UNIVERSIDADE ESTADUAL DE CAMPINAS
FACULDADE DE ENGENHARIA MECÂNICA

Marcel de Almeida

$\mathcal{H}_2$ feedback control for continuous semi-Markov jump linear systems

Controle $\mathcal{H}_2$ para sistemas lineares a tempo contínuo com saltos semimarkovianos

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Orientador: Prof. Dr. André Ricardo Fioravanti
Coorientador: Prof. Dr. Matheus Souza

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DISSERTAÇÃO DE MESTRADO ACADÊMICO

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Resumo

O objetivo deste trabalho é o desenvolvimento de controladores ótimos $\mathcal{H}_2$ por realimentação de estado e saída para sistemas semimarkovianos a tempo contínuo. Assim, esse objetivo é atingido por meio de duas abordagens distintas que resultam em técnicas baseadas em desigualdades matriciais lineares. A saber, a primeira abordagem está fundamentada na técnica de soma de quadrados, enquanto a segunda é desenvolvida por meio da obtenção de um sistema markoviano equivalente. Desse modo, a primeira parte da dissertação trata do desenvolvimento de condições de análise e de projeto de controladores por realimentação de estado para sistemas com taxas de transição racionais, utilizando a técnica baseada em soma de quadrados. Por fim, a segunda parte aborda o problema de estabilidade e realimentação de saída para um sistema semimarkoviano com tempos de permanência com distribuição Erlang, empregando a equivalência de sistemas.

Palavras-chave: Sistemas lineares, sistemas estocásticos, otimização convexa, desigualdades matriciais lineares, soma de quadrados.

Abstract

This work addresses the $\mathcal{H}_2$ state- and output-feedback control problem for continuous-time semi-Markov jump linear systems. Thus, this objective is achieved by means of two different approaches that result in techniques based on linear matrix inequalities. Namely, the first is based on the sum-of-squares technique, while the second is developed by transforming the original system into an equivalent Markov one. In this way, the first part of the thesis deals with the development of analysis and state-feedback design conditions for systems with rational transition rates, using the technique based on the sum of squares. Finally, the second part approaches the stability and output-feedback problem involving a semi-Markov jump linear system with Erlang-distributed dwell times, employing the equivalence of systems.

Keywords: Linear systems, stochastic systems, convex optimisation, linear matrix inequalities, sum of squares.

Abbreviations

MJLS	-	Markov Jump Linear System
S-MJLS	-	Semi-Markovian Jump Linear System
LMI	-	Linear Matrix Inequality
DLMI	-	Differential Linear Matrix Inequality
PDF	-	Probability Density Function
CDF	-	Cumulative Distribution Function
SDP	-	Semidefinite Programming

List of Symbols

$\mathbb{N}$	-	Set of natural numbers with zero.
$\mathbb{N}^*$	-	Set of natural numbers without zero.
$\mathbb{Z}$	-	Set of integers.
$\mathbb{R}$	-	Set of real numbers.
$\mathbb{R}_{+(+)}$	-	Set of non-negative (positive) real numbers.
$\mathbb{K}_N$	-	Set composed of the first N natural numbers excluding zero.
$\mathcal{E}[Z]$	-	Expected value of random variable Z.
$\mathbb{R}^n$	-	Set of real vectors of dimension n.
$\mathbb{R}^{m\times n}$	-	Set of real matrices of dimension $m \times n$.
A^T	-	Transpose of matrix A.
A^{-1}	-	Inverse of matrix A.
$\text{diag}(X, Y)$	-	Block diagonal matrix composed of blocks X and Y.
$*$	-	Symmetric block of a partitioned symmetric matrix.
$\text{tr}(A)$	-	Trace of matrix A.
$A \succ (\succeq)\ 0$	-	Matrix A is positive definite (semi-definite).
$A \prec (\preceq)\ 0$	-	Matrix A is negative definite (semi-definite).
$\mathbb{S}^n$	-	Set of symmetric matrices of dimension n.
$\mathbb{S}^n_{+(+)}$	-	Set of positive semi-definite (definite) matrices of dimension n.
$\|x(t)\|$	-	Euclidean norm of vector $x(t)$.
$\|\xi\|_2^2$	-	Squared norm of stochastic signal $\xi(t)$.
$\mathcal{L}_2^r$	-	Class of all signals of dimension r with finite squared norm.
$:=$	-	Equal by definition.
$a \wedge b$	-	Lesser of a, b.
$\square$	-	End of proof.

Contents

CHAPTER 1

Introduction

In many application fields of control system theory, dynamic systems exhibit a kind of behaviour that cannot be precisely determined nor even accurately modelled. In some cases, this behaviour is stochastic and characterised by abrupt changes, which can be caused, for instance, by alterations in operating conditions, failure of components such as sensors and actuators, and sudden external disturbances, among other factors [4].

One way of approaching a stochastic dynamic system consists in modelling the parameter responsible for these random and sudden events by means of a stochastic process, which can be defined as an indexed set of random variables [15]. Specifically, when considering continuous time, a stochastic process $\{\theta(t), t \geq 0\}$ consists of an infinite collection of random values, each of which is associated with a certain instant of time $t \in \mathbb{R}_+$. The set of all values that a stochastic process can assume is the *state space* of the process, denoted in this work by $\mathcal{M}$. In general, this set can be any subset of the reals, including the set of real numbers itself. However, this text considers only the finite set with the form $\mathcal{M} = \{1, 2, \ldots, N\}$. Furthermore, the time lapse between the time in which the process takes a certain value and the moment when it *jumps* to a new one is called *sojourn time*, or also *dwell time*, being represented throughout this thesis by τ.

About categorisation, one manner of classifying a stochastic process is based on the nature of the probability distribution of its sojourn time. Thus, processes with exponentially distributed sojourn times are termed Markov processes, whereas the others are called semi-Markov processes. More categorically, according to [35], a semi-Markov process may be viewed as a stochastic process that, after having entered a state i at a time T_k, randomly determines its length of stay τ_k for transition out of this state sampled from a probability density function $f_i(\tau)$, and also randomly determines the next state $j \neq i$ based on state transition probabilities $P = [p_{ij}]$, where $\sum_j p_{ij} = 1$ for all i, jumping thus to state j at time $T_{k+1} = T_k + \tau_k$. As mentioned above, the length of stay τ_k is known as the sojourn time. Moreover, it is convenient to define $T_0 = 0$ and for $t \in \mathbb{R}_+$, the elapsed time since the last jump, or *timer*, is set as

$$H(t) = \sum_{k=0}^{\infty}(t - T_k)1_{[T_k,\, T_{k+1})}(t), \tag{1.1}$$

in which the indicator function $1_{[T_k,\, T_{k+1})}(t)$ is defined as

$$1_{[T_k,\, T_{k+1})}(t) := \begin{cases} 1, & \text{if } \ t \in [T_k,\, T_{k+1}), \\ 0, & \text{otherwise.} \end{cases}$$

That being said, considering a probability space $(\Omega,\, \mathcal{F},\, \mathcal{P})$, this work addresses stochastic systems represented by the following mathematical model

$$\mathbb{G} : \begin{cases} \dot{x}(t) = A(\theta(t),\, H(t))x(t) + B(\theta(t))u(t) + E(\theta(t))w(t), \\ z(t) = C_z(\theta(t),\, H(t))x(t) + D_z(\theta(t))u(t) \end{cases} \tag{1.2}$$

in which, $\forall\ t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^q$ is the control input vector, $w(t) \in \mathbb{R}^m$ is the external perturbation vector and $z(t) \in \mathbb{R}^p$ is the output vector to be controlled. In addition, the state-space matrices $A(\theta(t),\, H(t)) \in \mathbb{R}^{n\times n}$, $B(\theta(t)) \in \mathbb{R}^{n\times q}$, $E(\theta(t)) \in \mathbb{R}^{n\times m}$, $C_z(\theta(t),\, H(t)) \in \mathbb{R}^{p\times n}$ and $D_z(\theta(t)) \in \mathbb{R}^{p\times q}$ depend on the continuous-time stochastic process $\{\theta(t),\, t \geq 0\}$ and, to ease the notation, are denoted by $A(\theta(t),\, H(t)) = A_i(h)$, $B(\theta(t)) = B_i$, $E(\theta(t)) = E_i$, $C_z(\theta(t),\, H(t)) = C_{zi}(h)$ and $D_z(\theta(t)) = D_{zi}$, whenever $\theta(t) = i\ \in \mathcal{M}$. In this way, the reader needs to notice that matrices $A_i(h)$, B_i, E_i, $C_{zi}(h)$ and D_{zi} define the i-th mode of operation of such a system.

A stochastic system is classified according to the process that drives its dynamics. Thus, semi-Markov Jump Linear Systems, or S-MJLS, are systems represented by the above equations, with the particularity of $\theta(t)$ being a semi-Markov process that exhibits the aforementioned characteristics.

Hence, by the nature of the process employed in its modelling, a semi-Markov system is capable of representing more general systems than those described in the Markov context [19] since the sojourn time of its process can be modelled by arbitrary distributions and, therefore, distinct from the exponential one. Nevertheless, the approach to such a system has some peculiarities, for instance, the presence of transition rates, denoted in this text by $\lambda(h)$, dependent on the sojourn time [18]. As a result, in this setting, developing techniques used in stability analysis and synthesis of controllers present adversities that do not exist in the Markov environment.

For that reason, the need to employ some tricks becomes evident when obtaining convex conditions for analysis and synthesis. One of these artifices is the modelling of the original S-MJLS utilising an equivalent MJLS when dealing with S-MJLS with Erlang-distributed dwell times, and the other is the use of techniques based on sum-of-squares decomposition in a more general context, in which probabilities distributions that yield process with rational transition rates are considered. Thus, the approach based on the equivalence of systems aims to explore the existing relations between the distributions of interest, which, in this work, originate stochastic processes with rational transition rates and the exponential one, to obtain an equivalent MJLS with a larger number of modes. As will be discussed in more depth in Chapter 3, the basic idea of this transformation resides in obtaining an equivalent system in which a set of modes of operation, termed a *cluster*, emulates the dynamic behaviour of a single mode of the original system. As a consequence, the application of this methodology yields linear matrix inequalities with the following structure

$$A_i' S_{ik} + S_{ik} A_i + \lambda_i (S_{ik+1} - S_{ik}) + C_i' C_i \prec 0, \tag{1.3}$$

$$A_i' S_{ik_i} + S_{ik_i} A_i + \lambda_i \left(\sum_{j=1}^{N} p_{ij} S_{j1} - S_{ik_i} \right) + C_i' C_i \prec 0, \tag{1.4}$$

which are deployed in verifying the stochastic stability of S-MJLS with the general form presented in (1.2), but with Erlang-distributed dwell times and timer-independent matrices, that is, $A(\theta(t), H(t)) = A(\theta(t))$ and $C_z(\theta(t), H(t)) = C_z(\theta(t))$. Furthermore, the matrix variables S_{ij} are positive definite matrices in these inequalities.

On the other hand, the method based on sum-of-squares decomposition is based on imposing a specific structure on the variables of the optimisation problem in such a way that when these variables are scalars, they must be non-negative polynomials, or else, in the matrix context, matrices composed of elements defined in this way. As a result, using this approach provides differential linear matrix inequalities with the form

$$\begin{aligned}&\rho_i(h)d_i(h)\left(A_i(h)^TQ_i(h)+Q_i(h)A_i(h)+\dot{Q}_i(h)+d_i(h)C_{zi}(h)^TC_{zi}(h)\right)-\\&-(\rho_i(h)\dot{d}_i(h)+\eta_i(h)d_i(h))Q_i(h)+d_i^2(h)\eta_i(h)\sum_{j\in\mathcal{M}\backslash\{i\}}p_{ij}Q_j(0)\prec-\epsilon\rho_i(h)d_i^2(h)I\end{aligned}\tag{1.5}$$

which are applied in assessing the stability of S-MJLS represented by (1.2), considering sojourn times with probability distributions that originate processes with rational transition rates. According to what will be expounded in Chapter 2, these constraints must hold for all $h \in \mathbb{R}_+$; that is, the optimisation problems restricted by them are infinite-dimensional. The matrix variables $Q_i(h)$ are then defined as polynomial matrices in h, making the optimisation problem convex and computationally tractable. At last, it is worth mentioning that in these inequalities, ϵ is a scalar, while $\rho_i(h)$ and $d_i(h)$ are scalar polynomials in h.

1.1 Work structure

This thesis is composed of four chapters. The first one provides a brief introduction to the problem that is the object of study, discusses how the text is structured and introduces the notation used throughout the work.

Since the development of the thesis resulted in the production of two scientific articles, both already published at the time of the conclusion of this text, its central chapters present these two documents. In this way, the second chapter's article exhibits the approach to S-MJLS with rational transition rates based on the technique involving sum-of-squares decomposition. Thus, Section 2.1 introduces the reader to the topic and presents a brief literature review on S-MJLS. Afterwards, the problem addressed in this work is formulated in Section 2.2, along with the introduction of some elementary results used throughout the text. Next, Section 2.3 discusses some probability distributions that generate processes with rational transition rates, such as the Rayleigh and Weibull ones. Later, the main results obtained are shown in Section 2.4, in which the reader will find the algorithms for the computation of an upper bound for the $\mathcal{H}_2$ norm of the systems under study, as well as those that provide the state-feedback gains that optimise the performance of the resulting closed-loop systems concerning the $\mathcal{H}_2$ norm. Moreover, some particularities related to the computational implementation of these algorithms are also discussed in this section. At last, the application of these results is illustrated by introducing some numerical examples, and the article is closed with some concluding remarks.

The third chapter presents the article concerned with the approach to S-MJLS with Erlang-distributed dwell times using the equivalence between these systems and their equivalent MJLS.

Similarly to the previous paper, Section 3.1 aims to contextualise the reader, which is accomplished by introducing some results available in the literature involving MJLS and S-MJLS. Soon after, Section 3.2 briefly presents the characteristics of the Erlang distribution, the formal definition of an S-MJLS, some basic definitions, the construction of the equivalent MJLS and the algorithm used to assess the stability and compute an upper bound for the $\mathcal{H}_2$ norm of the original S-MJLS. Subsequently, Section 3.3 approaches the object of study of the article, that is, the output-feedback cluster control. As a consequence, the central result of the text is contained in this section: Theorem 3.2 provides design conditions for optimal output-feedback gains that minimise the $\mathcal{H}_2$ norm of the closed-loop system. For illustrative purposes, Section 3.4 presents a set of numerical examples demonstrating the application of the results introduced in the preceding sections.

Moreover, by comparison, this section revisits one of the examples in the first article that composes this thesis. In this way, dealing with the same numerical problem using the results of the two papers demonstrates the subtle superiority of the methodology based on the sum of squares due to the controllers' dependence on the sojourn time. Lastly, the paper ends with a concluding section in which suggestions for future works are made.

Finally, the last chapter of the thesis is dedicated to closing remarks and a brief discussion of future works.

1.2 Notation

The notation employed in chapters two and three are discussed in their introductory sections. However, in general, in this text, scalars, vectors and realisations of random variables are denoted by lowercase letters, whereas matrices and random variables are represented by capital letters. The symbol $\star$ in a partitioned symmetric matrix denotes its symmetric blocks, that is

$$\begin{bmatrix} X & \star \\ Y & Z \end{bmatrix} = \begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix}. \tag{1.6}$$

The norm of a stochastic signal $\xi(t) \in \mathbb{R}^r$, with $t \geq 0$, is represented by $\|\xi\|_2$ and defined as

$$\|\xi\|_2 := \sqrt{\int_0^\infty \mathcal{E}\{\xi(t)^T \xi(t)\} dt}. \tag{1.7}$$

The other notations used in the text are more specific and, for this reason, are presented only in the list of symbols and also in the introductory sections mentioned earlier.

CHAPTER 2

$\mathcal{H}_2$ State-Feedback Control for Continuous Semi-Markov Jump Linear Systems with Rational Transition Rates

M. de Almeida, M. Souza, A. R. Fioravanti and O. L. V. Costa

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ABSTRACT

This paper presents the $\mathcal{H}_2$ State-Feedback Control for Continuous Semi-Markov Jump Linear Systems where the transition rates are given by the ratio of polynomials of the sojourn time. We show that, for instance, Rayleigh-, Erlang- and a class of Weibull-distributed sojourn time can be described by such a model. We provide Sum-of-Squares conditions for norm calculation and state-feedback control design problems. We conclude the paper with some examples and future directions.

KEYWORDS

Linear Systems, Stochastic Systems, Convex Optimisation, Sum-of-Squares.

2.1 Introduction

Several physical phenomena that present abrupt random changes in their dynamic behaviour can be modelled by dynamic systems characterised by subsystems (modes of operation) which are selected according to a stochastic process. Whenever each of the subsystems is linear and their choice is orchestrated by a continuous-time Markov chain (that is, the modes of operation will change state according to an exponential random variable and move to a different state as specified by the probabilities of a transition probabilities matrix), the dynamic model is named as a Markov jump linear system (MJLS). The theory of continuous-time MJLS has recently made considerable progress, since these systems model, in a satisfactory manner, the behaviour of many systems of practical interest. The reader is referred to [4, 3, 13, 10, 9, 31] and references therein for a sample of works in this field.

In some applications, however, the length of time the process remains in each mode of operation – called *sojourn time* – is not memoryless and this characteristic implies that the exponential distribution is no longer suitable to model this situation. In this case, the jump rates, represented by $\lambda_{ij}(h)$, are dependent on the sojourn time h (notice that in the exponential case, the jump rates are constant). Thus, such processes, known as *semi-Markov processes* [24], are used to model hybrid phenomena in a more general setting than the ones considered by continuous-time Markov chains.

The theory of linear systems subject to semi-Markov jumps, called semi-Markov jump linear systems (S-MJLS), still does not present the same progress as the one reached by the theory of MJLS. Nonetheless, analysis and synthesis results for S-MJLS can be easily found in literature, considering both time domains. For instance, both analysis and control synthesis for discrete-time S-MJLS were investigated in [36]. Now, considering the continuous-time case, techniques of robust stability analysis, and robust state feedback control for S-MJLS with norm-bounded uncertainty were developed in [19]. Still in the continuous-time setup, stability issues for S-MJLS with mode-dependent delays were addressed in [21]. Also, techniques of state estimation and sliding mode control for S-MJLS with mismatched uncertainties were studied in [22]. Besides, conditions to verify the stochastic stability of Ito differential equations with semi-markovian jump parameters were presented in [17]. Finally, different approaches to the problem of stability analysis of continuous-time S-MJLS can be found in [18] and [26], with [18] presenting a methodology based on linear matrix inequalities.

In this paper, we provide $\mathcal{H}_2$ analysis and state feedback control design conditions for S-MJLS with rational[1] transition rates. Both problems listed above are computationally solved using the sum-of-squares (SOS) technique [2], which consists of a mathematical optimisation model in which the variables and constraints are non-negative polynomials, or else matrices whose elements are polynomials. Specifically, the main contributions of this note are as follows:

- We present general results on the stability and on the $\mathcal{H}_2$ norm of S-MJLS systems (Lemmas 2.1 and 2.2).
- Testable SOS-based stability and $\mathcal{H}_2$ norm conditions have been developed for S-MJLS with rational transition rates (Lemma 2.3).
- Finally, computationally viable SOS-based design conditions for the $\mathcal{H}_2$ state-feedback mode-dependent control design problem for S-MJLS systems with rational transition rates have also been devised (Theorem 2.2).

Two numerical examples point out the main theoretical and computational features of these contributions.

This paper is structured as follows. The formulation of an S-MJLS and its basic properties are presented in Section 2.2, with a brief description of some sojourn time distributions in Section 2.3. Next, Section 2.4 presents SOS-based analysis and design conditions. Furthermore, Section 2.5 presents two numerical examples that validate the main results. Finally, concluding remarks are presented in Section 2.6.

At last, in terms of notation, for real matrices or vectors, $(.)^T$ indicates transpose. For a square matrix X, $\mathrm{Tr}(X)$ denotes its trace, and for partitioned symmetric matrices, the symbol $\star$ denotes its symmetric blocks. The set of natural numbers is denoted by $\mathbb{N}$. On a probability space $(\Omega,\mathcal{F},\mathcal{P})$, the symbol $\mathcal{E}$ denotes mathematical expectation and, for $A \in \mathcal{F}$, 1_A represents the indicator function

[1] ratio of polynomials

in A, that is, $1_A(\omega) = 1$ if $\omega \in A$, 0 otherwise. For any stochastic signal $\xi(t)$ defined in the continuous-time domain, the quantity $\|\xi\|_2^2 = \int_0^\infty \mathcal{E}\{\xi(t)^T\xi(t)\}dt$ is its squared norm. The class of all signals $\xi(t) \in \mathbb{R}^r$, $t \in \mathbb{R}_+$, such that $\|\xi\|_2^2$ is finite is denoted by $\mathcal{L}_2^r$. We recall that for an absolutely continuous function $g : \mathbb{R}_+ \to \mathbb{R}$ there exists an integrable function $\mathcal{X}g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$g(t) = g(s) + \int_s^t \mathcal{X}g(\ell)d\ell. \tag{2.1}$$

Note that $\mathcal{X}g$ is not unique. For simplicity we write $\dot{g} = \mathcal{X}g$. We say that an $n \times n$ matrix $P(t)$ is absolutely continuous in $\mathbb{R}_+$ if each element $P_{ij}(t)$ of $P(t)$, $i,j = 1,\ldots,n$, is an absolutely continuous function. We represent by $\mathcal{X}P(\ell)$ the $n \times n$ matrix such that (2.1) is satisfied replacing $g(.)$ and $\mathcal{X}g(.)$ by respectively $P_{ij}(.)$ and $\mathcal{X}P_{ij}(.)$ for each $i,j = 1,\ldots,n$, and write $\dot{P} = \mathcal{X}P$.

2.2 Problem Formulation and Basic Results

According to [35], a semi-Markov process may be viewed as a stochastic process that, after having entered a state i at a time T_k, it randomly determines its length of stay τ_k for transition out of this state sampled from a probability density function $f_i(\tau)$, and also randomly determines the next state $j \neq i$ based on state transition probabilities $P = [p_{ij}]$, where $\sum_j p_{ij} = 1$ for all i, jumping thus to state j at time $T_{k+1} = T_k + \tau_k$. The length of stay τ_k is known as the sojourn time. In what follows it is convenient to define $T_0 = 0$ and for $t \in \mathbb{R}_+$, the elapsed time since the last jump, or *timer*, is set as

$$H(t) = \sum_{k=0}^{\infty}(t - T_k)1_{[T_k,T_{k+1})}(t).$$

On a probability space $(\Omega,\mathcal{F},\mathcal{P})$ consider the continuous-time semi-Markovian jump linear system (S-MJLS):

$$\mathbb{G} : \begin{cases} \dot{x}(t) = A(\theta(t),H(t))x(t) + E(\theta(t))w(t), \\ z(t) = C_z(\theta(t),H(t))x(t) \end{cases} \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the external perturbation vector, $z(t) \in \mathbb{R}^p$ is the output vector to be controlled, and $\{\theta(t), t \geq 0\}$ is a semi-Markovian continuous-time process that takes value in the finite set $\mathcal{M} = \{1,2,\ldots,N\}$. For $h \in \mathbb{R}_+$, $i \in \mathcal{M}$, the *timer-dependent* state-space matrices $A(i,h) \in \mathbb{R}^{n\times n}$ and output matrices $C_z(i,h) \in \mathbb{R}^{p\times n}$ are continuous and uniformly bounded in h, and $E(i) \in \mathbb{R}^{n\times m}$. For notation simplicity, it is set $A(\theta(t),h) = A_i(h)$, $E(\theta(t)) = E_i$,

$C_z(\theta(t),h) = C_{zi}(h)$, whenever $\theta(t) = i \in \mathcal{M}$. We also assume that the system starts at $t = 0$ from $x(0) = x_0$, and the probability distribution of the Markov process at the initial time is given by $\mu = (\mu_1, \ldots, \mu_N)$ in such a way that $\mathrm{Prob}\{\theta(0) = i\} = \mu_i$.

Notice that for a semi-Markov process $\{\theta(t), t \geq 0\}$ we have that

$$\mathrm{Prob}\{\theta(t+\Delta) = j|\theta(t) = i, H(t) = h\} = \begin{cases} \lambda_{ij}(h)\Delta + o(\Delta), \forall i \neq j, \\ 1 + \lambda_{ii}(h)\Delta + o(\Delta), i = j, \end{cases} \tag{2.3}$$

where $\lim_{\Delta \to 0} o(\Delta)/\Delta = 0$, and $\lambda_{ij}(h) \geq 0$ for $i \neq j$, $\lambda_{ii}(h) = -\sum_{j \in \mathcal{M}\setminus\{i\}} \lambda_{ij}(h)$ are given transition rates. Notice also that a conventional MJLS is a particular case of the S-MJLS, obtained whenever the transition probabilities are timer independent.

Associated with the S-MJLS (2.2) there are two important issues to be analysed related to the stability and the $\mathcal{H}_2$ norm. The definitions of stability and the $\mathcal{H}_2$ norm for (2.2), introduced next, are similar to the ones presented in [4] for MJLS.

Definition 2.1. Consider the autonomous version of the system $\mathbb{G}$, that is, with $w := 0$. We have the following concepts of stability:

a) $\mathbb{G}$ is stochastically stable if there exists a positive definite matrix $M > 0$ such that

$$\mathcal{E}\left[\int_0^\infty x^T(t)x(t)dt \mid \theta_0, x_0\right] \leq x_0^T M x_0$$

for any initial condition $x(0) = x_0$ and $\theta(0) = \theta_0$.

b) $\mathbb{G}$ is mean square stable if

$$\lim_{t\to\infty} \mathcal{E}\left[\, x^T(t)x(t) \mid \theta_0, x_0\right] = 0$$

for any initial condition $x(0) = x_0$ and $\theta(0) = \theta_0$.

c) $\mathbb{G}$ is exponentially mean square stable if there exist $\alpha > 0$ and $\beta > 0$ such that

$$\mathcal{E}\left[\, x^T(t)x(t) \mid \theta_0, x_0\right] \leq \beta \|x_0\|^2 e^{-\alpha t} \tag{2.4}$$

for any initial condition $x(0) = x_0$ and $\theta(0) = \theta_0$.

It is clear that c) implies a) and b)

Definition 2.2. Consider a stochastically stable version of system $\mathbb{G}$. We denote by $\|\mathbb{G}\|_2$ the $\mathcal{H}_2$ norm of $\mathbb{G}$, which can be calculated by

$$\|\mathbb{G}\|_2^2 = \sum_{i\in\mathcal{M}} \sum_{s=1}^{m} \mu_i \|z_{s,i}\|_2^2, \tag{2.5}$$

in which $z_{s,i}$ is the output generated by the impulsive input $w(t) = e_s\delta(t)$ and by the initial conditions $x(0) = 0$ and $\theta_0 = i$; here, e_s is the s-th column of the identity matrix of order m.

We are going to use the theory of the piecewise deterministic Markov processes (PDMPs) to provide testable conditions for Definitions 2.1 and 2.2. We recall that PDMPs consist of a general family of non-diffusion stochastic models introduced in [6] and [7]. PDMPs are characterised by three local parameters: the flow ϕ, the jump rate λ, and the transition measure Q. The main idea is that the motion of a PDMP starting at the initial state $X_0 = (i,\xi)$ follows a deterministic flow $\phi_i(\xi,t)$ until the first jump time T_1, which occurs either spontaneously in a Poisson-like fashion with rate λ or when the flow $\phi_i(\xi,t)$ hits the boundary of the state space. In either case, the post-jump location of the process is selected by the transition measure $Q(.|\phi_i(\xi,T_1))$, and the motion restarts from this new point afresh.

Let $\mathcal{U}$ be the extended generator for a PDMP without boundaries and $\mathcal{D}(\mathcal{U})$ the domain of $\mathcal{U}$. From Theorem 26.14 in [7] if $f \in \mathcal{D}(\mathcal{U})$ then, for $X = (i,\xi)$, $\xi = (h,x)$, we have that $f(i,\phi_i(\xi,t))$ is absolutely continuous in $\mathbb{R}_+$, and

$$\mathcal{U}f(X) = \mathcal{X}f(X) + \lambda(X)\int_E (f(Y) - f(X))Q(dY|X). \tag{2.6}$$

The following result can be obtained (see Fact 37 and Theorem 70 in Chapter 2 of [1]).

Theorem 2.1. *For each $i \in \mathcal{M}$ and all $t_0 \in \mathbb{R}_+$ there exists a unique continuous matrix function (called state transition matrix) $\Psi_i(.,t_0)$ solution of the homogeneous linear matrix differential equation*

$$\frac{\partial \Psi_i(t,t_0)}{\partial t} = A_i(t)\Psi_i(t,t_0),$$
$$\Psi(t_0,t_0) = I.$$

Moreover the unique solution $x(t)$ of $\dot{x}(t) = A_i(t)x(t)$, $x(t_0) = x_0$ is given by $x(t) = \Psi_i(t,t_0)x_0$

Remark 2.1. For the case in which A_i is constant we have that the state transition matrix is given by $\Psi_i(t,t_0) = e^{A_i(t-t_0)}$.

We show next that system (2.2) is a particular case of a PDMP without boundaries. For that we define the state space of the PDMP $\{X(t)\}$ as $E = \mathcal{M} \times \mathbb{R}_+ \times \mathbb{R}^n$, and the PDMP is defined as follows:

$$X(t) = (\theta(t),\xi(t)),\ \xi(t) = (h(t),x(t))$$

where $h(t)$ represents the elapsed time since the last jump. The three local parameters, the flow ϕ, the jump rate λ, and the transition measure Q, are defined as follows: for $X_0 = (i,\xi_0)$, $\xi_0 = (h_0,x_0)$, the flow is given by

$$\phi_i(\xi_0,t) = (h_0 + t,\, \Psi_i(t + h_0,h_0)x_0),$$

the jump rate by (see (2.3))

$$\lambda(X_0) = \lambda(i,\xi_0) = \lambda(i,(h_0,x_0)) = -\lambda_{ii}(h_0)$$

and, for any measurable function $f : E \to \mathbb{R}$, the transition measure $Q(.|X_0)$ is defined as

$$Qf(X_0) = Qf(i,\xi_0) = Qf(i,(h_0,x_0)) = \sum_{j\in\mathcal{M},j\neq i} p_{ij}(h_0)f(j,(0,x_0))$$

where $p_{ij}(h_0)$ is defined as

$$p_{ij}(h_0) = \begin{cases} \frac{\lambda_{ij}(h_0)}{-\lambda_{ii}(h_0)} & \text{if } \lambda_{ij}(h_0) > 0; \\ 0 & \text{otherwise .} \end{cases}$$

Notice that, according to Q defined above, after a jump, the inter-arrival time is set to zero (that is, $h(T_k) = 0$). Between jumps $h(t)$ represents the elapsed time since the last jump. We introduce next a standard assumption related to the PDMPs.

Assumption (24.3) in [7]: For all $t \in \mathbb{R}_+$, we have that

$$\mathcal{E}_X \left(\sum_k 1_{\{T_k \le t\}} \right) < \infty. \tag{2.7}$$

From Proposition 24.6-1 in [7] we have that this Assumption is satisfied if for some $c > 0$ we have that $-\lambda_{ii}(h) \le c$ for all $i \in \mathcal{M}$ and $h \ge 0$.

Suppose now that there exists timer dependent matrices $P_i(h)$, with $i \in \mathcal{M}$, $h \ge 0$, such that: $P_i(h) > 0$, $P_i(h)$ is absolutely continuous in $\mathbb{R}_+$ and uniformly bounded in h, that is, for some $d > 0$,

$$P_i(h) \prec dI. \tag{2.8}$$

Consider a function $v(X)$, with $X = (i,\xi)$, $\xi = (h,x)$, defined as

$$v(X) = v(i,(h,x)) = x^T P_i(h)x. \tag{2.9}$$

We recall that the Condition (31.2) in [7] holds if for each $t \geq 0$ and any $X = (i_0,(h_0,x_0))$ we have that

$$\mathcal{E}_X \left(\sum_{\{T_i \leq t\}} |v(X(T_i)) - v(X(T_i^-))| \right) < \infty. \tag{2.10}$$

In what follows we set $\mu_{max} = \sup_{i \in \mathcal{M}, h \in \mathbb{R}_+} \|P_i(h)\|$. Notice that $\mu_{max} \geq \mu(A_i(h))$ where $\mu(.)$ represents the matrix measure (see [12]).

Proposition 2.1. *For the function v defined in (2.9) we have that Condition (31.2) in [7] is satisfied.*

Proof: We need to show that (2.10) holds. We notice that on the set $\{T_k \leq t\}$,

$$v(X(T_k)) - v(X(T_k^-)) = x(T_k)^T (P_{\theta(T_k)}(0) - P_{\theta(T_{k-1})}(T_k - T_{k-1}))x(T_k) \tag{2.11}$$

and from (2.8) and (2.11) we get that

$$|v(X(T_k)) - v(X(T_k^-))| \leq 2d\|x(T_k)\|^2. \tag{2.12}$$

Notice now that on the set $\{T_k \leq t\}$ we have that $x(X(T_k)) = x(X(T_k \wedge t))$. Set

$$h(s) = \sum_{j=0}^{k-1} 1_{\{T_j \leq s < T_{j+1}\}}(s - T_j).$$

From Theorem 27 in [12],

$$\|x(T_k \wedge t)\|^2 \leq \|x_0\|^2 e^{\int_0^{T_k \wedge t} \mu(A(\theta(s),h(s)))ds} \leq \|x_0\|^2 e^{\mu_{max} t} \tag{2.13}$$

since $\mu(A(\theta(s)),h(s)) \leq \mu_{max}$. Therefore from (2.12), on the set $\{T_k \leq t\}$,

$$|v(X(T_k)) - v(X(T_k^-))| \leq 2d\|x_0\|^2 e^{\mu_{max} t}, \tag{2.14}$$

and from (2.14) and (2.7)

$$\mathcal{E}_X \left(\sum_{\{T_i \leq t\}} |v(X(T_i)) - v(X(T_i^-))| \right) \leq \mathcal{E}_X \left(\sum_k 1_{\{T_k \leq t\}} \right) 2d\|x_0\|^2 e^{\mu_{max} t} < \infty,$$

completing the proof.

Proposition 2.2. *We have that $v \in \mathcal{D}(\mathcal{U})$ and for $X = (i,\xi)$, $\xi = (h,x)$,*

$$\mathcal{U}v(X) = x^T\Big(A_i(h)^T P_i(h) + P_i(h)A(h) + \mathcal{X}P_i(h) + \textstyle\sum_{j\in\mathcal{M}\setminus\{i\}} \lambda_{ij}(h)P_j(0) + \\ +\lambda_{ii}(h)P_i(h)\Big)x \tag{2.15}$$

Proof: From Proposition 2.1 we have that $v \in \mathcal{D}(\mathcal{U})$ (see [7], page 70). From the definition of the flow ϕ, we notice that

$$v(i,(\phi_i(\xi,t))) = (\Psi_i(t+h,h)x)^T P_i(h+t)\Psi_i(t+h,h)x$$

and thus, from Theorem 2.1,

$$\mathcal{X}v(i,\xi) = x^T(A_i(h)^T P_i(h) + P_i(h)A_i(h) + \mathcal{X}P_i(h))x.$$

We also have that

$$Qv(i,\xi) = x^T(\sum_{j\in\mathcal{M},j\neq i} \lambda_{ij}(h)P_j(0))x.$$

Combining these results and from (2.6) we obtain (2.15).

Proposition 2.3. *Let $X_0 = (\theta_0,(0,x))$, where $P(\theta_0 = i) = \mu_i$, $i \in \mathcal{M}$, and $x \in \mathbb{R}^n$. We have that*

$$\mathcal{E}(v(X(t))) - \mathcal{E}(v(X_0)) = \mathcal{E}(v(X(t))) - \sum_{i\in\mathcal{M}} \mu_i x^T P_i(0)x = \mathcal{E}\Big(\int_0^t \mathcal{U}v(X(s))ds\Big). \tag{2.16}$$

Proof: Equation (2.16) follows from the PDMP differential formula presented in Theorem 31.3 in [7], combined with Proposition 2.1.

We are now in a position to state and prove the following lemma, which provides a matrix inequality based condition able to guarantee stability for the S-MJLS (2.2).

Lemma 2.1. *Consider the autonomous version of S-MJLS* (2.2) *with given timer-dependent transition rates* (2.3)*. For any given $\epsilon > 0$, if there exist timer-dependent positive definite matrices $P_i(h) \succ \epsilon I$ absolutely continuous in $\mathbb{R}_+$ and uniformly bounded in h such that*

$$A_i(h)^T P_i(h) + P_i(h)A_i(h) + \dot{P}_i(h) + \sum_{j\in\mathcal{M}\setminus\{i\}} \lambda_{ij}(h)P_j(0) + \lambda_{ii}(h)P_i(h) \prec -\epsilon I, \tag{2.17}$$

for all $(i,h) \in \mathcal{M} \times \mathbb{R}_+$, then the system is exponentially mean square stable.

Proof. Set $f(t) = \mathcal{E}\Big(v(X(t))\Big)$. As before, uniform boundedness of P_i implies there exists $d > 0$ (as in (2.8)) such that $P_i(h) \prec dI$ holds for all $h \geq 0$ and all i. Taking this bound together with (2.15), (2.16) and (2.17) we obtain that

$$\dot{f}(t) = \mathcal{E}\Big(\mathcal{U}v(X(t))\Big) \leq -\frac{\epsilon}{d} f(t),$$

which implies that $f(t) \leq f(0)e^{-\alpha t}$, for $\alpha = \epsilon/d$. From the fact that $P_i(h) \succ \epsilon I$ we get that $f(t) \geq \epsilon\mathcal{E}\Big(\|x(t)\|^2\Big)$ and this allows us to define $\beta = \frac{d}{\epsilon}$, which yields (2.4), showing the exponential mean square stability as desired. □

The following lemma provides a matrix inequality based condition able to provide an upper-bound for $\|\mathbb{G}\|_2$.

Lemma 2.2. *Consider an S-MJLS* (2.2) *with given timer-dependent transition rates* (2.3). *For any given* $\epsilon > 0$, *if there exist timer-dependent positive definite matrices* $P_i(h) \succ \epsilon I$ *absolutely continuous in* $\mathbb{R}_+$ *and uniformly bounded in* h *such that*

$$\begin{aligned} A_i(h)^T P_i(h) + P_i(h)A_i(h) + \dot{P}_i(h) + \textstyle\sum_{j\in\mathcal{M}\backslash\{i\}} \lambda_{ij}(h)P_j(0) + \lambda_{ii}(h)P_i(h) + \\ +C_{zi}(h)^T C_{zi}(h) \prec -\epsilon I, \end{aligned} \tag{2.18}$$

for all $(i,h) \in \mathcal{M} \times \mathbb{R}_+$, *then the system is exponentially mean square stable. In this case, for any given initial distribution* μ, *the* $\mathcal{H}_2$ *norm of* (2.2) *is limited by*

$$\|\mathbb{G}\|_2^2 \leq \sum_{i\in\mathcal{M}} \mu_i \mathbf{tr}(E_i^T P_i(0)E_i). \tag{2.19}$$

Proof. Notice that whenever the conditions of the present lemma are satisfied, the exponential mean square stability is guaranteed. Moreover, for $x(0) = 0$, an impulse input $w(t) = e_i\delta(t)$ defines a discontinuity at $t = 0$, in such a way that $x(0^+) = E_{\theta_0}e_s$, so that, combining (2.15), (2.16) and (2.18) we obtain that

$$\mathcal{E}\Big(\int_0^t \|z_{s,\theta_0}(s)\|^2 ds\Big) \leq \sum_{i\in\mathcal{M}} \mu_i(E_ie_s)^T P_i(0)E_ie_s. \tag{2.20}$$

Taking the sum for $s = 1$ to m and making the limit as $t \to \infty$ it follows from (2.20) that $\|G\|^2 \leq \sum_{i\in\mathcal{M}} \mu_i \mathbf{tr}(E_i^T P_i(0)E_i)$, concluding the proof.

□

Clearly, when trying to obtain the least upper bound, one can define an optimisation problem with respect to the decision variables $\epsilon I \prec P_i(h) \prec \epsilon^{-1}I$, using (2.19) as objective function and (2.18) as constraints. Nonetheless, in such a case, the resulting optimisation problem is infinite-dimensional, so that to make it computationally tractable, some relaxation must be performed.

Remark 2.2. For the MJLS case, one can take constant positive definite matrices P_i in Lemma 2.1 and in Lemma 2.2. Moreover, in this case, the conditions are also necessary [4].

2.3 Processes with Rational Transition Rates

In this section, we will discuss some of the continuous probability distributions with support in $[0,\infty)$ that define rational transition rates. This aspect will be of great importance when deriving efficient computational methods able to test the conditions from Lemmas 2.1 and 2.2. The reader may refer to [29] for details.

It is important to notice that the transition rates $\lambda_{ij}(h)$ have the structure

$$\lambda_{ij}(h) = p_{ij}\frac{f_i(h)}{1 - F_i(h)} \tag{2.21}$$

for $(i,j) \in \mathcal{M} \times \mathcal{M}$, $i \neq j$, where p_{ij} is the jump probability from mode i into mode j after the end of its sojourn time, $f_i(h)$ is the $i-th$ mode probability density function for the sojourn time, and $F_i(h)$ its correspondent cumulative distribution function.

2.3.1 Exponential Distribution

The exponential distribution has one rate parameter and is the standard sojourn time distribution for the classic MJLS, as it is the probability distribution of the time between events in a Poisson point process, that is a process in which events occur continuously and independently at a constant average rate. It is the only continuous distribution that presents the memorylessness property, meaning that the distribution of the remaining sojourn time in one specific mode does not depend on how much time the mode is already active.

The probability density function and the cumulative distribution function of the Exponential distribution with parameter $\Lambda > 0$ are, respectively

$$f(x;\Lambda) = \begin{cases} 0, & x < 0 \\ \Lambda e^{-\Lambda x}, & x \geq 0 \end{cases} \qquad F(x;\Lambda) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\Lambda x}, & x \geq 0 \end{cases}$$

from where it is clear that the transition rates (2.21) calculated to this specific case provide constant terms.

Notice that whenever considering that all modes have exponentially distributed sojourn times, and the state-space matrices are timer-independent, we recover the classic MJLS results for stability and $\mathcal{H}_2$ norm calculation simply by considering that the P_i matrices in Lemmas 2.1 and 2.2 are timer-independent.

2.3.2 Rayleigh Distribution

The Rayleigh Distribution normally appears in two-dimensional (or complex) problems, whenever we are interested in the magnitude (or the absolute value) when each component is uncorrelated, normally distributed with equal variance, and zero mean. It presents one scale parameter $\sigma > 0$, and probability density function and the cumulative distribution function

$$f(x;\sigma) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}, & x \geq 0 \end{cases} \qquad F(x;\sigma) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^2/(2\sigma^2)}, & x \geq 0 \end{cases}$$

which implies that, for this case, the transition rates (2.21) are of the form $\lambda_{ij}(h) = (p_{ij}/\sigma_i^2)h$. Thus, the transition rate, from mode i to mode j, for a Rayleigh distributed sojourn time is not timer-independent, but linearly increasing with slope p_{ij}/σ_i^2.

2.3.3 Erlang Distribution

The Erlang distribution is a two-parameter family of continuous probability distributions. The two parameters are a positive integer k, which defines the overall structure of the distribution, and another positive real number Λ, which defines its rate of decay. For the specific case of $k = 1$, the Erlang distribution reduces itself to the exponential distribution with parameter Λ. Moreover, for the general case, it is the distribution of a sum of k independent exponential variables with the same parameter Λ each.

The probability density function and the cumulative distribution function of the Erlang distribution are, respectively

$$f(x;k,\Lambda) = \begin{cases} 0, & x < 0 \\ \dfrac{\Lambda^k x^{k-1} e^{-\Lambda x}}{(k-1)!}, & x \geq 0 \end{cases} \quad F(x;k,\Lambda) = \begin{cases} 0, & x < 0 \\ 1 - \displaystyle\sum_{n=0}^{k-1} \frac{1}{n!} e^{-\Lambda x} (\Lambda x)^n, & x \geq 0 \end{cases}$$

where $(\lambda x)^n := 1$ for $(x,n) = (0,0)$.

Thus, the transition rate, from mode i to mode j, for an Erlang distributed sojourn time is

$$\lambda_{ij}(h) = p_{ij} \frac{\Lambda_i^{k_i} h^{k_i - 1}}{(k_i - 1)! \sum_{n=0}^{k_i - 1} (\Lambda_i h)^n / n!}. \tag{2.22}$$

2.3.4 Chi Distribution

The Chi Distribution is a generalisation for the Rayleigh distribution. It has one natural parameter k representing the degrees of freedom ($k = 2$ for Rayleigh). When k is an even number, that is, $k = 2m$ for some natural number m, then the corresponding transition rate is rational.

To check this fact, we start with the probability density function and the cumulative distribution function of the Chi distribution

$$f(x;k) = \begin{cases} 0, & x < 0 \\ \dfrac{x^{k-1} e^{-x^2/2}}{2^{(k/2)-1} \Gamma(k/2)}, & x \geq 0 \end{cases} \qquad F(x;k) = \begin{cases} 0, & x < 0 \\ P(k/2, x^2/2), & x \geq 0 \end{cases}$$

where $\Gamma(\cdot)$ and $P(\cdot,\cdot)$ are the Gamma and the Regularised Gamma functions, respectively. Whenever $m = k/2$ is a natural number, we have

$$P(m,x) = 1 - \sum_{\ell=0}^{m-1} \frac{x^\ell e^{-x}}{\ell!} \tag{2.23}$$

and thus, setting m_i as the parameter associated to the Chi Distribution at the mode of operation i, the corresponding transition rates are

$$\lambda_{ij}(h) = p_{ij} \frac{h^{2m_i - 1}}{2^{m_i - 1} (m_i - 1)! \sum_{\ell=0}^{m_i - 1} (h^2/2)^\ell / \ell!}. \tag{2.24}$$

2.3.5 Weibull Distribution

The Weibull distribution is widely used in reliability engineering for *time-to-failure* modelling. Therefore, there exists a valuable practical appeal to consider Weibull distributed sojourn time for damage-tolerant control problems.

The Weibull distribution has two parameters, $k > 0$ known as the shape parameter, and $\Lambda > 0$ called the scale parameter. For the transition rate to be rational, we need to impose that $k \in \mathbb{N}$. This implies that we may only model the more usual cases where the failure rate increases with time, that is, an ageing process causes the parts to be more likely to fail as time passes.

The probability density function and the cumulative distribution function of the Weibull distribution are

$$f(x;k,\Lambda) = \begin{cases} 0, & x < 0 \\ \dfrac{k}{\Lambda}\left(\dfrac{x}{\Lambda}\right)^{k-1} e^{-(x/\Lambda)^k}, & x \geq 0 \end{cases} \qquad F(x;k,\Lambda) = \begin{cases} 0, & x < 0 \\ 1 - e^{-(x/\Lambda)^k}, & x \geq 0 \end{cases}$$

which implies that, setting k_i as the parameter associated to the Weibull Distribution at the mode of operation i, the transition rate from modes i to j is

$$\lambda_{ij}(h) = p_{ij}\frac{k_i h^{k_i-1}}{\Lambda_i^{k_i}} \tag{2.25}$$

that is rational (more specifically, polynomial) in h whenever $k_i \in \mathbb{N}$.

2.4 Sum-of-Squares Optimisation

In this section, we provide an approach to developing stability and $\mathcal{H}_2$ performance conditions for an S-MJLS with rational sojourn times. It is based on the so-called Sum-of-Squares (SOS) Optimisation (see for instance [2] and references therein). For that, we assume that

$$\lambda_{ij}(h) = p_{ij}\frac{\eta_i(h)}{\rho_i(h)} \tag{2.26}$$

for all $(i,j) \in \mathcal{M} \times \mathcal{M}$, $i \neq j$, where $\eta_i(h)$ and $\rho_i(h)$ are polynomial functions in h with $\rho_i(h) > 0$ for all $h \geq 0$. Clearly, $p_{ii} = 0$, $p_{ij} \geq 0$, $\sum_{j\in\mathcal{M}\setminus\{i\}} p_{ij} = 1$ and $\lambda_{ii}(h) = -\sum_{j\in\mathcal{M}\setminus\{i\}} \lambda_{ij}(h)$.

2.4.1 $\mathcal{H}_2$ Norm Calculation

One commonly used solution to Differential Inequalities, such as the ones that appear in Lemmas 2.1 and 2.2, is to discretise the timer domain $h \geq 0$ into a grid of points and approximate the derivative term inside the grid. This method is applied, for example, in [16] and [5]. One should notice that with the correct application of multiples inequalities per grid point, one might be able to apply robust stability arguments and show that stability (and guaranteed cost) is obtained for the whole timer domain.

Another interesting approach is to use SOS Optimisation. As we shall see in the sequel, such an approach requires a particular structure for matrices $P_i(h)$ in (2.18); in particular, that they must be themselves ratio of polynomials. Given this, we assume that

$$P_i(h) = Q_i(h)/d_i(h) \tag{2.27}$$

where $Q_i(h)$ are timer-dependent differentiable (but not yet polynomial) positive definite matrices and $d_i(h)$ are scalar polynomials in h, with $d_i(h) > 0$ for all $h \geq 0$. In order to ease further notation, we will also assume that $d_i(0) = 1$. Notice that $d_i(h)$ are differentiable since they are scalar polynomials in h.

Lemma 2.3. *Consider the S-MJLS* (2.2) *with given timer-dependent transition rates* (2.26)*. Given any $\epsilon > 0$ and N scalar polynomials $d_i(h)$ such that $d_i(h) > 0$ for all $h \geq 0$ and $d_i(0) = 1$, if there exist timer-dependent positive definite matrices $Q_i(h)$ absolutely continuous in $\mathbb{R}_+$ such that $Q_i(h)/d_i(h) < \epsilon^{-1}I$ and $Q_i(h)/d_i(h) > \epsilon I$ for all $h \geq 0$ and*

$$\begin{aligned}&\rho_i(h)d_i(h)\Big(A_i(h)^T Q_i(h) + Q_i(h)A_i(h) + \dot{Q}_i(h) + d_i(h)C_{zi}(h)^T C_{zi}(h)\Big) - \\ &- (\rho_i(h)\dot{d}_i(h) + \eta_i(h)d_i(h))Q_i(h) + d_i^2(h)\eta_i(h)\sum_{j\in\mathcal{M}\setminus\{i\}} p_{ij}Q_j(0) \prec -\epsilon\rho_i(h)d_i^2(h)I\end{aligned} \tag{2.28}$$

for all $(i,h) \in \mathcal{M} \times \mathbb{R}_+$, then the S-MJLS is exponentially mean square stable. In this case, for any given initial distribution μ, the $\mathcal{H}_2$ norm of (2.2) *is limited by*

$$\|\mathbb{G}\|_2^2 \leq \sum_{i\in\mathcal{M}} \mu_i \mathbf{tr}(E_i^T Q_i(0)E_i). \tag{2.29}$$

Proof. This is a direct application of the results from Lemma 2.2 assuming the particular structure of the matrices given in (2.27). For that, just recall that $d_i(h) > 0$, $d_i(0) = 1$ and $\rho_i(h) > 0$ for all $h \geq 0$. □

Remark 2.3. For $\epsilon > 0$ and N scalar polynomials $d_i(h)$ as in Lemma 2.3 set $\mathcal{S}$ such that $\Big\{Q_i(h)\Big\} \in \mathcal{S}$ if for every $i = 1,\ldots,N$, $Q_i(h)$ satisfy the conditions of Lemma 2.3. From the timer-dependent constraints (2.28) and the upper bound in (2.29) we can set the following optimisation problem:

$$\zeta^* = \inf_{\{Q_i(h)\}\in\mathcal{S}} \Big\{\sum_{i\in\mathcal{M}} \mu_i \mathbf{tr}(E_i^T Q_i(0)E_i)\Big\}, \tag{2.30}$$

so that from (2.29) we get that $\|\mathbb{G}\|_2^2 \leq \zeta^*$. In Section 2.4.3, we present more detail on the numerical solution of (2.30) using an SOS approach.

2.4.2 $\mathcal{H}_2$ State Feedback Design

We now focus on extending the previous results to the $\mathcal{H}_2$ state feedback control design problem. To that end, we consider a timer-independent version of system (2.2), that is

$$\mathbb{G}_c : \begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) + E(\theta(t))w(t), \\ z(t) = C_z(\theta(t))x(t) + D_z(\theta(t))u(t) \end{cases} \tag{2.31}$$

where, additionally to the signals and matrices described in (2.2), $u(t) \in \mathbb{R}^q$ is the control input and $B(\theta(t)) \in \mathbb{R}^{n\times q}$ and $D_z(\theta(t)) \in \mathbb{R}^{p\times q}$ also depend on the semi-Markovian continuous-time process $\{\theta(t),\, t \geq 0\}$. For notation simplicity, we also denote $B(\theta(t)) = B_i$, $D_z(\theta(t)) = D_{zi}$ whenever $\theta(t) = i \in \mathcal{M}$.

Theorem 2.2. *Consider the MJLS $\mathbb{G}_c$ given by* (2.31) *with transition rates given by* (2.26) *and zero initial conditions $x(0) = 0$. For some given $\rho > 0$, $\epsilon > 0$ and positive scalar polynomial functions $d_i(h) > 0$, $d_i(0) = 1$, $\mathbb{G}_c$ is stochastically stabilisable by a timer-dependent state-feedback control law*

$$u(t) = K_{\theta(t)}(h)x(t) \tag{2.32}$$

and the closed-loop system performance is upper-bounded by $\|\mathbb{G}_c\|_2^2 < \rho$ if, for all $h \geq 0$, there exist timer-dependent symmetric matrices $X_i(h) \succ 0$, $Z_{ij}(h) \succ 0$, symmetric matrices $W_i \succ 0$, and timer-dependent matrices $Y_i(h)$ with $Y_i(h)X_i(h)^{-1}$ uniformly bounded, such that $\epsilon I \prec X_i(h)d_i(h) \prec \epsilon^{-1}I$, the matrix inequalities

$$\sum_{i\in\mathcal{M}} \mu_i \mathbf{tr}(W_i) < \rho, \quad \begin{bmatrix} W_i & \star \\ E_i & X_i(0) \end{bmatrix} \succ 0, \tag{2.33a}$$

and the timer-dependent matrix inequalities

$$\begin{bmatrix} Z_{ij}(h) & \star \\ X_i(h) & X_j(0) \end{bmatrix} \succ 0, \quad \begin{bmatrix} \Xi_i(h) & \star & \star \\ \rho_i(h)(C_{zi}X_i(h) + D_{zi}Y_i(h)) & -\rho_i(h)I & \star \\ \rho_i(h)X_i(h) & 0 & -\rho_i(h)\epsilon^{-1}I \end{bmatrix} \prec 0 \tag{2.33b}$$

where

$$\begin{aligned} \Xi_i(h) =& d_i(h)\rho_i(h)(A_iX_i(h) + X_i(h)A_i^T + B_iY_i(h) + Y_i(h)^TB_i^T - \dot{X}_i(h)) + \\ &+ (\rho_i(h)\dot{d}_i(h) - \eta_i(h)d_i(h))X_i(h) + \eta_i(h)\sum_{j\in\mathcal{M}\setminus\{i\}} p_{ij}Z_{ij}(h), \end{aligned}$$

hold for all $h \geq 0$. *In the feasible case, the aforementioned control gains can be obtained by* $K_i(h) = Y_i(h)X_i(h)^{-1}$.

Proof. Let us assume that, for given ρ and $d_i(h) \geq 0$, with $d_i(0) = 1$, (2.33a) and (2.33b) are feasible. Then from inequality (2.33b), using Schur Complement, we have that

$$\begin{aligned} Z_{ij}(h) - X_i(h)X_j^{-1}(0)X_i(h) &\succ 0 \\ \Xi_i(h) + \rho_i(h)(C_{zi}X_i(h) + D_{zi}Y_i(h))^T(C_{zi}X_i(h) + D_{zi}Y_i(h)) &\prec -\epsilon\rho_i(h)X_i(h)^2. \end{aligned}$$

Pre- and post-multiplying both previous inequalities by $d_i(h)X_i(h)^{-1}$, denoting $Q_i(h)/d_i(h) = d_i(h)X_i(h)^{-1}$, and recalling that $Y_i(h) = K_i(h)X_i(h)$, provides

$$\begin{aligned} \frac{Q_i(h)}{d_i(h)}Z_{ij}(h)\frac{Q_i(h)}{d_i(h)} &\succ d_i(h)^2Q_j(0) \\ \widehat{\Xi}_i(h) + d_i(h)^2\rho_i(h)(C_{zi} + D_{zi}K_i(h))^T(C_{zi} + D_{zi}K_i(h)) &\prec -\epsilon\rho_i(h)d_i^2(h)I \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} \widehat{\Xi}_i(h) = {} & d_i(h)\rho_i(h)(Q_i(h)(A_i + B_iK_i(h)) + (A_i + B_iK_i(h))^TQ_i(h) + \dot{Q}_i(h)) - \\ & - (\rho_i(h)\dot{d}_i(h) + d_i(h)\eta_i(h))Q_i(h) + \eta_i(h)\sum_{j\in\mathcal{M}\setminus\{i\}} p_{ij}\frac{Q_i(h)}{d_i(h)}Z_{ij}(h)\frac{Q_i(h)}{d_i(h)} \end{aligned} \tag{2.35}$$

and noticing that substituting $Q_i(h)Z_{ij}(h)Q_i(h)/d_i(h)^2$ by $d_i(h)^2Q_j(0)$ in (2.35) does not change the sign of the second inequality in (2.34). But now this inequality is exactly (2.28) when replacing the open-loop matrices by the closed-loop ones: $A_i(h) \leftarrow A_i + B_iK_i(h)$ and $C_{zi}(h) \leftarrow C_{zi} + D_{zi}K_i(h)$. Moreover, as it was assumed that $Y_i(h)X_i(h)^{-1} = K_i(h)$ are uniformly bounded, so are the closed-loop matrices $A_i(h)$ and $C_{zi}(h)$.

Finally, from (2.33a), we have that

$$\sum_{i\in\mathcal{M}} \mu_i\mathbf{tr}(E_i^TQ_i(0)E_i) < \sum_{i\in\mathcal{M}} \mu_i\mathbf{tr}(W_i) < \rho$$

showing that $\|\mathbb{G}\|_2^2 < \rho$, and thus concluding the proof. □

Remark 2.4. For $\epsilon > 0$ and N scalar polynomials $d_i(h)$ as in Theorem 2.2 set $\mathcal{S}_c$ such that $\left\{X_i(h), Y_i(h), Z_{ij}(h), W_i, \rho\right\} \in \mathcal{S}_c$ if for every $i,j = 1,\ldots,N$, we have that $X_i(h), Y_i(h), Z_{ij}(h), W_i,\rho$ satisfy the conditions of Theorem 2.2. The least upper-bound of the closed-loop $\mathcal{H}_2$ norm of $\mathbb{G}$ can be obtained by solving the optimisation problem

$$\rho^* = \inf_{\left\{X_i(h), Y_i(h), Z_{ij}(h), W_i, \rho\right\} \in \mathcal{S}_c} \rho \quad . \tag{2.36}$$

From Theorem 2.2 we have that $\|\mathbb{G}\|_2^2 \leq \rho^*$. Notice that, for fixed $d_i(h)$, the objective function and the constraints are affine with respect to the decision variables.

Remark 2.5. The state-feedback control gains developed in this subsection are timer-dependent. If X_i and Y_i are forced to be timer-independent, the designed gains became timer-independent. Nonetheless, this might induce a great level of conservatism.

2.4.3 Implementation Details

In this brief subsection, we point out some aspects of the numerical implementation of the timer-dependent optimisation problems presented in this section. As stated before, our main numerical approach is based on SOS optimisation [2] and this motivated the structure imposed to the positive matrices P_i presented in (2.27). Yet another assumption must be made so that to make the optimisation problems like (2.30) or (2.36) convex and, therefore, numerically solvable: all variables must be polynomial with a given degree. Hence, all matrix variables, that is, Q_i in (2.30) and X_i, Y_i and Z_{ij} in (2.36) are polynomial matrices in h for all i,j.

Sum-of-squares problems, albeit still not as common as linear matrix inequalities optimisation problems, can be efficiently parsed and solved by a selection of software available in the literature. Indeed, algorithms implemented in parsers such as SOSTools [27] and Yalmip [23] convert SOS constraints with polynomial decision variables into a semidefinite optimisation problem, which, in turn, can be solved by several optimisation solvers, such as Mosek [25]. Hence, generally speaking, solving SOS or SDP problems can be seen as equivalent tasks and both can be quite challenging when dealing with dynamic systems with a large number of states or modes. In the examples of this note, we used Yalmip and Mosek [25] to solve the SOS optimisation problems.

In what follows, we present some specific remarks on the numerical implementation of the SOS problems proposed here.

Remark 2.6. One of the main parameters of the proposed approach is the degree of the polynomial variables. Indeed, higher degrees typically yield better solutions at the expense of computational time. It is important, however, to note that such degrees are not completely free for the designer to chose, as all variables are linked by the same constraints. For instance, the inequalities $\epsilon I \prec Q_i(h)/d_i(h) \prec \epsilon^{-1} I$, which must hold for all $h \geq 0$, imply that $\deg(Q_i) = \deg(d_i)$. We must also have that $\deg(Z_{ij}) = 2\deg(X_i)$, $\deg(Y_i) \leq \deg(X_i)$ and $\deg(X_i) = \deg(d_i)$ in order to ensure the feasibility of the constraints in (2.36).

Remark 2.7. Whenever all timer-dependent variables are polynomial matrices in h, the design conditions presented in the paper are matrix polynomial inequalities in $h \geq 0$. Therefore, through some usual arguments based on the Positivstellensatz [30], one might test its negativity inside the required domain through a convex optimisation problem involving its sum of squares decomposition, which is a convex optimisation problem, and thus computationally tractable. Indeed, the condition

$$\Phi_i(h) \prec 0, \quad \text{for } h \geq 0$$

may be replaced by

$$\Phi_i(h) \prec -h\Gamma_i(h)$$

for some $\Gamma_i(h) \succ 0$ for all h. Hence, computationally, we search for an SOS decomposition of $-\Phi_i(h) - h\Gamma_i(h)$ for some additional SOS polynomial variable $\Gamma_i(h)$. Note that, as to ensure the feasibility of the last inequality for $h \to \infty$, it is necessary that $\deg(\Gamma_i) \leq \deg(\Phi_i) - 1$, which means that results do not improve if $\deg(\Gamma_i)$ is increased beyond this point.

Remark 2.8. In the proposed formulation, the polynomial denominators $d_i(h)$ are not decision variables, but instead, they should be given to the optimisation method. This could be modified, in the expectation that better results are obtained, but with the additional cost of solving the resulting multi-convex problem in a block coordinate descent method.

Remark 2.9. If the realisation matrices A_i and C_i are themselves rational functions on h with positive denominators for $h \geq 0$, the results presented here can still be applied with minor modifications.

2.5 Numerical Examples

Example 2.1. To illustrate the results developed in this note, we consider a two-mode semi-Markovian jump system with realisation given by the state-space matrices

$$A_1 = \begin{bmatrix} 0.7 & -4 \\ 0 & -7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -7 & 4 \\ 0 & 0.7 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_1 = C_2 = I_2.$$

Notice that both matrices A_1 and A_2 are not Hurwitz. For this example, we consider that $\mu = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_i(h)$ is a fourth-order matrix polynomial, and $d_i(h) = 1 + h^4$.

Three possible distribution scenarios are considered for the sojourn time: Exponential, Rayleigh, and Erlang. Our first goal is to compare the H_2 guaranteed quadratic cost (squared norm) attained in each of these three cases considering the same mean sojourn time $\overline{S_t}$ for all distributions. Hence, we take $\Lambda_i = \overline{S_t}^{-1}$ for the Exponential case, $\sigma_i = \overline{S_t}\sqrt{2/\pi}$ for the Rayleigh distribution and $\Lambda_i = k_i/\overline{S_t}$ for the Erlang scenario; we take $k_i = 3$ for both modes in this last case.

The SOS optimisation problem presented in Lemma 2.3 was solved for each distribution (for the Exponential case, we used equivalent LMI-based conditions [4]), yielding the results shown in Figure 2.1. As each mode is itself unstable, it is expected that the H_2 performance index should increase with the mean sojourn time. For the Exponential case, this growth is monotonic, and at around $\overline{S_t} \approx 0.65$ the markovian system becomes unstable. For the Erlang case, the system becomes unstable at around $\overline{S_t} \approx 1.90$ which, as expected, is less than $k_i = 3$ times the value for which the markovian system becomes unstable. Finally, for the Rayleigh case, the system is stable for the whole interval considered here. As, in the present Erlang case, we have $\Lambda_i = 3/\overline{S_t}$, we considered only the scenarios where $\overline{S_t} \geq 0.1$ to minimise possible numerical issues of having large parameters.

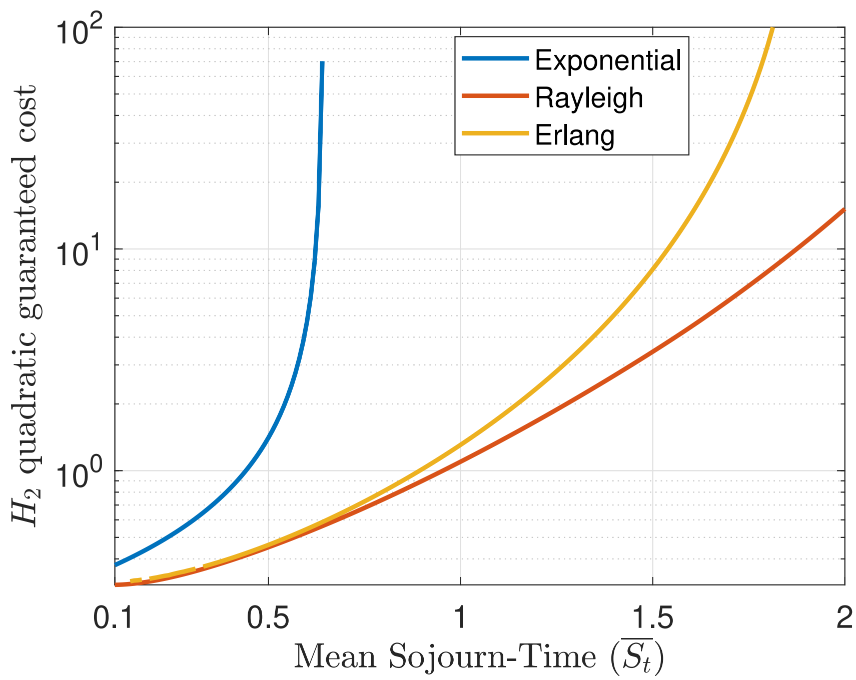

Figure 2.1: H_2 quadratic guaranteed cost (squared norm for the Exponential Distribution)

Considering $\overline{S_t} = 0.6$, the corresponding Q_i matrices for the Rayleigh distributed case are

$$Q_1(h) = \begin{bmatrix} Q_{1,11}(h) & Q_{1,12}(h) \\ Q_{1,12}(h) & Q_{1,22}(h) \end{bmatrix}, \quad Q_2(h) = \begin{bmatrix} Q_{2,11}(h) & Q_{2,12}(h) \\ Q_{2,12}(h) & Q_{2,22}(h) \end{bmatrix}$$

where

$$\begin{aligned}
Q_{1,11}(h) &= 0.828h^4 - 2.303h^3 + 3.856h^2 - 3.039h + 1.457 \\
Q_{1,12}(h) &= -1.726h^4 + 3.064h^3 - 2.243h^2 + 1.38h - 0.7059 \\
Q_{1,22}(h) &= 7.392h^4 + 0.4102h^3 + 2.492h^2 + 0.09725h + 0.4821 \\
Q_{2,11}(h) &= 1.798h^4 - 0.5252h^3 - 0.04118h^2 + 0.4058h + 0.1007 \\
Q_{2,12}(h) &= -0.8662h^4 + 0.2221h^3 + 0.08897h^2 - 0.2394h + 0.02569 \\
Q_{2,22}(h) &= 2.015h^4 - 3.991h^3 + 5.324h^2 - 3.714h + 1.791
\end{aligned}$$

To validate the results obtained so far, we performed a Monte Carlo statistical test, which involved $N_{sim} = 30000$ simulations, where we again considered the mean sojourn time $\overline{S_t} = 0.6$.

	Parameters	Guaranteed Cost	Sample Mean	Sample Variance
Rayleigh	$\sigma = 0.6\sqrt{2/\pi}$	0.5270	0.5223	0.0787
Erlang	$(k,\Lambda) = (3{,}5)$	0.5458	0.5411	0.1838

Table 2.1: Monte Carlo simulation results for Rayleigh- and Erlang-distributed sojourn times. Both the sample mean and the sample variance values refer to the H_2 cost computed in the 30000 simulations.

As pointed out in Table 2.1, the results obtained are consistent with the guaranteed cost calculated using Lemma 2.3, that is, the sample mean cost yielded by the 30000 Monte Carlo simulations performed is smaller than the guaranteed cost, as expected.

For the second part of the example, we focus only on the Rayleigh distribution and aim now to devise state-feedback control gains. We assume the same system as before, replacing the C matrices and constructing the B and D matrices as

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} I_2 \\ 0_{1\times 2} \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 0_{2\times 1} \\ 0.5 \end{bmatrix}.$$

Taking a mean sojourn time of 0.6, we design a timer-dependent state-feedback control law choosing the degrees of X and Y as 4 and 3, respectively. Finally, following the same strategy from the analysis part, we chose $d_i(h) = 1 + h^4$.

Solving the optimisation problem described in (2.36) provides a closed-loop guaranteed quadratic H_2 cost of 0.4480. Figure 2.2 shows the behaviour of the designed feedback gains with respect to the timer $h \in [0{,}2]$. For the given distribution and parameter σ, the probability of a sojourn time larger than 2 seconds is lower than 2×10^{-4}.

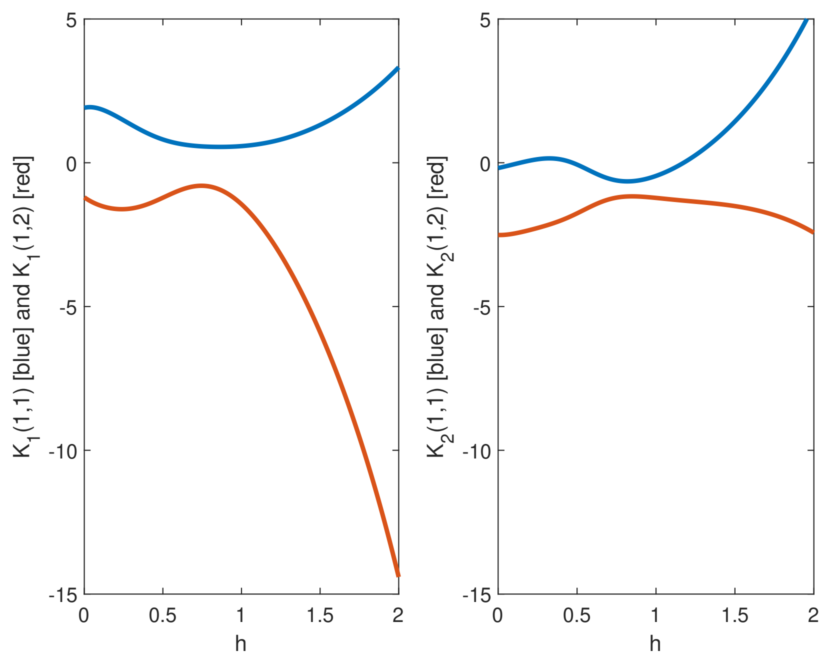

Figure 2.2: State-Feedback Gains

We conclude the Example by performing again a Monte-Carlo Simulation with $N_{sim} = 30000$ simulations for the closed-loop system. The resulting sample mean is 0.4460 with a sample variance of 0.0154, which, once again, is consistent with the values obtained by the SOS program.

Example 2.2. As the second example, inspired by [11], we adapted the linearized model of the unstable lateral dynamics of the unmanned aircraft model from [14]. The nominal state-space matrices are

$$\hat{A} = \begin{bmatrix} -11.4540 & 2.7185 & -19.4399 & 0 \\ 0.5068 & -2.9875 & 23.3434 & 0 \\ 0.0922 & -0.9957 & -0.4680 & 0.3256 \\ 1.0000 & 0.0926 & 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 78.4002 & -2.7282 \\ -3.4690 & 13.9685 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

and $\hat{E} = I_4$. See the aforementioned works for more information about this model.

We considered $N = 3$ modes of operation, where $\theta(t) = 1$ defines the nominal one, thus $(A,B,E)_1 = (\hat{A},\hat{B},\hat{E})$. Mode 2 considers a complete failure of the first actuator and mode 3 considers both a complete failure of the first actuator and a partial failure the second one. Thus $(A,B,C)_2 = (\hat{A},B_2,\hat{E})$ and $(A,B,E)_3 = (\hat{A},B_3,\hat{E})$, where

$$B_2 = \hat{B}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \hat{B}\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Finally, we also considered

$$C_i = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}, \quad i \in \{1,2,3\}.$$

We assumed that the transition rates between the modes increase linearly, that is $\lambda_{ij}(h) = (p_{ij}/\sigma_i^2)h$, which corresponds to Rayleigh distributed sojourn times. Specifically, we used the transition probability matrix $\Pi = [p_{ij}]$ where

$$\Pi = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 11/15 & 0 & 4/15 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

and $(\sigma_1,\sigma_2,\sigma_3) = (10/3, 2/3, 1/2) \times \sqrt{2/\pi}$. Lastly, we take that $\theta(0) = 1$ with probability one.

For different orders of the polynomial matrices $Q_i(h)$, we solved the state-feedback design problem and obtained the guaranteed costs presented in Table 2.2. In the same table, we provide the parser and the solver time of the corresponding semi-definite optimisation problem in an Intel Core i7-8750H CPU, 2.20 GHz, 16Gb RAM, running Windows 10 and Matlab R2020a. Parsing of the problem was performed by Yalmip [23] and solved by Mosek [25].

	$d = 2$	$d = 4$	$d = 8$	$d = 16$
H_2 Guaranteed Cost	1.3529	1.3378	1.2221	1.1301
Parser Time (a)	4.4590	4.5616	6.1803	12.2217
Solver Time (b)	4.7020	9.2894	34.8857	165.7873
Total Time (a+b)	9.161	13.851	41.0660	178.0090

Table 2.2: Guaranteed cost for the closed-loop H_2 performance, parser- and solver-time (in seconds), where $\deg(Q_i(h)) = d$.

One could argue that the true distribution of the sojourn time is not very relevant for the global performance of such system, and only its mean value is of importance. For testing such conjecture, we can propose a heuristic where we construct a Markov Jump Linear System with the same state-space matrices, but where the transition rates are now constant. That implies that the new sojourn time is exponentially distributed. We keep the same transition probability matrix Π and we choose the parameters of the exponential distribution in such a way that the mean sojourn times are the same as in the true Rayleigh case. For this case, we calculate the (timer-invariant) optimal H_2 state-feedback control law through standard LMIs.

Closing the loop with such controller, we performed $N_{sim} = 30000$ Monte-Carlo simulation of the original system where the sojourn time is Rayleigh distributed. The H_2 cost was estimated to be 1.3390. Observe that the proposed SOS design procedure in the present paper provides a guaranteed cost better than such value for any $d \geq 4$. In particular, for $d = 16$, our result provides a cost at least 15.6% smaller than such heuristic. Thus, the given argument that only the mean value of the sojourn time is of interest does not hold for this case.

2.6 Conclusions

In this paper, we presented convex optimisation based solutions for the $\mathcal{H}_2$ State-Feedback Control problem considering Continuous Semi-Markov Jump Linear Systems with rational transition rates. Some comparisons are provided. Future works will focus on more general sojourn time distributions, timer-independent gains, $\mathcal{H}_\infty$ norm, and filter and output-feedback problems.

Acknowledgements

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CHAPTER 3

$\mathcal{H}_2$ Output-Feedback Cluster Control for Continuous Semi-Markov Jump Linear Systems with Erlang Dwell Times

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ABSTRACT

This paper addresses the $\mathcal{H}_2$ output-feedback control problem for continuous-time semi-Markov jump linear systems (S-MJLS) with Erlang dwell times. To this end, we exploit the fact that any Erlang-distributed variable may be expressed as the sum of exponential random variables to devise a Markov-jump system equivalent to the original S-MJLS. This equivalence allows us to derive finite-dimensional control design conditions for Erlang S-MJLS. Numerical examples illustrate the main features of the results presented in the paper.

INDEX TERMS

Stochastic Systems, semi-Markov jump linear Systems, $\mathcal{H}_2$ control.

3.1 Introduction

Markov jump linear systems (MJLS) are stochastic dynamic models of great practical interest in several applications [4]. Such systems are composed of several modes of operation – or subsystems – which are selected by a random variable that *jumps* amongst these subsystems. There are two key stochastic properties these systems present: (1) jumps occur with probabilities that only depend on the current mode, as in a Markov chain, and (2) the *dwell times*, which correspond to the time a subsystem remains active (between two consecutive jumps), are random variables described by exponential distributions. Thus, both aspects present the memoryless property in this class of systems.

A more general class of stochastic systems is the one composed of semi-Markov jump linear systems (S-MJLS). In this class, dwell times may be modelled by arbitrary probability distributions whilst the stochastic process still respects the Markov property. Consequently, S-MJLS can be applied to a broader range of practical situations than those described by MJLS. However, this flexibility comes together with an increase in complexity, such as that the jump rates of S-MJLS are now dependent on the dwell time. Consequently, analysis and synthesis conditions for these systems become more challenging.

Among the distributions that can be used for the dwell times in S-MJLSs, the Erlang one draws attention due to its close relationship with the exponential distribution [28]. This paper exploits this relationship to devise an equivalent MJLS to an S-MJLS with Erlang distributed dwell times. As far as the authors are aware of, this is the first time that the control problem of Erlang S-MJLSs has been tackled considering this equivalence with MJLSs. We highlight next the main novelties with respect to existing results.

1) Our development leads to necessary and sufficient LMI-based stability and $\mathcal{H}_2$ performance conditions as well as LMI-based $\mathcal{H}_2$ control design conditions for Erlang S-MJLSs. In this sense, our results are less conservative than the ones presented in [33, 34], which provide only sufficient conditions for the stability problem, and than the ones in [33], which consider the state-feedback $\mathcal{H}_2$ control problem.

2) Our approach allows us to tackle the $\mathcal{H}_2$ static output feedback cluster control design problem, based on LMI optimisation problems. In this case, it is assumed that the state space of the semi-Markov process can be written as the union of disjoint sets (clusters), and the only information available to the controller is to which cluster the semi-Markov model belongs, as well as an output of the system. These assumptions make the problem more challenging and realistic from the practical point of view.

3) For the particular mode-dependent state-feedback case, the design conditions presented here are simpler than the ones proposed in [8], as this particular S-MJLS can be transformed into an equivalent MJLS with a larger number of modes. Hence, the resulting system can be analysed using the techniques available in the literature related to MJLS, for instance, [4], [9], and [3].

4) When compared with the approach in [20], it should be noticed that, although the phase-type approximation for holding-time distributions is rather general since it can be applied to any distribution on non-negative reals, it suffers from the fact it may need a vast number of phases, so that the control design and analysis of S-MJLS can be computationally infeasible (see [20]). On the other hand, the approach presented in this paper, being specific to the Erlang distribution, leads to very computationally efficient tools for the control design and analysis problems.

This paper is structured as follows. The formulation of an S-MJLS and its basic properties are presented in Section 3.2, together with a brief description of the Erlang distribution. This section also shows how stability and $\mathcal{H}_2$ performance conditions for an S-MJLS are obtained by solving a convex problem for an MJLS equivalent system, which is the approach proposed in this paper. In Section 3.3 the problem of $\mathcal{H}_2$ output feedback design for S-MJLS with Erlang distributed dwell times is addressed by using the equivalent MJLS approach and cluster-dependent design conditions. Then, illustrative examples are introduced in Section 3.4 to validate the main results presented in the previous sections. Finally, concluding remarks are presented in Section 3.5.

At last, the notation used throughout the text is now discussed. The transpose of real matrices or vectors is indicated by $(.)'$. The trace of a square matrix X is denoted by $\mathrm{tr}(X)$, and the symbol $\bullet$

represents the symmetric blocks in a partitioned matrix. For a square matrix G we define $\mathrm{He}(G) = G + G'$. The set of natural numbers is denoted by $\mathbb{N}$. On a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, the symbol $\mathcal{E}$ indicates mathematical expectation and, for $A \in \mathcal{F}$, 1_A represents the indicator function in A, that is, $1_A(\omega) = 1$ if $\omega \in A$, 0 otherwise. In addition, for any stochastic signal $\xi(t)$ defined in the continuous-time domain, the quantity $\|\xi\|_2^2 = \int_0^\infty \mathcal{E}\left[\xi(t)'\xi(t)\right] dt$ is its squared norm. The class of all signals $\xi(t) \in \mathbb{R}^r$, with $t \in \mathbb{R}_+$, such that $\|\xi\|_2^2$ is finite is denoted by $\mathcal{L}_2^r$. Finally we denote by $o(h)$ any function such that $\lim_{h\to 0} o(h)/h = 0$.

3.2 Erlang Distribution, Semi-MJLS and Equivalent MJLS

3.2.1 Erlang Distribution and Semi-MJLS

We begin this sub-section by recalling some properties of the Erlang distribution with parameters (κ,λ) [28]. These two positive parameters define a family of continuous probability distributions with support $x \in \mathbb{R}_+$. The parameter κ is a positive integer that specifies the structure of the distribution, and the positive real number λ defines its rate of decay. The probability density function of the Erlang distribution is

$$f(x;(\kappa,\lambda)) = \begin{cases} 0, & x < 0 \\ \dfrac{\lambda^\kappa x^{\kappa-1} e^{-\lambda x}}{(\kappa-1)!}, & x \geq 0. \end{cases}$$

where $(\lambda x)^n := 1$ for $(x,n) = (0,0)$. For the specific case of $\kappa = 1$, the Erlang distribution reduces itself to the exponential distribution with parameter λ. Moreover, for the general case, it is the distribution of the sum of κ independent exponential variables with the same parameter λ each. This is the key property that will be exploited in the sequel.

Let us now define the continuous semi-Markov jump linear system with Erlang dwell times that will be considered in this paper. According to [35], a semi-Markov process may be viewed as a stochastic process $\{\theta(t); t \geq 0\}$ that, after having entered a state i at a time T_k, randomly determines its length of stay τ_k for transition out of this state sampled from a probability density function $f_i(\tau)$, and also randomly determines the next state $j \neq i$ based on state transition probabilities $P = [p_{ij}]$, where $\sum_j p_{ij} = 1$ for all i, $p_{ii} = 0$, jumping thus to state j at time $T_{k+1} = T_k + \tau_k$. The length of stay τ_k is known as the *dwell time*.

In what follows we will assume that $\{\theta(t); t \geq 0\}$ is a semi-Markov process taking values in a finite set $\mathcal{N} = \{1,2,\ldots,N\}$ such that for each $i \in \mathcal{N}$, the dwell time after a jump to the state i has an Erlang distribution with parameters (κ_i,λ_i) (that is, $f_i(\tau) = f(\tau;(\kappa_i,\lambda_i))$). Therefore

$\{\theta(t); t \geq 0\}$ is characterised by the positive integer parameters κ_i, jump rates λ_i, $i \in \mathcal{N}$, and transition probability matrix $P = [p_{ij}]$. The initial probability distribution of $\theta(0)$ will be defined by $\mu = (\mu_1, \ldots, \mu_N)$, so that $\mathrm{Prob}\{\theta(0) = i\} = \mu_i$.

On a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ consider the continuous-time semi-Markovian jump linear system (S-MJLS):

$$\mathbb{G} : \begin{cases} \dot{x}(t) = A_{\theta(t)}x(t) + E_{\theta(t)}w(t), \\ z(t) = C_{\theta(t)}x(t) \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the external perturbation vector, $z(t) \in \mathbb{R}^p$ is the output vector to be controlled, and $\{\theta(t), t \geq 0\}$ is a semi-Markovian continuous-time process with Erlang dwell times as defined above. For each $i \in \mathcal{N}$, we have that $A_i \in \mathbb{R}^{n \times n}$, $C_i \in \mathbb{R}^{p \times n}$ and $E_i \in \mathbb{R}^{n \times m}$. Notice that a conventional MJLS is a particular case of the S-MJLS, obtained whenever $\kappa_i = 1 \; \forall i \in \mathcal{N}$.

Associated with the S-MJLS (3.1) there are two important definitions: stochastic stability and the $\mathcal{H}_2$ norm.

Definition 3.1. Consider the autonomous version of system $\mathbb{G}$, that is, $w := 0$. If $\|x\|_2^2 = \int_0^\infty \mathcal{E}(\|x(t)\|^2))dt < \infty$ for any initial condition $x(0) = x_0$ and $\theta(0) = \theta_0$, then we say that $\mathbb{G}$ is stochastically stable.

Definition 3.2. Consider that $\mathbb{G}$ is stochastically stable. We define $\|\mathbb{G}\|_2$, the $\mathcal{H}_2$ norm of $\mathbb{G}$, as

$$\|\mathbb{G}\|_2^2 = \sum_{i \in \mathcal{N}} \sum_{s=1}^{m} \mu_i \|z_{s,i}\|_2^2, \tag{3.2}$$

in which $z_{s,i}(t)$ is the output $z(t)$ in (3.1) generated by the impulsive input $w(t) = e_s\delta(t)$ and by the initial conditions $x(0) = 0$ and $\theta_0 = i$; here, e_s is the s-th column of the identity matrix of order m.

3.2.2 Equivalent MJLS

Based on the fact that the Erlang distribution is the sum of independent exponential distributions, our approach to developing stability and performance conditions for an S-MJLS with Erlang-driven dwell times $\mathbb{G}$ is to devise an *equivalent* MJLS $\mathbb{G}'$ and use it to evaluate stability and performance, as such properties are widely available for such stochastic systems [4]. We begin by defining the equivalent Markov process to an Erlang Semi-Markov process.

Definition 3.3. Let $\{\theta(t); t \geq 0\}$ be the Erlang semi-Markov process taking values in $\mathcal{N}$ as defined in sub-section 3.2.1, with parameters (κ_i,λ_i) and transition probability matrix $P = [p_{ij}]$. Set $\lambda_{ij} = \lambda_i p_{ij}$ for $i,j \in \mathcal{N}$, $i \neq j$, and $\lambda_{ii} = -\lambda_i$. Consider the new state space $\mathcal{N}_Z = \{(1,1),\ldots,(1,\kappa_1),\ldots,(N,1),\ldots,(N,\kappa_N)\}$ and define the *equivalent continuous-time Markov process* $Z(t) = (Z_1(t),Z_2(t))$ taking values in $\mathcal{N}_Z$ as follows. For $i \in \mathcal{N}$, $s = 1,\ldots,\kappa_i - 1$, the transition probabilities of $Z(t)$ are

$$\begin{aligned} &\mathrm{Prob}\{Z(t+h) = (i,k) \mid Z(t) = (i,s)\} = \\ &= \begin{cases} \lambda_i h + o(h), & k = s+1 \\ 1 - \lambda_i h + o(h), & k = s \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.3}$$

and, for $s = \kappa_i$,

$$\begin{aligned} &\mathrm{Prob}\{Z(t+h) = (j,\ell) \mid Z(t) = (i,\kappa_i)\} = \\ &= \begin{cases} \lambda_{ij} h + o(h), & j \neq i, \ell = 1 \\ 1 - \lambda_i h + o(h), & j = i, \ell = \kappa_i \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.4}$$

According to Definition 3.3, whenever the process is $Z(t) = (i,s)$, $s = 1,\ldots,\kappa_i - 1$, a jump to $(i,s+1)$ will occur with jump rate λ_i, and if $Z(t) = (i,\kappa_i)$, a jump to $(j,1)$ will occur with jump rate $\lambda_{ij} = p_{ij}\lambda_i$. Therefore, with $Z(t) = (Z_1(t),Z_2(t))$, we have that $\theta(t) = Z_1(t)$ represents the mode of the semi-Markov model, and $Z_2(t)$ represents the counter of the exponential jumps that have occurred since the last jump to $Z_1(t)$. Notice that if $Z(t) = (i,\kappa_i)$ then the next jump will occur at the rate λ_i to a state j with probability p_{ij} and the counter on the number of exponential jumps is re-set to 1. Thus, by coupling the state i with a counter of the κ_i exponentially distributed jump times with rate λ_i, we emulate the behaviour of semi-Markov process $\{\theta(t)\}$ with Erlang distributed dwell times with parameters (κ_i,λ_i), as $\theta(t) = Z_1(t)$. This is the key idea behind the Definition 3.3.

Example 3.1. Let us consider an Erlang semi-Markov process with 3 modes, that is, $\mathcal{N} = \{1,2,3\}$. Suppose that the Erlang distributed dwell times associated with these modes have parameters $\lambda_1 = 5$, $\lambda_2 = 8$, $\lambda_3 = 4$ and $\kappa_1 = 1$, $\kappa_2 = 2$, $\kappa_3 = 3$. Therefore for this case we have

$\mathcal{N}_Z = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$. The transition matrix is given by

$$P = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.4 & 0 & 0.6 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

and the transition rate matrix Λ_Z for the Markov process $Z(t)$, as defined in (3.3) and (3.4), is given by

$$\Lambda_Z = \begin{bmatrix} -5 & 3.5 & 0 & 1.5 & 0 & 0 \\ 0 & -8 & 8 & 0 & 0 & 0 \\ 3.2 & 0 & -8 & 4.8 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 \\ 2 & 2 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

Figure 3.1 shows the Markov diagram of $Z(t)$ where $S_1 = S'_1 = (1,1)$, $S'_2 = (2,1)$, $S'_3 = (2,2)$, $S_2 = S'_2 \cup S'_3$, $S'_4 = (3,1)$, $S'_5 = (3,2)$, $S'_6 = (3,3)$, $S_3 = S'_4 \cup S'_5 \cup S'_6$.

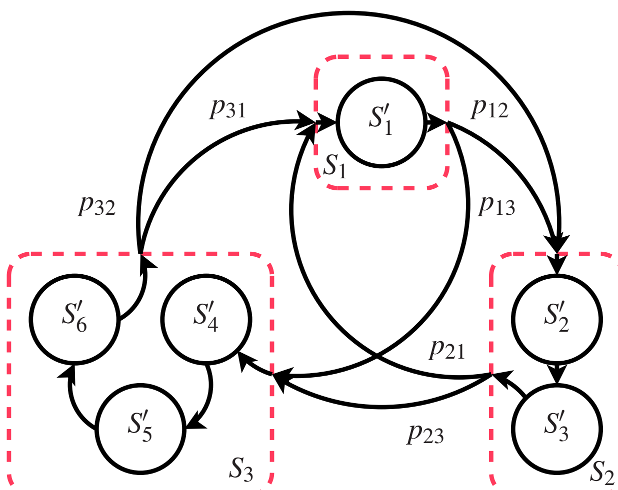

Figure 3.1: Markov diagram of the process $Z(t)$ presented in Example 3.1. The Erlang semi-Markov process $\theta(t) = Z_1(t)$ is represented by the red dashed "supermodes" S_1,S_2 and S_3 and by the transition probabilities p_{ij} (remember that at the state S_i the dwell time for the next jump has an Erlang distribution with parameters κ_i and λ_i). The Markov process $Z(t)$ is represented by the round modes $S'_1, \cdots, S'_6$, which now have exponentially distributed dwell times. Note that the inner arrows, i.e., the arrows that link modes associated with the same "supermode" S_i, are transitions with probability one, counting the number of exponentially distributed jumps.

Remark 3.1. It is important to note that this equivalence relationship has a trade-off in complexity. Indeed, the original Erlang semi-Markov process $\theta(t)$ has a smaller number of nodes but more complex stochastic properties; on the other hand, the Markov process $Z(t)$ has much simpler stochastic properties, but this gain comes with a toll of dealing with a more significant number of modes. Convex optimisation techniques allow us to deal with both the original formulation, as done in [8] with SoS-based conditions, as well as the MJLS-equivalent, as will be done in the sequel with LMIs.

Bearing Definition 3.3 in mind, and defining $\bar{A}_{ik} = A_i$, $\bar{E}_{ik} = E_i$, $\bar{C}_{ik} = C_i$, $(i,k) \in \mathcal{N}_Z$, we can re-write system (3.1), now in terms of the Markov process $Z(t)$, as follows:

$$\mathbb{G}' : \begin{cases} \dot{x}(t) = \bar{A}_{Z(t)}x(t) + \bar{E}_{Z(t)}w(t), \\ z(t) = \bar{C}_{Z(t)}x(t). \end{cases} \tag{3.5}$$

Notice that if the initial condition μ'_{ik} for $Z(t)$ satisfies

$$\begin{aligned} \mu'_{i1} &:= \mathrm{Prob}\{Z(0) = (i,1)\} = \mu_i = \mathrm{Prob}\{\theta(0) = i\}, \\ \mu'_{ik} &:= \mathrm{Prob}\{Z(0) = (i,k)\} = 0,\ k = 2, \ldots, \kappa_i, \end{aligned} \tag{3.6}$$

then $\theta(t) = Z_1(t)$ for all $t \geq 0$, so that $\bar{A}_{Z(t)} = A_{Z_1(t)} = A_{\theta(t)}$, and similarly $\bar{E}_{Z(t)} = E_{\theta(t)}$, $\bar{C}_{Z(t)} = C_{\theta(t)}$ for all $t \geq 0$. Therefore we conclude that system $\mathbb{G}$ and $\mathbb{G}'$ are **equivalent** in the sense that the dynamics of $x(t)$ and $z(t)$ can be equivalently described by (3.1) or (3.5) with initial condition given in (3.6). Finally, notice that, as the system dynamic behaviour does not change inside each cluster, such equivalences could be extended to a more general nonlinear scenario.

Conditions for the stochastic stability of (3.1) and to calculate an upper bound for the $\mathcal{H}_2$ norm are derived next, based on the results for continuous-time MJLS.

Theorem 3.1. *For some given $\rho > 0$, system $\mathbb{G}$ in (3.1) is stochastically stable and $\|\mathbb{G}\|_2^2 < \rho$ iff there exist $S_{ik} \succ 0$, $(i,k) \in \mathcal{N}_Z$, such that the linear matrix inequalities below are satisfied:*

$$\sum_{i \in \mathcal{N}} \mu_i tr(E_i' S_{i1} E_i) < \rho, \tag{3.7}$$

and for $i \in \mathcal{N}$, $k = 1, \ldots, \kappa_i - 1$,

$$A_i' S_{ik} + S_{ik} A_i + \lambda_i (S_{ik+1} - S_{ik}) + C_i' C_i \prec 0, \tag{3.8}$$

$$A_i' S_{i\kappa_i} + S_{i\kappa_i} A_i + \lambda_i \left(\sum_{j=1}^{N} p_{ij} S_{j1} - S_{i\kappa_i} \right) + C_i' C_i \prec 0. \tag{3.9}$$

Proof: From Lemma 3.37 and Theorem 5.4 in [4], considering the MJLS (3.5) with transition rates as in (3.3), we have that there exist $S_{ik} \succ 0$, $(i,k) \in \mathcal{N}_Z$ satisfying

$$\sum_{(i,k)\in\mathcal{N}_Z} \mu'_{ik} tr(\bar{E}'_{ik} S_{ik} \bar{E}_{ik}) < \rho, \tag{3.10}$$

$$\mathrm{He}\big(S_{ik}\bar{A}_{ik}\big) + \lambda_i(S_{ik+1} - S_{ik}) + \bar{C}'_{ik}\bar{C}_{ik} \prec 0, k = 1,\ldots,\kappa_i - 1, \tag{3.11}$$

$$\mathrm{He}\big(S_{i\kappa_i}\bar{A}_{i\kappa_i}\big) + \lambda_i \left(\sum_{j=1}^{N} p_{ij} S_{j1} - S_{i\kappa_i}\right) + \bar{C}'_{i\kappa_i}\bar{C}_{i\kappa_i} \prec 0, \tag{3.12}$$

iff $\mathbb{G}'$ is stochastically stable and $\|\mathbb{G}'\|_2^2 < \rho$. From $\bar{A}_{ik} = A_i$, $\bar{E}_{ik} = E_i$, $\bar{C}_{ik} = C_i$, $(i,k) \in \mathcal{N}_Z$, and the initial condition (3.6), it is immediate to see that (3.10), (3.11), (3.12) can be re-written respectively as in (3.7), (3.8), (3.9). From the equivalence between the systems $\mathbb{G}'$ and $\mathbb{G}$ we have that $\mathbb{G}'$ is stochastically stable iff $\mathbb{G}$ is stochastically stable. From Definition 3.2 and considering the initial condition as in (3.6), (3.2) implies that

$$\|\mathbb{G}'\|_2^2 = \sum_{(i,k)\in\mathcal{N}_Z} \sum_{s=1}^{m} \mu'_{ik} \|z_{s,(i,k)}\|_2^2 = \sum_{i\in\mathcal{N}} \sum_{s=1}^{m} \mu_i \|z_{s,(i,1)}\|_2^2 = \|\mathbb{G}\|_2^2,$$

so that $\|\mathbb{G}\|_2^2 = \|\mathbb{G}'\|_2^2$, completing the proof. □

Remark 3.2. Note that the analysis conditions for Erlang S-MJLS presented in Theorem 3.1 depend on the symmetric matrices S_{ik}, $(i,k) \in \mathcal{N}_Z$. If one wishes to work with less variables, more conservative conditions can be restated by imposing $S_{ik+1} - S_{ik} = \Delta_i$, $k = 1, \cdots, \kappa_i - 1$, to (3.11) and (3.12). In this case, the design variables become S_{i1} and Δ_i, $i \in \mathcal{N}$. The same argument can be adapted for the design conditions.

3.3 Output $\mathcal{H}_2$ Control for Erlang S-MJLS

In this section we consider the following controlled representation of system (3.1),

$$\mathbb{G}_c : \begin{cases} \dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t), \\ y(t) = F_{\theta(t)}x(t) \\ z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \end{cases} \tag{3.13}$$

where $u(t) \in \mathbb{R}^r$ is the control vector and $y(t) \in \mathbb{R}^s$ is the measurable output vector ($n \geq s$). For each $i \in \mathcal{N}$, we have that $B_i \in \mathbb{R}^{n\times r}$, $F_i \in \mathbb{R}^{s\times n}$ and $D_i \in \mathbb{R}^{p\times r}$. We recall that $\{\theta(t), t \geq 0\}$ is the Erlang semi-Markovian continuous-time process as defined in Section 3.2.

We will consider here an output cluster-dependent control problem for Erlang S-MJLS. By cluster we mean that $\mathcal{N}$ can be written as the union of M disjoint sets $\mathcal{N}_i$ (clusters), with $1 \leq M \leq N$, that is, $\mathcal{N} = \cup_{i=1}^{M} \mathcal{N}_i$, with $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$, $i \neq j$. We set $\mathcal{M} = \{1, \ldots, M\}$ and the function $g : \mathcal{N} \to \mathcal{M}$ such that $g(i) = \ell$ whenever $i \in \mathcal{N}_\ell$, that is, $g(i) = \sum_{\ell=1}^{M} \ell 1_{\mathcal{N}_\ell}(i)$. In other words, $g(\theta(t))$ represents the cluster where the semi-Markov process $\theta(t)$ belongs to at time t. We will assume that the controller $u(t)$ will only have access to $y(t)$ and $g(\theta(t))$, in the following form:

$$u(t) = K_{g(\theta(t))} y(t) = K_{g(\theta(t))} F_{\theta(t)} x(t), \tag{3.14}$$

where $K_\ell \in \mathbb{R}^{r \times s}$, $\ell \in \mathcal{M}$, are the feedback gains to be designed.

Remark 3.3. Note that the cluster control problem considered here includes, as particular cases, the following setups:

- *Mode-dependent control* if $M = N, \mathcal{N}_i = \{i\}$;
- *Mode-independent control* if $M = 1$ and $g(i) = 1\, \forall i \in \mathcal{N}$.

Furthermore, the state-feedback control problem can also be recovered whenever one has $F_i = I$, $i \in \mathcal{N}$.

Let us now introduce the following notation, for $i \in \mathcal{N}$:

$$\widetilde{A}_i = A_i + B_i K_{g(i)} F_i,\ \widetilde{C}_i = C_i + D_i K_{g(i)} F_i \tag{3.15}$$

so that (3.13) can be re-written as

$$\mathbb{G}_c : \begin{cases} \dot{x}(t) = \widetilde{A}_{\theta(t)} x(t) + E_{\theta(t)} w(t), \\ z(t) = \widetilde{C}_{\theta(t)} x(t). \end{cases} \tag{3.16}$$

The $\mathcal{H}_2$ output cluster-dependent control problem we want to study is as follows.

$\mathcal{H}_2$ output cluster-dependent control problem: Find $\{K_\ell; \ell \in \mathcal{M}\}$ such that system $\mathbb{G}_c$ as defined in (3.16) is stochastically stable and $\|\mathbb{G}_c\|_2^2 < \rho$ for some given $\rho > 0$.

Remark 3.4. Optimal $\mathcal{H}_2$ mode-dependent state-feedback control design conditions for MJLSs have been known for about 20 years (see for instance [3, 4] and references therein). As our main goal is to provide $\mathcal{H}_2$ optimal control design conditions for Erlang S-MJLS, it seems natural to exploit once again the relationship between an S-MJLS and its MJLS-equivalent to that end, as proposed in Section 3.2. However, each mode in the S-MJLS $\mathbb{G}$ is mapped onto κ_i modes in its

MJLS-equivalent $\mathbb{G}'$. That means the same feedback gain must be designed for those κ_i modes in this equivalent-based control design approach, as these subsystems are indistinguishable from the original problem.

The following assumption will be required to design the output cluster-dependent controller.

Assumption 3.1. For each $i \in \mathcal{N}_\ell$, there exist a non-singular matrix T_ℓ such that

$$F_i T_\ell = \begin{bmatrix} I_s & 0 \end{bmatrix}. \tag{3.17}$$

Notice that Assumption 3.1 will be satisfied if F_i is the same for each $i \in \mathcal{N}_\ell$ (that is, the output matrices F_i are the same within each cluster $\mathcal{N}_\ell$) and has full-row rank. The full-row rank condition is standard in output-feedback design problems. In the mode-dependent control setting, (3.17) is satisfied if F_i has full-row rank for all $i \in \mathcal{N}$.

Before presenting our LMI design conditions, we must introduce the following notation. Consider $n \times n$ matrices $X_{ik} \succ 0$, $(i,k) \in \mathcal{N}_Z$, and set for each $i \in \mathcal{N}$,

$$\begin{aligned} \Pi_i &= \begin{bmatrix} \sqrt{\lambda_i p_{i1}} I & \cdots & \sqrt{\lambda_i p_{ii-1}} I & \sqrt{\lambda_i p_{ii+1}} I & \cdots & \sqrt{\lambda_i p_{iN}} I \end{bmatrix}, \\ \mathcal{D}_i &= diag(X_{11}, \ldots, X_{i-11}, X_{i+11}, \ldots, X_{N1}). \end{aligned}$$

Notice that (recall that $p_{ii} = 0$)

$$\Pi_i \mathcal{D}_i^{-1} \Pi_i' - \lambda_i X_{i\kappa_i}^{-1} = \lambda_i (\sum_{j=1}^{N} p_{ij} X_{j1}^{-1} - X_{i\kappa_i}^{-1}). \tag{3.18}$$

Theorem 3.2. *For some given $\rho > 0$, suppose that there exist matrices $X_{ik} \succ 0$, $(i,k) \in \mathcal{N}_Z$, $W_i \succ 0$, $i \in \mathcal{N}$, G_ℓ and V_ℓ, $\ell \in \mathcal{M}$, and scalars $\epsilon_{ik} > 0$ such that the following inequalities are satisfied (LMIs for $\epsilon_{ik} > 0$ fixed):*

$$\sum_{i \in \mathcal{N}} \mu_i Tr(W_i) < \rho, \tag{3.19}$$

$$\begin{bmatrix} W_i & \bullet \\ E_i & X_{i1} \end{bmatrix} \succ 0, \tag{3.20}$$

$$\begin{bmatrix} -\lambda_i X_{i\kappa_i} & \bullet & \bullet & \bullet \\ 0 & -I & \bullet & \bullet \\ X_{i\kappa_i} & 0 & 0 & \bullet \\ \Pi_i' X_{i\kappa_i} & 0 & 0 & -\mathcal{D}_i \end{bmatrix} + \operatorname{He}\left(\begin{bmatrix} A_i T_{g(i)} G_{g(i)} - B_i \begin{bmatrix} V_{g(i)} & 0 \end{bmatrix} \\ C_i T_{g(i)} G_{g(i)} - D_i \begin{bmatrix} V_{g(i)} & 0 \end{bmatrix} \\ -T_{g(i)} G_{g(i)} \\ 0 \end{bmatrix} \begin{bmatrix} \epsilon_{i\kappa_i} I \\ 0 \\ I \\ 0 \end{bmatrix}' \right) \prec 0, \tag{3.21}$$

and for $k = 1, \dots, \kappa_i - 1$,

$$\begin{bmatrix} -\lambda_i X_{ik} & \bullet & \bullet & \bullet \\ 0 & -I & \bullet & \bullet \\ X_{ik} & 0 & 0 & \bullet \\ \sqrt{\lambda_i} X_{ik} & 0 & 0 & -X_{ik+1} \end{bmatrix} + \operatorname{He}\left(\begin{bmatrix} A_i T_{g(i)} G_{g(i)} - B_i \begin{bmatrix} V_{g(i)} & 0 \end{bmatrix} \\ C_i T_{g(i)} G_{g(i)} - D_i \begin{bmatrix} V_{g(i)} & 0 \end{bmatrix} \\ -T_{g(i)} G_{g(i)} \\ 0 \end{bmatrix} \begin{bmatrix} \epsilon_{ik} I \\ 0 \\ I \\ 0 \end{bmatrix}' \right) \prec 0, \tag{3.22}$$

with G_ℓ, for $\ell \in \mathcal{M}$, as

$$G_\ell = \begin{bmatrix} G_{\ell 1} & 0 \\ G_{\ell 2} & G_{\ell 3} \end{bmatrix}. \tag{3.23}$$

Then system $\mathbb{G}_c$ in (3.16) is stochastically stable and $\|\mathbb{G}_c\|_2^2 < \rho$ whenever the output cluster-dependent controller (3.14) is applied with the feedback controller matrices K_ℓ given by

$$K_\ell = -V_\ell G_{\ell 1}^{-1}, \;\; \ell \in \mathcal{M}. \tag{3.24}$$

Proof: From (3.21) (or (3.22)) it follows that $T_\ell G_\ell + G_\ell' T_\ell' \succ 0$ and, since T_ℓ is non-singular, we get that G_ℓ is non-singular. Therefore we have that $G_{\ell 1}$ is non-singular as well, so that the inverse in (3.24) is well defined. From (3.24) we have that $V_\ell = -K_\ell G_{\ell 1}$ so that (3.23) yields

$$K_\ell \begin{bmatrix} I_s & 0 \end{bmatrix} G_\ell = \begin{bmatrix} K_\ell & 0 \end{bmatrix} G_\ell = \begin{bmatrix} -V_\ell & 0 \end{bmatrix}. \tag{3.25}$$

Combining (3.17) with (3.25), and recalling (3.15), it follows that

$$\begin{aligned} & A_i T_{g(i)} G_{g(i)} - B_i \begin{bmatrix} V_{g(i)} & 0 \end{bmatrix} = A_i T_{g(i)} G_{g(i)} + B_i K_{g(i)} \begin{bmatrix} I_s & 0 \end{bmatrix} G_{g(i)} \\ & = A_i T_{g(i)} G_{g(i)} + B_i K_{g(i)} F_i T_{g(i)} G_{g(i)} \\ & = (A + B_i K_{g(i)} F_i) T_{g(i)} G_{g(i)} = \widetilde{A}_i T_{g(i)} G_{g(i)}. \end{aligned}$$

By using the same arguments as above we get that $C_i T_{g(i)} G_{g(i)} - D_i \begin{bmatrix} V_{g(i)} & 0 \end{bmatrix} = \widetilde{C}_i T_{g(i)} G_{g(i)}$.

From this we can re-write (3.21) as

$$\Phi_i + \mathrm{He}\left(W_i T_{g(i)} G_{g(i)} \mathcal{J}'_{i\kappa_i}\right) \prec 0, \tag{3.26}$$

where

$$\Phi_i = \begin{bmatrix} -\lambda_i X_{i\kappa_i} & \bullet & \bullet & \bullet \\ 0 & -I & \bullet & \bullet \\ X_{i\kappa_i} & 0 & 0 & \bullet \\ \Pi'_i X_{i\kappa_i} & 0 & 0 & -\mathcal{D}_i \end{bmatrix}, \ \mathcal{J}_{i\kappa_i} = \begin{bmatrix} \epsilon_{i\kappa_i} I \\ 0 \\ I \\ 0 \end{bmatrix}, \ W_i = \begin{bmatrix} \widetilde{A}_i \\ \widetilde{C}_i \\ -I \\ 0 \end{bmatrix}.$$

By defining

$$\mathbb{W}_i = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \widetilde{A}'_i & \widetilde{C}'_i & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{3.27}$$

we get that $\mathbb{W}_i$ has full rank and that $W'_i \mathbb{W}_i = 0$. Therefore from (3.26) it follows that $\mathbb{W}'_i \Phi_i \mathbb{W}_i \prec 0$, which allows us to conclude that

$$\begin{bmatrix} -\lambda_i X_{i\kappa_i} + \widetilde{A}_i X_{i\kappa_i} + X_{i\kappa_i} \widetilde{A}'_i & \bullet & \bullet \\ \widetilde{C}_i X_{i\kappa_i} & -I & \bullet \\ \Pi'_i X_{i\kappa_i} & 0 & -\mathcal{D}_i \end{bmatrix} \prec 0. \tag{3.28}$$

By applying the Schur's complement in (3.28) we get that

$$\begin{aligned} -\lambda_i X_{i\kappa_i} + & \widetilde{A}_i X_{i\kappa_i} + X_{i\kappa_i} \widetilde{A}'_i + X_{i\kappa_i} \Pi_i \mathcal{D}_i^{-1} \Pi'_i X_{i\kappa_i} + \\ & + X_{i\kappa_i} \widetilde{C}'_i \widetilde{C}_i X_{i\kappa_i} \prec 0. \end{aligned} \tag{3.29}$$

Multiplying (3.29) on the left and right hand side by $X_{i\kappa_i}^{-1}$ it follows, from (3.18), that (3.9) holds, after considering $S_{i1} = X_{i1}^{-1}$, $S_{i\kappa_i} = X_{i\kappa_i}^{-1}$. By repeating the same arguments as above we get from (3.22) that

$$\Phi_{ik} + \mathrm{He}\left(\widehat{W}_i T_{g(i)} G_{g(i)} \widehat{\mathcal{J}}'_{ik}\right) \prec 0, \tag{3.30}$$

where

$$\Phi_{ik} = \begin{bmatrix} -\lambda_i X_{ik} & \bullet & \bullet & \bullet \\ 0 & -I & \bullet & \bullet \\ X_{ik} & 0 & 0 & \bullet \\ \sqrt{\lambda_i} X_{ik} & 0 & 0 & -X_{ik+1} \end{bmatrix}$$

and $\widehat{W}_i$, $\widehat{\mathcal{J}}_{ik}$ are defined similarly to W_i, $\mathcal{J}_{i\kappa_i}$, after appropriately changing the dimension of the zero blocks, and replacing $\epsilon_{i\kappa_i}$ by ϵ_{ik}. Defining $\widehat{\mathbb{W}}_i$ as in (3.27), appropriately changing the dimension of the identity matrix in the 4×3 block, we get, as before, that $\widehat{\mathbb{W}}_i' \Phi_{ik} \widehat{\mathbb{W}}_i \prec 0$, implying that

$$\begin{bmatrix} -\lambda_i X_{ik} + \widetilde{A}_i X_{ik} + X_{ik} \widetilde{A}_i' & \bullet & \bullet \\ \widetilde{C}_i X_{ik} & -I & \bullet \\ \sqrt{\lambda_i} X_{ik} & 0 & -X_{ik+1} \end{bmatrix} \prec 0. \tag{3.31}$$

Applying the Schur's complement in (3.31) results into

$$-\lambda_i X_{ik} + \widetilde{A}_i X_{ik} + X_{ik} \widetilde{A}_i' + \lambda_i X_{ik} X_{ik+1}^{-1} X_{ik} + X_{ik} \widetilde{C}_i' \widetilde{C}_i X_{ik} \prec 0. \tag{3.32}$$

Multiplying (3.32) on the left and right hand side by X_{ik}^{-1} it follows that (3.8) holds, after considering $S_{ik} = X_{ik}^{-1}$, $S_{ik+1} = X_{ik+1}^{-1}$. Finally, by applying the Schur's complement in (3.20) we get that $W_i \succ E_i' X_{i1}^{-1} E_i = E_i' S_{i1} E_i$ so that from (3.19) we get that (3.7) is satisfied. Summing up, we have shown that for $S_{ik} = X_{ik}^{-1}$, the inequalities (3.7), (3.8) and (3.9) are satisfied, so that the result follows from Theorem 3.1. □

3.4 Examples

We present here several examples to illustrate the results obtained in the previous section.

Example 3.2. Let us consider the same S-MJLS presented in Example 3.1 with the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0.2 & -0.8 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ -1 & -0.3 \end{bmatrix},$$

$C_i = I$ and $E_i = \begin{bmatrix} 1 & 1 \end{bmatrix}'$ for $i \in \{1,2,3\}$. Let $\mu = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ denote the distribution of the initial mode, that is, $\mu_i = \mathrm{Prob}(\theta(0) = i)$. Note that the first mode in this system is unstable. In order to assess the stability of $\mathbb{G}$ and calculate its $\mathcal{H}_2$ norm, we use Theorem 3.1 to the MJLS-equivalent $\mathbb{G}'$ and consider an LMI optimisation problem in which it is desired to minimise ρ over the variables $S_{ik} \succ 0$, $\rho > 0$, satisfying (3.7)-(3.9). Solving this optimisation problem we obtain $\|\mathbb{G}'\|_2^2 =$

8.7862, implying that the S-MJLS $\mathbb{G}$ is stochastically stable and is such that $\|\mathbb{G}\|_2^2 = 8.7862$.

Example 3.3. Let us reconsider Example 3.2, including one state-feedback control input ($F_i = I$) such that $B_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$, $B_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}'$, and $B_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}'$. We also include one extra output channel, affected only by the control input, in a way that $D_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$, $i \in \{1,2,3\}$. We can again define an MJLS equivalent system, similarly to what was done in Example 3.2. For the design procedure, we have $\mathcal{N}_Z$ with 6 virtual modes. From the results of Theorem 3.2, we solve the LMI optimisation problem in which it is desired to minimise ρ over the variables $X_{ik} \succ 0$, $(i,k) \in \mathcal{N}_Z$, $W_i \succ 0$, $i \in \mathcal{N}$, G_i and V_i, $i \in \mathcal{N}$, satisfying (3.19)-(3.22). This yields

$$K_1 = \begin{bmatrix} -0.1740 \\ -0.9073 \end{bmatrix}', \; K_2 = \begin{bmatrix} -1.0238 \\ -1.1440 \end{bmatrix}', \; K_3 = \begin{bmatrix} -1.0786 \\ -0.0613 \end{bmatrix}'$$

and a guaranteed $\mathcal{H}_2$ cost of $\|\mathbb{G}_c'\|_2^2 \leq 2.5771$, whenever we take $\epsilon_{11} = 5$, $\epsilon_{2k} = 3$, $k = 1,2$, and $\epsilon_{3k} = 15$, $k = 1,2,3$. Closing the loop of the S-MJLS with these mode-dependent gains provide $\|\mathbb{G}_c\|_2^2 = 2.5342$. We can also devise a mode-independent controller by imposing $M = 1$ and $g(i) = 1 \forall i \in \mathcal{N}$ to the LMI optimisation problem described before. In this case, the design conditions yield the controller gain $K = \begin{bmatrix} -0.9884 & -0.2235 \end{bmatrix}$ and the bound $\|\mathbb{G}_c'\|_2^2 \leq 2.9698$, whenever we take $\epsilon_{11} = 12$, $\epsilon_{2k} = 18$, $k = 1,2$, and $\epsilon_{3k} = 18$, $k = 1,2,3$. For this gain, $\|\mathbb{G}_c\|_2^2 = 2.7348$.

Example 3.4. Finally, let us consider the two-mode S-MJLS described in [8], with realisation matrices

$$A_1 = \begin{bmatrix} 0.7 & -4.0 \\ 0.0 & -7.0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -7.0 & 4.0 \\ 0.0 & 0.7 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_1 = C_2 = \begin{bmatrix} I_2 \\ 0_{1\times 2} \end{bmatrix}, \; D_1 = D_2 = \begin{bmatrix} 0_{2\times 1} \\ 0.5 \end{bmatrix}.$$

Following [8], we consider Erlang dwell times with parameters $\kappa_1 = \kappa_2 = 3$ and $\lambda_1 = \lambda_2 = 5$; note that $p_{12} = p_{21} = 1$ in this example. We also take $\mu = [1 \;\; 0]$.

Let us first compare the state-feedback design conditions proposed in this paper to the SOS-based ones presented in [8]. As before, we solve the optimisation problem associated with Theorem 3.2 for $\epsilon_{11} = 2$, $\epsilon_{12} = \epsilon_{13} = 3$, $\epsilon_{2k} = 12$, $k = 1,2,3$, and obtain the bound $\|\mathbb{G}_c'\|_2^2 \leq 0.4773$, ensured by

$$K_1 = \begin{bmatrix} 1.2127 & -0.4145 \end{bmatrix}, \; K_2 = \begin{bmatrix} 0.0777 & -2.5930 \end{bmatrix}.$$

These gains yield the closed-loop performance $\|\mathbb{G}_c\|_2^2 = 0.4539$. Thus, even using simpler, static gains, we achieved performance similar to the one attained by the timer-dependent gains in [8], which ensured a guaranteed $\mathcal{H}_2$ cost of 0.4480 to the closed-loop system, validated by simulation.

In this example, we also illustrate the output-feedback problem. To this end, we consider the outputs associated with $F_1 = -[1 \ \ 0]$ and $F_2 = [1 \ \ 1]$. In this scenario, optimising the design conditions in Theorem 3.2 for $\epsilon_{1k} = 35$, $k = 1{,}2{,}3$, $\epsilon_{21} = 5$, $\epsilon_{22} = \epsilon_{23} = 10$, yields the guaranteed cost $\|\mathbb{G}_c'\|_2^2 \leq 1.8860$, achieved by the gains $K_1 = -2.2677$, $K_2 = -0.4890$. When we close the loop with these gains, we obtain the $\mathcal{H}_2$ norm of $\|\mathbb{G}_c\|_2^2 = 0.5855$, which is consistent with the performance bound obtained before.

3.5 Conclusions

In this paper, we presented a convex-optimisation based solution to the $\mathcal{H}_2$ Output and State-Feedback Control problem considering Continuous Semi-Markov Jump Linear Systems with Erlang dwell time. Our method is based on an equivalent MJLS and provides (static) output state feedback cluster-time-independent gains (when the only information for the process is on which cluster it belongs to). Using an academic example, we show the output-feedback design conditions proposed here can be applied to the state-feedback control problem and may obtain similar results to the timer-dependent state-feedback control gains provided by SOS-based design conditions. Future works will focus on more general dwell time distributions, the $\mathcal{H}_\infty$ norm, the filter and dynamic output-feedback problems, or the discrete-time semi-Markov case (see, for instance [32]).

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CHAPTER 4

Conclusions

The results obtained throughout the development of the research that culminated in this thesis show that it is possible to develop control strategies for an S-MJLS with rational transition rates in two different ways. The first one, introduced in Chapter 2, is straightforward and performed using sum-of-squares optimisation. In this method, the imposition of structure (2.27), with polynomial matrices $Q_i(h)$, results in the analysis conditions of Lemma 2.3, which provide an upper bound for the $\mathcal{H}_2$ norm of the system. Note that, in a similar way as demonstrated in Example 2.2, although the application of such conditions results in some level of conservatism, this can be attenuated by increasing the degree of the polynomial elements that form the matrices $Q_i(h)$. Furthermore, the structure mentioned above also enables the design of state-feedback control with gains that depend not only on the mode of the system but also on its sojourn time, which is achieved by employing the results of Theorem 2.2. Nevertheless, in the same way, as in the analysis, the application of these results generates conservatism, which can again be reduced by raising the degrees of the polynomials that constitute the matrix variables, as illustrated in Example 2.2. Moreover, the Example 2.1 shows that the dependence of the gains on the sojourn time, associated with the use of a suitable degree for the polynomials, provides a controller that presents a performance subtly superior to that designed applying the method based on equivalent MJLS.

The second approach, developed in Chapter 3, enables the stability analysis, the computation of a guaranteed $\mathcal{H}_2$ cost, and the design of output-feedback control for an S-MJLS with Erlang-distributed dwell times using the analysis and synthesis of an equivalent MJLS. Thus, even though the equivalent system has a more significant number of modes, as evidenced in Example 3.1, it has more elementary stochastic properties, which allows obtaining the analysis technique presented in Theorem 3.1, derived from the classical and computationally efficient results available in the literature related to MJLS. In addition, the existing relationship between the equivalent system and the original one permits the development of the output-feedback design conditions presented in Theorem 3.2, which provide cluster-dependent controllers.

Therefore, although each of these approaches has its particularities, advantages and limitations, both enable the stability analysis, the determination of an upper limit for the $\mathcal{H}_2$ norm and the design of controllers for S-MJLS with rational transition rates, which constitute a more general class of stochastic systems and, due to this, present a structure that increases the complexity of the development process of these techniques. Moreover, both methodologies result in convex optimisation problems, which can be solved efficiently using the methods already available for this class of optimisation problems.

Finally, the results presented in this thesis expand the horizon of possibilities for future works. For example, further research may extend the results developed here to the $\mathcal{H}_\infty$ norm context and address the filter problem. Furthermore, the interested reader can evaluate the possibility of obtaining equivalent MJLS from S-MJLS with dwell times distributed according to distributions different from the Erlang one, such as the Weibull distribution. At last, it is still possible to consider the application of these methods in more general S-MJLS, that is, in those with non-rational transition rates.

Bibliography

[1] F. M. Callier and C. A. Desoer. *Linear System Theory*. Springer-Verlag, 1999.

[2] G. Chesi. LMI techniques for optimization over polynomials in control: A survey. *IEEE Transactions on Automatic Control*, 55:2500 – 2510, Dec. 2010.

[3] O. L. V. Costa, J. B. R. do Val, and J. C. Geromel. Continuous-time state-feedback $\mathcal{H}_2$-control of markovian jump linear system via convex analysis. *Automatica*, 35(2):259–268, Feb. 1999.

[4] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov. *Continuous-Time Markov Jump Linear Systems*. Springer Berlin Heidelberg, 2012.

[5] R. F. Cunha, G. W. Gabriel, and J. C. Geromel. Partial sampled-data state feedback control of markov jump linear systems. *IFAC-PapersOnLine*, 51(25):222–227, Sep. 2018.

[6] M. H. A. Davis. Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society (B)*, 46(3):353–388, 1984.

[7] M. H. A. Davis. *Markov models and optimization*, volume 49 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, London, 1993.

[8] M. de Almeida, M. Souza, A. R. Fioravanti, and O. L. V. Costa. $\mathcal{H}_2$ state-feedback control for continuous semi-markov jump linear systems with rational transition rates. *International Journal of Control*, 96(1):1–11, 2021.

[9] D. P. de Farias, J. C. Geromel, J. B. R. do Val, and O. L. V. Costa. Output feedback control of markov jump linear systems in continuous-time. *IEEE Transactions on Automatic Control*, 45(5):944 – 949, May 2000.

[10] D. P. de Farias, J. C. Geromel, and J. B. R. do Val. A note on the robust control of markov jump linear uncertain systems. *Optimal Control Applications and Methods*, 23(2):105–112, Mar. 2002.

[11] A. M. de Oliveira, O. L. V. Costa, M. D. Fragoso, and F. Stadtmann. Dynamic output feedback control for continuous-time markov jump linear systems with hidden markov models. *International Journal of Control*, 95(3):1–13, 2020.

[12] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-output Properties*. Electrical Science. Academic Press, 1975.

[13] S. Dong, M. Liu, Z. G. Wu, and K. Shi. Observer-based sliding mode control for markov jump systems with actuator failures and asynchronous modes. *IEEE Transactions on Circuits and Systems II: Express Briefs*, pages 1–1, 2020.

[14] G. J. J. Ducard. *Fault-tolerant flight control and guidance systems*. Springer-Verlag London, 2009.

[15] R. M. Feldman and C. Valdez-Flores. *Applied Probability and Stochastic Processes*. Springer Publishing Company, Incorporated, 2nd edition, 2010.

[16] G. W. Gabriel, T. R. Gonçalves, and J. C. Geromel. Optimal and robust sampled-data control of Markov Jump Linear Systems: A differential LMI approach. *IEEE Transactions on Automatic Control*, 63(9):3054–3060, Sep. 2018.

[17] Z. Hou, J. Luo, P. Shi, and S. K. Nguang. Stochastic stability of ito differential equations with semi-markovian jump parameters. *IEEE Transactions on Automatic Control*, 51(8):1383–1387, Aug. 2006.

[18] J. Huang and Y. Shi. Stochastic stability of semi-markov jump linear systems: An lmi approach. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pages 4668–4673, Dec. 2011.

[19] J. Huang and Y. Shi. Stochastic stability and robust stabilization of semimarkov jump linear systems. *International Journal of Robust and Nonlinear Control*, 23(18):2028–2043, Dec. 2013.

[20] S. Jafari and K. Savla. A principled approximation framework for optimal control of Semi-Markov jump linear systems. *IEEE Transactions on Automatic Control*, 64(9):3616–3631, 2019.

[21] F. Li, L. Wu, and P. Shi. Stochastic stability of semi-markovian jump systems with mode-dependent delays. *International Journal of Robust and Nonlinear Control*, 24(18):3317–3330, Dec. 2014.

[22] F. Li, L. Wu, P. Shi, and C. C. Lim. State estimation and sliding mode control for semi-markovian jump systems with mismatched uncertainties. *Automatica*, 51:385–393, Jan. 2015.

[23] J. Löfberg. Pre- and post-processing sum-of-squares programs in practice. *IEEE Transactions on Automatic Control*, 54(5):1007–1011, 2009.

[24] J. Medhi. *Stochastic processes*. New Age International, 1994.

[25] ApS MOSEK. *The MOSEK optimization toolbox for MATLAB manual. Version 9.0.*, 2019. URL `http://docs.mosek.com/9.0/toolbox/index.html`.

[26] M. Ogura and C. F. Martin. Stability analysis of positive semi-markovian jump linear systems with state resets. *SIAM Journal on Control and Optimization*, 52(3):1809–1831, May 2014.

[27] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*. `http://arxiv.org/abs/1310.4716`, 2013.

[28] A. Papoulis. *Random variables and stochastic processes*. McGraw Hill, 1965.

[29] A. Papoulis and S. U. Pillai. *Probability, random variables, and stochastic processes*. Tata McGraw-Hill Education, 2002.

[30] P. A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96(2):293–320, Apr. 2003.

[31] G. Tartaglione, M. Ariola, G. De Tommasi, and F. Amato. Annular finite-time stability and stabilization of continuous-time markov jump linear systems. In *2019 18th European Control Conference*, pages 394–399, Jun. 2019.

[32] B. Wang and Q. Zhu. Stability analysis of discrete-time semi-markov jump linear systems. *IEEE Transactions on Automatic Control*, 65(12):5415–5421, 2020.

[33] B. Xin and D. Zhao. Generalized $\mathcal{H}_2$ control of the linear system with semi-markov jumps. *International Journal of Robust and Nonlinear Control*, 31(3):1005–1020, 2021.

[34] Z. Xu, H. Su, P. Shi, and Z. G. Wu. Asynchronous $\mathcal{H}_\infty$ control of semi-Markov jump linear systems. *Applied Mathematics and Computation*, 349:270–280, May 2019.

[35] S. Z. Yu. *Hidden Semi-Markov Models*. Elsevier, 2015.

[36] L. Zhang, Y. Leng, and P. Colaneri. Stability and stabilization of discrete-time semi-markov jump linear systems via semi-markov kernel approach. *IEEE Transactions on Automatic Control*, 61(2):503–508, Feb. 2016.

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