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Moduli Spaces of Neural Networks

Espaços Moduli de Redes Neurais

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"Pensé en un mundo sin memoria, sin tiempo,
consideré la posibilidad de un lenguaje que ignorara los sustantivos,
un lenguaje de verbos impersonales o de indeclinables epítetos.
Así fueron muriendo los días y con los días los años,
pero algo parecido a la felicidad ocurrió una mañana.
Llovió, con lentitud poderosa."
(Jorge Luis Borges, El inmortal)

Resumo

Neste trabalho usaremos a Teoria Geométrica dos Invariantes, formalizada em (MUMFORD; FOGARTY; KIRWAN, 1994), para definir e construir espaços moduli de representações de quivers que estão em relação com redes neurais. Mais formalmente, definiremos redes neurais em termos de quivers e estudaremos alguns espaços moduli para os tipos de quivers definidos em (ARMENTA; JODOIN, 2021), e cujas propriedades geométricas foram estudadas em (ARMENTA et al., 2022). Como passo prévio também generalizaremos a estabilidade necessária para definir tais espaços a uma mais geral e estudaremos algumas propriedades algébricas que elas tem baseados no trabalho por (RUDAKOV, 1997).

Palavras-chave: Espaço Moduli. Rede Neural. Teoria Geométrica dos Invariantes. Estabilidade. Representações de quivers.

Abstract

In this work we will use Geometric Invariant Theory, formalized in (MUMFORD; FOGARTY; KIRWAN, 1994), to define and construct the moduli spaces of quiver representations for quivers that are in relation with neural networks. More formally, we will define neural networks in terms of quivers and study some moduli spaces for those type of quivers as defined in (ARMENTA; JODOIN, 2021) and where its geometric properties were studied in (ARMENTA et al., 2022), via the common technique defined by (KING, 1994). As a previous step we will also generalize the stability needed for defining such spaces to a one more general and study some algebraic properties it has, based on the work by (RUDAKOV, 1997).

Keywords: Moduli Space. Neural Network. Geometric Invariant Theory. Stability. Quiver representations.

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Introduction

Classification problems are among the main interests in mathematics: the Fundamental Theorem of Linear Algebra gives the classification up to isomorphism of finite dimensional vector spaces, Sylow theorems give a classification up to isomorphism of finite groups, and the topological Euler characteristic gives a classification of compact orientable surfaces up to homeomorphism, just to mention some examples. In algebraic geometry, however, discrete invariants are not enough to get a good classifying space (*moduli space*) due that the set of (equivalence classes of) objects we would like to classify (for instance, vector bundles over a fixed algebraic variety) is often "too big". For the moduli space to enjoy some good properties such as compactness, connectedness, or smoothness, we have to not only fix enough discrete invariants, but also restrict ourselves to objects satisfying some type of condition (a *stability condition*).

One of the most approachable methods for constructing moduli spaces in Algebraic Geometry is Mumford's Geometric Invariant Theory (GIT) (MUMFORD; FOGARTY; KIRWAN, 1994). Ideally, one would find an algebraic variety that over-parametrizes the equivalence classes of objects we want to classify and then mod out such over-parametrization, if we are lucky and such over-parametrization is encoded as the action of a reductive group, Bingo!, GIT will give us a moduli space that, at least, is an algebraic variety.

The core of Mumford's GIT is that quotients of algebraic varieties by algebraic groups are not necessarily algebraic varieties, however if the group acting is reductive, then it is possible to choose and open subset of the variety (that is a union of orbits) so that the quotient of this open set by the restriction of the action is an algebraic variety. These "good orbits" are called *semistable orbits*. An interesting feature of Mumford's construction is that there is not a unique way to choose the open subset of semistable orbits, something that we will explore in several examples in this text.

In this dissertation we will treat the classifying problem of (equivalence classes of) representations of directed graphs (known as *quivers*). Briefly, a representation of a quiver consists of a vector space for each one of the vertices of the directed graph, and a linear transformation for each arrow.

For instance, if we want to study the representations of the quiver

$$1 \longrightarrow 2$$

we end up studying the linear transformations $V \longrightarrow W$, where V and W are two vector spaces. The discrete invariants that we fix in this case are precisely the dimensions of the corresponding vector spaces that we can encode in a dimension vector. Our GIT problem then is to classify quiver representations up to the action of changing basis on the vector spaces on each vertex. Fortunately for us, (KING, 1994) classifies all the semistable orbits in a very algebraic and intuitive way for this case, following the theory from (MUMFORD; FOGARTY; KIRWAN, 1994) and adapting it for the case when the action has a nontrivial kernel.

For doing so we show the equivalence between Mumford's and a $\theta-$stability, that depends on a **stability parameter**, a vector that has the same number of entries as vertices of the quiver, and that is orthogonal (in the usual sense) to the dimension vector of the representation we want to classify. So we just end up needing the parameter θ to obtain the semistable and stable orbits; more precisely, if the dot product between the representation and the stability parameter is positive for all subrepresentations we call the representation $\theta-$**semistable**.

$\theta-$stability and semistability are a particular case of a more algebraic notion of stability introduced in (RUDAKOV, 1997), which extends particularly for any abelian category, and then we can consider for instance, the category of coherent sheaves on a projective variety or the category of representations of quivers, thus obtaining a simpler way of identifying stability of objects and algebraic properties they inherit. Rudakov's stability is in particular an example of Bridgeland stability (BRIDGELAND, 2007), which is influential by itself on the modern theory of moduli spaces in algebraic geometry (see for example the related works of (TRAMEL; XIA, 2022), (SCHMIDT, 2020) and (MARTINEZ; SCHMIDT, 2019)).

One of the applications of the GIT for quiver representations has a relation with the study of neural networks, which is widely used in machine learning. In particular, we could define neural networks represented by quivers, and to study the data space in terms of the moduli space associated to it. This work was formalized in (ARMENTA; JODOIN, 2021), inspiring some results on computational aspects and bringing to the table the possible study of more moduli spaces of "mutations" of quivers. It was also seen in (ARMENTA et al., 2022), that this space has good geometric properties, given that the quiver associated to a neural network has a desirable structure.

In this dissertation we will use the basic theory from GIT to define moduli spaces of neural networks in the way as done by (ARMENTA; JODOIN, 2021), and (ARMENTA et al., 2022), and we will particularly formalize a variant version of the Manifold Hypothesis, a known conjecture in machine learning that could lead insights on why most neural networks always achieve a local minimum and hence are able to learn. Even with the huge computational implications this has, we will focus on the mathematical aspects to it; for example, giving more importance to the study of the associated moduli spaces than to the applications to different neural networks in the literature.

On the first chapter we give the preliminaries needed for the theory on the text, in particular, we briefly mention abelian categories, quiver representations, its generalities and a short review on GIT following the notes by Thomas (THOMAS, 2006) and King's seminal work (KING, 1994). In Chapter 2, we will generalize the stability for the one in (RUDAKOV, 1997), and show its relation with quiver representations. In Chapter 3, we formalize the theory of neural networks with the usage of quivers, we show how some basic operations on them work, how to identify via isomorphism neural networks and how to define a moduli space of them based on (ARMENTA; JODOIN, 2021). Lastly, Chapter 4, dedicates itself to the geometric properties of two main moduli spaces of a variant of the quiver, called the *double framed* quiver, and how it relates to neural networks via a restriction on the quiver worked and the dimension vector, following the initial work from (ARMENTA et al., 2022).

Unlike most of common works in mathematics, that have the intention of teach or to show new mathematics, all work in this text has already been developed, and we just expanded the theory and added some examples when we saw it as needed. With this in mind, we did not hesitate to be redundant with the explanation on some arguments, as this was one of our objectives. This election is merely personal, as some people would like to leave some details to the reader to make it more interesting or enticing. We tried to be as clear as possible (most of the time) using the help of examples.

1 Preliminaries

This chapter has the main objective to prepare the possible reader to the text, in the attempt of being as self-contained as possible. Here we will mention the theory needed for some parts of the text, which are not particularly linked one with another, so we recommend the reader to use this Chapter as a means of reference for any general fact that may be out of reach of the text, or whenever it is needed. With this in mind, we will just mention the principal results that are going to be of use, and we will refer to the articles or books in case of any particular proofs or for more details. One particular case that we would like to mention is Geometric Invariant Theory (GIT), which by itself is a very pretty theory, but with some heavy backgrounds, so we made some cuts when we saw it as appropriate. Throughout the text, we assume the reader is familiar with the general theory of algebraic geometry.

1.1 Abelian Categories and Grothendieck Group

Here we can use the preliminary definitions from (LANE, 1998). Suppose that we have a category $\mathcal{C}$, and a field k. We say that $\mathcal{C}$ is **abelian** if:

a. $\mathrm{Hom}(M, N)$ is a $k-$vector space for all $M, N \in \mathcal{C}$,

b. the composition of morphisms is bilinear,

c. $\mathcal{C}$ admits direct sums, and there is a zero element from $\mathcal{C}$ where the corresponding morphism between that space is the zero vector from $\mathrm{Hom}(0,0)$, and

d. each element in $\mathrm{Hom}_{\mathcal{C}}(M, N)$ admits a kernel $k \in \mathrm{Hom}_{\mathcal{C}}(K, M)$ and a cokernel $c \in \mathrm{Hom}_{\mathcal{C}}(N, C)$.

Examples of abelian categories are ample, in particular, we can mention the category of abelian groups GrpAb, the category of sheaves of abelian groups on a topological space X, Shv_X, and the category of finite dimensional vector spaces over a field k, FinVec_k.

Now, suppose that we have an abelian category $\mathcal{A}$. Let A be the free abelian group generated by the elements from $\mathrm{Ob}(\mathcal{A})$, The **Grothendieck group of** $\mathcal{A}$, denoted by $\mathcal{K}_0(\mathcal{A})$ is the quotient of A by the the following relation: $M - N + L = 0$ for every short exact sequence in $\mathcal{A}$:

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

For example, $\mathcal{K}_0(\text{FinVec}_k) \cong \mathbb{Z}$ by the rank nullity theorem. We won't use the group for anything different than FinVec_k or products of it, so we don't present any other examples.

1.2 Quiver Representations

In this section we present the general theory from quiver representations needed for the text. We will use mainly the definitions and notation from the text (SCHIFFLER, 2014), so we refer the reader for any more examples and for a broader approach of the topics treated here.

1.2.1 Generalities

A **quiver** $Q = (Q_0, Q_1, s, t)$ consists of: Q_0 a set of vertices, Q_1 a set of arrows, $s : Q_1 \longrightarrow Q_0$ a function that associates to each $\alpha \in Q_1$ its starting point $s(\alpha) \in Q_0$, and $t : Q_1 \longrightarrow Q_0$ a function that associates to each $\alpha \in Q_1$ its terminal point $t(\alpha) \in Q_0$. We will represent an $\alpha \in Q_1$ by

$$s(\alpha) \xrightarrow{\alpha} t(\alpha)$$

A vertex i is called a **sink** if there are not arrows $\alpha \in Q_1$ such that $s(\alpha) = i$, and will be called a **source** if there are not arrows $\beta \in Q_1$ such that $t(\beta) = i$. A sequence of arrows $(\alpha_1, \ldots, \alpha_n)$ such that $t(\alpha_k) = s(\alpha_{k+1})$ for $1 \leq k < n$ is called a **path**, and a path that starts at vertex i and ends at vertex j (this is, $s(\alpha_1) = i$ and $t(\alpha_n) = j$) will be denoted by $i \rightsquigarrow j$.

We say that Q is **connected** if for any decomposition of vertices $Q_0 = Q_0^1 \cup Q_0^2$, there exists an arrow $\alpha \in Q_1$ such that $s(\alpha) \in Q_0^1$ and $t(\alpha) \in Q_0^2$. Now we fix an algebraically closed field k. A **representation** of Q, denoted by $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a collection of $k-$vector spaces M_i one for each vertex $i \in Q_0$ and a collection of linear transformations:

$$\varphi_\alpha = \quad M_i \longrightarrow M_j$$

one for each arrow $i \xrightarrow{\alpha} j$. We say that the representation M is **finite dimensional** if M_i is a finite dimensional vector space for each $i \in Q_0$. In this case, its **dimension vector** is $\underline{\dim} M = (\dim_k M_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$. Suppose we have a fixed quiver Q and two representations $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ of Q. A **morphism of representations** $f : M \longrightarrow M'$ is a collection of linear transformations $f_i : M_i \longrightarrow M'_i$ such that the following diagram is commutative:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"] & M'_j \end{tikzcd}$$

for each arrow $i \xrightarrow{\alpha} j \in Q_1$. We say that such morphism is an **isomorphism** if each f_i is an isomorphism of vector spaces. The **isomorphism class** of M or the **isoclass** of M will be the set of representations of Q which are isomorphic to M. With this in mind, we can construct the **category of finite dimensional representations of the quiver** Q, noted by $\operatorname{rep} Q$, with objects and morphisms naturally defined.

Given two representations M and M' as before, its **direct sum** will be the representation of Q given by:

$$M \oplus M' = \left(M_i \oplus M'_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix} \right)_{i \in Q_0, \alpha \in Q_1}$$

So $\operatorname{rep} Q$ admits direct sums. And similarly, given a morphism between representations $f : M \longrightarrow M'$ its **kernel** will be the representation

$$\ker f = \left(\ker f_i, \varphi_\alpha|_{\ker f_i} \right)_{i \in Q_0, \alpha \in Q_1} \in \operatorname{rep} Q$$

and its **cokernel**:

$$\operatorname{coker} f = (\operatorname{coker} f_i, \chi_\alpha)_{i \in Q_0, \alpha \in Q_1}$$

with $\chi_\alpha : \operatorname{coker} f_i \longrightarrow \operatorname{coker} f_j$ defined by $\chi_\alpha (x + f_i(M_i)) = \varphi'_\alpha(x) + f_j(M_j)$. This shows that $\operatorname{rep} Q$ admits kernels and cokernels, and then it is an *abelian category*. Now, we say that a representation L is a **subrepresentation** of M if there is an injective morphism $i : L \hookrightarrow M$. By fixing a dimension vector $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$, we can see that the space of representations that have dimension $\mathbf{d}$ is given by

$$\mathcal{R}_{\mathbf{d}}(Q) = \bigoplus_{i \in Q_0} \operatorname{Hom}(V_i, V_j)$$

where V_i is a $k-$vector space of dimension d_i for each $i \in Q_0$. In particular, we have that

$$\operatorname{rep} Q = \bigoplus_{\mathbf{d} \in \mathbb{Z}^{|Q_0|}} \mathcal{R}_{\mathbf{d}}(Q)$$

For the next part we use the structure from (KIRILLOV JR., 2016). Suppose we have two representations M, N of a quiver Q, we define $\langle M, N \rangle \in \mathbb{Z}$ by

$$\langle M, N \rangle = \sum (-1)^i \dim \operatorname{Ext}^i(V, W) = \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}^1(M, N)$$

as all $\mathrm{Ext}^i = 0$ for $i > 1$. In particular, we have by ((KIRILLOV JR., 2016), Theorem 1.25) that if $\mathbf{u} = (u_i)_{i \in Q_0}$ and $\mathbf{v} = (v_i)_{i \in Q_0}$ are dimension vectors for M and N, the value $\langle M, N \rangle$ just depends on the vectors $\mathbf{u}$, $\mathbf{v}$. Then we can define a bilinear form on $\mathbb{Z}^{|Q_0|}$, which we call the **Euler form** of Q, and moreover:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i \in Q_0} u_i v_i + \sum_{\alpha \in Q_1} u_{s(\alpha)} v_{t(\alpha)}$$

1.2.2 A simple computation on Thin Quiver Representations

Here our objective is to give some criteria for the Hom between two type of manageable representations for Q. This will be used for a result on Chapter 3. We say that a representation M of Q with dimension vector $\mathbf{d} = (d_i)_{i \in Q_0}$ is **thin** if $d_i \leq 1$ for all $i \in Q_0$. Suppose that Q is a finite, acyclic, connected quiver and let $M = (M_i, \varphi_\alpha)$, $N = (N_i, \varphi'_\alpha)$ be two thin representations from Q. If there exists a vertex $i \in Q_0$ such that $M_i = N_i = \mathbb{C}$ and we have a section of the form:

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{C} \end{array} \qquad \text{or,} \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

we say that i has a **critic of type I** on the first case, or a **critic of type II** on the latter. Maybe the important thing is that the critics characterize the $\mathrm{Hom}(M, N)$; more specifically, $\mathrm{Hom}(M, N) = 0$ if and only if there is a vertex with a critic of type I or II.

1.3 A brief comment on the manifold hypothesis

Neural networks have been used for a variety of applications: language processing: (FATHI; SHOJA, 2018), voice recognition: (MELIN et al., 2006), (VENAYAGAMOORTHY; MOONASAR; SANDRASEGARAN, 1998), image processing: (HIJAZI et al., 2015), (TRAORE; KAMSU-FOGUEM; TANGARA, 2018), code analising and more recently, for language models for dialogue .

One leading question in machine learning is the one of how a determined neural network is always capable of learn. The most accepted possible answer is the famous Manifold Hypothesis . Briefly, it states that the input space of a neural network contains submanifolds of dimension strictly less where the real-world data lies, and even when its believed to be true, it has not been proved formally at the time. A known and extensive testing has already been done in (FEFFERMAN; MITTER; NARAYANAN, 2016).

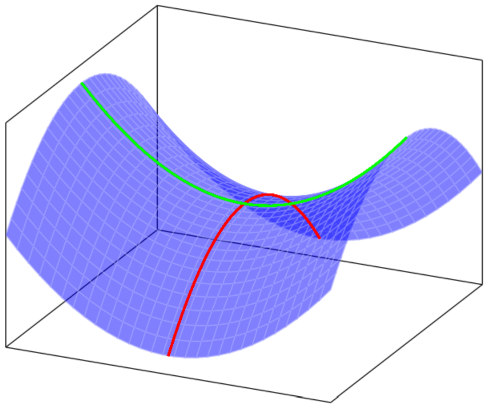


Figure 1 – An example of the MH. The blue surface could represent the space of all images, and the submanifold in red represents the images of cats (or any other object).

For example, if the blue surface represents all the images, the manifold hypothesis would imply the existence of submanifolds, the red curve could parameterize all cat images.

1.4 A quick (non-cute) review on GIT

1.4.1 Motivating problems for a quotient theory

On this part we will give try to explain why taking quotients on algebraic geometry is a task that may pose some problems when done in the more intuitive way. We believe that the treatment given in (THOMAS, 2006) is adequate and very geometrical for this segment, so we will follow his structure and motivation.

Supose we have a linear algebraic group G acting on a projective variety X through the group of matrices SL, and we want to form a quotient ${}^{X}\!/_{G}$ which parametrizes the orbits of the action of the group. In this case, we would like to consider such a quotient as a projective variety. We have certain problems with this:

1. **The quotient may not be Hausdorff.** There are non closed orbits which contain orbits of smaller dimension on its closures, and then the topological quotient is not Hausdorff. We have to reduce some orbits to obtain a separated (Hausdorff) quotient.

2. **It is not enough to remove smaller-dimension orbits.** Suppose we have the

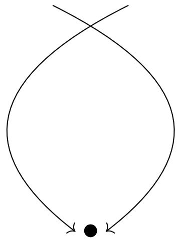


Figure 2 – A visual representation of condition 1.. The arrows and the dots represent different orbits, and the arrows have the dot as closure. Taken from (THOMAS, 2006).

action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by

$$\mathbb{C}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \tag{1.1}$$
$$(\lambda, (z_1, z_2)) \longmapsto (\lambda z_1, \lambda^{-1} z_2) \tag{1.2}$$

This is, acting through the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$. The orbit of an element $z = (z_1, z_2) \in \mathbb{C}^2$ will be:

$$\begin{aligned} \mathbb{C}^* \cdot z &= \left\{ \left(\lambda z_1, \lambda^{-1} z_2 \right) \mid \lambda \in \mathbb{C}^* \right\} \\ &= \{(a, b) \in \mathbb{C}^2 \mid ab = z_1 z_2\} \end{aligned}$$

In particular all orbits are parametrized by the constant α such that $z_1 z_2 = \alpha$. This gives us the hyperbolas on $\mathbb{C}^2$, the $x-$axis, $y-$axis and the point 0. If we remove the origin we would obtain two points at the origin, a nonseparated double point (corresponding to the orbits of the $x-$ and $y-$axis). Then the space wont be Hausdorff.

On this last example we would like to obtain as a topological quotient $\mathbb{C}$, where each complex number α represents one orbit (the hyperbola $z_1 z_2 = \alpha$). The quotient obtained from Geometric Invariant Theory (GIT) will identify the three orbits corresponding to zero, as when $\alpha \to 0$ such orbits intersect. More specifically, we identify some of the orbits to be "bad" and we remove them, and some that are "semistable" and identify them together. This compactifies the space, making it into a projective variety that we will note by ${}^{X}\!/\!_{G}$.

We proceed to construct such space. Let $\mathcal{O}_X(-1)$ the canonical line bundle of X, so the action of G can be lifted to one on $\mathcal{O}_X(-1)$, so we also work on the cone, and we will call the way on which the action acts on the fibers as a **linearisation** of the action. Tensoring we can also see that G acts on $\Gamma(\mathcal{O}_X(r)) = H^0(\mathcal{O}_X(r))$ for each $r \in \mathbb{Z}$. Then we construct ${}^{X}\!/\!_{G}$ via the ring of $G-$invariant sections

$$ {}^{X}\!/\!_{G} \longleftrightarrow \bigoplus_{r \in \mathbb{Z}} H^0(\mathcal{O}_X(r))^G $$

For having a good quotient we need the fact that $\bigoplus_{r\in\mathbb{Z}} H^0(\mathcal{O}_X(r))^G$ is finitely generated, and then we can define:

$$X/_G := \text{Proj} \bigoplus_{r\in\mathbb{Z}} H^0(\mathcal{O}_X(r))^G$$

Lastly we would like to explain what are the points from $X/_G$, which ones are the orbits that get thrown away and what happens if we change the linearisation of the action. In some cases the quotient may be trivial, but by composing with a character of the group it is possible to obtain more sensible quotients; for example $\mathbb{P}^n$, which can be constructed with a line bundle via a particular linearisation.

The next step is to try and look what are the points on the quotient above constructed. Consider the embedding of X into the quotient given by:

$$\begin{aligned} X &\dashrightarrow \mathbb{P}\left((H^0(\mathcal{O}_X(r))^G)^*\right) \\ x &\longmapsto ev_x \end{aligned}$$

where $ev_x := s(x)$ which in coordinates can be written as $(s_1(x) : \cdots : s_n(x)) \in \mathbb{P}^n$, and where the s_i generate $H^0(\mathcal{O}_X(r))^G$. This map is rational, so it is defined on the points there ev is not identically zero, this is, on the points where not all the s_i are zero. With this in mind, we say that a point $x \in X$ is **semistable** if there exists an integer $r > 0$ and section $s \in H^0(\mathcal{O}_X(r))^G$ such that $s(x) \neq 0$. We call a point that is not semistable as **unstable**.

This shows that the semistable points are those seen by the $G-$invariant functions. The embedding as above is defined in the set of semistable points (or **semistable locus**) denoted by $X^{\text{ss}} \subset X$, and is constant on $G-$orbits, as we are working with $G-$invariant functions. It allows to factor the space through the set $X^{\text{ss}}/_G$, but it can "eat" more than just the orbits, and then a point were this phenomena does not happens will be called **stable**. More formally, a point x is **stable** if it is semistable and $\bigoplus_{r\in\mathbb{Z}} H^0(\mathcal{O}_X(r))^G$ separates orbits near x and has finite stabilizer.

So we have a surjective map where we can pullback the line bundle obtained on the quotient and obtain the canonical line bundle:

$$\begin{tikzcd} \mathcal{O}_{X^{\text{ss}}}(-1) \arrow[d, dash] & \mathcal{O}_{X/_G} \arrow[l, dashed] \arrow[d, dash] \\ X^{\text{ss}} \arrow[r, two heads] & X/_G \end{tikzcd}$$

In general the map has good geometric properties on the stable locus $X^{\text{s}} \subseteq X^{\text{ss}} \subseteq X$, and is a *geometric quotient* in the sense of Mumford. In the following we will describe different characterizations of when a point is stable or semistable by looking at the induced point on the line bundle $\mathcal{O}_X(-1)$, or in the cone of X when seen as an affine variety.

1.4.2 General facts of GIT

On this part and for the next one we will follow the structure from (KING, 1994), and we will mention the main results without giving specifics on the proof of them. We refer the reader to the article or to ((KIRILLOV JR., 2016), Chapter 10) for a recent treatment of the topics. Suppose we have G as before and a finite-dimensional space V over k an algebraically closed field (for simplicity). Let $\mathcal{O}(-1)$ be the trivial line bundle for V, and χ an arbitrary character of G. By using χ we lift the $G-$action to $\mathcal{O}(-1)$ by

$$g \cdot (x, z) = \left(g \cdot x, \chi^{-1}(g)z\right)$$

A regular function $f \in k[V]$ is a **relative invariant of weight** χ if $f(g \cdot x) = \chi(g)f(x)$, and we will write $k[V]^{G,\chi}$ for the space of **relative invariant functions of weight** χ. By noting that the space of $\mathcal{O}(-1)$ is just $V \times k$, we say that an **invariant section** of $\mathcal{O}(-1)^n$ is a function $f(x)z^n \in k[V \times k]$ where $f(x)$ is a relative invariant of weight χ^n.

Now, let be $x \in V$, we say that x is $\chi-$**semistable** if there exists an $f \in k[V]^{G,\chi^n}$ with $n \geq 1$ such that $f(x) \neq 0$. If we denote by Δ the kernel of the representation of the group, we say that x is $\chi-$**stable** if it is $\chi-$**semistable**, $\dim G \cdot x = \dim {}^{G}\!/_{\Delta}$ and the $G-$action on the points where the function is non-zero is closed. With this we define

$$V /\!/ (G, \chi) := \operatorname{Proj}\left(\bigoplus_{n \in \mathbb{N}} k[V]^{G,\chi^n}\right)$$

which is projective over the affine quotient $V /\!/ G = \operatorname{Spec}\left(k[V]^G\right)$. Then if $k[V]^G$ is the field k the variety $V /\!/ (G, \chi)$ is projective. The central pillar of GIT ends up that we can see

$$\operatorname{Proj}\left(\bigoplus_{n \in \mathbb{N}} k[V]^{G,\chi^n}\right) = {}^{V_\chi^{\mathrm{ss}}}\!/_{\sim}$$

where V_χ^{ss} is the locus of $\chi-$semistable points and $x \sim y$ if and only if the closures of their correspoding orbits intersect, this is, when $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$ in V_χ^{ss}, in this case we say that x and y are **GIT equivalent**. Moreover, the points of the quotient are in $1-1$ correspondence with the closed orbits of V_χ^{ss}, we also say that the quotient parametrizes the closed orbits of V_χ^{ss}. In particular, we have an open subset of the quotient $V /\!/ (G, \chi)$ that corresponds to the $\chi-$stable orbits, which are closed.

Now we proceed to give different characterizations for $\chi-$stable and $\chi-$semistable points. Let $x \in V$ and we lift it to a point $\hat{x} = (x, z) \in V \times k$ with $z \neq 0$. Then we have that ((KING, 1994), Lemma 2.2) x is $\chi-$semistable if and only if

$$\overline{G \cdot \hat{x}} \cap 0 = \emptyset$$

where 0 is the zero section $V \times \{0\} \cong V$. We also require that $\chi(\Delta) = 1$. Similarly, we have that x is $\chi-$stable if and only if the orbit $G \cdot \hat{x}$ is closed and the stabilizer of $\hat{x}$ contains the kernel Δ with $[\mathrm{Stab}\, \hat{x} : \Delta] < \infty$. Now, we can also obtain a similar result when working with $1-$parameter subgroups, that give some particular orbits of the points: this is, we obtain that ((KING, 1994), Lemma 2.3) x is $\chi-$semistable if and only if for all $1-$parameter subgroups of G

$$\lim_{t \to 0} \lambda(t) \cdot \hat{x} \notin V \times \{0\}$$

and is $\chi-$stable if and only if all the one parameter subgroups for which such limit exist are in Δ. However, one radical point of GIT (for our main interest), comes from numerical characterisations of such points. For doing so, we consider the pairing between a character χ and a $1-$parameter subgroup of G λ given by $\langle \chi, \lambda \rangle = m \in \mathbb{Z}$ if and only if $\chi(\lambda(t)) = t^m$. Then by ((KING, 1994), Proposition 2.5) we say that x is $\chi-$semistable if and only if $\chi(\Delta) = 1$ and for each $1-$parameter subgroup for which the limit $\lim\limits_{t \to 0} \lambda(t) \cdot x$ exists we have that

$$\langle \chi, \lambda \rangle \geq 0$$

And it will be $\chi-$stable if and only if the only $1-$parameter subgroups λ for which such limit exists with $\langle \chi, \lambda \rangle = 0$ are in Δ. Those last two characterisations are known commonly as the **Hilbert-Mumford numerical criterion**, and we will use it for determining some conditions of those criteria for a particular case where the characters are well defined and the group acting has a relatively simple face: quiver representations.

1.4.3 GIT for quiver representations

On this part we are going to use the results mentioned above to the case of quiver representations. So, under our notation, let $\mathbf{d}$ be a dimension vector of a quiver Q, we fix $k-$vector spaces V_i of dimension d_i. We have that the isoclasses of representations of Q with dimension vector $\mathbf{d}$ are in a $1-1$ correspondence with the orbits of the **representation space** of Q:

$$\mathcal{R}_{\mathbf{d}}(Q) = \bigoplus_{\alpha \in Q_1 : i \to j} \mathrm{Hom}(V_i, V_j)$$

under the action of the group:

$$G_{\mathbf{d}}(Q) = \prod_{i \in Q_0} \mathrm{GL}(V_i)$$

acting via $(\tau \cdot \varphi)_\alpha = \tau_{s(\alpha)}\varphi_\alpha\tau_{t(\alpha)}^{-1}$ for each $\alpha \in Q_1$. This group contains the $1-$parameter subgroup

$$\Delta = \{(t\,\mathrm{id}, \dots, t\,\mathrm{id}) \mid t \in k^*\}$$

acting trivially. Now, the characters of $G_{\mathbf{d}}(Q)$, $\chi_\theta(\tau) : G_{\mathbf{d}} \longrightarrow k$, are all given by

$$\chi_\theta(\tau) = \prod_{i \in Q_0} \det(\tau_i)^{\theta_i}$$

for a choosing of $\theta \in \mathbb{Z}^{|Q_0|}$. We can see the election of the vector θ as an homeomorphism $\mathcal{K}_0(\mathrm{rep}\,Q) \to \mathbb{Z}$. Let $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \mathrm{rep}\,Q$, we write

$$\theta(M) = \sum_{i \in Q_0} \theta_i \dim M_i$$

And with this, the condition $\chi_\theta(\Delta) = 1$ becomes:

$$\begin{aligned} 1 = \chi_\theta(\Delta) &= \prod_{i \in Q_0} \det(\Delta_i)^{\theta_i} \\ &= \prod_{i \in Q_0} \det(t\,\mathrm{id})^{\theta_i} \\ &= \prod_{i \in Q_0} \left(t^{\dim M_i}\right)^{\theta_i} \\ &= t^{\sum \theta_i \dim M_i} \\ &= t^{\theta(M)} \end{aligned}$$

From where we get that $\theta(M) = 0$, so the dimension vector should be orthogonal to the vector θ. Now we define a more general type of stability, that we will call **King's stability** and our next objective is to give a sketch on why it coincides with the $\chi-$stability just given above. More formally, let $\mathcal{A}$ an abelian category and $\theta : \mathcal{K}_0 \longrightarrow \mathbb{R}$ an additive function. As in ((KING, 1994), Definition 1.1), we say that $M \in \mathcal{A}$ is $\theta-$**semistable** if $\theta(M) = 0$ and for every subobject $M' \subset M$ we have that $\theta(M') \geq 0$. Similarly, we say that M is $\theta-$**stable** if the only subobjects M' that $\theta(M') = 0$ are M and 0.

For our objective we will use the numerical characterisations given before, and we will see that any $1-$parameter subgroup of $G_{\mathbf{d}}(Q)$ can be seen as a filtration. Let $\lambda : k^* \longrightarrow G_{\mathbf{d}}(Q)$ a $1-$parameter subgroup, and for every $i \in Q_0$ we make the decomposition

$$V_i = \bigoplus_{n \in \mathbb{Z}} V_i^{(n)}$$

where $\lambda(t) \in G_{\mathbf{d}}(Q)$ acts on $V_i^{(n)}$ by t^n. We will write

$$V_i^{(\geq n)} = \bigoplus_{m \geq n} V_i^{(n)} = V_i^{(n)} \oplus V_i^{(n+1)} \oplus V_i^{(n+2)} \oplus \cdots$$

It is possible now to do a description of the transformations of the representation with this filtration, having in mind the action of λ, the linear map:

$$\varphi_\alpha^{(mn)} : V_{s(\alpha)}^{(n)} \longrightarrow V_{t(\alpha)}^{(m)}$$

acts by multiplication of t^{m-n}. Then $\lim\limits_{t\to 0} \lambda(t)\varphi_\alpha$ exists if and only if $\varphi_\alpha^{(mn)} = 0$ for all $m < n$. And we get this if φ_α defines a map

$$V_{s(\alpha)}^{(\geq n)} \longrightarrow V_{t(\alpha)}^{(\geq n)} \text{ for all } n \in \mathbb{Z}$$

This is, if all the subspaces $V_i^{(\geq n)}$ give subrepresentations M_n of M for all $n \in \mathbb{Z}$. With this, a $1-$parameter subgroup λ for which such limit exists determines a filtration of M:

$$\begin{gathered} \cdots \supseteq V_i^{(\geq 0)} \supseteq V_i^{(\geq 1)} \supseteq \cdots \supseteq V_i^{(\geq n)} \supseteq V_i^{(\geq n+1)} \supseteq \cdots \\ \cdots \supseteq M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots \end{gathered}$$

that is indexed by $\mathbb{Z}$ and for which $M_n = M$ for $n \ll 0$ and $M_n = 0$ for $n \gg 0$. Now let a $\mathbb{Z}-$filtration of M as above:

$$M \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k \subseteq 0$$

We define $\lambda(t)$ as acting on the complement of M^{n+1} in M^n by t^n, and we obtain some $1-$parameter subgroup λ (not necessarily unique), for which the limit $\lim\limits_{t\to 0} \lambda(t)\varphi_\alpha$ exists. We also have that

$$\lim_{t\to 0} \lambda(t)\varphi_\alpha = \bigoplus_{n\in\mathbb{Z}} \left(V_i^{(n)}, \varphi_\alpha^{(nn)}\right) = \bigoplus_{n\in\mathbb{Z}} M_n/M_{n+1}$$

Note that we also obtain a filtration on each one of the vector spaces:

$$M_i \supseteq M_i^{(1)} \supseteq M_i^{(2)} \supseteq \cdots \supseteq M_i^{(l)} \supseteq 0$$

one for each $i \in Q_0$. Unless the $1-$parameter subgroup is on the kernel Δ, we have that the filtration determined by λ is proper, so there is an index n such that $0 \neq M_n \neq M$. Suppose that we have an election of scalars $\theta \in \mathbb{Z}^{|Q_0|}$ that is orthogonal to $\mathbf{d}$, so we have

$$\theta(M) = \sum_{i\in Q_0} \theta_i d_i = 0$$

Then, we can obtain a relatively simple expression for the pairing between the characters and the $1-$parameter subgroups in terms of the filtration $\{M_n\}$ induced by λ. Noting that $\lambda(t) \in G_{\mathbf{d}}(Q)$, then $\lambda(t)_i \in \mathrm{GL}(V_i)$ for $i \in Q_0$. Therefore we can see:

$$\lambda(t)_i = \begin{bmatrix} \begin{bmatrix} t & & & \\ & t & & \\ & & \ddots & \\ & & & t \end{bmatrix}_{d_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} t^2 & & & \\ & t^2 & & \\ & & \ddots & \\ & & & t^2 \end{bmatrix}_{d_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \begin{bmatrix} t^n & & & \\ & t^n & & \\ & & \ddots & \\ & & & t^n \end{bmatrix}_{d_n} \end{bmatrix}$$

With this we make the following computation:

$$\begin{aligned}
\chi_\theta(\lambda(t)) &= \prod_{i \in Q_0} \det(\lambda(t)_i)^{\theta_i} \\
&= \prod_{i \in Q_0} \prod_{n \in \mathbb{Z}} \left(\det \varphi_\alpha^{(nn)} \right)^{\theta_i} \\
&= \prod_{i \in Q_0} \prod_{n \in \mathbb{Z}} \left(\underbrace{(t^n) \cdots (t^n)}_{\dim V_i^{(n)}} \right)^{\theta_i} \\
&= \prod_{i \in Q_0} \prod_{n \in \mathbb{Z}} \left(t^{n \dim V_i^{(n)}} \right)^{\theta_i} \\
&= \prod_{i \in Q_0} t^{\theta_i \sum_{n \in \mathbb{Z}} n \dim V^i_{(n)}} \\
&= t^{\sum_{i \in Q_0} \theta_i \sum_{n \in \mathbb{Z}} n \dim V_i^{(n)}}
\end{aligned}$$

An then this implies

$$\langle \chi_\theta, \lambda \rangle = \sum_{i \in Q_0} \theta_i \sum_{n \in \mathbb{Z}} n \dim V_i^{(n)} \tag{1.3}$$

Note that we also have that for each $n \in \mathbb{Z}$:

$$\sum_{i \in Q_0} \theta_i \dim V_i^{(n)} = \sum_{i \in Q_0} \theta_i \dim \left({}^{M_n}\!/\!_{M_{n+1}} \right) = \theta \left({}^{M_n}\!/\!_{M_{n+1}} \right)$$

And then

$$\sum_{n \in \mathbb{Z}} n\theta \left({}^{M_n}\!/\!_{M_{n+1}} \right) = \sum_{n \in \mathbb{Z}} n \sum_{i \in Q_0} \theta_i \dim V_i^{(n)} = \sum_{n \in \mathbb{Z}} \sum_{i \in Q_0} n\theta_i \dim V_i^{(n)} \tag{1.4}$$

As the sums are finite. Together (1.3) and (1.4) give

$$
\begin{aligned}
\langle \chi_\theta, \lambda \rangle &= \sum_{i \in Q_0} \theta_i \sum_{n \in \mathbb{Z}} n \dim V_i^{(n)} && \text{by (1.3)} \\
&= \sum_{n \in \mathbb{Z}} n\theta \left({}^{M_n}\!/\!{}_{M_{n+1}} \right) && \text{by (1.4)} \\
&= \sum_{n \in \mathbb{Z}} n \left(\theta(M_n) - \theta(M_{n+1}) \right) \\
&= \sum_{n \in \mathbb{Z}} n\theta(M_n) - \sum_{n \in \mathbb{Z}} n\theta(M_{n+1}) \\
&= \sum_{n \in \mathbb{Z}} n\theta(M_n) - \sum_{n \in \mathbb{Z}} (n-1)\theta(M_n) && \text{by changing the index} \\
&= \sum_{n \in \mathbb{Z}} \theta(M_n) \qquad (1.5)
\end{aligned}
$$

With this we have that ((KING, 1994), Proposition 1.1) any point in $\mathcal{R}_{\mathbf{d}}(Q)$ corresponding to a representation $M \in \operatorname{rep} Q$ is $\chi_\theta-$semistable (respectively $\chi_\theta-$stable) if and only if M is $\theta-$semistable (respectively $\theta-$stable). For proving this, suppose first that M is $\theta-$semistable, then by (1.5):

$$\langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} \theta(M_n) \geq 0$$

as $\theta(M_n) \geq 0$ by definition because $M_n \subseteq M$ for all $n \in \mathbb{Z}$ and the $1-$parameter subgroups for which the limit exists are those who induce such filtration M_n. Then M is $\chi_\theta-$semistable. Now suppose that M is $\theta-$stable, and suppose that exists a $1-$parameter subgroup λ such that $\lim \lambda(t)x$ exists and $\langle \chi_\theta, \lambda \rangle = 0$. Then λ indices the filtration M_n and

$$0 = \langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} \theta(M_n)$$

implies $\theta(M_n) = 0$ for all $n \in \mathbb{Z}$ as M is $\theta-$semistable. By its stability, $M_n = M$ or $M_n = 0$ for all n. As the filtration is descending it cannot be possible that after an index i where $M_i = M$ we have a $j > i$ with $M_j = 0$. This implies that exists an integer $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ we have $M_n = M$, and the filtration should look like this:

$$\cdots \subseteq 0 \subseteq \cdots \subseteq 0 \subseteq \underbrace{M}_{M_{n_0}} \subseteq \cdots \subseteq M$$

In particular, we know that $\lambda(t)$ acts on M by t^{n_0} and the matrix of $\lambda(t)$ will have the form

$$\lambda(t) = \begin{pmatrix} t^{n_0} & & & \\ & t^{n_0} & & \\ & & \ddots & \\ & & & t^{n_0} \end{pmatrix} = \operatorname{diag}(t^{n_0}, \ldots, t^{n_0}) \in \Delta$$

As $\Delta = \{(t\,\mathrm{id}, \dots, t\,\mathrm{id}) \,|\, t \in k^*\}$ by definition. By the Hilbert-Mumford numerical criterion, M is $\chi_\theta-$stable.

Let be M be $\chi_\theta-$semistable. A subrepresentation $M' \subset M$ induces a $\mathbb{Z}-$filtration with $M_i = M'_i$ for any value of i and $M_n = M$ or 0 accordingly. For example, we may have any of the following:

$$0 \subseteq \cdots \subseteq 0 \subseteq M' \subseteq M \subseteq M \subseteq \cdots \subseteq M$$
$$0 \subseteq \cdots \subseteq 0 \subseteq 0 \subseteq M' \subseteq M \subseteq \cdots \subseteq M$$

Which are two different filtrations that are essentially the same as $0 \subseteq M' \subseteq M$. Note that these filtrations are proper if and only if M' is a proper subrepresentation. As before, any $\mathbb{Z}-$filtration induces a $1-$parameter subgroup, and for the corresponding λ we have by (1.5)

$$\langle \chi_\theta, \lambda \rangle = \sum \theta(M_n) = \theta(M') \geq 0$$

as M' is the only element in the filtration and is greater than zero by the χ_θ semistability. This implies that M is $\theta-$semistable.

We suppose that M is $\chi_\theta-$stable, and that we have a $M' \subseteq M$ with $\theta(M') = 0$, we want to show that $M' = 0$ or $M' = M$. Let λ be the $1-$parameter subgroup that induces such filtration. Once again, $\langle \chi_\theta, \lambda \rangle = \theta(M') = 0$ and by the Hilbert-Mumford numerical criterion $\lambda \subseteq \Delta$. This is, $\lambda(t) \in \Delta$ for all $t \in k^*$, equivalently, $\lambda(t) = (t'\,\mathrm{id}, \dots, t'\,\mathrm{id})$ for some $t' \in k$. Then the induced filtration must be null or the trivial, thus M is $\theta-$stable.

With this we can compute χ_θ stability via the stability function θ. We will note by

$$\mathcal{M}_{\mathbf{d}}^{\theta-\mathrm{sst}}(Q) := \mathcal{R}_{\mathbf{d}}(Q) /\!/ G_{\mathbf{d}}(Q)$$

and will be called the **moduli space of $\theta-$semistable representations of dimension vector d**. In particular, $\mathcal{M}_{\mathbf{d}}^{\theta-\mathrm{sst}}(Q)$ is a projective variety ((KING, 1994), Proposition 4.3), and contains an open subset $\mathcal{M}_{\mathbf{d}}^{\theta-\mathrm{st}}(Q)$ of $\theta-$stable representations. If the quiver Q is finite, acyclic and connected then $\mathcal{M}_{\mathbf{d}}^{\theta-\mathrm{sst}}(Q)$ is irreducible and normal, and the subset $\mathcal{M}_{\mathbf{d}}^{\theta-\mathrm{st}}(Q)$ is smooth (this occurs because the path algebra is hereditary, ((ASSEM; SIMSON; SKOWROńSKI, 2006); Chapter VII, Theorem 1.7)).

2 Rudakov's Stability

The introduction of Geometric Invariant Theory not only gave an interesting notion of a space that parametrizes orbits but also a way on which we could see the changes of the action of a group in terms of the usual algebraic geometry. However, as we mentioned before, it is a really heavy background theory and we must impose a handful of prerequisites to even define the space as it is. King's equivalence of stabilities ((KING, 1994), Proposition 3.1) showed that in quiver representations there is a way of defining such space in just algebraic terms; and this leads the natural question on how much this can be done for another spaces.

In (RUDAKOV, 1997), was noted that one key ingredient needed for expanding this definition was the fact that rep Q was an abelian category, and the usage of a preorder between the elements of the category (this is, a way on which we can say where $M \preccurlyeq N$ in the objects of the category). Then there was a way of naturally defining stability for an arbitrary abelian category and where we could retrieve King's stability. In this chapter our objective is to develop the theory from (RUDAKOV, 1997), and to show how it relates to quiver representations. We also give some examples along the way and we compute a special case of all the $\theta-$semistable and stable representations for the $\mathbb{A}_2$ quiver.

2.1 Stability Structures

The concept of stability on a category is highly related with the notion of comparison between objects, so we will need a way of measuring different objects in an specific abelian category.

Definition 2.1.1 (Preorder)**.** *Let $\mathcal{A}$ be an abelian category. We say that there exists a **preorder over** $\mathcal{A}$ if we can compare objects in $\mathcal{A}$. This is, given $A, B \in \mathcal{A}$, then it occurs just one of the following three: $A > B$, $A < B$ or $A = B$.*

The following will be an example of measuring objects in $\mathcal{A}$.

Definition 2.1.2 (Additive Function)**.** *Let $\mathcal{A}$ an abelian category and $A, B, C \in Ob(\mathcal{A})$. We say that a function $\alpha : K_0(\mathcal{A}) \longrightarrow \mathbb{R}^+$ is **additive on** $\mathcal{A}$ if it is an homomorphism of groups and for any exact sequence of objects*

$$0 \to A \to B \to C \to 0$$

we have that $\alpha(B) = \alpha(A) + \alpha(C)$.

There may be a great number of additive functions on an abelian category, but here we present one that will be of common use.

Example 2.1.3. Let be FinVec_k the abelian category of finite-dimensional vector spaces over a field k. Then the function $\dim : \mathrm{FinVec}_k \to \mathbb{Z}^+$ is an additive function by the rank-nullity theorem.

Having a way of measure objects in a category, we can proceed to give a certain order between those objects that depends on the additive functions chosen.

Definition 2.1.4 (c:r slope)**.** *Let c and r be two additive function on an abelian category $\mathcal{A}$ such that $r(A) > 0$ for $A \in \mathcal{A}$. We define*

$$\alpha(A) = \frac{c(A)}{r(A)}$$

*and it will be called the **(c:r) slope** of $\mathcal{A}$. It also induces an order on objects as follows: if $A, B \in Ob(\mathcal{A})$ then we have one of the following*

- $\alpha(A) > \alpha(B)$
- $\alpha(A) = \alpha(B)$
- $\alpha(A) < \alpha(B)$

An interesting property of the slope is that it gives a preorder on the abelian category, and so being able to develop the theory of (RUDAKOV, 1997).

Corollary 2.1.5. *The (c:r) slope on $\mathcal{A}$ is a preorder in $\mathcal{A}$.*

From now on, we will call a $\mu-$**preorder** to the preorder induced by a (c:r) slope. Now, even when we do have a sense of order between objects, we still do not have the concept of stability, and walking towards that objective, we define a stability structure that will coincide with the usual concepts of stability defined by (KING, 1994).

Definition 2.1.6 (Stability Structure)**.** *Let be $\mathcal{A}$ an abelian category where there exists a preorder μ. We say that the preorder is a **stability structure** for $\mathcal{A}$ if for any objects $A, B, C \in Ob(\mathcal{A})$ and any short exact sequence: $0 \to A \to B \to C \to 0$ we have that:*

1. $\mu(A) < \mu(B) \Leftrightarrow \mu(B) < \mu(C) \Leftrightarrow \mu(A) < \mu(C)$
2. $\mu(A) > \mu(B) \Leftrightarrow \mu(B) > \mu(C) \Leftrightarrow \mu(A) > \mu(C)$
3. $\mu(A) = \mu(B) \Leftrightarrow \mu(B) = \mu(C) \Leftrightarrow \mu(A) = \mu(C)$

In essence, a stability structure defines an order that behaves well with respect to short exact sequences. It is noteworthy that the usage of one additive function is not enough for constructing a stability structure. Our only example already shows this behaviour.

Example 2.1.7. Consider $\mathcal{A} = \mathrm{FinVec}_k$. Then, with the preorder above defined we have that dim is not a stability structure for $\mathcal{A}$, because if $k = \mathbb{C}$, and we have the following exact sequence:

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{T_1} \mathbb{C}^5 \xrightarrow{T_2} \mathbb{C}^3 \longrightarrow 0$$

where,

$$T_1(x,y) = (x,y,0,0,0) \text{ and } T_2(a,b,c,d,e) = (a,b,c)$$

Then $\dim_{\mathbb{C}}\left(\mathbb{C}^2\right) = 2$, $\dim_{\mathbb{C}}\left(\mathbb{C}^5\right) = 5$. However, $\dim_{\mathbb{C}}\left(\mathbb{C}^5\right) =\not< \dim_{\mathbb{C}}\left(\mathbb{C}^3\right)$.

But not everything is lost, we have an example.

Proposition 2.1.8 ((RUDAKOV, 1997), Lemma 3.2)**.** *The slope preorder in $\mathcal{A}$ is a stability structure for $\mathcal{A}$.*

Proof. Let $A, B, D \in Ob(\mathcal{A})$ and $0 \longrightarrow A \longrightarrow B \longrightarrow D \longrightarrow 0$ a short exact sequence. We have to prove 1., 2. and 3.. Suppose fisrt that $\mu(A) < \mu(B)$, then $\mu(A) - \mu(B) < 0$, which implies that

$$\frac{c(A)}{r(A)} - \frac{c(B)}{r(B)} < 0$$

This is,

$$\frac{c(A)r(B) - c(B)r(A)}{r(A)r(B)} < 0$$

We can write the fraction as:

$$\begin{vmatrix} c(A) & r(A) \\ c(B) & r(B) \end{vmatrix} \frac{1}{r(A)r(B)} < 0$$

If we denote by U the matrix above, we have that as $r(A) > 0$ and $r(B) > 0$ by definition, $\det U < 0$. By the additivity of the functions:

$$\begin{aligned} 0 > \det U &= \begin{vmatrix} c(A) & r(A) \\ c(B) & r(B) \end{vmatrix} \\ &= \begin{vmatrix} c(A) & r(A) \\ c(A) + c(D) & r(A) + r(D) \end{vmatrix} = \begin{vmatrix} c(A) & r(A) \\ c(D) & r(D) \end{vmatrix} \end{aligned}$$

where on the last equation we used the properties of determinants. Hence, remembering that also $r(D) > 0$ we get that

$$\frac{c(A)}{r(A)} - \frac{c(D)}{r(D)} < 0$$

which implies $\mu(A) < \mu(D)$. In a similar way,

$$0 > \begin{vmatrix} c(A) & r(A) \\ c(D) & r(D) \end{vmatrix}$$
$$= \begin{vmatrix} c(A)+c(D) & r(A)+c(D) \\ c(D) & +r(D) \end{vmatrix} = \begin{vmatrix} c(B) & r(B) \\ c(D) & r(D) \end{vmatrix}$$

from where we get $\mu(B) < \mu(D)$. This proves 1., and the same argument holds for 2. and for 3. Thus, the slope preorder is a stability structure. □

Example 2.1.9. Let us consider again $\mathcal{A} = \mathrm{FinVec}_k$, and we define $c : K_0(\mathcal{A}) \to \mathbb{R}$ given by $c(V) = k\dim(V)$, with $k \in \mathbb{R}^+$, $V \in \mathcal{A}$, and $r : K_0(\mathcal{A}) \to \mathbb{R}$ by $r(V) = \dim(V)$. Then $\mu(V) = k$ for all $V \in \mathcal{A}$, and the stability structure is trivial. In particular, for any two $V, W \in \mathcal{A}$, $\mu(V) = \mu(W)$.

Example 2.1.10. Now let be $\mathcal{A} = \operatorname{rep} Q$, where $Q = \; 1 \longrightarrow 2 \longrightarrow 3$, and given $M \in \operatorname{rep} Q$ of the form

$$M = \; V_1 \longrightarrow V_2 \longrightarrow V_3$$

we define $c(M) = \sum k_i \dim(V_i)$, $r(M) = \sum \dim(V_i)$ where $k_i \in \mathbb{R}^+$ for $i = 1,2,3$. Suppose we have the representations:

$$M_1 : \; \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \;, \quad M_2 : \; \mathbb{C}^2 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^2$$

Then,

$$\mu(M_1) = \frac{c(M_1)}{r(M_1)} = \frac{k_1+k_2+k_3}{3}, \text{ and } \mu(M_2) = \frac{2k_1+k_2+2k_2}{3}$$

And thus $\mu(M_2) \geq \mu(M_1)$. However, if we define

$$M_3 : \; \mathbb{C} \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}$$

then there exist scalars k_i for which $\mu(M_3) > \mu(M_2)$ and for which $\mu(M_3) < \mu(M_2)$ (for example, the pairs $(0,1,0)$ and $(1,0,1)$).

Another good property of the stability structure is that it places the objects of a short exact sequence in a well-desired ordering, even when we compare the elements with an object outside the sequence, as the following lemma shows.

Lemma 2.1.11 ((RUDAKOV, 1997), Lemma 1.2)**.** *Let $\mathcal{A}$ an abelian category and μ a stability structure over $\mathcal{A}$. If $A, B \in Ob(\mathcal{A})$ are such that $\; 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \;$ is a short exact sequence, then given a $0 \neq D \in Ob(\mathcal{A})$ we have:*

a. if $\mu(A) > \mu(D)$ and $\mu(C) > \mu(D)$, then $\mu(B) > \mu(D)$,

b. if $\mu(A) < \mu(D)$ and $\mu(C) < \mu(D)$, then $\mu(B) < \mu(D)$,

c. *if* $\mu(A) = \mu(D)$ *and* $\mu(C) = \mu(D)$, *then* $\mu(B) = \mu(D)$.

Proof. Suppose without lose of generality that $\mu(A) < \mu(B)$, then by the stability structure, $\mu(B) < \mu(C)$ and $\mu(A) < \mu(C)$, we obtain that $\mu(A) < \mu(B) < \mu(C)$. We will prove b., so we suppose $\mu(A) < \mu(D)$ and $\mu(C) < \mu(D)$, thus $\mu(A) < \mu(B) < \mu(C) < \mu(D)$, in particular $\mu(B) < \mu(D)$. A similar argument holds for proving a. and c.. □

From now on, and until the end of the section, we will assume that $\mathcal{A}$ is an abelian category that has stability structure μ. Now we proceed to prove a generalization of the last lemma, on which we can compare objects given a filtration and some relations between the factor objects of such filtration.

Lemma 2.1.12 ((RUDAKOV, 1997), Lemma 1.3)**.** *Given* $B, D \in Ob(\mathcal{A})$ *non-zero, and a filtration of* B *of the type*

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m \subset F_{m+1} = B$$

where $F_i \subset F_{i+1}$ *means that* F_i *is a subobject of* F_{i+1}. *Defining* $G_i = F_i/F_{i-1}$ *for* $i \in \{1, \ldots, m+1\}$, *we have that:*

a. *if* $\mu(G_i) < \mu(D)$ *for* $i \in \{1, \ldots, m+1\}$, *then* $\mu(B) < \mu(D)$,

b. *if* $\mu(G_i) > \mu(D)$ *for* $i \in \{1, \ldots, m+1\}$, *then* $\mu(B) > \mu(D)$,

c. *if* $\mu(G_i) = \mu(D)$ *for* $i \in \{1, \ldots, m+1\}$, *then* $\mu(B) = \mu(D)$.

Proof. We proceed by induction on the length of the filtration m. If we have a filtration of the type $0 \subset F_1 \subset B$, such that $\mu(F_1/0) < \mu(D)$ and $\mu(B/F_1) < \mu(D)$, we construct the following exact sequence:

$$0 \longrightarrow F_1 \longrightarrow B \longrightarrow B/F_1 \longrightarrow 0$$

Then, by Lemma 2.1.11 with D, we obtain that $\mu(B) < \mu(D)$. This shows the result for the case $m = 1$. In a similar way we can show that every F_i holds $\mu(F_i) < \mu(D)$, and by constructing the following exact sequence:

$$0 \longrightarrow F_m \longrightarrow B \longrightarrow G_{m+1} \longrightarrow 0$$

and applying once again Lemma 2.1.11 with D, we obtain $\mu(B) < \mu(D)$. This proves a., and the same argument follows for proving b. and c.. □

Remark 2.1.13 ((RUDAKOV, 1997), Lemma 1.4). We also have a stronger property, we can compare any two objects in a filtration. More formally, if B is an object of $\mathcal{A}$

with a filtration: $0 = F_0 \subset F_1 \subset \ldots \subset F_m \subset F_{m+1} = B$ with factors $G_i = F_i/F_{i-1}$ for $i \in \{1, \ldots, m+1\}$ and

$$\mu(G_{m+1}) < \mu(G_m) < \cdots < \mu(G_1)$$

Given integers k, p with $k \geq 0$, $p \geq 1$, $k + p \geq m + 1$ we denote $G_p^k = F_k/F_{k-p}$. Then for another pair $n, q \in \mathbb{Z}$ with $n \geq 0$, $q \geq 1$, $n + q \geq m + 1$ we have:

$$\mu\left(G_p^k\right) < \mu\left(G_q^n\right) \text{ if and only if } (k, p) > (n, q)$$

where the order between ordered pairs of integers is lexicographic.

Next we introduce a concept that will be fundamental henceforth. Until now, the concepts of subset and preorder were disconnected, and there could be subobjects B of an object $A \in \mathcal{A}$ such that its order was bigger, and this is, in a way, counterintuitive. Those objects where all its subobjects behave "nicely" in the preorder of the stability structure will be called *stable*. We define them formally.

Definition 2.1.14 ((RUDAKOV, 1997), Definition 1.5 and 1.6)**.** *Given a non-zero object $A \in Ob(\mathcal{A})$ we say that A is **stable under the stability structure** μ (or simply, **stable**) if for any subobject $B \subset A$ we have that $\mu(B) < \mu(A)$, and we say that A is **semistable under the stability structure** μ if $\mu(B) \leq \mu(A)$.*

Example 2.1.15. Consider the Example 2.1.9. Here any object $V \in \mathrm{FinVec}_k$ is semistable as $\mu(V) = \mu(W)$ for any subobject W of V. However, there are no stable objects.

Example 2.1.16 ((RUDAKOV, 1997), p. 244). We consider the example 2.1.10 with $k_i = 1$ for $i = 1, 2, 3$. The non-zero subobjects of $M_1 : \ \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}$ are:

$$M_1' : \ 0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \quad \text{and } M_1'' : \ 0 \longrightarrow 0 \longrightarrow \mathbb{C}$$

Where the morphisms between the $\mathbb{C}$ vector spaces are isomorphisms. Then, $\mu(M_1) = \mu(M_1') = \mu(M_1'') = 1$ and thus M_1 is semistable. If $k_1 = 1, k_2 = 2$ and $k_3 = 3$ we obtain

$$\mu(M_1) = \frac{6}{3} = 2, \quad \mu(M_1') = \frac{5}{2}, \quad \mu(M_1'') = 3$$

Which implies that M_1 is not stable. If, otherwise $k_1 = 3$, $k_2 = 2$ and $k_3 = 1$, then M_1 is stable.

As we saw the election of scalars changes the stability structure and with it, the objects that are stable or not. Even if it seems like it is arbitrary, that election is important for determining on which stable objects we are interested, and those where the stability coincides with (KING, 1994), for example. We will discuss those details with more depth later. Now we proceed to show a useful criterion for deciding when a determined object is stable by comparing it with a subset of the subobjects.

Lemma 2.1.17. *Let $A \in Ob(\mathcal{A})$, then:*

a. *A is stable if and only if for any non-zero factor object B we have $\mu(A) < \mu(B)$,*

b. *A is semistable if and only if for any non-zero factor object B we have $\mu(A) \leq \mu(B)$.*

We remember that a factor object of $A \in \mathcal{A}$ is any object of the type A/B for a subobject $B \subset A$.

Proof. Suppose A is stable, and let B a factor object. Then $B = A/A'$ for some $A' \subset A$, and we can construct the following exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A/A' \longrightarrow 0$$

As A is stable and $A' \subset A$, then $\mu(A') < \mu(A)$ and by Definition 2.1.6, $\mu(A') < \mu(A) < \mu(A/A') = \mu(B)$.

Conversely, suppose that for any non-zero factor object B we have $\mu(A) < \mu(B)$ and let C be a subobject of A. Constructing

$$0 \longrightarrow C \longrightarrow A \longrightarrow A/C \longrightarrow 0$$

short exact, we have by hypothesis $\mu(A) < \mu(A/C)$. Again, by Definition 2.1.6 this implies $\mu(C) < \mu(A)$ and as C was chosen arbitrarily, this shows the stability of A. This shows a., and we follow a similar argument for proving b.. $\square$

The following theorem answers the question on non-zero morphisms between stable or semistable objects, and relates the order of those objects for which such morphism exists. In a certain way this is similar to the Schur lemma for representations, as it states that whenever a non-zero morphism between stable objects exists, it must be an isomorphism.

Theorem 2.1.1 ((RUDAKOV, 1997), Theorem 1)**.** *Let $A, B \in Ob(\mathcal{A})$ semi-stable objects such that $\mu(A) \geq \mu(B)$ and suppose that there exists a non-zero morphism $\varphi : A \to B$. Then the following holds:*

a. *$\mu(A) = \mu(B)$,*

b. *if A is stable, then φ is a monomorphism,*

c. *if B is stable, then φ is an epimorphism,*

d. *if A and B are stable, then φ is an isomorphism.*

Proof. We will prove a., so we consider the following exact sequences:

$$0 \longrightarrow \ker\varphi \longrightarrow A \longrightarrow \operatorname{im}\varphi \longrightarrow 0 \tag{2.1}$$

$$0 \longrightarrow \operatorname{im}\varphi \longrightarrow B \longrightarrow \operatorname{coker}\varphi \longrightarrow 0 \tag{2.2}$$

As $\operatorname{im}\varphi \subset B$ and B is semi-stable, $\mu(\operatorname{im}\varphi) \leq \mu(B)$, and as $\ker\varphi \subset A$ with A semistable, $\mu(\ker\varphi) \leq \mu(A)$. By Definition 2.1.6, $\mu(A) \leq \mu(\operatorname{im}\varphi)$, which implies

$$\mu(A) \leq \mu(\operatorname{im}\varphi) \leq \mu(B)$$

But, as by hypothesis $\mu(A) \geq \mu(B)$ we get $\mu(A) = \mu(\operatorname{im}\varphi) = \mu(B)$. This shows a..

For b., let us suppose that φ is not a monomorphism, then $\ker\varphi \neq 0$. This is, $\mu(\ker\varphi) < \mu(A)$. Definition 2.1.6 on the exact sequence (2.1) implies $\mu(\ker\varphi) < \mu(A) < \mu(\operatorname{im}\varphi)$, a contradiction with the fact that $\mu(A) = \mu(\operatorname{im}\varphi) = \mu(B)$. Similarly, for c., if φ is not an epimorphism, $\operatorname{im}\varphi \neq 0$ which implies $\mu(\operatorname{im}\varphi) < \mu(B)$, a contradiction. Lastly, if A and B are stable, then φ is a monomorphism and an epimorphism, thus, an isomorphism. $\square$

Remark 2.1.18. Theorem 2.1.1 says, in particular, that if A is an stable object, then every non-zero automorphism is an isomorphism.

2.2 King's Stability

In this section we will construct another definition of stability that will be equivalent to the one defined on the first section, but with the advantage of being easier to calculate stable (or semi-stable) objects on $\operatorname{rep} Q$, where Q is a quiver. We will also give a characterization of stable objects for the $\mathbb{A}_2$ quiver. We start with the definition of King's stability for an abelian category.

Definition 2.2.1 ((KING, 1994), Definition 1.1)**.** *Let $\mathcal{A}$ be an abelian category and $\Theta : K_0(\mathcal{A}) \to \mathbb{R}$ an additive function on the Grothendieck group. An object $M \in \mathcal{A}$ is called $\Theta-$**semi-stable** if $\Theta(M) = 0$ and every subobject $M' \subset M$ satisfies $\Theta(M') \geq 0$. Such an M is called $\Theta-$**stable** if the only subobjects M' with $\Theta(M') = 0$ are M and 0.*

Following Example 2.1.10 we saw that there are objects M that are stable (or semi-stable) with a stability structure μ, even if $\mu(M) \neq 0$, which goes against the hypothesis of the stability of Definition 2.2.1. We proceed to construct an additive function that depends on μ where that condition (and even the stability) holds.

Proposition 2.2.2 ((RUDAKOV, 1997), Proposition 3.4)**.** *Let $\mathcal{A}$ be an abelian category with stability structure $\mu = \dfrac{c}{r}$. Given $A \in \mathcal{A}$, we consider the following additive function:*

$$\theta = -c + \mu(A)r$$

Then $\theta(A) = 0$ and A is stable on the stability structure μ if and only if it is $\theta-$stable in the sense of Definition 2.2.1.

Proof. Let $M \in \mathcal{A}$ be stable on the stability structure μ, and consider a subobject $B \subset A$. By the definition of stability, $\mu(B) < \mu(A)$, which is equivalent to

$$\frac{c(B)}{r(B)} < \mu(A)$$

Then $c(B) < \mu(A)r(B)$, and $0 < -c(B) + \mu(A)r(B)$, thus, $0 < \theta(B)$ and A is $\theta-$stable. Similarly, if A is $\theta-$semistable then A is stable on the stability structure μ. □

Before we give some examples, we will make a brief discussion about additive functions on the Grothendieck group of $\operatorname{rep} Q$, where $Q = (Q_0, Q_1)$, Q_0 being its set of vertices and Q_1 its set of arrows. We remember that $K_0(\mathrm{FinVec}_k) \cong \mathbb{Z}$, and then, $K_0(\operatorname{rep} Q) \cong \mathbb{Z}^{|Q_0|}$. Then, an additive function on $K_0(\operatorname{rep} Q)$ will be an homomorphism of groups $\varphi \in \operatorname{Hom}\left(\mathbb{Z}^{|Q_0|}, \mathbb{R}\right)$, but as $\operatorname{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$ because $(\mathbb{Z}, +)$ is cyclic, we obtain

$$\operatorname{Hom}\left(\mathbb{Z}^{|Q_0|}, \mathbb{R}\right) \cong \bigoplus_{i=1}^{|Q_0|} \operatorname{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}^{|Q_0|}$$

This means that an additive function on $K_0(\operatorname{rep} Q)$ is completely determined by the election of $|Q_0|$ real scalars (a similar phenomena happened on the Example 2.1.10, when choosing the k_i), then every election gives a different homomorphism on the Grothendieck group and with it, a different $\Theta-$stability on the sense of Definition 2.2.1. This motivates us to define stability on the sense of King via the election of some scalars θ_i, for $i \in \{1, \ldots, |Q_0|\}$, and that will be the way on which we will define it for $\operatorname{rep} Q$ henceforth.

Remark 2.2.3. We note that the election of the Θ vector should be orthogonal to the dimension vector of M, as in Subsection 1.4.3.

The above example shows that it is enough to look at an election of scalars that is orthogonal to the dimension vector of the quiver and it will give a notion of stability in the King's sense, and hence, one in the Rudakov's sense. Now we are going to try to characterize the stable and semistable representations for an $\mathbb{A}_2$ quiver. The principal result is the following:

Theorem 2.2.1. *Consider the quiver* $Q: \ 1 \longrightarrow 2$ *, and a representation*

$$\gamma: \ \mathbb{C}^m \xrightarrow{M} \mathbb{C}^n$$

of Q *in the usual sense, where* $(m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ *and* $M \in M_{n,m}(\mathbb{C})$*. If* $m = n$*, then* $\dim \ker M \geq 1$ *if and only if* γ *is* $(-m,m)-$*unstable and* $\ker M = 0$ *if and only if* γ *is* $(-m,m)-$*semistable.*

Proof. We will start with the case $m = n$. In this proof we will refer to $(-m,m)-$stability by simply saying stability. Given $p, q \in \mathbb{Z}_{\geq 0}$ we define the following representations for the sake of the proof:

$$A_q: \ 0 \longrightarrow \mathbb{C}^q$$

$$B_p: \ \mathbb{C}^p \longrightarrow 0$$

$$C_{(p,q)}: \ \mathbb{C}^p \longrightarrow \mathbb{C}^q$$

We know that the representations of type A_q are always subrepresentations of γ, as the following diagram commutes:

$$\begin{tikzcd} \gamma: & \mathbb{C}^m \arrow[r] & \mathbb{C}^m \\ A_q: & 0 \arrow[u, hook] \arrow[r] & \mathbb{C}^q \arrow[u, hook] \end{tikzcd}$$

However, they do not unstabilize. Choosing an orthogonal vector $(-m,m)$ we obtain:

$$\left(\dim A_q\right) \cdot (-m,m) = -m(0) + mq \geq 0$$

Now if $\dim \ker M \geq 1$, the representation B_1 is a subrepresentation of γ, because taking $0 \neq \overrightarrow{v} \in \ker M$ the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}^m \arrow[r, "M"] & \mathbb{C}^m \\ \mathbb{C} \arrow[u, hook, "\overrightarrow{v}"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & 0 \arrow[u] \end{tikzcd}$$

as $M\overrightarrow{v} = 0$. Nevertheless, this subrepresentation unstabilizes because:

$$(\dim B_1) \cdot (-m,m) = (1,0) \cdot (-m,m) = -m < 0$$

And hence γ is unstable.

Now we are going to consider the case where $\ker M = 0$, or M is one-to-one. We assert that if there exists a subrepresentation $\gamma' = C_{(p,q)}$ with $p > q$ then M is unstable as:

$$\left(\dim \gamma'\right) \cdot (-m,m) = -mp + mq = m(-p+q) < 0$$

Thus, M will be stable if there are no subrepresentations γ' with $p > q$. We will proceed to show that this in fact does not happens. Suppose that $\gamma' : \mathbb{C}^p \to \mathbb{C}^q$ is a subrepresentation of γ, obtaining then:

$$\begin{tikzcd}
\mathbb{C}^m \arrow[r, "M"] & \mathbb{C}^m \\
\mathbb{C}^p \arrow[u, hook, "N_1"] \arrow[r, "N"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{C}^q \arrow[u, hook, "N_2"']
\end{tikzcd}$$

As N_1 and N_2 are one-to-one, $\dim(\mathrm{im}(N_1)) = p$ and $\dim(\mathrm{im}(N_2)) = q$. Also, having in mind that the above diagram commutes, $MN_1 = N_2N$ and

$$\dim\left(\mathrm{im}(MN_1)\right) = \dim\left(\mathrm{im}(N_1)\right) = p$$

Because M is bijective (one-to-one between finite-dimensional vector spaces). Similarly, we obtain that:$\dim\left(\mathrm{im}(N_2N)\right) \leq \min\{p, q\} = q$, which is a contradiction, because the dimensions should coincide. Hence, γ is semistable.

Lastly, we note that γ is strictly semi-stable when $m > 1$, given that $\eta : \mathbb{C} \longrightarrow \mathbb{C}$ will always be a subrepresentation and moreover,

$$(\dim \eta) \cdot (-m, m) = (1, 1) \cdot (-m, m) = 0$$

And thus, there exists a subrepresentation η different from γ and 0 such that $\Theta(\eta) = 0$. $\square$

2.3 The Harder-Narasimhan and Jordan-Hölder filtrations

The first section gives a hint on how important are stable objects in an abelian category, as they not only behave well on the stability structure but even more with morphisms between them. As the example of the $\mathbb{A}_2$ quiver showed, not every object is stable, so we would like to study those objects and how far from being stable they are. A way to see this for a special category is the Harder-Narasimhan filtration, a filtration where all the factors are semi-stable and with monotone increasing order on the stability structure. In this section we will describe it formally and we will calculate it for an unstable representation of the $\mathbb{A}_2$ quiver. Finally, we give a similar notion for a strictly semi-stable object, the so called Jordan-Hölder filtration, where each factor ends up being stable.

2.3.1 The First Object

We start by giving some definitions that will serve as the hypothesis of our "special" category, and remembering that a **noetherian** object A is one where any ascending chaing of subobjects of A stabilizes, and is **artinian** if any descending chain of subobjects stabilizes.

Definition 2.3.1 (Quasi and weakly-noetherian categories). *Let $\mathcal{A}$ be an abelian category with stability structure μ and $A \in Ob(\mathcal{A})$. Given a family of subobjects of A, $\{A_i\}_{i=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$, we consider the following chains:*

a. $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ *such that* $\mu(A_1) \leq \mu(A_2) \leq \mu(A_3) \leq \cdots$

b. $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ *such that* $\mu(B_1) \geq \mu(B_2) \geq \mu(B_3) \geq \cdots$

*In any chain of type a. stabilizes (this is, there exists a $j \in \mathbb{Z}$ such that $A_k = A_{k+1}$ and $\mu(A_k) = \mu(A_{k+1})$ for $k \geq j$) we say that A is **quasi-noetherian**; and if any chain of type a. and of type b. stabilizes we call A **weakly-noetherian**. With this in mind, if any object of $\mathcal{A}$ is quasi-noetherian (resp. weakly-noetherian) we say that the category $\mathcal{A}$ is **quasi-noetherian** (resp. **weakly-noetherian**).*

Similarly, for descending chains we define,

Definition 2.3.2 (Weakly-artinian category). *Given $\mathcal{A}$ an abelian category, $A \in Ob(\mathcal{A})$, and a family of subobjects $\{A_i\}_{i=1}^{\infty}$ such that we have:*

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \text{ such that } \mu(A_1) \leq \mu(A_2) \leq \mu(A_3) \leq \cdots$$

*If for any of subobjects of A the above chain stabilizes, we say that A is **weakly-artinian**, and if every object of the category $\mathcal{A}$ is weakly-artinian, we will call the category $\mathcal{A}$ **weakly-artinian**.*

Remark 2.3.3. In particular, if A is weakly-artinian, any chain of type

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \text{ such that } \mu(A_1) < \mu(A_2) < \mu(A_3) < \cdots$$

must be finite, or else we would have a contradiction with the above definition.

The categories we just defined are extremely particular in nature, and there are not so many examples of them. We gave those definitions as they will be the "least" amount of hypotheses needed for assuring the existence of a filtration of the type we want. Now we start doing the preparations for our filtration, on which our first result says us that any subobject of a quasi-noetherian and weakly-artinian category is semi-stable or that there exists a semi-stable object with greater order. From now on, let $A \in Ob(\mathcal{A})$ be quasi-noetherian and weakly-artinian.

Proposition 2.3.4 ((RUDAKOV, 1997), Lemma 1.10). *Let $A_1 \subseteq A$ a non-zero subobject of A. Then just one of the following happens:*

a. A_1 *is semi-stable, or*

b. *exists a semi-stable* $0 \neq A'_1 \subseteq A$ *such that* $\mu(A'_1) > \mu(A_1)$.

Proof. Let $B_1 = A_1$, and suppose that a. does not happens, this is, that B_1 is not semi-stable. Then there exists a nonzero $B_2 \subseteq B_1$ such that $\mu(B_2) > \mu(B_1)$. If B_2 is semi-stable, then we obtain our object, if not, there would exist a $0 \neq B_3 \subseteq B_2$ such that $\mu(B_3) > \mu(B_2)$. If none of the B_i is semi-stable, we would obtain a chain of the type:

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \text{ such that } \mu(B_1) < \mu(B_2) < \mu(B_3) < \cdots$$

But as A is weakly-artinian, then that chain must stabilize by Remark 2.3.3. □

Example 2.3.5. Consider the quiver $Q: \ 1 \longrightarrow 2$, and the category rep Q with the stability structure given by:

$$\mu(A) = \frac{c(A)}{r(A)} = -\frac{\sum \theta_i \dim(A_i)}{\sum \dim(A_i)}$$

where A is a subobject of an object M, and $\theta = (\theta_1, \theta_2) \in (\dim M)^{\perp}$. If we have

$$M: \ k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} k^2$$

Then the representation $M' : \ k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} k$ is a subrepresentation of M, and is unstable by Theorem 2.2.1. Choosing the orthogonal vector $\theta = (-2, 2)$, by the Proposition above, there should exist a semi-stable subobject M'_1 of M such that $\mu(M'_1) > \mu(M_1)$. In this case, we see that the representation $M'_1 := \ k \longrightarrow 0$ is a subobject of M as the following diagram commutes:

$$\begin{tikzcd}
k^2 \arrow[r, "{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}"] & k^2 \\
k \arrow[u, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

and we have

$$2 = \mu(M'_1) > \mu(M_1) = \frac{2}{3}$$

Which shows the existence of such object.

Lemma 2.3.6 ((RUDAKOV, 1997), Lemma 1.11)**.** *Let A_1 be a non-zero subobject if A, and suppose that exists a semi-stable subobject A_{sst} of A such that*

$$\mu(A_{\mathrm{sst}}) > \mu(A_1)$$

Then just one of the following happens:

a. A_{sst} is a subobject of A_1, this is, $A_{\text{sst}} \subseteq A_1$, or

b. there exists a $A_1' \subseteq A$ such that $A_1 \subseteq A_1'$ and $\mu(A_1) < \mu(A_1')$.

Note 2.3.7. The above lemma can be seen in diagrams as follows. If A_1 and A_{sst} are subobjects of A, we have two injective morphisms:

$$\begin{tikzcd} A & A_{\text{sst}} \arrow[l, hook'] \\ A_1 \arrow[u, hook] & \end{tikzcd}$$

If $\mu(A_{\text{sst}}) > \mu(A_1)$, the Lemma 2.3.6 assures that just one of the following happens:

a. there exists an injective morphism φ_1 such that the diagram commutes:

$$\begin{tikzcd} A & A_{\text{sst}} \arrow[l, hook'] \arrow[dl, hook', dashed, "\exists\varphi_1"] \\ A_1 \arrow[u, hook] & \end{tikzcd}$$

b. exists a A_1' such that $\mu(A_1) < \mu(A_1')$ and injective morphisms φ_1 and φ_2 as shown:

$$\begin{tikzcd} A & A_{\text{sst}} \arrow[l, hook'] \\ A_1 \arrow[u, hook] \arrow[r, hook, "\varphi_1"] & A_1' \arrow[ul, hook', dashed, "\exists\varphi_2"'] \end{tikzcd}$$

Proof. We can construct the following short exact sequences:

$$0 \longrightarrow A_1 \cap A_{\text{sst}} \longrightarrow A_{\text{sst}} \longrightarrow A_{\text{sst}} \big/ A_1 \cap A_{\text{sst}} \longrightarrow 0 \tag{2.3}$$

$$0 \longrightarrow A_1 \longrightarrow A_1 \oplus A_{\text{sst}} \longrightarrow A_1 \oplus A_{\text{sst}} \big/ A_1 \longrightarrow 0 \tag{2.4}$$

Let us suppose that a. does not happen, then $A_{\text{sst}} \not\subseteq A_1$, which implies that $A_1 \oplus A_{\text{sst}} \neq A_1$, thus $A_1 \oplus A_{\text{sst}} \neq A_1$ and $A_1 \oplus A_{\text{sst}} \big/ A_1 \neq 0$.

From now on, we will call $F := A_{\text{sst}} \big/ A_1 \cap A_{\text{sst}} = A_1 \oplus A_{\text{sst}} \big/ A_1$. We have two cases then, $A_1 \cap A_{\text{sst}} = 0$ or $A_1 \cap A_{\text{sst}} \neq 0$. If $A_1 \cap A_{\text{sst}} = 0$, then $F = A_{\text{sst}}$, which implies $\mu(F) = \mu(A_{\text{sst}})$. If $A_1 \cap A_{\text{sst}} \neq 0$, then it is a subobject of A_{sst} and as it is semi-stable, we have that $\mu(A_1 \cap A_{\text{sst}}) \leq \mu(A_{\text{sst}})$. By the stability structure applied to the exact sequence (2.3) we obtain $\mu(A_{\text{sst}}) \leq \mu(F)$. Then, for any case, $\mu(A_{\text{sst}}) \leq \mu(F)$.

As by hypothesis we have $\mu(A_{sst}) > \mu(A_1)$, this implies $\mu(A_1) < \mu(F)$, and applying the definition of stability structure on the exact sequence (2.4) we obtain $\mu(A_1) < \mu(A_1 \oplus A_{sst})$. Then the element $A_1' := A_1 \oplus A_{sst}$ is one such that:

$$A_1 \subseteq A_1 \oplus A_{sst} = A_1' \text{ and } \mu(A_1) < \mu(A_1 \oplus A_{sst}) = \mu(A_1')$$

Which proves the lemma. □

In order to prove the main result from this part, we give a specific name for an object who "eats" all semi-stables with greater order.

Definition 2.3.8 (Greedy Subobject). *A subobject $0 \neq B$ of A is said to be **greedy** if for any semi-stable $B_{sst} \subseteq A$ such that $\mu(B_{sst}) > \mu(B)$, we have $B_{sst} \subseteq B$.*

Remark 2.3.9. Note that A considered as a subobject of itself is trivially greedy.

The main importance of those objects is that we can always construct a greedy subobject who measures more than any object in A.

Lemma 2.3.10. *Given a non-zero subobject B of A, there exists a greedy subobject G such that $\mu(G) \geq \mu(B)$.*

Proof. Let be $G_0 := B$. If such an object does not exist for G_0, then exists a semi-stable B_{sst} with $\mu(B_{sst}) > \mu(G_0)$ such that B_{sst} is not a subobject of G_0. By Lemma 2.3.6, exists a $G_1 := G_0' \supseteq G_0$ with $\mu(G_1) > \mu(G_0)$. Similarly to before, if G_1 is not greedy, then exists a $G_2 \supseteq G_1$ with $\mu(G_2) > \mu(G_1)$. We can then construct a sequence of the type:

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \text{ such that } \mu(G_0) < \mu(G_1) < \mu(G_2) < \cdots$$

of subobjects of A, which stabilizes as A is quasi-noetherian. □

The following proposition is important because it will allow us to construct the Harder-Narasimhan filtration we talked about. In particular, it gives the existence of a subobject of maximum order in the stability structure, which will turn out to be the first element of the filtration, as we shall see on the following subsection.

Proposition 2.3.11 ((RUDAKOV, 1997), Proposition 1.9)**.** *Let A be quasi-noetherian and weakly-artinian. Then there exists a unique subobject $F^+(A)$ of A such that:*

a. if $0 \neq B \subseteq A$ is a subobject of A, then $\mu(B) \leq \mu(F^+(A))$, and

b. if $0 \neq B \subseteq A$ and $\mu(B) = \mu(F^+(A))$, then $B \subseteq F^+(A)$.

Proof. For the *uniqueness*, suppose that there exists two objects $F^+(A)$ and $F^2(A)$ with properties a. and b.. Then, by property a. as $F^+(A) \subseteq A$ we obtain $\mu(F^2(A)) \leq \mu(F^+(A))$ and similarly $\mu(F^+(A)) \leq \mu(F^2(A))$. This implies $\mu(F^2(A)) = \mu(F^+(A))$. By property b. applied twice we get that $F^+(A) \subseteq F^2(A)$ and $F^2(A) \subseteq F^+(A)$, which indicates $F^+(A) = F^2(A)$.

Let us show *existence*. Suppose that A does not satisfies a., by Lemma 2.3.10 (applied to A) there exists a greedy subobject $A_1 \subseteq A$ such that $\mu(A_1) > \mu(A)$.

Showing the existence for A_1 of an object with property a. implies the existence of such object for A. Suppose that A_1 has the property a., then exists a subobject $A_1' \subseteq A_1$ such that if $0 \neq S \subseteq A_1$, then $\mu(S) \leq \mu(A_1')$. In particular, $\mu(A_1) \leq \mu(A_1')$. Let $0 \neq B \subseteq A$ any subobject, and we want to prove that $\mu(B) \leq \mu(A_1')$. By Proposition 2.3.4 we have two cases:

a. B is semi-stable. If $\mu(B) \leq \mu(A_1)$, we are done, but if $\mu(B) > \mu(A_1)$, as A_1 is greedy we have $B \subseteq A_1$, which implies that $\mu(B) \leq \mu(A_1')$ as A_1 satisfies a..

b. There exists a B' semi-stable such that $\mu(B') > \mu(B)$. If $\mu(B') \leq \mu(A_1)$, we obtain $\mu(B) < \mu(A_1')$ by the transitivity of the preorder. If $\mu(B') > \mu(A_1)$, then as B' is semi-stable and A_1 is greedy, $B' \subseteq A_1$, and as A_1 satisfies a., $\mu(B) < \mu(B') \leq \mu(A_1')$.

In any case, we have that if we prove a. for A_1, then it is proved for A. If A_1 does not satisfy a., there is a greedy object $A_2 \subseteq A_1$ with $\mu(A_2) > \mu(A_1)$. The statement is proved, because if we do not find such element, we would have an infinite chain of the type:

$$A_1 \supseteq A_2 \supseteq \cdots \text{ such that } \mu(A_1) < \mu(A_2) < \cdots$$

which does not exist because A is weakly-artinian. This assures the existence of an element with property a..

Let A_0 an object with property a. but that does not satisfies b., then it exists some B with $\mu(B) = \mu(A_0)$ but $B \not\subseteq A_0$. By Proposition 2.3.4 or B is semi-stable or there exists a $B' \subseteq A$ semi-stable such that $\mu(B') > \mu(B) = \mu(A_0)$, but $\mu(B') \leq \mu(A_0)$ by property a., then $\mu(B') = \mu(B) = \mu(A_0)$. Hence, we can assume that B is semi-stable. Let $A_1 := A_0 \oplus B$. Using a reasoning similar to the one on the proof of Lemma 2.3.6, we can show that:

$$\mu(A_1) = \mu(A_0 \oplus B) \geq \mu(A_0)$$

As $B \not\subseteq A_0$, then $A_0 \subsetneq A_1$, and by following an analogous argument to the first part, we have that A_1 also has the property a.. If A_1 does not satisfy b., then we can construct an A_2 where a. is valid and with an strict inclusion $A_1 \subsetneq A_2$ such that $\mu(A_2) \geq \mu(A_1)$. We would get a subobject that satisfies both a. and b., if not we obtain an infinite chain of the type:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \text{ such that } \mu(A_0) \leq \mu(A_1) \leq \mu(A_2) \leq \cdots$$

And this does not happens in the quasi-noetherian object A. $\square$

Remark 2.3.12. Note that $F^+(A)$ is semi-stable and A is semi-stable if and only if $A = F^+(A)$.

2.3.2 The Harder-Narasimhan Filtration

We just saw that there is always a subobject of a quasi-noetherian and weakly-artinian object that has the biggest order in the stability structure (among all its subobjects). On this section we will use it for constructing a filtration of an object with semi-stable factors and where the factors are ordered strictly decreasing. Such filtration will be called Harder-Narasimhan, and we proceed to define it formally.

Definition 2.3.13 (Harder-Narasimhan Filtration)**.** *Let $\mathcal{A}$ an abelian category and $A \in Ob(\mathcal{A})$. If there exists a filtration*

$$0 = F_0A \subseteq F_1A \subseteq \cdots \subseteq F_mA \subseteq F_{m+1}A = A$$

such that

a. *the factors $G_iA = {}^{F_{i+1}A}\!/_{F_iA}$ for $i = 0, 1, \ldots, m$ are semi-stable, and*

b. $\mu(G_mB) < \mu(G_{m-1}B) < \cdots < \mu(G_0B)$*,*

Then we say that the filtration $\{F_iA\}_{i=0}^{m+1}$ is a ***Harder-Narasimhan filtration (or HN filtration) for** A **of length** m.*

Our first result is that if there is a HN filtration, then the most desestabilizing object (the one from Proposition 2.3.11) is the first object of the filtration, as we anticipated!

Proposition 2.3.14 ((RUDAKOV, 1997), Proposition 1.13)**.** *Let A be a quasi-noetherian and weakly-artinian object for which there exists a HN filtration $\{F_iA\}_{i=0}^{m}$. Then $F^+(A) = F_1A$, where $F^+(A)$ is the object from Proposition 2.3.11.*

Proof. We will proceed by induction over the length of the filtration. If $m = 0$, we have a filtration $0 \subseteq F_0A \subseteq F_1A = A$, where $G_0A = {}^{F_1A}\!/_{F_0A} = A$ is semi-stable. Then, $F_1A = A = F^1(A)$. Let us suppose that the proposition holds for any HN filtration of length $m-1$. Given a HN filtration $\{F_iA\}_{i=0}^{m+1}$ of length m, we will prove that $F_1A = F^+(A)$. During this proof, we will call $F_i := F_iA$. Let $0 \neq B \subseteq A$, we construct the filtration:

$$0 = {F_1}/{F_1} \subseteq {F_2}/{F_1} \subseteq \cdots \subseteq {F_m}/{F_1} \subseteq {A}/{F_1}$$

which is a HN filtration of length $m-1$. By induction hypothesis, we have that:

$$F^+\left({A}/{F_1}\right) = {F_2}/{F_1}$$

Now, by the second isomorphism theorem we get:

$${B \oplus F_1}/{F_1} = {B}/{F_1 \cap B}$$

And as ${B \oplus F_1}/{F_1} \subseteq {A}/{F_1}$; then by definition of F^+,

$$\mu\left({B}/{F_1 \cap B}\right) \leq \mu\left(F^+\left({A}/{F_1}\right)\right) = \mu\left({F_2}/{F_1}\right)$$

But ${F_2}/{F_1} = G_1$, thus,

$$\mu\left({B}/{F_1 \cap B}\right) \leq \mu(G_1) \tag{2.5}$$

By definition of the filtration, $\mu(G_1) < \mu(G_0) = \mu(F_1)$ as $G_0 = F_1$, then,

$$\mu\left({B}/{F_1 \cap B}\right) \leq \mu(F_1)$$

We also note that as F_1 is semi-stable and $F_1 \cap B \subseteq F_1$, $\mu(F_1 \cap B) \leq \mu(F_1)$. Now, considering the following filtration for B:

$$0 \subseteq F_1 \cap B \subseteq B$$

using the Lemma 2.1.12 with $\mu(F_1)$ we obtain that $\mu(B) \leq \mu(F_1)$, which implies that F_1 satisfies property a. from Proposition 2.3.11.

Now we will prove that F_1 satisfies b.. Suppose that $\mu(B) = \mu(F_1)$, and we want to show that $B \subseteq F_1$. We have that $\mu(F_1 \cap B) \leq \mu(F_1) = \mu(B)$. By Definition 2.1.6 applied to the following exact sequence:

$$0 \longrightarrow F_1 \cap B \longrightarrow B \longrightarrow {B}/{F_1 \cap B} \longrightarrow 0$$

We get

$$\mu(B) \leq \mu\left({B}/{F_1 \cap B}\right) \text{ if } {B}/{F_1 \cap B} \neq 0$$

As $\mu(B) = \mu(F_1) = \mu(G_0) > \mu(G_1)$, we deduce $\mu\left({B}/{F_1 \cap B}\right) > \mu(G_1)$, a contradiction with (2.5). Then ${B}/{F_1 \cap B} = 0$, from where we get $B = F_1 \cap B$ and thus $B \subseteq F_1$. This is, F_1 satisfies property b. of Proposition 2.3.11. By the uniqueness of the element, we get that $F_1 = F^+(A)$, as we wanted. □

We proceed now to the main theorem of the section, which says that such filtration always exists in our "special" category.

Theorem 2.3.1 ((RUDAKOV, 1997), Theorem 2)**.** *Let $\mathcal{A}$ be a weakly-artinian and weakly-noetherian category. Then, for any object $A \in \mathcal{A}$ exists a unique Harder-Narasimhan filtration of A.*

Proof. Let $\mathcal{A}$ weakly-artinian and weakly-noetherian. First we prove *existence*. For simplifying the notation, we will write F_i and G_i instead of F_iA and G_iA. Now, to construct the filtration we define:

$$F_0 = 0, \quad F_1 = F^+(A), \quad \ldots, \quad F_{i+1} = \pi_i^{-1}\left(F^+\left({}^{B}\!/\!_{F_i}\right)\right)$$

where π_i is the canonical projection from B to ${}^{B}\!/\!_{F_i}$. We note that

$$G_i = {}^{F_{i+1}}\!/\!_{F_i} = F^1\left({}^{B}\!/\!_{F_i}\right)$$

is semi-stable by Remark 2.3.12, and that $\mu(G_{i+1}) < \mu(G_i)$ by Definition 2.1.6 applied to the short exact sequence:

$$0 \longrightarrow G_i \longrightarrow {}^{F_{i+2}}\!/\!_{F_i} \longrightarrow G_{i+1} \longrightarrow 0$$

because as ${}^{F_{i+2}}\!/\!_{F_i}$ is a subobject of ${}^{B}\!/\!_{F_i}$, by definition of F^+ we get

$$\mu\left({}^{F_{i+2}}\!/\!_{F_i}\right) < \mu\left(F^+\left({}^{B}\!/\!_{F_i}\right)\right) = \mu(G_i)$$

Now, from Remark 2.1.13 it is possible to conclude that $\mu(F_1) > \mu(F_2) > \cdots$ as we can write every $F_i = {}^{F_i}\!/\!_{F_0} = G_i^i$, then $\mu(F_i) < \mu(F_j)$ if and only if $i > j$. We note that we have a chain of the type:

$$F_1 \subseteq F_2 \subseteq \cdots \text{ such that } \mu(F_1) > \mu(F_2) > \cdots$$

and as A is weakly-noetherian, that chain stabilizes, so there exists an m for which $F_{m+1} = B$.

For the *uniqueness*, as the elements of the filtration are defined by the existence of the elements given by the Proposition 2.3.11 and 2.3.14, then such filtration must be unique. □

One of the benefits of the presented proof is that it is constructive, so it gives a recursive way of building the Harder-Narasimhan filtration.

Example 2.3.15. Let be $Q: \; 1 \longrightarrow 2$, $M \in \operatorname{rep} Q$ with $\dim M = (5,3)$, a maximum rank morphism and $\theta = (\theta_1, \theta_2) = (-3,5) \in (\dim M)^{\perp}$ an additive function on $K_0(\operatorname{rep} Q)$. We consider the stability structure given on Example 2.3.5, so $\mu(M) = 0$. We know that the representations that desestabilize (this is, those $S \subseteq M$ with $\mu(S) > 0$) are those with strictly decreasing vector dimension. Now, there cannot be subrepresentations with decreasing order such that the difference between dimensions is greater than 2. If that happens, then the dimensions of the kernel of the composition in the diagram won't coincide. Among the possible subobjects with difference at most 2, the most desestabilizing is:

$$M' : \; k^2 \longrightarrow 0$$

with $\mu(M') = 6$. Then $F^+(M) = M'$. Now,

$$M/F_1 : \; k^3 \xrightarrow{\tilde{M}} k^3$$

where $\tilde{M}$ is injective because we constructed the injection $k^2 \xhookrightarrow{N} k^5$ from the generators in the kernel. Then M/F_1 is stable and

$$F^+\left(M/F_1M\right) = M/F_1M$$

which means that:

$$F_2M = \pi_1^{-1}\left(F^+\left(M/F_1M\right)\right) = \pi_1^{-1}\left(M/F_1M\right) = M$$

And the HN filtration of M has length one and is:

$$0 \subseteq \left(\; k^2 \longrightarrow 0 \; \right) \subseteq \left(\; k^5 \longrightarrow k^3 \; \right) = M$$

Example 2.3.16 ((RUDAKOV, 1997), p. 244). Consider the Example 2.1.16. If $a_i = i$, then as $\mu(M_1'') = 3$ is the most desestabilizing subobject, we obtain that $F^+(M_1) = M_1'' = F_1M_1$. Now,

$$M_1/F_1M_1 = \; \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0$$

Its unique subobject is $S: \; 0 \longrightarrow \mathbb{C} \longrightarrow 0$, and as:

$$\mu\left(M_1/F_1M_1\right) = \frac{1k_1 + 1k_2 + 0k_3}{1+1} = \frac{3}{2} < \mu(S) = 2$$

Then, $F^+\left(M_1/F_1M_1\right) = S$. Now, if $\pi_1 : M_1 \longrightarrow M_1/F_1$, we have that:

$$F_2M_1 = \pi_1^{-1}(S) = \; 0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \; = M_1'$$

Following a similar argument to the above, we can show that $F_3M_1 = M_1$ and hence we have a HN filtration of length 2:

$$0 \subseteq M_1'' \subseteq M_1' \subseteq M_1$$

2.3.3 The Jordan-Hölder Filtration

On this last section, we will construct a filtration for strictly semi-stable objects. This one, unlike the HN filtration is not unique, but here the factors are uniquely determined. We also show an example for the $\mathbb{A}_2$ quiver.

Definition 2.3.17 (Jordan-Hölder Filtration)**.** *Let $\mathcal{A}$ be a weakly-artinian and quasi-noetherian category, and $A \in Ob(\mathcal{A})$. If there exists a filtration:*

$$0 = F_0A \subseteq F_1A \subseteq \cdots \subseteq F_mA \subseteq F_{m+1}A = A$$

such that

a. the factors $G_iA = {}^{F_{i+1}A}\!\diagup_{F_iA}$ for $i = 0, 1, \ldots, m$ are stable, and

b. $\mu(A) = \mu(G_0A) = \mu(G_1A) = \cdots = \mu(G_mA)$

*we will call the filtration $\{F_iA\}_{i=0}^{m+1}$ a **Jordan-Hölder filtration for** A **of length** m, or also **JH filtration for** A.*

Example 2.3.18. Consider the Example 2.3.5. By Theorem 2.2.1 we know that when $\ker T = 0$, the representation $N: \ k^n \xrightarrow{T} k^n$ is strictly semi-stable for $n \in \mathbb{Z}^+$. We define the objects $F_iN: \ k^i \longrightarrow k^i$ for $i = 0, \ldots, n$. Then, the collection $\{F_iN\}_{i=0}^{n}$ is a JH filtration for N of length $n-1$. Note that,

$$G_iN = {}^{F_{i+1}N}\!\diagup_{F_iN} = \ k \longrightarrow k$$

Here when we quotient we end up removing i linearly independent vectors from the transformation of the representation $F_{i+1}N$. Also by Theorem 2.2.1 we have that G_iN is stable for $i = 0, \ldots, n-1$, and

$$\mu(N) = \mu(G_0N) = \mu(G_1N) = \cdots = \mu(G_{n-1}N) = 0$$

We note that different elections of linearly independent vectors give rise to different Jordan-Hölder filtrations for N.

Theorem 2.3.2 ((RUDAKOV, 1997), Theorem 3)**.** *Suppose that $\mathcal{A}$ is a weakly-artinian and quasi-noetherian category- Then any semi-stable object A has a JH filtration $\{F_iA\}_{i=0}^{m}$, where its set of factors is uniquely defined by the properties a. and b. from Definition 2.3.17.*

3 Neural Networks and Weak Manifold Hypothesis

In the attempt of give a mathematical framework for neural networks and to prove some intuitions involving them, (ARMENTA; JODOIN, 2021) introduces a new and innovative way of representing a neural network via quivers. This approach is effective as we can encode each one of the neurons in a determined neural network, we can represent the operations via representations of such a quiver, it gently adapts to the different types of neural networks used in practice (convolutional, fully connected and so on), and it also has deep computational consequences ((ARMENTA; JODOIN, 2021), Chapter 8).

Mathematically, it can be used to formalize a variant of the manifold hypothesis ((GOODFELLOW; BENGIO; COURVILLE, 2016), Section 5.11.3), which briefly states that the data space lies in a submanifold of dimension strictly smaller inside the input space. However, for defining the space of neural networks and the input space, and to parameterize them adequately (when working with quiver and quiver representations), is only natural to consider GIT. In this chapter our intention is to develop the mathematical theory from (GOODFELLOW; BENGIO; COURVILLE, 2016), and to use King's stability for constructing and formalizing what we called the *weak manifold hypothesis*. As such, we will skip most of the computational implications that this approach has, and we focus on its algebraic nature.

3.1 Neural Networks in terms of quivers

In this section we will introduce the concept of neural networks with the language of quivers and we will show the way on which the information passes through a determined neural network. Our main objective is to show that there is invariance on how the information passes between two neural networks that are isomorphic. This illustrates why we would focus our attention on some "kind" of isoclasses of neural networks, as there won't be any alteration on the feed-forward of the information. We end the section with a short comment on architecture, that is useful when classifying different types of networks.

3.1.1 Network quivers and network functions

We start with some previous definitions of vertices and of an arranged by layers and network quiver, from which we will construct a neural network.

Definition 3.1.1 (Input, output and bias vertices). *Let Q be a quiver, we choose a subset of d source vertices of Q that will be called **input vertices**. All the other source vertices that are not input will be called **bias vertices**. The set of all sinks of Q will be called **output vertices**, and the remaining vertices will be called **hidden vertices**.*

Definition 3.1.2 ((ARMENTA; JODOIN, 2021), Definition 4.1 and 4.2). *A quiver Q is said to be a **network quiver** if it can be drawn from left to right arranging its vertices into columns such that:*

a. There are no oriented edges from vertices to the right to vertices to the left.

b. There are no oriented edges between vertices in the same column, different than loops and edges from bias vertices.

c. There are no loops on source (i.e. input and bias) nor sink vertices.

d. There is just one loop on each hidden vertex.

*If a certain quiver Q holds a. and b., we say that it is **arranged by layers**. We will call the first layer, with d source vertices, the **input layer**; the last layer on the right, with k output vertices, the **output layer**; and the layers that are nor input nor output will be called **hidden layers**, and are ordered from left to right.*

Example 3.1.3. The quiver

$$\begin{tikzcd}
 & & 2 \arrow[r] \arrow[ddr] & 5 \arrow[dr] & \\
 & & & & 8 \\
Q: & 1 \arrow[uur] \arrow[r] \arrow[ddr] & 3 \arrow[uur] \arrow[r] \arrow[ddr] & 6 \arrow[ur] \arrow[dr] & \\
 & & & & 9 \\
 & & 4 \arrow[uur] \arrow[r] & 7 \arrow[ur] & \\
 & & 10 \arrow[ur] & &
\end{tikzcd}$$

is an arranged by layers quiver that it is not a network quiver. However,

$$\begin{tikzcd}
 & & 2 \arrow[loop above] \arrow[r] \arrow[ddr] & 5 \arrow[loop above] \arrow[dr] & \\
 & & & & 8 \\
Q' : & 1 \arrow[uur] \arrow[r] \arrow[ddr] & 3 \arrow[loop above] \arrow[uur] \arrow[r] \arrow[ddr] & 6 \arrow[loop above] \arrow[ur] \arrow[dr] & \\
 & & & & 9 \\
 & & 4 \arrow[loop above] \arrow[uur] \arrow[r] & 7 \arrow[loop above] \arrow[ur] & \\
 & & 10 \arrow[ur] & &
\end{tikzcd}$$

is a network quiver with 1 input vertex, 1 output vertex 8, 9, two hidden layers of 3 vertices each, and a bias vertex.

Example 3.1.4. A simple example of network quivers are those obtained by a subset of the Dynkin diagrams, the quivers of finite type that are linearly oriented, which we will call **Dynkin Network Quivers**. Those are not that useful in practice, but they help to understand simple fluxes of neural networks in small amounts of vertices.

$$\begin{tikzcd}
\mathbb{A}_n \text{ network quiver: } 1 \arrow[r] & 2 \arrow[loop above] \arrow[r] & \cdots \arrow[r] & n-1 \arrow[loop above] \arrow[r] & n
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & n-1 \\
\mathbb{D}_n \text{ network quiver: } 1 \arrow[r] & 2 \arrow[loop above] \arrow[r] & \cdots \arrow[r] & n-2 \arrow[loop above] \arrow[ur] \arrow[dr] & \\
 & & & & n
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E}_6 \text{ network quiver: } 1 \arrow[r] & 2 \arrow[loop above] \arrow[r] & 3 \arrow[loop above] \arrow[r] & 4 \arrow[loop above] \arrow[r] & 5 \\
 & & 6 \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E}_7 \text{ network quiver: } 1 \arrow[r] & 2 \arrow[loop above] \arrow[r] & 3 \arrow[loop above] \arrow[r] & 4 \arrow[loop above] \arrow[r] & 5 \arrow[loop above] \arrow[r] & 6 \\
 & & 7 \arrow[u] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E}_8 \text{ network quiver: } 1 \arrow[r] & 2 \arrow[loop above] \arrow[r] & 3 \arrow[loop above] \arrow[r] & 4 \arrow[loop above] \arrow[r] & 5 \arrow[loop above] \arrow[r] & 6 \arrow[loop above] \arrow[r] & 7 \\
 & & 8 \arrow[u] & & & &
\end{tikzcd}$$

We carry on to defining the concept of an activation function, as they will be needed along with the "neuron" to compute a forward pass.

Definition 3.1.5 ((ARMENTA; JODOIN, 2021), Definition 4.4)**.** *An **activation function** is a one variable non-linear function $f : \mathbb{C} \to \mathbf{C}$ differentiable except on a set of measure zero.*

Example 3.1.6. The functions ReLU, tanh, sigmoid and others serve as usual examples of activation functions.

Now, when a determined neural network makes a forward pass, it takes the weight associated to two neurons and it uses it with the output value of the first neuron to obtain the input of the second neuron. These processes can be encoded neuron by neuron via a linear map between two $\mathbf{C}-$vector spaces for each weight between the neurons. If we do this for all the network quiver Q, we obtain a representation where all vector spaces are $\mathbb{C}$, but without maps on the loops. This is, a thin representation of a new quiver, called *delooped*, and we proceed to define it.

Definition 3.1.7 ((ARMENTA; JODOIN, 2021), Definition 4.3)**.** *Let Q be a network quiver. The **delooped quiver** Q° is the quiver obtained from Q by removing all loops, so the set of vertices remains invariant. It can also be written as*

$$Q^\circ = \{Q_0^\circ, Q_1^\circ, s^\circ, t^\circ\}$$

Example 3.1.8. The delooped quivers of the Dynkin Network Quivers are the corresponding linearly oriented Dynkin quivers.

What we said before can be rephrased as: "the weights of a neural networks Q define a thin quiver representation of the quiver Q°". Still, we have not said what happens to the output of a neuron with respect to the activation function. This hints that whenever we want to make a forward pass we use two things: the activation functions and the thin representation of Q°; which leads to the following definition.

Definition 3.1.9 ((ARMENTA; JODOIN, 2021), Definition 4.6)**.** *Let Q be a quiver network with d input vertices and k output vertices. A **neural network** over Q consists of a pair (W, f), where W is a thin quiver representation of Q° and $f = (f_v)_{v \in Q_0}$ are activation functions, one for each loop on Q.*

Example 3.1.10 ((ARMENTA; JODOIN, 2021), Remark 6.2)*.* The easiest way to construct a neural network over a network quiver may be to take all the activation functions as the identity. This is, every thin quiver representation M of Q° induces a neural network, denoted by $(M, 1)$, where $1 = (\mathrm{id}, \dots, \mathrm{id})$ is the identity activation function.

Example 3.1.11. Consider the network quiver Q' from the Example 3.1.3. The following is a neural network over Q',

$$
Q' : \quad
\begin{tikzcd}
 & \mathbb{C} \arrow[loop above, "f_2"] \arrow[r] \arrow[ddr] & \mathbb{C} \arrow[loop above, "f_5"] \arrow[dr] & \\
 & & & \mathbb{C} \\
\mathbb{C} \arrow[uur] \arrow[r] \arrow[ddr] & \mathbb{C} \arrow[loop above, "f_3"] \arrow[uur] \arrow[r] \arrow[ddr] & \mathbb{C} \arrow[loop above, "f_6"] \arrow[ur] \arrow[dr] & \\
 & & & \mathbb{C} \\
 & \mathbb{C} \arrow[loop above, "f_4"] \arrow[uur] \arrow[r] & \mathbb{C} \arrow[loop above, "f_7"] \arrow[ur] & \\
 & \mathbb{C} \arrow[ur] & &
\end{tikzcd}
$$

where the (f_i) are activation functions.

Example 3.1.12. Another simple example of neural networks are the ones obtained from Dynkin Network Quivers on Example 3.1.4. We will call them **Dynkin Neural Networks**. Note that the $\mathbb{A}_n$ neural network has 1 input vertex and 1 output vertex, the $\mathbb{D}_n$ neural network has 1 input vertex and 2 output vertices and $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ have all 1 input, 1 output and 1 bias.

There is some common terminology that appears on the literature around neural networks. Let (W, f) be a neural network; in our case, we will refer to a **neuron** or **unit** interchangeably to the combinatorial properties of a vertex and its activation function. We will call a **weight** to the number that defines the linear map W_α, for an $\alpha \in Q_1$. Our next step will be to describe how to compute on a determined network, one consistent with the usual computations that are done in practice. This leads to the following.

Definition 3.1.13 ((ARMENTA; JODOIN, 2021), Definition 4.7)**.** *Let (W, f) be a neural network over a network quiver Q; and let $\mathbf{x} \in \mathbb{C}^d$ a vector, which we will call **input vector of the network**. Given $i \in Q_0$, we write:*

$$\mathcal{E}_i = \{\alpha \in Q_1 \mid t(\alpha) = i\}$$

*the set of all edges with target i. Then the **activation output** of the vertex $i \in Q_0$ with respect to $\mathbf{x}$ after applying a forward pass will be denoted by $\mathrm{a}(W, f)_i(x)$ and will be given by:*

a. x_i *if $i \in Q_0$ is an input vertex,*

b. 1 *if $i \in Q_0$ is a bias vertex,*

c. $f_i\left(\sum_{\alpha \in \mathcal{E}_i} W_\alpha \, \mathrm{a}(W, f)_{s(\alpha)}(x)\right)$ *if $i \in Q_0$ is a hidden vertex,* *d.* $\sum_{\alpha \in \mathcal{E}_i} W_\alpha \, \mathrm{a}(W, f)_{s(\alpha)}(x)$ *if* $i \in Q_0$ *is an output vertex,*

Example 3.1.14. Consider the neural network from Example 3.1.11. We will write the weights between the layers as matrices. For example,

$$W_1 = \begin{pmatrix} 0.3 \\ -0.7 \\ -0.2 \end{pmatrix}$$

means that the weights $W_{\alpha_1} = 0.3$, $W_{\alpha_2} = -0.7$, $W_{\alpha_3} = -0.2$ where $\alpha_1 : \ 1 \longrightarrow 2$, $\alpha_2 : \ 1 \longrightarrow 3$ and $\alpha_3 : \ 1 \longrightarrow 4$. Similarly, the weights between the first and second layer will be given (by definition) as:

$$W_2 = \begin{pmatrix} -0.3 & 1.3 & * \\ 0.6 & -1.2 & 0.5 \\ * & -0.7 & 0.2 \end{pmatrix}$$

where $*$ means that there is no connection between the corresponding neurons, hence we don't assign any weight to such connection; lastly,

$$W_2 = \begin{pmatrix} 0.4 & 0.5 & * \\ * & -0.7 & -0.6 \end{pmatrix}$$

Assume moreover that the activation functions are all ReLU, then, if $-1 = x \in \mathbf{C}$ is an input vector of (W, f),

$$\begin{aligned}
\mathrm{a}(W, f)_1(x) &= x = -1 \\
\mathrm{a}(W, f)_3(x) &= \mathrm{ReLU}(-0.7x) = \mathrm{ReLU}(0.7) = 0.7 \\
\mathrm{a}(W, f)_2(x) &= \mathrm{ReLU}(0.3x) = 0 \\
\mathrm{a}(W, f)_4(x) &= \mathrm{ReLU}(-0.2x) = 0.2
\end{aligned}$$

Then the output of the first layer will be:

$$\mathrm{ReLU} \begin{pmatrix} 0.3x \\ -0.7x \\ -0.2x \end{pmatrix} = \begin{pmatrix} 0 \\ 0.7 \\ 0.2 \end{pmatrix}$$

Now,

$$\begin{aligned}
\mathrm{a}(W,f)_5(x) &= \mathrm{ReLU}\left(\sum_{\alpha\in\mathcal{E}_5} W_\alpha\, \mathrm{a}(W,f)_{s(\alpha)}(x)\right)\\
&= \mathrm{ReLU}\left(-0.3\,\mathrm{ReLU}(0.3x) + 1.3\,\mathrm{ReLU}(-0.7x)\right)\\
&= \mathrm{ReLU}\left(-0.3\cdot 0 + 1.3\cdot 1.7\right)\\
&= 0.91\\
\mathrm{a}(W,f)_6(x) &= \mathrm{ReLU}(0.6\cdot 0 - 1.2\cdot 0.7 + 0.5\cdot 0.2)\\
&= \mathrm{ReLU}(-0.84 + 0.1) = 0\\
\mathrm{a}(W,f)_7(x) &= \mathrm{ReLU}(-0.7\cdot 0.7 + 0.2\cdot 0.2 + 1)\\
&= \mathrm{ReLU}(-0.49 + 0.4 + 1) = 0.91
\end{aligned}$$

As $\mathrm{a}(W.f)_{10} = 1$, with $W_{\alpha_{11}} = 1$, $\alpha_{11} : 10 \to 7$. Then, the output of the second layer will be:

$$\begin{pmatrix} 0.91 \\ 0 \\ 0.91 \end{pmatrix}$$

Lastly,

$$\begin{aligned}
\mathrm{a}(W,f)_8(x) &= 0.4\cdot 0.91 + 0.5\cdot 0\\
&= 0.364\\
\mathrm{a}(W,f)_9(x) &= -0.7\cdot 0.91 = -0.637
\end{aligned}$$

Which implies that the output of the neural network will be $\begin{pmatrix} 0.364 \\ -0.637 \end{pmatrix}$.

The example above showed the basic operations of a neural network. According to (ARMENTA; JODOIN, 2021), the two main advantages of using this definition are that (1) any architecture may be representable, and (2) simplifies the notation on proofs relating network functions. For defining formally what this means, we note that in the example we obtained a 2×1 vector, that represented how the input vector -1 runs through the net. This motivates the following definition.

Definition 3.1.15 ((ARMENTA; JODOIN, 2021), Definition 4.8)**.** *Given a network quiver Q, let (W,f) a neural network over Q, with d input vertices, k output vertices and l layers. The **network function** of the neural networj is the function*

$$\Psi(W,f)(x) : \mathbb{C}^d \longrightarrow \mathbb{C}^k$$

such that

$$\Psi(W,f)(x) = (\mathrm{a}(W,f)_i(x))_{i\in\mathcal{E}_{l+1}}$$

that is, the activation outputs of the output vertices of (W,f) with respecto to an input vector $x \in \mathbb{C}^d$.

Example 3.1.16. Consider the Example 3.1.14. Then we obtained that

$$\Psi(W,f)(-1) = \begin{pmatrix} 0.364 \\ -0.637 \end{pmatrix}$$

and following a similar process we get

$$\Psi(W,f)(1) = \begin{pmatrix} 0.09 \\ -0.808 \end{pmatrix}$$

3.1.2 Isomorphisms of neural networks

By just looking at the combinatorial approach until now, it is easy to create a huge amount of neural networks over a network quiver, as any election of scalars for the weights will determine one. With this in mind, we want to study how different are the neural networks over Q, when given separate scalars, and we would like to measure such change via the algebraic nature induced in the network quiver; more specifically, via an isomorphism of quivers. However, by noting the presence of the activation functions on a neural network, we may impose an additional condition.

Definition 3.1.17 ((ARMENTA; JODOIN, 2021), Definition 4.9)**.** *Let Q a network quiver and (W,f), (V,g) two neural networks over Q. A* ***morphism of neural networks*** *$\tau : (W,f) \to (V,g)$ is a morphism of thin quiver representations $\tau : W \to V$ such that $\tau_i = 1$ for all $i \in Q_0$ that is not a hidden vertex, and for every $v \in Q_0$, the following diagram commutes:*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f_v"] \arrow[d, "\tau_v"'] & \mathbb{C} \arrow[d, "\tau_v"] \\ \mathbb{C} \arrow[r, "g_v"] & \mathbb{C} \end{tikzcd}, \quad \text{or} \quad \begin{tikzcd} \mathbb{C} \arrow[loop above, "f_v"] \arrow[d, "\tau_v"] \\ \mathbb{C} \arrow[loop below, "g_v"] \end{tikzcd}$$

Given a morphism of neural networks

$$\tau : (W,f) \longrightarrow (V,g)$$

we say that τ is an ***isomorphism of neural networks*** *if the induced morphism of thin representations $\tau : W \to V$ is an isomorphism. That being said, two neural networks are* ***isomorphic*** *if there exists an isomorphism of neural networks between them.*

Example 3.1.18. Consider the $\mathbb{A}_4-$network quiver and the neural networks with activation functions:

$$(W,f_1): \quad \begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C} \arrow[loop above, "\mathrm{ReLU}"] \arrow[r] & \mathbb{C} \arrow[loop above, "\mathrm{ReLU}"] \arrow[r] & \mathbb{C} \end{tikzcd}$$

$$(V, f_2): \quad \begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C} \arrow[loop above, "\mathrm{ReLU}"] \arrow[r] & \mathbb{C} \arrow[loop above, "g"] \arrow[r] & \mathbb{C} \end{tikzcd}$$

where $g(x) = \min(x, 0)$ and with weights $\begin{pmatrix} 0.2 & 0.4 & 0.6 \end{pmatrix}$ for W and $\begin{pmatrix} 0.08 & -0.3 & -2 \end{pmatrix}$ for V. Then the morphism $\tau : W \to V$ given by:

$$\tau = (1, 0.4, -0.3, 1)$$

is a morphism of thin quiver representations.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "0.2"] \arrow[d, "1"'] & \mathbb{C} \arrow[loop above, "\mathrm{ReLU}"] \arrow[r, "0.4"] \arrow[d, "0.4"] & \mathbb{C} \arrow[loop above, "\mathrm{ReLU}"] \arrow[r, "0.6"] \arrow[d, "-0.3"] & \mathbb{C} \arrow[d, "1"] \\ \mathbb{C} \arrow[r, "0.08"] & \mathbb{C} \arrow[loop below, "\mathrm{ReLU}"] \arrow[r, "-0.3"] & \mathbb{C} \arrow[loop below, "g"] \arrow[r, "-2"] & \mathbb{C} \end{tikzcd}$$

Now we look into the transformations of the activation functions. For the vertex 2, we have

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{ReLU}"] \arrow[d, "0.4"'] & \mathbb{C} \arrow[d, "0.4"] \\ \mathbb{C} \arrow[r, "\mathrm{ReLU}"] & \mathbb{C} \end{tikzcd}$$

If $x \leq 0$, then $\mathrm{ReLU}(0.4x) = 0 = 0.4\,\mathrm{ReLU}(x)$, and if $x > 0$, $0.4\,\mathrm{ReLU}(x) = 0.4x = \mathrm{ReLU}(0.4x)$. For vertex 3,

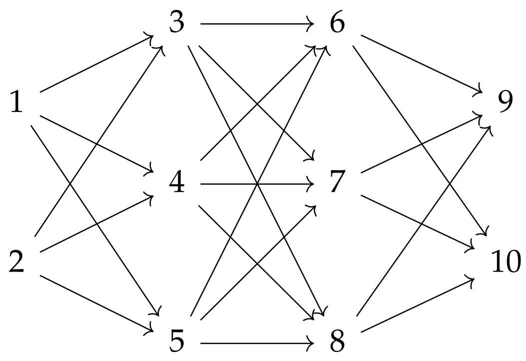
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{ReLU}"] \arrow[d, "-0.3"'] & \mathbb{C} \arrow[d, "-0.3"] \\ \mathbb{C} \arrow[r, "g"] & \mathbb{C} \end{tikzcd}$$

Here, if $x \leq 0$, $\mathrm{ReLU}(-0.3x) = -0.3x = -0.3\min(x, 0) = -0.3g(x)$; and if $x > 0$, $\mathrm{ReLU}(-0.3x) = 0 = -0.3\min(x, 0) = -0.3g(x)$. This says that the activation functions commute, and then:

$$\widetilde{\tau} : (W, f_1) \longrightarrow (V, f_2)$$

is an morphism of neural networks. As the morphisms are non-zero, they are invertible and thus $\widetilde{\tau}$ is an isomorphism.

Example 3.1.19 ((ARMENTA; JODOIN, 2021), Appendix A). Consider the ReLU multilayer perceptron (W, f) with 2 hidden layers of 3 neurons, and 2 neurons on the input and output layers:



And consider the following weight matrices:

$$W_1 = \begin{pmatrix} 1.3 & -0.1 \\ 0.9 & 0.5 \\ -1.0 & 0.7 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 0.3 & -1.2 & 0.9 \\ 0.7 & 0.2 & -0.3 \\ 0.6 & 0.4 & -0.1 \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 0.4 & -1.1 & -0.7 \\ -0.8 & 0.6 & 0.2 \end{pmatrix}$$

Then the neural network (W, ReLU) with the weight matrices above defined is isomorphic to the neural network (V, g) with weights:

$$V_1 = \begin{pmatrix} -0.13 & 0.01 \\ -0.36 & -0.2 \\ -1.2 & 0.84 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0.9 & 0.9 & 0.225 \\ -6.3 & -0.45 & -0.225 \\ 4.2 & 0.7 & 0.058\bar{3} \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 1.\bar{3} & -1.\bar{2} & 1.0 \\ -2.\bar{6} & 0.\bar{6} & \frac{2}{7} \end{pmatrix}$$

and with activation functions given by the following matrix:

$$g = \begin{pmatrix} \min(x,0) & \text{ReLU} \\ \min(x,0) & \text{ReLU} \\ \text{ReLU} & \min(x,0) \end{pmatrix}$$

on where each entry of the matrix has the activation function associated to the corresponding vertex.

It may be cumbersome to do all the calculations for verifying that two different neural networks are isomorphic, and for this we will describe a way to construct isomorphic neural networks when given one, and it will also show in a more intuitive way why the two neural networks from the example above are isomorphic (without brute force). In order to do so, we define an important class of quiver.

Definition 3.1.20 ((ARMENTA; JODOIN, 2021), Definition 4.11)**.** *Let Q be a network quiver. The **hidden quiver** of Q, denoted by $\widetilde{Q} = \left(\widetilde{Q_0}, \widetilde{Q_1}, \widetilde{s}, \widetilde{t}\right)$ is the quiver obtained from Q, where $\widetilde{Q_0}$ is the subset of Q_0 which consists of all hidden vertices and $\widetilde{Q_1}$ is the subset of arrows between those hidden vertices that are not loops.*

Example 3.1.21. Consider the network quiver Q' from Example 3.1.3, then

$$\widetilde{Q'}: \quad \begin{tikzcd} 2 \arrow[r] \arrow[dr] & 5 \\ 3 \arrow[r] \arrow[ur] \arrow[dr] & 6 \\ 4 \arrow[r] \arrow[ur] & 7 \end{tikzcd}$$

Example 3.1.22. Consider the network quivers from Example 3.1.4, then $\widetilde{\mathbb{A}_n} = \mathbb{A}_{n-2}$, $\widetilde{\mathbb{D}_n} = \mathbb{A}_{n-3}$ and $\widetilde{\mathbb{E}_6} = \mathbb{A}_3$, $\widetilde{\mathbb{E}_7} = \mathbb{A}_4$, $\widetilde{\mathbb{E}_8} = \mathbb{A}_5$.

Now we define the group of change basis, which acts on neural networks, and that will have an important property: all elements in any orbit are isomorphic.

Definition 3.1.23 ((ARMENTA; JODOIN, 2021), Definition 4.12)**.** *The **group of change basis** for Q is denoted as:*

$$\widetilde{G} = \prod_{i \in \widetilde{Q_0}} \mathbb{C}^*$$

*An element $\tau \in \widetilde{G}$ is called a **change of basis** for the network (W, f).*

Example 3.1.24. If Q is the network quiver from Example 3.1.3, then, by Example 3.1.21,

$$\widetilde{G} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \simeq (\mathbb{C}^*)^6$$

And if Q is the $\mathbb{A}_n$ network quiver, then by Example 3.1.22

$$\widetilde{G} = (\mathbb{C}^*)^{n-2}$$

Note that this group has factors the number of hidden vertices of Q. We proceed to describe the action of such a group on the set of neural networks of a quiver Q, and as the action will be given as change of basis, we would like to describe it on each one of the vertices. However, the group only "shows" such a change for a hidden neuron, yet this can be solved if we don't change any vertex that is not hidden. In other words, given a $\widetilde{\tau} \in \widetilde{G}$, there exists a $\tau \in G$ induced by $\widetilde{\tau}$, where

$$\tau_i = 1 \text{ for all } i \in Q_0 - \mathcal{H}_0$$

where $\mathcal{H}_0$ is the set of hidden vertices of Q. This means that we can see elements of $\widetilde{G}$ as elements of G. With this in mind,

Definition 3.1.25 ((ARMENTA; JODOIN, 2021)). *Let (W, f) be a neural network over Q. The **action of** $\widetilde{G}$ on (W, f) is given element-wise by:*

$$\tau \cdot (W, f) = (\tau \cdot W, \tau \cdot f)$$

where $\tau \in \widetilde{G}$, and

a. *$\tau \cdot W$ is the thin representation where for every $\alpha \in Q_1$,*

$$(\tau \cdot W)_\alpha = W_\alpha \frac{\tau_{t(\alpha)}}{\tau_{s(\alpha)}},$$

b. *$\tau \cdot f$, the activation on the hidden vertex $i \in Q_0$ is given by*

$$(\tau \cdot f)_i (x) = \tau_i f_i \left(x \tau_i^{-1} \right) \text{ for all } x \in \mathbb{C}$$

Remark 3.1.26. Note that $\Gamma : (W, f) \longrightarrow (\tau \cdot W, \tau \cdot f)$ is an isomorphism of neural networks.

Example 3.1.27 ((ARMENTA; JODOIN, 2021), Appendix A). Consider again Example 3.1.19, and the neural network (W, f) where f is ReLU as there. If we consider $\tau \in \widetilde{G} = (\mathbb{C}^*)^6$ given by:

$$\tau = \begin{pmatrix} -0.1 & 0.3 \\ -0.4 & 0.9 \\ 1.2 & -0.7 \end{pmatrix}$$

then the action of τ over (W, f) will be, on weights as:

$$\tau W_1 = \begin{pmatrix} 1.3 \left(\frac{-0.1}{1.0} \right) & -0.1 \left(\frac{-0.1}{1.0} \right) \\ 0.9 \left(\frac{-0.4}{1.0} \right) & 0.5 \left(\frac{-0.4}{1.0} \right) \\ -1.0 \left(\frac{1.2}{1.0} \right) & 0.7 \left(\frac{1.2}{1.0} \right) \end{pmatrix} = \begin{pmatrix} -0.13 & 0.01 \\ -0.36 & -0.2 \\ -1.2 & 0.84 \end{pmatrix} = V_1$$

Similarly,

$$\tau W_2 = \begin{pmatrix} 0.3 \left(\frac{0.3}{-0.1} \right) & -1.2 \left(\frac{0.3}{-0.4} \right) & 0.9 \left(\frac{0.3}{1.2} \right) \\ 0.7 \left(\frac{0.9}{-0.1} \right) & 0.2 \left(\frac{0.9}{-0.4} \right) & -0.3 \left(\frac{0.9}{1.2} \right) \\ 0.6 \left(\frac{-0.7}{-0.1} \right) & 0.4 \left(\frac{-0.7}{-0.4} \right) & -0.1 \left(\frac{-0.7}{1.2} \right) \end{pmatrix}$$

$$= \begin{pmatrix} -0.9 & 0.9 & 0.225 \\ -6.3 & -0.45 & -0.225 \\ 4.2 & 0.7 & 0.058\bar{3} \end{pmatrix} = V_2$$

And lastly,

$$\tau W_3 = \begin{pmatrix} \frac{0.4}{0.3} & \frac{-1.1}{0.9} & \frac{-0.7}{-0.7} \\ \frac{-0.8}{0.3} & \frac{0.6}{0.9} & \frac{0.2}{-0.7} \end{pmatrix} = \begin{pmatrix} 1.\bar{3} & -1.\bar{2} & 1.0 \\ -2.\bar{6} & 0.\bar{6} & \frac{2}{7} \end{pmatrix} = V_3$$

Now, for the activations we have that $(\tau \cdot f)_i(x) = (\tau \cdot \mathrm{ReLU})_i(x) = \tau_i \mathrm{ReLU}(x\tau_i^{-1})$. So, if $\tau_i > 0$,

$$(\tau \cdot f)_i(x) = \tau_i \mathrm{ReLU}(x\tau_i^{-1}) = \tau_i\tau_i^{-1} \mathrm{ReLU}(x) = \mathrm{ReLU}(x)$$

But if $\tau_i < 0$,

$$(\tau \cdot f)_i(x) = \tau_i\tau_i^{-1} \min(x,0) = \min(x,0)$$

Thus, considering the values of τ, the matrix of activations will be

$$\tau \cdot f = \begin{pmatrix} \min(x,0) & \mathrm{ReLU} \\ \min(x,0) & \mathrm{ReLU} \\ \mathrm{ReLU} & \min(x,0) \end{pmatrix} = g$$

By Remark 3.1.26, we have that $(W, f) = (\tau \cdot W, \tau \cdot f)$, but we just saw that $(\tau \cdot W, \tau \cdot f) = (V, g)$, which shows the isomorphism of the Example 3.1.19.

We end this section with the following result, a relation between the network outputs of two isomorphic neural networks, which has special importance for this approach.

Theorem 3.1.1 ((ARMENTA; JODOIN, 2021), Theorem 4.13)**.** *Let $\tau : (W, f) \longrightarrow (V, g)$ an isomorphism of neural networks, then its network functions are the same, i.e.,*

$$\Psi(W, f) = \Psi(V, g)$$

Remark 3.1.28. Before we start, we make a brief comment on isomorphisms of neural networks. Let τ be one of such, then as the representations are thin, we can see (V, g) as

$$\tau \cdot (W, f) \text{ for some } \tau \in \widetilde{G}$$

because the election of a morphism requires a choosing of scalars that are on $\widetilde{G}$, and the conditions hold because of the commutativity of the diagrams. In other words, all representations isomorphic to (W, f) are in the $\widetilde{G}-$orbit of (W, f), $\widetilde{G} \cdot (W, f)$.

Proof. Let $\tau : (W, f) \longrightarrow (V, g)$ an isomorphism of neural networks over Q, and let $\alpha \in Q_1$, with $\alpha : s(\alpha) \longrightarrow t(\alpha)$. We will compare the outputs of the neural networks on a given input vector $x \in \mathbb{C}^d$. Let $i \in Q_0$ be any source vertex, $i \xrightarrow{\alpha} t(\alpha)$. We have,

$$\begin{aligned} \mathrm{a}(W,f)_i(x) &= \begin{cases} x_i & \text{if } i \text{ is a source vertex} \\ 1 & \text{if } i \text{ is a bias vertex} \end{cases} \\ &= \mathrm{a}(V,g)_i(x) \end{aligned}$$

Now let i be a vertex on the first hidden layer, $s(\alpha) \xrightarrow{\alpha} i$. By definition, we have

$$\mathrm{a}(W,f)_i(x) = f_i\left(\sum_{\epsilon\in\mathcal{E}_i} W_\epsilon\, \mathrm{a}(W,f)_{s(\epsilon)}(x)\right)$$

And also, as $V_\alpha = W_\alpha \dfrac{\tau_{s(\alpha)}}{\tau_{s(\alpha)}} = W_\alpha \tau_{t(\alpha)}$, we obtain by Definition 3.1.25 b.:

$$\begin{aligned} \mathrm{a}(V,g)_i(x) &= \tau_i f_i\left(\frac{1}{\tau_i}\sum_{\epsilon\in\mathcal{E}_i} V_\epsilon\, \mathrm{a}(V,g)_{s(\epsilon)}(x)\right) \\ &= \tau_i f_i\left(\frac{1}{\tau_i}\sum_{\epsilon\in\mathcal{E}_i} W_\epsilon \tau_{t(\epsilon)}\, \mathrm{a}(V,g)_{s(\epsilon)}(x)\right) \\ &= \tau_i f_i\left(\frac{1}{\tau_i}\sum_{\epsilon\in\mathcal{E}_i} W_\epsilon \tau_i\, \mathrm{a}(V,g)_{s(\epsilon)}(x)\right) \\ &= \tau_i f_i\left(\sum_{\epsilon\in\mathcal{E}_i} W_\epsilon\, \mathrm{a}(V,g)_{s(\epsilon)}(x)\right) \end{aligned}$$

As by definition, all arrows ϵ in $\mathcal{E}_i$ hold $t(\epsilon) = i$:

$$\begin{tikzcd} \epsilon_1 \arrow[dr] & \\ \vdots & i \\ \epsilon_k \arrow[ur] & \end{tikzcd}$$

Also note that $s(\epsilon)$ is a source vertex for all $\epsilon \in \mathcal{E}_i$, then, $\mathrm{a}(V,g)_{s(\epsilon)} = \mathrm{a}(W,f)_{s(\epsilon)}$, and we get

$$\mathrm{a}(V,g)_i(x) = \tau_i\, \mathrm{a}(W,f)_i(x) \tag{3.1}$$

Now we assume that i is a hidden vertex on the second layer, with $s(\alpha) \longrightarrow i$, then,

$$\begin{aligned}
\mathsf{a}(V,g)_i(x) &= \tau_i f_i\left(\frac{1}{\tau_i}\sum_{\epsilon\in\mathcal{E}_i} V_\epsilon\,\mathsf{a}(V,g)_{s(\epsilon)}(x)\right)\\
&= \tau_i f_i\left(\frac{1}{\tau_i}\sum_{\epsilon\in\mathcal{E}_i} \frac{W_\epsilon\tau_i}{\tau_{s(\epsilon)}}\,\mathsf{a}(V,g)_{s(\epsilon)}(x)\right)\\
&= \tau_i f_i\left(\sum_{\epsilon\in\mathcal{E}_i} \frac{W_\epsilon}{\tau_{s(\epsilon)}}\,\mathsf{a}(W,f)_{s(\epsilon)}(x)\right)\\
&= \tau_i\,\mathsf{a}(W,f)_i(x)
\end{aligned}$$

where the next to last equation is due to 3.1. We can proceed by induction to obtain

$$\mathsf{a}(W,f)_j(x) = \tau_j\,\mathsf{a}(V,g)_j(x) \text{ for all } j\in Q_0$$

With this, we find that

$$\Psi(W,f)(x) = \Psi(V,g)(z)$$

as by definition they are the activation outputs of the output vertices, and as τ is an isomorphism of neural networks, $\tau_j = 1$ for all output vertex j. The arbitrary choice of $x\in\mathbb{C}^d$ shows the equality desired. □

Example 3.1.29. Take the Example 3.1.27, this is, the neural networks (W,f) and (V,g) that we know by Example 3.1.19 and 3.1.27 that are isomorphic. Here we will calculate its network functions on $x = \begin{pmatrix}0.6\\0.8\end{pmatrix}$. The output of the first layer is:

$$\begin{aligned}
\mathrm{ReLU}(W_1x) &= \mathrm{ReLU}\begin{pmatrix}1.3(0.6)-0.1(0.8)\\0.9(0.6)+0.5(0.8)\\-1.0(0.6)+0.7(0.8)\end{pmatrix}\\
&= \mathrm{ReLU}\begin{pmatrix}0.7\\0.94\\-0.04\end{pmatrix} = \begin{pmatrix}0.7\\0.94\\0\end{pmatrix}
\end{aligned}$$

The output of the second layer is:

$$\begin{aligned}
\mathrm{ReLU}(W_2\,\mathrm{ReLU}(W_1x)) &= \mathrm{ReLU}\begin{pmatrix}0.3(0.7)-1.2(0.94)\\0.7(0.7)+0.2(0.94)\\0.6(0.7)+0.4(0.94)\end{pmatrix}\\
&= \mathrm{ReLU}\begin{pmatrix}-0.918\\0.678\\0.796\end{pmatrix} = \begin{pmatrix}0\\0.678\\0.796\end{pmatrix}
\end{aligned}$$

Therefore the output of the network is:

$$\begin{aligned}\Psi(W,f)(x) &= W_3(\mathrm{ReLU}(W_2\,\mathrm{ReLU}(W_1x)))\\ &= W_3\begin{pmatrix}0\\0.678\\0.796\end{pmatrix} = \begin{pmatrix}-1.303\\0.566\end{pmatrix}\end{aligned}$$

Now lets do a forward pass on $(\tau W, \tau f) = (V, g)$ for the same x. On the first layer we have:

$$\begin{aligned}\begin{pmatrix}\min(0,x)\\ \min(0,x)\\ \mathrm{ReLU}\end{pmatrix}\tau W_1 x &= \begin{pmatrix}\min(0,x)\\ \min(0,x)\\ \mathrm{ReLU}\end{pmatrix}\begin{pmatrix}-0.13 & 0.01\\ -0.36 & -0.2\\ -1.2 & 0.84\end{pmatrix}\begin{pmatrix}0.6\\0.8\end{pmatrix}\\ &= \begin{pmatrix}\min(0,x)\\ \min(0,x)\\ \mathrm{ReLU}\end{pmatrix}\begin{pmatrix}-0.07\\-0.376\\-0.048\end{pmatrix} = \begin{pmatrix}-0.07\\-0.376\\0\end{pmatrix}\end{aligned}$$

On the second hidden layer we obtain:

$$\begin{aligned}\begin{pmatrix}\mathrm{ReLU}\\ \mathrm{ReLU}\\ \min(0,x)\end{pmatrix}\tau W_2\begin{pmatrix}-0.07\\-0.376\\0\end{pmatrix} &= \begin{pmatrix}\mathrm{ReLU}\\ \mathrm{ReLU}\\ \min(0,x)\end{pmatrix}\begin{pmatrix}-0.9 & 0.9 & 0.225\\ -6.3 & -0.45 & -0.225\\ 4.2 & 0.7 & 0.058\bar{3}\end{pmatrix}\begin{pmatrix}-0.07\\-0.376\\0\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{ReLU}\\ \mathrm{ReLU}\\ \min(0,x)\end{pmatrix}\begin{pmatrix}-0.2754\\0.6102\\-0.5572\end{pmatrix} = \begin{pmatrix}0\\0.6102\\-0.5572\end{pmatrix}\end{aligned}$$

And lastly,

$$\begin{aligned}\Psi((\tau W,\tau f)(x)) &= \tau W_3\begin{pmatrix}0\\0.6102\\-0.5572\end{pmatrix}\\ &= \begin{pmatrix}1.\bar{3} & -1.\bar{2} & 1.0\\ -2.\bar{6} & 0.\bar{6} & -\frac{2}{7}\end{pmatrix}\begin{pmatrix}0\\0.6102\\-0.5572\end{pmatrix}\\ &= \begin{pmatrix}-1.303\\0.566\end{pmatrix} = \Psi(W,f)(x)\end{aligned}$$

as the theorem showed. Note also that the activation outputs of a layer for (W, f) are the same as the outputs for (V, g) just multiplied by the corresponding constant that appears on τ, as we also showed on the proof of the Theorem.

We end up with a property of ReLU networks, called the positive scale invariance or positive homogeneity.

Corollary 3.1.30 ((ARMENTA; JODOIN, 2021), Corollary 4.6)**.** *Let* (W, f) *be a neural network over a network quiver* Q *over* $\mathbb{R}$ *where* f *is the* ReLU *activation function. Then if* $\tau = (\tau_v)_{v \in Q_0}$ *where* $\tau_i = 1$ *if* i *is not a hidden vertex and* $\tau_j > 0$ *for all other* j *we have*

$$\tau(W, f) = (\tau \cdot W, f)$$

In particular,

$$\Psi(W, f) = \Psi(\tau \cdot W, f)$$

Proof. We know that ReLU satisfies $\mathrm{ReLU}(\tau_i x) = \tau_i \mathrm{ReLU}(x)$ when $\tau_i > 0$, and any real x. Now, as

$$(\tau \cdot f) = \tau_i f_i \left(\frac{x}{\tau_i} \right) \text{ on } i \in Q_0$$

we obtain,

$$\tau_i f_i \left(\frac{x}{\tau_i} \right) = \tau_i \mathrm{ReLU} \left(\frac{x}{\tau_i} \right) = \mathrm{ReLU}(x)$$

Thus, $\tau \cdot f = f$, from which

$$\tau \cdot (W, f) = (\tau \cdot W, \tau \cdot f) = (\tau \cdot W, f)$$

as desired. □

Example 3.1.31*.* Consider Example 3.1.19, if we choose τ as a matrix with all positive real numbers, we would have obtained $\tau \cdot f = f$ as the Corollary showed, and then there is just necessary to do the transformations on the weight matrices.

3.1.3 Types of architectures

At the moment we have described the concepts of neural networks and how the input vectors flow through it. One important topic to discuss is the one that expresses how the neural network is constructed, and this is what we call by architecture.

Definition 3.1.32 ((GOODFELLOW; BENGIO; COURVILLE, 2016))**.** *The* ***architecture of a neural network*** *refers to the overall structure of it; it says how many units it should have and how these units are connected to each other.*

In particular, we have the following

Definition 3.1.33 ((ARMENTA; JODOIN, 2021), Definition 5.2)**.** *Let* (W, f) *be a neural network over* Q*.*

1. *The* ***combinatorial architecture*** *is the network quiver* Q*.*

2. *The **weight architecture** is given by the restrictions on how the weights are chosen, this is, the values W_ϵ for edges ϵ.*

3. *The **activation architecture** refers to the set of activation functions f_i associated to the loops of Q.*

Different types of neural networks may have different combinatorial, weight and activation architectures, but we do know that isomorphic neural networks have the same combinatorial architecture by definition. However, an isomorphism may change the weight or the activation architecture, as the Example 3.1.29 showed.

3.2 Data representations in a Neural Network

In this section our objective will be to describe a way of representing the data in a different framework and perspective, more specifically, via the neuron outputs that are obtained by a forward pass. This will allow to encode each vector from a data set in a thin quiver representation, which then will have a mathematical description of such data in terms of the architecture of the neural network. Such representation will have a great advantage, it won't need the existence of activation functions, and we will describe this phenomena in detail. As in (ARMENTA; JODOIN, 2021), here we will focus on how the encoding of the representations happens rather than on how such representations are learned by a determined neural network. We start by defining what we mean by data.

Definition 3.2.1 ((ARMENTA; JODOIN, 2021), Definition 6.1)**.** *A **labeled data set** is a finite set $\{(x_i, t_i)\}_{i=1}^n$ of pairs such that $x_i \in \mathbb{C}^d$ (which we will call a **data vector**) and t_i will be called a **data target** (or **target**).*

Throughout the text we will say indistinctively dataset or data sample. For example, t_i can be a number or an element of a set (for regression and classification, respectively).

Now let x be a data vector from a data sample $D = \{(x_i, t_i)\}_{i=1}^n$. We will construct a thin quiver representation with the outputs of the neuron values when we make x flow through the neural network. As such, the object that originates is specific to the data vector, and it will have in mind the activation functions and each one of the neuron values (also called *feature maps*). Those are important as they can be visualized (YOSINSKI et al., 2015). We will call such thin quiver representation by W_x^f, with identity activations as in Example 3.1.10. As W_x^f will take in mind the activation outputs and the values before doing the activation, this inspires the following:

Definition 3.2.2 ((ARMENTA et al., 2022), p. 116)**.** *Let Q a network quiver and (W, f) a neural network over Q. Given a vertex $i \in Q_0$ and a vector $x \in \mathbf{C}^d$, the* ***preactivation*** *of (W, f) at i with respect to x will be:*

$$\text{pre-a}(W,f)_i(x) := \begin{cases} 1 & \text{if } i \text{ is a source vertex} \\ \sum\limits_{\alpha \in \mathcal{E}_v} W_\alpha \, \text{a}(W,f)_{s(\alpha)}(x) & \text{in any other case} \end{cases}$$

Remark 3.2.3. Note that if i is a hidden vertex and $x \in \mathbb{C}^d$,

$$f_i\left(\text{pre-a}(W,f)_i(x)\right) = \text{a}(W,f)_i(x)$$

With this, we can define

Definition 3.2.4 ((ARMENTA; JODOIN, 2021))**.** *Let (W, f) be a neural network and x a data vector from a data sample D. Then the* ***representation of*** $(\mathbf{W}, \mathbf{f})$ ***at*** $\mathbf{x}$ *will be an identity neural network $(W_x^f, 1)$, where*

$$\left(W_x^f\right)_\epsilon = W_\epsilon \frac{\text{a}(W,f)_{s(\epsilon)}(x)}{\text{pre-a}(W,f)_{s(\epsilon)}(x)}$$

and $\epsilon \in Q_1 : s(\epsilon) \longrightarrow t(\epsilon)$. We will denote it by W_x^f, as there is no confusion with the activation functions.

Remark 3.2.5. Note that if $s(\epsilon)$ is an input vertex, $\left(W_x^f\right)_\epsilon = W_\epsilon x_i$ and if $s(\epsilon)$ is a bias vertex, $\left(W_x^f\right)_\epsilon = W_\epsilon$.

Example 3.2.6. Consider the $\mathbb{A}_5$ neural network (W, f) given by,

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "W_\alpha"'] & \mathbb{C} \arrow[loop above, "f_1"] \arrow[r, "W_\beta"'] & \mathbb{C} \arrow[loop above, "f_2"] \arrow[r, "W_\gamma"'] & \mathbb{C} \arrow[loop above, "f_3"] \arrow[r, "W_\delta"'] & \mathbb{C}
\end{tikzcd}$$

Given a data vector $x \in \mathbb{C}$ we obtain that W_x^f will have the form:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "W_\alpha x"'] & \mathbb{C} \arrow[loop above, "1"] \arrow[r, "W_\beta \frac{\text{a}(W,f)_2(x)}{\text{pre-a}(W,f)_2(x)}"'] & \mathbb{C} \arrow[loop above, "1"] \arrow[r, "W_\gamma \frac{\text{a}(W,f)_3(x)}{\text{pre-a}(W,f)_3(x)}"'] & \mathbb{C} \arrow[loop above, "1"] \arrow[r, "W_\delta \frac{\text{a}(W,f)_3(x)}{\text{pre-a}(W,f)_3(x)}"'] & \mathbb{C}
\end{tikzcd}$$

Applying the definition we get:

$$\begin{aligned} \text{a}(W,f)_2(x) &= f_1(W_\alpha \, \text{a}(W,f)_1(x)) \\ &= f_1(W_\alpha x) \\ \text{pre-a}(W,f)_2(x) &= W_\alpha x \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{a}(W,f)_3(x) &= f_2\left(W_\beta\, \mathrm{a}(W,f)_2(x)\right) \\
&= f_2\left(W_\beta f_1(W_\alpha x)\right) \\
\text{pre-a}(W,f)_3(x) &= W_\beta f_1(W_\alpha x)
\end{aligned}$$

And lastly,

$$\begin{aligned}
\mathrm{a}(W,f)_4(x) &= f_3\left(W_\gamma\, \mathrm{a}(W,f)_3(x)\right) \\
&= f_3\left(W_\gamma f_2\left(W_\beta f_1(W_\alpha x)\right)\right) \\
\text{pre-a}(W,f)_4(x) &= W_\gamma f_2\left(W_\beta f_1(W_\alpha x)\right)
\end{aligned}$$

Which implies that W_x^f has the form:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "W_\alpha x"'] & \mathbb{C} \arrow[loop above, "1"] \arrow[r, "W_\beta \frac{f_1(W_\alpha x)}{W_\alpha x}"'] & \mathbb{C} \arrow[loop above, "1"] \arrow[r, "W_\gamma \frac{f_2(W_\beta f_1(W_\alpha x))}{W_\beta f_1(W_\alpha x)}"'] & \mathbb{C} \arrow[loop above, "1"] \arrow[r, "W_\delta \frac{f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x)))}{W_\gamma f_2(W_\beta f_1(W_\alpha x))}"'] & \mathbb{C}
\end{tikzcd}$$

Example 3.2.7. Consider the $\mathbb{D}_4$ neural network given by:

$$(W,f) = \begin{tikzcd}
 & & \mathbb{C} \\
\mathbb{C} \arrow[r, "W_\alpha"] & \mathbb{C} \arrow[loop above, "f_1"] \arrow[ur, "W_\beta"] \arrow[dr, "W_\gamma"] & \\
 & & \mathbb{C}
\end{tikzcd}$$

If x is a data vector,

$$(W_x^f, 1) = \begin{tikzcd}
 & & \mathbb{C} \\
\mathbb{C} \arrow[r, "W_\alpha x"] & \mathbb{C} \arrow[loop above, "1"] \arrow[ur, "W_\beta \frac{\mathrm{a}(W,f)_2(x)}{\text{pre-a}(W,f)_2(x)}"] \arrow[dr, "W_\gamma \frac{\mathrm{a}(W,f)_2(x)}{\text{pre-a}(W,f)_2(x)}"'] & \\
 & & \mathbb{C}
\end{tikzcd}$$

and doing the same computations as before, we get:

$$(W_x^f, 1) = \begin{tikzcd}
 & & \mathbb{C} \\
\mathbb{C} \arrow[r, "W_\alpha x"] & \mathbb{C} \arrow[loop above, "1"] \arrow[ur, "W_\beta \frac{f_1(W_\alpha x)}{W_\alpha x}"] \arrow[dr, "W_\gamma \frac{f_1(W_\alpha x)}{W_\alpha x}"'] & \\
 & & \mathbb{C}
\end{tikzcd}$$

There is a possibility where

$$\text{pre-a}(W,f)_{s(\epsilon)}(x) = 0$$

and then the representation of (W,f) at x won't be defined at such a vertex. Note nevertheless that the set where it happens is of measure zero, and then, in such cases we can add a value $\eta \neq 0$ (sufficiently small but computable) to the preactivation and consider that value as the preactivation. With this in mind, we will suppose $\text{pre-a}(W,f)_i(x) \neq 0$ from now on.

The biggest property if the representation W_x^f is that it does not alter the network function of (W,f) when evaluated on a set of ones. This means that W_x^f encodes all the combinatorics done by the neural network when making x flow. More formally,

Theorem 3.2.1 ((ARMENTA; JODOIN, 2021), Theorem 6.4)**.** *Let (W,f) be a neural network over a network quiver Q, $x \in \mathbb{C}^d$ a data vector from a labeled data set D, and W_x^f the representation of (W,f) at x. Then,*

$$\Psi(W,f)(x) = \Psi(W_x^f, 1)(1^d)$$

Proof. We will use a similar argument to the one in Theorem 3.1.1. Let $i \in Q_0$ be a source vertex. Then we have by definition

$$\text{a}(W_x^f, 1)_i(1^d) = 1$$

If $j \in Q_0$ is in the *first hidden layer*, we have that for every $\alpha \in \mathcal{E}_j$,

$$\text{a}(W_x^f, 1)_{s(\alpha)}(1^d) = 1 \tag{3.2}$$

Given such a $j \in \mathcal{E}_j$, we can do a decomposition $\mathcal{E}_j = \mathcal{E}_j^B \cap \mathcal{E}_j^I$, where

$$\begin{aligned} \mathcal{E}_j^B &= \{\alpha \in \mathcal{E}_j \,|\, s(\alpha) \text{ is a bias vertex}\} \\ \mathcal{E}_j^I &= \{\alpha \in \mathcal{E}_j \,|\, s(\alpha) \text{ is an input vertex}\} \end{aligned}$$

where such union is clearly disjoint. Therefore,

$$\begin{aligned} \text{a}(W_x^f, 1)_j(1^d) &= \sum_{\epsilon \in \mathcal{E}_j} \left(W_x^f\right)_\epsilon \text{a}(W_x^f, 1)_{s(\epsilon)}(1^d) \\ &= \sum_{\epsilon \in \mathcal{E}_j} \left(W_x^f\right)_\epsilon && \text{by (3.2)} \\ &= \sum_{\epsilon \in \mathcal{E}_j^B} \left(W_x^f\right)_\epsilon + \sum_{\epsilon \in \mathcal{E}_j^I} \left(W_x^f\right)_\epsilon \\ &= \sum_{\epsilon \in \mathcal{E}_j^B} W_\epsilon + \sum_{\epsilon \in \mathcal{E}_j^I} W_\epsilon x_\epsilon && \text{by Remark 3.2.5} \\ &= \text{pre-a}(W,f)_j(x) \end{aligned}$$

Then we have that

$$f_j\left(a(W_x^f,1)_j(1^d)\right) = \mathrm{a}(W,f)_j(x)$$

Now, if $j \in Q_0$ is in the *second hidden layer* we get:

$$\begin{aligned}
\mathrm{a}(W_x^f,1)_j(1^d) &= \sum_{\epsilon\in\mathcal{E}_j} \left(W_x^f\right)_\epsilon \mathrm{a}(W_x^f,1)_{s(\epsilon)}(1^d) \\
&= \sum_{\epsilon\in\mathcal{E}_j} W_\epsilon \frac{\mathrm{a}(W,f)_{s(\epsilon)}(x)}{\text{pre-a}(W,f)_{s(\epsilon)}(x)} \mathrm{a}(W_x^f,1)_{s(\epsilon)}(1^d) \\
&= \sum_{\epsilon\in\mathcal{E}_j} W_\epsilon \frac{\mathrm{a}(W,f)_{s(\epsilon)}(x)}{\text{pre-a}(W,f)_{s(\epsilon)}(x)} \text{pre-a}(W,f)_{s(\epsilon)}(x) \\
&= \sum_{\epsilon\in\mathcal{E}_j} W_\epsilon\, \mathrm{a}(W,f)_{s(\epsilon)}(x) \\
&= \text{pre-a}(W,f)_j(x)
\end{aligned}$$

This says that

$$f_j\left(\mathrm{a}(W_x^f,1)(1^d)\right) = \mathrm{a}(W,f)_j(x)$$

Proceeding by induction we obtain:

$$\Psi(W_x^f,1)(1^d) = \Psi(W,f)(x)$$

as the output layer does not have activation functions and the function Ψ is the evaluation of the activation outputs at the output vertices. □

Example 3.2.8. Consider the Example 3.2.6. Then

$$\begin{aligned}
\Psi(W,f)(x) &= \mathrm{a}(W,f)_5(x) \\
&= W_\delta\, \mathrm{a}(W,f)_4(x) \\
&= W_\delta f_3\left(W_\gamma\, \mathrm{a}(W,f)_3(x)\right) \\
&= W_\delta f_3\left(W_\gamma f_2\left(W_\beta\, \mathrm{a}(W,f)_2(x)\right)\right) \\
&= W_\delta f_3\left(W_\gamma f_2\left(W_\beta f_1\left(W_\alpha\, \mathrm{a}(W,f)_1(x)\right)\right)\right) \\
&= W_\delta f_3\left(W_\gamma f_2\left(W_\beta f_1\left(W_\alpha x\right)\right)\right)
\end{aligned}$$

Similarly, noting that here $d = 1$, the output of representation of the data at x will be:

$$\begin{aligned}
\Psi(W_x^f,1)(1) &= \mathrm{a}(W_x^f,1)_5(1) \\
&= W_\delta \frac{f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x)))}{W_\gamma f_2(W_\beta f_1(W_\alpha x))} \mathrm{a}(W_x^f,1)_4(1) \\
&= W_\delta \frac{f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x)))}{W_\gamma f_2(W_\beta f_1(W_\alpha x))} W_\gamma \frac{f_2(W_\beta f_1(W_\alpha x))}{W_\beta f_1(W_\alpha x)} \mathrm{a}(W_x^f,1)_3(1) \\
&= W_\delta \frac{f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x)))}{W_\beta f_1(W_\alpha x)} \mathrm{a}(W_x^f,1)_3(1)
\end{aligned}$$

$$
\begin{aligned}
&= W_\delta \frac{f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x)))}{W_\beta f_1(W_\alpha x)} W_\beta \frac{f_1(W_\alpha x)}{W_\alpha x} \, \mathrm{a}(W_x^f, 1)_2(1) \\
&= W_\delta \frac{f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x)))}{W_\alpha x} W_\alpha x \, \mathrm{a}(W_x^f, 1)_1(1) \\
&= W_\delta f_3(W_\gamma f_2(W_\beta f_1(W_\alpha x))) \\
&= \Psi(W, f)(x)
\end{aligned}
$$

As the Theorem showed.

Example 3.2.9 ((ARMENTA; JODOIN, 2021), Appendix B). Consider Example 3.1.29. There we obtained that

$$
\Psi(W, f)(x) = \begin{pmatrix} -1.303 \\ 0.566 \end{pmatrix}
$$

We denote by V_1, V_2 and V_3 the weight matrices of W_x^f. Then V_1 is given by

$$
\begin{aligned}
V_1 &= \begin{pmatrix} 1.3(0.6) & -0.1(0.8) \\ 0.9(0.6) & 0.5(0.8) \\ -1.0(0.6) & 0.7(0.8) \end{pmatrix} \\
&= \begin{pmatrix} 0.78 & -0.08 \\ 0.54 & 0.4 \\ -0.6 & 0.56 \end{pmatrix}
\end{aligned}
$$

For V_2 and V_3 we will have to calculate the activations and the corresponding pre-activations on each layer. However, we already did that in Example 3.1.29. More specifically,

$$
\begin{aligned}
\mathrm{a}(W, f)_3(x) &= 0.7 = \text{pre-a}(W, f)_3(x) \\
\mathrm{a}(W, f)_4(x) &= 0.94 = \text{pre-a}(W, f)_4(x) \\
\mathrm{a}(W, f)_5(x) &= 0 \\
\text{pre-a}(W, f)_5(x) &= -0.04
\end{aligned}
$$

Which indicates that:

$$
\begin{aligned}
V_2 &= \begin{pmatrix} 0.3\left(\frac{0.7}{0.7}\right) & -1.2\left(\frac{0.94}{0.94}\right) & 0.9\left(\frac{0}{-0.04}\right) \\ 0.7\left(\frac{0.7}{0.7}\right) & 0.2\left(\frac{0.94}{0.94}\right) & -0.3\left(\frac{0}{-0.04}\right) \\ 0.6\left(\frac{0.7}{0.7}\right) & 0.4\left(\frac{0.94}{0.94}\right) & -0.1\left(\frac{0}{-0.04}\right) \end{pmatrix} \\
&= \begin{pmatrix} 0.3 & -1.2 & 0 \\ 0.7 & 0.2 & 0 \\ 0.6 & 0.4 & 0 \end{pmatrix}
\end{aligned}
$$

Similarly, we obtain:

$$
\begin{aligned}
V_3 &= \begin{pmatrix} 0.4\left(\frac{0}{-0.918}\right) & -1.1\left(\frac{0.678}{0.678}\right) & -0.7\left(\frac{0.796}{0.796}\right) \\ -0.8\left(\frac{0}{-0.918}\right) & 0.6\left(\frac{0.678}{0.678}\right) & 0.2\left(\frac{0.796}{0.796}\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1.1 & -0.7 \\ 0 & 0.6 & 0.2 \end{pmatrix}
\end{aligned}
$$

Now we will compute $\Psi(W_x^f, 1)$ on $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have that

$$
V_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.78 - 0.08 \\ 0.54 + 0.04 \\ -0.6 + 0.56 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.94 \\ -0.04 \end{pmatrix} = W_1 x
$$

On the second layer,

$$
V_2 \begin{pmatrix} 0.7 \\ 0.94 \\ -0.04 \end{pmatrix} = \begin{pmatrix} 0.3(0.7) - 1.2(0.94) \\ 0.7(0.7) + 0.2(0.94) \\ 0.6(0.7) + 0.4(0.94) \end{pmatrix} = \begin{pmatrix} -0.918 \\ 0.678 \\ 0.796 \end{pmatrix} = W_2 \left(\mathrm{ReLU}(W_1 x)\right)
$$

And on the third layer we obtain,

$$
\begin{aligned}
\Psi(W_x^f, 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= V_3 \begin{pmatrix} -0.918 \\ 0.678 \\ 0.796 \end{pmatrix} \\
&= \begin{pmatrix} -1.1(0.678) - 0.7(0.796) \\ 0.6(0.678) + 0.2(0.796) \end{pmatrix} \\
&= \begin{pmatrix} -1.303 \\ 0.566 \end{pmatrix} \\
&= \Psi(W, f)(x)
\end{aligned}
$$

Which again verifies the theorem.

Note that the combinatorial architectures of (W, f) and $(W_x^f, 1)$ are equal, and the weight and activation architectures of $(W_x^f, 1)$ are determined by the corresponding architectures of (W, f), and the outputs of (W, f) when x is percolating through it. In particular, we see that all the nonlinear parts of the activation functions can be encoded in a representation (a linear object). We end this section with a result on two isomorphic representations.

Corollary 3.2.10 ((ARMENTA; JODOIN, 2021), Corollary 6.5)**.** *Let x_1 and x_2 are two data vectors from a data sample D. If $W_{x_1}^f \cong W_{x_2}^f$ via $\tilde{G}$, then*

$$
\Psi(W, f)(x_1) = \Psi(W, f)(x_2)
$$

Proof. As $(W^f_{x_1}, 1) \cong (W^f_{x_2}, 1)$, we have that

$$\Psi(W^f_{x_1}, 1)(1^d) = \Psi(W^f_{x_2}, 1)(1^d)$$

Using Theorem 3.2.1 twice we get:

$$\begin{aligned} \Psi(W, f)(x_1) &= \Psi(W^f_{x_1}, 1)(1^d) \\ &= \Psi(W^f_{x_2}, 1)(1^d) \\ &= \Psi(W, f)(x_2) \end{aligned}$$

As desired. □

3.3 Stable Neural Networks

The last Corollary and Theorem in the prior section show that the isomorphism classes of thin quiver representations $\left[W^f_x\right]$ under the action of the group $\tilde{G}$ of neural networks represent the data and the output of (W, f). This induces the construction of a space whose points will be such isoclasses of quiver representations, which ends up being a *moduli space*.

In this section our objective is to define it, and with it to propose a different version of the manifold hypothesis ((GOODFELLOW; BENGIO; COURVILLE, 2016), Chapter 5.11.3) in the same way as did by (ARMENTA; JODOIN, 2021). The manifold hypothesis roughly states that many data of high-dimensions lie on manifolds of less dimension inside the input space; however, here we will describe an explicit map from the input space to the moduli space of a neural network and then we can translate the data manifold to the moduli space. More specifically, we will prove that the network function factors through the moduli space.

For doing so, we will put some restrictions.

Remark 3.3.1. We will assume that the weights of (W, f) and W^f_x are non-zero. We can assume this as the set where it happens is of measure zero, and we could add a number to make it non-zero sufficiently small so that it is still computable.

Remark 3.3.2. We will also assume for simplicity, that there are no bias vertices in Q. Remember that the group of change basis $\tilde{G}$ does not change the bias vertex, and as any input vertex can be extended to a vertex with one's the number of those vertices, such assumption can be made. It is also noteworthy that all the results from this section are also applied to neural networks with bias vertices.

For doing such formalization, we will use framed quiver representations and their dual concepts, co-framed representations; following the philosophy of (REINEKE, 2008a) for our case of neural networks.

Definition 3.3.3 ((ARMENTA; JODOIN, 2021), Definition 7.2)**.** *Let Q be a network quiver and $\widetilde{Q}$ its hidden quiver as in Definition 3.1.20. We will call **input vertices of $\widetilde{\mathbf{Q}}$** to the subset:*

$$\{v \in \widetilde{Q_0} \mid \exists\, \alpha \in Q_1 \text{ such that } t(\alpha) = v \text{ and } s(\alpha) \text{ is input of } Q\}$$

*and we call **output vertices of $\widetilde{\mathbf{Q}}$** to the set:*

$$\{w \in \widetilde{Q_0} \mid \exists\, \alpha \in Q_1 \text{ such that } s(\alpha) = w \text{ and } t(\alpha) \text{ is output of } Q\}$$

This is, the input vertices of $\widetilde{Q}$ are those vertices connected to the input vertices of Q, and the output vertices of $\widetilde{Q}$ are those connected to the output vertices of Q.

Example 3.3.4. Consider the network quiver Q:

$$Q: \quad \begin{tikzcd}
1 \arrow[dr] \arrow[drrr, bend left] & & & & 9 \\
& 4 \arrow[rr] \arrow[dr] & & 7 \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] \arrow[dr] & & 6 \arrow[ur] \arrow[dr] & & 10 \\
& 5 \arrow[rr] \arrow[ur] \arrow[drrr, bend right] & & 8 \arrow[ur] \arrow[dr] & \\
3 \arrow[ur] & & & & 11
\end{tikzcd}$$

We know that

$$\widetilde{Q}: \quad \begin{tikzcd}
4 \arrow[rr] \arrow[dr] & & 7 \\
& 6 \arrow[ur] \arrow[dr] & \\
5 \arrow[rr] \arrow[ur] & & 8
\end{tikzcd}$$

Then, according to the above definition, the vertices 4, 5 and 7 are input vertices of $\widetilde{Q}$ and the vertices 7, 8 and 5 are output vertices of $\widetilde{Q}$. Note also that 7 is not a source vertex nor vertex 5 is a sink.

Now, given $\widetilde{W}$ a thin representation of $\widetilde{Q}$, we fix the following families of vector spaces:

- $\{V_i\}_{i \in \widetilde{Q_0}}$, indexed by vertices of $\widetilde{Q}$, $V_i = \mathbb{C}^k$ when i is an output vertex of $\widetilde{Q}$ and 0 for any other $i \in \widetilde{Q_0}$.
- $\{U_j\}_{j \in \widetilde{Q_0}}$, also indexed by $\widetilde{Q_0}$, where $U_j = \mathbb{C}^d$ when j is an output vertex of $\widetilde{Q}$ and $U_j = 0$ for any other $j \in \widetilde{Q_0}$.

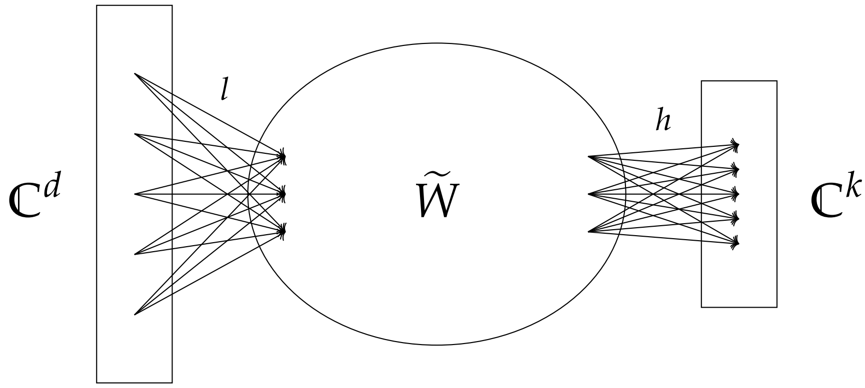


Figure 3 – An illustration of a double framed thin quiver representation. In the lateral boxes there are the vector spaces of the framing and co-framing. Taken from (ARMENTA; JODOIN, 2021).

With this in mind, we proceed to define our building blocks for the main objects of the section.

Definition 3.3.5 ((ARMENTA; JODOIN, 2021), Definition 7.4)**.** *Let $\widetilde{W}$ be a thin representation of $\widetilde{Q}$. Then,*

a. *A pair $(\widetilde{W}, h)$, where $h = \{h_i\}_{i \in \widetilde{Q_0}}$ is given by $h_i : \widetilde{W}_i \longrightarrow V_i$* for each $i \in \widetilde{Q_0}$ *will be called a **framed quiver representation** of $\widetilde{Q}$ by the family of vector spaces $\{V_i\}_{i \in \widetilde{Q_0}}$.*

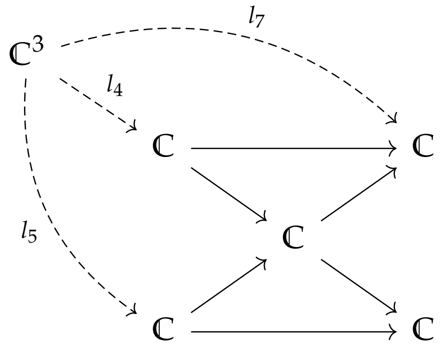
b. *A pair $(\widetilde{W}, l)$, where $h = \{l_j\}_{j \in \widetilde{Q_0}}$ is given by $l_j : U_j \longrightarrow \widetilde{W}_j$* for each $j \in \widetilde{Q_0}$ *is called a **co-framed quiver representation** of $\widetilde{Q}$ by the family of vector spaces $\{U_j\}_{j \in \widetilde{Q_0}}$.*

Note that by the definition of the vector spaces $\{V_i\}_{i \in \widetilde{Q_0}}$ and $\{U_j\}_{j \in \widetilde{Q_0}}$, we have that $h_i = 0$ when i is not an output vertex of $\widetilde{Q}$ and $l_j = 0$ when $j \in \widetilde{Q_0}$ is not an input vertex of $\widetilde{Q}$.

Example 3.3.6. Consider Example 3.3.4. Then, given a thin representation of $\widetilde{Q}$, called $\widetilde{W}$ and a choice of 3 non-zero maps $h_i : \mathbb{C} \longrightarrow \mathbb{C}^3$ for $i = 5, 7, 8$ will give a framed quiver representation for $\widetilde{Q}$. In a diagram,

$$\begin{tikzcd}
\mathbb{C} \arrow[rr] \arrow[dr] & & \mathbb{C} \arrow[dddr, dashed, bend left, "h_7"] & \\
& \mathbb{C} \arrow[ur] \arrow[dr] & & \\
\mathbb{C} \arrow[rr] \arrow[ur] \arrow[drrr, dashed, bend right, "h_5"'] & & \mathbb{C} \arrow[dr, dashed, "h_8"] & \\
& & & \mathbb{C}^3
\end{tikzcd}$$

where we omitted he zero morphisms. Also, a choosing if non-zero maps $l_j : \mathbb{C} \longrightarrow \mathbb{C}$ for $j = 4, 5, 7$ will give a co-framed representation for $\widetilde{Q}$, this is



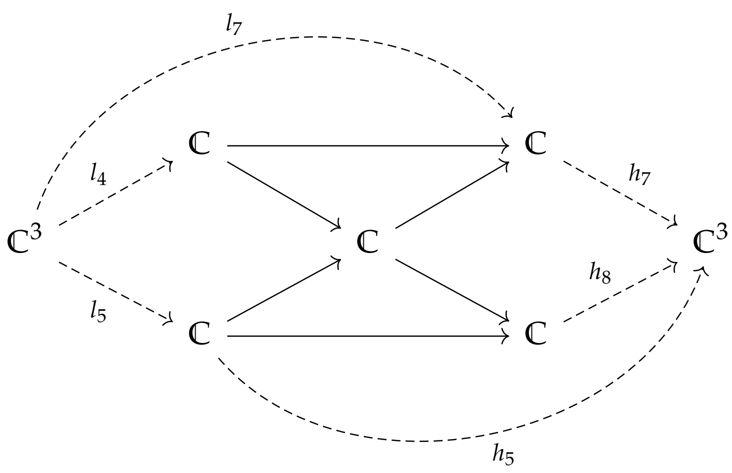
where again, we omitted the zero morphisms.

We define then our main objects.

Definition 3.3.7 ((ARMENTA; JODOIN, 2021), Definition 7.6)**.** *A **double framed thin quiver representation** for $\widetilde{Q}$ is a triple $(l, \widetilde{W}, h)$, where:*

- *$\widetilde{W}$ is a thin quiver representation of $\widetilde{Q}$, the hidden quiver;*
- *$(\widetilde{W}, l)$ is a co-framed representation of $\widetilde{Q}$ and*
- *$(\widetilde{W}, h)$ is a framed representation of $\widetilde{Q}$.*

Example 3.3.8. Consider the Example 3.3.6. Then $(l, \widetilde{W}, h)$ with $l = (l_4, l_5, l_6, l_7, l_8)$ and $h = (h_4, h_5, h_6, h_7, h_8)$ is a double framed thin quiver representation for Q,



where the zero morphisms are missing.

As before, there is group that acts on such representations in a "natural" way, but we should be a bit careful with the details.

Definition 3.3.9 ((ARMENTA; JODOIN, 2021), Definition 7.8)**.** *The **group of change basis** for double framed thin quiver representations is $\widetilde{G}$, the group of change of basis for neural networks. Given a $(l, \widetilde{W}, h)$, with $l_j = (l_j^1, \ldots, l_j^d)$, $h_i = (h_i^1, \ldots, h_i^k)$ and $\tau \in \widetilde{G}$, the **action of the group** will be given by*

$$\tau \cdot \left(l, \widetilde{W}, h\right) = \left(\tau \cdot l, \tau \cdot \widetilde{W}, \tau \cdot h\right)$$

where,

- *$\tau \cdot \widetilde{W}$ is the same as defined in Definition 3.1.25 a..*
- *$(\tau \cdot l)_j = \left(l_j^1 \tau_j, \cdots, l_j^d \tau_j\right)$ for all $j \in \widetilde{Q_0}$, and*
- *$(\tau \cdot h)_i = \left(\dfrac{h_i^1}{\tau_i}, \cdots, \dfrac{h_i^k}{\tau_i}\right)$ for all $i \in \widetilde{Q_0}$.*

We note that every double framed thin quiver representation for $\widetilde{Q}$ can be seen as a

$$\tau \cdot \left(l, \widetilde{W}, h\right) \text{ for some } \tau \in \widetilde{G}$$

With this we proceed to prove the first big result of the section: we can study isoclasses of double framed thin representations in the attempt of studying isoclasses of thin quiver representations $\left[W_x^f\right]$.

Theorem 3.3.1 ((ARMENTA; JODOIN, 2021), Theorem 7.9)**.** *There is a 1-1 correspondence between the set of isomorphism classes of thin representations $[W]$ over the delooped quiver Q° via $\widetilde{G}$ and the set of isomorphism classes $\left[(l, \tilde{W}, h)\right]$ of double framed thin quiver representations of $\tilde{Q}$.*

Proof. We first note that the change of basis group is the same, so the isoclasses will be given by the same action. The correspondence is as follows:

- Suppose we have a thin representation W of Q°. Then, by removing the input and output layers of Q we obtain a thin representation $\widetilde{W}$ of $\widetilde{Q}$. We just need to construct the maps l and h, but these are given by the weights starting on input vertices of Q for the map l, and by the weights ending on output vertices of Q for the map h (here we considered the input vertices as coordinates of $\mathbf{C}^d$, and the output as coordinates of $\mathbb{C}^k$).
- Now, given a double framed thin quiver representation $(l, \widetilde{W}, h)$, the entries l become the weights of a thin quiver representation starting on input vertices, and dually for the entries of h. The weights of $\widetilde{W}$ will define the hidden weights of W.

□

The proof of this theorem allows to do two things:

1. Identify easily a thin representation W of Q° with a double framed thin quiver representation of $\widetilde{Q}$.

2. Identify the isoclasses
$$[W] = \left[(l, \widetilde{W}, h)\right]$$

We will use such identifications indistinctively whenever there is no risk to confusion.

Our objective becomes to study the space of isomorphism classes of all double framed thin quiver representations of $\widetilde{Q}$. By a work (NAKAJIMA, 1998), it is known that such varieties of quivers don't behave well, its topology is not Hausdorff in most cases. Then we proceed to use the stability from (KING, 1994), as in Definition 2.2.1, one that he showed to have a good structure. We become interested now to study the *stable* double framed thin quiver representations.

In (REINEKE, 2008a) was given a good description of those spaces for a framing (called framed moduli spaces), and was introduced a definition of stability that coincides with King's one. Here we give such definition as in (ARMENTA; JODOIN, 2021), where we do the framing and the co-framing at the same time, and we use the dual stability for the co-framing.

Definition 3.3.10 ((ARMENTA; JODOIN, 2021), Definition 7.12)**.** *Let $(l, \widetilde{W}, h)$ a double framed thin quiver representation. We say that $(l, \widetilde{W}, h)$ is* ***stable*** *if:*

a. The only subrepresentation U of $\widetilde{W}$ contained in $\ker(h)$ *is* 0*, and*

b. The only subrepresentation V of $\widetilde{W}$ containing $\operatorname{im}(l)$ *is* $\widetilde{W}$*.*

We make a clarification on the above definition. Suppose we have a double framed thin representation $(l, \widetilde{W}, h)$. Then,

- l is defined by a family of maps of the type
$$\{l_i : \mathbb{C}^{n_i} \longrightarrow \widetilde{W_i} \,|\, i \in \widetilde{Q_0}\}$$
where $n_i = 0$ if i is not an input vertex of $\widetilde{Q}$ and $n_i = d$ when i is an input vertex of $\widetilde{Q}$. Then $\operatorname{im}(l)$ will be
$$\operatorname{im}(l) = (\operatorname{im}(l_i))_{i \in \widetilde{Q_0}}, \text{ where } \operatorname{im}(l_i) \subset \widetilde{W_i}$$
a family of vector spaces indexed by $\widetilde{Q_0}$.

- h is also defined as a family of maps

$$\{h_j : \widetilde{W_j} \longrightarrow \mathbb{C}^{n_j} \mid j \in \widetilde{Q_0}\}$$

where $n_j = k$ when j is an output vertex of $\widetilde{Q}$ and 0 in any other case. Note that $\ker(h)$ is inside the representation $\widetilde{W}$. Moreover,

$$\ker(h) = \big(\ker(h_j)\big)_{j\in\widetilde{Q_0}}, \text{ where } \ker(h_j) \subset \widetilde{W}_j$$

is a family of vector spaces indexed by $\widetilde{Q}$.

On Definition 3.3.10 we assumed without loss of generality that the weights on the input layer of W_x^f (see Remark 3.3.1), and that all maps h_j are non-zero for every output vertex j of $\widetilde{Q}$, as the set where those events occur are of measure zero, and it is possible (again) to add an $\eta > 0$ sufficiently small and computable. Similarly, we will assume that all maps l_i are non-zero for every input vertex i of $\widetilde{Q}$.

Theorem 3.3.2 ((ARMENTA; JODOIN, 2021), Theorem 7.13)**.** *Let (W, f) be a neural network and $x \in \mathbb{C}^d$ a data vector from a labeled data set D. Then the double framed thin quiver representation W_x^f is stable.*

Proof. For proving that $W_x^f = (l, \widetilde{W}, h)$ is stable, we are going to prove a. and b..

Let $i \in Q_0$ an output vertex, then $h_i : \mathbb{C} \longrightarrow \mathbb{C}^k$ is a linear map, so $\ker h_i \in \{0, \mathbb{C}\}$. If $\ker h_i = \mathbb{C}$, then $h_i = 0$, but by hypothesis we have $h_i \neq 0$. This implies $\ker(h_i) = 0$. Let us suppose that $U = (U_j)_{j\in Q_0}$ is a subrepresentation of $\widetilde{W}$ such that $U \subset \ker(h)$. We obtain that $U_i \subset \ker(h_i) = 0$ for each output vertex i. If all the vertices are output vertices of $\widetilde{W}$ we are done, as we showed that each vector space (at an output vertex) of the subrepresentation U must be zero.

Let l be a vertex that is not output. As $U_l \subset \ker(h_l)$, it could be 0 or $\mathbb{C}$. If all vertices have as vector spaces 0 then we are also done, as U must be the zero representation. Suppose that there are vertices such that its corresponding vector space is $\mathbb{C}$. Note that among those vertices, there exists a vertex l and an arrow $\alpha : l \longrightarrow t(\alpha)$ such that $U_{t(\alpha)} = 0$, as the network quiver is connected and W_x^f has at least one output vertex (where its corresponding vector space is 0). This means that we have a critic of type I at the vertex l, as shown by the Figure 4. A contradiction because U is a subrepresentation, and there exists an injective morphism from U to $\widetilde{W}$. Then, there are no vertices where its corresponding vector space is $\mathbb{C}$, and this shows a..

Given an input vertex $j \in Q_0$ we have $l_j : \mathbb{C}^d \longrightarrow \mathbb{C}$ a linear map, and then $\operatorname{im}(l_j) \in \{0, \mathbb{C}\}$. If $\operatorname{im}(l_j) = 0$, $l_j = 0$, and as by hypothesis all $l_j \neq 0$ for output vertices j, we get

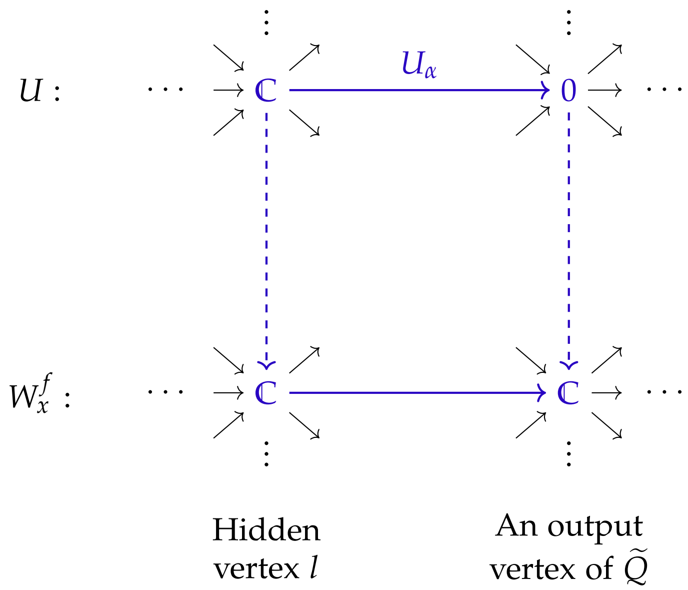


Figure 4 – The subrepresentation $U \subset \ker(h)$ of W_x^f, and its corresponding critic of type I, shown in blue.

$\mathrm{im}(l_j) = \mathbb{C}$. Suppose we are given a $V \subset \widetilde{W}$ such that $\mathrm{im}(l) \subset V$, we know then that for any input vertex j,

$$\mathbb{C} = \mathrm{im}(l_j) \subset V_j$$

which implies $V_j = \mathbb{C}$ for each input vertex j.

Once again, if all vector spaces of V are $\mathbb{C}$, or if all vertices from $\widetilde{W}$ are input vertices, we are done. Suppose that there are vertices such that its corresponding vector space is 0. We can find then an input vertex j' and an arrow $\beta : j' \longrightarrow t(\beta)$ such that $V_{t(\beta)} = 0$. We obtain again a critic of type I at the vertex j', as shown by the Figure 5. As before this is a contradiction and with it b. is proved. $\square$

Now we proceed to define some last notions that will help us to achieve our expected objective.

Definition 3.3.11 ((ARMENTA; JODOIN, 2021), Definition 7.14)**.** *We will denote by $_d\mathcal{R}_k(\widetilde{Q})$ the space of* ***all double framed thin quiver representations****. Similarly, the* ***moduli space of double framed thin quiver representations*** *of $\widetilde{Q}$ is*

$$_d\mathcal{M}_k(\widetilde{Q}) := \{[V] \mid V \in {}_d\mathcal{R}_k(\widetilde{Q}) \text{ is stable}\}$$

Note that the elements of the moduli space are isoclasses of stable double framed thin quiver representations of $\widetilde{Q}$ over the action of the group $\widetilde{G}$ of neural networks. Now,

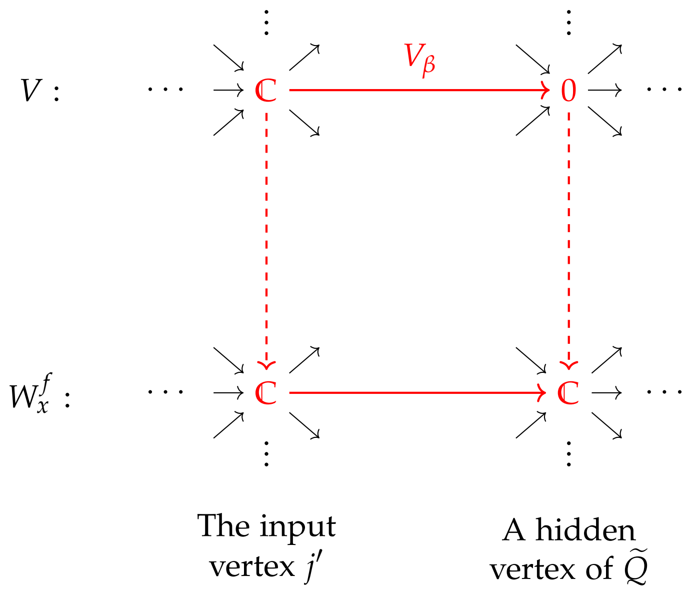


Figure 5 – The subrepresentation $\operatorname{im}(l) \subset V$ of W_x^f, and its corresponding critic of type I, shown in red.

Definition 3.3.12 ((ARMENTA et al., 2022), p. 16)**.** *Let (W, f) be a neural network. Then, the* ***knowledge map*** *of (W, f) is*

$$\begin{aligned} \varphi(W,f)\colon \mathbb{C}^d &\longrightarrow {}_d\mathcal{R}_k(\widetilde{Q}) \\ x &\longmapsto W_x^f \end{aligned}$$

By Theorem 3.3.2, if all the weights are non-zero then the knowledge map takes values into the moduli space:

$$\begin{aligned} \varphi(W,f)\colon \mathbb{C}^d &\longrightarrow {}_d\mathcal{M}_k(\widetilde{Q}) \\ x &\longmapsto \left[W_x^f\right] \end{aligned}$$

Now given $[V] \in {}_d\mathcal{R}_k(\widetilde{Q})$, we can define a map:

$$\widehat{\Psi} : {}_d\mathcal{M}_k(\widetilde{Q}) \longrightarrow \mathbb{C}$$

by $\widehat{\Psi}\left([V]\right) := \Psi(V,1)(1^d)$. We see that $\widehat{\Psi}$ is well-defined as the election of the representative does not alter the value of the network function by Theorem 3.1.1. With this,

Corollary 3.3.13 ((ARMENTA; JODOIN, 2021), Corollary 7.18)**.** *The network function of a neural network can be written as:*

$$\Psi(W,f) = \widehat{\Psi} \circ \varphi(W,f)$$

Which is equivalent to say that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}^d \arrow[r, "\Psi(W,f)"] \arrow[d, "\varphi(W,f)"'] & \mathbb{C}^k \\
{}_d\mathcal{M}_k(\widetilde{Q}) \arrow[ur, "\widehat{\Psi}"'] &
\end{tikzcd}$$

Proof. By Theorem 3.2.1, given an $x \in \mathbb{C}^d$ we have:

$$\begin{aligned}
\widehat{\Psi} \circ \varphi(W,f)(x) &= \widehat{\Psi}\left[W_x^f\right] \\
&= \Psi(W_x^f, 1)(1^d) \\
&= \Psi(W,f)(x)
\end{aligned}$$

Which shows the corollary. □

The last result shows that all the decisions of (W,f) pass through the moduli space and that this does not depend on the architecture, nor the data. We can also mention that this has been generalized to a suitable functoriality on the neural network ((ARMENTA et al., 2022), Theorem 7.3).

One interesting remark of last corollary ((ARMENTA; JODOIN, 2021), Section 7.1, Consequence 3) is that the data manifold in the input space of (W,f), denoted $\mathcal{M}$ is taken via $\varphi(W,f)$ to

$$\varphi(W,f)(\mathcal{M}) \subset {}_d\mathcal{M}_k(\widetilde{Q})$$

That ends up being a subspace of the moduli space, which parametrizes all the outputs that the neural network (W,f) produces on $\mathcal{M}$. In particular, we want to compute the dimension of such a space, and this will give an insight into the variation of the manifold hypothesis. However, even if we don't know exactly what it is, we have a bound given by the following and last theorem of the chapter, which will be proved with more generality in Chapter 4.

Theorem 3.3.3 ((ARMENTA; JODOIN, 2021), Theorem 7.16)**.** *Let Q a network quiver. There exists a geometric quotient ${}_d\mathcal{M}_k(\widetilde{Q})$ of ${}_d\mathcal{R}_k(\widetilde{Q})$ by the action of $\widetilde{Q}$, which is non-empty and with complex dimension*

$$\dim_{\mathbb{C}}\left({}_d\mathcal{M}_k(\widetilde{Q})\right) = |Q_1^\circ| - |\widetilde{Q_0}|$$

4 The Moduli Space of Neural Networks

The last results from Chapter 3 say in particular that the space constructed is a quotient as in Subsection 1.4.3, and then it induces the question of studying the properties of such a space via some algebraic geometry tools. In (REINEKE, 2008a) some study of moduli spaces of quiver representations were studied, introducing the concept of a **framed moduli space**, describing also different interpretations of the space. We briefly mentioned this word in Definition 3.3.7, which is in fact related to the results obtained in (REINEKE, 2008a).

Using those tools, and inspired on the study of moduli spaces that appears in the context of neural networks, in (ARMENTA et al., 2022) an extensive study of the geometric properties of a more general space was made (Q does not need to be a network quiver, for example). In this last chapter, our intention is to develop the first part of such text corresponding to Chapters 2, 3 and 4, to present some examples on the theory and to make a relation with the space ${}_d\mathcal{M}_k(Q)$. Lastly, we comment on some possible implications of the results presented in this text.

4.1 Deframing

Inspired by the GIT approach (mentioned on the preliminaries) and the construction of the moduli space of neural networks as in the chapter before, we want to study the space of classes of isomorphisms of representations under the action of a determined group. However, as we mentioned before, the group may be large and then such study may be hard. In this section our objective is to show that in the case of finite acyclic quivers (for example, the hidden quiver of a network quiver) we can study the corresponding representation space as one of a *deframed* quiver. This means then that we can reduce the problem of studying the moduli space for the quiver in terms of another with no sinks or sources, and with oriented cycles.

We start by fixing some notation and expanding some definitions from last chapter.

From now on, let Q be an acyclic finite quiver, s_Q its set of sources and t_Q its set of sinks.

Definition 4.1.1. *The **hidden quiver** $\widetilde{Q}$ is the subquiver of Q with all arrows between the set of vertices: $\widetilde{Q_0} = Q_0 - s_Q - t_Q$. This is, the full subquiver without sinks or sources.*

We already gave some examples of hidden quivers when Q is a network quiver. In this case, we have that s_Q is the set of bias and input vertices and t_Q is the set of output vertices. However, note that this definition is wider as gives the option of having multiple arrows, and the quivers may not be arranged by layers, for example. We fix a dimension vector $\mathbf{d} = (d_i)_{i \in Q_0}$ of Q and complex vector spaces V_i, with $\dim(V_i) = d_i$ for $i \in Q_0$.

Definition 4.1.2. *The* ***representation space of*** *Q will be*

$$\mathcal{R}_{\mathbf{d}}(Q) = \bigoplus_{\alpha: i \to j} \mathrm{Hom}(V_i, V_j)$$

also called the ***variety of complex representations of dimension vector d****. Its* ***base change group*** *is*

$$G_{\mathbf{d}}(Q) = \prod_{i \in Q_0} \mathrm{GL}(V_i)$$

which acts on $\mathcal{R}_{\mathbf{d}}(Q)$ by

$$\begin{aligned} G_{\mathbf{d}}(Q) \times \mathcal{R}_{\mathbf{d}}(Q) &\longrightarrow \mathcal{R}_{\mathbf{d}}(Q) \\ \big((\tau_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1}\big) &\longmapsto (\tau_j V_\alpha \tau_i^{-1})_{\alpha: i \to j} \end{aligned}$$

And this can be seen in a diagram as follows:

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "\tau_i"'] & V_j \arrow[d, "\tau_j"] \\ V_i \arrow[r, dashed, "\tau_j V_\alpha \tau_i^{-1}"'] & V_j \end{tikzcd}$$

We also have the corresponding subgroup

$$G_{\mathbf{d}}(\widetilde{Q}) = \{(\tau_i)_{i \in Q_0} \mid \tau_i = 1 \text{ for all } i \notin \widetilde{Q_0}\}$$

So this group can only change the space when one of the spaces corresponds to a non-hidden vertex. Note the similarity with the corresponding group in last chapter. The main objective of this chapter is to study $G_{\mathbf{d}}(\widetilde{Q})-$orbits in $\mathcal{R}_{\mathbf{d}}(Q)$, and to give a description on the space of the orbits along with some of its properties. We explore a little bit on the representation spaces of the hidden quiver and its relation with Q.

Definition 4.1.3 ((ARMENTA et al., 2022), Definition 2.1). *Let $i \in \widetilde{Q_0}$, and $s_Q = \{s_1, \dots, s_k\}$, $t_Q = \{t_1, \dots, t_q\}$*

$$U_i = \bigoplus_{\substack{\alpha:\, t(\alpha)=i \\ s(\alpha)\in s_Q}} V_{s(\alpha)} = \bigoplus_{\alpha:\, s_k \to i} V_{s_k}$$

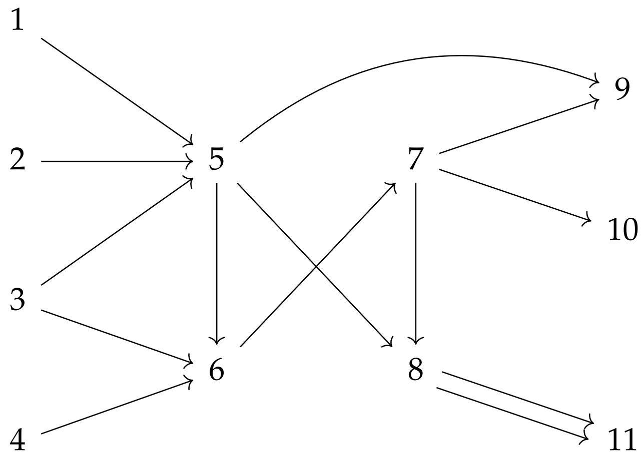
then U_i has a copy of each V_s whenever there is an arrow from a source to the vertex i. Similarly,

$$W_i = \bigoplus_{\substack{\alpha:\, s(\alpha)=i \\ t(\alpha)\in t_Q}} V_{t(\alpha)} = \bigoplus_{\alpha:\, i \to t_l} V_{t_l}$$

and W_i has a copy of each space V_t whenever there is an arrow from i to a sink vertex t. We write $u_i := \dim U_i$ and $w_i = \dim W_i$.

Remark 4.1.4. Note that in particular, if s is a source from $\widetilde{Q}$ then $U_s \neq 0$, or else it would be a source from Q, and it will not belong to $\widetilde{Q}$. Similarly, $W_t \neq 0$ for every sink t from $\widetilde{Q}$.

Example 4.1.5. Consider the quiver Q shown:



We know that $\widetilde{Q}$ is given by:

$$\widetilde{Q} = \begin{tikzcd} 5 \arrow[d] \arrow[dr] & 7 \arrow[d] \\ 6 \arrow[ur] & 8 \end{tikzcd}$$

Suppose we have a thin representation of Q, this is $\mathbf{d} = (1, \ldots, 1)$, or equivalently, $M = (V_i)_{i \in Q_0} = (\mathbb{C})_{i \in Q_0}$. Noting that $s_Q = \{1, 2, 3, 4\}$, $t_Q\{9, 10, 11\}$:

$$U_5 = \bigoplus_{\substack{\alpha\colon t(\alpha)=5 \\ s(\alpha)\in s_Q}} V_{s(\alpha)} = \bigoplus_{\substack{\alpha\colon t(\alpha)=5 \\ s(\alpha)\in s_Q}} \mathbb{C} = \mathbb{C}^3$$

as there are only 3 arrows with that condition. Similarly,

$$U_6 = \bigoplus_{\substack{\alpha\colon t(\alpha)=6 \\ s(\alpha)\in s_Q}} \mathbb{C} = \mathbb{C}^2$$

Lastly, we have

$$U_7 = \bigoplus_{\substack{\alpha\colon t(\alpha)=7 \\ s(\alpha)\in s_Q}} \mathbb{C} = 0 = U_8$$

And we obtain directly that $u_5 = 3$, $u_6 = 2$, $u_7 = u_8 = 0$. Following a similar process we can also get the W vector spaces, namely,

$$W_5 = \bigoplus_{\substack{\alpha:\, s(\alpha)=5 \\ t(\alpha)\in t_Q}} V_{t(\alpha)} = \bigoplus_{\substack{\alpha:\, s(\alpha)=5 \\ t(\alpha)\in t_Q}} \mathbb{C} = \mathbb{C}$$
$$W_6 = \bigoplus_{\substack{\alpha:\, s(\alpha)=6 \\ t(\alpha)\in t_Q}} \mathbb{C} = 0$$
$$W_7 = \bigoplus_{\substack{\alpha:\, s(\alpha)=7 \\ t(\alpha)\in t_Q}} \mathbb{C} = \mathbb{C}^2 = W_8$$

This also implies that $w_5 = 1$, $w_6 = 0$, $w_7 = w_8 = 2$.

Note that if we changed our thin representation for another one, it directly changes the spaces U and W. When the representation is thin, we can obtain the values u, v by just counting the arrows on the quiver that have the property under the sum. Now we can group all maps that go from a source to any vertex i in $\widetilde{Q}_0$ into one:

$$l_i = (V_\alpha)_{\alpha:\, s_k \to i} : U_i \longrightarrow V_i$$

and similarly all the maps that go from a vertex $j \in \widetilde{Q}_0$ to a sink:

$$h_j = (V_\alpha)_{\alpha:\, j \to t_l} : V_j \longrightarrow W_j$$

In our running example we have,

Example 4.1.6. Consider Example 4.1.5, then we can compress the morphisms into:

$$\begin{tikzcd}
& & & \mathbb{C} \\
\mathbb{C}^3 \arrow[r, "f_5"] & \mathbb{C} \arrow[urr, bend left, "h_5"] \arrow[dr] & \mathbb{C} \arrow[r, "h_7"] & \mathbb{C}^2 \\
\mathbb{C}^2 \arrow[r, "f_6"] & \mathbb{C} \arrow[ur] & \mathbb{C} \arrow[r, "h_8"] & \mathbb{C}
\end{tikzcd}$$

and where we did not draw the zero morphisms. We note the similarity with Example 3.3.8.

We remember that we have a dimension vector $\mathbf{d}$ of Q fixed, and then we can obtain a dimension vector for $\widetilde{Q}$ by just restricting it, and it will be called $\widetilde{\mathbf{d}}$. With this construction, we can make a description of the representation space of Q with dimension vector $\mathbf{d}$.

Lemma 4.1.7 ((ARMENTA et al., 2022), Lemma 2.2)**.** *We obtain an isomorphism of affine spaces:*

$$\mathcal{R}_{\boldsymbol{d}}(Q) \cong \mathcal{R}_{\widetilde{\boldsymbol{d}}}(\widetilde{Q}) \times \bigoplus_{i\in\widetilde{Q_0}} \operatorname{Hom}(U_i, V_i) \times \bigoplus_{i\in\widetilde{Q_0}} \operatorname{Hom}(V_i, W_i)$$

and we describe the action of $G_{\mathbf{d}}(\widetilde{Q})$ in these terms as:

$$(\tau_i)_{i\in\widetilde{Q}_0}\cdot\Big((V_\alpha)_{\alpha\in\widetilde{Q}_1},(l_i)_{i\in\widetilde{Q}_0},(h_i)_{i\in\widetilde{Q}_0}\Big) \;=\; \Big((\tau_j V_\alpha \tau_i^{-1})_{\alpha\colon i\to j},(\tau_i l_i)_{i\in\widetilde{Q}_0},(h_i\tau_i^{-1})_{i\in\widetilde{Q}_0}\Big)$$

Proof. The isomorphism is naturally given by:

$$(V_\alpha)_{\alpha\in Q_1}\longmapsto\Big((V_\alpha)_{\alpha\in\widetilde{Q}_1},(l_i)_{i\in\widetilde{Q}_0},(h_i)_{i\in\widetilde{Q}_0}\Big)$$

as the spaces $\mathrm{Hom}(U_i,V_i)$ and $\mathrm{Hom}(V_i,W_i)$ compensate for all the morphisms between the hidden quiver and the sink and sources vertices. The action is obtained by its definition, and using the fact that on sinks and sources the corresponding τ_i is the identity. □

Example 4.1.8. Consider the quiver

$$Q=\begin{tikzcd} 1 \arrow[dr] & & & & 7 \\ 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[ur] \arrow[dr] & \\ 3 \arrow[ur] & & & & 8 \end{tikzcd}$$

and the dimension vector $\mathbf{d}=(1,\dots,1)$. This is, a thin representation. Then we have that:

$$\mathcal{R}_{\mathbf{d}}(Q)=\bigoplus_{\alpha\colon i\to j}\mathrm{Hom}(\mathbb{C},\mathbb{C})=\bigoplus_{\alpha\colon i\to j}\mathbb{C}\cong\mathbb{C}^7$$

as $|Q_1|=7$. We can see that $\widetilde{Q}=\mathbb{A}_3$, and then, by restricting the dimension vector we obtain $\mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q})\cong\mathbb{C}^2$ as $\mathbb{A}_3$ has 2 arrows. In particular, $\dim\mathcal{R}_{\mathbf{d}}(Q)=7$ and $\dim\mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q})=2$. On other side,

$$\begin{aligned}\bigoplus_{i\in\widetilde{Q}_0}\mathrm{Hom}(U_i,V_i)&=\mathrm{Hom}(U_4,V_4)\oplus\mathrm{Hom}(U_5,V_5)\oplus\mathrm{Hom}(U_6,V_6)\\ &=\mathrm{Hom}(\mathbb{C}^3,\mathbb{C})\oplus\mathrm{Hom}(0,\mathbb{C})\oplus\mathrm{Hom}(0,\mathbb{C})\\ &=\mathrm{Hom}(\mathbb{C}^3,\mathbb{C})\cong\mathbb{C}^3\end{aligned}$$

and we obtain that its dimension is 3. Similarly,

$$\begin{aligned}\bigoplus_{i\in\widetilde{Q}_0}\mathrm{Hom}(V_i,W_i)&=\mathrm{Hom}(V_4,W_4)\oplus\mathrm{Hom}(V_5,W_5)\oplus\mathrm{Hom}(V_6,W_6)\\ &=\mathrm{Hom}(\mathbb{C},0)\oplus\mathrm{Hom}(\mathbb{C},0)\oplus\mathrm{Hom}(\mathbb{C},\mathbb{C}^2)\\ &=\mathrm{Hom}(\mathbb{C},\mathbb{C}^2)\cong\mathbb{C}^2\end{aligned}$$

and its dimension is 2. This shows that

$$\begin{aligned}7=\dim\mathcal{R}_{\mathbf{d}}(Q)&=\dim\mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q})+\dim\bigoplus_{i\in\widetilde{Q}_0}\mathrm{Hom}(U_i,V_i)+\dim\bigoplus_{i\in\widetilde{Q}_0}\mathrm{Hom}(V_i,W_i)\\ &=2+3+2\end{aligned}$$

And which shows the isomorphism of the theorem.

Example 4.1.9. Consider again Example 4.1.5, then we obtain that $\dim \mathcal{R}_{\mathbf{d}}(Q) = 14$, and

$$\begin{aligned} \dim \mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q}) &= 4 \\ \dim \bigoplus_{i\in\widetilde{Q}_0} \mathrm{Hom}(U_i, V_i) &= 3+2+0+0 = 5 \\ \dim \bigoplus_{i\in\widetilde{Q}_0} \mathrm{Hom}(V_i, W_i) &= 1+0+2+2 = 5 \end{aligned}$$

We get,

$$\begin{aligned} 14 = \dim \mathcal{R}_{\mathbf{d}}(Q) &= \dim \mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q}) + \dim \bigoplus_{i\in\widetilde{Q}_0} \mathrm{Hom}(U_i, V_i) + \dim \bigoplus_{i\in\widetilde{Q}_0} \mathrm{Hom}(V_i, W_i) \\ &= 4+5+5 \end{aligned}$$

Which shows once again the desired isomorphism.

As we said, we reduce the problem of studying those orbits of the quiver $\widetilde{Q}$ to another quiver. We present its definition.

Definition 4.1.10 ((ARMENTA et al., 2022), Definition 2.3)**.** *The* ***deframed quiver*** Q' ***with respect to the dimension*** $\mathbf{d}$ *is the quiver with vertices:*

$$Q'_0 = \widetilde{Q}_0 \cup \{\infty\}$$

and set of arrows:

$$Q'_1 = \widetilde{Q}_1 \cup \beta \cup \gamma$$

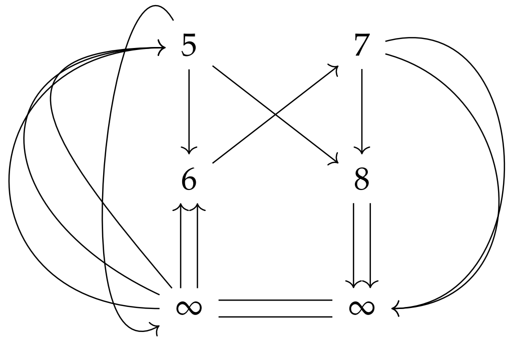
Where

$$\begin{aligned} \beta &= \{\beta_{i,k} : \infty \longrightarrow i \,|\, i \in \widetilde{Q}_0,\, k = 1, \dots, u_i\} \\ \gamma &= \{\gamma_{i,l} : i \longrightarrow \infty \,|\, i \in \widetilde{Q}_0,\, l = 1, \dots, w_i\} \end{aligned}$$

And its ***dimension vector*** $\mathbf{d}'$ *will be* $d'_i = d_i$ *for* $i \in \widetilde{Q}_0$ *and* $d'_\infty = 1$.

Note that the definition is independent of the representation, as we just use the vector $\mathbf{d}$ for counting the number of arrows that enter or leave the vertex ∞, and this just makes use of the definition of U_i and W_i, whose dimensions can be counted only based on the vector $\mathbf{d}$.

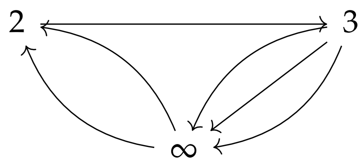
Example 4.1.11. Let again be under the conditions of Example 4.1.5. In this case, if we have a thin representation (i.e. $\mathbf{d}$ is the vector of 11 ones), we obtain that

$Q' =$ 

And where the zero morphisms are not drawn.

In the following example we show the importance of the dimension vector when constructing the deframed quiver.

Example 4.1.12. Let Q be the $\mathbb{A}_4$ quiver, then $\widetilde{Q}$ is the $\mathbb{A}_2$ quiver. If we have $\mathbf{d} = (2, 1, 0, 3)$, then $u_2 = 2$, $u_3 = 0$ and $w_2 = 0$, $w_3 = 3$. This implies that

$Q' =$ 

In particular, the representation spaces of dimension vector $\mathbf{d}$ for Q and of the deframed quiver with the corresponding dimension vector are the same.

Lemma 4.1.13 ((ARMENTA et al., 2022), Lemma 2.4)**.** *There exists an isomorphism*

$$\mathcal{R}_{\boldsymbol{d}}(Q) \cong \mathcal{R}_{\boldsymbol{d'}}(Q')$$

and this isomorphism preserves the orbits of the action of $G_{\boldsymbol{d}}(\widetilde{Q})$*, as a subgroup of* $G_{\boldsymbol{d'}}(Q')$*.*

Proof. We can obtain a representation of Q' of dimension vector $\mathbf{d}'$ by giving:

(1) a representation from $\mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q})$,

(2) vectors $v_{i,k} \in V_i$ that represents the arrows $\beta_{i,k}$ from β,

(3) covectors $\varphi_{i,l} \in \mathrm{Hom}(V_i, \mathbb{C}) = V_i^*$ that represents the arrows $\gamma_{i,l}$ from γ.

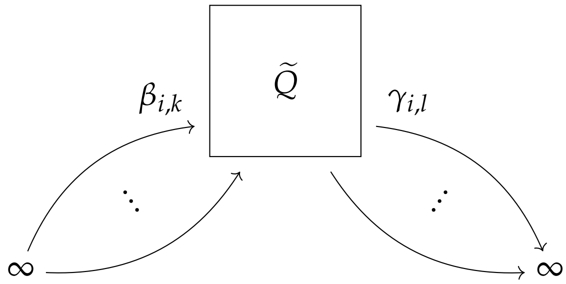


Figure 6 – Obtaining a representation of $\mathcal{R}_{\mathbf{d}'}(Q')$ by the choosing of an element in $\mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q})$ with vectors and covectors, represented at the vertex i.

This can be seen in Figure 6, where the vectors and covectors define the arrows of the corresponding representation. Now choose bases for the spaces U_i and W_i for each $i \in \widetilde{Q_0}$. Then, we can collect all the vectors $v_{i,k} \in V_i$ into a function $l_i : U_i \longrightarrow V_i$, as there are u_i elements of type $v_{i,k}$, and then we obtain a

$$l = (l_i)_{i \in \widetilde{Q_0}} \in \bigoplus_{i \in \widetilde{Q_0}} \operatorname{Hom}(U_i, V_i)$$

Similarly, we can collect all covectors $\varphi_{i,l} \in V_i^*$ and construct a map $h_i : V_i \longrightarrow W_i$ and we get a

$$h = (h_i)_{i \in \widetilde{Q_0}} \in \bigoplus_{i \in \widetilde{Q_0}} \operatorname{Hom}(V_i, W_i)$$

This induces a natural morphism

$$\mathcal{R}_{\mathbf{d}'}(Q') \longrightarrow \mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q}) \times \bigoplus_{i \in \widetilde{Q_0}} \operatorname{Hom}(U_i, V_i) \times \bigoplus_{i \in \widetilde{Q_0}} \operatorname{Hom}(V_i, W_i)$$

And we can obtain the inverse morphism in a similar way. Then, by Lemma 4.1.7, $\mathcal{R}_{\mathbf{d}}(Q) \cong \mathcal{R}_{\mathbf{d}'}(Q')$, and the first part of the theorem is proved.

For the second part, we just have to look at the action of the base change group on $\mathcal{R}_{\mathbf{d}'}(Q')$. The base change group $G_{\mathbf{d}'}(Q')$ can be seen as $\mathbb{C}^* \times G_{\mathbf{d}}(\widetilde{Q})$, and the action will be given by

$$(\lambda, (\tau_i))_{i \in \widetilde{Q_0}} \cdot \Big((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (v_{i,k})_{i,k}, (\varphi_{i,l})_{i,l} \Big)$$
$$= \Big((\tau_j V_\alpha \tau_i^{-1})_{\alpha \in \widetilde{Q_1}}, (\lambda^{-1} v_{i,k} \tau_i)_{i,k}, (\lambda \varphi_{i,l} \tau_i^{-1})_{i,l} \Big)$$

We note that the action on the vertices for arrows in $\widetilde{Q_1}$ is the same, and it is possible to see the action on the vectors and covectors with the following diagram

$$\begin{tikzcd}
\mathbb{C} & \mathbb{C} \arrow[l, "\lambda"'] \arrow[r, "\lambda"] \arrow[d, bend right, "v_{i,k}"'] & \mathbb{C} \\
& V_i \arrow[u, bend right, "\varphi_{i,l}"'] \arrow[d, "\tau_i"] & \\
& V_i \arrow[uur, dashed, bend right, "\lambda\varphi_{i,l}\tau_i^{-1}"'] &
\arrow[from=1-1, to=3-2, dashed, bend right, "\lambda^{-1}v_{i,k}\tau_i"']
\end{tikzcd}$$

whenever $i \in \widetilde{Q_0}$. $\square$

We end this section by showing that the orbits of the spaces are the same, so we can reduce our objective to the one of studying $G_{\mathbf{d'}}(Q')-$orbits on $\mathcal{R}_{\mathbf{d'}}(Q')$:

Lemma 4.1.14. *The groups $G_{\mathbf{d}}(\widetilde{Q})$ and $G_{\mathbf{d'}}(Q')$ have the same orbits in $\mathcal{R}_{\mathbf{d}}(Q) \cong \mathcal{R}_{\mathbf{d'}}(Q')$.*

Proof. We note that as before, $G_{\mathbf{d'}}(Q') \cong \mathbb{C}^* \times G_{\mathbf{d}}(\widetilde{Q})$, and then it is enough to check that any additional action by a $\lambda \in \mathbb{C}^*$ of $G_{\mathbf{d}}(\widetilde{Q})$ can be seen as an element of $G_{\mathbf{d}}(\widetilde{Q})$. In fact, given a λ as before, define $g \in G_{\mathbf{d}}(\widetilde{Q})$ by $g_i = \lambda^{-1}\mathrm{id}_{V_i}$ for each $i \in \widetilde{Q_0}$.

An action of $G_{\mathbf{d'}}(Q') \cong \mathbb{C}^* \times G_{\mathbf{d}}(\widetilde{Q})$ in that way is given by:

$$(\lambda, \mathrm{id}) \cdot \Big((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (v_{i,k})_{i,k}, (\varphi_{i,l})_{i,l} \Big) = \Big((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (\lambda^{-1} v_{i,k})_{i,k}, (\lambda \varphi_{i,l})_{i,l} \Big)$$

and we can see it as the following action of $G_{\mathbf{d}}(\widetilde{Q})$,

$$\Big(\lambda, (g_i)_{i \in \widetilde{Q_0}}\Big) \cdot \Big((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (v_{i,k})_{i,k}, (\varphi_{i,l})_{i,l} \Big) = \Big((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (\lambda^{-1} v_{i,k})_{i,k}, (\lambda \varphi_{i,l})_{i,l} \Big)$$

So the actions are the same. If for instance, we would have some τ_i instead of the identity on the first equation, we can compose the two actions, and it still can be seen as an action of $G_{\mathbf{d}}(\widetilde{Q})$. This shows the lemma. $\square$

In the following sections we will proceed to study the space of isoclasses of $G_{\mathbf{d'}}(Q')-$orbits on $\mathcal{R}_{\mathbf{d'}}(Q')$, via the usual mode of a quotient space: that will give us the semisimple representations, or by eliminating the "bad" orbits as in Chapter 1, that will gives us the stable representations.

4.2 Moduli of Semisimple Representations

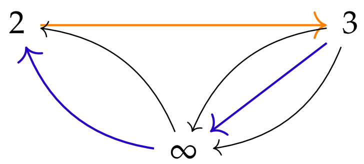
Usually, when the problem of studying orbits of a space appears it is common the technique of stability, and for quiver representations the moduli space defined by (KING, 1994) is more than enough to give good geometric properties, and to parameterize in an elegant way such spaces. However, when the quiver has oriented cycles, we saw that the space of semisimple representations is non-trivial, and that we can parameterize such space via those cycles. In this section we define them and give some coordinates of the space.

We start by giving our motivation for studying them.

Lemma 4.2.1 ((ARMENTA et al., 2022), Lemma 3.1)**.** *There exists at least one oriented cycle in Q'.*

Proof. It is enough to consider the composition of a path in $\widetilde{Q}$ with an arrow $\beta_{i,k}$ and $\gamma_{j,l}$. We can always construct such a path in $\widetilde{Q}$ as Q is connected. This is particularly illustrated in Figure 6. □

Example 4.2.2. Consider Example 4.1.12. We illustrate an oriented cycle of Q'. In blue the arrows from β and γ, and in orange a path (the only one) in $\widetilde{Q}$:

$Q' =$


Note that this is also possible in the Example 4.1.11.

With this, we can define:

Definition 4.2.3. *The* ***moduli space of semisimple representations of*** Q'*, noted by* $\mathcal{M}^{\text{ssimp}}_{\boldsymbol{d'}}(Q')$ *is:*

$$\mathcal{M}^{\text{ssimp}}_{\boldsymbol{d'}}(Q') := \mathcal{R}_{\boldsymbol{d'}}(Q') / G_{\boldsymbol{d'}}(Q')$$

which parametrizes by definition the isomorphism classes of semisimple representations of Q' *of dimension vector* $\boldsymbol{d'}$*.*

Using the results from Chapter 1, we obtain the following properties.

Theorem 4.2.1 ((ARMENTA et al., 2022), Theorem 3.2)**.** *The space* $\mathcal{M}^{\text{ssimp}}_{\boldsymbol{d'}}(Q')$*:*

a. Is an affine irreducible variety.

b. Parametrizes the closed orbits of $G_{\boldsymbol{d'}}(Q')$ *in* $\mathcal{R}_{\boldsymbol{d'}}(Q')$*; this is, the closed orbits of* $G_{\boldsymbol{d}}(\widetilde{Q})$ *in* $\mathcal{R}_{\boldsymbol{d}}(Q)$*.*

c. Has dimension $\dim \mathcal{R}_{\boldsymbol{d}}(Q) - \dim G_{\boldsymbol{d}}(\widetilde{Q})$*, if exists a simple representation of* Q' *of dimension* $\boldsymbol{d'}$*.*

Proof. The first two statements are by the general theory given in Chapter 1, and by Lemma 4.1.14. We obtain item c. as we know that the dimension is given by $1 - \langle \mathbf{d}', \mathbf{d}' \rangle_{Q'}$, see for example ((REINEKE, 2008b), Theorem 2.2), and this is:

$$\begin{aligned}
&1 - \langle \mathbf{d}', \mathbf{d}' \rangle_{Q'} \\
&= 1 - \sum_{i \in Q'_0} (d'_i)^2 + \sum_{(\alpha: i \to j) \in Q'_1} d'_i d'_j \\
&= 1 - \left((d'_\infty)^2 + \sum_{i \in \widetilde{Q}_0} (d_i)^2 \right) + \sum_{(\alpha: i \to j) \in Q'_1} d'_i d'_j \\
&= 1 - \left(1 + \sum_{i \in \widetilde{Q}_0} d_i^2 \right) + \left(\sum_{(\alpha: i \to j) \in \widetilde{Q}_1} d_i d_j + \sum_{\alpha: s(\alpha) = \infty} d'_\infty d'_{t(\alpha)} + \sum_{\alpha: t(\alpha) = \infty} d'_{s(\alpha)} d'_\infty \right) \\
&= - \sum_{i \in \widetilde{Q}_0} d_i^2 + \sum_{(\alpha: i \to j) \in \widetilde{Q}_1} d_i d_j + \sum_{i \in \widetilde{Q}_0} u_i d_i + \sum_{i \in \widetilde{Q}_0} d_i w_i \\
&= \sum_{i \in \widetilde{Q}_0} (u_i + w_i) d_i - \left(\sum_{i \in \widetilde{Q}_0} d_i^2 - \sum_{(\alpha: i \to j) \in \widetilde{Q}_1} d_i d_j \right) \qquad (*) \\
&= \sum_{i \in \widetilde{Q}_0} (u_i + w_i) d_i + \langle \widetilde{\mathbf{d}}, \widetilde{\mathbf{d}} \rangle_{\widetilde{Q}}
\end{aligned}$$

Where we have that

$$\sum_{\alpha: s(\alpha) = \infty} d'_\infty d'_{t(\alpha)} = \sum_{i \in \widetilde{Q}_0} u_i d_i$$

as we have u_i arrows $\infty \longrightarrow i$, for each $i \in \widetilde{Q}_0$, and $d'_\infty = 1$ in each case when exists such an arrow. The same argument holds for

$$\sum_{\alpha: t(\alpha) = \infty} d'_{s(\alpha)} d'_\infty = \sum_{i \in \widetilde{Q}_0} d_i w_i$$

On the other hand,

$$\begin{aligned}
&\dim \mathcal{R}_{\mathbf{d}}(Q) \\
&\quad = \dim \left(\mathcal{R}_{\widetilde{\mathbf{d}}}(\widetilde{Q}) \times \bigoplus_{i \in \widetilde{Q}_0} \mathrm{Hom}(U_i, V_i) \times \bigoplus_{i \in \widetilde{Q}_0} \mathrm{Hom}(V_i, W_i) \right) \\
&\quad = \dim \bigoplus_{(\alpha: i \to j) \in \widetilde{Q}_1} \mathrm{Hom}(V_i, V_j) + \dim \bigoplus_{i \in \widetilde{Q}_0} \mathrm{Hom}(U_i, V_i) + \dim \bigoplus_{i \in \widetilde{Q}_0} \mathrm{Hom}(V_i, W_i) \\
&\quad = \sum_{(\alpha: i \to j) \in \widetilde{Q}_1} \dim \mathrm{Hom}(V_i, V_j) + \sum_{i \in \widetilde{Q}_0} \dim \mathrm{Hom}(U_i, V_i) + \sum_{t \in \widetilde{Q}_0} \dim \mathrm{Hom}(V_i, W_i) \\
&\quad = \sum_{(\alpha: i \to j) \in \widetilde{Q}_1} d_i d_j + \sum_{i \in \widetilde{Q}_0} u_i d_i + \sum_{i \in \widetilde{Q}_0} d_i w_i \\
&\quad = \sum_{(\alpha: i \to j) \in \widetilde{Q}_1} d_i d_j + \sum_{i \in \widetilde{Q}_0} (u_i + w_i) d_i
\end{aligned}$$

And similarly, we obtain

$$\dim G_{\mathbf{d}}(\widetilde{Q}) = \dim\left(\prod_{i\in\widetilde{Q}_0} \mathrm{GL}(V_i)\right) = \sum_{i\in\widetilde{Q}_0} \dim \mathrm{GL}(V_i) = \sum_{i\in\widetilde{Q}_0} d_i^2$$

Then Equation $(*)$ becomes:

$$\begin{aligned}(*) &= \dim \mathcal{R}_{\mathbf{d}}(Q) - \dim G_{\mathbf{d}}(\widetilde{Q}) \\ &= 1 - \langle \mathbf{d'}, \mathbf{d'} \rangle_{Q'}\end{aligned}$$

Which shows what we wanted. $\square$

Example 4.2.4. Consider the quiver $Q = \mathbb{A}_3$, with dimension vector $\mathbf{d} = (1,1,1)$, we know then that $\widetilde{Q} = \mathbb{A}_1$ and with vertex 2. Moreover,

$$\dim \mathcal{R}_{\mathbf{d}}(Q) = 2 \text{ and } \dim G_{\mathbf{d}}(\widetilde{Q}) = 1$$

as there are two arrows in Q, and just one vertex with dimension 1 in $\widetilde{Q}$. Then, by Theorem 4.2.1 we have $\dim \mathcal{M}^{\mathrm{ssimp}}_{\mathbf{d'}}(Q') = 1$. This is, $\mathcal{M}^{\mathrm{ssimp}}_{\mathbf{d'}}(Q') \cong \mathbb{C}$.

Example 4.2.5. Consider again Example 4.1.8. We saw that $\mathcal{R}_{\mathbf{d}}(Q) = 7$. Now,

$$\dim G_{\mathbf{d}}(\widetilde{Q}) = \sum_{i\in\widetilde{Q}_0} d_i^2 = 3$$

Which implies $\dim \mathcal{M}^{\mathrm{ssimp}}_{\mathbf{d'}}(Q') = 7 - 3 = 4$. On the other hand we also have:

$$Q' = \begin{tikzcd} 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[dl, bend right] \arrow[dl, bend left] \\ & \infty \arrow[ul, bend left] \arrow[ul] \arrow[ul, bend right] & \end{tikzcd}$$

Note the similarity with Example 4.1.12. And with this,

$$\begin{aligned}1 - \langle \mathbf{d'}, \mathbf{d'} \rangle_{Q'} &= 1 - \left(\sum_{i\in Q'_0} d_i^2 - \sum_{(\alpha: i\to k)\in Q'_1} d'_i d'_j\right) \\ &= 1 - 4 + 7 = 4\end{aligned}$$

Which verifies the theorem. Note that $|Q'_1| = 7$ and the dimensions are 1 on each case.

Now we proceed to give coordinates of the space of semisimple representations. In general, we see that there are easy to describe in terms of the oriented cycles from Q':

Theorem 4.2.2 ((ARMENTA et al., 2022), Theorem 3.3)**.** *The moduli space $\mathcal{M}^{\text{ssimp}}_{\boldsymbol{d}'}(Q')$ is isomorphic to the image of the map*

$$\mathcal{R}_{\boldsymbol{d}}(Q) \cong \mathcal{R}_{\widetilde{\boldsymbol{d}}}(\widetilde{Q}) \times \bigoplus_{i\in\widetilde{Q}_0} \operatorname{Hom}(U_i, V_i) \times \bigoplus_{i\in\widetilde{Q}_0} \operatorname{Hom}(V_i, W_i) \longrightarrow \bigoplus_{\omega: i \rightsquigarrow j} \operatorname{Hom}(U_i, W_j)$$

given by

$$\left((V_\alpha)_{\alpha\in\widetilde{Q}_1}, (l_i)_{i\in\widetilde{Q}_0}, (h_i)_{i\in\widetilde{Q}_0}\right) \longmapsto ((h_j V_\omega l_i))_{\omega: i\rightsquigarrow j}$$

Proof. We know that the coordinates are given by the traces along oriented cycles. Now, as Q is acyclic, $\widetilde{Q}$ also is, and we get that the only possible oriented cycles in $\widetilde{Q}$ are those described in the proof of Lemma 4.2.1, and then given by $\gamma_{j,l}\omega\beta_{i,k}$, where ω is a path $i \rightsquigarrow j$ in $\widetilde{Q}$, and $k = 1, \ldots, u_i$, $l = 1, \ldots, w_j$. This implies that we can describe such coordinates by the following functions

$$\left((V_\alpha)_{\alpha\in\widetilde{Q}_1}, (v_{i,k})_{i,k}, (\varphi_{i,l})_{i,l}\right) \xrightarrow{F_{\omega,l,k}} \varphi_{j,l} V_\omega v_{i,k} \tag{4.1}$$

Where we used the isomorphism from Lemma 4.1.13. Now, gluing the values for each one of the paths into the corresponding h_j and l_i we obtain the result from the Theorem. □

Remark 4.2.6. We note that the map as above separates closed orbits, and then equivalently separates isoclasses of semisimple representations by Theorem 4.2.1. More specifically, it also separates isoclasses of simple representations.

Example 4.2.7 ((ARMENTA et al., 2022), p. 103). Consider as hidden quiver $\widetilde{Q}$ the extended Dynkin quiver $\widetilde{\mathbb{D}}_4$, with dimension vector $\widetilde{\mathbf{d}} = (1,1,1,1,1)$. This is,

$$\begin{tikzcd}
1 \arrow[dr] & & 4 \\
& 3 \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] & & 5
\end{tikzcd}$$

We also consider $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5) = (2,2,0,0,1)$ and $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5) = (0,0,0,2,2)$. An element of $\mathcal{R}_{\mathbf{d}}(Q)$ is given by:

$$\begin{tikzcd}
\mathbb{C}^2 \cong U_1 \arrow[r, "\varphi_1"] & \mathbb{C} \arrow[dr, "\lambda_1"] & & \mathbb{C} \arrow[r, "v_4"] & W_4 \cong \mathbb{C}^2 \\
& & \mathbb{C} \arrow[ur, "\lambda_3"] \arrow[dr, "\lambda_4"] & & \\
\mathbb{C}^2 \cong U_2 \arrow[r, "\varphi_2"] & \mathbb{C} \arrow[ur, "\lambda_2"] & & \mathbb{C} \arrow[r, "v_5"] & W_5 \cong \mathbb{C}^2 \\
& & & U_5 \cong \mathbb{C} \arrow[u, "\varphi_5"] &
\end{tikzcd} \tag{4.2}$$

where $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, 5$, $\varphi_5 \in \mathbb{C}$, $v_4, v_5 \in \mathbb{C}^2$ and $\varphi_1, \varphi_2 \in \left(\mathbb{C}^2\right)^*$. The map from Theorem 4.2.2 has codomain:

$$\bigoplus_{\omega : i \rightsquigarrow j} \mathrm{Hom}(U_i, W_j) = \mathrm{Hom}(U_1, W_4) \oplus \mathrm{Hom}(U_1, W_5) \oplus \mathrm{Hom}(U_2, W_4) \oplus \mathrm{Hom}(U_2, W_5) \oplus \mathrm{Hom}(U_5, W_5)$$

which we can embed into the space $\mathrm{Hom}(U_1 \oplus U_2 \oplus U_5, W_4 \oplus W_5)$, isomorphic to a space of 4×5 matrices with complex entries. We can also write the representation (4.2) as

$$\left((\lambda_l)_{l=1}^5, (\varphi_l)_{l=1}^2, (v_l)_{l=4}^5\right)$$

Then the functions from (4.1) can be rewritten:

$$\left((\lambda_l)_{l=1}^5, (\varphi_l)_{l=1}^2, (v_l)_{l=4}^5\right) \longmapsto \left(v_j V_\omega \varphi_i\right)_{\omega : i \rightsquigarrow j}$$

whenever ω is a path from i to j, and then V_ω becomes a multiplication of the corresponding λ_i. The map from the Theorem then assigns to such representation the following matrix (seen as embedded into the space $\mathrm{Hom}(U_1 \oplus U_2 \oplus U_5, W_4 \oplus W_5)$):

$$\begin{bmatrix} v_4\lambda_1\lambda_3\varphi_1 & v_4\lambda_2\lambda_3\varphi_2 & 0 \\ v_5\lambda_1\lambda_4\varphi_1 & v_5\lambda_2\lambda_4\varphi_2 & v_5\varphi_5 \end{bmatrix}$$

Before ending this section we give two criteria for the existence of a simple representation for the deframed quiver Q', as its existence will imply the non-emptiness of the corresponding moduli space.

Lemma 4.2.8 ((ARMENTA et al., 2022), Lemma 4.1). *Let $V = \left((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (l_i)_{i \in \widetilde{Q_0}}, (h_i)_{i \in \widetilde{Q_0}}\right)$ an element from $\mathcal{R}_{\mathbf{d}'}(Q') \cong \mathcal{R}_{\mathbf{d}}(Q)$. Then V is simple if and only if the following hols:*

a. *The largest subrepresentation (in $\widetilde{Q}$) $V' = (V'_i)_{i \in \widetilde{Q_0}}$ of V where $V'_i \subset \ker(h_i)$ for all $i \in \widetilde{Q_0}$ is the representation 0.*

b. *The smallest subrepresentation (in $\widetilde{Q}$) $V' = (V'_i)_{i \in \widetilde{Q_0}}$ of V where $\mathrm{im}(l_i) \subset V'_i$ for all $i \in \widetilde{Q_0}$ is V.*

Proof. Suppose V is simple. Let V' a subrepresentation of V such that $V'_i \subseteq \ker(h_i)$ for all $i \in \widetilde{Q_0}$. As V is simple, then $V' = 0$ or $V' = V$. If $V' = V$, we obtain that $V_i \subseteq \ker(h_i)$ for all $i \in \widetilde{Q_0}$, which implies $\ker(h_i) = V_i$. Then $h_i = 0$ for all $i \in \widetilde{Q_0}$. As the morphisms

$$h_i : V_i \longrightarrow \bigoplus_{\alpha : i \to t_l} V_{t_l}$$

are zero, then V' is isomorphic to one where all $W_i = 0$ for all $i \in \widetilde{Q_0}$. By Remark 4.1.4 we have that there are no sinks vertices on $\widetilde{Q}$. Thus it exists an oriented cycle on $\widetilde{Q}$, a contradiction, as Q does not have any oriented cycles by definition. This proves a.. For b. we use a similar argument, as if there are no sources on $\widetilde{Q}$ we also have an oriented cycle on $\widetilde{Q}$, again a contradiction.

Now suppose we have a representation V with properties a. and b.. We will prove that it is simple, so suppose that V' is a subrepresentation. The vector space V'_∞ may be or 0 or $\mathbb{C}$, if it's 0, $\ker(h_i) = V_i$ for all $i \in \widetilde{Q_0}$, and then

$$V'_i \subseteq \ker(h_i) = V_i \text{ for all } i \in \widetilde{Q_0}$$

By a. we get that $V' = 0$. If $V'_\infty = \mathbb{C}$, with the following commutative diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "l_i"] \arrow[d, equal] & V_i \\ \mathbb{C} \arrow[r, "l'_i"] & V'_i \arrow[u, hook] \end{tikzcd}$$

we get that $\operatorname{im}(l_i) \subseteq V'_i$ and by b., $V' = V$. Then V is simple. □

The characterization of semisimple representations and its moduli space, altogether with a description of the coordinates of it (a bit on the spirit of Theorem 4.2.2) was given in (BRUYN; PROCESI, 1990). In particular, we can adapt some numerical results in terms of the Euler form to give one of the characterizations for the existence of simple representations. More specifically,

Theorem 4.2.3 ((ARMENTA et al., 2022), Theorem 4.2)**.** *Suppose that $\widetilde{Q}$ is not the extended Dynkin quiver $\widetilde{\mathbb{A}}_n$. Then exists a simple representation of Q' with dimension vector $\mathbf{d'}$ if and only if:*

a. There is a vertex $i \in \widetilde{Q_0}$ such that $d_i(u_i + w_i) \neq 0$,

b. $u_i \geq \langle \mathbf{d}, e_i \rangle_{\widetilde{Q}}$ for all $i \in \widetilde{Q_0}$, and

c. $w_i \geq \langle e_i, \mathbf{d} \rangle_{\widetilde{Q}}$ for all $i \in \widetilde{Q_0}$.

If $Q' = \widetilde{\mathbb{A}}_n$, then exists a simple representation if and only if the representation is thin, or $d_i = 1$ for all $i \in \widetilde{Q_0}$.

Proof. The proof is out of the purposes of this text, and then is omitted. The original numerical criterion can be found in ((BRUYN; PROCESI, 1990), Theorem 4). □

In particular, if the representation for the hidden quiver is thin, we have:

Corollary 4.2.9 ((ARMENTA et al., 2022), Corollary 4.3)**.** *Suppose that $d_i = 1$ for all $i \in \widetilde{Q}_0$. Then $\mathcal{M}^{\text{ssimp}}_{\mathbf{d}'}(Q')$ is an affine irreducible variety with*

$$\dim \mathcal{M}^{\text{ssimp}}_{\mathbf{d}'}(Q') = |Q_1| - |\widetilde{Q_0}|$$

Proof. Let $i \in \widetilde{Q}$ such that i is a source, then $d_i(u_i + w_i) = u_i + w_i \neq 0$ and this proves a.. Now, as

$$\begin{aligned} \langle \mathbf{d}, e_i \rangle_{\widetilde{Q}} &= \sum_{j \in \widetilde{Q_0}} d_j(e_i)_j - \sum_{(\alpha: a \to b) \in \widetilde{Q_1}} d_a(e_i)_b \\ &= \sum_{j \in \widetilde{Q_0}} (\delta_i)_j - \sum_{(\alpha: a \to b) \in \widetilde{Q_1}} (\delta_i)_b \\ &= 1 - |\widetilde{Q_1^{\to i}}| \leq 0 \leq u_i \end{aligned}$$

Where $Q_1^{\to i}$ is the subset of arrows that end at the vertex i. This shows b., and we obtain c. with a similar argument. Thus there exists a simple representation of Q' of dimension vector $\mathbf{d}'$ by Theorem 4.2.3, and then by Theorem 4.2.1 we get that

$$\begin{aligned} \dim \mathcal{M}^{\text{ssimp}}_{\mathbf{d}'}(Q') &= \dim \mathcal{R}_{\mathbf{d}}(Q) - \dim G_{\mathbf{d}}(\widetilde{Q}) \\ &= \sum_{(\alpha: i \to j) \in Q_1} d_i d_j - \sum_{i \in \widetilde{Q_0}} d_i^2 \\ &= \sum_{(\alpha: i \to j) \in Q_1} 1 - \sum_{i \in \widetilde{Q_0}} 1 \\ &= |Q_1| - |\widetilde{Q_0}| \end{aligned}$$

Which shows the dimension computation. The other properties are direct consequence of Theorem 4.2.1. □

Before we end this section we make a small comment on the space of stable double framed thin quiver representations from Chapter 3. Let Q be a network quiver as in Definition 3.1.2, and suppose it has d input vertices and k output vertices. As $\widetilde{Q}$ does not have loops (see Definition 3.1.20) we obtain that

$${}_d\mathcal{M}_k(\widetilde{Q}) = \mathcal{M}^{\text{ssimp}}_{\mathbf{d}'}((Q^\circ)')$$

as the conditions of stability in ${}_d\mathcal{M}_k(\widetilde{Q})$ (see Definition 3.3.10) and simpleness in $\mathcal{M}^{\text{ssimp}}_{\mathbf{d}'}(Q')$ (see Lemma 4.2.8) coincide. Thus by Corollary 4.2.9 we get:

$$\dim_d \mathcal{M}_k(\widetilde{Q}) = |Q_1^\circ| - |\widetilde{Q_0}|$$

Which proves Theorem 3.3.3.

4.3 Moduli of Stable Representations and geometric properties

In this section we will define two different types of moduli spaces arising from King's stability: one from a construction of a "double framed" quiver which won't have oriented cycles, and with a trivial stability function; and one derived from the moduli space of semisimple representations. Particularly, the advantage of the last one is its smoothness, and all the geometric properties shared with $\mathcal{M}^{\text{ssimp}}_{\mathbf{d'}}(Q')$. And for concluding, we give some relations with the moduli space defined in Chapter 3, we prove Theorem 3.3.3 as a corollary of the results given, and we make comments on some topics around the results and its possible relation with a few similar researches.

4.3.1 Constructions and some equivalences

We will define an alternate quiver from Q', having another vertex, and which has an essentially different moduli space.

Definition 4.3.1 ((ARMENTA et al., 2022), p. 104)**.** *The **double framed quiver** Q'' is the quiver associated to Q, where $Q''_0 = \widetilde{Q_0} \cup \{0, \infty\}$, set of arrows: $Q''_1 = \widetilde{Q_1} \cup \beta \cup \gamma$. With:*

$$\beta = \{\beta_{i,k} : 0 \longrightarrow i \,|\, i \in \widetilde{Q_0},\, k = 1, \ldots, u_i\}$$
$$\gamma = \{\gamma_{i,l} : i \longrightarrow \infty \,|\, i \in \widetilde{Q_0},\, l = 1, \ldots, w_i\}$$

*And its **dimension vector d″** will be $d''_i = d_i$ for all $i \in \widetilde{Q_0}$ and $d''_0 = 1 = d''_\infty$.*

Example 4.3.2. Let $Q = \mathbb{A}_3$, if we have $\mathbf{d} = (1,1,1)$ then we obtain:

$$Q'' = \;\; 0 \longrightarrow 2 \longrightarrow \infty$$

the $\mathbb{A}_3$ quiver once again, with relabeled vertices.

Example 4.3.3. Consider the quiver from Example 3.3.4, and where we computed its hidden quiver. If we take thin representations, we obtain that:

$$Q'' = \begin{tikzcd}
& 4 \arrow[rr] \arrow[dr] & & 7 \arrow[dr, bend left] \arrow[dr, bend right] & \\
0 \arrow[ur, bend left] \arrow[ur, bend right] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[urrr, bend left=60] & & 6 \arrow[ur] \arrow[dr] & & \infty \\
& 5 \arrow[rr] \arrow[ur] \arrow[urrr, bend right=60] & & 8 \arrow[ur, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

Example 4.3.4. Lastly, consider Example 4.1.12, with the same dimension vector $\mathbf{d} = (2,1,0,3)$ we get:

$$\begin{tikzcd} Q'' = 0 \arrow[r, bend left] \arrow[r, bend right] & 2 \arrow[r] & 3 \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \infty \end{tikzcd}$$

We would like to consider stability in the King's sense, (with respect to Definition 2.2.1) for a determined stability function: $\theta_i = 0$ for all $i \in \widetilde{Q_0}$, $\theta_0 = 1$, $\theta_\infty = -1$. We note that if $M \in \mathcal{R}_{\mathbf{d}''}(Q'')$, then

$$\theta(M) = \theta \cdot \dim M = (1, 0, \ldots, 0, -1) \cdot (1, d_1, \ldots, d_{|Q_0''|}, 1) = 0$$

And it is reasonable to consider such stability as the dimension vector is orthogonal to θ. With this in mind, we will call $\mathcal{M}_{\mathbf{d}''}^{\theta-\mathrm{sst}}(Q'')$ to the space of **semi-stable representations of dimension vector d″** of Q''. This space is essentially different from $\mathcal{M}_{\mathbf{d}'}^{\mathrm{ssimp}}(Q')$:

Lemma 4.3.5 ((ARMENTA et al., 2022), p. 104)**.** *The expected dimension of $\mathcal{M}_{\mathbf{d}''}^{\theta-\mathrm{sst}}(Q'')$ is*

$$\dim \mathcal{R}_d(Q) - \dim G_d(\widetilde{Q}) - 1$$

Proof. We give an sketch, for more details, we refer the reader to (ARMENTA et al., 2022). Note that here the acting group will have one additional vertex (namely, 0), and it will separate the dilations from the input and output vertices into the respective vertices. Then,

$$\begin{aligned} \dim \mathcal{M}_{\mathbf{d}''}^{\theta-\mathrm{sst}}(Q'') &= \dim \mathcal{R}_{\mathbf{d}}(Q) - \dim \left(G_{\mathbf{d}}(\widetilde{Q}) \times \mathbb{C}^* \right) \\ &= \dim \mathcal{R}_{\mathbf{d}}(Q) - \dim G_{\mathbf{d}}(\widetilde{Q}) - 1 \end{aligned}$$

which in particular is equal to $\mathcal{M}_{\mathbf{d}'}^{\mathrm{ssimp}}(Q') - 1$. □

Example 4.3.6 ((ARMENTA et al., 2022), p. 104). Consider Example 4.2.4, where we found that $\dim \mathcal{M}_{\mathbf{d}'}^{\mathrm{ssimp}}(Q') = 1$. We are going to compute the stable representations with dimension vector $\mathbf{d} = (1,1,1)$, and with stability function

$$\theta = (\theta_0, \theta_2, \theta_\infty) = (1, 0, -1)$$

We use the contrary definition of King's stability where for any subrepresentation U, we have $\theta(U) \leq 0$. There are 4 possiblities of representations with such dimension vector under the action of $G_{\mathbf{d}}(Q)$, those are:

$$M_{\lambda_1, \lambda_2} : \ \mathbb{C} \xrightarrow{\lambda_1} \mathbb{C} \xrightarrow{\lambda_2} \mathbb{C}$$

If $\lambda_1 = 0$, then $N_1 = \ \mathbb{C} \longrightarrow 0 \longrightarrow 0$ is a subrepresentation with

$$\theta(N_1) = \theta \cdot \dim N = (1, 0, -1) \cdot (1, 0, 0) = 1 \geq 0$$

then M_{0,λ_2} is unstable. Similarly, if $\lambda_2 = 0$, $N_2 = \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0$ is a subrepresentation with

$$\theta(N_2) = \theta \cdot \dim N = (1, 0, -1) \cdot (1, 1, 0) = 1 \geq 0$$

and hence unstable again. However, we can verify that if $\lambda_1, \lambda_2 \neq 0$ the only subrepresentations (see Example 2.1.16) are

$$M_1 = 0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \quad \text{and} \quad M_2 = 0 \longrightarrow 0 \longrightarrow \mathbb{C}$$

with $\theta(M_1) = \theta(M_2) = 1$ and then stable. This implies that $\mathcal{M}_{\mathbf{d}''}^{\theta-\text{sst}}(Q'')$ is just one point and then has dimension zero, as the Lemma showed. Note that by Example 4.3.2, Q'' is the quiver $\mathbb{A}_3$, and we also have $Q' = \begin{tikzcd} \infty \arrow[r, bend left] & 2 \arrow[l, bend left] \end{tikzcd}$.

4.3.2 The main theorem

Now we construct a variant of the moduli space $\mathcal{M}_{\mathbf{d}'}^{\text{ssimp}}(Q')$ which will always be smooth. So, we define a stability function Θ for Q' as

$$\Theta_\infty = \sum_{i \in \widetilde{Q_0}} d_i, \text{ and } \Theta_i = -1 \text{ for all } i \in \widetilde{Q_0}$$

Then, if $M' \in \mathcal{R}_{\mathbf{d}'}(Q')$ is the representation associated to $M \in \mathcal{R}_{\mathbf{d}}(Q)$,

$$\begin{aligned}
\Theta(M) &= \Theta \cdot \dim M' = \Theta \cdot \mathbf{d}' \\
&= \left(\sum_{i \in \widetilde{Q_0}}, -1, \dots, -1 \right) \cdot \left(1, d_1, \dots, d_{|\widetilde{Q_0}|}\right) \\
&= \sum_{i \in \widetilde{Q_0}} d_i - \sum_{i \in \widetilde{Q_0}} d_i = 0
\end{aligned}$$

And then again it makes sense to consider it as a stability function because it belongs to the set of orthogonal vectors of the dimension. Now, remembering that our semistability is $\Theta(U) \leq 0$ for all U subrepresentations, we obtain the following:

Lemma 4.3.7 ((ARMENTA et al., 2022), Lemma 4.4). *Let $V = \left((V_\alpha)_{\alpha \in \widetilde{Q_1}}, (l_i)_{i \in \widetilde{Q_0}}, (h_i)_{i \in \widetilde{Q_0}} \right)$ an element from $\mathcal{R}_{\mathbf{d}'}(Q')$, then the following are equivalent:*

a. *V is $\Theta-$semistable.*

b. *V is $\Theta-$stable.*

c. *The smallest representation (in $\widetilde{Q}$) $V' = (V'_i)_{i \in \widetilde{Q_0}}$ such that $\operatorname{im}(l_i) \subseteq V'_i$ for all $i \in \widetilde{Q_0}$ is V.*

Proof. We know that b. implies a. by definition. Suppose that V is $\Theta-$semistable, and let V' a subrepresentation of V with dimension vector $\mathbf{d'} = (d_i)_{i\in\widetilde{Q_0}}$ and such that $\Theta(V') = 0$, then

$$\begin{aligned} 0 = \Theta(V') &= \Theta \cdot \dim V' \\ &= \left(\sum_{i\in\widetilde{Q_0}} d_i, -1, \ldots, -1\right) \cdot \left(1, d'_1, \ldots, d'_{|\widetilde{Q_0}|}\right) \\ &= \sum_{i\in\widetilde{Q_0}} d_i - d'_i \end{aligned}$$

But as V' is a subrepresentation, each one of the morphisms must be injective and then $d_i - d'_i \geq 0$. So, the only way on which the above equation is zero occurs when $d_i = d'_i$, this is, when $V' = V$. This shows that V is $\Theta-$stable, and we obtain that a. implies b..

Now suppose that V is $\Theta-$stable, and let V' be a subrepresentation of V in $\widetilde{Q}$ such that $\operatorname{im}(l_i) \subseteq V'_i$ for all $i \in \widetilde{Q_0}$. As V is $\Theta-$stable we get $\Theta(V') \leq 0$. We can construct a representation associated to V' in Q' such that its vector space on the vertex ∞ is $\mathbb{C}$, because if not then we would obtain a contradiction with the hypothesis, as we can glue the $v_{i,k}$ to get the l_i and the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{v_{i,k}} & V_i \\ \Big\| & & \uparrow \\ \mathbb{C} & \xrightarrow{v'_{i,k}} & V'_i \end{array}$$

is commutative. By the $\Theta-$semistability of V,

$$\Theta(V') = \Theta \cdot \dim V' = \sum_{i\in\widetilde{Q_0}} (d_i - d'_i) \leq 0$$

But by definition $d_i - d'_i \geq 0$, so $\sum(d_i - d'_i) \geq 0$. And this implies $\Theta(V') = 0$, and the $\Theta-$stability implies that $V' = V$ or $V' = 0$. It cannot be 0 as it does not holds the initial condition $\operatorname{im}(l_i) \subseteq V'_i$, so $V' = V$ and we obtain c.. We just proved that b. implies c..

Following a very similar argument to the ones just given above we obtain that c. implies b., and this proves the lemma. □

We can apply the theory from Chapter 1 and Theorem 4.2.1 to obtain our main result, the properties of the moduli spaces for representations of Q:

Theorem 4.3.1 ((ARMENTA et al., 2022), Theorem 4.5)**.** *The space $\mathcal{M}^{\Theta-\mathrm{sst}}_{\mathbf{d'}}(Q')$:*

1. *Is a smooth affine irreducible variety.*

2. *Parametrizes the isomorphism classes of stable representations of Q' with dimension vector $\boldsymbol{d'}$.*

3. *Its dimension is* $\dim \mathcal{R}_{\boldsymbol{d}}(Q) - \dim G_{\boldsymbol{d}}(\widetilde{Q})$ *if it is nonempty.*

4. *It admits a projective morphism*

$$\mathcal{M}_{\boldsymbol{d'}}^{\Theta-\mathrm{sst}}(Q') \longrightarrow \mathcal{M}_{\boldsymbol{d'}}^{\mathrm{ssimp}}(Q')$$

Proof. This happens as the quiver Q is finite, acyclic and without oriented cycles. It is smooth as stability coincides with semistability (see Preliminaries, Subsection 1.4.3). □

4.3.3 Ok, so what's next? Final comments

Theorem 4.3.1 says that we have good geometric properties on the space of $\Theta-$stable representations of Q with dimension vector $\mathbf{d}$. In particular, its smoothness which $\mathcal{M}_{\mathbf{d'}}^{\mathrm{ssimp}}(Q')$ does not have. It allows us to compute its Betti numbers for example, and to consider it as a complex manifold. One of the possibilities that could be done with the space, for instance, happens when Q is a network quiver, and we can make an analogy similar to the one on the last section.

In this case, we do have that the space is a toric variety ((ARMENTA et al., 2022), Theorem 4.5), and then we can expect to be able to study the toric geometry of the space along with the corresponding GIT properties inherited from it. It may be interesting to study what happens with them and if they can explain some properties that the neural networks have, using for example, the theory in ((COX; LITTLE; SCHENCK, 2011), Chapter 14).

One of the main similarities that (REINEKE, 2008a) and (ARMENTA et al., 2022) have in common is that on the first, just the framed quiver moduli are defined, and on the second the same tools are used to defined the double framed quivers, even for stability. We would expect to obtain some of the results that were obtained on the first, like computation of Chern classes and Cox rings for those new defined type of spaces. We expect that there may be some results that are directly obtained from the theory made by Reineke.

Lastly, we didn't mention some applications with Neural Networks on the computational aspect, and the approach for defining mathematical neural networks in the sense of Chapter 3, introduced in (ARMENTA; JODOIN, 2021) has several consequences on the interpretation of those and the corresponding encoding of information that passes on them. There are some questions that are expected to be solved with that approach, and it is a natural question to look after the similarities that may appear with the moduli spaces and the inherited geometric nature of such spaces. A huge part of the computational questions and possible leading researches were commented on (ARMENTA; JODOIN, 2021), which are nice and interesting topics to look at.

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