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SIMILARITY RENORMALIZATION GROUP EVOLUTION OF NUCLEAR FORCES

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In this work we study the similarity renormalization group (SRG) evolution of chiral effective field theory (ChEFT) nucleon-nucleon (NN) interactions derived within the framework of the subtracted kernel method (SKM) approach. We apply the SRG transformation to evolve the leading-order (LO) ChEFT NN potential in the ${}^1S_0$ and the ${}^3S_1 - {}^3D_1$ partial-wave channels and calculate the corresponding phase-shifts.

Keywords: Renormalization group; chiral effective field theory.

1. Introduction

The similarity renormalization group (SRG) formalism, developedby Glazek and Wilson[1,2] and independently by Wegner,[3] is an approach based on a series of continuous unitary transformations that evolve hamiltonians with a cutoff on energy differences. Viewing the hamiltonian as a matrix in a given basis, the SRG transformations suppress off-diagonal matrix elements as the cutoff is lowered, forcing the hamiltonian towards a band-diagonal form. Recently, the SRG approach has been applied to evolve phenomenological and chiral effective field theory (ChEFT) NN potentials to phase-shift equivalent softer forms,[4,5,6,7,8] effectively decoupling low-energy observables from high-energy degrees of freedom. It has been shown that such a decoupling leads to more perturbative NN potentials, greatly simplifying calculations in nuclear few and many-body problems.

In this work we study the SRG evolution of ChEFT NN interactions derived within the framework of the subtracted kernel method (SKM), a renormalization approach based on a subtracted scattering equation.[9,10,11,12] We apply the SRG transformation to evolve the SKM leading-order (LO) ChEFT NN potential in the ${}^1S_0$ and the ${}^3S_1 - {}^3D_1$ partial-wave channels and calculate the corresponding phase-shifts. In previous work[13] we have described in detail the SKM renormalization

procedure and the SRG evolution of the SKM-LO ChEFT NN potential in the $^1S_0$ partial-wave channel.

2. Similarity Renormalization Group Formalism

Consider an initial hamiltonian in the center of mass frame for a system of two nucleons, which can be written in the form $H = T_{\rm rel} + V$, where $T_{\rm rel}$ is the relative kinetic energy and V is the NN potential. Here and in what follows we use units such that $\hbar = c = M = 1$, where M is the nucleon mass. The SRG transformation is defined by a unitary operator designed to act on the hamiltonian and evolve it with a cutoff λ on energy differences at the interaction vertices,

$$H_\lambda = U(\lambda)\, H\, U^\dagger(\lambda) \equiv T_{\rm rel} + V_\lambda \ . \tag{1}$$

In this work, we employ the formulation of the SRG developed by Wegner,[3] based on a flow equation that governs the unitary evolution of the hamiltonian with a flow parameter s,

$$\frac{dH_s}{ds} = [\eta_s, H_s] \ , \tag{2}$$

which is to be solved with the boundary condition $H_s|_{s\to s_0} \equiv H_{s_0}$.

The flow parameter s has dimensions of (energy)$^{-2}$ and ranges from 0 to ∞. In terms of a similarity cutoff λ, here with dimensions of momentum, the flow parameter is given by the relation $s = \lambda^{-4}$. The SRG transformation is generated by the anti-hermitian operator $\eta_s = [G_s, H_s]$, which is defined by the choice of the operator G_s. Wegner's choice in the original formulation is the full diagonal part of the hamiltonian in a given basis, $G_s = {\rm diag}(H_s)$. Here we use the free hamiltonian, $G_s = T_{\rm rel}$, which yields the flow equation

$$\begin{aligned}\frac{dV_s(p,p')}{ds} &= -(p^2 - p'^2)^2\, V_s(p,p') \\ &\quad + \frac{2}{\pi}\int_0^\infty dq\, q^2\, (p^2 + p'^2 - 2q^2)\, V_s(p,q)\, V_s(q,p'),\end{aligned} \tag{3}$$

where $V_s(p,p')$ is a brief notation for the projected NN potential matrix elements $V_s^{(JLL'S;I)}(p,p')$ in a partial-wave relative momentum space basis, $\mid q(LS)J;I\,\rangle$, with normalization given by

$$1 = \frac{2}{\pi}\int_0^\infty dq\, q^2 \mid q(LS)J;I\,\rangle\,\langle\, q(LS)J;I\mid. \tag{4}$$

The superscripts J, $L(L')$, S and I denote respectively the total angular momentum, the orbital angular momentum, the spin and the isospin quantum numbers of the NN state. For non-coupled channels ($L = L' = J$), the NN potential matrix elements $V_s(p,p')$ are simply given by $V_s(p,p') = V_s^{(JJJS;I)}(p,p')$. For coupled channels ($L, L' = J \pm 1$), the $V_s(p,p')$ represent 2×2 matrices of matrix elements

for the different combinations of L and L':

$$V_s(p,p') = \begin{pmatrix} V_s^{(JLLS;I)}(p,p') & V_s^{(JLL'S;I)}(p,p') \\ V_s^{(JL'LS;I)}(p,p') & V_s^{(JL'L'S;I)}(p,p') \end{pmatrix} . \tag{5}$$

The scattering observables for each NN interaction channel can be calculated by iterating the corresponding SRG evolved potential through the LS equation for the partial-wave T-matrix (for simplicity, we drop the subscript s denoting the flow parameter and the superscripts denoting the quantum numbers of the NN state):

$$T(p,p';k^2) = V(p,p') + \frac{2}{\pi} \int_0^\infty dq\, q^2\, \frac{V(p,q)}{k^2 - q^2 + i\,\epsilon}\, T(q,p';k^2) . \tag{6}$$

The phase-shifts are then obtained from the relation between the T-matrix and the S-matrix "on-shell", $S(k^2) = 1 - 2ik\; T(k,k;k^2)$.

3. Subtracted Kernel Method Approach

We begin by considering the formal Lippmann-Schwinger (LS) equation for the T-matrix of a two-nucleon system, written in operator form as

$$T(E) = V + V\; G_0^+(E)\; T(E) , \tag{7}$$

where V is the NN potential and $G_0^+(E) = [E - H_0 + i\epsilon]^{-1}$ is the free Green's function for the NN system with outgoing-wave boundary conditions. For singular NN potentials, Eq. (7) becomes ill-defined due to the ultraviolet divergencies that appear in the momentum integrals. In the SKM approach,[9,10,11,12] a regularized and renormalized LS equation is derived by performing subtractions in the propagator at a certain energy scale.

Consider a singular potential containing a regular term plus a Dirac-delta contact interaction. In momentum space such a potential is given by $V = V_{\rm reg} + C_0$, where C_0 is the strength of the contact interaction. Using Eq. (7), the potential V can be formally written in terms of the T-matrix at a given energy scale $-\mu^2$:

$$V = \left[1 + T(-\mu^2)\; G_0^+(-\mu^2)\right]^{-1}\; T(-\mu^2) . \tag{8}$$

Replacing the potential V in Eq. (7) by its expression in terms of $T(-\mu^2)$ given in Eq. (8) we obtain (after some manipulation) the subtracted kernel LS equation for the T-matrix

$$T(E) = T(-\mu^2) + T(-\mu^2)\; \left[G_0^+(E) - G_0^+(-\mu^2)\right]\; T(E) . \tag{9}$$

The subtracted kernel LS equation provides a finite solution for the T-matrix at any given energy E, once its value at the subtraction scale $-\mu^2$ is known. Thus, the input for the solution of Eq. (9) is $T(-\mu^2)$, which is called "driving term" and contains the physical information apparently lost due to the removal of the

propagation through intermediate states at the scale $-\mu^2$. A simple *ansatz* consists in considering that the driving term is given by

$$T(-\mu^2) = V(-\mu^2) = V_{\rm reg} + C_0(-\mu^2) \; , \tag{10}$$

where $C_0(-\mu^2)$ is the renormalized strength of the contact interaction at the subtraction scale $-\mu^2$, which is fixed by fitting data for scattering observables and, therefore, encodes the physical information.

Once $C_0(-\mu^2)$ in Eq. (10) is fixed, and so the driving term $T(-\mu^2)$ is known, a renormalized potential $V_{\mathcal{R}}$ can be formally defined from Eq. (8):[11]

$$V \to V_{\mathcal{R}} \equiv \left[1 + T(-\mu^2)\; G_0^+(-\mu^2)\right]^{-1} \; T(-\mu^2) \; . \tag{11}$$

Replacing V by $V_{\mathcal{R}}$ in Eq. (7), we obtain the LS equation for the renormalized T-matrix:

$$T_{\mathcal{R}}(E) = V_{\mathcal{R}} + V_{\mathcal{R}}\; G_0^+(E)\; T_{\mathcal{R}} \; . \tag{12}$$

Although $V_{\mathcal{R}}$ is not well defined for singular interactions, for a driving term $T(-\mu^2)$ containing a regular term plus a Dirac-delta contact interaction, Eq. (12) gives a finite solution for the T-matrix that is equivalent to the one obtained from the subtracted kernel LS equation Eq.(9), i.e. $T_{\mathcal{R}}(E) = T(E)$.

4. Numerical Results and Discussion

We solve Eq. (3) numerically, obtaining a non-perturbative solution for the evolution of the SKM-LO ChEFT potential in the ${}^1S_0$ and the ${}^3S_1 - {}^3D_1$ partial-wave channels. The relative momentum space is discretized on a grid of gaussian integration points, leading to a system of non-linear first-order coupled differential equations which is solved using an adaptative fifth-order Runge-Kutta algorithm.

The boundary condition for Eq. (3) is set at $s = 0$ ($\lambda \to \infty$), such that the initial potential is given by the fixed-point potential $V_{\mathcal{R}}(p, p')$ derived by implementing the SKM scheme for the LO ChEFT interaction, which consists of the one-pion exchange potential (OPEP) plus Dirac-delta contact interactions,

$$V(\vec{p}, \vec{p'}) = V_{1\pi}(\vec{p}, \vec{p'}) + \frac{1}{2\pi^2}\left[C_0^s\;\left(\frac{1 - \vec{\tau}_1 \cdot \vec{\tau}_2}{4}\right) + C_0^t\left(\frac{3 + \vec{\tau}_1 \cdot \vec{\tau}_2}{4}\right)\right] \; . \tag{13}$$

The coefficients C_0^s and C_0^t correspond to the strengths of the contact interactions respectively for the ${}^1S_0$ and ${}^3S_1$ channels and $V_{1\pi}(\vec{p}, \vec{p'})$ is the OPEP, given by

$$V_{1\pi}(\vec{p}, \vec{p'}) = -\frac{g_a^2}{4(2\pi)^3 f_\pi^2}\vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{\sigma}_1 \cdot (\vec{p'} - \vec{p})\; \vec{\sigma}_2 \cdot (\vec{p'} - \vec{p})}{(\vec{p'} - \vec{p})^2 + m_\pi^2} \; , \tag{14}$$

where σ_i and τ_i are the usual spin and isospin Pauli matrices for nucleon i, $g_a = 1.25$ is the axial coupling constant, $f_\pi = 93$ MeV is the pion weak-decay constant and $m_\pi = 138$ MeV is the pion mass.

For convenience, in our calculations we use the K-matrix (reactance matrix). The LS equation for the K-matrix is similar to the one for the T-matrix, except

that standing-wave boundary conditions are imposed for the Green's function. In this way, the $i\epsilon$ prescription used in the LS equation for the T-matrix is replaced by the principal value such that the K-matrix is real.

The relation between the T-matrix and the K-matrix "on-shell" is given by:

$$K^{-1}(k,k;k^2) = T^{-1}(k,k;k^2) + i\ k\ . \tag{15}$$

We also introduce an ultraviolet momentum cutoff Λ which is convenient for the numerical calculations when we further consider the evolution of the SKM renormalized potential through the SRG transformation. It is important to emphasize that such a cutoff acts just as an instrumental regulator for the numerical integrations.

The subtracted kernel LS equations for the K-matrices in the $^1S_0$ and the $^3S_1 - ^3D_1$ channels are respectively given by:

$$K_s^{00}(p,p';k^2) = V_s^{(1),00}(p,p';-\mu^2) + \frac{2}{\pi}\ \mathcal{P}\int_0^\Lambda dq\ q^2\ \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)\times$$
$$\times\ \frac{V_s^{(1),00}(p,q;-\mu^2)}{k^2-q^2} K_s^{00}(q,p';k^2)\ , \tag{16}$$

$$K_t^{l_1l_2}(p,p';k^2) = V_t^{(1),l_1l_2}(p,p';-\mu^2) + \frac{2}{\pi}\ \sum_{l_3}\mathcal{P}\int_0^\Lambda dq\ q^2\ \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)\times$$
$$\times\ \frac{V^{(1),l_1l_3}(p,q;-\mu^2)}{k^2-q^2} K_t^{l_3l_2}(q,p';k^2)\ , \tag{17}$$

where the driving terms $V_s^{(1),00}(p,p';-\mu^2)$ and $V_t^{(1),l_1l_2}(p,p';-\mu^2)$ are defined by the *ansatz*

$$V_s^{(1),00}(p,p';-\mu^2) \equiv K_s^{00}(p,p';-\mu^2) = V_{1\pi,s}^{00}(p,p') + C_0^s(-\mu^2)\ , \tag{18}$$

$$V_t^{(1),l_1l_2}(p,p';-\mu^2) \equiv K_t^{l_1l_2}(p,p';-\mu^2) = V_{1\pi,t}^{l_1l_2}(p,p') + C_0^t(-\mu^2)\ \delta_{l_2,0}\ \delta_{l_1,0}\ . \tag{19}$$

The renormalized strengths of the contact interactions $C_0^s(-\mu^2)$ and $C_0^t(-\mu^2)$ are fixed at the subtraction scale $\mu = 25$ fm^{-1} by fitting the experimental values of the scattering lengths for the $^1S_0$ and the $^3S_1$ channels, respectively given by $a_s = -23.7$ fm and $a_s = 5.43$ fm.

Once the driving terms are known, we can obtain the renormalized potentials $V_{\mathcal{R},s}^{(1),00}(p,p')$ and $V_{\mathcal{R},t}^{(1),l_1l_2}(p,p')$ by solving the integral equations,

$$V_{\mathcal{R},s}^{(1),00}(p,p') = V_s^{(1),00}(p,p';-\mu^2) -$$
$$-\ \frac{2}{\pi}\mathcal{P}\ \int_0^\Lambda dq\ q^2\ \frac{V_s^{(1),00}(p,q;-\mu^2)}{-\mu^2-q^2}\ V_{\mathcal{R},s}^{(1),00}(q,p')\ , \tag{20}$$

$$V_{\mathcal{R},t}^{(1),l_1l_2}(p,p') = V_t^{(1),l_1l_2}(p,p';-\mu^2) -$$
$$-\ \frac{2}{\pi}\sum_{l_3}\mathcal{P}\ \int_0^\Lambda dq\ q^2\ \frac{V_t^{(1),l_1l_3}(p,q;-\mu^2)}{-\mu^2-q^2}\ V_{\mathcal{R},t}^{(1),l_3l_2}(q,p')\ , \tag{21}$$

which are derived by manipulating Eq. (11).

In Fig. 1 we show the results obtained for the SRG evolution of the SKM-LO ChEFT potential in the $^1S_0$ channel. As one can observe, the off-diagonal matrix elements are systematically suppressed as the similarity cutoff λ is lowered, such that the potential is driven towards a band-diagonal form.

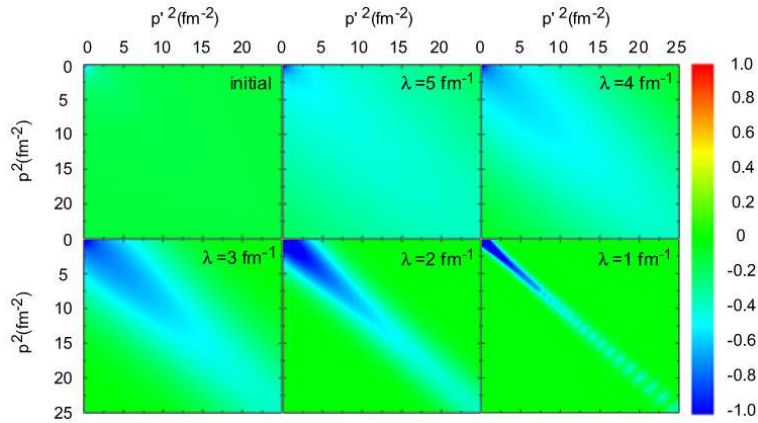

Fig. 1. (Color online) SRG evolution of the SKM-LO ChEFT potential in the $^1S_0$ channel.

In Fig. (2) we show the phase-shifts in the $^1S_0$ channel obtained for the initial potential $V_{\mathcal{R}}(p, p')$ and for the SRG potentials evolved up to several values of the similarity cutoff λ. As expected for a unitary transformation, the results obtained for the initial potential and for the SRG evolved are the same (apart from relative numerical errors smaller than 10^{-9}).

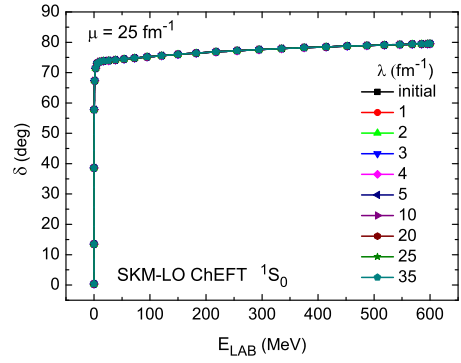

Fig. 2. (Color online) Phase-shifts in the $^1S_0$ channel as a function of the laboratory energy.

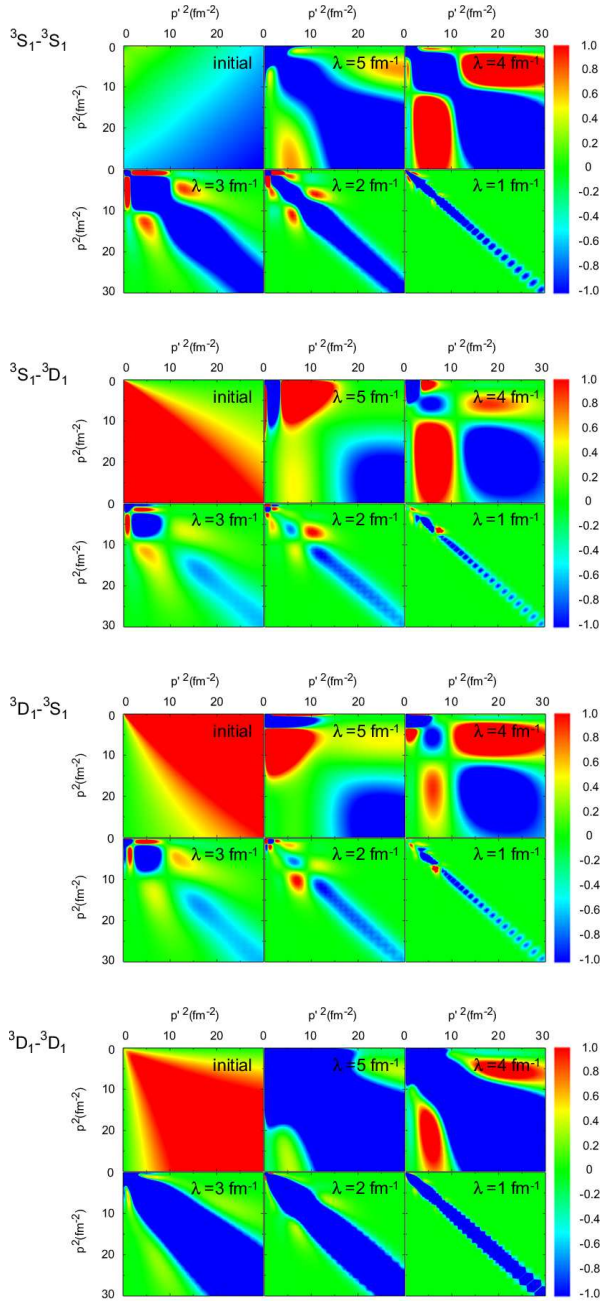

Fig. 3. (Color online) SRG evolution of the SKM-LO ChEFT potential in the $^3S_1 - ^3D_1$ channel.

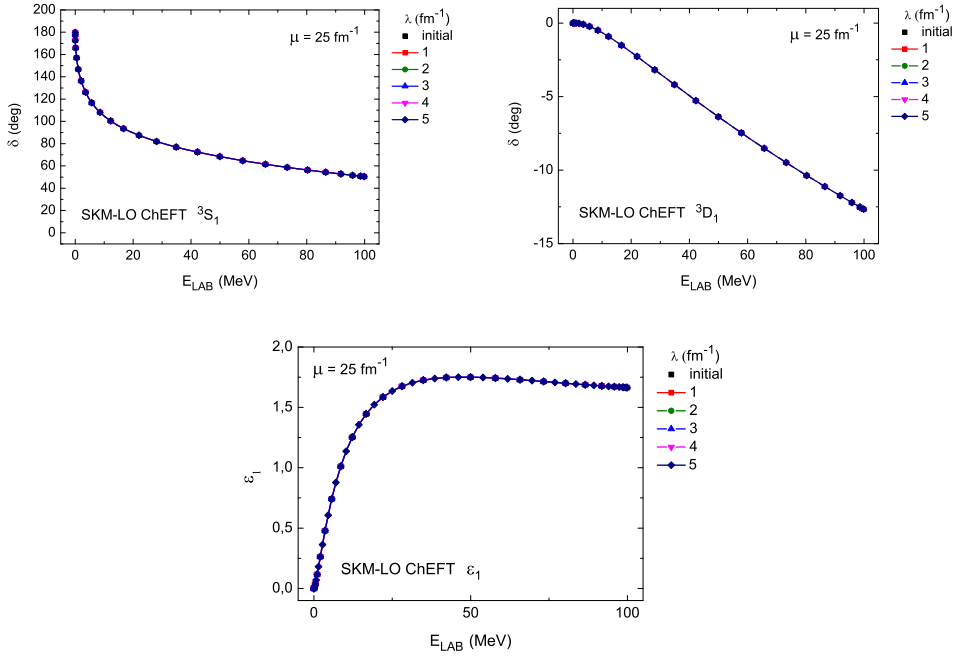

Fig. 4. (Color online) Phase-shifts and mixing parameter in the $^3S_1 - ^3D_1$ channel as a function of the laboratory energy.

In Fig. 3 we show the results obtained for the SRG evolution of the SKM-LO ChEFT potential in the $^3S_1 - ^3D_1$ channel. In Fig. 4 we show the phase-shifts in the $^3S_1$ and $^3D_1$ channels and the mixing parameter ϵ_1 obtained for the initial potential $V_{\mathcal{R}}(p, p')$ and for the SRG potentials evolved up to several values of the similarity cutoff λ. As one can observe, the results are similar to those obtained for the $^1S_0$ channel: the potential is driven towards a band-diagonal form and the phase-shifts are preserved as the SRG cutoff is lowered.

Acknowledgments

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