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Celestial Holography from the Flat Space Limit of AdS/CFT

Holografia Celestial a partir do Limite de Espaço Plano de AdS/CFT

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I dedicate this work to my dear and beloved father Waldir de Gioia

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Resumo

A formulação de uma teoria completa de gravitação quântica é um fenomenal problema em aberto na Física Teórica contemporânea, com o princípio holográfico, precisamente formulado para gravitação quântica em espaços-tempo assintoticamente negativamente curvados na forma da correspondência AdS/CFT, permanecendo uma das principais ferramentas para abordá-lo. Mais geralmente, ele propõe que uma teoria quântica da gravitação deva admitir uma formulação equivalente em termos de uma teoria não gravitacional de menor dimensão. Nesse contexto, Holografia Celestial emergiu na última década como uma proposta para holografia em espaços-tempo assintoticamente planos, conjecturando que a gravitação quântica em tais backgrounds deva ser dual a uma Teoria de Campos Conforme Celestial (CCFT) de codimensão dois vivendo na esfera celestial no infinito nulo.

Se por um lado diversas entradas no dicionário foram descobertas e novos insights sobre a teoria do bulk tenham emergido por essa proposta, desenvolver uma definição inrínseca de uma CCFT e propor a construção concreta de pares duais de teorias no bulk e no boundary permanece um grande desafio. Nesta tese de doutorado, pretendemos contribuir para esse problema propondo uma conexão entre CCFT e CFT, motivada pelo limite de espaço plano da correspondência AdS/CFT.

Mostramos que diagramas de Witten em AdS, com operadores colocados em faixas infinitesimais de largura $\Delta\tau \propto O(R^{-1})$ ao redor de dois time-slices separados por π no

tempo global, reduzem à amplitudes celestiais no limite $R \to \infty$, mediante a introdução de uma apropriada identificação antípoda dos dois time-slices. Esse resultado é verificado por cálculo explícito para a função de dois pontos não perturbativa de um operador escalar em uma onda de choque em AdS, sugerindo sua validade como um resultado não perturbativo.

Motivados por essa análise, consideramos uma CFT genérica no cilindro Lorentziano e estudamos a simetria conforme de uma faixa infinitesimal genérica ao redor de um time-slice, encontrando que no limite $R \to \infty$ um aprimoramento de dimensão infinita da simetria conforme $\mathfrak{so}(3,2)$ emerge, com os vetores de Killing conformes associados parameterizados por uma função $f(z,\bar{z})$ e um vetor de Killing conforme $Y(z,\bar{z})$ em S^2. Notavelmente, mostramos que no limite $R \to \infty$ esses campos vetoriais podem ser reorganizados em campos vetoriais que obedecem a álgebra de BMS_4 estendida.

Um operador primário com dimensão Δ e autovalor de spin s na CFT é mostrado então se transformar como um operador primário bidimensional com dimensão efetiva $\hat{\Delta} = \Delta + u\partial_u$ e spin bidimensional $\ell = s$ pela ação da subálgebra de superrotações, sugerindo a introdução de uma transformada integral para diagonalizar tais dimensões. Mostramos que isso é realizado por uma transformada tipo Mellin temporal que extrai modos do operador primário na CFT que se transformam como operadores primários sob superrotações. O processo pelo qual esses modos são construídos é análogo à redução dimensional de Kaluza-Klein, em que uma torre de campos de diferentes massas é obtida pela redução dimensional de um único campo no espaço de dimensão maior. Assim, observamos que estamos de fato estudando uma redução dimensional de uma CFT no cilindro para uma teoria bidimensional em S^2.

Procedemos então a mostrar que em um número arbitrário de dimensões as componentes transversas da transformação shadow de uma corrente $\widetilde{J}_a$ e da transformação shadow da parte simétrica sem traço do tensor de energia-momentum $\widetilde{T}_{\{ab\}}$ reproduzem o teorema do gluon soft em ordem dominante, o teorema do graviton soft em ordem dominante e o teorema do graviton soft em ordem sub-dominante, quando inseridos em funções de correlação, estabelecendo a emergência das simetrias soft através do processo de redução dimensional. Os resultados sugerem que qualquer CFT no cilindro Lorentziano possui um setor celestial, caracterizado por uma redução dimensional à time-slices, governado por simetrias soft. Essa observação sugere que seja possível construir CCFT a partir de CFT por redução dimensional, por conseguinte provendo uma possível ferramenta para investigar a definição intrínseca de CCFT e a construção de pares duais em holografia celestial a partir de exemplos conhecidos de AdS/CFT.

Abstract

The formulation of a complete theory of quantum gravity is an outstanding problem in contemporary Theoretical Physics, with the holographic principle, precisely formulated for quantum gravity in asymptotically negatively curved spacetimes in the form of the AdS/CFT correspondence, remaining one of the main tools to approach it. More generally it states that a quantum theory of gravity should admit one equivalent formulation in terms of a lower-dimensional non-gravitational theory. In that context, Celestial Holography emerged in the last decade as a proposal for holography in asymptotically flat spacetimes, conjecturing that quantum gravity in such backgrounds should be dual to a codimension two Celestial Conformal Field Theory (CCFT) living in the celestial sphere at null infinity.

While several entries of the dictionary have been uncovered and new insights about the bulk theory have emerged from this proposal, developing one intrinsic definition of a CCFT and proposing the construction of concrete dual pairs of bulk and boundary theories remains a major challenge. In this Ph.D. thesis, we intend to contribute to this problem by proposing a connection between CCFT and standard CFT, motivated by the flat space limit of the AdS/CFT correspondence.

We show that AdS Witten diagrams, with operators placed on infinitesimal strips of width $\Delta\tau \propto O(R^{-1})$ about two time-slices separated by π in global time, reduce to celestial amplitudes in the $R \to \infty$ limit provided a suitable antipodal identification of the two

slices is introduced. This result is verified by explicit calculation for a non-perturbative two-point function of a scalar operator in an AdS shockwave background, suggesting its validity as a non-perturbative result.

Motivated by this analysis, we consider a generic CFT on the Lorentzian cylinder and study the conformal symmetry of a generic infinitesimal strip about a time-slice, finding that in the strict $R \to \infty$ limit an infinite-dimensional enhancement of the standard global conformal algebra $\mathfrak{so}(3,2)$ arises, with the associated conformal Killing vectors parameterized by a function $f(z,\bar{z})$ and a conformal Killing vector $Y(z,\bar{z})$ on S^2. Remarkably, we show that in the $R \to \infty$ limit these vector fields can be reorganized into vector fields obeying the extended BMS_4 algebra.

A primary operator with dimension Δ and spin eigenvalue s in the CFT is then shown to transform as a two-dimensional primary operator with effective dimension $\hat{\Delta} = \Delta + u\partial_u$ and two-dimensional spin $\ell = s$ under the action of the superrotation subalgebra, suggesting the introduction of an integral transform to diagonalize the dimensions. We show this is accomplished by a time Mellin-like transform that extracts modes from the parent CFT primary operator that transform as two-dimensional primary operators. The procedure by which these modes are constructed is analogous to the Kaluza-Klein dimensional reduction in which a tower of fields of different masses is obtained from the dimensional reduction of a single field in a higher-dimensional space. As such, we observe that we are in fact studying a dimensional reduction of a three-dimensional CFT on the cylinder to a two-dimensional theory on the S^2 time-slices.

We then proceed to show in arbitrary dimensions that the transverse components of the shadow current $\widetilde{J}_a$ and of the symmetric traceless part of the shadow stress tensor $\widetilde{T}_{\{ab\}}$ reproduce the leading soft gluon theorem, leading soft graviton theorem and subleading soft graviton theorem when inserted into correlation functions, establishing the emergence of soft symmetries from the dimensional reduction procedure. The results suggest that any CFT on the Lorentzian cylinder has a celestial sector, characterized by a dimensional reduction to time-slices, governed by soft symmetries. This realization suggests that it may be possible to construct CCFT from CFT by dimensional reduction, thereby giving a possible tool to investigate the intrinsic definition of CCFT and the construction of AFS/CCFT dual pairs from known AdS/CFT examples.

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List of Abbreviations

AdS	**A**nti-**d**e **S**itter
AFS	**A**symptotically **F**lat **S**pacetime
BMS	**B**ondi **M**etzner **S**achs
CCFT	**C**elestial **C**onformal **F**ield **T**heory
CFT	**C**onformal **F**ield **T**heory
CPS	**C**ovariant **P**hase **S**pace
QFT	**Q**uantum **F**ield **T**heory
SYM	**S**uper **Y**ang **M**ills

Contents

1

Introduction

The holographic principle is the statement that a quantum theory of gravity in a D-dimensional spacetime (M, g) should be equivalently encoded in a non-gravitational theory in its boundary ∂M [3]. This general property of gravity can be argued on general grounds from the black hole area-entropy law

$$S = \frac{A}{4G}, \tag{1.1}$$

and the observation that if a black hole were big enough to contain all of the universe, the total amount of information contained in it should scale as the area of the boundary and not as its volume. A precise realization of this principle has been given for the first time by Maldacena in Anti-de Sitter (AdS) spacetime in the form of the duality between Type IIB Superstring Theory in $\mathrm{AdS}_5 \times S^5$ with N units of five form flux and $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $\mathrm{SU}(N)$ living in its conformal boundary in a suitable large N limit and subject to a specific relation between the coupling constants of the theories [4].

Today this is known as one example of the broader $\text{AdS}_{d+1}/\text{CFT}_d$ dualities, which suggest that quantum gravity in asymptotically negatively curved spacetimes should be dual to conformal field theories living in their conformal boundaries, see [5, 6] for reviews.

On the one hand, AdS/CFT has proven a powerful duality that has given many insights both into the nature of quantum gravity, as well as in properties of conformal field theories which possess gravitational duals. On the other hand, the argument in favor of a holographic principle is general enough that it suggests that spacetimes with different asymptotic structures should also admit a holographic description. In that regard, one particular case of interest is clearly the class of asymptotically flat spacetimes (AFS), which can be invoked in several scenarios as a good approximation of reality, especially if one avoids discussing phenomena at cosmological scales.

Compared to the AdS case, flat space holography poses additional challenges. In particular, while the conformal boundary of AdS is timelike, and hence is able to house a standard quantum field theory, the boundary of asymptotically flat spacetimes is comprised of two null surfaces, future and past null infinities $\mathcal{I}^{\pm}$, together with singular points, future and past timelike infinities $i^{\pm}$ and spacelike infinity i^0. These singular points may be described as codimension one slices by the introduction of appropriate hyperbolic slicings [7–9]. Quantum field theories living in this kind of manifold are much less understood than those defined in standard Lorentzian manifolds, presenting one of the challenges to the construction of flat space holography. In fact, it has been anticipated in [10], by relying upon an uplift of $\text{AdS}_3/\text{CFT}_2$ holography through a hyperbolic slicing of Minkowski spacetime, that in asymptotically flat spacetimes the boundary theory should live in two dimensions lower than the bulk theory and have Euclidean signature, in stark contrast with AdS/CFT in which case the boundary theory lives in one dimension lower than the bulk theory and shares the Lorentzian signature with the bulk.

Despite the challenges, remarkable progress has been achieved in the past decade, propelled by the seminal work of Strominger [11] in which the symmetries of the gravitational scattering problem in AFS have been revisited. BMS supertranslations have been shown to be a symmetry of the gravitational $\mathcal{S}$-matrix, whose Ward identity takes the form of a U(1) Kac-Moody current algebra in the celestial sphere at null infinity [11]. Not only that, such Ward identity is remarkably equivalent to the statement of Weinberg's leading soft graviton theorem [12], an observation which lies at the heart of being able to ascertain that such transformations are really symmetries of the gravitational $\mathcal{S}$-matrix. In the following years, it has been further demonstrated that the gravitational memory effect measures transition between inequivalent vacua connected by supertranslations and is nothing but a Fourier transform of the soft graviton theorem [13], thereby establishing one triangular equivalence between *asymptotic symmetries*, *soft theorems* and *memory effects*, which became known as an instance of the *IR Triangle* [14]. In the years following Strominger's original analysis, the relation between asymptotic symmetries, soft theorems, and memory effects has been extended to abelian [9, 15–21] and non-abelian gauge theories [22–25].

Central to the development of the holographic proposal is the fact that in the same way as the leading soft graviton theorem is a statement of supertranslation symmetry of the gravitational $\mathcal{S}$-matrix, there exists a subleading soft graviton theorem which implies the superrotation symmetry of the gravitational $\mathcal{S}$-matrix [26, 27]. While BMS supertranslations comprise one infinite-dimensional enhancement of the standard translations, superrotations comprise one infinite-dimensional enhancement of the Lorentz group. Geometrically, the Lorentz group $\mathrm{SO}(1,3) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, acts on the cross-sectional spheres at future and past null infinities $\mathcal{I}^{\pm}$ by means of the global conformal transformations. Superrotations, on the other hand, act as the infinite-dimensional local enhancement of the global conformal algebra to the full local conformal Virasoro algebra. The superrotation Ward identity then implies that $\mathcal{S}$-matrix elements behave as correlation functions of primary operators in a two-dimensional conformal field theory with operator-valued weights [28].

Choosing a basis of external scattering states to be a so-called conformal primary bases diagonalizes these conformal weights and recasts the $\mathcal{S}$-matrix as an object sharing the properties of a conformal correlator on the celestial sphere [29, 30]. Celestial Holography then conjectures that there exists a two-dimensional Celestial Conformal Field Theory (CCFT) living in the celestial sphere whose correlators holographically encode the bulk quantum gravity $\mathcal{S}$-matrix in such a basis. Much of the research to date has then focused on understanding the imprints of asymptotic symmetries and universal aspects of bulk scattering on CCFT [31–50].

An important development has been the understanding of the celestial Operator Product Expansion (OPE). It is well-known that tree-level scattering amplitudes of massless particles develop singularities in the limit in which two of the particles are taken to be collinear [51]. Once recast in a conformal primary basis, this collinear singularity was shown to translate into a structure that resembles an OPE for the CCFT. It was later understood that the celestial OPE could be equivalently derived from symmetry arguments, at first by imposing conformal symmetry and soft symmetries [52], and later by imposing just a subset of the full Poincaré symmetry [53].

By manipulating the celestial OPE as a standard CFT_2 OPE, it was understood that it implies in a whole tower of soft symmetries generated by various positive-helicity soft gluons and gravitons to various subleading orders in the low energy expansion [54]. In turn, after a suitable redefinition of the currents amounting to taking a light-transform, this tower of soft symmetries was shown to organize, in the gravitational case, in the form of the wedge subalgebra of the loop algebra of $w_{1+\infty}$, with an analogous result for the gauge theory version [55]. It is remarkable that this rich symmetry structure encoded in the $w_{1+\infty}$ algebra, unveiled by manipulating celestial correlators as true CFT_2

correlators, has then been explicitly shown to be realized in the Einstein equations of General Relativity [56] thereby providing one extremely non-trivial check of the validity of the holographic dictionary.

Despite its many successes, Celestial Holography is really in its infancy, with many open problems remaining to be understood. Firstly, most of the analyses so far have been conducted in perturbation theory and at tree level. Secondly, contrary to AdS/CFT, that since its early days had concrete dual pairs construable from string theories, like the type IIB superstring theory in $\text{AdS}_5 \times S^5$ dual to $\mathcal{N} = 4$ SYM proposed by Maldacena, up until recently there were no proposals of intrinsic CCFT constructions, neither of their gravitational duals. Very recently a concrete proposal of a dual pair has emerged [57], but it still has the shortcoming of relying on restricting to self-dual gravity. Moreover, the connection between Celestial Holography and string theory is still far from being completely understood.

In this thesis we aim to contribute to both of these concerns. Firstly, we study the eikonal approximation to celestial four-point functions. It is well-known that in momentum space, $2 \to 2$ scattering amplitudes of massless particles, mediated by arbitrarily spinning exchanges, have a regime known as eikonal regime, characterized by high energies $s \gg 1$ and small scattering angle $\frac{t}{s} \ll 1$, in which the amplitude is dominated by t-channel exchanges and can be written in terms of an eikonal phase which resums an infinite number of Feynman diagrams including all orders in the coupling constant. In that regime one can access non-perturbative Physics of scattering through the eikonal phase.

In particular, it is possible to equivalently formulate the problem as the propagation of one of the incoming particles in the semiclassical background created by the other. Such background is an Aichelburg-Sexl shockwave, and the propagation of the first particle in this background is known to be characterized by the Shapiro time delay acquired by the particle when it crosses the shock. The eikonal phase then computes exactly this time delay, from an analysis of $2 \to 2$ scattering mediated by gravitons. We therefore are motivated to extend the eikonal approximation to CCFT in order to have a regime in which we can compute a non-perturbative celestial correlator which may have applications for the study of causality constraints in Celestial Holography.

Secondly, we establish a precise connection between AFS/CCFT and the flat space limit of AdS/CFT, which in turn is tantamount to a precise connection between CCFT and CFT. Establishing a tight connection between AdS/CFT and AFS/CCFT is an interesting goal firstly because it brings Celestial Holography closer to the well-established AdS/CFT, allowing for the employment of known AdS/CFT results in the study of flat space holography. Secondly because it can potentially lead to a prescription to construct AFS/CCFT dual pairs from known AdS/CFT ones.

The construction of flat space $\mathcal{S}$-matrices from a flat space limit of AdS/CFT has been studied in the past from several perspectives [58–63], relying on the observation that AdS spacetime of radius R turns into flat spacetime in the large R limit. All of the previous analyses tried to reconstruct momentum space $\mathcal{S}$-matrix elements. In particular, one such prescription based on the HKLL buk reconstruction has been recently put forward in [62, 63] where it was argued that conformal correlators restricted to infinitesimal time intervals of width $\Delta\tau \propto R^{-1}$ about global times $\tau = \pm\frac{\pi}{2}$ on the Lorentzian cylinder give rise to momentum space $\mathcal{S}$-matrix elements after being subject to a certain transform. Motivated by this analysis, we study this kinematic configuration in the CFT.

In AdS/CFT correlation functions of the boundary theory are computed through a bulk calculation in perturbation theory by means of AdS Witten diagrams. This method is analogous to how $\mathcal{S}$-matrices in flat space are evaluated in terms of Feynman diagrams. In the Witten diagram case, external lines are bulk-to-boundary propagators corresponding to operator insertions in the correlator, internal lines are bulk-to-bulk propagators, and vertices are integrals over AdS with factors of the coupling, corresponding to interaction terms in a bulk effective Lagrangian expanded about AdS, see e.g. [5,6] for a more comprehensive discussion. By studying the large R behavior of AdS Witten diagrams we find that the CFT correlators in this flat space limit configuration directly reduce to celestial correlators in the large AdS radius limit, a priori without the necessity of any non-trivial transform. We verify this explicitly for the two-point function in an AdS shockwave background, providing one non-perturbative check of the validity of this proposal. This suggests that there is a connection between a dimensional reduction of CFTs on the Lorentzian cylinder to constant time slices and CCFTs on the sphere. We then show that for a generic CFT carefully studying this dimensional reduction it is possible to derive the CCFT conformally soft theorems, a key signature of CFT.

This thesis is organized as follows: in Chapter 2 we give a comprehensive review of celestial holography. In Chapter 3 we study the eikonal approximation in Celestial CFT and the associated celestial propagation on shockwave backgrounds. We also study the flat space limit of scalar AdS Witten diagrams and show that they reduce to scalar celestial amplitudes in the large AdS radius limit, with the flat space limit of an AdS shockwave two-point function being an explicit verification of the result. In Chapter 4 we study the emergence of CCFT symmetries from a dimensional reduction to time-slices of a CFT on the Lorentzian cylinder, obtaining both the extended BMS_4 algebra and the conformally soft symmetries. Finally in Chapter 5 we discuss the results and ponder on their impact on the understanding of the AFS/CCFT correspondence.

2

Celestial Holography Review

2.1 Introduction

What are the observables of quantum gravity in asymptotically flat spacetimes and what are its symmetries? These are the zeroth order questions one would ask when trying to formulate flat space holography since the observables must be encoded in the dual theory and the symmetries of the two theories must match. On the one hand, since a quantum theory of gravity precludes the existence of local observables, the most natural observable to consider is the $\mathcal{S}$-matrix. On the other hand, the question of what the symmetries of the quantum gravitational $\mathcal{S}$-matrix are is subtle. This is already illustrated in classical GR: in the 1960s, Bondi, Metzner, van der Burg, and Sachs have given a precise formulation of asymptotic flatness and found that, contrary to expectations, the diffeomorphisms preserving asymptotic flatness provide one infinite-dimensional enlargement of the Poincaré group of Minkowski isometries, where translations are enhanced to supertranslations [64–66]. The resulting symmetry group takes the form of a semi-direct product of supertranslations and Lorentz transformations and became known as the BMS group.

It was only in the last decade, however, that the BMS group has been shown to give rise to a symmetry of the gravitational $\mathcal{S}$-matrix [11] and that the supertranslation Ward identity was shown to be, in fact, equivalent to Weinberg's soft graviton theorem [12]. This led to the realization that soft theorems in general are symmetry statements. Indeed, a subleading soft graviton theorem has been shown to hold at tree level [26] and shown to imply that superrotations, proposed earlier in [67] as an enhancement of the Lorentz symmetry in the asymptotically flat bulk, are indeed a symmetry of the gravitational $\mathcal{S}$-matrix [27].

Remarkably, such Ward identities of bulk asymptotic symmetries, obeyed by $\mathcal{S}$-matrix elements of massless particles, take the form of current algebra Ward identities in a two-dimensional CFT on the celestial sphere $\mathcal{CS}^2$ obeyed by conformal correlators of primary operators, with non-standard operator-valued dimensions, whose insertion points on the sphere are the angles at which the particles enter and exit spacetime through past and future null infinities $\mathcal{I}^{\pm}$. Indeed the supertranslation Ward identity takes the form of a U(1) Kac-Moody current algebra where the role of the charges of the various operators is played by the bulk energies [68], while the superrotation Ward identity takes the form of a stress tensor Ward identity where the role of the two-dimensional spin is played by the four-dimensional helicities, and the role of the two-dimensional conformal dimensions is played by an operator $\hat{\Delta} = -\omega\partial_\omega$ [28].

While from a bulk perspective, it is interesting to describe the $\mathcal{S}$-matrix in a basis of momentum eigenstates, diagonalizing the action of translation symmetry, what this showed is that from a boundary perspective, a basis that diagonalize the operator-valued conformal dimensions is more interesting. This led to the introduction of *conformal primary bases* [29, 30], which from the boundary perspective diagonalize conformal dimensions and from a bulk perspective, diagonalizes the boost generator in the direction of the momentum. Upon transforming to a conformal primary basis, the $\mathcal{S}$-matrix behaves like a conformal correlator in a two-dimensional CFT on $\mathcal{CS}^2$, with each asymptotic particle corresponding to a continuum of fields of various possible conformal dimensions. The $\mathcal{S}$-matrix elements in a conformal primary basis are then called *celestial amplitudes* or equivalently *celestial correlators*.

In this chapter, we review the basics of Celestial Holography. We start in section 2.2 by discussing the general framework of asymptotic symmetries, in order to give a precise definition of the asymptotic symmetry group and lay the foundations of why gauge transformations with non-trivial action at the boundary of spacetime are true symmetries of the theory and not mere redundancies. In section 2.3 we then discuss the asymptotic structure of Minkowski spacetime by Penrose's conformal compactification description of $\mathcal{I}^+$ and $\mathcal{I}^-$. In section 2.4 we introduce the more general concept of asymptotically flat spacetimes, introducing the BMS group as a group of asymptotic symmetries and its proposed extension to the extended BMS group, with the associated canonical surface charges. Then in section 2.5 we discuss how in the quantum scattering theory the Ward identities of these symmetries are implied by soft theorems. The natural appearance of Ward identities characteristic of a two-dimensional U(1) Kac-Moody symmetry and of a two-dimensional stress tensor motivate us to introduce the conformal primary bases and recast the $\mathcal{S}$-matrix as a celestial correlator in section 2.6 along with some of its consequences. Finally, in section 2.7 we review how the investigation of the symmetry structure of celestial CFT has been improved by the understanding of the celestial OPE and how employing this framework reveals the $w_{1+\infty}$ symmetry of gravity.

2.2 Asymptotic Symmetries

We start by discussing the definition of asymptotic symmetries. We consider field theories in some generic D-dimensional spacetime (M, g) in which there are local symmetries, also known as gauge symmetries. The formalism which allows us to better discuss these symmetries is the Covariant Phase Space (CPS) formalism, which permits us to construct a phase space $(\Gamma, \mathbf{\Omega})$ in a covariant manner starting from a given Lagrangian form $\mathbf{L}$ in (M, g). See [25, 69, 70] for detailed reviews.

The construction of Γ starts with $\mathcal{A}$, the space of allowed field configurations, defined by a choice of boundary conditions on the fields. Given $\mathcal{S} \subset \mathcal{A}$ the space of solutions to the classical equations of motion, the CPS formalism provides a prescription to construct a pre-symplectic form $\mathbf{\Omega}$ in $\mathcal{S}$. This pre-symplectic form will be a closed two-form in $\mathcal{S}$, but because of trivial gauge transformations, it will in general be degenerate. The actual phase space is therefore a subset $\Gamma \subset \mathcal{S}$ defined by a gauge-fixing condition that disallows trivial gauge transformations and renders the pullback of $\mathbf{\Omega}$ to Γ a closed and non-degenerate two form, which is finally chosen as the symplectic form of the theory [25].

We now outline the prescription. First one observes that by using the Liebnitz rule several times, a generic variation $\delta\mathbf{L}$ of the Lagrangian form can always be put in the form

$$\delta\mathbf{L} = E[\Phi]\delta\Phi + d\mathbf{\Theta}[\Phi, \delta\Phi], \tag{2.1}$$

where $E[\Phi]$ are the equations of motion of the theory and $\mathbf{\Theta}$ is a $(d-1)$-form on spacetime, known as *pre-sympletic potential density*, which depends on the field configuration Φ being varied and depends linearly on the variation $\delta\Phi$. For that reason, $\mathbf{\Theta}$ is also interpreted as a 1-form in $\mathcal{A}$. In general, an object which is a p-form in M and is also a q-form in $\mathcal{A}$ is called a (p, q)-form, so that $\mathbf{\Theta}$ is a $(D-1, 1)$-form. A more rigorous definition can be provided by employing jet bundles and the variational bicomplex [69].

From $\boldsymbol{\Theta}$ one defines the *pre-sympletic density* $\boldsymbol{\omega}$, which is a $(D-1,2)$-form, by taking one anti-symmetric variation:

$$\boldsymbol{\omega}[\Phi, \delta_1\Phi, \delta_2\Phi] \;=\; \delta_1\boldsymbol{\Theta}[\Phi, \delta_2\Phi] - \delta_2\boldsymbol{\Theta}[\Phi, \delta_1\Phi]. \tag{2.2}$$

Finally, choosing a Cauchy slice $\Sigma \subset M$, one defines the *pre-sympletic form* $\boldsymbol{\Omega}$ which is finally a 2-form in $\mathcal{A}$:

$$\boldsymbol{\Omega}[\Phi, \delta_1\Phi, \delta_2\Phi] \;=\; \int_\Sigma \boldsymbol{\omega}[\Phi, \delta_1\Phi, \delta_2\Phi]. \tag{2.3}$$

The study of symmetries in this language now follows from standard Hamiltonian mechanics. Let $\delta_1\Phi = \delta_\varepsilon\Phi$ be a symmetry transformation with parameter ε and let $\delta_2\Phi = \delta\Phi$ be a generic field variation. The condition that $\delta_\varepsilon\Phi$ be a symmetry is the condition that it be a canonical transformation. It translates into the existence of a Hamiltonian charge Q_ε such that [25]

$$\boldsymbol{\Omega}[\Phi, \delta_\varepsilon\Phi, \delta\Phi] \;=\; -\delta Q_\varepsilon[\Phi]. \tag{2.4}$$

This equation is just the statement that using the Poisson bracket defined by $\boldsymbol{\Omega}$, the charge Q_ε generates the symmetry, i.e.

$$\{Q_\varepsilon, f\} \;=\; \delta_\varepsilon f, \tag{2.5}$$

for a generic phase space function $f \in C^\infty(\Gamma)$. In that case, in the CPS formalism, to study a certain symmetry, one studies $\boldsymbol{\Omega}[\Phi, \delta_\varepsilon\Phi, \delta\Phi]$ and tries to write it in the form (2.4).

In particular, one may follow this prescription for a gauge symmetry that the theory admits. When the symmetry is local, Noether's second theorem applies and it directly leads to the *fundamental theorem of the covariant phase space* [69], which says that for a local symmetry δ_ε there exists a $(D-2,1)$-form $\mathbf{k}_\varepsilon[\Phi,\delta\Phi]$, unique up to exact differentials, such that on $\mathcal{S}$,

$$\boldsymbol{\omega}[\Phi,\delta_\varepsilon\Phi,\delta\Phi] \;=\; d\mathbf{k}_\varepsilon[\Phi,\delta\Phi], \quad \forall\Phi\in\mathcal{S}. \tag{2.6}$$

Employing Stokes theorem, an immediate corollary is that $\boldsymbol{\Omega}[\Phi,\delta_\varepsilon\Phi,\delta\Phi]$ is a corner term at $\partial\Sigma$:

$$\boldsymbol{\Omega}[\Phi,\delta_\varepsilon\Phi,\delta\Phi] \;=\; \int_{\partial\Sigma}\mathbf{k}_\varepsilon[\Phi,\delta\Phi]. \tag{2.7}$$

At this stage one may encounter three possibilities:

1. The contraction of the pre-symplectic form with the gauge transformation vanishes identically for all $\delta\Phi$. In that case, δ_ε is a degenerate vector of $\boldsymbol{\Omega}$ and therefore $\boldsymbol{\Omega}$ does not yet qualify as a true symplectic form. These are the *trivial gauge transformations* and they must be gauge fixed to give rise to a true phase space $\Gamma\subset\mathcal{S}$.

2. The integral does not vanish, and it can be written exactly as $-\delta Q_\varepsilon[\Phi]$. In that particular case, the gauge transformation δ_ε is a true canonical transformation. It has non-trivial action on phase space and has physical consequences. As such it is not a mere redundancy in the description, but rather a symmetry of the theory. These are called *large gauge transformations.*

3. The integral does not vanish, but it cannot be written as $-\delta Q_\varepsilon[\Phi]$ for some $Q_\varepsilon\in C^\infty(\Gamma)$ because there is an additional inexact term, i.e., which cannot be written as δ of some phase space function. In that case we say that we have one non-integrable charge and a prescription is necessary to select one integrable part.

The asymptotic symmetry group of the theory is defined to be the quotient:

$$\text{ASG} \quad = \quad \frac{\text{Allowed gauge transformations}}{\text{Trivial gauge transformations}}. \tag{2.8}$$

Moreover, upon using a gauge condition to eliminate the trivial gauge transformations so as to have a well-defined phase space, one may identify

$$\text{ASG} \quad = \quad \text{Residual gauge transformations.} \tag{2.9}$$

We notice that residual gauge transformations may never contain trivial gauge transformations, otherwise, one merely observes that the gauge-fixing condition was not enough to ensure the existence of a well-defined phase space.

The definition of $\boldsymbol{\Theta}$ from $\delta\mathbf{L}$ leaves some residual ambiguities unfixed [69]. In fact, $\delta\mathbf{L}$ only defines $d\boldsymbol{\Theta}$ and therefore we may redefine $\boldsymbol{\Theta} \to \boldsymbol{\Theta} + d\mathbf{B}$ which in turn leads to $\boldsymbol{\omega} \to \boldsymbol{\omega} + d\boldsymbol{\eta}$. This reflects on the charge by transforming $\mathbf{k}_\varepsilon \to \mathbf{k}_\varepsilon + \boldsymbol{\eta}$. These ambiguities can be harnessed to perform a renormalization procedure when one encounters a divergent symplectic structure and a divergent canonical charge, see for example [71].

Finally, we remark that what characterizes a gauge transformation as trivial or non-trivial is its behavior at $\partial\Sigma$. Since Σ has been chosen as a Cauchy slice, it must be such that any causal curve in M intersects it exactly once. In that case, $\partial\Sigma$ necessarily lies at the boundary of spacetime, at infinity. We shall illustrate that concretely in the case of asymptotically flat spacetimes to which we turn next.

2.3 Asymptotic Structure of Minkowski Spacetime

2.3.1 Penrose's Conformal Compactification and Null Infinities

Our starting point will be a review of the structure of the asymptotic boundary of Minkowski spacetime. With that in mind, we shall introduce the standard Penrose compactification construction which introduces null infinities $\mathcal{I}^{\pm}$ as codimension one surfaces. These are adequate for the study of massless fields, but inadequate for the study of massive ones, in which case one needs to introduce the hyperbolic slicing of Minkowski spacetime. In this thesis we are not going to discuss massive fields, see [14,72] for reviews of including the massive case.

The intuitive idea behind the Penrose compactification is a simple one: we follow light rays traveling radially. These will fall into two categories: they are either incoming or outgoing. If they are incoming, when we follow these light rays to large distances we are going to reach the *place where light rays come from* - this is what we *define* as $\mathcal{I}^-$. If they are outgoing, when we follow these light rays to large distances we are going to reach the *place where light rays go to* - this is what we *define* as $\mathcal{I}^+$. Mathematically what we need to do is to find coordinates on spacetime whose coordinate lines coincide with these light rays. In other words we need to find functions which are constant along outgoing radial null geodesics or along incoming radial null geodesics. One way to do so is to first introduce spherical coordinates (t, r, x^A), where t is the usual inertial time, r is the radial coordinate and x^A are coordinates on S^2. The metric in these coordinates reads

$$ds^2 = -dt^2 + dr^2 + r^2\gamma_{AB}dx^A dx^B, \tag{2.10}$$

where γ is the S^2 round metric.

Next we introduce the *retarded time* $u = t - r$, which is constant along the outgoing radial null geodesics, and the *advanced time* $v = t+r$, which is constant along the incoming radial null geodesics. If we switch to coordinates (u, r, x^A) and keep (u, x^A) fixed we will have one outgoing radial null geodesic and r works as a parameter along it. If we take $r \to \infty$ we are following such a geodesic to large distances. This is what we define as $\mathcal{I}^+$. Likewise if we switch to coordinates (v, r, x^A) and keep (v, x^A) fixed we will have one incoming radial null geodesic with r a parameter along it. Taking $r \to \infty$ we follow such a geodesic to large distances and the resulting set of points is what we define as $\mathcal{I}^-$.

While $\mathcal{I}^\pm$ are not true sets of points in Minkowski spacetime, Penrose's conformal compactification realizes these surfaces in a larger *unphysical* spacetime. Consider the coordinates (u, v, x^A) where we have eliminated both t and r in favor of u and v. Observe that the condition $r \geq 0$ in the original coordinates translates into $v \geq u$. Now $\mathcal{I}^+$ is reached keeping (u, x^A) fixed and taking $v \to +\infty$ while $\mathcal{I}^-$ is reached keeping (v, x^A) fixed and taking $u \to -\infty$. The idea now is to *pull $\mathcal{I}^\pm$ to finite coordinate values preserving the causal structure*. One simple transformation that achieves the two goals is to set $u = \tan \widetilde{u}$ and $v = \tan \widetilde{v}$. In that case $u, v \to \pm\infty$ corresponds to $\widetilde{u}, \widetilde{v} \to \pm\frac{\pi}{2}$. Moreover, this transformations is conformal, and the metric tensor in the new coordinates has the form

$$ds^2 = \frac{1}{4\cos^2\tilde{u}\cos^2\tilde{v}}\bigg(- d\tilde{u}d\tilde{v} + \sin^2(\tilde{u}-\tilde{v})\gamma_{AB}dx^A dx^B\bigg). \tag{2.11}$$

In the chart $(\widetilde{u}, \widetilde{v}, x^A)$ the new coordinates have ranges $-\frac{\pi}{2} < \widetilde{u}, \widetilde{v} < \frac{\pi}{2}$ subject to the constraint $\widetilde{v} \geq \widetilde{u}$ and by our discussion we would like to identify $\mathcal{I}^+$ with $\widetilde{v} = \frac{\pi}{2}$ and $\mathcal{I}^-$ with $\widetilde{u} = -\frac{\pi}{2}$. It is again clear that these points are not a part of Minkowski spacetime. Not only that, it is clear that we cannot extend the Minkowski metric to a larger spacetime described by the coordinates in the compactified range $-\frac{\pi}{2} \leq \widetilde{u}, \widetilde{v} \leq \frac{\pi}{2}$ as we can see from (2.11) that it diverges in the boundary. More technically, Minkowski spacetime is already a geodesically complete Lorentzian manifold and therefore cannot be further extended.

On the other hand, we see that the Minkowski metric has been writen as a Weyl rescaling of another metric, which in these coordinates reads

$$d\tilde{s}^2 = -d\tilde{u}d\tilde{v} + \sin^2(\tilde{u} - \tilde{v})\gamma_{AB}dx^A dx^B. \tag{2.12}$$

The spacetime described by the same coordinates, but with this metric instead of the Minkowski one, is not geodesically complete and can be extended by compactifying the ranges of the coordinates $\widetilde{u}, \widetilde{v}$. It turns out, however, that this metric is not uniquely determined by the physical Minkowski spacetime geometry: any Weyl rescaling of it would be equally good. What this tells us is that this manifold on which we have embedded Minkowski spacetime does not really carry a metric, it carries a *conformal structure*: an equivalence class of metrics where two metrics are equivalent if and only if they are Weyl rescalings of one another.

This tells us that we can choose a representative of the equivalence class of the metric but it will be an *unphysical metric*. Fixing one choice of this unphysical metric, the bigger spacetime $(\widetilde{M}, \widetilde{g})$, which in the aforementioned sense extends Minkowski spacetime to include null infinity, is called the associated *unphysical spacetime*. Its boundary $\partial\widetilde{\mathcal{M}}$ now contains $\mathcal{I}^{\pm}$ as true null hypersurfaces with topology $\mathcal{I}^{\pm} \simeq \mathbb{R}\times S^2$. The cross-sectional spheres at $\mathcal{I}^{\pm}$ are what we call *celestial spheres* denoted by $\mathcal{CS}^2{}_{\pm}$. We observe that at first, we have two separate spheres, one at $\mathcal{I}^+$ and one at $\mathcal{I}^-$, but there is one natural *antipodal identification* between them which allows us to talk about a single celestial sphere $\mathcal{CS}^2$ [11].

The coordinates on the sphere will be chosen to be complex stereographic projection coordinates given by:

$$\hat{x}^1 = \frac{z+\bar{z}}{1+z\bar{z}}, \tag{2.13}$$

$$\hat{x}^2 = \frac{-i(z-\bar{z})}{1+z\bar{z}}, \tag{2.14}$$

$$\hat{x}^3 = \frac{1-z\bar{z}}{1+z\bar{z}}. \tag{2.15}$$

The boundary $\partial\widetilde{M}$ has more than just $\mathcal{I}^\pm$. We have described the loci $\widetilde{u} = -\frac{\pi}{2}$ and $\widetilde{v} = \frac{\pi}{2}$, but we also have introduced with the compactification $\widetilde{u} = \frac{\pi}{2}$ and $\widetilde{v} = -\frac{\pi}{2}$. Because of the constraint $\widetilde{v} \geq \widetilde{u}$ we see that when $\widetilde{u} = \frac{\pi}{2}$ the only possible value of $\widetilde{v}$ is $\widetilde{v} = \frac{\pi}{2}$ also. This would at first seem to correspond to a surface diffeomorphic to S^2, but observe in the unphysical metric $\widetilde{g}$ that the scaling factor $\sin^2(\widetilde{u} - \widetilde{v})$ defining the radius of this sphere *vanishes identically* and that this clearly also happens for all metrics in the conformal equivalence class of $\widetilde{g}$. This means that this sphere is shrunk to a point in $\widetilde{M}$. This point characterized by $\widetilde{u} = \widetilde{v} = \frac{\pi}{2}$ is what we call *future timelike infinity* and denote by i^+. Physically it is the place where massive particles asymptote to in the far future. The same applies to $\widetilde{v} = -\frac{\pi}{2}$ that is a point i^- called *past timeline infinity*, identified as the place where massive particles come from. Finally, we also have $\widetilde{u} = -\frac{\pi}{2}$ and $\widetilde{v} = \frac{\pi}{2}$, which is also a point i^0 called *spacelike infinity*, corresponding to far away distances at a fixed time, in other words, $r \to \infty$ with constant t.

In summary the Penrose conformal compactification tells us that Minkowski spacetime can be conformally embedded into one unphysical spacetime whose metric is determined only up to conformal transformations and inside of which we can identify its asymptotic boundary which is comprised of two null surfaces $\mathcal{I}^\pm$ corresponding to the places where massless particles come from and go to, two points $i^\pm$ corresponding to places where massive particles come from and go to, and a point i^0 corresponding to far away places in space.

2.3.2 Coordinates near $\mathcal{I}^\pm$

Instead of directly employing the Penrose conformal compactification procedure and working in the unphysical spacetime it is common to work directly in Minkowski spacetime and treat $\mathcal{I}^\pm$ by taking limits. In this section, we present two commonly employed coordinate systems in this analysis, the retarded and advanced coordinates. These coordinates are then generalized by the definition of Bondi gauge coordinates in the next section when gravity is present.

We start with retarded coordinates $(u, r, z, \bar{z})$, useful near $\mathcal{I}^+$, in which the inertial time t is traded by retarded time u and on which the metric takes the form

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{2.16}$$

The transformation between Cartesian coordinates x^μ and retarded coordinates is given by

$$t = u + r, \quad x^1 + ix^2 = \frac{2rz}{1 + z\bar{z}}, \quad x^3 = \frac{r(1 - z\bar{z})}{1 + z\bar{z}}, \tag{2.17}$$

and $\gamma_{z\bar{z}}$ is the only non-vanishing component of the S^2 round metric in the $(z, \bar{z})$ coordinates, $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$.

As we have already observed, in the Penrose conformal compactification picture, at the two ends of $\mathcal{I}^+$ there are the points i^+ and i^0. We shall define $\mathcal{I}^+_\pm$ by taking a point $(u, z, \bar{z})$ at $\mathcal{I}^+$ and taking the limit $u \to \pm\infty$. This gives us two spheres, which characterize the way one may approach i^+ and i^0 from within $\mathcal{I}^+$.

Near $\mathcal{I}^-$ on the other hand we shall employ the advanced coordinates $(v, r, z, \bar{z})$ in which the metric takes the form

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \tag{2.18}$$

and whose transformation to Cartesian coordinates is given by

$$t = v - r, \quad x^1 + ix^2 = -\frac{2rz}{1 + z\bar{z}}, \quad x^3 = -\frac{r(1 - z\bar{z})}{1 + z\bar{z}}. \tag{2.19}$$

In that scenario $\mathcal{I}^-$ is reached fixing $(v, z, \bar{z})$ and taking $r \to \infty$. As we we did with $\mathcal{I}^+$ we define the boundaries $\mathcal{I}^-_\pm$ obtained by fixing $(z, \bar{z})$ and taking $v \to \pm\infty$. In that sense, $\mathcal{I}^-_+$ is what we get approaching i^0 from within $\mathcal{I}^-$ and $\mathcal{I}^-_-$ is what we get approaching i^- from within $\mathcal{I}^-$.

It is important to observe that the $(z, \bar{z})$ coordinates in the retarded chart $(u, r, z, \bar{z})$ and $(z, \bar{z})$ coordinates in the advanced chart $(v, r, z, \bar{z})$ are not the same coordinates on S^2, but are rather *antipodally matched*. This is a convention taken for convenience to simplify formulas and is connected to our observation in the last section that $\mathcal{I}^+$ and $\mathcal{I}^-$ are antipodally matched across i^0 giving rise to a single celestial sphere $\mathcal{CS}^2$. Put differently the two charts are related by the product of parity and time reversal elements of the Lorentz group PT.

2.4 Asymptotically Flat Spacetimes

2.4.1 Asymptotically Flat Gravity

Now that we have elucidated the asymptotic structure of Minkowski spacetime we consider asymptotically flat spacetimes, in which we have gravity. The intuitive idea is that asymptotically flat spacetimes should have the same asymptotic structure as Minkowski spacetime, but can be totally different in their interior, in particular having non-trivial topology. Asymptotically flat spacetimes can be defined either in the gauge-fixing approach, following the work of Bondi, van der Burg, Metzner and Sachs [64–66] or in the geometrical formulation of Penrose. Here we review the definition in the gauge-fixing approach. In that case one first defines a Bondi coordinate system to be a chart (u, r, x^A) such that

$$g_{rr} = g_{rA} = 0, \quad \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0. \tag{2.20}$$

When these conditions are met we also say that the metric is in Bondi gauge. A spacetime is said to be asymptotically flat at future null infinity $\mathcal{I}^+$ when there exists an open set in which the metric can be put in Bondi gauge by choosing coordinates (u, r, x^A) in which r is unbounded from above, x^A parameterize a complete S^2, and where the metric takes the form

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du) \tag{2.21}$$

with the functions $\beta(u, r, x^A)$, $V(u, r, x^A)$, $g_{AB}(u, r, x^A)$ and $U^A(u, r, x^A)$ obeying the falloff conditions:

$$\frac{V(u, x^A)}{r} = -1 + O(r^{-1}), \tag{2.22}$$

$$\beta(u, x^A) = O(r^{-2}) \tag{2.23}$$

$$U_A(u, x^A) = O(r^{-2}), \tag{2.24}$$

$$g_{AB}(u, x^A) = r^2\gamma_{AB} + O(r), \tag{2.25}$$

Likewise, it is said to be asymptotically flat at past null infinity $\mathcal{I}^-$ when there is a region with Bondi coordinates (v, r, x^A) in which r is unbounded from above, with the metric taking the form:

$$ds^2 = e^{2\beta^-}\frac{V^-}{r}dv^2 + 2e^{2\beta^-}dvdr + g_{AB}(dx^A - U^{-A}du)(dx^B - U^{-B}du). \quad (2.26)$$

We shall consider spacetimes that are asymptotically flat at both future and past null infinities, so that both $\mathcal{I}^+$ and $\mathcal{I}^-$ can be defined. Henceforth we are going to focus on $\mathcal{I}^+$ with the understanding that all we say has a parallel at $\mathcal{I}^-$. We shall then discuss how these two are connected in the definition of the scattering problem in General Relativity.

The geometry is assumed to be sourced by a matter stress tensor $T^M_{\mu\nu}$ of massless matter assumed to obey the fall-off conditions near $\mathcal{I}^+$:

$$T^M_{uu} = O(r^{-2}), \quad T^M_{ur} = O(r^{-4}), \quad T^M_{rr} = O(r^{-4}), \quad (2.27)$$

$$T^M_{uA} = O(r^{-2}), \quad T^M_{rA} = O(r^{-3}), \quad T^M_{AB} = O(r^{-1}). \quad (2.28)$$

Imposing Einstein's equations the metric can be shown to have an expansion of the form [69]

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + r^2\gamma_{AB}dx^Adx^B \\ &\quad + \frac{2m_B}{r}du^2 + rC_{AB}dx^Adx^B + D^BC_{AB}dudx^A \\ &\quad + \frac{1}{r}\left[\frac{4}{3}(N_A + u\partial_A m_B) - \frac{1}{8}D_A(C_{BC}C^{BC})\right]dudx^A \\ &\quad + \cdots \end{aligned} \quad (2.29)$$

where $m_B(u,z,\bar{z})$, $C_{AB}(u,z,\bar{z})$ and $N_A(u,z,\bar{z})$ are parameters identifying the solution known as *Bondi mass aspect*, *gravitational shear* and *angular momentum aspect* and where D_A is the (S^2,γ) covariant derivative. We further define $N_{AB} = \partial_u C_{AB}$ the Bondi news tensor. The parameters $m_B(u,z,\bar{z})$ and $N_A(u,z,\bar{z})$ are further constrained by Einstein's equation to obey

$$\begin{aligned} \partial_u m_B &= \frac{1}{4} D^A D^B N_{AB} - T_{uu}, & (2.30) \\ \partial_u N_A &= -\frac{1}{4} D^B \left(D_B D^C C_{AC} - D_A D^C C_{BC} \right) - u \partial_u \partial_A m_B - T_{uA}, & (2.31) \end{aligned}$$

where T_{uu} and T_{uA} contain contributions from the matter source $T^M_{\mu\nu}$ and from the energy-momentum carried away by gravitational radiation as well

$$\begin{aligned} T_{uu} &= \frac{1}{8} N_{AB} N^{AB} + 4\pi G \lim_{r\to\infty} r^2 T^M_{uu}, & (2.32) \\ T_{uA} &= -\frac{1}{4} \partial_A (C_{BC} N^{BC}) + \frac{1}{4} D_B \left(C^{BC} N_{CA} \right) - \frac{1}{2} C_{AB} D_C N^{BC} + 8\pi G \lim_{r\to\infty} r^2 T^M_{uA}. & (2.33) \end{aligned}$$

We therefore see that prescribing the Bondi news $N_{AB}(u, z, \bar{z})$ together with the initial values of $m_B(u, z, \bar{z})$, $N_A(u, z, \bar{z})$ and $C_{AB}(u, z, \bar{z})$ as $u \to -\infty$ determines the metric components near $\mathcal{I}^+$. Indeed, knowing N_{AB} we may integrate $\partial_u C_{AB} = N_{AB}$ and determine C_{AB} up to its initial value while knowing N_{AB} and C_{AB} together with the matter sources, the constraint equations determine $m_B(u, z, \bar{z})$ and $N_A(u, z, \bar{z})$ up to their initial values. Notice, in particular, that owing to the Bondi gauge conditions, the gravitational shear is a traceless field, i.e., $\gamma^{AB} C_{AB} = 0$, and as a consequence so is N_{AB}. The field N_{AB}, therefore, encodes two degrees of freedom corresponding to the two polarizations of gravitons in four dimensions.

2.4.2 BMS and Extended BMS Algebras

With the specification of the theory given in the last section, we turn to its asymptotic symmetries. The gauge transformations of the theory are diffeomorphisms generated by spacetime vector fields ξ^μ and under which the metric varies by its Lie derivative $\delta_\xi g = \mathcal{L}_\xi g$. The asymptotic symmetries of the theory are therefore characterized by all the vector fields ξ which preserve the Bondi gauge condition (2.20) and the falloff conditions (2.22), i.e.,

$$\begin{aligned} \mathcal{L}_\xi g_{rr} &= \mathcal{L}_\xi g_{rA} = 0, \quad g^{AB}\mathcal{L}_\xi g_{AB} = 0, && (2.34)\\ \mathcal{L}_\xi g_{uu} &= O(r^{-1}), \quad \mathcal{L}_\xi g_{ur} = O(r^{-2}), \quad \mathcal{L}_\xi g_{uA} = O(1), \quad \mathcal{L}_\xi g_{AB} = O(r). && (2.35)\end{aligned}$$

The solutions to these conditions are vector fields $\xi_{(f,Y)}$ given by [69]

$$\begin{aligned} \xi_{(f,Y)} &= \left(\frac{u}{2}(D\cdot Y) + f\right)\partial_u + \left(-\frac{r}{2}(D\cdot Y) - \frac{u}{2}(D\cdot Y) + \frac{1}{2}D^2 f + O(r^{-1})\right)\partial_r \\ &\quad + \left(Y^A - \frac{D^A f + \frac{u}{2}D^A(D\cdot Y)}{r} + O(r^{-2})\right)\partial_A \end{aligned} \tag{2.36}$$

parameterized by an arbitrary function $f(z,\bar{z})$ on S^2 and a vector field $Y^A(z,\bar{z})$ on S^2 constrained to obey the conformal Killing equation

$$D_A Y_B + D_B Y_A = \gamma_{AB}(D\cdot Y). \tag{2.37}$$

It is useful to split the space of these vector fields in a subspace $\xi_f = \xi_{(f,0)}$ and another $\xi_Y = \xi_{(0,Y)}$. The vector fields ξ_f are the generators of supertranslations. The four rigid translations are obtained when f is taken to be:

$$f_0 = 1, \quad f_1 = \frac{z+\bar{z}}{1+z\bar{z}}, \quad f_2 = \frac{-i(z-\bar{z})}{1+z\bar{z}}, \quad f_3 = \frac{1-z\bar{z}}{1+z\bar{z}}. \tag{2.38}$$

On the other hand, it is known that the conformal Killing equation on S^2 admits both globally-defined solutions, which span the global conformal algebra $\mathfrak{sl}(2,\mathbb{C})$, and locally-defined solutions, which have point singularities, giving rise to two copies of the Witt algebra $\mathfrak{witt}_L \oplus \mathfrak{witt}_R$. Indeed in the $(z,\bar{z})$ coordinates, the conformal Killing equation reduces to

$$\partial_{\bar{z}} Y^z = 0, \quad \partial_z Y^{\bar{z}} = 0, \tag{2.39}$$

whose solutions are holomorphic and anti-holomorphic vector fields $Y^z(z)$ and $Y^{\bar{z}}(\bar{z})$. If the vector fields are allowed to have point singularities, the BMS asymptotic flatness boundary conditions are violated at these point singularities, and for that reason on the original BMS analysis, the vector fields Y have been demanded to be globally defined. In that case, the vector fields ξ_Y are the generators of Lorentz transformations, with the three rotations given by the choices

$$Y_{12} = -i(z\partial_z - \bar{z}\partial_{\bar{z}}), \quad Y_{23} = -i\frac{z^2-1}{2}\partial_z + i\frac{\bar{z}^2-1}{2}\partial_{\bar{z}}, \quad Y_{31} = -\frac{1+z^2}{2}\partial_z - \frac{1+\bar{z}^2}{2}\partial_{\bar{z}}, \tag{2.40}$$

and the three boosts given by the choices

$$Y_{01} = \frac{1-z^2}{2}\partial_z + \frac{1-\bar{z}^2}{2}\partial_{\bar{z}}, \quad Y_{02} = \frac{i(1+z^2)}{2}\partial_z - \frac{i(1+\bar{z}^2)}{2}\partial_{\bar{z}}, \quad Y_{03} = -z\partial_z - \bar{z}\partial_{\bar{z}}. \tag{2.41}$$

In [67] it was proposed that one should indeed drop the requirement that Y be globally defined, allowing for meromorphic vector fields with poles on the sphere. When that happens the vector fields ξ_Y generate superrotations, which we see to be an infinite-dimensional enhancement of the Lorentz transformation inasmuch as the supertranslations are an infinite-dimensional enhancement of the translations. We will see later that soft theorems indeed give the proper justification to consider this enlargement which is central to the celestial holography proposal. Another proposal, also motivated by soft theorems is to allow Y to generate arbitrary diffeomorphisms on S^2 [73, 74].

Regardless of the choice of which Y should be allowed, if one projects the vector field $\xi_{(f,Y)}$ to $\mathcal{I}^+$ the result is

$$\xi_{(f,Y)}|_{\mathcal{I}^+} = f\partial_u + \frac{1}{2}(D\cdot Y)u\partial_u + Y^A\partial_A \tag{2.42}$$

Using the Lie bracket at $\mathcal{I}^+$ one may then show that these vector fields obey the algebra

$$\begin{gathered}[\xi_f,\xi_{f'}]=0,\quad [\xi_Y,\xi_f]=\xi_{\hat{f}},\quad [\xi_Y,\xi_{Y'}]=\xi_{\hat{Y}},\\ \hat{f}=Y(f)-\frac{1}{2}(D\cdot Y)f,\quad \hat{Y}=[Y,Y'].\end{gathered} \tag{2.43}$$

When Y is demanded to be globally defined the resulting algebra is known as the BMS algebra $\mathfrak{bms}_4$, when Y is demanded just to be meromorphic with possible point singularities the resulting algebra is known as the extended BMS algebra $\mathfrak{ebms}_4$, and when Y is an arbitrary diffeomorphism generator the resulting algebra is known as the generalized BMS algebra $\mathfrak{gbms}_4$. Henceforth we are going to focus on the extended BMS algebra.

Having the spacetime diffeomorphisms ξ_f and ξ_Y we can study the variation of the gravitational data by acting on the metric through the Lie derivative. The supertranslation action on gravitational data is [75]

$$\delta_f C_{AB} = f\partial_u C_{AB} + (-2D_AD_B + \gamma_{AB}D^2)f, \tag{2.44}$$

$$\delta_f N_{AB} = f\partial_u N_{AB} \tag{2.45}$$

while the superrotation action is [75]

$$\delta_Y C_{AB} = \left[\frac{(D\cdot Y)}{2}(u\partial_u - 1) + \mathcal{L}_Y\right]C_{AB} + \frac{u}{2}(-2D_AD_B + \gamma_{AB}D^2)(D\cdot Y), \tag{2.46}$$

$$\delta_Y N_{AB} = \left[\frac{(D\cdot Y)}{2}u\partial_u + \mathcal{L}_Y\right]C_{AB} + \frac{1}{2}(-2D_AD_B + \gamma_{AB}D^2)(D\cdot Y) \tag{2.47}$$

Following the Covariant Phase Space prescription one may then derive the surface charges associated with the extended BMS transformations that implement the above symmetry actions through the Poisson brackets in phase space[1]. What one finds is [14]

$$Q^+(f) = \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\sqrt{\gamma} f m_B, \tag{2.48}$$

$$Q^+(Y) = \frac{1}{8\pi G}\int_{\mathcal{I}^+_-} d^2z\sqrt{\gamma} Y^A N_A. \tag{2.49}$$

2.4.3 Boundary Conditions at $\mathcal{I}^+_\pm$ and $\mathcal{I}^-_\pm$

To have a well-defined phase space in the classical theory, in which the charges $Q^\pm(f)$ and $Q^\pm(Y)$ generate supertranslations and superrotations through the Poisson bracket, one also imposes boundary conditions at $u \to \pm\infty$ and $v \to \pm\infty$. The conditions considered in [11] are the *Christodoulou-Kleinerman* (CK) boundary conditions characterized by $N_{AB} \sim O(|u|^{-1-\epsilon})$ and $N_{AB} \sim O(|v|^{-1-\epsilon})$ for $\epsilon > 0$ at $\mathcal{I}^+$ and $\mathcal{I}^-$, together with m_B and N_A finite as $u \to \pm\infty$ and $v \to \pm\infty$. One also imposes the conditions that $m_B|_{\mathcal{I}^-_-} = m_B|_{\mathcal{I}^+_+} = 0$ and that $N_A|_{\mathcal{I}^-_-} = N_A|_{\mathcal{I}^+_+} = 0$. These boundary conditions have been motivated by the analysis of Christodoulou and Kleinerman of the non-linear stability of Minkowski spacetime, in which it was shown that it exists solutions to the Einstein equations with appropriate matter sources which give rise to geodesically complete solutions verifying these properties [76].

Assuming that $\partial_u m_B$ and $\partial_u N_A$ are also finite at large $|u|$, the finiteness of m_B and N_A in this limit implies that actually $\partial_u m_B \to 0$ and $\partial_u N_A \to 0$ at large $|u|$. In that case, the CK boundary conditions together with the constraint equation (2.31) imply that

$$D^B\left(D_B D^C C_{AC} - D_A D^C C_{BC}\right)|_{\mathcal{I}^+_\pm} = 0 \tag{2.50}$$

[1]The derivation is subtle, however, because the charges are non-integrable and because for superrotations the charge diverges. A renormalization is thus necessary together with a prescription to select an integrable part [69, 71].

and this equation in turn implies that [69]

$$C_{AB}|_{\mathcal{I}^+_\pm} = (-2D_A D_B + \gamma_{AB} D^2) C_\pm, \tag{2.51}$$

where $C_\pm(x^A)$ are two scalar functions on the sphere parameterizing the boundary values of the shear. We may equivalently trade $C_\pm(x^A)$ by $C(x^A) = C_-(x^A)$ and the shift $\Delta C = C_+(x^A) - C_-(x^A)$. It is then clear that a supertranslation $f(x^A)$ acts on these variables by shifting the field at $\mathcal{I}^+_-$, namely $C(x^A)$:

$$C(x^A) \to C(x^A) + f(x^A). \tag{2.52}$$

With these boundary conditions, one obtains a phase space in which the action of the supertranslation charges is well-defined and generates the correct symmetry action through the Poisson bracket. The action of superrotations, however, is not well-defined in this phase space. This can already be seen from the fact that $\delta_Y C_{AB}$ has a term that is linear in u. As a result, configurations in the phase space defined by CK boundary conditions are moved out of this phase space by superrotations [14]. To have a phase space in which superrotations can be defined demands altering these boundary conditions so that the news tensor does not vanish at $\mathcal{I}^+_\pm$, but rather diverges linearly in u. See [71,75] for developments in this direction.

2.4.4 Scattering Problem and Charge Conservation

The scattering problem in gravity can be described in the classical theory in finding a map between the phase space at $\mathcal{I}^-$ to the phase space at $\mathcal{I}^+$ mapping in configurations to out configurations. In the quantum theory, which we discuss in the next section, we instead seek to define one $\mathcal{S}$-matrix mapping the in Hilbert space defined at $\mathcal{I}^-$ to the out Hilbert space defined at $\mathcal{I}^+$.

The key insight due to Strominger is that in the absence of an extra boundary condition linking data at $\mathcal{I}^-$ and data at $\mathcal{I}^+$ the scattering problem is, in fact, ill-defined, and that one such simple boundary condition can be proposed by demanding it to be compatible with Lorentz and CPT symmetries [11]. The reason is that the gravitational data at $\mathcal{I}^\pm$ is defined in terms of one specific BMS frame. If we choose some BMS frame at $\mathcal{I}^-$ and specify the in data, the Einstein equations only determine the out data up to BMS transformations acting at $\mathcal{I}^+$.

The proposal in [11] was to antipodally identify data at $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$, motivated by the fact that it respects Lorentz and CPT symmetries:

$$m_B|_{\mathcal{I}^+_-}(\Omega) = m_B|_{\mathcal{I}^-_+}(-\Omega), \tag{2.53}$$

$$C_{AB}|_{\mathcal{I}^+_-}(\Omega) = C_{AB}|_{\mathcal{I}^-_+}(-\Omega), \tag{2.54}$$

$$N_A|_{\mathcal{I}^+_-}(\Omega) = N_A|_{\mathcal{I}^-_+}(-\Omega). \tag{2.55}$$

where $\Omega \in S^2$. This proposal is further *a posteriori* justified by the fact that supertranslation and superrotation symmetries of the gravitational $\mathcal{S}$-matrix follow from the leading and subleading soft graviton theorems. At the classical level, from the expression of the charges as integrals over $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ it is clear that this proposal leads to charge conservation in the scattering problem once the symmetries are demanded to preserve this condition.

2.5 Ward Identities of Asymptotic Symmetries

2.5.1 Quantum Scattering Preliminaries

All the analysis done so far pertains to a classical theory. We now consider the quantum version, by connecting the formalism of asymptotic symmetries, where massless fields are studied through their data at $\mathcal{I}^\pm$, to the formalism of scattering in QFT. In this section we briefly review how scattering amplitudes are calculated through the LSZ prescription in QFT, establishing notation and conventions in doing so. Notice that in this section we assume the standard setting of QFT in which fields vanish at infinity.

The possible one-particle Hilbert spaces of relativistic particles are given by Wigner's classification [77]. For a massless particle of spin s, this state space is spanned by states $|p,\ell\rangle$ where p is the null four-momenta, $p^2=0$, $p^0>0$, and $\ell=\pm s$ is the particle's helicity. From the one-particle Hilbert spaces one can then construct the many-particle Fock spaces with basis states

$$|p_1,\ell_1;\dots;p_n,\ell_n\rangle = a^\dagger_{\ell_1}(p_1)\cdots a^\dagger_{\ell_n}(p_n)|0\rangle. \tag{2.56}$$

Free states are normalized according to the covariant normalization convention in which the creation and annihilation operators obey the oscillator algebra in the form

$$[a_\ell(p),a^\dagger_{\ell'}(p')] = (2\pi)^3 2p^0\delta_{\ell\ell'}\delta^{(3)}(\vec{p}-\vec{p}\,'). \tag{2.57}$$

The spaces of in/out scattering states $\mathcal{H}_{\text{in/out}}\subset\mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of the interacting theory of interest, are related to the free Fock spaces by the isomorphism established by Møller operators $\Omega_\pm$ [77]:

$$|p_1,\ell_1;\dots;p_n,\ell_n;\pm\rangle = \Omega_\pm|p_1,\ell_1;\dots;p_n,\ell_n\rangle,\quad \Omega_\pm\equiv\lim_{t\to\pm\infty}e^{iHt}e^{-iH_0t}, \tag{2.58}$$

where H_0 and H are respectively the free and interacting Hamiltonians and the label $\pm$ indicates the out/in states. The overlap between an element of $\mathcal{H}_{\text{in}}$ and one element of $\mathcal{H}_{\text{out}}$ defines the $\mathcal{S}$-matrix, which can be written as a matrix element of a certain $\mathcal{S}$-operator between free states

$$\langle p_1',\ell_1';\dots;p_n',\ell_n'|\mathcal{S}|p_1,\ell_1;\dots;p_m,\ell_m\rangle = \langle p_1',\ell_1';\cdots;p_n',\ell_n';+|p_1,\ell_1;\dots;p_m,\ell_m;-\rangle \tag{2.59}$$

where $\mathcal{S} = \Omega_+^\dagger \Omega_-$. The way in which the $\mathcal{S}$-matrix is then evaluated in Quantum Field Theory is by means of the LSZ reduction formula. The starting point is to rewrite the scattering amplitudes in terms of creation and annihilation operators

$$\langle p_i', \ell_i'; +|p_i, \ell_i; -\rangle = \langle \Omega| \prod_{i=1}^{n} a_{\text{out}}(p_i', \ell_i') \prod_{i=1}^{n} a_{\text{in}}^\dagger(p_i, \ell_i)|\Omega\rangle, \tag{2.60}$$

where we assume the same in/out vacuum $|\Omega\rangle$. This is the standard assumption in the LSZ derivation in consonance with the QFT assumption that fields vanish at i^0. When this assumption is dropped and fields are allowed to be non-trivial at i^0, their boundary values at $\mathcal{I}_-^+$ and $\mathcal{I}_+^-$ parameterize out and in vacua, which become distinct. This has important implications in the context of IR divergences [25, 78].

One then observes that any free or in/out creation/annihilation operators can be embedded into quantum fields transforming on specific representations of the universal cover of the Lorentz group:

$$\Phi_{\text{in/out}}(x) = \sum_{\ell} \int_{H_0^+} \frac{d^3p}{(2\pi)^3 2\omega_p} \left(\varepsilon_\ell^*(p) a_\ell^{\text{in/out}}(p) e^{ipx} + \varepsilon_\ell(p) a_\ell^{\text{in/out}}(p)^\dagger e^{-ipx} \right), \tag{2.61}$$

where $\varepsilon_\ell(p)$ are the polarization vectors/tensors/spinors, which take values in the representation space of $\Phi_{\text{in/out}}(x)$. One then relates $\Phi_{\text{in/out}}(x)$ to the full bulk interacting fields by means of the scattering assumption: in the asymptotic regions of spacetime, the bulk field should become free

$$\Phi(x) \to \sqrt{Z}\Phi_{\text{in/out}}(x), \quad \text{as } t \to \pm\infty \tag{2.62}$$

where Z is the wavefunction renormalization [79]. The operators $a_\ell^{\text{out}}(p)$ and $a_\ell^{\text{in}}(p)^\dagger$ can then be extracted directly from $\Phi(x)$ by taking an appropriate inner product with a corresponding position space wavefunction living in the space of positive-frequency solutions to the linearized field equations. To be more precise, we consider the cases of spin zero, one, and two. In these cases we have the following inner products:

$$(\phi, \phi') = -i \int_\Sigma d\Sigma^\mu \left(\phi \nabla_\mu \phi'^* - \phi'^* \nabla_\mu \phi\right), \tag{2.63}$$

$$(A, A') = -i \int_\Sigma d\Sigma^\mu \left[A^\nu \left(\nabla_\mu A'^*_\nu - \nabla_\nu A'^*_\mu\right) - A'^\nu \left(\nabla_\mu A^*_\nu - \nabla_\nu A^*_\mu\right)\right], \tag{2.64}$$

$$(h, h') = -i \int_\Sigma d\Sigma^\rho \left[h^{\mu\nu} \left(\nabla_\rho h'^*_{\mu\nu} - 2\nabla_\mu h'^*_{\rho\nu}\right) - h'^{\mu\nu} \left(\nabla_\rho h^*_{\mu\nu} - 2\nabla_\mu h'^*_{\rho\nu}\right)\right], \tag{2.65}$$

where Σ is a Cauchy surface. These inner products are constructed from the symplectic structure $\boldsymbol{\Omega}$ whose construction we outlined in section 2.2 [80]. In that regard, in the same way as $\boldsymbol{\Omega}$, they can be shown to be independent of the choice of Cauchy surface.

We can then extract $a^{\text{out}}_\ell(p)$ and $a^{\text{in}}_\ell(p)^\dagger$ by taking the inner product with wavefunctions that are plane waves dressed with the appropriate polarizations:

$$a^{\text{in}}_\ell(p)^\dagger = -(\varepsilon_\ell(p) e^{-ipx}, \Phi_{\text{in}}), \quad a^{\text{out}}_\ell(p) = (\varepsilon^*_\ell(p) e^{ipx}, \Phi_{\text{out}}). \tag{2.66}$$

Since the inner products are then independent of the Cauchy surface used in their evaluation, one may push that surface to the asymptotic regions of spacetime, where (2.62) is assumed to hold and then one can trade $\Phi_{\text{in/out}}$ in these inner products by the full bulk interacting field Φ. As a result, one may eliminate each $a^{\text{in}}_\ell(p)^\dagger$ and each $a^{\text{out}}_\ell(p)$ in (2.60) in favor of the interacting bulk field using this procedure. This leads to the LSZ reduction formula:

$$\langle p'_i, \ell'_i; +|p_i, \ell_i, -\rangle = \left(\prod_i \prod_j \int d^4x_i d^4x_j \varepsilon^*_{\ell_j}(p_j) e^{ip'_j \cdot x'_j} \varepsilon_{\ell_i}(p_i) e^{-ip_i \cdot x_i}\right) \mathrm{C}(x_i, x'_j), \tag{2.67}$$

where $\mathrm{C}(x_1, \ldots, x_n)$ is the connected component of the interacting time-ordered correlator $\langle \Omega | T\{\Phi(x_1) \cdots \Phi(x_n)\} | \Omega \rangle$ with external legs removed, and where we are suppressing the Lorentz representation indices in both the polarizations and the time-ordered correlator, which are assumed to be appropriately contracted with one another.

We are going to parameterize massless momenta in terms of variables $(\omega, z, \bar{z})$ as

$$p^\mu(\omega, z, \bar{z}) = \eta\omega\hat{q}^\mu(z, \bar{z}), \tag{2.68}$$

where $\eta = +1$ for outgoing particles and $\eta = -1$ for incoming ones, ω is the energy and $\hat{q}^\mu(z, \bar{z})$ is the embedding of S^2 on the lightcone of $\mathbb{R}^{1,3}$ given by

$$\hat{q}^\mu(z, \bar{z}) = (1, \Omega(z, \bar{z})), \tag{2.69}$$

$$\Omega(z, \bar{z}) = \left(\frac{z + \bar{z}}{1 + z\bar{z}}, \frac{-i(z - \bar{z})}{1 + z\bar{z}}, \frac{1 - z\bar{z}}{1 + z\bar{z}}\right). \tag{2.70}$$

On the other hand, in the integer spin case, to parameterize the polarization tensors we first define the polarization vectors $\varepsilon^\alpha_\mu(p)$ by

$$\varepsilon^+(p(\omega, z, \bar{z})) = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \quad \varepsilon^-(p(\omega, z, \bar{z})) = \frac{1}{\sqrt{2}}(z, 1, i, -z), \tag{2.71}$$

and then we define $\varepsilon^\pm_{\mu\nu} = \varepsilon^\pm_\mu \varepsilon^\pm_\nu$. We shall then use the shorthand notation

$$\varepsilon^\ell(z, \bar{z}) = \varepsilon^\ell(p(\omega, z, \bar{z})), \tag{2.72}$$

$$a_\ell(\omega, z, \bar{z}) = a_\ell(p(\omega, z, \bar{z})), \quad a_\ell(\omega, z, \bar{z})^\dagger = a_\ell(p(\omega, z, \bar{z}))^\dagger. \tag{2.73}$$

2.5.2 Soft Theorems and Asymptotic Symmetries

In the quantum theory, the statement of supertranslation and superrotation symmetry of the scattering problem can be written as

$$\langle \text{out}| Q^+(f, Y)\mathcal{S} - \mathcal{S}Q^-(f, Y)|\text{in}\rangle = 0, \tag{2.74}$$

where by $Q^{\pm}(f,Y)$ we mean either $Q^{\pm}(f)$ or $Q^{\pm}(Y)$. We are going to call these equations the *asymptotic symmetry Ward identities* in accordance to the terminology employed in the literature[2]. We are now going to review the argument by which one shows these Ward identities are implied by the leading and subleading soft graviton theorems. To shorten some equations we are going to introduce the notation

$$:Q(f,Y)\mathcal{S}: \;\equiv\; Q^{+}(f,Y)\mathcal{S} - \mathcal{S}Q^{-}(f,Y). \tag{2.75}$$

The starting point is to rewrite $Q^{+}(f)$ and $Q^{+}(Y)$ as integrals over $\mathcal{I}^{+}$, using the constraint equations for $\partial_u m_B$ and $\partial_u N_A$, and observing that the boundary conditions imply in particular that $m_B|_{\mathcal{I}^+_+} = 0$ and $N_A|_{\mathcal{I}^+_+} = 0$:

$$\begin{aligned} Q^{+}(f) &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}}(D_z^2 fN^{zz} + D_{\bar{z}}^2 fN^{\bar{z}\bar{z}}) \\ &\quad +\frac{1}{4\pi G}\int_{\mathcal{I}^+} dud^2z\sqrt{\gamma}fT_{uu}, \end{aligned} \tag{2.76}$$

$$\begin{aligned} Q^{+}(Y) &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}}(D_z^3 Y^z uN^{zz} + D_{\bar{z}}^3 Y^{\bar{z}} uN^{\bar{z}\bar{z}}) \\ &\quad +\frac{1}{8\pi G}\int_{\mathcal{I}^+} dud^2z\sqrt{\gamma}\left[Y^z(T_{uz} + u\partial_z T_{uu}) + Y^{\bar{z}}(T_{u\bar{z}} + u\partial_{\bar{z}} T_{uu})\right] \end{aligned} \tag{2.77}$$

The first term in both $Q^{+}(f)$ and $Q^{+}(Y)$ depends only on the news tensor N_{AB}, is linear in the fields and is called the *soft term*, denoted respectively $Q_S^{+}(f)$ and $Q_S^{+}(Y)$, whereas the second term is quadratic in the fields, includes dependence on matter fields, and is called the *hard term*, denoted respectively $Q_H^{+}(f)$ and $Q_H^{+}(Y)$. Under this splitting the Ward identities take the form of a balance between soft and hard contributions:

$$\langle \text{out}|:Q_S(f,Y)\mathcal{S}:|\text{in}\rangle = -\langle \text{out}|:Q_H(f,Y)\mathcal{S}:|\text{in}\rangle. \tag{2.78}$$

Since the news tensor is the $\mathcal{I}^{+}$ data of the gravitational field, it can be related to the

[2] This should not be confused with the Ward-Takahashi identity encountered in QFT.

creation and annihilation operators of outgoing gravitons given that the bulk gravitational field in the asymptotic region becomes free and can be matched to the in/out fields according to (2.62)[3]. The precise relation is that the Fourier modes of the news tensor,

$$N^{\omega}_{AB}(z,\bar{z}) = \int_{-\infty}^{\infty} du N_{AB}(u,z,\bar{z})e^{i\omega u}. \tag{2.79}$$

are given by [12]

$$N^{\omega}_{zz}(z,\bar{z}) = -\frac{\kappa\omega a_{\text{out}}(\omega,z,\bar{z},+2)}{2\pi(1+z\bar{z})^2}, \tag{2.80}$$

$$N^{-\omega}_{zz}(z,\bar{z}) = -\frac{\kappa\omega a^{\dagger}_{\text{out}}(\omega,z,\bar{z},-2)}{2\pi(1+z\bar{z})^2}, \tag{2.81}$$

where $\omega > 0$ is assumed and where $\kappa^2 = 32\pi G$. The zero mode is defined by an averaging procedure so that the resulting mode is hermitian in the quantum theory:

$$N^{0}_{zz} = \lim_{\omega\to 0^+} \frac{1}{2}[N^{\omega}_{zz} + N^{-\omega}_{zz}]. \tag{2.82}$$

[3]When allowing for non-trivial fields at $\mathcal{I}^+_\pm$ it is necessary to be careful about relation (2.62). The reason is that the interacting field $\mathcal{I}^+$ data need not have boundary values at $\mathcal{I}^+_+$ and $\mathcal{I}^+_-$ summing to zero, whereas a field possessing a Fourier transform, such as the out field, does indeed satisfy this property. A resolution is to split $\mathcal{I}^+$ data as

$$\phi(u,z,\bar{z}) = \widehat{\phi}(u,z,\bar{z}) + \chi(z,\bar{z})$$

where $\widehat{\phi}(u,z,\bar{z})$ has a Fourier transform and $\chi(z,\bar{z}) = \frac{1}{2}[\phi(\infty,z,\bar{z}) + \phi(-\infty,z,\bar{z})]$. Then $\partial_u\phi(u,z,\bar{z})$ always has a Fourier transform and can be matched to the corresponding $\mathcal{I}^+$ data constructed from the out field. For gravity, in particular, this means that we should match the news tensors of the bulk interacting field and the out field. More generally, (2.62) should be understood as a matching between the *radiative data* at $\mathcal{I}^\pm$ of interacting and in/out fields.

These relations imply that $Q_S^+(f)$ and $Q_S^+(Y)$ depend respectively on the following modes of the news tensor

$$\begin{aligned} N_{zz}^{(0)+} &\equiv \int_{-\infty}^{\infty} du N_{zz} \\ &= -\frac{\kappa}{4\pi(1+z\bar{z})^2} \lim_{\omega\to 0} \omega \left[a_-^{\text{out}}(\omega,z,\bar{z}) + a_+^{\text{out}}(\omega,z,\bar{z})^\dagger \right], \quad (2.83) \\ N_{zz}^{(1)+} &\equiv \int_{-\infty}^{\infty} du u N_{zz} \\ &= \frac{i\kappa}{4\pi(1+z\bar{z})^2} \lim_{\omega\to 0} \partial_\omega \left[\omega \left(a_-^{\text{out}}(\omega,z,\bar{z}) - a_+^{\text{out}}(\omega,z,\bar{z})^\dagger \right) \right], \quad (2.84) \end{aligned}$$

with analogous results for $N_{\bar{z}\bar{z}}^{(0)+}$ and $N_{\bar{z}\bar{z}}^{(1)+}$ and for the various $N_{AB}^{(0)-}$ and $N_{AB}^{(1)-}$ at $\mathcal{I}^-$. In turn, invoking crossing symmetry, this means that the soft side of the supertranslation Ward identity reads [12]

$$\langle \text{out}| : Q_S(f)\mathcal{S} : |\text{in}\rangle = -\frac{1}{2\pi\kappa} \int d^2z D_z^2 f \lim_{\omega\to 0} \omega \langle \text{out}| a_-^{\text{out}}(\omega,z,\bar{z})\mathcal{S}|\text{in}\rangle, \quad (2.85)$$

whereas the soft side of the superrotation Ward identity reads [27]

$$\langle \text{out}| : Q_S(Y)\mathcal{S} : |\text{in}\rangle = -\frac{i}{2\pi\kappa} \int d^2z D_z^3 Y^z \lim_{\omega\to 0} \partial_\omega \left(\omega \langle \text{out}| a_-^{\text{out}}(\omega,z,\bar{z})\mathcal{S}|\text{in}\rangle \right). \quad (2.86)$$

Likewise, the hard contributions are constructed from the news tensor and the $\mathcal{I}^+$ radiative data of the other massless fields in the theory. They can be connected to creation and annihilation operators of the corresponding outgoing particles in the same way as the news tensor is connected to outgoing gravitons in (2.80), (2.81) and (2.82). As a result, it is possible to show that for supertranslations we have [12]

$$-\langle \text{out}| : Q_H(f)\mathcal{S} : |\text{in}\rangle = \sum_i f(z_i,\bar{z}_i)\omega_i \langle \text{out}|\mathcal{S}|\text{in}\rangle, \quad (2.87)$$

and for superrotations we have [27]

$$-\langle\text{out}|\!:Q_H(Y)\mathcal{S}:\!|\text{in}\rangle = -i\sum_k \left(Y^{z_k}\partial_{z_k} + Y^{\bar{z}_k}\partial_{\bar{z}_k} + \mathfrak{h}_k D_{z_k}Y^{z_k} + \bar{\mathfrak{h}}_k D_{\bar{z}_k}Y^{\bar{z}_k}\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle, \tag{2.88}$$

where we have defined the following operators, with a suggestive notation which will be fully explained in the next section

$$\mathfrak{h}_i = \frac{\widehat{\Delta}_i + \ell_i}{2}, \quad \bar{\mathfrak{h}}_i = \frac{\widehat{\Delta}_i - \ell_i}{2}, \quad \widehat{\Delta}_i = -\omega_i\partial_{\omega_i}. \tag{2.89}$$

We thus notice that establishing the Ward identities of asymptotic symmetries demands us to study scattering amplitudes in which the outgoing state has an additional graviton whose energy is taken to be small and show that (2.85) reproduces (2.87) and (2.86) reproduces (2.88) . Fortunately, in the 1960's Weinberg established the leading soft graviton theorem, which exhibits the leading behavior of such amplitudes in the $\omega \to 0$ limit [81], while in the last decade, Strominger and Cachazo extended Weinberg's result to the sub-leading and sub-subleading orders in the small energy expansion, at tree level [26].

In general, soft graviton theorems state that an $\mathcal{S}$-matrix element with an additional outgoing or incoming graviton whose energy is taken to be sufficiently small, factorize into one soft term, admitting one Laurent expansion in the graviton energy, multiplied by the $\mathcal{S}$-matrix element without the graviton. The soft term can be an operator acting on the remaining $\mathcal{S}$-matrix element, as it happens for the subleading and sub-subleading soft graviton theorems. The soft graviton theorem with leading and subleading contributions can be written as

$$\langle\text{out}|a_\ell^{\text{out}}(p)\mathcal{S}|\text{in}\rangle = \left(S_0^{(\ell)} + S_1^{(\ell)} + O(\omega)\right)\langle\text{out}|\text{in}\rangle, \tag{2.90}$$

where the leading and subleading soft factors are

$$S_0^{(\ell)} = \frac{\kappa}{2}\sum_{k=1}^{n}\frac{p_k^\mu p_k^\nu \varepsilon^\ell_{\mu\nu}(p)}{p_k\cdot p}, \quad S_1^{(\ell)} = -\frac{i\kappa}{2}\sum_{k=1}^{n}\frac{\varepsilon^\ell_{\mu\nu}(p)p_k^\mu p_\lambda}{p_k\cdot p}\mathcal{J}_k^{\lambda\nu}, \tag{2.91}$$

and we have $S_0^{(\ell)} = O(\omega^{-1})$ and $S_1^{(\ell)} = O(1)$. Observe that we have

$$\lim_{\omega\to 0}\omega\langle \text{out}|a_-^{\text{out}}(\omega,z,\bar z)\mathcal{S}|\text{in}\rangle = S_0^{(-)}\langle\text{out}|\text{in}\rangle, \tag{2.92}$$

$$\lim_{\omega\to 0}\partial_\omega\left[\omega\langle \text{out}|a_-^{\text{out}}(\omega,z,\bar z)\mathcal{S}|\text{in}\rangle\right] = S_1^{(-)}\langle\text{out}|\text{in}\rangle, \tag{2.93}$$

so that the soft theorem allows for the explicit evaluation of the soft side of the Ward identities in terms of the soft factor.

To proceed we use the parameterization (2.68) of the momenta, so that we can show that the leading and subleading soft factors take the following form [28]

$$S_0^{(+)} = -\frac{\kappa(1+z\bar z)}{2\omega}\sum_k\frac{\omega_k(\bar z-\bar z_k)}{(z-z_k)(1+z_k\bar z_k)}, \tag{2.94}$$

$$S_0^{(-)} = -\frac{\kappa(1+z\bar z)}{2\omega}\sum_k\frac{\omega_k(z-z_k)}{(\bar z-\bar z_k)(1+z_k\bar z_k)}, \tag{2.95}$$

$$S_1^{(+)} = \frac{\kappa}{2}\sum_k\frac{(\bar z-\bar z_k)^2}{z-z_k}\left[\frac{2\bar{\mathfrak{h}}_k}{\bar z-\bar z_k}-\Gamma^{\bar z_k}_{\bar z_k\bar z_k}\bar{\mathfrak{h}}_k-\partial_{\bar z_k}+|s_k|\Omega_{\bar z_k}\right], \tag{2.96}$$

$$S_1^{(-)} = \frac{\kappa}{2}\sum_k\frac{(z-z_k)^2}{\bar z-\bar z_k}\left[\frac{2\mathfrak{h}_k}{z-z_k}-\Gamma^{z_k}_{z_kz_k}\mathfrak{h}_k-\partial_{z_k}+|s_k|\Omega_{z_k}\right], \tag{2.97}$$

where Γ^z_{zz}, $\Gamma^{\bar z}_{\bar z\bar z}$ are the Christoffel symbols of the Levi-Civita connection on S^2 with round metric γ_{AB} and where $\Omega_z = \frac{1}{2}\Gamma^z_{zz}$ parameterizes the associated spin connection [28].

It is now possible to evaluate the soft side of the Ward identities explicitly. We integrate by parts on $\mathcal{CS}^2$ transfering the covariant derivatives D_z^2 and D_z^3 to the soft factors. We then observe that

$$\gamma^{z\bar{z}} D_z^2 \left(\gamma_{z\bar{z}} S_0^{(-)}\right) = -2\pi\kappa \sum_k \omega_k \delta^{(2)}(z, z_k), \tag{2.98}$$

$$\gamma^{z\bar{z}} D_z^3 \left(\gamma_{z\bar{z}} S_1^{(-)}\right) = -2\pi\kappa \sum_k \left[\delta^{(2)}(z, z_k)\partial_{z_k} - D_z \delta^{(2)}(z, z_k)\mathfrak{h}_k\right]. \tag{2.99}$$

It is then immediate to evaluate the integral on the soft side of the Ward identities using the delta distributions and check explicitly that the hard side of the Ward identities are reproduced. As a result, the leading and subleading soft graviton theorems imply respectively on the supertranslation and superrotation symmetry of the gravitational $\mathcal{S}$-matrix[4].

2.5.3 Kac-Moody Current and Stress Tensor Ward Identities

The supertranslation and superrotation Ward identities are, by definition, statements of symmetries of the bulk gravitational theory. In this section, we show that they take a form that is reminiscent of statements of symmetry of two-dimensional conformal correlators, that being the starting point of Celestial Holography.

Starting with supertranslations, we are going to define a two-dimensional vector on the celestial sphere $\mathcal{P}_z^{\pm}$ to be the soft supertranslation charge for the choice of supertranslation parameter $f(w) = \frac{1}{z-w}$ where $(z, \bar{z})$ is a fixed point:

$$\mathcal{P}_z^{\pm} = Q_S^{\pm}(f), \quad f(w) = \frac{1}{z-w}, \tag{2.100}$$

[4]The leading soft graviton theorem is exact and has no loop corrections, but the subleading soft graviton theorem indeed has quantum corrections [82]. The implications of these corrections to the superrotation symmetry have been discussed in [83] with evidence that symmetry survives at loop level after renormalizing the generators

with the anti-holomorphic component defined similarly. One then may show by explicit computation

$$\mathcal{P}_z^{\pm} = \frac{8\pi}{\kappa^2}\gamma^{z\bar{z}}\partial_{\bar{z}}N_{zz}^{(0)\pm}. \tag{2.101}$$

Substituting this particular supertranslation on the supertranslation Ward identity we observe it reads

$$\langle \text{out}|: \mathcal{P}_z\mathcal{S} :|\text{in}\rangle = \sum_i \frac{\omega_i}{z - z_i}\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{2.102}$$

This takes the form of a U(1) Kac-Moody Ward identity for a current $(\mathcal{P}_z, \mathcal{P}_{\bar{z}})$ on a two-dimensional CFT on the sphere, where the current is inserted at $(z, \bar{z})$ and each of the charged operators inserted at $(z_i, \bar{z}_i)$ and have charges ω_i. Observe that just as in the proof that soft theorems imply in the Ward identities, after writing $\mathcal{P}_z^{\pm}$ explicitly in terms of graviton creation and annihilation operators we can invoke crossing symmetry in order to write everything just in terms of the outgoing field $\mathcal{P}_z^{+}$.

Likewise, we consider superrotations. Suppose we choose the superrotation parameter to be the holomorphic field $Y^w = \frac{1}{z-w}$ for some fixed point $(z, \bar{z})$. Define

$$\mathcal{T}_{zz}^{\pm} = 2iQ_S^{\pm}(Y), \quad (Y^w, Y^{\bar{w}}) = \left(\frac{1}{z-w}, 0\right), \tag{2.103}$$

with the anti-holomorphic component $\mathcal{T}_{\bar{z}\bar{z}}^{\pm}$ defined similarly. The corresponding superrotation Ward identity then takes the form [28]

$$\langle \text{out}|: \mathcal{T}_{zz}\mathcal{S} :|\text{in}\rangle = \sum_k \left[\frac{\mathfrak{h}_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z - z_k}\mathfrak{h}_k + \frac{1}{z-z_k}(\partial_{z_k} - |\ell_k|\Omega_{z_k})\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{2.104}$$

This equation takes the form of a stress tensor Ward identity for a stress tensor in a two-dimensional conformal field theory, where the stress tensor is inserted at $(z, \bar{z})$ and each primary operator is inserted at $(z_i, \bar{z}_i)$ and has operator-valued conformal weights $(\mathfrak{h}_i, \bar{\mathfrak{h}}_i)$. In particular, such primary operators have dimensions $\widehat{\Delta}_i = -\omega_i \partial_{\omega_i}$ and two-dimensional spins equal to the four-dimensional helicities ℓ_i.

2.6 Conformal Primary Bases and Celestial Correlators

2.6.1 Diagonalizing Dilatations on $\mathcal{CS}^2$

The subleading soft graviton theorem implies the superrotation Ward identity, which for a particular choice of superrotation parameter takes the form of a CFT$_2$ stress tensor Ward identity where the operators have operator-valued dimensions. This suggests recasting the $\mathcal{S}$-matrix in a basis in the asymptotic Hilbert spaces $\mathcal{H}_{\text{in/out}}$ diagonalizing these dimensions. To do so, in the one-particle state one must diagonalize the operator $-\omega\partial_\omega$. This in turn is accomplished by a Mellin transform:

$$|\Delta, z, \bar{z}, \ell\rangle \;=\; \int_0^\infty d\omega \omega^{\Delta-1} |\omega, z, \bar{z}, \ell\rangle, \tag{2.105}$$

so that the resulting state is an eigenstate of $\widehat{\Delta}$ with eigenvalue Δ. At the level of wavefunctions appearing in the LSZ prescription, the in/out scalar plane waves get mapped to

$$\varphi^{\eta}_{\Delta}(x; z, \bar{z}) = \frac{(i\eta)^{\Delta}\Gamma(\Delta)}{(-\hat{q}(z, \bar{z}) \cdot x + i\eta\epsilon)^{\Delta}}, \tag{2.106}$$

where the $i\epsilon$ prescription distinguishes in/out wavefunctions. When dressed with appropriate polarization vectors/tensors/spinors these are the position space wavefunctions corresponding to the states (2.105) which are obtained by diagonalizing the operator

$-\omega\partial_\omega$. These are examples of a more general class of wavefunctions called *conformal primary wavefunctions*.

Before giving the general definition of conformal primary wavefunctions let us review the connection between bulk Lorentz transformations and boundary conformal transformations. The key observation is the well-known fact that the global conformal group of $\mathbb{R}^{p,q}$ is isomorphic to $\mathrm{SO}(p+1,q+1)$. In particular, the global conformal group of the sphere S^{d-1}, which is the conformal compactification of $\mathbb{R}^{d-1}$, is isomorphic to $\mathrm{SO}(1,d)$.

In particular, the global conformal group of S^2 is the Möbius group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ known to be isomorphic to the Lorentz group $\mathrm{SO}(1,3)$. Geometrically, let there be given an embedding $\hat{q}(z,\bar{z})$ of the sphere into the ligthcone of $\mathbb{R}^{1,3}$. In the previous sections, we have chosen the round embedding

$$\hat{q}(z,\bar{z}) \;=\; (1,\Omega(z,\bar{z})), \tag{2.107}$$

which is used together with the standard advanced and retarded Bondi coordinates near $\mathcal{I}^\pm$. In this section, we are going to choose instead

$$\hat{q}(z,\bar{z}) \;=\; (1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1+z\bar{z}), \tag{2.108}$$

which effectively flattens the sphere to a plane, and is used together with flat null coordinates near $\mathcal{I}^\pm$, see for example [25] for an analysis of $\mathcal{I}^\pm$ in these coordinates in the context of non-abelian gauge theory theory.

The vector $\hat{q}(z,\bar{z})$ then connects bulk Lorentz transformations and bondary conformal transformations:

$$\hat{q}^\mu\left(z',\bar{z}'\right) \;=\; \left|\frac{\partial(z',\bar{z}')}{\partial(z,\bar{z})}\right|^{1/2} \Lambda^\mu{}_\nu \hat{q}^\nu(z,\bar{z}), \tag{2.109}$$

where $(z,\bar{z}) \to (z',\bar{z}')$ is a global conformal transformation on the sphere:

$$z' = \frac{az+b}{cz+d}, \quad \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \tag{2.110}$$

and $\Lambda \in \mathrm{SO}(1,3)$ is an associated Lorentz transformation by means of the isomorphism $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{SO}(1,3)$ [72, 84]. At the level of generators, the Lorentz generators $J_{\mu\nu}$ are reorganized into the global conformal generators $\{L_n, \bar{L}_n : n = -1,0,1\}$ of global conformal transformations on the sphere as [72, 84]

$$\begin{aligned}
L_{-1} &= \frac{1}{2}(-J_1 + iJ_2 + iK_1 + K_2), \quad \bar{L}_{-1} = \frac{1}{2}(J_1 + iJ_2 + iK_1 - K_2),\\
L_0 &= \frac{1}{2}(J_3 - iK_3), \quad \bar{L}_0 = \frac{1}{2}(-J_3 - iK_3),\\
L_1 &= \frac{1}{2}(J_1 + iJ_2 - iK_1 + K_2), \quad \bar{L}_1 = \frac{1}{2}(-J_1 + iJ_2 - iK_1 - K_2),
\end{aligned} \tag{2.111}$$

where we have split the Lorentz generators into rotations J_i and boosts K_i. Under these definitions the Lorentz algebra of $J_{\mu\nu}$ becomes equivalent to the $\mathfrak{sl}(2,\mathbb{C})$ algebra of the $L_n, \bar{L}_n$:

$$[L_n, L_m] = (n-m)L_{n+m}, \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}, \quad [L_n, \bar{L}_m] = 0. \tag{2.112}$$

After these preliminaries we are ready to define in generality the concept of *conformal primary wavefunctions*:

Definition: A *radiative conformal primary wavefunction* of dimension Δ and spin ℓ is a wavefunction $\Phi_{\Delta,\ell}(x;z,\bar{z})$ in the spin $s = |\ell|$ representation of the Lorentz group $D_s : \mathrm{Spin}(1,3) \to \mathrm{GL}(V)$ that satisfies the spin s linearized equations of motion and which transforms as a conformal quasi-primary operator of dimension Δ and 2D spin ℓ under

Lorentz transformations:

$$\Phi_{\Delta,\ell}\left(\Lambda^{\mu}{}_{\nu}x^{\nu};\frac{aw+b}{cw+d},\frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right)=(cw+d)^{\Delta+\ell}(\bar{c}\bar{w}+\bar{d})^{\Delta-\ell}D_s(\Lambda)\Phi_{\Delta,\ell}(x;w,\bar{w}). \quad (2.113)$$

The terminology *radiative* in the above definition means that the wavefunction solves the linearized field equations and has the 2D and 4D spins related by $s=|\ell|$. This is to be contrasted to *generalized* conformal primary wavefunctions that do not need to obey the field equations and can have $|\ell|<s$ [48]. When we just talk about conformal primary wavefunctions it is to be understood that we are talking about radiative ones.

Conformal primary wavefunctions have been introduced in [30,85], in which they were constructed and analyzed for massive and massless scalars, as well as gauge fields and metric perturbation. The construction was a posteriori extended to other spins [48]. To write these wavefunctions it is convenient to introduce a null tetrad $\{\ell,n,m,\bar{m}\}$ given by

$$\ell^{\mu} = \frac{\hat{q}^{\mu}}{-\hat{q}\cdot x}, \quad n^{\mu}=x^{\mu}+\frac{x^2}{2}\ell^{\mu}, \quad (2.114)$$

$$m^{\mu} = \epsilon^{+\mu}+(\epsilon^{+}\cdot x)\ell^{\mu}, \quad \bar{m}^{\mu}=\epsilon^{-\mu}+(\epsilon^{-}\cdot x)\ell^{\mu}, \quad (2.115)$$

that are null vectors satisfying $\ell\cdot n=-1$ and $m\cdot\bar{m}=1$. In terms of these quantities, the spin one and spin two conformal primary wavefunctions are

$$\begin{aligned} A^{\eta}_{\Delta,J=+1}&=m\varphi^{\eta}_{\Delta}, \quad A^{\eta}_{\Delta,J=-1}=\bar{m}\varphi^{\eta}_{\Delta},\\ h^{\eta}_{\Delta,J=+2}&=mm\varphi^{\eta}_{\Delta}, \quad h^{\eta}_{\Delta,J=-2}=\bar{m}\bar{m}\varphi^{\eta}_{\Delta}. \end{aligned} \quad (2.116)$$

These wavefunctions transform in the necessary manner and they can be shown to be gauge-equivalent to the Mellin transforms of plane waves dressed with polarization vectors and tensors [30]. The half-integer spin wavefunctions can be constructed by decomposing the null tetrad $\{\ell,n,m,\bar{m}\}$ into a spin frame, see e.g. [48] for the details and explicit wavefunctions. These are not the only possible conformal primary wavefunctions though, as it is possible to construct a different set of such wavefunctions by taking a two-dimensional shadow transform in the boundary variables $(z, \bar{z})$ [30].

In a conformal primary wavefunction $\Phi^{\pm}_{\Delta,\ell}(x; z, \bar{z})$ the quantities $(\Delta, z, \bar{z}, \ell)$ label the state of the asymptotic particle. In particular, $(z, \bar{z})$ labels the point at $\mathcal{CS}^2$ at which the particle enters or exits spacetime, thereby encoding the direction of its momentum, ℓ is the particle's helicity and Δ is a conformal dimension which replaces the energy eigenvalue and can be interpreted as a boost weight.

It is then possible to show that the Mellin transforms of plane waves form a basis of solutions to the linearized field equations of positive-frequency when the dimensions lie in the *principal series* $\Delta \in 1 + i\mathbb{R}$ [30]. More recently it has been shown that for wavefunctions that as a function of retarded time belongs to Schwartz space $\mathcal{S}(\mathbb{R})$ a set of conformal primary wavefunctions with integer dimensions also form a basis [86].

From conformal primary wavefunctions, one may extract the associated annihilation and creation operators. In the LSZ prescription, outgoing states are created by outgoing annihilation operators acting to the left, while incoming states are created by incoming creation operators acting to the right. These operators corresponding to conformal primary wavefunctions are denoted $\mathcal{O}^{\pm}_{\Delta,\ell}(z, \bar{z})$. We are also going to define the conformal weights

$$h = \frac{\Delta + \ell}{2}, \quad \bar{h} = \frac{\Delta - \ell}{2}, \tag{2.117}$$

and equivalently denote the operators by $\mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})$. Reorganizing the Lorentz generators into $\{L_n, \bar{L}_n\}$ as in (2.111) they act on these operators as

$$\begin{aligned}
[L_{-1}, \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})] &= \partial \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z}), \quad [\bar{L}_{-1}, \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})] = \bar{\partial}\mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z}),\\
[L_0, \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})] &= (h + z\partial)\mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z}), \quad [\bar{L}_0, \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})] = (\bar{h} + \bar{z}\bar{\partial})\mathcal{O}^{\pm}(z,\bar{z}),\\
[L_1, \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})] &= \left(2hz + z^2\partial_z\right)\mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z}), \quad [\bar{L}_1, \mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z})] = \left(2\bar{h}\bar{z} + \bar{z}^2\partial_{\bar{z}}\right)\mathcal{O}^{\pm}_{h,\bar{h}}(z,\bar{z}).
\end{aligned} \tag{2.118}$$

An operator transforming under the action of $\{L_n, \bar{L}_n\}$ like this is called a *quasi-primary operator*. Any operator generated from it by acting with L_{-1} and $\bar{L}_{-1}$ a certain number of times is then called a *descendant*. As such, Lorentz symmetry alone implies that creation and annihilation operators in a conformal primary basis transform as quasi-primary operators on the celestial sphere. Superrotation symmetry then implies that they actually behave as primary operators in a two-dimensional CFT [87].

In the case of gravitons, we denote the corresponding primary operator of dimension Δ and spin $\ell = \pm 2$ by $G^{\pm,\eta}_{\Delta}(z,\bar{z})$ where as usual $\eta = \pm$ indicates whether the particle is outgoing/incoming. Likewise, in the case of gluons, we denote the corresponding primary operator by $O^{\pm a,\eta}_{\Delta}(z,\bar{z})$, where a is the corresponding color index in the adjoint representation of the structure group of the theory.

In summary, taking a Mellin transform with respect to the external energies, the $\mathcal{S}$-matrix elements are then transformed into objects that behave as conformal correlators:

$$\prod_{i=1}^{n}\int_0^\infty \frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\langle \text{out}|\mathcal{S}|\text{in}\rangle = \left\langle \mathcal{O}^{\pm}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\cdots\mathcal{O}^{\pm}_{h_n,\bar{h}_n}(z_n,\bar{z}_n)\right\rangle, \tag{2.119}$$

where the right-hand side is then called a *celestial amplitude* or *celestial correlator*.

2.6.2 Poincaré constraints

Let us denote by $\mathcal{A}_n(p_i,\ell_i)$ one momentum space $\mathcal{S}$-matrix element of n massless particles with momenta $p_i=\eta_i\omega_i\hat{q}(z_i,\bar{z}_i)$ and helicities ℓ_i. Taking a Mellin transform we obtain the associated celestial amplitude

$$\widetilde{\mathcal{A}}_n(\Delta_i,z_i,\bar{z}_i,\ell_i) \;=\; \prod_{i=1}^n\int_0^\infty\frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\mathcal{A}(\eta_i\omega_i\hat{q}(z_i,\bar{z}_i),\ell_i). \tag{2.120}$$

This amplitude is subject to the constraints of Poincaré symmetry. In particular, we have the Lorentz constraints

$$\sum_{i=1}^n L_m^{(i)}\widetilde{\mathcal{A}}_n \;=\; \sum_{i=1}^n \bar{L}_m^{(i)}\widetilde{\mathcal{A}}_n=0, \tag{2.121}$$

where $L_m^{(i)}$ and $\bar{L}_m^{(i)}$ are the Lorentz/conformal generators acting on the i-th particle, and we also have the translation constraints

$$\sum_{i=1}^n P_\mu^{(i)}\widetilde{\mathcal{A}}_n \;=\; 0, \tag{2.122}$$

where $P_\mu^{(i)}$ are the generators of translations acting on the i-th particle. In a plane wave basis, $P_\mu^{(i)}$ acts diagonally, extracting the momentum eigenvalue

$$P_\mu^{(i)}\mathcal{A}_i(p_i,\ell_i)=\eta_i\omega_i\hat{q}_\mu(z_i,\bar{z}_i)\mathcal{A}_i(p_i,\ell_i). \tag{2.123}$$

This action can be transformed to a conformal primary basis by means of the Mellin transform and it follows that the translation generator acts on conformal primary wavefunctions as weight-shifting operators [33, 40]

$$P_\mu^{(i)}=\eta_i\hat{q}_\mu(z_i,\bar{z}_i)e^{\partial_{\Delta_i}}. \tag{2.124}$$

This means that when transforming to a conformal basis, the energy maps to a weight-shifting operator according to $\omega \to e^{\partial_\Delta}$. Considering the lightcone component $P^+ = P^0 + P^3$ which is given by $P^+ = 2\eta e^{\partial_\Delta}$ it thus follows that celestial amplitudes must obey the constraint

$$\sum_{i=1}^{n} \eta_i \widetilde{\mathcal{A}}_n(\Delta_1, \ldots, \Delta_i + 1, \ldots, \Delta_n) \;=\; 0, \tag{2.125}$$

where we have suppressed dependence on $(z_i, \bar{z}_i)$ and ℓ_i for conciseness.

On the one hand, (2.121) are the constraints of global conformal invariance that are well-known to constrain the form of conformal correlators in standard conformal field theories [88]. On the other hand, (2.125) are new constraints that come from translations in the asymptotically flat bulk, which do not appear in standard conformal field theories. These extra constraints imply, in particular, that two, three and four-point functions are singular in their dependence on $(z_i, \bar{z}_i)$ [40].

One important example is that of the four-point function. It can be shown using Poincaré symmetry that a generic celestial four-point function can be put in the form

$$\widetilde{\mathcal{A}}_4(h_i, \bar{h}_i, z_i, \bar{z}_i) \;=\; K_{h_i,\bar{h}_i}(z_i, \bar{z}_i)\delta(z - \bar{z}) f^{h_i,\bar{h}_i}(z, \bar{z}), \tag{2.126}$$

where $K_{h_i,\bar{h}_i}(z_i, \bar{z}_i)$ is a conformally-covariant prefactor

$$K_{h_i,\bar{h}_i}(z_i, \bar{z}_i) \;=\; \prod_{i<j=1}^{4} z_{ij}^{h/3 - h_i - h_j} \bar{z}_{ij}^{\bar{h}/3 - \bar{h}_i - \bar{h}_j}, \tag{2.127}$$

with $z_{ij} = z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$, $h = \sum_i h_i$, $\bar{h} = \sum_i \bar{h}_i$ and where z and $\bar{z}$ are the conformal cross-ratios:

$$z \;=\; \frac{z_{13} z_{24}}{z_{12} z_{34}}, \quad \bar{z} = \frac{\bar{z}_{13} \bar{z}_{24}}{\bar{z}_{12} \bar{z}_{34}}, \tag{2.128}$$

which are invariant under conformal transformations and further related to the Mandelstam invariants $s = -(p_1 + p_2)^2$ and $t = -(p_1 + p_3)^2$ by

$$z = -\frac{t}{s}. \tag{2.129}$$

It can be further shown that $f^{h_i,\bar{h}_i}(z, \bar{z})$ just depends on the $(h_i, \bar{h}_i)$ through the combination $\beta = \sum_i \Delta_i$ and the individual helicities ℓ_i, so that we can also write $f^{h_i,\bar{h}_i}(z, \bar{z}) = f^{\beta,\ell_i}(z, \bar{z})$ [72].

It is important to observe that while standard conformal symmetry implies that a four-point function of a conformal field theory takes the form of a conformally-covariant prefactor such as $K_{h_i,\bar{h}_i}(z_i, \bar{z}_i)$ multiplied by a function of the cross-ratios $(z, \bar{z})$, the presence of a delta distribution $\delta(z - \bar{z})$ enforcing reality of the cross ratios comes from momentum conservation in the bulk: the four particles must scatter on a plane which intersecting the celestial sphere imposes that the four-points must lie on a circle. This behavior is a non-standard feature of CCFT in comparison to CFT and is a manifestation of the extra symmetries it enjoys, in this case the bulk translation symmetry. Recent investigations have considered the possibility of breaking translation invariance in the bulk in a controlled manner so as to obtain regular celestial amplitudes, whose behavior is closer to that of standard CFT [89, 90].

2.6.3 Conformally Soft Limits

The change of basis from momentum eigenstates to conformal primary wavefunctions has been motivated by the existence of soft symmetries, namely, by the fact that certain soft insertions in the $\mathcal{S}$-matrix behave as currents in a two-dimensional CFT and, moreover, express asymptotic symmetries in the bulk scattering. Since the change of basis trades energies by conformal dimensions through the Mellin transform it is not a priori obvious how these results carry over to the conformal primary basis. It turns out that the soft mode to subleading order s can be accessed as a $\Delta = 1 - s$ conformal primary operator [36, 38, 91, 92].

To get started, assume the soft theorem for the insertion of a massless particle of helicity ℓ holds good to arbitrary subleading orders, so that we have the momentum space identity

$$\langle \text{out}| a_\ell^{\text{out}}(\omega, z, \bar{z}) \mathcal{S} |\text{in}\rangle = \left[\frac{1}{\omega} S_0^{(\ell)} + S_1^{(\ell)} + \cdots \right] \langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{2.130}$$

We call the operator $N_s^{(\ell)}(z, \bar{z})$ defined by

$$N_s^{(\ell)}(z, \bar{z}) \equiv \frac{1}{s!} \lim_{\omega \to 0} \partial_\omega^s \left[\omega a_\ell^{\text{out}}(\omega, z, \bar{z}) \right] \tag{2.131}$$

an outgoing sub$^{(s)}$-leading soft particle because its insertions in the $\mathcal{S}$-matrix projects out the sub$^{(s)}$-leading soft factor by

$$\langle \text{out}| N_s^{(\ell)}(z, \bar{z}) \mathcal{S} |\text{in}\rangle = S_s^{(\ell)} \langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{2.132}$$

The non-trivial observation is that $N_s^{(\ell)}(z, \bar{z})$ can be also accessed in the conformal primary basis because of the Dirac delta identity

$$\frac{(-1)^s}{s!} \delta^{(s)}(x) = \lim_{\epsilon \to 0} \frac{\epsilon}{2} |x|^{\epsilon - s - 1}, \quad s \in \mathbb{Z}, \quad s \geq 0 \tag{2.133}$$

valid on a space of functions that decay to zero sufficiently fast as $|x| \to \infty$ [36]. Indeed, consider such a function $f(\omega)$ and observe that

$$\lim_{\Delta \to 1-s} (\Delta - 1 + s) \int_0^\infty \frac{d\omega}{\omega} \omega^\Delta f(\omega) = \lim_{\epsilon \to 0} \int_0^\infty d\omega \left(\epsilon \omega^{\epsilon - s - 1} \right) [\omega f(\omega)], \tag{2.134}$$

where we set $\epsilon = \Delta - 1 + s$. We can now use (2.133) to evaluate the integral, obtaining

$$\lim_{\Delta \to 1-s} (\Delta - 1 + s) \int_0^\infty \frac{d\omega}{\omega} \omega^\Delta f(\omega) = \frac{1}{s!} \lim_{\omega \to 0} \partial_\omega^s [\omega f(\omega)]. \tag{2.135}$$

In particular this means that if $\mathcal{O}^+_{\Delta,\ell}(z,\bar{z})$ is a celestial operator obtained as the Mellin transform of $a^{\text{out}}_\ell(\omega, z, \bar{z})$ we can identify

$$\lim_{\Delta \to 1-s} (\Delta - 1 + s) \mathcal{O}^+_{\Delta,\ell}(z,\bar{z}) = N^{(\ell)}_s(z,\bar{z}). \tag{2.136}$$

For this reason, we call the $\Delta \to 1 - s$ limit a sub$^{(s)}$-leading conformally soft limit and we refer to a graviton with dimension $\Delta = 1 - s$ as a sub$^{(s)}$-leading conformally soft graviton, with the same terminology applied for other massless particles such as the gluon.

Using these ideas the standard momentum space soft theorems directly imply into conformally soft theorems obeyed by celestial correlators. More precisely, whenever insertions of a massless particle of helicity ℓ obeys a sub$^{(s)}$-leading energetic soft theorem, insertions of $\mathcal{O}^+_{\Delta,\ell}(z,\bar{z})$ exhibit a corresponding pole as $\Delta \to 1 - s$ and the residue on that pole is a conformally soft factor times the amplitude without the conformally soft particle. Equivalently, insertions of the conformally soft operator $N^{(\ell)}_s(z,\bar{z})$ factorize in terms of a conformally soft factor.

2.7 Celestial Holographic Algebras

2.7.1 OPE in Conformal Field Theory

Generically in QFT quantum fields are operator-valued distributions. As such, the product of fields develops singularities in the coincidence limit, as is well-known to happen for standard scalar-valued distributions. The *Operator Product Expansion* (OPE) aims to precisely characterize this singular structure, being the statement that in some suitable set of fields, the product of two elements of the set can be approximated in terms of other members of the set as [93]

$$\Phi_i(x)\Phi_j(y) \sim \sum_k C_{ij}^k(x,y)\Phi_k(y), \tag{2.137}$$

where the $\sim$ sign means that the right-hand side well approximates the left-hand side in the coincidence limit. The series is generally understood as asymptotic, instead of convergent, and understood to be valid inside of correlation functions [93].

In CFT the OPE can be formulated precisely, with the set of fields being the set of primary fields and their descendants, and in that case, it becomes a convergent series, valid in correlation functions, inside an open ball containing the two operators and no other insertions. What lies behind the existence of an OPE in CFT is the *state-operator map* [88]. The OPE in a CFT is then heavily constrained by conformal symmetry. Indeed, writing the OPE as[5]

$$\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2) = \sum_{h,\bar{h}}\sum_{m,\bar{m}} C_{h,\bar{h}}^{(m,\bar{m})}(z_1,\bar{z}_1,z_2,\bar{z}_2)\partial^m\bar{\partial}^{\bar{m}}\mathcal{O}_{h,\bar{h}}(z_2,\bar{z}_2), \tag{2.138}$$

we may impose symmetry constraints by choosing a symmetry generator Q and demanding its adjoint action through the commutator commutes with taking the OPE. This gives rise to constraints on the OPE coefficients $C_{h,\bar{h}}^{(m,\bar{m})}(z_1,\bar{z}_1,z_2,\bar{z}_2)$.

The constraints imposed by L_{-1} and $\bar{L}_{-1}$ are

$$\begin{aligned}
(\partial_{z_1}+\partial_{z_2})\, C_{h,\bar{h}}^{(m,\bar{m})}(z_1,\bar{z}_1,z_2,\bar{z}_2) &= 0,\\
(\partial_{\bar{z}_1}+\partial_{\bar{z}_2})\, C_{h,\bar{h}}^{(m,\bar{m})}(z_1,\bar{z}_1,z_2,\bar{z}_2) &= 0,
\end{aligned} \tag{2.139}$$

and together they imply that $C_{h,\bar{h}}^{(m,\bar{m})}(z_1,\bar{z}_1,z_2,\bar{z}_2) = C_{h,\bar{h}}^{(m,\bar{m})}(z_{12},\bar{z}_{12})$. In particular this expresses translation invariance of the OPE and means we can study the OPE without

[5] In this section we consider a generic CFT_2. As such we denote its primary operators by $\mathcal{O}_{h,\bar{h}}(z,\bar{z})$ in order not to confuse with the celestial primaries $\mathcal{O}_{h,\bar{h}}(z,\bar{z})$ in CCFT_2.

loss of generality by choosing $z_2 = 0$ and $z_1 = z$. Doing so, the L_0 and $\bar{L}_0$ constraints are

$$\begin{aligned}(h_1 + h_2 + z\partial_z)C^{(m,\bar{m})}_{h,\bar{h}}(z,\bar{z}) &= (h+m)C^{(m,\bar{m})}_{h,\bar{h}}(z,\bar{z}),\\ (\bar{h}_1 + \bar{h}_2 + \bar{z}\partial_{\bar{z}})C^{(m,\bar{m})}_{h,\bar{h}}(z,\bar{z}) &= (\bar{h}+\bar{m})C^{(m,\bar{m})}_{h,\bar{h}}(z,\bar{z}),\end{aligned} \tag{2.140}$$

and assuming $(z, \bar{z})$ as independent complex variables we have that

$$C^{(m,\bar{m})}_{h,\bar{h}}(z,\bar{z}) = C^{(m,\bar{m})}_{h,\bar{h}}\, z^{h+m-h_1-h_2}\,\bar{z}^{\bar{h}+\bar{m}-\bar{h}_1-\bar{h}_2}. \tag{2.141}$$

Finally, the L_1 and $\bar{L}_1$ constraints are

$$\begin{aligned}(m+1)(2h+m)C^{(m+1,\bar{m})}_{h,\bar{h}} &= (h-h_1-h_2+m)C^{(m,\bar{m})}_{h,\bar{h}},\\ (\bar{m}+1)(2\bar{h}+\bar{m})C^{(m,\bar{m}+1)}_{h,\bar{h}} &= (\bar{h}-\bar{h}_1-\bar{h}_2+\bar{m})C^{(m,\bar{m})}_{h,\bar{h}},\end{aligned} \tag{2.142}$$

and they are recursion relations fixing the contributions of descendants in terms of the corresponding primary contribution. In particular they are special cases of the general recursion relation

$$(m+1)(x+m)a_{m+1} \quad = \quad (y+m)a_m, \tag{2.143}$$

that once iterated has the unique solution

$$a_m = \frac{1}{m!}\frac{\Gamma(x)}{\Gamma(y)}\frac{\Gamma(y+m)}{\Gamma(x+m)}a_0. \tag{2.144}$$

As such we find that the solution to the L_1 and $\bar{L}_1$ constraints is

$$C^{(m,\bar{m})}_{h,\bar{h}} = \frac{1}{m!\bar{m}!}\frac{\Gamma(2h_3)\Gamma(2\bar{h}_3)\Gamma(h_1-h_2+h_3+m)\Gamma(\bar{h}_1-\bar{h}_2+\bar{h}_3+\bar{m})}{\Gamma(h_1-h_2+h_3)\Gamma(\bar{h}_1-\bar{h}_2+\bar{h}_3)\Gamma(2h_3+m)\Gamma(2\bar{h}_3+\bar{m})}C^{(0,0)}_{h,\bar{h}}. \tag{2.145}$$

In summary, a CFT has a convergent OPE among primary operators and their descendants, valid inside correlation functions in an open ball containing the two operators being multiplied and no other insertions, with the property that all contributions from the descendants of a primary operator are determined in terms of the primary contribution. We then often write the OPE as

$$\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2) \sim C_{h,\bar{h}} z_{12}^{h-h_1-h_2} \bar{z}_{12}^{\bar{h}-\bar{h}_1-\bar{h}_2} \mathcal{O}_{h,\bar{h}}(z_2,\bar{z}_2), \tag{2.146}$$

where $\sim$ means up to the descendants' contributions. We next turn to the analysis of how an OPE structure arises in CCFT.

2.7.2 Collinear Limits as a Celestial OPE

Scattering amplitudes of gluons and gravitons develop singularities when two of the external particles are taken to be collinear [94–99], see also [100] for a review. These singularities arise from Feynman diagrams in which the two external lines are connected to a three-point vertex. If they have momenta p_1 and p_2 the third, internal, line connected to the vertex will have momentum $P = p_1 + p_2$ and its propagator will behave as

$$\frac{1}{(p_1+p_2)^2} = \frac{1}{2p_1 \cdot p_2}, \tag{2.147}$$

with a pole singularity in the colinear limit $p_1 \cdot p_2 \to 0$. When that happens the third line becomes on-shell and the diagram factorizes into a contribution coming from the vertex in the collinear configuration times the diagram in which the two external particles are replaced by a single one.

If the particles have momenta $p_1 = \omega_1 \hat{q}(z_1,\bar{z}_1)$ and $p_2 = \omega_2 \hat{q}(z_2,\bar{z}_2)$ the collinear singularity occurs when $|z_{12}|^2 \to 0$, which corresponds to the coincidence limit on the celestial sphere. As such it is natural to conjecture that the collinear limit in the bulk corresponds to the OPE limit in the boundary. This was indeed shown to be the case in gauge theory [34] and gravity [52] using known results of collinear factorization, with the analysis extended to any massless particles coupling by a three-point vertex using a BCFW shift [53].

It is important to remark that when studying the celestial OPE one often considers either the holomorphic limit $z_{12} \to 0$ with $\bar{z}_{12}$ fixed, or the anti-holomorphic limit $\bar{z}_{12} \to 0$ with z_{12} fixed, instead of just $|z_{12}|^2 \to 0$. The reason for doing this is that when both $z_{12} \to 0$ and $\bar{z}_{12} \to 0$ order of limit issues arise [52]. Very importantly, in order to make sense of these holomorphic and anti-holomorphic limits, it is necessary to analytically continue the spacetime signature from Lorentzian $(1,3)$ to Kleinian $(2,2)$, because in $(2,2)$ signature the variables $(z,\bar{z})$ parameterizing null momenta become real and independent instead of complex conjugates of one another. In particular, the $\mathrm{SL}(2,\mathbb{C})$ symmetry is continued to $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ where the left factor acts on z and the right factor acts on $\bar{z}$.

For gravitons, the collinear limit can be written as

$$\lim_{z_{12}\to 0} \mathcal{A}_{s_1 s_2 \cdots s_n}(p_1, p_2 \ldots, p_n) = \sum_{s=\pm 2} \mathrm{Split}^{s}_{s_1 s_2}(p_1,p_2)\mathcal{A}_{s\cdots s_n}(P, \cdots, p_n). \tag{2.148}$$

where $\mathrm{Split}^{s}_{s_1 s_2}(p_1,p_2)$ are known as splitting functions and the singular ones in the holomorphic limit $z_{12} \to 0$ can be shown to take the form [52]

$$\mathrm{Split}^{2}_{22}(p_1,p_2) \;=\; -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\frac{\omega_P^2}{\omega_1\omega_2}, \quad \mathrm{Split}^{-2}_{2-2}(p_1,p_2) = -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\frac{\omega_2^3}{\omega_1\omega_P}. \tag{2.149}$$

The Mellin transform of $\mathcal{A}_{s_1\cdots s_n}(p_1,\ldots,p_n)$ in the collinear limit can be taken by changing the variables (ω_1,ω_2) to (t,ω_P) where

$$\omega_1 = t\omega_P, \quad \omega_2 = (1-t)\omega_P. \tag{2.150}$$

In terms of these variables, $\mathcal{A}_{s\cdots s_n}(P,\ldots,p_n)$ depends just on ω_P while all the t dependence lies in the splitting functions. In particular one observes that the integral over t can be evaluated explicitly in terms of Euler beta functions, while the integral over ω_P can be combined with the remaining energy integrals to transform $\mathcal{A}_{s\cdots s_n}(P,\ldots,p_n)$ into a celestial amplitude. Using the celestial correlator notation, this means that we have

$$\begin{aligned}
\langle G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2)\cdots\rangle &\sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)\langle G^+_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)\cdots\rangle,\\
\langle G^+_{\Delta_1}(z_1,\bar{z}_1)G^-_{\Delta_2}(z_2,\bar{z}_2)\cdots\rangle &\sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+3)\langle G^-_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)\cdots\rangle,
\end{aligned} \tag{2.151}$$

in which by $\sim$ we mean up to terms that are regular in the collinear limit. Since these equations are true for any other insertions into the celestial correlator, they can be written as operator equations

$$\begin{aligned}
G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2) &\sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)G^+_{\Delta_1+\Delta_2}(z_2,\bar{z}_2),\\
G^+_{\Delta_1}(z_1,\bar{z}_1)G^-_{\Delta_2}(z_2,\bar{z}_2) &\sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+3)G^-_{\Delta_1+\Delta_2}(z_2,\bar{z}_2).
\end{aligned} \tag{2.152}$$

Observing the powers of z_{12} and $\bar{z}_{12}$ it is possible to see that they are consistent with the ones corresponding to contributions of primaries of dimension $\Delta_1+\Delta_2$ and spin $\ell=\pm 2$ to the OPEs of the primaries on the left-hand side. As such, we see that indeed collinear singularities in scattering amplitudes translate into an OPE-like structure on celestial correlators.

At this point, it is already possible to appreciate the advantage of the holographic perspective. Assuming that an OPE obeying the constraints of conformal symmetry exists, and knowing the primary contribution, it is possible to include all anti-holomorphic descendants using (2.145), or equivalently by employing the OPE block construction [54, 101, 102].

In the next section, we are going to review how even the primary contribution, which in this section we argued can be computed from collinear singularities of scattering amplitudes, can also be constrained and determined from symmetry.

2.7.3 Celestial OPE from Symmetry

The celestial OPE suggested by collinear limits can remarkably be fixed by symmetry. Indeed, in [52] it was shown that assuming that the celestial operators have an operator product expansion and proposing a sensible ansatz, translation symmetry together with conformally soft symmetries were capable of fixing the contributions of primary operators to the celestial OPE. The results match the ones obtained by the analysis of collinear singularities, such as the one we have exemplified in the last section. Even more surprisingly, it was then later shown in [53] that just a subset of Poincaré symmetry alone is capable of achieving the same result. In this section, we briefly review this argument and write down the celestial OPE with $\mathrm{SL}(2,\mathbb{R})_R$ descendants constructed in [53].

The starting point of the analysis is to recall that ultimately the celestial OPE captures bulk collinear singularities. As such, given two celestial operators $\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)$ and $\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2)$ each contribution to the product $\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2)$ is associated to a particular three-point vertex in a bulk effective Lagrangian. If the vertex has dimension d_V, a dimensional analysis argument shows that the corresponding primary contribution to the $\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2)$ OPE must have $\Delta = \Delta_1 + \Delta_2 + d_V - 5$ (see Appendix A of [52]). Defining $p = d_V - 4$ and assuming such primary has spin ℓ, it's conformal weights are

$$h = h_1 + h_2 + \frac{p-1+(\ell-\ell_1-\ell_2)}{2}, \quad \bar{h} = \bar{h}_1 + \bar{h}_2 + \frac{p-1-(\ell-\ell_1-\ell_2)}{2}. \tag{2.153}$$

Since at tree-level scattering amplitudes can have only simple poles as singularities one must rule out branch points and higher-order poles. This imposes a constraint that there must be $n \in \mathbb{Z}$ such that

$$p - 1 + \ell - \ell_1 - \ell_2 = 2(n-1), \quad 0 \leq n \leq p. \tag{2.154}$$

In that case the primary contribution to the OPE takes the form

$$\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2) \sim C^{(0)}_{p,n}(\bar{h}_1,\bar{h}_2) z_{12}^{n-1} \bar{z}_{12}^{p-n} \mathcal{O}_{h,\bar{h}}(z_2,\bar{z}_2), \tag{2.155}$$

where we understand $C^{(0)}_{p,n}(\bar{h}_1,\bar{h}_2)$ as a function of $(\bar{h}_1,\bar{h}_2)$ and of the spins ℓ_1 and ℓ_2, the dependence on which we leave implicit. In [53] the case $n = 0$ has been considered and the contributions of all $\mathrm{SL}(2,\mathbb{R})_R$ descendants have been included, keeping only the leading singularity in the holomorphic collinear limit $z_{12} \to 0$, which in this case is a simple pole. To that end one proposes the ansatz

$$\mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2) \sim \frac{1}{z_{12}} \sum_{\bar{m}=0}^{\infty} C^{(\bar{m})}_{p}(\bar{h}_1,\bar{h}_2) \bar{z}_{12}^{p+\bar{m}} \partial_{\bar{z}_2}^{\bar{m}} \mathcal{O}_{h,\bar{h}}(z_2,\bar{z}_2). \tag{2.156}$$

We already know that all $C^{(\bar{m})}_p(\bar{h}_1,\bar{h}_2)$ are fixed in terms of $C^{(0)}_p(\bar{h}_1,\bar{h}_2)$ by conformal symmetry. The key element of the analysis is to further constrain the OPE with the translation charges P_μ to determine $C^{(0)}_p(\bar{h}_1,\bar{h}_2)$. Writing the momentum operators in terms of spinor components $P_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} P_\mu$, where $\sigma^\mu = (1,\sigma^i)$, with σ^i being the Pauli matrices, we observe that the charges $P_{-\frac{1}{2},\pm\frac{1}{2}}$ do not mix $\mathrm{SL}(2,\mathbb{R})_L$ descendants and can be consistently imposed while studying just the leading pole in z_{12}. The constraint imposed by $P_{-\frac{1}{2},-\frac{1}{2}}$ is

$$C^{(0)}_p\left(\bar{h}_1 + \frac{1}{2}, \bar{h}_2\right) + C^{(0)}_p\left(\bar{h}_1, \bar{h}_2 + \frac{1}{2}\right) = C^{(0)}_p(\bar{h}_1,\bar{h}_2), \tag{2.157}$$

while the constraint imposed by $P_{-\frac{1}{2},\frac{1}{2}}$ is

$$C^{(\bar{m})}_p\left(\bar{h}_1 + \frac{1}{2}, \bar{h}_2\right) = (\bar{m}+1) C^{(\bar{m})}_p(\bar{h}_1,\bar{h}_2). \tag{2.158}$$

Combining this constraint with the $\bar{L}_1$ constraint gives a recursion relation in $\bar{h}_1$ at fixed $\bar{m}$ which for the special case $\bar{m} = 0$ reduces to

$$(2\bar{h}_1 + p)C_p^{(0)}(\bar{h}_1, \bar{h}_2) = (2\bar{h}_1 + 2\bar{h}_2 + 2p)C_p^{(0)}\left(\bar{h}_1 + \frac{1}{2}, \bar{h}_2\right). \tag{2.159}$$

Further combining with the $P_{-\frac{1}{2},-\frac{1}{2}}$ constraint gives a similar recursion relation in $\bar{h}_2$

$$(2\bar{h}_2 + p)C_p^{(0)}(\bar{h}_1, \bar{h}_2) = (2\bar{h}_1 + 2\bar{h}_2 + 2p)C_p^{(0)}\left(\bar{h}_1, \bar{h}_2 + \frac{1}{2}\right). \tag{2.160}$$

The two recursion relations can be combined to show that under suitable assumptions on the behavior of the OPE coefficients on $(\bar{h}_1, \bar{h}_2)$ there is a unique solution [52, 53]

$$C_p^{(0)}(\bar{h}_1, \bar{h}_2) = \gamma_p^{\ell_1,\ell_2} B(2\bar{h}_1 + p, 2\bar{h}_2 + p), \tag{2.161}$$

where $\gamma_p^{\ell_1,\ell_2}$ is a constant that captures the remaining dependence on spin, that can be shown to be proportional to the coupling constant of the vertex by comparing to the collinear limit analysis [53]. Conformal symmetry then fixes

$$C_p^{(\bar{m})}(\bar{h}_1, \bar{h}_2) = \gamma_p^{\ell_1,\ell_2} \frac{1}{\bar{m}!} B(2\bar{h}_1 + p + \bar{m}, 2\bar{h}_2 + p), \tag{2.162}$$

and therefore the celestial OPE with complete dependence on $\bar{z}_{12}$ takes the form

$$\mathcal{O}_{h_1,\bar{h}_1}(z_1, \bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2, \bar{z}_2) \sim \frac{\gamma_p^{\ell_1,\ell_2}}{z_{12}} \sum_{\bar{m}=0}^{\infty} \frac{B(2\bar{h}_1 + p + \bar{m}, 2\bar{h}_2 + p)}{\bar{m}!} \bar{z}_{12}^{p+\bar{m}} \partial_{\bar{z}_2}^{\bar{m}} \mathcal{O}_{h,\bar{h}}(z_2, \bar{z}_2). \tag{2.163}$$

It is then straightforward to check that the symmetry-based derivation recovers the collinear singularity result from the previous section.

2.7.4 Conformally soft currents and $w_{1+\infty}$

We finally turn to the subject of how the celestial OPE allows for the organization of conformally soft symmetries to arbitrary subleading orders [54]. In the particular case of gravity that we are going to consider, this reveals a $w_{1+\infty}$ symmetry of gravity [53, 55]. We review this analysis by following [53–56], in particular we follow closely the analysis of [56], that carefully keeps track of contact terms. We consider positive-helicity gravitons, however, while in [56] negative-helicity ones have been considered. We connect to the notation used in the other papers in the end.

We start by recalling that a primary field $\phi(z,\bar{z})$ with weights $(h,\bar{h})$ in a two-dimensional conformal field theory admits a mode expansion near $(z,\bar{z})=(\infty,\infty)$ [56]

$$\phi(z,\bar{z}) = z^{-2h}\bar{z}^{-2\bar{h}} \sum_{n,m=0}^{\infty} \frac{\phi_{n,m}}{z^n \bar{z}^m}. \tag{2.164}$$

We can equivalently resum the complete holomorphic dependence and write this as

$$\phi(z,\bar{z}) = \bar{z}^{-2\bar{h}} \sum_{m=0}^{\infty} \frac{\phi_m(z)}{\bar{z}^m}. \tag{2.165}$$

Note that this differs from the standard mode expansion used in CFT_2 [88] that has been employed to study the soft algebra in [54] by a shift $n \to n+h$ and $m \to m+\bar{h}$.

Let us then consider $N_s(z,\bar{z}) \equiv N_s^{(+2)}(z,\bar{z})$ a positive-helicity sub$^{(s)}$-leading soft graviton with weights $(h,\bar{h}) = (\frac{3-s}{2}, -\frac{s+1}{2})$. In particular we have $-2\bar{h} = s+1$ and it follows that when $0 \le m \le s+1$ we have a polynomial contribution to the mode expansion and when $m > s+1$ we have a Laurent series contribution. We thus split [56]

$$N_s(z,\bar{z}) = H_s(z,\bar{z}) + \check{N}_s(z,\bar{z}), \tag{2.166}$$

where

$$H_s(z,\bar{z}) = \sum_{m=0}^{s+1} \bar{z}^m N_s^{-m}(z), \quad \check{N}_s(z,\bar{z}) = \sum_{m=1}^{\infty} \frac{N_s^m(z)}{\bar{z}^m}. \tag{2.167}$$

In particular, we observe that $\partial_{\bar{z}}^{s+2}$ annihilates $H_s(z,\bar{z})$. As such, defining[6]

$$q_s^1(z,\bar{z}) \equiv \partial_{\bar{z}}^{s+2} \check{N}_s(z,\bar{z}), \tag{2.168}$$

this field gives an equivalent encoding of the modes $N_s^m(z)$ of $\check{N}_s(z,\bar{z})$, while the field $H_s(z,\bar{z})$ encodes currents $N_s^{-m}(z)$ that together form a $(s+2)$-dimensional $\mathrm{SL}(2,\mathbb{R})_R$ multiplet with highest weight $\frac{1+s}{2}$ and lowest weight $-\frac{1+s}{2}$.

The OPE of $N_s(z,\bar{z})$ with an arbitrary field $\mathcal{O}_{h,\bar{h}}(z,\bar{z})$ is obtained by taking a conformally soft limit of the OPE in (2.163). In this limit, $(\Delta_1 + s - 1)B(2\bar{h}_1 + p + \bar{m}, 2\bar{h}_2 + p)$ is nonzero only when $m \le s+1-p$ and then we have

$$\begin{aligned} N_s(z_1,\bar{z}_1)\mathcal{O}_{h_2,\bar{h}_2}(z_2,\bar{z}_2) \sim -\frac{\gamma_p^{\ell_1,\ell_2}}{z_{12}} \sum_{\bar{m}=0}^{s+1-p} & \frac{(-1)^{\bar{m}+p+s}\Gamma(2\bar{h}_2+p)}{(1-\bar{m}-p+s)!\Gamma(-1+2\bar{h}_2+\bar{m}+2p-s)} \\ & \times \bar{z}_{12}^{p+\bar{m}} \partial_{\bar{z}_2}^{\bar{m}} \mathcal{O}_{h,\bar{h}}(z_2,\bar{z}_2). \end{aligned} \tag{2.169}$$

The contribution from the minimal coupling vertex is obtained for $p=1$ [53], and henceforth we focus on that particular case. In particular, we note an important difference between $(1,3)$ and $(2,2)$ bulk signatures. In $(1,3)$ signature, z_{12} and $\bar{z}_{12}$ are complex conjugates of one another while in $(2,2)$ signature they are real and independent. In that case we have

$$\partial_{\bar{z}_1} \frac{1}{z_{12}} = \begin{cases} 2\pi\delta^{(2)}(z_{12}), & (1,3) \text{ bulk signature}, \\ 0, & (2,2) \text{ bulk signature}. \end{cases} \tag{2.170}$$

[6]The notation $q_s^1(z,\bar{z})$ has been introduced in [56] because it is the linear contribution to a quantity $q_s(z,\bar{z})$ that generalizes the Bondi mass aspect and angular momentum aspect at $\mathcal{I}_-^+$ to higher spins. As such, $q_s^1(z,\bar{z})$ is the soft part of the spin s charge aspect, that once integrated against a corresponding spin s parameter gives rise to a spin s soft charge at $\mathcal{I}_-^+$.

In that case we observe that the OPE of $q^1_s(z,\bar{z})$ with other fields will identically vanish in $(2,2)$ signature while it will just give contact terms in $(1,3)$ signature. In particular, we observe that $q^1_s(z,\bar{z})$ encodes the same mode expansion of $\check{N}_s(z,\bar{z})$, so that $\check{N}_s(z,\bar{z})$ can be omitted in $(2,2)$ signature or if we disregard operators that just produce contact terms in OPE's, that being the reason it doesn't appear in [54].

Now the $w_{1+\infty}$ symmetry of gravity can be unveiled by taking a light-transform of the positive-helicity gravitons. More precisely, the light-transform is defined by

$$\mathbf{L}[\mathcal{O}_{h,\bar{h}}](z,\bar{z}) \equiv \int \frac{d\bar{w}}{2\pi i}\frac{1}{(\bar{z}-\bar{w})^{2-2\bar{h}}}\mathcal{O}_{h,\bar{h}}(z,\bar{w}), \tag{2.171}$$

and it is justified in $(2,2)$ signature. In that case one may show by taking the light-transform of the soft graviton mode expansion that (see Appendix G of [56])

$$\frac{1}{\kappa}(-1)^{s+3}\Gamma(s+3)\mathbf{L}[G^+_{1-s+\epsilon}](z,\bar{z}) = \frac{1}{\kappa}\frac{q^1_s(z,\bar{z})}{\epsilon} + W_s(z,\bar{z}) + O(\epsilon), \tag{2.172}$$

where

$$W_s(z,\bar{z}) = \frac{1}{\kappa}\sum_{n=0}^{s+1}\frac{(-1)^{n+s}N_s^{-n}(z)}{\bar{z}^{s+2-n}}n!(s+1-n)!, \tag{2.173}$$

$$q^1_s(z,\bar{z}) = \sum_{n=0}^{\infty}\frac{(-1)^s N^n_s(z)}{\bar{z}^{s+2+n}}\frac{(s+1+n)!}{(n-1)!}. \tag{2.174}$$

Both $W_s(z,\bar{z})$ and $q^1_s(z,\bar{z})$ have $(h,\bar{h}) = (\frac{3-s}{2},\frac{s+3}{2})$. In order to match the notation of [53, 55] we define $q = \frac{3+s}{2}$ so that $(h,\bar{h}) = (3-q,q)$. We then define $w^q(z,\bar{z})$ by

$$w^q(z,\bar{z}) \equiv W_{2q-3}(z,\bar{z}). \tag{2.175}$$

In order to write the mode expansion of $w^q(z,\bar{z})$ in the standard conformally covariant form we need to shift $n \to q-1-n$. In that case one obtains

$$w^q(z,\bar{z}) = \sum_{n=1-q}^{q-1} \frac{w_n^q(z)}{\bar{z}^{n+q}}, \tag{2.176}$$

where the currents $w_n^q(z)$ are related to the modes of $H_s(z,\bar{z})$ by

$$w_n^q(z) = \frac{1}{\kappa}(-1)^{3q-n}(q-n-1)!(q+n-1)!N_{2q-3}^{n+1-q}(z). \tag{2.177}$$

Finally, if we denote by $H_n^k(z)$ the modes of $H_{1-k}(z,\bar{z})$ in a conformally covariant expansion, as done in [54, 55], we find $N_s^m(z) = H^{1-s}_{m+\frac{s+1}{2}}(z)$. As a result

$$w_n^q(z) = \frac{1}{\kappa}(-1)^{3q-n}(q-n-1)!(q+n-1)!H_n^{4-2q}(z). \tag{2.178}$$

Up to the factor of $(-1)^{3q-n}$, this is the definition of the w-currents given in [55]. Finally it is possible to show that defining the commutator of holomorphic fields

$$[A,B](z) = \oint \frac{dw}{2\pi i} A(w)B(z), \tag{2.179}$$

and invoking the celestial OPE, the currents $w_n^q(z)$ obey the algebra

$$[w_m^p, w_n^q] = [m(q-1) - n(p-1)]w_{m+n}^{p+q-2}. \tag{2.180}$$

Given the restricted ranges $1-p \leq m \leq p-1$ and $1-q \leq n \leq q-1$, this is known as the wedge subalgebra of the loop algebra of $w_{1+\infty}$, which is often called just the $w_{1+\infty}$ algebra for simplicity.

Remarkably, in [56] the $w_{1+\infty}$ symmetry of gravity, derived originally using CFT$_2$ methods together with the holographic perspective, has been shown to be encoded in the

Einstein equations of classical GR. The analysis was based on the generalization of the supertranslation and superrotation charges to a whole family of higher-spin charges and the $w_{1+\infty}$ symmetry was derived from the Poisson bracket in the classical phase space. This is an extremely non-trivial check of the AFS/CCFT dictionary and also an example in which the holographic duality has taught something new about the bulk.

3

Eikonal Approximation in Celestial CFT

3.1 Introduction

Advances in understanding the asymptotic structure of asymptotically flat spacetimes (AFS) [11, 12, 64, 65, 103–106] have recently crystallized into the proposal that gravity in four-dimensional (4D) AFS may be dual to a conformal field theory (CFT) living on the celestial sphere at null infinity [26–30]. A central aspect of the holographic dictionary is the identification of asymptotic massless fields at $\mathcal{I}^{\pm}$ with operator insertions on the celestial sphere upon exchanging their dependence on retarded/advanced times for conformal scaling dimensions via a Mellin transform. The resulting observables on the sphere, also known as celestial amplitudes, compute overlaps between past and future asymptotic boost, instead of the standard energy-momentum, eigenstates. As such, celestial amplitudes carry the same information as the $\mathcal{S}$-matrix while making the Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetries manifest [29, 30].

As anticipated in [10], the proposed holographic correspondence in AFS distinguishes itself from its counterparts in asymptotically negatively and positively curved spacetimes in that the boundary conformal theory lives in two lower dimensions compared to the gravitational theory. Consequently, familiar aspects of standard CFTs with bulk gravity duals such as the state operator correspondence, unitarity or the relationship between entanglement and bulk geometry are obscured. As a first step in gaining intuition about celestial CFT (CCFT), much of the research to date has focused on studying the imprints of asymptotic symmetries and universal aspects of bulk scattering on celestial amplitudes [31–50]. Remarkably, at tree-level, the symmetry structure of CCFT appears to be much richer than anticipated, including global shift symmetries associated with bulk translations and their local enhancements [11, 12], a Virasoro enhancement of the Lorentz $\mathrm{SL}(2,\mathbb{C})$ [27, 28], all of which are further promoted to a $w_{1+\infty}$ symmetry associated with the tower of subleading soft graviton theorems [53–55].

Taking a leap of faith, one hope is that celestial CFT will ultimately provide a non-perturbative completion of gravity in AFS (see [107] for recent evidence in this direction in a 2D model of gravity), while a complete understanding of celestial symmetries would serve as a guiding principle for extracting non-perturbative details of scattering processes. Evidence for the latter is already manifest, on the one hand in the realization that large gauge symmetries suggest a prescription to eliminate infrared divergences at the $\mathcal{S}$-matrix level to all orders in perturbation theory in abelian [78, 108–110] and possibly non-abelian gauge theory [25, 111, 112], and gravity [109, 113–115], on the other hand in that CFT machinery such as operator product expansion (OPE) blocks [54, 102, 116] allows for the resummation of the leading holomorphic or antiholomorphic collinear divergences – a key element in the identification of the $w_{1+\infty}$ higher spin symmetry of classical gravitational scattering [56].

One of the goals of this thesis is to provide a new entry in the AFS/CCFT dictionary related to a universal, non-perturbative property of 2-2 scattering amplitudes in four-dimensional AFS, namely the leading eikonal exponentiation of t-channel exchanges at high energy [117–119]. Naively, one challenge is that celestial amplitudes scatter boost eigenstates involving integrals over all energies and hence it is a-priori not clear how to take a high-energy limit. However, as shown in [109] low- and high-energy features of massless 4-point scattering are reflected in the analytic structure of the corresponding celestial amplitudes in the net boost weight β. While low energy features are captured by the poles at negative even β (see also [36–38] for similar behavior in conformally soft limits), the high-energy regime can be accessed in the limit of large β [32, 109]. It is natural to suspect then that at large β and small cross-ratio $z \equiv -t/s \ll 1$, celestial amplitudes are dominated by t-channel exchanges. In section 3.3 we present arguments in favor of this proposal by revisiting the position-space calculation of the flat-space eikonal amplitude [119] in a conformal primary basis. As a result we obtain a celestial version of the eikonal exponentiation of t-channel exchanges of arbitrary spin.

Interestingly, the celestial eikonal phase is in general[1] operator valued and each term in its small-coupling expansion acts as a weight-shifting operator [33] on the external scaling dimensions. This is expected as spinning operators couple to scalars via higher derivative interactions which in a conformal primary basis result in shifted weights and resonates with results found in the exponentiation of IR divergences in gauge theory and gravity [109, 120]. Note however that our analysis is complementary, since the eikonal phase discussed here is related to the imaginary part of the exponent of soft $\mathcal{S}$-matrix that results from virtual particle exchanges, rather than the typically discussed real part which arises when the exchanges become on-shell [81].

The eikonal exponentiation of graviton exchanges is particularly interesting as in flat

[1] For exchanges of spin $j \neq 1$.

space it is well known to be reproduced by the propagation of a probe particle in a shockwave background [121–123]. The eikonal exponentiation of graviton exchanges has also been generalized to scattering on black hole backgrounds where the scattering is governed by virtual graviton exchanges on the horizon and is non-perturbatively described in $\hbar$ and $\gamma \sim M_{\rm Pl}/M_{\rm BH}$ in terms of a black-hole eikonal phase, in the regime $s \gg \gamma^2 M_{\rm Pl}^2$, where $M_{\rm Pl}$ is the Planck mass and $M_{\rm BH}$ the black hole mass [124–129]. More recently, the scattering problem in non-perturbative backgrounds has been approached with modern amplitude methods [130–132] including double copy constructions [133–137]. This motivates us to compute the celestial two-point function in a shockwave background. The result is strikingly similar to the analog formula in AdS$_4$ [123] and we establish a relation between the two by demonstrating that the celestial result can be directly recovered as a flat space limit of the AdS result. This observation is a special case of a more general relation between celestial amplitudes and flat space limits of Witten diagrams which we discuss in section 3.6. In particular, we present a general argument that scalar $(d+1)$-dimensional AdS Witten diagrams reduce to $(d-1)$-dimensional CCFT amplitudes to leading order in the limit of large AdS radius provided the boundary operators are placed on certain past and future time-slices. While it is well known that flat space $\mathcal{S}$-matrices in 4D can be extracted from CFT$_3$ correlators either via the HKLL prescription [61–63] or via the flat space limit of Mellin space correlators [60, 138–140] (see also [141] for a recent review of the connection between the two), what we find here instead is that celestial amplitudes arise directly as flat space limits of CFT$_3$ correlators with particular kinematics and with analytically continued dimensions. We regard this as additional evidence that celestial amplitudes are natural candidate holographic observables for quantum gravity in 4D AFS.

This chapter is organized as follows. In section 3.3 we identify an eikonal regime in celestial CFT and derive the celestial eikonal amplitude for the scattering of 4 massless scalars mediated by massive scalar exchanges. In section 3.3.1 we show that the same result is reproduced by the direct Mellin transform of the flat-space eikonal amplitude, while in section 3.3.2 we explicitly check that the first term in a small coupling expansion precisely reproduces the t-channel celestial amplitude in the celestial eikonal limit. We generalize our result to exchanges of arbitrary spin in section 3.3.3. Section 3.4 is devoted to the study of the celestial propagator in a shockwave background. After a review of the momentum space phase shift acquired by a particle crossing a shockwave in section 3.4.1, we express this in a conformal primary basis in section 3.4.2. We identify the CCFT source that relates this to the celestial eikonal formula for graviton exchange in section 3.4.3. In section 4.5.3 we show that the same formula can be obtained as the flat space limit of the CFT_3 correlator associated with propagation through a shock in AdS_4. We establish a general relation between AdS_{d+1} Witten diagrams in the flat space limit and CCFT_{d-1} amplitudes in section 3.6. Various technical details are collected in the appendices.

3.2 Preliminaries

The momentum space scattering amplitude of 4 massless scalars in 4D Minkowski spacetime takes the general form

$$\mathbb{A}_4(p_1, \cdots, p_4) = \mathcal{A}_4(s,t)(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4} p_i\right). \tag{3.1}$$

Here the Mandelstam invariants s, t are defined as

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_3)^2 \tag{3.2}$$

and we parameterize massless on-shell momenta as

$$p_i = \eta_i \omega_i \hat{q}(z_i, \bar{z}_i), \tag{3.3}$$

where ω_i are external energies, $\hat{q}$ are null vectors towards a point $(z_i, \bar{z}_i)$ on the celestial sphere[2]

$$\hat{q}(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}) \tag{3.4}$$

and $\eta_i = +1$ ($\eta_i = -1$) for outgoing (incoming) particles.

The amplitudes (3.1) are mapped to celestial amplitudes[3] or 2D CCFT observables $\widetilde{\mathcal{A}}$ by a Mellin transform [29, 30],

$$\widetilde{\mathcal{A}}(\Delta_j, z_j, \bar{z}_j) = \left(\prod_{i=1}^{4} \int_0^\infty d\omega_i \omega_i^{\Delta_i - 1}\right) \mathbb{A}_4(p_j). \tag{3.5}$$

This map effectively trades asymptotic energy-momentum eigenstates for states that diagonalize boosts towards the point $(z_i, \bar{z}_i)$ on the celestial sphere. As such, the resulting celestial amplitudes transform covariantly under the Lorentz $\mathrm{SL}(2, \mathbb{C})$.

In the following, it will be convenient to recall that the momentum space amplitude (3.1) and the celestial amplitude (3.5) can be obtained directly by integrating the connected component of the time-ordered bulk correlation function $\mathrm{C}(x_1, \cdots, x_4)$ with amputated external legs against different external wavefunctions $\psi(x_i; p_i)$. While (3.1) is defined by integrating C against plane wave eigenstates $\psi(x_i; p_i) = e^{-ip_i \cdot x_i}$ [119][4]

$$\mathbb{A}_4(p_j) = \left(\prod_{i=1}^{4} \int d^4 x_i e^{-ip_i \cdot x_i}\right) \mathrm{C}(x_j), \tag{3.6}$$

[2] Technically in this parameterization the celestial sphere is flattened to a plane.

[3] Unless otherwise stated, celestial amplitudes will refer to observables on the 2D celestial sphere.

[4] We work in the mostly + signature in which the mode expansion of a massless scalar field takes the form $\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left(a^\dagger_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}} e^{ik\cdot x}\right)$. Then (3.6) with $p_i = \eta_i \omega_i \hat{q}_i$ is such that positive (negative) energy modes are created in the in ($\eta_i = -1$) (out ($\eta_i = 1$)) states (see eg. [142]).

celestial amplitudes arise from choosing the external wavefunctions $\psi(x_i;p_i)$ to be instead conformal primary solutions to the scalar wave equation $\varphi_{\Delta_i}(x_i;\eta_i\hat{q}_i)$ [29, 30]

$$\varphi_{\Delta_i}(x_i;\eta_i\hat{q}_i)=\frac{(i\eta_i)^{\Delta_i}\Gamma(\Delta_i)}{(-\hat{q}_i\cdot x_i+i\eta_i\epsilon)^{\Delta_i}}, \tag{3.7}$$

namely,

$$\widetilde{\mathcal{A}}(\Delta_j,z_j,\bar{z}_j)=\left(\prod_{i=1}^{4}\int d^4x_i\varphi_{\Delta_i}(x_i;\eta_i\hat{q}_i)\right)\mathrm{C}(x_j). \tag{3.8}$$

Indeed, (3.5) follows immediately upon noticing that plane waves and massless conformal primaries are related by a Mellin transform [29, 30],

$$\varphi_{\Delta}(x;\eta\hat{q})\equiv\int_0^\infty d\omega\omega^{\Delta-1}e^{-i\omega\eta\hat{q}\cdot x}=\frac{(i\eta)^{\Delta}\Gamma(\Delta)}{(-\hat{q}\cdot x+i\eta\epsilon)^{\Delta}}. \tag{3.9}$$

One of the aims of this work is to explore the relationship between celestial amplitudes and correlation functions of CFT_3 with bulk AdS_4 gravity duals. Such a relation was first proposed in [143], where it was argued that amplitudes in d-dimensional celestial CFT should be related to CFT_{d+1} correlators in the bulk point limit. There, this correspondence was studied explicitly for the case of 4-point scalar scattering in AdS_3 mediated by massive and massless scalar exchanges in which case the corresponding Witten diagrams in the bulk-point configuration were found to reduce to amplitudes in 1-dimensional CCFT. In the next chpater we extend the relationship between celestial amplitudes and AdS Witten diagrams by showing that generic scalar AdS_{d+1} Witten diagrams with particular kinematics reduce to CCFT_{d-1} amplitudes in the flat space limit.[5] We will check this explicitly in the example of propagation of a particle in a shockwave background, related to the eikonal exponentiation of t-channel graviton exchanges [117, 118, 122, 123]. As we will

[5]See also [90] for a different kind of relation between celestial amplitudes and AdS_3 Witten diagrams in the particular case of Yang-Mills CCFT with a marginal deformation involving a chirally coupled massive scalar. Celestial Yang-Mills theory in the presence of a spherical dilaton shockwave has also been studied in [89, 144]

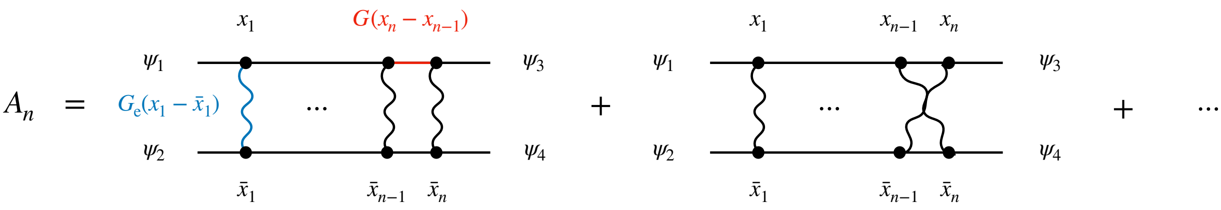

FIGURE 3.1: Contributions from ladder diagrams involving n t-channel exchanges to the scattering of 4 scalars.

see, it is the representation (3.8) that makes the connection between celestial amplitudes and CFT correlators in the flat space limit most manifest. In the next section we start by deriving a formula for the eikonal exponentiation of arbitrary spinning exchanges in celestial 4-point massless scalar scattering.

3.3 Eikonal regime in celestial CFT

In this section we propose that celestial 4-point amplitudes of massless particles have universal behavior in the limit of large net conformal dimension $\beta \gg 1$ and small cross ratio $z \ll 1$. We argue that in this kinematic regime, the CCFT encodes the eikonal physics [117, 119] of bulk 4-point scattering amplitudes. We present a formula for the eikonal exponentiation of arbitrary spinning t-channel exchanges in a conformal primary basis. We find that the eikonal exponent is in general operator valued, with weight-shifting operators replacing powers of the center-of-mass energy in a momentum space basis. Our formula shares similarities with the eikonal amplitude in AdS_4 suggesting a relation between celestial amplitudes and CFT_3 correlators with particular kinematics.

Consider the 4-point scalar scattering amplitude associated with the sum over crossed ladder diagrams with n massive exchanges of arbitrary spin j in Figure 3.1

$$A_n = (ig)^{2n} \int d^4x_1 \cdots d^4x_n d^4\bar{x}_1 \cdots d^4\bar{x}_n \psi(x_n; p_3) G(x_n - x_{n-1}) \cdots G(x_2 - x_1) \psi(x_1; p_1) \\ \times \psi(\bar{x}_n; p_4) G(\bar{x}_n - \bar{x}_{n-1}) \cdots G(\bar{x}_2 - \bar{x}_1) \psi(\bar{x}_1; p_2) \sum_{\sigma \in S_n} G_{\mathrm{e}}(x_1 - \bar{x}_{\sigma(1)}) \cdots G_{\mathrm{e}}(x_n - \bar{x}_{\sigma(n)}). \tag{3.10}$$

As indicated in Figure 3.1, G and G_{e} are internal position-space propagators corresponding to the external legs and exchanges respectively, while $\psi(x; p)$ are external wavefunctions. Each vertex comes with a factor of ig, where g is the coupling constant. As reviewed in section 3.2, the momentum space amplitude associated with n crossed ladder exchanges is obtained by taking $\psi(x; p)$ to be plane waves. Resuming the amplitudes in (3.10) for all $n > 0, n \in \mathbb{Z}$ (which excludes the disconnected contribution from $n = 0$) in the approximation where G are on-shell valid at high energies $s \gg -t$, one obtains the standard eikonal amplitude [119, 145]

$$\mathcal{A}_{\mathrm{eik}}(s, t = -p_\perp^2) \simeq 2s \int_{\mathbb{R}^2} d^2x_\perp e^{ip_\perp \cdot x_\perp} \left(e^{\frac{ig^2}{2} s^{j-1} G_\perp(x_\perp)} - 1 \right). \tag{3.11}$$

Here $G_\perp(x_\perp)$ is the transverse propagator

$$G_\perp(x_\perp) \equiv \int \frac{d^2k_\perp}{(2\pi)^2} \frac{e^{ik_\perp \cdot x_\perp}}{k_\perp^2 + m^2}, \tag{3.12}$$

$p_\perp \equiv p_{3,\perp} + p_{1,\perp}$ is the net momentum transfer and j is the spin of the exchanged particles. (3.11) is expected to approximate 4-point massless scalar scattering amplitudes in the high energy $s \gg -t$ limit [146]. It is natural to expect a similar regime to exist in celestial CFT, in which celestial amplitudes are dominated by a phase. The $s \gg -t$ regime immediately maps to a small cross-ratio $z \equiv -\frac{t}{s} \ll 1$ limit in the CCFT. Moreover, we will see in the next section that in a conformal primary basis external lines become approximately on-shell in the limit of large external dimensions Δ_1, Δ_2, or equivalently $\beta \equiv \sum_{i=1}^{4} \Delta_i - 4 \gg 1$. This resonates with the results of [33,109] where it was shown that Mellin integrals are dominated by high energies in the limit of large net boost weight. We will therefore identify a universal eikonal regime in CCFT characterized by

$$\beta \gg 1, \qquad z \ll 1. \tag{3.13}$$

3.3.1 Celestial eikonal exponentiation of scalar exchanges

The celestial counterpart of (3.11) can be obtained by evaluating (3.10) with the external wavefunctions replaced by conformal primary wavefunctions $\psi(x_i; q_i) \to \varphi_{\Delta_i}(x_i; \eta_i \hat{q}_i)$, where $\varphi_{\Delta_i}(x_i; \eta_i \hat{q}_i)$ were defined in (3.7). By construction, the resulting celestial amplitudes transform covariantly under Lorentz transformations $x \to \Lambda \cdot x$, $z \to z' = \frac{az+b}{cz+d}$ like 2D correlation functions of scalar primary operators since the measure, G and G_{e} in (3.10) are Lorentz invariant while [29]

$$\varphi_{\Delta_i}(\Lambda \cdot x_i; \eta_i \hat{q}_i(z_i', \bar{z}_i')) = \left| \frac{\partial \vec{z}_i'}{\partial \vec{z}_i} \right|^{-\Delta_i/2} \varphi_{\Delta_i}(x_i; \eta_i \hat{q}_i(z_i, \bar{z}_i)). \tag{3.14}$$

This implies that the celestial amplitude for n crossed ladder t-channel exchanges will be of the form

$$\widetilde{\mathcal{A}}_n(\Delta_i, z_i, \bar{z}_i) = I_{13-24}(z_i, \bar{z}_i) f_n(z, \bar{z}), \tag{3.15}$$

where I_{13-24} is a 4-point conformally covariant factor[6] and f_n is a function of the conformally invariant cross-ratio z. Motivated by the center of mass kinematics (see appendix A.3.1), it is convenient to parameterize the null vectors $\hat{q}_i$ as[7]

$$\begin{aligned} \hat{q}_i &= (1+q_i, q_{i,\perp}, 1-q_i), \quad i=1,3 \\ \hat{q}_i &= (1+q_i, q_{i,\perp}, -1+q_i), \quad i=2,4, \end{aligned} \tag{3.17}$$

where $q_{i,\perp}$ are 2-component vectors and $\hat{q}_i^2 = 0 \implies 4q_i = |q_{i,\perp}|^2$. At high energies, $\omega_1 \simeq \omega_3,\ \omega_2 \simeq \omega_4$ and $p_i^+ = 2\eta_i\omega_i \gg p_{i,\perp},\ p_i^- \simeq 0,$ for $i = 1,3$ and vice-versa for $2,4$ meaning that $q_i \propto |q_{i,\perp}|^2 \ll 1$. In this case the cross-ratio reduces to

$$z = -\frac{t}{s} = \frac{\omega_3}{\omega_2}\frac{\hat{q}_1\cdot\hat{q}_3}{\hat{q}_1\cdot\hat{q}_2} \simeq (q^1_{24,\perp}+iq^2_{24,\perp})(q^1_{13,\perp}-iq^2_{13,\perp}), \tag{3.18}$$

where we used momentum conservation

$$\frac{\omega_3}{\omega_2} = \frac{q^1_{24,\perp}+iq^2_{24,\perp}}{q^1_{13,\perp}+iq^2_{13,\perp}} = \frac{q^1_{24,\perp}-iq^2_{24,\perp}}{q^1_{13,\perp}-iq^2_{13,\perp}}. \tag{3.19}$$

We hence see that eikonal kinematics imply small z. Note that in the $z \to 0$ limit, (3.17) are a special case (up to a Jacobian factor) of (3.3) where the momenta of $1,3$ and $2,4$ are respectively expanded around antipodal points on the celestial sphere. This kinematic configuration is illustrated in Figure 3.2.

[6] For $n \geq 1$ it takes the form

$$I_{13-24}(z_i,\bar{z}_i) = \frac{\left(\frac{z_{34}}{z_{14}}\right)^{h_{13}}\left(\frac{z_{14}}{z_{12}}\right)^{h_{24}}}{z_{13}^{h_1+h_3}z_{24}^{h_2+h_4}}\frac{\left(\frac{\bar{z}_{34}}{\bar{z}_{14}}\right)^{\bar{h}_{13}}\left(\frac{\bar{z}_{14}}{\bar{z}_{12}}\right)^{\bar{h}_{24}}}{\bar{z}_{13}^{\bar{h}_1+\bar{h}_3}\bar{z}_{24}^{\bar{h}_2+\bar{h}_4}} \tag{3.16}$$

with $h_i = \bar{h}_i = \frac{\Delta_i}{2}$, but it may also involve singular conformally covariant structures as will be the case for the disconnected $n=0$ contribution.

[7] The complex coordinates $(z_i,\bar{z}_i)$, $(w_i,\bar{w}_i)$ in the parameterizations $q_{i,\perp} = (z_i+\bar{z}_i, -i(z_i-\bar{z}_i))$ for $i=1,3$, and $q_{i,\perp} = (w_i+\bar{w}_i, -i(w_i-\bar{w}_i))$ for $i=2,4$ are in different patches. Writing both in the same patch introduces Jacobian factors in the celestial amplitudes.

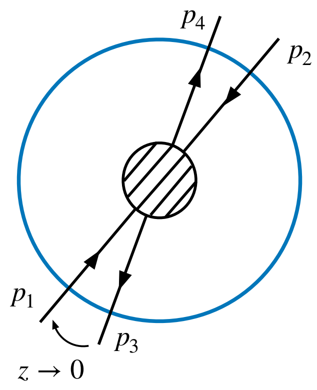

FIGURE 3.2: Eikonal kinematics in which the operators associated with particles 1, 3 and 2, 4 are respectively inserted around antipodal points on the celestial sphere.

To evaluate the integrals in (3.10) we employ light-cone coordinates,

$$x^- = x^0 - x^3, \quad x^+ = x^0 + x^3, \quad x_\perp^i = x^i, \ i = 1, 2, \tag{3.20}$$

in which the Minkowski metric takes the form

$$ds^2 = -dx^- dx^+ + ds_\perp^2. \tag{3.21}$$

In the limit $q_i \ll 1$, $\hat{q}_i \cdot x$ are approximated by [119]

$$\hat{q}_i \cdot x = -x^- + q_{i,\perp} \cdot x_\perp - q_i x^+ \simeq -x^- + q_{i,\perp} \cdot x_\perp, \quad i = 1, 3, \tag{3.22}$$

$$\hat{q}_i \cdot x = -x^+ + q_{i,\perp} \cdot x_\perp - q_i x^- \simeq -x^+ + q_{i,\perp} \cdot x_\perp, \quad i = 2, 4 \tag{3.23}$$

and the conformal primary wavefunctions are therefore given by

$$\varphi_{\Delta_1}(x; -\hat{q}_1) = \frac{(-i)^{\Delta_1}\Gamma(\Delta_1)}{(x^- - q_{1,\perp} \cdot x_\perp - i\epsilon)^{\Delta_1}}, \quad \varphi_{\Delta_3}(x; \hat{q}_3) = \frac{i^{\Delta_3}\Gamma(\Delta_3)}{(x^- - q_{3,\perp} \cdot x_\perp + i\epsilon)^{\Delta_3}}, \tag{3.24}$$

$$\varphi_{\Delta_2}(x;-\hat{q}_2) = \frac{(-i)^{\Delta_2}\Gamma(\Delta_2)}{(x^+ - q_{2,\perp}\cdot x_\perp - i\epsilon)^{\Delta_2}}, \quad \varphi_{\Delta_4}(x;\hat{q}_4) = \frac{i^{\Delta_4}\Gamma(\Delta_4)}{(x^+ - q_{4,\perp}\cdot x_\perp + i\epsilon)^{\Delta_4}}. \tag{3.25}$$

In a momentum space basis it can be argued that in the high energy limit, the internal 1-3 and 2-4 propagators are well approximated by on-shell ones (corresponding to classical particle trajectories). In a conformal primary basis, energies are traded for conformal dimensions and it is not obvious whether an analogous argument can be made. Nevertheless, we show in appendix A.1 that a similar approximation holds instead at large $\Delta_1, \Delta_2 \gg 1$, in which case these propagators become

$$G_{13}(x_i, x_j) = -\frac{i(x_i^- - q_{1,\perp}\cdot x_{i,\perp} + i\epsilon)}{2\Delta_1}\delta(x_i^- - x_j^-)\Theta(x_i^+ - x_j^+)\delta^{(2)}(x_{i,\perp} - x_{j,\perp}), \tag{3.26}$$

$$G_{24}(\bar{x}_i, \bar{x}_j) = -\frac{i(\bar{x}_i^+ - q_{2,\perp}\cdot \bar{x}_{i,\perp} + i\epsilon)}{2\Delta_2}\Theta(\bar{x}_i^- - \bar{x}_j^-)\delta(\bar{x}_i^+ - \bar{x}_j^+)\delta^{(2)}(\bar{x}_{i,\perp} - \bar{x}_{j,\perp}). \tag{3.27}$$

As for the propagators for scalar exchanges of mass m, we use the standard formula [142]

$$G_e(x-\bar{x}) = -i\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-\bar{x})}}{k^2+m^2-i\epsilon}. \tag{3.28}$$

We now have all ingredients needed to evaluate (3.10). We refer the reader to appendix A.2 for the lengthy yet straightforward calculation and simply state the result. For n crossed scalar exchanges of mass m we find

$$\widetilde{\mathcal{A}}_n = 4(2\pi)^2\int d^2x_\perp d^2\bar{x}_\perp \frac{(i\hat{\chi})^n}{n!}\frac{i^{\Delta_1+\Delta_3}\Gamma(\Delta_1+\Delta_3)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3}}\frac{i^{\Delta_2+\Delta_4}\Gamma(\Delta_2+\Delta_4)}{(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4}}, \tag{3.29}$$

where we defined

$$\hat{\chi} \equiv \frac{g^2}{8}e^{-\partial_{\Delta_1}}e^{-\partial_{\Delta_2}}G_\perp(x_\perp, \bar{x}_\perp), \tag{3.30}$$

and $G_\perp$ is the position space transverse propagator in (3.12).

Summing all connected diagrams with $n > 0$ yields the eikonal celestial amplitude

$$\widetilde{\mathcal{A}}_{\text{eik}} \simeq 4(2\pi)^2 \int d^2x_\perp d^2\bar{x}_\perp \left(e^{i\hat{\chi}} - 1\right) \frac{i^{\Delta_1+\Delta_3}\Gamma(\Delta_1+\Delta_3)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3}} \frac{i^{\Delta_2+\Delta_4}\Gamma(\Delta_2+\Delta_4)}{(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4}}, \tag{3.31}$$

where $\simeq$ stands for the leading terms in the celestial eikonal regime of large Δ_1, Δ_2 and small z. This formula (together with its generalization to arbitrary spinning exchanges where (3.30) is simply replaced by (3.59)) is one of the main results of this chapter. It has two interesting features. First, the eikonal phase $\hat{\chi}$ is operator valued for all spins $j \neq 1$. This feature of CCFT is familiar from both celestial double copy constructions [45,147] and the conformally soft exponentiation of infrared divergences in gravity [109,114,148,149]. Second, it looks remarkably similar to the eikonal amplitude in AdS [119]. Indeed, we will later establish a relation between its cousin, the celestial two-point function in a shockwave background, and the flat-space limit of its AdS counterpart. A general argument for the relation between AdS_{d+1} Witten diagrams in the flat-space limit and CCFT_{d-1} amplitudes will be given in section 3.6.

The eikonal formula can also be directly derived as a Mellin transform of the momentum space amplitude (3.11). While this is to be expected from the standard relation between conformal primary wavefunctions and plane waves (3.9), we find it nevertheless instructive to provide this alternate derivation in the remainder of this section.

Mellin transform of the eikonal amplitude

We now show that the celestial eikonal amplitude (3.31) is simply a Mellin transform of the scalar momentum space eikonal amplitude (3.11) including the momentum conserving delta function,

$$\mathbb{A}_{\text{eik}} = 2s \int_{\mathbb{R}^2} d^2x_\perp e^{ip_\perp \cdot x_\perp} \left[\exp\left(\frac{ig^2}{2s} G_\perp(x_\perp)\right) - 1\right] (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^4 p_i\right). \tag{3.32}$$

Our strategy is to start with the celestial eikonal formula (3.31) and show that it can be recast as a Mellin transform of (3.32) with respect to the external energies. To this end, consider the Taylor expansion of (3.31) in powers of g^2,

$$\begin{aligned}\widetilde{\mathcal{A}}_{\text{eik}} &= 4(2\pi)^2 \sum_{n=1}^{\infty} \frac{1}{n!} \int d^2x_\perp d^2\bar{x}_\perp \left(\frac{ig^2}{8} G_\perp(x_\perp, \bar{x}_\perp) \right)^n \\ &\times \frac{i^{\Delta_1+\Delta_3-n}\Gamma(\Delta_1+\Delta_3-n)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3-n}} \frac{i^{\Delta_2+\Delta_4-n}\Gamma(\Delta_2+\Delta_4-n)}{(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4-n}}. \end{aligned} \tag{3.33}$$

Introducing parameters ω_1, ω_2 and using the Mellin representation (3.9) for each term in the sum,

$$\begin{aligned}\widetilde{\mathcal{A}}_{\text{eik}} &= 4(2\pi)^2 \sum_{n=1}^{\infty} \frac{1}{n!} \int d^2x_\perp d^2\bar{x}_\perp \left(\frac{ig^2}{8} G_\perp(x_\perp, \bar{x}_\perp) \right)^n \int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_2} \omega_1^{\Delta_1+\Delta_3-n} \omega_2^{\Delta_2+\Delta_4-n} \\ &\times e^{-i\omega_1 q_{13,\perp}\cdot x_\perp} e^{-i\omega_2 q_{24,\perp}\cdot \bar{x}_\perp} \\ &= \int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_2} \omega_1^{\Delta_1+\Delta_3} \omega_2^{\Delta_2+\Delta_4} 4(2\pi)^2 \int d^2\bar{x}_\perp e^{-i(\omega_1 q_{13,\perp}+\omega_2 q_{24,\perp})\cdot \bar{x}_\perp} \\ &\times \sum_{n=1}^{\infty} \frac{1}{n!} \int d^2x_\perp \left(\frac{ig^2}{2\cdot 4\omega_1\omega_2} G_\perp(x_\perp) \right)^n e^{-i\omega_1 q_{13,\perp}\cdot x_\perp}, \end{aligned} \tag{3.34}$$

where the last line follows from shifting $x_\perp \to x_\perp + \bar{x}_\perp$ under which $G(x_\perp, \bar{x}_\perp) \to G(x_\perp)$. The integrals over $x_\perp$ and $\bar{x}_\perp$ are now decoupled and the latter evaluates to a delta function

$$\begin{aligned}\widetilde{\mathcal{A}}_{\text{eik}} &= \int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_2} \omega_1^{\Delta_1+\Delta_3} \omega_2^{\Delta_2+\Delta_4} 4(2\pi)^4 \delta^{(2)}(\omega_1 q_{1,\perp} + \omega_2 q_{2,\perp} - \omega_1 q_{3,\perp} - \omega_2 q_{4,\perp}) \\ &\times \sum_{n=1}^{\infty} \frac{1}{n!} \int d^2x_\perp \left(\frac{ig^2}{2\cdot 4\omega_1\omega_2} G_\perp(x_\perp) \right)^n e^{-i\omega_1 q_{13,\perp}\cdot x_\perp}. \end{aligned} \tag{3.35}$$

Inserting the identity

$$\int_0^\infty d\omega_3 d\omega_4 \delta(\omega_3 - \omega_1)\delta(\omega_4 - \omega_2) = 1, \tag{3.36}$$

(3.35) reduces to

$$\begin{aligned}\widetilde{\mathcal{A}}_{\text{eik}} = \int_0^\infty \left(\prod_{i=1}^4 \frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\right)(2\pi)^4\delta(\omega_1-\omega_3)\delta(\omega_2-\omega_4)\delta^{(2)}(\omega_1 q_{1,\perp}+\omega_2 q_{2,\perp}-\omega_3 q_{3,\perp}-\omega_4 q_{4,\perp})\\ \times 4\omega_1\omega_2\sum_{n=1}^\infty \frac{1}{n!}\int d^2x_\perp\left(\frac{ig^2}{2\cdot 4\omega_1\omega_2}G_\perp(x_\perp)\right)^n e^{-i\omega_1 q_{13,\perp}\cdot x_\perp}.\end{aligned} \tag{3.37}$$

Using the parameterizations of momenta (3.3) in the eikonal configuration (3.17) with $q_i \ll 1$,[8]

$$p_1^+ = -2\omega_1, \quad p_2^- = -2\omega_2, \quad p_3^+ = 2\omega_3, \quad p_4^- = 2\omega_4, \tag{3.38}$$

while the components with $+ \leftrightarrow -$ vanish to leading order. Then

$$\begin{aligned}\widetilde{\mathcal{A}}_{\text{eik}} &= \int_0^\infty \prod_{i=1}^4 \frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}(2\pi)^4 4\delta(p_1^+ + p_3^+)\delta(p_2^- + p_4^-)\delta^{(2)}(p_{1,\perp}+p_{2,\perp}+p_{3,\perp}+p_{4,\perp})\\ &\times\; 4\omega_1\omega_2\sum_{n=1}^\infty\frac{1}{n!}\int d^2x_\perp\left(\frac{ig^2}{2\cdot 4\omega_1\omega_2}G_\perp(x_\perp)\right)^n e^{i(p_{1,\perp}+p_{3,\perp})\cdot x_\perp},\end{aligned} \tag{3.39}$$

and since

$$\delta^{(4)}(p) = 2\delta(p^+)\delta(p^-)\delta^{(2)}(p_\perp), \quad s \simeq 4\omega_1\omega_2 \tag{3.40}$$

we find

$$\widetilde{\mathcal{A}}_{\text{eik}} = \prod_{i=1}^4\left(\int_0^\infty \frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\right)\mathbb{A}_{\text{eik}}. \tag{3.41}$$

[8] Here $p^+ = p^0 + p^3$, $p^- = p^0 - p^3$.

This shows that the celestial eikonal amplitude (3.31) is precisely the Mellin transform of the momentum space eikonal formula (3.11). On the one hand, this result seems to follow from the defining relations (3.6), (3.8), (3.9). On the other hand, our first derivation in appendix A.2 invokes the approximations (3.26), (3.27) for the external line propagators in a conformal primary basis which are valid at large Δ_1, Δ_2. Here, we see instead that Δ_1, Δ_2 need to be large in order for the integrand of (3.41) to be dominated by eikonal kinematics. We regard this perfect match as evidence that (3.31) describes the behavior of scalar celestial 4-point scattering to leading order in the celestial eikonal limit (3.31) and to all orders in the coupling g. We conclude by pointing out that the Mellin transform of the eikonal phase (3.41) is not strictly convergent. This issue can be cured by allowing for the phase-shift to acquire an imaginary part. Physically, this could be for example due to black hole production [150], or due to radiation-reaction effects [151, 152]. It would be interesting to further study the implications of the convergence of celestial amplitudes, in relation to the conjecture proposed in [36] (for recent work in this direction see also [153]).

In the next section we show that the leading term in an expansion of (3.31) in powers of g reproduces the tree-level celestial scalar 4-point amplitude with a massive t-channel exchange in the $z \to 0$ limit.

3.3.2 Perturbative expansion

As a warm up, let us start by evaluating the disconnected contribution

$$\widetilde{\mathcal{A}}_0 = 4(2\pi)^2 \int d^2x_\perp \frac{i^{\Delta_1+\Delta_3}\Gamma(\Delta_1+\Delta_3)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3}} \int d^2\bar{x}_\perp \frac{i^{\Delta_2+\Delta_4}\Gamma(\Delta_2+\Delta_4)}{(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4}} \tag{3.42}$$

given by setting $n = 0$ in (3.29). While this term has been removed in our formulas, we expect it to reduce to the product of two scalar celestial two point functions with the correct normalization given in [30]. These integrals can be evaluated by writing the integrands in their Mellin representations and evaluating the integrals over the transverse coodinates which give rise to delta functions,

$$\widetilde{\mathcal{A}}_0 = 4(2\pi)^2 \left[(2\pi)^2\delta^{(2)}(q_{13,\perp}) \int_0^\infty d\omega_1 \omega_1^{(\Delta_1+\Delta_3-2)-1}\right] \left[(2\pi)^2\delta^{(2)}(q_{24,\perp}) \int_0^\infty d\omega_2 \omega_2^{(\Delta_2+\Delta_4-2)-1}\right]. \tag{3.43}$$

The remaining Mellin transforms follow from [30][9]

$$\delta(i\Delta) = \frac{1}{2\pi}\int_0^\infty d\omega\omega^{\Delta-1}, \tag{3.44}$$

therefore $\widetilde{\mathcal{A}}_0$ factorizes as

$$\widetilde{\mathcal{A}}_0 = \left[(2\pi)^4\delta^{(2)}(z_{13})\delta(\Delta_1+\Delta_3-2)\right]\left[(2\pi)^4\delta^{(2)}(w_{24})\delta(\Delta_2+\Delta_4-2)\right]. \tag{3.45}$$

(3.45) agrees with the product of two celestial two-point functions, or equivalently the disconnected contribution to massless scalar 4-point t-channel scattering.

We now turn to the leading contribution to (3.31) in a small g expansion. This should reproduce the celestial amplitude for massive t-channel exchange [35, 155]. We start with

$$\widetilde{\mathcal{A}}_1 = 2\pi^2 ig^2 \int d^2x_\perp d^2\bar{x}_\perp G_\perp(x_\perp,\bar{x}_\perp)\frac{i^{\Delta_1+\Delta_3-1}\Gamma(\Delta_1+\Delta_3-1)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3-1}}\frac{i^{\Delta_2+\Delta_4-1}\Gamma(\Delta_2+\Delta_4-1)}{(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4-1}}. \tag{3.46}$$

Replacing $G_\perp(x_\perp,\bar{x}_\perp)$ by its Fourier representation (3.12), and using the Mellin representation of the conformal primary wavefunctions, the integrals over $x_\perp$ and $\bar{x}_\perp$ decouple

[9] Such integrals are formally valid for $\Delta_i \in 1+i\lambda$ for $\lambda \in \mathbb{R}$, violating our eikonal conditions $\Delta_1, \Delta_2 \gg 1$. We regard the dimensions in (3.45) as analytically continued away from the principal series, see [154] for a prescription to do so. Note that the eikonal conditions on Δ_1, Δ_2 only translate into a condition on β for connected celestial amplitudes.

and again become delta functions

$$\widetilde{\mathcal{A}}_1 = \frac{(2\pi)^4 i g^2}{2} \int_0^\infty \frac{d\omega_1}{\omega_1} \omega_1^{\Delta_1+\Delta_3-1} \int_0^\infty \frac{d\omega_2}{\omega_2} \omega_2^{\Delta_2+\Delta_4-1} \times \int d^2 k_\perp \frac{1}{k_\perp^2 + m^2} \delta^{(2)}(k_\perp - \omega_1 q_{13,\perp}) \delta^{(2)}(k_\perp + \omega_2 q_{24,\perp}). \tag{3.47}$$

The remaining integrals are evaluated in appendix A.3 and result in

$$\widetilde{\mathcal{A}}_1 = \frac{(2\pi)^4 i g^2}{\sin \pi\beta/2} \frac{\pi m^{\beta-2}}{4} \left(-\frac{q^1_{24,\perp}}{q^1_{13,\perp}} \right)^{\Delta_2+\Delta_4-2} |q_{13,\perp}|^{-\beta} \, \delta(q^1_{24,\perp} q^2_{13,\perp} - q^2_{24,\perp} q^1_{13,\perp}). \tag{3.48}$$

From (3.18) we immediately see that the delta function imposes reality of the cross-ratio, $z - \bar{z} = 0$. Moreover, in the center of mass frame with z allowed to be complex,

$$\begin{aligned} &q_{1,\perp} = (0,0), \;\; q_{2,\perp} = (0,0), \\ &q_{3,\perp} = \left(\sqrt{z} + \sqrt{\bar{z}}, -i(\sqrt{z} - \sqrt{\bar{z}})\right), \;\; q_{4,\perp} = \left(-\sqrt{z} - \sqrt{\bar{z}}, -i(\sqrt{z} - \sqrt{\bar{z}})\right) \end{aligned} \tag{3.49}$$

we find that

$$\widetilde{\mathcal{A}}_1 = (2\pi)^4 i \left(\sqrt{z}\right)^{-\beta} \delta(z - \bar{z}) \frac{g^2 \pi}{8m^2} \frac{(m/2)^\beta}{\sin \pi\beta/2} + \cdots . \tag{3.50}$$

Here $\cdots$ denote subleading terms in the small z limit which don't contribute at leading order in the eikonal approximation (3.13).

To compare to the expected result (see [155] for the formula in the same conventions used here up to a factor of $(2\pi)^4 i$)

$$\widetilde{\mathcal{A}}_{\text{t-channel}}(\Delta_i, z_i, \bar{z}_i) = I_{13-24}(z_i, \bar{z}_i) N_{gm}(\beta) \delta(z - \bar{z}) |z|^2 |z-1|^{h_{13}-h_{24}}, \tag{3.51}$$

$$N_{gm}(\beta) = \frac{g^2 \pi}{8m^2} \frac{(m/2)^\beta}{\sin \pi\beta/2} \tag{3.52}$$

we now evaluate (3.51) in the corresponding kinematic configuration

$$z_1 = 0, \quad z_2 = \infty, \quad z_3 = \sqrt{z}, \quad z_4 = -\frac{1}{\sqrt{z}}. \tag{3.53}$$

We find[10]

$$\lim_{z_2\to\infty, z\to 0} |z_2|^{2\Delta_2} \left|\sqrt{z}\right|^{-2\Delta_4} \widetilde{\mathcal{A}}_{\text{t-channel}}\left(0,\infty,\sqrt{z},-\frac{1}{\sqrt{z}}\right) = \frac{g^2\pi}{8m^2}\frac{(m/2)^\beta}{\sin\pi\beta/2}\delta(z-\bar{z})(\sqrt{z})^{-\beta}. \tag{3.54}$$

We hence see that the tree-level contribution to the eikonal expansion (3.31) agrees with the t-channel massive scalar exchange celestial amplitude as it should.

3.3.3 Generalization to spinning exchanges

In this section we generalize the celestial eikonal formula (3.31) to the case where the exchanges have arbitrary spin j.

Spinning propagators $G_e^{\mu_1\ldots\mu_j\nu_1\ldots\nu_j}(x,\bar{x})$ couple to the external lines via derivative interactions. As argued in section 3.3.1, in the eikonal limit external propagators are approximated by (3.26) and (3.27). This implies that, in analogy to the derivation in [119], the dominant contribution from (celestial) spinning propagators in the eikonal limit is

$$\widetilde{G}_{\text{e}}(x_i,\bar{x}_{\sigma(i)}) = (-2)^j P^1_{\mu_1}\cdots P^1_{\mu_j} G_{\text{e}}^{\mu_1\ldots\mu_j\nu_1\ldots\nu_j}(x_i,\bar{x}_{\sigma(i)}) P^2_{\nu_1}\cdots P^2_{\nu_j}, \tag{3.55}$$

with[11]

$$G_{\text{e}}^{\mu_1\ldots\mu_j\nu_1\ldots\nu_j}(x,\bar{x}) \simeq \eta^{(\mu_1\nu_1}\cdots\eta^{\mu_j\nu_j)} G_{\text{e}}(x,\bar{x}). \tag{3.56}$$

Here the indices μ,ν are separately symmetrized, $G_e(x,\bar{x})$ is the scalar propagator given

[10] Note that $1,3$ and $2,4$ are evaluated in patches around the north and south poles respectively, hence the Jacobian factor is needed in (3.54) for comparison with (3.50).

[11] We stick to the convention in [119] that the external particles are oppositely charged with respect to odd j fields. Trace terms vanish since P_1, P_2 are on-shell, while terms where the derivatives are distributed over all $1,3$ and the propagator are subleading in the eikonal limit.

in (3.28) and we defined the celestial massless momentum operators P^1_μ and P^2_μ acting on external particles 1 and 2 [33]

$$P^i_\mu = -(\hat{q}_i)_\mu e^{\partial_{\Delta_i}}, \quad i = 1, 2. \tag{3.57}$$

One can therefore follow through the same derivation in appendix A.2 with the simple replacement

$$G_{\rm e}(x_i, \bar{x}_{\sigma(i)}) \to \widetilde{G}_{\rm e}(x_i, \bar{x}_{\sigma(i)}) \simeq (-2P^1 \cdot P^2)^j G_{\rm e}(x_i, \bar{x}_{\sigma(i)}). \tag{3.58}$$

Recalling that the eikonal kinematics are such that $\hat{q}_1 \cdot \hat{q}_2 \approx -2$, the final result is of the same form as (3.31) with $\hat{\chi} \to \hat{\chi}_j$, where

$$\hat{\chi}_j = \frac{g^2 (4 e^{\partial_{\Delta_1}} e^{\partial_{\Delta_2}})^{j-1}}{2} G_\perp(x_\perp, \bar{x}_\perp). \tag{3.59}$$

For $j = 0$, we recover precisely (3.31). The same derivation goes through for massless exchanges, with the transverse propagator (3.12) replaced by its massless counterpart with $m = 0$. In this case the transverse propagator develops a logarithmic divergence which can be regulated by introducing an IR cutoff. The AdS radius R provides a natural cutoff in AdS. The IR divergence reappears in the limit as $R \to \infty$, as we will see in the related analysis of shockwave two-point functions in section 4.5.3. We finally note that, for $j = 2$, the celestial eikonal amplitude (3.41) is analytic in the right-hand complex net boost-weight β plane. A similar analytic structure was found in [109] for celestial 2-2 scattering dominated by black hole production at high energies. It would be interesting to further explore the relation between non-perturbative aspects of gravitational scattering and the analytic properties of celestial amplitudes.

In the remainder of this chapter, we will focus on the formula for graviton exchanges, namely $j = 2$ and $m = 0$, in which case $g^2 = 8\pi G$. We will see that the celestial eikonal exponentiation of graviton exchanges is related to the celestial two-point function of a particle in a shockwave background. In particular, we will identify the source in the CCFT that relates the two to leading order in perturbation theory. Interestingly, this relation is analogous to the one in AdS/CFT and will be shown in section 4.5.3 to be directly recovered in a flat space limit of the AdS result.

3.4 Celestial scattering in shockwave background

In this section we study the celestial amplitude describing the propagation of a scalar field in the presence of a shock $h_{--}(x^-, x_\perp) = \delta(x^-)h(x_\perp)$. We compare the leading term in an expansion of this two-point function in powers of h with the leading connected contribution to the eikonal celestial amplitude involving a spin 2 exchange computed in section 3.3 and find perfect agreement. Moreover, we show that this formula arises as the flat-space limit of the scalar two-point function in the presence of a shock in AdS_4. This establishes a relation between celestial propagation in a shockwave background and the flat space limit of four-point functions in CFT_3 with operators inserted in small time windows around future and past boundary spheres.

3.4.1 Review: scalar field in shockwave background

We consider the shockwave geometry

$$ds^2 \quad = \quad -dx^-dx^+ + ds_\perp^2 + h(x_\perp)\delta(x^-)(dx^-)^2 \tag{3.60}$$

sourced by a stress tensor whose only non-vanishing component is

$$T_{--} = \delta(x^-)T(x_\perp), \tag{3.61}$$

localized along the null surface $x^- = 0$. The metric (3.60) solves the full non-linear Einstein's equations provided that [118, 122, 123]

$$\partial_\perp^2 h(x_\perp) = -\frac{\kappa^2}{2} T(x_\perp), \tag{3.62}$$

where $\kappa^2 = 32\pi G$.[12]

On the other hand, the propagation of a scalar field in the background (3.60) is governed by the wave equation

$$\Box_{\text{shock}} \phi(x) = 0 \tag{3.63}$$

which reduces to

$$-4\partial_- \partial_+ \phi - 4\delta(x^-) h(x_\perp) \partial_+^2 \phi + \partial_\perp^2 \phi = 0. \tag{3.64}$$

In a neighborhood of $x^- = 0$, the transverse part can be neglected and (3.64) simplifies to

$$\partial_+ \partial_- \phi = -h(x_\perp) \delta(x^-) \partial_+^2 \phi. \tag{3.65}$$

Taking a Fourier transform of both sides with respect to x^+ and integrating by parts, we find

$$\partial_- \widetilde{\phi}(x^-, k, x_\perp) \;\; = \;\; -ikh(x_\perp) \delta(x^-) \widetilde{\phi}(x^-, k, x_\perp), \tag{3.66}$$

where we defined the Fourier transform of ϕ with respect to x^+

$$\widetilde{\phi}(x^-, k, x_\perp) \equiv \int_{-\infty}^{\infty} dx^+ \widetilde{\phi}(x^-, x^+, x_\perp) e^{-ikx^+}. \tag{3.67}$$

[12] Our conventions follow from the Einstein-Hilbert action coupled to matter $S_{g+m} = \int d^4x \sqrt{-g} \left(\frac{2}{\kappa^2} R + \mathcal{L}_M \right)$.

The solution is obtained by integrating (3.66) over x^- with $x^- \in [-\epsilon, \epsilon]$ for infinitesimal $\epsilon > 0$. One finds that the scalar modes before and after the shock are simply related by a phase shift

$$\widetilde{\phi}(\epsilon, k, x_\perp) = \widetilde{\phi}(-\epsilon, k, x_\perp) e^{-ikh(x_\perp)}. \tag{3.68}$$

Equivalently, upon inverting the Fourier transform we find the matching condition

$$\phi(\epsilon, x^+, x_\perp) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \widetilde{\phi}(-\epsilon, k, x_\perp) e^{-ikh(x_\perp)+ikx^+} = \phi(-\epsilon, x^+ - h(x_\perp), x_\perp). \tag{3.69}$$

We hence recover the well known result [122] that upon crossing a shockwave, probe particles acquire a time shift $\Delta x^+ = h(x_\perp)$.

3.4.2 Celestial shock two-point function

Equipped with this result, it can be shown (see appendix A.4) that the scalar propagator in the background of the shock (3.60) takes the form

$$A_{\text{shock}}(p_2, p_4) = 4\pi p_4^- \delta(p_4^- + p_2^-) \int d^2 x_\perp e^{i(p_{4,\perp} + p_{2,\perp})\cdot x_\perp} e^{i\frac{h(x_\perp)}{2} p_2^-}. \tag{3.70}$$

To express this in a conformal primary basis, we parameterize p_i as in (3.3), (3.17) in which case

$$p_i^- = 2\eta_i \omega_i, \qquad p_{i,\perp} = \eta_i \omega_i (z_i + \bar{z}_i, -i(z_i - \bar{z}_i)) \equiv \eta_i \omega_i q_{i,\perp} \tag{3.71}$$

and the momentum space amplitude (3.70) becomes

$$A_{\text{shock}}(p_2, p_4) \;\; = \;\; 4\pi\omega_4 \delta(\omega_4 - \omega_2) \int d^2 x_\perp e^{i(\omega_4 q_{4,\perp} - \omega_2 q_{2,\perp})\cdot x_\perp} e^{-i\omega_2 h(x_\perp)}. \tag{3.72}$$

The celestial propagator is then found by evaluating Mellin transforms with respect to ω_2 and ω_4,

$$\widetilde{A}_{\text{shock}}(\Delta_2, z_2, \bar{z}_2; \Delta_4, z_4, \bar{z}_4) \;=\; \int_0^\infty d\omega_2 \omega_2^{\Delta_2-1} \int_0^\infty d\omega_4 \omega_4^{\Delta_4-1} A_{\text{shock}}(p_2, p_4). \quad (3.73)$$

One of the Mellin transforms is easily computed due to the delta function in energy and the remaining Mellin integral reduces to the standard Mellin transform of an exponential, namely

$$\begin{aligned}\widetilde{A}_{\text{shock}}(\Delta_2, z_2, \bar{z}_2; \Delta_4, z_4, \bar{z}_4) &= 4\pi \int_0^\infty d\omega_2 \omega_2^{\Delta_2+\Delta_4-1} \int d^2x_\perp e^{-i\omega_2\left[q_{24,\perp}\cdot x_\perp + h(x_\perp)\right]} \\ &= 4\pi \int d^2x_\perp \frac{i^{\Delta_2+\Delta_4}\Gamma(\Delta_2+\Delta_4)}{[-q_{24,\perp}\cdot x_\perp - h(x_\perp) + i\epsilon]^{\Delta_2+\Delta_4}}.\end{aligned} \quad (3.74)$$

This formula is remarkably similar to its counterpart in AdS_4 [123]

$$\langle \mathcal{O}_\Delta(\mathbf{p}_2)\mathcal{O}_\Delta(\mathbf{p}_4)\rangle_{\text{shock}} = \mathcal{C}_\Delta \int_{H_2} d^2\mathbf{x}_\perp \frac{\Gamma(2\Delta)}{(2q\cdot\mathbf{x}_\perp - \mathbf{h}(\mathbf{x}_\perp) + i\epsilon)^{2\Delta}}, \quad (3.75)$$

where $\mathbf{p}_2 = -(0,1,0),\ \mathbf{p}_4 = (q^2,1,q)$[13] are embedding space (here $\mathbb{R}^{1,1}\times\mathbb{R}^{1,2}$) coordinates, $\mathbf{h}(\mathbf{x}_\perp)$ is a solution to the AdS counterpart of (3.62) and $\mathcal{C}_\Delta$ is a normalization constant given by

$$\mathcal{C}_\Delta \equiv \frac{1}{\pi^2}\frac{R^{2(\Delta-1)}}{\Gamma(\Delta-\frac{1}{2})^2}. \quad (3.76)$$

In section 4.5.3 we explain how it can be obtained from a flat space limit. Before that, we clarify the relation between (3.74) and the celestial amplitude that resums the eikonal spin 2 exchanges.

[13] Our h is defined with respect to a future-pointing x^+ hence the apparent sign difference with respect to [123]

3.4.3 Relation to eikonal amplitude

The momentum space scalar propagator (3.70) reproduces the plane wave basis four-point eikonal amplitude of massless scalars interacting by graviton exchange, given an appropriate choice for the shockwave source [122]. In this section we identify the shockwave source in the CCFT following a similar procedure to that of [123] in the AdS context.

To this end, we consider the leading term in the expansion of the celestial eikonal amplitude for graviton exchange, namely

$$\widetilde{\mathcal{A}}_1^{j=2} = 8\pi^2 i\kappa^2 \int d^2x d^2\bar{x}_\perp G_\perp^{m=0}(x_\perp, \bar{x}_\perp) \frac{i^{\Delta_1+\Delta_3+1}\Gamma(\Delta_1+\Delta_3+1)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3+1}} \frac{i^{\Delta_2+\Delta_4+1}\Gamma(\Delta_2+\Delta_4+1)}{(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4+1}}. \tag{3.77}$$

On the other hand, expanding (3.74) to linear order in $h(x_\perp)$, we find

$$\widetilde{A}^1_{\text{shock}} = -4\pi i \int d^2x_\perp \frac{i^{\Delta_2+\Delta_4+1}\Gamma(\Delta_2+\Delta_4+1)}{(-q_{24,\perp}\cdot x_\perp + i\epsilon)^{\Delta_2+\Delta_4+1}} h(x_\perp). \tag{3.78}$$

Upon choosing

$$h(x_\perp) = -2\pi\kappa^2 \int d^2\bar{x}_\perp G_\perp^{m=0}(x_\perp, \bar{x}_\perp) \frac{i^{\Delta_1+\Delta_3+1}\Gamma(\Delta_1+\Delta_3+1)}{(-q_{13,\perp}\cdot \bar{x}_\perp)^{\Delta_1+\Delta_3+1}}, \tag{3.79}$$

with

$$T = T(\bar{x}_\perp) = -4\pi \frac{i^{\Delta_1+\Delta_3+1}\Gamma(\Delta_1+\Delta_3+1)}{(-q_{13,\perp}\cdot \bar{x}_\perp)^{\Delta_1+\Delta_3+1}}, \tag{3.80}$$

we see that (3.78) reproduces (3.77). Note that while in a momentum space basis, the energy-momentum tensor carries a scale associated with the energy of the source,[14] (3.80) provides a definition of the source intrinsic to the CCFT. Up to normalization, (3.80) is analogous to the CFT_3 source found in [123] when relating the AdS shockwave two-point function to the AdS eikonal amplitude. It is obtained by comparing the leading term

[14] We thank Tim Adamo for an interesting discussion on this point.

in the perturbative expansion of (3.74) in $h(x_\perp)$ to the celestial amplitude (3.46) for t-channel graviton exchange. As such the stress tensor (3.80) corresponds to a point source in a conformal primary basis. In the next chapter we clarify this connection by showing that the celestial formulas can be obtained directly as flat space limit of CFT_3 correlators with particular kinematics.

3.5 Flat space limit of shockwave two-point function in AdS_4

The symmetries of celestial amplitudes inherited from 4D Lorentz invariance are the same as the symmetries that preserve codimension-1 slices of CFT_3. Since in the flat space limit, CFT_3 operators are known to localize on such global time slices [61,63,138,141], it is natural to expect a direct relation between CFT_3 correlation functions in the flat space limit and celestial amplitudes. In this section we illustrate how this works in the case of the shockwave two-point function (3.74). Specifically, after reviewing the calculation of the shockwave two-point function in AdS_4, we show that for particular kinematics, in the limit of large AdS radius R, this two-point function reduces to the celestial propagator in a shockwave background (3.74).

Consider the embedding of a 4-dimensional hyperboloid

$$-(X^0)^2-(X^1)^2+\sum_{i=2}^{4}(X^i)^2=-R^2 \tag{3.81}$$

in $\mathbb{R}^{1,1}\times\mathbb{R}^{1,2}$ with metric

$$ds^2=-dX^+dX^- -(dX^1)^2+\sum_{i=2}^{3}(dX^i)^2 \tag{3.82}$$

and where

$$X^\pm=X^0\pm X^4 \tag{3.83}$$

are lightcone coordinates in $\mathbb{R}^{1,1}$.

Parameterizing

$$\begin{aligned} X^+ &= -R\frac{\cos\tau - \sin\rho\Omega_4}{\cos\rho}, \quad & X^- &= -R\frac{\cos\tau + \sin\rho\Omega_4}{\cos\rho}, \\ X^1 &= -R\frac{\sin\tau}{\cos\rho}, & X^i &= R\tan\rho\Omega_i,\ i=2,3, \end{aligned} \tag{3.84}$$

with $\sum_{i=2}^4 \Omega_i^2 = 1$, (3.82) becomes the AdS_4 metric in global coordinates

$$ds^2 = \frac{R^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_{\mathbb{S}^2}^2\right). \tag{3.85}$$

The (τ,ρ) coordinates cover the ranges $\rho \in [0, \frac{\pi}{2}]$, $\tau \in [-\pi, \pi]$ and the boundary is approached as $\rho \to \frac{\pi}{2}$. Up to conformal rescaling, points on the boundary are parameterized by

$$\mathbf{p} = \lim_{\rho\to\pi/2} \frac{1}{2} R^{-1}\cos\rho \mathbf{X} \tag{3.86}$$

with $\mathbf{p}^2 = 0$. We denote AdS_4 bulk points by $\mathbf{X} = (X^+, X^-, X^i)$ and boundary points by $\mathbf{p}$.

Following [123] we consider the AdS_4 shock geometry

$$ds^2_{\text{shock}} = -ds^2_{\text{AdS}_4} + dX^- dX^- \delta(X^-)\mathbf{h}(X^i), \tag{3.87}$$

where for $X^- = 0$,

$$-(X^1)^2 + \sum_{i=2}^{3}(X^i)^2 = -R^2 \tag{3.88}$$

and hence on the shock front, $\mathbf{h}$ depends only on transverse directions $\mathbf{x}_\perp \in H_2$ in the 2-dimensional hyperbolic space H_2 defined by (3.88). Einstein's equations imply $\mathbf{h}$ is a solution to the sourced wave equation on H_2 [123]

$$\left[\Box_{H_2} - \frac{2}{R^2}\right] \mathbf{h}(\mathbf{x}_\perp) = -\frac{\kappa^2}{2}\mathbf{T}(\mathbf{x}_\perp). \tag{3.89}$$

Note also that the shock front is chosen to lie along the Poincaré horizon as illustrated in Figure 3.3.

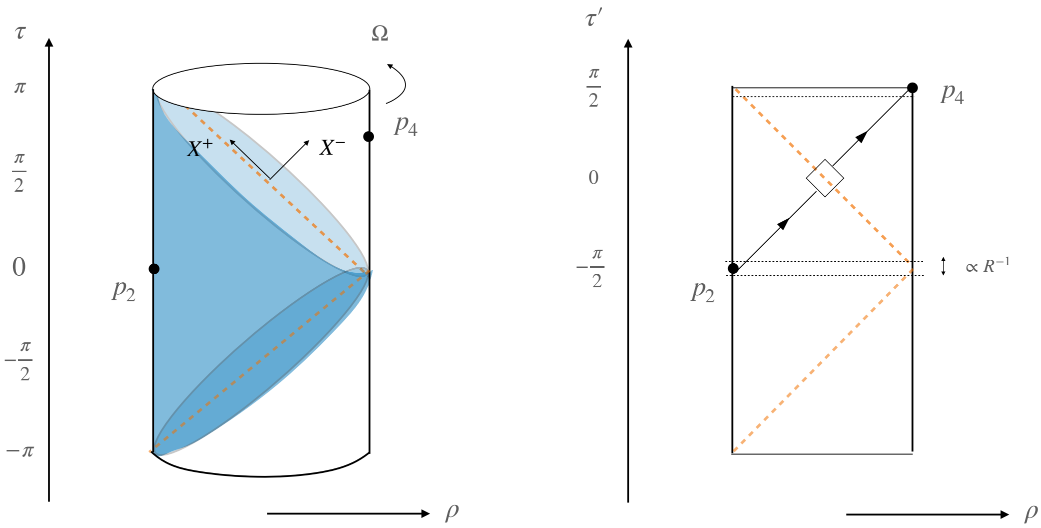

FIGURE 3.3: Left: Poincaré patch of AdS_4 with a shockwave along the horizon at $X^- = 0$. The boundary is approached as $\rho \to \frac{\pi}{2}$ and Ω parameterize S^2 constant τ boundary slices. Right: Zooming into a bulk flat space region of AdS around the shock at $\rho = 0$. As $R \to \infty$, the AdS_4 shockwave two-point function with $\mathbf{p}_2, \mathbf{p}_4$ inserted around $\tau_2' = -\frac{\pi}{2}$ and $\tau_4' = \frac{\pi}{2}$ respectively becomes the celestial shockwave two-point function.

The two-point function in this shockwave background takes the form [123]

$$\langle \mathcal{O}_\Delta(\mathbf{p}_2)\mathcal{O}_\Delta(\mathbf{p}_4)\rangle_{\rm shock} = \mathcal{C}_\Delta \int_{H_2} d^2\mathbf{x}_\perp \frac{\Gamma(2\Delta)}{\left(2\sum_{i=1}^3 q^i X_i(\mathbf{x}_\perp) - \mathbf{h}(\mathbf{x}_\perp)\right)^{2\Delta}}, \tag{3.90}$$

with $\mathcal{C}_\Delta$ given in (3.76) and and without loss of generality, the boundary operators are inserted at

$$\mathbf{p}_2 = -\left(0,1,0\right), \quad \mathbf{p}_4 = \left(q^2,1,q\right). \tag{3.91}$$

The relative sign is chosen such that the operators are inserted on opposite sides of the shock, otherwise the two point function can be shown to take the same form as in empty AdS.

We would like to zoom in around the flat space region around $\tau = \frac{\pi}{2}, \rho = 0$. To this end we consider the shifted coordinate

$$\tau' = \tau - \frac{\pi}{2} \tag{3.92}$$

and take the limit $R \to \infty$ with

$$\tau' = \frac{t}{R}, \quad \rho = \frac{r}{R} \tag{3.93}$$

and (t, r) fixed, as illustrated in Figure 3.3. It is straightforward to show that in this limit

$$\begin{aligned} X^+ &\to t + r\Omega_4 + \mathcal{O}(R^{-1}) = x^+, \quad & X^- &\to t - r\Omega_4 + \mathcal{O}(R^{-1}) = x^-, \\ X^1 &\to -R + \mathcal{O}(1), & X^i &\to r\Omega_i = x^i_\perp, \ \ i = 2,3, \end{aligned} \tag{3.94}$$

and hence the shockwave metric becomes that of a planar shock in Minkowski space

$$ds^2 = -dx^+dx^- + ds^2_\perp + (dx^-)^2\delta(x^-)h(x_\perp) \tag{3.95}$$

with

$$\Box_\perp h(x_\perp) = -\frac{\kappa^2}{2}T(x_\perp). \tag{3.96}$$

Finally, parametrizing

$$q = (-\cos\tau'_q, \tilde{\Omega}_2, \tilde{\Omega}_3), \tag{3.97}$$

where $\tau'_q \in [0, \pi]$ we find

$$\lim_{R\to\infty} \langle \mathcal{O}_\Delta(\mathbf{p}_2)\mathcal{O}_\Delta(\mathbf{p}_4)\rangle_{\text{shock}} = \mathcal{C}_\Delta \int d^2x_\perp \frac{\Gamma(2\Delta)}{\left(-R\cos\tau'_q + x_\perp\cdot\tilde{\Omega} - h(x_\perp)\right)^{2\Delta}}. \tag{3.98}$$

Unless $\tau'_q = \frac{\pi}{2} + \mathcal{O}(R^{-1})$, we see that (3.98) is suppressed[15] by a factor $R^{-2\Delta}$ and the amplitude will vanish. This is to be expected as otherwise the point in the bulk at which $\mathcal{O}$ interacts with the shockwave will be outside the flat space region we are zooming into (see Figure 3.3). It is also consistent with the HKLL prescription that relates bulk scattering states in the flat space limit to boundary operators localized in windows of width $\Delta\tau \sim R^{-1}$ around $\tau' = \pm\frac{\pi}{2}$ [62, 63]. It follows that for this configuration, the shockwave two-point function reduces to

$$\lim_{R\to\infty} \langle \mathcal{O}_\Delta(\mathbf{p}_2)\mathcal{O}_\Delta(\mathbf{p}_4)\rangle_{\text{shock}} = \mathcal{C}_\Delta \int d^2x_\perp \frac{\Gamma(2\Delta)}{(-x_\perp\cdot q_{24,\perp} - h(x_\perp))^{2\Delta}}, \tag{3.99}$$

which precisely agrees with the celestial result (3.74). Placing $\mathcal{O}_\Delta(\mathbf{p}_4)$ anywhere else in the $\Delta\tau = \mathcal{O}(R^{-1})$ window results in a constant shift that can be absorbed in the definition of h.[16]

We conclude that

$$\lim_{R\to\infty} \langle \mathcal{O}^-_\Delta(\mathbf{p}_2)\mathcal{O}^+_\Delta(\mathbf{p}_4)\rangle_{\text{shock}} = \frac{R^{2(\Delta-1)}}{4\pi^3 i^{2\Delta}}\Gamma\left(\Delta - \frac{1}{2}\right)^{-2} \widetilde{A}_{\text{shock}}(\Delta, \hat{q}_2; \Delta, \hat{q}_4), \tag{3.100}$$

where the $+$ $(-)$ labels on the LHS indicate that the CFT_3 boundary operators are to be inserted at global times $\tau = \frac{\pi}{2} + \tau_0$ $(\tau = -\frac{\pi}{2} + \tau_0)$ provided that the bulk flat space region

[15] It is assumed that the "parent" boundary CFT_3 is unitary and hence the operators have positive dimensions.

[16] Recall that (3.96) determines h up to solutions of $\Box_\perp h = 0$.

of interest lies at τ_0. It would be interesting to generalize this analysis for the scattering of arbitrary spin particles in spherical shock backgrounds (see [90] in the massless background limit for a recent example). It would also be interesting to study the flat space limit of scattering in AdS black hole backgrounds and in particular its implications for signatures of chaos in CCFT [156, 157].

3.6 Celestial amplitudes from flat space limits of Witten diagrams

The discussion in the previous section is a particular instance of a general result namely, that celestial amplitudes arise naturally as the leading term in a large radius expansion of $\mathrm{AdS}_4/\mathrm{CFT}_3$ Witten diagrams. More generally, in this section we show that scalar Witten diagrams in $\mathrm{AdS_{d+1}}/\mathrm{CFT_d}$ reduce to CCFT_{d-1} amplitudes in the flat space limit. We restrict to non-derivative interactions for simplicity. In establishing this correspondence we assume the following:

- The boundary CFT_d operators $\mathcal{O}_{\Delta_i}(\mathbf{p}_i)$ are inserted on global time slices $\tau = \pm\frac{\pi}{2}$.
- The two spheres at $\tau = \pm\frac{\pi}{2}$ on the boundary of AdS are antipodally matched.[17]

We start by studying the individual building blocks of AdS_{d+1} Witten diagrams - *external lines*, *vertices* and *internal lines* - and their expansion in a large R limit. We will see that they map precisely to $(d+1)$-dimensional flat space Feynman diagrams computed in a basis of external conformal primary wavefunctions, or equivalently, CCFT_{d-1} celestial amplitudes.

[17] It would be interesting to understand the physical meaning of such a matching condition in AdS, perhaps by studying asymptotic field configurations as the boundary is approached along different null directions. We thank Laurent Freidel for a discussion on this point.

3.6.1 External lines

Let $K_\Delta(\mathbf{p},\mathbf{x})$ be the bulk-to-boundary propagator in the embedding space representation [60],[18]

$$K_\Delta(\mathbf{p},\mathbf{x}) = \frac{C^d_\Delta}{(-2\mathbf{p}\cdot\mathbf{x}+i\epsilon)^\Delta} \tag{3.101}$$

and

$$C^d_\Delta \equiv \frac{\Gamma(\Delta)}{2\pi^{d/2}\Gamma(\Delta-\frac{d}{2}+1)R^{(d-1)/2-\Delta}}. \tag{3.102}$$

Parameterizing respectively bulk and boundary points $\mathbf{x}$ and $\mathbf{p}$ with (τ,ρ,Ω) and (τ_p,Ω_p) as in (3.84), (3.86) where $\Omega_p,\Omega\in S^{d-1}$, setting $\tau=t/R$ and $\rho=r/R$ and expanding at large R, we find

$$K_\Delta(\mathbf{p},\mathbf{x}) = C^d_\Delta\left[\frac{1}{(R\cos\tau_p+t\sin\tau_p-r\Omega_p\cdot\Omega+O(R^{-1})+i\epsilon)^\Delta}\right]. \tag{3.103}$$

Like in the shockwave analysis, we see that assuming $\Delta\geq 0$, unless $\tau_p=\pm\frac{\pi}{2}$, the leading contribution to the bracket in (3.103) vanishes as $R\to\infty$. On the other hand, choosing $\tau_p=\frac{\pi}{2}$ we have

$$K_\Delta(\mathbf{p},\mathbf{x}) \;=\; C^d_\Delta\left[\frac{1}{(-\tilde{q}\cdot x+i\epsilon)^\Delta}+O(R^{-1})\right], \tag{3.104}$$

where $x=(t,r\Omega)\in\mathbb{R}^{1,d}$ is the point in flat space and where $\tilde{q}=(1,\Omega_p)\in\mathbb{R}^{1,d}$ is a null vector in the direction Ω_p. As a result, up to normalization, $K_\Delta(\mathbf{p},\mathbf{x})$ maps (up to a phase) under $R\to\infty$ to an outgoing conformal primary wavefunction, when $\tau_p=\frac{\pi}{2}$.

[18] This representation of $K_\Delta(\mathbf{p},\mathbf{x})$ is valid only in particular Poincaré patches [119]. It is sufficient in our case since we restrict to configurations with boundary insertions at $\tau=\pm\frac{\pi}{2}$ and bulk points close to the center of AdS.

Likewise if we choose $\tau_p = -\frac{\pi}{2}$,

$$K_\Delta(\mathbf{p},\mathbf{x}) \;=\; C^d_\Delta \left[\frac{1}{(\tilde{q}\cdot x + i\epsilon)^\Delta} + O(R^{-1})\right] \tag{3.105}$$

where x is the same, but now $\tilde{q} = (1, \Omega^A_p)$ with $\Omega^A_p = -\Omega_p$ the antipodal point of Ω_p on the sphere. In this case we see that the bulk-to-boundary propagator maps (up to a phase) to an incoming conformal primary wavefunction.

Outgoing or incoming $i\epsilon$ prescriptions are obtained depending on the sign of $\tau_p = \pm\frac{\pi}{2}$. Moreover, the antipodal identification is needed to ensure Lorentz covariance of the resulting conformal primary wavefunctions. Note that placing the operators at other global times $\tau_p = \pm\frac{\pi}{2} + \Delta\tau_p$ with $\Delta\tau_p \propto R^{-1}$ leads, in the flat space limit, to conformal primary wavefunctions that diagonalize boosts with respect to different origins in spacetime.

3.6.2 Vertices

For the particular case of non-derivative coupling we are considering, AdS_{d+1} vertices take the form

$$ig\int_{\mathrm{AdS_{d+1}}} d^{d+1}\mathbf{x}. \tag{3.106}$$

Writing the measure explicitly in global coordinates (τ,ρ,Ω), and transforming to $\tau = t/R$ and $\rho = r/R$, we have its large R expansion

$$d^{d+1}\mathbf{x} \;=\; d^{d+1}x + O(R^{-2}). \tag{3.107}$$

Moreover since $t = R\tau$ and $r = R\rho$, it follows that $t \in (-\infty,\infty)$ and $r \in [0,\infty)$ in the flat space limit. Hence

$$ig\int_{\mathrm{AdS_{d+1}}} d^{d+1}\mathbf{x} = ig\int_{\mathbb{R}^{1,d}} d^{d+1}x + O(R^{-2}), \tag{3.108}$$

and the rule for the vertex in AdS_{d+1} maps to the rule for the vertex in $\mathbb{R}^{1,d}$.

3.6.3 Internal lines

To discuss the internal lines we recall that the AdS_{d+1} bulk-to-bulk propagator of dimension Δ obeys the equation [123]

$$\left(\Box_{AdS_{d+1}} - \frac{\Delta(\Delta-d)}{R^2}\right)\mathbf{\Pi}_\Delta(\mathbf{x},\bar{\mathbf{x}}) \quad = \quad i\delta_{AdS_{d+1}}(\mathbf{x},\bar{\mathbf{x}}). \tag{3.109}$$

On the one hand the Laplacian is

$$\begin{aligned}\Box_{AdS_{d+1}} &= \frac{-\cos^2\rho}{R^2}\partial_\tau^2 + \frac{\cos^{d+1}\rho}{\sin^{d-1}\rho}\partial_\rho\left(\frac{\sin^{d-1}\rho}{\cos^{d+1}\rho}\sqrt{\gamma}\frac{\cos^2\rho}{R^2}\partial_\rho\right) + \frac{\cos^2\rho}{R^2\sin^2\rho}\frac{1}{\sqrt{\gamma}}\partial_A\left(\sqrt{\gamma}\gamma^{AB}\partial_B\right)\\ &= \Box_{\mathbb{R}^{1,d}} + O(R^{-2}),\end{aligned} \tag{3.110}$$

where γ is the round S^{d-1} metric and $\Box_{\mathbb{R}^{1,d}}$ is the flat space Laplacian. On the other hand the delta function is

$$\begin{aligned}\delta_{AdS_{d+1}}(\mathbf{x},\bar{\mathbf{x}}) &= \frac{\delta(\tau-\bar{\tau})\delta(\rho-\bar{\rho})\delta^{d-1}(\Omega-\bar{\Omega})}{\sqrt{-g_{AdS_{d+1}}}}\\ &= \delta_{\mathbb{R}^{1,d}}(x,\bar{x}) + O(R^{-2}),\end{aligned} \tag{3.111}$$

where $\delta_{\mathbb{R}^{1,d}}(x,\bar{x})$ is the Minkowski space delta distribution. Altogether the large R expansion of the defining equation for the bulk-to-bulk propagator is

$$\left[\left(\Box_{\mathbb{R}^{1,d}} + O(R^{-2})\right) - \frac{\Delta(\Delta-d)}{R^2}\right]\mathbf{\Pi}_\Delta(\mathbf{x},\bar{\mathbf{x}}) \quad = \quad i\delta_{\mathbb{R}^{1,d}}(x,\bar{x}) + O(R^{-2}). \tag{3.112}$$

It follows that the AdS_{d+1} propagator has a large-R expansion

$$\mathbf{\Pi}_\Delta(\mathbf{x},\bar{\mathbf{x}}) \quad = \quad G(x,\bar{x}) + O(R^{-2}), \tag{3.113}$$

where $G(x, \bar{x})$ ought to obey

$$(\Box_{\mathbb{R}^{1,d}} - m^2)G(x, \bar{x}) = i\delta_{\mathbb{R}^{1,d}}(x, \bar{x}), \quad m \equiv \lim_{R\to\infty} \frac{\Delta}{R}. \tag{3.114}$$

Therefore, we either recover massive exchanges when $\Delta = O(R)$ or massless exchanges when $\Delta = O(1)$.

A final remark is that while equation (3.114) does not have a unique solution, the fact that $\mathbf{\Pi}_\Delta(\mathbf{x}, \bar{\mathbf{x}})$ computes time-ordered two-point functions in AdS$_{d+1}$ implies that its leading behavior $G(x, \bar{x})$ also computes time-ordered two-point functions in $\mathbb{R}^{1,d}$. This imposes one additional condition on (3.114) which singles out the Feynman propagator.

3.6.4 Forming the diagrams

Combining all of the ingredients, we find that none of the large-R corrections contribute at leading order. As a result, the leading term in a large R expansion of a Witten diagram reduces to the position space Feynman diagram for the same interaction in flat space with external wavefunctions taken to be conformal primaries. By the definition (3.8), this coincides with the corresponding celestial amplitude!

We exemplify by considering a t-channel exchange Witten diagram

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1}(\mathbf{p}_1)\mathcal{O}_{\Delta_2}(\mathbf{p}_2)\mathcal{O}_{\Delta_3}(\mathbf{p}_3)\mathcal{O}_{\Delta_4}(\mathbf{p}_4)\rangle &= (ig)^2 \int_{AdS_{d+1}} d^{d+1}\mathbf{x} d^{d+1}\mathbf{y} \mathbf{\Pi}_\Delta(\mathbf{x}, \mathbf{y}) \\ &\quad K_{\Delta_1}(\mathbf{p}_1, \mathbf{x})K_{\Delta_3}(\mathbf{p}_3, \mathbf{x})K_{\Delta_2}(\mathbf{p}_2, \mathbf{y})K_{\Delta_4}(\mathbf{p}_4, \mathbf{y}).\end{aligned} \tag{3.115}$$

Taking $\mathbf{p}_1$ and $\mathbf{p}_2$ inserted at $\tau = -\frac{\pi}{2}$ and $\mathbf{p}_3$ and $\mathbf{p}_4$ inserted at $\tau = \frac{\pi}{2}$, we find

$$K_{\Delta_i}(\mathbf{p}_i, \mathbf{x}) = \mathcal{N}^d_{\Delta_i} \left[\varphi_{\Delta_i}(x; -\hat{q}_i) + O(R^{-1})\right], \quad i = 1, 2, \tag{3.116}$$

$$K_{\Delta_i}(\mathbf{p}_i, \mathbf{x}) = \mathcal{N}^d_{\Delta_i} \left[\varphi_{\Delta_i}(x; \hat{q}_i) + O(R^{-1})\right], \quad i = 3, 4, \tag{3.117}$$

where $\mathcal{N}_{\Delta_i}$ are given by

$$\mathcal{N}^d_{\Delta_i} = \frac{C^d_{\Delta_i}}{i^{\Delta_i}\Gamma(\Delta_i)} = \frac{R^{-(d-1)/2+\Delta_i}}{2\pi^{d/2} i^{\Delta_i}\Gamma(\Delta_i - \frac{d-1}{2})}. \tag{3.118}$$

Assuming further that the exchanged operator has $\Delta = mR + O(1)$, then

$$\langle \mathcal{O}^-_{\Delta_1}(\mathbf{p}_1)\mathcal{O}^-_{\Delta_2}(\mathbf{p}_2)\mathcal{O}^+_{\Delta_3}(\mathbf{p}_3)\mathcal{O}^+_{\Delta_4}(\mathbf{p}_4)\rangle = \left(\prod_{i=1}^4 \mathcal{N}^d_{\Delta_i}\right)\Bigg((ig)^2 \int_{\mathbb{R}^{1,d}} d^{d+1}x d^{d+1}y G_{\mathrm{e}}(x,y) \\ \times \varphi_{\Delta_1}(x;-\hat{q}_1)\varphi_{\Delta_2}(y;-\hat{q}_2)\varphi_{\Delta_3}(x;\hat{q}_3)\varphi_{\Delta_4}(y;\hat{q}_4) + O(R^{-1})\Bigg), \tag{3.119}$$

and up to normalization the leading term in the large R expansion is the corresponding flat space Feynman diagram computed with position space Feynman rules and conformal primary external wavefunctions. More generally, in the flat space limit, CFT_d correlators with operators inserted at $\tau_i = \pm\frac{\pi}{2} + O(R^{-1})$ are related to CCFT_{d-1} amplitudes of in/out operators with the same dimensions, namely

$$\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = \lim_{R\to\infty}\left(\prod_{i=1}^4 \mathcal{N}^d_{\Delta_i}\right)^{-1} \langle \mathcal{O}^-_{\Delta_1}(\mathbf{p}_1)\mathcal{O}^-_{\Delta_2}(\mathbf{p}_2)\mathcal{O}^+_{\Delta_3}(\mathbf{p}_3)\mathcal{O}^+_{\Delta_4}(\mathbf{p}_4)\rangle. \tag{3.120}$$

Celestial amplitudes of operators with arbitrary dimensions (such as conformally soft ones) may then be obtained by analytic continuation.

At the operator level, what we have shown is that a generic CFT_d quasi-primary operator $\mathcal{O}_\Delta(\mathbf{p})$ inserted on past/future global time slices S^{d-1} maps in the flat space limit to an incoming/outgoing celestial operator $\mathcal{O}^\pm_\Delta(\vec{z})$ in CCFT_{d-1} via

$$\mathcal{O}^\pm_\Delta(\vec{z}) \equiv \lim_{R\to\infty} (\mathcal{N}^d_\Delta)^{-1}\mathcal{O}^\pm_\Delta\left(\tau = \pm\frac{\pi}{2}, \vec{z}\right), \tag{3.121}$$

where the limit holds in the weak sense,

$$\langle \mathcal{O}^{\pm}_{\Delta}(\vec{z}) \cdots \rangle = \lim_{R\to\infty} (\mathcal{N}^{d}_{\Delta})^{-1} \left\langle \mathcal{O}^{\pm}_{\Delta}\left(\tau = \pm\frac{\pi}{2}, \vec{z}\right) \cdots \right\rangle . \tag{3.122}$$

This prescription beautifully matches with the relation between two-point functions in a shock background found by explicit calculation in (3.100).

4

Celestial Sector in CFT: Conformally Soft Symmetries

4.1 Introduction

Celestial holography proposes a correspondence between theories of gravity in 4-dimensional (4D) asymptotically flat spacetimes and conformal field theories (CFT) living on the 2D celestial sphere at infinity [29, 158]. In particular, scattering observables in the 4D theory are computed by correlation functions in the 2D theory, also known as celestial amplitudes,[1] and are subject to a wide range of symmetries [33, 35–38, 52, 91, 154] (see also [159] for a recent review). This correspondence appears to be very different from other instances of holography. Most notably, it relates a bulk theory to a boundary theory in *two* lower dimensions, while the bulk soft theorems imply the existence of towers of negative

[1] Celestial amplitudes will be assumed to be defined in 2D whenever the dimension is not explicitly specified.

dimension operators in the celestial CFT [39], naively rendering the boundary theory non-unitary.

On the other hand, for massless[2] scattering, a simple flat space limit of holographic CFT_d correlators was found in [1] to yield $(d-1)$-dimensional celestial amplitudes. This suggests that at least some of the unique features of celestial CFT should arise in a certain limit of conventional CFT in one higher dimension. The goal of this paper is to explain how leading and subleading conformally soft symmetries [35–38] emerge precisely in this way.

Motivated by the configuration of boundary operators for which CFT_3 correlators reduce to celestial amplitudes, we first study the symmetries of an interval on the Lorentzian cylinder of small width $\Delta\tau \propto R^{-1}$ in global time. We show that in the limit $R \to \infty$, the conformal isometries of this strip are enhanced to an infinite dimensional symmetry parameterized by a function and a local conformal Killing vector on a two-sphere. For finite large R (corresponding to a strip of small, but finite width), the infinite dimensional symmetry is broken by $O(R^{-1})$ terms. We show explicitly via a procedure that mimics the Inonu-Wigner contraction [163] of the conformal algebra to Poincaré, that the enhanced conformal isometries of the intervals around $\tau = \pm\frac{\pi}{2}$ generate an extended BMS_4 algebra to leading order at large R. Moreover, under these symmetries, CFT_3 primary operators of dimension Δ at $\tau = \pm\frac{\pi}{2} + \frac{u}{R}$ transform as 2D primary operators of effective dimension $\hat{\Delta} = \Delta + u\partial_u$. $\hat{\Delta}$ can be diagonalized by an integral transform with respect to u analogous to that relating Carrollian and celestial operators [164, 165].

This analysis suggests that conformally soft symmetries in 2D CCFT are generated by certain modes of the 3D stress tensor in the strips. In the second part of the paper

[2] It has been long known that massive and in some cases massless momentum space scattering amplitudes can be extracted from correlation functions of unitary CFT_d with holographic AdS_{d+1} duals in various flat space limits [58–60, 160, 161]. Interestingly, it was recently shown that such CFT_d 4-point correlators exhibit conjectured properties of $(d+1)$-dimensional scattering amplitudes, including dispersion relations, unitarity and the Froissart bound in a flat-space limit [162].

we show that the *shadow* stress tensor Ward identities in CFT_d allow one to extract both the leading and subleading conformally soft graviton operators in CCFT_{d-1}. We establish this by lifting the method used in [166] to derive stress tensor Ward identities from the subleading soft graviton theorem in arbitrary dimensions to the embedding space. This allows us to derive the shadow stress tensor Ward identities on the Lorentzian cylinder $\mathbb{R} \times S^{d-1}$ and study their restriction to an infinitesimal global time strip. Specifically, we find that

$$\lim_{u\to 0}\partial_u \widetilde{T}_{ab} \text{ and } \lim_{u\to 0}(1-u\partial_u)\widetilde{T}_{ab}, \tag{4.1}$$

where $\widetilde{T}_{ab}$ is the shadow transform of the CFT_d stress tensor and a, b are indices on S^{d-1} become respectively, upon subtracting the trace, the leading and subleading conformally soft gravitons in CCFT_{d-1}!

Our results are interesting for several reasons. Firstly, they demonstrate that celestial CFT may not be as exotic of a theory as anticipated. On the contrary, the leading and subleading conformally soft symmetries arise universally in a simple limit of any CFT_3, irrespective of whether or not it is holographic. In this sense, our approach is complementary to that in [63, 167, 168] which relies on the existence of an AdS bulk dual. More generally, we find that any CFT_d contains a $(d-1)$-dimensional "celestial" sector characterized by an emergent BMS-like symmetry.[3] Secondly, our results suggest that holographic CFT_d correlators encode information about gravity in $(d+1)$-dimensional asymptotically flat spacetimes (AFS) that need not be lost in the flat space limit. It would be extremely interesting to understand the further implications, as well as the limitations of this approach.

This paper is organized as follows. In section 4.2 we review the relation between AdS Witten diagrams and celestial amplitudes at large AdS radius. We show how each operator in an infinitesimal time interval around $\tau = \pm\frac{\pi}{2}$ in a CFT_d on the Lorentzian

[3] In $d > 3$ the vector fields are parameterized by a function on the sphere and a CKV on S^{d-1}, in particular there is no local enhancement of the latter like for $d = 3$.

cylinder maps to a continuum of operators in CCFT_{d-1} via an integral transform over the interval. In section 4.3 we generalize the relation between AdS Witten diagrams and celestial amplitudes to massless spinning external states. In particular, we demonstrate that, at large AdS radius, spinning bulk-to-boundary propagators in AdS_{d+1} with fixed dimensions become massless spinning conformal primary wavefunctions in $\mathbb{R}^{1,d}$. In section 4.4 we analyze the conformal Killing equations in a global time strip of the 3D Lorentzian cylinder of infinitesimal width $\Delta\tau \sim R^{-1}$. We find an emergent infinite dimensional symmetry in the limit $R \to \infty$ labelled by a function and a vector field on the sphere. We show in section 4.4.1 that the associated vector fields reorganize into the generators of an extended BMS_4 algebra after a Inonu-Wigner-like contraction. In section 4.4.2 we show that CFT_3 operators in the strips around $\tau = \pm\frac{\pi}{2}$ transform like conformal primary operators in CCFT_2 under these symmetries.

In section 4.5 we derive the conformally soft gluon and graviton theorems in CCFT_{d-1} as a limit of the Ward identities of a shadow current and the stress tensor in CFT_d. In sections 4.5.1, 4.5.2 we revisit the derivation of these Ward identities using the embedding space formalism. The large-R limits of these identities are worked out in section 4.5.3. After projection to the Lorentzian cylinder, we demonstrate in section 4.5.3 that the leading conformally soft gluon is obtained from the components of the shadow current transverse to the S^{d-1} at $\tau = \frac{\pi}{2}$. The leading and subleading conformally soft gravitons are similarly extracted from an expansion of the transverse traceless component of the shadow stress tensor around $\tau = \frac{\pi}{2}$ in section 4.5.3. We collect various technical results in the appendices.

4.2 Preliminaries

In this section we review how, in the large AdS radius limit, scalar AdS Witten diagrams reduce to Feynman diagram constituents of celestial amplitudes. This result will be extended to account for massless spinning external states, as well as exchanges of arbitrary mass and spin in section 4.3. Importantly, we clarify the relation between insertions of CFT operators at different global times τ_0 in a strip of width $\Delta\tau = O(R^{-1})$ and the continuum of celestial operators corresponding to an asymptotic state in 4D AFS.

Conformal correlation functions in CFT_d are obtained by summing over all possible AdS_{d+1} Witten diagrams [169]. The building blocks of the latter are bulk-to-boundary and bulk-to-bulk propagators. It will be convenient to express the bulk-to-boundary propagators in the embedding space formalism [170, 171]. We denote points or vectors in the embedding space $\mathbb{R}^{2,d}$ by capital letters $X, P, \cdots$. Points in bulk AdS_{d+1} are constrained to obey $X^2 := \eta_{\mu\nu}X^\mu X^\nu = -R^2$, where $\eta_{\mu\nu} = (-, +, \cdots, +, -)$ and can be parameterized by global coordinates $(\tau, \rho, \vec{z})$ as

$$X^0(\tau,\rho,\vec{z}) = R\frac{\sin\tau}{\cos\rho}, \quad X^{d+1}(\tau,\rho,\vec{z}) = R\frac{\cos\tau}{\cos\rho}, \quad X^i(\tau,\rho,\vec{z}) = R\tan\rho\,\Omega^i(\vec{z}). \tag{4.2}$$

Here $\Omega(\vec{z}) \in S^{d-1}$ are unit normals to the sphere parameterized by coordinates $\vec{z}$ with

$$\Omega(\vec{z}) = \left(\frac{2z^1}{1+|\vec{z}|^2}, \ldots, \frac{2z^{d-1}}{1+|\vec{z}|^2}, \frac{1-|\vec{z}|^2}{1+|\vec{z}|^2}\right). \tag{4.3}$$

In these coordinates the boundary is located at $\rho = \frac{\pi}{2}$ and boundary points correspond to null vectors $P^2 = 0$, where

$$P(\tau,\vec{z}) = \lim_{\rho\to\frac{\pi}{2}} \frac{\cos\rho}{R} X(\tau,\rho,\vec{z}), \tag{4.4}$$

or equivalently

$$P^0(\tau,\vec{z}) = \sin\tau, \quad P^{d+1}(\tau,\vec{z}) = \cos\tau, \quad P^i(\tau,\vec{z}) = \Omega^i(\vec{z}). \tag{4.5}$$

The correlation functions $\langle \mathcal{O}_{\Delta_1}(P_1)\cdots\mathcal{O}_{\Delta_n}(P_n)\rangle$ of scalar operators $\mathcal{O}_{\Delta_i}(P_i)$ in a holographic CFT$_d$ can be computed by summing over AdS$_{d+1}$ Witten diagrams (see [5] for a review). Motivated by the relation between scattering amplitudes and AdS/Witten diagrams in the flat space limit [60,62,63], a limit was proposed in [1] in which AdS/Witten diagrams reduce to celestial amplitudes. In this prescription, boundary operators are placed at

$$\tau_i = \pm\frac{\pi}{2} + \frac{u_i}{R}, \tag{4.6}$$

while bulk global coordinates are redefined as

$$\tau = \frac{t}{R}, \quad \rho = \frac{r}{R}, \tag{4.7}$$

before taking $R \to \infty$ with (t, r) fixed. One of the main observations of [1] is that to leading order at large R, scalar bulk to boundary propagators in AdS$_{d+1}$

$$K_\Delta(X, P) = \frac{C_\Delta}{(-P\cdot X + i\epsilon)^\Delta}, \tag{4.8}$$

with C_Δ a normalization constant, become proportional to $\mathbb{R}^{1,d}$ conformal primary wavefunctions [29]

$$\varphi_\Delta(x;\eta\hat{q}) = \frac{(i\eta)^\Delta\Gamma(\Delta)}{(-\hat{q}\cdot x + i\eta\epsilon)^\Delta}. \tag{4.9}$$

Here $\eta = \pm 1$ depending on whether the boundary operators are placed around $\tau = \pm\frac{\pi}{2}$ with the spheres at $\tau = \pm\frac{\pi}{2}$ assumed to be antipodally related, x is a point in $(d+1)$-dimensional flat space and

$$\hat{q}(\vec{z}) \;\; = \;\; (1, \Omega(\vec{z}))\,. \tag{4.10}$$

Analyzing the other elements of the AdS/Witten diagrams, one concludes that these reduce to the building blocks of celestial amplitudes to leading order at large R.

The correspondence established in [1] left an important question open. A bulk scalar field in AdS corresponds to an operator of definite dimension in CFT, while massless asymptotic states in flat space should map to a continuum of operators of dimensions $\Delta = \frac{d-1}{2} + i\lambda$ in CCFT_{d-1} [30]. In contrast, according to (4.8), (4.9) the celestial amplitudes appear to simply inherit the dimension of the primary operator in the parent CFT. We conclude this section by explaining how one can in fact extract a continuum of operators in CCFT from the large R expansion of (4.8).

Recall that the conformal primary wavefunctions obtained from bulk-to-boundary propagators in the large R limit depend on the position at which the CFT_d operators are inserted within the global time strip of infinitesimal width $\propto R^{-1}$. In particular,

$$\lim_{R\to\infty} K_\Delta(X,P)|_{\tau_p=\frac{\pi}{2}+\frac{u_0}{R}} \propto \frac{1}{(t-u_0-r\Omega\cdot\Omega_p+i\epsilon)^\Delta} + O(R^{-1}). \tag{4.11}$$

This result corresponds to an outgoing conformal primary wavefunction defined with respect to a different origin in spacetime, namely

$$\varphi_\Delta(x-x_0;\hat{q}) \propto \frac{1}{(-\hat{q}\cdot(x-x_0)+i\epsilon)^\Delta}, \tag{4.12}$$

where $x_0 = (u_0,0,0,0)$. Now note that this shift in origin can be traded for a shift in the conformal dimension Δ by an integral transform on u_0. Specifically,

$$\begin{aligned}\int_{-\infty}^{\infty} du_0 u_0^{-\Delta_0} \frac{i^\Delta}{(t-u_0-r\Omega\cdot\Omega_p+i\epsilon)^\Delta} &= \frac{1}{\Gamma(\Delta)}\int_{-\infty}^{\infty} du_0 u_0^{-\Delta_0}\int_0^\infty d\omega\omega^{\Delta-1}e^{i\omega(t-u_0-r\Omega\cdot\Omega_p+i\epsilon)} \\ &= \frac{2i^{\Delta-1}\sin(\pi\Delta_0)B(\Delta+\Delta_0-1,1-\Delta_0)}{(t-r\Omega\cdot\Omega_p+i\epsilon)^{\Delta+\Delta_0-1}}, \quad \text{Re}\Delta_0\in(0,1),\end{aligned} \tag{4.13}$$

where $B(x, y)$ is the Euler beta function. Similar to calculations involving conformal primary wavefunctions in CCFT, the integral formally converges only for $\Delta_0 = c + i\lambda$, with $c \in (0, 1)$ and $\lambda \in \mathbb{R}$. Nevertheless the result may be analytically continued away from this line in the complex Δ_0 plane [86, 154, 172]. Following [30], these conformal primary wavefunctions can then be shown to form a complete basis for asymptotic scattering states in $\mathbb{R}^{1,d}$ provided that Δ_0 takes the appropriate continuum of values.

We conclude that up to an interesting normalization,[4] insertions of CFT_d operators at different points in the infinitesimal global time intervals generate the expected continuum of CCFT_{d-1} operators. The transformation (4.13) is the same that maps operators in a Carrollian conformal field theory to celestial operators [164, 165]. We will return to this in section 4.4.2. A complementary approach is to keep the u_0 dependence and then relate the $R \to \infty$ limit of AdS Witten diagrams to Carrollian correlators instead of celestial ones [176].

4.3 Spinning celestial amplitudes from flat space limit

We now discuss the extension of the result reviewed in the previous section to external spinning operators. We analyze in turn the flat space limit of massless spinning bulk-to-boundary propagators, spinning bulk-to-bulk propagators and vertices.

[4]In (4.13) we assumed that one can exchange the order of integrals over u_0 and ω. It would be important, yet beyond the scope of this paper, to study under what conditions this is allowed. It is possible that different prescriptions will yield celestial amplitudes that differ by Poincaré invariant structures as observed for example in [90, 173]. We thank Walker Melton and Sruthi Narayanan for a discussion on this point. It would also be interesting to understand the precise relation between our prescription and those proposed in [174, 175] based on an AdS/dS slicing of flat space.

4.3.1 Bulk-to-boundary propagators

We start by considering the spinning bulk-to-boundary propagators for fields of dimension Δ and spin J [171]

$$K^{\Delta,J}_{\vec{\mu};\vec{\nu}}(X;P) = C_{\Delta;J}\partial_{\mu_1}X^{A_1}\cdots\partial_{\mu_J}X^{A_J}\partial_{\nu_1}P^{B_1}\cdots\partial_{\nu_J}P^{B_J}\frac{I_{\{A_1;\{B_1}(X;P)\cdots I_{A_J\};B_J\}}(X;P)}{(-P\cdot X+i\epsilon)^{\Delta}}, \tag{4.14}$$

where

$$I_{A;B}(X;P) = \frac{-P\cdot X\eta_{AB}+P_A X_B}{-P\cdot X+i\epsilon}. \tag{4.15}$$

Here A_i, B_i are $\mathbb{R}^{2,d}$ embedding space indices, μ_i run over the rescaled coordinates (t,r,Ω) defined in (4.3), (4.7) and ν_i run over the boundary coordinates (u,Ω) in (4.6). $\partial_{\mu_i}X^{A_i}$, $\partial_{\nu_i}P^{B_i}$ hence implement projections onto the corresponding bulk and boundary tensors respectively and $\{\cdot\}$ denotes the symmetric traceless component. We collect some useful results on the embedding space formalism in appendix B.1. $C_{\Delta,J}$ is a normalization constant [171]

$$C_{\Delta,J} \;=\; \frac{(J+\Delta-1)\Gamma(\Delta)}{2\pi^{d/2}(\Delta-1)\Gamma(\Delta+1-\frac{d}{2})R^{(d-1)/2-\Delta+J}}. \tag{4.16}$$

We see that spinning bulk-to-boundary propagators are obtained from the scalar ones defined in (4.8) by dressing with the conformally covariant tensors in (4.15). It then suffices to analyze the behavior of these tensors in the flat space limit.

Using the large R expansions

$$X(\tau,\rho,\vec{z}) \;=\; (0,R)+(x,0)+O(R^{-1}), \tag{4.17}$$

$$P(\tau_i,\vec{z}_i) \;=\; \pm(\hat{q}(\vec{z}_i),0)\mp\left(0,\frac{u_i}{R}\right)+O(R^{-2}) \tag{4.18}$$

of the bulk and boundary embedding space vectors, where $x=(t,r\Omega(\vec{z}))$ are Cartesian coordinates and $\hat{q}$ is defined in (4.10), one obtains the expansions of the projectors $\partial_\mu X^A$ and $\partial_\nu P^B$. From these expansions it immediately follows that

$$\eta_{AB}\partial_\mu X^A \partial_\nu P^B = \begin{cases} O(R^{-2}), & \nu = u, \\ \pm\partial_a \hat{q}_\mu(\vec{z}) + O(R^{-1}), & \nu = z^a, \end{cases} \tag{4.19}$$

$$P_A X_B \partial_\mu X^A \partial_\nu P^B = \begin{cases} \hat{q}_\mu(\vec{z}) + O(R^{-1}), & \nu = u, \\ (\partial_a \hat{q}(\vec{z}) \cdot x)\, \hat{q}_\mu(\vec{z}) + O(R^{-1}), & \nu = z^a. \end{cases} \tag{4.20}$$

The expansion of the conformally covariant tensors (4.15) projected onto bulk and boundary indices follows directly from these results. We distinguish between two cases. First, when the boundary index is $\nu = u$ we have

$$I_{\mu,u}(X,P) = \pm \lim_{\Delta\to 0} \frac{1}{\Delta}\left[\partial_\mu\left(\frac{1}{(-\hat{q}\cdot x \pm i\epsilon)^\Delta}\right) + O(R^{-1})\right], \tag{4.21}$$

which we recognize as the derivative of a scalar conformal primary wavefunction. Likewise, if the boundary index is $\nu = z^a$ we have

$$I_{\mu,a}(X,P) = \pm\left[\partial_a \hat{q}_\mu(\vec{z}) + \frac{\partial_a \hat{q}(\vec{z})\cdot x}{(-\hat{q}\cdot x \pm i\epsilon)}\hat{q}_\mu(\vec{z}) + O(R^{-1})\right]. \tag{4.22}$$

Hence, up to normalization and a phase, the flat space limit of $I_{\mu,a}(X,P)$ corresponds to the conformally covariant tensor used in the construction of spinning conformal primary wavefunctions given in [85].[5] Putting everything together, we conclude that general massless spinning conformal primary wavefunctions are obtained from flat space limits of the spinning bulk-to-boundary propagators (4.14) with transverse indices. Note however that the dimensionally reduced bulk to boundary propagators have a non-vanishing trace. In order to obtain conformal primary wavefunctions in CCFT_{d-1} the trace has to

[5] The polarization vectors $\partial_a \hat{q}$ are gauge equivalent to the ones defined in [30].

be subtracted. For example, in the spin two case this is implemented by applying the projector [30]

$$P^{b_1 b_2}_{a_1 a_2} \equiv \delta^{b_1}_{\{a_1}\delta^{b_2}_{a_2\}} - \frac{1}{d-1}\delta_{a_1 a_2}\delta^{b_1 b_2}. \tag{4.23}$$

Finally, (4.21) implies that bulk-to-boundary propagators with time indices on the boundary result in pure gauge conformal primary wavefunctions. We leave a better understanding of this, as well as additional data resulting from the dimensional reduction to future work.

4.3.2 Bulk-to-bulk propagators and vertices

The spin J bulk-to-bulk propagator in AdS$_{d+1}$ obeys the equations [171]

$$\begin{gathered}\left(\Box_{AdS} - \frac{\Delta(\Delta-d)}{R^2} + \frac{J}{R^2}\right)\Pi_{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(X,\bar{X}) = -g_{\mu_1\{\nu_1}\cdots g_{|\mu_J|\nu_J\}}\delta_{AdS}(X,\bar{X}),\\ \nabla^{\mu_1}\Pi_{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(X,\bar{X}) = 0.\end{gathered} \tag{4.24}$$

To take the flat space limit we assume that all of the components are in the chart (t,r,Ω), in which the AdS metric $g_{\mu\nu}$ becomes the Minkowski metric $\eta_{\mu\nu}$ to leading order at large R

$$g_{\mu\nu} = \eta_{\mu\nu} + O(R^{-2}). \tag{4.25}$$

On the other hand, the Laplace operator behaves as $\Box_{AdS} = \Box_{\mathbb{R}^{1,d}} + O(R^{-2})$ and the Dirac delta behaves as $\delta_{AdS}(X,\bar{X}) = \delta_{\mathbb{R}^{1,d}}(x,\bar{x}) + O(R^{-2})$ [1]. Therefore the first equation turns into the equation for the propagator of a spin J field of mass $m = \lim_{R\to\infty}\frac{\Delta}{R}$ in flat space. The second equation can be treated in the same way since $g_{\mu\nu} = \eta_{\mu\nu} + O(R^{-2})$ and the AdS covariant derivative becomes the flat spacetime covariant derivative when $R\to\infty$.

As a result, the bulk-to-bulk propagator must have an expansion of the form

$$\Pi_{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(X,\bar{X}) \;=\; G_{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(x,\bar{x})+O(R^{-2}), \tag{4.26}$$

where $G_{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(x_1,x_2)$ is the Feynman propagator for a symmetric traceless tensor of spin J in $\mathbb{R}^{1,d}$.

Since vertices are simply integrals over AdS which become integrals over $\mathbb{R}^{1,d}$ in the flat space limit, we conclude that AdS-Witten diagrams for spinning particles reduce to CCFT_{d-1} amplitudes of spinning massless particles in the flat space configuration (4.6).

4.4 From conformal to infinite dimensional symmetry

Consider a d-dimensional CFT on the Lorentzian cylinder with metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + d\Omega_{d-1}^2, \tag{4.27}$$

where $d\Omega_{d-1}^2$ is the metric on the $(d-1)$-sphere of unit radius. Conformal transformations are coordinate transformations that preserve the metric up to a Weyl rescaling. Specifically, infinitesimal conformal transformations are obtained by finding the diffeomorphisms

$$x'^\mu = x^\mu + \epsilon^\mu(x) \tag{4.28}$$

under which the metric transforms as

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = \sigma(x) g_{\mu\nu}(x). \tag{4.29}$$

Such diffemorphisms are subject to the conformal Killing equations

$$\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu = \frac{2}{d}\nabla\cdot\epsilon(x) g_{\mu\nu}. \tag{4.30}$$

The solutions to these equations generate the conformal algebra $\mathfrak{so}(d,2)$ for $d \geq 3$, while for $d = 2$ this algebra admits a Virasoro enhancement.

The relation between celestial amplitudes on the $(d-1)$-dimensional celestial sphere and conformal correlation functions of primary operators localized to strips of infinitesimal width $\Delta\tau \propto \frac{1}{R}$ as $R \to \infty$ suggests that, on short global time scales, d-dimensional conformal field theories should develop an infinite dimensional symmetry. In this section we show that this is indeed the case by analyzing the conformal Killing equations (4.30) in this limit. We specialize to $d = 3$ in which case the emergent "celestial" CFT is 2-dimensional and expected to be governed by the extended BMS symmetries of 4D asymptotically flat spacetimes (AFS) [11, 12, 27, 67].

For $d = 3$, (4.27) reduces to

$$ds^2 = -d\tau^2 + 2\gamma_{z\bar{z}}dzd\bar{z}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}, \tag{4.31}$$

where we introduced stereographic coordinates $(z, \bar{z})$ on the unit 2-sphere with metric $\gamma_{z\bar{z}}$. We would like to zoom into a region of the 3-dimensional Lorentzian cylinder of infinitesimal width centered around a global time slice at τ_0. To this end, we introduce the coordinate u defined by

$$\tau = \tau_0 + \frac{u}{R}, \tag{4.32}$$

in which case the metric (4.31) becomes

$$ds^2 = -R^{-2}du^2 + 2\gamma_{z\bar{z}}dzd\bar{z}. \tag{4.33}$$

The conformal Killing equations associated with (4.33) take the form

$$\partial_u \epsilon^u = \frac{1}{3}\nabla\cdot\epsilon, \tag{4.34}$$

$$\partial_u \epsilon_z + \partial_z \epsilon_u = 0, \tag{4.35}$$

$$D_{\bar z}\epsilon_z + D_z \epsilon_{\bar z} = \frac{2}{3}\nabla\cdot\epsilon\gamma_{z\bar z}, \quad D_z\epsilon_z = 0, \tag{4.36}$$

where D_A is the covariant derivative on the sphere and we denote indices tangent to the sphere by A.

The last equation in (4.36) is solved by

$$\gamma_{z\bar z}\partial_z\epsilon^{\bar z} = \gamma_{z\bar z}\partial_{\bar z}\epsilon^z = 0 \implies \epsilon^A = F(u)Y^A(z,\bar z), \tag{4.37}$$

where Y^A are conformal Killing vectors on the sphere. Moreover (4.34) and the first equation in (4.36) yield[6]

$$2\partial_u\epsilon^u = F(u)D\cdot Y \implies \epsilon^u = \frac{1}{2}\int^u du' F(u')D\cdot Y + f(z,\bar z). \tag{4.38}$$

Finally, $F(u)$ is determined from (4.38) and (4.35). In the limit as $R\to\infty$ we distinguish between two cases. If $D\cdot Y = 0$ we immediately find

$$\partial_u F(u) = O(R^{-2}) \implies F(u) = c + O(R^{-2}), \tag{4.39}$$

where c is a constant. For future convenience we chose $c=1$ which reproduces the standard Lie algebra of rotation generators to leading order at large R. On the other

[6]Note that $f(z,\bar z)$ may depend on R. As we show later, the global translations are obtained from an Inonu-Wigner contraction of vector fields with $f(z,\bar z) = R$. Supertranslations may also be obtained by allowing $f(z,\bar z) = Rf_0(z,\bar z)$ and directly applying (4.56) to the local generators.

hand, if $D \cdot Y \neq 0$, taking a u derivative of (4.35) we find

$$\partial_u^2 F(u) Y_A - \frac{F(u)\partial_A D \cdot Y}{2R^2} = 0, \tag{4.40}$$

or upon taking the divergence on the sphere,[7]

$$\left[\partial_u^2 F(u) + \frac{1}{R^2} F(u)\right] D \cdot Y = 0. \tag{4.42}$$

(4.42) is solved by

$$F(u) = e^{\pm i(\tau_0 + \frac{u}{R})}. \tag{4.43}$$

Since we have taken a u derivative and a divergence on the sphere in order to arrive at (4.39) and (4.43), it is important to verify whether these solutions also obey the original conformal Killing equation (4.35). In fact (4.39), (4.43) fail to obey (4.35) away from the $R \to \infty$ limit. For $D \cdot Y \neq 0$

$$\delta_{\epsilon^\pm} g_{uA} = \pm \frac{i e^{\pm i(\tau_0 + \frac{u}{R})}}{R} \alpha_A(z, \bar{z}) - \frac{\partial_A f(z, \bar{z})}{R^2}, \qquad \alpha_A = Y_A + \frac{1}{2} D_A (D \cdot Y). \tag{4.44}$$

Therefore the violation is $O(R^{-1})$ for the local CKV on the sphere, while in the special case $D \cdot Y = 0$ the violation is $O(R^{-2})$. The enhanced conformal Killing symmetry in the strip is therefore broken at $O(R^{-1})$. Singularities in the local CKVs on the sphere also lead to a violation of the conformal Killing equations by contact terms.

[7] Recall that conformal Killing vectors on the sphere obey

$$D_A D^A D_B Y^B = -2 D \cdot Y. \tag{4.41}$$

The vector fields that preserve the metric of a 3D Lorentzian cylinder in an infinitesimal time interval $\propto R^{-1}$ in the limit $R \to \infty$ are hence

$$\epsilon^{\pm} = \left[\mp \frac{iR}{2} F_{\pm}(u) D \cdot Y + f(z, \bar{z}) \right] \partial_u + F_{\pm}(u) Y^A \partial_A, \tag{4.45}$$

where

$$\begin{cases} F_{\pm}(u) = e^{\pm i(\tau_0 + \frac{u}{R})}, \quad D \cdot Y \neq 0, \\ F_{\pm}(u) = 1, \quad D \cdot Y = 0. \end{cases} \tag{4.46}$$

It may be interesting, yet beyond the scope of this paper, to systematically understand whether (4.33) and (4.45) admit subleading corrections[8] at large R that allow for an enhancement of conformal symmetry in a strip of small yet finite size.

A few comments are in order. Just like the generators of the extended BMS group in 4D AFS, the vector fields (4.45) are labelled by a function $f(z, \bar{z})$ and a local conformal Killing vector $Y^A(z, \bar{z})$ on the sphere. The resulting symmetry group is infinite dimensional, in contrast to the conformal group in 3 dimensions. At first glance this may seem surprising, however we ought to keep in mind that (4.45) are *not* symmetries of full 3D CFT but only of infinitesimal time intervals.

Moreover, note that in the $R \to \infty$ limit the metric (4.33) develops a "null direction" reflected by the vanishing of the g_{uu} component. As such, the restriction to short global timescales shares similarities with the Carrollian limit [177, 178]. In the next section we show how the extended BMS_4 algebra is recovered from the enhanced conformal symmetries (4.45) of the strip by a Inonu-Wigner contraction [163].

[8]Unfortunately this naively appears to require coupling the boundary CFT to gravity. We thank Jan de Boer for a discussion on this point.

4.4.1 Extended BMS_4 algebra in CFT_3

We now show that the extended BMS_4 algebra can be extracted from the algebra generated by the vector fields (4.45). This procedure is analogous to Inonu-Wigner contraction of the conformal algebra to Poincaré [163].

We start by noting that appropriate linear combinations of (4.45) generate an $\mathfrak{so}(3,2)$ algebra for constant $f(z,\bar{z})$ and $Y = Y^A\partial_A$ restricted to the global conformal Killing vectors of the sphere [27],

$$Y_{12} = -i(z\partial_z - \bar{z}\partial_{\bar{z}}), \quad Y_{23} = -i\frac{z^2-1}{2}\partial_z + i\frac{\bar{z}^2-1}{2}\partial_{\bar{z}}, \quad Y_{31} = -\frac{1+z^2}{2}\partial_z - \frac{1+\bar{z}^2}{2}\partial_{\bar{z}}, \tag{4.47}$$

$$Y_{01} = \frac{1-z^2}{2}\partial_z + \frac{1-\bar{z}^2}{2}\partial_{\bar{z}}, \quad Y_{02} = \frac{i(1+z^2)}{2}\partial_z - \frac{i(1+\bar{z}^2)}{2}\partial_{\bar{z}}, \quad Y_{03} = -z\partial_z - \bar{z}\partial_{\bar{z}}. \tag{4.48}$$

(4.47) correspond to rotations of the 2-sphere and have vanishing divergence $D\cdot Y_{ij} = 0$ while (4.48) have non-vanishing divergence

$$D\cdot Y_{0i} = -2\Omega_i, \tag{4.49}$$

where $\Omega = \frac{1}{1+z\bar{z}}(z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})$ is the unit normal to the sphere at $(z,\bar{z})$. Specifically, identifying

$$D = -i\epsilon_{f=R}, \quad J_{ij} = i\epsilon_{Y_{ij}}, \tag{4.50}$$

$$P_i = i\epsilon^+_{Y_{0i}}, \ K_i = i\epsilon^-_{Y_{0i}}, \tag{4.51}$$

we find the commutation relations [5]

$$\begin{aligned}
&[D, J_{ij}] = 0, \quad [D, P_i] = P_i, \quad [D, K_i] = -K_i, \\
&[J_{ij}, P_k] = i(\delta_{ik} P_j - \delta_{jk} P_i), \quad [J_{ij}, K_k] = i(\delta_{ik} K_j - \delta_{jk} K_i), \\
&[P_i, K_j] = 2i(i\delta_{ij} D - J_{ij}), \quad [J_{ij}, J_{k\ell}] = i\,[\delta_{ik} J_{j\ell} + \delta_{j\ell} J_{ik} - \delta_{jk} J_{i\ell} - \delta_{i\ell} J_{jk}].
\end{aligned} \tag{4.52}$$

These generators can be reorganized in terms of Lorentz generators M_{AB} of the embedding space $\mathbb{R}^{2,3}$ [9]

$$M_{40} = -D, \quad M_{i4} = \frac{P_i + K_i}{2}, \tag{4.53}$$

$$M_{ij} = J_{ij}, \quad M_{i0} = \frac{P_i - K_i}{2i}, \quad i = 1, 2, 3. \tag{4.54}$$

Explicit computation shows that (4.52) imply that M_{AB} obey the $\mathfrak{so}(3,2)$ algebra

$$[M_{AB}, M_{CD}] = i(\eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC}) \tag{4.55}$$

with $\eta_{00} = \eta_{44} = -1, \eta_{ii} = 1$ and all other components vanishing. The Inonu-Wigner contraction is implemented by redefining

$$\mathcal{P}^\mu = \frac{1}{R} M^{4\mu}, \quad \mu = 0, \cdots, 3 \tag{4.56}$$

and taking $R \to \infty$ while keeping $\mathcal{P}^\mu$ and $M_{\mu\nu}$ fixed. It is straightforward to show that in this limit, (4.52) reduce to the Poincaré algebra, with $\mathcal{P}^\mu$ and $M_{\mu\nu}$ the translation and Lorentz generators in $\mathbb{R}^{1,3}$ respectively.

[9] Our conventions differ slightly from those in [60] and are simply related by exchanging the 0 and 4 directions or equivalently shifting $\tau \to \tau + \frac{\pi}{2}$ in (4.2).

We now demonstrate that an analogous Inonu-Wigner contraction of the local vector fields (4.45) leads to the extended BMS$_4$ algebra $\mathfrak{ebms}_4$. In analogy with (4.56) we define

$$T_Y = i\frac{\epsilon_Y^+ + \epsilon_Y^-}{2R}, \quad L_Y = \frac{\epsilon_Y^+ - \epsilon_Y^-}{2} \tag{4.57}$$

for arbitrary conformal Killing vector fields Y[10] and take the limit $R \to \infty$. Setting $\tau_0 = \frac{\pi}{2} + O(R^{-1})$, we find from (4.45) and (4.57)

$$-iT_Y = \frac{1}{2}D \cdot Y\partial_u + O(R^{-2}), \tag{4.58}$$

$$-iL_Y = Y^A\partial_A + \frac{u}{2}D \cdot Y\partial_u + O(R^{-2}). \tag{4.59}$$

Together with the vector fields with $Y = 0$, parametrized by an arbitrary function f on the sphere

$$T_f \equiv i\epsilon_f = if(z,\bar{z})\partial_u + O(R^{-2}), \tag{4.60}$$

L_Y generate $\mathfrak{ebms}_4$

$$\begin{aligned}
&[T_{f_1}, T_{f_2}] = O(R^{-2}),\\
&[L_{Y_1}, L_{Y_2}] = iL_{[Y_1,Y_2]} + O(R^{-2}),\\
&[T_f, L_Y] = \left[Y(f) - \frac{1}{2}(D\cdot Y)f(z,\bar{z})\right]\partial_u + O(R^{-2}) = iT_{f'=\frac{1}{2}(D\cdot Y)f - Y(f)} + O(R^{-2}).
\end{aligned} \tag{4.61}$$

Note that

$$\lim_{R\to\infty} T_Y = \lim_{R\to\infty} T_{f=\frac{1}{2}D\cdot Y} \tag{4.62}$$

which means that T_Y correspond to a special class of supertranslation vector fields T_f

[10] Note that the rotation generators with $D \cdot Y = 0$ are obtained directly as $M_{ij} = J_{ij}$, hence no linear combination is necessary.

with $f = \frac{1}{2} D \cdot Y$ and are hence redundant. Analogous results are obtained by expanding (4.57) around $\tau_0 = -\frac{\pi}{2}$. The results of this section are summarized in Figure 4.1.

$$\begin{array}{ccc} \mathfrak{so}(3,2)\text{ generators} & \xrightarrow{\text{short times}} & \text{local enhancement (4.45)} \\ {\scriptstyle\text{Inonu}-\text{Wigner}}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{Inonu}-\text{Wigner}} \\ \text{Poincaré} & \longrightarrow & \mathfrak{ebms}_4 \end{array}$$

FIGURE 4.1: The metric of a CFT_d on the Lorentzian cylinder develops an approximately null direction over infinitesimal global time intervals $\Delta\tau \sim R^{-1}$. In the limit $R \to \infty$, the conformal Killing equations admit an infinite dimensional set of solutions parameterized by a function on S^{d-1} and a conformal Killing vector on S^{d-1}. In particular, for $d = 3$, an Inonu-Wigner contraction in the intervals around $\tau = \pm\frac{\pi}{2}$ leads to vector fields that obey the extended BMS_4 algebra.

Finally, consider the shift $\tau_0 \to \tau_0 + \pi$ in $\epsilon_Y^{\pm}$ defined in (4.45). Under this transformation, $\epsilon_Y^{\pm} \to -\epsilon_Y^{\pm}$. The same transformation can be implemented for the globally defined vector fields by keeping τ fixed and considering instead an antipodal map on S^2. Therefore, the action of L_Y and T_Y on S^2 slices of the Lorentzian cylinder separated by π in global time becomes the same provided the slices are antipodally related. This is compatible with the observation in [1] that in order to respect Lorentz invariance in the flat space limit of AdS Witten diagrams it is necessary to antipodally identify the time-slices corresponding to in/out states. It further suggests that the antipodal matching condition between $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ employed in AFS [11] arises naturally in the flat space limit proposed in [1]. Note that similar arguments led to a derivation of the matching conditions via a resolution of i^0 with hyperbolic slices [179, 180].

4.4.2 Transformation of CFT_3 primary operators in the strip

We now study the action of the conformal Killing vectors on CFT_3 primary operators and show that when restricted to global time slices, these operators transform as quasi-primary operators in CCFT_2. We work in Euclidean signature and Wick rotate at the end.

A primary operator $\mathcal{O}_\Delta(x)$ of arbitrary spin transforms in some representation D : $\mathrm{SO}(3) \to \mathrm{GL}(V)$. The action action of a conformal Killing vector ϵ on such an operator is [181]

$$\delta_\epsilon \mathcal{O}_\Delta(x) = -\left[(\nabla\cdot\epsilon)\frac{\Delta}{3} + \epsilon^\mu \nabla_\mu + \frac{i}{2}\nabla_\mu \epsilon_\nu S^{\mu\nu}\right]\mathcal{O}_\Delta(x), \tag{4.63}$$

where ∇_μ is the spin covariant derivative [25] [11]

$$\nabla_\mu = \partial_\mu + \frac{i}{2}\omega_\mu{}^{ab} S_{ab}. \tag{4.64}$$

Here $\omega_\mu{}^{ab}$ is the torsion-free spin connection defined in terms of a vielbein e^a_μ

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}, \tag{4.65}$$

where $g_{\mu\nu}$ is the 3-dimensional metric, S_{ab} are the generators of the representation D and $S_{\mu\nu} = e^a_\mu e^b_\nu S_{ab}$. Note that $\mathcal{O}_\Delta(x)$ are defined to only carry internal indices. As an example, in appendix B.2 we demonstrate that (4.64) reduces to the standard Levi-Civita connection when acting on Lorentz vectors. The (Wick rotated) metric (4.33) is recovered with the following choice of vielbein e^a_μ

$$e^1 = \sqrt{\frac{\gamma_{z\bar{z}}}{2}}(dz + d\bar{z}), \quad e^2 = -i\sqrt{\frac{\gamma_{z\bar{z}}}{2}}(dz - d\bar{z}), \quad e^3 = \frac{du}{R}. \tag{4.66}$$

Taking $\epsilon = L_Y$, namely

$$L_Y \equiv \frac{\epsilon^+_Y - \epsilon^-_Y}{2} \tag{4.67}$$

$$= \frac{i}{2}(D\cdot Y)u\partial_u + iY^A\partial_A + O(R^{-1}), \quad \tau_0 = \frac{\pi}{2}, \tag{4.68}$$

[11] This agrees with the definition involving Σ in [25] upon setting $\Sigma^{\mu\nu} = iS^{\mu\nu}$, with $S^{\mu\nu}$ obeying (4.55).

we show in appendix B.3 that (4.63) becomes

$$\delta_{L_Y}\mathcal{O}_\Delta(x) = -i\left[D_zY^z\mathfrak{h} + D_{\bar{z}}Y^{\bar{z}}\bar{\mathfrak{h}} + Y^z(\partial_z - \Omega_z J_3) + Y^{\bar{z}}(\partial_{\bar{z}} - \Omega_{\bar{z}}J_3) + O(R^{-1})\right]\mathcal{O}_\Delta(x), \tag{4.69}$$

where we defined the operator-valued weights

$$\mathfrak{h} \equiv \frac{\hat{\Delta} + J_3}{2}, \quad \bar{\mathfrak{h}} \equiv \frac{\hat{\Delta} - J_3}{2}, \quad \hat{\Delta} \equiv \Delta + u\partial_u. \tag{4.70}$$

Finally given that J_3 acts diagonally on a primary operator,

$$J_3\mathcal{O}_\Delta = s\mathcal{O}_\Delta, \tag{4.71}$$

the operator-valued weights simplify to

$$\mathfrak{h} = \frac{\hat{\Delta} + s}{2}, \quad \bar{\mathfrak{h}} = \frac{\hat{\Delta} - s}{2}. \tag{4.72}$$

On the other hand, note that the dilatation operator in the two-dimensional theory is not diagonal in the basis of primary operators of the CFT_3. Indeed, only operators placed at $u = 0$ diagonalize the two-dimensional weights (4.72). For this special case, one obtains operators transforming like two-dimensional primary operators with respect to conformal transformations of the slices, whose dimensions agree with those of the corresponding CFT_3 operators. More generally $\hat{\Delta}$ can be diagonalized by the time Mellin-like transform discussed at the level of the bulk-to-boundary propagators in section 4.2, namely

$$\widehat{\mathcal{O}}_\Delta(z,\bar{z};\Delta_0) \equiv N(\Delta,\Delta_0)\int_{-\infty}^{\infty} du\; u^{-\Delta_0}\mathcal{O}_\Delta(u,z,\bar{z}), \tag{4.73}$$

where $N(\Delta, \Delta_0)$ is chosen to reproduce the standard normalization of CCFT operators. Under this transformation we have

$$u\partial_u \;\; \to \;\; \Delta_0 - 1 \tag{4.74}$$

and therefore it follows that $\widehat{\mathcal{O}}_\Delta(z, \bar{z}; \Delta_0)$ transforms as a two-dimensional quasi-primary operator with weights

$$h \;\; = \;\; \frac{(\Delta + \Delta_0 - 1) + s}{2}, \quad \bar{h} = \frac{(\Delta + \Delta_0 - 1) - s}{2}. \tag{4.75}$$

The transformation of $\widehat{\mathcal{O}}_\Delta$ under L_Y is therefore

$$\delta_{L_Y} \widehat{\mathcal{O}}_\Delta(z, \bar{z}; \Delta_0) = -i \left[D_z Y^z h + D_{\bar{z}} Y^{\bar{z}} \bar{h} + Y^z(\partial_z - s\Omega_z) + Y^{\bar{z}}(\partial_{\bar{z}} - s\Omega_{\bar{z}}) + O(R^{-1}) \right] \widehat{\mathcal{O}}_\Delta. \tag{4.76}$$

As an example consider a CFT_3 current J_μ of dimension $\Delta = 2$ and spin $s = 1$. According to (4.76) its restriction to an equal time slice, $(\widehat{J}_z, \widehat{J}_{\bar{z}})$, transforms under 2d conformal transformations of the slice as an operator of dimension $\Delta_{CCFT} = 1 + \Delta_0$ and spin $s = 1$. Choosing $\Delta_0 = 0$ then yields a 2D current. Likewise the stress tensor $T_{\mu\nu}$ has $\Delta_{CFT} = 3$ and spin $s = 2$. In this case its 2D counterpart $\widehat{T}$ has $\Delta_{CCFT} = 2 + \Delta_0$. Therefore choosing $\Delta_0 = 0$ again yields an operator that transforms as the stress tensor in two-dimensions. Currents in the dimensionally reduced theory can be equivalently obtained from currents in the parent CFT_3 by performing a 3D shadow transform followed by restriction to the $u = 0$ slice and a 2D shadow transform. It can be easily checked that this prescription lowers the dimension of the operator by 1. This is detailed in appendix B.4 and motivates our calculations in the following section.

This discussion brings the proposed projection from CFT_d to CCFT_{d-1} closer to the standard dimensional reduction procedure. The starting point in dimensional reduction is a manifold $M \times K$, where K is usually taken to be compact. A field Φ in this higher-dimensional space can be decomposed into modes that diagonalize a differential operator on K. The coefficients in the expansion of Φ in terms of these modes are then a tower of fields Φ_m in M [182]. This is analogous to what happens here. Explicitly, we start with a CFT_3 on $\mathbb{R} \times S^2$ and note that the operator $\mathcal{O}_\Delta(u, z, \bar{z})$ can be expanded in terms of eigenfunctions of the differential operator $u\partial_u$ in $\mathbb{R}$ and a continuum of modes $\widehat{\mathcal{O}}_\Delta(z, \bar{z}; \Delta_0)$. In this case the role of K is played by the non-compact $\mathbb{R}$ and therefore we obtain a continuum instead of a discrete set of fields in the dimensionally-reduced theory on S^2. Similar ideas applied to the distinct context of relating celestial holography to holography for the continuum of $\mathrm{AdS}_3/\mathrm{CFT}_2$ slices of the future/past Milne wedges of Minkowski spacetime have been put forward in [10, 31]. It would be interesting to establish a precise equivalence between these two approaches.

Finally, note that the transformation (4.73) is the same as the one recently employed in [164, 165] to relate Carrollian and celestial holography. This transformation appears here in a novel context and we believe it deserves further study. One difference here is that the effective dimension of the CCFT operator is not simply Δ_0, but instead $\Delta + \Delta_0 - 1$. One hence has to account for the shift by the dimension Δ of the operator in the parent CFT_3 when taking conformally soft limits for example. The additional shift by 1 is due to the fact the CFT_3 vector field (4.59) has no radial component. In the case of superrotation vector fields in AFS this is known to induce a shift by 1 in the conformal primary dimension of on asymptotic field with respect to its action [27]. It would be interesting to further explore how radial evolution in AFS arises from the perspective of the flat space limit of CFT_3.

We conclude this section by noting that in the case when Y is a globally defined CKV on S^2, the vector fields L_Y are also globally-defined on the cylinder and therefore must be linear combinations of $\mathfrak{so}(3, 2)$ generators. In this case, conformal symmetry of the CFT_3

implies the Ward identity

$$\sum_{i=1}^{n} \delta_{L_{Y_i}} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = 0. \tag{4.77}$$

In the large R limit this reduces to

$$\sum_{i=1}^{n} \left[D_{z_i} Y^{z_i} h_i + D_{\bar{z}_i} Y^{\bar{z}_i} \bar{h}_i + Y^{z_i}(\partial_{z_i} - s_i \Omega_{z_i}) + Y^{\bar{z}_i}(\partial_{\bar{z}_i} - s_i \Omega_{\bar{z}_i}) + O(R^{-1}) \right] \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = 0, \tag{4.78}$$

which corresponds to the global $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ symmetry of the CCFT_2 as expected. When Y are not globally defined, we expect the symmetry action on the correlator (4.77) to reduce in the large R limit to an insertion of the CCFT_2 stress tensor. In the next section we will show that the subleading conformally soft graviton theorem in CCFT and the associated stress tensor Ward identity follow from the flat limit of the CFT_3 shadow stress tensor Ward identities. Remarkably, the large-R expansion of the shadow stress tensor Ward identity in CFT_3 allows us to also directly recover the *leading* conformally soft graviton theorem.

4.5 CCFT_{d-1} conformally soft theorems from CFT_d

In this section we describe how soft symmetries in CCFT_{d-1} emerge from the higher-dimensional CFT_d upon dimensional reduction. As a first step, we identify the operators in CFT_d that become conformally soft operators. In particular, we show that the leading conformally soft gluon in CCFT_{d-1} arises in the flat limit[12] from a shadow-transformed conserved current in CFT_d. Similarly, the leading and subleading conformally soft gravitons are obtained from the CFT_d stress tensor.

[12] Defined here as the localization of the operator at $u = 0$ in a time strip $\tau = \tau_0 + \frac{u}{R}$ of infinitesimal width. As we show in appendix B.4 one can equivalently start from the time-Mellin transformed shadow current (4.73) in the strip and take $\Delta_0 = 1$. In this paper, the flat space limit, while motivated by holography, doesn't require the CFT_d to have a holographic dual.

The relation between soft theorems in $\mathbb{R}^{1,d+1}$ and shadow stress tensor Ward identities in CFT$_d$ was first observed in [166]. Here we combine this general correspondence with the flat space limit to derive CCFT$_{d-1}$ conserved operators (associated instead with soft theorems in $\mathbb{R}^{1,d}$) from CFT$_d$ ones.

Particularly relevant will be the shadow transform of a spin J tensor field in CFT$_d$ which is defined in the embedding space (see appendix B.1) as

$$\widetilde{\Phi}^{A_1\cdots A_J}(P) \equiv \int D^dY \frac{\prod_i(\eta^{A_iB_i}(P\cdot Y) - Y^{A_i}P^{B_i})}{(-2P\cdot Y)^{d-\Delta+J}}\Phi_{B_1\cdots B_J}(Y). \tag{4.79}$$

The shadow transform squares to the identity up to normalization [183]. This integral transform maps a primary of dimension and spin (Δ, J) to another primary of dimension and spin $(d-\Delta, J)$. In the remainder of this section we lift the analysis of [166] to the embedding space $\mathbb{R}^{1,d+1}$ and evaluate shadow current and shadow stress tensor insertions

$$\langle \widetilde{J}_A(P)\mathcal{O}_1(P_1)\cdots\mathcal{O}_n(P_n)\rangle, \quad \langle \widetilde{T}_{AB}(P)\mathcal{O}_1(P_1)\cdots\mathcal{O}_n(P_n)\rangle. \tag{4.80}$$

Our approach is therefore independent on the choice of lightcone section or conformally flat manifold (Σ, g). In order to take the flat space limit we project and analytically continue to CFT$_d$ on the Lorentzian cylinder. To simplify formulas we introduce the notation $\mathbb{X}$ for a string of primary field insertions in correlation functions

$$\langle \mathbb{X}\rangle \equiv \langle \mathcal{O}_1(P_1)\cdots\mathcal{O}_n(P_n)\rangle. \tag{4.81}$$

Since the dimensions of the leading conformally soft gluon and subleading conformally soft gravitons are $\Delta = 1$ and $\Delta = 0$ respectively in any number of dimensions, it is perhaps to be expected that the flat limit will lead the corresponding conformally soft theorems. What we find remarkable is that this approach also allows us to easily recover the the leading conformally soft graviton! This can be obtained by acting on the CFT$_d$ shadow stress tensor with ∂_u in the strip. We will see that in the limit $R \to \infty$ this indeed precisely reproduces the leading conformally soft graviton theorem in CCFT$_{d-1}$.

4.5.1 Shadow current

Using the defining relation (4.79), the shadow transform of a spin-1 field in the embedding space can be written as

$$\widetilde{J}_A(P) = \frac{1}{4}\int D^dY \frac{\partial_{P^A}\partial_{Y^B}\log(-2P\cdot Y)}{(-2P\cdot Y)^{d-\Delta-1}} J^B(Y). \tag{4.82}$$

Here we have used the following identities

$$\frac{\partial}{\partial P^A}\log(-2P\cdot Y) = \frac{Y_A}{P\cdot Y}, \quad \frac{\partial}{\partial P^A}\frac{\partial}{\partial Y^B}\log(-2P\cdot Y) = \frac{\eta_{AB}(P\cdot Y) - P_B Y_A}{(P\cdot Y)^2}. \tag{4.83}$$

We now consider a $\mathfrak{g}$-valued current where $\mathfrak{g}$ is the Lie algebra of a Lie group G which is a global symmetry of the CFT$_d$. Omitting color indices and recalling that the dimension of a current is $\Delta = d-1$, (4.82) reduces to

$$\widetilde{J}_A(P) = \frac{1}{4}\int D^dY \partial_{P^A}\partial_{Y^B}\log(-2P\cdot Y) J^B(Y) \tag{4.84}$$

$$= -\frac{1}{4}\int D^dY \partial_{P^A}\log(-2P\cdot Y)\partial_{Y^B} J^B(Y), \tag{4.85}$$

where in the last line we have integrated by parts.[13] We now invoke the Ward identity[14] [170]

$$\partial_B\langle J^B(Y)\mathbb{X}\rangle = \sum_{i=1}^{n}\delta(Y,P_i)T_i\langle\mathbb{X}\rangle, \tag{4.86}$$

[13]Recall that on the lightcone $J^B(Y) \sim J^B(Y) + Y^B f(Y)$.
[14]The embedding space delta function $\delta(Y,P_i)$ is defined by $\int D^dY\delta(Y,P_i) = 1$.

where T_i are the generators of the representation of G in which $\mathcal{O}_i$ transforms. It follows immediately that

$$\langle \widetilde{J}_A(P)\mathbb{X}\rangle = -\frac{1}{4}\sum_{i=1}^{n}\frac{(P_i)_A}{P\cdot P_i}T_i\langle\mathbb{X}\rangle. \tag{4.87}$$

Finally, we can project (4.87) to a particular section of the lightcone parameterized by $P^A(x)$. In this case we find

$$\langle \widetilde{J}_\mu(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = -\frac{1}{4}\sum_{i=1}^{n}\frac{\partial_\mu P(x)\cdot P(x_i)}{P(x)\cdot P(x_i)}T_i\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle. \tag{4.88}$$

Equivalently, as described in appendix B.1 we can choose a set of orthogonal polarization tensors $\varepsilon_a^A(x)$ (B.7) and project the components of the shadow current to an orthogonal basis obtaining

$$\langle \widetilde{J}_a(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = -\frac{1}{4}\sum_{i=1}^{n}\frac{\varepsilon_a(x)\cdot P(x_i)}{P(x)\cdot P(x_i)}T_i\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle, \tag{4.89}$$

which coincides with the leading soft gluon theorem in the embedding space $\mathbb{R}^{1,d+1}$ with the soft gluon operator given by [166] [15]

$$\mathcal{S}_a(x) \equiv -4\widetilde{J}_a(x). \tag{4.91}$$

Our main result will be to demonstrate that analytic continuation to Lorentzian signature followed by the flat limit prescription of [1] will yield the leading conformally soft gluon.

[15] Note that we normalize the shadow transform (4.79) according to [183]. This normalization differs from the one in [166] by a factor of $(-1/2)^J$. To see this, note that when contracted onto lightcone tensors,

$$\begin{aligned}\frac{1}{4}\frac{\eta_{AB}(P\cdot Y)-P_BY_A}{(P\cdot Y)}J^B(Y) &= \frac{1}{4}\frac{\eta_{AB}(P\cdot Y)-P_BY_A-Y_BP_A}{(P\cdot Y)}J^B(Y)\\ &= -\frac{1}{2(P-Y)^2}\left[\eta_{AB}-2\frac{(P-Y)_A(P-Y)_B}{(P-Y)^2}\right]J^B(Y).\end{aligned} \tag{4.90}$$

The leading and subleading conformally soft gravitons in CCFT_{d-1} (or equivalently the soft graviton in $\mathbb{R}^{1,d}$) can be recovered in a similar way from the CFT_d stress tensor. To show this, we first need to generalize the embedding space analysis herein to the shadow stress tensor.

4.5.2 Shadow stress tensor

For a spin two field the shadow transform takes the form

$$\widetilde{T}_{AB}(P) = \frac{1}{16}\int D^dY \frac{\partial_{P^A}\partial_{Y^C}\log(-2P\cdot Y)\partial_{P^B}\partial_{Y^D}\log(-2P\cdot Y)}{(-2P\cdot Y)^{d-\Delta-2}}T^{CD}(Y). \quad (4.92)$$

For the stress tensor, $\Delta = d$ and so

$$\widetilde{T}_{AB}(P) = \frac{1}{16}\int D^dY(-2P\cdot Y)^2\partial_{P^A}\partial_{Y^C}\log(-2P\cdot Y)\partial_{P^B}\partial_{Y^D}\log(-2P\cdot Y)T^{CD}(Y). \quad (4.93)$$

While the steps involved in the derivation of the relation between the shadow transform of the stress tensor and the soft graviton theorem are similar to those in [166], we find it instructive to repeat the significantly simpler calculation here in the embedding space. Integrating by parts and using (4.83) this can be written as

$$\widetilde{T}_{AB}(P) = -\frac{1}{8}\int D^dY\frac{Y_A}{P\cdot Y}\partial_{Y^C}\left\{[\eta_{BD}(P\cdot Y) - P_DY_B]T^{CD}(Y)\right\} + (A\leftrightarrow B) \quad (4.94)$$

and further evaluating the derivative with respect to Y one finds

$$\begin{aligned}\widetilde{T}_{AB}(P) = \frac{1}{4}\int D^dY\frac{Y_A}{P\cdot Y}\eta_{B[C}P_{D]}T^{CD}(Y)\\ -\frac{1}{8}\int D^dY\frac{Y_A}{P\cdot Y}[\eta_{BD}(P\cdot Y) - P_DY_B]\partial_{Y^C}T^{CD}(Y) + (A\leftrightarrow B),\end{aligned} \quad (4.95)$$

where $[.,.]$ stands for antisymmetrization. We ensured that the manifest symmetry of (4.92) under $A \leftrightarrow B$ is preserved upon integration by parts.

The insertions of both terms on the RHS of (4.95) in correlation functions are determined by the uplift of the stress tensor Ward identities to the embedding space [170]. In particular, the first line involves $T^{[CD]}$ whose insertions are related to the spin component $\mathcal{S}^{CD}$ of the Lorentz generators in the embedding space

$$\langle T^{[CD]}(Y)\mathbb{X}\rangle = -\frac{i}{2}\sum_{i=1}^{n}\delta(Y,P_i)\mathcal{S}_i^{CD}\langle\mathbb{X}\rangle. \tag{4.96}$$

We then find that inside correlation functions, the first line in (4.95) simplifies to

$$\begin{aligned}\frac{1}{4}\int D^dY\frac{Y_A}{P\cdot Y}\eta_{B[C}P_{D]}\langle T^{CD}(Y)\mathbb{X}\rangle &= -\frac{i}{8}\sum_{i=1}^{n}\frac{(P_i)_AP_D}{P\cdot P_i}\eta_{BC}\mathcal{S}_i^{CD}\langle\mathbb{X}\rangle\\ &= \frac{i}{8}\sum_{i=1}^{n}\frac{(P_i)_AP^D}{P\cdot P_i}(\mathcal{S}_i)_{DB}\langle\mathbb{X}\rangle.\end{aligned} \tag{4.97}$$

On the other hand, the second term in (4.95) is determined by the stress tensor Ward identity

$$\langle\partial_{Y^C}T^{CD}(Y)\mathbb{X}\rangle = -\eta^{DE}\sum_{i=1}^{n}\delta(Y,P_i)\partial_{P_i^E}\langle\mathbb{X}\rangle. \tag{4.98}$$

Using this Ward identity, insertions of the second term in (4.95) can then be shown to be related to the orbital part of the embedding space Lorentz generators, $\mathcal{L}_{DB}$, namely

$$\mathcal{L}_{DB} \equiv -i(P_D\partial_{P^B} - P_B\partial_{P^D}). \tag{4.99}$$

Specifically, we find that inside correlation functions the second term in (4.95) reduces to

$$-\frac{1}{8}\int D^dY\frac{Y_A}{P\cdot Y}[\eta_{BD}(P\cdot Y)-P_DY_B]\langle\partial_{Y^C}T^{CD}(Y)\mathbb{X}\rangle = \frac{i}{8}\sum_{i=1}^{n}\frac{(P_i)_AP^D}{P\cdot P_i}(\mathcal{L}_i)_{DB}\langle\mathbb{X}\rangle. \tag{4.100}$$

Combining the two contributions from equation (4.95) we find the embedding space formula for insertions of the stress tensor in CFT$_d$

$$\begin{aligned}\langle \widetilde{T}_{AB}(P)\mathbb{X}\rangle &= \frac{i}{8}\sum_{i=1}^{n}\frac{(P_i)_A P^D}{P\cdot P_i}[(\mathcal{L}_i)_{DB}+(\mathcal{S}_i)_{DB}]\langle\mathbb{X}\rangle + (A\leftrightarrow B)\\ &\equiv \frac{i}{8}\sum_{i=1}^{n}\frac{(P_i)_A P^D}{P\cdot P_i}(\mathcal{J}_i)_{DB}\langle\mathbb{X}\rangle + (A\leftrightarrow B).\end{aligned} \tag{4.101}$$

As before, we can now project to a particular section parameterized by $P^A(x)$

$$\begin{aligned}\langle \widetilde{T}_{\mu\nu}(x)\mathbb{X}\rangle &= \frac{\partial P^A}{\partial x^\mu}\frac{\partial P^B}{\partial x^\nu}\langle \widetilde{T}_{AB}(P(x))\mathbb{X}\rangle\\ &= \frac{i}{4}\sum_{i=1}^{n}\frac{\partial_{\{\mu}P^A(x)\partial_{\nu\}}P^B(x)P_A(x_i)P^D(x)}{P(x)\cdot P(x_i)}(\mathcal{J}_i)_{DB}\langle\mathbb{X}\rangle.\end{aligned} \tag{4.102}$$

Alternatively, using the orthogonal set of polarization vectors ε_a^A (B.7) to construct the spin two tensors $\varepsilon_{ab}^{AB}=\varepsilon^A_{\{a}\varepsilon^B_{b\}}$ and projecting to the associated orthonormal basis, we find [166]

$$\langle \widetilde{T}_{ab}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{i}{4}\sum_{i=1}^{n}\frac{\varepsilon_{ab}^{AB}(x)P_A(x_i)P^D(x)}{P(x)\cdot P(x_i)}(\mathcal{J}_i)_{DB}\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle, \tag{4.103}$$

which upon defining[16]

$$\mathcal{G}_{ab} = -4\widetilde{T}_{ab} \tag{4.104}$$

we recognize as the formula for a subleading soft graviton insertion in the embedding space $\mathbb{R}^{1,d+1}$.

4.5.3 Large R expansions

We now apply these results to a CFT$_d$ on the Lorentzian cylinder and show that the conformally soft theorems in the dimensionally reduced CCFT$_{d-1}$ arise naturally from

[16] Working in units where $\kappa=\sqrt{32\pi G}=2$.

the flat space limit prescription proposed in [1]. We work with the analytic continuation to Lorentzian signature of the Euclidean results derived in the previous sections.

Consider the embedding

$$P(\tau, \vec{z}) \;\;=\;\; (\sin\tau, \Omega(\vec{z}), \cos\tau) \tag{4.105}$$

of the d-dimensional Lorentzian cylinder in $\mathbb{R}^{2,d}$ with metric $\eta_{AB} = (-1, 1, \cdots, -1)$ introduced in section 4.2. Here $\Omega^2 = 1$ are unit normals to S^{d-1}. We also consider the polarization tensors

$$\varepsilon_a(\tau, \vec{z}) \;\;=\;\; (z_a \sin\tau, \delta_a^b, -z_a, z_a \cos\tau), \quad a = 1, \ldots, d-1, \tag{4.106}$$

$$\varepsilon_d(\tau, \vec{z}) \;\;=\;\; (\cos\tau, \vec{0}, -\sin\tau), \tag{4.107}$$

where δ_a^b denotes a vector with vanishing components except for an entry equal to 1 at $b = a$. These are such that $\varepsilon_a \cdot P = \varepsilon_d \cdot P = 0$ provided that

$$z_a = \frac{\Omega_a}{1 + \Omega_d}, \quad a = 1, \cdots d-1. \tag{4.108}$$

Moreover, $\varepsilon_a \cdot \varepsilon_b = \eta_{ab}$ where $\eta_{dd} = -1$. They also enjoy the property that setting $\tau = \frac{\pi}{2} + \frac{u}{R}$ and expanding at large R

$$\begin{aligned} \varepsilon_a &= (z_a, \delta_a^b, -z_a, 0) + O(R^{-1}), \\ \varepsilon_d &= (0, \vec{0}, -1) + O(R^{-1}). \end{aligned} \tag{4.109}$$

We therefore see that $\varepsilon_a = (\epsilon_a, 0) + O(R^{-1})$ where ϵ_a are polarization vectors in $\mathbb{R}^{1,d}$ [166]. In the case of CFT$_3$ ($d = 3$), it will be convenient to trade the coordinates (z^1, z^2) for complex coordinates $(z, \bar{z}) \equiv (z_1 + iz_2, z_1 - iz_2)$, and $\varepsilon_1(\tau, \vec{z})$ and $\varepsilon_2(\tau, \vec{z})$ for the following linear combinations

$$\varepsilon_z(\tau,z,\bar{z})=\frac{1}{\sqrt{2}}(\bar{z}\sin\tau,1,-i,-\bar{z},\bar{z}\cos\tau),\quad \varepsilon_{\bar{z}}(\tau,z,\bar{z})=\frac{1}{\sqrt{2}}(z\sin\tau,1,i,-z,z\cos\tau). \tag{4.110}$$

In the flat space limit, (4.110) become $\varepsilon_a=(\epsilon_a,0)+O(R^{-1})$ with ϵ_z and $\epsilon_{\bar{z}}$ the polarization vectors associated respectively with positive and negative helicities in $\mathbb{R}^{1,3}$, namely

$$\epsilon_z(z,\bar{z}) = \frac{1}{\sqrt{2}}(\bar{z},1,-i,-\bar{z}),\quad \epsilon_{\bar{z}}(z,\bar{z})=\frac{1}{\sqrt{2}}(z,1,i,-z). \tag{4.111}$$

For simplicity we will assume that all of the operators are placed at $\tau=\frac{\pi}{2}$, which holographically would amount to considering all bulk particles to be outgoing. If one of the particles is taken to be incoming, following [1] we insert the corresponding operator at $(-\frac{\pi}{2},\vec{z}^A)$ where $\vec{z}^A$ denotes the antipodal map. In that case we observe that $P(-\frac{\pi}{2},\vec{z}^A)=-P(\frac{\pi}{2},\vec{z})$. Taking this into account therefore produces the required sign difference in the corresponding contribution to the leading soft graviton factor. Finally, recall that at large R and $\tau=\frac{\pi}{2}+\frac{u}{R}$

$$P(\tau,\vec{z}) = (q(\vec{z}),0)+O(R^{-1}), \tag{4.112}$$

where $q(\vec{z})=(1,\Omega(\vec{z}))$ is a null vector in $\mathbb{R}^{1,d}$.

Leading conformally soft gluon theorem

Equipped with these results, consider a $\mathfrak{g}$-valued conserved current J in a CFT$_d$ with global symmetry group G. Insertions of the shadow transform of this current into correlation functions on the Lorentzian cylinder are obtained from the embedding space formula (4.89) by projecting with the polarization tensors $\{\varepsilon_a,\varepsilon_d\}$ in (4.106). Expanding at large

R and using (4.109) together with (4.112) we find

$$\langle \mathcal{S}_a(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \sum_{i=1}^{n} \frac{\epsilon_a(x)\cdot q(x_i)}{q(x)\cdot q(x_i)} T_i \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle + O(R^{-1}), \quad (4.113)$$

which reproduces the leading conformally soft gluon theorem in CCFT_{d-1}. Note that in the limit $u \to 0$ the large R corrections drop out. In the particular case of CFT_3 using the set of polarizations $\{\varepsilon_z, \varepsilon_{\bar{z}}, \varepsilon_3\}$ we find

$$\frac{\epsilon_z(x)\cdot q(x_i)}{q(x)\cdot q(x_i)} = \frac{1}{\sqrt{2}}\frac{1+z\bar{z}}{z-z_i}, \quad \frac{\epsilon_{\bar{z}}(x)\cdot q(x_i)}{q(x)\cdot q(x_i)} = \frac{1}{\sqrt{2}}\frac{1+z\bar{z}}{\bar{z}-\bar{z}_i}, \quad (4.114)$$

and therefore we recover

$$\langle \mathcal{S}_z(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{1+z\bar{z}}{\sqrt{2}} \sum_{i=1}^{n} \frac{T_i}{z-z_i} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle + O(R^{-1}), \quad (4.115)$$

$$\langle \mathcal{S}_{\bar{z}}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{1+z\bar{z}}{\sqrt{2}} \sum_{i=1}^{n} \frac{T_i}{\bar{z}-\bar{z}_i} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle + O(R^{-1}), \quad (4.116)$$

which are the holomorphic and antiholomorphic $\mathfrak{g}$-Kac-Moody Ward identities [184].

The time component of the CFT_3 shadow current leads to an identity that resembles a soft scalar theorem [185]

$$\langle \widetilde{J}_u(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \sim \frac{u}{R}\sum_{i=1}^{n} \frac{T_i}{q(x)\cdot q(x_i)} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle + O(R^{-3}). \quad (4.117)$$

Note that the leading term in (4.117) is of a different order in a large R expansion compared to (4.115), (4.116). Such soft theorems were argued in [186,187] to arise from conservation laws associated with higher form symmetries in 4D AFS. From a boundary perspective, we find that they are a simple consequence of dimensional reduction. It would be interesting yet beyond the scope of this paper to understand the relation between these different perspectives, as well as the role of these additional symmetries in CCFT_{d-1}.

Leading and subleading conformally soft graviton theorems

Next we consider the shadow stress tensor $\widetilde{T}_{AB}(P)$ whose insertions are given by (4.101) or, upon projection to the Lorentzian cylinder, by (4.103). As we show in details in Appendix B.5 restricting to components on a constant time slice $a, b \in \{1, \ldots, d-1\}$, we find in the flat limit that

$$\partial_u \langle \mathcal{G}_{\{ab\}} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \;=\; \sum_{i=1}^{n} \frac{\epsilon_{ab}^{AB}(x) q_A(x_i) q_B(x_i)}{q(x) \cdot q(x_i)} \partial_{u_i} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle + O(R^{-1}). \quad (4.118)$$

Here ϵ_{ab} is the transverse, traceless polarization tensor in $\mathbb{R}^{1,d}$. Upon switching to a basis that diagonalizes the dilatation operator on S^{d-1} via the transform (4.73), ∂_{u_i} becomes the weight-shifting operator $e^{\partial_{\Delta_i}}$. Note that in the limit $u \to 0$, the large R corrections to (4.118) drop out. We hence see that insertions of $\lim_{u\to 0} \partial_u \mathcal{G}_{\{ab\}}$ reproduce the leading conformally soft graviton theorem in $\mathbb{R}^{1,d}$ with $\mathcal{N}_{ab}^{(0)} \equiv \lim_{u\to 0} \partial_u \mathcal{G}_{\{ab\}}$ the leading soft graviton operator.

Moreover, we show in Appendix B.5, that

$$(1 - u\partial_u) \langle \mathcal{G}_{\{ab\}} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = i \sum_{i=1}^{n} \frac{\epsilon_{ab}^{AB}(x) q_A(x_i) q^C(x)}{q(x) \cdot q(x_i)} (\mathcal{J}_i)_{BC} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle + O(R^{-1}), \quad (4.119)$$

where $(\mathcal{J}_i)_{BC}$ have indices restricted to $B, C < d+1$ due to $\epsilon_a^{d+1} = q^{d+1} = 0$. In this case, $(\mathcal{J}_i)_{BC}$ coincide with the $\mathfrak{so}(d,2)$ generators whose action on conformal primary operators restricted to the strip (4.33) was worked out in section 4.4.2. Their action hence coincides with that of the Lorentz generators in $(d+1)$-dimensional AFS, or equivalently, conformal $\mathfrak{so}(d,1)$ transformations. Therefore insertions of $\lim_{u\to 0}(1 - u\partial_u)\mathcal{G}_{\{ab\}}$ reproduce the subleading conformally soft graviton theorem in $\mathbb{R}^{1,d}$ and the subleading conformally soft graviton operator is related to the CFT$_d$ shadow stress tensor via $\mathcal{N}_{ab}^{(1)} \equiv \lim_{u\to 0}(1 - u\partial_u)\mathcal{G}_{\{ab\}}$. The constructions of the supertranslation current and the stress tensor from $\mathcal{N}_{ab}^{(0)}$ and $\mathcal{N}_{ab}^{(1)}$

then follow directly from respectively [11, 12] and [27, 28] .

We now specialize to CFT$_3$. Using the large R expansions 4.109 of the polarization tensors $\{\varepsilon_z, \varepsilon_{\bar z}, \varepsilon_3\}$ we construct the transverse traceless spin 2 polarization tensors $\epsilon_{ab} = \epsilon_{\{a}\epsilon_{b\}}$. The only non-vanishing components are $\epsilon^{AB}_{zz} = \epsilon^A_z \epsilon^B_z$ and $\epsilon^{AB}_{\bar z\bar z} = \epsilon^A_{\bar z}\epsilon^B_{\bar z}$. Therefore the expressions for the leading soft factors reduce to those derived in [12],

$$\frac{\epsilon^{AB}_{zz}(x)q_A(x_i)q_B(x_i)}{q(x)\cdot q(x_i)} = -\frac{\bar z-\bar z_i}{z-z_i}\frac{1+z\bar z}{1+z_i\bar z_i}, \tag{4.120}$$

$$\frac{\epsilon^{AB}_{\bar z\bar z}(x)q_A(x_i)q_B(x_i)}{q(x)\cdot q(x_i)} = -\frac{z-z_i}{\bar z-\bar z_i}\frac{1+z\bar z}{1+z_i\bar z_i}, \tag{4.121}$$

and consequently

$$\langle \mathcal{N}^{(0)}_{zz}\mathcal{O}_1\cdots\mathcal{O}_n\rangle = -\sum_{i=1}^{n}\frac{\bar z-\bar z_i}{z-z_i}\frac{1+z\bar z}{1+z_i\bar z_i}\partial_{u_i}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle, \tag{4.122}$$

$$\langle \mathcal{N}^{(0)}_{\bar z\bar z}\mathcal{O}_1\cdots\mathcal{O}_n\rangle = -\sum_{i=1}^{n}\frac{z-z_i}{\bar z-\bar z_i}\frac{1+z\bar z}{1+z_i\bar z_i}\partial_{u_i}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle. \tag{4.123}$$

Insertions of $\mathcal{N}^{(1)}_{zz}$ and $\mathcal{N}^{(1)}_{\bar z\bar z}$ can be treated similarly. Relegating the complete calculation to Appendix B.6, we find that

$$\begin{aligned}\langle \mathcal{N}^{(1)}_{zz}\mathcal{O}_1\cdots\mathcal{O}_n\rangle &= \sum_{i=1}^{n}\left[\frac{(\bar z-\bar z_i)(1+\bar z z_i)}{(z-z_i)(1+z_i\bar z_i)}2\bar{\mathfrak{h}}_i - \frac{(\bar z-\bar z_i)^2}{z-z_i}(\partial_{\bar z_i}-\Omega_{\bar z_i}J_3)\right]\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle,\\ \langle \mathcal{N}^{(1)}_{\bar z\bar z}\mathcal{O}_1\cdots\mathcal{O}_n\rangle &= \sum_{i=1}^{n}\left[\frac{(z-z_i)(1+z\bar z_i)}{(\bar z-\bar z_i)(1+z_i\bar z_i)}2\mathfrak{h}_i - \frac{(z-z_i)^2}{\bar z-\bar z_i}(\partial_{z_i}-\Omega_{z_i}J_3)\right]\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle,\end{aligned} \tag{4.124}$$

which agrees with the formula for the subleading soft factor [27, 28] with external weights $(\mathfrak{h}_i, \bar{\mathfrak{h}}_i)$ and helicities J_3 as defined in (4.72). Taking a two-dimensional shadow transform of $\mathcal{N}^{(1)}_{ab}$ as in [28] yields the CCFT$_2$ stress tensor.

4.6 Discussion

In this paper we studied the symmetries of CFT_3 on the Lorentzian cylinder over short time intervals. We showed that strips of infinitesimal width $\propto R^{-1}$ around any time-slice admit an infinite-dimensional set of locally-defined solutions in the $R \to \infty$ limit. These can be reorganized into vector fields obeying the $\mathfrak{ebms}_4$ algebra. The extended BMS_4 symmetry emerges via a Inonu-Wigner contraction which for the global subalgebra reduces to the contraction of the $\mathfrak{so}(3,2)$ algebra to Poincaré. We studied the transformation properties of CFT_3 primary operators in the strip under the superrotation subalgebra of $\mathfrak{ebms}_4$ and found that they transform as two-dimensional conformal primaries with operator-valued effective dimensions $\hat{\Delta} = \Delta + u\partial_u$.

The two-dimensional dilatation can be diagonalized by a time Mellin-like transform. Consequently each CFT_3 primary operator results in a continuum of CCFT_2 primary operators of the same spin and with dimensions $\Delta_{\mathrm{CCFT}} = \Delta + \Delta_0 - 1$ where Δ is the CFT_3 dimension and Δ_0 is the dual Mellin dimension. We argued that the special case $\Delta_0 = 1$ implements a restriction to the $u = 0$ time-slice, in agreement with previous results [1].

We showed that, inside the strip, the transverse components $\widetilde{T}_{ab}$ of the $\Delta = 0$ shadow stress tensor give rise to operators $\mathcal{N}_{ab}^{(0)}$ and $\mathcal{N}_{ab}^{(1)}$ whose insertions into correlation functions reproduce the leading and subleading conformally soft graviton theorems. Likewise, the transverse components $\widetilde{J}_a$ of the $\Delta = 1$ shadow current provide an operator $\mathcal{S}_a$ whose insertions reproduce the leading soft gluon theorem. As such, conformally soft theorems and the corresponding infinite-dimensional CCFT_{d-1} symmetries effectively emerge from the dimensional reduction of the CFT_d.

There are several aspects of our dimensional reduction or flat space limit that we believe deserve further investigation. The conformal Killing vectors (4.45) giving rise to the $\mathfrak{ebms}_4$ algebra violate the conformal Killing equation at finite R. This appears to be in stark contrast to the asymptotic symmetries of 4D AFS that are exact and can be extended into the bulk. It would be interesting to understand whether the symmetries can be preserved in the strip beyond the $R \to \infty$ limit and relate this to the emergence of a bulk radial direction from the CFT. Interestingly, both large r corrections to the asymptotic charges in 4D AFS and corrections away from the large AdS radius limit have been linked to loop corrections [168, 188]. It would also be interesting to connect our enhanced conformal Killing symmetries (4.45) in the strip to the bulk Λ-BMS algebra [189] which similarly arises, subject to certain boundary conditions, in the limit of infinite AdS radius.

More generally, our analysis provides motivation for looking for boundary conditions in AdS that turn on shadow operators on the boundary. These operators are dual to modes in AdS that are in general non-normalizable near the boundary, but normalizable deep inside the bulk. This seems consistent with the flat space limit prescription which amounts to zooming in close to the center of AdS [62, 63], as well as proposals suggesting that flat space physics may be obtained via a $T\bar{T}$ deformation [190, 191]. It would also be interesting to understand if the whole tower of $w_{1+\infty}$ currents in celestial CFT [55] can similarily arise from a limit of CFT$_3$.

The approach we have adopted in this paper proposes a connection between CCFT and standard CFT. In principle these ideas may allow for an understanding of how general features of CFT, such as the existence of an associative OPE, are reflected in the dimensionally reduced theory, potentially allowing for a better understanding of the corresponding features of CCFT. In particular, our results suggest that the stress tensor of the reduced theory is closely related to the stress tensor of the parent CFT, so that it may be possible to extract a CCFT central charge from this procedure. This may shed light on previous proposals based on a hyperbolic slicing of Minkowski spacetime [31, 156, 192].

Finally, the shadow transform played an important role in this analysis, since it allowed for the construction of the soft operators from the stress tensor and current. In Lorentzian signature, the shadow transform constructed by Wick rotating the Euclidean shadow is just one member out of a group of transformations preserving the Casimirs of the conformal group [193]. It therefore seems plausible that the other transforms will also play meaningful roles in the dimensionally reduced CCFT. We hope to address some of these issues in future work.

5

Conclusion

In this thesis we have studied non-perturbative aspects of celestial amplitudes in the high-energy regime in terms of the eikonal approximation, and the connection between Celestial Holography and the flat space limit of AdS/CFT.

We first identified the celestial eikonal regime of large net conformal dimension β and small cross-ratio z in which massless 4-point celestial amplitudes are governed by a simple formula (3.31) in terms of a celestial eikonal phase, shown to be the standard momentum space eikonal phase written in a conformal primary basis. Our formula shares similarities with the analog eikonal formula that approximates four-point functions in AdS_4.

The expected connection between the eikonal amplitude for graviton exchanges and the two-point function in a shockwave background motivated us to compute the celestial shockwave two-point function of scalars. Like for the eikonal amplitude, the celestial shockwave two-point function is remarkably similar to the AdS shockwave two-point function studied in [123] and, as in the AdS case, we were able to find a source for the shockwave that reproduces the celestial eikonal amplitude at tree level.

The similarity between the celestial shockwave two-point and the corresponding AdS shockwave two-point function led us to prove that the celestial result is obtained as the $R \to \infty$ limit of the AdS result when the operators are placed on time slices in the AdS boundary separated by π in global time. We have then shown by studying the constituents of Witten diagrams that more generally scalar $\text{AdS}_{d+1}/\text{CFT}_d$ correlators reduce in the $R \to \infty$ limit to CCFT_{d-1} correlators of celestial operators of the same dimension as the CFT_d operators when the later are again placed on global time slices separated by π.

We generalized this prescription to spinning Witten diagrams and for operators placed on strips of width $\Delta\tau \propto \frac{1}{R}$ about the time slices separated by π in global time, obtaining spinning conformal primary wavefunctions from the first generalization, and the continuum of dimensions, characteristic of CCFT, from the second one. Motivated by these results we then studied the conformal symmetry of the Lorentzian cylinder on such infinitesimal time intervals. Remarkably we found that the finite-dimensional $\mathfrak{so}(3,2)$ global conformal symmetry admits one infinite-dimensional enhancement in the strict $R \to \infty$ limit. Moreover, the associated vector fields were then shown to reorganize into vector fields obeying a $\mathfrak{ebms}_4$ algebra up to corrections that vanish in the $R \to \infty$ limit, generalizing the Inonu-Wigner contraction from $\mathfrak{so}(3,2)$ to Poincaré, well-known from the flat space limit of bulk AdS_4.

The infinitesimal transformation of a CFT_3 primary operator with respect to the $\mathfrak{ebms}_4$ vector fields generating superrotations was shown to reduce to the transformation of a two-dimensional primary operator on the spherical time-slices of the cylinder with effective dimension $\hat{\Delta} = \Delta + u\partial_u$. The diagonalization of the dimensions is performed by the time Mellin-like transform that we introduced at the level of Witten diagrams to generate the continuum of celestial dimensions, and in this new context has been motivated by symmetry. In particular the $\Delta_0 = 1$ mode has been shown to essentially reproduce the restriction of the CFT_3 primary operator to the slices, consistently recovering our earlier results. This analysis suggests that our flat space limit prescription hints at CCFT_2 emerging from a dimensional reduction of CFT_3 on the Lorentzian cylinder to the spherical time-slices, being very similar to the standard Kaluza-Klein reduction.

We then showed that starting from a $\mathfrak{g}$-valued conserved current J_μ, taking its 3d shadow transform and restricting to its transverse components $\widetilde{J}_a$, we obtain a field $\mathcal{S}_a$ that is a $\Delta = 1$ gluon on the reduced theory. Its insertions into correlators are determined from the Ward identity of J_μ and reproduce the conformally soft gluon theorem. Likewise, starting from the CFT_3 stress tensor $T_{\mu\nu}$ and taking the transverse components of its shadow $\widetilde{T}_{ab}$ it is possible to obtain fields $\mathcal{N}^{(0)}_{ab}$ and $\mathcal{N}^{(1)}_{ab}$ that are respectively $\Delta = 1$ and $\Delta = 0$ gravitons on the reduced theory. Their insertions are determined by the stress tensor Ward identities and have been shown to reproduce respectively the leading and subleading conformally soft graviton theorems. Remarkably, the subleading soft factor was then shown to match the standard soft factor, with $\hat{\Delta} = \Delta + u\partial_u$ in place of the flat space dimension $-\omega\partial_\omega$. These results show that conformally soft symmetries are present in the reduced theory, further suggesting that it is indeed a CCFT_2.

These results suggest that a manner in which CCFT_{d-1} can be intrinsically constructed from a CFT_d by a dimensional reduction procedure to time-slices. On the one hand, this brings Celestial Holography into the broader framework of holography that has been developed based on AdS/CFT since Maldacena's original work, by showing that it is a natural consequence of AdS/CFT. On the other hand, this seems to provide a prescription for the construction of CCFT that has the potential to overcome a major shortcoming in the field, namely, the lack of methods to construct examples of dual pairs. Moreover, linking Celestial Holography and AdS/CFT can potentially help to establish a connection between Celestial Holography and string theory that is still lacking. In particular, there exists a duality between M-theory in $\text{AdS}_4 \times S^7$ and ABJM theory in the AdS_4 boundary. As such it may be possible to construct a complete CCFT_2 by dimensionally reducing the ABJM theory according to our prescription, and this would possibly describe a bulk asymptotically flat quantum gravity theory that originates form M-theory.

There are many questions to answer about this dimensional reduction procedure still. Is the reduced theory a CCFT? It appears that to give a definite answer to this question in terms of a mathematical proof would require us to have an intrinsic definition of CCFT, other than a theory whose correlators reproduce a bulk $\mathcal{S}$-matrix in a conformal primary basis. In that case, possibly our best approach to it is to understand what other features of CCFT can be obtained via dimensional reduction and whether there are limitations in the sense that some known CCFT features do not arise from this procedure. In this line of thought, the natural next step would be then to understand if we can obtain the $w_{1+\infty}$ symmetry of gravity encoded in the further subleading conformally soft gravitons.

Can the dimensional reduction teach us something new about CCFT? Linking CCFT to a full CFT certainly has the potential to allow for a better understanding of CCFT features. In particular, we may harness the dimensional reduction perspective to better understand a possible CCFT central charge and even to better comprehend the corrections to the $w_{1+\infty}$ algebra, supposing that it indeed emerges from the dimensional reduction.

In summary, we believe that this work provides good evidence that dimensionally reducing CFT_d on the cylinder to time-slices produces CCFT_{d-1}. As such, it potentially provides a new tool with which non-perturbative aspects of CCFT as well as the intrinsic construction of CCFT can be studied.

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Appendices

A

Appendix to Chapter 3

A.1 Celestial propagators in eikonal regime

In this appendix we show that in a conformal primary basis, in a limit of large external dimensions, the external leg propagators become nearly on-shell. For massless scalars the Klein-Gordon equation in $(x^-, x^+, x_\perp)$ coordinates (3.20) reads

$$\left(-4\partial_-\partial_+ + \partial_\perp^2\right) G_\Delta(x;\hat{q}) = 2i\delta(x^+)\delta(x^-)\delta^{(2)}(x_\perp). \tag{A.1}$$

Integrating this equation against a generalized conformal primary wavefunction [48] with eikonal kinematics like in (3.24), we find

$$\begin{aligned}\int d^4x \frac{f(x^2)}{(x^- - q_{i,\perp}\cdot x_\perp)^{\Delta_i}} \Big[&\left(-4\partial_-\partial_+ + \partial_\perp^2\right) G_{\Delta_i}(x, x_0; \hat{q}_i) \\ &- 2i\delta(x^- - x_0^-)\delta(x^+ - x_0^+)\delta^{(2)}(x_\perp - x_{\perp,0})\Big] = 0.\end{aligned} \tag{A.2}$$

Upon integration by parts,

$$\begin{aligned}&\int d^4x \left(\frac{-4\partial_-\partial_+ f(x^2) + \partial_\perp^2 f}{(x^- - q_{i,\perp}\cdot x_\perp)^{\Delta_i}} + \Delta_i \frac{4\partial_+ f(x^2) + 2q_{i,\perp}\cdot\partial_\perp f}{(x^- - q_{i,\perp}\cdot x_\perp)^{\Delta_i+1}}\right) G_{\Delta_i}(x, x_0; \hat{q}_i)\\ &- 2i\int d^4x \frac{f(x^2)}{(x^- - q_{i,\perp}\cdot x_\perp)^{\Delta_i}}\delta(x^- - x_0^-)\delta(x^+ - x_0^+)\delta^{(2)}(x_\perp - x_{\perp,0})\Big] = 0.\end{aligned} \tag{A.3}$$

For $\Delta_i \gg 1$, and $|q_{i,\perp}| = 2\sqrt{q_i} \ll 1$, the only term that survives in the first line is

$$\begin{aligned}&\int d^4x \Delta_i \frac{4\partial_+ f(x^2)}{(x^- - q_{i,\perp}\cdot x_\perp)^{\Delta_i+1}} G_{\Delta_i}(x, x_0; \hat{q}_i)\\ &- 2i\int d^4x \frac{f(x^2)}{(x^- - q_{i,\perp}\cdot x_\perp)^{\Delta_i}}\delta(x^- - x_0^-)\delta(x^+ - x_0^+)\delta^{(2)}(x_\perp - x_{\perp,0})\Big] = 0\end{aligned} \tag{A.4}$$

and so

$$-4\Delta_i(x^- - q_{i,\perp}\cdot x_\perp)^{-1}\partial_+ G_{\Delta_i}(x, x_0; \hat{q}_i) = 2i\delta(x^- - x_0^-)\delta(x^+ - x_0^+)\delta^{(2)}(x_\perp - x_{\perp,0}),\ i = 1, 3. \tag{A.5}$$

Repeating the same calculation with wavefunctions as in (3.25) we find that the propagators for the external lines can therefore be approximated in the celestial eikonal limit by

$$\begin{aligned}G_{\Delta_i}(x, x_0; \hat{q}_i) &= -\frac{i(x^- - q_{i,\perp}\cdot x_\perp)}{2\Delta_i}\delta(x^- - x_0^-)\Theta(x^+ - x_0^+)\delta^{(2)}(x_\perp - x_{\perp,0}), \quad i = 1, 3,\\ G_{\Delta_i}(x, x_0; \hat{q}_i) &= -\frac{i(x^+ - q_{i,\perp}\cdot x_\perp)}{2\Delta_i}\Theta(x^- - x_0^-)\delta(x^+ - x_0^+)\delta^{(2)}(x_\perp - x_{\perp,0}), \quad i = 2, 4,\end{aligned} \tag{A.6}$$

as promised.

A.2 Eikonal amplitude in CCFT

Applying position space Feynman rules to the ladder diagrams with n exchanges we have

$$\begin{aligned}\widetilde{\mathcal{A}}_n = (ig)^{2n}\int d^4x_1\cdots d^4x_n d^4\bar{x}_1\cdots d^4\bar{x}_n \varphi_{\Delta_3}(x_n;\hat{q}_3)G(x_n-x_{n-1})\cdots G(x_2-x_1)\varphi_{\Delta_1}(x_1;-\hat{q}_1)\\ \times\varphi_{\Delta_4}(\bar{x}_n;\hat{q}_4)G(\bar{x}_n-\bar{x}_{n-1})\cdots G(\bar{x}_2-\bar{x}_1)\varphi_{\Delta_2}(\bar{x}_1;-\hat{q}_2)\\ \times\sum_{\sigma\in S_n}G_{\mathrm{e}}(x_1-\bar{x}_{\sigma(1)})\cdots G_{\mathrm{e}}(x_n-\bar{x}_{\sigma(n)}).\end{aligned} \tag{A.7}$$

The propagators $G(x_k-x_{k-1})$ connecting particles 1 and 3 and $G(\bar{x}_k-\bar{x}_{k-1})$ connecting particles 2 and 4 can respectively be approximated by (A.6). In this approximation, writing the integrals in the (3.20) coordinates, we find

$$\begin{aligned}\widetilde{\mathcal{A}}_n &= \left(\frac{ig}{2}\right)^{2n}\int\varphi_{\Delta_1}(x_1;-\hat{q}_1)\varphi_{\Delta_2}(\bar{x}_1;-\hat{q}_2)\varphi_{\Delta_3}(x_n;\hat{q}_3)\varphi_{\Delta_4}(\bar{x}_n;\hat{q}_4)\\ &\times\prod_{k=2}^{n}\frac{-i(x_k^- - q_{1,\perp}\cdot x_{\perp,k}-i\epsilon)}{2\Delta_1}\delta(x_k^- - x_{k-1}^-)\Theta(x_k^+ - x_{k-1}^+)\delta^{(2)}(x_{\perp,k}-x_{\perp,k-1})\\ &\times\prod_{k=2}^{n}\frac{-i(\bar{x}_k^+ - q_{2,\perp}\cdot\bar{x}_{\perp,k}-i\epsilon)}{2\Delta_2}\Theta(\bar{x}_k^- - \bar{x}_{k-1}^-)\delta(\bar{x}_k^+ - \bar{x}_{k-1}^+)\delta^{(2)}(\bar{x}_{\perp,k}-\bar{x}_{\perp,k-1})\\ &\times\sum_{\sigma\in S_n}\prod_{k=1}^{n}G_{\mathrm{e}}(x_k,\bar{x}_{\sigma(k)})\prod_{k=1}^{n}\left(dx_k^- dx_k^+ d^2x_{\perp,k}d\bar{x}_k^- d\bar{x}_k^+ d^2\bar{x}_{\perp,k}\right).\end{aligned} \tag{A.8}$$

Integrating over the delta functions sets $x_k^- = x_1^-$, $x_{\perp,k}=x_{\perp,1}$, $\bar{x}_k^+ = \bar{x}_1^+$ and $\bar{x}_{\perp,k}=\bar{x}_{\perp,1}$ for all k and (A.8) reduces to

$$\begin{aligned}\widetilde{\mathcal{A}}_n &= \left(\frac{ig}{2}\right)^{2n}\left(\frac{-1}{4\Delta_1\Delta_2}\right)^{n-1}\int\frac{(-i)^{\Delta_1}\Gamma(\Delta_1)}{(x^- - q_{1,\perp}\cdot x_\perp - i\epsilon)^{\Delta_1+1-n}}\frac{(-i)^{\Delta_2}\Gamma(\Delta_2)}{(\bar{x}^+ - q_{2,\perp}\cdot\bar{x}_\perp - i\epsilon)^{\Delta_2+1-n}}\\ &\times\frac{i^{\Delta_3}\Gamma(\Delta_3)}{(x^- - q_{3,\perp}\cdot x_\perp + i\epsilon)^{\Delta_3}}\frac{i^{\Delta_4}\Gamma(\Delta_4)}{(\bar{x}^+ - q_{4,\perp}\cdot\bar{x}_\perp + i\epsilon)^{\Delta_4}}dx^- d^2x_\perp d\bar{x}^+ d^2\bar{x}_\perp\\ &\times\int\prod_{k=2}^{n}\Theta(x_k^- - x_{k-1}^-)\Theta(\bar{x}_k^+ - \bar{x}_{k-1}^+)\sum_{\sigma\in S_n}\prod_{k=1}^{n}G_{\mathrm{e}}(x_k,\bar{x}_{\sigma(k)})\prod_{k=1}^{n}(dx_k^+ d\bar{x}_k^-).\end{aligned} \tag{A.9}$$

Now thanks to the theta functions the integrals on the third line decouple [119] and

$$\widetilde{\mathcal{A}}_n = \left(\frac{ig}{2}\right)^{2n}\left(\frac{-1}{4\Delta_1\Delta_2}\right)^{n-1}\frac{1}{n!}\int\frac{(-i)^{\Delta_1}\Gamma(\Delta_1)}{(x^- - q_{1,\perp}\cdot x_\perp - i\epsilon)^{\Delta_1+1-n}}\frac{(-i)^{\Delta_2}\Gamma(\Delta_2)}{(\bar{x}^+ - q_{2,\perp}\cdot\bar{x}_\perp - i\epsilon)^{\Delta_2+1-n}}$$
$$\times\frac{i^{\Delta_3}\Gamma(\Delta_3)}{(x^- - q_{3,\perp}\cdot x_\perp + i\epsilon)^{\Delta_3}}\frac{i^{\Delta_4}\Gamma(\Delta_4)}{(\bar{x}^+ - q_{4,\perp}\cdot\bar{x}_\perp + i\epsilon)^{\Delta_4}}\left(\int d\bar{x}^- dx^+ G_{\rm e}(x,\bar{x})\right)^n dx^- d\bar{x}^+ d^2x_\perp d^2\bar{x}_\perp. \tag{A.10}$$

Using the Fourier representation (3.28) of $G_e(x,\bar{x})$ one can show that

$$\int d\bar{x}^- dx^+ G_{\rm e}(x,\bar{x}) = -2iG_\perp(x_\perp,\bar{x}_\perp), \tag{A.11}$$

where

$$G_\perp(x_\perp,\bar{x}_\perp) \equiv \int\frac{d^2k_\perp}{(2\pi)^2}\frac{e^{ik_\perp\cdot(x_\perp-\bar{x}_\perp)}}{k_\perp^2+m^2-i\epsilon}. \tag{A.12}$$

Further combining everything to the power n we have

$$\widetilde{\mathcal{A}}_n = 4\int dx^- d\bar{x}^+ d^2x_\perp d^2\bar{x}_\perp\frac{(-i)^{\Delta_1+1}\Gamma(\Delta_1+1)}{(x^- - q_{1,\perp}\cdot x_\perp - i\epsilon)^{\Delta_1+1-n}}\frac{(-i)^{\Delta_2+1}\Gamma(\Delta_2+1)}{(\bar{x}^+ - q_{2,\perp}\cdot\bar{x}_\perp - i\epsilon)^{\Delta_2+1-n}}$$
$$\times\frac{i^{\Delta_3}\Gamma(\Delta_3)}{(x^- - q_{3,\perp}\cdot x_\perp + i\epsilon)^{\Delta_3}}\frac{i^{\Delta_4}\Gamma(\Delta_4)}{(\bar{x}^+ - q_{4,\perp}\cdot\bar{x}_\perp + i\epsilon)^{\Delta_4}}\frac{(-1)^n}{n!}\left(\frac{ig^2}{8\Delta_1\Delta_2}G_\perp(x_\perp,\bar{x}_\perp)\right)^n,$$

which at large Δ_1, Δ_2 can be approximated by

$$\widetilde{\mathcal{A}}_n = 4\int dx^- d\bar{x}^+ d^2x_\perp d^2\bar{x}_\perp\frac{(-i)^{\Delta_1+1-n}\Gamma(\Delta_1+1-n)}{(x^- - q_{1,\perp}\cdot x_\perp - i\epsilon)^{\Delta_1+1-n}}\frac{(-i)^{\Delta_2+1-n}\Gamma(\Delta_2+1-n)}{(\bar{x}^+ - q_{2,\perp}\cdot\bar{x}_\perp - i\epsilon)^{\Delta_2+1-n}}$$
$$\times\frac{i^{\Delta_3}\Gamma(\Delta_3)}{(x^- - q_{3,\perp}\cdot x_\perp + i\epsilon)^{\Delta_3}}\frac{i^{\Delta_4}\Gamma(\Delta_4)}{(\bar{x}^+ - q_{4,\perp}\cdot\bar{x}_\perp + i\epsilon)^{\Delta_4}}\frac{1}{n!}\left(\frac{ig^2}{8}G_\perp(x_\perp,\bar{x}_\perp)\right)^n,$$

since

$$(\Delta_i)_n = \Delta_i(\Delta_i-1)\cdots(\Delta_i-n+1)\simeq\Delta_i^n, \quad i=1,2. \tag{A.13}$$

The shifts in n can then be written in terms of weight-shifting operators $e^{-n\partial_{\Delta_1}}$, $e^{-n\partial_{\Delta_2}}$ and therefore the connected eikonal celestial amplitude is

$$\begin{aligned}\widetilde{\mathcal{A}}_{eik.} = \sum_{n=1}^{\infty} \widetilde{\mathcal{A}}_n = 4 \int dx^- d\bar{x}^+ d^2x_\perp d^2\bar{x}_\perp \left(e^{i\hat{\chi}} - 1\right) \frac{(-i)^{\Delta_1+1}\Gamma(\Delta_1+1)}{(x^- - q_{1,\perp}\cdot x_\perp - i\epsilon)^{\Delta_1+1}} \\ \times \frac{(-i)^{\Delta_2+1}\Gamma(\Delta_2+1)}{(\bar{x}^+ - q_{2,\perp}\cdot \bar{x}_\perp - i\epsilon)^{\Delta_2+1}} \frac{i^{\Delta_3}\Gamma(\Delta_3)}{(x^- - q_{3,\perp}\cdot x_\perp + i\epsilon)^{\Delta_3}} \frac{i^{\Delta_4}\Gamma(\Delta_4)}{(\bar{x}^+ - q_{4,\perp}\cdot \bar{x}_\perp + i\epsilon)^{\Delta_4}},\end{aligned} \tag{A.14}$$

where the eikonal phase is now an operator

$$\hat{\chi} \equiv \frac{ig^2}{8} e^{-\partial_{\Delta_1} - \partial_{\Delta_2}} G_\perp(x_\perp, \bar{x}_\perp). \tag{A.15}$$

Note that (A.15) is the same as the momentum space formula with the center of mass energy promoted to an operator $s \to \hat{s} \simeq 4e^{\partial_{\Delta_1}+\partial_{\Delta_2}}$.

Since $\hat{\chi}$ is independent of $x^-, \bar{x}^+$ we can further evaluate these integrals upon shifting $x^- \to x^- + q_{1,\perp}\cdot x_\perp$ and $\bar{x}^+ \to \bar{x}^+ + q_{2,\perp}\cdot \bar{x}_\perp$ and then rescaling $x^- \to (q_{13,\perp}\cdot x_\perp)x^-$ and $\bar{x}^+ \to (q_{24,\perp}\cdot \bar{x}_\perp)\bar{x}^+$. The resulting integrals can be evaluated in terms of the standard identity [194]

$$\int_{-\infty}^{\infty} dz \, \frac{1}{z^x} \frac{1}{(1-z)^y} = \frac{2ix \sin(\pi y)}{1-x-y} B(x+y, 1-y), \tag{A.16}$$

yielding

$$\widetilde{\mathcal{A}}_{eik.} = 4 \times (2\pi)^2 \int d^2x_\perp d^2\bar{x}_\perp \left(e^{i\hat{\chi}} - 1\right) \frac{i^{\Delta_1+\Delta_2} i^{\Delta_3+\Delta_4}\Gamma(\Delta_1+\Delta_3)\Gamma(\Delta_2+\Delta_4)}{(-q_{13,\perp}\cdot x_\perp)^{\Delta_1+\Delta_3}(-q_{24,\perp}\cdot \bar{x}_\perp)^{\Delta_2+\Delta_4}}. \tag{A.17}$$

A.3 t-channel exchange

In this section we evaluate the tree-level contribution to the eikonal celestial amplitude. We start with (A.21) and compute the integral over $k_\perp$,

$$\widetilde{\mathcal{A}}_1 = \frac{(2\pi)^4 ig^2}{2}\int_0^\infty \frac{d\omega_1}{\omega_1}\omega_1^{\Delta_1+\Delta_3-1}\int_0^\infty \frac{d\omega_2}{\omega_2}\omega_2^{\Delta_2+\Delta_4-1}\frac{1}{(\omega_1 q_{13,\perp})^2+m^2}\delta^{(2)}(\omega_1 q_{13,\perp}+\omega_2 q_{24,\perp}) \tag{A.18}$$

The integral over ω_2 can be done by first noting that given two two-dimensional vectors $v=(v^1,v^2)$ and $w=(w^1,w^2)$,

$$\begin{aligned}\delta^{(2)}(\xi v+\xi' w) &= \delta(\xi v^1+\xi' w^1)\delta(\xi v^2+\xi' w^2)\\ &= \frac{1}{\xi}\delta\left(\xi'+\xi\frac{v^1}{w^1}\right)\delta(w^1v^2-v^1w^2).\end{aligned} \tag{A.19}$$

As a result,

$$\delta^{(2)}(\omega_1 q_{13,\perp}+\omega_2 q_{24,\perp}) = \frac{1}{\omega_1}\delta\left(\omega_2+\omega_1\frac{q^1_{24,\perp}}{q^1_{13,\perp}}\right)\delta(q^1_{24,\perp}q^2_{13,\perp}-q^2_{24,\perp}q^1_{13,\perp}) \tag{A.20}$$

and we can integrate over ω_2

$$\begin{aligned}\widetilde{\mathcal{A}}_1 &= \frac{(2\pi)^4 ig^2}{2}\left(-\frac{q^1_{24,\perp}}{q^1_{13,\perp}}\right)^{\Delta_2+\Delta_4-2}\delta(q^1_{24,\perp}q^2_{13,\perp}-q^2_{24,\perp}q^1_{13,\perp})\\ &\times\int_0^\infty\frac{d\omega_1}{\omega_1}\omega_1^{\Delta_1+\Delta_2+\Delta_3+\Delta_4-4}\frac{1}{(\omega_1 q_{13,\perp})^2+m^2}.\end{aligned} \tag{A.21}$$

Relabeling $\beta = \sum_i \Delta_i - 4$ and changing variables by rescaling $\omega_1 \to \frac{1}{|q_{13,\perp}|}\omega_1$, we find

$$\begin{aligned}\widetilde{\mathcal{A}}_1 &= \frac{(2\pi)^4 i g^2}{2}\left(-\frac{q^1_{24,\perp}}{q^1_{13,\perp}}\right)^{\Delta_2+\Delta_4-2} \delta(q^1_{24,\perp}q^2_{13,\perp} - q^2_{24,\perp}q^1_{13,\perp}) \\ &\quad\times\left(\frac{1}{|q_{13,\perp}|}\right)^{\beta}\int_0^\infty d\omega_1 \omega_1^{\beta-1}\frac{1}{\omega_1^2+m^2}.\end{aligned} \tag{A.22}$$

Finally, the remaining integral is a standard Mellin transform [1]

$$\int_0^\infty d\omega_1 \omega_1^{\beta-1}\frac{1}{\omega_1^2+m^2} = \frac{\pi m^{\beta-2}}{2}\frac{1}{\sin\pi\beta/2}, \tag{A.23}$$

and (A.21) can be put into the form

$$\widetilde{\mathcal{A}}_1 = \frac{\pi m^{\beta-2}}{4}\frac{(2\pi)^4 i g^2}{\sin\pi\beta/2}\left(-\frac{q^1_{24,\perp}}{q^1_{13,\perp}}\right)^{\Delta_2+\Delta_4-2}|q_{13,\perp}|^{-\beta}\delta(q^1_{24,\perp}q^2_{13,\perp} - q^2_{24,\perp}q^1_{13,\perp}). \tag{A.24}$$

A.3.1 Eikonal kinematics

By studying the small scattering angle kinematics in a center of mass frame one finds that the momenta of the particles can be written as

$$p_1 = -\frac{\sqrt{s}}{2}(1,0,0,1), \quad p_2 = -\frac{\sqrt{s}}{2}(1,0,0,-1), \tag{A.25}$$

$$p_3 = \frac{\sqrt{s}}{2}(1,2\sqrt{z},0,1), \quad p_4 = \frac{\sqrt{s}}{2}(1,-2\sqrt{z},0,-1). \tag{A.26}$$

This motivates us to define

$$\hat{q}_i = (1+q_i, q_{i,\perp}, 1-q_i), \quad i = 1,3, \tag{A.27}$$

$$\hat{q}_i = (1+q_i, q_{i,\perp}, -1+q_i), \quad i = 2,4. \tag{A.28}$$

[1] While the integral converges for $\beta \in (0,2)$, the result can be analytically continued.

In this case small z kinematics are equivalent to $q_i \ll 1$. Note that setting

$$\hat{q}_i = (1 + z_i\bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i\bar{z}_i), \quad i = 1, 3, \tag{A.29}$$

$$\hat{q}_i = (1 + w_i\bar{w}_i, w_i + \bar{w}_i, -i(w_i - \bar{w}_i), -1 + w_i\bar{w}_i), \quad i = 2, 4, \tag{A.30}$$

implies that $(z_i, \bar{z}_i)$ and $(w_i, \bar{w}_i)$ are coordinates in different charts of S^2, namely, the stereographic projections based respectively on the north and the south poles of the sphere. To express the momenta in the same chart, we perform an inversion, $(w_i, \bar{w}_i) = \left(\frac{1}{\bar{z}_i}, \frac{1}{z_i}\right)$ which yields

$$\hat{q}_i = (1 + z_i\bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i\bar{z}_i), \quad i = 1, 3, \tag{A.31}$$

$$\hat{q}_i = \frac{1}{z_i\bar{z}_i}(1 + z_i\bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i\bar{z}_i), \quad i = 2, 4. \tag{A.32}$$

In particular, one sees that the center of mass momenta (A.25) are obtained by choosing

$$z_1 = 0, \quad z_2 = \infty, \quad z_3 = \sqrt{z}, \quad z_4 = -\frac{1}{\sqrt{z}}. \tag{A.33}$$

Notice that it immediately follows from (A.25) and (A.33), that in the eikonal approximation, z is indeed $-t/s$ and also the two-dimensional cross-ratio:

$$z = -\frac{t}{s} = \frac{z_{13}z_{24}}{z_{12}z_{34}}. \tag{A.34}$$

Our derivation of the celestial eikonal amplitude will therefore assume external conformal primary wavefunctions $\varphi_{\Delta_i}(x; \eta_i\hat{q}_i)$ with null vectors of the form (A.27) satisfying $q_i \ll 1$. This kinematic configuration is illustrated in Figure 3.2

A.4 Propagator in shockwave background

In this section we review the evaluation of the momentum space scalar propagator

$$A(p_1, p_2) \equiv \langle 0|a_{\text{out}}(p_2)a^\dagger_{\text{in}}(p_1)|0\rangle \tag{A.35}$$

in a shockwave background. Let $v_p^{\text{in/out}}(x)$ and $u_p^{\text{in/out}}(x)$ be solutions to the Klein-Gordon equation behaving respectively as e^{-ipx} and e^{ipx} in the in/out regions of the spacetime under consideration. Define the Bogoliubov coefficients $\alpha(p,q)$ and $\beta(p,q)$ by the expansion

$$v_q^{\text{in}}(x) = \int_{H_0^+} d\Omega(p)\left[\alpha(p,q)v_p^{\text{out}}(x) + \beta(p,q)u_p^{\text{out}}(x)\right], \tag{A.36}$$

where H_0^+ is the zero mass shell and $d\Omega(q) = \frac{d^3q}{(2\pi)^3 2q^0}$ is the Lorentz invariant measure. Recalling that in/out fields are defined by

$$\phi_{\text{in/out}}(x) = \int_{H_0^+} d\Omega(p)\left(a_{\text{in/out}}(p)u_p^{\text{in/out}}(x) + a^\dagger_{\text{in/out}}(p)v_p^{\text{in/out}}(x)\right), \tag{A.37}$$

and that they are related to interacting fields through

$$\phi(x) \to \sqrt{Z}\phi_{\text{in/out}}(x), \quad \text{as } t \to \pm\infty, \tag{A.38}$$

where Z is the wavefunction renormalization, one may show that

$$a^\dagger_{\text{in}}(q) = \int_{H_0^+} d\Omega(p)\left[\alpha(p,q)a^\dagger_{\text{out}}(p) - \beta(p,q)a_{\text{out}}(p)\right]. \tag{A.39}$$

For the particular case $\beta(p,q)=0$, in which case the in/out vacua coincide, this immediately allows one to show that

$$A(p_1,p_2)=\alpha(p_2,p_1). \tag{A.40}$$

To evaluate $\alpha(p,q)$ consider $v_q^{\text{in}}(x)=e^{-iq\cdot x}$ when $x^-<0$. Using the boundary condition relating the solution at $x^-<0$ and $x^->0$ one finds that

$$\begin{aligned} v_q^{\text{in}}(\epsilon,x^+,x_\perp) &= v_q^{\text{in}}(-\epsilon,x^+-h(x_\perp),x_\perp) \\ &= \int_{H_0^+} d\Omega(p)\left(4\pi p^-\delta(p^--q^-)\int d^2x'_\perp e^{-i\frac{h(x'_\perp)}{2}q^-}e^{ix'_\perp\cdot(p_\perp-q_\perp)}\right)e^{i\frac{x^+}{2}p^-}e^{-ip_\perp\cdot x_\perp}. \end{aligned} \tag{A.41}$$

Comparison with the definition of the Bogoliubov coefficients shows that $\beta(p,q)=0$ and allows one to read off the propagator:

$$A_{\text{shock}}(p_1,p_2) \;=\; 4\pi p_2^-\delta(p_2^--p_1^-)\int d^2x_\perp e^{i(p_{2,\perp}-p_{1,\perp})\cdot x_\perp}e^{-i\frac{h(x_\perp)}{2}p_1^-}. \tag{A.42}$$

B

Appendix to Chapter 4

B.1 Embedding space primer

A Euclidean CFT_d is defined on the projective null cone in the embedding space $\mathbb{R}^{1,d+1}$ with metric η_{AB}.[1] The projective null cone is parametrized by a vector P obeying

$$P^2 \;=\; 0, \quad P \sim \lambda P, \quad \lambda \neq 0. \tag{B.1}$$

Choosing a representative from each equivalence class yields a section of the lightcone $\Sigma \subset \mathbb{R}^{1,d+1}$ corresponding to a conformally flat manifold on which the CFT_d is realized. The non-linear action of the conformal group on Σ is realized through the combination of Lorentz transformations $\mathrm{SO}(d+1,1)$ and rescalings of the null cone that preserves the chosen section. Let $P(x)$ be an embedding of Σ into $\mathbb{R}^{1,d+1}$. Then the metric it inherits

[1]Lorentzian CFT_d are instead lifted to $\mathbb{R}^{2,d}$.

from the ambient space is

$$ds^2_\Sigma = \eta_{AB}\frac{\partial P^A}{\partial x^\mu}\frac{\partial P^B}{\partial x^\nu}dx^\mu dx^\nu. \tag{B.2}$$

A different section Σ' embedded by $P'(x')$ is related to Σ by a rescaling

$$P'(x') = \omega(x)P(x). \tag{B.3}$$

The metrics on the two sections Σ, Σ' can then be shown to be related by a Weyl rescaling

$$ds^2_{\Sigma'} = \omega^2(x)ds^2_\Sigma. \tag{B.4}$$

We conclude that conformal maps between different conformally flat manifolds are represented in the embedding space by Weyl rescalings and Lorentz transformations of the embeddings of the corresponding lightcone sections (see [181] for a review).

A primary field of dimension Δ and spin J in a CFT$_d$ on a given section can be lifted to a field on the lightcone as follows. If $\phi_{\mu_1\cdots\mu_J}(x)$ is a spin J symmetric traceless tensor, its lift to a tensor $\Phi_{A_1\cdots A_J}(P)$ defined on the embedding space lightcone has to obey the following properties [170]

1. $\Phi_{A_1\cdots A_J}(P)$ is symmetric, traceless and transverse $P^{A_i}\Phi_{A_1\cdots A_J}(P) = 0$,

2. $\Phi_{A_1\cdots A_J}(P)$ is defined up to terms $P_{A_i}\Lambda_{A_1\cdots\hat{A}_i\cdots A_J}(P)$, where $\hat{A}_i$ denotes a missing index,

3. $\Phi_{A_1\cdots A_J}(P)$ is homogenous of degree $-\Delta$: $\Phi_{A_1\cdots A_J}(\omega P) = \omega^{-\Delta}\Phi_{A_1\cdots A_J}(P)$.

If Σ is parameterized by $P(x)$, $\phi_{\mu_1\cdots\mu_J}(x)$ is then recovered by the projection [170]

$$\phi_{\mu_1\cdots\mu_J}(x) = \frac{\partial P^{A_1}}{\partial x^{\mu_1}}\cdots\frac{\partial P^{A_J}}{\partial x^{\mu_J}}\Phi_{A_1\cdots A_J}(P(x)). \tag{B.5}$$

Projecting using the Jacobian of the embedding as done above reproduces the coordinate components of the tensor field. Alternatively, we can introduce a set of polarization vectors $\varepsilon_a^A(x)$ in the embedding space obeying

$$\varepsilon_a \cdot P \;=\; 0, \quad \varepsilon_a \cdot \varepsilon_b = \delta_{ab}. \tag{B.6}$$

The pullback of ε_a to the section (Σ, g) can then be shown to give rise to a vielbein in (Σ, g), namely

$$e^a_\mu = \frac{\partial P^A}{\partial x^\mu}\varepsilon^a_A, \quad \varepsilon^A_a = e^\mu_a \frac{\partial P^A}{\partial x^\mu} - (\varepsilon_a \cdot \bar{q})q^A, \tag{B.7}$$

where [181]

$$g^{\mu\nu}\frac{\partial P^A}{\partial x^\mu}\frac{\partial P^B}{\partial x^\nu} = \eta^{AB} + q^A\bar{q}^B + q^B\bar{q}^A, \tag{B.8}$$

with $g_{\mu\nu} = (P^+)^2\eta_{\mu\nu}$, $q^A = P^A/P^+$ and $\bar{q}^A = -2\delta^A_-$.

As a result, the symmetric, traceless combination $\varepsilon^{A_1\cdots A_J}_{a_1\cdots a_J} = \varepsilon^{A_1}_{\{a_1}\cdots\varepsilon^{A_j}_{a_J\}}$ can be used as projectors which allow us to recover the components of the tensor field with respect to the orthonormal basis

$$\phi_{a_1\cdots a_J}(x) \;=\; \varepsilon^{A_1\cdots A_J}_{a_1\cdots a_J}(x)\Phi_{A_1\cdots A_J}(P(x)). \tag{B.9}$$

Primary fields in more general representations of SO(d) can be handled in the same way. They are lifted to fields in representations of SO$(1, d+1)$ defined on the lightcone with homogeneity of degree $-\Delta$ which are transverse in the appropriate sense and which can be projected back to the original representation by introducing appropriate projection matrices. These fields are again only defined modulo terms that lie in the kernel of the projection matrices. The particular case of Dirac spinors in several dimensions is discussed for example in [181].

It will also be useful to recall the definition of conformal integrals on the space of homogeneous functions $f(X)$ of degree $-d$ on the lightcone [183]

$$\int D^d X f(X) \;=\; \frac{1}{\mathrm{Vol}(\mathrm{GL}(1,\mathbb{R})^+)} \int d^{d+2} X \delta(X^2) f(X). \tag{B.10}$$

In practice such integrals are evaluated by gauge-fixing the rescaling freedom and introducing an appropriate Faddeev-Popov determinant.

B.2 Properties of the spin covariant derivative

In this section we show that the spin-covariant derivative (4.64) reduces to the Levi-Civita connection when acting on fields transforming in the vector representation of $SO(3)$, namely if

$$(S_{ab})^c{}_d = -i\left(\delta^c_a \delta_{bd} - \delta_{ad}\delta^c_b\right) \tag{B.11}$$

then

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma. \tag{B.12}$$

To see this we evaluate $\nabla_\mu V^a$ where V^a are the vielbein components of the vector field, and then transform to the coordinate components $\nabla_\mu V^\nu$. We start with

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b. \tag{B.13}$$

The coordinate components are defined by

$$\nabla_\mu V^\nu \equiv e^\nu_a \nabla_\mu V^a. \tag{B.14}$$

Evaluating $\nabla_\mu V^\nu$,

$$\nabla_\mu V^\nu = e^\nu_a \partial_\mu V^a + e^\nu_a \omega_\mu{}^a{}_b V^b. \tag{B.15}$$

We now transform $V^a = e^a_\sigma V^\sigma$ on the RHS

$$\nabla_\mu V^\nu = e^\nu_a \partial_\mu (e^a_\sigma V^\sigma) + e^\nu_a e^b_\sigma \omega_\mu{}^a{}_b V^\sigma \quad \text{(B.16)}$$
$$= (e^\nu_a \partial_\mu e^a_\sigma) V^\sigma + e^\nu_a e^a_\sigma \partial_\mu V^\sigma + e^\nu_a e^b_\sigma \omega_\mu{}^a{}_b V^\sigma \quad \text{(B.17)}$$

and recall that $e^\nu_a e^a_\sigma = \delta^\nu_\sigma$ and $e^\nu_a e^b_\sigma \omega_\mu{}^a{}_b = \omega_\mu{}^\nu{}_\sigma$, where $\omega_\mu{}^\nu{}_\sigma$ is given by (B.25). In this case

$$\nabla_\mu V^\nu = (e^\nu_a \partial_\mu e^a_\sigma) V^\sigma + \partial_\mu V^\nu + (\Gamma^\nu_{\mu\sigma} - e^\nu_a \partial_\mu e^a_\sigma) V^\sigma. \quad \text{(B.18)}$$

The terms with $e^\nu_a \partial_\mu e^a_\sigma$ cancel and we are left with

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma, \quad \text{(B.19)}$$

which agrees with the Levi-Civita covariant derivative of the vector field with respect to the coordinate components.

B.3 Conformal Killing vector field action in the strip

The components of the rotation generators with respect to the vielbein

$$e^1 = \sqrt{\frac{\gamma_{z\bar{z}}}{2}}(dz + d\bar{z}), \quad e^2 = -i\sqrt{\frac{\gamma_{z\bar{z}}}{2}}(dz - d\bar{z}), \quad e^3 = \frac{du}{R} \quad \text{(B.20)}$$

are $S_{\mu\nu} = e^a_\mu e^b_\nu S_{ab}$. Explicitly, we find

$$S_{uz} = \frac{i}{R}\sqrt{\frac{\gamma_{z\bar{z}}}{2}} J_-, \quad \text{(B.21)}$$
$$S_{u\bar{z}} = -\frac{i}{R}\sqrt{\frac{\gamma_{z\bar{z}}}{2}} J_+, \quad \text{(B.22)}$$
$$S_{z\bar{z}} = i\gamma_{z\bar{z}} J_3, \quad \text{(B.23)}$$

where

$$J_- = S_{23} - iS_{31}, \quad J_+ = S_{23} + iS_{31}, \quad J_3 = S_{12}. \tag{B.24}$$

The coordinate components of the torsion-free spin connection ${\omega_\mu}^\sigma{}_\nu$ are given by

$${\omega_\mu}^\sigma{}_\nu = \Gamma^\sigma_{\mu\nu} - e^\sigma_a \partial_\mu e^a_\nu \tag{B.25}$$

and therefore, we see that its only non-vanishing components are

$${\omega_z}^z{}_z = -{\omega_z}^{\bar{z}}{}_{\bar{z}} = \frac{1}{2}\Gamma^z_{zz}, \tag{B.26}$$

$${\omega_{\bar{z}}}^{\bar{z}}{}_{\bar{z}} = -{\omega_{\bar{z}}}^z{}_z = \frac{1}{2}\Gamma^{\bar{z}}_{\bar{z}\bar{z}}, \tag{B.27}$$

where

$$\Gamma^z{}_{zz} = -\frac{2\bar{z}}{1+z\bar{z}}, \quad \Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = -\frac{2z}{1+z\bar{z}}. \tag{B.28}$$

As a result, defining

$$\Omega_z \equiv \frac{1}{2}\Gamma^z_{zz}, \quad \Omega_{\bar{z}} \equiv -\frac{1}{2}\Gamma^{\bar{z}}_{\bar{z}\bar{z}} \tag{B.29}$$

we find that the spin covariant derivative of $\mathcal{O}_\Delta$ is given by

$$\nabla_u \mathcal{O}_\Delta = \partial_u \mathcal{O}_\Delta, \tag{B.30}$$

$$\nabla_z \mathcal{O}_\Delta = \partial_z \mathcal{O}_\Delta - \Omega_z J_3 \mathcal{O}_\Delta, \tag{B.31}$$

$$\nabla_{\bar{z}} \mathcal{O}_\Delta = \partial_{\bar{z}} \mathcal{O}_\Delta - \Omega_{\bar{z}} J_3 \mathcal{O}_\Delta. \tag{B.32}$$

Now fix $\tau_0 = \frac{\pi}{2}$ and take $\epsilon = L_Y$ given by

$$\begin{aligned} L_Y &\equiv \frac{\epsilon_Y^+ - \epsilon_Y^-}{2} \\ &= \frac{i}{2}(D \cdot Y)u\partial_u + iY^A\partial_A + O(R^{-1}). \end{aligned} \tag{B.33}$$

We will show that $\delta_{L_Y}\mathcal{O}_\Delta$ reproduces the action of Y on a 2D primary operator in the large R limit. To this end observe from (B.30)-(B.32) and (B.21)-(B.23) that for this vector field we have

$$\begin{aligned} \nabla \cdot L_Y &= i\frac{3}{2}D \cdot Y + \mathcal{O}(R^{-1}), \\ L_Y^\mu \nabla_\mu \mathcal{O}_\Delta &= i\left[\frac{1}{2}D \cdot Y u\partial_u + Y^z(\partial_z - \Omega_z J_3) + Y^{\bar{z}}(\partial_{\bar{z}} - \Omega_{\bar{z}} J_3) + O(R^{-1})\right]\mathcal{O}_\Delta, \\ \frac{i}{2}\nabla_\mu (L_Y)_\nu S^{\mu\nu} &= \frac{i}{2}(D_z Y^z - D_{\bar{z}} Y^{\bar{z}})J_3 + O(R^{-1}). \end{aligned} \tag{B.34}$$

From this we immediately see that the expansion of $\delta_{L_Y}\mathcal{O}_\Delta(x)$ is

$$\delta_{L_Y}\mathcal{O}_\Delta(x) = -i\left[D_z Y^z \mathfrak{h} + D_{\bar{z}} Y^{\bar{z}} \bar{\mathfrak{h}} + Y^z(\partial_z - \Omega_z J_3) + Y^{\bar{z}}(\partial_{\bar{z}} - \Omega_{\bar{z}} J_3) + O(R^{-1})\right]\mathcal{O}_\Delta(x). \tag{B.35}$$

Here we have defined the operator-valued weights

$$\mathfrak{h} \equiv \frac{\hat{\Delta} + J_3}{2}, \quad \bar{\mathfrak{h}} \equiv \frac{\hat{\Delta} - J_3}{2}, \quad \hat{\Delta} \equiv \Delta + u\partial_u. \tag{B.36}$$

This agrees precisely with the transformation of a 2D primary operator, as given for example in [28].

B.4 Shadows and dimensional reduction

In this appendix we discuss the connection between the d-dimensional shadow transform on the cylinder and the Mellin-like transform on an infinitesimal time strip that implements the dimensional reduction to S^{d-1}. All embedding space fields are assumed to obey the properties described in appendix B.1. We begin by projecting the embedding space formula for the shadow transform to a particular section. Starting from (4.79), we find

$$\begin{aligned}\widetilde{\Phi}_{\mu_1\cdots\mu_J}(x) &= \prod_i \frac{\partial P^{A_i}}{\partial x^{\mu_i}} \widetilde{\Phi}_{A_1\cdots A_J}(P(x)) \\ &= \prod_i \frac{\partial P^{A_i}}{\partial x^{\mu_i}} \int D^d P(y) \frac{\prod_i (\eta_{A_i B_i} P(x)\cdot P(y) - P_{A_i}(y) P_{B_i}(x))}{(-2P(x)\cdot P(y))^{d-\Delta+J}} \prod_i \eta^{B_i C_i} \Phi_{C_1\cdots C_J}(P(y)),\end{aligned} \tag{B.37}$$

where the conformal integral is gauge-fixed to a particular section $Y = P(y)$. We now use (B.8) to eliminate $\eta^{B_i C_i}$, noting that the $q^{(B_i}\bar{q}^{C_i)}$ contributions contract to zero, namely

$$\begin{aligned}\widetilde{\Phi}_{\mu_1\cdots\mu_J}(x) = &\prod_i \frac{\partial P^{A_i}}{\partial x^{\mu_i}} \int D^d P(y) \frac{\prod_i (\eta_{A_i B_i} P(x)\cdot P(y) - P_{A_i}(y) P_{B_i}(x))}{(-2P(x)\cdot P(y))^{d-\Delta+J}} \\ &\times \prod_i g^{\sigma_i \rho_i}(y) \frac{\partial P^{B_i}}{\partial y^{\sigma_i}} \frac{\partial P^{C_i}}{\partial y^{\rho_i}} \Phi_{C_1\cdots C_J}(P(y)) \\ &= \int D^d P(y) \frac{\prod_i \frac{\partial P^{A_i}}{\partial x^{\mu_i}} \frac{\partial P^{B_i}}{\partial y^{\nu_i}} (\eta_{A_i B_i} P(x)\cdot P(y) - P_{A_i}(y) P_{B_i}(x))}{(-2P(x)\cdot P(y))^{d-\Delta+J}} \Phi^{\nu_1\cdots\nu_J}(y).\end{aligned} \tag{B.38}$$

We finally observe that owing to (4.83) we can write

$$\widetilde{\Phi}_{\mu_1\cdots\mu_J}(x) = \int d^d y \sqrt{g(y)} \frac{\prod_i \partial_{x^{\mu_i}} \partial_{y^{\nu_i}} \log(-2P(x)\cdot P(y))}{(-2P(x)\cdot P(y))^{d-\Delta}} \Phi^{\nu_1\cdots\nu_J}(y), \tag{B.39}$$

which is the shadow transform restricted to a section of lightcone [183].

Now we consider the particular case of the cylinder section parameterized by (4.105) and expand at large R. In this case taking $x = (\tau, \Omega)$ and $y = (\tau', \Omega')$ we have

$$P(x) \cdot P(y) \;\; = \;\; -\cos(\tau - \tau') + \Omega \cdot \Omega'. \tag{B.40}$$

Setting $\tau = \pm\frac{\pi}{2} + \frac{u}{R}$, expanding at large R and taking the time Mellin-like transform (4.73) we find

$$\begin{aligned}
&\Gamma(\Delta_0) \int_{-\infty}^{\infty} du u^{-\Delta_0} \widetilde{\Phi}^{\pm}_{\mu_1 \cdots \mu_J}(u, \Omega) = \Gamma(\Delta_0) \int_{-\infty}^{\infty} du u^{-\Delta_0} \\
&\times \int d\tau' d^{d-1}\vec{z}\,' \frac{\prod_i \partial_{x^{\mu_i}} \partial_{y^{\nu_i}} \log(\pm 2\sin\tau' \mp 2\frac{u}{R}\cos\tau' - 2\Omega \cdot \Omega')}{(\pm 2\sin\tau' \mp 2\frac{u}{R}\cos\tau' - 2\Omega \cdot \Omega')^{d-\Delta}} \Phi^{\nu_1 \cdots \nu_J}(y) \\
&= -i \frac{\Gamma(\Delta_0)}{\Gamma(d-\Delta)} \int_{-\infty}^{\infty} du u^{-\Delta_0} \int d\tau' d^{d-1}\vec{z}\,' \int_0^{\infty} d\omega (-i\omega)^{d-\Delta-1} e^{i\omega(\pm 2\sin\tau' \mp 2\frac{u}{R}\cos\tau' - 2\Omega \cdot \Omega')} \\
&\times F_{\mu_1 \cdots \mu_i}(x, y),
\end{aligned} \tag{B.41}$$

where

$$F_{\mu_1 \cdots \mu_i}(x, y) = \prod_i \partial_{x^{\mu_i}} \partial_{y^{\nu_i}} \log(\pm 2\sin\tau' - 2\Omega \cdot \Omega') \Phi^{\nu_1 \cdots \nu_J}(y) + \mathcal{O}(R^{-1}) \tag{B.42}$$

and μ_i, ν_i are restricted to Ω, Ω'. We also defined

$$\Phi^{\pm}(u, \Omega) \equiv \Phi(\pm\frac{\pi}{2} + \frac{u}{R}, \Omega). \tag{B.43}$$

In general, $\int du u^{-\Delta_0} \widetilde{\Phi}$ is an operator in CFT$_d$ with dimension $d - \Delta + \Delta_0 - 1$ (see section 4.4). Setting $\Delta_0 = 0$ should then yield an operator of dimension $d - \Delta - 1$ in CCFT$_{d-1}$. Note that for $\Delta_0 = 0$, (B.41) is singular which suggests one should take a residue [36].

Indeed, the residue of (B.41) at $\Delta_0 = 0$ reduces to

$$\int_{-\infty}^{\infty} du \widetilde{\Phi}^{\pm}_{\mu_1\cdots\mu_J}(u,\Omega) = -\frac{1}{\Gamma(d-\Delta)} \int d\tau' d^{d-1}\vec{z}\,' \int_0^{\infty} d\omega (-i\omega)^{d-1-\Delta-1} \frac{R}{2} \sum_{\tau_0=\pm\frac{\pi}{2}} \delta(\tau'-\tau_0)$$
$$\times\, e^{i\omega(\pm 2\sin\tau' - 2\Omega\cdot\Omega')} F_{\mu_1\cdots\mu_i}(x,y)$$
$$= -\frac{i}{2}\frac{R}{d-1-\Delta} \int d^{d-1}\vec{z}\,' \sum_{\alpha\in\{0,1\}} \frac{\prod_i \partial_{x^{\mu_i}}\partial_{y^{\nu_i}} \log(\pm e^{i\pi\alpha}2 - 2\Omega\cdot\Omega')}{(\pm e^{i\pi\alpha}2 - 2\Omega\cdot\Omega')^{d-1-\Delta}}$$
$$\times\, \Phi^{\nu_1\cdots\nu_J}(e^{i\pi\alpha}\frac{\pi}{2}, \Omega') + \mathcal{O}(R^0), \tag{B.44}$$

which we recognize as proportional to a linear combination of $(d-1)$-dimensional shadow transforms in the strips around $\pm\frac{\pi}{2}$. Note the appearance of a linear combination of incoming and outgoing insertions. It may be interesting to understand this better, perhaps in relation to the proposal of [195].

On the other hand, taking the residue at $\Delta_0 = 1$ of (B.41) and using the identity [36]

$$\lim_{\epsilon\to 0} \epsilon x^{\epsilon-1} = 2\delta(x), \tag{B.45}$$

we find

$$\operatorname*{Res}_{\Delta_0=1} \Gamma(\Delta_0) \int_{-\infty}^{\infty} du u^{-\Delta_0} \widetilde{\Phi}^{\pm}_{\mu_1\cdots\mu_J}(u,\Omega) = 2\widetilde{\Phi}^{\pm}_{\mu_1\cdots\mu_J}(0,\Omega). \tag{B.46}$$

This operator is a primary of dimension $d-\Delta$ in the CFT$_d$ as well as in the CCFT$_{d-1}$. For $d=3$, taking a 2D shadow then yields an operator of dimension $\Delta - 1$, which in the special case of the CFT$_3$ stress tensor reduces to the stress tensor in the CCFT$_2$.

More generally, given operators $\mathcal{O}^{\pm}_{\Delta}(u,\Omega)$ in strips around $\pm\frac{\pi}{2}$,

$$\operatorname*{Res}_{\Delta_0=1} \int_{-\infty}^{\infty} du u^{-\Delta_0} \mathcal{O}^{\pm}_{\Delta}(u,\Omega) = 2\mathcal{O}^{\pm}_{\Delta}(0,\Omega). \tag{B.47}$$

Since $\Delta_{\text{CCFT}} = \Delta + \Delta_0 - 1$ we get an operator of $\Delta_{\text{CCFT}} = \Delta$. We conclude that placing an operator at $u = 0$ inside a small time interval corresponds in CCFT to an operator that inherits the dimension Δ of the operator in the parent CFT, as found in [1].

B.5 Derivation of CCFT$_{d-1}$ conformally soft theorems from CFT$_d$

In this appendix, we give the derivation of the leading and subleading conformally soft graviton theorems from the higher dimensional shadow stress tensor correlator. We start by defining

$$S_{ab}^{(d)} = \sum_{i=1}^{n} \frac{\varepsilon_a^A \varepsilon_b^B(x) P_A(x_i) P^C(x)}{P(x)\cdot P(x_i)} (\mathcal{J}_i)_{CB}, \tag{B.48}$$

so that the shadow stress tensor correlator in the CFT$_d$ becomes

$$\langle \mathcal{G}_{ab} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = -i S_{\{ab\}}^{(d)} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle. \tag{B.49}$$

To compute the flat space limit of $S_{ab}^{(d)}$ we expand at large R keeping the first subleading contributions. To keep track of them we introduce the following notation:

$$P = q + \delta q, \quad \varepsilon_a = \epsilon_a + \delta\epsilon_a, \quad a \in \{1, \ldots, d-1\}, \tag{B.50}$$

where $q = (q^0, q^i, 0)$ denotes the leading term in P and $\epsilon_a = (\epsilon_a^0, \epsilon_a^i, 0)$ the leading term in ε_a. These correspond to the flat space counterparts of P and ε_a. δq and $\delta\epsilon_a$ are the deviations from the flat space limit and take the form

$$\delta q = (\sin\tau - 1, \vec{0}, \cos\tau), \quad \delta\epsilon_a = z_a \delta q, \quad a \in \{1, \cdots, d-1\}. \tag{B.51}$$

We restrict our attention to the components of the shadow stress tensor tangent to the S^{d-1} on which the CCFT is defined, namely with $a \in \{1, \cdots, d-1\}$.

We need to evaluate

$$\frac{\varepsilon_a(x) \cdot P(x_i)}{P(x) \cdot P(x_i)}, \quad P^A(x)\varepsilon_b^B(x)(\mathcal{J}_i)_{AB}. \tag{B.52}$$

The first quantity is immediate to expand and yields

$$\frac{\varepsilon_a(x) \cdot P(x_i)}{P(x) \cdot P(x_i)} = \frac{\epsilon_a(x) \cdot q(x_i)}{q(x) \cdot q(x_i)} + O(R^{-1}). \tag{B.53}$$

For the second one we have

$$P^A(x)\varepsilon_b^B(x)(\mathcal{J}_i)_{AB} = q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB} + z_b q^A(x)\delta q^B(x)(\mathcal{J}_i)_{AB} + \delta q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB}. \tag{B.54}$$

We now study the second and third terms observing that for $\tau = \frac{\pi}{2} + \frac{u}{R}$ and large R, $(\mathcal{J}_i)_{A,d+1} = iRq_A(x_i)\partial_{u_i} + O(1)$

$$\begin{aligned} q^A(x)\delta q^B(x)(\mathcal{J}_i)_{AB} &= -(\sin\tau - 1)q^j(x)(\mathcal{J}_i)_{0j} + \cos\tau q(x) \cdot q(x_i)\left(iR\partial_{u_i} + O(R^0)\right), \\ \delta q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB} &= (\sin\tau - 1)\epsilon_b^j(x)(\mathcal{J}_i)_{0j} - \cos\tau\epsilon_b(x) \cdot q(x_i)\left(iR\partial_{u_i} + O(R^0)\right). \end{aligned} \tag{B.55}$$

As a result, we have

$$\begin{aligned} P^A(x)\varepsilon_b^B(x)(\mathcal{J}_i)_{AB} &= q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB} - z_b(\sin\tau - 1)q^j(x)(\mathcal{J}_i)_{0j} \\ &+ z_b\cos\tau q(x) \cdot q(x_i)\left(iR\partial_{u_i} + O(R^0)\right) + (\sin\tau - 1)\epsilon_b^j(x)(\mathcal{J}_i)_{0j} \\ &- \cos\tau\epsilon_b(x) \cdot q(x_i)\left(iR\partial_{u_i} + O(R^0)\right). \end{aligned} \tag{B.56}$$

At this point, we can further expand at large R. In particular, we notice that the first term is $O(1)$ because $A, B < d+1$. For the others we write $\tau = \frac{\pi}{2} + \frac{u}{R}$ and expand at large R to find

$$\begin{aligned} P^A(x)\varepsilon_b^B(x)(\mathcal{J}_i)_{AB} &= q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB} - iuz_b q(x)\cdot q(x_i)\partial_{u_i} + iu\epsilon_b(x)\cdot q(x_i)\partial_{u_i} \\ &\quad + O(R^{-1}). \end{aligned} \tag{B.57}$$

Combining with (B.53) we find

$$\begin{aligned} S_{ab}^{(d)} &= \sum_{i=1}^{n} \frac{\varepsilon_a(x)\cdot P(x_i)}{P(x)\cdot P(x_i)} P^A(x)\varepsilon_b^B(x)(\mathcal{J}_i)_{AB} \\ &= \sum_{i=1}^{n} \left[\frac{\epsilon_a(x)\cdot q(x_i)}{q(x)\cdot q(x_i)} \Big(q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB} - iuz_b q(x)\cdot q(x_i)\partial_{u_i} + iu\epsilon_b(x)\cdot q(x_i)\partial_{u_i} \Big) + O(R^{-1}) \right]. \end{aligned} \tag{B.58}$$

Taking one derivative in u we get

$$\begin{aligned} \partial_u S_{ab}^{(d)} &= i\sum_{i=1}^{n} \left[\frac{\epsilon_a(x)\cdot q(x_i)}{q(x)\cdot q(x_i)} \Big(-z_b q(x)\cdot q(x_i)\partial_{u_i} + \epsilon_b(x)\cdot q(x_i)\partial_{u_i} \Big) + O(R^{-1}) \right] \\ &= i\sum_{i=1}^{n} \left[\Big(-z_b\epsilon_a(x)\cdot q(x_i)\partial_{u_i} + \frac{\epsilon_a(x)\cdot q(x_i)\epsilon_b(x)\cdot q(x_i)}{q(x)\cdot q(x_i)}\partial_{u_i} \Big) + O(R^{-1}) \right]. \end{aligned} \tag{B.59}$$

Now observe that the first term is proportional to the operator $\sum_i q^A(x_i)\partial_{u_i}$ which annihilates conformal correlators by the global conformal symmetry of the CFT$_d$ to leading order at large R (or equivalently by momentum conservation in the flat limit). Specifically

$$\sum_{i=1}^{n} \mathcal{J}_{j,d+1}(x_i)\langle\mathbb{X}\rangle = \sum_{i=1}^{n} \left(-iP_j(x_i)\partial_{P^{d+1}(x_i)} + iP_{d+1}(x_i)\partial_{P^j(x_i)} + \mathcal{S}_{j,d+1} \right)\langle\mathbb{X}\rangle = 0, \quad j = 0, \cdots d, \tag{B.60}$$

and therefore

$$\sum_{i=1}^{n} iq_j(x_i)\partial_{u_i}\langle\mathbb{X}\rangle = \frac{1}{R}\sum_{i=1}^{n}\left(-iP_{d+1}(x_i)\partial_{P^j(x_i)} - \mathcal{S}_{j,d+1}\right)\langle\mathbb{X}\rangle = \mathcal{O}(R^{-1}), \quad j = 0, \cdots d. \tag{B.61}$$

As such, only the second term remains

$$\partial_u S_{ab}^{(d)} = i\sum_{i=1}^{n}\left[\frac{\epsilon_a(x)\cdot q(x_i)\epsilon_b(x)\cdot q(x_i)}{q(x)\cdot q(x_i)}\partial_{u_i} + O(R^{-1})\right], \tag{B.62}$$

which coincides with the leading soft factor. Moreover, it is also clear that

$$(1-u\partial_u)S_{ab}^{(d)} = \sum_{i=1}^{n}\frac{\epsilon_a(x)\cdot q(x_i)}{q(x)\cdot q(x_i)}q^A(x)\epsilon_b^B(x)(\mathcal{J}_i)_{AB} + O(R^{-1}), \tag{B.63}$$

where since $a, b \in \{1, \ldots, d-1\}$ it follows that $A, B \in \{0, \ldots, d\}$ and in this range $(\mathcal{J}_i)_{AB}$ act as the $\mathbb{R}^{1,d}$ Lorentz generators in the flat space limit. Finally we take the $(d-1)$-dimensional symmetric traceless component of $S_{ab}^{(d)}$ with $a, b \in \{1, \ldots, d-1\}$ by applying the projector (4.23). Then

$$\epsilon_{ab} \equiv \epsilon_{\{a}^A\epsilon_{b\}}^B = \frac{1}{2}[\epsilon_a^A\epsilon_b^B + \epsilon_a^B\epsilon_b^A] - \frac{\eta_{ab}}{d-1}[\eta^{cd}\epsilon_c^A\epsilon_d^B]. \tag{B.64}$$

However, since $\epsilon_a^{d+1} = 0$ it follows that $\eta^{cd}\epsilon_c^A\epsilon_d^B = \delta^{cd}\epsilon_c^A\epsilon_d^B$ and that $\epsilon_{\{a}^A\epsilon_{b\}}^B = 0$ when either A or B are $d+1$. As a result, for $a, b \in \{1, \ldots, d-1\}$,

$$\epsilon_{\{a}^A\epsilon_{b\}}^B = \frac{1}{2}[\epsilon_a^A\epsilon_b^B + \epsilon_a^B\epsilon_b^A] - \frac{\delta_{ab}}{d-1}[\delta^{cd}\epsilon_c^A\epsilon_d^B], \quad A, B < d+1, \tag{B.65}$$

which coincide with the symmetric traceless polarizations in $\mathbb{R}^{1,d}$. As a result, the operators $\mathcal{N}_{ab}^{(0)} = \lim_{u\to 0}\partial_u\mathcal{G}_{\{ab\}}$ and $\mathcal{N}_{ab}^{(1)} = \lim_{u\to 0}(1-u\partial_u)\mathcal{G}_{\{ab\}}$ play the role of leading and subleading conformally soft gravitons in $\mathbb{R}^{1,d}$. It is immediate to see that they have the expected dimensions $\Delta = 1$ and $\Delta = 0$ respectively.

We conclude this appendix with a comment on the timelike components of the shadow stress tensor. For $d = 3$ one can construct from the u, A components of the shadow stress tensor operators which coincide with the supertranslation currents in the dimensionally reduced theory. This is perhaps to be expected, as conservation of the CFT_3 stress tensor leads to relations among its transverse and time components. It may be interesting to further explore these constraints in relation to the asymptotic Einstein equations in 4D AFS.

B.6 Subleading soft factor in CCFT_2

In this appendix we calculate the subleading soft factor

$$(1 - u\partial_u)S^{(d)}_{ab} = \sum_{i=1}^{n} \frac{\epsilon_a(x) \cdot q(x_i)}{q(x) \cdot q(x_i)} q^A(x)\epsilon^B_b(x)(\mathcal{J}_i)_{AB} + O(R^{-1}), \tag{B.66}$$

in the specific case of reduction from CFT_3 to CCFT_2. We need to evaluate $q^A(x)\epsilon^B_b(x)(\mathcal{J}_i)_{AB}$ using the complex polarization vectors $\{\epsilon_z, \epsilon_{\bar{z}}\}$. We recall that $(\mathcal{J}_i)_{AB}$ are the $\mathfrak{so}(3,2)$ generators acting on the i-th primary operator. The actions of such conformal Killing vectors and their large R expansion have been studied in section 4.4.2. In particular, we note that since $q^4 = \epsilon^4_b = 0$, only $(\mathcal{J}_i)_{AB}$ with $A, B < 4$ appear. For this range of indices, we have[2]

$$\mathcal{J}_{AB}\mathcal{O}_i = -\delta_{L_{Y_{AB}}}\mathcal{O}_i, \quad A, B = 0, \cdots 3, \tag{B.67}$$

[2] It is possible to check by explicit computation that $\mathcal{J}_{AB}$ reproduces the conformal Killing vector action by studying its action on lightcone fields in coordinates adapted to the cylinder section. Indeed, parameterizing the lightcone as $X = (r\sin\tau, r\Omega, r\cos\tau)$, so that the cylinder section is obtained by gauge-fixing $r = 1$, and evaluating $\mathcal{J}_{AB}\mathcal{O}_\Delta(X)$, we find due to the homogeneity of $\mathcal{O}_\Delta(X)$ under rescalings that $-r\partial_r\mathcal{O}_\Delta = \Delta\mathcal{O}_\Delta$. Then (B.67) follows by straightforward computation.

where L_Y has been defined in (4.59) and Y_{AB} are the S^2 conformal Killing vectors (4.47) and (4.48). We have computed the large R expansion of $\delta_{L_{Y_{AB}}}\mathcal{O}_i$ in (4.69), which yields

$$(\mathcal{J}_i)_{AB}\mathcal{O}_i = i\left(D_{z_i}Y^{z_i}_{AB}\mathfrak{h}_i + D_{\bar{z}_i}Y^{\bar{z}_i}_{AB}\bar{\mathfrak{h}}_i + Y^{z_i}_{AB}(\partial_{z_i} - \Omega_{z_i}J_3) + Y^{\bar{z}_i}_{AB}(\partial_{\bar{z}_i} - \Omega_{\bar{z}_i}J_3) + O(R^{-1})\right)\mathcal{O}_i. \tag{B.68}$$

Now using the explicit parametrization of q and $\{\epsilon_z, \epsilon_{\bar{z}}\}$ it is straightforward to compute the following contractions

$$q^A(x)\epsilon^B_{\bar{z}}(x)Y_{AB}(z_i,\bar{z}_i) = -\frac{(z-z_i)^2}{1+z\bar{z}}\partial_{z_i}, \tag{B.69}$$

$$q^A(x)\epsilon^B_{z}(x)Y_{AB}(z_i,\bar{z}_i) = -\frac{(\bar{z}-\bar{z}_i)^2}{1+z\bar{z}}\partial_{\bar{z}_i}, \tag{B.70}$$

from which we immediately obtain

$$\begin{aligned}
-iq^A(x)\epsilon^B_{\bar{z}}(x)(\mathcal{J}_i)_{AB}\mathcal{O}_i &= \left[\frac{(z-z_i)(1+z\bar{z}_i)}{(1+z\bar{z})(1+z_i\bar{z}_i)}2\mathfrak{h}_i - \frac{(z-z_i)^2}{1+z\bar{z}}(\partial_{z_i}-\Omega_{z_i}J_3) + O(R^{-1})\right]\mathcal{O}_i,\\
-iq^A(x)\epsilon^B_{z}(x)(\mathcal{J}_i)_{AB}\mathcal{O}_i &= \left[\frac{(\bar{z}-\bar{z}_i)(1+\bar{z}z_i)}{(1+z\bar{z})(1+z_i\bar{z}_i)}2\bar{\mathfrak{h}}_i - \frac{(\bar{z}-\bar{z}_i)^2}{1+z\bar{z}}(\partial_{\bar{z}_i}-\Omega_{\bar{z}_i}J_3) + O(R^{-1})\right]\mathcal{O}_i.
\end{aligned} \tag{B.71}$$

In turn, this means that we have

$$\begin{aligned}
(1-u\partial_u)S^{(3)}_{\bar{z}\bar{z}} &= i\sum_{i=1}^{n}\left[\frac{(z-z_i)(1+z\bar{z}_i)}{(\bar{z}-\bar{z}_i)(1+z_i\bar{z}_i)}2\mathfrak{h}_i - \frac{(z-z_i)^2}{\bar{z}-\bar{z}_i}(\partial_{z_i}-\Omega_{z_i}J_3)\right] + O(R^{-1}),\\
(1-u\partial_u)S^{(3)}_{zz} &= i\sum_{i=1}^{n}\left[\frac{(\bar{z}-\bar{z}_i)(1+\bar{z}z_i)}{(z-z_i)(1+z_i\bar{z}_i)}2\bar{\mathfrak{h}}_i - \frac{(\bar{z}-\bar{z}_i)^2}{z-z_i}(\partial_{\bar{z}_i}-\Omega_{\bar{z}_i}J_3)\right] + O(R^{-1}),
\end{aligned} \tag{B.72}$$

which take the form of the standard CCFT_2 soft factors [27,28] with the operator-valued weights $(\mathfrak{h}, \bar{\mathfrak{h}})$ in place of the standard weights.