UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

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Sliding Shilnikov Orbit: theoretical aspects and applications in biological systems

Órbita Deslizante de Shilnikov: aspectos teóricos e aplicações em sistemas biológicos

Campinas

2023

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Dissertação apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestra em Matemática.

Dissertation presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Mathematics.

Supervisor: Douglas Duarte Novaes

Este trabalho corresponde à versão final da Dissertação defendida pela aluna Letícia Souza Cândido e orientada pelo Prof. Dr. Douglas Duarte Novaes.

Campinas

2023

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

C161s Cândido, Letícia Souza, 1999-
Sliding Shilnikov orbit : theoretical aspects and applications in biologicial systems / Letícia Souza Cândido. – Campinas, SP : [s.n.], 2023.

Orientador: Douglas Duarte Novaes.
Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Órbitas homoclínicas. 2. Esquemas de Bernoulli. 3. Campos vetoriais. 4. Caos. I. Novaes, Douglas Duarte, 1988-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações Complementares

Título em outro idioma: Órbita deslizante de Shilnikov : aspectos teóricos e aplicações em sistemas biológicos
Palavras-chave em inglês:
Homoclinic orbits
Bernoulli schemes
Vector fields
Chaos
Área de concentração: Matemática
Titulação: Mestra em Matemática
Banca examinadora:
Douglas Duarte Novaes [Orientador]
Gabriel Ponce
Regilene Delazari dos Santos Oliveira
Data de defesa: 03-08-2023
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0009-0009-9700-7025
- Currículo Lattes do autor: http://lattes.cnpq.br/7958658565249654

Dissertação de Mestrado defendida em 03 de agosto de 2023 e aprovada pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). DOUGLAS DUARTE NOVAES

Prof(a). Dr(a). GABRIEL PONCE

Prof(a). Dr(a). REGILENE DELAZARI DOS SANTOS OLIVEIRA

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

To me, to the voices in my head
and to my syster Lavínia.

Acknowledgements

Primeiramente devo agradecer aos meus pais que me deram tanto apoio, mesmo sabendo das dificuldades que iríamos passar. Não consigo colocar em palavras o quanto fui e sou abençoada por ter essas pessoas incríveis acreditando em mim e me ajudando a seguir meus sonhos. Minhas consquistas não são apenas minhas, são nossas. Do fundo do coração, obrigada!

Agradeço também à minha irmãzinha Lavínia que me tira tantos sorrisos e faz tudo se tornar mais leve. Sei que você não tem idade pra entender isso agora, mas estou longe hoje para estar muito pertinho daqui um tempo, te dando tudo de melhor desse mundo. Espero te ajudar a trilhar seus caminhos e te apoiar sempre, minha vidinha.

Agradeço à minha família que sempre acreditou em mim, que me apoia, que me entende e que torce para o meu bem, em especial ao meu irmão Felipe, minha vovó Geny, minha madrinha Wanusa, meu padrinho Alessandro e meus afilhados, Miguel e Gabriel (e Maria Cecília que está por vir).

À Ana Paula não tenho palavras para demonstrar tamanha gratidão por todo o apoio que me deu. Por me ouvir, me incentivar, por acreditar em mim quando nem eu acreditava mais. Não sei onde estaria sem sua ajuda e não quero pensar nessa possibilidade. Ter te conhecido foi, sem sombra de dúvidas, um dos melhores presentes que já ganhei (top 3).

Ao Professor Douglas agradeço pela paciência, dedicação e ensinamentos ao longo desses poucos anos de mestrado. Obrigada por ter me acolhido como sua orientanda. Espero que tenha paciência por mais quatro anos :).

Não posso deixar de agradecer aos amigos que fiz aqui em Campinas, pessoas com quem desabafo, falo besteiras, estudo junto, paro pra tomar um cafezinho... pessoas que se uniram e formaram uma grande família no predinho do IMECC. Sou grata por terem me acolhido aqui com tanto carinho, o caminho se tornou mais leve tendo vocês na minha vida e vou levá-los no coração para sempre. Obrigada Alejandro, Cássia, Crys, Daniel, Gabi, Gabriel, Gui, Julianita, Leo Garcia, Leo Silva, Lucas, Matheus, Mayara, Mel, Pehuen, Robson, Samuel, Sidney, Tenilson, Tiago e Walteir.

Agradeço também aos meus amigos de Goiânia que, mesmo com a distância, não deixaram de torcer por mim em especial à Tereza, Djovana, Wellimar, Geraldo e Renata. Sinto a falta de vocês diariamente.

Às minhas irmãs acadêmicas Gabriella, Marina e Amanda, vocês são incríveis. Torço muito pelo sucesso de vocês!! Obrigada por todo o apoio.

Além disso, agradeço à Kamilinha por ter me mostrado essa área linda que estudo e ter me incentivado desde cedo na carreira acadêmica (e fora também) e ao Professor João Carlos Medrado por ter topado me orientar durante a graduação e me aguentar por dois anos. Por fim, agradeço ao Professor Ricardo por ter me ajudado tanto desde que me mudei para Campinas.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001.

Agradeço à UNICAMP por ter se tornado minha segunda casa.

“The perfect gift.
A girl trapped in a box.
She only dances when someone else opens the lid,
when someone else winds her up.
If this is a story I’m telling,
I must be telling it to someone.
There’s always someone, even when there is no one.
***I will not be that girl in the box**.”*
-(The Handmaid’s Tale.)

Resumo

Esta dissertação é inspirada no trabalho desenvolvido por Shilnikov, onde é observado um comportamento interessante em campos vetoriais suaves, onde uma trajetória conecta um ponto de equilíbrio a si mesmo. Este tipo de trajetória é chamada de *órbita homoclínica*. Shilnikov estudou um tipo especial de órbita homoclínica, que conecta um ponto de "sela-foco" a si mesmo e, além disso, estudou o comportamento do campo vetorial próximo a tal órbita. Como o estudo de campos vetoriais suaves por partes é de nosso interesse, tendo em vista a grande utilidade desses campos para desenvolvimento de modelos aplicados, Novaes e Teixeira buscaram generalizar o conceito de órbita homoclínica, (particularmente a órbita homoclínica de Shilnikov) para o contexto de campos suaves por partes no artigo *Shilnikov problem in Filippov dynamical systems*, onde observaram um resultado análogo ao de Shilnikov. Posteriormente, Novaes, Ponce e Varão provaram a existência de um comportamento caótico próximo a tal orbita em *Chaos induced by sliding phenomena in Filippov systems*. Por fim, Carvalho, Novaes e Gonçalves mostraram a existência desta órbita em um modelo biológico em *Sliding Shilnikov connection in Filippov-type predator–prey model*.

Palavras-chave: Órbita Deslizante de Shilnikov, Campos Vetoriais Suaves por Partes, Bernoulli Shift, Caos.

Abstract

This dissertation is inspired by the work developed by Shilnikov, where an interesting behavior is observed in smooth vector fields, where a trajectory connects an equilibrium point to itself. This type of trajectory is called a *homoclinic orbit*. Shilnikov studied a special type of homoclinic orbit, which connects a "saddle-focus" point to itself, and furthermore, he studied the behavior of the vector field near such an orbit. As the study of piecewise smooth vector fields is of interest, given their great utility in the development of applied models, Novaes and Teixeira sought to generalize the concept of homoclinic orbit (particularly Shilnikov's homoclinic orbit) to the context of piecewise smooth fields in the article *Shilnikov problem in Filippov dynamical systems*, where they observed a result analogous to Shilnikov's. Subsequently, Novaes, Ponce, and Varão proved the existence of chaotic behavior near such an orbit in *Chaos induced by sliding phenomena in Filippov systems*. Finally, Carvalho, Novaes, and Gonçalves demonstrated the existence of this orbit in a biological model in *Sliding Shilnikov connection in Filippov-type predator–prey model*.

Keywords: Sliding Shilnikov Orbit, Piecewise Vector Field, Bernoulli Shift, Chaos.

Contents

Introduction

In classical theory, differential equations are closely related to vector fields. To study real-world problems, many researchers develop mathematical models using ODEs (Ordinary Differential Equations). However, in order to achieve a better approximation, it was necessary to develop the study of piecewise smooth vector fields. In the case of smooth vector fields, linear algebra provides tools for the qualitative study of these solutions. On the other hand, to understand the concepts of trajectory, singularities and stability of a piecewise vector field, we assume the Filippov convention, (1), which is necessary for a good notion of trajectory.

So we can approximate real problems by numerical models, even if these problems are "strange". Some interesting behavior was observed by Shilnikov (2, 3), where he studied a "homoclinic" connection, i.e., a trajectory of a smooth vector field, which connects an equilibrium point to itself. Then, he establishes the homoclinic Shilnikov orbit, which we are going to introduce formally. First, consider the vector field given by

$$\dot{x} = f(x),$$

where f $\in C^r(U),\ r \geqslant 1$, and $U \subset \mathbb{R}^3$. Assume that the vector field admits an equilibrium point $p \in \mathbb{R}^3$ such that the eigenvalues associated with the point p are given by $\lambda_1,\ \lambda_2$ and λ_3, with $\lambda_1 \in \mathbb{R}$ and $\lambda_2,\ \lambda_3$ are complex conjugate with part real nonzero. If $sign\lambda_1 \neq sign(Re(\lambda_2)) = sign(Re(\lambda_3))$, then we say that p is a **saddle-focus.**

Definition 0.1 (Shilnikov's homoclinic orbit)**.** *Let $\dot{x} = f(x)$ as given above and suppose that this vector field admits a saddle-focus equilibrium point. Associated with the eigenvalue λ_1 there exists an invariant curve that we call W_1 and, associated with the complex eigenvalues λ_2 and λ_3, there exists a sub-manifold of dimension 2 that we call W_2. A **Shilnikov homoclinic orbit** is a solution of the vector field that connects the saddle-focus to itself.*

Now we introduce the result that inspired this work, which determines a chaotic behavior on a smooth differential equation system.

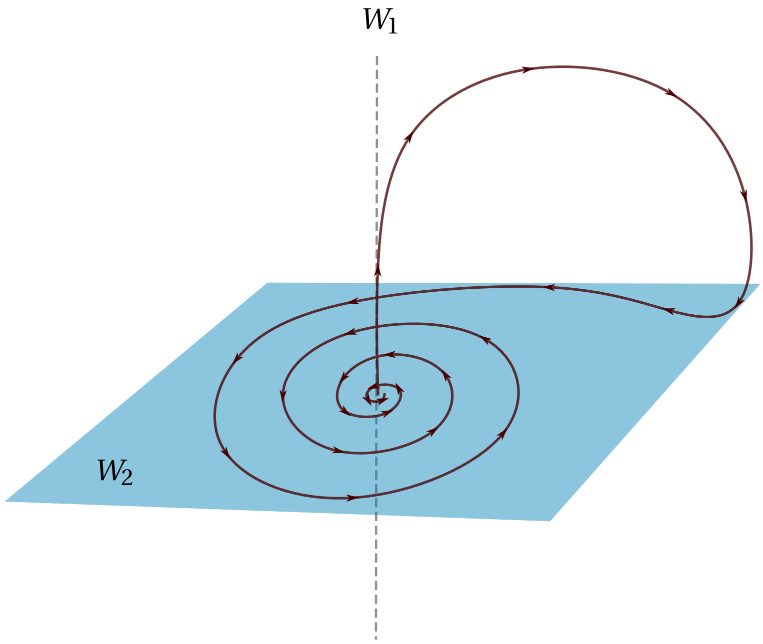

Figure 1 – Example of a Shilnikov homoclinic orbit.

Theorem 0.2 (Shilnikov)**.** *Consider the following differential equation system*

$$\begin{cases} \dot{x} &= \rho x - \omega y + P(x,y,z,\mu) \\ \dot{y} &= \omega x + \rho y + Q(x,y,z,\mu) \\ \dot{z} &= \lambda z + R(x,y,z,\mu) \end{cases} \tag{1}$$

where μ is the bifurcation parameter, $\dot{P}(0,0,0,\mu) = \dot{Q}(0,0,0,\mu) = \dot{R}(0,0,0,\mu) = 0$, $\lambda\rho < 0$ and $\omega \neq 0$. Assuming that the system 1 admits a Shilnikov homoclinic orbit namely Γ, if $\mid \frac{\rho}{\lambda} \mid < 1$ then there exists infinitely many periodic orbits in a neighborhood of Γ. If $\mid \frac{\rho}{\lambda} \mid \geqslant 1$, then Γ is isolated of periodic orbits. The condition $\mid \frac{\rho}{\lambda} \mid < 1$ is called ***Shilnikov Condition****.*

In order to generalize the study of Shilnikov, we define an equivalent result for piecewise vector fields, the *sliding Shilnikov orbit*. Furthermore, we can verify that a vector field with such an orbit presents a chaotic behavior. This dissertation is organized as follows:

1. In chapter 1, we introduce some basic concepts for the study of smooth vector fields;
2. In chapter 2 we generalize the concepts in chapter 1.1 for piecewise vector fields;
3. In chapter 3 we define our main object of study, the *sliding Shilnikov orbit*, and study the behavior of a piecewise smooth vector field close to the orbit;
4. In chapter 4 we study, using ergodic theory, the chaotic behavior of a *sliding Shilnikov orbit*;

5. Finally, in chapter 5, we prove that a biological model admits such orbit.

Chapter 1

Preliminary

The objective of this chapter is to provide an introduction to fundamental concepts and findings in the theory of smooth vector fields. To accomplish this, we will explore the local behavior of vector fields, establish the connections between differential equations and dynamical systems, and review essential results mentioned in section 1.1.In section 1.2, we delve into the examination of the global behavior of a dynamical system, focusing on its stability properties. Moving on to section 1.3, we introduce the concept of bifurcation and specifically explore a notable type known as the *saddle-node* bifurcation. In both section 1.4 and section 1.5, we provide precise definitions for a first integral of the vector field and establish formal definitions for periodic orbits and limit cycles. Additionally, we present Dulac's criteria. Finally, in section 1.6, we investigate the behavior of a predator-prey system.

Let's begin with a classical definition.

Let U be an open subset of $\mathbb{R}^n$, and $C^1(U)$ represent the set of C^1-vector fields defined on U. We define a *dynamical system* as a function $\phi(t,x)$ that is defined for all $t \in \mathbb{R}$ and $x \in U$. This function describes the manner in which points $x \in U$ "move" or evolve over time. To formalize this concept, we state the following definition:

Definition 1.1. *Let $U \subset \mathbb{R}^n$ be an open subset. A* ***dynamical system*** *in U is a C^1-map*

$$\phi : \mathbb{R} \times U \to U,$$

where $\phi_t(x) := \phi(t,x)$ satisfies the following properties:

- $\phi_0(x) = x \ \forall x \in U;$
- $(\phi_t \circ \phi_s)(x) = \phi_{t+s}(x).$

It is worth noting that in general, if $\phi_t(x)$ represents a dynamical system on U, then the function

$$f(x) = \frac{d}{dt}\phi_t(x)\ |_{t=0}$$

defines a C^1-vector field on U.

In this context, the function ϕ_t can be interpreted as the time map associated with the flow $\dot{x} = f(x)$. Conversely, if we consider the differential equation

$$\dot{x} = f(x), \tag{1.1}$$

it generates a smooth dynamical system, also known as a smooth vector field. This is because the time map of the flow, given by ϕ_t, is well-defined and continuously differentiable for all $t \in \mathbb{R}$.

1.1 Local Behavior of a Vector Field

In this section, we recall some basic definitions and results that will be used in this paper. Let's introduce the following definition:

Definition 1.2. *Consider an open subset $U \subset \mathbb{R}^n$ and suppose that f is continuously differentiable on U. A function $x(t)$ is a* ***solution of the differential equation*** $\dot{x} = f(x)$ *on an interval $I \subset \mathbb{R}$ if $x(t)$ is differentiable on I and satisfies $\dot{x}(t) = f(x(t))$ for all $t \in I$ where $x(t) \in U$.*

Given $x_0 \in U$, we say that $x(t)$ is a ***solution of the initial value problem***

$$\begin{cases} \dot{x} &= f(x) \\ x(t_0) &= x_0 \end{cases}, \tag{1.2}$$

on the interval I, if $t_0 \in I$, $x(t_0) = x_0$, and $x(t)$ is a solution of the differential equation $\dot{x} = f(x)$ on I.

This definition establishes the concept of a solution to a differential equation and an initial value problem associated with the differential equation $\dot{x} = f(x)$. It specifies that a function $x(t)$ is considered a solution if it satisfies the equation $\dot{x}(t) = f(x(t))$ for all t in the given interval I, where $x(t)$ belongs to the open subset U. Additionally, for an initial value problem, a solution $x(t)$ should fulfill both the differential equation and the specified initial condition $x(t_0) = x_0$ at a specific time t_0.

In general, the system will have a solution if f is continuous, but the continuity of f is not sufficient to guarantee the uniqueness of the solution. Additional conditions or assumptions may be necessary to establish the uniqueness of solutions.

Example 1.3. *The initial value problem*

$$\begin{cases} \dot{x} &= 3x^{\frac{2}{3}} \\ x(0) &= 0 \end{cases}, \tag{1.3}$$

has two different solutions through the origin. Indeed, we have that $x_1(t) \equiv 0$ and $x_2(t) = t^3$ are two solutions of the system 1.3.

Now we recall the **Fundamental Existence-Uniqueness Theorem**.

Theorem 1.4. *Let $U \subset \mathbb{R}^n$ be an open subset containing a point x_0 and assume that $f \in C^1(U)$. Then there exists an $\varepsilon > 0$ such that the initial value problem 1.2 has a unique solution $x(t)$ on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.*

A proof of this theorem can be found in (4).

The existence-uniqueness theorem is typically formulated in a local context. To apply this theorem to a more general and "interesting" space, it is necessary to consider the additional condition that the solution $x(t)$ depends continuously on the initial condition x_0. In light of this, we can consider the following complementary theorems:

Theorem 1.5 (Dependence on Initial Condition)**.** *Let U be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $f \in C^1(U)$. Then there exists an $\varepsilon > 0$ and $V \subset U$ such that, for all $y \in V$, the initial value problem*

$$\begin{cases} \dot{x} &= f(x) \\ x(t_0) &= y \end{cases},$$

has a unique solution $x(t, y)$ in $[t_0 - \varepsilon, t_0 + \varepsilon] \times V$.

Theorem 1.6 (Dependence on Parameters)**.** *Let U be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, μ_0), with $x_0 \in \mathbb{R}^n$ and $\mu_0 \in \mathbb{R}^m$. Assuming that $f \in C^1(U)$, then there exists an $\varepsilon > 0$ and $V \subset U$ such that for all $(y, \mu) \in V$, the initial value problem*

$$\begin{cases} \dot{x} &= f(x, \mu) \\ x(t_0) &= y \end{cases},$$

has a unique solution $x(t, y, \mu)$ in $[t_0 - \varepsilon, t_0 + \varepsilon] \times V$.

In addition, we have the unique solution of 1.1 defined on a maximal interval of existence. These results can be found in (4).

The following local theorem demonstrates that in the vicinity of a hyperbolic equilibrium point x_0, the nonlinear system $\dot{x} = f(x)$ exhibits similar qualitative behavior as the linear system $\dot{x} = Ax$, where $A = Df(x_0)$.

Let's establish some concepts regarding the local structure of a system of differential equations.

Definition 1.7. *Two autonomous systems of differential equations, such as $\dot{x} = f(x)$ and $\dot{x} = Ax$ (where $A = Df(x_0)$), are said to be **topologically equivalent** in a neighborhood of x_0 if there exists an open set U containing x_0, and a homeomorphism $h : U \to V$, where $x_0 \in V$ and V is an open set, such that trajectories of $\dot{x} = f(x)$ in U are mapped to trajectories of $\dot{x} = Ax$ in V by h, while preserving the orientation with respect to time.*

Studying the behavior of a nonlinear vector field can sometimes be challenging. However, there are certain conditions under which the local behavior of a nonlinear vector field is topologically equivalent to that of a linear vector field. In this context, the following results hold.

Theorem 1.8 (Hartman-Grobman, (4))**.** *Let U be an open subset of $\mathbb{R}^n$ containing x_0, $f \in C^1(U)$ and ϕ_t be the flow of the nonlinear system 1.1. Suppose that $f(x_0) = 0$ and that the matrix $A = Df(x_0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of a neighborhood V of x_0 onto an open set W containing the origin such that, for each $y \in V$, there is an open interval $I_y \subset \mathbb{R}$, containing zero, and*

$$H \circ \phi_t(y) = e^{At}H(y),\ \forall t \in I_y.$$

Theorem 1.9 (Hartman, (4))**.** *Let U be an open subset of $\mathbb{R}^n$ containing x_0, $f \in C^2(U)$ and ϕ_t be the flow of the nonlinear system 1.1. Suppose that $f(x_0) = 0$ and that the matrix $A = Df(x_0)$ has all eigenvalues with a negative (or positive) real part. Then there exists a C^1-diffeomorphism H of a neighborhood V of x_0 onto an open set W containing the origin such that, for each $y \in V$, there is an open interval $I_y \subset \mathbb{R}$, containing zero, and*

$$H \circ \phi_t(y) = e^{At}H(y),\ \forall t \in I_y.$$

This theorem is of great importance to the study of the derivative of a first return map defined on a neighborhood of the object that we are interested in, the *sliding Shilnikov orbit.*

1.2 Global Behavior of a Smooth Vector Field

This section aims to extend our concepts and realize a global study of the vector fields. Now we generalize the notions of topologically equivalent and topologically conjugate vector fields.

Definition 1.10. *Consider open subsets $U_1, U_2 \subset \mathbb{R}^n$, and let $f \in C^1(U_1)$ and $g \in C^1(U_2)$. The two autonomous systems of differential equations*

$$\dot{x} = f(x) \tag{1.4}$$

and

$$\dot{x} = g(x), \tag{1.5}$$

*are said to be **topologically equivalent** if there exists a homeomorphism $h : U_1 \to U_2$ that maps trajectories of 1.4 onto trajectories of 1.5 while preserving their orientation with respect to time.*

*If $U_1 = U_2 = U$, then the systems 1.4 and 1.5 are said to be **topologically equivalent on** U, and the vector fields f and g are said to be topologically equivalent on U.*

This definition clarifies that two systems of differential equations are topologically equivalent if there exists a homeomorphism that maps trajectories of one system to trajectories of the other system while preserving their orientation in time. Furthermore, if the systems are defined on the same open subset U, they are considered topologically equivalent on that subset, and their vector fields are also regarded as topologically equivalent on U.

In the given context, it is important to note that while the homeomorphism h preserves the orientation of trajectories over time, it is not necessarily required to preserve the parameterization of trajectories. However, if h does preserve the parameterization by time, then the vector fields f and g are said to be *topologically conjugate*, which can be defined as follows:

Definition 1.11. *Let $U, V \subset \mathbb{R}^n$ and $f \in C^1(U)$, $g \in C^1(V)$. We say that f and g are **conjugate** if there is a homeomorphism $h : U \to V$ such that h satisfies the conjugacy equation $h \circ f = g \circ h$. We say that the vector fields f and g are C^r**-conjugate**, if the homeomorphism h is C^r.*

In order to study the global behavior of a dynamical system, we have to extend our results. The following theorem guarantees the global existence of a solution on a topologically equivalent system to Equation 1.1.

Theorem 1.12 ((4))**.** *For $f \in C^1(\mathbb{R}^n)$ and each $x_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{cases} \dot{x} &= \dfrac{f(x)}{1+|f(x)|} \\ x(t_0) &= x_0, \end{cases} \tag{1.6}$$

has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$. So 1.6 defines a dynamical system on $\mathbb{R}^n$, and 1.6 is topologically equivalent to 1.1 on $\mathbb{R}^n$.

Theorem 1.13 ((4))**.** *Suppose that U is an open subset of $\mathbb{R}^n$ and that $f \in C^1(U)$. Then there exists a function $g \in C^1(U)$ such that*

$$\dot{x} = g(x) \tag{1.7}$$

defines a dynamical system on U and 1.7 is topologically equivalent to 1.1 on U.

The study of topological equivalency between vector fields guarantees certain "stability" on a neighborhood of some vector field. In the next chapter, we generalize this concept to *piecewise vector fields.*

1.3 Saddle-node Bifurcation

In this section, we explore some illustrative examples of bifurcations that arise in nonlinear systems. We begin by noting that the concept of *bifurcation* involves perturbing a vector field and observing the resulting changes in its behavior.

Consider a system described by the equation:

$$\dot{x} = f_\mu(x) = f(x, \mu), \tag{1.8}$$

where $\mu \in \mathbb{R}$. We assume that f_μ is a C^∞ function that depends on the parameter μ. A *bifurcation* occurs when a significant alteration in the system's structure can be observed, typically resulting from changes in the parameter μ.

Example 1.14. *Consider the first-order equation*

$$\dot{x} = f_\mu(x) = x^2 + \mu.$$

Then, if $\mu = 0$*, the equation has a single eigenvalue at* $x = 0$*. Since* $f_0'(0) = 0$ *and* $f_0''(0) = 2 \neq 0$*, for* $\mu > 0$ *this equation has no equilibrium points and, for* $\mu < 0$ *has a pair of equilibrium points* $\pm\sqrt{-\mu}$*. Then we have the following bifurcation diagram:*

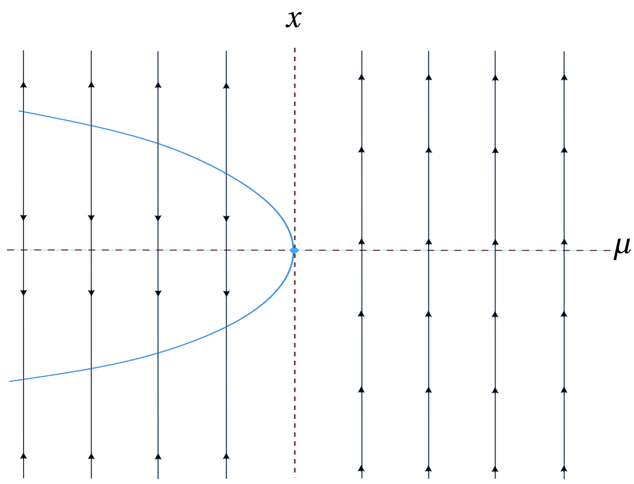

Figure 2 – Bifurcation diagram for the example 1.14.

We focus on a certain type of bifurcation, called *saddle-node* bifurcation. In a saddle-node bifurcation, a distinct behavior can be observed based on the location of the parameter value μ_0 and a specific interval I along the x-axis. The behavior of the differential equation in this bifurcation can be summarized as follows:

- If $\mu < \mu_0$, there exist two equilibrium points within the interval I.
- If $\mu = \mu_0$, there is a single equilibrium point within the interval I.

- If $\mu > \mu_0$, there are no equilibrium points within the interval I.

To specifically identify this type of bifurcation, we rely on the following theorem:

Theorem 1.15 (Saddle-Node Bifurcation, (5))**.** *Suppose that*

$$\dot{x} = f_\mu(x)$$

is a first-order differential equation for which the following holds:

1. $f_{\mu_0}(x_0) = 0$;
2. $f'_{\mu_0}(x_0) = 0$;
3. $f''_{\mu_0}(x_0) \neq 0$;
4. $\dfrac{\partial f_{\mu_0}}{\partial \mu}(x_0) \neq 0$.

Then, this differential equation undergoes a saddle-node bifurcation at $\mu = \mu_0$.

Consider the following example:

Example 1.16. *Let the system*

$$\begin{cases} \dot{x} &= x^2 + \mu \\ \dot{y} &= -y \end{cases} \tag{1.9}$$

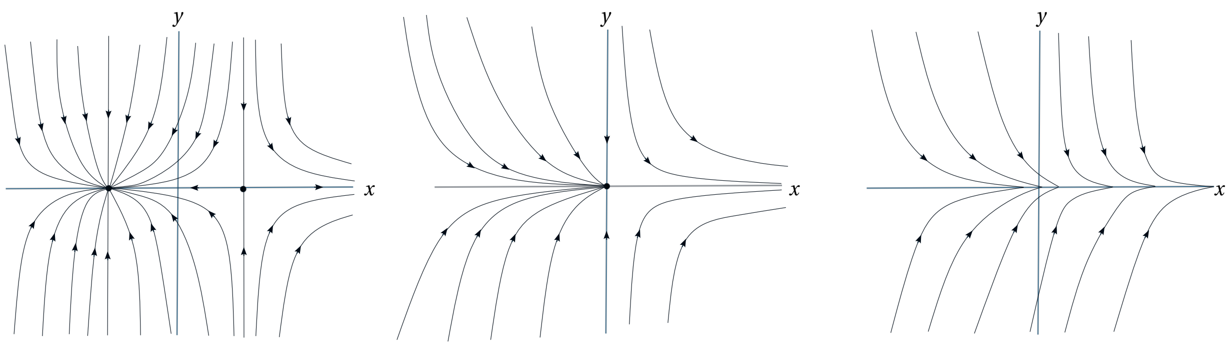

Figure 3 – Phase-portrait of the system 1.9 when $\mu < 0$, $\mu = 0$ and $\mu > 0$, respectively.

When $\mu = 0$, the system described by equation 1.9 possesses a single equilibrium point, specifically located at the origin $(0,0)$. Notably, trajectories along the y-axis converge towards the origin as t approaches infinity. On the other hand, the remaining solutions move towards the right and diverge to infinity as t tends to infinity.

In the case where $\mu > 0$, the system exhibits $\dot{x} > 0$, causing all solutions to move towards the right. As a result, the equilibrium point ceases to exist. On the *other hand, when $\mu < 0$, a pair of equilibrium points emerges: $\lambda_1 = (\sqrt{-\mu}, 0)$ and $\lambda_2 = (-\sqrt{-\mu}, 0)$. The point λ_1 represents a saddle point, while λ_2 corresponds to a sink, attracting nearby trajectories.*

1.4 First Integrals

In this section, we introduce the concept of a first integral, which plays a crucial role in the analysis of differential equation systems with non-hyperbolic equilibrium points. Specifically, when dealing with two-dimensional vector fields, the phase portraits of the system can be fully characterized by the presence of a first integral. Consider the vector field of the form

$$\dot{x} = f(x),$$

where $x \in U \subset \mathbb{R}^n$ and $f \in C^r(U)$, $r \geqslant 1$. A function with scalar values $F : U \to \mathbb{R}$ is called a **first integral** of the vector field above, if it is constant along its trajectories, that is, $F(x(t)) = c$, where $x(t)$ is a trajectory and $c \in \mathbb{R}$. Then, $F(x)$ is a first integral if, and only if, satisfies

$$\langle \nabla F(x), f(x) \rangle = 0,$$

for all points x where F is defined.

Additionally, it is worth noting that the level surfaces of the function F remain unchanged under the flow of the vector field. As a result, for two-dimensional vector fields that possess a first integral, the trajectories of the vector field lie entirely within the level curves of the function F. This property further aids in understanding and visualizing the behavior of the system.

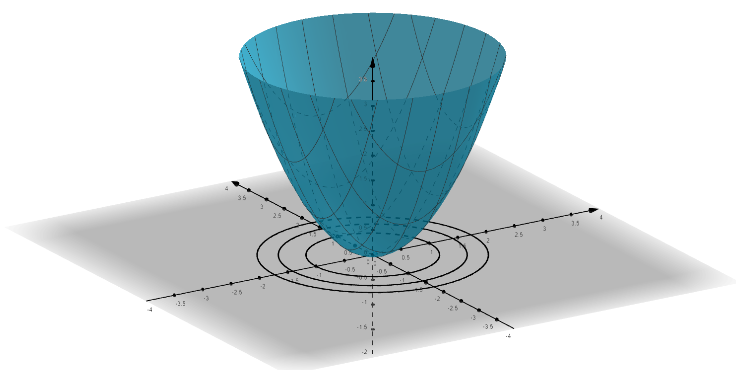

Figure 4 – 3D plot of the first integral F of the vector field 1.10.

Example 1.17. *Consider the following differential equation system on* $\mathbb{R}^2$

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x \end{cases} \tag{1.10}$$

and the function $F : \mathbb{R}^2 \to \mathbb{R}$ *given by* $F(x, y) = x^2 + y^2$. *Let* $(x(t), y(t))$ *be a solution of our system and note that*

$$\langle \nabla F(x, y), f(x, y) \rangle = \langle (2x, 2y), (y, -x) \rangle = 2xy - 2yx = 0.$$

Therefore, the function F *is the first integral of the system.*

1.5 Periodic Orbits and Limit Cycles

As observed earlier, it is convenient to consider the system 1.1 as defining a dynamical system or a smooth vector field $\phi(t, x)$ on the open subset U. For any point $x \in U$, the function $\phi(\cdot, x) : \mathbb{R} \to U$ represents a trajectory of the system 1.1 passing through the initial point x_0 in U. We can define the notion of ω-limit points and α-limit points for trajectories of the system 1.1 as follows:

Definition 1.18. *A point* $p \in U$ *is an* ω***-limit point*** *of the trajectory* $\phi(\cdot, x)$ *of the system 1.1 if there exists a sequence* $(t_n) \in \mathbb{R}$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \phi(t_n, x) = p$.

Similarly, a point $q \in U$ *is an* α***-limit point*** *of the trajectory* $\phi(\cdot, x)$ *if there exists a sequence* $(t_n) \in \mathbb{R}$ *such that* $\lim_{n\to\infty} t_n = -\infty$ *and* $\lim_{n\to\infty} \phi(t_n, x) = q$.

The set of all ω*-limit points of a trajectory* Γ *is called the* ω*-limit set of* Γ*, denoted by* $\omega(\Gamma)$. *Similarly, the set of all* α*-limit points of a trajectory* Γ *is called the* α*-limit set of* Γ*, denoted by* $\alpha(\Gamma)$.

These concepts allow us to analyze the long-term behavior of trajectories and provide insights into the stability and asymptotic properties of the system.

We can define the concept of a cycle or periodic orbit in the system 1.1 as follows:

Definition 1.19. *A* ***cycle*** *or* ***periodic orbit*** *of the system 1.1 is a closed solution curve that is not an equilibrium point of 1.1.*

A periodic orbit Γ *is called* ***stable*** *if for every* $\varepsilon > 0$*, there exists a neighborhood* U_ε *of* Γ *such that for all* $x \in U_\varepsilon$ *and* $t > 0$, $d(\phi(t, x), \Gamma) < \varepsilon$. *In other words, all trajectories starting sufficiently close to* Γ *remain close to* Γ *for all future time.*

A periodic orbit is called ***unstable*** *if it is not stable, meaning that there exist trajectories starting arbitrarily close to* Γ *that eventually diverge from* Γ.

*A periodic orbit is called **asymptotically stable** if it is stable and for any point x in a neighborhood U of Γ, the distance between $\phi(t,x)$ and Γ approaches zero as t tends to infinity, i.e.,* $\lim_{t\to\infty} d(\phi(t,x),\Gamma) = 0$.

The stability properties of a periodic orbit provide information about the long-term behavior of trajectories in the system and can help us understand the stability and convergence of solutions. Notice that cycles of the system 1.1 are periodic solutions of this system. The minimal value T for which $\phi(t+T,x_0) = \phi(t,x_0)$ is called the **period** of the periodic orbit $\phi(\cdot,x_0)$. We now consider periodic orbits of a planar system 1.1.

Definition 1.20. *A **limit cycle** Γ in a planar system 1.1 is a cycle of 1.1 that serves as the α-limit or ω-limit set of a trajectory of 1.1, excluding Γ itself.*

*A limit cycle Γ is classified as **stable** if it is the ω-limit set for every trajectory within a certain neighborhood of Γ. This means that all trajectories starting sufficiently close to Γ converge to Γ as t tends to infinity.*

*Conversely, if Γ is the α-limit set for every trajectory within a neighborhood of Γ, then Γ is considered an **unstable** limit cycle. In this case, trajectories starting sufficiently close to Γ diverge from Γ as t tends to negative infinity.*

Limit cycles play a crucial role in the dynamics of nonlinear systems, representing recurring patterns or periodic behavior. Their stability properties determine whether the system's trajectories converge to or diverge from the cycle, providing valuable insights into the long-term behavior of the system.

Example 1.21. *Consider the following system of differential equations with $\omega > 0$*

$$\begin{cases} \dot{x} = -y + \omega x(1 - x^2 - y^2) \\ \dot{y} = x + \omega y(1 - x^2 - y^2) \end{cases}.$$

Making a polar change of coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$ and the system on the new coordinates is:

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - rsen\theta\dot{\theta}, & (*) \\ \dot{y} = \dot{r}sen\theta + rcos\theta\dot{\theta}. & (**) \end{cases}$$

Multiplying $(*)$ by $\cos\theta$ and $(**)$ by $\sin\theta$, we can find $\dot{r}$:

$$\dot{r} = \frac{\dot{x}x + \dot{y}y}{r}.$$

Similarly,

$$\dot{\theta} = \frac{\dot{y}x - \dot{x}y}{r^2}.$$

Replacing $x, \dot{x}, y, \dot{y}$, we have the new system

$$\begin{cases} \dot{r} = \omega r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}.$$

Solving these ODE's, we get

$$r(t, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-2\omega t}\right]^{-1/2} \quad \text{and } \theta = t + c, c \in \mathbb{R}.$$

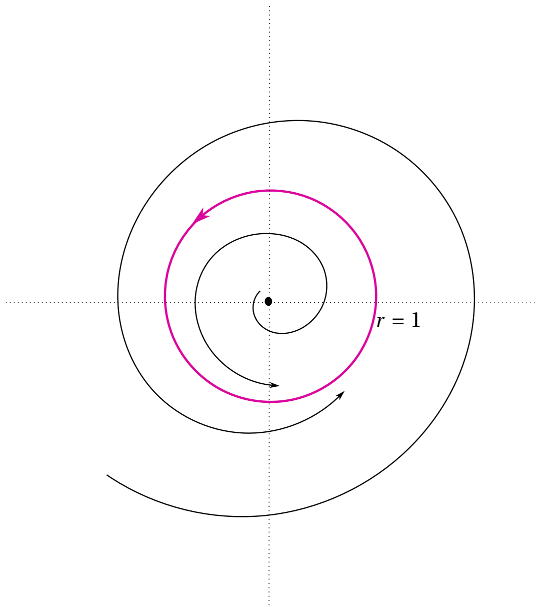

Figure 5 – Phase-portrait of the example1.21.

When $r_0 = 1$, *then* $r(t, r_0) = \left[1 + \left(\frac{1}{1^2} - 1\right) e^{-2\omega t}\right]^{-1/2} = 1 = r_0$. *On this initial value* $r(t, 1) = 1$ *for all* $t \in \mathbb{R}$, *so the circle of radius* $r = 1$ *is a closed orbit of our system of period* 2π. *For* $r < 1$, $\dot{r}(r) > 0$ *and for* $r > 1$, $\dot{r}(r) < 0$. *So the trajectory starting on a point* $r < 1$ *is increasing. On the other hand, the trajectory starting on a point* $r > 1$ *is decreasing. So, the limit cycle is stable.*

Our last theorem of this section establishes conditions under which the planar system has no limit cycles.

Theorem 1.22 (Dulac´s Criteria, (4))**.** *Let* U *be a simply connected region on* $\mathbb{R}^2$ *and* $f \in C^1(U)$. *If there exists a function* $g \in C^1(U)$ *such that* $\nabla \cdot (gf)$ *is not identically zero and does not change its sign in* U, *then the system 1.1 has no closed orbit lying entirely in* U.

1.6 Predator-Prey Systems

Mathematicians Alfred Lotka and Vito Volterra developed a model that seeks to solve a very interesting type of problem, the prey-predator problem. This model deals with two species that share an environment, where prey have plenty of food available animals and predators feed on prey. Note that a model represented by only two populations cannot express the different types of events existing in the environment, thus ignoring important factors such as climate or human actions, but the study of a simple model is what makes it possible, later, to advance in research to understand advanced phenomena.

To model the interaction between predators and prey, we consider a system involving two species: the predators (denoted by y) and the prey (denoted by x). We make the following assumptions:

- The prey population x is the total food supply for the predators;
- In the absence of predators, x grows at a rate proportional to the current population, i.e., if $y = 0$, $\dot{x} = ax$ where $a > 0$;
- When $y \neq 0$, x decreases at a rate proportional to the number of predator-prey encounters.

We can model this system using the differential equation $\dot{x} = ax - bxy$, where $b > 0$ represents the interaction strength between predators and prey. This equation describes the dynamics of the prey population.

Next, let's consider the predator population (y) and make the following assumptions:

- In the absence of prey, y declines at a rate proportional to the current population, i.e., if $x = 0$, $\dot{y} = -cy$ where $c > 0$;
- When $x \neq 0$, y increases at a rate proportional to the number of predator-prey encounters.

Combining these assumptions, we arrive at a simplified predator-prey system:

$$\begin{cases} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy. \end{cases} \tag{1.11}$$

This system captures the dynamics of predator-prey interactions. To study the properties of such systems, we can refer to the result presented in (5). It provides valuable insights into the behavior and stability of predator-prey systems.

Theorem 1.23 ((5))**.** *Every solution of the predator-prey system, except for the equilibrium point and the coordinate axes, is a closed orbit.*

Chapter 2

Filippov System

The objective of this chapter is to introduce the fundamental concepts of Filippov systems. The dynamics of these systems cannot be determined using the classical theory of smooth dynamical systems. Therefore, the first step in understanding these systems is to establish the notions of trajectory, orbit, and singularity. In section 2.1, we define Filippov's convention, and in section 2.2, we examine the singularities of a piecewise vector field. Finally, in section 2.3, we extend the concepts of topological equivalence and topological conjugacy to vector fields.

2.1 Filippov's Convention

Let U be an open bounded subset of $\mathbb{R}^3$, and let $K = \bar{U}$ be its closure. We define the set $C^r(K, \mathbb{R}^3)$ as the collection of all C^r vector fields $X : K \to \mathbb{R}^3$. Additionally, we introduce $\Omega_h^r(K, \mathbb{R}^3)$ as the space of piecewise vector fields given by:

$$Z(x) = \begin{cases} X(x) & \text{if} \quad h(x) > 0, \\ Y(x) & \text{if} \quad h(x) < 0, \end{cases} \tag{2.1}$$

Here, X and Y belong to $C^r(K, \mathbb{R}^3)$, and $h : K \to \mathbb{R}$ is a differentiable function with zero as a regular value. The notation often used for Equation 2.1 is $Z = (X, Y)$, and it is referred to as a **Filippov system**. The set $\Sigma = h^{-1}(0)$ represents the switching surface of codimension 1.

It is worth noting that $C^r(K, \mathbb{R}^3)$ is equipped with the C^r topology, while $\Omega_h^r(K, \mathbb{R}^3)$ is endowed with the product topology. The switching manifold Σ can be divided into regions exhibiting different dynamical behaviors. For any p belonging to Σ, we define $Xh(p)$ as the **Lie derivative** of $h(p)$ in the direction of the vector $X(p)$. Considering the notation in Equation 2.1, we can state the following:

- $\Sigma^c \subset \Sigma$ is referred to as a *crossing* region if $Xh(p)Yh(p) > 0$ for $p \in \Sigma$.

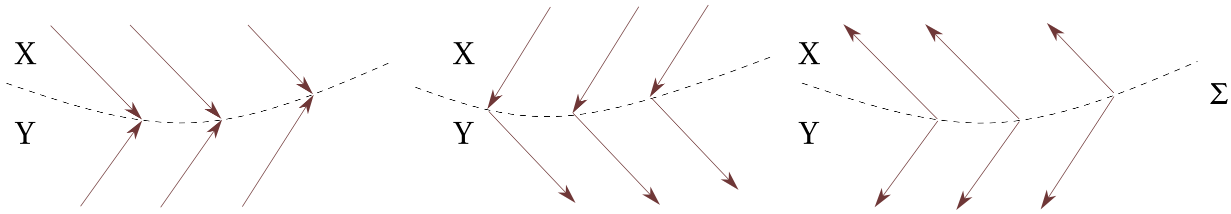

Figure 6 – Sliding, crossing, and escape regions, respectively.

- $\Sigma^e \subset \Sigma$ is referred an *escaping* region if $Xh(p)Yh(p) < 0$ with $Xh(p) > 0$ and $Yh(p) < 0$.
- $\Sigma^s \subset \Sigma$ is referred as a *sliding* region if $Xh(p)Yh(p) < 0$ with $Xh(p) < 0$ and $Yh(p) > 0$.

This definition does not include the points of **tangency**, which are points where one of the two vector fields is tangent to Σ. These points are on the boundary of the regions Σ^e, Σ^s and Σ^c, that we will denote by $\partial\Sigma^e, \partial\Sigma^s$ and $\partial\Sigma^c$, respectively. If $p \in \Sigma^c$, then the trajectory passing through p is the concatenation of the trajectories of the vectors X and Y by that point.

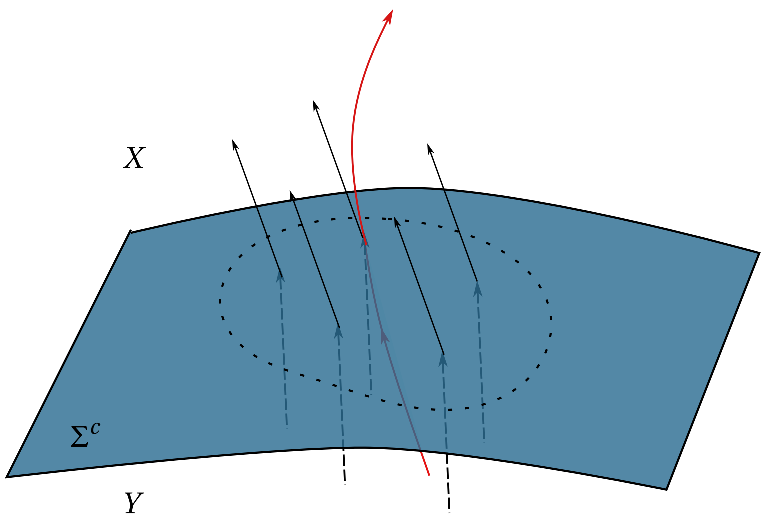

Figure 7 – Sketch of the crossing region.

To understand the flow of $Z = (X, Y)$ on the region $\Sigma^e \cup \Sigma^e$ we define a new vector field:

Definition 2.1. *In the context above, for all $p \in \Sigma^e \cup \Sigma^s$ we define the* ***sliding vector field*** *of Z, called $\widetilde{Z}$, by:*

$$\widetilde{Z}(p) = \frac{Yh(p)X(p) - Xh(p)Y(p)}{Yh(p) - Xh(p)}. \tag{2.2}$$

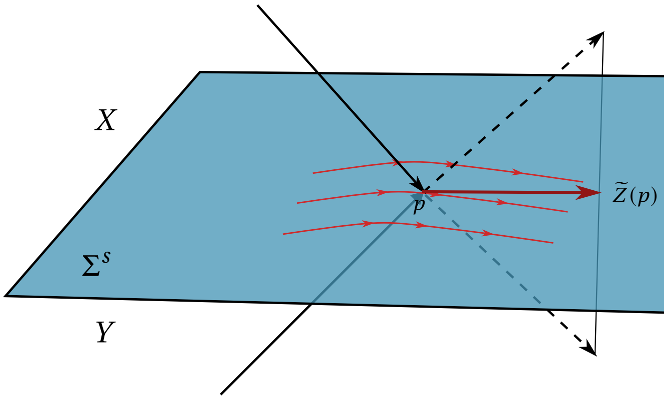

Figure 8 – Sketch of the sliding vector field

To facilitate the analysis of the orbits of the sliding vector fields, it is advantageous to introduce the normalized sliding vector field,

$$\widetilde{\widetilde{Z}}(p) = Yh(p)X(p) - Xh(p)Y(p). \tag{2.3}$$

The normalized sliding vector field has the same phase portrait as $\widetilde{Z}$ but with the direction of the flow reversed in the escape region. So we can define the trajectory of the Filippov system given by 2.1.

Definition 2.2 (See (6)). *Let $\psi_Z(t,p)$ be the local trajectory of $\dot{x} = Z(x)$ passing through a point $p \in \mathbb{R}^3$ at a time $t \in I_p \subset \mathbb{R}$, where $\psi_Z(0,p) = p$ and $0 \in I_p$ denotes the interval of definition of $\psi_Z(t,p)$. The Filippov's conventions can be summarized as follows:*

- *If $h(p) > 0$ (resp. $h(p) < 0$), then $Z(p) = X(p)$ (resp. $Z(p) = Y(p)$), and the trajectory is defined as $\psi_Z(t,p) = \psi_X(t,p)$ (resp. $\psi_Z(t,p) = \psi_Y(t,p)$) for $t \in I_p$.*

- *For $p \in \Sigma^c$ such that $Xh(p) > 0$ and $Yh(p) > 0$, the trajectory is defined as $\psi_Z(t,p) = \psi_Y(t,p)$ for $t \in I_p \cap t < 0$ and $\psi_Z(t,p) = \psi_X(t,p)$ for $t \in I_p \cap t > 0$. When $Xh(p) < 0$ and $Yh(p) < 0$, the definition is the same but with reversed time.*

- *For $p \in \Sigma^e \cup \Sigma^s$ such that $\widetilde{Z}(p) \neq 0$, the trajectory is defined as $\psi_Z(t,p) = \psi_{\widetilde{Z}}(t,p)$ for $t \in I_p$.*

- *For $p \in \partial\Sigma^c \cup \partial\Sigma^e \cup \partial\Sigma^s$ such that the definitions of trajectories for points in Σ on both sides of p can be extended to p and coincide, the trajectory passing through p is considered to be the same as this extended trajectory. We refer to p as a **tangency regular point**.*

- *For any other point p where $\psi_Z(t,p) = p$ for all $t \in \mathbb{R}$, we have singularities of the vector fields X, Y, and $\widetilde{Z}$. This includes tangency points in Σ that are not regular, which we refer to as **singular tangency points**.*

2.2 Singularities

Now we present the definitions of singular points of the Filippov system $Z = (X, Y) \in \Omega^r$.

Definition 2.3. *A point $p \in \Sigma$ is a **tangency point** of X if $Xh(p) = 0$. We say that a tangency point $x \in \Sigma$ is a **visible fold** of X (resp. Y) if $(X)^2h(x) > 0$ (resp. $(Y)^2h(x) < 0$).*

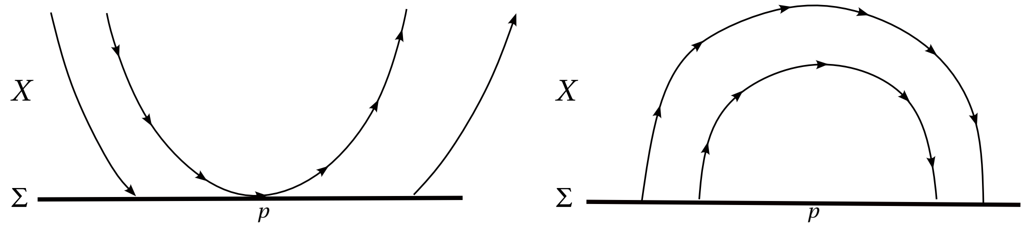

Figure 9 – A visible and invisible fold regular points of X, respectively.

*Reversing the inequalities we call the tangency point **invisible fold**.*

Definition 2.4. *A point $p \in \Sigma$ is a Σ-**regular point** of the vector field X (resp. Y), if $Xh(p) \neq 0$ (resp. $Yh(p) \neq 0$).*

In this case, for a visible fold $p \in \Sigma$ of X such that p is a Σ-regular point of Y, we say that p is a **fold-regular point of Z with respect to** X or a **fold-regular point of** X.

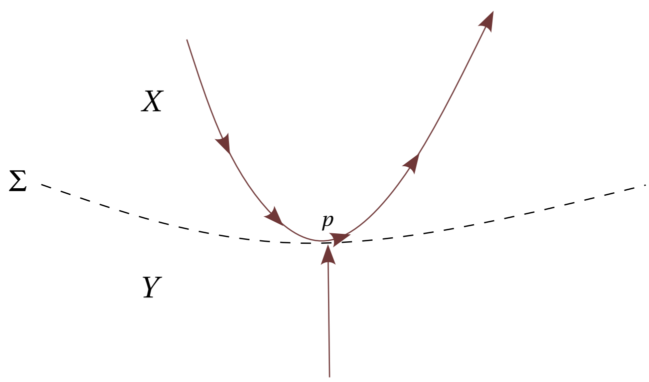

Figure 10 – A fold-regular point of Z with respect to X

So, the singularities of the Filippov system 2.1 can be classified as follows:

- $p \in X$ (resp. Y) such that $h(p) > 0$ (resp. $h(p) < 0$) and $X(p) = 0$ (resp. $Y(p) = 0$);
- $p \in \Sigma^e \cup \Sigma^s$ such that $\widetilde{Z}(p) = 0$ and we say that p is a **pseudo-equilibrium** of Z;
- $p \in \partial\Sigma^c \cup \partial\Sigma^e \cup \partial\Sigma^s$, the tangency points of Z where $Xh(p) = 0$ or $Yh(p) = 0$.

Any other point of Z is called a **regular point**.

Definition 2.5. *When p^* is a hyperbolic singularity of $\widetilde{Z}$, it is referred to as a **hyperbolic pseudo-equilibrium**. Specifically, if $p^* \in \Sigma^s$ (resp. Σ^e) is an unstable (stable) hyperbolic focus of $\widetilde{Z}$, then p^* is classified as a **hyperbolic pseudo-saddle-focus**.*

2.3 Structural Stability and Topological Conjugacy

We can generalize the concepts of vector fields topologically equivalent of the section 1.2.

Definition 2.6. *Let Z_1 and $Z_2 \in \Omega^r$. We say that Z_1 and Z_2 are **topologically equivalent** if there exists a homeomorphism $\phi : U \to U$ satisfying $\phi(\Sigma_{Z_1}) = \Sigma_{Z_2}$ and sending orbits of Z_1 to orbits of Z_2 preserving time orientation.*

Definition 2.7. *We say that a Filippov System $Z_0 = (X_0, Y_0) \in \Omega^r$ is **structurally stable** if there exists an open neighborhood $B \subset \Omega^r$ of Z_0 such that, if $Z = (X, Y) \in B$ is a Filippov System, then Z and Z_0 are topologically equivalent.*

Proposition 2.8 (See (7))**.** *Given $Z = (X, Y) \in \Omega^r_h(K, \mathbb{R}^3)$, if $q \in \partial\Sigma^e \cup \partial\Sigma^s$ is a fold-regular point of Z with respect to X, then the sliding vector field $\widetilde{Z}$ is transverse to $\partial\Sigma$ at q and there exists a neighborhood V of q such that $U := V \cap (\partial\Sigma^e \cup \partial\Sigma^s)$ and $Z\,|_U$ is structurally stable.*

Proof. For $Z = (X, Y)$ we denote $X, Y \in C^r$ by $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$. Consider $h(x, y, z) = z$ and $q \in \Sigma = h^{-1}(0)$ such that $Xh(q) = 0$, $X^2h(q) > 0$ and $Yh(q) \neq 0$. For $p \in \Sigma^e \cup \Sigma^s$ we have the sliding vector field

$$\widetilde{Z}(p) = \frac{Yh(p)X(p) - Xh(p)Y(p)}{Yh(p) - Xh(p)} = \frac{Y_3(p)X(p) - X_3(p)Y(p)}{Y_3(p) - X_3(p)},$$

and $Xh(p) = X_3(p)$. Now, for q we have

$$\widetilde{Z}(q) = \frac{Y_3(q)X(q)}{Y_3(q)}$$

$$\therefore \widetilde{Z}(q) = (X_1(q), X_2(q), 0).$$

Now,

$$\langle \widetilde{Z}(q), \nabla Xh(q) \rangle = \langle (X_1(q), X_2(q)), (X_3^1(q), X_3^2(q)) \rangle.$$

On the other hand, we have

$$X^2h(q) = \langle \nabla Xh(q), X(q) \rangle = \langle (X_3^1(q), X_3^2(q), 0), (X_1(q), X_2(q), X_3(q)) \rangle.$$

Since $X^2h(q) > 0$, we have $\langle \widetilde{Z}(q), \nabla Xh(q) \rangle = X^2h(q) > 0$ and the sliding vector field $\widetilde{Z}$ is transverse to $\partial\Sigma$ at q. The structural stability property follows from Proposition 5.2 found in (7). ■

Chapter 3

Sliding Shilnikov Orbit

In order to generalize the study of Shilnikov, Novaes and Teixeira ((8)) defined an equivalent result for piecewise vector fields, the *sliding Shilnikov orbit* and analyzed the behavior of the systems that exhibits such an orbit. In section 3.1, is presented the sliding Shilnikov orbit and state the main results for this chapter. In section 3.2, we construct a first return map in the neighborhood of the sliding Shilnikov orbit, which will be used to prove the theorems in section 3.3.

3.1 Statement of the Results

The following definition introduces the concept of sliding Shilnikov orbit.

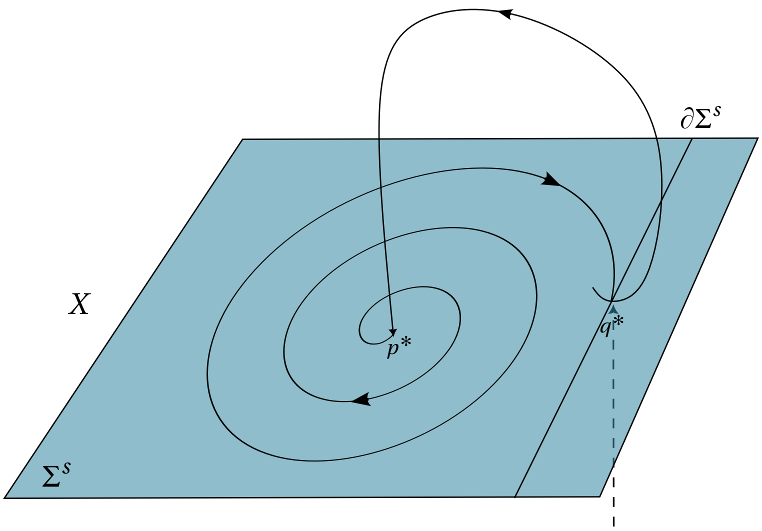

Figure 11 – Sliding Shilnikov orbit.

Definition 3.1 (See (8))**.** *Let $Z = (X, Y)$ be a piecewise continuous vector field having a hyperbolic pseudo-saddle-focus $p^* \in \Sigma^s$ and let $q^* \in \partial\Sigma^s$ be a visible fold-regular point of Z with respect to X such that:*

- *the backward trajectory of Z starting at q^* follows the sliding vector field $\widetilde{Z}$ and converges to p^*.*
- *the forward trajectory of Z, starting at q^*, intersects the switching surface only at crossing points before reaching p^*. It eventually reaches p^* at a finite time $t_0 > 0$*

*Through p^* and q^* we can characterize a sliding loop Γ. We call Γ a **sliding Shilnikov orbit**. We denote $\Gamma^+ = \Gamma \cap X$ and $\Gamma^s = \Gamma \cap \Sigma^s$.*

If $Z_\mu = (X_\mu, Y_\mu) \in \Omega^r$ is a 1-parameter family of Filippov systems breaking the sliding Shilnikov orbit for $\mu \neq 0$, then this family is referred to as a **splitting** of Γ_0. The first

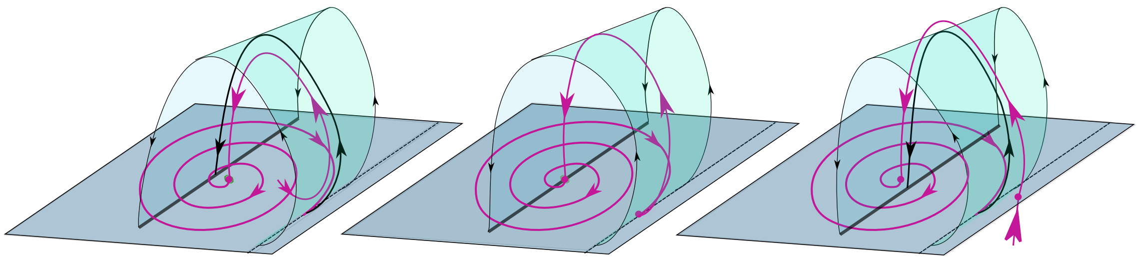

Figure 12 – A splitting of Γ_0.

main result in this part demonstrates that if $Z_0 \in \Omega^r$ exhibits a sliding Shilnikov orbit, then any neighborhood $U \subset \Omega^r$ of Z_0 contains infinitely many topological equivalence classes of vector fields. This result is proven in subsection 3.3.1.

Theorem 3.2 ((8))**.** *Assuming that $Z_0 = (X_0, Y_0) \in \Omega^r$ belongs to Ω^r and has a sliding Shilnikov orbit Γ_0, and considering $U \subset \Omega^r$ be a small neighborhood of Z_0, we can conclude the following:*

There exists a function $f : U \to \mathbb{R}$, with 0 as a regular value, such that $Z \in U$ admits a sliding Shilnikov orbit if and only if $f(Z) = 0$. Moreover, there are in U infinitely many topological equivalence classes of Filippov vector fields present.

The second main result in this part is a version of Shilnikov's Theorem concerning the existence of sliding periodic orbits of Z_μ that are 1-periodic within a neighborhood of a sliding Shilnikov orbit. The proof of this result is presented in subsection 3.3.2.

Theorem 3.3 ((8))**.** *Assuming that $Z_0 = (X_0, Y_0) \in \Omega^r$ and has a sliding Shilnikov orbit Γ_0, and considering $Z_\mu = (X_\mu, Y_\mu) \in \Omega^r$ as a splitting of Γ_0, the following statements hold:*

- *for $\mu = 0$, every neighborhood $U \subset \mathbb{R}^3$ of Γ_0 contains countably infinitely many sliding periodic orbits of Z_0;*

- *Let $W \subset \mathbb{R}^3$ be a small neighborhood of Γ_0. Then, for each $|\mu| \neq 0$ sufficiently small, W contains at least a finite number $N(\mu) > 0$ of sliding periodic orbits of Z_μ. Moreover, $N(\mu) \to \infty$ when $\mu \to 0$.*

3.2 First Return Map

Studying the first return map in a small neighborhood $V \subset \partial\Sigma^s$ of q^* allows us to comprehend the behavior of a system near a sliding Shilnikov orbit. Therefore, consider $Z_0 \in \Omega_h^r$ as a Filippov system that possesses a sliding Shilnikov orbit Γ_0, and let Z_μ represent a splitting of Γ_0.

Consider $x_s \in \Sigma^s$ and $v \in \mathbb{R}^3$ with $h(v) > 0$. Let $\psi_{\widetilde{Z}}(t, x_s)$ and $\psi_X(t, v)$ denote the solutions of the differential systems induced by $\widetilde{Z}$ and X, respectively. These solutions satisfy the initial conditions:

$$\psi_{\widetilde{Z}}(0, x_s) = x_s \text{ and } \psi_X(0, v) = v.$$

Take $r > 0$ sufficiently small such that $\eta_r := \overline{B_r(q^*) \cap \partial\Sigma^s}$ is a subset of the fold line, where $B_r(q^*) \subset \Sigma$ is the ball with center q^* and radius r.

We have the following claim:

Claim 3.4. *For $r > 0$ sufficiently small, we can find a function $\widetilde{t} : \eta_r \to \mathbb{R}_+^*$, such that $\psi_X(\widetilde{t}(x_s), x_s) \in \Sigma^s$ and this contact is transversal for all $x_s \in \eta_r$.*

Proof. From Definition 3.1, there exists $t_0 > 0$ such that $\psi_X(t_0, q^*) = p^* \in \Sigma^s$ and the intersection of this flow with Σ^s is transversal at p^*.

So we define the function

$$\begin{aligned} f: \quad \mathbb{R} \times \eta_r &\to \mathbb{R}^+ \\ (t, x_s) &\mapsto h(\psi_X(t, x_s)). \end{aligned}$$

Then, $f(t_0, q^*) = h(\psi_X(t_0, q^*)) = h(p^*) = 0$ since $p^* \in \Sigma$.

Furthermore, $\dfrac{\partial}{\partial t} f(t_0, q^*) = \nabla h(p^*) \cdot X(p^*) \neq 0$ since p^* is a regular point of X. Then, by the Implicit function theorem, taking an appropriate r, there exists a function $\widetilde{t}$ defined on η_r such that $\widetilde{t}(q^*) = t_0$ and $f(\widetilde{t}, x_s) = 0$ for all $x_s \in \eta_r$. Then, $\psi_X(\widetilde{t}(x_s), x_s) \in \Sigma^s$ for every $x_s \in \eta_r$. This function $\widetilde{t}$ is just the time that η_r takes to get to Σ^s under the flow of X. So we proved the claim. ■

Now, we define $\alpha_r := \{\psi_X(\widetilde{t}(x_s), x_s); x_s \in \eta_r\}$ and consider $\gamma : \eta_r \to \alpha_r$ the diffeomorphism

$$\gamma(x_s) = \psi_X(\widetilde{t}(x_s), x_s). \tag{3.1}$$

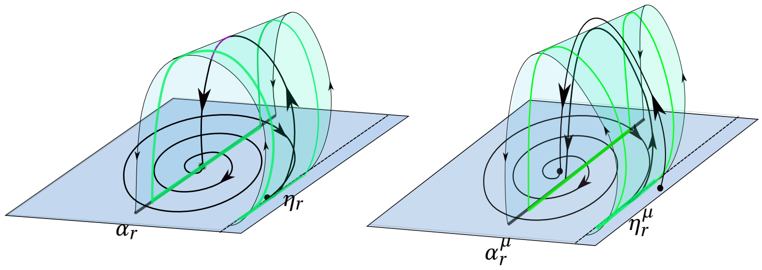

Figure 13 – Sketch of the diffeomorphisms γ and γ_μ, respectively.

Here this diffeomorphism is well defined because α_r is taken as the forward saturation of η_r. In a similar manner, we can construct a diffeomorphism $\gamma_\mu : \eta_r^\mu \to \alpha_r^\mu$. However, it should be noted that the pseudo-saddle-focus is no longer included within α_r^μ.

Claim 3.5. *For sufficiently small values of $r > 0$, the backward saturation of η_r through the flow of $\widetilde{Z}$ intersects the curve α_r an infinite countable number of times. Moreover, a first return map π can be defined on a subset $\mathcal{V}_r \subset \eta_r$ that maps into η_r.*

Proof. Proposition 2.8 establishes that the intersection of Γ^s and $\partial\Sigma^s$ at q^* is transversal. Thus, by choosing a sufficiently small $r > 0$, the backward saturation $\mathcal{B}_r$ of η_r through the flow of $\widetilde{Z}$ converges to p^* due to the fact that $\psi_{\widetilde{Z}}(t, x_s)$ is a diffeomorphism. Therefore

$$\mathcal{B}_r \cap \alpha_r = \bigcup_{i=1}^{\infty} J_i, \tag{3.2}$$

where $J_i \cap J_j = \emptyset$ if $i \neq j$ and, since $p^* \in \alpha_r$, $\lim_{i\to\infty} J_i = \{p^*\}$.

For each $i \in \mathbb{N}$ we define $I_i := \gamma^{-1}(J_i) \subset \eta_r$. Since γ is a diffeomorphism, then $I_i \cap I_j = \emptyset$ if $i \neq j$ and

$$\lim_{i\to\infty} I_i = \lim_{i\to\infty}(\gamma^{-1}(J_i)) = \gamma^{-1}(\lim_{i\to\infty} J_i) = \{q^*\}.$$

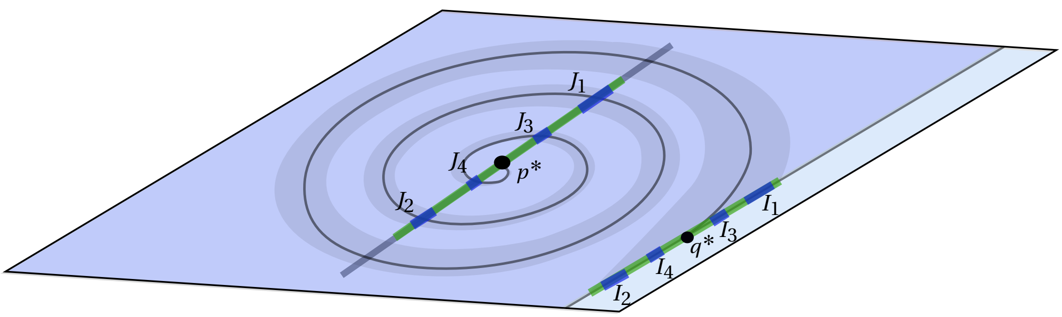

Figure 14 – Construction of the intervals J_i and I_i.

We point out that there exist a function $\widetilde{t}_{\widetilde{Z}} : \alpha_r \backslash \{p^*\} \to \mathbb{R}^*_+$ such that, for each $y \in \alpha_r \backslash \{p^*\}$, $\widetilde{t}_{\widetilde{Z}}$ is the time such that $\psi_{\widetilde{Z}}(\widetilde{t}_{\widetilde{Z}}(y_s), y_s) \in \eta_r$. Defining

$$\mathcal{V}_r := \bigcup_{i \in \mathbb{N}} I_i, \tag{3.3}$$

we finally can establish the first return map $\pi : \mathcal{V}_r \to \eta_r$ as

$$\pi(x_s) = \psi_{\widetilde{Z}}(\widetilde{t}_{\widetilde{Z}}(\psi_X(\widetilde{t}(x_s), x_s)), \psi_X(\widetilde{t}(x_s), x_s)). \tag{3.4}$$

When $|\mu| \neq 0$ is sufficiently small, we can follow the previous procedure to construct a first return map $\pi_\mu : \mathcal{V}_r^\mu \to \eta_r^\mu$ using the splitting system Z_μ. However, in this case, the backward saturation of η_r^μ through the flow of $\widetilde{Z}_\mu$ intersects α_r^μ in a finite number of connected components. As a result, the set $\mathcal{V}_r^\mu$ will be a finite union of n_μ intervals:

$$\mathcal{V}_r^\mu = \bigcup_{i=1}^{n_\mu} I_i^\mu,$$

Here, each interval I_i^μ corresponds to the inverse image of J_i^μ under the diffeomorphism γ_μ.

■

The subsequent result aims to comprehend the behavior of the aforementioned first return map.

Proposition 3.6 (See (9))**.** *Considering $\mathcal{V}_r$ as defined in 3.3, there exists a sufficiently small $r > 0$ such that $|\pi'(x_s)| > 1$ for every $x_s \in \mathcal{V}_r$. As a result, for $|\mu| \neq 0$ sufficiently small, $|\pi'(x_s)| > 1$ holds for every $x_s \in \mathcal{V}_r^\mu$.*

Proof. We saw that, for $r > 0$ small, that the curve α_r is transversal to the flow of the vector field $\widetilde{Z}$ and the focus p^* is contained in this curve.

Claim 3.7. *There exists $R_1, R_2 > 0$ such that, for each $0 < r \leqslant R_1$, a first return map $\zeta_r : \alpha_r \to \alpha_{R_2}$ is well defined.*

Proof. Indeed, since the point p^* is a focus of the sliding vector field, we know that, for a small neighborhood of p^*, the trajectories spiral around p^* tending to p^* in negative time and, for positive time, these trajectories are increasing. Then, for $R_1 > 0$ sufficiently small, there exist $t > 0$ and $R_2 > 0$ such that $\psi_{\widetilde{Z}}(t, y_s) \in \alpha_{R_2}$ for every point $y_s \in \alpha_{R_1}$. Since $\alpha_r \subset \alpha_{R_1}$ for all $0 < r \leqslant R_1$, then we can define the first return map

$$\zeta_r : \alpha_r \to \alpha_{R_2},$$

by composing the flow of the sliding vector field $\psi_{\widetilde{Z}}$. ■

Since p^* is a hyperbolic fixed point of ζ_r, define

$$\widetilde{\zeta}_r(x_s) = \zeta_r(x_s) - p^*,$$

and we can see that $\widetilde{\zeta}_r(p^*) = 0$. By Theorem 1.9 there exists a neighborhood V' of the origin and a C^1-diffeomorphism $\widetilde{H} : V' \to V'$ such that

$$\widetilde{\zeta}_r(x_s) = \widetilde{H} \circ \lambda \circ \widetilde{H}^{-1}(x_s),$$

with $|\lambda| > 1$. As we are just moving our point, the behavior is the same close to p^*, so, without loss of generality, we can assume that there exists a neighborhood $V \subset \alpha_{R_1}$ of p^* and a C^1-diffeomorphism $H : V \to V$ such that

$$\zeta_r(x_s) = H \circ \lambda \circ H^{-1}(x_s),$$

for all $x_s \in V$, where $|\lambda| > 1$.

We can choose $0 < r < R_1$ sufficiently small such that $\alpha_r \subset V$. So, if $\zeta_r^{k-1}(x_s) \in V$, then $\zeta_r^k(x_s) = H(\lambda^k H^{-1}(x_s))$.

Given that the backward saturation $\mathcal{B}_r$ of η_r through the flow of $\widetilde{Z}$ converges to $p^* \in V$, we can identify the first connected component of α_r that lies entirely within $V \cap \mathcal{B}_r$ as $\mathcal{S}$. The flow of $\widetilde{Z}$ induces a diffeomorphism ρ between $\mathcal{S}$ and η_r, while the flow of X induces a diffeomorphism γ (see 3.1) between η_r and α_r.

Since ρ and γ are diffeomorphism and $\mathcal{S}$ and η_r are compact subsets, so ρ' and γ' admits minimum points. Then, there exists $\delta > 0$ and $\delta_X > 0$ such that $\delta = min\{|\rho'(x_s)|; x_s \in \mathcal{S}\}$ and $\delta_X = min\{|\gamma'(x_s)|; x_s \in \eta_r\}$.

Given $i_0 \in \mathbb{N}$, there exists a sufficiently small $0 < r < R_1$ such that $\zeta_r^i \cap \mathcal{S} = \varnothing$ for every $0 < i < i_0$. Since $|\lambda| > 1$ and i_0 is arbitrary, we may assume that $\delta_X \delta |\lambda|^{i_0} > 1$.

Finally take $\mathcal{V}_r$ as 3.3, for every $x_s \in \mathcal{V}_r$, let m be a positive integer such that $\zeta_r^m(\gamma(x_s)) \in \mathcal{S}$. From the continuity of the map ζ_r, there exists a neighborhood $W \subset \mathcal{V}_r$ of x_s such that $\zeta_r^m(\gamma(w)) \in \mathcal{S} \subset V$ for every $w \in W$. You can see that $m \geqslant i_0$. Therefore, for every $w \in W$, the first return map reads

$$\begin{aligned} \pi(w) &= \rho \circ \zeta_r^m \circ \gamma(w) \\ &= \rho(H(\lambda^m H^{-1}(\gamma(w)))). \end{aligned}$$

Then,

$$\begin{aligned} \pi'(x_s) &= \rho'(H(\lambda^m H^{-1}(\gamma(w)))) \cdot (H(\lambda^m H^{-1}(\gamma(w))))' \\ &= \rho'(H(\lambda^m H^{-1}(\gamma(w)))) \cdot (H'((\lambda^m H^{-1}(\gamma(w))) \cdot \lambda^m (H^{-1})'(\gamma(w)) \cdot \gamma'(w) \\ \therefore \quad & |\pi'(x_s)| \geqslant \delta_X \delta |\lambda|^m > \delta_X \delta |\lambda|^{i_0} > 1. \end{aligned}$$

Then we proved our result. For a splitting the proof is similar. ■

3.3 Proof of the main results

To establish a common framework for the proofs, we consider the following assumptions:

Without loss of generality, we set $h(x, y, z) = z$ and consider $\Sigma = \{(x, y, 0); x, y \in \mathbb{R}\}$ as our switching surface. Let $Z_0 = (X_0, Y_0)$ be a system that possesses a sliding Shilnikov orbit Γ_0. This orbit connects the hyperbolic pseudo-saddle-focus $p^* = (0,0,0) \in \Sigma^s$ (or Σ^e) to itself and includes the visible fold-regular point q^* of Z_0 with respect to X_0. Additionally, we assume that the orbit of Z_0 connecting p^* to q^* intersects the switching surface Σ only at p^* and q^*.

Let $U \subset \Omega^r$ be a small neighborhood of Z_0. According to the structural stability property of a fold-regular point (see Proposition 2.8), for each $Z = (X, Y) \in U$, there exists a fold-regular point q_Z^* that lies on a curve of fold-regular points η_r^Z, where $\eta_r^Z \to \eta_r$ and $q_Z^* \to q^*$ as $Z \to Z_0$.

Similar to the approach used in section 3.2, we can deduce that the backward trajectories of $\widetilde{Z}$ starting at points on η_r^Z converge to p_Z^*, and the forward trajectories of X, starting at points on η_r^Z, intersect the switching surface transversely along a curve denoted as α_r^Z. Thus, as $Z \to Z_0$, we have $\alpha_r^Z \to \alpha_r$. It follows that Z possesses a sliding Shilnikov orbit if, and only if, $p_Z^* \in \alpha_r^Z$.

3.3.1 Proof of Theorem 3.2

Proof. For simplicity, let us consider α_r as the curve $\{(x, 0, 0) \mid -r \leqslant x \leqslant r\}$. Now, for any $Z \in U$, the curve α_r^Z can be represented as a graph $y = g_Z(x) = c_1^Z + c_2^Z x + O_2(x)$, where $c_1^{Z_0} = c_2^{Z_0} = 0$, and c_1 and c_2 are chosen to be sufficiently small in $\mathbb{R}$.

Denote the pseudo-equilibrium $p_Z^* = (x_z, y_z, 0)$, then we define the function f by:

$$\begin{aligned} f: \ U &\to \ \mathbb{R} \\ Z &\mapsto \ g_Z(x_Z) - y_Z. \end{aligned} \tag{3.5}$$

Then f is a C^1 function and $f(Z_0) = g_{Z_0} - y_{Z_0} = 0$. To finish the proof, we have to show that 0 is a regular value of f, i.e, the linear transformation f' is surjective for every $Z \in f^{-1}(0)$.

Let $\bar{Z} \in U$ satisfying $f(\bar{Z}) = 0$ and take $V \in \Omega^r$. Then, there exist a smooth curve $Z(\tau) \in \Omega^r$ such that $Z(0) = \bar{Z}$ and $Z'(0) = V$. So the derivative of f at $\bar{Z}$ in the direction V is given by:

$$f'(\bar{Z}) \cdot V = \frac{d}{d\tau} f(Z(v)) \mid_{\tau=0} = \lim_{\tau \to 0} \frac{f(Z(\tau)) - f(\bar{Z})}{\tau}.$$

Now, exists $\tau > 0$ such that $p^*_{Z(\tau)} = (0,0,0)$ and $g_{Z(\tau)}(x_{Z(\tau)}) = \tau$ because the pseudo-equilibrium $p^*_{Z(\tau)}$ is not contained in $\alpha_r^{Z(\tau)}$. Then we get that $f(Z(\tau)) = \tau$ and, therefore, $f'(\bar{Z}) \cdot V = 1$. This implies that $f'(\bar{Z})$ is surjective for every $\bar{Z} \in f^{-1}(0)$, and 0 is a regular value of f.

Consider the splitting Z_μ of the sliding Shilnikov orbit Γ_0. Since the pseudo equilibrium p^*_μ of $\widetilde{Z}_\mu$ is not contained in α_r^μ for $\mu \neq 0$, we can observe that the saturation of η_r^μ through the backward flow of $\widetilde{Z}_\mu$ intersects α_R^μ in a finite number of disjoint sets, as shown in 3.2. Let n_μ denote the number of such intersections. Thus, we can find trajectories of $\widetilde{Z}_\mu$ starting at η_r^μ that intersect α_r^μ in exactly n_μ points. Furthermore, it can be noted that the intersection between α_r^μ and any trajectory of $\widetilde{Z}_\mu$ starting at η_r^μ contains at most n_μ points.

If $Z_1, Z_2 \in U$ are topologically equivalent, then the curves $\eta_r^{Z_1}$ and $\alpha_r^{Z_1}$ are mapped to the curves $\eta_r^{Z_2}$ and $\alpha_r^{Z_2}$, respectively. As a result, we have $n_{\mu_1} = n_{\mu_2}$, indicating that the number of intersections between $\alpha_r^{Z_1}$ and trajectories of $\widetilde{Z}_1$ starting at $\eta_r^{Z_1}$ is equal to the number of intersections between $\alpha_r^{Z_2}$ and trajectories of $\widetilde{Z}_2$ starting at $\eta_r^{Z_2}$. This implies the existence of infinitely many topological equivalence classes of Filippov vector fields in any neighborhood $U \subset \Omega^r$ of Z_0. Therefore, the proof is concluded.

■

3.3.2 Proof of Theorem 3.3

Proof. Using the notation of section 3.2, we are going to prove the first statement of Theorem 3.3. For $x_s \in \Sigma^s$ and $v \in \mathbb{R}^3$ let $\psi_{\widetilde{Z}}(t, x_s)$ and $\psi_X(t, v)$ be the flows of $\widetilde{Z}$ and X, respectively. If we consider $x_s \in I_i$, then there exists $t_i^s(x_s) < 0$ and $t_i^X(x_s) < 0$ such that $\xi_i(x_s) := \psi_{\widetilde{Z}}(t_i^s(x_s), x_s) \in J_i$ and $\psi_X(t_i^X(x_s), \xi_i(x_s)) \in I_i$. So, define

$$\phi_i(x_s) := \psi_X(t_i^X(x_s), \xi_i(x_s)).$$

Notice that ϕ_i is a C^r function and, for each $i \in \mathbb{N}$, if $y \in I_i$ and $x = \phi_i(y)$, then

$$\pi(x) = \pi(\phi_i(y)) = \pi(\psi_X(t_i^X(y), \xi_i(y))) = y.$$

This property implies that a fixed point of ϕ_i is also a fixed point of π. By the Brouwer fixed point theorem, since ϕ_i is defined from a compact set to itself, for each i there exists $q_i \in I_i$ such that $\phi_i(q_i) = q_i$. Consequently, we have $\pi(q_i) = q_i$, which means that q_i is a fixed point of π. Therefore, we obtain a sliding periodic orbit in I_i.

Continuing with this construction, we obtain a sequence $(q_i)_{i=1}^\infty$, where $q_i \in I_i$ and $\pi(q_i) = q_i$. It is worth noting that q_i converges to q^*. Thus, every neighborhood of Γ_0 contains countably infinitely many sliding periodic orbits of Z_0.

For the second statement, consider a splitting of Γ_0 denoted as Z_μ, where p^*_μ is not contained in α^μ_r. Let us define the intersection $S^\mu_r \cap \alpha^\mu_r \cap W$ as a collection of finite disjoint sets J_i, with n_μ sets in total. As μ approaches zero, the value of n_μ tends to infinity. By observing that the sliding periodic orbits for each Z_μ correspond to the fixed points of $\phi^\mu_i(x^\mu_s) := \psi_{X_\mu}(t^{X_\mu}_i(x^\mu_s), \xi^\mu_i(x^\mu_s))$, we can conclude the proof. ■

Chapter 4

Chaotic Behavior Close to a Sliding Shilnikov Orbit

The purpose of this chapter is to demonstrate that a vector field exhibiting a sliding Shilnikov orbit displays a peculiar behavior referred, in some contexts, to as *chaos*. In Section 4.1, we state the main result for this chapter developed by Novaes, Ponce and Varão, see (9), and introduce fundamental definitions that will be utilized to assess the *chaotic* nature of a system. In Section 4.2, we review properties of functions that preserve measure on a probability space. Additionally, in Section 4.2.1, we establish a class of measure-preserving functions known as *ergodic* functions and provide an example of such functions known as the *Bernoulli shift*. Finally, in Section 4.3, we present a proof of the main result.

4.1 Statement of the Results

The chaotic behavior of a dynamical system is typically characterized by the presence of an invariant set where the dynamics are transitive, sensitivity to initial conditions, and set of periodic points is dense. The main result of this chapter guarantees that a Filippov system exhibiting a sliding Shilnikov orbit indeed possesses these properties and thus has this type of chaotic behavior.

Theorem 4.1 ((9)). *Let $Z = (X, Y) \in \Omega^r$ be a Filippov system and assume that Z admits a sliding Shilnikov orbit Γ. Denote by π the first return map defined on $\mathcal{V}_r$ in the vicinity of q^*. For sufficiently small $r > 0$, there exists a set $\Lambda \subset \mathcal{V}r$ satisfying the following properties:*

- *For each $n \in \mathbb{N}$, there exists a π-invariant Cantor set $\Lambda_n \subset \Lambda$ such that $\pi \mid \Lambda_n$ is conjugate to the shift on $\Sigma_2 \times \Sigma_n^*$. In other words, there exists a homeomorphism*

$h_n : \Sigma_2 \times \Sigma_n^* \to \Lambda_n$ such that $h_n \circ \sigma_n = \pi \circ h_n$. Consequently, the dynamics on Λ_n are transitive, sensitive to initial conditions, and have dense periodic points.

- *There exists a homeomorphism $h : \Sigma_2 \times \Sigma^b \to \Lambda := \bigcup_n \Lambda_n$ that conjugates the dynamics of σ and π. Moreover, $\Lambda \cup q^*$ is a compact set. In particular, the topological entropy of π is infinite.*

In light of the aforementioned theorem, let us introduce the following definitions:

1. **Invariant Set**: A subset Λ of the state space is said to be an invariant set under a map or flow if, once a point belongs to Λ, its image under the map or flow also belongs to Λ;

2. **Topologically Transitivity**: A dynamical system is topologically transitive on an invariant set Λ if for every pair of nonempty open sets U and V in Λ, there exist $x \in U$, a trajectory of $\phi(t, x)$ of the system and $t_0 > 0$ such that $\phi(t_0, x) \in V$;

3. **Sensitivity to Initial Conditions**: Sensitivity to initial conditions refers to the property of a dynamical system where arbitrarily close initial conditions can lead to significantly different trajectories over time;

4. **Dense Periodic Points**: A set of periodic points is said to be dense if it is densely distributed in the state space, meaning that for any open subset of the state space, there exists at least one periodic point within that subset.

These definitions lay the foundation for understanding the behavior of the first return map π in relation to the sliding Shilnikov orbit and establish the chaotic nature of the Filippov system.

We are going to introduce some basic notions and results that we use to prove Theorem 4.1.

4.2 Measure-Preserving Maps

In this section we present the concepts of probability space and measure preserving maps.

Definition 4.2. *A **probability space** is a triple $(X, \mathcal{A}, \lambda)$, where X is a topological space, $\mathcal{A}$ is a σ-algebra and λ is a measure on $(X, \mathcal{A})$ such that $\lambda(X) = 1$.*

Given $(X, \mathcal{A}, \lambda)$, the probability of an event in some set B occurring is $\lambda(B)$. We recall the definition of a measurable map:

Definition 4.3. *Let $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{M}, \nu)$ be probability spaces and $f : X \to Y$. We say that f **is measurable** if for all $B \in \mathcal{M}$, $f^{-1}(B) \in \mathcal{A}$ and we say that f **is measure-preserving** if for all $B \in \mathcal{M}$, $\lambda(f^{-1}(B)) = \nu(B)$.*

To understand the notion of a measure-preserving map, let us consider an intuitive example:

Example 4.4. *Consider $S^1 = \mathbb{R}/\mathbb{Z}$ the unit circle. We can put the normal Lebesgue measure from the real line onto S^1. Define $\mathcal{R}_\alpha : S^1 \to S^1$ where $\mathcal{R}_\alpha([x]) = [x + \alpha]$, for $\alpha \in \mathbb{R}$. So $\mathcal{R}_\alpha$ is the rotation of the circle by α.*

We observe that $\mathcal{R}_\alpha$ is measurable. Furthermore, $\mathcal{R}_\alpha$ is measure-preserving with respect to the Lebesgue measure. Indeed, notice that rotation does not change arc length. Thus, the pre-image of any measurable set ifs just a collection of arcs of the same length.

4.2.1 Ergodicity

In this subsection, we will define a sub-class of measure-preserving maps called **ergodic maps**. Ergodicity is a crucial property in dynamical systems as it signifies that the dynamics cannot be "broken down" into simpler or reducible measurable dynamics. It implies a specific type of chaos within the system with respect to a given measure. Let $(X, \mathcal{A}, \lambda)$ be a probability space.

Definition 4.5. *A map $f : X \to X$ is **ergodic** if f is measure-preserving and for every $B \in \mathcal{A}$ such that $f^{-1}(B) = B(mod\ (0))$, then either $\lambda(B) = 0$ or $\lambda(B) = 1$.*

Ergodicity is clearly a stronger condition for maps than being measure-preserving. In fact, let us consider what happens if f is measure-preserving but not ergodic. Then, there exist at least one set $B \in \mathcal{A}$ such that $\lambda(B) \in (0, 1)$ and $f^{-1}(B) = B$. But note that $\lambda(X \backslash B) \in (0, 1)$ and $f^{-1}(X \backslash B) = X \backslash B$. Thus, we can break such map f into two simpler maps $f\ |_B$ and $f\ |_{X \backslash B}$.

With the notation of Theorem 4.4 we have the following lemma.

Lemma 4.6. *Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure and the standard Borel σ-algebra. Let $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ an irrational number. Then the rotation by α, denoted by $\mathcal{R}_\alpha$, is ergodic.*

Proof. It is an elementary result in dynamics that the orbit under $\mathcal{R}_\alpha$ of every point in S^1 is dense. Thus, $n\alpha$ mod $\mathbb{Z}$ is dense in S^1 for all $n \in \mathbb{N}$. Let $B \subset S^1$ such that $\mathcal{R}_\alpha(B) = B$. Fix $\varepsilon > 0$, then there exists a continuous function $f \in L^1(S^1)$ (because continuous functions are dense in L^1) such that

$$\| f - \chi_B \|_1 < \varepsilon.$$

Because B is invariant under $\mathcal{R}_\alpha$, we can apply the triangle inequality to see that

$$\| f \circ \mathcal{R}_\alpha^n - f \|_1 \leqslant \| f \circ \mathcal{R}_\alpha^n - \chi_B \|_1 + \| \chi_B - f \|_1 \leqslant 2\varepsilon.$$

Notice that, $\mathcal{R}_\alpha^n = x + n\alpha \mod \mathbb{Z}$, so we have that

$$\| f(x + n\alpha) - f(x) \|_1 \leqslant 2\varepsilon$$

for all $n \in \mathbb{N}$ and $x \in S^1$. Because the orbit $n\alpha$ is dense in S^1 and f is continuous, we have that

$$\| f(x + t - f(x) \|_1 \leqslant 2\varepsilon \tag{4.1}$$

for all $t \in S^1$. We can apply Fubini's theorem to see that

$$\begin{aligned} \| f - \int f(t)dt \|_1 &= \int \mid \int f(x) - f(x+t)dt \mid dx \\ &\leqslant \int \int \mid f(x) - f(x+t) \mid dxdt \\ &\leqslant 2\varepsilon \end{aligned}$$

where the last inequality holds by 4.1. Notice that our choice of f gives us that

$$\| \chi_B - f \|_1 < \varepsilon \text{ and } \| f(t)dt - \mu(B) \|_1 < \varepsilon.$$

Therefore, applying the triangle inequality we have

$$\| \chi_B - \mu(B) \|_1 \leqslant \| \chi_B - f \| + \| f - \int f(t)dt \|_1 + \| \int f(t)dt - \mu(B) \|_1 \leqslant 4\varepsilon.$$

Because our choice of ε was arbitrary, we must have that χ_B is constant almost everywhere. Hence, $\mu(B)$ is either 0 or 1 and $\mathcal{R}_\alpha$ is ergodic. ■

4.2.2 Bernoulli Shifts

In this subsection, we will examine the behavior of the one-sided Bernoulli shift and establish its ergodicity. Let us begin by defining the relevant sets. Given a natural number $n \in \mathbb{N}^* := \mathbb{N} \backslash \{0\}$, we define $X = \{0, 1, \ldots, n-1\}$ and $X^* = \{1, \ldots, n\}$. We denote the space of all sequences of natural numbers between 0 and $n-1$ as $\Sigma_n = \{0, 1, \ldots, n-1\}^{\mathbb{N}}$, and the set $\Sigma_n^* = \{1, \ldots, n\}^{\mathbb{N}}$.

These sets are countable product spaces, where each coordinate takes values from a discrete compact set. By Tychonoff's theorem, see (10), both Σ_n and Σ_n^* are compact, equipped with the product topology induced by the discrete topology on the spaces X and X^*, respectively.

We can define a metric in this space that generates the product topology as follows:

$$d : \Sigma_n \times \Sigma_n \to \mathbb{R}$$
$$d\left((x_j)_{j\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}\right) = \begin{cases} 0, & \text{if } x_j = y_j \text{ for all } j \in \mathbb{N}, \\ \dfrac{1}{2^m}, & \text{where } m = \max\{a \in \mathbb{N} : x(i) = y(i), |i| < a\} \end{cases}$$

This metric quantifies the difference between two sequences by considering the first index at which they differ. If the sequences are identical, the distance is zero. Otherwise, the distance is determined by the reciprocal of the power of 2 corresponding to the first differing index.

We can interpret each element of X as a possible outcome of an experiment. Let each outcome have probability $p_0, ..., p_{n-1}$, respectively. In particular we have:

$$\sum_{i=0}^{n-1} p_i = 1.$$

We will use these probabilities to construct a probability measure on Σ_n. Let $\mathcal{A}$ be our σ-algebra (the set of all subsets of X) and for $B \in \mathcal{A}$ we define

$$p(B) = \sum_{x \in B} p_x,$$

where p_x is the probability of event x. A probability measure μ on Σ_n can be constructed by extending our measure of X.

Definition 4.7. *Given $j \in \mathbb{N}$ and m values $a_1, ..., a_m \in X$, a* ***cylinder*** *is the set defined by*

$$C(j; a_1, ..., a_m) := \{(x_i)_i \in \mathbb{N} : x_{j+1} = a_1, x_{j+2} = a_2, ..., x_{j+m} = a_m\}.$$

The disjoint union of such cylinder sets form an algebra which generates the Borel σ-algebra on Σ_n. Thus, we can take μ as the product measure $p^{\mathbb{N}}$. This measure is characterized by its values on cylinders. So, the measure μ of some cylinder $C(j; a_1, ..., a_m)$ is given by

$$\mu(C(j; a_1, ..., a_m)) = \prod_{i=1}^{m} p_{a_i}.$$

Such μ is called a **Bernoulli measure**. Acting on Σ_n we will consider a class of maps that "shift" the position of a sequence to the left or the right by some integer amount.

Definition 4.8. *The one-sided Bernoulli shift $\sigma : \Sigma_n \to \Sigma_n$ is the transformation where, for any $(x_j) \in \Sigma_n$, $\sigma((x_j)) = (x_{j+1})$.*

Now we can see that σ is measure-preserving for any Bernoulli shift $\sigma : \Sigma_n \to \Sigma_n$. In fact, fix $k \in \mathbb{N}$ and $a_1, ..., a_m \in X$. Let $C(k; a_1, ..., a_m)$ be a cylinder set. Then

$$\sigma^{-1}(C(k; a_1, ..., a_m)) = C(k+1; a_1, ..., a_m),$$

so,

$$\mu(C(k+1; a_1, ..., a_m)) = \prod_{i=1}^{m} p_{a_i} = \mu(C(k; a_1, ..., a_m)).$$

So σ is measure-preserving.

Theorem 4.9. *In the above notation, the left shift σ is ergodic under the Bernoulli measure μ.*

A proof of this result can be found at (11) but we shall prove it for the reader's convenience.

Proof. Suppose A in the σ-algebra generated by unions of cylinder and suppose $\sigma^{-1}(A) = A$. We will show that $\mu(A) = 0$ or $\mu(A) = 1$ by showing that $\mu(A)^2 = \mu(A)$. Fix $\varepsilon > 0$, so we can choose a finite union of cylinder sets $C := \bigcup_{j=1}^{k} \{C_j\}$ such that

$$\mu(A \triangle C) < \frac{\varepsilon}{4},$$

where $A \triangle C = (A \backslash C) \cup (C \backslash A)$ is the symmetric difference. Now we can see that

$$\begin{aligned} |\mu(A) - \mu(C)| &= |\mu(A\backslash C) + \mu(A \cap C) - (\mu(C\backslash A) + \mu(A \cap C))| \\ &= |\mu(A\backslash C) + \mu(A \cap C) - \mu(C\backslash A) - \mu(A \cap C)| \\ &= |\mu(A\backslash C) - \mu(C\backslash A)| \\ &\leqslant \mu(A\backslash C) + \mu(C\backslash A) \\ &\leqslant \frac{\varepsilon}{4}. \end{aligned} \tag{4.2}$$

How C is a finite union of cylinders, which have finite coordinates that contribute to the measure of C, then there exists $m \in \mathbb{N}$ and $B \in \Sigma_n$ such that $B = \sigma^{-m}(C)$ and

$$\mu(A \triangle B) = \mu(A \triangle \sigma^{-m}(C)) = \mu(\sigma^{-m}(A) \triangle \sigma^{-m}(C)) = \mu(A \triangle C) \tag{4.3}$$

because $\sigma^{-1}(A) = A$ and σ is measure-preserving. We point out that $\mu(B \cap C) = \mu(B)\mu(C) = \mu(C)^2$ because C and B are disjoint in the portions that contribute to their measures. Now we observe that

$$A \triangle (C \cap B) = (A \backslash (C \cap B)) \cup ((C \cap B) \backslash A) \subset (A \triangle C) \cup (A \triangle B) \tag{4.4}$$

and by 4.2 and 4.3, $\mu(A\triangle(C\cap B)) < \dfrac{\varepsilon}{2}$. Finally,

$$\begin{aligned}
|\mu(A)-\mu(A)^2| &= |\mu(A)-\mu(C\cap B)+\mu(C\cap B)-\mu(A)^2| \\
&\leqslant |\mu(A)-\mu(C\cap B)|+|\mu(C\cap B)-\mu(A)^2| \\
&< \frac{\varepsilon}{2}+|\mu(C)^2-\mu(A)^2| \\
&= \frac{\varepsilon}{2}+|\mu(C)^2-\mu(A)\mu(C)+\mu(A)\mu(C)-\mu(A)^2| \\
&\leqslant \frac{\varepsilon}{2}+|\mu(C)||\mu(C)-\mu(A)|+|\mu(A)||\mu(C)-\mu(A)| \\
&\leqslant \frac{\varepsilon}{2}+2|\mu(C)-\mu(A)| \\
&< \varepsilon.
\end{aligned}$$

Because ε is arbitrary, we have that $\mu(A)=\mu(A)^2$ and the shift map σ is ergodic. ■

In the context of a probability space, a measurable automorphism $f : X \to X$ is referred to as a **Bernoulli automorphism** if it is measurable isomorphic to a Bernoulli shift $\sigma : \Sigma_n \to \Sigma_n$, that is, f exhibits similar dynamical properties as the Bernoulli shift on a finite symbolic space Σ_n.

To prove our main result of this chapter we will work with the spaces $\Sigma_2 \times \Sigma_n^*$. For each $n \in \mathbb{N}^*$, on each $\Sigma_2 \times \Sigma_n^*$ we have the shift on two-coordinates

$$\sigma((x_j),(y_m)) = ((x_{j+1}),(y_{m+1})).$$

Let us define

$$\Sigma^b := \bigcup_{n\in\mathbb{N}^*} \Sigma_n^* = \{\{x_i\}_i \mid \exists L \in \mathbb{R}; \mid x_i \mid \leqslant L, x_i \in \mathbb{N}^* \forall i\}.$$

We consider the two coordinates shift

$$\sigma : \Sigma_2 \times \Sigma^b \to \Sigma_2 \times \Sigma^b.$$

To simplify the notation, we will denote the restriction of the Bernoulli shift σ to the space $\Sigma_2 \times \Sigma_n^*$ as σ_n.

The topological entropy is a crucial numerical invariant that is closely associated with the growth of orbits in a dynamical system. It quantifies the exponential rate at which the number of distinguishable orbit segments grows when observed with arbitrary but finite precision. In essence, the topological entropy provides a measure of the overall exponential complexity of the orbit structure using a single numerical value. It offers valuable insights into the intricate dynamics and complexity of a system's orbits.

Definition 4.10. *For a compact σ-invariant set $\Upsilon \subset \Sigma_2 \times \Sigma^b$ we define the **topological entropy** of $\sigma\mid_\Upsilon$ as*

$$h_{\sigma|_\Upsilon} := \lim_{n\to\infty} \frac{1}{n} \# Per_n(\sigma\mid_\Upsilon),$$

where $\#Per_n(\sigma|_\Upsilon)$ *means the number of periodic points of period* n*.*

As a observation, in the case of the Bernoulli shift $\sigma : \Sigma_n \to \Sigma_n$, $h_\sigma = \log(n)$.

4.3 Proof of Theorem 4.1

Proof. Considering the Filippov system $\dot{x} = Z(x)$ given by 2.1, we denote by $\phi(t, x)$ its solution satisfying the initial value $\phi(0, x) = x$. Assuming that Z contains a sliding Shilnikov orbit Γ, let $p^* \in \Sigma^s$ be a hyperbolic pseudo-saddle-focus, and $q^* \in \partial\Sigma^s$ be a visible fold point of Z with respect to X, as defined in Definition 3.1.

Consider a neighborhood $\eta \subset \partial\Sigma^s$ of q^* for which the first return map π is well-defined. Let α be the forward saturation of η through the flow of X that intersects Σ. We will denote the endpoints of η by q_0 and q_1, and represent η as $[q_0, q_1]$. According to Theorem 3.3, the set of points in η that return infinitely many times to η through the forward (Filippov) flow of Z is non-empty. Let us denote this set by Λ, i.e.,

$$\Lambda = \{x_s \in \eta \mid \exists\{t_n\}_n, t_n \to \infty, \phi(t_n, x_s) \in \eta\}.$$

We define $\eta_0 = [q_0, q^*]$ and $\eta_1 = [q^*, q_1]$, and consider α_0 and α_1 as the intersections of Σ with the forward saturation of η_0 and η_1, respectively. For a point $x_s \in \Lambda$, we denote by $\mu_\star(x_s)$, where $\star \in \{0, 1\}$, as the number of intersections that the forward flow orbit of x_s has with $\alpha_\star$ before returning to Λ. In other words,

$$\mu_\star(x_s) := \#\{\phi(t, x_s) \cap \alpha_\star; 0 < t < t_{x_s}\},$$

where t_{x_s} is the first return time of $\phi(t_{x_s}, x_s)$ on Λ.

Notice that $\mu_\star(x_s)$ counts the amount of times that the flow of $\widetilde{Z}$ intersects $\alpha_\star$. Our aim is to construct a map

$$h_n : \Sigma_2 \times \Sigma_n^* \to \Lambda,$$

that will conjugate the dynamics of σ_n with π. Fix a natural number $n > 0$ and take a point $(u, v) = ((u_j), (v_j)) \in \Sigma_2 \times \Sigma_n^*$. We will define $h_n((u, v))$ by a limit process.

Define $P_0(u, v)$ as the points which are in η_{u_0}, that is, $P_0(u, v) = \eta_{u_0}$. So, if $u_0 = 0$ (resp. $u_0 = 1$), then $\eta_{u_0} = \eta_0$ (resp. $\eta_{u_0} = \eta_1$). Now, define $P_1(u, v)$ as the points which are in η_{u_0} and, before arriving by the first return map to η_{u_1}, touches v_0 times the segment α_{u_1}, that is,

$$P_1(u, v) = \{x_s \in P_0(u, v); \mu_{u_1}(x_s) = v_0, \pi(x_s) \in \eta_{u_1}\}.$$

For example, if we consider $(u, v) = ((0, 1, \cdots), (3, \cdots))$, we have $P_0(u, v) = \eta_0$ and $P_1(u, v) = \{x_s \in \eta_0; \mu_1(x_s) = 3, \pi(x_s) \in \eta_1\}$.

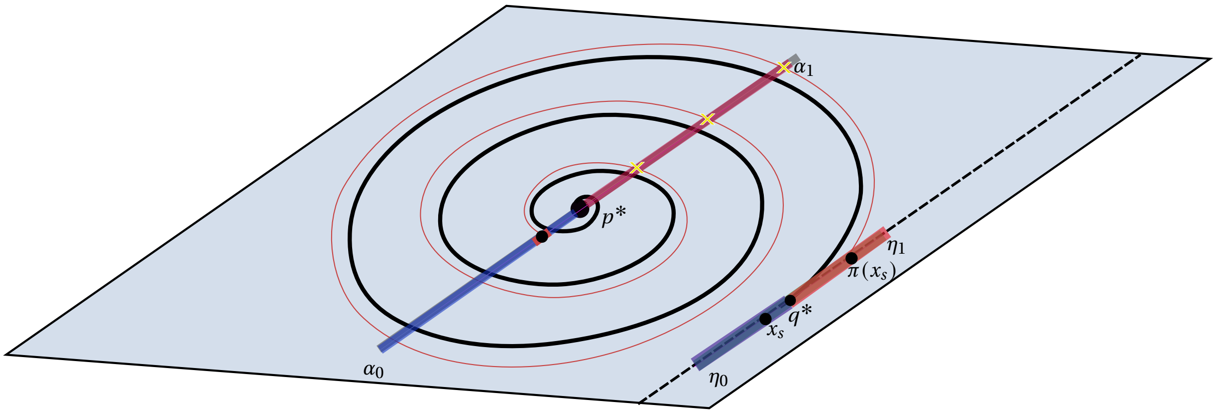

Figure 15 – Example for $u = (0, 1, \cdots)$ and $v = (3, \cdots)$.

In general, we define

$$P_{m+1}(u, v) = \{x_s \in P_m(u, v); \mu_{u_{m+1}}(x_s) = v_m, \pi^{m+1}(x_s) \in \eta_{u_{m+1}}\}.$$

Now, if we define the following set

$$P(u, v) := \bigcap_{i \in \mathbb{N}} P_i(u, v), \tag{4.5}$$

we get that $P(u, v)$ is a point or a interval since $P_i(u, v) \subset P_{i-1}(u, v)$ and each $P_i(u, v)$ is a closed interval. We want to show that $P(u, v)$ is, in fact, a point.

Claim 4.11. *If $P(u, v) \cap P(u', v') \neq \emptyset$, then $P(u, v) = P(u', v')$. In particular, $(u, v) = (u', v')$.*

Proof. Indeed, we can see that the set $P(u, v)$ is uniquely defined by the behavior of a orbit passing through the point (u, v). So, if there exists a point $(x, y) \in P(u, v) \cap P(u', v')$, then $P(u', v')$ there must be the same set of $P(u, v)$. ■

Claim 4.12. *$\pi(P(u, v))$ is of the form $P(u', v')$.*

Proof. In fact

$$\begin{aligned} \pi(P(u,v)) &= \pi\left(\bigcap_i P_i((u_j), (v_j))\right) = \left(\bigcap_i \pi(P_i((uj), (v_j)))\right) \\ &= \bigcap_i P_i((\pi(u_j), \pi(v_j))) = \bigcap_i P_i((u_{j+1}), (v_{j+1})) \\ &= P(\sigma(u, v)). \end{aligned} \tag{4.6}$$

And this concludes our lemma with $(u', v') = \sigma(u, v)$. ■

Claim 4.13. *If $\pi : \Lambda \to \Lambda$ is such that $|\pi'(x_s)| > 1$ for all $x_s \in \Lambda$, then $P(u, v)$ is a point for all $(u, v) \in \Sigma_2 \times \Sigma_n^*$.*

Proof. Let us consider ℓ as the length measure. We have to show that $\ell(P(u,v)) = 0$. Suppose by absurd that $\ell(P(u,v)) > 0$, then $\ell(\pi(P(u,v))) > \ell(P(u,v)) > 0$. But, by construction, since $\pi^n(P(u,v)) \subset \eta$ and $\ell(\eta) = |q_1 - q_0| < \infty$, then the family $\{\pi^n(P(u,v))\}$ cannot be pairwise disjoint. In fact, if it could be, we would have

$$|q_1 - q_0| = \ell(\eta) \geqslant \ell(\bigcup_n \pi^n(P(u,v))) = \sum_n \ell(\pi^n(P(u,v))) > \sum_n \ell(P(u,v)) = \infty,$$

and it is an absurd.

Therefore, there must exist n_0 and n_1 such that

$$\pi^{n_0}(P(u,v)) \cap \pi^{n_1}(P(u,v)) \neq \varnothing.$$

So by the above lemmas

$$\pi^{n_0}(P(u,v)) = \pi^{n_1}(P(u,v)),$$

and $\pi^{n_0 - n_1}(P(u,v)) = P(u,v)$ which cannot happen since $|\pi'| > 1$. Hence $\ell(P(u,v)) = 0$ and $P(u,v)$ is a point. ■

Defining

$$h_n : \Sigma_2 \times \Sigma_n^* \to \Lambda$$
$$(u,v) \mapsto P(u,v),$$

by Theorem 3.6 we know that if q_0 and q_1 are sufficiently close to q^*, then $|\pi'| > 1$ on η. This ensures that the functions h_n are well-defined, as stated in the lemma.

Now, considering that the domain of h_{n+1} contains the domain of h_n and these two functions coincide on the domain of h_n by construction, we can define a function $h : \Sigma_2 \times \Sigma^b \to \Lambda$ as follows:

$$(u,v) \mapsto h_n(u,v), \text{ if } (u,v) \in \Sigma_2 \times \Sigma_n^*,$$

Therefore, the function h is well-defined. Moreover, since we have proven that $\pi(P(u,v)) = P(\sigma(u,v))$ implies $\pi \circ h_n = h_n \circ \sigma_n$, it follows that $\pi \circ h = h \circ \sigma$.

Claim 4.14. *The maps h_n and h are continuous.*

Proof. Let $(u,v) \in \Sigma_2 \times \Sigma_n^*$ and $\varepsilon > 0$. So we have

$$h_n(u,v) = P(u,v) = \bigcap_i P_i(u,v),$$

and consider n_ε such that $P_{n_\varepsilon}(u,v) = (-\varepsilon + h_n(u,v), h_n(u,v) + \varepsilon) \cap \Lambda \subset \eta$, which is a open set in Λ. We have to show that the preimage of P_{n_ε} is an open set in $\Sigma_2 \times \Sigma_n^*$. Indeed, let $\mathcal{W}_{(u,v)}$ be a neighborhood of (u,v) in $\Sigma_2 \times \Sigma_n^*$ given by the cylinder

$$\mathcal{W}_{(u,v)} := \{(x,y); x_i = u_i, y_{i+1} = v_{i+1}, i \in \{0,1,...,n_\varepsilon\}\}.$$

And the continuity follows since

$$h_n(\mathcal{W}_{(u,v)}) = P_{n_\varepsilon}.$$

The proof for the continuity of h is the same. ■

Claim 4.15. *Maps h_n and h are homeomorphisms onto their image.*

Proof. To establish the homeomorphism property of $h_n : \Sigma_2 \times \Sigma_n^* \to \Lambda_n$, we can demonstrate its injectivity based on Theorem 4.11. Additionally, we can show that the inverse of h_n is continuous.

To illustrate the continuity of the inverse, let's consider a point $h_n(u, v)$ and a neighborhood $\mathcal{U}(u, v)$ containing (u, v). Here, $(x, y) \in \mathcal{U}(u, v)$ indicates that the first digits of the sequences (u, v) and (x, y) coincide.

By exploiting the continuity of the flow, we can assert that for points sufficiently close to $h_n(u, v)$, their trajectories must align with the predetermined trajectory associated with (u, v). As a result, the continuity property naturally follows.

In a similar manner, we can establish the homeomorphism property for h, analogous to the case of h_n.

■

The above lemmas imply Theorem 4.1.

■

Chapter 5

Sliding Shilnikov Orbit in Filippov Predator-Prey Model

In this chapter is presented a biological model that exhibits a sliding Shilnikov Orbit. In the field of ecology, the concept of *prey switching* refers to the adaptive behavior of predators, where they adjust their habitat or diet based on the abundance of available prey.

This model focuses on ciliates, a type of protist characterized by their animal-like behavior. These single-celled organisms are commonly found in aquatic environments and play a significant role in linking the lower and higher levels of marine and freshwater food webs. The investigation is specifically centered around Lake Constance, located on the border of Germany, Switzerland, and Austria. This lake has been a subject of scientific study for many years, and based on the existing data, Pilts et al.,(12), developed a model that captures the dynamics between a predator, its preferred prey, and alternative prey. This model incorporates a linear trade-off mechanism that quantifies the predator's preference between the two types of prey.

Carvalho, Gonçalves and Novaes proved in (13) that a dynamical predator-prey model that incorporates a linear trade-off in the predator's preference for different types of prey admites a Sliding Shilnikov Orbit, which is the main result of this chapter.

First, we consider $\dot{u} = (\dot{p}_1, \dot{p}_2, \dot{P})^T$, where $(p_1, p_2, P) \in \mathbb{R}^3_+$, then

$$\dot{u} = Z(u)$$

where

$$Z(u) = \begin{cases} ((r_1 - \beta_1 P)p_1,\ r_2 p_2,\ (eq_1\beta_1 p_1 - m)P) & if \quad h(p_1, p_2, P) > 0, \\ (r_1 p_1,\ (r_2 - \beta_2 P)p_2,\ (eq_2\beta_2 p_2 - m)P) & if \quad h(p_1, p_2, P) < 0. \end{cases} \tag{5.1}$$

with $h(p_1, p_2, P) = \beta_1 p_1 - a_q \beta_2 p_2$.

The switching manifold here is the plane $h^{-1}(0)$ and the variables of model 5.1, $p_1,\ p_2$ and P represents the density of preferred prey, alternative prey and predator population, respectively. The parameters $q_i \geqslant 0$ represent the preference for prey $i \in \{1,2\}$, and $a_q > 0$ is the slope of the preference trade-off. The intercept of the preference trade-off $b_q = q_2 - a_q q_1$ is assumed to satisfy $b_q \geqslant 0$. Furthermore, $e > 0$ is the proportion of predation that goes into predator growth, $\beta_1 > 0$ and $\beta_2 > 0$ are the death rates of the preferred prey and alternative prey due to predation, respectively. Now, $m > 0$ is the predator per capita death rate per day and $r_1 > r_2 > 0$ are the per capita growth rates of the preferred and alternative prey, respectively.

We can see that the parameters of the system 5.1 are of the form

$$\xi = (r_1, r_2, a_q, q_1, q_2, \beta_1, \beta_2, m, e) \in \mathbb{R}^2_r \times \mathbb{R}^3_{b_q} \times \mathbb{R}^4_+ = \mathcal{M},$$

where $\mathbb{R}^2_r = \{(r_1, r_2) \in \mathbb{R}^2_+; r_1 < r_2\}$ and $\mathbb{R}^3_{b_q} = \{(a_q, q_1, q_2) \in \mathbb{R}^3_+; q_2 \geqslant a_q q_1\}$.

We introduce the main result of this chapter:

Theorem 5.1 ((13))**.** *There exists a codimension one submanifold $\mathcal{N}$ of $\mathcal{M}$ such that the Filippov system 5.1 possesses a sliding Shilnikov orbit whenever $\xi \in \mathcal{N}$. Moreover, there exists a neighborhood $\mathcal{U} \subset \mathcal{M}$ of $\mathcal{N}$ such that the Filippov system 5.1 behaves chaotically whenever $\xi \in \mathcal{U}$.*

5.1 Proof of Theorem 5.1

First, we make the following change of variables:

$$x = \beta_1 p_1, \ y = a_q \beta_2 p_2 \text{ and } z = \beta_1 P.$$

So, we have

$$\dot{x} = \beta_1 \dot{p}_1, \ \dot{y} = a_q \beta_2 \dot{p}_2 \text{ and } \dot{z} = \beta_1 \dot{P}.$$

In these new variables, the system 5.1 is given by:

$$Z(x,y,z) = \begin{cases} X^+(x,y,z) & ,if \quad h(x,y,z) > 0, \\ X^-(x,y,z) & ,if \quad h(x,y,z) < 0. \end{cases} \tag{5.2}$$

where

$$\begin{aligned} X^+(x,y,z) &= ((r_1 - z)x, \ r_2 y, \ (eq_1 x - m)z), \\ X^-(x,y,z) &= \left(r_1 x, \left(r_2 - \frac{\beta_2}{\beta_1} z\right) y, \left(\frac{eq_2}{a_q} y - m\right) z\right), \end{aligned}$$

and $h(x,y,z) = x - y$.

Now the switching manifold Σ is given by the plan $\{(x,x,z) \in \mathbb{R}^3; x \geqslant 0, z \geqslant 0\}$. Since we are searching for a sliding Shilnikov orbit, we wnat to show that the system 5.2 satisfy the properties given in Definition 3.1. First, we are going to study the behavior of the vector fields and their contacts with the switching manifold Σ.

5.1.1 Singularities of the system 5.2.

We point out that the behavior of a flow of the vector field $X^+(x,y,z)$ can be understood by studying the behavior of the restricted vector field

$$\hat{X}^+(x,z) = ((r_1 - z)x,\ (eq_1x - m)z)\,,$$

since the system given by $X^+(x,y,z)$ is decoupled with respect to y. The equilibrium points of $\hat{X}^+(x,z)$ are $E_0 = (0,0)$ and $E_1 = \left(\dfrac{m}{eq_1}, r_1\right)$. The Jacobian matrix associated with this vector field is given by

$$D\hat{X}^+(x,z) = \begin{pmatrix} r_1 - z & -x \\ eq_1z & eq_1x - m \end{pmatrix}, \tag{5.3}$$

and $D\hat{X}^+(E_0)$ have eigenvalues $\lambda_0^1 = r_1$ and $\lambda_0^2 = -m$ with eigenvectors $(1,0)$ and $(0,1)$, respectively. By Theorem 1.8, the equilibrium E_0 is topologically equivalent to a saddle point. On the other hand, the equilibrium E_1 has pure imaginary eigenvalues, $\lambda_1^1 = i\sqrt{mr_1}$ and $\lambda_1^2 = -i\sqrt{mr_1}$, so we cannot use Theorem 1.8 is this case. To study the behavior of the equilibrium point E_1 we shall construct a first integral for the vector field as in section 1.4. Define

$$\begin{aligned} F : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x,z) &\mapsto eq_1x + z - m\log\left(\frac{eq_1x}{m}\right) - r_1\log\left(\frac{z}{r_1}\right) - (m + r_1), \end{aligned} \tag{5.4}$$

So we can calculate

$$\begin{aligned} \langle \nabla F(x), \hat{X}^+(x,z)\rangle &= \left\langle \left(\frac{eq_1x - m}{x}, \frac{z - r_1}{z}\right), ((r_1 - z)x, (eq_1x - m)z)\right\rangle \\ &= (eq_1x - m)(r_1 - z) + (z - r_1)(eq_1x - m) \\ &= 0. \end{aligned}$$

It implies that the behavior close to the point E_1 is a center type. Since the system X^+ is decoupled with respect to y and the solution of $\dot{y} = r_2y$ is given by $y(t) = y_0e^{r_2t}$ with $y_0 > 0$, then when $t \rightarrow \infty$, $y(t) \rightarrow \infty$ and the trajectories of X^+ spiral from $\{(x,0,z); x \geqslant 0, z \geqslant 0\}$ on the y direction until cross Σ.

5.1.2 Contact points of X^+ and X^- with Σ.

Now we have to calculate the Lie derivative of our vector fields with respect to the function h to determine tangency points. Note that we are looking for visible fold regular points of Z with respect to X^+. Let $p = (x,x,z) \in \Sigma$, then the Lie derivative of our vector fields are given by

$$X^+h(p) = \langle (1,-1,0), X^+(p)\rangle = (r_1 - z)x - r_2x = (r_1 - r_2 - z)x,$$

and

$$X^-h(p) = \langle (1,-1,0), X^-(p)\rangle = r_1 x - \left(r_2 - \frac{\beta_2}{\beta_1}z\right)x = \left(r_1 - r_2 + \frac{\beta_2}{\beta_1}z\right)x.$$

Solving $X^+h(p) = 0$, the tangency points of the vector field X^+ are living on the lines $S_1^+ = \{(0,0,z); z \geqslant 0\}$ and $S_2^+ = \{(x,x,r_1-r_2); x > 0\}$.

In the same way, the tangency points of the vector X^- are on the lines $S_1^- = S_1^+$ and $S_2^- = \left\{\left(x, x, \frac{\beta_1(r_2-r_1)}{\beta_2}\right); x > 0\right\}$. How $r_1 > r_2 > 0$ and we are considering points with positive coordinates, the curve S_2^- is not contained on our interested domain. Studying the sign of the Lie derivative, we have, for $p \in \Sigma$, the following:

- $X^+h(p) > 0 \Leftrightarrow z < r_1 - r_2$;
- $X^+h(p) < 0 \Leftrightarrow z > r_1 - r_2$;
- $X^-h(p) > 0 \Leftrightarrow z > \dfrac{\beta_1(r_2-r_1)}{\beta_2}$;
- $X^-h(p) < 0 \Leftrightarrow z < \dfrac{\beta_1(r_2-r_1)}{\beta_2} < 0.$

Then the switching manifold can be partitioned into two open regions:

$$\Sigma^c = X^+h(p)X^-h(p) > 0 = \{(x,x,z); 0 < z < r_1 - r_2\},$$

and

$$\Sigma^s = X^+h(p)X^-h(p) < 0 = \{(x,x,z); z > r_1 - r_2\}.$$

The tangency points that we are interested in are on the boundary of the sliding region, so they are in $S_2^+$to to. Solving the Lie derivative of order 2 with respect of the vector field X^+ for a point $q \in S_2^+$, we have:

$$(X^+)^2h(q) = \langle \nabla X^+h(q), X^+(q)\rangle = \langle (0,0,-x), X^+(q)\rangle = -x(eq_1x - m)(r_1 - r_2).$$

So

$$(X^+)^2h(q) = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{m}{eq_1}.$$

Then, for $0 < x < \dfrac{m}{eq_1}$, we have that $(X^+)^2h(q) > 0$ and we define a subset of the curve S_2^+ of visible fold points as $S_v^+ = \left\{(x,x,r_1-r_2); 0 < x < \dfrac{m}{eq_1}\right\}$. Our first result of this section proves that the forward saturation of any point in S_v^+ through the flow of X^+ intersects transversely the switching manifold Σ in finite time:

Claim 5.2. *Let* $x_0 \in \left(0, \dfrac{m}{eq_1}\right)$. *The forward trajectory of* X^+ *passing through* $(x_0, x_0, r_1 - r_2)$ *intersects the switching manifold* Σ *transversely at a point denoted as*

$$\mu(x_0) = (u(x_0), u(x_0), v(x_0)).$$

In other words, the saturation of S_v^+ *through the forward flow of* X^+ *intersects* Σ *transversely along the curve* $\{\mu(x_0); 0 < x_0 < \dfrac{m}{eq_1}\}$. *Furthermore, the following statements hold:*

1. *For* $x_0 < \dfrac{m}{eq_1}$ *sufficiently close to* $\dfrac{m}{eq_1}$, *we have*

$$\begin{aligned} u(x_0) &= \frac{m}{eq_1} - 3\left(x_0 - \frac{m}{eq_1}\right) + O\left(x_0 - \frac{m}{eq_1}\right)^2, \\ v(x_0) &= r_1 - r_2 + O\left(x_0 - \frac{m}{eq_1}\right)^2; \end{aligned}$$

2. *Given* $x_0 \in \left(0, \dfrac{m}{eq_1}\right)$, *for* $r_2 > 0$ *sufficiently small, we have*

$$\begin{aligned} u(x_0) &= x_0 + O(r_2), \\ v(x_0) &= r_1 + ((2r_1 T(x_0)(m - eq_1 x_0))r_2)^{1/2} + O(r_2^{3/2}), \end{aligned}$$

Here, $T(x_0)$ *represents the period, when* $r_2 = 0$, *of the solution* $(x(t, x_0, r_2), z(t, x_0, r_2))$.

Proof. Take $(x_0, x_0, r_1 - r_2) \in S_v^+$. The construction of the proof uses the parameter r_2 to control the trajectory of the vector field X^+ and it will be a bifurcation parameter value.

Let $\phi(t, x_0, r_2) := (x(t, x_0, r_2), y(t, x_0, r_2), z(t, x_0, r_2))$ be the solution of X^+ such that $\phi(0, x_0, r_2) = (x_0, x_0, r_1 - r_2)$. Notice that:

$$\begin{aligned} \frac{\partial x}{\partial t}(0, x_0, r_2) &= ((r_1 - z(t, x_0, r_2))x(t, x_0, r_2)) \mid_{t=0} = r_2 x_0; \\ \frac{\partial^2 x}{\partial t^2}(0, x_0, r_2) &= \left(-\frac{\partial z}{\partial t}(t, x_0, r_2)x(t, x_0, r_2) + (r_1 - z(t, x_0, r_2))\frac{\partial x}{\partial t}(t, x_0, r_2)\right) \mid_{t=0} \\ &= -(eq_1 x_0 - m)(r_1 - r_2)x_0 + r_2^2 x_0 \\ &= r_2^2 x_0 + x_0(r_1 - r_2)(m - eq_1 x_0); \\ \frac{\partial y}{\partial t}(0, x_0, r_2) &= r_2 y(0, x_0, r_2) = r_2 x_0; \\ \frac{\partial^2 y}{\partial t^2}(0, x_0, r_2) &= r_2 \frac{\partial y}{\partial t}(0, x_0, r_2) = r_2^2 x_0. \end{aligned}$$

Then we have

- $x(0, x_0, r_2) = y(0, x_0, r_2)$,
- $\dfrac{\partial x}{\partial t}(0, x_0, r_2) = \dfrac{\partial}{\partial t}(0, x_0, r_2)$,
- $\dfrac{\partial^2 x}{\partial t^2}(0, x_0, r_2) > \dfrac{\partial^2 y}{\partial t^2}(0, x_0, r_2)$.

For $t > 0$ sufficiently small, we have $x(t, x_0, r_2) > y(t, x_0, r_2)$. We point out that the solution $x(t, x_0, r_2)$ is bounded for every $0 < x_0 < \dfrac{m}{eq_1}$ since the equilibrium point E_1 is a center, and the solution $y(t, x_0, r_2)$ is unbounded increasing. Then, it must exist $t_1(x_0, r_2) > 0$ such that

$$\begin{aligned} x(t_1(x_0, r_2), x_0, r_2) &= y(t_1(x_0, r_2), x_0, r_2) \\ &= x_0 e^{r_2 t_1(x_0, r_2)}. \end{aligned} \tag{5.5}$$

So for each x_0 we have that exists a time $t_1(x_0, r_2)$ such that the trajectory passing tangentially by $(x_0, x_0, r_1 - r_2)$ reaches transversely the switching manifold Σ at a point $\phi(t_1(x_0, r_2), x_0, r_2)$.

Then we define $\mu(x_0) := \phi(t_1(x_0, r_2), x_0, r_2)$, where $u(x_0) = x(t_1(x_0, r_2), x_0, r_2)$ and $v(x_0) = z(t_1(x_0, r_2), x_0, r_2)$.

For the second part of the lemma we have to prove that the Taylor series of $u(x_0)$ around $\dfrac{m}{eq_1}$ is given by

$$u(x_0) = \frac{m}{eq_1} - 3\left(x_0 - \frac{m}{eq_1}\right) + O\left(x_0 - \frac{m}{eq_1}\right)^2.$$

First, we study the difference $x(t, x_0, r_2) - y(t, x_0, r_2) = x(t, x_0, r_2) - x_0 e^{r_2 t}$ around $t = 0$. Defining $\widetilde{\theta}(t, x_0) = x(t, x_0, r_2) - x_0 e^{r_2 t}$, we have:

$$\begin{aligned} \widetilde{\theta}(0, x_0) &= \quad x_0 - x_0 = 0; \\ \widetilde{\theta}'(0, x_0) &= \quad r_2 x_0 - r_2 x_0 = 0; \\ \widetilde{\theta}''(0, x_0) &= \quad x_0(r_1 - r_2)(m - eq_1 x_0); \\ \widetilde{\theta}'''(0, x_0) &= \quad (r_1 - r_2)x_0(eq_1 x_0(-eq_1 x_0 - 4r_2 + 2m) + m(3r_2 - m)); \\ &\quad \vdots \end{aligned} \tag{5.6}$$

and then, around $t = 0$ we have:

$$\frac{x_0(r_1 - r_2)(m - eq_1 x_0)}{2} t^2 + \frac{(r_1 - r_2)x_0(eq_1 x_0(-eq_1 x_0 - 4r_2 + 2m) + m(3r_2 - m))}{6} t^3 + O_4(t).$$

So we can define a function $\theta(t, x_0) := \dfrac{\widetilde{\theta}(t, x_0)}{t^2}$.

Since we want to study the behavior when x_0 is sufficiently close to $\dfrac{m}{eq_1}$, we apply the implicit function theorem for the point $\left(0, \dfrac{m}{eq_1}\right)$:

We have

$$\theta\left(0, \frac{m}{eq_1}\right) = 0 \text{ and } \frac{\partial\theta}{\partial t}\left(0, \frac{m}{eq_1}\right) = \frac{-r_2 m^2(r_1 - r_2)}{6eq_1} \neq 0.$$

Then, there exists a unique function $t_2(x_0)$ defined on a neighborhood of $\frac{m}{eq_1}$ such that $t_2\left(\frac{m}{eq_1}\right) = 0$ and $\theta(t_2(x_0), x_0) = 0$ for every x_0 in this neighborhood. Since we have $u(x_0) = x(t_1(x_0, r_2), x_0, r_2)$, from the uniqueness of t_2, for x_0 sufficiently close to $\frac{m}{eq_1}$, $t_1(x_0, r_2) = t_2(x_0)$. Using that,

$$t_2'\left(\frac{m}{eq_1}\right) = -\frac{\frac{\partial\theta}{\partial x_0}\left(0, \frac{m}{eq_1}\right)}{\frac{\partial\theta}{\partial t}\left(0, \frac{m}{eq_1}\right)} = \frac{-3eq_1}{mr_2},$$

we calculate $u(x_0)$ and $v(x_0)$ around $x_0 = \frac{m}{eq_1}$:

$$\begin{aligned} u(x_0) &= u\left(\frac{m}{eq_1}\right) + \frac{\partial u}{\partial x_0}\left(\frac{m}{eq_1}\right)\left(x_0 - \frac{m}{eq_1}\right) + O_2\left(x_0 - \frac{m}{eq_1}\right)^2 \\ &= \frac{m}{eq_1} - 3\left(x_0 - \frac{m}{eq_1}\right) + O_2\left(x_0 - \frac{m}{eq_1}\right)^2; \\ v(x_0) &= (r_1 - r_2) + O_2\left(x_0 - \frac{m}{eq_1}\right)^2. \end{aligned} \tag{5.7}$$

Finally, for the second statement, we shall prove that given $\widetilde{x_0} \in \left(0, \frac{m}{eq_1}\right)$, there exists a neighborhood $\mathcal{U}$ of $\widetilde{x_0}$ and $\widetilde{r_2} > 0$ such that $u(x_0) < \frac{m}{eq_1}$ and $v(x_0) > r_1$ for every $(x_0, r_2) \in \mathcal{U} \times (0, \widetilde{r_2}]$. So, we define the following function

$$\vartheta(t, r_2) = x(t, x_0, r_2) - x_0 e^{r_2 t}.$$

Since $(x(t, x_0, r_2), z(t, x_0, r_2))$ is periodic in the variable t (because we have predator-prey system), for $r_2 = 0$, denote by $T(x_0) > 0$ the period of the solution

$$(x(t, x_0, 0), z(t, x_0, 0)),$$

i.e,

$$(x(T(x_0), x_0, 0), z(T(x_0), x_0, 0)) = (x_0, r_1).$$

We can see that $\vartheta(T(x_0), 0) = x(T(x_0), x_0, 0) - x_0 = 0$.

Now, using Theorem 1.9, we are going to show that there is a saddle-node bifurcation occurring at $t = T(x_0)$, where the bifurcation parameter is $r_2 = 0$. Notice that we have the first integral given by 5.4 of the decoupled system, so,

$$F(x(t, x_0, r_2), z(t, x_0, r_2)) = F(x_0, r_1 - r_2).$$

Computing the derivative in the variable r_2 of the expression above at $t = T(x_0)$ and $r_2 = 0$, we get

$$\frac{\partial x}{\partial r_2}(T(x_0), x_0, 0) = 0.$$

Thus, we get

- $\dfrac{\partial \vartheta}{\partial t}(T(x_0), 0) = (r_1 - r_1)x_0 = 0;$
- $\dfrac{\partial^2 \vartheta}{\partial t^2}(T(x_0), 0) = r_1(m - eq_1x_0)x_0 > 0;$

and

$$\frac{\partial \vartheta}{\partial r_2}(T(x_0), 0) = -x_0T(x_0) < 0. \tag{5.8}$$

Then, by Theorem 1.9, we get the existence of a saddle-node bifurcation at $t = T(x_0)$. Now, to conclude this proof, we shall explicitly compute the solutions bifurcating from $t = T(x_0)$.

Since $\vartheta(T(x_0), 0) = 0$ and $\dfrac{\partial \vartheta}{\partial r_2}(T(x_0), 0) < 0$, from the implicit function theorem, exist a neighborhood I_1 of $T(x_0)$ and I_2 of 0 and a unique differentiable function $\varrho : I_1 \to I_2$ such that $\varrho(T(x_0)) = 0$ and $\vartheta(t, \varrho(t)) = 0$ for all $t \in I_1$.

Moreover,

$$\varrho'(T(x_0)) = -\frac{\frac{\partial \vartheta}{\partial t}(T(x_0), 0)}{\frac{\partial \vartheta}{\partial r_2}(T(x_0), 0)} = \frac{(r_1 - z(T(x_0), x_0, 0))x(T(x_0), x_0, 0)}{-x_0T(x_0)} = 0$$

and

$$\varrho''(T(x_0)) = \frac{r_1(m - eq_1x_0)}{x_0T(x_0)}.$$

For t sufficiently close to $T(x_0)$, $r_2 = \varrho(t)$, and then,

$$r_2 = \frac{r_1(m - eq_1x_0)}{2x_0T(x_0)}(t - T(x_0))^2 + O_3(t - T(x_0))^3. \tag{5.9}$$

Consider now the change $s = (t - T(x_0))$, then equation 5.9 gives $r_2 = \dfrac{r_1(m - eq_1x_0)}{2x_0T(x_0)}s + O_3(|s|^{3/2})$.

Define the function $\widetilde{\rho} : U_1 \to U_2$ by

$$\widetilde{\rho}(s) = \frac{r_1(m - eq_1x_0)}{2x_0T(x_0)}s + O_3(|s|^{3/2}),$$

where $s \in U_1$ and $r_2 \in U_2$ are open neighborhoods containing 0. Since $\widetilde{\rho}$ is differentiable and $\widetilde{\rho}'$ is an isomorphism, then, by the inverse function theorem, we get the existence of neighborhoods V_1 and V_2 of 0 and a unique differentiable function $\rho : V_1 \to V_2$ such that

$$s = \rho(r_2), \rho(0) = 0 \text{ and } \rho'(0) = \frac{2x_0 T(x_0)}{r_1(m - eq_1x_0)} > 0.$$

Then, going back through the change $s = (t - T(x_0))^2$, we get two distinct positive times $t = T(x_0) \pm \sqrt{\rho(r_2)}$ bifurcating from $t = T(x_0)$.

Since $t_1(x_0, r_2)$ is the first return time, we conclude that

$$\begin{aligned} t_1(x_0, r_2) &= T(x_0) - \sqrt{\rho(r_2)} \\ &= T(x_0) - \left(\rho(0) + \rho'(o)r_2 + O(r_2^2)\right)^{\frac{1}{2}} \\ &= T(x_0) - \left(r_2(\rho'(0) + O(r_2))\right)^{\frac{1}{2}} \\ &= T(x_0) - r_2^{\frac{1}{2}}\left(\rho'(0) + O(r_2)\right)^{\frac{1}{2}} \\ &= T(x_0) - (\rho'(0)r_2)^{\frac{1}{2}} + O(r_2^{\frac{3}{2}}) \\ &= T(x_0) - \left(\frac{2x_0T(x_0)}{r_1(m - eq_1x_0)}r_2\right)^{\frac{1}{2}} + O(r_2^{\frac{3}{2}}). \end{aligned} \tag{5.10}$$

Now, we compute, for $r_2 > 0$ sufficiently small and t close to $T(x_0)$, $v(x_0)$ is given by

$$\begin{aligned} z(t, x_0, r_2) &= z(t, x_0, 0) + \frac{\partial z}{\partial r_2}(t, x_0, 0)r_2 + O(r_2^2) \\ &= z(T(x_0), x_0, 0) + \frac{\partial z}{\partial t}((T(x_0), x_0, 0)(t - T(x_0)) + O((t - T(x_0))^2) \\ &= r_1 + (eq_1x(T(x_0), x_0, 0) - m)r_1(t - T(x_0)). \end{aligned}$$

And, finally, we can compute $u(x_0)$ and $v(x_0)$ for $r_2 > 0$ sufficiently small:

$$\begin{aligned} u(x_0) &= x_0e^{r_2t_1(x_0,r_2)} = x_0exp\left(r_2T(x_0) - \left(\frac{2x_0T(x_0)r_2}{r_1(m - eq_1x_0)}r_2\right)^{1/2} + r_2O(r_2^{3/2})\right) \\ &= x_0 + O(r_2); \\ v(x_0) &\quad r_1 + (eq_1x(T(x_0), x_0, 0) - m)r_1(t_1(x_0, r_2) - T(x_0)) \\ &= r_1 + (eq_1x_0 - m)r_1\left(T(x_0) - \left(\frac{2x_0T(x_0)}{r_1(m - eq_1x_0)}r_2\right)^{\frac{1}{2}} + O(r_2^{\frac{3}{2}}) - T(x_0)\right) \\ &= r_1 + (m - eq_1x_0)r_1\left(\left(\frac{2x_0T(x_0)}{r_1(m - eq_1x_0)}r_2\right)^{\frac{1}{2}} + O(r_2^{\frac{3}{2}})\right) \\ &= r_1 + (2x_0T(x_0)(m - eq_1x_0)r_2)^{\frac{1}{2}} + O(r_2^{\frac{3}{2}}). \end{aligned} \tag{5.11}$$

And this concludes the proof. ■

Our next result establishes some conditions that we are going to use to find the sliding Shilnikov orbit.

Claim 5.3. *There exists $a,\ b,\ c$ and d, with $0 < a < b < \dfrac{m}{eq_1}$ and $0 < c < d$, such that $0 < u(x_0) < \dfrac{m}{eq_1}$ and $v(x_0) > r_1$ for every $(x_0, r_2) \in [a,b] \times [c,d]$. Moreover, for $r_2 \in [c,d]$, $\mu(x_0)$ is differentiable on $[a,b]$ and $u'(x_0)^2 + v'(x_0)^2 \neq 0$ for every $(x_0, r_2) \in [a,b] \times [c,d]$.*

Proof. From the above result, we have that $u(x_0) > \dfrac{m}{eq_1}$ for x_0 sufficiently close to $\dfrac{m}{eq_1}$ and, for a fixed $x_0 \in \left(0, \dfrac{m}{eq_1}\right)$, there exists $\widetilde{r_2} > 0$ such that $u(x_0) < \dfrac{m}{eq_1}$ for every $r_2 \in (0, \widetilde{r_2}]$.

Therefore, since $u(x_0)$ is continuous, for $r_2 < \widetilde{r_2}$ there exists $x_0^* \in \left(0, \dfrac{m}{eq_1}\right)$ such that $u(x_0^*) = \dfrac{m}{eq_1}$ and $u(x_0) < \dfrac{m}{eq_1}$ for $x_0 < x_0^*$ sufficiently close to x_0^*.

Moreover, $v(x_0^*) > r_1$, and, consequently, $v(x_0) > r_1$ for $x_0 < x_0^*$ because $\left(\dfrac{m}{eq_1}, r_1\right)$ is a critical point for the first integral 5.4.

Take now $x_1 < x_0^*$ such that $v(x_1) > r_1$ and $u(x_1) < \dfrac{m}{eq_1}$. Thus, from the continuous dependence of solutions on the initial conditions and parameters, we get the existence of a, b, c and d, with $0 < a < x_1 < b < \dfrac{m}{eq_1}$ and $0 < c < r_{2_*} < d$ such that $0 < u(x_0) < \dfrac{m}{eq_1}$ and $v(x_0) > r_1$ for every $(x_0, r_2) \in [a,b] \times [c,d]$.

Now, we define

$$\Theta(t, x_0, r_2) = x(t, x_0, r_2) - x_0 e^{r_2 t},$$

and we have to show that μ is differentiable.

Indeed, from claim 5.2, we know that, for each $(\bar{x_0}, \bar{r_2}) \in [a,b] \times [c,d]$, we get the existence of $t_1(\bar{x_0}, \bar{r_2}) > 0$ such that $\Theta(t_1(\bar{x_0}, \bar{r_2}), \bar{x_0}; \bar{r_2}) = 0$.

Observe that

$$\begin{aligned} \frac{\partial \Theta}{\partial t}(t_1(\bar{x_0}, \bar{r_2}), \bar{x_0}\bar{r_2}) &= \frac{\partial x}{\partial t}(t_1(\bar{x_0}, \bar{r_2}), \bar{x_0}\bar{r_2}) - x_0 r_2 e^{r_2 t_1(\bar{x_0}, \bar{r_2})} \\ &= (r_1 - r_2 - v(\bar{x_0}))u(\bar{x_0}) \neq 0, \end{aligned}$$

then, by the implicit function theorem, there exists a unique differentiable function $t_2(x_0, r_2)$ defined in a neighborhood V of $(\bar{x_0}, \bar{r_2})$, such that

$$t_2(\bar{x_0}, \bar{r_2}) = t_1(\bar{x_0}, \bar{r_2}),$$

and $\Theta(t_2(x_0, r_2), x_0, r_2) = 0$ for every $(x_0, r_2) \in V$. According to the uniqueness property, it follows that $t_1 = t_2$, which implies the differentiability of t_1 at $(\bar{x_0}, \bar{r_2})$ and, consequently, the differentiability of μ at $x_0 = \bar{x_0}$ and $r_2 = \bar{r_2}$. Since $(\bar{x_0}, \bar{r_2})$ was taken arbitrarily in the compact set $[a,b] \times [c,d]$, we conclude the differentiability of μ for every (x_0, r_2).

Finally, notice that $u'(x_0) = v'(x_0) = 0$ and computing the derivative of the last identity in the variable x_0, we get that $x_0 = \dfrac{m}{eq_1}$. Then, $u'(x_0)^2 + v'(x_0)^2 \neq 0$ for every $(x_0, r_2) \in [a, b] \times [c, d]$. ■

5.1.3 The sliding vector field of the system 5.2.

The goal of this subsection is to study the behavior of the sliding vector field seeking to guarantee the conditions for the existence of the sliding Shilnikov orbit. Considering the system 5.2, we are going to do another change of variables to make our work easier. Let $w = x - y$ and $p = (x, w, z)$, then 5.2 is written as

$$Z(p) = \begin{cases} \left((r_1 - z)x,\ r_2 w + x(r_1 - r_2 - z),\ (eq_1 x - m)z\right), & \text{if } h(p) > 0, \\ \left(r_1 x,\ r_1 x - (x - w)\left(r_2 - \dfrac{\beta_2}{\beta_1} z\right),\ \left(\dfrac{eq_2}{a_q}(x - w) - m\right) z\right), & \text{if } h(p) < 0. \end{cases} \tag{5.12}$$

where $h(p) = h(x, w, z) = w$ and $(x, w, z) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R} \times \mathbb{R}_{\geqslant 0}$. As we defined before, we can calculate the sliding vector field given by the formula 2.2 and

$$\widetilde{Z}(p) = \begin{pmatrix} x\left(\dfrac{\beta_1 r_2 + \beta_2 r_1}{\beta_1 + \beta_2} - \dfrac{\beta_2 z}{\beta_1 + \beta_2}\right) \\ -mz + x\left(\dfrac{e(a_q q_1 - q_2)(r_1 - r_2)\beta_1}{a_q(\beta_1 + \beta_2)} + \dfrac{e(\beta_1 q_2 + a_q \beta_2 q_1)z}{a_q(\beta_1 + \beta_2)}\right) \end{pmatrix}. \tag{5.13}$$

The vector field 5.13 have two equilibrium points, $(0, 0)$ and

$$p^* = (x^*, z^*) = \left(\frac{a_q m(\beta_1 r_2 + \beta_2 r_1)}{e(\beta_1 q_2 r_2 + a_q \beta_2 q_1 r_1)}, r_1 + \frac{\beta_1 r_2}{\beta_2}\right). \tag{5.14}$$

Now, we have

$$0 < x^* < \frac{m}{eq_1} \text{ and } z^* > r_1.$$

In order to study the behavior of p^* and determine when it can be a hyperbolic pseudo-saddle-focus, we have the following lemma.

Lemma 5.4. *Let $\xi \in \mathcal{M}$ and assume that*

$$m < \frac{4(\beta_1 + \beta_2)(r_2\beta_1 + r_1\beta_2)(q_2 r_2 \beta_1 + a_q q_1 r_1 \beta_2)^2}{(q_2 - a_q q_1)^2 (r_1 - r_2)^2 \beta_1^2 \beta_2^2}. \tag{5.15}$$

As such, the following statements hold:

- *the equilibrium p^* is a repulsive focus;*
- *there exists $\widetilde{x} \in \left[0, \dfrac{m}{eq_1}\right)$ such that the backward orbit of $\widetilde{Z}$ of any point of the straight segment $L = \left\{(x, r_1 - r_2); \widetilde{x} < x \leqslant \dfrac{m}{eq_1}\right\} \subset S_v^+$ is contained in Σ^s and converges asymptotically to the equilibrium p^*.*

Proof. The Jacobian matrix of $\widetilde{Z}$ at (x^*, z^*) is given by

$$\begin{pmatrix} 0 & -\dfrac{a_q m\beta_2(r_2\beta_1 + r_1\beta_2)}{e(\beta_1+\beta_2)(q_2r_2\beta_1 + a_q q_1 r_1\beta_2)} \\ e\left(q_1r_1 + \dfrac{q_2r_2\beta_1}{a_q\beta_2}\right) & -\dfrac{m(a_q q_1 - q_2)(r_1-r_2)\beta_1\beta_2}{(\beta_1+\beta_2)(q_2r_2\beta_1 + a_q q_1 r_1\beta_2)} \end{pmatrix}$$

Let $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ be the eigenvalues of the matrix given above. So we have

$$\widetilde{\lambda}_1 = \frac{A-B}{C} \text{ and } \widetilde{\lambda}_2 = \frac{A+B}{C},$$

where

$$\begin{aligned} A &= a_q e\beta_1\beta_2^2 m(r_1-r_2)(-a_q q_1 + q_2), \\ B &= a_q\beta_2 e\Bigg[m\Bigg(a_q^2\beta_2^2q_1^2\left(\beta_1^2 m(r_1-r_2)^2 - 4r_1^2(\beta_1+\beta_2)(\beta_2r_1+\beta_1r_2)\right) \\ &\quad - 2a_q\beta_1\beta_2q_1q_2\left(\beta_1\beta_2\left(m(r_1-r_2)^2 + 4r_1r_2(r_1+r_2)\right) + 4\beta_2^2r_1^2r_2 + 4\beta_1^2r_1r_2^2\right) \\ &\quad + \beta_1^2q_2^2\left(\beta_2^2\left(m(r_1-r_2)^2 - 4r_1r_2^2\right) - 4\beta_1\beta_2r_2^2(r_1+r_2) - 4\beta_1^2r_2^3\right)\Bigg)\Bigg]^{1/2}, \\ C &= 2a_q\beta_2 e(\beta_1+\beta_2)(a_q\beta_2q_1r_1 + \beta_1q_2r_2). \end{aligned}$$

By condition 5.15, we have $Im(\widetilde{\lambda}_j) \neq 0$, and $Re(\widetilde{\lambda}_j) = \dfrac{A}{C} > 0,\ j \in \{1,2\}$. Since the real part is nonzero, by Theorem 1.8, the equilibirum p^* is a hyperbolic repulsive focus of $\widetilde{Z}$ and, then, a hyperbolic pseudo saddle-focus. Now we have to show the second part of the lemma.

Claim 5.5. *The vector field $\widetilde{Z}$ does not admit a limit cycle lying on the open region $\mathbb{R}^2_+ = \{(x,z); x > 0, z > 0\}$.*

Proof. Let be the function

$$\begin{aligned} g:\quad \mathbb{R}^2_+ &\to \mathbb{R} \\ (x,z) &\mapsto \frac{1}{xz}. \end{aligned} \tag{5.16}$$

Since $x > 0$ and $z > 0$, the function g is well defined and is C^1 in $\mathbb{R}^2_+$. Now, $\nabla\cdot(g\widetilde{Z})$ is given by the expression

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{1}{z}\left(\frac{\beta_1r_2+\beta_2r_1-\beta_2z}{\beta_1+\beta_2}\right)\right) + \frac{\partial}{\partial z}\left(-\frac{m}{x}+\frac{1}{z}\left(\frac{e(a_qq_1-q_2)(r_1-r_2)\beta_1 + e(\beta_1q_2+a_q\beta_2q_1)z}{a_q(\beta_1+\beta_2)}\right)\right) \\ &= \frac{e(q_2-a_qq_1)(r_1-r_2)\beta_1}{a_q(\beta_1+\beta_2)z^2} > 0, \text{ for all } (x,z)\in\mathbb{R}^2_+. \end{aligned} \tag{5.17}$$

So, by Theorem 1.22, $\widetilde{Z}$ does not admits a limit cycle in $\mathbb{R}^2_+$. ■

We observe that the sliding vector field $\widetilde{Z}$ can be written in the following way

$$\widetilde{Z}(x,z) = \hat{Z}(x,z) + \left(0, \frac{e(a_q q_1 - q_2)(r_1 - r_2)\beta_1 x}{a_q(\beta_1+\beta_2)}\right),$$

where

$$\hat{Z}(x,z) = \left(\frac{(\beta_1 r_2 + \beta_2 r_1 - \beta_2 z)x}{\beta_1+\beta_2}, -mz + \frac{e(\beta_1 q_2 + a_q\beta_2 q_1)xz}{a_q(\beta_1+\beta_2)}\right).$$

Then, $\hat{Z}(x,v)$ is a predator-prey system with the following first integral:

$$\begin{aligned} F_0(x,z) = \quad & -m - \frac{r_2\beta_1 + r_1\beta_2}{\beta_1+\beta_2} + \frac{e(a_q q_1\beta_2 + q_2\beta_1)x}{a_q(\beta_1+\beta_2)} + \frac{\beta_2 z}{\beta_1+\beta_2} \\ & -m\log\left(\frac{e(a_q q_1\beta_2 + q_2\beta_1)x}{a_q m(\beta_1+\beta_2)}\right) - \frac{r_2\beta_1 + r_1\beta_2}{\beta_1+\beta_2}\log\left(\frac{\beta_2 z}{r_2\beta_1 + r_1\beta_2}\right). \end{aligned}$$

Now, if we take $a = \dfrac{(\beta_1+\beta_2)(q_2 r_2\beta_1 + a_q q_1 r_1\beta_2)}{(q_2\beta_1 + a_q q_1\beta_2)(r_2\beta_1 + r_1\beta_2)} > 0$, we can see that $F_0(ax^*, z^*) = 0$.

How $\langle \nabla F_0(ax,z), \widetilde{Z}(x,z)\rangle = \dfrac{(q_2 - a_q q_1)(r_1 - r_2)\beta_1(r_2\beta_1 + (r_1 - z)\beta_2)^2}{r_2\beta_1 + r_1\beta_2} > 0$ for every $(x,z) > (0,0)$ with $z \neq z^*$, we get that the level curves of $F_0(ax,z)$ are negatively invariant, i.e., the trajectories of the sliding vector field $\widetilde{Z}$ points outward to the level curves of F_0. By claim 5.5, the vector field $\widetilde{Z}$ has no limit cycles, then the focus (x^*, z^*) must attract the orbits of every point in the positive quadrant when $t \to -\infty$. Then we have proved that the focus p^* attracts the orbits in the first quadrant. To finish, we have to show that

Consider $\phi(t)$ the trajectory of $\widetilde{Z}$ passing through $\left(r_1 - r_2, \dfrac{m}{eq_1}\right)$. We define $\widetilde{x}$ as follows:

- If there exists t_s>0 such that $\phi(t_s) \in S_v^+$, then take $\widetilde{x} = \phi(t_s)$;
- Otherwise, take $\widetilde{x} = 0$.

In both cases, $\widetilde{x} < \dfrac{m}{eq_1}$.

Indeed, if $\widetilde{x} \geqslant \dfrac{m}{eq_1}$, there would exist a periodic solution passing through $\left(r_1 - r_2, \dfrac{m}{eq_1}\right)$ and the forward flow of the point $\left(r_1 - r_2, \dfrac{m}{eq_1}\right)$ by the vector field X^+ which is an absurd since there is no closed orbit in this region. So we have $\widetilde{x} < \dfrac{m}{eq_1}$.

■

5.1.4 Existence of a sliding Shilnikov orbit and chaotic behavior of 5.2.

Let's put all the pieces of this puzzle together and guarantee the existence of a sliding Shilnikov orbit as in definition 3.1. By claim 5.2, we know that the saturation of S^v_+ through the forward flow of X^+ reaches Σ transversely in a curve $\mu(x_0) = (u(x_0), u(x_0), v(x_0))$, where $0 < x_0 < \dfrac{m}{eq_1}$.

Moreover, from claim 5.3, there exist a, b, c and d, with $0 < a < b < \dfrac{m}{eq_1}$ and $0 < c < d$, such that $0 < u(x_0) < \dfrac{m}{eq_1}$ and $v(x_0) > r_1$ for every $(x_0, r_2) \in [a, b] \times [c, d]$.

Then, there exists $x_0 \in [a, b]$ such that $(u(x_0), v(x_0)) = (x^*, z^*)$. We assume $c < r_2 < d$ (to use 5.3) and

$$m < \frac{4r_2 v(x_0)(v(x_0) - (r_1 - r_2))(a_q q_1 r_1 + q_2(v(x_0) - r_1))^2}{(r_1 - r_2)^2 (q_2 - a_q q_1)^2 (v(x_0) - r_1)^2}.$$

Then, from lemma 5.4, $(x^*, z^*) \in \Sigma_s$ is a repulsive focus of $\widetilde{Z}$ and there exists $\widetilde{x} \in \left[0, \dfrac{m}{eq_1}\right)$ such that the backward orbit of any point in the straight segment $L = \{(x, r_1 - r_2); \widetilde{x} < x \leqslant \dfrac{m}{eq_1}\}$ is contained in Σ^s and converges asymptotically to p^*.

If $\widetilde{x} = 0$, then from lemma 5.4, we have characterized a sliding Shilnikov connection through the fold-regular point $(x_0, x_0, r_1 - r_2)$ and the pseudo-equilibrium provided by $(u(x_0), u(x_0), v(x_0))$ and our job here is done.

Suppose now that $\widetilde{x} \neq 0$.

Claim 5.6. *In the context above, if $(x^*, z^*) = (u(x_0), v(x_0))$, then $\widetilde{x} < x_0$.*

Proof. We can see that the points $(x^*, z^*) = (u(x_0), v(x_0))$ and $(x_0, r_1 - r_2)$ lie in the same level set of the first integral F given by 5.4. Define

$$C := F^{-1}(F(x^*, z^*)).$$

Firstly, we have to study the behavior of $\widetilde{Z}$ on C. Since we are interested in qualitative behavior, we just have to study the sign of the following product

$$\langle \nabla F(x, z), \widetilde{Z}(x, z)\rangle = \frac{(r_1 - r_2 - z)(eq_2 x(r_1 - z) + a_q(-eq_1 r_1 x + mz))\beta_1}{a_q z(\beta_1 + \beta_2)},$$

where $(x, z) \in C$. But we observe that $a_q(\beta_1 + \beta_2) > 0$ and, since $z \in \Sigma^s \cup S^+_X$, $z \geqslant r_1 - r_2$, then we just have to study the sign of

$$eq_2 x(r_1 - z) + a_q(-eq_1 r_1 x + mz).$$

So, if we take the equation $eq_2x(r_1 - z) + a_q(-eq_1r_1x + mz) = 0$, we have a hyperbole containing points $(0,0)$ and (x^*, z^*) and each connected component of the hyperbole intersects C at most in two points. If we solve the equation in z, then we have:

$$eq_2x(r_1 - z) + a_q(-eq_1r_1x + mz) = 0 \Rightarrow z = \widetilde{z}(x) = \frac{e(q_2 - a_qq_1)r_1x}{eq_2x - a_qm}.$$

Now, we observe that

- $F(x, \widetilde{z}(x)) > F(x^*, z^*)$ if $(x, \widetilde{z}(x)) \in ext(C)$,
- $F(x, \widetilde{z}(x) < F(x^*, z^*)$ if $(x, \widetilde{z}(x)) \in int(C)$,
- $F(x, \widetilde{Z}(x)) = F(x^*, z^*)$ if $(x, \widetilde{z}(x)) \in C$,

and $F(x^*, \widetilde{z}(x^*)) - F(x^*, z^*) = 0$. Then, for every $x \in (0, x^*)$, we have

$$F(x, \widetilde{z}(x)) < F(x^*, z^*) \text{ and } eq_2x(r_1 - z) + a_q(-eq_1r_1x + mz) < 0.$$

So, $\langle \nabla F(x,z), \widetilde{Z}(x,z)\rangle > 0$ and the vector field $\widetilde{Z}$ points outward C since $(x,z) \in C$ and $x \in (0, x^*)$.

Finally, let $\phi(t)$ be the trajectory of $\widetilde{Z}$ passing through $\left(r_1 - r_2, \frac{m}{eq_1}\right)$. Since $\widetilde{x} = \pi_X\phi(t_s)$ for some $t_s > 0$ and $\phi(t_s) \in S_X^+$, there exists $t_s' \in (0, t_s)$ such that $\pi_X\phi(t_s') = x^*$. Hence, $\phi(t) \in ext(C)$ for $t \in [t_s', t_s]$ and $\widetilde{x} < x_0$. ∎

By lemma 5.4, there exists a sliding Shilnikov orbit. To establish the conclusion of our main result, we need to demonstrate the existence of a codimension one submanifold in $\mathcal{M}$ such that the system 5.1 possesses a sliding Shilnikov orbit for any point ξ lying on this submanifold.

In fact, we can define the set $\widetilde{\mathcal{N}}$ as the collection of parameter vectors $\xi = (r_1, r_2, a_q, q_1, q_2, \beta_1, \beta_2, e, m) \in \mathcal{M}$ that satisfy all the inequalities specified in the preceding constructions. Consequently, for every $\xi \in \widetilde{\mathcal{N}}$, the Fillipov system 5.1 exhibits a sliding Shilnikov orbit.

Claim 5.7. *A codimension one submanifold exists within $\widetilde{\mathcal{N}}$ such that Theorem 5.1 holds.*

Proof. By the inequalities constructed on the proofs above and the fact that $(x^*, z^*) = (u(x_0), v(x_0))$, we define the following functions:

$$\varpi(x_0, r_2) = \frac{4r_2v(x_0)(v(x_0) - (r_1 - r_2))(a_qq_1r_1 + q_2(v(x_0) - r_1))^2}{(r_1 - r_2)^2(q_2 - a_qq_1)^2(v(x_0) - r_1)^2};$$

$$\iota(x_0) = \frac{a_q m v(x_0)}{(a_q q_1 r_1 + q_2(v(x_0) - r_1))u(x_0)},$$

and

$$\kappa(x_0) = \frac{r_2 \beta_1}{v(x_0) - r_1}.$$

Let us begin by observing that ϖ is a positive continuous function and, as a result, it assumes a minimum value $M > 0$ on the compact set $[a, b] \times [c, d]$. Additionally, we have $\iota'(x_0)^2 + \kappa'(x_0)^2 \neq 0$ for every $x_0 \in (0, \tau)$. Indeed, we observe that $\iota'(x_0)^2 + \kappa'(x_0)^2 = 0$ if and only if $u'(x_0)^2 + v'(x_0)^2 = 0$, which would contradict 5.3.

Let us assume, without loss of generality, that there exists $x_0 \in (a, b)$ such that $\kappa'(x_0) = 0$. By the inverse function theorem, we can locally invert the function κ. This means that there exists a neighborhood B of $\kappa(x_0)$ and a unique function κ^{-1} : $B \to (a, b)$ such that $\kappa \circ \kappa^{-1}(\beta_2) = \beta_2$ for any $\beta_2 \in B$.

Now, consider $c \leqslant r_2 \leqslant d$, $m \leqslant M$, $\beta_2 \in B$, and $e = \iota \circ \kappa^{-1}(\beta_2)$. We can verify that the inequalities of the construction mentioned above are satisfied.

Therefore, for $\xi = (r_1, r_2, a_q, q_1, q_2, \beta_1, \beta_2, e, m) \in \mathcal{M}$, we define $\mathcal{N} \subset \widetilde{\mathcal{N}}$ as follows:

$$\mathcal{N} = \{\xi \in \mathcal{M} : c < r_2 < d, m < M, \beta_2 \in B \text{ and } e = \iota \circ \kappa^{-1}(\beta_2)\}.$$

It is important to note that $\mathcal{N}$ represents a codimension one submanifold of $\mathcal{M}$ since it is a graph defined in an open domain. Furthermore, we can establish the existence of a neighborhood $U \subset \mathcal{M}$ around $\mathcal{N}$ such that whenever $\xi \in U$, the Filippov system 5.1 exhibits chaotic behavior.

■

This concludes the proof of our main result of this chapter.

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