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ALEXANDER MUÑOZ GARCIA

On persistence and polynomial decay of solutions to nonlinear dispersive equations and applications

Sobre a persistência e decaimento polinomial de soluções de equações dispersivas não lineares e aplicações

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Alexander Muñoz Garcia

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- ORCID do autor: https://orcid.org/0000-0001-9031-4203
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Prof(a). Dr(a). ADEMIR PASTOR FERREIRA

Prof(a). Dr(a). MAHENDRA PRASAD PANTHEE

Prof(a). Dr(a). BIANCA MORELLI RODOLFO CALSAVARA

Prof(a). Dr(a). JOSÉ FELIPE LINARES RAMIREZ

Prof(a). Dr(a). JOSÉ MANUEL JIMÉNEZ URREA

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“The most exciting phrase to hear in science,
the one that heradls the most discoveries,
is not "Eureka" (I found it)
but "That’s funny...".
(Isaac Asimov)

RESUMO

Na primeira parte deste trabalho é estabelecido um método amplo para obter boa colocação local de equações dispersivas nos espaços de Sobolev com peso $H^s(\mathbb{R}) \cap L^2(|x|^{2b}dx)$. Aplicamos este método a várias equações dispersivas como a equação de Ostrovsky-Stepanyams-Tsimring, a equação de Kawahara, um modelo de quinta ordem e o sistema de Hirota-Satsuma. A segunda parte do trabalho é dedicada a demonstrar a optimalidade da relação entre o decaimento e a regularidade obtida no método desenvolvido; usando a equação de Korteweg-de Vries modificada como exemplo.

Porfim, como uma aplicação direta da teoria desenvolvida nos espaços com peso, na parte final são obtidos resultados de tipo *blow-up* dispersivo de soluções para a equação de Kawahara e para o sistema de Hirota-Satsuma.

Palavras-chave: Espaços de Sobolev com peso, Decaimento polinomial, *Blow-up* dispersivo, Boa colocação local.

ABSTRACT

In the first part of this work we establish a wide method to obtain local well-posedness of dispersive equations in the weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2b}dx)$. We apply this method for several dispersive equations such as the Ostrovsky-Stepanyams-Tsimring equation, the Kawahara equation, a fifth order model, and the Hirota-Satsuma system. The second part of this work is devoted to show that the relation between decay and regularity obtained with the developed method is optimal; using the modified Korteweg-de Vries equation as example.

Finally, as a direct application of the theory in weighted spaces, we obtain results related to the dispersive blow-up of solutions to the Kawahara equation and Hirota-Satsuma system.

Keywords: Weighted Sobolev spaces, Polynomial decay, Dispersive blow-up, Local well-posedness.

LIST OF SYMBOLS

$A \backslash B$	The set of elements in A that are not in B.
$\mathbb{R}^n$	n-dimensional Euclidean space.
$\lfloor x \rfloor$	The greatest integer less than or equal to x.
Δf	The Laplacian of f.
$\widehat{f}$	The Fourier transform of f.
$f^\vee$	The inverse Fourier transform of f.
$\text{sgn}(x)$	Sign function.
$D^s(f)$	Fractional derivative of order s of f.
$J^s(f)$	The operator $(I - \Delta)^s$.
$\mathcal{H}(f)$	Hilbert transform of f.
$\mathcal{D}^s(f)$	Stein derivative of order s of f.
$L^p(\mathbb{R}^n)$	Lebesgue space on $\mathbb{R}^n$ of order p.
$H^s(\mathbb{R}^n)$	L^2-based Sobolev space of order s.
$L^p(wdx)$	Weighted Lebesgue space of order p.
$Z_{s,b}$	Weighted Sobolev space of functions in $H^s(\mathbb{R}^n)$ and in $L^2(\|x\|^{2b}dx)$.
$L^p_s(\mathbb{R}^n)$	L^p-based Sobolev space of order s.
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space on $\mathbb{R}^n$.
$C(I, Y)$	Set of continuous functions on I with values in Y.

$C^k(A)$	Set of functions on A with k continuous derivatives.
$C^\infty(A)$	Set of infinitely differentiable functions on A.
$C_0^\infty(A)$	Set of infinitely differentiable functions on A having compact support.
$\\|\cdot\\|_{b,p}$	Norm in the space $L_s^p(\mathbb{R}^n)$.
$\\|\cdot\\|_{L_t^q L_x^p}$	Norm in the mixed Lebesgue space.

Contents

INTRODUCTION

Nonlinear dispersive equations are equations of the form

$$\partial_t u - im(D)u + N(u) = 0, \tag{1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $m(D)$ is an operator defined as a Fourier multiplier based on a real-valued m and N is a nonlinear function. These kind of equations have been extensively studied in the last decades due to the fact they are models that arise from several physical phenomena, specially in the case of wave propagation. A fundamental aspect in the theory of dispersive equations is the study of well-posedness. Following Kato (see [34]), we say that the initial-value problem (IVP)

$$\begin{cases} \partial_t u(x,t) = f(u), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

is *locally well-posed* in the Banach space Y if the next two conditions are satisfied:

1. For each initial datum $u_0 \in Y$ there exist $T > 0$ and a unique solution u in the space $C([0,T];Y)$.

2. The data-solution map $u_0 \mapsto u$ is continuous from Y to $C([0,T];Y)$.

In case T can be selected arbitrarily large, we say the IVP is *globally well-posed* in Y. It is worth to emphasize that condition 1 above is actually requiring two things: the existence of a unique solution and its persistence in the functional space Y along time. The persistence property is one of the main concerns of this work.

Initial value problems associated to several dispersive equations have been considered extensively in the literature. Classical methods such as the contraction principle have been employed to obtain local well-posedness in functional spaces measuring regularity of the solutions (see for instance [12], [34], [37] and the references therein). In [34], when studying the well-known Korteweg-de Vries (KdV) equation (see (4) below), Kato also considered spaces that, in addition to smoothness, also measure the decay of the solutions. Among the possibilities, persistence in the spaces $Z_{s,b} := H^s(\mathbb{R}^n) \cap L^2(|x|^{2b}dx)$ plays an important role. The relation between decay and regularity displayed by the Fourier transform suggest the study of the persistence in such spaces. Several classical results support the existence of a natural bond between the two spaces involved in the definition of $Z_{s,b}$.

In the past years, new techniques based on Besov or Bourgain spaces have been used to address the IVP associated to many dispersive equations in low regularity spaces, unfortunately, the relation between decay and regularity under these new technologies is not well understood yet. Earlier works dealing with persistence in the spaces $Z_{s,b}$ are based on formulas that interchange weights with the group associated to the linear part of the underlying equation. In [26], [27] and [28], based on the commutative properties of the operators $\Gamma_j = x_j + 2it\partial_j$, the authors used the equality

$$x^\alpha e^{it\Delta}u_0 = e^{it\Delta}\Gamma^\alpha u_0, \quad \text{where } \Gamma = (\Gamma_1, \ldots, \Gamma_n) \text{ and } \alpha \in \mathbb{N}^n;$$

together with calculus inequalities for the operators Γ_j to show that if $u_0 \in Z_{m,k}$, with m, k integers, then the IVP associated with the Schrödinger equation,

$$\begin{cases} i\partial_t u + \Delta u + \mu|u|^{a-1}u = 0, & a > 1, \\ u(x,0) = u_0(x), \end{cases} \tag{3}$$

has a unique solution

$$u \in C([0,T]; Z_{m,k}) \cap L^q([0,T]; L^p_k(\mathbb{R}^n) \cap L^p(|x|^k dx)),$$

for appropriate m and k. Here (p,q) is some admissible pair. This result for indices m, k not necessarily integers was obtained by Nahas and Ponce in [46, Theorem 1].

In [34], Kato considered the following IVP

$$\begin{cases} \partial_t u + \partial_x^3 u + a(u)\partial_x u = 0, & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,t) = u_0(x), \end{cases} \tag{4}$$

where $a(u)$ is a real-valued C^∞ function. If $a(u) = u$ we have the Korteweg-de Vries (KdV) equation. The KdV equation was derived in [38] as a model describing the propagation of waves in one dimensional dispersive media. From the mathematical point of view, this equation has been widely studied in the literature, see for instance [15], [34], [37] and the references therein. Also, if $a(u) = u^2$ we obtain the modified Korteweg-de Vries (mKdV) equation and if $a(u) = u^k$ for $k > 2$ a positive integer, we obtain the generalized Korteweg-de Vries (gKdV) equation . Using the operators $\Gamma_t := x - 3t\partial_x^2$ and $A := \partial_t + \partial_x^3$, Kato showed that $[\Gamma_t, A] = 0$ and used this to note that both, $U(t)(xu_0)$ and $\Gamma_t U(t)u_0$, are solutions of the problem

$$\begin{cases} Au = 0, \\ u(0) = xu_0. \end{cases}$$

Therefore, it follows that $U(t)(xu_0) = \Gamma_t U(t)u_0$. The latter can be translated into the formula

$$xU(t)u_0(x) = U(t)xu_0(x) + 3tU(t)(\partial_x^2 u_0)(x). \tag{5}$$

Using (5), Kato proved the local well-posedness of (4) in $Z_{2r,r}$ for $r \geqslant 1$ integer. This was later extended by Nahas [44] to non-integer indices.

Another example of a model studied in the spaces $Z_{s,b}$ is the Benjamin-Ono equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + uu_x = 0,$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\widehat{\mathcal{H}f}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi).$$

For integers s and b, persistence in these spaces was first studied by Iorio in [31]. For non-integer indices the persistence properties were established by Fonseca and Ponce in [24]. For the study of the IVP associated with other dispersive equations we refer the reader to [7], [8], [18], [19], [21], [23], [33], [48] and references therein.

Our first concern in this thesis is to study decay properties of solutions for linear problems for several dispersive equations. More precisely, we are interested in discussing the problem

$$\begin{cases} \partial_t u + Lu = 0, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{6}$$

where L is a linear operator satisfying $\widehat{Lf}(\xi) = i\phi(\xi)\widehat{f}(\xi)$ for some continuous real-valued function ϕ and $u_0 \in Z_{s,b}$. Via Fourier transform, the solution of (6) is given by

$$U(t)u_0(x) = u(x,t) = (e^{-it\phi(\xi)}\widehat{u}_0)^\vee(x),$$

where $\{U(t)\}_{t\in\mathbb{R}}$ is the associated linear group. We shall assume that the phase $\phi : \mathbb{R}^n \to \mathbb{R}$ is regular enough (see conditions (A) and (B) below) in order to define a group in $H^s(\mathbb{R})$.

The KdV case (when $\phi(\xi) = -\xi^3$) is one of the best understood due to the physical significance of the model. Denote with $U(t)$ the linear group associated to the linear part of (4), that is,

$$U(t)f(x) := \left(e^{it\xi^3}\widehat{f}(\xi)\right)^\vee(x). \tag{7}$$

To study u in the spaces $L^2(|x|^{2b})$ it is necessary to look at expressions of the form $|x|^b u(x,t)$. Taking into account the Duhamel formulation of solutions

$$u(x,t) = U(t)u_0(x) - \int_0^t U(t-t')u(x,t')^k \partial_x u(x,t')dt', \tag{8}$$

the relation between the unitary group $U(t)$ and the weights $|x|^b$ needs to be understood. Recently, using one version of the Stein derivative (which is not directly depending on the Fourier transform), the formula in (5) was later generalized to weights with fractional powers in [22]. It was proven that for $b \in (0,1)$ we have

$$|x|^b U(t)u_0(x) = U(t)(|x|^b u_0)(x) + U(t)\{\Phi_{t,b}(\widehat{u}_0(\xi))\}^\vee(x), \tag{9}$$

where the residual term $\{\Phi_{t,b}(\widehat{u}_0(\xi))\}^\vee$ can be estimated in terms of the L^2 norm of the fractional derivative $D^{2b}u_0$. The authors used the following version of Stein's derivative:

$$D_\alpha f(x) = \lim_{\epsilon \to 0^+} \frac{1}{c_\alpha} \int_{|y| \geqslant \epsilon} \frac{f(x+y) - f(x)}{|y|^{n+\alpha}}\, dy. \tag{10}$$

The main advantage of this version relies on the fact that for suitable functions f, it follows that $\widehat{D_\alpha(f)} = |\xi|^\alpha \widehat{f}$. This allowed the authors to recover the unitary group after a convenient application of a Leibniz-type rule for Stein derivatives.

Note that a close inspection of both, (5) and (9), suggest that one can expect a bond between the regularity index s and the twice the decay $2b$.

In Theorem 1.5 below we prove that if u_0 is in $Z_{s,b} := H^s(\mathbb{R}) \cap L^2(|x|^{2b}dx)$ for $b \leqslant s/K$ then the IVP (6) has a solution u satisfying the inequality

$$\||x|^b u(t)\|_{L^2} \leqslant C\left\{(1+|t|)\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2}\right\} \tag{11}$$

where $\|\cdot\|_{s,2}$ denotes the norm in $H^s(\mathbb{R}^n)$. The parameter K is related to the greatest dispersion present in L. Note that (11) indeed establishes that the solution of (6) persists in $Z_{s,b}$ for any time interval. A similar result to (11) was obtained in [9, Theorem 1.11]. The authors considered a phase function given by

$$\phi(\xi) = \sum_{j=1}^p C_j \xi^{\beta_j}, \quad \xi \in \mathbb{R}^n, \quad \beta_j \in (\mathbb{Z}^+)^n,$$

and established the inequality

$$\||x|^b u(t)\|_{L^2} \leqslant C\||x|^b u_0\|_{L^2} + A(\|u_0\|_{H^{a(b)}}),$$

where $b \geqslant 1$, A is a non negative continuous function and $a(b) := \max_{1,\dots,p}(|\beta_j| - 1)b$. Their proof relies on estimates based on the differential equation itself. On the other hand, our approach to prove (11) follows the ideas of Nahas and Ponce [46] and relies on estimates based on Stein's derivative $\mathcal{D}^b$ of the phase function (see (12) below). In consequence, we are able to include weights with $0 < b < 1$ and establish the same interpolation inequality with $A(x) = (1 + |t|)x$ and $a(b) = Kb$.

We point out that (11) may be seen as an alternative to (9) in the sense that it interchanges weights with the group but also accepts several dimensions and a wide variety of phase functions. On the other hand, in contrast with (9) we lose the punctual identity. A disadvantage of (11) compared to (9), is the impossibility of using Strichartz type estimates once (11) has been applied. This prevents the application of the theory developed here in the context of estimates that do not rely on the L^2-based Sobolev spaces. An example of this situation is the nonlinear Schrödinger equation in which the inequalities used to prove local well-posedness are based on the spaces $L^p_b(\mathbb{R}^n)$.

The main tool to prove (11) is the estimate presented in Lemma 1.9 (below), which in turn is based on previous results that faced persistence properties for particular equations such as in [8], [22] and [46]. Some other works in which related computations have been done are [23] and [33]. In [46], the authors dealt with the Schrödinger equation; using the Stein derivative defined as

$$\mathcal{D}^b(f)(x) := \left(\int_{\mathbb{R}^n} \frac{|f(y) - f(x)|^2}{|x-y|^{n+2b}} dy \right)^{\frac{1}{2}}. \tag{12}$$

They estimated $\mathcal{D}^b(e^{it|\xi|^2})(x)$ by exploiting the radial behavior of the integral

$$\int_{\mathbb{R}^n} \frac{|e^{i(-2\sqrt{t}x\cdot y + |y|^2)} - 1|^2}{|y|^{n+2b}} dy.$$

This estimate was later extended in [8] for $\mathcal{D}^b(e^{it\xi^3})(x)$ when dealing with the Ostrovsky equation. We follow these ideas to generalize it for phase function satisfying weak regularity conditions. Roughly speaking, we require ϕ to be locally Lipschitz with some conditions on how the Lipschitz constant varies in space (see conditions (A) and (B) below).

It is worth mentioning that the modulus present in the definition of $\mathcal{D}^b$ generates cancellation of oscillations when f is of the form $e^{it\phi}$, preventing estimate (11) to be in terms of the group associated to ϕ, in contrast with (9). This issue restrict optimal applications of (11) for some nonlinear equations, in which the problem can be resolved using regularization via Sobolev embedding but that might imply extra constraints in the regularity index s that may not match the best local well-posedness result available.

As a direct application of (11) we prove local well-posedness results in weighted spaces for several physical models, including local and non local models and a system of equations. The first model we are interested in is the Kawahara equation. Consider the IVP

$$\begin{cases} \partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u + \gamma \partial_x^5 u = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{13}$$

where α, β, γ are real numbers with $\alpha\gamma \neq 0$. The Kawahara equation was derived in [35] as a model equation describing solitary-wave propagation in media where the first-order dispersion coefficient is anomalously small. It also arises in modeling gravity-capillary waves on a shallow layer and magneto-sound propagation in plasma. Several results for the IVP (13) can be found in the current literature. In particular, the local well-posedness in the Sobolev spaces was established in [17] for $s > 1/4$. By using Bourgain's spaces, the Sobolev index for the local well-posedness of (13) may be pushed down to $s > -7/4$ (see, for instance, [13]). However, since the work in [17] was established with the technique introduced in [37] (which uses Strichartz's estimates, smoothing effects, and a maximal function estimate combined with the contraction mapping principle), we use it instead to establish in Chapter 2 that (13) is locally well-posed in $Z_{s,b}$ for $s > 1/4$ and $0 \leqslant b \leqslant s/4$.

Now, consider the Hirota-Satsuma system

$$\begin{cases} \partial_t u - a(\partial_x^3 u + 6u\partial_x u) = 2rv\partial_x v, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ \partial_t v + \partial_x^3 v + 3u\partial_x v = 0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \end{cases} \tag{14}$$

where u and v are real-valued functions of the variables $x, t \in \mathbb{R}$ and a, r are nonzero real constants. The system (14) was derived in [29] and describes interactions of two long waves with different dispersion relations. Concerning local well-posedness in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$ via contraction principle, in [1, Theorem 2.1] it was proved to be locally well-posed for $s' = s$ with $s > 3/4$. By performing a natural modification of the Banach space, (11) was used to establish the system is locally well-posed in $Z_{s,b} \times Z_{s,b}$ for $s > 3/4$ and $0 \leqslant b \leqslant s/2$.

Next we consider the IVP associated with the so-called Ostrovsky-Stepanyams-Tsimring (OST for short) equation

$$\begin{cases} \partial_t u + \partial_x^3 u - \eta(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u) + u\partial_x u = 0, & x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{15}$$

where $\eta > 0$ is a real constant and $\mathcal{H}$ is the Hilbert transform. The equation in (15) was derived by Ostrovsky, Stepanyams and Tsimring [47] to describe the radiational instability of long waves in a stratified shear flow. The IVP (15) in classical Sobolev spaces was considered in [11, Theorem 1.1]. The authors used an improved smoothing effect to prove local well-posedness in $H^s(\mathbb{R})$ for $s \geqslant 0$. In Chapter 2 we prove it is also locally well-posed in $Z_{s,b}$ for $s > 0$ and $0 \leqslant b \leqslant s/2$.

Next we consider another fifth-order model by replacing the first-order derivative in the nonlinear part of (13) by a second-order derivative. More precisely, we consider the following IVP

$$\begin{cases} \partial_t u + \alpha u\partial_x^2 u + \beta\partial_x^3 u + \gamma\partial_x^5 u = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{16}$$

where again α, β, γ are real numbers with $\alpha\gamma \neq 0$. The local well-posedness of (16) in $H^s(\mathbb{R})$ was established in [51] for $s > 5/4$. We prove it is also locally well-posed in $Z_{s,b}$ for $s > 5/4$ and $0 \leqslant b \leqslant s/4$.

Note from (9), (11) and the results mentioned above that the upper bound for decay $s/K \geqslant b$ seems to be mandatory. In this work we also study if such condition is optimal. A positive answer was obtained in the sense that if the decay b of the solution exceeds s/K, say $b = s/K + \varepsilon$, then the solution is actually more regular than initially considered and the regularity index seems to be $s + K\varepsilon$. The first result in this direction was proved for the KdV equation and it is due to Isaza, Linares and Ponce [32]. The authors proved that if the solution of the KdV equation in $L^2(\mathbb{R})$ is so that in two different times it accepts a decay of $|x|^\alpha$ for some $\alpha > 0$, then the solution, which was initially only in $L^2(\mathbb{R})$, is in $H^{2\alpha}(\mathbb{R})$. A similar result was later obtained by Bustamante, Jiménez and Mejía in [6] for the fifth-order KdV equation

$$\partial_t u + \partial_x^5 u + u\partial_x u = 0,$$

starting with a solution in $H^2(\mathbb{R})$.

The idea used in [32], is to develop a bootstrap argument depending on the size of α. In each step the authors perform energy estimates in an accurate way to obtain decay for the solution and its derivative. More precisely, for $\alpha > 0$ they established that for almost every $t \in [t_0, t_1]$ it follows that $\partial_x u(t)\langle x\rangle^{\alpha-1/2} \in L^2(\mathbb{R})$ and for all $t \in [t_0, t_1]$ that $u(t)\langle x\rangle^\alpha \in L^2(\mathbb{R})$. By considering $f := \langle x\rangle^{\alpha-1/2}u(t^*)$, for some $t^* \in [t_0, t_1]$, the latter implies that $Jf := (\langle\xi\rangle\widehat{f})^\vee$ and $\langle x\rangle^{1/2}f$ are in $L^2(\mathbb{R})$. Using interpolation (see Lemma 1.4 below), the following bound was obtained:

$$\left\| J^{1\theta}\left(\langle x\rangle^{(1-\theta)\frac{1}{2}} f\right)\right\|_2 \leqslant c\|Jf\|_2^\theta \|\langle x\rangle^{1/2}f\|_2^{1-\theta}, \qquad \theta \in (0,1). \tag{17}$$

In order to give a conclusion about $J^s u$, for some $s \in \mathbb{R}$, it was required the equality $(1-\theta)/2 + \alpha - 1/2 = 0$, that is, $\theta = 2\alpha$ and therefore they proved $u(t^*) \in H^{2\alpha}(\mathbb{R})$.

In [32] the authors suggested that the proof extends to the mKdV equation. It appears that some adjustments are required to raise according to the size of α. For instance, if $\alpha > 1/2$ the same choice of θ done in (17) might not be the best due to the constraint $\theta \leqslant 1$. Moreover, regardless of the choice of θ, the most regular scenario for J^θ leads to $H^1(\mathbb{R})$. This concern exhibits the needing of increase the regularity over f, that is, to get a higher estimate than Jf; while the role of α is re-escalated to fit $(1-\theta)/2 + \tilde{\alpha} - 1/2 = 0$ with $\theta \in (0,1)$. One natural way of doing this reads as follows: in case $\alpha \in \left(\frac{r}{2}, \frac{r+1}{2}\right]$ we can take ∂_x^r to the mKdV equation and prove that $\partial_x^{r+1}u(t)\langle x\rangle^{\tilde{\alpha}-1/2}$ is in $L^2(\mathbb{R})$, where $\tilde{\alpha} = \alpha - r/2 \in (0, 1/2]$. By setting $f := \langle x\rangle^{\tilde{\alpha}-1/2}u(t^*)$ it might be seen that $J^{r+1}f$ and $\langle x\rangle^{1/2}f$ are in $L^2(\mathbb{R})$. Interpolating as in (17) we would get

$$\left\| J^{(r+1)\theta}\left(\langle x\rangle^{(1-\theta)\frac{1}{2}} f\right)\right\|_2 \leqslant c\|J^{r+1}f\|_2^\theta \|\langle x\rangle^{1/2}f\|_2^{1-\theta}, \qquad \theta \in (0,1).$$

From the condition $(1-\theta)/2 + \tilde{\alpha} - 1/2 = 0$ we obtain $\theta = 2\alpha - r$, which is in $(0,1)$. Unfortunately this interpolation will not lead to the best estimate for $J^s u$ because $(r+1)\theta$ is less than or equal to the expected gain of regularity 2α and only attains it when $\alpha = (r+1)/2$.

Given that only an increase in the regularity does not resolve on its own, in the proof of Theorem 3.7 (below) we not only increase the spatial derivatives of u but we also consider some decay for them. We manage to prove that in the general case $\alpha \in \left(\frac{r}{2}, \frac{r+1}{2}\right]$, for almost every t in a subinterval of $[t_0, t_1]$ we have

$$\langle x\rangle^{\tilde{\alpha}-1/2}\partial_x^{r+1}u(t) \in L^2(\mathbb{R}) \quad \text{and} \quad \langle x\rangle^{\tilde{\alpha}}\partial_x^{r}u(t) \in L^2(\mathbb{R}). \tag{18}$$

By considering $f := \langle x\rangle^{\tilde{\alpha}-1/2}\partial_x^r u(t^*)$ it is noted that Jf and $\langle x\rangle^{1/2}f$ are in $L^2(\mathbb{R})$. Therefore interpolating analogous to (17) with $\theta = 2\tilde{\alpha}$ we would obtain $J^{2\tilde{\alpha}}\partial_x^r u(t^*) \in L^2(\mathbb{R})$, that is to say, $u(t^*) \in H^{2\alpha}(\mathbb{R})$.

The optimal relation in the general case $s/K \geqslant b$ is discussed in Chapter 5. Partial results for the Kawahara equation, the OST equation and some perturbations of the KdV are possible to be extended as in the mKdV case. A generalized setting (such as the one in (6)) is one of the topics of current research.

Finally, to stand out the importance of studying dispersive equations in weighted Sobolev spaces, this work carry on with a study of dispersive blow-up properties. The phenomenon of dispersive blow-up was first identified by Benjamin, Bona and Mahony in [2] for the linear KdV equation. Roughly speaking the authors proved the existence of an infinitely smooth bounded initial data such that the corresponding solution blows-up in finite time in the L^∞ norm. The pioneer mathematical work studying the existence of solutions for nonlinear dispersive equations presenting a behavior similar to the linear KdV is due to Bona and Saut [4]. In that paper the authors considered the generalized nonlinear KdV equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad k \in \mathbb{Z}^+,$$

and constructed initial data in $H^\ell(\mathbb{R}) \cap C^\infty(\mathbb{R})$, for a suitable choice of ℓ, such that the corresponding solution satisfies

$$\lim_{(x,t)\to(x_*,t_*)} |u(x,t)| = +\infty, \tag{19}$$

where (x_*, t_*) is a point in $\mathbb{R}\times(0,\infty)$; moreover, the solution u is continuous except at (x_*, t_*). The strategy of the authors was first to construct a solution of the linear problem satisfying (19) and then, using the decay properties of the solutions in weighted spaces, they showed that the nonlinear part do not destroy that behavior. This emphasizes the linear feature of this kind of singularity and makes it different, for instance, of the blow-up in Sobolev norms where the effects of the nonlinearity are stronger.

After that, in [42], the authors addressed the same question for $k \geqslant 2$ but with a simplified approach. Indeed, the authors showed that in this situation is sufficient to show that the integral part of the solution in the Duhamel formulation is more regular than the linear one. More precisely, they established if the initial datum belong to $H^s(\mathbb{R}), s \geqslant 1$, then the corresponding integral part belongs to $C(\mathbb{R}; H^{s+1}(\mathbb{R}))$. This was enough to prove the existence of dispersive blow-up. More recently, in [41], using fractional weighted spaces, the authors also improved the results of [4] in the case $k = 1$, i.e., for the KdV equation. For additional results concerning dispersive blow-up we refer the reader to [3], [5], [39] and [40].

Although the ideas employed below may be applied to several models, we will pay particular attention to the Kawahara equation and the Hirota-Satsuma system. Similar results to the ones we prove in Chapter 4 were obtained in [41] for the KdV equation, in [40] for the two-dimensional Zakharov-Kuznetsov equation, and in [39] for the Schrödinger-KdV system. We first emulate the ideas of [41] to construct a smooth initial data such that the global solution of the associated linear IVP has an infinite number of discontinuities; at these times the linear flow cannot be smooth, which is then identified as the dispersive blow-up taking place at $x = 0$. Then it is shown that the integral term in Duhamel formulation of solutions is smoother than the linear part, which unleash regularity on the linear term.

This thesis is organized as follows. In Chapter 1 we introduce notation and some preliminary and linear results used through this thesis. In particular, (11) is proved. Chapter 2 is devoted to prove local well-posedness in weighted Sobolev spaces for several models. Chapter 3 analyzes the relation between decay and regularity using the mKdV equation as example. In Chapter 4 the weighted theory developed in Chapter 2 is applied to obtain dispersive blow-up of solutions to the Kawahara equation and Hirota-Satsuma system. Finally, in Chapter 5 further results and current research topics are discussed.

Part of this thesis is already published in a scientific article format (see [43]).

CHAPTER 1

PRELIMINARIES AND LINEAR ESTIMATES

This chapter is devoted to develop the linear theory related to the study of persistence in weighted Sobolev spaces, including the proof of (11). We first introduce the notation used in this thesis.

1.1 Notation

We use C and M to denote several constants that may vary from line to line. Sometimes we use subscript or parenthesis to indicate dependence of parameters; for instance $C_\phi = C(\phi)$ means that the constant C depends on ϕ. We shall write $a \simeq b$, where a and b are two positive numbers, when there exists a constant $C > 0$ such that $C^{-1}a \leqslant b \leqslant Ca$. Given a real number r, we use r^+ (respect. r^-) to mean $r + \varepsilon$ (respect. $r - \varepsilon$) for some sufficiently small $\varepsilon > 0$.

By $L^p = L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant +\infty$, we denote the standard Lebesgue space endowed with the usual norm. If w is a weight (a non negative measurable function), by $L^p(wdx)$ (or $L^p(w)$ for short) we denote the space L^p with respect to the measure $w(x)dx$. Given a function f defined on $\mathbb{R}^n$, $\widehat{f}$ and $f^\vee$ stand, respectively, for the Fourier and inverse Fourier transforms of f. The operators D^s and J^s are defined via Fourier transform as

$$\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi) \quad \text{and} \quad \widehat{J^s f}(\xi) = \langle \xi \rangle^s \widehat{f}(\xi),$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$. Given $s \in \mathbb{R}$, by $H^s = H^s(\mathbb{R}^n)$ we mean the L^2-based Sobolev space of order s. For $1 < p < \infty$ and $b \in \mathbb{R}$, the space $L^p_b(\mathbb{R}^n)$ is defined as $L^p_b(\mathbb{R}^n) = (1 - \Delta)^{-b/2} L^p(\mathbb{R}^n)$. Note that in the case $p = 2$ and $b = s$, $L^2_s(\mathbb{R}^n)$ is nothing but the Sobolev space $H^s(\mathbb{R}^n)$. In particular, the norm in $H^s(\mathbb{R}^n)$ is given by

$$\|f\|_{H^s} := \|f\|_{s,2} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

We write $Z_{s,b}$ to denote to the weighted Sobolev space $H^s(\mathbb{R}) \cap L^2(|x|^{2b}dx)$. The so-called Schwartz space is denoted by $\mathcal{S}(\mathbb{R})$.

Given a function $f = f(x,t)$ of the variables x and t, sometimes we use $\|f\|_{L^p_x}$ to indicate that we are taking the L^p norm with respect to the variable x only. Also, given $T > 0$ we use L^p_T to denote the L^p space over the interval $[0,T]$. For $1 \leqslant q, r \leqslant \infty$, the norm in the mixed space $L^q_T L^r_x$ is given by

$$\|f\|_{L^q_T L^r_x} = \big\| \|f(t,\cdot)\|_{L^r_x} \big\|_{L^q_T}.$$

Similar considerations apply to the space $L^r_x L^q_T$. In the case both indices agree, that is $q = r$, we have $\|f\|_{L^r_x L^q_T} = \|f\|_{L^q_T L^r_x} =: \|f\|_{L^r_{xT}}$.

1.2 Commutator and interpolation estimates

In this section we recall some commutator and interpolation estimates which will be useful below. We start with the following commutator estimate for homogeneous derivatives.

Lemma 1.1. *Let $s \in (0,1)$. Then*

(i) For $1 < p < \infty$,

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \leqslant C\|g\|_{L^\infty}\|D^s f\|_{L^p}.$$

(ii) For $1 < r, p_1, p_2, q_1, q_2 < \infty$ satisfying

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2},$$

it holds

$$\|D^s(fg)\|_{L^r} \leqslant C\|f\|_{L^{p_1}}\|D^s g\|_{L^{q_1}} + C\|D^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}.$$

(iii) For $1 < p_1, p_2, q_1, q_2 < \infty$ satisfying

$$1 = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2},$$

we have

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^1_x L^2_T} \leqslant C\|g\|_{L^{p_1}_x L^{q_1}_T}\|D^s f\|_{L^{p_2}_x L^{q_2}_T}.$$

Proof. For part (i) see Theorem A.12 in [37]. For part (ii) see Proposition 3.3 in [14]. For (iii) see Theorem A.13 in [37]. □

Denote with A_p the Muckenhoupt class on $\mathbb{R}^n$. More precisely, given $1 < p < \infty$, the Muckenhoupt class A_p consists of all weights ω such that

$$[\omega]_p = \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(y)dy\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}(y)dy\right)^{p-1} < \infty, \tag{1.1}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$; additional details and properties may be seen in [16] and [30]. The next result is a version of the Kato-Ponce commutator estimate in weighted spaces.

Lemma 1.2. *Let $1 < p, q < \infty$ and $1/2 < \ell < \infty$ be such that $\frac{1}{\ell} = \frac{1}{p} + \frac{1}{q}$. If $v \in A_p$, $w \in A_q$ and $s > \max\{0, n(\frac{1}{\ell} - 1)\}$ or s is a non-negative even integer, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\|D^s(fg) - fD^s g\|_{L^\ell(v^{\frac{\ell}{p}}w^{\frac{\ell}{q}})} \leqslant C\|D^s f\|_{L^p(v)}\|g\|_{L^q(w)} + \|\nabla f\|_{L^p(v)}\|D^{s-1}g\|_{L^q(w)}. \tag{1.2}$$

Proof. See Theorem 1.1 in [16]. □

We also need the following characterization for the boundedness of the Hilbert transform in weighted spaces.

Lemma 1.3. *The Hilbert transform is bounded in $L^p(wdx)$, $1 < p < \infty$, if and only if $w \in A_p$.*

Proof. See Theorem 9 in [30]. □

We finally introduce two interpolation inequalities.

Lemma 1.4. *Assume $a, b > 0$, $1 < p < \infty$ and $\theta \in (0, 1)$. If $J^a f \in L^p(\mathbb{R}^n)$ and $\langle x\rangle^b f \in L^p(\mathbb{R}^n)$ then*

$$\|\langle x\rangle^{(1-\theta)b} J^{\theta a} f\|_{L^p(\mathbb{R}^n)} \leqslant C\|\langle x\rangle^b f\|^{1-\theta}_{L^p(\mathbb{R}^n)}\|J^a f\|^{\theta}_{L^p(\mathbb{R}^n)}. \tag{1.3}$$

The same holds for D instead of J. Moreover, for $p = 2$ we have

$$\left\|J^{\theta a}\left(\langle x\rangle^{(1-\theta)b} f\right)\right\|_{L^2(\mathbb{R}^n)} \leqslant C\|\langle x\rangle^b f\|^{1-\theta}_{L^2(\mathbb{R}^n)}\|J^a f\|^{\theta}_{L^2(\mathbb{R}^n)}. \tag{1.4}$$

Proof. Inequality (1.4) follows from (1.3) in view of Plancherel's identity. The proof of (1.3) follows using Hadamard's three lines theorem. See Lemma 4 in [46]. □

1.3 Weighted inequalities

As briefly mentioned in the introduction, we consider the linear problem

$$\begin{cases} \partial_t u + Lu = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{1.5}$$

where $\widehat{Lf}(\xi) = i\phi(\xi)\widehat{f}(\xi)$ and ϕ is a continuous real-valued function satisfying the regularity conditions

(A) There exists a continuous function $g : \mathbb{R}^n \to \mathbb{R},\ g > 0$ except maybe at $x = 0$, so that for all $x, y \in \mathbb{R}^n$ with $|x - y| \leqslant |x|$ we have $|\phi(x) - \phi(y)| \leqslant g(x)|x - y|$.

(B) There exists $C > 0$ such that for all $x, y \in \mathbb{R}^n$ satisfying $|x - y| \geqslant |x|$ we have $|\phi(x) - \phi(y)| \leqslant C|x - y|^a$, for some $a \geqslant 1$.

Note that by taking $x = 0$ in condition (B) we deduce that $|\phi(y)| \leqslant C(1 + |y|^a)$ for any $y \in \mathbb{R}^n$. In particular, from Stone's theorem one can see that L generates a unitary group in $H^s(\mathbb{R}^n)$, for any $s \in \mathbb{R}$.

Some examples of phase functions satisfying (A) and (B) are given below. We will present the proofs to the Appendix A. Assume $k \in \mathbb{Z}^+$ and $i \in \{1, \ldots, n\}$.

1. Let $\phi_1 : \mathbb{R} \to \mathbb{R}$ be given by $\phi_1(x) = x^k$. In this case we may take $g(x) := C_k|x|^{k-1}$ and $a = k$. In the particular case $k = 3$ we see that $\phi_1(x) = x^3$ is the phase function associated to the linear KdV equation.
2. Let $\phi_2 : \mathbb{R}^n \to \mathbb{R}$ be given by $\phi_2(x) = |x|^k$. Here we may take again $g(x) = C_k|x|^{k-1}$ and $a = k$. Note that for $k = 2$ we obtain $\phi_2(x) = |x|^2$ which is the phase function associated to the linear Schrödinger equation.
3. Denote by $\widehat{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. The functions $\phi_3^i : \mathbb{R}^n \to \mathbb{R}$ defined by $\phi_3^i(x) := x_i|\widehat{x}_i|^2$, also satisfy (A) and (B). In this case $g(x) = C|x|^2$ and $a = 3$.
4. Define $\phi_4^i : \mathbb{R}^n \to \mathbb{R}$ as $\phi_4^i(x) = x_i^k$. Then ϕ_4^i also satisfies (A) and (B) with $g(x) = C_k|x|^{k-1}$ and $a = k$. Alternatively, we may also take $g(x) = C_k|x_i|^{k-1}$ (see [7]). By taking $\phi(x) = \phi_4^1(x) + \phi_3^1(x) = x_1^3 + x_1|\widehat{x}_1|^2$, we see that ϕ is the phase function associated with the linear n-dimensional Zakharov-Kuznetsov equation,

$$\partial_t u + \partial_x \Delta u + = 0.$$

5. More generally, for $\beta \in \mathbb{N}^n$, by taking $\phi_5 : \mathbb{R}^n \to \mathbb{R}$ as $\phi_5(x) = x^\beta$ we obtain that it satisfies (A) and (B) with $g(x) = C_\beta|x|^{|\beta|-1}$ and $a = |\beta|$.

The main theorem of this thesis, concerning persistence of solutions of linear IVPs in weighted Sobolev spaces is proved in Section 1.5 (below) and reads as follows.

Theorem 1.5. *Let $p \in \mathbb{Z}^+$ and assume that $\phi_1, \ldots, \phi_p$ satisfy conditions (A) and (B) with $g_i(x) \leqslant C_i(1 + |x|^{k_i})$, for some $k_i \in \mathbb{Z}^+$ and $C_i > 0$, $i = 1 \ldots, p$. Set*

$$\Phi(\xi) := \sum_{i=1}^{p} \phi_i(\xi)$$

and $K := \max\{k_i, i = 1, \dots, p\}$*. Let* L *be the linear operator defined by* $Lf = \left(i\Phi(\xi)\widehat{f}\right)^{\vee}$ *and assume* $0 < s < K$*. If* $u \in C([-T,T], H^s(\mathbb{R}^n))$ *is the solution of the IVP*

$$\begin{cases} \partial_t u + Lu = 0, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\ u(0) = u_0 \in Z_{s,b} := H^s(\mathbb{R}^n) \cap L^2(|x|^{2b}dx), \end{cases} \tag{1.6}$$

with $0 < b \leqslant s/K$*, then* u *satisfies the inequality*

$$\||x|^b u(t)\|_{L^2} = \||x|^b U(t)u_0\|_{L^2} \leqslant C\left\{(1+|t|)\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2}\right\}, \tag{1.7}$$

where $\|\cdot\|_{s,2}$ *denotes the norm in* $H^s(\mathbb{R}^n)$ *and* C *depends on* K*,* p*,* s *and* n*.*

The condition $s < K$ in Theorem 1.5 can be eliminated as described in Section 1.5 below.

1.4 Stein derivative

In this section we discuss the technical tools involving Stein's derivatives. Let us begin by recalling the definition of Stein derivative $\mathcal{D}^b$. For any real number $b \in (0,1)$ and a measurable function f define

$$\mathcal{D}^b(f)(x) := \left(\int_{\mathbb{R}^n} \frac{|f(y)-f(x)|^2}{|x-y|^{n+2b}}dy\right)^{\frac{1}{2}}.$$

The next theorem gives a useful characterization of the spaces $L^p_b(\mathbb{R}^n)$ due to Stein [49].

Theorem 1.6. *Let* $b \in (0,1)$ *and* $\dfrac{2n}{n+2b} < p < \infty$*. A function* f *belongs to* $L^p_b(\mathbb{R}^n)$ *if and only if* $f \in L^p(\mathbb{R}^n)$ *and* $\mathcal{D}^b(f) \in L^p(\mathbb{R}^n)$*. In addition,*

$$\|f\|_{b,p} := \|f\|_{L^p_b} \simeq \|f\|_{L^p} + \|D^b(f)\|_{L^p} \simeq \|f\|_{L^p} + \|\mathcal{D}^b(f)\|_{L^p}. \tag{1.8}$$

From (1.8) one sees that the norm in $L^p_b(\mathbb{R}^n)$ may be given in terms of either D^b or $\mathcal{D}^b$. The advantage of using Stein's derivative is that it is useful to perform pointwise computations.

Next we estate a Leibniz type rule for $\mathcal{D}^b$.

Lemma 1.7. *For* $b \in (0,1)$ *and measurable functions* f *and* g*, we have*

$$\mathcal{D}^b(fg)(x) \leqslant \|f\|_{L^\infty}\mathcal{D}^b(g)(x) + |g(x)|\mathcal{D}^b(f)(x), \tag{1.9}$$

and

$$\|\mathcal{D}^b(fg)\|_{L^2} \leqslant \|f\mathcal{D}^b(g)\|_{L^2} + \|g\mathcal{D}^b(f)\|_{L^2}. \tag{1.10}$$

Proof. This was proved in Proposition 1 in [46]. The idea is to use the triangle inequality after an addition and subtraction of $f(y)g(x)$ to get

$$\begin{aligned}\mathcal{D}^b(fg)(x) &= \left(\int_{\mathbb{R}^n} \frac{|fg(y) - fg(x)|^2}{|y|^{n+2b}} dy\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{\mathbb{R}^n} \frac{|f(y)(g(x) - g(y))|^2}{|y|^{n+2b}} dy\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^n} \frac{|g(x)(f(x) - f(y))|^2}{|y|^{n+2b}} dy\right)^{\frac{1}{2}} \\ &= A(fg)(x) + |g(x)|\mathcal{D}^b(f)(x) \qquad (1.11) \\ &\leqslant \|f\|_\infty \mathcal{D}^b(g)(x) + |g(x)|\mathcal{D}^b(f)(x),\end{aligned}$$

where

$$A(fg) = \left(\int_{\mathbb{R}^n} \frac{|f(y)(g(x) - g(y))|^2}{|y|^{n+2b}} dy\right)^{\frac{1}{2}},$$

which proves (1.9).

Now, using Fubini-Tonelli's Theorem we obtain

$$\begin{aligned}\|A(fg)\|_{L^2} &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)(g(x) - g(y))|^2}{|y|^{n+2b}} dydx\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^n} |f(y)|^2 \int_{\mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|y|^{n+2b}} dydx\right)^{\frac{1}{2}} \\ &= \|f\mathcal{D}^b(g)\|_{L^2}.\end{aligned}$$

Therefore, (1.10) follows from (1.11). □

We also may prove the following.

Proposition 1.8. *Let $b \in (0,1)$ and $p \in \mathbb{Z}^+$, $p \geqslant 2$. Assume $h_i : \mathbb{R}^n \to \mathbb{C}$, $i = 1, \ldots, p$, are measurable. Then*

$$\mathcal{D}^b\left(\prod_{i=1}^p h_i\right)(x) \leqslant \sum_{i=1}^p \mathcal{D}^b(h_i)(x) \prod_{\substack{j=1 \\ j\neq i}}^p \|h_j\|_{L^\infty}. \qquad (1.12)$$

Proof. Note that $\mathcal{D}^b(f)(x)$ is always a positive quantity. So the proposition follows by induction on p just by iterating (1.9). □

Now we establish a pointwise estimate for the Stein derivative of phase functions satisfying (A) and (B).

Lemma 1.9. *Let $b \in (0,1)$. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions (A) and (B). For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ we have*

$$\mathcal{D}^b(e^{it\phi(\cdot)})(x) \leqslant C\left\{1 + (1 + |t|)g(x)^b\right\}, \qquad (1.13)$$

where $g(x)$ is as in (A) and the constant $C > 0$ depends on n, b and ϕ.

Proof. We follow a similar strategy to the one applied in [46, Proposition 2]. Let $x \in \mathbb{R}^n$ be nonzero. Then,

$$\mathcal{D}^b(e^{it\phi(\cdot)})(x) = \left(\int_{\mathbb{R}^n} \frac{\left|e^{it\phi(x)} - e^{it\phi(y)}\right|^2}{|x-y|^{n+2b}} dy\right)^{1/2} = \left(\int_{\mathbb{R}^n} \frac{\left|e^{it(\phi(x)-\phi(y))} - 1\right|^2}{|x-y|^{n+2b}} dy\right)^{1/2} \equiv I.$$

To simplify notation, by $B(a, R)$ we mean the closed ball of radius $R > 0$ centered at the point a in $\mathbb{R}^n$. Split $\mathbb{R}^n$ into the following three sets:

$$E_1 := B(x, g(x)^{-1})^c, \; E_2 := B(x, g(x)^{-1}) \cap B(x, |x|) \text{ and } E_3 := B(x, g(x)^{-1}) \cap B(x, |x|)^c,$$

where A^c means the complement of the set A in $\mathbb{R}^n$. Let I_j, $j = 1, 2, 3$, be the integral I with the integration over $\mathbb{R}^n$ replaced by the integration over E_j. Since, clearly, $I \leqslant C(I_1+I_2+I_3)$ we see that it suffices to estimate I_j.

In what is coming after, the inequalities

$$|e^{i\theta} - 1| \leqslant 2 \quad \text{and} \quad |e^{i\theta} - 1| \leqslant |\theta|, \quad \theta \in \mathbb{R}, \tag{1.14}$$

shall be used repeatedly without being mentioned.

The idea to estimate I_j is to use (1.14) and then to explore the radial feature of the resulting function. We begin by estimating I_1:

$$\begin{aligned} I_1 &\leqslant \left(\int_{E_1} \frac{4}{|x-y|^{n+2b}} dy\right)^{1/2} \leqslant C_n \left(\int_{g(x)^{-1}}^{\infty} \frac{r^{n-1}}{r^{n+2b}} dr\right)^{1/2} \\ &= C_{n,b} \left(\int_{g(x)^{-1}}^{\infty} r^{-1-2b} dr\right)^{1/2} \leqslant C_{n,b} \left(g(x)^{2b}\right)^{1/2} = C_{n,b} g(x)^b. \end{aligned} \tag{1.15}$$

For I_2 we need to divide into two cases.

<u>Case 1, $g(x)^{-1} \leqslant |x|$.</u> In this case, $E_2 = B(x, g(x)^{-1})$. So, by using condition (A) we deduce

$$\begin{aligned} I_2 &\leqslant C_\phi \left(\int_{E_2} \frac{|tg(x)|x-y||^2}{|x-y|^{n+2b}} dy\right)^{1/2} \leqslant C_\phi |t| g(x) \left(\int_{B(x,g(x)^{-1})} |x-y|^{2-n-2b} dw\right)^{1/2} \\ &\leqslant C_{\phi,n} |t| g(x) \left(\int_0^{g(x)^{-1}} r^{1-2b} dr\right)^{1/2} \\ &= C_{\phi,n,b} |t| g(x) \left(g(x)^{2b-2}\right)^{1/2} = C_{\phi,n,b} |t| g(x)^b. \end{aligned} \tag{1.16}$$

<u>Case 2, $|x| < g(x)^{-1}$.</u> Here we have $E_2 = B(x, |x|) \subset B(x, g(x)^{-1})$. Hence, we can use the same calculations done in Case 1 to obtain

$$I_2 \leqslant \left(\int_{B(x,g(x)^{-1})} \frac{\left|e^{it(\phi(x)-\phi(y))} - 1\right|^2}{|x-y|^{n+2b}} dy\right)^{1/2} \leqslant C_{\phi,n,b} |t| g(x)^b. \tag{1.17}$$

Finally we estimate I_3. Note that E_3 is an annulus and it is empty if $|x| \geqslant g(x)^{-1}$. So we will always assume that $|x| \leqslant g(x)^{-1}$. Here we divide the proof into three cases.

Case 1, $1 \leqslant |x|$. In this case we promptly obtain

$$\begin{aligned} I_3 &\leqslant \left(\int_{E_3} \frac{4}{|x-y|^{n+2b}} dy \right)^{1/2} = C_n \left(\int_{|x|}^{g(x)^{-1}} r^{n-1-n-2b} dr \right)^{1/2} \\ &\leqslant C_n \left(\int_1^{g(x)^{-1}} r^{-1-2b} dr \right)^{1/2} \leqslant C_{n,b} \left(1 - g(x)^{2b}\right)^{1/2} \leqslant C_{n,b}. \end{aligned} \tag{1.18}$$

Case 2, $|x| < 1 < g(x)^{-1}$. We split E_3 into the sets

$$E_{31} := E_3 \cap B(x,1) \quad \text{and} \quad E_{32} := E_3 \cap B(x,1)^c.$$

Using condition (B), since $2a - 1 \geqslant 1$, we get

$$\begin{aligned} I_3 &\leqslant \left(C_\phi \int_{E_{31}} \frac{|x-y|^{2a}}{|x-y|^{n+2b}} dy + \int_{E_{32}} \frac{2^2}{|x-y|^{n+2b}} dy \right)^{1/2} \\ &\leqslant C_{\phi,n} \left(\int_{|x|}^{1} r^{2a-1-2b} dr + \int_1^{g(x)^{-1}} r^{-1-2b} dr \right)^{1/2} \\ &\leqslant C_{\phi,n} \left(\int_{|x|}^{1} r^{1-2b} dr + \int_1^{g(x)^{-1}} r^{-1-2b} dr \right)^{1/2} \\ &\leqslant C_{\phi,n,b} \left((1 - |x|^{2-2b}) + (1 - g(x)^{2b}) \right)^{1/2} \\ &\leqslant C_{\phi,n,b}. \end{aligned} \tag{1.19}$$

Case 3, $g(x)^{-1} \leqslant 1$. Here we use condition (B) again to obtain

$$\begin{aligned} I_3 &\leqslant C_\phi \left(\int_{E_3} \frac{|x-y|^{2a}}{|x-y|^{n+2b}} dy \right)^{1/2} = C_{\phi,n} \left(\int_{|x|}^{g(x)^{-1}} \frac{r^{2a+n-1}}{r^{n+2b}} dr \right)^{1/2} \\ &\leqslant C_{\phi,n} \left(\int_{|x|}^{1} r^{2a-1-2b} dr \right)^{1/2} \leqslant C_{\phi,n} \left(\int_{|x|}^{1} r^{1-2b} dr \right)^{1/2} \\ &= C_{\phi,n,b} \left(1 - |x|^{2-2b}\right)^{1/2} \leqslant C_{\phi,n,b}. \end{aligned} \tag{1.20}$$

From estimates (1.15)-(1.20) we obtain (1.13), which proves the theorem for $x \neq 0$. Finally, if $x = 0$ and $g(0) > 0$, the proof above remains equal. In case $g(0) = 0$, we divide $\mathbb{R}^n$ into $E_1^0 = B(0, 1/2)^c$ and $E_2^0 = B(0, 1/2)$. Note that, as in (1.15), it can be seen that $I_1^0 \leqslant C_{\phi,n,b}$. Also, the argument in (1.20) remains equal for I_2^0. We therefore have $\mathcal{D}^b(e^{it\phi(\cdot)})(0) \leqslant C_{\phi,n,b}$ and the proof of the lemma is completed. □

Remark 1.10. *It is worth mentioning that Lemma 1.9 is still valid if we impose only the weaker condition*

(A') *There exists $g : \mathbb{R}^n \to \mathbb{R}$ measurable, $g > 0$ except maybe at 0, such that for all $x \in \mathbb{R}^n$ if $|x-y| \leqslant 1$ then $|\phi(x) - \phi(y)| \leqslant g(x)|x-y|$.*

instead of (A) and (B). The idea is to consider the sets

$$E_1 := B(x, g(x)^{-1})^c, \quad E_2 := B(x, g(x)^{-1}) \cap B(x,1) \quad \text{and} \quad E_3 := B(x, g(x)^{-1}) \cap B(x,1)^c,$$

and to note that the estimate of E_3 is the exactly (1.18).

1.5 Proof of Theorem 1.5

This section is devoted to prove Theorem 1.5. The main tool is the point-wise estimate established in Lemma 1.9.

Proof of Theorem 1.5. It suffices to prove the theorem with $b = s/K$. So, assume $f := u_0 \in L^2(|x|^{2b}dx) \cap H^{Kb}(\mathbb{R}^n)$. We already now that L generates a unitary group, say, $\{U(t)\}$ in $H^{Kb}(\mathbb{R}^n)$ such that $U(t)f = (e^{-it\Phi(\cdot)}\widehat{f})^{\vee}$. From Plancherel's theorem and (1.8) we have

$$\begin{aligned} \||x|^b U(t)f\|_{L^2} &= \|D^b(e^{-it\Phi(\cdot)}\widehat{f})\|_{L^2} \\ &\leqslant C\|e^{-it\Phi(\cdot)}\widehat{f}\|_{L^2} + C\|\mathcal{D}^b(e^{-it\Phi(\cdot)}\widehat{f})\|_{L^2} \\ &\leqslant C\|f\|_{L^2} + C\|\mathcal{D}^b(e^{-it\Phi(\cdot)}\widehat{f})\|_{L^2}. \end{aligned} \tag{1.21}$$

Hence we need to estimate the quantity $\|\mathcal{D}^b(e^{-it\Phi(\cdot)}\widehat{f})\|_{L^2}$. According to (1.10) and (1.12) we have

$$\begin{aligned} \|\mathcal{D}^b(e^{-it\Phi(\cdot)}\widehat{f})\|_{L^2} &\leqslant \|\widehat{f}\mathcal{D}^b(e^{-it\Phi(\cdot)})\|_{L^2} + \|e^{-it\Phi(\cdot)}\mathcal{D}^b(\widehat{f})\|_{L^2} \\ &\leqslant \left\|\widehat{f}\mathcal{D}^b\left(\prod_{i=1}^p e^{-it\phi_i(\cdot)}\right)\right\|_{L^2} + \|\mathcal{D}^b(\widehat{f})\|_{L^2} \\ &\leqslant \left\|\widehat{f}\sum_{i=1}^p \mathcal{D}^b(e^{-it\phi_i(\cdot)})\cdot 1\right\|_{L^2} + \|\mathcal{D}^b(\widehat{f})\|_{L^2}. \end{aligned} \tag{1.22}$$

In view of Lemma 1.9,

$$\begin{aligned} \left\|\widehat{f}\sum_{i=1}^p \mathcal{D}^b(e^{-it\phi_i(\cdot)})\right\|_{L^2} &\leqslant C\left\|\widehat{f}\sum_{i=1}^p \{1+(1+|t|)g_i(x)^b\}\right\|_{L^2} \\ &\leqslant C\left\|\widehat{f}\sum_{i=1}^p \{1+(1+|t|)(1+|x|^{bk_i})\}\right\|_{L^2} \\ &\leqslant C\left\|\widehat{f}\sum_{i=1}^p \{1+(1+|t|)(2+|x|)^{bK}\}\right\|_{L^2} \\ &\leqslant C(1+|t|)\left\|(1+|x|)^{bK}\widehat{f}\right\|_{L^2} \\ &\leqslant C(1+|t|)\|f\|_{bK,2}, \end{aligned} \tag{1.23}$$

where the constant C depends on n, b, K and p. Moreover, since $f \in L^2(|x|^{2b}dx) \cap L^2(\mathbb{R}^n)$ we have $\widehat{f} \in H^b(\mathbb{R}^n)$ and by Theorem 1.6,

$$\|\mathcal{D}^b(\widehat{f})\|_{L^2} \leqslant C\|\widehat{f}\|_{L^2} + C\|D^b(\widehat{f})\|_{L^2} = C\|f\|_{L^2} + C\||x|^b f\|_{L^2}. \tag{1.24}$$

Gathering together estimates (1.21)-(1.24) the proof of Theorem 1.5 is complete. □

Under the hypotheses of Theorem 1.5, the condition $s < K$ can be eliminated. That is, a similar estimate to (1.7) holds when $b > 1$. The idea is to obtain a formula interchanging weights with integer powers with the associated group. Denote $\Gamma_t := x - t(\Phi'(\xi)^\wedge)^\vee$. Consider $A := \partial_t + L$. It can be seen that $\widehat{A\Gamma_t} = \widehat{\Gamma_t A}$ and therefore $[\Gamma_t, A] = 0$. Note that both, $U(t)xu_0$ and $\Gamma_t U(t)u_0$, are solutions of the problem

$$\begin{cases} Au = 0, \\ u(0) = xu_0, \end{cases}$$

and therefore $U(t)xu_0 = \Gamma_t U(t)u_0$. The latter can be translated into the formula

$$xU(t)u_0(x) = U(t)xu_0(x) + tU(t)\{\Phi'(\xi)\widehat{u}_0\}^\vee(x). \tag{1.25}$$

Define the operators E and R as follows: for a suitable f (a Schwartz function for instance), set $Ef = xf$ and $Rf = t\{\Phi'(\xi)\widehat{f}_0\}^\vee$. Under this notation, (1.25) can be rewritten as $xU(t)u_0 = U(t)(E+R)u_0$. By iterating this formula, for $m \in \mathbb{Z}^+$ we have

$$x^m U(t)u_0(x) = U(t)(E+R)^m u_0(x). \tag{1.26}$$

Let $s > 0$ be arbitrary. Write $s = mK + \tilde{a}$, where $m \in \mathbb{Z}^+$ and $\tilde{a} \in [0, K)$. It is enough to consider the case $b = s/K$, that is, $b = m + \tilde{a}/K =: m + a$, where $a \in [0, 1)$.

Let us assume first $a = 0$. Define $J = \left\{ A \subset \{0, 1, \ldots, m\} \mid \sum_{i \in A} i = m \right\}$. We have

$$\begin{aligned} \||x|^b U(t)u_0\|_2 &= \|x^m U(t)u_0\|_2 = \|U(t)(E+R)^m u_0\|_2 = \|(E+R)^m u_0\|_2 \\ &\leqslant \sum_{\{j_0,\ldots,j_m\} \in J} \|R^{j_0} E^{j_1} R^{j_2} \cdots R^{j_m} u_0\|_2. \end{aligned} \tag{1.27}$$

Note the sum above include all the terms present in the expansion of $(E+R)^m$. To simplify the exposition we are going to use $\Phi(\xi) = c_K \xi^{K+1}$ so that $Rf = t\partial_x^K f$. For any $\{j_0, j_1, \ldots, j_m\} \in J$ we have

$$\begin{aligned} \|R^{j_0} E^{j_1} R^{j_2} \cdots R^{j_m} u_0\|_2 &= |t|^{J^e_{m-1} + j_m} \|\partial_x^{Kj_0} \left(x^{j_1} \partial_x^{Kj_2} \left(\cdots \partial_x^{Kj_m} u_0 \right) \right) \|_2 \\ &= |t|^{J^e_{m-1} + j_m} \|\partial_x^{Kj_0} \left(x^{j_1} f_2 \right) \|_2, \end{aligned} \tag{1.28}$$

where $J^e_x := \left\{ \sum j_i \mid i \text{ even and } 0 \leqslant i \leqslant x \right\}$ and $f_i := \partial_x^{Kj_i} \left(x^{j_{i+1}} \partial_x^{Kj_{i+2}} (\cdots \partial_x^{Kj_m} u_0) \right)$; i even. By the product rule and the fact $|\partial_x^{l_0} x^{j_1}| \leqslant c\langle x \rangle^{j_1 - l_0}$ for $l_0 \leqslant j_1$ we have

$$\begin{aligned} \|\partial_x^{Kj_0}(x^{j_1} f_2)\|_2 &\leqslant C_{j_0} \sum_{\substack{l_0=0 \\ l_0 \leqslant j_1}}^{Kj_0} \|\langle x \rangle^{j_1 - l_0} \partial_x^{Kj_0 - l_0} f_2\|_2 \\ &\leqslant C_{j_0} \sum_{\substack{l_0=0 \\ l_0 \leqslant j_1}}^{Kj_0} \|\langle x \rangle^{j_1 - l_0} \partial_x^{K(j_0+j_2) - l_0} (x^{j_3} f_4)\|_2. \end{aligned} \tag{1.29}$$

Using the product rule again we have

$$\begin{aligned}\|\langle x\rangle^{j_1-l_0}\partial_x^{K(j_0+j_2)-l_0}(x^{j_3}f_4)\| &\leqslant C_{j_0,j_2}\sum_{\substack{l_2=0\\ l_2\leqslant j_3}}^{KJ_2^e-l_0}\|\langle x\rangle^{j_1+j_3-l_0-l_2}\partial_x^{K(j_0+j_2)-l_0-l_2}f_4\|_2\\ &\leqslant C_{J_2^e}\sum_{\substack{l_2=0\\ l_2\leqslant j_3}}^{KJ_2^e-L_0}\|\langle x\rangle^{J_3^o-L_2}\partial_x^{K(J_2^e)-L_2}f_4\|_2,\end{aligned} \tag{1.30}$$

where $L_x := \left\{\sum l_i \mid i \text{ even and } 0\leqslant i\leqslant x\right\}$ and $J_x^o := \left\{\sum j_i \mid i \text{ odd and } 1\leqslant i\leqslant x\right\}$. From (1.29) and (1.30)

$$\|\partial_x^{Kj_0}(x^{j_1}f_2)\|_2 \leqslant C_{J_2^e}\sum_{\substack{l_0=0\\ l_0\leqslant j_1}}^{KJ_0^e}\sum_{\substack{l_2=0\\ l_2\leqslant j_3}}^{KJ_2^e-L_0}\|\langle x\rangle^{J_3^o-L_2}\partial_x^{KJ_2^e-L_2}f_4\|. \tag{1.31}$$

We continue this process $m^* := \lfloor m/2\rfloor$ times to get

$$\begin{aligned}\|\partial_x^{Kj_0}(x^{j_1}f_2)\|_2 &\leqslant C_{J_{m-1}^e}\sum_{\substack{l_0=0\\ l_0\leqslant j_1}}^{KJ_0^e}\sum_{\substack{l_2=0\\ l_2\leqslant j_3}}^{KJ_2^e-L_0}\cdots\sum_{\substack{l_{2m^*}=0\\ l_{2m^*}\leqslant j_{2m^*+1}}}^{KJ_{m-1}^e-L_{2m^*-2}}\|\langle x\rangle^{J_{m-1}^o-L_{m-1}}\partial_x^{KJ_{m-1}^e-L_{m-1}}\partial_x^{Kj_m}u_0\|_2\\ &\leqslant C_m\sum_{i=0}^{m^*}\sum_{\substack{l_{2i}=0\\ l_{2i}\leqslant j_{2i+1}}}^{KJ_{2i}^e-L_{2i-2}}\|\langle x\rangle^{J_{m-1}^o-L_{m-1}}\partial_x^{KJ_{m-1}^o+Kj_m-L_{m-1}}u_0\|_2.\end{aligned} \tag{1.32}$$

We proceed to estimate each of the terms $\|\langle x\rangle^{J_{m-1}^o-L_{m-1}}\partial_x^{KJ_{m-1}^o+Kj_m-L_{m-1}}u_0\|_2$. First recall $u_0\in H^{Km}(\mathbb{R})\cap L^2(|x|^{2m}dx)$. We interpolate as follows: take θ satisfying the condition $2m\theta = K(J_{m-1}^e+j_m)-L_{m-1}$, that is,

$$\theta = \frac{K(J_{m-1}^e+j_m)-L_{m-1}}{Km-K(J_{m-1}^e+j_m)+L_{m-1}}.$$

Note $\theta\in(0,1)$ since $KJ_{m-1}^e-L_{m-1}>0$ and $Km>K(J_{m-1}^e+j_m)$. Moreover,

$$1-\theta = \frac{Km-K(J_{m-1}^e+j_m)+L_{m-1}}{Km}.$$

Consider

$$\beta = \frac{(J_{m-1}^o-L_{m-1})Km}{Km-K(J_{m-1}^e+j_m)+L_{m-1}}.$$

This way $(1-\theta)\beta = J_{m-1}^o-L_{m-1}$. Also, since $J_{m-1}^e+J_{m-1}^o+j_m=m$ it is easy to see $\beta\leqslant m$. Interpolating using (1.3) we get

$$\begin{aligned}\|\langle x\rangle^{J_{m-1}^o-L_{m-1}}D_x^{K(J_{m-1}^e+j_m)-L_{m-1}}u_0\|_2 &\leqslant C\|D_x^{Km}u_0\|_2^{\theta}\|\langle x\rangle^{\beta}u_0\|_2^{1-\theta}\\ &\leqslant C\|u_0\|_{Km,2}+C\|\langle x\rangle^{b}u_0\|_2.\end{aligned} \tag{1.33}$$

Note (1.33) can be used to continue estimate (1.32) after an application of Lemma 1.3. Taking this into account, from (1.27), (1.28), (1.32) and (1.33) we get

$$\||x|^b U(t)u_0\|_2 \leqslant C_b(1+|t|)^b \left\{\|u_0\|_{Kb,2} + \|\langle x\rangle^b u_0\|_2\right\}. \tag{1.34}$$

Let us consider now $a \in (0,1)$. Using Theorem 1.5, we have

$$\begin{aligned}\||x|^b U(t)u_0\|_2 &= \||x|^a x^m U(t)u_0\|_2 = \||x|^a U(t)(E+R)^m u_0\|_2 \\ &\leqslant C(1+|t|)\left\{\|(E+R)^m u_0\|_{Ka} + \||x|^a (E+R)^m u_0\|_2\right\}.\end{aligned} \tag{1.35}$$

Arguing as done to get (1.32) we have that

$$\begin{aligned}\||x|^a(E+R)^m u_0\|_2 &\leqslant \sum_{\{j_0\dots,j_m\}\in J} \|\langle x\rangle^a R^{j_0}E^{j_1}R^{j_2}\cdots R^{j_m}u_0\|_2 \\ &\leqslant C_m|t|^{J^e_{m-1}+j_m} \sum_{\{j_0\dots,j_m\}\in J}\sum_{i=0}^{m^*}\sum_{\substack{l_{2i}=0\\ l_{2i}\leqslant j_{2i+1}}}^{KJ^e_{2i}-L_{2i-2}} \|\langle x\rangle^{J^o_{m-1}+a-L_{m-1}}(\mathcal{H}D_x)^{K(J^e_{m-1}+j_m)-L_{m-1}}u_0\|_2\end{aligned} \tag{1.36}$$

Recall $u_0 \in H^{Ka+Km}(\mathbb{R}) \cap L^2(|x|^{2a+2m}dx)$. We interpolate as follows: take θ so that $K(m+a)\theta = K(J^e_{m-1}+j_m) - L_{m-1}$, that is,

$$\theta = \frac{K(J^e_{m-1}+j_m-L_{m-1})}{Km+Ka}.$$

Since $Km+Ka > K(J^e_{m-1}+j_m)$ and $KJ^e_{m-1}-L_{m-1} > 0$ we guarantee $\theta \in (0,1)$. The latter forces

$$1-\theta = \frac{Ka+Km+L_{m-1}-(J^e_{m-1}+j_m)}{Km+Ka}$$

and therefore

$$\beta = \frac{(J^o_{m-1}+a-L_{m-1})(Ka+Km)}{Km+Ka+L_{m-1}-K(J^e_{m-1}+j_m)}$$

is so that $(1-\theta)\beta = J^o_{m-1}+a-L_{m-1}$ with $\beta \leqslant m+a$. Thus, using Lemma 1.3 and interpolating using (1.3) we get

$$\begin{aligned}\|\langle x\rangle^{J^o_{m-1}+a-L_{m-1}}(\mathcal{H}D_x)^{K(J^e_{m-1}+j_m)-L_{m-1}}u_0\|_2 &\leqslant C\|D^{Ka+Km}u_0\|_2^\theta \|\langle x\rangle^\beta u_0\|_2^{1-\theta} \\ &\leqslant C\|u_0\|_{H^{K(m+a)}} + c\|\langle x\rangle^b u_0\|_2.\end{aligned} \tag{1.37}$$

Hence

$$\||x|^a(E+R)^m u_0\|_2 \leqslant C_m(1+|t|)^m\left\{\|u_0\|_{H^{K(m+a)}} + \|\langle x\rangle^b u_0\|_2\right\}. \tag{1.38}$$

Now, for $\|(E+R)^m u_0\|_{H^{Ka}}$ we estimate $\|J^{Ka}(E+R)^m u_0\|_2$ as follows.

$$\begin{aligned}\|J^{Ka}(E+R)^m u_0\|_2 &\leqslant \sum_{\{j_0,\dots,j_m\}\in J} |t|^{J^e_{m-1}+j_m}\|J^{Ka}\left(\partial_x^{Kj_0}(x^{j_1}\partial_x^{Lj_2}\cdots\partial_x^{Kj_m}u_0)\right)\|_2 \\ &\leqslant C_{m,K}(1+|t|)^m \sum_{\{j_0,\dots,j_m\}\in J} \|\langle\xi\rangle^{Ka}\partial_\xi^{Kj_0}\partial_\xi^{j_1}(\xi^{Kj_2}\cdots\xi^{Kj_m}\widehat{u}_0)\|_2.\end{aligned} \tag{1.39}$$

Arguing as in (1.29)-(1.32) we have

$$\|\langle\xi\rangle^{Ka}\partial\xi^{Kj_0}\partial_\xi^{j_1}(\xi^{Kj_2}\cdots\xi^{Kj_m}\widehat{u}_0)\|_2 \leqslant C_m\sum_{i=0}^{m*}\sum_{\substack{l_{2i+1}=0\\ l_{2i+1}\leqslant Kj_{2i+2}}}^{j_{2i+1}}\|\langle\xi\rangle^{Ka+KJ^e_{m-1}+Kj_m-L_{m-1}}\partial_\xi^{J^o_{m-1}-L_{m-1}}\widehat{u}_0\|_2. \tag{1.40}$$

Note each of the terms on the right right-hand side of (1.40) can be estimated via interpolation. Take θ so that $Ka+KJ^e_{m-1}+Kj_m-L_{m-1}=(1-\theta)(Ka+Km)$, that is,

$$\theta=\frac{Km-(KJ^e_{m-1}+Kj_m-L_{m-1})}{Km+Ka}.$$

It can be seen that θ is in $(0,1)$. This forces to set

$$\alpha=\frac{(J^o_{m-1}-L_{m-1})(Ka+Km)}{Km-(kJ^e_{m-1}+j_m-L_{m-1})},$$

which is less that or equal to $m+a$. Interpolating and using Lemma 1.3 we have

$$\begin{aligned}\|\langle\xi\rangle^{Ka+KJ^e_{m-1}+kj_m-L_{m-1}}\partial_\xi^{J^o_{m-1}-L_{m-1}}\widehat{u}_0\|_2 &\leqslant C\|\langle\xi\rangle^{Ka+Km}\widehat{u}_0\|_2^{\theta}\|D_\xi^\alpha\widehat{u}_0\|_2^{1-\theta}\\ &\leqslant C\|u_0\|_{H^{K(a+m)}}+C\|\langle x\rangle^{m+a}u_0\|_2.\end{aligned} \tag{1.41}$$

Combining (1.40) and (1.41) we obtain,

$$\|(E+R)^m u_0\|_{H^{Ka}}\leqslant C_{m,K}(1+|t|)^m\left\{\|u_0\|_{Kb}+\|\langle x\rangle^b u_0\|_2\right\}. \tag{1.42}$$

Finally, from (1.35), (1.38) and (1.42) we conclude

$$\||x|^bU(t)u_0\|_2\leqslant C_{b,K}(1+|t|)^{\lfloor b/K\rfloor+1}\left\{\|u_0\|_{Kb,2}+\|\langle x\rangle^b u_0\|_2\right\}. \tag{1.43}$$

CHAPTER 2

WELL-POSEDNESS IN WEIGHTED SPACES

This chapter is devoted to prove local well-posedness results in the spaces $Z_{s,b}$. In all cases, the main idea is to use the technique introduced in [37] which combines Strichartz-type estimates, Kato's smoothing effects and a maximal function estimate with the contraction mapping principle to obtain a unique fixed point (the solution) of the corresponding integral equation.

2.1 Kawahara equation

We first recall the local theory in $H^s(\mathbb{R})$.

Theorem 2.1. *([17, Theorem 3.5]) Let $u_0 \in H^s(\mathbb{R})$, $s > 1/4$. There exists $T > 0$, depending on α, β, γ and $\|u_0\|_{s,2}$, such that (13) has a unique solution satisfying $u \in C([-T,T]; H^s(\mathbb{R}))$ and*

$$\|u\|_{L^4_x L^\infty_T} + \|\partial_x u\|_{L^4_T L^\infty_x} + \|D^{s+2}_x u\|_{L^\infty_x L^2_T} + \|D^s \partial_x u\|_{L^4_x L^2_T} < \infty. \tag{2.1}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^s(\mathbb{R})$ such that the map $\widetilde{u}_0 \mapsto \widetilde{u}(t)$, from V into the class defined by (2.1), with T' instead of T, is Lipschitz.

We briefly describe how Theorem 2.1 is proved. Denote by $W(t)$ the unitary group associated to the linear part of the problem (13), that is,

$$W(t)u_0(x) = \left(e^{it(-\gamma\xi^5+\beta\xi^3)}\widehat{u_0}\right)^\vee(x). \tag{2.2}$$

For $M, T > 0$ and $s > 1/4$, consider the space

$$X^T_M := \{w \in C([-T,T]; H^s(\mathbb{R})) \mid \Lambda^T(w) \leqslant M\},$$

where

$$\Lambda^T(w) := \max_{[-T,T]} \|w\|_{s,2} + \|\partial_x w\|_{L^4_T L^\infty_x} + \|w\|_{L^4_x L^\infty_T} + \|D^{s+2} w\|_{L^\infty_x L^2_T} + \|D^s_x \partial_x w\|_{L^4_x L^2_T}.$$

In [17] the authors showed that the integral equation

$$\Psi(u)(t) = W(t)u_0 + \alpha \int_0^t W(t-t')(u\partial_x u)(t')dt' \tag{2.3}$$

is a contraction in X^T_M with

$$\Lambda^T(\Psi(u)) \leqslant C\|u_0\|_{s,2} + CT^{1/2}\Lambda^T(u)^2, \tag{2.4}$$

for some $C > 0$ and any $u \in X^T_M$.

Moreover, the following lemma was established:

Lemma 2.2. *Let $s > 1/4$ and $0 < T \leqslant 1$. If $\Lambda^T(u) < \infty$ then $u\partial_x u \in L^2([-T,T]; H^s(\mathbb{R}))$ and*

$$\left(\int_{-T}^T \|(u\partial_x u)(t')\|^2_{s,2}\, dt'\right)^{1/2} \leqslant C\Lambda^T(u)^2.$$

Proof. See Lemma 3.3 in [17]. □

Note that the phase function $\Phi(x) = -\gamma x^5 + \beta x^3$ satisfies the conditions of Theorem 1.5. Hence, we are in a position to prove the following.

Theorem 2.3. *In addition to hypotheses of Theorem 2.1, assume $u_0 \in L^2(|x|^{2b}dx)$ for $b \leqslant s/4$. There exists $T = T(\|u_0\|_{Z_{s,b}}) > 0$ such that (13) has a unique solution u in the class defined by (2.1) with $Z_{s,b}$ instead of $H^s(\mathbb{R})$. Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of $u_0 \in Z_{s,b}$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$, from V into the class defined by (2.1), with $Z_{s,b}$ instead of $H^s(\mathbb{R})$ and T' instead of T, is Lipschitz.*

Proof. Set $\lambda^T_6(w) := \max_{[-T,T]} \||x|^b w\|_{L^2_x}$ and consider the space

$$Y^T_M := \{w \in C([-T,T]; Z_{s,b}) \mid \Omega^T(w) \leqslant M\}, \quad \text{where} \quad \Omega^T(w) = \Lambda^T(w) + \lambda^T_6(w).$$

To see that Ψ maps Y^T_M into itself we need to estimate it in the norm λ^T_6. For any $u \in X^T_M$, using (1.7) and Hölder's inequality we get

$$\||x|^b\Psi(u)\|_{L^2_x} \leqslant C\Big\{(1+T)\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2_x} + T^{1/2}(1+T)\|u\partial_x u\|_{L^2_T H^s_x} \\ + T^{1/2}\||x|^b u\partial_x u\|_{L^2_T L^2_x}\Big\}. \tag{2.5}$$

According to Lemma 2.2 we have $\|u\partial^2_x u\|_{L^2_T H^s_x} \leqslant C\Lambda^T(u)^2$. Besides, using Hölder's inequality we obtain

$$\||x|^b u\partial_x u\|_{L^2_T L^2_x} \leqslant \max_{[-T,T]} \||x|^b u\|_{L^2_x} \|\partial_x u\|_{L^2_T L^\infty_x} \leqslant T^{1/4}\lambda^T_6(u)\|\partial_x u\|_{L^4_T L^\infty_x} \leqslant T^{1/4}\Omega^T(u)^2. \tag{2.6}$$

Hence, from (2.5) and (2.6) we conclude

$$\lambda_6^T(u) \leqslant C\left\{(1+T)\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2_x} + (1+T)(T^{3/4}+T^{1/2})\Omega^T(u)^2\right\}. \tag{2.7}$$

Finally, by combining (2.4) and (2.7) we obtain

$$\Omega^T(\Psi(u)) \leqslant C\left\{(1+T)\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2_x} + (1+T)(T^{3/4}+T^{1/2})\Omega^T(u)^2\right\}.$$

By taking $M = 2C\left\{2\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2_x}\right\}$ and $0 < T < 1$ such that

$$C(1+T)(T^{3/4}+T^{1/2})M < \frac{1}{2},$$

we infer that $\Psi : Y_M^T \to Y_M^T$ is well defined. The rest of the proof carry on from standard arguments. □

Theorem 2.3 is a local well-posedness result. Taking a careful look to the way in which λ_6^T was estimated, a persistence result can be obtained. More precisely we have the following corollary.

Corollary 2.4. *Let $u \in C([-T,T];H^s(\mathbb{R}))$ be the solution provided by Theorem 2.1. Suppose there exists $t^* \in [-T,T]$ such that $|x|^b u(t^*) \in L^2(\mathbb{R})$ for $0 < b \leqslant s/4$. Then $u \in C([-T,T];Z_{s,b})$.*

Proof. We first recall that T in Theorem 2.1 is such that $4CT^{1/2}M = 1$, where $M = 2C\|u_0\|_{s,2}$. That means

$$T = \min\left\{1, \ \frac{1}{64C^4\|u(0)\|_{s,2}^2}\right\}. \tag{2.8}$$

Note from (2.5), (2.6) and (2.7) we have that for any $T' \leqslant T$:

$$\begin{aligned}\lambda_6^{T'}(u) &\leqslant C(1+T')\|u(0)\|_{s,2} + C\||x|^b u(0)\|_{L^2} + C(T')^{1/2}(1+T')\Lambda^{T'}(u)^2 \\ &\quad + C(T')^{1/2}(T')^{1/4}\Lambda^{T'}(u)\lambda_6^{T'}(u).\end{aligned} \tag{2.9}$$

Combining the latter with (2.8) we have that

$$\begin{aligned}\lambda_6^{T'}(u) &\leqslant C(1+T)\|u(0)\|_{s,2} + c\||x|^b u(0)\|_{L^2} + \frac{1}{4}(1+T)M + \frac{1}{4}T^{1/4}\lambda_6^{T'}(u) \\ &\leqslant 2C\|u(0)\|_{s,2} + C\||x|^b u(0)\|_{L^2} + \frac{1}{2}M + \frac{1}{4}\lambda_6^{T'}(u).\end{aligned} \tag{2.10}$$

From (2.10) we obtain the *a priori* bound

$$\lambda_6^{T'}(u) \leqslant \frac{12}{3}C\|u(0)\|_{s,2} + \frac{4}{3}C\||x|^b u(0)\|_{L^2}. \tag{2.11}$$

Now, consider $\tilde{u}_0(x) := u(t^*)$ and the IVP (13) with initial data $\tilde{u}_0$. Applying Theorem 2.3 there exists a local solution $\tilde{u} \in C([-T_0+t^*, t^*+T_0]; Z_{s,b})$. By uniqueness we have

$$u \in C([-T,T];H^s(\mathbb{R})) \cap C([-T_0+t^*, t^*+T_0]; Z_{s,b}),$$

where $0 < T_0 < 1$ is chosen so that

$$2C(1+T_0)(T_0^{3/4}+T_0^{1/2})M_0 < 1 \text{ with } M_0 = 4C\|\tilde{u}_0\|_{s,2} + 2C\||x|^b\tilde{u}_0\|_{L^2}. \tag{2.12}$$

Therefore

$$T_0 = \frac{1}{32C^2M_0^2}. \tag{2.13}$$

Note

$$\|\tilde{u}_0\|_{s,2} = \|u(t^*)\|_{s,2} \leqslant \max_{[-T,T]} \|u(t)\|_{s,2} \leqslant M = 2C\|u(0)\|_{s,2}$$

and

$$\||x|^b\tilde{u}_0\|_{L^2} = \||x|^b u(t^*)\|_{L^2} \leqslant \lambda_6^T(u) \leqslant \frac{12}{3}C\|u(0)\|_{s,2} + \frac{4}{3}C\||x|^b u(0)\|_{L^2}.$$

Hence

$$M_0 \leqslant \frac{48}{3}C^2\|u(0)\|_{s,2} + \frac{8}{3}\||x|^b u(0)\|_{L^2} =: MM_0. \tag{2.14}$$

According to the definition of T_0 we have

$$T_0(M_0) = \frac{1}{32C^2M_0^2} \geqslant \frac{1}{32C^2(MM_0)^2} =: T^* > 0. \tag{2.15}$$

Since (2.15) is valid in the whole interval $[-T,T]$, we can reapply the weighted local theory $2T/T^*$-times to obtain $u \in C([-T,T];Z_{s,b})$. □

2.2 The Hirota-Satsuma system

We first state the local well-posedness result in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$.

Theorem 2.5. *([1, Theorem 2.1]) Let $a \neq 0$ and $s > 3/4$. Then for any $u_0, v_0 \in H^s(\mathbb{R})$, there exist $T = T(\|u_0\|_{s,2}, \|v_0\|_{s,2}) > 0$ and a unique solution (u,v) of problem (14) such that*

$$\begin{aligned} &u,v \in C([-T,T];H^s(\mathbb{R})), && \partial_x u, \partial_x v \in L_T^4 L_x^\infty, && D_x^s\partial_x u, D_x^s\partial_x v \in L_x^\infty L_T^2,\\ &u,v \in L_x^2 L_T^\infty, && \partial_x u, \partial_x v \in L_x^\infty L_T^2. \end{aligned} \tag{2.16}$$

Moreover, for any $T' \in (0,T)$ there exist neighborhoods V of u_0 in $H^s(\mathbb{R})$ and V' of v_0 in $H^s(\mathbb{R})$ such that the map $(\tilde{u}_0,\tilde{v}_0) \mapsto (\tilde{u},\tilde{v})$ from $V \times V'$ into the class defined by (2.16), with T' instead of T, is Lipschitz.

Denote by $U_a(t)$ the unitary group associated with the linear part of the first equation in (14), that is, $U_a(t)f = (e^{-ita\xi^3}\widehat{f})^\vee$ and set $U(t) \equiv U_{-1}(t)$. It is clear that conditions (A) and (B) are satisfied by the phase function $\Phi(x) = ax^3$, for any $a \neq 0$.

We recall the strategy to prove Theorem 2.5. For $T > 0$ set

$$\Lambda_s^T(w) := \max_{[-T,T]} \|w(t)\|_{s,2} + \|\partial_x w\|_{L_T^4 L_x^\infty} + \|D_x^s\partial_x w\|_{L_x^\infty L_T^2} + (1+T)^{-1/2}\|w\|_{L_x^2 L_T^\infty} + \|\partial_x w\|_{L_x^\infty L_T^2}.$$

In [1] it was shown that the map $\Psi(u,v) = (\Psi_1(u,v), \Psi_2(u,v))$ defined by

$$\begin{cases} \Psi_1(u,v)(t) = U_a(t)u_0 + \int_0^t U_a(t-t')(6au\partial_x u - 2rv\partial_x v)(t')dt', \\ \Psi_2(u,v)(t) = U(t)v_0 - 3\int_0^t U(t-t')(u\partial_x v)(t')dt', \end{cases}$$

is a contraction in the space

$$X_M^T := \{(u,v) \in C([-T,T], H^s(\mathbb{R})) \times C([-T,T], H^s(\mathbb{R})) \mid \Lambda_s^T(u) + \Lambda_s^T(v) \leqslant M\},$$

for a suitable choice of the parameters T and M with

$$\Lambda_s^T(\Psi_1(u,v)) + \Lambda_s^T(\Psi_2(u,v)) \leqslant C\|u_0\|_{s,2} + C\|v_0\|_{s,2} + CT^{1/2}(T^{1/4} + (1+T)^{1/2})M^2, \quad (2.17)$$

for some universal constant $C > 0$ and any $(u,v) \in X_M^T$. From the contraction mapping principle one obtains the unique solution.

Theorem 2.6. *Assume, in addition to the hypotheses in Theorem 2.5, that* $u_0, v_0 \in L^2(|x|^{2b}dx)$ *with* $b \leqslant s/2$. *Then there exist* $T = T(\|u_0\|_{Z_{s,r}}, \|v_0\|_{Z_{s,r}}) > 0$ *and a unique solution* (u,v) *of* (14) *such that* u, v *are in the class defined by* (2.16) *with* $Z_{s,b}$ *instead of* $H^s(\mathbb{R})$.

Moreover, for any $T' \in (0,T)$ *there exist neighborhoods* V *of* u_0 *in* $Z_{s,b}(\mathbb{R})$ *and* V' *of* v_0 *in* $Z_{s,b}$ *such that the map* $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ *from* $V \times V'$ *into the class defined by* (2.16) *with* $Z_{s,b}$ *instead of* $H^s(\mathbb{R})$ *and* T' *instead of* T, *is Lipschitz.*

Remark 2.7. *In case* $a = 0$, *the idea developed below can be carried on with simpler computations and lead to a similar result. See Theorem 2.2 in [1].*

Proof of Theorem 2.6. We follow the same strategy described above. Consider

$$\lambda^T(w) := \max_{[-T,T]} \||x|^b w\|_{L^2_x}.$$

We are going to prove that $\Psi(u,v)$ is a contraction in the space

$$Y_M^T := \{(u,v) \in C([-T,T], Z_{s,b}) \times C([-T,T], Z_{s,b}) \mid \Omega_s^T(u) + \Omega_s^T(v) \leqslant M\},$$

endowed with the norm $|||(u,v)||| := \Omega_s^T(u) + \Omega_s^T(v)$, where $\Omega_s^T(w) = \Lambda_s^T(w) + \lambda^T(w)$ and $T, M > 0$ will be determined later.

We begin by estimating $\Psi_1(u,v)$ for $(u,v) \in Y_M^T$. In view of (2.17) it suffices to estimate $\lambda^T(\Psi_1(u,v))$. Using Minkowski's inequality we obtain

$$\begin{aligned} \||x|^b\Psi_1(u,v)\|_{L^2_x} &\leqslant \||x|^bU_a(t)u_0\|_{L^2_x} + \int_0^T \||x|^bU_a(t-t')(6au\partial_x u + 2rv\partial_x v)(t')\|_{L^2_x}dt' \\ &\leqslant \||x|^bU_a(t)u_0\|_{L^2_x} + \int_0^T \|(|x|^bU_a(t-t')6au\partial_x u)(t')\|_{L^2_x}dt' \\ &\quad + \int_0^T \|(|x|^bU_a(t-t')2rv\partial_x v)(t')\|_{L^2_x}dt' \\ &\leqslant I + II + III. \end{aligned} \tag{2.18}$$

In view of (1.7),

$$I \leqslant C\||x|^b u_0\|_{L^2_x} + C(1+T)\|u_0\|_{s,2}. \tag{2.19}$$

for some positive constant C (depending on s). Another application of (1.7) combined with Hölder's inequality gives

$$\begin{aligned} II &\leqslant \int_0^T C\|(|x|^b u\partial_x u)(t')\|_{L^2_x} + C(1+T)\|(u\partial_x u)(t')\|_{s,2}dt' \\ &\leqslant CT^{1/2}(1+T)\left\{\||x|^b u\partial_x u\|_{L^2_TL^2_x} + \|u\partial_x u\|_{L^2_TL^2_x} + \|D^s_x(u\partial_x u)\|_{L^2_TL^2_x}\right\} \end{aligned}$$

Since Λ_s^T contains the $L^\infty_T H^s$ norm, the last two terms in the above inequality have already been estimated in [1, Theorem 2.1]; more precisely,

$$\|u\partial_x u\|_{L^2_TL^2_x} + \|D^s_x(u\partial_x u)\|_{L^2_TL^2_x} \leqslant CT^{1/2}(T^{1/4} + (1+T)^{1/2})M^2. \tag{2.20}$$

To bound the remaining term we use Hölder's inequality to deduce

$$\||x|^b u\partial_x u\|_{L^2_TL^2_x} \leqslant T^{1/4} \max_{[-T,T]} \||x|^b u\|_{L^2_x}\|\partial_x u\|_{L^4_TL^\infty_x} \leqslant T^{1/4}(\Omega^T_s(u))^2 \leqslant T^{1/4}M^2.$$

Hence

$$II \leqslant CT^{1/2}(1+T)(T^{1/4} + (1+T)^{1/2})M^2. \tag{2.21}$$

A similar computation establishes

$$III \leqslant CT^{1/2}(1+T)(T^{1/4} + (1+T)^{1/2})M^2. \tag{2.22}$$

Estimates (2.19)-(2.22) yield

$$\lambda^T(\Psi_1(u,v)) \leqslant C\||x|^b u_0\|_{L^2_x} + C(1+T)\|u_0\|_{s,2} + CT^{1/2}(1+T)(T^{1/4} + (1+T)^{1/2})M^2.$$

By using the same argument it can be seen that

$$\begin{aligned} \lambda^T(\Psi_2(u,v)) &\leqslant \||x|^bU(t)v_0\|_{L^2_x} + \int_0^T \||x|^bU(t-t')(u\partial_x v)(t')\|_{L^2_x}dt' \\ &\leqslant C\||x|^b v_0\|_{L^2_x} + C(1+T)\|v_0\|_{s,2} + CT^{1/2}(1+T)(T^{1/4} + (1+T)^{1/2})M^2. \end{aligned}$$

Collecting these estimates we get

$$\Omega_s^T(\Psi_1(u,v)) + \Omega_s^T(\Psi_2(u,v)) \leqslant C\Big\{\||x|^b u_0\|_{L^2_x} + |x|^b v_0\|_{L^2_x} + (1+T)(\|u_0\|_{s,2} + \|v_0\|_{s,2})\Big\} + CT^{1/2}(1+T)(T^{1/4} + (1+T)^{1/2})M^2.$$

By choosing

$$M = 2C\Big\{\||x|^b u_0\|_{L^2} + \||x|^b v_0\|_{L^2} + 2(\|u_0\|_{s,2} + \|v_0\|_{s,2})\Big\}$$

and $0 < T \leqslant 1$ sufficiently small such that

$$2CT^{1/2}(1+T)(T^{1/4} + (1+T)^{1/2})M \leqslant 1,$$

we deduce that $\Psi : Y_M^T \to Y_M^T$ is well defined. Moreover, similar arguments show that Ψ is a contraction. The rest of the proof follows from standard arguments; thus we omit the details. □

Corollary 2.8. *Let $u \in C([-T,T]; H^s(\mathbb{R}))$ be the solution of (14) provided by Theorem 2.5. Suppose there exists $t^* \in [-T,T]$ such that $|x|^b u(t^*) \in L^2(\mathbb{R})$ for $0 < b \leqslant s/2$. Then $u \in C([-T,T]; Z_{s,b})$.*

Proof. The proofs follow in a similar manner as done in Corollary 2.4. □

2.3 The OST equation

In [11], the authors proved the following theorem.

Theorem 2.9. *([11, Theorem 1.1]) Let $u_0 \in H^s(\mathbb{R})$ with $s \geqslant 0$. Then there exist $T > 0$ and a unique solution of the IVP (15) such that*

$$\begin{aligned} &u \in C([0,T]; H^s(\mathbb{R})), \qquad \|\partial_x u\|_{L^2_T L^{p_1}_x} + \|D^s \partial_x u\|_{L^2_T L^{p_1}_x} < \infty, \\ &\|u\|_{L^2_T L^{q_1}_x} + \|D^s u\|_{L^2_T L^{q_1}_x} < \infty, \end{aligned} \tag{2.23}$$

for $2 < p_1 < \infty$ and q_1 defined through the relation $\dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{2}$. Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^s(\mathbb{R})$ such that the map $\widetilde{u}_0 \mapsto \widetilde{u}(t)$, from V into the class defined by (2.23), with T' instead of T, is Lipschitz.

Besides Strichartz's estimates, the authors used the contraction principle with a refined smoothing effect for the group

$$V(t)u_0 = \left(e^{-it\Phi(\cdot)}\widehat{u_0}\right)^\vee, \quad \text{where } \Phi(\xi) = -\xi^3 - \eta(|\xi| - |\xi|^3). \tag{2.24}$$

In particular the next lemma was established.

Lemma 2.10. *If $u_0 \in H^s(\mathbb{R})$, $0 < s \leqslant 1$, $0 < T < 1$ and $\gamma := \min\{\frac{1}{2}, \frac{2s}{3}\}$, then*

$$\|\partial_x V(t)u_0\|_{L^2_T L^\infty_x} \leqslant CT^\gamma \|D^s_x u_0\|_{L^2},$$

for some constant $C > 0$ depending on η and s.

Proof. See Corollary 2.2 in [11]. □

As before, we rush an overview of the proof of Theorem 2.9. Consider the space

$$X^T_M = \{w \in C([0,T]; H^s(\mathbb{R})) \mid \Lambda^T(w) \leqslant M\},$$

with

$$\Lambda^T(w) = \sum_{i=1}^{5} \lambda^T_i(w) := \max_{[0,T]} \|w\|_{s,2} + \|\partial_x w\|_{L^2_T L^{p_1}_x} + \|D^s \partial_x w\|_{L^2_T L^{p_1}_x} + T^{-\gamma(p_1)}\|w\|_{L^2_T L^{q_1}_x} + T^{-\gamma(p_1)}\|D^s w\|_{L^2_T L^{q_1}_x},$$

where $\gamma(p_1)$ is a positive constant depending only on p_1. The authors, in [11] then proved that the map $\Psi : X^T_M \to X^T_M$, defined by

$$\Psi(u)(t) = V(t)u_0 - \int_0^t (V(t-t')u\partial_x u)(t')dt',$$

is a contraction, for a suitable choice of T and M satisfying

$$\Lambda^T(\Psi(u)) \leqslant C\|u_0\|_{s,2} + CT^{\gamma(p_1)}M^2, \tag{2.25}$$

for some positive constant C and any $u \in X^T_M$.

Theorem 2.11. *Let $u_0 \in Z_{s,b}$ with $s > 0$ and $b \leqslant s/2$. There exist $T = T(\|u_0\|_{Z_{s,b}}) > 0$ and a unique u in the class defined by (2.23), with $Z_{s,b}$ instead of $H^s(\mathbb{R})$, which is the solution of the IVP (15). Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $Z_{s,b}$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$, from V into the class defined by (2.23), with $Z_{s,b}$ instead of $H^s(\mathbb{R})$ and T' instead of T, is Lipschitz.*

In order to prove Theorem 2.11, note that the phase Φ in (2.24) satisfy the conditions of Theorem 1.5 because it is a combination of particular cases of functions ϕ mentioned in Chapter 1. We therefore may use (1.7).

Proof of Theorem 2.11. We provide details for the computations when $0 < s < 1$. Set $\gamma = \min\{1/2, 2s/3\}$. For $0 < T < 1$, in addition to the norms in Λ^T, consider $\lambda^T_6(w) := T^{-\gamma}\|\partial_x w\|_{L^2_T L^\infty_x}$ and $\lambda^T_7(w) := \||x|^b w\|_{L^\infty_T L^2_x}$. Define

$$Y^T_M := \{w \in C([0,T]; Z_{s,b}) \mid \Omega^T(w) \leqslant M\} \quad \text{where} \quad \Omega^T(w) = \Lambda^T(w) + \lambda^T_6(w) + \lambda^T_7(w).$$

We will show that for suitable choices of M and T, the map $\Psi : Y_M^T \to Y_M^T$ is well defined and is a contraction. From (2.25) it remains to estimate the norms λ_6^T and λ_7^T. In view of Lemma 2.10 we have

$$\begin{aligned}\lambda_6^T(\Psi(u)) &\leqslant T^{-\gamma}\|\partial_x V(t)u_0\|_{L_T^2 L_x^\infty} + T^{-\gamma}\left\|\partial_x V(t)\int_0^t (V(-t')u\partial_x u)(t')dt'\right\|_{L_T^2 L_x^\infty} \\ &\leqslant C\|D^s u_0\|_{L_x^2} + C\left\|D_x^s \int_0^t (V(-t')u\partial_x u)(t')dt'\right\|_{L_x^2} \\ &\leqslant C\|u_0\|_{s,2} + \int_0^T \|D_x^s(u\partial_x u)(t')\|_{L_x^2} dt' \\ &\equiv C(\|u_0\|_{s,2} + I).\end{aligned} \tag{2.26}$$

According to the fractional Leibniz rule (see Lemma 1.1) we have

$$\|D_x^s(u\partial_x u)\|_{L^2} \leqslant C\|u\|_{L^{q_1}}\|D_x^s\partial_x u\|_{L^{p_1}} + C\|\partial_x u\|_{L^{p_1}}\|D_x^s u\|_{L^{q_1}}.$$

Therefore, from Hölder's inequality, we deduce

$$I \leqslant C\|u\|_{L_T^2 L_x^{q_1}}\|D_x^s\partial_x u\|_{L_T^2 L_x^{p_1}} + C\|\partial_x u\|_{L_T^2 L_x^{p_1}}\|D_x^s u\|_{L_T^2 L_x^{q_1}} \leqslant CT^{\gamma(p_1)}\Omega^T(u)^2.$$

We conclude from (2.26) that

$$\lambda_6^T(\Psi(u)) \leqslant C\|u_0\|_{s,2} + CT^{\gamma(p_1)}\Omega^T(u)^2.$$

Besides, using Theorem 1.5, we get

$$\begin{aligned}\||x|^b\Psi(u)\|_{L_x^2} &\leqslant \||x|^b V(t)u_0\|_{L_x^2} + \int_0^T \left\||x|^b V(t-t')(u\partial_x u)(t')\right\|_{L^2} dt' \\ &\leqslant C(1+T)\|u_0\|_{s,2} + C\||x|^b u_0\|_{L_x^2} + C\int_0^T (1+T)\|u\partial_x u\|_{s,2} dt' \\ &\quad + C\int_0^T \||x|^b u\partial_x u\|_{L_x^2} dt' \\ &= C(1+T)\|u_0\|_{s,2} + C_s\||x|^b u_0\|_{L_x^2} + II + III.\end{aligned} \tag{2.27}$$

The term II can be estimated as done with I (actually, this term has already been estimated in the $H^s(\mathbb{R})$ local theory). In particular, we obtain

$$II \leqslant C(1+T)T^{\gamma(p_1)}\Lambda^T(u)^2 \leqslant C(1+T)T^{\gamma(p_1)}\Omega^T(u)^2. \tag{2.28}$$

In what comes to III we use Hölder's inequality as follows:

$$\begin{aligned}III &\leqslant CT^{1/2}\||x|^b u\partial_x u\|_{L_T^2 L_x^2} \leqslant CT^{1/2}\max_{[0,T]}\||x|^b u\|_{L_x^2}\|\partial_x u\|_{L_T^2 L_x^\infty} \\ &\leqslant CT^{1/2+\gamma}\lambda_6^T(u)\lambda_7^T(u) \leqslant CT^{1/2+\gamma}\Omega^T(u)^2.\end{aligned} \tag{2.29}$$

From (2.27)-(2.29) we conclude

$$\lambda_7^T(u) \leqslant C(1+T)\|u_0\|_{s,2} + C\||x|^b u_0\|_{L_x^2} + C(1+T)(T^{1/2+\gamma} + T^{\gamma(p_1)})\Omega^T(u)^2.$$

Gathering together the above estimates we finally obtain

$$\Omega^T(\Psi(u)) \leqslant C(1+T)\|u_0\|_{s,2} + C\||x|^b u_0\|_{L^2_x} + C(1+T)(T^{1/2+\gamma} + T^{\gamma(p_1)})\Omega^T(u)^2.$$

By setting $M = 2C\left\{2\|u_0\|_{s,2} + \||x|^b u_0\|_{L^2}\right\}$ and taking $0 < T < 1$ such that

$$C(1+T)(T^{1/2+\gamma} + T^{\gamma(p_1)})M \leqslant \frac{1}{2}$$

it can be seen that $\Phi : Y^T_M \to Y^T_M$ is well defined. Moreover, similar arguments show that Ψ is a contraction. To finish the proof we use standard arguments, thus, we omit the details. □

Remark 2.12. *In [20], using a purely dissipative method, the author established the local well-posedness of (15) in $H^s(\mathbb{R})$ for $s > -3/2$. However, as we already said, the relation between decay and low regularity is not well understood; so, we are not able to establish a local well-posedness result in $Z_{s,b}$ for indices $s \leqslant 3/4$.*

Corollary 2.13. *Let $u \in C([-T,T]; H^s(\mathbb{R}))$, with $s > 0$, be the solution of (15) provided by Theorem 2.9. Suppose there exists $t^* \in [-T,T]$ such that $|x|^b u(t^*) \in L^2(\mathbb{R})$ for $0 < b \leqslant s/2$. Then $u \in C([-T,T]; Z_{s,b})$.*

Proof. The proof follows in a similar manner as done in Corollary 2.4. □

2.4 A fifth-order equation

In what comes to (16), the following was established in [51, Theorem 1.1].

Theorem 2.14. *Suppose $\beta\gamma < 0$. Let $u_0 \in H^s(\mathbb{R})$, $s \geqslant 5/4$. There exists $T = T(\|u_0\|_{s,2}) > 0$ such that (16) has a unique solution satisfying*

$$\begin{aligned} &u \in C([-T,T]; H^s(\mathbb{R})), \quad \|u\|_{L^2_x L^\infty_T} < \infty, \quad \|\partial_x^2 u\|_{L^4_T L^\infty_x} < \infty, \\ &\text{and} \quad \|D^s \partial_x^2 u\|_{L^\infty_x L^2_T} < \infty. \end{aligned} \tag{2.30}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^s(\mathbb{R})$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$, from V into the class defined by (2.30), with T' instead of T, is Lipschitz.

The idea is to consider positive constants T and M and the space

$$X^T_M := \{w \in C([-T,T]; H^s(\mathbb{R})) \mid \Lambda^T(w) \leqslant M\},$$

where

$$\Lambda^T(w) := \max_{[-T,T]} \|w\|_{s,2} + \|\partial_x^2 w\|_{L^4_T L^\infty_x} + \|w\|_{L^2_x L^\infty_T} + \|D^{s+2} w\|_{L^\infty_x L^2_T}.$$

Let $W(t)$ be as in (2.2). For $s \geqslant 5/4$, and suitable choices of T and M, in [51] the authors showed that the integral equation

$$\Psi(u)(t) = W(t)u_0 + \alpha \int_0^t W(t-t')(u\partial_x^2 u)(t')dt'$$

maps X_M^T into itself, is a contraction and satisfies

$$\Lambda^T(\Psi(u)) \leqslant C\|u_0\|_{s,2} + CT^{1/2}\Lambda^T(u)^2,$$

for some $C > 0$ and any $u \in X_M^T$.

Moreover, they showed the following lemma:

Lemma 2.15. *Let $0 \leqslant T < 1$ and $s \geqslant 5/4$. If $\Lambda^T(u) < \infty$ then $u\partial_x^2 u \in L^2([-T,T]; H^s(\mathbb{R}))$ and*

$$\left(\int_{-T}^{T} \left\|(u\partial_x^2 u)(t')\right\|_{s,2}^2 dt'\right)^{1/2} \leqslant C\Lambda^T(u)^2,$$

where $C > 0$ depends only on α, β, γ, and s.

Proof. See Lemma 3.2 in [51]. □

Theorem 2.16. *In addition to the hypotheses of Theorem 2.14, assume $u_0 \in L^2(|x|^{2b}dx)$ for $b \leqslant s/4$. There exists $T = T(\|u_0\|_{Z_{s,b}}) > 0$ such that (16) has a unique solution u in the class defined by (2.30) with $Z_{s,b}$ instead of $H^s(\mathbb{R})$. Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of $u_0 \in Z_{s,b}$ such that the map $\widetilde{u}_0 \mapsto \widetilde{u}(t)$, from V into the class defined by (2.30), with $Z_{s,b}$ instead of $H^s(\mathbb{R})$ and T' instead of T, is Lipschitz.*

Proof of Theorem 2.16. The proof follows by setting $\lambda_5^T(w) := \max_{[-T,T]} \||x|^b w\|_{L_x^2}$ and arguing as in the proof of Theorem 2.3. □

Corollary 2.17. *Let $u \in C([-T,T]; H^s(\mathbb{R}))$ be the solution of (16) provided by Theorem 2.14. Suppose there exists $t^* \in [-T,T]$ such that $|x|^b u(t^*) \in L^2(\mathbb{R})$ for $0 < b \leqslant s/4$. Then $u \in C([-T,T]; Z_{s,b})$.*

CHAPTER 3

REGULARITY VERSUS DECAY

In this chapter we discuss the relation between regularity and decay displayed in Theorem 1.5, using the mKdV equation as example. More precisely, we consider the IVP

$$\begin{cases} \partial_t v + \partial_x^3 v + v^2 \partial_x v = 0, & x, t \in \mathbb{R}. \\ v(x,0) = v_0(x). \end{cases} \tag{3.1}$$

The idea is to work somehow in the opposite direction to what was done in Chapter 2. Instead of starting with a regular initial data having some decay, we start from an initial data having decay and we show this would imply an increment in its regularity in the sense of Theorem 3.7 below.

3.1 Linear estimates

We first recall one result related to the linear group $U(t)$ defined in (7).

Lemma 3.1. *For $u_0 \in L^2(\mathbb{R})$ we have*

$$\|D_x^{-1/4} U(t) u_0\|_{L_x^4 L_T^\infty} \leqslant c \|u_0\|_{L_x^2}. \tag{3.2}$$

Proof. See Theorem 3.7 in [37]. □

Here we introduce the precise formulation of (9).

Theorem 3.2. *Let $b \in (0,1)$. If $u_0 \in Z_{2b,b}$ then for all $t \in \mathbb{R}$ and for almost every $x \in \mathbb{R}$*

$$|x|^b U(t) u_0(x) = U(t)(|x|^b u_0)(x) + U(t)\{\Phi_{t,b}(\widehat{u}_0(\xi))\}^\vee(x) \tag{3.3}$$

with

$$\|U(t)\{\Phi_{t,b}(\widehat{u}_0(\xi))\}^\vee\|_2 \leqslant c(1+|t|)\|u_0\|_{H^{2b}}. \tag{3.4}$$

Moreover, if in addition one has that for $\beta \in (0, b)$

$$D^{\beta}(|x|^b u_0) \in L^2(\mathbb{R}) \text{ and } u_0 \in H^{\beta+2b}(\mathbb{R}) \tag{3.5}$$

then for all $t \in \mathbb{R}$ *and for almost every* $x \in \mathbb{R}$

$$D_x^{\beta}(|x|^b U(t)u_0)(x) = U(t)(D^{\beta}|x|^b u_0)(x) + U(t)(D^{\beta}(\{\Phi_{t,b}(\widehat{u}_0)(\xi)\}^{\vee})(x) \tag{3.6}$$

with

$$\|D^{\beta}(\{\Phi_{t,b}(\widehat{u}_0)(\xi)\}^{\vee})\|_2 \leqslant c(1+|t|)\|u_0\|_{H^{\beta+2b}}. \tag{3.7}$$

Proof. See Theorem 1 in [22]. □

3.2 Local theory results

We gather some useful results regarding the existence of solutions to (3.1) in Sobolev and weighted Sobolev spaces.

Theorem 3.3 (Local theory). *Let* $s \geqslant 1/4$. *Then for any* $v_0 \in H^s(\mathbb{R})$ *there exists* $T = T(\|D_x^{1/4} v_0\|_{L^2}) = c(\|D_x^{1/4} v_0\|_{L^2}^{-4})$ *and a unique solution* $v(t)$ *of* (3.1) *such that*

$$v \in C\left([0,T]; H^s(\mathbb{R})\right), \tag{3.8}$$

$$\|D_x^s \partial_x v\|_{L_x^\infty L_T^2} + \|D_x^{s-1/4}\partial_x v\|_{L_x^{20} L_T^{5/2}} + \|D_x^s v\|_{L_x^5 L_T^{10}} + \|v\|_{L_x^4 L_T^\infty} < \infty \tag{3.9}$$

and

$$\|\partial_x v\|_{L_x^\infty L_T^2} < \infty. \tag{3.10}$$

Moreover, there exists a neighborhood $\mathcal{V}$ *of* v_0 *in* $H^s(\mathbb{R})$ *such that the map* $\tilde{v}_0 \mapsto \tilde{v}(t)$ *from* $\mathcal{V}$ *into the class defined by* (3.8) - (3.10) *is smooth.*

We briefly mention a few aspects related to the proof of Theorem 3.3; for the details we refer the reader to [37, Theorem 2.4]. The authors used the Banach fixed point theorem in the space

$$X_a^T := \left\{ v \in C([0,T]; H^s(\mathbb{R})) \mid \Lambda^T(v) \leqslant a \right\}, \tag{3.11}$$

where

$$\begin{aligned} \Lambda^T(v) = {} & \|v\|_{L_T^\infty H^s} + \|D^s \partial_x v\|_{L_x^\infty L_T^2} + \|D_x^{s-1/4}\partial_x v\|_{L_x^{20} L_T^{5/2}} \\ & + \|D_x^s v\|_{L_x^5 L_T^{10}} + \|v\|_{L_x^4 L_T^\infty} + \|\partial_x v\|_{L_x^\infty L_T^2}. \end{aligned} \tag{3.12}$$

It was shown that, for $a = 2c\|v_0\|_{H^{1/4}}$ and T such that $2ca^2 T^{1/2} < 1$, the operator

$$\Psi_{v_0}(v) = U(t)v_0 - \int_0^t U(t-t')(v^2\partial_x v)(t')dt' \tag{3.13}$$

has a unique fixed point in X_a^T. In particular, it was proved that

$$\int_0^T \|J^s(v^2\partial_x v)\|_{L_x^2} dt' \leqslant cT^{1/2}\Lambda^T(v)^3. \tag{3.14}$$

Theorem 3.4. *The IVP (3.1) is globally well-posed for initial data $u_0 \in H^s(\mathbb{R})$, $s \geqslant 1/4$.*

Proof. See Theorem 3 in [15] and Theorem 1.2 in [25]. □

Theorem 3.5 (Local theory in weighted spaces)**.** *Let $v \in C([0,T]; H^s(\mathbb{R}))$ denote the solution of (3.1) provided by Theorem 3.3. If $v_0 \in Z_{s,b}$ with $s \geqslant 2b$ then the solution satisfies*

$$v \in C([0,T]; Z_{s,b}) \quad \text{and} \quad \||x|^{s/2}v\|_{L_x^5 L_T^{10}} < \infty. \tag{3.15}$$

For any $T' \in (0,T)$ there exists a neighborhood $\mathcal{V}$ of v_0 in $Z_{s,b}$ such that the map $\tilde{v}_0 \mapsto \tilde{v}(t)$ from $\mathcal{V}$ into the class defined by (3.8)-(3.10) and (3.15) with T' instead of T is smooth.

Essentially, the proof follows in a similar fashion as the one in Theorem 3.3 by considering the norm

$$\mu^T(v) := \Lambda^T(v) + \||x|^{s/2}v\|_{L_x^5 L_T^{10}} + \||x|^{s/2}v\|_{L_T^\infty L_x^2} \tag{3.16}$$

and using Theorem 3.2 (we refer the reader to Theorem 1 in [45]).

As breifly mentioned in the introduction, in [32] it was shown that

Theorem 3.6. *Let $u \in C(\mathbb{R}, L^2(\mathbb{R}))$ be the global solution of the IVP (4), with $a(u) = u$, provided by the local well-posedness theory in $L^2(\mathbb{R})$. If there exist $\alpha > 0$ and two different times $t_0,\ t_1 \in \mathbb{R}$ such that*

$$|x|^\alpha u(x,t_0),\ |x|^\alpha u(x,t_1) \in L^2(\mathbb{R}), \tag{3.17}$$

then $u \in C(\mathbb{R}, H^{2\alpha}(\mathbb{R}))$.

Based on the proof of Theorem 3.6 and taking some ideas from teh proof of Theorem 1.6 in [6], we proved the following result.

Theorem 3.7. *Let $v_0 \in H^{1/4}(\mathbb{R})$. Let $v \in C\left([0,T]; H^{1/4}(\mathbb{R})\right)$ be the solution provided by Theorem 3.3. Assume there exist $t_0, t_1 \in [0,T]$ with $t_0 < t_1$ and $\alpha > 0$ such that*

$$|x|^\alpha v(t_0) \in L^2(\mathbb{R}) \text{ and } |x|^\alpha v(t_1) \in L^2(\mathbb{R}). \tag{3.18}$$

Then $v \in C\left([0,T]; H^{2\alpha}(\mathbb{R})\right)$.

Note that in Theorem 3.7 when $\alpha \in (0, 1/8]$ there is no gain of extra regularity. This is consistent with the weighted local theory in which the solution persists in $H^{1/4}(\mathbb{R}) \cap$

$L^2(|x|^{2b}dx)$ whenever $b \leqslant 1/8$. Another way to formulate Theorem 3.7 is to replace (3.18) with

$$|x|^{1/8+\alpha}v(t_0) \in L^2(\mathbb{R}) \text{ and } |x|^{1/8+\alpha}v(t_1) \in L^2(\mathbb{R})$$

and the conclusion $v \in C\left([0,T]; H^{2\alpha}(\mathbb{R})\right)$ with $v \in C\left([0,T]; H^{\frac{1}{4}+2\alpha}(\mathbb{R})\right)$ instead.

From the proof of Theorem 3.7 it will be clear that it remains valid for $v_0 \in H^s(\mathbb{R})$ for any $s \geqslant 1/4$. In fact, the following corollary is a direct consequence of this fact.

Corollary 3.8. *Let $v_0 \in H^s(\mathbb{R})$, $s \geqslant 1/4$. Let v be the global in-time solution provided by Theorem 3.4. Assume there exist $t_0, t_1 \in \mathbb{R}$ and $\alpha > 0$ such that*

$$|x|^{\frac{s}{2}+\alpha}v(t_0) \in L^2(\mathbb{R}) \text{ and } |x|^{\frac{s}{2}+\alpha}v(t_1) \in L^2(\mathbb{R}).$$

Then $v \in C\left([-T,T]; H^{s+2\alpha}(\mathbb{R})\right)$ for any $T > 0$.

Proof. Without loss of generality assume $T > \max\{|t_0|, |t_1|\}$ be arbitrary. Note that $u_0 \in H^{\frac{1}{4}}(\mathbb{R})$ with the respective solution $v \in C([-T,T]; H^{\frac{1}{4}}(\mathbb{R}))$ so that $|x|^{\tilde{\alpha}}v(t_i)$ are in $L^2(\mathbb{R})$ for $i = 0,1$ and $\tilde{\alpha} = \dfrac{s}{2} + \alpha > 0$. By Theorem 3.7 we obtain that $v \in C([-T,T]; H^{2\tilde{\alpha}}(\mathbb{R}))$. The result follows since $2\tilde{\alpha} = s + 2\alpha$. □

Remark 3.9. *We may interpret Theorem 3.7 and Corollary 3.8 as an ill-posedness result in the following sense: assume v_0 belongs to $Y_{s,b} = \left(H^s(\mathbb{R}) \backslash H^{s^+}(\mathbb{R})\right) \cap L^2(|x|^{2b}dx)$ for $s, b \in \mathbb{R}$ with $b > s/2 \geqslant 1/8$, then the corresponding solution of the mKdV equation does not belong to $Y_{s,b}$.*

3.3 Proof of Theorem 3.7

The proof is presented in four cases, in which the first three are required to be done explicitly because of technical details involving the size of the truncated weights with respect to the regularity of the solution. The fourth case provides a construction of the solution in a general setting.

Without loss of generality assume $t_0 = 0$.

3.3.1 Case $\alpha \in (0, 1/2]$.

Let $\{v_{0m}\}_m \subset C_0^\infty(\mathbb{R})$ be a sequence converging to v_0 in $H^{1/4}(\mathbb{R})$. Denote with $v_m(\cdot) \in H^\infty(\mathbb{R})$ the corresponding solution provided by Theorem 3.3 with initial data v_{0m}. By regularity of the data-solution map we can assume all the v_m's are defined in $[0,T]$ with v_m converging to v in $C\left([0,T]; H^{1/4}(\mathbb{R})\right)$.

For $N \in \mathbb{N}$, denote

$$\tilde{\tilde{\phi}}_{0,N}^{\alpha}(x) := \begin{cases} \langle x\rangle^{2\alpha} - 1 & x \in [0,N], \\ (2N)^{2\alpha} & x \in [3N,\infty). \end{cases}$$

Let $\tilde{\phi}_{0,N}^{\alpha}$ be a regularization of $\tilde{\tilde{\phi}}_{0,N}^{\alpha}$ such that for $j = 1,2,\dots$ we have $\left|\partial_x^j \tilde{\phi}_{0,N}^{\alpha}\right| \leqslant c$ with c independent on N. Set $\phi_N \equiv \phi_{0,N}^{\alpha}$ to be the odd extension of $\tilde{\phi}_{0,N}^{\alpha}$, that is,

$$\phi_N(x) := \begin{cases} \tilde{\phi}_{0,N}^{\alpha}(x) & x \in [0,\infty], \\ -\tilde{\phi}_{0,N}^{\alpha}(-x) & x \in (-\infty,0). \end{cases} \tag{3.19}$$

Note that $\phi_N' \geqslant 0$.

Take the mKdV equation for v_m and multiply it by $v_m\phi_N$. After integration by parts in space we get:

$$\frac{1}{2}\frac{d}{dt}\int v_m^2\phi_N dx + \frac{3}{2}\int(\partial_x v_m)^2\phi_N' dx - \frac{1}{2}\int v_m^2\phi_N^{(3)}dx - \frac{1}{4}\int v_m^4\phi_N' dx = 0. \tag{3.20}$$

Integrating over $[0,t_1]$ we get:

$$\int v_m^2(t_1)\phi_N dx - \int v_{0m}^2\phi_N dx + 3\int_0^{t_1}\int(\partial_x v_m)^2\phi_N' dxdt - \int_0^{t_1}\int v_m^2\phi_N^{(3)}dxdt - \frac{1}{2}\int_0^{t_1}\int v_m^4\phi_N' dxdt = 0. \tag{3.21}$$

For m large enough we have

$$\left|-\int_0^{t_1}\int v_m^2\phi_N^{(3)}dxdt\right| \leqslant c\int_0^{t_1}\int v_m^2 dxdt \leqslant c\|v_m\|^2_{L^2_{xt_1}} \leqslant c\|v_{0m}\|^2_{L^2_{xt_1}} \leqslant 2ct_1\|v_0\|_2^2. \tag{3.22}$$

Also,

$$\left|-\frac{1}{2}\int_0^{t_1}\int v_m^4\phi_n' dxdt\right| \leqslant c\int_0^{t_1}\int v_m^4 dxdt \leqslant ct_1\|v_m\|^4_{L^4_xL^\infty_{t_1}} \leqslant c\|v_m\|^4_{L^4_xL^\infty_{t_1}} \leqslant 2\|v\|^4_{L^4_xL^\infty_{t_1}}. \tag{3.23}$$

From (3.21)-(3.23) we get that for $m \gg 1$

$$\begin{aligned} \int_0^{t_1}\int(\partial_x v_m)^2\phi_N' dxdt &\leqslant c(\|v_0\|_2^2 + \|v\|^4_{L^4_xL^\infty_{t_1}}) + \|v_0\phi_N^{1/2}\|^2_{L^2_x} + \|v(t_1)\phi_N^{1/2}\|^2_{L^2_x} \\ &\leqslant c(\|v_0\|_2^2 + \|v\|^4_{L^4_xL^\infty_{t_1}} + \|\langle x\rangle^\alpha v_0\|^2_{L^2_x} + \|\langle x\rangle^\alpha v(t_1)\|^2_{L^2_x}). \end{aligned} \tag{3.24}$$

Thus

$$\limsup_{m\to\infty}\int_0^{t_1}\int(\partial_x v_m)^2\phi_N' dxdt \leqslant M, \tag{3.25}$$

where $M > 0$ depends on $\|v_0\|_{Z_{1/4,\alpha}}$, $\|v(t_1)\langle x\rangle^\alpha\|_{L^2_x}$ and $\|v\|_{L^4_xL^\infty_{t_1}}$.

We claim that, for any fixed $N \in \mathbb{N}$, the left hand side of (3.24) converges to

$$\int_0^{t_1}\int(\partial_x v)^2\phi_N' dxdt$$

as $m \to \infty$. Indeed,

$$\begin{aligned}\left|\int_0^{t_1}\int \left[(\partial_x v_m)^2 - (\partial_x v)^2\right]\phi_N' dxdt\right| &\leqslant c\int_0^{t_1}\int_{[-3N,3N]} |\partial_x v_m - \partial_x v||\partial_x v_m + \partial_x v| dxdt \\ &\leqslant c\|\partial_x v_m - \partial_x v\|_{L^2_{t_1}L^2_x[-3N,3N]}\|\partial_x v_m + \partial_x v\|_{L^2_{t_1}L^2_x[-3N,3N]} \\ &\leqslant c_N\|\partial_x v_m - \partial_x v\|_{L^\infty_x L^2_{t_1}}\left(\|\partial_x v_m\|_{L^\infty_x L^2_{t_1}} + \|\partial_x v\|_{L^\infty_x L^2_{t_1}}\right) \\ &\xrightarrow[m\to\infty]{} 0,\end{aligned} \tag{3.26}$$

where we used (3.10) together with the continuous dependence on the initial data.

In view of (3.25), we therefore have

$$\int_0^{t_1}\int (\partial_x v)\phi_N' dxdt \leqslant M.$$

Since ϕ_N' is even and for $x > 1$ we have that $\phi_N'(x) \xrightarrow[N\to\infty]{} 2\alpha\langle x\rangle^{2\alpha-2}x \sim \langle x\rangle^{2\alpha-1}$, we deduce

$$\int_0^{t_1}\int_{|x|>1}(\partial_x v)^2\langle x\rangle^{2\alpha-1}dxdt \leqslant c\liminf_{N\to\infty}\int_0^{t_1}\int_{|x|>1}(\partial_x v)^2\phi_N' dxdt \leqslant M. \tag{3.27}$$

Moreover,

$$\int_0^{t_1}\int_{|x|<1}(\partial_x v)^2\langle x\rangle^{2\alpha-1}dxdt = \int_{|x|<1}\int_0^{t_1}(\partial_x v)^2\langle x\rangle^{2\alpha-1}dtdx \leqslant c\|\partial_x v\|^2_{L^\infty_x L^2_{t_1}} < \infty. \tag{3.28}$$

From (3.27) and (3.28) we conclude

$$\int_0^{t_1}\int (\partial_x v)^2\langle x\rangle^{2\alpha-1}dxdt < \infty,$$

which implies

$$\langle x\rangle^{\alpha-1/2}\partial_x v \in L^2(\mathbb{R}) \text{ for almost every } t \in [0,t_1]. \tag{3.29}$$

Arguing in a similar fashion as done to obtain (3.29) but using ϕ_N as the even extension of $\tilde{\phi}_N^\alpha$ instead, it can be seen that for any $t \in [0,t_1]$ we have $\langle x\rangle^\alpha v \in L^2(\mathbb{R})$ (see also [32, page 143]). Set $t_* \in [0,t_1]$ so that $\langle x\rangle^{\alpha-1/2}\partial_x v(t_*) \in L^2(\mathbb{R})$. By writing $f := \langle x\rangle^{\alpha-1/2}v(t_*)$, from (3.29) it can be seen that $J^1 f \in L^2(\mathbb{R})$. From Lemma 1.4, for $\theta \in (0,1)$ we have that

$$\|J^\theta(\langle x\rangle^{(1-\theta)/2}f)\|_2 \leqslant c\|J^1 f\|_2^\theta\|\langle x\rangle^{1/2}f\|_2^{1-\theta} < \infty. \tag{3.30}$$

Setting $\theta = 2\alpha$ we have that $\alpha - 1/2 + (1-\theta)/2 = 0$ and therefore we conclude $J^{2\alpha}v(t_*) \in L^2(\mathbb{R})$, that is,

$$v(t_*) \in H^{2\alpha}(\mathbb{R}). \tag{3.31}$$

An iterative argument involving the proof of Theorem 3.3 with initial data $\tilde{v}_0 = v(t_*)$ shows that $v \in C([0,T];H^{2\alpha}(\mathbb{R}))$. Moreover, using Theorem 3.5 it can be seen that the fact $\langle x\rangle^\alpha v(t_*) \in L^2(\mathbb{R})$ imply v is in the class defined by (3.8) - (3.10) and (3.15) with $Z_{2\alpha,\alpha}$ instead of $H^{2\alpha}(\mathbb{R})$.

3.3.2 Case $\alpha \in (1/2, 1]$.

Since $\alpha > 1/2$, it can be seen that $|x|^{1/2}v(t_i)$ is in $L^2(\mathbb{R})$ for $i = 0, 1$. Therefore, the conclusion of the previous case holds in $H^1(\mathbb{R})$. In particular we know $v \in C\left([0,T]; Z_{1,1/2}\right)$ with $\langle x\rangle^{\tilde{\alpha}-1/2}\partial_x v$ in $L^2(\mathbb{R})$ for almost every $t \in [0, t_1]$ and $\tilde{\alpha} \in (0, 1/2]$. We claim the latter now also holds for $\alpha \in (1/2, 1]$.

Let $\{v_{0m}\}_m \subset C_0^\infty(\mathbb{R})$ be a sequence converging to v_0 in $Z_{1,1/2}$. Denote with $v_m(\cdot) \in H^\infty(\mathbb{R})$ the corresponding solution provided by Theorem 3.5 with initial data v_{0m}. By regularity of the data-solution map we can assume all the v_m's are defined in $[0, T]$ with v_m converging to v in $C\left([0,T]; Z_{1,1/2}\right)$. By following the same steps done to obtain (3.29) we note that all remains equal except for the fact $|(\phi^\alpha_{0,N})'|$ is no longer bounded above independent on N. So, what is left is to estimate the terms involving $(\phi^\alpha_{0,N})'$. In fact, instead of estimate (3.23) we argue as follows:

$$\int_0^{t_1}\int v_m^4\phi_N' dxdt \leqslant c\int_0^{t_1}\|v_m\|^2_{L^\infty_x}\int v_m^2\langle x\rangle dxdt$$
$$\leqslant c\|v_m\|^2_{L^\infty_{t_1}H^1}\|v_m\langle x\rangle^{1/2}\|^2_{L^2_{xt_1}} \leqslant ct_1\|v_m\|^3_{L^\infty_{t_1}Z_{1,1/2}} \leqslant 2c\|v\|^3_{L^\infty_{t_1}Z_{1,1/2}}.$$

Also, for estimate (3.26) we note that $|\phi_N'| \leqslant c_N$ with c_N depending on N. Since N is fixed, the same computations remain valid.

Under these considerations, arguing as in (3.20)-(3.28) it can be seen that for almost every $t \in [0, t_1]$ we have

$$\langle x\rangle^{\alpha-1/2}\partial_x v \in L^2(\mathbb{R}), \tag{3.32}$$

which proves our claim. Moreover, a similar analysis (without evaluating the limit as $m \to \infty$) shows that for m large enough

$$\|\langle x\rangle^{\alpha-1/2}\partial_x v_m\|_{L^2_{xt_1}} \leqslant M, \tag{3.33}$$

with M depending on $\|v_0\|_{Z_{1,1/2}}$, $\|\langle x\rangle^{\alpha-1/2}v(t_1)\|_{L^2_x}$ and $\|v\|_{L^\infty_{t_1}Z_{1,1/2}}$. Assume without loss of generality that

$$\langle x\rangle^{\alpha-1/2}\partial_x v(t_i) \text{ are in } L^2(\mathbb{R}) \text{ for } i = 0, 1. \tag{3.34}$$

Note in case (3.34) is not true, we can take a smaller subinterval $[t_0^*, t_1^*] \subset [0, t_1]$ in which the end points satisfy (3.34).

Consider now $\phi_N \equiv \phi^\alpha_{1,N}$ build as done in (3.19) but based on the function

$$\tilde{\tilde{\phi}}^\alpha_{1,N}(x) := \begin{cases} \langle x\rangle^{2\alpha-1} - 1 & x \in [0, N], \\ (2N)^{2\alpha-1} & x \in [3N, \infty). \end{cases}$$

Take ∂_x to the mKdV equation for v_m and multiply it by $\partial_x v_m\phi_N$ to get

$$\partial_t w_m w_m\phi_N + 3\partial_x^3 w_m w_m\phi_N + 2v_m w_m^3\phi_N + v_m^2 w_m\partial_x w_m\phi_N = 0, \tag{3.35}$$

where we denote $w_m := \partial_x v_m$. Integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\int w_m^2\phi_N dx+\frac{3}{2}\int(\partial_x w_m)^2\phi_N' dx-\frac{1}{2}\int w_m^2\phi_N^{(3)}dx+\int 2v_m w_m^3\phi_N dx+\int v_m^2 w_m\partial_x w_m\phi_N dx = 0.$$

Note that, in terms of w_m, the first three terms above remain the same as in (3.21) for the mKdV equation and the respective estimates are analogous. The only difference relies on the terms coming from the derivative of the non-linearity $v_m^2\partial_x v_m$. Integrating by parts in the space variable we have that for $t \in [0, t_1]$:

$$2\int v_m w_m^3\phi_N dx = -2\int v_m^2 w_m\partial_x w_m\phi_N dx - \int v_m^2 w_m^2\phi_N' dx.$$

Since, for m large enough we have

$$\left|\int v_m^2 w_m^2\phi_N' dx\right| \leqslant c\|v_m\|_{L_x^\infty}^2\|w_m\|_{L_x^2}^2 \leqslant c\|v_m\|_{L_{t_1}^\infty H^1}^4 \leqslant 2c\|v\|_{L_{t_1}^\infty H^1}^4, \tag{3.36}$$

we only need to focus on the term

$$\left|-\int_0^{t_1}\int v_m^2 w_m\partial_x w_m\phi_N dxdt\right|.$$

We proceed as follows: first we note that

$$\left|-\int_0^{t_1}\int v_m^2 w_m\partial_x w\phi_N dxdt\right| \leqslant \|\partial_x w_m\|_{L_x^\infty L_{t_1}^2}\|w_m\phi_N^{1/2}\|_{L_{xt_1}^2}\|v_m^2\phi_N^{1/2}\|_{L_x^2 L_{t_1}^\infty}. \tag{3.37}$$

Let us estimate each term on the right-hand side of (3.37). From the continuous dependence (see (3.9) with $s = 1$), for m large enough

$$\|\partial_x^2 v_m\|_{L_x^\infty L_{t_1}^2} \leqslant 2c\|\partial_x^2 v\|_{L_x^\infty L_{t_1}^2}.$$

Also, from (3.33) we know that for $m \gg 1$,

$$\|w_m\phi_N^{1/2}\|_{L_{xt_1}^2} \leqslant c\|\langle x\rangle^{\alpha-1/2}\partial_x v_m\|_{L_{xt_1}^2} \leqslant M, \tag{3.38}$$

Finally, for the term $\|v_m^2\phi_N^{1/2}\|_{L_x^2 L_{t_1}^\infty}$, we note that $\|v_m^2\phi_N^{1/2}\|_{L_x^2 L_{t_1}^\infty} \leqslant \|v_m\phi_N^{1/4}\|_{L_x^4 L_{t_1}^\infty}^2$. Using the integral equation and the fact $|\phi_N|^{1/4} \leqslant c\langle x\rangle^{3/8}$ we have that

$$\|v_m\phi_N^{1/4}\|_{L_x^4 L_{t_1}^\infty} \leqslant \|\langle x\rangle^{3/8}U(t)v_{0m}\|_{L_x^4 L_{t_1}^\infty} + \int_0^{t_1}\|\langle x\rangle^{3/8}U(t-t')v_m^2\partial_x v_m\|_{L_x^4 L_{t_1}^\infty}dt'. \tag{3.39}$$

In view of (3.2) and (3.3) we obtain

$$\begin{aligned}\|\langle x\rangle^{3/8}U(t)v_{0m}\|_{L_x^4 L_{t_1}^\infty} &\leqslant \|U(t)\langle x\rangle^{3/8}v_{0m}\|_{L_x^4 L_{t_1}^\infty} + \|U(t)\{\Phi_{t,3/8}(\widehat{u}_0)(\xi)\}^\vee\|_{L_x^4 L_{t_1}^\infty}\\ &\leqslant c\|D^{1/4}\langle x\rangle^{3/8}v_{0m}\|_{L_x^2} + c\|D^{1/4}\{\Phi_{t,3/8}(\widehat{u}_0)(\xi)\}^\vee\|_{L_x^2}\\ &\leqslant c\|J^{1/4}\langle x\rangle^{3/8}v_{0m}\|_{L_x^2} + c\|D^{1/4}\{\Phi_{t,3/8}(\widehat{u}_0)(\xi)\}^\vee\|_{L_x^2}.\end{aligned} \tag{3.40}$$

Interpolating using Lemma 1.4 gives

$$\|J_x^{1/4}\langle x\rangle^{3/8}f\|_{L_x^2} \leqslant c\|J_x^1 f\|_{L^2}^{1/4}\|\langle x\rangle^{1/2}f\|_{L^2}^{3/4}. \tag{3.41}$$

Also, according to (3.7) we get

$$\|D^{1/4}\{\Phi_{t,3/8}(\widehat{f})(\xi)\}^{\vee}\|_{L^2_x} \leqslant c(1+t_1)\|f\|_{H^1}. \tag{3.42}$$

We combine (3.41) and (3.42) into (3.40) to obtain for m large that

$$\begin{aligned}\|\langle x\rangle^{3/8}U(t)v_{0m}\|_{L^4_xL^\infty_{t_1}} &\leqslant c\|J^1_x v_{0m}\|^{1/4}_{L^2_x}\|\langle x\rangle^{1/2}v_{0m}\|^{3/4}_{L^2_x} + c\|v_{0m}\|_{H^1}\\ &\leqslant 2c\|v_0\|_{Z_{1,1/2}}.\end{aligned} \tag{3.43}$$

For the other term in (3.39), we argue in a similar fashion applying (3.41) and (3.42) to get

$$\begin{aligned}&\int_0^{t_1}\|\langle x\rangle^{3/8}U(t-t')v_m^2\partial_x v_m\|_{L^4_xL^\infty_{t_1}}dt'\\ &\qquad\leqslant c\int_0^{t_1}\|J^1_x(v_m^2\partial_x v_m)\|^{1/4}_{L^2_x}\|\langle x\rangle^{1/2}v_m^2\partial_x v_m\|^{3/4}_{L^2_x}dt' + c(1+t_1)\int_0^{t_1}\|v_m^2\partial_x v_m\|_{H^1}dt'\\ &\qquad\leqslant c\int_0^{t_1}\left(\|v_m^2\partial_x v_m\|_{H^1} + c\|\langle x\rangle^{1/2}v_m^2\partial_x v_m\|_{L^2_x}\right)dt'.\end{aligned} \tag{3.44}$$

According to (3.9), (3.15) and (3.16), for $m \gg 1$,

$$\begin{aligned}\int_0^{t_1}\|\langle x\rangle^{1/2}v_m^2\partial_x v_m\|_{L^2_x}dt' &\leqslant c\|\langle x\rangle^{1/2}\partial_x v_m v_m^2\|_{L^2_{xt_1}}\\ &\leqslant c\|\langle x\rangle^{1/2}v_m\|_{L^5_xL^{10}_{t_1}}\|v_m\|_{L^4_xL^\infty_{t_1}}\|\partial_x v_m\|_{L^{20}_xL^{5/2}_{t_1}}\\ &\leqslant c\mu(v_m)^3 \leqslant 2c\mu(v)^3,\end{aligned} \tag{3.45}$$

where we used the continuous dependence on the initial data.

Combining (3.14) and (3.45) it follows from (3.44) that

$$\int_0^{t_1}\|\langle x\rangle^{3/8}U(t-t')v_m^2\partial_x v_m\|_{L^4_xL^\infty_{t_1}}dt' \leqslant M, \tag{3.46}$$

where M depends on several norms in which v is known to be finite. From (3.39)-(3.46) we conclude

$$\|v_m^2\phi_N^{1/2}\|_{L^2_xL^\infty_{t_1}} \leqslant M. \tag{3.47}$$

Gathering (3.35)-(3.47) we can emulate the argument in (3.20) -(3.25) to conclude

$$\limsup_{m\to\infty}\int_0^{t_1}\int(\partial_x w_m)^2\phi_N' dxdt \leqslant M, \tag{3.48}$$

with M depending on $\|v_0\|_{Z_{1,1/2}}$ and $\|\langle x\rangle^\alpha v(t_1)\|_{L^2_x}$ among other norms in which v is known to be finite.

Note that the convergence argument done in (3.26) can be mimicked with w_m instead of v_m and using the continuous dependence in the norm $\|\partial_x^2 v_m\|_{L^\infty_xL^2_{t_1}}$ provided by the local theory.

Continuing as in (3.27) and (3.28) it can be seen that

$$\langle x\rangle^{\alpha-1}\partial_x w \in L^2(\mathbb{R}) \text{ for almost every } t \in [0,t_1]. \tag{3.49}$$

From (3.32) and (3.49) there exists $t_* \in [0,t_1]$ such that $\langle x\rangle^{1/2} f$ and $J^1 f$ are in $L^2(\mathbb{R})$; where f denotes the function $\langle x\rangle^{\alpha-1}\partial_x v(t_*)$. Interpolating as in (1.3) with $\theta = 2\alpha - 1$ we conclude $v(t_*) \in H^{2\alpha}(\mathbb{R})$. Once more, an iterative argument shows that $v \in C([0,T]; H^{2\alpha}(\mathbb{R}))$. Moreover, it can be seen that v is in the class defined by (3.8) - (3.10) and (3.15) with $Z_{2\alpha,\alpha}$ instead of $H^{2\alpha}(\mathbb{R})$.

3.3.3 Case $\alpha \in (1, 3/2]$.

By setting $\alpha = 1$ in the result of the previous case and using the local theory in weighted spaces we have that $v \in C([0,T]; Z_{2,1})$ with $\langle x\rangle^{1/2}\partial_x^k v(t) \in L^2(\mathbb{R})$ for almost every $t \in [0,t_1]$ and $k = 1, 2$. We claim that $\langle x\rangle^{\alpha-1}\partial_x^2 v(t)$ is in $L^2(\mathbb{R})$ for almost every $t \in [0,t_1]$. Indeed, let $\{v_{0m}\}_m \subset C_0^\infty(\mathbb{R})$ be a sequence converging to v_0 in $Z_{2,1}$. Denote with $v_m(\cdot) \in H^\infty(\mathbb{R})$ the corresponding solution provided by Theorem 3.5 with initial data v_{0m}. By regularity of the data-solution map we can assume all the v_m's are defined in $[0,T]$ with v_m converging to v in $C([0,T]; Z_{2,1})$.

The idea is to emulate what was done from (3.35) to (3.49) noticing that $|\phi'_{1,N}|$ is not longer uniformly bounded above independent on N. We re-estimate the related terms as follows.

Instead of inequality (3.36) we do

$$\begin{aligned}\left|-\int_0^{t_1}\int v_m^2(\partial_x v_m)^2\phi'_{1N}dxdt\right| &\leqslant c\int_0^{t_1}\int v_m^2(\partial_x v_m)^2\langle x\rangle dxdt \leqslant c\|\langle x\rangle^{1/2}v_m\|_{L^2_{xt_1}}\|\partial_x v_m\|^2_{L^\infty_{xt_1}}\\ &\leqslant c\|\langle x\rangle^{1/2}v_m\|_{L^\infty_{t_1}L^2_x}\|v_m\|_{L^\infty_{t_1}H^2} \leqslant c\|v_m\|^2_{Z_{2,1}} \leqslant 2c\|v\|^2_{Z_{2,1}}.\end{aligned} \tag{3.50}$$

Instead of (3.37) we argue as follows:

$$\left|-\int_0^{t_1}\int v_m^2\partial_x v_m\partial_x^2 v_m\phi_{1N}dxdt\right| \leqslant \|\langle x\rangle^{3/2}v_m^2\|_{L^2_xL^\infty_{t_1}}\|\langle x\rangle^{1/2}\partial_x v_m\|_{L^2_{xt_1}}\|\partial_x^2 v_m\|_{L^\infty_xL^2_{t_1}}. \tag{3.51}$$

Note that for m large we have $\|\partial_x^2 v_m\|_{L^\infty_xL^2_{t_1}} \leqslant c\|\partial_x^2 v\|_{L^\infty_xL^2_{t_1}}$, which is finite because of the local theory.

Using (3.33) with $\alpha = 1$, for $m \gg 1$ it follows that $\|\langle x\rangle^{1/2}\partial_x v_m\|_{L^2_{xt_1}} \leqslant M$ where M depends on $\|v_0\|_{Z_{1,1/2}}$ and $\|\langle x\rangle^{1/2}v(t_1)\|_{L^2_x}$ among other norms for v.

For the term $\|\langle x\rangle^{3/2}v_m^2\|_{L^2_xL^\infty_{t_1}}$ we have that

$$\begin{aligned}\|\langle x\rangle^{3/2}v_m^2\|^{1/2}_{L^2_xL^\infty_{t_1}} &\leqslant \|\langle x\rangle^{3/4}v_m\|_{L^4_xL^\infty_{t_1}}\\ &\leqslant \|\langle x\rangle^{3/4}U(t)v_{0m}\|_{L^4_xL^\infty_{t_1}} + \int_0^{t_1}\|\langle x\rangle^{3/4}U(t-t')v_m^2\partial_x v_m\|_{L^4_xL^\infty_{t_1}}dt'.\end{aligned} \tag{3.52}$$

According to (3.3) and (3.2),

$$\begin{aligned}\|\langle x\rangle^{3/4}U(t)v_{0m}\|_{L^4_xL^\infty_{t_1}} &\leqslant \|U(t)\langle x\rangle^{3/4}v_{0m}\|_{L^4_xL^\infty_{t_1}} + \|U(t)\{\Phi_{t,3/4}(\widehat{v}_{0m}(\xi))\}^\vee\|_{L^4_xL^\infty_{t_1}}\\ &\leqslant c\|D_x^{1/4}\langle x\rangle^{3/4}v_{0m}\|_{L^2_x} + \|D_x^{1/4}\{\Phi_{t,3/4}(\widehat{v}_{0m}(\xi))\}^\vee\|_{L^2_x}\\ &\leqslant c\|J_x^{1/4}\langle x\rangle^{3/4}v_{0m}\|_{L^2_x} + \|D_x^{1/4}\{\Phi_{t,3/4}(\widehat{v}_{0m}(\xi))\}^\vee\|_{L^2_x}.\end{aligned} \tag{3.53}$$

Interpolating with $\theta = 1/8$, it follows

$$\begin{aligned}\|J_x^{1/4}\langle x\rangle^{3/4}v_{0m}\|_{L^2_x} &\leqslant c\|J_x^2v_{0m}\|_{L^2_x}^{1/8}\|\langle x\rangle^{6/7}v_{0m}\|_{L^2_x}^{7/8}\\ &\leqslant c\|v_{0m}\|_{H^2} + c\|\langle x\rangle v_{0m}\|_{L^2_x}\\ &\leqslant c\|v_{0m}\|_{Z_{2,1}}.\end{aligned} \tag{3.54}$$

Also, by (3.7) we have

$$\|D_x^{1/4}\{\Phi_{t,3/4}(\widehat{v}_{0m}(\xi))\}^\vee\| \leqslant c(1+t_1)\|v_{0m}\|_{H^{7/4}} \leqslant c\|v_{0m}\|_{Z_{2,1}}. \tag{3.55}$$

Thus, for m large enough $\|\langle x\rangle^{3/4}U(t)v_{0m}\|_{L^4_xL^\infty_{t_1}} \leqslant 2c\|v_0\|_{Z_{2,1}}$.

For the remaining term in (3.52), we use (3.41) and (3.42) to get

$$\begin{aligned}&\int_0^{t_1}\|\langle x\rangle^{3/4}U(t-t')v_m^2\partial_xv_m\|_{L^4_xL^\infty_{t_1}}dt'\\ &\quad\leqslant c\int_0^{t_1}\|J_x^2(v_m^2\partial_xv_m)\|_{L^2_x}^{1/8}\|\langle x\rangle v_m^2\partial_xv_m\|_{L^2_x}^{7/8}dt' + c(1+t_1)\int_0^{t_1}\|v_m^2\partial_xv_m\|_{H^2}dt'\\ &\quad\leqslant c\int_0^{t_1}\left(\|v_m^2\partial_xv_m\|_{H^2} + c\|\langle x\rangle v_m^2\partial_xv_m\|_{L^2_x}\right)dt'.\end{aligned} \tag{3.56}$$

Using the continuous dependence on the initial data, for m large enough we have

$$\begin{aligned}\int_0^{t_1}\|\langle x\rangle v_m^2\partial_xv_m\|_{L^2_x}dt' &\leqslant c\|\langle x\rangle\partial_xv_mv_m^2\|_{L^1_{t_1}L^2_x} \leqslant c\|\langle x\rangle^{1/2}\partial_xv_m\|_{L^2_{xt_1}}\|\langle x\rangle^{1/2}v_m\|_{L^\infty_{xt_1}}\|v_m\|_{L^2_{t_1}L^\infty_x}\\ &\leqslant c\|\langle x\rangle^{1/2}\partial_xv_m\|_{L^2_{xt_1}}\|v_m\|_{L^\infty_{t_1}H^1}\|J^1\langle x\rangle^{1/2}v_m\|_{L^\infty_{t_1}L^2_x}\\ &\leqslant c\|\langle x\rangle^{1/2}\partial_xv_m\|_{L^2_{xt_1}}\|v_m\|_{L^\infty_{t_1}Z_{1,1/2}}(\|v_m\|_{L^\infty_{t_1}H^2} + \|\langle x\rangle v_m\|_{L^\infty_{t_1}L^2_x}) \leqslant M,\end{aligned} \tag{3.57}$$

where we interpolated with $\theta = 1/2$ and used (3.33) with $\alpha = 1$. Note M depends on $\|v\|_{L^\infty_{t_1}Z_{2,1}}$ among other norms in which v is known to be finite. Combining (3.14) and (3.57) it follows that for $m \gg 1$

$$\int_0^{t_1}\|\langle x\rangle^{3/4}U(t-t')v_m^2\partial_xv_m\|_{L^4_xL^\infty_{t_1}}dt' \leqslant M, \tag{3.58}$$

where M depends on $\|v_0\|_{Z_{2,1}}$ and $\|\langle x\rangle^\alpha v(t_1)\|_{L^2_x}$ among other norms in which v is known to be finite.

From (3.52)-(3.58) we conclude

$$\|\langle x\rangle^{3/2} v_m^2\|_{L_x^2 L_{t_1}^\infty}^{1/2} \leqslant \|\langle x\rangle^{3/4} v_m\|_{L_x^4 L_{t_1}^\infty} \leqslant M. \tag{3.59}$$

Under these two modifications, an analogous argument as the one developed in (3.35)-(3.49) supports that for almost every $t \in [0, t_1]$ we have

$$\langle x\rangle^{\alpha-1} \partial_x^2 v(t) \in L^2(\mathbb{R}), \tag{3.60}$$

which proves our claim. Note also that, if in the proof of (3.60) the convergence in m is skipped, we may that for $m \gg 1$

$$\|\langle x\rangle^{\alpha-1} \partial_x^2 v_m\|_{L_{xt_1}^2} \leqslant M, \tag{3.61}$$

with M depending on norms in which v is finite.

In what follows, without loss of generality, assume we have that $\langle x\rangle^{\alpha-1}\partial_x^2 v(t_i)$ are in $L^2(\mathbb{R})$ for $i = 0, 1$. Next we claim that $\langle x\rangle^{\alpha-3/2}\partial_x^3 v \in L^2(\mathbb{R})$ for almost every $t \in [0, t_1]$.

Indeed, Denote $w_m := \partial_x^2 v_m$ and define $\phi_N \equiv \phi_{2,N}^\alpha$ as done in (3.19) based on

$$\tilde{\tilde{\phi}}_{2,N}^\alpha(x) := \begin{cases} \langle x\rangle^{2\alpha-2} - 1 & x \in [0, N], \\ (2N)^{2\alpha-2} & x \in [3N, \infty). \end{cases}$$

Applying ∂_x^2 to the mKdV equation for v_m and multiplying it by $w_m\phi_N$ we have

$$\partial_t w_m w_m \phi_N + \partial_x^3 w_m w_m \phi_N + \left(2(\partial_x v_m)^3 + 6 v_m \partial_x v_m w_m + v_m^2 \partial_x w_m\right) w_m \phi_N = 0. \tag{3.62}$$

Integrate (3.62) by parts in space. After an extra integration over $[0, t_1]$ we get that

$$\begin{aligned}\int w_m^2(0)\phi_N dx - \int w_m^2(t_1)\phi_N dx + 3\int_0^{t_1}\int (\partial_x w_m)^2 \phi_N' dxdt - \int_0^{t_1}\int w_m^2 \phi_N^{(3)} dxdt + \\ 4\int_0^{t_1}\int (\partial_x v_m)^3 w_m \phi_N dxdt + 12\int_0^{t_1}\int v_m \partial_x v_m w_m^2 \phi_N dxdt + 2\int_0^{t_1}\int v_m^2 \partial_x w_m w_m \phi_N dxdt = 0\end{aligned} \tag{3.63}$$

Note

$$\left|4\int_0^{t_1}\int (\partial_x v_m)^3 w_m \phi_N dxdt\right| \leqslant c\|\langle x\rangle^{1/2}\partial_x v_m\|_{L_{xt_1}^2} \|\langle x\rangle^{\alpha-1} w_m\|_{L_{xt_1}^2} \|\partial_x v_m\|_{L_{xt_1}^\infty}^2, \tag{3.64}$$

where the terms $\|\langle x\rangle^{1/2}\partial_x v_m\|_{L_{xt_1}^2}$ and $\|\langle x\rangle^{\alpha-1}\partial_x^2 v_m\|_{L_{xt_1}^2}$ were estimated in (3.33) and (3.61). The term $\|\partial_x v_m\|_{L_{xt_1}^\infty}$ may estimated via Sobolev embedding.

We conclude for m large enough that

$$\left|\int_0^{t_1}\int (\partial_x v_m)^3 w_m \phi_N dxdt\right| \leqslant M, \tag{3.65}$$

where M depends on several norms related to v such as $\|v_0\|_{Z_{2,1}}$ and $\|\langle x\rangle^{1/2}\partial_x v(t_1)\|_{L^2_x}$.

For the next term, using (3.61), we have that for m large that

$$\left|12\int_0^{t_1}\int v_m\partial_x v_m w_m^2\phi_N dxdt\right| \leqslant c\|v_m\|_{L^\infty_{xt_1}}\|\partial_x v_m\|_{L^\infty_{xt_1}}\|w_m^2\phi_N\|_{L^1_{xt_1}} \\ \leqslant c\|v_m\|^2_{L^\infty_{t_1}H^2}\|\langle x\rangle^{\alpha-1}w_m\|^2_{L^2_{xt_1}} \leqslant M. \tag{3.66}$$

On the other hand,

$$\left|2\int_0^{t_1}\int v_m^2\partial_x w_m w_m\phi_N dxdt\right| \leqslant \|\partial_x w_m\|_{L^\infty_x L^2_{t_1}}\|\langle x\rangle v_m^2\|_{L^2_x L^\infty_{t_1}}\|w_m\|_{L^2_{xt_1}}.$$

By the local theory we have that for $m \gg 1$

$$\|\partial_x w_m\|_{L^\infty_x L^2_{t_1}} \leqslant c\|\partial_x^3 v_m\|_{L^\infty_x L^2_{t_1}} \leqslant 2c\|v\|_{L^\infty_{t_1}H^2}.$$

and

$$\|w_m\|_{L^2_{xt_1}} \leqslant c\|v\|_{L^\infty_{t_1}Z_{2,1}}.$$

Finally, for $\|\langle x\rangle v_m^2\|_{L^2_x L^\infty_{t_1}} \leqslant \|\langle x\rangle^{1/2}v_m\|^2_{L^4_x L^\infty_{t_1}}$ we note

$$\|\langle x\rangle^{1/2}v_m\|_{L^4_x L^\infty_{t_1}} \leqslant \|\langle x\rangle^{3/4}v_m\|_{L^4_x L^\infty_{t_1}} \leqslant M, \tag{3.67}$$

where we used (3.59).

We conclude that for m large enough we have

$$\left|2\int_0^{t_1}\int v_m^2\partial_x w_m w_m\phi_N dxdt\right| \leqslant M. \tag{3.68}$$

Combining (3.65) - (3.68) into (3.63) and arguing as in (3.20)-(3.28) it can be seen that

$$\int_0^{t_1}\int(\partial_x w)^2\langle x\rangle^{2\alpha-2}dx < \infty;$$

that is,

$$\langle x\rangle^{\alpha-3/2}\partial_x^3 v(t) \in L^2(\mathbb{R}) \text{ for almost every } t \in [0,t_1]. \tag{3.69}$$

Using (3.60) and (3.69) it follows via interpolation that for some $t_* \in [0,t_1]$ we have

$$v(t_*) \in H^{2\alpha}(\mathbb{R}). \tag{3.70}$$

The latter implies $v \in C\left([0,T]; H^{2\alpha}(\mathbb{R}) \cap L^2(|x|^{2\alpha}dx)\right)$ as before.

3.3.4 Case $\alpha \in (r/2, (r+1)/2]$, $r \geqslant 3$.

It is enough to prove by induction that for any $n \in \{2, 3, \dots, r\}$ it follows that for all β in $\left(\frac{n}{2}, \frac{n+1}{2}\right]$ with $\beta \leqslant \alpha$, we have

$$\begin{cases} \langle x \rangle^{\beta - n/2} \partial_x^n v(t) \in L^2(\mathbb{R}), \\ \langle x \rangle^{\beta - n/2 - 1/2} \partial_x^{n+1} v(t) \in L^2(\mathbb{R}), \end{cases} \tag{3.71}$$

for almost every t in a closed subinterval I of $[0, t_1]$. In such case, setting $n = r$ and $\beta = \alpha$ and interpolating as in (3.30), there would exists $t_* \in [0, t_1]$ such that $v(t_*) \in H^{2\alpha}(\mathbb{R})$, which leads to the desired conclusion.

Note the "base" case ($n = 2$) follows from (3.60) and (3.69). For the inductive step, assume (3.71) is valid for $n - 1$. We will prove it is also valid for n. Using the inductive hypothesis with $\beta = n/2$, it can be seen that, for $f = \langle x \rangle^{\beta - n/2 + 1/2} \partial_x^n v$, we have $J^1 f$ and $\langle x \rangle f$ are in $L^2(\mathbb{R})$. Interpolating as in (1.3), there exists $t_{**} \in [0, t_1]$ such that $v(t_{**}) \in H^n(\mathbb{R})$. It can be seen that this implies $v \in C([0, T]; H^n(\mathbb{R}))$. Moreover, since $\alpha > n/2$, using Theorem 3.5 it can be seen that $v \in C([0, T]; Z_{n, n/2})$.

Recall, via Sobolev embedding, that for $k = 1, 2, \dots, n-1$:

$$\|\partial_x^k v\|_{L_x^\infty} \leqslant c \|v\|_{H^n}. \tag{3.72}$$

In addition, using Lemma 1.3 and interpolation, for any $t \in [0, T]$ we have that

$$\|\langle x \rangle^j \partial_x^k v\|_{L_x^2} \leqslant c \|v\|_{H^n}^\theta \|\langle x \rangle^{n/2} v\|_{L_x^2}^{1-\theta} \leqslant c \|v\|_{Z_{n,n/2}}, \tag{3.73}$$

where $\theta = k/n$ and $j = (n-k)/2$.

Without loss of generality assume $I = [0, t_1]$. Let $\beta \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$ with $\beta \leqslant \alpha$. we claim that $\langle x \rangle^{\beta - n/2} \partial_x^n v(t) \in L^2(\mathbb{R})$ for almost every $t \in [0, t_1]$. In fact, let $\{v_{0m}\}_m \subset C_0^\infty(\mathbb{R})$ be a sequence converging to v_0 in $Z_{n,n/2}$. Denote with $v_m \in S(\mathbb{R})$ the corresponding solution provided by Theorem 3.5 with initial data v_{0m}. By regularity of the data-solution map we can assume all the v_m's are defined in $[0, T]$ with v_m converging to v in $C\left([0, T]; Z_{n,n/2}\right)$.

Consider $\phi_N \equiv \phi_{(n-1),N}^\beta$ built as in (3.19) from the function

$$\tilde{\tilde{\phi}}_{(n-1),N}^\beta(x) := \begin{cases} \langle x \rangle^{2\beta - n + 1} - 1 & x \in [0, N], \\ (2N)^{2\beta - n + 1} & x \in [3N, \infty). \end{cases}$$

Applying ∂_x^{n-1} to the mKdV equation for v_m we get

$$\partial_t z_m + \partial_x^3 z_m + \partial_x^{n-1}(v_m^2 \partial_x v_m) = 0, \quad z_m := \partial_x^{n-1} v_m.$$

We focus only on the non-linear terms. Note

$$\partial_x^{n-1}(v_m^2\partial_x v_m) = \sum_{k\leqslant n-1}\sum_{j\leqslant k}\binom{n-1}{k}\binom{k}{j}\partial_x^j v_m \partial_x^{k-j}v_m\partial_x^{n-k}v_m,$$

after multiplication by $\partial_x^{n-1}v_m\phi_N$ and integration we get

$$\begin{aligned}\int_0^{t_1}\int \partial_x^{n-1}v_m\partial_x^{n-1}(v_m^2\partial_x v_m)\phi_N dxdt \\ &= \sum_{k\leqslant n-1}\sum_{j\leqslant k}\binom{n-1}{k}\binom{k}{j}\int_0^{t_1}\int \partial_x^{n-1}v_m\partial_x^{n-k}v_m\partial_x^j v_m\partial_x^{k-j}v_m\phi_N dxdt. \\ &\equiv \sum_{k\leqslant n-1}\sum_{j\leqslant k}\binom{n-1}{k}\binom{k}{j}A_{k,j}.\end{aligned} \tag{3.74}$$

We estimate the $A_{k,j}$'s by cases.

Case A1, if $k=0$.

$$\begin{aligned}|A_{0,0}| &\leqslant \|\partial_x^{n-1}v_m\|_{L^\infty_{xt_1}}\|v_m\phi_N^{1/4}\|_{L^\infty_{xt_1}}\|\partial_x^n v_m\|_{L^2_{xt_1}}\|v_m\phi_N^{3/4}\|_{L^2_{xt_1}} \\ &\leqslant c\|v_m\|^2_{L^\infty_{t_1}H^2}\|J^1\langle x\rangle^{1/2}v_m\|_{L^\infty_{t_1}L^2_x}\|\langle x\rangle^{3/2}v_m\|_{L^\infty_{t_1}L^2_x}.\end{aligned} \tag{3.75}$$

Note that interpolating with $\theta = 1/n$ we have

$$\|J^1\langle x\rangle^{1/2}v_m\|_{L^\infty_{t_1}L^2_x} \leqslant c\|v_m\|^\theta_{L^\infty_{t_1}H^n}\|\langle x\rangle^{n/(2(n-1))}v_m\|^{1-\theta}_{L^\infty_{t_1}L^2_x} \leqslant c\|v_m\|_{L^\infty_{t_1}Z_{n,n/2}}. \tag{3.76}$$

Using that $3/2 \leqslant n/2$, (3.75) and (3.76) we conclude that for m large enough

$$|A_{0,0}| \leqslant c\|v\|^4_{L^\infty_{t_1}Z_{n,n/2}}. \tag{3.77}$$

Case A2, if $k \neq 0$.

Note in this case, since $n \geqslant 3$ we have $|\phi_N| \leqslant c\langle x\rangle^2 \leqslant c\langle x\rangle^{n-k/2}$. Using this and (3.72) - (3.73) we have, for any $j \in \{0,1,\ldots,n-1\}$ and $m \gg 1$, that

$$\begin{aligned}|A_{k,j}| &\leqslant c\|\partial_x^{n-1}v_m\|_{L^\infty_{xt_1}}\|\partial_x^{n-k}v_m\|_{L^\infty_{xt_1}}\|\langle x\rangle^{(n-j)/2}\partial_x^j v_m\|_{L^2_{xt_1}}\|\langle x\rangle^{(n-k+j)/2}\partial_x^{k-j}v_m\|_{L^2_{xt_1}} \\ &\leqslant c\|v_m\|^2_{L^\infty_{t_1}H^n}\|v_m\|^{2\theta}_{L^\infty_{t_1}H^n}\|\langle x\rangle^{n/2}v_m\|^{2-2\theta}_{L^\infty_{t_1}L^2_x} \leqslant c\|v\|^4_{L^\infty_{t_1}Z_{n,n/2}}.\end{aligned} \tag{3.78}$$

From (3.77) and (3.78) and arguing for z_m analogous to what was done in (3.20)-(3.29) for v_m it can be seen that

$$\langle x\rangle^{\beta-n/2}\partial_x^n v(t) \in L^2(\mathbb{R}) \text{ for almost every } t \in [0,t_1]. \tag{3.79}$$

Also, for m large enough

$$\|\langle x\rangle^{\beta-n/2}\partial_x^n v_m(t)\|_{L^2_{xt_1}} \leqslant M, \tag{3.80}$$

where M depends on $\|v\|_{L^\infty_{t_1}Z_{n,n/2}}$, $\|v_0\|_{Z_{n,n/2}}$ and $\|\langle x\rangle^{\beta-n/2}z_m(t_1)\|_{L^2_x}$.

From (3.79) there exist $t_0^* < t_1^*$ so that $\langle x\rangle^{\alpha-n/2}\partial_x^n v(t_0^*)$ and $\langle x\rangle^{\alpha-n/2}\partial_x^n v(t_1^*)$ are in $L^2(\mathbb{R})$. For simplicity we denote $[t_0^*, t_1^*]$ with $[0, t_1]$ again.

Let us now prove $\langle x\rangle^{\beta-(n+1)/2}\partial_x^{n+1}v(t)$ is in $L^2(\mathbb{R})$ for almost every $t \in [0, t_1]$. Construct $\phi_N \equiv \phi_{n,N}^\beta$ as in (3.19) but based on the function

$$\tilde{\tilde{\phi}}_{n,N}^\beta(x) := \begin{cases} \langle x\rangle^{2\beta-n} - 1 & x \in [0, N], \\ (2N)^{2\beta-n} & x \in [3N, \infty). \end{cases}$$

Denote $w_m := \partial_x^n v_m$. Take ∂_x^n to the mKdV equation for v_m and multiply it by $w_m\phi_N$. We get for the non-linear part

$$\begin{aligned} \int_0^{t_1}\int w_m\partial_x^n(v_m^2\partial_x v_m)\phi_N dxdt &= \sum_{k\leqslant n}\sum_{j\leqslant k}\binom{n}{k}\binom{k}{j}\int_0^{t_1}\int w_m\partial_x^{n+1-k}v_m\partial_x^j v_m\partial_x^{k-j}v_m\phi_N dxdt \\ &\equiv \sum_{k\leqslant n}\sum_{j\leqslant k}\binom{n}{k}\binom{k}{j}B_{k,j}. \end{aligned} \tag{3.81}$$

Let us consider some cases.

Case B1, if $k = 0$.

Using (3.9), (3.47) and (3.80) we have that for m large enough

$$\begin{aligned} |B_{0,0}| &\leqslant c\|\partial_x^{n+1}v_m\|_{L_x^\infty L_{t_1}^2}\|w_m\phi_N^{1/2}\|_{L_{xt_1}^2}\|v_m^2\phi_N^{1/2}\|_{L_x^2L_{t_1}^\infty} \\ &\leqslant c\|\partial_x^{n+1}v_m\|_{L_x^\infty L_{t_1}^2}\|\langle x\rangle^{\beta-n/2}w_m\|_{L_{xt_1}^2}\|v_m\langle x\rangle^{1/4}\|_{L_x^4L_{t_1}^\infty}^2 \leqslant M, \end{aligned} \tag{3.82}$$

where M depends on $\|v\|_{L_{t_1}^\infty Z_{n,n/2}}$ among other norms in which v is finite.

Case B2, if $k = n$.

Note that in case $j = 0$ or $j = n$, using (3.80) we have for $m \gg 1$

$$\begin{aligned} |B_{n,n}| = |B_{n,0}| &\leqslant c\|\partial_x v_m\|_{L_{xt_1}^\infty}\|v_m\|_{L_{xt_1}^\infty}\|\langle x\rangle^{\beta-n/2}w_m\|_{L_{xt_1}^2}^2 \\ &\leqslant c\|v_m\|_{L_{t_1}^\infty H^n}^2\|\langle x\rangle^{\beta-n/2}w_m\|_{L_{xt_1}^2}^2 \leqslant M \end{aligned} \tag{3.83}$$

Now, if $j \in \{1, \ldots, n-1\}$ we use (3.72), (3.73) and (3.79) to get for m large that

$$\begin{aligned} |B_{n,j}| &\leqslant c\|\langle x\rangle^{\beta-n/2}w_m\|_{L_{xt_1}^2}^2\|\langle x\rangle^{1/2}\partial_x v_m\|_{L_{xt_1}^2}\|\partial_x^j v_m\|_{L_{xt_1}^\infty}\|\partial_x^{n-j}v_m\|_{L_{xt_1}^\infty} \\ &\leqslant c\|v_m\|_{L_{t_1}^\infty H^n}^2\|\langle x\rangle^{\beta-n/2}w_m\|_{L_{xt_1}^2}^2\|v_m\|_{L_{t_1}^\infty Z_{n,n/2}} \\ &\leqslant M. \end{aligned} \tag{3.84}$$

Case B3, if $k \in \{1, \ldots, n-1\}$.

In this case we argue in a similar manner as done in case B2. Namely, for $m \gg 1$:

$$\begin{aligned}|B_{k,j}| &\leqslant c\|\langle x\rangle^{\beta-n/2}w_m\|^2_{L^2_{xt_1}}\|\langle x\rangle^{\beta-n/2}\partial_x^{n+1-k}v_m\|_{L^2_{xt_1}}\|\partial_x^j v_m\|_{L^\infty_{xt_1}}\|\partial_x^{k-j}v_m\|_{L^\infty_{xt_1}}\\ &\leqslant c\|v_m\|^2_{L^\infty_{t_1}H^n}\|\langle x\rangle^{\beta-n/2}w_m\|^2_{L^2_{xt_1}}\|\langle x\rangle^{\beta-n/2}\partial_x^{n+1-k}v_m\|_{L^2_{xt_1}}\\ &\leqslant M.\end{aligned} \tag{3.85}$$

With these estimates of the non-linear part, it can be seen (as done in (3.20)-(3.28)) that

$$\int_0^{t_1}\int(\partial_x^{n+1}v)^2\langle x\rangle^{2\beta-(n+1)}dxdt < \infty, \tag{3.86}$$

which implies

$$\langle x\rangle^{\beta-(n+1)/2}\partial_x^{n+1}v(t) \in L^2(\mathbb{R}) \text{ for almost every } t \in [0,t_1]. \tag{3.87}$$

From (3.79) and (3.87) the inductive step follows.

□

CHAPTER 4 DISPERSIVE BLOW-UP

In this section we use the local theory developed in Chapter 2 to study dispersive blow-up properties regarding the Kawahara equation (13) and the Hirota-Satsuma system (14). Weighed local theory is crucial to overcome some estimates that end up being done in $L^1_x(\mathbb{R})$.

4.1 The Kawahara Equation

The main result of this section is Theorem 4.1 below. It is proved in two steps: we first build a C^∞ initial data whose associated solution to the linear part of the equation fails to be C^3 in a sequence of times. Then it is proved that the nonlinear part of the solution will not make it worse in the sense we can obtain a nonlinear smoothing effect that reduces the regularity properties of the solution to the linear term.

Theorem 4.1. *Assume $\gamma < 0$ and $3\beta + 10\gamma > 0$. There exists an initial data $u_0 \in C^\infty(\mathbb{R}) \cap H^{7/2^-}(\mathbb{R})$ such that the solution $u \in C([0,T]; H^{7/2^-}(\mathbb{R}))$ given by Theorem 2.3 satisfies*

$$u(\cdot, t^*) \in C^3(\mathbb{R}\backslash\{0\}) \quad \textit{and} \quad u(\cdot, t^*) \notin C^3(\mathbb{R}),$$

for some $t^* \in (0, T)$.

4.1.1 Construction of the initial data

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := e^{-2|x|}$. Set $\phi(x) := (f * f)(x) = \dfrac{1}{2}e^{-2|x|}(1 + 2|x|)$. It is not difficult to see that $\phi \in H^{7/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{7/4^-}dx)$, $\phi \in C^3(\mathbb{R}\backslash\{0\})\backslash C^3(\mathbb{R})$, $e^x\phi \in L^2(\mathbb{R})$, and $e^{-x}\phi \in L^2(\mathbb{R})$.

Assume for the moment that u_0 has the form

$$u_0(x) := \sum_{j=1}^{\infty} \alpha_j W(-\sigma j)\phi(x), \tag{4.1}$$

where $W(t)$ is the unitary group defined in (2.2), $\sigma > 0$ is fixed and α_j will be defined later.

Proposition 4.2. *Assume $\gamma < 0$ and $3\beta + 10\gamma > 0$. For any $\sigma > 0$ there exists a sequence $\{\alpha_j\}$ such that the function u_0 in (4.1) belongs to $C^\infty(\mathbb{R}) \cap H^{7/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{7/4^-}dx)$. In addition, the associated global-in-time solution $u \in C(\mathbb{R}; H^{7/2^-}(\mathbb{R}))$ of the linear part of the IVP associated to the Kawahara equation (13) satisfies*

(i) For any $t > 0$ with $t \notin \sigma\mathbb{Z}^+$ we have $u(\cdot, t) \in C^\infty(\mathbb{R})$.

(ii) For any $t \in \sigma\mathbb{Z}^+$ we have $u(\cdot, t) \in C^3(\mathbb{R}\backslash\{0\})\backslash C^3(\mathbb{R})$.

Proof. The proof is based on Section 3 of [41]. For the sake of completeness we carry on the details here. We first prove $u_0 \in C^\infty(\mathbb{R})$. For that it suffices to show that $e^{-x}u_0 \in C^\infty(\mathbb{R})$. Thus, in view of Sobolev's embedding, it suffices to prove that $\partial_x^m(e^{-x}u_0) \in L^2(\mathbb{R})$, for any $m \in \mathbb{Z}^+$. To prove this, let us consider the IVP

$$\begin{cases} \partial_t w + Lw = 0, & t < 0, \\ w(x,0) = e^{-x}\phi, \end{cases}$$

where

$$Lw = \gamma\partial_x^5 w + 5\gamma\partial_x^4 w + (\beta + 10\gamma)\partial_x^3 w + (3\beta + 10\gamma)\partial_x^2 w + (3\beta + 5\gamma)\partial_x w + (\beta + \gamma)w.$$

For one hand, since $\partial_t(e^x w) + \beta\partial_x^3(e^x w) + \gamma\partial_x^5(e^x w) = 0$ and $e^x w(x,0) = \phi(x)$ we deduce that $W(t)\phi(x) = e^x w(x,t)$. On the other hand, it is easy to see that the solution of the above IVP is

$$w(x,t) = W(t)e^{-5\gamma t\partial_x^4}e^{-10\gamma t\partial_x^3}e^{-(3\beta+10\gamma)t\partial_x^2}\left(e^{-x+(2\beta+4\gamma)t}\phi(x - (3\beta + 5\gamma)t)\right).$$

Hence,

$$e^{-x}W(t)\phi(x) = W(t)e^{-5\gamma t\partial_x^4}e^{-10\gamma t\partial_x^3}e^{-(3\beta+10\gamma)t\partial_x^2}\left(e^{-x+(2\beta+4\gamma)t}\phi(x - (3\beta + 5\gamma)t)\right).$$

Next using Plancherel's theorem and the facts that $\gamma < 0$ and $3\beta + 10\gamma > 0$ we deduce

$$\begin{aligned} \|\partial_x^m(e^{-x}W(t)\phi)\|_{L^2} &\leqslant \|\xi^m e^{(3\beta+10\gamma)t\xi^2}\|_{L^\infty}\|e^{-x+(2\beta+4\gamma)t}\phi(x - (3\beta + 5\gamma)t)\|_{L^2} \\ &\leqslant \frac{c_m e^{-(\beta+\gamma)t}}{((3\beta + 10\gamma)|t|)^{k/2}}, \end{aligned} \tag{4.2}$$

where c_m is a constant depending on m and we have used that $e^{-x}\phi \in L^2(\mathbb{R})$.

Inequality (4.2) now yields

$$\begin{aligned}\|\partial_x^m(e^{-x}u_0)\|_{L^2} &\leqslant \sum_{j=1}^{\infty}\alpha_j\|\partial_x^m(e^{-x}W(-\sigma j)\phi)\|_{L^2}\\ &\leqslant \sum_{j=1}^{\infty}\alpha_j\frac{c_m e^{-(\beta+\gamma)\sigma j}}{((3\beta+10\gamma)\sigma j)^{m/2}}.\end{aligned}$$

By choosing α_j such that the above series converges for any $m \in \mathbb{Z}^+$ (for instance, take $\alpha_j := e^{-j^2}$). we conclude that $u_0 \in C^\infty(\mathbb{R})$.

Since $W(t)$ is bounded in $H^s(\mathbb{R})$, the fact $u_0 \in H^{7/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{7/4^-}dx)$ follows directly from inequality (1.7) and the properties of ϕ.

Before proving (i) and (ii), let us now consider the IVP

$$\begin{cases}\partial_t w + Lw = 0, \quad t > 0,\\ w(x,0) = e^x\phi,\end{cases}$$

where

$$Lw = \gamma\partial_x^5 w - 5\gamma\partial_x^4 w + (\beta+10\gamma)\partial_x^3 w - (3\beta+10\gamma)\partial_x^2 w + (3\beta+5\gamma)\partial_x w - (\beta+\gamma)w.$$

Here we have $W(t)\phi(x) = e^{-x}w(x,t)$ and w is given by the expression

$$w(x,t) = W(t)e^{5\gamma t\partial_x^4}e^{-10\gamma t\partial_x^3}e^{(3\beta+10\gamma)t\partial_x^2}\left(e^{x-(2\beta+4\gamma)t}\phi(x-(3\beta+5\gamma)t)\right).$$

Thus,

$$\partial_x^m(e^xW(t)\phi(x)) = \partial_x^m W(t)e^{5\gamma t\partial_x^4}e^{-(2\beta+10\gamma)t\partial_x^3}e^{(3\beta+10\gamma)t\partial_x^2}\left(e^{x-(2\beta+4\gamma)t}\phi(x-(3\beta+5\gamma)t)\right)$$

with

$$\begin{aligned}\|\partial_x^m(e^xW(t)\phi)\|_{L^2} &\leqslant \|\xi^m e^{-(3\beta+10\gamma)t\xi^2}\|_{L^\infty}\|e^{x-(2\beta+4\gamma)t}\phi(x-(3\beta+5\gamma)t)\|_{L^2}\\ &\leqslant \frac{c_m e^{(\beta+\gamma)t}}{((3\beta+10\gamma)t)^{m/2}},\end{aligned} \tag{4.3}$$

where we used that $e^x\phi \in L^2(\mathbb{R})$.

We now establish conditions (i) and (ii). To see that (i) holds, assume $t > 0$ is so that $t \notin \sigma\mathbb{Z}^+$. As before, it is enough to prove $e^{-x}W(t)u_0 \in H^m(\mathbb{R})$ for all $m \in \mathbb{Z}^+$. From (4.2), (4.3) and the fact that ϕ is symmetric, we get

$$\begin{aligned}\left\|\partial_x^m\left(e^{-x}W(t)u_0\right)\right\|_{L^2} &\leqslant \sum_{j=1}^{\infty}\alpha_j\left\|\partial_x^m\left(e^{-x}W(t-\sigma j)\phi\right)\right\|_{L^2}\\ &\leqslant c_m\sum_{j=1}^{\infty}\alpha_j\frac{e^{(\beta+\gamma)|t-\sigma j|}}{((3\beta+10\gamma)|t-\sigma j|)^{m/2}}.\end{aligned} \tag{4.4}$$

By our choice of α_j, the rightmost series in (4.4) is finite for all $m \in \mathbb{Z}^+$.

Finally, to prove (ii), assume $t = \sigma n$, for some $n \in \mathbb{Z}^+$. We have

$$W(t)u_0 = \alpha_n \phi + \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \alpha_j W(\sigma(n-j))\phi.$$

Using the above arguments, we may show that the series belongs to $C^\infty(\mathbb{R})$. The conclusion then follows because $\phi \in C^3(\mathbb{R}\backslash\{0\})\backslash C^3(\mathbb{R})$. □

4.1.2 Nonlinear smoothing

The goal of this section is to prove that the integral term in the Duhamel formulation (2.3) of the solution of (13) is more regular than the solution of the corresponding linear equation.

We begin by recalling some useful inequalities.

Lemma 4.3. *Assume $T \in (0,1)$ and let $W(t)$ be as in (2.2).*

(i) For any $\varphi \in L^2(\mathbb{R})$,

$$\|D^2 W(t)\varphi\|_{L^\infty_x L^2_T} \leqslant C\|\varphi\|_{L^2_x}. \tag{4.5}$$

(ii) If $f \in L^1_T L^2_x$ then

$$\sup_{[0,T]} \left\| D^2 \int_0^t W(t-t')f(\cdot,t')dt' \right\|_{L^2_x} \leqslant C\|f\|_{L^1_x L^2_T}. \tag{4.6}$$

(iii) For any $\theta \in (0,1)$, $-1 < \alpha \leqslant \dfrac{3}{2}$ and $\varphi \in L^2(\mathbb{R})$,

$$\|D^{\frac{\theta\alpha}{2}} W(t)\varphi\|_{L^q_T L^p_x} \leqslant C\|\varphi\|_{L^2_x}, \tag{4.7}$$

where $p = 2/(1-\theta)$ and $q = 10/\theta(\alpha+1)$.

Proof. For (4.5) see [17, Theorem 2.6]. Estimate (4.6) follows from (4.5) and a duality argument. For (4.7) see [17, Theorem 2.4]. □

With the above inequalities in hand we are able to prove the following result.

Proposition 4.4. *Let $s > \dfrac{13}{6}$ and assume $u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{s/2}dx)$. Let $u(t)$ be the solution of the IVP (13) provided by Theorem 2.3,*

$$u(t) = W(t)u_0 + \int_0^t W(t-t')(u\partial_x u)dt' =: W(t)u_0 + \mathcal{Z}(t), \quad t \in [0,T]. \tag{4.8}$$

Then, $\mathcal{Z}(t) \in H^{s+1}(\mathbb{R})$ for any $t \in [0,T]$.

Proof. From Theorem 2.3 we already know that $\mathcal{Z}(t) \in H^s(\mathbb{R})$. So we only need to prove that $D^{s+1}\mathcal{Z}(t) \in L^2(\mathbb{R})$. Note that in the proof of Theorem 2.3 we have assumed $0 < T < 1$; thus, in view of (4.6) we have

$$\begin{aligned}\left\| D^{s+1} \int_0^t W(t-t')(u\partial_x u)dt' \right\|_{L^2_x} &\leqslant C\|D^{s-1}(u\partial_x u)\|_{L^1_x L^2_T} \\ &\leqslant C\left(\|uD^{s-1}\partial_x u\|_{L^1_x L^2_T} + \left\|\left[D^{s-1}, u\right]\partial_x u\right\|_{L^1_x L^2_T}\right) \\ &\equiv C\,(I + II)\,,\end{aligned}$$

where $\left[D^{s-1}, u\right]\partial_x u = D^{s-1}(u\partial_x u) - uD^{s-1}\partial_x u$. According to Hölder's inequality,

$$I \leqslant \|u\|_{L^{6/5}_x L^3_T}\|D^{s-1}\partial_x u\|_{L^6_{xT}} \equiv I_1 I_2.$$

In order to estimate I_1 we use Hölder's inequality again to get

$$I_1 \leqslant \|\langle x\rangle^{-r}\|_{L^2_x}\|\langle x\rangle^r u\|_{L^3_T L^3_x} \leqslant CT^{1/3}\|\langle x\rangle^r u\|_{L^\infty_T L^3_x},$$

where $r > \dfrac{1}{2}$. Using the embedding $H^{1/6}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ together with the interpolation (1.4) we get

$$I_1 \leqslant CT^{1/3}\|J^{1/6}\langle x\rangle^r u\|_{L^\infty_T L^2_x} \leqslant CT^{1/3}\|J^s u\|^{\theta}_{L^\infty_T L^2_x}\|\langle x\rangle^{s/4} u\|^{1-\theta}_{L^\infty_T L^2_x}, \tag{4.9}$$

where $\theta = 1/6s$. Note that to apply (1.4) we have written $r = (1-\theta)\dfrac{s}{4}$, therefore the condition $r > \dfrac{1}{2}$ forces $s > \dfrac{13}{6}$. According to the weighted local theory the right-hand side of (4.9) is finite.

On the other hand, from Hölder's inequality and the Strichartz estimate (4.7) with $\theta = \dfrac{2}{3}$ and $\alpha = 0$ it follows that

$$\begin{aligned}I_2 &\leqslant T^{1/10}\|D^{s-1}\partial_x u\|_{L^{15}_T L^6_x} \leqslant T^{1/10}\|D^{s-1}\partial_x u_0\|_{L^{15}_T L^6_x} \\ &\qquad\qquad + T^{1/10}\left\| D^{s-1}\partial_x W(t)\int_0^t W(-t')u\partial_x u dt'\right\|_{L^{15}_T L^6_x} \\ &\leqslant CT^{1/10}\|D^s u_0\|_{L^2_x} + CT^{1/10}\int_0^T \|D^s(u\partial_x u)\|_{L^2_x}\,dt' \\ &\leqslant CT^{1/10}\|D^s u_0\|_{L^2_x} + CT^{3/5}\|D^s(u\partial_x u)\|_{L^2_{xT}},\end{aligned} \tag{4.10}$$

which is finite according to Lemma 2.2.

From (4.9) and (4.10) we conclude that I is finite.

It remains to prove II is finite. For that let us introduce the weights $v = w = \langle x\rangle^r$, with $r > 1$ to be determined latter. By setting $\ell = 2$ and $p = q = 4$ we see that $v^{\frac{\ell}{p}} w^{\frac{\ell}{q}} = \langle x\rangle^r$. Since $\langle x\rangle^r \in A_4$, from Hölder's inequality and Lemma 1.2 we obtain

$$\begin{aligned}II &\leqslant C\|\langle x\rangle^{-r/2}\|_{L^2_x}\|\langle x\rangle^{r/2}[D^{s-1}, u]\partial_x u\|_{L^2_{xT}} \\ &\leqslant C\left\|\|\langle x\rangle^{r/4}D^{s-1}u\|_{L^4_x}\|\langle x\rangle^{r/4}\partial_x u\|_{L^4_x} + \|\langle x\rangle^{r/4}\partial_x u\|_{L^4_x}\|\langle x\rangle^{r/4}D^{s-2}\partial_x u\|_{L^4_x}\right\|_{L^2_T}.\end{aligned} \tag{4.11}$$

Because $D = \mathcal{H}\partial_x$ we infer from Lemma 1.3 that

$$\|\langle x\rangle^{r/4} D^{s-2}\partial_x u\|_{L^4_x} = \|\langle x\rangle^{r/4}\mathcal{H}D^{s-1}u\|_{L^4_x} \leqslant C\|\langle x\rangle^{r/4}D^{s-1}u\|_{L^4_x},$$

which, from (4.11), yields

$$\begin{aligned} II &\leqslant C\left\|\|\langle x\rangle^{r/4}D^{s-1}u\|_{L^4_x}\|\langle x\rangle^{r/4}\partial_x u\|_{L^4_x}\right\|_{L^2_T} \\ &\leqslant C\left\|\|\langle x\rangle^{r/4}D^{s-1}u\|^2_{L^4_x}\right\|_{L^2_T} + C\left\|\|\langle x\rangle^{r/4}\partial_x u\|^2_{L^4_x}\right\|_{L^2_T} \\ &\leqslant C\left\|\|\langle x\rangle^{r/4}D^{s-1}u\|_{L^4_x}\right\|^2_{L^4_T} + C\left\|\|\langle x\rangle^{r/4}\partial_x u\|_{L^4_x}\right\|^2_{L^4_T} \equiv CII_1 + CII_2. \end{aligned}$$

We begin estimating II_1 by using (1.3) (with D instead of J):

$$\begin{aligned} II_1^{1/2} = \left\|\|\langle x\rangle^{r/4}D^{s-1}u\|_{L^4_x}\right\|_{L^4_T} &\leqslant C\left\|\|\langle x\rangle^{b}u\|^{\theta}_{L^4_x}\|D^a u\|^{1-\theta}_{L^4_x}\right\|_{L^4_T} \\ &\leqslant C\left\|\|\langle x\rangle^{b}u\|_{L^4_x} + \|D^a u\|_{L^4_x}\right\|_{L^4_T} \\ &\leqslant CT^{1/4}\|\langle x\rangle^{b}u\|_{L^\infty_T L^4_x} + CT^{1/8}\|D^a u\|_{L^8_T L^4_x}, \end{aligned} \tag{4.12}$$

where

$$\theta \in (0,1), \quad a = \frac{s-1}{1-\theta}, \quad \text{and} \quad b = \frac{r}{4\theta}. \tag{4.13}$$

For the term $\|\langle x\rangle^{b}u\|_{L^\infty_T L^4_x}$ we use the embedding $H^{1/4}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ and (1.4) to obtain

$$\|\langle x\rangle^{b}u\|_{L^\infty_T L^4_x} \leqslant C\|J^{1/4}(\langle x\rangle^{b}u)\|_{L^\infty_T L^2_x} \leqslant C\|J^s u\|^{1-\lambda}_{L^\infty_T L^2_x}\|\langle x\rangle^{s/4}u\|^{\lambda}_{L^\infty_T L^2_x} < \infty, \tag{4.14}$$

with

$$\lambda \in (0,1), \quad \lambda s = \frac{1}{4}, \quad \text{and} \quad (1-\lambda)\frac{s}{4} = b. \tag{4.15}$$

Conditions (4.13) and (4.15) leave $\theta = \dfrac{4r}{4s-1}$, which is in the interval $(0,1)$ provided $s > r + \dfrac{1}{4}$. Hence, if $r = 1+\varepsilon$ for some $0 < \varepsilon < 11/12$ we see that (4.14) holds for any $s > 13/6$.

For the second term on the right-hand side of (4.12), according to the choice of θ and r above, we have

$$a = \frac{s-1}{1-\theta} = \frac{4s^2-5s+1}{4s-(5+4\varepsilon)} = s + \varepsilon + \frac{1}{4s-5-4\varepsilon} + \frac{\varepsilon(5+4\varepsilon)}{4s-5-4\varepsilon}.$$

Therefore, by assuming ε sufficiently small we may write

$$a = s + \frac{1}{4s-5} + \tilde{\varepsilon}$$

where $\tilde{\varepsilon} > 0$ is also small enough. By setting $\delta = \dfrac{3}{8} - \tilde{\varepsilon} - \dfrac{1}{4s-5}$, our assumption $s > \dfrac{13}{6}$ gives $\delta > 0$ and we may write $a = 3/8 + s - \delta$. We employ the Strichartz estimate (4.7) with $\theta = \dfrac{1}{2}$ and $\alpha = \dfrac{3}{2}$ to get

$$\begin{aligned}\|D^a u\|_{L^8_T L^4_x} &\leqslant \|D^{3/8}W(t)D^{s-\delta}u_0\|_{L^8_T L^4_x} + \left\|D^{3/8}W(t)\int_0^t W(-t')D^{s-\delta}(u\partial_x u)dt'\right\|_{L^8_T L^4_x}\\ &\leqslant \|D^{s-\delta}u_0\|_{L^2_x} + \int_0^T \|D^{s-\delta}u\partial_x u\|_{L^2_x}dt'\\ &\leqslant \|u_0\|_{H^s} + CT^{1/2}\|u\partial_x u\|_{L^2_T H^s} < \infty,\end{aligned} \tag{4.16}$$

where the right-hand side of the above inequality is finite thanks to Lemma 2.2. This proves $II_1 < \infty$.

To see that II_2 is finite we proceed in exactly the same manner by noticing that II_2 is almost the same as II_1 but with less derivatives. Indeed, from Lemma 1.3 and (1.2),

$$\begin{aligned}II_2^{1/2} = \|\langle x\rangle^{r/4}\mathcal{H}Du\|_{L^4_{xT}} &\leqslant C\|\langle x\rangle^{r/4}Du\|_{L^4_{xT}}\\ &\leqslant CT^{1/4}\|\langle x\rangle^{r/4\theta}u\|_{L^\infty_T L^4_x} + CT^{1/8}\|D^{\frac{1}{1-\theta}}u\|_{L^8_T L^4_x}\end{aligned}$$

with (as in (4.14))

$$\|\langle x\rangle^{r/4\theta}u\|_{L^\infty_T L^4_x} \leqslant C\|J^{1/4}(\langle x\rangle^{r/4\theta}u)\|_{L^\infty_T L^2_x} \leqslant C\|J^s u\|^{1-\lambda}_{L^\infty_T L^2_x}\|\langle x\rangle^{s/4}u\|^{\lambda}_{L^\infty_T L^2_x} < \infty.$$

Besides, since

$$\frac{1}{1-\theta} = \frac{4s-1}{4s-5-4\varepsilon} = 1 + \frac{4}{4s-5} + \tilde{\tilde{\varepsilon}} = \frac{3}{8} + \eta,$$

for $\tilde{\tilde{\varepsilon}} > 0$ small and $\eta = \dfrac{5}{8} + \dfrac{4}{4s-5} + \tilde{\tilde{\varepsilon}} < s$, as done in (4.16) we deduce

$$\begin{aligned}\|D^{\frac{1}{1-\theta}}u\|_{L^8_T L^4_x} &= \|D^{3/8}D^{\eta}u\|_{L^8_T L^4_x}\\ &\leqslant C\|u_0\|_{H^\eta} + CT^{1/2}\|u\partial_x u\|_{L^2_T H^\eta}\\ &\leqslant C\|u_0\|_{H^s} + CT^{1/2}\|u\partial_x u\|_{L^2_T H^s} < \infty.\end{aligned}$$

This shows that II_2 is finite and completes the proof of the proposition. □

Proof of Theorem 4.1. Let $u_0 \in C^\infty(\mathbb{R}) \cap H^{7/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{7/4^-}dx)$ be the initial data constructed in Proposition 4.2. Let $u(t)$, $t \in [0, T]$, be the corresponding solution. We may assume that σ is sufficiently small such that $\sigma \in (0, T)$. Thus, for $t^* = \sigma$,

$$u(t^*) = W(t^*)u_0 + \int_0^{t^*} W(t^* - t')(u\partial_x u)dt' =: W(t^*)u_0 + \mathcal{Z}(t^*).$$

From Proposition 4.4 we know that $\mathcal{Z}(t^*) \in H^{\frac{9}{2}^-}(\mathbb{R}) \hookrightarrow C^3(\mathbb{R})$. Since $W(t^*)u_0 \in C^3(\mathbb{R}\backslash\{0\})\backslash C^3(\mathbb{R})$, the conclusion then follows from Proposition 4.2. □

4.2 The Hirota-Satsuma system

In a similar fashion as done in Section 4.1, the main result of this section (Theorem 4.5 below) is proved in two steps. We first construct an appropriate initial data for the linear IVP associated to (14) and then we show that the integral part of the solution is smoother than the linear one.

Theorem 4.5. *There exists an initial data* $(u_0, v_0) \in \left(C^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R})\right)^2$ *such that the solution* $(u, v) \in \left(C([0,T]; H^{3/2^-}(\mathbb{R}))\right)^2$ *of the IVP (14) given by Theorem 2.6 satisfies*

$$(u,v)(\cdot,t^*) \in \left(C^1(\mathbb{R}\backslash\{0\})\right)^2 \quad \text{and} \quad (u,v)(\cdot,t^*) \notin \left(C^1(\mathbb{R})\right)^2,$$

for some $t^* \in (0,T)$.

4.2.1 Construction of the initial data

Let $\{U_a(t)\}$ and $\{U(t)\}$ be the unitary groups introduced in Section 2.2. In [41, Section 3] the authors showed that, for some suitable sequence $\{\alpha_j\}$,

$$w_0(x) := \sum_{j=1}^{\infty} \alpha_j U(-j) e^{-2|x|}$$

belongs to

$$C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{3/2^-} dx)$$

and satisfies:

(i) for any $t \notin \mathbb{Z}$, $U(t)w_0 \in C^1(\mathbb{R})$;

(ii) for any $t \in \mathbb{Z}$, $U(t)w_0 \in C^1(\mathbb{R}\backslash\{0\})\backslash C^1(\mathbb{R})$.

Here, with a slightly modification of their proof and by taking

$$u_0(x) := \sum_{j=1}^{\infty} \alpha_j U_a(-\sigma j) e^{-2|x|} \quad \text{and} \quad v_0(x) := \sum_{j=1}^{\infty} \alpha_j U(-\sigma j) e^{-2|x|}, \tag{4.17}$$

for some real constant σ, we can show the following.

Proposition 4.6. *The functions in (4.17) satisfy*

$$(u_0, v_0) \in \left(C^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{3/2^-} dx)\right)^2.$$

Moreover, the associated global-in-time solution $(u,v) \in \left(C(\mathbb{R}; H^{3/2^-}(\mathbb{R}))\right)^2$ *of the linear part of the IVP (14) satisfy*

(i) For any $t > 0$ *with* $t \notin \sigma\mathbb{Z}^+$ *we have* $(u,v)(\cdot,t) \in (C^\infty(\mathbb{R}))^2$.

(ii) For any $t \in \sigma\mathbb{Z}^+$ *we have* $(u,v)(\cdot,t) \in \left(C^1(\mathbb{R}\backslash\{0\})\backslash C^1(\mathbb{R})\right)^2$.

Proof. See Section 3 in [41] (see also Lemma 3.2 in [39]). □

4.2.2 Nonlinear smoothing

Let us start by recalling some linear estimates.

Lemma 4.7. *For any $a \neq 0$ and $u_0 \in L^2(\mathbb{R})$ we have*

$$\|D_x^{-1/4} U_a(t) u_0\|_{L^4_x L^\infty_T} \leqslant C_a \|u_0\|_{L^2}, \tag{4.18}$$

$$\|\partial_x U_a(t) u_0\|_{L^\infty_x L^2_T} \leqslant C_a \|u_0\|_{L^2} \tag{4.19}$$

and

$$\|D_x^{-1/12} U_a(t) u_0\|_{L_x^{60/13} L_T^{15}} \leqslant C_a \|u_0\|_{L^2}. \tag{4.20}$$

Proof. For (4.18) and (4.19) see Theorems 3.5 and 3.7 in [37]. Estimate (4.20) follows interpolating (4.18) and (4.19); indeed, it suffices to define the family of analytic operators $T_z u_0 = D^{z/4} D^{1-z} U_a(t) u_0$, $0 \leqslant \mathrm{Re}(z) \leqslant 1$ and apply the Stein interpolation theorem ([50, Theorem 4.1] with $z = \dfrac{13}{15}$ (see a similar result in Corollary 3.8 of [37]). □

We also recall the following Strichartz estimate

Lemma 4.8. *For any $a \neq 0$ and $u_0 \in L^2(\mathbb{R})$,*

$$\|D^{\alpha\theta/2} U_a(t) u_0\|_{L^q_T L^p_x} \leqslant C \|u_0\|_{L^2}, \tag{4.21}$$

where $(q,p) = \left(\dfrac{6}{\theta(\alpha+1)}, \dfrac{2}{1-\theta}\right)$ *and* $(\theta, \alpha) \in (0,1) \times [0, 1/2]$.

Proof. See Lemma 2.4 in [36]. □

With these tools in hand we can prove the following smoothing property.

Proposition 4.9. *Let* $\dfrac{7}{6} < s < \dfrac{11}{6}$ *and* $(u_0, v_0) \in \left(H^s(\mathbb{R}) \cap L^2(|x|^s dx)\right)^2$. *Let $(u,v)(t)$ be the solution of the Hirota-Satsuma system (14) provided by Theorem 2.6 and given by*

$$\begin{cases} u(t) = U_a(t) u_0 + \displaystyle\int_0^t U_a(t-t')(6au\partial_x u - 2rv\partial_x v)(t')dt' := U_a(t)u_0 + \mathcal{Z}_1(t) \\ v(t) = U(t) v_0 + 3\displaystyle\int_0^t U(t-t')(u\partial_x v)(t')dt' := U(t)v_0 + \mathcal{Z}_2(t). \end{cases}$$

Then, $\mathcal{Z}_i(t) \in H^{s+\frac{1}{6}}(\mathbb{R})$, $i = 1,2$, $t \in [0,T]$.

Proof. We show the computations for $\mathcal{Z}_2$, same procedure apply to $\mathcal{Z}_1$. Since $\mathcal{Z}_2(t) \in H^s(\mathbb{R})$ it suffices to show that $\|D^{s+\frac{1}{6}} \mathcal{Z}_2(t)\|_{L^2_x}$ is finite. We follow partially the ideas in [39, Lemma 5.2]. For that, first note that (4.19) and duality give

$$\sup_{[0,T]} \left\| \partial_x \int_0^t U(t-t') f(\cdot, t') dt' \right\|_{L^2_x} \leqslant C \|f\|_{L^1_x L^2_T}. \tag{4.22}$$

Hence, using (4.22), Lemma 1.1 (part (iii)), and Hölder's inequality we infer

$$\begin{aligned}\left\| D^{s+\frac{1}{6}}\mathcal{Z}_2(t)\right\|_{L^2_x} &\leqslant C\|D^{s-\frac{5}{6}}(u\partial_x v)\|_{L^1_xL^2_T}\\ &\leqslant C\|D^{s-\frac{5}{6}}(u\partial_x v) - uD^{s-\frac{5}{6}}\partial_x v - \partial_x v D^{s-\frac{5}{6}}u\|_{L^1_xL^2_T}\\ &\quad + C\|uD^{s-\frac{5}{6}}\partial_x v\|_{L^1_xL^2_T} + C\|\partial_x v D^{s-\frac{5}{6}}u\|_{L^1_xL^2_T}\\ &\leqslant C\|u\|_{L^{6/5}_xL^3_T}\|D^{s+\frac{1}{6}}v\|_{L^6_{xT}} + C\|\partial_x v\|_{L^{60/13}_xL^{15}_T}\|D^{s-\frac{5}{6}}u\|_{L^{60/47}_xL^{30/13}_T}\\ &\leqslant C\{I + II_1II_2\}.\end{aligned}$$

To see that I is finite we combine the ideas developed in Section 2.2 together with [41] and the proof of Lemma 5.2 in [39]. Indeed, using (4.21) with $\alpha = \frac{1}{2}$, $\theta = \frac{2}{3}$ and $p = q = 6$, we obtain

$$\begin{aligned}\|D^{s+\frac{1}{6}}v\|_{L^6_{xT}} &\leqslant \|D^{\frac{1}{6}}U(t)D^s v_0\|_{L^6_{xT}} + \left\| D^{\frac{1}{6}}U(t)\int_0^t U(-t')D^s(u\partial_x v)(t')dt'\right\|_{L^6_{xT}}\\ &\leqslant C\|D^s u_0\|_{L^2_x} + C\int_0^T \|u\partial_x v\|_{L^2_x}dt'\\ &\leqslant C\|u_0\|_{s,2} + CT^{1/2}\|u\partial_x v\|_{L^2_T H^s}.\end{aligned}$$

The last term in the above inequality has already been shown to be finite in the local theory (see for instance (2.20)). This shows that $\|D^{s+\frac{1}{6}}v\|_{L^6_{xT}}$ is finite. To see that $\|u\|_{L^{6/5}_xL^3_T}$ is finite we need to use the local theory in weighted spaces. In fact, from Hölder's inequality, Sobolev embedding and (1.4) we deduce, for some $r = \frac{1}{2}^+$,

$$\begin{aligned}\|u\|_{L^{6/5}_xL^3_T} &\leqslant C\|\langle x\rangle^r u\|_{L^3_{xT}} \leqslant CT^{1/3}\|\langle x\rangle^r u\|_{L^\infty_T L^3_x}\\ &\leqslant C\|J^{1/6}(\langle x\rangle^r u)\|_{L^\infty_T L^2_x}\\ &\leqslant C\|J^s u\|^{1-\lambda}_{L^\infty_T L^2_x}\|\langle x\rangle^{s/2^-}u\|^{\lambda}_{L^\infty_T L^2_x},\end{aligned}$$

with $\lambda\frac{s}{2}^- = r$ and $\frac{1}{6} < (1-\lambda)s$. Since $s > 7/6$ we may take $\lambda = \frac{1}{s}^+$ to conclude that I is finite.

In view of (4.20),

$$\begin{aligned}II_1 &\leqslant \|\partial_x U(t)v_0\|_{L^{60/13}_xL^{15}_T} + \left\|\partial_x U(t)\int_0^t U(-t')u\partial_x v dt'\right\|_{L^{60/13}_xL^{15}_T}\\ &\leqslant C\|\mathcal{H}D^{13/12}v_0\|_{L^2} + C\int_0^T \left\|\mathcal{H}D^{13/12}(u\partial_x v)\right\|_{L^2}dt'\\ &\leqslant C\|v_0\|_{s,2} + CT^{1/2}\left\|u\partial_x v\right\|_{L^2_T H^s} < \infty,\end{aligned}$$

where in the last inequality we used that $\mathcal{H}$ is bounded in L^2 and the fact that $s > \frac{7}{6} > \frac{13}{12}$. Again, the term $\|u\partial_x v\|_{L^2_T H^s}$ may be bounded as done in the local theory.

In what comes to II_2 we argue as follows. For $\gamma > 7/20$ (to be chosen latter) we have

$$II_2 \leqslant \|\langle x\rangle^{-\gamma}\|_{L_x^{20/7}}\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}u\|_{L_{xT}^{30/13}} \leqslant C\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}u\|_{L_{xT}^{30/13}}.$$

Set $\dot{\mathcal{Z}}_1(t) = U_a(-t)\mathcal{Z}_1(t)$. Using Hölder's inequality in time and (3.3) we get

$$\begin{aligned}II_2 \leqslant CT^{37/90}\Bigg\{&\left\|U_a(t)\left(\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}u_0\right)\right\|_{L_T^{45}L_x^{30/13}} + \left\|U_a(t)\{\Phi_{t,\gamma}\widehat{D^{s-\frac{5}{6}}u_0}\}^{\vee}\right\|_{L_T^{45}L_x^{30/13}}\\ &+\left\|U_a(t)\left(\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}\dot{\mathcal{Z}}_1\right)\right\|_{L_T^{45}L_x^{30/13}} + \left\|U_a(t)\{\Phi_{t,\gamma}\widehat{D^{s-\frac{5}{6}}\dot{\mathcal{Z}}_1}\}^{\vee}\right\|_{L_T^{45}L_x^{30/13}}\Bigg\}.\end{aligned}$$

Next, by setting $\gamma = 5/12$ and using Strichartz estimate (4.21) with $\alpha = 0$ and $\theta = 2/15$ we deduce

$$\begin{aligned}II_2 &\leqslant CT^{37/90}(1+T)\Big\{\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}u_0\|_{L^2} + \|D^{2\gamma+s-\frac{5}{6}}u_0\|_{L^2} + \|D^{s-\frac{5}{6}}u_0\|_{L^2}\\ &\qquad+\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}\dot{\mathcal{Z}}_1\|_{L^2} + \|D^{2\gamma+s-\frac{5}{6}}\dot{\mathcal{Z}}_1\|_{L^2} + \|D^{s-\frac{5}{6}}\dot{\mathcal{Z}}_1\|_{L^2}\Big\}\\ &\leqslant CT^{37/90}(1+T)\Big\{\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}u_0\|_{L^2} + \|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}\dot{\mathcal{Z}}_1\|_{L^2} + \|u_0\|_{H^s} + \|\dot{\mathcal{Z}}_1\|_{H^s}\Big\}.\end{aligned} \tag{4.23}$$

Since

$$\|\dot{\mathcal{Z}}_1\|_{H^s} \leqslant C\int_0^T \|(6au\partial_x u - 2rv\partial_x v)\|_{H^s}dt',$$

we can prove that $\|\dot{\mathcal{Z}}_1\|_{H^s}$ is finite in a similar fashion as done in the local theory. Therefore, to conclude II_2 is finite it only remains to bound the first two terms on the right-hand side of (4.23), which can be estimated using (1.3) and the weighted local theory. In fact, first note that from (1.3),

$$\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}u_0\|_{L^2} \leqslant C\|\langle x\rangle^{s/2}u_0\|_{L^2}^{1-\lambda}\|D^s u_0\|_{L^2}^{\lambda} < \infty,$$

where $\lambda = \dfrac{6s-5}{6s}$ and $(1-\lambda)\dfrac{s}{2} = \gamma = \dfrac{5}{12}$. Also, setting $N(u,v) = 6au\partial_x u - 2rv\partial_x v$ and using (1.7) we have

$$\begin{aligned}\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}\dot{\mathcal{Z}}_1\|_{L^2} &\leqslant \int_0^T \|\langle x\rangle^{\gamma} U_a(-t')D^{s-\frac{5}{6}}N(u,v)\|_{L^2}dt'\\ &\leqslant C(1+T)\int_0^T \|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}N(u,v)\|_{L^2} + \|N(u,v)\|_{H^s}dt'\\ &\leqslant C(1+T)\left\{\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}N(u,v)\|_{L_T^1L_x^2} + T^{1/2}\|N(u,v)\|_{L_T^2H^s}\right\}.\end{aligned}$$

The second term in the right-hand side of the above inequality can be bounded as it was done in (2.20). For the first term, note that for $t \in [0,T]$ and $\lambda = \dfrac{6s-5}{6s}$ defined above, we have

$$\begin{aligned}\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}}N(u,v)\|_{L_x^2} &\leqslant C\|\langle x\rangle^{s/2}N(u,v)\|_{L_x^2}^{1-\lambda}\|D^sN(u,v)\|_{L_x^2}^{\lambda}\\ &\leqslant C\left(\|\langle x\rangle^{s/2}N(u,v)\|_{L_x^2} + \|D^sN(u,v)\|_{L_x^2}\right).\end{aligned}$$

Hence,

$$\|\langle x\rangle^{\gamma} D^{s-\frac{5}{6}} N(u,v)\|_{L^1_T L^2_x} \leqslant C(1+T)T^{1/2}\left\{\|\langle x\rangle^{s/2} N(u,v)\|_{L^2_T L^2_x} + \|D^s N(u,v)\|_{L^2_T L^2_x}\right\},$$

where both terms, $\|\langle x\rangle^{s/2} N(u,v)\|_{L^2_T L^2_x}$ and $\|D^s N(u,v)\|_{L^2_T L^2_x}$, can be estimated using the local theory in weighted spaces as in (2.21). This completes the proof of the Proposition. □

With Proposition 4.9 in hand, following the same idea as in the proof of Theorem 4.1, we can prove Theorem 4.5; so we omit the details.

CHAPTER 5

FURTHER RESULTS AND FUTURE RESEARCH

In this chapter we briefly discuss some further research topics that can be derived and understood from the methods and theorems proved in the previous chapters of this thesis.

In Chapter 3 it was shown for the mKdV equation that $s/2 \geqslant b$ is an optimal relation between decay and regularity. Such relation is natural once (5) and (9) are considered in $L^2(\mathbb{R})$. A more general version of (5) was mentioned in (1.25) in terms of $\Phi'(\xi)$. When this equation is seen in $L^2(\mathbb{R})$ the inequality (1.7) arises. This inequality suggest that the relation $s/K \geqslant b$ should be optimal, at least, for the dispersive models in the scope of conditions (A) and (B).

Partial results were obtained for the Kawahara and OST equation. Their extension to a full result such as Theorem 3.7 is currently under research. For instance, in the case of the Kawahara equation, if the solution $u \in C\left([-T,T]; H^2(\mathbb{R})\right)$ provided by Theorem 2.1 is so that there exist two times $t_1 \neq t_2$ such that $u(t_1)|x|^{1/2+\alpha}$ and $u(t_2)|x|^{1/2+\alpha}$ are in $L^2(\mathbb{R})$ it is expected that $u \in C\left([-T,T]; H^{2+4\alpha}(\mathbb{R})\right)$. A first approach arguing in a similar fashion as done in Chapter 3 suggest this holds for $\alpha \in (0, 7/32]$.

Note in this case $K = 4$ and therefore s/K agrees with $1/2$ when $s = 2$. Basically, such first approach should be the base case to raise within the size of α as done in the proof of Theorem 3.7; the main issue is that the range $(0, 7/32]$ is slightly less than the expected $(0, 1/4]$.

For the OST equation, a similar result can be obtained from $L^2(\mathbb{R})$. If $u \in C([-T,T]; L^2(\mathbb{R}))$ is the solution provided by Theorem 2.9 and if for any $\alpha \in (0, 1/2]$ there are $t_0 \neq t_1$ such that $|x|^{\alpha} u(t_i) \in L^2(\mathbb{R})$, $i = 0, 1$; then $u \in C([-T,T]; H^{2\alpha}(\mathbb{R}))$.

The proof of the latter is done pretty much in the same manner exposed in Section 3.3.1 except for the smoothing effect which in this case is really stronger. It is expected that this can be easily extended as in Chapter 3 for a full range of α, even in a more general setting when the regularity of the solution is small compared to the smoothing effect in Lemma 2.10. Note the OST equation can be seen as a member of a general class of perturbation of the KdV equation introduced in [10]. The authors considered the IVP associated to the equation

$$\partial_t u + \partial_x^3 u + \eta L u + u\partial_x u = 0,\ x \in \mathbb{R}, \tag{5.1}$$

where $\eta > 0$ is a parameter and L is defined via Fourier transform by $\widehat{Lu}(\xi) = -\Phi(\xi)\widehat{u}(\xi)$ with

$$\Phi(\xi) = \sum_{j=0}^{n}\sum_{i=0}^{2m} c_{ij}\xi_i|\xi|^j,\ c_{ij} \in \mathbb{R},\ c_{2m,n} = -1; \tag{5.2}$$

and Φ bounded above.

When $\Phi(\xi)$ is of the form $\Phi_k(\xi) = |\xi|^{k+2} - |\xi|^k$ with $k = 1$ we would have the OST equation. Theorem 2.11 can be extended to these perturbation when $\Phi = \Phi_k$ with $k \in \mathbb{Z}^+$. Note for this phase function the value of K defined in Theorem 1.5 is $K = k+1$. More precisely, if we consider the IVP

$$\begin{cases} \partial_t u + \partial_x^3 + \eta L u + u\partial_x u = 0,\ t > 0,\ x \in \mathbb{R}. \\ u(x,0) = u_0(x), \end{cases} \tag{5.3}$$

where $\eta > 0$ and $\widehat{Lu}(\xi) = -\Phi_k(\xi)\widehat{u}(\xi)$ with $k \in \mathbb{Z}^+$ and assume $u_0 \in Z_{s,b}$ for $s > 0$ with $0 \leqslant b \leqslant s/(k+1)$, there would exist $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap \cdots$.

A relation of type $s/(k+1) \geqslant b$ is expected to be optimal, which is particularly curious because a first approach suggested that an extra decay of α in two different times would imply a gain of regularity of 2α rather than the expected gain of $(k+1)\alpha$. This interesting fact is not contradictory but yet being subject of research at this moment.

In the more general case, at least for high regularity, one can expect a similar behaviour with respect for the persistence of solutions in weighted Sobolev spaces. In fact, if we consider the IVP (5.3) where $\widehat{Lu}(\xi) = -\Phi(\xi)\widehat{u}(\xi)$ and Φ is a real-valued bounded function defined by

$$\Phi(\xi) := \sum_{j=j_0}^{n}\sum_{i=i_0}^{m} c_{ij}\xi^{2i}|\xi|^{2j+1}, \tag{5.4}$$

for $i_0,\ j_0,\ m,\ n \in \mathbb{Z}^+ \cup \{0\}$, $1 \leqslant j_0 \leqslant m$, with $c_{i_0 j_0} \neq 0,\ c_{m,n} = -1$. Then for an initial data $u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2b}dx)$ where $m+n < s \leqslant (m+n)(2(j_0+i_0)+1)$ and $b \leqslant \dfrac{s}{2(m+n)}$ the solution $u \in C([0,T]; H^s(\mathbb{R})$ provided by Theorem 1.1 in [10] persists in $Z_{s,b}$.

It might result clarifying to address the question of the optimal relation between s and b in high regularity due to the fact the method used to establish such persistence in $Z_{s,b}$ is also an energy estimate when compared to the proof of Theorem 3.7.

Finally, in what comes to dispersive blow-up of solutions, to complement the results presented in Chapter 4, a non-local model should be the natural continuation of the work. One first step is to consider (5.3) in which the strong smoothing effect presented by the linear group should imply a similar strong nonlinear smoothing effect in the sense of Proposition 4.4. It also would require a generalization to the construction of the initial data whose solution associated to the linear part of the equation present infinitely many times with lack of regularity and adapts itself to the equation in consideration.

The attainability of results such as the existence of a nonlinear smoothing effect presented in Proposition 4.4 seems to be plausible in a more general context. From the computations done in Chapter 4, it appears that these kind of results might be obtained in a reasonably similar fashion for dispersive equations whose phase function fit in the description of (A) and (B) and whose local well-posedness in weighted Sobolev spaces is done using the classical Kenig-Ponce-Vega method. A unified approach is then being a subject of research.

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APPENDIX A

SOME PHASES SATISFYING (A) AND (B)

We give some examples of functions satisfying conditions (A) and (B).

A multivariate polynomial

Let $\beta \in \mathbb{N}^n$ with $|\beta| \geqslant 1$. Consider $\Phi(x) = x^\beta$. We have

$$\begin{aligned} |\phi(x-w) - \phi(x)| = |(x-w)^\beta - x^\beta| &= \left| \sum_{\alpha \leqslant \beta} \binom{\beta}{\alpha} x^{\beta-\alpha}(-w)^\alpha - x^\beta \right| \\ &\leqslant \left| \sum_{\substack{\alpha \leqslant \beta \\ \alpha \neq 0}} \binom{\beta}{\alpha} x^{\beta-\alpha}(-w)^\alpha \right| \leqslant \sum_{\substack{\alpha \leqslant \beta \\ \alpha \neq 0}} \binom{\beta}{\alpha} |x|^{|\beta|-|\alpha|}|w|^{|\alpha|} \\ &\leqslant |w| \sum_{\substack{\alpha \leqslant \beta \\ \alpha \neq 0}} \binom{\beta}{\alpha} |x|^{|\beta|-|\alpha|}|w|^{|\alpha|-1} \end{aligned} \tag{A.1}$$

From (A.1), if $|w| < |x|$ we have

$$|\phi(x-w) - \phi(x)| \leqslant |w| \sum_{\substack{\alpha \leqslant \beta \\ \alpha \neq 0}} \binom{\beta}{\alpha} |x|^{|\beta|-|\alpha|}|x|^{|\alpha|-1} = |w| \sum_{\substack{\alpha \leqslant \beta \\ \alpha \neq 0}} \binom{\beta}{\alpha} |x|^{|\beta|-1} = C_\beta |x|^{|\beta|-1}|w|.$$

Similarly, in case $|x| \leqslant |w|$ from (A.1) it follows that

$$|\phi(x-w) - \phi(x)| \leqslant |w| \sum_{\substack{\alpha \leqslant \beta \\ \alpha \neq 0}} \binom{\beta}{\alpha} |w|^{|\beta|-|\alpha|}|w|^{|\alpha|-1} = C_\beta |w|^{|\beta|}.$$

□

Notice that in this case are included the phases ϕ_1, ϕ_4 and ϕ_5 presented in Chapter 1.

Modulus raised to an integer power

Set $k \in \mathbb{Z}^+$. We claim $\phi(x) = |x|^k$ satisfy conditions (A) and (B). Notice that

$$|x-w|^k - |x|^k = (|x-w| - |x|) \sum_{j=0}^{k-1} |x-w|^{k-j-1}|x|^j. \tag{A.2}$$

Suppose that for all $\alpha \in [0,1]$ we have $0 \neq \alpha x + (1-\alpha)(x-w)$. By the mean value theorem we have $|x-w| - |x| = \nabla|\xi_{xw}| \cdot w$ where $(\nabla|\cdot|)_i = x_i|x|^{-1}$. Hence

$$\begin{aligned} |\phi(x-w) - \phi(x)| &= \left|\nabla|\xi_{xw}| \cdot w\right| \sum_{j=0}^{k-1} |x-w|^{k-j-1}|x|^j \leqslant \left|\nabla|\xi_{xw}|\right| |w| \sum_{j=0}^{k-1} |x-w|^{k-j-1}|x|^j \\ &\leqslant |w| \sum_{j=0}^{k-1} (|x| + |w|)^{k-j-1} |x|^j \end{aligned} \tag{A.3}$$

Now, if $|w| < |x|$, from (A.3) we conclude

$$|\phi(x-w) - \phi(x)| \leqslant |w| \sum_{j=0}^{k-1} (2|x|)^{k-j-1} |x|^j \leqslant C_k |x|^{k-1}|w|.$$

Similarly, in case $|x| \leqslant |w|$, from (A.3) we conclude

$$|\phi(x-w) - \phi(x)| \leqslant |w| \sum_{j=0}^{k-1} (2|w|)^{k-j-1} |w|^j = c_k |w|^k.$$

On the other hand, if for some $\alpha \in [0,1)$ we have $0 = \alpha x + (1-\alpha)(x-w)$, then we have $x = (1-\alpha)w$. The latter implies $|x| \leqslant |w|$. In this situation we only need to prove condition (B). Namely,

$$\begin{aligned} |\phi(x-w) - \phi(x)| &= ||x-w| - |x|| \sum_{j=0}^{k-1} |x-w|^{k-j-1}|x|^j \\ &\leqslant (|x-w| + |x|) \sum_{j=0}^{k-1} |x-w|^{k-j-1}|x|^j \\ &\leqslant (|w| + 2|x|) \sum_{j=0}^{k-1} (|x| + |w|)^{k-j-1}|x|^j \\ &\leqslant 3|w| \sum_{j=0}^{k-1} 2^{k-j-1}|w|^{k-1} = C_k|w|^k. \end{aligned}$$

Finally, if $\alpha = 1$ we would have $x = 0$, we obviously have $|x| \leqslant |w|$ and (B) holds trivially. □

The computation above cover the phase function ϕ_2 defined in Chapter 1

Phase ϕ_3

Denote by $\widehat{x_i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Consider $\phi(x) = \phi_3^i(x) := x_i|\widehat{x_i}|^2$, for $i \in \{1, \dots, n\}$. We need to show conditions (A) and (B) are satisfied.
We have

$$\begin{aligned} |\phi(x-w) - \phi(x)| &= \left|(x_i - w_i)|\widehat{x_i} - \widehat{w_i}|^2 - x_i|\widehat{x_i}^2|\right| \\ &= \left|(x_i - w_i)\left(|\widehat{x_i}|^2 - 2\widehat{x_i}\cdot\widehat{w_i} + |\widehat{x_i}|^2\right) - x_i|\widehat{x_i}|^2\right| \\ &= \left| -2x_i\widehat{x_i}\cdot\widehat{w_i} + x_i|\widehat{w_i}|^2 - w_i|\widehat{x_i}|^2 + 2w_i\widehat{x_i}\cdot\widehat{w_i} - w_i|\widehat{w_i}|^2 \right| \\ &\leqslant 2|x_i||\widehat{x_i}||\widehat{w_i}| + |x_i||\widehat{w_i}|^2 + |w_i||\widehat{x_i}|^2 + 2|w_i||\widehat{x_i}||\widehat{w_i}| + |w_i||\widehat{w_i}|^2 \end{aligned} \tag{A.4}$$

Using Young's inequality we have

$$2|x_i||\widehat{x_i}||\widehat{w_i}| \leqslant 2|\widehat{w_i}|\left(\frac{|x_i|^2}{2} + \frac{|\widehat{x_i}|^2}{2}\right) = |\widehat{w_i}||x|^2$$

and

$$2|w_i||\widehat{x_i}||\widehat{w_i}| \leqslant 2|\widehat{x_i}|\left(\frac{|w_i|^2}{2} + \frac{|\widehat{w_i}|^2}{2}\right) = |\widehat{x_i}||w|^2.$$

We continue (A.4) with

$$\begin{aligned} |\phi(x-w) - \phi(x)| &\leqslant |\widehat{w_i}||x|^2 + |x_i||\widehat{w_i}|^2 + |w_i||\widehat{x_i}|^2 + |\widehat{x_i}||w|^2 + |w_i||\widehat{w_i}|^2 \\ &\leqslant |w||x|^2 + |x||w|^2 + |w||x|^2 + |x||w|^2 + |w||w|^2 \\ &\leqslant |w|(|x|^2 + 2|x||w| + |w|^2). \end{aligned} \tag{A.5}$$

Now, if $|w| < |x|$ we have

$$|\phi(x-w) - \phi(x)| \leqslant |w|(|x|^2 + 2|x|^2 + |x|^2) = C|x|^2|w|.$$

Similarly, if $|x| \leqslant |w|$ we have

$$|\phi(x-w) - \phi(x)| \leqslant |w|(|w|^2 + 2|w|^2 + |w|^2) = C|w|^3.$$