\}$ is relatively compact in H. Without loss of generality, we can assume that $\{u_m(t)\}$ is a Cauchy sequence in H. In what follows, we prove that $\{u_m(t)\}$ is a Cauchy sequence in V_p.

We first observe that, from the p-coercivity of $\mathbb{S}$ (2) and the Korn inequality,

$$\frac{\nu_1 \mu^{p-2}}{\tilde{c}_0^p}\|u-v\|_{1,p}^p \le \big(\mathbb{S}(Du)-\mathbb{S}(Dv), Du-Dv\big)$$

for all $u, v \in V_p$.

Therefore, by Lemmas 29 and 30, we have that

$$\begin{aligned}\frac{\nu_1 \mu^{p-2}}{\tilde{c}_0^p}\|u_k-u_l\|_{1,p}^p &\le \big(\mathbb{S}(Du_k)-\mathbb{S}(Du_l), Du_k-Du_l\big)\\ &= -\Big(\frac{\partial u_k}{\partial t}-\frac{\partial u_l}{\partial t}, u_k-u_l\Big)+\Big(\big(u_k\otimes u_k-u_l\otimes u_l\big), \nabla(u_k-u_l)\Big)\\ &\le \Big|\frac{\partial u_k}{\partial t}-\frac{\partial u_l}{\partial t}\Big||u_k-u_l| + \|u_k\|_{1,p}\|u_k-u_l\|_{2p'}^2\\ &\le 2\varrho_5(t)|u_k-u_l| + \varrho_3(t)\|u_k-u_l\|_{2p'}^2.\end{aligned}$$

Lemma 29 and the interpolation inequality, for $n=3$, give us

$$\begin{aligned}\|u_k-u_l\|_{2p'}^2 &\le \widehat{C}|u_k-u_l|^{\frac{2p-3}{p}}\|u_k-u_l\|_{1,2}^{\frac{3}{p}}\\ &\le \widehat{C}_p|u_k-u_l|^{\frac{2p-3}{p}}2(\varrho_3(t))^{\frac{3}{p}}\end{aligned}$$

and, for $n=2$,

$$\begin{aligned}\|u_k-u_l\|_{2p'}^2 &\le \widehat{C}|u_k-u_l|^{\frac{2(p-1)}{p}}\|u_k-u_l\|_{1,2}^{\frac{2}{p}}\\ &\le \widehat{C}_p|u_k-u_l|^{\frac{2(p-1)}{p}}(\varrho_3(t))^{\frac{2}{p}}.\end{aligned}$$

Plugging these estimates into the previous one completes the proof. □

We are able to obtain the existence of minimal pullback attractors for the process U in V_p.

Theorem 32. *Assume that $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\Omega)^n)$ satisfies $f \in \mathcal{I}_\eta^{p'}$ for some $\eta > 0$. Then, there exist the minimal pullback $\mathcal{D}_F^{V_p}$-attractor $\mathcal{A}_{\mathcal{D}_F^{V_p}} = \{\mathcal{A}_{\mathcal{D}_F^{V_p}}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_\eta^{H,V_p}$-attractor $\mathcal{A}_{\mathcal{D}_\eta^{H,V_p}} = \{\mathcal{A}_{\mathcal{D}_\eta^{H,V_p}}(t) : t \in \mathbb{R}\}$ for the closed process $U : \mathbb{R}^2_d \times V_p \to V_p$. The minimal pullback $\mathcal{D}_\eta^{H,V_p}$-attractor belongs to $\mathcal{D}_\eta^{H,V_p}$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^{V_p}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{H}}(t) \subset \mathcal{A}_{\mathcal{D}_\eta^{H}}(t) = \mathcal{A}_{\mathcal{D}_\eta^{H,V_p}}(t) \quad \forall t \in \mathbb{R}, \tag{31}$$

where $\mathcal{A}_{\mathcal{D}_F^H}$ and $\mathcal{A}_{\mathcal{D}_\eta^H}$ are the minimal pullback $\mathcal{D}_F^H$-attractor and the minimal pullback $\mathcal{D}_\eta^H$-attractor, respectively, for the process $U : \mathbb{R}^2_d \times H \to H$, whose existence is guaranteed by Theorem 23.

Finally, if $f \in \mathcal{I}_\eta^{p'}$ satisfies (18), then all the above inclusions are actually equalities.

Proof. The existence of the pullback attractor for the closed process U on V_p, in the universe $\mathcal{D}_\eta^{H,V_p}$, follows from Theorem 8, Corollary 28, Proposition 31, and Remark 11. The existence of pullback attractor in the universe $\mathcal{D}_F^{V_p}$ with the inclusion (31) is given by Corollary 10 and Theorem 9. The last assessment is a consequence of Corollary 10. □

Remark 33. By (31), in particular, the following pullback attraction result in V_p holds

$$\lim_{\tau\to-\infty} dist_{V_p}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\eta^H}(t)) = 0 \quad \text{for all } t \in \mathbb{R} \text{ and any } \widehat{D} \in \mathcal{D}_\eta^H,$$

and, if f satisfies (18), then

$$\lim_{\tau\to-\infty} dist_{V_p}(U(t,\tau)D, \mathcal{A}_{\mathcal{D}_F^H}(t)) = 0 \text{ for all } t\in\mathbb{R} \text{ and any } D\in\mathcal{D}_F^H.$$

Taking into account Remarks 24 and 25, since all throughout this section $p > 2$, we conclude

Corollary 34. *Under the assumptions of Theorem 32, there exists pullback $\mathcal{D}_\mu^{H,V_p}$-attractors $\mathcal{A}_{\mathcal{D}_\mu^{H,V_p}} \subset \mathcal{D}_\mu^{H,V_p}$ for any $\mu \geq \eta$, and analogous relationships as in Theorem 32 hold. Moreover, their time-sections satisfy $\mathcal{A}_{\mathcal{D}_{\mu_1}^{H,V_p}}(t) \subset \mathcal{A}_{\mathcal{D}_{\mu_2}^{H,V_p}}(t)$ for any $t\in\mathbb{R}$ and $\eta\leq\mu_1\leq\mu_2$.*

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