A NEW SIMPLE PROOF FOR THE LUM-CHUA'S CONJECTURE

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Abstract. In this paper, using the theory of inverse integrating factor, we provide a new simple proof for the Lum-Chua's conjecture, which says that a continuous planar piecewise linear differential system with two zones separated by a straight line has at most one limit cycle. In addition, we prove that if this limit cycle exists, then it is hyperbolic and its stability is characterized in terms of the parameters. To the best of our knowledge, the hyperbolicity of the limit cycle has not been pointed out before.

1. Introduction

The study of limit cycles in planar piecewise linear differential systems dates back to Minorsky [13] in 1962, and Andronov et al. [1] in 1966. Since them, these systems have received a lot of attention by the scientific community mainly because of their wide range of application in applied science as idealization of nonlinear phenomenon. The following continuous planar piecewise linear differential system with two zones separated by a straight line is the simplest possible configuration for a piecewise linear differential system,

$$\dot{\mathbf{x}} = \begin{cases} A_L\mathbf{x} + \mathbf{b}, & \text{if} \quad x_1 \leqslant 0, \\ A_R\mathbf{x} + \mathbf{b}, & \text{if} \quad x_1 \geqslant 0. \end{cases} \tag{1}$$

Here, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $A_{L,R} = (a_{ij}^{L,R})_{2\times 2}$, with $a_{12}^L = a_{12}^R = a_{12}$ and $a_{22}^L = a_{22}^R = a_{22}$, and $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$. In 1991, after computer experimentations, Lum and Chua [11] stated the following conjecture:

Lum-Chua's Conjecture. ([11]) *A continuous planar piecewise vector field with two zones separated by a straight line has at most one limit cycle. The limit cycle, if it exists, is either attracting or repelling.*

This conjecture was proved in 1998 by Freire et al. [4]. Their proof is performed on a large casuistic, which distinguishes every possible configurations depending on the spectrums of the matrices A_L and A_R. In 2013, Llibre et al. [10] made use of Massera's approach [12] for proving a particular case of

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this conjecture. Our main goal in this paper is to provide a new and simple proof for the Lum-Chua's conjecture. Our proof is based on the *Inverse Integrating Factor* for linear differential systems (see [2]), which provides a unified way to deal with the problem, avoiding the large casuistic of the former proof. In addition, we also prove that the limit cycle, if it exists, is hyperbolic and, consequently, either attracting or repelling. To the best of our knowledge, the hyperbolicity of the limit cycle has not been pointed out before.

Accordingly, the Lum-Chua's Conjecture follows straightforwardly from the next theorem, which is the main result of this paper.

Theorem 1. *The continuous planar piecewise vector field* (1) *has at most one limit cycle, which is hyperbolic if it exists. Moreover, in this case,* $(a_{12}b_2 - a_{22}b_1)\mathrm{tr}(A_L) \neq 0$ *and the limit cycle is attracting* (*resp. repelling*) *provided that* $(a_{12}b_2 - a_{22}b_1)\mathrm{tr}(A_L) < 0$ (*resp.* $(a_{12}b_2 - a_{22}b_1)\mathrm{tr}(A_L) > 0$), *where* tr *stands for the trace of the matrix.*

This paper is structured as follows. First, in Section 2 we introduce all the preliminary results needed to prove Theorem 1. More specifically, in Section 2.1, we introduce the *Liénard Normal Form* for continuous piecewise linear differential systems; and, in Section 2.2, we introduce an *Inverse Integrating Factor* for linear differential systems in the Liénard form and we show how to use it for describing the Poincaré half-maps. Finally, Section 3 is completly devoted to the proof of Theorem 1.

2. Normal Form and Inverse Integrating Factor

This section is devoted to introduce some preliminary results. First, we introduce the *Liénard Normal Form* for continuous piecewise linear differential systems. Then, we discuss an application of the *Inverse Integrating factor* for linear differential systems.

2.1. Liénard Normal Form.

One can readily see that $a_{12} \neq 0$ is a necessary condition for the existence of periodic solutions of system (1). In this case, from [3], the linear change of variables $(x, y) = (x_1, a_{22}x_1 - a_{12}x_2 - b_1)$ transforms system (1) into

$$(2) \qquad \begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a \end{cases} \quad \text{for} \quad x < 0, \qquad \begin{cases} \dot{x} = T_R x - y \\ \dot{y} = D_R x - a \end{cases} \quad \text{for} \quad x \geqslant 0,$$

where $T_{L,R} = \mathrm{tr}(A_{L,R})$, $D_{L,R} = \det(A_{L,R})$, and $a = a_{12}b_2 - a_{22}b_1$. Notice that any limit cycle of system (2) is anti-clockwise oriented and crosses the switching set $\Sigma = \{x = 0\}$ twice.

2.2. **Inverse Integrating Factors and Poincaré Half-Maps.** Inverse integrating factors have been used to study limit cycles of planar smooth differential systems. In [9, 8], the relationship between limit cycles and the zero set of an inverse integrating factor is stablished. More information on this topic can be found in the survey [5]. Here, inverse integrating factors are used to describe Poincaré half-maps of piecewise linear differential systems as performed in [2].

Consider the following linear differential system

$$\begin{cases} \dot{x} = & Tx - y, \\ \dot{y} = & Dx - a, \end{cases} \tag{3}$$

and take the section $\Sigma = \{x = 0\}$. Here, we are interested in characterizing Poincaré half-maps of system (3) associated to Σ. Roughly speaking, given $y_0 \geqslant 0$, let $y_1^+(y_0) \leqslant 0$ be defined as the y-coordinate of the first return to Σ of the forward trajectory of system (3) starting at $(0, y_0)$. Analogously, we define the map $y_1^-(y_0)$ for the backward trajectory. We call y_1^+ and y_1^- by *Forward Poincaré Half-Map* and *Backward Poincaré Half-Map*, respectively.

It is easy to see that if $a = D = 0$, the Poincaré half-maps $y_1^\pm$ cannot be defined. In [2], assuming that $a^2 + D^2 \neq 0$, an *Inverse Integrating Factor* for (3) is given by

$$V(x, y) = D\big(Dx^2 - Txy + y^2\big) + a\big((T^2 - 2D)x - Ty + a\big). \tag{4}$$

There, it is proved that

$$PV\left\{\int_{y_1^\pm(y_0)}^{y_0} \frac{-y}{V(0, y)} dy\right\} = c^\pm, \tag{5}$$

where $PV\{\cdot\}$ stands for the *Cauchy Principal Value* and contants $c^\pm$ depend only on the parameters of system (3). Moreover, for $a \neq 0$, $V(0, y) > 0$ for every $y \in [y_1^\pm(y_0), y_0]$, whenever $y_1^\pm(y_0)$ are defined.

It is worthwhile to mention that the above integral diverges when $a = 0$. Thus, in this case, the Cauchy principal value is necessary to overcome this difficulty (see, for instance, [6, 7] for more applications of the Cauchy integral value to planar vector fields). When $a \neq 0$, the above integral does not diverge and, consequently, the Cauchy principal value just takes the value of the integral.

Computing the derivative of (5) with respect to y_0, one can see that the graph of each Poincaré half-map $y_1^\pm(y_0)$ is an orbit the following vector field

$$X(y_0, y_1) = -\big(y_1 V(0, y_0), y_0 V(0, y_1)\big). \tag{6}$$

3. Proof of Theorem 1

Consider system (1). As discussed in Section 2.1, assuming $a_{12} \neq 0$, which is a necessary condition for the existence of periodic solutions, system (1) is transformed into system (2) through a linear change of variables. In this case, we are denoting $T_{L,R} = \mathrm{tr}(A_{L,R})$, $D_{L,R} = \det(A_{L,R})$, and $a = a_{12}b_2 - a_{22}b_1$.

It is a simple consequence of the Green's Theorem that no periodic orbit exists when $T_L T_R > 0$ (see [4]). It is also easy to see that no limit cycle can exist when either the system is homogeneous, i.e. $a = 0$, or $T_L T_R = 0$. Thus, for the sake of our interest, it is sufficient to assume that $a \neq 0$ and $T_L T_R < 0$.

Now, from (4), we define the inverse integrating factor for the linear systems defining the piecewise linear vector field (2) for $x < 0$ and $x > 0$, respectively, as

$$
\begin{aligned}
V_L(x,y) &= D_L\big(D_L x^2 - T_L xy + y^2\big) + a\big((T_L^2 - 2D_L)x - T_L y + a\big),\\
V_R(x,y) &= D_R\big(D_R x^2 - T_R xy + y^2\big) + a\big((T_R^2 - 2D_R)x - T_R y + a\big).
\end{aligned}
\tag{7}
$$

Let $y_L(y_0)$ (resp. $y_R(y_0)$) be the forward (resp. backward) Poincaré half-map associated with the planar system (2) for $x < 0$ (resp. $x > 0$) and let $I_L \subset \mathbb{R}_{\geqslant 0}$ (resp. $I_R \subset \mathbb{R}_{\geqslant 0}$) be its interval of definition. In addition, for $a \neq 0$, $V_{L,R}(0,y) > 0$ for every $y \in [y_{L,R}(y_0), y_0]$, $y_0 \in I_{L,R}$. Obviously, no periodic solution can exist when $I_L \cap I_R = \emptyset$. Thus, for $y_0 \in I := I_L \cap I_R$, define the displacement function $\delta(y_0) = y_R(y_0) - y_L(y_0)$. Clearly, $\delta(y_0^*) = 0$ if, and only if, there exists a periodic orbit passing through $(0, y_0^*)$ and $(0, y_1^*)$, $y_1^* = y_R(y_0^*) = y_L(y_0^*)$. Furthermore, it is a hyperbolic limit cycle if, and only if, $\delta'(y_0^*) \neq 0$. In this case, the limit cycle is attracting (resp. repelling) provided that $\delta'(y_0^*) < 0$ (resp. $\delta'(y_0^*) > 0$). Accordingly, assuming $\delta(y_0^*) = 0$, the direction of the flow of system (2) along Σ implies that $y_0^* > 0$ and $y_1^* < 0$. From (3),

$$\delta'(y_0^*) = C(y_0^*, y_1^*) F(y_0^*, y_1^*),$$

where the functions C and F are real functions defined as

$$C(y_0, y_1) = \frac{-y_0(y_0 - y_1)}{y_1 V_R(0, y_0) V_L(0, y_0)}$$

and

$$F(y_0, y_1) = \frac{V_L(0, y_1) V_R(0, y_0) - V_L(0, y_0) V_R(0, y_1)}{y_0 - y_1}. \tag{8}$$

Substituting (7) into $F(y_0, y_1)$, we get

$$F(y_0, y_1) = a^3(T_L - T_R) + a(D_L T_R - D_R T_L) y_0 y_1 + a^2(D_R - D_L)(y_0 + y_1).$$

Since $C(y_0^*, y_1^*) > 0$, one has the sign of $\delta'(y_0^*)$ is determined by F. Notice that the curve $F^{-1}(0)$ describes a hyperbola, possibly degenerate. Denote

$$Q = \{(y_0, y_1) \in \mathbb{R}^2 : \ y_0 \geqslant 0 \text{ and } y_1 \leqslant 0\} \quad \text{and}$$
$$R_\pm = \{(y_0, y_1) \in Q : \ \text{sign}(F(y_0, y_1)) = \pm\text{sign}(aT_L)\}$$

Notice that $F(0,0) = a^3(T_L - T_R)$. Since $T_L T_R < 0$, we get $(0,0) \in R_+$.

Now, consider the curves $(y_0, y_{L,R}(y_0))$, with $y_0 \in I_{L,R}$, defined in Q. Clearly, if a periodic solution contains the points $(0, y_0)$ and $(0, y_1)$, then the point (y_0, y_1) is contained in both curves. Moreover, from (6), the graph of $y_{L,R}(y_0)$ is an orbit the following vector field

$$X_{L,R}(y_0, y_1) = -\big(y_1 V_{L,R}(0, y_0), y_0 V_{L,R}(0, y_1)\big).$$

In the sequel, we shall study the vector fields $X_{L,R}$ along the curve $F^{-1}(0)$ for $(y_0, y_1) \in \text{int}(Q)$. From (8), for $(y_0, y_1) \in \text{int}Q$, the equation $F(y_0, y_1) = 0$ is equivalent to $V_L(0, y_1)V_R(0, y_0) = V_L(0, y_0)V_R(0, y_1)$. So, substituting this last equality into $\langle \nabla F(y_0, y_1), X_{L,R}(y_0, y_1)\rangle$ and using expression (8) of F we get

$$\begin{aligned} G_{L,R}(y_0, y_1) = & \ \Big\langle \nabla F(y_0, y_1), X_{L,R}(y_0, y_1)\Big\rangle\Big|_{F^{-1}(0)} \\ = & \ V_{L,R}(0, y_1) a\big(T_L V_R(0, y_0) - T_R V_L(0, y_0)\big). \end{aligned}$$

Since $T_L T_R < 0$, we conclude that $\text{sign}(G_{L,R}(y_0, y_1)) = \text{sign}(aT_L)$, whenever both $y_{L,R}$ are defined. This means that the curves $y_1 = y_{L,R}(y_0)$ intersects $F^{-1}(0)$ in $\text{int}(Q)$, at most once, from R_- to R_+. Since $a \neq 0$, the origin is a quadratic contact point of the continuous piecewise linear system (2). Thus, one of the Poincaré half-maps $y_1 = y_{L,R}(y_0)$ is defined for $y_0 > 0$ sufficiently small and can be continuously extended to $y_0 = 0$ as $y_1 = 0$. Consequently, the graph of such a Poincaré half-map does not intersect the set $F^{-1}(0)$. Hence, any point $(y_0^*, y_1^*) \in \text{Int}(Q)$ corresponding to an existing limit cycle is contained in R_+, which implies that its stability is determined by $\text{sign}(aT_L)$. Namely, it is attracting (resp. repelling) provided that $aT_L < 0$ (resp. $aT_L > 0$). Therefore, if a limit cycle exists, it is unique and hyperbolic.

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