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Ergodic measure disintegration along foliations with invariant arc-lengths

Desintegração ergódica de medida ao longo de folheações com comprimentos de arco invariantes

Campinas

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Resumo

Seja $(M, \mathcal{A}, \mu)$ um espaço de probabilidade e $f : M \to M$ um homeomorfismo que preserva uma medida de probabilidade ergódica μ. Dada $\mathcal{F}$ uma foliação f-invariante contínua de dimensão 1 em M com folhas de classe C^1, mostramos que se f preserva um $\mathcal{F}$-sistema de comprimentos de arcos contínuo $\{l_x\}_{x \in M}$, então podemos classificar as medidas condicionais de μ ao longo de $\mathcal{F}$ em três possibilidades: elas são ou atômicas para quase toda folha, ou são equivalentes à medida λ_x que é induzida pelo $\mathcal{F}$-sistema de comprimento de arco, ou o seu suporte é um conjunto de Cantor da folha, para quase toda folha.

Além disso, mostramos que se $f : M \to M$ é um C^1-difeomorfismo transitivo parcialmente hiperbólico com direção central topologicamente neutra que preserva uma medida ergódica, então a desintegração dessa medida é atômica ou equivalente a medida induzida pelo sistema de comprimentos de arco ao longo de cada folha central.

Palavras-chave: Medidas condicionais, medidas ergódicas, dinâmica hiperbólica, conjunto de Cantor, centro topologicamente neutro.

Abstract

Let $(M, \mathcal{A}, \mu)$ be a probability space and let $f : M \to M$ be a homeomorphism that preserves an ergodic probability measure μ. Given a continuous f-invariant foliation $\mathcal{F}$ of dimension 1 in M with C^1 leaves, we show that if f preserves a continuous $\mathcal{F}$-arc length system $\{l_x\}_{x \in M}$, then the conditional measures of μ along $\mathcal{F}$ can be classified into three possibilities: they are either atomic for almost every leaf, or equivalent to the measure λ_x induced by the $\mathcal{F}$-arc length system, or their support is a Cantor set on almost every leaf.

Furthermore, we prove that if $f : M \to M$ is a C^1 transitive, partially hyperbolic diffeomorphism with a topologically neutral central direction that preserves an ergodic measure, then the disintegration of this measure is either atomic or equivalent to the measure induced by the arc lengths along each central leaf.

Keywords: Conditional measures, ergodic measures, hyperbolic dynamics, Cantor set, topologically neutral center.

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1 Introduction

A general notion of a dynamical system is given by a pair (M, f), where M is an ambient manifold and $f : M \to M$ maps are continuous or discrete time that relates the current state of the system to its past and future states. The theory of dynamical systems aims to identify patterns and understand the asymptotic behavior of orbits in such systems.

A useful approach to understand the temporal evolution of dynamical systems in spaces with invariant measures is to study their statistical and geometrical properties. This is precisely the objective of *ergodic theory*, which considers an ambient manifold M equipped with a probability measure μ, a σ-algebra $\mathcal{A}$, and a map $f : M \to M$ that preserves the measure; that is, for every $A \in \mathcal{A}$, we have $\mu(A) = \mu(f^{-1}(A))$. Ergodic theory seeks to identify properties that are valid for almost all trajectories of the system with respect to the measure μ. We say that a system is *ergodic* if the f-invariant subsets only belong to sets with measure zero or one; that is, if for every measurable set $B \subset M$ such that $f^{-1}(B) = B$, we have $\mu(B) = 0$ or $\mu(B) = 1$.

Even though ergodicity implies unpredictability from a measure standpoint, there are several degrees of unpredictability that make up the ergodic hierarchy. The ergodic hierarchy differentiates systems based on how quickly they mix sets over time. Among the many fine ergodic properties, *the Bernoulli property* is the most robust form of unpredictability in terms of measure. This means that we can find a symbolic representation of the system that is equivalent to a shift, and we can find a finite partition of the system where the symbolic representation generated by this partition is measurably equivalent to a standard Bernoulli shift. In regards to orbit information, this indicates that, using this finite partition, it is impossible to determine the partition element of the initial point even if all past and future orbit information is available. This level of unpredictability is known as "chaos in terms of measure". Linear toral automorphisms without eigenvalues of norm one are natural examples of Bernoulli automorphisms, as shown in [33].

Measure disintegration

Measure disintegration techniques have been a crucial tool in ergodic theory. We can disintegrate a probability measure μ over the partition induced by any countable generated sub-σ-algebra $\mathcal{E} \subset \mathcal{B}$ for a topological space X with its Borel σ-algebra $\mathcal{B}$ and a probability measure μ on X. This disintegration involves finding a collection of probability measures $\{\mu_x\}_{x\in X}$, such that $\mu_x([x]) = 1$ for the element $[x]$ of the partition induced by $\mathcal{E}$ containing x. The function $x \mapsto \mu_x$ is measurable with respect to the Borel σ-algebra. Moreover, for any $B \in \mathcal{B}$, $\mu(B) = \int \mu_x(B)\, d\mu(x)$. We refer to this collection of probability measures as the *disintegration of* μ *along* $\mathcal{E}$.

If we have a one-dimensional continuous foliation $\mathcal{F}$ and any (small) foliation box U, we can disintegrate the restriction of μ to the foliation box into conditional probabilities along the local leaves $\mathcal{F}|U(x)$ using the disintegration $\{\mu_x^U : x \in U\}$. This disintegration is called the *disintegration of* μ *along* $\mathcal{F}|U$.

We say that a probability measure μ has Lebesgue disintegration along $\mathcal{F}$ if, for μ-almost every x, any representative of the conditional probability measure $\{\mu_x\}_{x\in M}$ is equivalent to Riemannian volume on $\mathcal{F}(x)$ in terms of their zero sets. On the other hand, μ has atomic disintegration if $\{\mu_x\}_{x\in M}$ is an atomic class for μ-almost every x. Some results obtained with this tool include, for example, the proof that the stable and unstable foliations of globally hyperbolic (or Anosov) systems are absolutely continuous, despite not being C^1 in general, given by Anosov and Sinai [2, 3] in the 1960s. This result crucial in Anosov's celebrated proof that the geodesic flow for any compact manifold with negative curvature is ergodic.

In certain recent investigations, some properties of measure disintegration have been obtained for some systems, such as for partially hyperbolic diffeomorphisms, that is, diffeomorphism f such that the tangent bundle TM admits Df-invariant splitting $E^s \oplus E^c \oplus E^u$ such that $Df|_{E^s}$ is a uniformly contracting, $Df|_{E^u}$ is uniformly expanding, and $Df|_{E^c}$ is dominated by both: vectors in E^c are neither as contracted as vector in E^s, nor as expanded as vectors in E^u. When the center direction is integrable we have a foliation tangent to E^c, called *center foliation*.

Some authors had have carateristics for the disintegration along of center foliation, for example, in the work of D. Ruelle and A. Wilkinson they proved in [31] that certain partially hyperbolic dynamics with negative fiberwise Lyapunov exponent have atomic disintegration of the preserved measure along the fibers. Later, A. Homburg proved in [21] that some of the examples considered in [31] have disintegration consisting of only one Dirac measure. A. Avila, M. Viana, and A. Wilkinson proved in [4] that C^1-volume preserving perturbations of the time-1 map of geodesic flows on negatively curved surfaces have either atomic or absolutely continuous disintegration of the volume measure along the center foliation. Moreover, in the latter case, the perturbation must be the time-1 map of an Anosov flow. Within the class of diffeomorphisms derived from Anosov diffeomorphisms, G. Ponce, A. Tahzibi, and R. Varão exhibited an open class of volume preserving diffeomorphisms in [28] that have atomic disintegration along the center foliation. Recently, A. Tahzibi and J. Zhang answered a question from [27] and proved in [32] that non-hyperbolic measures of diffeomorphisms derived from Anosov diffeomorphisms on $\mathbb{T}^3$ may also have atomic disintegration along the center foliation.

Note that all of the results mentioned above have as hypotheses some kind of hyperbolicity. In this work, one of our main goals is to better understand the disintegration of an invariant measure along an invariant foliation for the dynamics without requiring hyperbolicity or partial hyperbolicity for f, but assuming that the invariant foliation has some type of rigidity metric with respect to f. In other words, we aim to investigate what the possible characterizations are of the conditional measures obtained when we disintegrate μ over a foliation $\mathcal{F}$, assuming that the behavior of f along $\mathcal{F}$ is far from being hyperbolic

Setting and statement of results.

Our work will take place in a probability space $(M, \mathcal{A}, \mu)$, where M is a compact Riemannian manifold with at least two dimensions, μ is a non-atomic Borel measure, and $\mathcal{A}$ is the completion of the Borel σ-algebra $\mathcal{B}$ of M with respect to the measure μ. Essentially, this means that the space $(M, \mathcal{A}, \mu)$ is equivalent to the probability space $([0,1], \mathcal{A}_{[0,1]}, \mathrm{Leb}_{[0,1]})$, where $\mathrm{Leb}_{[0,1]}$ is the standard Lebesgue measure on $[0,1]$ and $\mathcal{A}_{[0,1]}$ is the σ-algebra of Lebesgue measurable sets of $[0,1]$.

The following is the main result of this work.

Theorem A. *[25] Let $f : M \to M$ be a homeomorphism over a compact smooth manifold and $\mathcal{F}$ be a f-invariant one dimensional continuous foliation of M by C^1-submanifolds and $\{l_x\}_{x\in M}$ a $\mathcal{F}$-arc length system. If f is ergodic with respect to an f-invariant measure μ then one of the following holds:*

a) the disintegration of μ along $\mathcal{F}$ is atomic.

b) for almost every $x \in M$, the conditional measure on $\mathcal{F}(x)$ is equivalent to the measure λ_x defined on simple arcs of $\mathcal{F}(x)$ by:

$$\lambda_x(\gamma([0,1])) = l_x(\gamma), \text{ where } \gamma \text{ is a simple arc.}$$

c) for almost every $x \in M$, the conditional measure on $\mathcal{F}(x)$ is supported in a Cantor subset of $\mathcal{F}(x)$.

The existence of invariant systems of metrics was obtained in [11] for the context of transitive partially hyperbolic diffeomorphisms with topological neutral center, meaning that f and f^{-1} have Lyapunov stable center direction (see [19, section 7.3.1]), i.e, given any $\varepsilon > 0$ there exists $\delta > 0$ for which, given any C^1 path γ tangent to the center direction, one has

$$\text{lenght}(\gamma) < \delta \Rightarrow \text{lenght}(f^n(\gamma)) < \varepsilon, \quad \forall n \in \mathbb{Z}.$$

According to [30, Corollary 7.6], the diffeomorphisms in question have a continuously integrable center direction, which gives rise to a foliation $\mathcal{F}^c$ of M.

This theorem has a useful application in the case of partially hyperbolic diffeomorphisms with a one-dimensional topologically neutral center direction. Bonatti and Zhang proved in [11, Theorem A] the existence of a $\mathcal{F}^c$-arc length system for such diffeomorphisms, where $\mathcal{F}^c$ denotes the center foliation.

We are able to show a dichotomy for this class of diffeomorphisms, where the case of conditional measures supported on a Cantor set is not necessary.

Theorem B. *[20] Let $f : M \to M$ be a transitive C^1 partially hyperbolic diffeomorphism on a closed manifold M. Assume that f has one-dimensional topologically neutral center. If f is ergodic with respect to a f-invariant total support measure μ then either:*

a) the disintegration of μ along of center foliation, $\mathcal{F}^c$, is atomic.

b) for almost every $x \in M$, the conditional measure on $\mathcal{F}^c(x)$ is equivalent to the measure λ_x defined on simple arcs of $\mathcal{F}^c(x)$ by:

$$\lambda_x(\gamma([0,1])) = l_x(\gamma), \textit{ where } \gamma \textit{ is a simple arc.}$$

An important application of this theorem is [26] where the author proved in Theorem A that if f is a $C^{1+\alpha}$, $\alpha \geqslant 1$, partially hyperbolic diffeomorphism with an orientable one-dimensional center bundle, whose orientation is preserved by f, and f preserves a smooth ergodic measure μ while being topologically neutral along the center direction, then the conditional measures of the disintegration of μ along the center foliation $\mathcal{F}^c$ are atomic or the center foliation, $\mathcal{F}^c$, is leafwise absolutely continuous and f is Bernoulli.

2 Preliminaries

In this chapter, we aim to briefly introduce some relevant concepts that are essential to understanding our results. Our primary objective is to make this text as self-contained as possible by providing references that will facilitate comprehension. While we will state most of the results, we will omit their proofs, which can be found in the provided references.

2.1 Measure-Theoretical Properties of Partitions

A partition $\mathcal{P}$ of a measurable space $(X, \mathcal{A}, \mu)$ is a collection of a measurables subsets of X, $\mathcal{P}$ called atoms of $\mathcal{P}$, satisfying

- $P \cap Q = \emptyset$ for every pair of distinct atoms $P, Q \in \mathcal{P}$;
- $\bigcup\limits_{P \in \mathcal{P}} P = X$.

Given a sub-σ-algebra $\mathcal{E} \subset \mathcal{B}$ generated by a countable family of subsets in $\mathcal{B}$, $\{E_n\}_{n \in \mathbb{N}}$, associated to $\mathcal{E}$ we define the equivalence relation $\sim_{\mathcal{E}}$ as for $x, y \in M$ we write $x \sim_{\mathcal{E}} y$ if $\chi_E(x) = \chi_E(y)$ for every $E \in \mathcal{E}$, where χ_E is the characteristic function. The equivalence classes under $\sim_{\mathcal{E}}$ are measurable and can be represented as intersections of sets F_n of the form $F_n \in E_n, X \backslash E_n$. In other words, for every $x \in X$, the equivalence class of x is given by

$$[x] := \bigcap_{E \in \mathcal{E} \colon x \in E} E = \bigcap \{F_n \;:\; F_n \in \{E_n, X \smallsetminus E_n\} \text{ and } x \in F_n\}$$

Consequently, $[x]$ is a Borel set for every $x \in X$ and $\{[x] : x \in X\}$ is a partition of X.

We will call the partition $[x] : x \in X$ associated with the countably generated σ-algebra a *countably generated partition*.

Definition 2.1. *Given a sub-σ-algebra $\mathcal{E} \subset \mathcal{B}$ generated by a countable family $\{E_n\}_{n\in\mathbb{N}}$. A family of measures $\{\mu_x\}_{x\in X}$ is called a system of conditional measures or a disintegration of μ associated to $\mathcal{E}$, if*

1. *Given $\varphi \in C^0(X)$, then $x \mapsto \int \varphi \, d\mu_x$ is $\mathcal{E}$-measurable;*

2. *$\mu_x([x]) = 1$, μ-almost every x;*

3. *if $B \subset X$ is measurable, then*
$$\mu(B) = \int_{y\in B} \mu_y(B) d\,\mu(y).$$

For simplicity, we also say that $\{\mu_x\}_{x\in X}$ is a disintegration with respect to the partition $\mathcal{P} = \{[x] : x \in X\}$. The existence of disintegration with respect to such partitions is garantee by the following result, which we prove here for the sake of the reader.

Theorem 2.2. *[15] Let X be a metric compact space, $\mathcal{B}$ the Borelian σ-algebra and $\mathcal{E} \subset \mathcal{B}$ a sub-σ-algebra generated by a countable family of Borelian subset, then there exists a system of conditional measures with respect to $\mathcal{E}$.*

Proof. Since X is a compact metric space, we can choose a countable, dense, and $\mathbb{Q}$-linear subset $V = \{f_0, f_1, ...\} \subset C(X)$ such that $f_0 = 1$. Let $g_0 = f_0 = 1$ and for every $i \geqslant 1$ we define g_i as the conditional expectation of f on $\mathcal{E}$, this is,
$$g_i = \mathbb{E}(f_i|\mathcal{E}) \in L^1(X, \mathcal{B}, \mu)$$
where
$$\mathbb{E}(f_i|\mathcal{E})(y) = \frac{1}{\mu([y])} \int_{[y]} f_i \, d\mu, \text{ almost every } y \in X.$$
Actually here g_i denotes a representative of the equivalence class of integrable functions. Let $X_0 \subset X$ be a full measure subset such that for every $\alpha, \beta \in \mathbb{Q}$ and every $f_i, f_j \in V$ the following conditions are satisfied,

1. $\mathbb{E}(\alpha f_i + \beta f_j|\mathcal{E})(x) = \alpha\mathbb{E}(f_i|\mathcal{E})(x) + \beta\mathbb{E}(f_j|\mathcal{E})(x)$, for every $x \in X_0$,

2. $\min f_i \leqslant \mathbb{E}(f_i|\mathcal{E})(x) \leqslant \max f_i$, for every $x \in X_0$.

Thus, for every $x \in X_0$ we can consider the functional $\mathcal{H}_x : C(X) \to \mathbb{R}$ defined by
$$\mathcal{H}_x(f_i) = g_i(x), \text{ for } i = 0, 1, \ldots$$
Notice that from Item 2 above, it follows that $\|\mathcal{H}_x\| \leqslant 1$ for almost every $x \in X$. Hence by the Riesz Representation Theorem, for every $y \in X_0$ there exists a probability measure μ_y on X such that
$$\mathcal{H}_y(f) = \int f(x) \, d\mu_y(x).$$
For $y \in X \setminus X_0$ we define μ_y to be some fixed measure to ensure measurability.

Claim 1. The function $y \mapsto \mu_y$ is $\mathcal{E}$-measurable and $\mathbb{E}(\chi_A|\mathcal{E}) = \mu_y(A)$ for almost every $y \in X$.

Proof. Let $\mathcal{C} \subset \mathcal{B}$ be the family of subsets $A \in \mathcal{B}$ such that the function $y \mapsto \mu_y(A)$ is $\mathcal{E}$-measurable and $\mathbb{E}(\chi_A|\mathcal{E})(y) = \mu_y(A)$ for almost every $y \in X$. We want to show that $\mathcal{C} = \mathcal{B}$. To do this, we use the Monotone Class Theorem, which states that a family of monotone classes that contains a algebra that generates the Borel σ-algebra $\mathcal{B}$ is equal to the σ-algebra $\mathcal{B}$. We show that $\mathcal{C}$ is a family of monotone classes and contains an algebra that generates $\mathcal{B}$.

Consider $\mathcal{C}_0 \subset \mathcal{B}$ the collection of set A such that χ_A is pointwise limit of a uniformly bounded sequence of continuous functions. Notice that for any open subset B, the indicator function χ_B is the pointwise limit of a sequence of continuous functions $0 \leqslant h_n \leqslant 1$. Therefore, the collection $\mathcal{C}_0$ is not empty. First, let us show that $\mathcal{C}_0$ is an algebra:

- Clearly $X, \varnothing \in \mathcal{C}_0$.
- Let $A \in \mathcal{C}_0$, then there exists a sequence of continuous functions $0 \leqslant h_n \leqslant 1$, such that h_n converges pointwise to χ_A, consider $\overline{h}_n = 1 - h_n$ we have that $\overline{h}_n$ converges pointwise to $\chi_{X\setminus A}$, thus $X \setminus A \in \mathcal{C}_0$.
- Given $A, B \in \mathcal{C}_0$, then there exist uniformly bounded sequences of continuous functions h_n and g_n such that $h_n(x) \to \chi_A(x)$ and $g_n(x) \to \chi_B(x)$, then $(h_n \cdot g_n)(x) \to (\chi_A \cdot \chi_B)(x) = \chi_{A\cap B}(x)$, consequently $A \cap B \in \mathcal{C}_0$.

Now, we will show that the algebra $\mathcal{C}_0$ is contained in $\mathcal{C}$. Indeed, let $A \in \mathcal{C}_0$ and $0 \leqslant h_n \leqslant 1$ be a sequence of continuous functions such that $h_n \to \chi_A$, thus by the Dominate Convergence Theorem we have

$$\lim_{n\to\infty} \int h_n \, d\mu_y = \int \chi_A \, d\mu_y = \mu_y(A).$$

This implies that the function $y \mapsto \mu_y(A)$ is the pointwise limit of the sequence of functions $\overline{h}_n : y \mapsto \int h_n \, d\mu_y$, we already know that $\mathbb{E}(h_n|\mathcal{E}) = \int h_n \, d\mu_y$ almost every $y \in X$. Since $\mathbb{E}(h_n|\mathcal{E})$ is $\mathcal{E}$-measurable for every $n \in \mathbb{N}$, follows that $y \mapsto \mu_y$ is $\mathcal{E}$-measurable and $\mathbb{E}(\chi_A|\mathcal{E})(y) = \mu_y(A)$, then we have that $\mathcal{C}_0 \subset \mathcal{C}$.

On the other hand, notice that it contains the closed subsets, given any closed $A \subset X$, consider for $n \in \mathbb{N}$

$$h_n(x) = exp(-n \cdot \mathrm{dist}(x, A)),$$

notice that $h_n \to \chi_A$, then we have $A \in \mathcal{C}_0$. Thus $\mathcal{C}_0$ generates the Borel σ-algebra $\mathcal{B}$.

Finally, let us see that $\mathcal{C}$ is a family of monotone classes. Let $A_1 \subset A_2 \subset \cdots$ be a countable collection of increasing sets belonging to $\mathcal{C}$ and $A = \cup A_i$. We have that $\chi_{A_n} \to \chi_A$

in L^1. Then for almost all $y \in X$, $\mu_y(A) = \lim \mu_y(A_n)$. Thus, the function $y \mapsto \mu_y(A)$ is the pointwise limit of the sequence of measurable functions, $y \mapsto \mu_y(A_n) = \mathbb{E}(\chi_{A_n}|\mathcal{E})(y)$. By continuity of conditional expectation, we have that $\mathbb{E}(\chi_{A_n}|\mathcal{E}) \to \mathbb{E}(\chi_A|\mathcal{E})$. Therefore, $\mu_y(A) = \mathbb{E}(\chi_A|\mathcal{E})$, hence $A = \cup A_i \in \mathcal{C}$. Analogously, we show that if $B_1 \supset B_2 \supset \cdots$ is a sequence of decreasing sets belonging to $\mathcal{C}$, then $B = \cap B_i$ belongs to $\mathcal{C}$.

Thus, we have shown that $\mathcal{C}$ is a family of monotone classes containing an algebra $\mathcal{C}_0$, which generates the σ-algebra $\mathcal{B}$. Then, by the Monotone Class Theorem, we have that $\mathcal{C} = \mathcal{B}$, as we wanted to show. □

Claim 2. For all $f \in L^1$ and almost every point $y \in X$

$$\mathbb{E}(f|\mathcal{E})(y) = \int f \, d\mu_y \tag{2.1}$$

Proof. We know that (2.1) holds for $f = \chi_A$ by the previous Claim. Since we can approximate any function $f \in L^1$ by simple functions, and both sides of the equality (2.1) are linear and continuous under monotone increasing sequences, we conclude the proof of Claim 2. □

Claim 3. For almost every $y \in X$, $\mu_y([y]) = 1$.

Proof. Let $\{E_n\}$ generate $\mathcal{E}$, for each $n \in \mathbb{N}$ and for every $y \in X$, by definition of $[y]$ we have $[y] \subset E_n$ or $[y] \cap E_n = \varnothing$, then by Claim 2 we have that for almost every point $y \in X$

$$\mu_y(E_n) = \mathbb{E}(\chi_{E_n}|\mathcal{E})(y) = \frac{1}{\mu([y])} \int_{[y]} \chi_{E_n} \, d\mu = \chi_{E_n}(y), \tag{2.2}$$

Now, for each $n \in \mathbb{N}$ consider $F_n \in \{E_n, X \setminus E_n\}$, we have that

$$[y] = \bigcap_{y \in F_n} F_n,$$

using F_n in equation (2.2), it follows that $\mu_y(F_n) = 1$, for every $n \in \mathbb{N}$ such that $y \in F_n$, thus $\mu_y([y]) = 1$. □

Therefore, we consider the family of mesures $\{\mu_y\}_{y \in X}$, and it is a system of contitional measure for μ. □

It is natural to wonder whether we may have two distinct disintegrations for the measure μ with respect to $\mathcal{E}$. The following result shows that, in terms of measure, the system is indeed unique.

Proposition 2.3. *[15] If $\{\mu_y\}$ and $\{\nu_y\}$ are systems of contitional measures for μ then $\mu_y = \nu_y$ μ-almost every $x \in X$.*

Proof. Assume that there exists a subset $Q \subset X$ with $\mu(Q) > 0$ such that $\mu_y \neq \nu_y$ for all $y \in Q$. Consider a dense countable set of functions $\{\varphi_k\} \subset C^0(X)$. Define the sets

$$A_k = \left\{ x \in Q \, : \, \int_{[x]} \varphi_k \, d\mu_x \neq \int_{[x]} \varphi_k \, d\nu_x \right\}$$

Since $\mu(\bigcup A_k) = \mu(Q) > 0$, there exists k_0 such that $\mu(A_{k_0}) > 0$. Consider the set $Q_0 \subset A_{k_0}$ given by

$$Q_0 = \left\{ x \in A_{k_0} \, : \, \int_{[x]} \varphi_{k_0} \, d\mu_x > \int_{[x]} \varphi_{k_0} \, d\nu_x \right\}$$

Without loss of generality, assume that $\mu(Q_0) > 0$. Then

$$\begin{aligned} \int \varphi_{k_0} \cdot \chi_{Q_0} \, d\mu &= \int (\varphi_{k_0} \cdot \chi_{Q_0} \, d\mu_x) \, d\mu(x) = \int_{Q_0} (\varphi_{k_0} \, d\mu_x) \, d\mu(x) \\ &> \int_{Q_0} (\varphi_{k_0} \, d\nu_x) \, d\nu(x) = \int \varphi_{k_0} \cdot \chi_{Q_0} \, d\mu. \end{aligned}$$

This is a contradiction, which implies that the assumption that $\mu_y \neq \nu_y$ for all $y \in Q$ is false.

□

2.2 Basics on Foliations

In this section we recall some geometric and measurable properties of foliations and disintegration of measure on a foliated box.

Let M be a smooth manifold of dimension m. A *C^r-foliation*, $r \geqslant 0$, of dimension $0 < n < m$ by C^1-manifolds is defined as a maximal atlas $\mathcal{F}$ of class C^r on M with the following properties:

1. if $(U, \varphi) \in \mathcal{F}$ then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$, where U_1, U_2 are open disks;
2. If $(U, \varphi), (V, \phi) \in \mathcal{F}$ such that $U \cap V \neq \emptyset$, then the function $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ satisfies:
$$\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y)).$$

Let $\mathcal{F}$ be a C^r foliation of dimension $0 < n < m$ of a manifold M of dimension n. Given $(U, \varphi) \in \mathcal{F}$ with

$$\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n},$$

the subsets $\varphi^{-1}(U_1 \times \{c\})$, for some $c \in U_2$ are called the *plaques of* $\mathcal{F}$.

We will say that M *is foliated by* $\mathcal{F}$ and the chart $(U, \varphi) \in \mathcal{F}$ is called the *foliated box.*

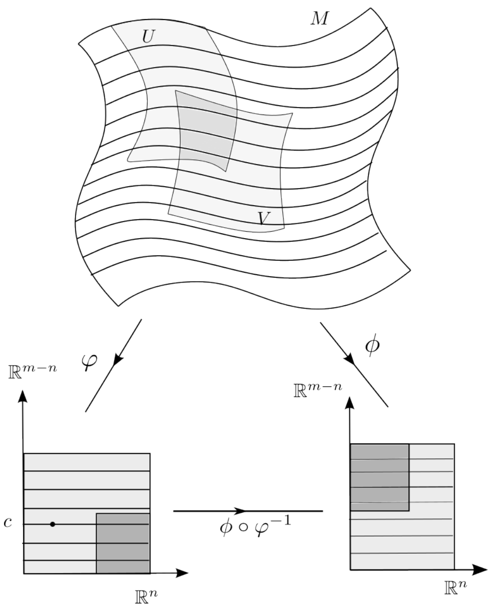

Figure 1 – Foliation

By the definition, we have that $f = \varphi\,|_{U_1 \times \{c\}} \colon U_1 \times \{c\} \to U$ is an C^r embedding. Therefore the plaques are connected submanifolds of class C^r, $r \geqslant 0$, and dimension n. In addition, given two plaques α and β in U either $\alpha \cap \beta = \emptyset$ or $\alpha = \beta$. We will use the notation $\mathcal{F}|U(x)$ to denote *the plaque in U that contains x.*

From now on, $\mathcal{F}$ will denote a continuous and f-invariant one dimensional foliation for the manifold M.

Proposition 2.4. *Let $(M, \mathcal{A}, \mu)$ be a probability space, where M is a manifold and $\mathcal{A}$ is the completation of the Borel σ-algebra $\mathcal{B}$. If $\mathcal{F}$ is a one-dimensional continuous foliation of M and given a finite open cover $\mathcal{U}$ of M by local charts of $\mathcal{F}$ then, for every $U \in \mathcal{U}$ the family of plaques $\{\mathcal{F}|U(x)\}_{x \in U}$ is a countably generated measurable partition for U.*

Proof. Given $\mathcal{F}$ a one-dimensional continuous foliation $\mathcal{F}$ and a local chart $\varphi : U \to (0,1) \times B_1^{n-1}(0)$, we define the countable collection of subsets $\{\widetilde{E}_{q,k}\}$ by:

$$\widetilde{E}_{q,k} = (0,1) \times B(q, 1/k) \subset \mathbb{R} \times \mathbb{R}^{n-1}, \quad q \in B_1^{n-1}(0) \cap \mathbb{Q}^{n-1},\ k \in \mathbb{N}.$$

Note that every $\widetilde{E}_{q,k}$ is a Borelian subset of $\mathbb{R}^n$ and since φ is an homeomorphism we have that $E_{q,k} = \varphi^{-1}(\widetilde{E}_{q,k})$ is a Borelian subset in U. Let $\mathcal{E} \subset \mathcal{B}$ be the sub-σ-algebra generated by the family of Borelian sets $\{E_{q,k}\}$. Notice that for every $y \in M$ the atom, $[y]$ is the connected component of $\mathcal{F}(y) \cap U$ that contains y, this is, $\mathcal{F}|U(y)$. Therefore $\{\mathcal{F}|U(x)\}_{x \in U}$ is a countably generated partition for U. □

By the proposition above, for each $U \in \mathcal{U}$, there exists a family of measures $\{\mu_x^U\}_{x \in U}$ that is the disintegration of the measure $\mu(\cdot|U)$ in U.

In this case we also call $\{\mu_y^U\}_{y\in U}$, *the system of conditional measures of* μ or *the disintegration of* μ *along* $\mathcal{F}$ *restricted to* U.

A significant observation to make here is that the conditional measures for distinct foliation boxes match on their intersection. That is, if U_1 and U_2 are foliation boxes such that $U_1 \cap U_2 \neq \emptyset$, then for almost every $x \in U_1 \cap U_2$, we have $\mu_x^{U_1} = \mu_x^{U_2}$ up to a constant factor. In other words, the conditional measures μ_x^U are compatible across different foliation boxes, which reflects the coherence of the foliation structure.

Proposition 2.5. *[4]. If* U_1 *and* U_2 *are domains of two local charts* φ_1 *and* φ_2 *of* $\mathcal{F}$*, then for almost every* x *the conditional measures* $\mu_x^{U_1}$ *and* $\mu_x^{U_2}$ *coincide up to a constant on* $U_1 \cap U_2$.

Proof. For the sake of the reader, we will recall the proof given in [4]. Let Σ be a cross-section of U_1, this is, Σ is a submanifold of dimension $m - n$ intersecting every local leaf at exactly one point. Let μ_{U_1} be the measure on Σ obtained by projecting $\mu(\cdot|U_1)$ along the local leaves. Consider any $C = U_1 \cap U_2 \subset U_1$ and let μ_C be the image of $\mu(\cdot|C)$ under the projection along the local leaves. The Radon-Nikodym derivative

$$\frac{d\mu_C}{d\mu_{U_1}} \text{ at } \mu_C\text{-almost every point.}$$

Then for any measurable set $E \subset C$,

$$\mu(E) = \int_\Sigma \mu_x^{U_1}(E)\, d\mu_{U_1}(x) = \mu(E) = \int_\Sigma \mu_x^{U_1}(E) \frac{d\mu_C}{d\mu_{U_1}}\, d\mu_C(x)$$

By essential uniqueness, this proves that the disintegration of $\mu(\cdot|C)$ along the local leaves is given by

$$\mu_x^C = \frac{d\mu_{U_1}}{d\mu_C}(y)(\mu_x^{U_1}|C) \text{ for } \mu\text{-almost every } x \in C$$

where $y \in \mathcal{F}|U_1(x) \cap \Sigma$. Doing the same for $C = U_1 \cap U_2 \subset U_2$, we have that

$$\frac{d\mu_{U_1}}{d\mu_C}(y)(\mu_x^{U_1}|C) = \frac{d\mu_{U_2}}{d\mu_C}(y)(\mu_x^{U_2}|C) \text{ for } \mu\text{-almost every point } x \in C.$$

□

As observed in [4], Proposition 2.5 allows us to define a family of classes of measures $\{\Omega_x : x \in M\}$, such that

- $\omega_x(M \setminus \mathcal{F}(x)) = 0$ for every $x \in M$ and every representative $\omega_x \in \Omega_x$,
- the function $x \mapsto \Omega_x$ is constant on the leaves of $\mathcal{F}$,

- for a foliated box U we have that the conditional measures μ_x^U along the plaques $\mathcal{F}|U(x)$ coincide almost everywhere with the normalized restrictions of the Ω_x to the plaques in U, this is, for almost every point $x \in U$ we have

$$\mu_x^U = \omega_x(\cdot|\mathcal{F}|U(x)),$$

where ω_x denotes a representant of Ω_x.

2.3 Lebesgue differentiation theorem

In this section, we will recall an important result of measure theory known as the Lebesgue differentiation theorem. The theorem applies to classes of metric measure spaces (X, d, μ) that meet certain conditions.

Definition 2.6. *Given a metric space (X, d), a measure ν on X is said to be a doubling measure if there exists a constant $R > 0$ such that for any $x \in X$ and any $r > 0$ we have*

$$\nu(B(x, 2r)) \leqslant R \cdot \nu(B(x, r)).$$

We say that (X, d, μ) is a doubling metric measure space if μ is a doubling measure for the metric space (X, d).

The following theorem is a Lebesgue differentiation theorem for general doubling metric measure spaces.

Theorem 2.7. *[18, Lebesgue differentiation Theorem] Let (X, d, μ) be a doubling metric measure space, and $f : X \to \mathbb{R}$ a localy integrable function. Then, almost every point $x \in X$ is a Lebesgue density point of f, that is,*

$$\lim_{r\to 0} \frac{1}{\mu(B[x, r])} \int_{B[x,r]} |f(y) - f(x)|\, d\mu(y) = 0,$$

where $B[x, r]$ denote the closed ball in X with center x and radius r. In particular,

$$\lim_{r\to 0} \frac{1}{\mu(B[x, r])} \int_{B[x,r]} f(y)\, d\mu(y) = f(x),$$

for almost every $x \in X$.

Note that for every $x \in \operatorname{supp}\mu$ we have $\mu(B[x, r]) > 0$ for every $r > 0$, therefore the two expressions given in the theorem make sense almost everywhere.

To present the following result, let us recall the definition of a regular measure.

Definition 2.8. *Let X be a topological space, and let $\mathcal{A}$ be a σ-algebra on X. Let μ be a measure on $(X, \mathcal{A})$. We say that μ is a regular measure if, for every measurable subset $A \subset X$, we have:*

$$\mu(A) = \sup\{\mu(F) \,:\, F \subset A,\, F \text{ compact and measurable,}\}$$

and

$$\mu(A) = \inf\{\mu(G) \,:\, G \supset A,\, G \text{ open and measurable.}\}$$

Using Theorem 2.7, we can find a characterization for the Radon-Nikodym derivative associated to two measures, one of them being a doubling measure.

Theorem 2.9. *[18, Lebesgue–Radon–Nikodym theorem] Let (X, d, μ) be a doubling metric measure space and ν be a locally finite Borel-regular measure on X. If $\nu << \mu$ then for almost every $x \in X$ the Radon-Nikodym derivate of ν with respect μ, $\frac{d\nu}{d\mu}(x)$, is given by the limit*

$$\frac{d\nu}{d\mu}(x) = \lim_{r\to 0} \frac{\nu(B[x,r])}{\mu(B[x,r])} \tag{2.3}$$

Proof. Consider the locally integrable function $f : X \to \mathbb{R}$, given by

$$f(x) = \frac{d\nu}{d\mu}(x), \text{ for almost every } x \in X.$$

Since μ is a doubling measure in (X, d) by the Lebesgue differentiation Theorem, we have that

$$\begin{aligned} f(x) &= \lim_{r\to 0} \frac{1}{\mu(B[x,r])} \int_{B[x,r]} f(y)\, d\mu(y) \\ &= \lim_{r\to 0} \frac{1}{\mu(B[x,r])} \int_{B[x,r]} \frac{d\nu}{d\mu}(y)\, d\mu(y) \\ &= \lim_{r\to 0} \frac{\nu(B[x,r])}{\mu(B[x,r])}. \end{aligned}$$

□

3 Metric System

In this chapter, we introduce the definition of an $\mathcal{F}$-arc length system for a continuous and f-invariant one-dimensional foliation $\mathcal{F}$. This definition was motivated by the notion of *center metric* given by Bonatti and Zhang in [11].

3.1 Invariant arc-lengths systems

From now on, $\mathcal{F}$ will denote a continuous and f-invariant one-dimensional foliation for the manifold M.

Definition 3.1. *Given a foliation $\mathcal{F}$ of M and $x \in M$, we say that a C^1-curve $\gamma : [0,1] \to \mathcal{F}(x)$ is a simple arc if $\gamma(t) \neq \gamma(s)$ for all $t \neq s$ with $(t,s) \notin \{(0,1),(1,0)\}$.*

By convention, by degenerate arc we mean a point. Given two simple arcs γ and σ, we write $\gamma \sim \sigma$ to indicate that σ is a reparametrization of γ. Clearly, this defines an equivalence relation on the space of simple arcs. Abusing the notation, the simple arcs γ, is any representative of the class $[\gamma]$.

Definition 3.2. *We say that a sequence of simple arcs γ_n converges to γ (in the C^0-topology) if γ_n converges pointwise to γ.*

Definition 3.3. *We call $\{l_x\}$ an $\mathcal{F}$-arc length system, if for each $x \in M$, l_x is a real valued function defined on the simple arcs on $\mathcal{F}(x)$ and satisfies the following properties:*

1. *l_x is strictly positive on the non-degenerate arcs, and vanish on degenerate arcs,*
2. *let $\gamma : [0,1] \to \mathcal{F}(x)$ be a simple arc and $a \in (0,1)$, then*
$$l_x(\gamma[0,a]) + l_x(\gamma[a,1]) = l_x(\gamma[0,1]);$$
3. *let $\gamma : [0,1] \to \mathcal{F}(x)$ a simple arc, then*
$$l_x(\gamma[0,1]) = l_{f(x)}(f(\gamma[0,1]));$$

4. *given a sequence of simple arcs* $\gamma_n : [0,1] \to \mathcal{F}(x_n)$ *converging to a simple arc* $\gamma : [0,1] \to \mathcal{F}(x)$*, then*

$$l_{x_n}(\gamma_n) \to l_x(\gamma), \quad \text{as } n \to +\infty.$$

Below we present some examples of systems with invariant foliations that admit $\mathcal{F}$-arc length systems.

Example 3.4. Assume $d \geqslant 2$ and let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation given by a matrix with integer entries, such that 1 is an eigenvalue of L. Let v be an eigenvector of L associated with the eigenvalue 1. Consider the subspace $E = \mathbb{R} \cdot v$. Since L is linear, it induces a linear map $f_L : \mathbb{T}^d \to \mathbb{T}^d$, where $\mathbb{T}^d$ is the d-dimensional torus, and E induces a one-dimensional foliation $\mathcal{F}$ on $\mathbb{T}^d$ that is f_L-invariant. It follows that f_L preserves the arc length along the leaves of $\mathcal{F}$, and therefore the standard arc-lengths on the leaves form an $\mathcal{F}$-arc length system.

Question 3.1. *Is there an ergodic measure* μ *preserved by* f_L *whose conditional measures are supported on a Cantor subset of the leaves?*

Example 3.5. Let $\varphi : \mathbb{R} \times M \to M$ be any C^1 flow. The foliation $\mathcal{F}$ given by the orbits of φ is a φ_t-invariant C^1-foliation of M for any fixed $t \in \mathbb{R}$. There is a natural $\mathcal{F}$-arc length system in this case given by:

$$l_x(\gamma) := l, \quad \text{with} \quad \varphi(l, \gamma(0)) = \gamma(1).$$

Assume that almost every $x \in M$ is not a periodic point of φ. Given any φ_t-ergodic invariant measure μ, it follows from [22, Example 7.4] that the disintegration of μ along $\mathcal{F}$ is either Lebesgue or atomic.

Example 3.6. Some skew-products also provide interesting examples. For instance, consider the function $f : \mathbb{T}^d \times S^1 \to \mathbb{T}^d \times S^1$, given by

$$f(x, y) = (g(x), R_\alpha(y)),$$

where $g : \mathbb{T}^d \to \mathbb{T}^d$ is any homeomorphism and $R_\alpha : S^1 \to S^1$ is a rotation of angle α.

In this example, the foliation $\mathcal{F}$ is defined as the set of leaves of the form $\{x\} \times S^1$, where x is an element of $\mathbb{T}^d$. These leaves are invariant under f, meaning that if (x, y) is on a leaf $\mathcal{F}(x_0, y_0)$, then $f(x, y)$ is on the leaf $\mathcal{F}(f(x_0, y_0))$. Moreover, if for every $x \in \mathbb{T} \times S^1$ we take l_x to be the usual arc length measure on the circle S^1, but defined on the leaf $\{x\} \times S^1$, then we get a collection of arc length measures $\{l_x\}$ that forms an $\mathcal{F}$-arc length system. This means that the lengths of curves along the leaves of $\mathcal{F}$ are well-defined and can be measured consistently using these arc length measures.

In this example it is easy to determine the measurable properties of $\mathcal{F}$ in the sense that, given a Borel g-invariant measure ν, the measure $\nu \times \lambda_{S^1}$ is f-invariant, and a direct application of the Fubini Theorem shows that the disintegration of μ along $\mathcal{F}$ has the Lebesgue measures λ_{S^1} as its conditional measures.

Center metric for transitive partially hyperbolic diffeormorphism with topological neutral center

One of the main motivations of this work was to understand the measurable properties of the center foliation preserved by partially hyperbolic diffeomorphisms with topological neutral center. Our results imply that the disintegration of any f-invariant ergodic probability measure of such maps falls into one of two possible cases. When the conditional measures have full support, the occurrence of an invariance principle is proven by the author in [26]. Furthermore, if the measure is smooth, full support of the conditional measures implies the Bernoulli property for f.

In order to present the following example, which was provided by recent results of Bonatti-Zhang [11], first we are going to give some important definitions of properties for dynamical systems.

Definition 3.7. *A C^1 diffeomorphism $f : M \to M$, on a compact Riemannian manifold M, is said to be partially hyperbolic if there is a nontrivial splitting*

$$TM = E^s \oplus E^c \oplus E^u$$

such that

$$Df(x)E^\tau(x) = E^\tau(f(x)),\ \tau \in \{s, c, u\}$$

and a Riemannian metric for which there are continuous positive functions $\mu, \hat{\mu}, \nu, \hat{\nu}, \gamma, \hat{\gamma}$ with

$$\nu(p), \hat{\nu}(p) < 1, \quad \text{and} \quad \mu(p) < \nu(p) < \gamma(p) < \hat{\gamma}(p)^{-1} < \hat{\nu}(p)^{-1} < \hat{\mu}(p)^{-1},$$

such that for any vector $v \in T_pM$,

$$\mu(p)||v|| < ||Df(p) \cdot v|| < \nu(p)||v||,\ \text{if}\, v \in E^s(p)$$

$$\gamma(p)||v|| < ||Df(p) \cdot v|| < \hat{\gamma}(p)^{-1}||v||,\ \text{if}\, v \in E^c(p)$$

$$\hat{\nu}(p)^{-1}||v|| < ||Df(p) \cdot v|| < \hat{\mu}(p)^{-1}||v||,\ \text{if}\, v \in E^u(p).$$

If a partially hyperbolic diffeomorphism f has invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ that are tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$ respectively, then we say that f *is dynamically coherent*. In such cases, the intersection of $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ forms the *center foliation.*

While the strong stable and strong unstable bundles of partially hyperbolic diffeomorphisms are always integrable, and they are integrated into unique f-invariant foliations known as strong foliations, the center bundle presents a more complex situation. Even in the one-dimensional center case, center foliations may not exist, as proved by examples in [10] and [29].

Definition 3.8. *We say that f has topological neutral center if, for any $\varepsilon > 0$, there exists $\delta > 0$ for which: given any C^1 curve $\gamma : [0,1] \to M$ with $\gamma'(t) \in E^c(\gamma(t))$, $0 \leqslant t \leqslant 1$, if length$(\gamma) < \delta$ then length$(f^n(\gamma)) < \varepsilon$, for all $n \in \mathbb{Z}$.*

In [11] the authors study partially hyperbolic diffeomorphisms that are transitive and have a one-dimensional topologically neutral center, resulting in dynamically coherent systems. They proved the existence of a continuous metric along the center foliation that remains invariant under the dynamical systems.

Theorem 3.9. *[11, Theorem A] Let $f : M \to M$ be a transitive C^1 partially hyperbolic diffeomorphism with one-dimensional topologically neutral center direction. Then f admits a center metric system which is invariant under f.*

Outline of proof. We will construct metrics $\{\ell_L\}$ along the center leaves in a residual subset $\mathcal{N}$ of M. Then, we will show that the metric we construct is f-invariant, continuous, and invariant under the holonomies of the strong stable and strong unstable foliations. Thus, we can extend this metric system to the entire manifold M.

Before describing the set $\mathcal{N}$, we will define and present the main properties of the limit center maps, which will be crucial in constructing this system of metrics.

Let L_1 and L_2 be two center leaves of f. We say that a map $F : L_1 \to L_2$ is a *limit center map* if there exists a sequence $\{n_i\} \subset \mathbb{Z}$ with $|n_i| \to \infty$ such that $\{f^{n_i}\}$ pointwise converges to F. We denote by $\mathscr{L}(L_1, L_2)$ (resp. $\mathscr{L}(L)$) the set of all limit center maps from L_1 to L_2 (resp. from L to L).

Some important properties of the set of limit center maps are:

1. *Uniformly topologically neutrality*: For any $\varepsilon > 0$ small, there exist $\delta > 0$ and $\eta > 0$ such that for any $F \in \mathscr{L}(L_1, L_2)$, and any two points $x, y \in L_1$, we have
   - If $\mathrm{d}_1^c(x,y) < \delta$, then $\mathrm{d}_2^c(F(x), F(y)) < \varepsilon$, where d_i^c denotes the distance on center leaf L_i. In particular $F : (L_1, \mathrm{d}_1^c) \to (L_2, \mathrm{d}_2^c)$ is continuous;
   - If for $x, y \in L_1$ we have $\varepsilon_0/4 > \mathrm{d}_1^c(x,y) > \varepsilon$, then $\mathrm{d}_2^c(F(x), F(y)) > \eta$, where $\varepsilon_0 > 0$ is a lower bound for the length of center leaves.
2. For each $F \in \mathscr{L}(L_1, L_2)$, F is a surjective local homeomorphism.
3. If $F : L_1 \to L_2$ and $G : L_2 \to L_3$ are a limit center maps, then the composition $G \circ F$ is a limit center map from L_1 to L_3.

4. Let $F \in \mathscr{L}(L)$, and suppose that F has a fixed point $x \in L$:
    - if F is orientation preserving, then $F = \mathrm{Id}_L$;
    - if F is orientation reversing, then F is an involution on L, this is, $F^2 = \mathrm{Id}_L$.
5. If $F \in \mathscr{L}(L)$, then F is a homeomorphism.
6. For any $x \in M$, if there exists a sequence $\{n_i\} \subset \mathbb{N}$ such that $f^{n_i} \longrightarrow y \in M$, we can use the transitivity and continuity of the center foliation, together with the topologically neutral property along the one-dimensional center bundle, to construct a limit center map $F : L_x \to L_y$ that satisfies $F(x) = y$.

On the other hand, consider the set

$$\mathcal{N} = \{x \in M \,:\, \alpha(x) = \omega(x) = M\}, \tag{3.1}$$

where,

$$\omega(x) = \{y \in M \,:\, \exists n_k \overset{k\to\infty}{\longrightarrow} \infty \text{ such that } f^{n_k}(x) \overset{k\to\infty}{\longrightarrow} y\},$$

is the ω-limit of x; and

$$\alpha(x) = \{y \in M \,:\, \exists n_k \overset{k\to\infty}{\longrightarrow} -\infty \text{ such that } f^{n_k}(x) \overset{k\to\infty}{\longrightarrow} y\},$$

is the α-limit of x.

Note that $\mathcal{N}$ is f-invariant and since f is transitive, $\mathcal{N}$ is a residual subset of M. Assuming that L is a center leaf and $L \cap \mathcal{N} \neq \varnothing$, it follows from item (6) above that L is a subset of $\mathcal{N}$. This means that $\mathcal{N}$ contains all the center leaves, so the set $\mathcal{N}$ is saturated by the center leaves.

Additionally, for any center leaf L containing a point in $\mathcal{N}$, we have that $\mathscr{L}^+(L)$ is a group. Indeed, we already have $\mathrm{Id}_L \in \mathscr{L}^+(L)$ and for any $F, G \in \mathscr{L}^+(L)$, $G \circ F \in \mathscr{L}^+(L)$.

In order to show that $\mathscr{L}^+(L)$ is a group, it is necessary to demonstrate that for any $F \in \mathscr{L}^+(L)$, there exists a $G \in \mathscr{L}^+(L)$ such that $F \circ G = G \circ F = \mathrm{Id}_L$. Consider an $F \in \mathscr{L}^+(L)$ such that $F(x) = y$ for some $y \in L$. Since $L \subset \mathcal{N}$, there exists a $G \in \mathscr{L}^+(L)$ such that $G(y) = x$. Then, the limit center map $G \circ F$ has a fixed point. Using the item (4), we have $G \circ F = \mathrm{Id}_L$. By the item (5), both F and G are homeomorphisms on L.

Furthermore, using the properties of the limit center map, we can conclude that for any center leaf L containing a point in $\mathcal{N}$, the action on L given by the group $\mathscr{L}^+(L)$ is both free and transitive. By the Hölder theorem (see [24]), the group $\mathscr{L}^+(L)$ is isomorphic to the group of translations (respectively, rotations) on $\mathbb{R}$ (respectively, S^1).

Since every orientation-reversing limit center map from L to L is an involution (see [11, Proposition 4.7]), we can conclude that $\mathscr{L}(L)$ forms a group. Moreover, $\mathscr{L}(L)$ is either equivalent to $\mathscr{L}^+(L)$ or can be generated by the union of $\mathscr{L}^+(L)$ and $-\mathrm{Id}_L$.

Remark 3.10. *The properties mentioned above play a crucial role in proving Theorem 3.9 since the group formed by combining translations and* $-\mathrm{Id}_{\mathbb{R}}$ *(resp. rotations and* $-\mathrm{Id}_{\mathbb{S}^1}$*) maintains the Euclidean metric on* $\mathbb{R}$ *(resp.* $\mathbb{R}/\mathbb{Z}$*) unaltered. Any metric on* $\mathbb{R}$ *(resp.* $\mathbb{R}/\mathbb{Z}$*) that is invariant under the set of translations (resp. rotations) can be obtained by scaling the Euclidean metric by a constant factor.*

Definition of the family of center metrics $\{\ell_L\}_{L\subset\mathcal{N}\text{ center leaf}}$

The item (6) above establishes that for any $x, y \in M$, if y belongs to $\omega(x)$, then there exists a limit center map from L_x to L_y. This enables us to establish connections between the limit center maps on distinct center leaves, and we can prove that for any two center leaves L_1, L_2 such that L_1 and L_2 have non-empty intersection with $\mathcal{N}$, we have:

- Each limit center map $F : L_1 \to L_2$ is a homeomorphism.
- For any limit center maps $F, G \in \mathscr{L}(L_1, L_2)$, there exist $F_1 \in \mathscr{L}(L_1)$ and $F_2 \in \mathscr{L}(L_2)$ so that
$$G = F \circ F_1 = F_2 \circ F.$$

Let ℓ be an $\mathscr{L}(L)$-invariant metric on a center leaf $L \subset \mathcal{N}$. Consider a metric $F_*(\ell)$ in L_1, given by

$$F_*(\ell)(F(\sigma_{[a,b]})) = \ell(\sigma_{[a,b]}), \text{ for every center map } \sigma : [a, b] \to L.$$

Since $F : L \to L_1$ is a homeomorphism, $F_*(\ell)$ is well defined.

Furthermore, the second point above establishes that the metric $F_*(\ell)$ on L_1 remains unaffected by the choice of F and is $\mathscr{L}(L_1)$-invariant. That is, given a leaf center $L \cap \mathcal{N} \neq \varnothing$ and a $\mathscr{L}(L)$-invariant metric ℓ_L on L, for any center leaf $L_1 \cap \mathcal{N} \neq \varnothing$ and any two limit center maps $F_1, F_2 \in \mathscr{L}(L, L_1)$, we have

$$(F_1)_*(\ell_L) = (F_2)_*(\ell_L).$$

Moreover, note that $f(L) \subseteq \mathcal{N}$, which implies that for any $F \in \mathscr{L}(L, f(L))$, the composition $f^{-1} \circ F \in \mathscr{L}(L)$, and hence $F_*(\ell_L) = f_*(\ell_L)$.

Thus, we can ensure the existence of a family of metrics $\{\ell_L\}_{L\subset\mathcal{N}\text{ center leaf}}$ in the center leaves that are contained within $\mathcal{N}$, such that

- For any two center leaves L_1, L_2 contained in $\mathcal{N}$ and any $F \in \mathscr{L}(L_1, L_2)$, we have
$$F_*(\ell_{L_1}) = \ell_{L_2};$$
- For any leaf $L \subset \mathcal{N}$, we have
$$f_*(\ell_L) = \ell_{f(L)},$$

- If $\{\widetilde{\ell}_L\}_{L\subset\mathcal{N}\text{ center leaf}}$ is another family of metric satisfying the two properties above, then there exist $\lambda > 0$ such that

$$\widetilde{\ell}_L = \lambda \cdot \ell_L.$$

Therefore, in order to prove Theorem 3.9, it is necessary to establish that the family of metrics $\{\ell_L\}_{L\subset\mathcal{N}\text{ center leaf}}$ can be extended smoothly as a center metric on all of M. The main tool for proving this is to check that the family $\{\ell_L\}_{L\subset\mathcal{N}\text{ center leaf}}$ is invariant under the holonomies of the strong stable and strong unstable foliations.

Holonomy of the strong stable foliation

First, we show that given a center leaf L and points $x, y \in L$ with $y \in \omega(x)$, every strong stable leaf intersects L in at most one point. Indeed, by item (6), there exists a limit center map $F : L \to L$ such that $F(x) = y$. Assume that there are $z_1, z_2 \in L$ such that $z_1 \neq z_2$ and $z_1 \in \mathcal{F}^{ss}(z_2)$. Since F is a limit center map, there exists a sequence $n_i \longrightarrow \infty$ such that f^{n_i} converges to F. This implies that $F(z_1) = F(z_2)$, contradicting the fact that F is a homeomorphism. This observation is important for defining the holonomy between two leaves.

Now, let $L_1, L_2 \subset \mathcal{N}$ be two center leaves in the same center-stable leaf L^{cs}. By Proposition 2.8 in [11], since f is a C^1 partially hyperbolic diffeomorphism with topologically neutral center, the center stable foliation has the completeness property. Thus, L_2 is contained in the union of the strong stable leaves through L_1 which coincides with L^{cs} and vice versa. According to the above observation, each strong stable leaf cuts L_1 in at most one point, and the same for L_2. Thus,

- The map $H^{ss} : L_1 \to L_2$ induced by the holonomy of the strong stable foliation is a homeomorphism from L_1 to L_2, because each strong stable leaf in L^{cs} intersects L_1 and L_2 in exactly one point, creating a unique correspondence between the two sets.

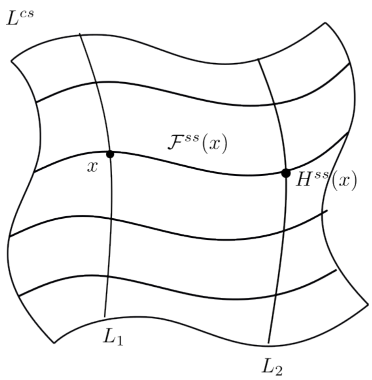

Figure 2 – Holonomy of the strong stable foliation

- Let $H^{ss} : L_1 \to L_2$ be the holonomy of the strong stable foliation and $\{\ell_L\}_{L \subset \mathcal{N} \text{ center leaf}}$ be a family of metrics in the center leaves in $\mathcal{N}$. Then

$$\ell_{L_2} = (H^{ss})_*(\ell_{L_1}).$$

Finally, it was proved in [11, Proposition 4.18], that the family of metrics in the center leaves in $\mathcal{N}$, $\{\ell_L\}_{L \subset \mathcal{N} \text{ center leaf}}$ can be extended in a unique way, by continuity, to all the center leaves, defining a center metric on M.

□

As the proof of Theorem 3.9 strongly relies on the fact that the central direction is one-dimensional, a natural question arises:

Question 3.2. *What happens for partially hyperbolic systems with a* 2*-dimensional center foliation and a topologically neutral center? Are there invariant metrics?*

The previous theorem is a very important part of the proof of the following result, which is a classification of partially hyperbolic diffeomorphisms on a closed 3-manifold with topological neutral center.

Theorem 3.11. *[11, Theorem C] Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism and M a closed manifold of dimension* 3*. Assume that f has a one-dimensional topologically neutral center and f is transitive. Then, up to finite lifts and iterates, f is C^0-conjugate to one of the following:*

1. *Skew products over a linear Anosov on $\mathbb{T}^2$ with rotations of the circle.*
2. *The time* 1*-map of a transitive topological Anosov flow.*

3.2 Invariant $\mathcal{F}$-metric systems

Let $\mathcal{F}$ be a one-dimensional continuous foliation for M and $\{l_x\}_{x \in M}$ be an $\mathcal{F}$-arc length system as defined in the previous section. For every $x \in M$ in this section we are going to define a metric d_x in the leaf $\mathcal{F}(x)$. The metric system $\{d_x\}_{x \in M}$ is important throughout this work because based on it we define the measure within the plaque of the foliation $\mathcal{F}$.

Since we will show that the metric system is additive, first we define what it means for a point to be between two points within a leaf $\mathcal{F}(x)$ for all $x \in M$.

Definition 3.12. *Let $\mathcal{F}$ be an f-invariant one-dimensional foliation of M. For any $x \in M$, given $y, z, w \in \mathcal{F}(x)$ we say that y is between z and w, if there exists a simple arc*

$\gamma : [0,1] \to \mathcal{F}(x)$ such that $\gamma(0) = z$, $\gamma(1) = w$, $\gamma(t) = y$ for some $t \in (0,1)$, and γ has the least length of the simple arcs at $\mathcal{F}(x)$ connecting the points y and z, this is,

$$l_x(\gamma) = \min\{l_x(\alpha) \,:\, \alpha : [0,1] \to \mathcal{F}(x) \text{ such that } \alpha \text{ is a simple arc, with } \alpha(0) = z \text{ and } \alpha(1) = w\}.$$

Next, we show that for each $x \in M$, one can define an additive metric d_x in $\mathcal{F}(x)$. Furthermore, we show that the metric system $\{d_x\}_{x \in M}$ is an f-invariant metric system along the foliation $\mathcal{F}$, that is, given any $x \in M$, the equality $d_{f(x)}(f(z), f(w)) = d_x(z, w)$ holds for every $z, w \in \mathcal{F}(x)$.

Lemma 3.13. *Consider the $\mathcal{F}$-arc length system, $\{l_x\}_{x \in M}$. For every $x \in M$ the function d_x in $\mathcal{F}(x)$, given by*

$$d_x(y, z) := \min\{l_x(\gamma) \,:\, \gamma : [0,1] \to \mathcal{F}(x) \text{ is simple arc, with } \gamma(0) = y \text{ and } \gamma(1) = z\},$$

has the following properties:

1. *d_x is an additive metric, that is, given $y, z, w \in \mathcal{F}(x)$ such that y is between z and w, then*
$$d_x(z, w) = d_x(z, y) + d_x(y, w);$$
2. *d_x is invariant by f, that is,*
$$d_{f(x)}(f(z), f(y)) = d_x(z, y).$$

Proof. First we show that for $x \in M$, d_x is a metric on $\mathcal{F}(x)$, let $y, z \in \mathcal{F}(x)$:

- $d_x(y,z) \geqslant 0$: from the definition it is clear that $d_x(y, z) \geqslant 0$ and $d_x(y, z) = 0$ if, and only if, $x = y$.
- $d_x(x, y) = d_x(y, x)$:
$$\begin{aligned} d_x(y, z) &= \min\{l_x(\gamma) \,:\, \gamma : [0,1] \to \mathcal{F}(x) \text{ is simple with } \gamma(0) = y,\, \gamma(1) = z\} \\ &= \min\{l_x(\gamma) \,:\, \gamma : [0,1] \to \mathcal{F}(x) \text{ is simple with } \gamma(0) = z,\, \gamma(1) = y\} \\ &= d_x(z, y). \end{aligned}$$
- $d_x(z, w) \leqslant d_x(z, y) + d_x(y, w)$: we will actually show item (1), that is, the metric is additive.

1. Since for every $x \in M$ we have that $\mathcal{F}(x)$ is a one-dimensional manifold, then $\mathcal{F}(x)$ is homeomorphic either to $\mathbb{R}$ or $\mathbb{S}^1$, which is why we divide the proof into two parts.

First we assume that $\mathcal{F}(x)$ is homeomorphic to $\mathbb{R}$. Given $z, w, y \in \mathcal{F}(x)$ such that y is between z and w, there exists a single simple arc (modulo parameterization), $\alpha : [0,1] \to \mathcal{F}(x)$ and $t_0 \in (0,1)$ such that $\alpha(0) = z$, $\alpha(1) = w$ and $\alpha(t_0) = y$. Then $\alpha_{|[0,t_0]}$ and $\alpha_{|[t_0,1]}$ are the only simple connected paths (modulo reparametritation) from z to y, and from y to w, respectively. Thus, by Definition 3.3 we have that

$$\begin{aligned} d_x(z,w) &:= \min\{l_x(\gamma) : \gamma : [0,1] \to \mathcal{F}(x),\ \gamma(0) = z \text{ and } \gamma(1) = w\} \\ &= l_x(\alpha) = l_x(\alpha_{|[0,t_0]}) + l_x(\alpha_{|[t_0,1]}) \\ &= d_x(z,y) + d_x(y,w). \end{aligned}$$

Now, suppose that $\mathcal{F}(x)$ is homeomorphic to S^1. Then, modulo reparamentrization, there are only two paths $\alpha_1, \alpha_2 : [0,1] \to \mathcal{F}(x)$ that connect the points z and w. Assume, without loss of generality, that $l_x(\alpha_1) < l_x(\alpha_2)$. This implies that

$$d_x(z,w) = \min\{l_x(\gamma) : \gamma : [0,1] \to \mathcal{F}(x),\ \gamma(0) = z \text{ and } \gamma(1) = w\} = l_x(\alpha_1)$$

and there exists $t_0 \in (0,1)$ such that $\alpha_1(t_0) = y$.

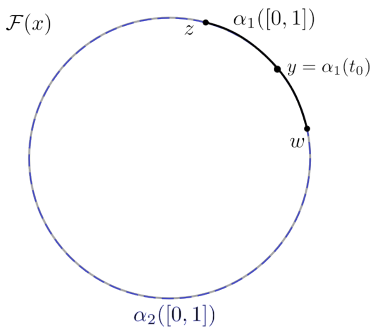

Figure 3 – Simple arcs α_1 and α_2

We will show that $d_x(z,y) = l_x(\alpha_{1|[0,a]})$, and with an analogous proof we also obtain that $d_x(y,w) = l_x(\alpha_{1|[a,0]})$. Indeed, suppose there is an other path $\alpha_3 : [0,1] \to \mathcal{F}(x)$ for which $\alpha_3(0) = z$, $\alpha_3(1) = y$ and $d_x(z,y) = l_x(\alpha_3)$. Since there are only two paths connecting points z and y (unless reparameterized), we have that α_3 is the concatenation of α_2 with $-\alpha_{1|[-1,-a]}$, where $-\alpha_1$ denotes the curve $-\alpha_1 : [-1,0] \to \mathcal{F}(x)$, $-\alpha_1(t) := \alpha_1(-t)$. Thus,

$$l_x(\alpha_{1|[0,a]}) > d_x(z,y) = l_x(\alpha_3) \geqslant l_x(\alpha_2) \geqslant l_x(\alpha_1),$$

which is a contradiction. Therefore, $l_x(\alpha_{1|[0,a]}) = d_x(z,y)$ and, analogously, $l_x(\alpha_{1|[a,1]}) = d_x(y,w)$. By the second item of Definition 3.3, we have

$$d_x(z,w) = l_x(\alpha_1) = l_x(\alpha_{1|[0,a]}) + l_x(\alpha_{1|[a,1]}) = d_x(z,y) + d_x(y,w),$$

concluding that d_x is an additive metric, as we wanted to show.

2. By the definition of d_x and the $\mathcal{F}$-arc length system $\{l_x\}$, we have that

$$\begin{aligned} d_{f(x)}(f(z), f(y)) &= \min\{l_{f(x)}(\gamma) : \gamma : [0,1] \to \mathcal{F}(f(x)) \text{ with } \gamma(0) = f(y), \gamma(1) = f(z)\} \\ &= \min\{l_x(\gamma) : \gamma : [0,1] \to \mathcal{F}(x) \text{ with } \gamma(0) = y, \gamma(1) = z\} \\ &= d_x(z,y). \end{aligned}$$

□

Definition 3.14. *We call $\{d_x\}_{x\in M}$ the $\mathcal{F}$-metric system if $\{d_x\}_{x\in M}$ is the family of metric associated to the $\mathcal{F}$-arc length system $\{l_x\}_{x\in M}$, given in Lemma 3.13.*

Remark 3.15. *It is important to note that, in general, from the definition of d_x we cannot guarantee the convergence $d_{x_n}(x_n, y_n) \to d_x(x,y)$ for two sequences $x_n \to x$ and $y_n \to y$ with $y_n \in \mathcal{F}(x_n)$, $y \in \mathcal{F}(x)$. This problem motivates the definition of the property we call plaque-continuous.*

Definition 3.16. *Consider $\mathcal{F}$ a one-dimensional continuous foliation of M. We say that a function $F : \bigcup_{x\in M} \mathcal{F}(x) \times \mathcal{F}(x) \to [0,\infty)$ is plaque-continuous if given any $p \in M$, there exists a foliated box $p \in U$, such that for any sequences $x_n \to x$, $y_n \to y$ with $y_n \in \mathcal{F}|U(x_n)$, $x \in U$ and $y \in \mathcal{F}|U(x)$, we have*

$$\lim_{n\to\infty} F(x_n, y_n) = F(x,y).$$

Any such foliated box U will be called a continuity-domain of F.

Definition 3.17. *We say that a family of metrics $\{d_x : x \in M\}$ is plaque-continuous, if the function $F : \bigcup_{x\in M} \mathcal{F}(x) \times \mathcal{F}(x) \to [0,\infty)$ defined by*

$$F(x,y) := d_x(x,y),$$

is plaque continuous. In this case, if U is a continuity-domain of F, we will also say that U is a continuity-domain of $\{d_x\}$.

In the following proposition we will show that the metric system $\{d_x\}$ is plaque-continuous, which guarantees that the problem mencioned in Remark (3.15) does not occur restricted to plaques.

Proposition 3.18. *The $\mathcal{F}$-metric system $\{d_x\}_{x\in M}$, from Definition 3.14 is plaque-continuous.*

Proof. Let (φ, U) be a local chart of $\mathcal{F}$, where $\varphi : U \to (0,1) \times B(0,r) \subset \mathbb{R}^n$ for some $r > 0$. We know that the plaques of $\mathcal{F}$ in U are given by $\varphi^{-1}((0,1) \times \{z\})$, $z \in B(0,r)$. For any $p \in U$, consider $\phi : V \subset U \to (0,1) \times B(0,s) \subset \mathbb{R}^n$ another local chart centered in p such that

$$x \in V \Rightarrow l_x(\mathcal{F}|U(x)) > 3 \cdot l_x(\mathcal{F}|V(x)).$$

This can be done by the continuity of l_x in $\mathcal{F}(x)$.

Consider $x \in V$, $y \in \mathcal{F}|V(x) = \phi^{-1}((0,1), z')$ and sequences $x_n \in V$, $y_n \in \mathcal{F}|V(x_n) = \phi^{-1}((0,1), z_n)$ with $x_n \to x$ and $y_n \to y$. For each $n \in \mathbb{N}$ we define the simple curve $\gamma_n(t) := \phi^{-1}((1-t)\phi(x_n) + t\phi(y_n), z_n)$. Note that this curve minimizes the l_x-length connecting x_n and y_n, that is, $d_{x_n}(x_n, y_n) = l_{x_n}(\gamma_n)$.

By the convergence of the sequences x_n and y_n, we have that $\gamma_n \to \gamma$, where γ is the simple curve given by $\gamma(t) = \phi^{-1}((1-t)\phi(x) + t\phi(y), z')$, and by the choice of the local chart V, we have

$$d_{x_n}(x_n, y_n) = l_{x_n}(\gamma_n) \quad \text{and} \quad d_x(x,y) = l_x(\gamma).$$

Therefore, using the continuity of $\{l_x\}$ we conclude that $\lim_{n\to\infty} d_{x_n}(x_n, y_n) = \lim_{n\to\infty} l_{x_n}(\gamma_n) = l_x(\gamma) = d_x(x,y)$, this is, $\{d_x\}$ is plaque-continuous. □

Proposition 3.19. *Let $\mathcal{U}$ be a finite open cover of M by local charts of $\mathcal{F}$. There exists $\mathfrak{r} > 0$ such that for all $x \in M$, there is $U \in \mathcal{U}$ with*

$$B_{d_x}(x, \mathfrak{r}) \subset U.$$

Proof. For each $x \in M$, take any $U_x \in \mathcal{U}$ with $x \in U_x$. Since $U_x \cap \mathcal{F}(x)$ is an open set in $\mathcal{F}(x)$, there exists $r_x > 0$ for which $B_{d_x}(x, r_x) \subset U_x \cap \mathcal{F}(x)$. By plaque continuity of $\{d_x\}$, there exists a neighborhood $x \in V_x \subset U_x$ for which

$$y \in V_x \Rightarrow B_{d_y}(y, r_x) \subset U_x.$$

Since M is compact, we may cover M with a finite number of neighborhoods V_{x_i}, $1 \leqslant i \leqslant l$. Take $\mathfrak{r} = \min\{r_{x_i} : 1 \leqslant i \leqslant l\}$. □

Now using the fact that $\{d_x\}$ is plaque continuous, we are able to show that the union of open balls $B_{d_x}(x, r) \subset \mathcal{F}(x)$, for $r > 0$ small enough and x varying along a transversal to $\mathcal{F}$, is an open set in M.

Lemma 3.20. *Given any local open transversal T to $\mathcal{F}$, for any $r > 0$ small enough, the set*

$$S := \bigcup_{x \in T} B_{d_x}(x, r)$$

is open.

Proof. Let $\mathfrak{r} > 0$ be the number as in Proposition 3.19. Given $r > 0$ and $r < \mathfrak{r}$ small enough, if $x \in M$, there exists a local chart $(U, \varphi) \in \mathcal{U}$.

Consider $x \in T$, we can assume that T is a local transversal associated to the local chart (U, φ), then by the plaque continuity of $\{d_x\}$, for $r > 0$ small enough, we have that $B_{d_y}(y, r) \subset U$ for every $y \in T$. In particular $U \setminus \overline{T}$ has two open connected components, U_1 and U_2, with $\overline{U_1} \cap \overline{U_2} = \overline{T}$.

Since U is a local chart for a one-dimensional foliation, we may consider an orientation on the $\mathcal{F}|U$-plaques. Now we assume that S is not open. Then, there exists $y \in S$ and a sequence $y_k \notin S$, with $y_k \to y$. Consider $x \in T$ such that $y \in B_{d_x}(x, r) \subset \mathcal{F}(x) \cap U$. Denote by $\Phi = \{\varphi_t\}$ the flow on the $\mathcal{F}|U$-plaques induced by the orientation fixed before and such that

$$d_p(\varphi_t(p), p) = |t|,$$

whenever $\varphi_t(p)$ is defined. Let $t_0 \in \mathbb{R}$ such that $x = \varphi_{t_0}(y)$. As $y \in B_{d_x}(x, r)$ and $B_{d_x}(x, r)$ is an open set in $\mathcal{F}|U(x)$, there exists $\delta > 0$ for which

$$\varphi_t(y) \in B_{d_x}(x, r) \subset S, \quad t \in [t_0 - \delta, t_0 + \delta].$$

Now, by the plaque continuity and the fact that $y_k \to y$, we have

$$\varphi_{t_0-\delta}(y_k) \to \varphi_{t_0-\delta}(y), \quad \varphi_{t_0+\delta}(y_k) \to \varphi_{t_0+\delta}(y).$$

Observe that $\varphi_{t_0-\delta}(y)$ and $\varphi_{t_0+\delta}(y)$ belong to different connected components, thus, for k large enought the same happens for $\varphi_{t_0-\delta}(y_k)$ and $\varphi_{t_0+\delta}(y_k)$.

Since $\gamma_k := \{\varphi_t(y_k) : \quad t \in [t_0 - \delta, t_0 + \delta]\}$ is an arc with points in both the interior and the exterior of the connected component U_1, it must intersect its boundary, namely T. Then there exists $t_1 \in [t_0 - \delta, t_0 + \delta]$ such that $\varphi_{t_1}(y_k) \in T$. By the choices of t_0 and δ we have that $y_k \in S$ for large k, yielding a contradition.

That is, S is open, as we wanted to show. □

Lemma 3.21. *Let $\mathfrak{r} > 0$ be a real number given in Proposition 3.19. For every $0 < t \leqslant \mathfrak{r}/2$ and any Borel subset $B \subset M$, the set defined by*

$$\Phi_t(B) := \{x \in M : d_x(x, B) < t\}, \tag{3.2}$$

is a measurable set.

Proof. Let $\mathcal{U}$ be a finite cover of M by local charts which are continuity-domains of $\{d_x\}$. Consider $\mathfrak{r}$ the number given by Proposition 3.19. In particular the family $\{U_{\mathfrak{r}/2} : U \in \mathcal{U}\}$, defined by

$$U_{\mathfrak{r}/2} = \{x \in U : d_x(x, \partial U) \geqslant \mathfrak{r}/2\},$$

is still a cover of M. Let $B \subset M$ be a Borel subset. Observe that

$$\Phi_t(B \cap U_{\mathfrak{r}/2}) \subset U, \quad U \in \mathcal{U}, \quad t < \mathfrak{r}/2.$$

We will prove that, for $U \in \mathcal{U}$, the subset $\Phi_t(B \cap U_{\mathfrak{r}/2})$ is measurable and, since $\mathcal{U}$ is a finite cover of M, we conclude that $\Phi_t(B)$ is measurable.

Let $\varphi_U : U \to B_1^{n-1}(0) \times (0,1)$ be a local chart of $\mathcal{F}$. Since the foliation is of dimension one inside U, we can consider the orientation in the plaques $\mathcal{F}|U(x)$, which is induced by the orientation in the line segments of the form $\{\overline{x}\} \times (0,1) \subset \mathbb{R}^{n-1} \times \mathbb{R}$. This orientation induces, at each plaque, an order relation, which we will denote by $\prec$ (the plaque being implicit in the context).

Again, as in the proof of Proposition 3.19, we consider the flow along the plaques. Explicitly, for $s \in [-t,t]$, with $0 < t < \mathfrak{r}/2$ fixed, we define $\phi_s^U : U_{\mathfrak{r}/2} \to U$ by:

- for $s > 0$, $\phi_s^U(x)$ is the only point of the plaque $\mathcal{F}|U(x)$ such that $d_x(x, \phi_s^U(x)) = s$ and $x \prec \phi_s^U(x)$;
- for $s < 0$, $\phi_s^U(x)$ is the unique point of the plaque $\mathcal{F}|U(x)$ such that $d_x(x, \phi_s^U(x)) = -s$ and $\phi_s^U(x) \prec x$.

Due to the fact that cover $\mathcal{U}$ was chosen as being a finite collection of local charts which are a continuity-domain of $\{d_x\}$, we have that, for every $|s| < t$, the function ϕ_s^U is continuous and, consequently, by the definition of ϕ_s^U, it is a homeomorphism. Thus $\phi_s^U(B \cap U_{\mathfrak{r}/2})$ is a measurable subset of M for every $s \in [-t,t]$.

Now, for each $1 \leqslant i \leqslant n$, take

$$\Phi_t^U(B) := \bigcup_{\substack{q \in \mathbb{Q} \\ q < t}} \phi_q^U(B \cap U_{\mathfrak{r}/2}), \quad 0 \leqslant t < \mathfrak{r}/2.$$

Notice that $\Phi_t^U(B)$ is a measurable set, since each set in the countable union is measurable as we have proved before. Consequently,

$$\Phi_t(B) = \bigcup_{U \in \mathcal{U}} \Phi_t^U(B),$$

is a measurable set, as we wanted to show. □

In the following definition we will use the identification $S^1 = [0,1]/\sim$ where $0 \sim 1$, thus the point 0 stands for the equivalence class of 0 in S^1.

Definition 3.22. *Let $\mathcal{F}$ be a one-dimensional foliation of M. Given an $\mathcal{F}$-arc length system, $\{l_x\}_{x \in M}$, for $x \in M$ we have a well defined homeomorphism*

$$h_x : \mathcal{F}(x) \to F,$$

where $F = \mathbb{R}$ or $F = S^1$, $h_x(x) = 0$, and such that, for any simple arc $\gamma : [0,1] \to \mathcal{F}(x)$ we have

$$l_x(\gamma[0,1]) = \lambda(h_x(\gamma[0,1])),$$

where λ denotes the Lebesgue measure on F. In particular $\lambda(h_x(\gamma[0,1]))$ is the size of the interval $h_x(\gamma[0,1])$. We now define the measure λ_x on $\mathcal{F}(x)$ given by:

$$\lambda_x = (h_x^{-1})_* \lambda.$$

Note that, if $\gamma[0,1]$ is a simple arc in $\mathcal{F}(x)$, then

$$\lambda_x(\gamma[0,1]) = \lambda(h_x(\gamma[0,1])) = l_x(\gamma[0,1]).$$

Consequently, the measure λ_x is a doubling measure.

3.3 Properties of non-atomic disintegrations

For the proof of the main theorem, it is necessary to understand the topological structure of $\sup \mu_x^U$, where $\{\mu_x^U\}_{x\in M}$ is a disintegration of μ on a local chart U, as well as the properties of the measure $\mu(\cdot|U)$ with respect to this disintegration. To this end, we consider the specific context of our case.

Let $\mathcal{F}$ be a one-dimensional continuous foliation of M that is invariant under f. We consider $\mathcal{U}$ to be a finite cover of M by local charts U of $\mathcal{F}$ such that $\overline{U}$ is also contained within a local chart of $\mathcal{F}$, and each $U \in \mathcal{U}$ is a continuity domain of the $\mathcal{F}$-metric system $\{d_x\}_{x\in M}$.

For each fixed $U \in \mathcal{U}$, we consider $\{\mu_x\}_{x\in U}$ to be the disintegration of $\mu(\cdot|U)$ along the plaques of $\mathcal{F}$ in the local chart U. We say that the disintegration of μ along $\mathcal{F}$ is *atomic* if for any local chart U, for almost every $y \in U$ there exists $a(y)$ in the plaque $\mathcal{F}|U(y)$ with $\mu_y^U(a(y)) > 0$. If we assume that the disintegration is not atomic, then for each $U \in \mathcal{U}$ there exists a null measure subset $\mathcal{A}_U$ such that μ_x^U is not atomic for every $x \notin \mathcal{A}_U$.

We also fix the following notation: For $\mathfrak{r} > 0$, we denote the constant obtained in Proposition 3.19, and for any subset $X \subset M$, we denote by $\mathcal{B}_X$ the Borel sigma algebra of X given by the topology induced by that of M. It is important to observe that, by definition, for any $U \in \mathcal{U}$, the set $\mathcal{A}_U$ is $\mathcal{F}|U$-saturated in U.

Lemma 3.23. *If the disintegration of μ along $\mathcal{F}$ is not atomic, then for each $0 < r < \mathfrak{r}$, $U \in \mathcal{U}$ and $x \in U \setminus \mathcal{A}_U$, the map*

$$y \mapsto \mu_x^U(B_{d_x}(y,r)),$$

is continuous when restricted to the subset $V_x \subset \mathcal{F}|U(x)$ given by

$$V_x = \{y \in \mathcal{F}|U(x) :\ B_{d_x}(y,r) \subset \mathcal{F}|U(x)\}.$$

Proof. Let $U \in \mathcal{U}$ be a fixed local chart and take $x \in U$. It should be noted that the set $V_x \subset \mathcal{F}(x)$ containing x is connected and open subset in $\mathcal{F}|U(x)$. By definition of continuity, we want to show that

$$\lim_{n\to\infty} \mu_x^U(B_{d_x}(y_n,r)) = \mu_x^U(B_{d_x}(y,r)),$$

for any sequence $y_n \to y$, and $y_n, y \in V_x$.

Let y belong to V_x and denote by $\partial B_{d_x}(y,r)$ the boundary of the set $B_{d_x}(y,r)$ inside the plaque $\mathcal{F}|U(x)$. Since μ_x^U is not atomic and $\mathcal{F}$ is a one-dimensional foliation, for $0 < r < \mathfrak{r}$, we have that

$$\mu_x^U(\partial B_{d_x}(y,r)) = 0 \text{ and } \mu_x^U(\partial B_{d_x}(y_n,r)) = 0, \forall n \in \mathbb{N}.$$

Now, consider the set B_n given by $B_n := B_{d_x}(y_n,r)\Delta B_{d_x}(y,r)$, where $Y\Delta Z$ denotes the symmetric difference of the sets Y and Z. From standard measure theory, we have that:

$$\limsup_{n\to\infty} \mu_x^U(B_n) \leqslant \mu_x^U\left(\limsup_{n\to\infty} B_n\right).$$

Thus, since we are assuming that the disintegration is not atomic, this implies that the conditional measure of boundaries of balls is zero. Then we have

$$\begin{aligned}\limsup_{n\to\infty} \mu_x^U(B_n) &\leqslant \mu_x^U\left(\bigcap_{m=1}^{\infty}\bigcup_{n\geqslant m} B_n\right)\\ &\leqslant \mu_x^U(\partial B_{d_x}(y,r)) = 0.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \mu_x^U(B_{d_x}(y,r) \smallsetminus B_{d_x}(y_n,r)) \leqslant \lim_{n\to\infty} \mu_x^U(B_{d_x}(y_n,r) \smallsetminus B_{d_x}(y,r)) = 0.$$

Consequently,

$$\lim_{n\to\infty} \mu_x^U(B_{d_x}(y_n,r)) = \mu_x^U(B_{d_x}(y,r)),$$

as we wanted to show. □

Proposition 3.24. *Let $(U,\varphi) \in \mathcal{U}$ be a fixed local chart and $0 < r < \mathfrak{r}$. For every open subset $V \subset U$ such that*

$$x \in V \Rightarrow B_{d_x}(x,r) \subset U,$$

the map given by

$$x \mapsto \mu_x^U(B_{d_x}(x,r)),$$

is $\mathcal{B}_{V\smallsetminus\mathcal{A}_U}$-measurable when it is considered restricted to $V \smallsetminus \mathcal{A}_U$, and consequently, it is also $\mathcal{B}_{U\smallsetminus\mathcal{A}_U}$-measurable as $V \subset U$.

Proof. Let $(U,\varphi) \in \mathcal{U}$ be a fixed local chart. If $x, y \in U$ belong to the same $\mathcal{F}$-plaque in U, then $\mu_x^U = \mu_y^U$. By the definition of the family of measures μ_x^U, we already know that for all Borel subset $B \subset U$ the function

$$x \in V \mapsto \mu_x^U(B),$$

is Borel measurable.

Since we are considering (U, φ) as a local chart, we have that $\varphi\mid_V$ is an homeomorphism, and we can assume that it is of the form $\varphi = \varphi\mid_V : V \to (0,1) \times B_1^{n-1}(0)$, where $B_1^{n-1}(0) \subset \mathbb{R}^{n-1}$ is an open ball in $\mathbb{R}^{n-1}$.

Setting $g_r : V \to [0, \infty)$ to be

$$g_r(x) = \mu_x^U(B_{d_x}(x,r)),$$

we have

$$g_r \circ \varphi^{-1}(x_1, x_2) = \mu^U_{\varphi^{-1}(x_1,x_2)}(B_{d_{\varphi^{-1}(x_1,x_2)}}(\varphi^{-1}(x_1,x_2), r)).$$

In the first part of the proof, we will show that the function $g_r \circ \varphi^{-1} : (0,1) \times B_1^{n-1}(0) \to [0,1]$ is continuous in the variable x_1 and Borel measurable in the variable x_2.

Since the second coordinate being fixed $\{x_2\}$, we are evaluation the function on a single plaque. Note, that the function $g_r \circ \varphi^{-1}(\cdot, x_2)$ corresponds to the function $y \mapsto \mu^U_{x_2}(B_{d_{x_2}}(y,r))$. Then from Lemma 3.23, restricted to $V \backslash \mathcal{A}_U$ we have the continuity of $g_r \circ \varphi^{-1}$ the first coordinate, where the conditional measure is non-atomic.

Now, fix the first coordinate $x_1 \in (0,1)$, and consider the transversal $T = \{x_1\} \times B_1^{n-1}(0) \subset (0,1) \times B_1^{n-1}(0)$. By Lemma 3.20, we have that the set

$$S := \bigcup_{x \in \varphi^{-1}(T)} B_{d_x}(x,r),$$

is an open subset of M. Thus, the definition of measure disintegration implies that $y \mapsto \mu_y^U(S)$ is a Borel measurable function on V, which proves that $g_r \circ \varphi^{-1}(x_1, \cdot)$ is a $\mathcal{B}_T$-measurable function. In particular, its restriction to $T \cap \varphi(V \smallsetminus \mathcal{A}_U) = T \smallsetminus \varphi(\mathcal{A}_U)$ is a $\mathcal{B}_{T \smallsetminus \varphi(\mathcal{A}_U)}$-measurable map. But observe that by the definition of the set S, for $x \in T$ we have that

$$\mu_x^U(S) = \mu_x^U(B_{d_x}(x,r)).$$

Therefore, for fixed x_1, the map $x_2 \in G \smallsetminus \pi_2(\varphi(\mathcal{A}_U)) \mapsto \mu^U_{\varphi^{-1}(x_1,x_2)}(B_{d_{\varphi^{-1}(x_1,x_2)}}(\varphi^{-1}(x_1,x_2), r))$ is $\mathcal{B}_{G \smallsetminus \pi_2(\varphi(\mathcal{A}_U))}$-measurable, where $\pi_2 : (0,1) \times G \mapsto G$ is the projection onto the second coordinate.

Consequently, $g_r \circ \varphi^{-1}$ restricted to $((0,1) \times G) \smallsetminus \varphi(\mathcal{A}_U)$ is a jointly measurable function with respect to the product sigma-algebra $\mathcal{B}_{(0,1)} \times \mathcal{B}_{G \smallsetminus \pi_2(\varphi(\mathcal{A}_U))}$ (see for example [1, Lemma 4.51]). As φ is a homeomorphism, we conclude that g_r is $\mathcal{B}_{V \smallsetminus \mathcal{A}_U}$-measurable, as we wanted to show.

□

In the following Lemma, we prove that the subset of M consisting of all points $x \in M$ for which there is a ball in $\mathcal{F}(x)$ with null μ_x^U measure, is a relatively Borel set. This set will be essential for the proof of the main theorem.

Lemma 3.25. *For each $U \in \mathcal{U}$, the set*

$$\mathcal{Z}_U = \bigcup_{x \in U \setminus \mathcal{A}_U} \mathcal{F}|U(x) \setminus supp\ \mu_x^U,$$

is $\mathcal{B}_{U \setminus \mathcal{A}_U}$-measurable set.

Proof. First let us give a better formulation for the definition of $\mathcal{Z}_U$. Observe that

$$\mathcal{Z}_U = \{x \in U \setminus \mathcal{A}_U : \mu_x^U(I) = 0 \text{ for some open ball } x \in I \subset \mathcal{F}(x)\}.$$

Consider an enumeration $\{q_1, q_2, \ldots\}$ of the $\mathbb{Q} \cap [0,1]$, and let $\mathcal{U}$ be the given finite family of local charts covering M associated to foliation $\mathcal{F}$. For each $U \in \mathcal{U}$ and $i \in \mathbb{N}$, consider also the function $\phi_i^U : U_i \setminus \mathcal{A}_U \to \mathbb{R}$ given by

$$\phi_i^U(x) = \mu_x^U(B_{d_x}(x, q_i)),$$

where $U_i = \{x \in U : B_{d_x}(x, q_i) \subset U\}$.

Observe that we may cover U_i with a countable number of local charts $V_i^j \subset U_i$, $j \in \mathbb{N}$ and, by Proposition 3.24, we know that $\phi_i^U|V_i^j$ is a $\mathcal{B}_{V_i^j \setminus \mathcal{A}_U}$-measurable function for every j. In particular ϕ_i^U is $\mathcal{B}_{U_i \setminus \mathcal{A}_U}$-measurable for every i. Now for every $i \in \mathbb{N}$ define

$$\mathcal{Z}_i^U := (\phi_i^U)^{-1}(\{0\}) \subset M.$$

By the definition of ϕ_i^U, we have that $\mathcal{Z}_i^U$ is a $\mathcal{B}_{U_i \setminus \mathcal{A}_U}$-measurable subset,(in particular, a $\mathcal{B}_{U \setminus \mathcal{A}_U}$-measurable subset) and $\mu(\mathcal{Z}_i^U) = 0$. Note that, by definition of the set $\mathcal{Z}_U$, we can describe $\mathcal{Z}_U$ as an enumerable union of the sets $\mathcal{Z}_i^U$, that is,

$$\mathcal{Z}_U = \bigcup_{i=1}^{\infty} \mathcal{Z}_i^U. \tag{3.3}$$

Therefore $\mathcal{Z}_U$ is a $\mathcal{B}_{U \setminus \mathcal{A}_U}$-measurable subset, as we wanted to show. Moreover, since $\mu(\mathcal{Z}_i^U) = 0$ for any $i \in \mathbb{N}$, we have that $\mu(\mathcal{Z}_U) = 0$. □

Note that, for every $x \in U \setminus \mathcal{A}_U$, we have

$$\mathcal{Z}_U \cap \mathcal{F}|U(x) = \mathcal{F}|U(x) \setminus supp\, \mu_x^U,$$

and by definition the support of the measure μ_x^U is a closed set, which implies that the set $\mathcal{Z}_U \cap \mathcal{F}|U(x)$ is open in $\mathcal{F}|U(x)$. On the other hand, consider the null measure subset $\mathcal{P}$ given by

$$\mathcal{P} = \{x : \exists U, V \in \mathcal{U},\ x \in U \cap V,\ \mu_x^U(\cdot|U \cap V) \nsim \mu_x^V(\cdot|U \cap V)\},$$

that is, $\mathcal{P}$ is the set of points x for which there exists two local charts U and V in $\mathcal{U}$, both containing x, where the respective conditional measures at the plaque of x, μ_x^U and μ_x^V, are not equivalent on the intersection $\mathcal{F}|U(x) \cap \mathcal{F}|V(x)$. In particular this set has zero measure by Proposition 2.5. Set

$$\widetilde{M} := M \setminus \left(\bigcup_{U \in \mathcal{U}} (\mathcal{Z}_U \cup \mathcal{A}_U) \cup \mathcal{P} \right),$$

and note that $\mu(\widetilde{M}) = 1$. Now we define the full measure f-invariant subset given by

$$M_0 := \bigcap_{n \in \mathbb{Z}} f^n(\widetilde{M}). \tag{3.4}$$

To guarantee that the conditional measures are defined in all the leaves $\mathcal{F}(x)$, we define the measure μ_x in the following way: for each $x \in M_0$, we denote by μ_x the measure on $\mathcal{F}(x)$ given by the conditional measure μ_x^U, where $U \in \mathcal{U}$ is such that $x \in B(x, \mathfrak{r}) \subset U$. We then normalize μ_x^U so that it assigns weight exactly one to $B_{d_x}(x, \mathfrak{r})$. In other words, for a measurable set $F \subset \mathcal{F}(x)$, we have

$$\mu_x(F) = \mu_x^U(F | B_{d_x}(x, \mathfrak{r})). \tag{3.5}$$

This guarantees that μ_x is defined in all leaves $\mathcal{F}(x)$ and has the desired properties.

Given any $y \in B_{d_x}(x, \mathfrak{r}) \cap M_0$, the measures μ_y and μ_x are proportional to each other at the intersection $B_{d_x}(x, \mathfrak{r}) \cap B_{d_y}(y, \mathfrak{r})$. In fact, let U and V be different local charts associated with $\mathcal{F}$ such that $x \in B(x, \mathfrak{r}) \subset U$ and $y \in B(y, \mathfrak{r}) \subset V$.

Since $x, y \in M_0$, by choosing M_0 and Proposition 2.5, we have that the conditional measures μ_x^U and μ_y^V coincide up to a constant on $U \cap V$. In particular, these measures coincide up to a constant on $B_{d_x}(x, \mathfrak{r}) \cap B_{d_y}(y, \mathfrak{r})$. This means that there exists a constant $\beta > 0$ for which $\mu_y = \beta \cdot \mu_x$ restricted to $B_{d_x}(x, \mathfrak{r}) \cap B_{d_y}(y, \mathfrak{r})$.

We can see the form of this constant by evaluating both sides of equality $\mu_y = \beta \cdot \mu_x$ at the set $B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r})$, which yields

$$\beta \cdot \mu_x(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r})) = \mu_y(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r}))$$
$$\Rightarrow \beta = \frac{\mu_y(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r}))}{\mu_x(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r}))}.$$

We also call the family of measure $\{\mu_x\}$ the disintegration of μ along $\mathcal{F}$. Since the family of measures $\{\mu_x\}$ is described in terms of the conditional measures μ_x^U, for $U \in \mathcal{U}$, we obtain the following results that guarantee the continuity and measurability of the functions $y \mapsto \mu_y(B_{d_x}(y, r))$ and $r \mapsto \mu_y(B_{d_x}(y, r))$, respectively.

Corollary 3.26. *For each* $0 < r < \mathfrak{r}$, $x \in M_0$

$$y \in B_{d_x}(x, \mathfrak{r}) \cap M_0 \mapsto \mu_y(B_{d_x}(y, r)),$$

is continuous.

Proof. For a certain fixed $0 < r < \mathfrak{r}$, take any $x \in M_0$. Let $y \in B_{d_x}(x, \mathfrak{r}) \cap M_0$ and $U \in \mathcal{U}$ with $B_{d_x}(y, \mathfrak{r}) \subset U$, take $y_n \in B_{d_x}(x, \mathfrak{r}) \cap M_0$ with $y_n \longrightarrow y$ as $n \longrightarrow \infty$ and $B_{d_x}(y_n, \mathfrak{r}) \subset U$.

By definition,

$$\mu_y = \mu_y^U(\cdot | B_{d_x}(y, \mathfrak{r})), \ \mu_{y_n} = \mu_{y_n}^U(\cdot | B_{d_x}(y_n, \mathfrak{r})).$$

Therefore, for $n \in \mathbb{N}$ such that $B_{d_x}(y, r) \subset B_{d_x}(y_n, \mathfrak{r})$ and $B_{d_x}(y_n, r) \subset B_{d_x}(y, \mathfrak{r})$,

$$\mu_{y_n}(B_{d_x}(y_n, r)) = \frac{\mu_{y_n}^U(B_{d_x}(y_n, r))}{\mu_{y_n}^U(B_{d_x}(y_n, \mathfrak{r}))} = \frac{\mu_y^U(B_{d_x}(y_n, r))}{\mu_y^U(B_{d_x}(y_n, \mathfrak{r}))}. \tag{3.6}$$

By Lemma 3.23 we have $\mu_y^U(B_{d_x}(y_n, r)) \longrightarrow \mu_y^U(B_{d_x}(y, r))$ and $\mu_y^U(B_{d_x}(y_n, \mathfrak{r})) \longrightarrow \mu_y^U(B_{d_x}(y, \mathfrak{r}))$ as $n \longrightarrow \infty$. Therefore $\mu_{y_n}(B_{d_x}(y_n, r)) \longrightarrow \mu_y(B_{d_x}(y, r))$, as we wanted to show.

□

Corollary 3.27. *For each $x \in M_0$, the map*

$$r \in [0, \mathfrak{r}] \mapsto \mu_x(B_{d_x}(x, r)),$$

is continuous. Furthermore, the map

$$(x, r) \in M_0 \times [0, \mathfrak{r}] \mapsto \mu_x(B_{d_x}(x, r)), \tag{3.7}$$

is jointly measurable.

Proof. First, let us prove that for any $x \in M_0$ the function $r \in [0, \mathfrak{r}] \mapsto \mu_x(B_{d_x}(x, r))$ is a continuous. Let $0 = r < \mathfrak{r}$ and $r_n \in [0, \mathfrak{r}] \searrow r$ (if $r = \mathfrak{r}$ the argument is analogous). Hence, $\mu_x(B_{d_x}(x, r_n)) = \mu_x(B_{d_x}(x, r)) + \mu_x(B_{d_x}(x, r_n) \setminus B_{d_x}(x, r))$. As μ_x is non-atomic we have

$$\lim_{n \to \infty} \mu_x(B_{d_x}(x, r_n) \setminus B_{d_x}(x, r)) = 0.$$

Then,

$$\mu_x(B_{d_x}(x, r_n)) \to \mu_x(B_{d_x}(x, r)),$$

showing the first part of the statement.

Let us show the second statement. For each $x \in M_0$, let $x \in V_x$ a local chart with

$$y \in V_x \Rightarrow B_{d_y}(y, \mathfrak{r}) \subset U_x, \quad \text{for some } U_x \in \mathcal{U}.$$

As M is compact, we may cover M with a finite number of such local charts, say $V_1, V_2, \ldots, V_l$, and call $U_1, U_2, \ldots, U_l$ the associated local charts in $\mathcal{U}$. For any j, consider

$$y \in V_j \mapsto \mu_y(B_{d_y}(y, r)).$$

Observe that

$$\mu_y(B_{d_y}(y, r)) = \frac{\mu_y^{U_j}(B_{d_y}(y, r))}{\mu_y^{U_j}(B_{d_y}(y, \mathfrak{r}))}.$$

Therefore, by Proposition 3.24, $y \in V_j \cap M_0 \mapsto \mu_y(B_{d_y}(y,r))$ is a $\mathcal{B}_{U_j \setminus \mathcal{A}_{U_j}}$-measurable map. As j is arbitrary, $y \in M_0 \mapsto \mu_y(B_{d_x}(y,r))$ is a $\mathcal{B}_{U \cap M_0}$-measurable map.

Thus, the map given by (3.7) is jointly measurable, as it is continuous in the first coordinate and $\mathcal{B}_{M_0}$-measurable in the second.

□

For the rest of this section, we will show that the disintegration $\{\mu_x\}x \in M$ is a family of f-invariant measures on $\mathcal{F}$.

By the definition of f-metric system $\{d_x\}$, we have that this family of metrics is $\mathcal{F}$-invariant, which implies

$$f(B_{d_x}(x,r)) = B_{d_{f(x)}}(f(x),r), \text{ for } r \leqslant \mathfrak{r}. \tag{3.8}$$

We also know that the conditional measures are also invariant. Thus, it makes sense to talk about the invariance of $\{\mu_x\}$, which is the Lemma that we will show next.

Lemma 3.28. *The system of measure disintegration $\{\mu_x\}_{x \in M_0}$ is f-invariant in the following sense, for every $0 < \varepsilon \leqslant \mathfrak{r}$*

$$f_*\mu_x(B_{d_{f(x)}}(f(x),\varepsilon)) = \mu_{f(x)}(B_{d_{f(x)}}(f(x),\varepsilon)).$$

In other words, $f_*\mu_x = \mu_{f(x)}$ on $B_{f(x)}(f(x),\mathfrak{r})$.

Proof. Given $x \in M_0$, there exists two local charts $U, V \in \mathcal{U}$ such that $x \in B_{d_x}(x,\mathfrak{r}) \subset U$ and $f(x) \in B_{d_{f(x)}}(f(x),\mathfrak{r}) \subset V$. From *(ii)* in Lemma 3.13 we obtain

$$f(B_{d_x}(x,\varepsilon)) = B_{d_{f(x)}}(f(x),\varepsilon) \subset B_{d_{f(x)}}(f(x),\mathfrak{r}) \subset V.$$

From the properties of measure disintegration, we know that $f_*\mu_x^U$ are measures equivalent to $\{\mu_x^V\}$ at the intersection $f(U) \cap V$, thus we have

$$\begin{aligned} f_*\mu(B_{d_{f(x)}}(f(x),\varepsilon)) = \mu_x(f^{-1}(B_{d_{f(x)}}(f(x),\varepsilon))) &= \frac{\mu_x^U(f^{-1}(B_{d_{f(x)}}(f(x),\varepsilon)))}{\mu_x^U(f^{-1}(B_{d_{f(x)}}(f(x),\mathfrak{r})))} \\ &= \frac{f_*\mu_x^U(B_{d_{f(x)}}(f(x),\varepsilon))}{f_*\mu_x^U(B_{d_{f(x)}}(f(x),\mathfrak{r}))} = \frac{\alpha\mu_{f(x)}^V(B_{d_{f(x)}}(f(x),\varepsilon))}{\alpha\mu_{f(x)}^V(B_{d_{f(x)}}(f(x),\mathfrak{r}))} \\ &= \mu_{f(x)}(B_{d_{f(x)}}(f(x),\varepsilon)). \end{aligned}$$

□

4

Classification of conditional measures

Taking into account the properties of the metric system $\{d_x\}$, in the last section of this chapter we consider an atlas of local charts associated with the foliation $\mathcal{F}$, denoted by $\mathcal{U}$. We prove the properties of the disintegration of the measure μ in U, where $U \in \mathcal{U}$ is a local chart. Specifically, we show how the measure μ decomposes along the leaves of $\mathcal{F}$ within each chart $U \in \mathcal{U}$.

4.1 Proof Theorem A

Let $(M, \mathcal{A}, \mu)$ be a probability space and $\mathcal{F}$ be a continuous one-dimensional foliation of M. Note that if the conditional measures $\{\mu_x\}$ are atomic we have nothing to do. Then, from now on the results that will be shown are assuming that the disintegration of μ along the foliation $\mathcal{F}$ is not atomic.

Let M_0 the full measure subset given by (3.4) and consider $\{\mu_x\}_{x\in M_0}$ the system of conditional measures along $\mathcal{F}$, and let $d = \{d_x\}_{x\in M}$ be the $\mathcal{F}$-metric system induced by the $\mathcal{F}$-arc length system $\{l_x\}_{x\in M}$ as in Definition 3.14.

Definition 4.1. *We define the distortion of the conditional measures μ_x with respect to the measures λ_x induced by the $\mathcal{F}$-metric system by*

$$\Delta(x) = \begin{cases} \limsup\limits_{\varepsilon\to 0} \dfrac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} & \text{if} \quad x \in M_0, \\ 0 & \text{if} \quad x \notin M_0. \end{cases}$$

Recall that $B_{d_x}(x, \varepsilon)$ is the ball inside $\mathcal{F}(x)$ with respect to the metric d_x, centered at the point x and with radius ε, in particular $\mu_x(B_{d_x}(x, \varepsilon)) > 0$ for all $x \in supp\, \mu_x$ (where the support here is inside $\mathcal{F}(x)$). Thus, it makes sense to evaluate the quantity above.

Observe that, the Corollary 3.26 we have that the function Δ is measurable, but it is not immediately true that $\Delta(x) < \infty$ for μ-almost every x. Also note that from

Lemma 3.28 and the f-invariance of the $\mathcal{F}$-metric system we have

$$f_*\mu_x = \mu_{f(x)} \text{ and } f(B_{d_x}(x,\varepsilon)) = B_{d_{f(x)}}(f(x),\varepsilon),$$

and

$$d_{f(x)}(f(x), f(y)) = d_x(x,y),$$

Therefore, we conclude that $\Delta(x)$ is f-invariant. By ergodicity of f, it follows that $\Delta(x)$ is constant almost everywhere. Let us call that constant by Δ, this is, there exists a Borel f-invariant full measure set $\widehat{M} \subset M_0$ such that

$$\Delta(x) = \Delta \quad \text{for every } x \in \widehat{M}. \tag{4.1}$$

Note that Δ can be finite or $\Delta = \infty$. Since the proofs are different, in the following sections, we will divide the proof for these two cases. In the sequel, we will prove some technical Lemmas for the case $\Delta = \infty$ and the technical Lemmas for the case $\Delta < \infty$.

4.1.1 Technical Lemmas for the case $\Delta = \infty$

Lemma 4.2. *If $\Delta = \infty$, there exists a sequence $\varepsilon_k \to 0$, as $k \to +\infty$, and a full measure subset $R^\infty \subset \widehat{M}$ such that*

1. *R^∞ is f-invariant;*
2. *for all $x \in R^\infty$ we have*
$$\frac{\mu_x(B_{d_x}(x,\varepsilon_k))}{2\varepsilon_k} \geqslant k. \tag{4.2}$$

Proof. Let $k \in \mathbb{N}^*$ arbitrary. Since $\Delta(x) = \Delta$ for every $x \in \widehat{M}$, define

$$\varepsilon_k(x) := \begin{cases} \sup\left\{\varepsilon \leqslant \mathfrak{r} : \dfrac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} \geqslant k\right\}, & \text{if} \quad x \in \widehat{M} \\ 0, & \text{if} \quad x \in M \setminus \widehat{M}. \end{cases}$$

Claim: The function $\varepsilon_k(x)$ is measurable for all $k \in \mathbb{N}$.

Proof. Define the function $w : M_0 \times [0,\mathfrak{r}] \to [0,\infty)$ given by:

$$w(x,\varepsilon) = \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon}.$$

By Corollary 3.27, for any $x \in M_0$, the function $w(x,\cdot) : (0,\mathfrak{r}) \to [0,\infty)$ is continuous, and from Proposition 3.24, for any fixed $0 < \varepsilon < \mathfrak{r}$, the function $w(\cdot,\varepsilon) : M_0 \to [0,\infty)$ is a measurable.

Given any $k \in \mathbb{N}$ and $\beta > 0$, by the definitions of ε_k and the function w, we have that $x \in \varepsilon_k^{-1}((0,\beta))$ implies

$$\sup\left\{\varepsilon \leqslant \mathfrak{r} \,:\, \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} \geqslant k\right\} \in (0,\beta).$$

This is equivalent to

$$\begin{aligned} &\frac{\mu_x(B_{d_x}(x,r))}{2r} < k \text{ for every } \beta \leqslant r \leqslant \mathfrak{r} \\ \Longleftrightarrow\, & w(x,r) \in [0,k) \text{ for every } \beta \leqslant r \leqslant \mathfrak{r} \\ \Longleftrightarrow\, & x \in \bigcap_{\beta \leqslant r \leqslant 1} w(\cdot,r)^{-1}([0,k)) \end{aligned}$$

Then, by the continuity of $w(x,\cdot)$ and the density of $\mathbb{Q}$ at $(\beta,\mathfrak{r})$, it follows that

$$\bigcap_{\beta \leqslant r \leqslant 1} w(\cdot,r)^{-1}([0,k)) = \bigcap_{\beta \leqslant r \leqslant 1, r\in\mathbb{Q}} w(\cdot,r)^{-1}([0,k)).$$

So we have just shown that

$$\varepsilon_k^{-1}((0,\beta)) = \bigcap_{\beta \leqslant r \leqslant 1, r\in\mathbb{Q}} w(\cdot,r)^{-1}([0,k)).$$

Therefore, $\varepsilon_k^{-1}((0,\beta))$ is measurable, as it is a countable intersection of measurable subsets of M_0, and by definition $\varepsilon_k^{-1}(\{0\}) = M \setminus M_0$ also a measurable subset. Consequently ε_k is a measurable function for every k. □

Note that $\varepsilon_k(x)$ is f-invariant. In fact, by Lemma 3.28 and the f-invariance of $\{d_x\}$, we obtain

$$\begin{aligned} &\mu_{f(x)}(B_{d_{f(x)}}(f(x),\varepsilon)) = f_*\mu_x(B_{d_{f(x)}}(f(x),\varepsilon)) \\ =&\mu_x(f^{-1}(B_{d_{f(x)}}(f(x),\varepsilon))) \\ =&\mu_x(B_{d_x}(x,\varepsilon)). \end{aligned}$$

this implies

$$\begin{aligned} \varepsilon_k(f(x)) =& \sup\{\varepsilon \leqslant 1 \,:\, \frac{\mu_{f(x)}(B_{d_{f(x)}}(f(x),\varepsilon))}{2\varepsilon} \geqslant k\} \\ =& \sup\{\varepsilon \leqslant 1 \,:\, \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} \geqslant k\} \\ =&\varepsilon_k(x). \end{aligned}$$

Thus, for any $k \in \mathbb{N}*$, the function ε_k is f-invariant and by ergodicity, ε_k is constant almost everywhere. Let R_k^∞ be a full measure set such that $\varepsilon_k(x)$ is constant equal to ε_k for every $x \in R_k^\infty$.

Now, let us see that the sequence $\{\varepsilon_k\}$ satisfies $\varepsilon_k \to 0$ as $k \to \infty$. Since $[0,\mathfrak{r}]$ is a compact set of $\mathbb{R}$, we can take the sequence $\{\varepsilon_k\} \subset [0,\mathfrak{r}]$ to be convergent.

Suppose that there exists $\varepsilon > 0$ such that $\varepsilon_k \to \varepsilon$. Then by continuity of $x \to \mu_x(B_{d_x}(x,r))$ (see Corollary 3.27), we have that $\mu_x(B_{d_x}(x,\varepsilon_k)) \to \mu_x(B_{d_x}(x,\varepsilon))$ as $k \to \infty$, then

$$\frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} = \lim_{k\to\infty} \frac{\mu_x(B_{d_x}(x,\varepsilon_k))}{2\varepsilon_k} \geqslant \lim_{k\to\infty} k,$$

which is a contradiction. Therefore, we conclude that $\varepsilon_k \to 0$ as $k \to \infty$.

Take $\widetilde{R}^\infty := \bigcap_{k=1}^{+\infty} R_k^\infty$. Since each R_k^∞ has full measure, $\widetilde{R}^\infty$ has full measure and clearly satisfies what we want for the sequence $\{\varepsilon_k\}_k$. Finally, take $R^\infty = \bigcap_{i\in\mathbb{Z}} f^i(\widetilde{R}^\infty)$. The set R^∞ is f-invariant, has full measure and satisfies (1) and (2). □

We now set, for each $U \in \mathcal{U}$, $x \in U \setminus (\mathcal{Z}_U \cup \mathcal{A}_U)$,

$$\Pi_{x,U}^\infty := \left\{ y \in \mathcal{F}|U(x) \setminus \mathcal{Z}_U : \frac{1}{2\varepsilon_k} \cdot \frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{\mu_y^U(B_{d_x}(y,\mathfrak{r}))} \geqslant k, \forall k \text{ with } B_{d_x}(y,\varepsilon_k) \subset U \right\},$$

and

$$\Pi_U^\infty := \bigcup_{x\in U\setminus(\mathcal{Z}_U\cup\mathcal{A}_U)} \Pi_{x,U}^\infty.$$

Observe that if $x \in R^\infty$ then $x \in \Pi_{x,U}^\infty$, therefore $R^\infty \cap U \subset \Pi_U^\infty$. In particular, $U \setminus \Pi_U^\infty \subset U \setminus R^\infty$. Since $\mu(R^\infty) = 1$, Π_U^∞ is measurable.

Lemma 4.3. *Let $U \in \mathcal{U}$ be a local chart. For every $x \in R^\infty \cap U$, consider $\delta = \delta(x) > 0$ for which*

$$B_{d_x}[x, 2\cdot\delta + \mathfrak{r}] \subset U.$$

The set $\Pi_{x,U}^\infty \cap B_{d_x}[x,\delta]$ is a closed subset on the plaque $\mathcal{F}|U(x)$.

Proof. Let $y_n \to y$, $y_n \in \Pi_{x,U}^\infty \cap B_{d_x}[x,\delta]$, $y \in \mathcal{F}|U(x)$. In particular, $B_{d_x}(y_n,\mathfrak{r}) \subset B_{d_x}(x,\delta+\mathfrak{r}) \subset U$, and taking the limit over n, we also have $B_{d_x}(y,\mathfrak{r}) \subset B_{d_x}(x,\delta+\mathfrak{r}) \subset U$. Furthermore, it is clear that $y \in B_{d_x}[x,\delta]$ since this is a closed set. By Lemma 3.23, for each $k \in \mathbb{N}$ the map

$$y \in B_{d_x}[x,\delta] \subset F_{\varepsilon_k}(x) \mapsto \mu_y^U(B_{d_x}(y,\varepsilon_k)),$$

is continuous and the same holds for

$$y \in B_{d_x}[x,\delta] \subset F_{\mathfrak{r}}(x) \mapsto \mu_y^U(B_{d_x}(y,\mathfrak{r})).$$

Thus,

$$\lim_{n\to\infty} \frac{\mu_{y_n}^U(B_{d_x}(y_n,\varepsilon_k))}{\mu_{y_n}^U(B_{d_x}(y_n,\mathfrak{r}))} = \frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{\mu_y^U(B_{d_x}(y,\mathfrak{r}))}, \quad k \geqslant 1.$$

Which implies that for all $k \geqslant 1$ we have

$$\frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{2\varepsilon_k \cdot \mu_y^U(B_{d_x}(y,\mathfrak{r}))} = \lim_{n\to\infty} \frac{\mu_{y_n}^U(B_{d_x}(y_n,\varepsilon_k))}{2\varepsilon_k \cdot \mu_{y_n}^U(B_{d_x}(y_n,\mathfrak{r}))} \geqslant k,$$

that is, $y \in \Pi^{\infty}_{x,U}$, as we wanted.

$\square$

For $A \subset U$ we define the $\mathcal{F}|U$-saturate of A on U given by

$$\mathcal{F}|U(A) = \bigcup_{x \in A} \mathcal{F}|U(x)$$

Now, we consider the following set

$$D^{\infty}_{U} := \mathcal{F}|U(\Pi^{\infty}_{U}) \setminus (\mathcal{F}|U)(\mathcal{Z}_{\mathcal{U}}).$$

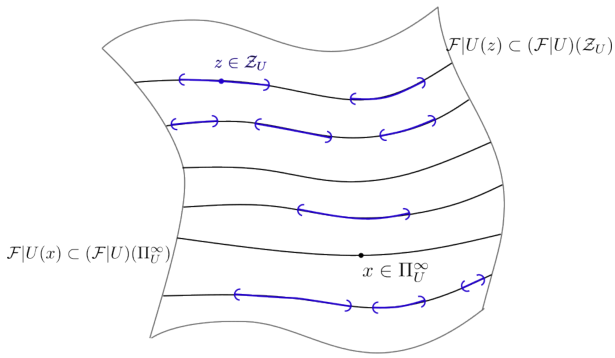

Figure 4 – Set D^{∞}_{U}

In other words, D^{∞}_{U} is the union of the plaques $\mathcal{F}|U(x)$ for $x \in \Pi^{\infty}_{U}$ that do not have intervals of μ^{U}_{x}-null measure.

Lemma 4.4. *The set D^{∞}_{U} defined above is a measurable subset.*

Proof. Consider the natural projection associated to the foliation $\mathcal{F}$ given by

$$\begin{aligned} \pi : U &\to U/\mathcal{F} \\ x &\mapsto \pi(x) := \mathcal{F}|U(x). \end{aligned}$$

As U is an open subset of a manifold, in particular, it is a Polish space thus, $U/\mathcal{F}$ with the quotient topology is also a Polish space. By Lemma 3.3, we know that $\mathcal{Z}_U = \chi_U \cap (U \setminus \mathcal{A}_U)$ for some Borel subset $\chi_U \subset U$ and $U \setminus \mathcal{A}_U$ is $\mathcal{F}|U$-saturated, then

$$\pi(\mathcal{Z}_U) = \pi(\chi_U) \cap \pi(U \setminus \mathcal{A}_U),$$

where $\pi(\chi_U)$ is a Souslin set[1] by [8, Corollary 1.10.9]. Therefore

$$\mathcal{F}|(\mathcal{Z}_U) = \pi^{-1}(\pi(\chi_U) \cap \pi(U \setminus \mathcal{A}_U)) = \pi^{-1}(\pi(\chi_U)) \cap (U \setminus \mathcal{A}_U),$$

[1] A subset of a Polish space Y is called a Souslin set, or an analytical set, if it is the image of a Polish space X by a continuous map from X to Y.

is a measurable set.

Since $R^\infty \cap U \subset \mathcal{F}|U(\Pi_U^\infty)$ and $\mu(R^\infty) = 1$ we have that $\mathcal{F}|U(\Pi_U^\infty)$ is a measurable subset of U, this implies that $D_U^\infty = \mathcal{F}|U(\Pi_U^\infty) \setminus \mathcal{F}|U(\mathcal{Z}_U)$ is a measurable set, as we wanted. □

Now we define the set

$$D^\infty := \bigcup_{n\in\mathbb{Z}, U\in\mathcal{U}} f^n\left(\bigcup D_U^\infty\right).$$

Since D_U^∞ is measurable, by ergodicity of f and the f-invariance of D^∞, must satisfy either $\mu(D^\infty) = 0$ or $\mu(D^\infty) = 1$.

4.1.2 Technical Lemmas for the case $\Delta < \infty$

Lemma 4.5. *If $\Delta < \infty$, there exists a sequence $\varepsilon_k \to 0$, as $k \to +\infty$, and a full measure subset $R \subset \widehat{M}$ such that*

1. *R is f-invariant;*
2. *for every $x \in R$, then*
$$\left|\frac{\mu_x(B_{d_x}(x,\varepsilon_k))}{2\varepsilon_k} - \Delta\right| \leqslant \frac{1}{k}. \tag{4.3}$$

Proof. The proof is very similar to the proof of Lemma 4.2. Let $k \in \mathbb{N}^*$ arbitrary. Since $\Delta(x) = \Delta$ for every $x \in \widehat{M}$ define

$$\varepsilon_k(x) := \begin{cases} \sup\left\{\varepsilon \leqslant \mathfrak{r} : \left|\frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} - \Delta\right| \leqslant \frac{1}{k}\right\} & \text{if } x \in \widehat{M} \\ 0 & \text{if } x \in M \setminus \widehat{M} \end{cases}.$$

Observe that for $x \in \widehat{M}$ such $\varepsilon_k(x)$ exists, in fact, since

$$\Delta = \limsup_{\varepsilon\to 0} \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} \text{ for } x \in \widehat{M},$$

we can take a sequence $\varepsilon_l \to 0$ such that the given ratio approaches Δ.

Claim: The function $\varepsilon_k(x)$ is measurable for all $k \in \mathbb{N}$.

Proof. Define the function $w : M_0 \times [0, \mathfrak{r}] \to [0, \infty)$ given by

$$w(x, \varepsilon) = \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon}.$$

As observed in the proof of Lemma 4.2, for fixed $x \in M_0$, the function $r \mapsto w(x,r)$ is continuous and for any $0 < r \leqslant \mathfrak{r}$, the function $x \mapsto w(x,r)$ is measurable. Given any $k \in \mathbb{N}$, $k > 0$ and $\beta > 0$, let us show that $\varepsilon_k^{-1}((0,\beta))$ is a measurable set. Note that:

$$\begin{aligned} x \in \varepsilon_k^{-1}((0,\beta)) &\Longleftrightarrow \varepsilon_k(x) \in (0,\beta) \\ &\Longleftrightarrow \sup\left\{\varepsilon \leqslant \mathfrak{r} : \left|\frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} - \Delta\right| \leqslant \frac{1}{k}\right\} \in (0,\beta) \\ &\Longleftrightarrow \left|\frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} - \Delta\right| > \frac{1}{k}, \quad \text{for every} \quad \varepsilon > \beta \\ &\Longleftrightarrow \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} - \Delta > \frac{1}{k} \quad \text{or} \quad \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} - \Delta < -\frac{1}{k}, \text{ for } \varepsilon > \beta \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} \varepsilon_k^{-1}((0,\beta)) =& \{x : \varepsilon_k(x) \in (0,\beta)\} \\ =& \bigcap_{\beta \leqslant r \leqslant 1} w(\cdot,r)^{-1}\left(\left[\Delta + \frac{1}{k}, \infty\right)\right) \cup w(\cdot,r)^{-1}\left(\left[0, \Delta - \frac{1}{k}\right)\right). \end{aligned}$$

Now, the continuity of $w(x,\cdot)$ and the density of $\mathbb{Q}$ on $[\beta,\mathfrak{r}]$, implies that

$$\varepsilon_k^{-1}((0,\beta)) = \bigcap_{\beta \leqslant r \leqslant \mathfrak{r}, r \in \mathbb{Q}} w(\cdot,r)^{-1}\left(\left[\Delta + \frac{1}{k}, \infty\right)\right) \cup w(\cdot,r)^{-1}\left(\left[0, \Delta - \frac{1}{k}\right)\right).$$

Therefore, $\varepsilon_k^{-1}((0,\beta))$ is measurable, as it is a countable intersection of measurable subsets of M_0 and by definition $\varepsilon_k^{-1}(\{0\}) = M \setminus M_0$ also a measurable subset, and consequently ε_k is a measurable function for every k. □

With a similar argument used in Lemma 4.5, we show that $\varepsilon_k(x)$ is f-invariant, by ergodicity we may take the full measure set R_k where $\varepsilon_k(x)$ is constant equal to ε_k. The sequence $\varepsilon_k \to 0$ as $k \to \infty$.

Consider $\tilde{R} := \bigcap_{k=1}^{+\infty} R_k$. Since each R_k has full measure, $\tilde{R}$ has full measure and clearly satisfies what we want for the sequence $\{\varepsilon_k\}_k$. The set $R = \bigcap_{i \in \mathbb{Z}} f^i(\tilde{R})$ is f-invariant, has full measure and satisfies (1) and (2), as we wanted. □

Similar to the definitions made in Section 4.1.1, we set

$$\Pi_U := \bigcup_{x \in U \setminus (\mathcal{Z}_U \cup \mathcal{A}_U)} \Pi_{x,U},$$

where

$$\Pi_{x,U} := \left\{y \in \mathcal{F}|U(x) \setminus \mathcal{Z}_U : \left|\frac{1}{2\varepsilon_k} \cdot \frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{\mu_y^U(B_{d_x}(y,\mathfrak{r}))} - \Delta\right| \leqslant \frac{1}{k}, \forall k \text{ with } B_{d_x}(y,\mathfrak{r}) \subset U\right\}.$$

Lemma 4.6. *For every* $x \in R \cap U$*, consider* $\delta(x) > 0$ *for which*

$$B_{d_x}[x, 2 \cdot \delta + \mathfrak{r}] \subset U.$$

The set $\Pi_{x,U} \cap B_{d_x}[x, \delta]$ *is a closed subset on the plaque* $\mathcal{F}|U(x)$*.*

Proof. Identical to the proof of Lemma 4.3. □

Similar to the definitions made in Section 4.1.1, we consider the set

$$D_U := \mathcal{F}|U(\Pi_U) \setminus (\mathcal{F}|U)(\mathcal{Z}_{\mathcal{U}}).$$

Lemma 4.7. *The set* D_U *definided above is a measurable subset.*

Proof. The proof uses the same arguments as those in Lemma 4.4. □

Now, we define the f-invariant subset

$$D := \bigcup_{n \in \mathbb{Z}, U \in \mathcal{U}} f^n \left(\bigcup D_U \right).$$

As D_U is measurable for all $U \in \mathcal{U}$, D is measurable, and again by ergodicity, we have $\mu(D) = 0$ or $\mu(D) = 1$.

After proving the auxiliary lemmas for Δ^∞ (resp. Δ) and obtaining the sets D (resp. D^∞) we divide the next part of the proof into four cases.

- *Case 1:* $\Delta < \infty$ *and* $\mu(D) = 0$: In this case, we will prove that almost every leaf has the conditional measure μ_x supported on a Cantor set.
- *Case 2:* $\Delta = \infty$ *and* $\mu(D^\infty) = 0$: As in the Case 1, we will prove that almost every leaf has the conditional measure μ_x supported on a Cantor set.
- *Case 3:* $\Delta = \infty$ *and* $\mu(D^\infty) = 1$: we will show that this case does not occur.
- *Case 4:* $\Delta < \infty$ *and* $\mu(D) = 1$: In this case, we will prove that for almost every leaf $\mathcal{F}(x)$, the conditional measure μ_x is equivalent to the measure λ_x.

4.1.3 Case 1: $\Delta < \infty$ and $\mu(D) = 0$.

As was said above, in this case, we will show that for every local chart $U \in \mathcal{U}$ and almost every $x \in U$, the support of the conditional measures μ_x^U is a Cantor set on the plaque

$\mathcal{F}|U(x)$. Note that by the definition of D, $\mu(D) = 0$ implies that $\mu(\mathcal{F}|U(\mathcal{Z}_U)) = 1$, which means that there are many leaves of $\mathcal{F}$ containing intervals of null measure.

As fixed in Section 2.2, consider $\{\Omega_x\}_{x\in M}$ the disintegration of μ along $\mathcal{F}$. By definition for this family of mesures, we can consider $\mathcal{G}$ a full measure $\mathcal{F}$-saturated set of points where any representative ω_x of the class Ω_x is allways invariant under f, this is

$$f^j_*\Omega_x = \Omega_{f^j(x)}, \quad \forall j \in \mathbb{Z}.$$

Let $U \in \mathcal{U}$ be a local chart, we also know from the definition of $\{\Omega_x\}_{x\in M}$ and $\{\mu^U_x\}$ that measures μ^U_x and ω_x coincide almost every point $x \in M$. Since we are assuming that the disintegration is not atomic, we can define the full-measure set in U given by

$$\mathcal{G}^U := \{x \in U : \mu^U_x = \omega_x(\cdot|\mathcal{F}|U(x))\} \cap \{x : \mu^U_x \text{ is non-atomic}\}.$$

Now, we consider $\hat{n} \in \mathbb{N}$ such that $1/\hat{n} < \mathfrak{r}$. Note that $\hat{n}$ does not depend on the local chart U. For $n \in \mathbb{N}$, $n \geqslant \hat{n}$, consider the following set

$$\Phi^U_{1/n}(\mathcal{Z}_U) := \{x \in U : d_x(x, \mathcal{Z}_U) < 1/n\}.$$

We assert that $\Phi^U_{1/n}(\mathcal{Z}_U)$ is measurable. In fact, by Lemma 3.25, the set $\mathcal{Z}_U \subset U$ is $\mathcal{B}_{U\setminus\mathcal{A}_U}$-measurable set, which implies that there exists a Borelian subset χ_U such that $\mathcal{Z}_U = \chi_U \cap (U \setminus \mathcal{A}_U)$. Since $\mathcal{A}_U$ is $\mathcal{F}$-saturated in U, we have

$$\Phi^U_{1/n}(\mathcal{Z}_U) = \{x \in M : d_x(x, \chi_U) < 1/n\} \cap (U \setminus \mathcal{A}_U).$$

By Lemma 3.21, as χ_U is a Borel subset of U we have that $\{x \in M : d_x(x, \chi_U) < 1/n\}$ is measurable, therefore $\Phi^U_{1/n}(\mathcal{Z}_U)$ is measurable, as we wanted to show.

Now, for $n \in \mathbb{N}$ such that $n \geqslant \hat{n}$ we consider the set

$$\mathcal{E}_n = \bigcup_{j,U} \mathcal{G}^U \cap f^j(\Phi^U_{1/n}(\mathcal{Z}_U) \cap \mathcal{G}^U). \tag{4.4}$$

Since $\Phi^U_{1/n}(\mathcal{Z}_U)$ is measurable, we have that $\mathcal{E}_n$ is an f-invariant measurable subset of M, thus by the ergodicity of f, the set $\mathcal{E}_n$ either has full or null measure.

Lemma 4.8. $\mu(\mathcal{E}_n) = 1$ *for every* $n \in \mathbb{N}$ *and* $n \geqslant \hat{n}$.

Proof. If we assume that there exists $N_0 \in \mathbb{N}$ with $N_0 \geqslant \hat{n}$ such that $\mu(\mathcal{E}_{N_0}) = 0$. In fact, $\mu(\mathcal{E}_{N_0}) = 0$ implies that $\mu(\mathcal{E}^U_{N_0}) = 0$ for any local chart $U \in \mathcal{U}$, and moreover, since $\mathcal{E}^U_N \subset \mathcal{E}^U_{N_0}$, we have $\mu(\mathcal{E}^U_N) = 0$ for any $N \geqslant N_0$.

By definition of $\mathcal{E}^U_{N_0}$, we know that

$$\mathcal{G}^U \cap \mathcal{G} \cap \varphi_{1/N}(\mathcal{Z}_U) \subset \mathcal{E}^U_{N_0} \ \text{ for } \ N \geqslant N_0,$$

as $\mu(\mathcal{G}^U \cap \mathcal{G}) = \mu(U)$, it should be noted that

$$\mu(\Phi^U_{1/N}(\mathcal{Z}_U)) = \mu(\mathcal{G}^U \cap \mathcal{G} \cap \Phi_{1/N}(\mathcal{Z}_U)) \leqslant \mu(\mathcal{E}^U_n) = 0, \quad \forall N \geqslant N_0.$$

From definition of system of conditional measures, we notice that

$$\mu(\Phi^U_{1/N}(\mathcal{Z}_U)) = \int_{y \in \Phi^U_{1/N}(\mathcal{Z}_U)} \mu^U_y(\mathcal{F}|U(y) \cap \Phi^U_{1/N}(\mathcal{Z}_U))\, d\mu(y) = 0,$$

and

$$\mathcal{F}|U(y) \cap \Phi^U_{1/N}(\mathcal{Z}_U) = \Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(y)).$$

In particular, for almost every $x \in U$ we have

$$\mu^U_x(\Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x))) = 0. \tag{4.5}$$

As $\Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x)))$ is an open subset of $\mathcal{F}|U(x)$, for every $y \in \Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x)))$ there exists $r > 0$ such that $B_{d_x}(y, r) \subset \Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x)))$ then by (4.5) we have

$$\mu^U_x(B_{d_x}(y, r)) = 0.$$

Terefore, for almost every $x \in U$ we find that

$$\Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x)) \subset \mathcal{Z}_U \cap \mathcal{F}|U(x).$$

But clearly the other continence holds, thus $\mathcal{Z}_U \cap \mathcal{F}|U(x) = \Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x)))$ and this implies $\mathcal{Z}_U \cap \mathcal{F}|U(x)) = \mathcal{F}|U(x)$. As this happens for almost every $x \in U$ we fall in contradiction with the fact that $\mu(\mathcal{Z}_U \cap U) = 0$. Therefore $\mu(\mathcal{E}_n) = 1$ for every $n \in \mathbb{N}$ and $n \geqslant \hat{n}$. □

Since $\mu(\mathcal{E}_n) = 1$ for every $n \in \mathbb{N}$ and $n \geqslant \hat{n}$. We will prove that for every set $U \in \mathcal{U}$, the measure μ^U_x is supported on a Cantor set.

First, we prove that the intersection of set $\mathcal{Z}_U \cap \mathcal{F}|U(x)$ is dense in $\mathcal{F}|U(x)$ for almost all $x \in U$. In fact, for $U \in \mathcal{U}$ we consider the subset $\mathcal{E}^U_n \subset U$, given by

$$\mathcal{E}^U_n = \mathcal{G}^U \cap \left(\bigcup_{j \in \mathbb{Z}} \mathcal{G}^U \cap f^j(\Phi^U_{1/n}(\mathcal{Z}_U) \cap \mathcal{G}^U) \right).$$

Since $\mu(\mathcal{E}_n) = 1$ the set $\mathcal{E}^U_n$ has full measure in U for every $n \geqslant \hat{n}$.

For $z \in \mathcal{E}^U = \bigcap_{n \geqslant \hat{n}} \mathcal{E}^U_n$, as $n \geqslant \hat{n}$ and $1/\hat{n} \leqslant \mathfrak{r}$, we have that $z \in \mathcal{F}|U(x)$ for some $x \in U$ and $B_{d_x}(z, 1/n) \subset U$. For $n \geqslant \hat{n}$, let $j \in \mathbb{Z}$ with

$$f^{-j}(z) \in \Phi^U_{1/n}(\mathcal{Z}_U) \cap \mathcal{G}^U,$$

and since $f^{-j}(z) \in \Phi^U_{1/n}(\mathcal{Z}_U)$ there exists a point $p \in \mathcal{F}(f^{-j}(z)) \cap \mathcal{Z}_U$ such that

$$d_{f^{-j}(x)}(p, f^{-j}(z)) < 1/n.$$

Then, by the f-invariance of the $\mathcal{F}$-metric system $\{d_x\}$, we have $d_x(f^j(p), z) < 1/n$, which implies $f^j(p) \in U$. If $\mu_p^U(B_{d_x}(p,\delta)) = 0$, for some $\delta > 0$ small, since $\mu_p^U \sim \omega_{f^{-j}(z)}$ (because $f^{-j}(z) \in \mathcal{G}^U$) and $z \in \mathcal{G}$, it follows that $\omega_z(f^{-j}(I_p)) = \omega_{f^j(z)}(I_p) = 0$ and $z \in \mathcal{G}^u$. Thus $\mu_z^U(f^{-j}(I_p)) = 0$, therefore, $f^j(p) \in \mathcal{Z}_U$ and $z \in \Phi_{1/n}(\mathcal{Z}_U)$. Consequently $z \in \overline{\mathcal{Z}_U}$, for almost every $z \in \mathcal{E}^U$.

If $\overline{\mathcal{Z}_U} \neq \mathcal{F}|U(z)$, then we would be able to find an open arc in the complement of $\overline{\mathcal{Z}_U}$ (in particular in the complement of $\mathcal{Z}_U$), which must have positive μ_x^U measure. Then, for almost every $z \in \mathcal{E}^U$, this contradicts the fact that $\mu_z^U(\mathcal{E}^U) = 1$. That is, the set $\mathcal{Z}_U \cap \mathcal{F}|U(z)$ is dense in $\mathcal{F}|U(z)$, for almost every $z \in \mathcal{E}^U$.

Lemma 4.9. *$\mathcal{C}_x := \mathcal{F}|U(x) \setminus \mathcal{Z}_U$ is a Cantor set in $\mathcal{F}|U(x)$, for every $x \in \mathcal{E}^U$.*

Proof. To prove that $\mathcal{C}_x$ is a Cantor set, we will show that it is nowhere dense and perfect. Note that, by the definition of $\mathcal{Z}_U$, we have that $\mathcal{C}_x$ is closed subset in $\mathcal{F}|U(x)$ (see Lemma 3.25), and given that $\mathcal{C}_x := \mathcal{F}|U(x) \setminus \mathcal{Z}_U$ and $\mathcal{Z}_U \cap \mathcal{F}|U(x)$ is a dense set in $\mathcal{F}|U(x)$, it follows that $\mathcal{C}_x$ is nowhere dense.

Given $x \in \mathcal{E}^U$, let us see that $\mathcal{C}_x$ has no isolated points. Suppose by contradiction that there is an isolated point in $\mathcal{C}_x$, say $y \in \mathcal{C}_x$. Then, there exists $r > 0$ such that $B(y,r) \subset U$ and $B_{d_x}(y,r) \cap \mathcal{C}_x = \{y\}$. As $y \in \mathcal{C}_x$ it follows that $0 < \mu_x^U(B_{d_x}(y,r))$. Then, by the basic properties of probability measures, we have that

$$0 < \mu_x^U(B_{d_x}(y,r)) = \mu_x^U(B_{d_x}(y,r) \setminus \mathcal{C}_x) + \mu_x^U(\{y\}) = \mu_x^U(\{y\}),$$

therefore $\mu_x^U(\{y\}) > 0$, which contradicts the assumption that $x \in \mathcal{E}^U \subset U \setminus \mathcal{A}_U$. This means that μ_x^U is not atomic, which implies that $\{y\}$ cannot have positive measure with respect to μ_x^U.

Thus, we have shown that $\mathcal{C}_x$ is a set that is never dense and perfect. Therefore, we have that $\mathcal{C}_x$ is a Cantor set, as we wanted to show. □

Thus, for almost every $x \in M$ and any local chart U, the conditional measures μ_x^U is supported in the Cantor set $\mathcal{C}_x$. We remark that for the Lemma 4.9, we did not use the fact that $\Delta = \infty$, so the same argument will work when $\Delta < \infty$.

4.1.4 Case 2: $\Delta = \infty$ and $\mu(D^\infty) = 0$.

In this scenario, we follow a similar approach to the previous case. Specifically, we show, as in Case 1, that for almost every $x \in U$, the measure μ_x is supported in a Cantor set.

Note that $\mu(D^\infty) = 0$ implies that for every local chart $U \in \mathcal{U}$, $\mu(D_U^\infty) = 0$. Since $D_U^\infty = \mathcal{F}|U(\Pi_U^\infty) \setminus (\mathcal{F}|U)(\mathcal{Z}_{\mathcal{U}})$ and $\mu(\mathcal{F}|U(\Pi_U^\infty)) = \mu(U)$, we have that $\mu(\mathcal{F}|U(\mathcal{Z}_{\mathcal{U}})) = 0$.

Consequently, almost every plaque $\mathcal{F}|U(x)$ does not contain intervals of null measure with respect to μ_x^U.

Again, as in the Case 1, for every $n \geqslant \hat{n}$, we define the set $\mathcal{E}_n$ given in (4.4). If $\mu(\mathcal{E}_n) = 1$ for every $n \in \mathbb{N}$, then there exists a subset of U, namely $\mathcal{E}^U$ with $\mu(\mathcal{E}^U) = \mu(U)$, such that $z \in \mathcal{E}^U$ implies $\mathcal{Z}_U \cap \mathcal{F}|U(z)$ is dense is $\mathcal{F}|U(z)$. Hence, as showed by the Lemma 4.9, it follows that the support of μ_x^U is a Cantor subset of the plaque $\mathcal{F}|U(x)$ for almost every $x \in U$.

Otherwise, if $\mu(\mathcal{E}_{N_0}) = 0$ for some $N_0 \in \mathbb{N}$, then as in Case 1, we conclude that $\mathcal{Z}_U \cap \mathcal{F}|U(x)) = \mathcal{F}|U(x)$, contradicting the fact that $\mu(\mathcal{Z}_U \cap U) = 0$. Thus, this case does not occur.

4.1.5 Case 3: $\Delta = \infty$ and $\mu(D^\infty) = 1$.

Let us prove that this case cannot occur. Suppose that $\mu(D^\infty) = 1$. Take $U \in \mathcal{U}$ such that $\mu(D^\infty \cap U) > 0$ and for $\mathfrak{r} > 0$ given in Proposition 3.19 the set $\{x \in U : B_{d_x}(x, \mathfrak{r}) \subset U\} \neq \varnothing$. Since $\mu(\mathcal{F}|U(\Pi_U^\infty)) = \mu(U)$ and by definition of the system of conditional measures, for almost every point $y \in U$, we have $\mu_y^U(\Pi_U^\infty \cap \mathcal{F}|U(y)) = 1$. In particular, for almost every $y \in D^\infty \cap U$, we have that

$$\mu_y^U(\Pi_U^\infty \cap \mathcal{F}|U(y)) = 1. \tag{4.6}$$

Since $\{x \in U : B_{d_x}(x, \mathfrak{r}) \subset U\} \neq \varnothing$ we can to consider $y \in D^\infty$ and $x \in \mathcal{F}|U(y)$ such that $B_{d_x}(x, \mathfrak{r}) \subset U$. Also, from Lemma 4.3, we have that $\Pi_{x,U}^\infty \cap B_{d_x}[x, \delta]$ is a closed subset in $\mathcal{F}|U(x)$ for some $\delta > 0$ small ($\delta < \mathfrak{r}$). Next, using the properties of the sets D_U^∞ and $\Pi_{x,U}^\infty$ we are going to prove that

$$B_{d_x}[x, \delta] \subset \Pi_{x,U}^\infty \quad \text{for} \quad x \in \mathcal{F}|U(y) \cap \Pi_U^\infty.$$

Suppose, by contradiction that there exists $z \in B_{d_x}[x, \delta] \smallsetminus \Pi_{x,U}^\infty \cap B_{d_x}[x, \delta]$. Since $\Pi_{x,U}^\infty \cap B_{d_x}[x, \delta]$ is closed in $\mathcal{F}|U(y)$, there is $\delta_2 > 0$ such that

$$B_{d_x}(z, \delta_2) \subset B_{d_x}[x, \delta] \smallsetminus \Pi_{x,U}^\infty \cap B_{d_x}[x, \delta].$$

Then, by (4.6), we have $\mu_x^U(B_{d_x}(z, \delta_2)) = 0$. This means that $z \in \mathcal{Z}_U$, note that $\mathcal{F}|U(z) = \mathcal{F}|U(y)$, then $\mathcal{F}|U(y) \subset \mathcal{F}|U(\mathcal{Z}_U)$. Consequently, $\mathcal{F}|U(\mathcal{Z}_U) \cap D_U^\infty \neq \varnothing$, which contradicts the definition of D_U^∞. Therefore,

$$\Pi_{x,U}^\infty \cap B_{d_x}[x, \delta] = B_{d_x}[x, \delta],$$

as we wanted to show.

On the other hand, consider $0 < r_0 < \delta$ small enough such that $B_{d_x}(x, \mathfrak{r}+2\cdot r_0) \subset U$. Let $k \in \mathbb{N}$ be large enough such that $\varepsilon_k < r_0$. By Lemma 3.13, we have that $\{d_x\}$ is an additive metric system. This implies that we can take $\lfloor r_0/\varepsilon_k \rfloor$ disjoint balls of radius ε_k inside $B_{d_x}(x, r_0)$, say with center $a_1, a_2, \ldots, a_{\lfloor r_0/\varepsilon_k \rfloor} \in \Pi^{\infty}_{x,U}$. Then, we have

$$\sum_{i=1}^{\lfloor r_0/\varepsilon_k \rfloor} \mu_x^U(B_{d_x}(a_i, \varepsilon_k)) \leqslant \mu_x^U(B_{d_x}(x, r_0))$$

$$\Rightarrow \sum_{i=1}^{\lfloor r_0/\varepsilon_k \rfloor} \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \leqslant \frac{\mu_x^U(B_{d_x}(x, r_0))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))}.$$

Thus, we can write

$$\sum_{i=1}^{\lfloor r_0/\varepsilon_k \rfloor} \frac{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \cdot \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))} \leqslant \frac{\mu_x^U(B_{d_x}(x, r_0))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))}. \tag{4.7}$$

By Lemma 3.23, we have that the function $w \in B_{d_x}[x, r_0] \subset F_{\mathfrak{r}} \mapsto \mu_x^U(B_{d_x}(w, \mathfrak{r}))$ is continuous. Therefore, this function is bounded from below, which means that there exists an $\eta > 0$ such that

$$\frac{\mu_x^U(B_{d_x}(w, \mathfrak{r}))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \geqslant \eta, \quad \text{for every} \quad w \in B_{d_x}[x, r_0].$$

On the other hand, for each $i = 1, ..., \lfloor r_0/\varepsilon_k \rfloor$, we have that $a_i \in B_{d_x}[x, \delta]$. As $B_{d_x}[x, \delta] \subset \Pi^{\infty}_{x,U}$, by the definition of the set $\Pi^{\infty}_{x,U}$, it follows that

$$\frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))} \geqslant 2\varepsilon_k \cdot k \quad \text{for all} \quad i = 1, \ldots, \lfloor r_0/\varepsilon_k \rfloor.$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{\lfloor r_0/\varepsilon_k \rfloor} \frac{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \cdot \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))} \\
&\geqslant \eta \cdot \sum_{i=1}^{\lfloor r_0/\varepsilon_k \rfloor} \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))} \\
&\geqslant \eta \cdot (2\varepsilon_k \cdot k) \cdot \left\lfloor \frac{r_0}{\varepsilon_k} \right\rfloor
\end{aligned}$$

So, from (4.7) we have

$$\eta \cdot (2\varepsilon_k \cdot k) \cdot \left\lfloor \frac{r_0}{\varepsilon_k} \right\rfloor \leqslant \frac{\mu_x^U(B_{d_x}(x, r_0))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))}.$$

Taking $k \to \infty$, the left side goes to infinity from where we conclude that

$$\mu_x((B_{d_x}(x, r_0)) = \frac{\mu_x^U(B_{d_x}(x, r_0))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} = \infty,$$

which is a contradiction. Thus, this case does not occur.

4.1.6 Case 4: $\Delta < \infty$ and $\mu(D) = 1$.

Given any local chart $U \in \mathcal{U}$, if $\mu(D) = 1$ and $\Delta < \infty$, we will show that for almost every $x \in U$ the conditional measure μ_x^U in the plaque $\mathcal{F}|U(x)$ is equivalent to the measure λ_x given in Definition 3.22.

First, we show that for almost every $x \in M$, the coditional measure μ_x is absolutely continuous with respect to the measure λ_x.

Lemma 4.10. *The constant $\Delta \neq 0$ and*

$$\mu_x \ll \lambda_x,$$

for μ-almost every $x \in M$.

Proof. Let $y \in D$. Then, for some $n_0 \in \mathbb{Z}$ and $U \in \mathcal{U}$ we have $f^{n_0}(y) \in D_U$. Call $x = f^{n_0}(y)$. As $x \in \mathcal{F}|U(\Pi_U) \setminus \mathcal{F}|U(\mathcal{Z}_U)$, we have $\mu_x^U(\mathcal{F}|U(\Pi_U) \cap \mathcal{F}|U(x)) = 1$. Therefore, we conclude (by the same argument used in Case 3 - Section 4.1.5) that

$$\Pi_{x,U} \cap B_{d_x}[x, \delta] = B_{d_x}[x, \delta],$$

for some small δ. Consequently, $\Pi_U \cap \mathcal{F}|U(x) \supset F_{\mathfrak{r}}(x)$, where $F_{\mathfrak{r}}(x)$ is an connected subset in the plaque $\mathcal{F}|U(x)$ defined by

$$F_{\mathfrak{r}}(x) = \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geqslant \mathfrak{r}\}. \tag{4.8}$$

Furthermore, we have $F_{\mathfrak{r}} \subset \Pi_{x,U}$.

On the other hand, the definition of set Π_U implies that for any $k \in \mathbb{N}^*$, $U \in \mathcal{U}$ and $x \in \Pi_U$, the following inequality holds:

$$\left| \frac{\mu_x^U(B_{d_x}(x, \varepsilon_k))}{2\varepsilon_k \cdot \mu_x^U(B_{d_x}(x, \mathfrak{r}))} - \Delta \right| \leqslant \frac{1}{k}. \tag{4.9}$$

Given a constant $r > 0$ such that $B_{d_x}(x, r) \subset F_{\mathfrak{r}}(x)$. Take $k_0 \in \mathbb{N}$ such that $k_0^{-1} < r$. Since $\{d_x\}$ is a $\mathcal{F}$-metric system, for any $k > k_0$ we need at most $s(k) = \lfloor r/\varepsilon_k \rfloor + 1$ points, say $a_1, a_2, ..., a_{s(k)} \in B_{d_x}(x, r)$, to cover the ball $B_{d_x}(x, r)$ with balls of radius ε_k.

Again by continuity (see Lemma 3.23), there exists $\beta > 0$ such that

$$\alpha_i := \frac{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \leqslant \beta \quad \text{for all} \quad i = 1, \ldots, s(k).$$

Since $a_1, a_2, ..., a_{s(k)} \in B_{d_x}(x, r) \subset \Pi_{x,U}$, we also have

$$\frac{\mu_x^U(B_{d_x}(x, \varepsilon_k))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \leqslant \frac{2\varepsilon_k}{k} + 2\varepsilon_k \cdot \Delta.$$

Therefore,

$$\begin{aligned}\frac{\mu_x^U(B_{d_x}(x,r))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))} \leqslant \sum_{i=1}^{s(k)} \frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))} &= \sum_{i=1}^{s(k)} \alpha_i \cdot \frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))} \\ &\leqslant \beta \cdot \sum_{i=1}^{s(k)} \frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))} \\ &\leqslant \beta \sum_{i=1}^{s(k)} \frac{2\varepsilon_k}{k} + \Delta 2\varepsilon_k \\ &= \beta \cdot s(k) \frac{2\varepsilon_k}{k} + \beta \cdot s(k) 2\varepsilon_k \cdot \Delta.\end{aligned}$$

Since $\lim s(k)\varepsilon_k \to r$ as $k \to \infty$, we have that

$$\lim_{k\to\infty} \beta \cdot s(k) \frac{\varepsilon_k}{k} = 0$$

Therefore, taking the limit when $k \to \infty$, we have that

$$\frac{\mu_x^U(B_{d_x}(x,r))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))} \leqslant 2\Delta \cdot \beta \cdot r.$$

This means that

$$\frac{\mu_x(B_{d_x}(x,r))}{\nu_x(B_{d_x}(x,r)} \leqslant \Delta \cdot \beta.$$

Therefore, $\mu_x \ll \lambda_x$ when restricted to $F_{\mathfrak{r}}$ and $\Delta > 0$. As $r > 0$ can be taken to be arbitrarily small, it follows that $\mu_x \ll \lambda_x$, as we wanted to show.

□

Next, we are able to conclude that μ_x^U is equivalent to the measure λ_x.

Lemma 4.11. *For μ almost every $x \in M$*

$$\mu_x^U \sim \lambda_x.$$

Proof. By Lemma 4.10, we know that $\mu_x^U \ll \lambda_x$. Since λ_x is a doubling measure, as stated in Theorem 2.9, for λ_x-almost every point $y \in F_{\mathfrak{r}}(x)$ defined in 4.8, we have that the Radon-Nikodym derivative $d\mu_x^U/d\lambda_x$ exists and is given by

$$\frac{d\mu_x^U}{d\lambda_x}(y) = \lim_{r\to 0} \frac{\mu_x^U(B_{d_x}(y,r))}{\lambda_x(B_{d_x}(y,r))}.$$

In particular, by taking the limit along the subsequence ε_k, $k \to \infty$, we conclude that

$$\frac{d\mu_x^U}{d\lambda_x}(y) = \lim_{k\to\infty} \frac{\mu_x^U(B_{d_x}(y,\varepsilon_k))}{\lambda_x(B_{d_x}(y,\varepsilon_k))}, \quad \lambda_x - a.e \quad y \in F_{\mathfrak{r}}(x),$$

which implies

$$\frac{d\mu_x^U}{d\lambda_x}(y) = \beta(y) \cdot \Delta, \quad \lambda_x - a.e \quad y \in F_{\mathfrak{r}}(x),$$

where $\beta(y) = \mu_x^U(B_{d_x}(y, \mathfrak{r}))$. Since β is a continuous function when restricted to the plaque $\mathcal{F}|U(x)$, given any compact $I \subset F_{\mathfrak{r}}(x)$ we have

$$\beta_1 \Delta \leqslant \frac{d\mu_x}{d\lambda_x}(y) \leqslant \beta_2 \Delta, \; \lambda_x \text{ a.e } \; y \in I.$$

Then,

$$\beta_1 \Delta \cdot \lambda_x \leqslant \mu_x, \; \lambda_x \text{ a.e } \; y \in I.$$

This implies that the set $E \subset \mathcal{F}|U(x)$ if such that $\mu_x^U(E) = 0$, then for any compact subset $I \subset \{y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geqslant \mathfrak{r}\}$, we have $\lambda_x(E \cap I) = 0$.

Since $\mathcal{F}(x)$ can be expressed as a countable union of increasing compact subsets, we can conclude that $\lambda_x(E \cap y \in \mathcal{F}|U(x) : d_x(y, \partial \mathcal{F}|U(x)) \geqslant \mathfrak{r}) = 0$. Furthermore, since $\mathfrak{r}$ can be arbitrarily small, we can conclude that $\lambda_x(E) = 0$. Thus, we have shown that $\lambda_x \ll \mu_x^U$ as desired. □

5 Proof of Theorem B

5.1 Proof Theorem B

First, we recall the Hölder theorem for actions on one-dimensional manifolds. The action given by a group G acting on a manifold M is a *free action* if each non-trivial element in G has no fixed points.

Theorem 5.1 (Hölder Theorem). *[24] Let G be a group of orientation preserving homeomorphisms acting freely on $\mathbb{R}$ (resp.S^1). Then G is isomorphic to a subgroup of translations on $\mathbb{R}$ (resp. of the rotations in S^1).*

As in the proof of the Theorem 3.9 in [11], consider the set $\mathcal{N} = \{x \in M : \alpha(x) = \omega(x) = M\}$, since f is transitive, $\mathcal{N}$ is full measure residual subset.

Definition 5.2. *We say that a map $F : \mathcal{F}^c(x) \to \mathcal{F}^c(x)$ is a limit center map if there exists a sequence $\{n_i\} \subset \mathbb{Z}$ with $|n_i| \to \infty$ such that $\{f^{n_i}\}$ pointwise converges to F.*

For each $x \in \mathcal{N}$ we consider

$$\mathscr{L}(\mathcal{F}^c(x)) := \{F : \mathcal{F}^c(x) \to \mathcal{F}^c(x) : F \text{ is a limit center map}\} \text{ and}$$
$$\mathscr{L}^+(\mathcal{F}^c(x)) := \{F \in \mathscr{L}(\mathcal{F}^c(x)) \ : \ F \text{ preserves the orientation of } \mathcal{F}^c(x)\}.$$

Remark 5.3. *(see [11, Proposition 4.7] For every $x \in \mathcal{N}$, we have:*

- *If $\mathcal{F}^c(x)$ is not compact, then there is a homeomorphism $\psi_x : \mathcal{F}^c(x) \to \mathbb{R}$, such that*

  $$\mathscr{L}^+(\mathcal{F}^c(x)) = \{\psi_x^{-1} \circ T_t \circ \psi_x,\ t \in \mathbb{R}\},$$

  where T_t is the translation $T_t : \mathbb{R} \to \mathbb{R}$, $s \mapsto s + t$.

- *If* $\mathcal{F}^c(x)$ *is compact, then there is a homeomorphism* $\psi_x : \mathcal{F}^c(x) \to S^1$*, such that*

$$\mathscr{L}^+(\mathcal{F}^c(x)) = \{\psi_x^{-1} \circ R_t \circ \psi_x,\ t \in S^1 = \mathbb{R}/\mathbb{Z}\},$$

  where R_t *is the rotation* $R_t : S^1 \to S^1$, $s \mapsto s + t(\mathrm{mod}\mathbb{Z})$.

- $\mathscr{L}^+(\mathcal{F}^c(x))$ *is a group whose action on* $\mathcal{F}^c(x)$ *is transitive.*

- $\mathscr{L}(\mathcal{F}^c(x))$ *either coincides with* $\mathscr{L}^+(\mathcal{F}^c(x))$ *or is the group generated by* $\mathscr{L}^+(\mathcal{F}^c(x))$ *and* $-Id|\mathcal{F}^c(x)$.

- *If* $F : \mathcal{F}^c(x) \to \mathcal{F}^c(x)$ *is a limit center map having a fixed point* $x \in \mathcal{F}^c(x)$*, if* F *is orientation preserving, then* F *is the identity map of* $\mathcal{F}^c(x)$*. This is, the action on* $\mathcal{F}^c(x)$ *given by the group* $\mathscr{L}^+(\mathcal{F}^c(x))$ *is free.*

Lemma 5.4. *For* μ*-almost every* $x \in M$*, for* μ_x*-almost every* $y \in \mathcal{F}(x)$*, there is a limit center map* $F \in \mathscr{L}^+(\mathcal{F}^c(x))$ *such that* $F(x) = y$ *and* $F_*\mu_x = \alpha\mu_x$*.*

Proof. We have that the map $x \mapsto \mu_x$ is measurable, restricted to a full measure subset of M (see for example 3.27). Using Lusin's theorem takes $\{K_i\}$ an increasing sequence of compact sets for which the map $x \mapsto \mu_x$ is continuous when restricted to each K_i. As μ is ergodic, for μ-almost every point $x \in K_i$, the orbit of x is dense in a full measurable subset of K_i. Let $P_i = \mathcal{N} \cap K_i$.

Let $x \in P_i$ and $y \in \mathcal{F}^c(x) \cap P_i$. From [11, Proposition 4.7], we know that there exists $F \in \mathscr{L}^+(\mathcal{F}^c(x))$ such that $F(x) = y$. Since F is a limit center map, there exists a sequence $n_k \subset \mathbb{Z}$ with $|n_k| \to \infty$ so that the sequence $f^{n_k}|_{\mathcal{F}^c(x)}$ point wise converges to F.

By the continuity of $x \mapsto \mu_x$ on the set P_i and since $y \in P_i$, we have that

$$F_*\mu_x = \lim_{k\to\infty} f_*^{n_k}\mu_x = \lim_{k\to\infty} \mu_{f^{n_k}(x)} = \mu_{F(x)} = \mu_y$$

By taking the limit set of K_i, we conclude that for μ-almost every point $x \in P$ (where $P = \lim P_i$) and $y \in \mathcal{F}^c(x) \cap P$, there exists $F \in \mathscr{L}^+(\mathcal{F}^c(x))$ such that $F(x) = y$ and $F_*\mu_x = \mu_y$, and by definition of μ_x, we have $F_*\mu_x = \alpha\mu_x$. □

Lemma 5.5. *For each* $x \in P$*, the set given by*

$$G_x := \{F \in \mathscr{L}^+(\mathcal{F}^c(x)) \,:\, F_*\mu_x = \alpha\mu_x \text{ for some } \alpha > 0\},$$

is a closed subgroup of $\mathscr{L}^+(\mathcal{F}^c(x))$.

Proof. Using Lemma 5.4, we can conclude that G_x is nonempty. Additionally, we can prove that G_x is a subgroup of $\mathscr{L}^+(\mathcal{F}^c(x))$ by showing that for any $F, G \in G_x$, $F \circ G^{-1} \in G_x$.

To prove that G_x is a subgroup of $\mathscr{L}^+(\mathcal{F}^c(x))$, we only need to show that $(F \circ G^{-1})_*\mu_x = \alpha\mu_x$ for some $\alpha \in \mathbb{R}$. Since F and G are both elements of G_x, we know that there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $G_*\mu_x = \alpha_1\mu_x$ and $F_*\mu_x = \alpha_2\mu_x$.

Let A be any measurable subset of $\mathcal{F}^c(x)$, we have

$$\begin{aligned}(G^{-1})_*\mu_x(A) =&\mu_x((G^{-1})^{-1}(A)) = \mu_x(G(A))\\ =&\frac{1}{\alpha_1}G_*\mu_x(G(A)) = \frac{1}{\alpha_1}\mu_x(G^{-1}(G(A)))\\ =&\frac{1}{\alpha_1}\mu_x(A).\end{aligned}$$

This implies that $G^{-1} \in G_x$. Then

$$\begin{aligned}(F \circ G^{-1})_*\mu_x(A) &= \mu_x((F \circ G^{-1})^{-1}(A)) = \mu_x(G(F^{-1}(A))\\ &= (G^{-1})_*\mu_x(F^{-1}(A)) = \alpha_1\mu_x(F^{-1}(A))\\ &= \alpha_1 F_*\mu_x(A) = \alpha_1\alpha_2\mu_x(A)\\ &= \alpha\mu_x(A).\end{aligned}$$

Therefore, $F \circ G^{-1} \in G_x$, and we have shown that G_x is a subgroup of $\mathscr{L}^+(\mathcal{F}^c(x))$.

Consider a sequence $\{F_n\}_{n\in\mathbb{N}} \subset G_x$ that converges to $F \in \mathscr{L}^+(\mathcal{F}^c(x))$. We aim to show that $F \in G_x$, which means $F * \mu_x = \alpha\mu_x$ for some $\alpha \in \mathbb{R}$.

For any bounded subset $A \subset \mathcal{F}^c(x)$, we have $F_n(A) \to F(A)$ in the Hausdorff metric. Suppose that $\mu_x(\partial F(A)) = 0$, we have that $\mu_x(F_n(A))$ converges to $\mu_x(F(A))$. To see this, consider $B_n = F(A_n)\Delta F(A)$, which is the symmetric difference between the sets $B_n = F(A_n)$ and $F(A)$.

$$\begin{aligned}\limsup_{n\to\infty}\mu_x(B_n) &\leqslant \mu_x(\limsup_{n\to\infty}\ B_n)\\ &= \mu_x\left(\bigcap_{m=1}^{\infty}\bigcup_{n\geqslant m} B_n\right)\\ &\leqslant \mu_x(\partial F(A)) = 0.\end{aligned}$$

Thus,

$$\lim_{n\to 0}\mu_x(F(A_n) \setminus F(A)) = \lim_{n\to\infty}\mu_x(F(A) \setminus F(A_n)) = 0.$$

This implies that

$$\lim_{n\to\infty}\mu_x(F_n(A)) = \mu_x(F(A)).$$

Therefore, for every measurable bounded subset $A \subset \mathcal{F}^c(x)$ such that $F_*\mu_x(\partial A) = 0$, we have $(F_n)_*\mu_x(A) \to F_*\mu_x(A)$. This implies that $(F_n)_*\mu_x \to F_*\mu_x$. We also have $(F_n)_*\mu_x = \alpha_n\mu_x$, where $\alpha_n = \mu_x(A)/\mu_x(F_n(A))$ for any measurable subset $A \subset \mathcal{F}^c(x)$ such that $\mu_x(F(A)) \neq 0$ and $n \in \mathbb{N}$ large enough.

Since F is a homemomorphism, we can take $A \subset \mathcal{F}^c(X)$ such that $\mu_x(A) > 0$ and $\mu_x(F(A)) > 0$. Furthermore, since μ_x is not atomic, $\mu_x(\partial(F(A))) = 0$, where $\partial(F(A))$

denotes the boundary of $F(A)$. It follows that $\alpha_n \to \alpha = \mu_x(A)/\mu_x(F(A))$, and hence $F_*\mu_x = \alpha\mu_x$. Therefore, $F \in G_x$, as we wanted to show. □

Proof of Theorem B. Let $x \in P$ and consider the action on $\mathcal{F}^c(x)$ given by the group $\mathscr{L}^+(\mathcal{F}^c(x))$. This action is free and transitive for every x. Applying the Hölder Theorem 5.1, we conclude that $\mathscr{L}^+(\mathcal{F}^c(x))$ is isomorphic to the group of translations (resp. rotations) on $\mathbb{R}$ (resp. S^1) if $\mathcal{F}^c(x)$ is homeomorphic to $\mathbb{R}$ (resp. S^1). We denote by T_x the group of translations (resp. rotations) on $\mathbb{R}$ (resp. S^1).

By Lemma 5.5, we have that G_x is a closed subgroup of $\mathscr{L}^+(\mathcal{F}^c(x))$. Therefore, G_x must be either the whole group T_x, of translations (rotations) of $\mathcal{F}^c(x)$, or the discrete group generated by a single element.

First, suppose G_x is isomorphic to a discrete subgroup of T_x. In this scenario, μ_x would imply the existence of only a countable number of points in $\mathcal{F}^c(x)$ with full measure, since each atom of μ_x must be mapped to an atom by a fixed translation. Additionally, the support cannot be a Cantor set because the atoms must be equidistant from each other.

Now, suppose that G_x is isomorphic to the group of translations in $\mathbb{R}$ (resp. rotations in S^1), which implies that the support of μ_x is full. This, in turn, implies that μ_x is equivalent to the measure λ_x generated by the metric length system. This completes the proof of the theorem. □

6 Discussions

In this section, we will make some remarks on the one-dimensional hypothesis and then state some directions for further investigation.

If the dimension of the foliation $\mathcal{F}$ is greater than one, the arguments used in the proof of the main theorem cannot be applied. This is due to the inability to utilize the structure of a manifold of dimension one. For example, the proof of Lemma 3.13 heavily depends on the fact that the leaves of the foliation $\mathcal{F}$ are homeomorphic to either S^1 or $\mathbb{R}$. Another significant result that utilizes the fact that the foliation is one-dimensional, is Proposition 3.18, where, we prove that the metric system is plaque continuous, which is a critical result for the proof of Theorem A. This raises the natural question:

Question 6.1. *If the foliation $\mathcal{F}$ has dimension $\geqslant 2$ with an f-invariant metric system, can we have a classification for the conditional measure, in the same sense as Theorem A?*

One of the challenges in this scenario is the presence of multiple directions. When dealing with a foliation $\mathcal{F}$ of dimension greater than one, it becomes impossible to establish a ordering inside of the each leaf.

On the other hand, in [11] the authors provided a classification for C^1 partially hyperbolic diffeomorphisms on a closed 3-manifold M, which a topologically neutral center of one-dimension and transitive. This classification mentioned above raised an important question:

Question 6.2. *Is it possible to classify functions in $\mathbb{T}^3$ based on their topological or metric properties while preserving 1-dimensional foliations with a $\mathcal{F}$-arc length system, as stipulated by Theorem C presented by Bonatti-Zhang in [11]?*

Another natural question arises regarding the existence of the $\mathcal{F}$-arc length system for a partially hyperbolic diffeomorphism with a central direction of dimension greater than 1.

Question 6.3. *If the center foliation $\mathcal{F}^c$ for a partially hyperbolic diffeomorphism has dimension $\geqslant 2$, is there an $\mathcal{F}$-metric system?*

The literature identifies a crucial property called *quasi-isometric in the center*, which falls under the class of partially hyperbolic diffeomorphisms with a central direction of dimension 1. This property holds significant value and is extensively studied in the field.

Definition 6.1. *The partially hyperbolic and dynamically coherent diffeomorphism f is quasi-isometric in the center if there exist $K_0 \geqslant 1$ and $c_0 >$ such that for every $x, y \in M$ satisfying $\mathcal{F}^c(x) = \mathcal{F}^c(y)$ and every $n \in \mathbb{Z}$,*

$$K_0^{-1}\mathrm{d}^c(x,y) - c_0 \leqslant \mathrm{d}^c(f^n(x), f^n(y)) \leqslant K_0\mathrm{d}^c(x,y) + c_0,$$

where d^c is the distance along the leaves of $\mathcal{F}^c$ induces by the Riemannian metric.

For example, if the center leaves are compact and arranged in a fiber bundle, this property holds. However, there is a significant subset of systems known as discretized Anosov flows [6, 23], which have non-compact one-dimensional center leaves that are quasi-isometric in the center. In these systems, each center leaf is individually fixed such that $f(x) \in \mathcal{F}^c(x)$ for all $x \in M$. Perturbations of the time-one map of Anosov flows fall into this category.

Definition 6.2. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with a one-dimensional central direction. We say that f is a discretized Anosov flow if there exist*

1. *an orientation foliation $\mathcal{F}^c$ such that for every $x \in M$, the leaf $\mathcal{F}^c(x) \in \mathcal{F}^c$ is C^1, tangent to E^c and satisfies $f(\mathcal{F}^c(x)) = \mathcal{F}^c(x)$, and*
2. *a continuous map $\tau : M \to \mathbb{R}$, such that*

$$f(x) = \varphi_{\tau(x)}(x)$$

for every $x \in M$, where $\varphi_t : M \to M$ denotes a unit speed flow whose orbits are the leaves of $\mathcal{F}^c$.

Discretized Anosov flows have received extensive attention in the literature, albeit sometimes under different names. For instance, the first instances of robustly transitive diffeomorphisms isotopic to the identity were obtained in [9], which were constructed to be arbitrarily close to the time 1 map on any Anosov flow. These examples are categorized as discretized Anosov flows.

Recent work in [5] demonstrated that discretized Anosov flows account for every dynamically coherent homotopic to the identity partially hyperbolic diffeomorphism of many 3-manifolds. Additionally, [17] (and [16]) showed that in most 3-manifolds, discretized Anosov flows are accessible and ergodic when they preserve a volume form.

Other notable dynamical results related to discretized Anosov flows, include the rigid results found by [4], the measurements of maximal entropy by [12], the centralizers rigidity for partially hyperbolic diffeomorphisms examined by [14] and [7], and the invariant principle demonstrated in [13].

Question 6.4. *If f is a discretized Anosov flow, can we construct an $\mathcal{F}^c$-arc length system that is measurable, at least?*

Question 6.5. *In general, do systems with quasi-isometric centers admit a measurable $\mathcal{F}^c$-metric system?*

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