UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

UNIVERSITÉ DE BOURGOGNE

Institut de Mathématiques de Bourgogne

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Moduli spaces of quasitrivial sheaves on the three dimensional projective space

Espaços de módulos de feixes quasitrivais sobre o espaço projetivo tri-dimensional

Campinas

2022

Douglas Manoel Guimarães

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática no âmbito do Acordo de Cotutela firmado entre a UNICAMP e a UNIVERSITÉ DE BOURGOGNE (FRANÇA).

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics under the Agreement for the Joint Supervision signed between UNICAMP and the UNIVERSITÉ DE BOURGOGNE (FRANCE).

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Este trabalho corresponde à versão final da Tese defendida pelo aluno Douglas Manoel Guimarães e orientada pelos Profs. Drs. Marcos Benevenuto Jardim e Daniele Faenzi.

Campinas

2022

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

G947m

Guimarães, Douglas Manoel, 1992-
Moduli spaces of quasitrivial sheaves on the three dimensional projective space / Douglas Manoel Guimarães. – Campinas, SP : [s.n.], 2022.

Orientadores: Marcos Benevenuto Jardim e Daniele Faenzi.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.
Em cotutela com: Université de Bourgogne, França.

1. Esquemas (Geometria algébrica). 2. Teoria de módulos. 3. Teoria de feixes. I. Jardim, Marcos Benevenuto, 1973-. II. Faenzi, Daniele. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. V. Título.

Informações para Biblioteca Digital

Título em outro idioma: Espaços de módulos de feixes quasitrivais sobre o espaço projetivo tri-dimensional
Palavras-chave em inglês:
Schemes (Algebraic geometry)
Moduli theory
Sheaf theory
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Abdelmoubine Amar Henni
Ugo Bruzzo
Alexander Sergeyevich Tikhomirov
Charles Aparecido de Almeida
Gaia Comaschi
Data de defesa: 16-02-2022
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0003-2503-0655
- Currículo Lattes do autor: http://lattes.cnpq.br/8310958448973960

Tese de Doutorado defendida em 16 de fevereiro de 2022 e aprovada pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Este trabalho é dedicado à minha familía,
minha mãe, meu pai (em memória) e meus irmãos.

Acknowledgements

I thank God for all the opportunities I had which made this work possible.

I thank my mother, father (in memory), and brothers who never let me lack for anything and always supported me unconditionally.

Thanks to all my Ph.D. colleagues who have made this journey a little lighter. Especially to Mauro, Thiago, and Ronaldo, my party partners. To Hugo, Danilo, and Alan, my orienteering brothers who were always open to discuss any subject and play a domino in the IMECC square. Also, thanks to my friends at discord who made quarantine more acceptable.

Special thanks to my advisors Marcos Jardim and Daniele Faenzi for all the patience and dedication they had with me, and that this work would certainly be impossible without all their help.

To all the professors I had contact with on this journey, both at UNICAMP and at UFSC, who made my path in algebraic geometry a little easier. Especially to Amar, who agreed to guide me in my master's degree at UFSC and who introduced me to algebraic geometry, and Simone for all the classes, advice and conversations I had at UNICAMP.

Also, thanks to the professors Ugo Bruzzo, Alexander Tikhomirov, Gaia Comaschi, Abdelmoubine Amar Henni and Charles Almeida to accepting being the jury for this work.

I also thank the entire staff at IMECC, who were always very attentive and always resolved issues as quickly as possible. Thanks to the whole team in the mathematics department at the Université de Bourgogne, who made my stay and adaptation in Dijon easier.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

"Quando a inteligência conduz o amor,
há lógica e razão.
Mas quando o amor dirige a inteligência,
a compaixão expressa-se e a caridade
toma conta dos comportamentos humanos."
(Pensamentos de Joanna de Ângelis)

Resumo

Um feixe sem torção E em $\mathbb{P}^3$ é chamado de quasitrivial se $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ e $\dim(E^{\vee\vee}/E) = 0$. Enquanto esses feixes são sempre μ-semiestáveis, eles nem sempre são semiestáveis. Nós estudamos o espaço de módulos de Gieseker–Maruyama $\mathcal{N}(r,n)$ de feixes semiestáveis de posto r em $\mathbb{P}^3$ com $h^0(E^{\vee\vee/E}) = n$ via o esquema Quot de pontos $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$. Nós mostramos que $\mathcal{N}(r,n)$ é vazio quando $r > n$, enquanto $\mathcal{N}(n,n)$ não têm pontos estáveis e é isomorfo ao produto simétrico $\mathrm{Sym}^n(\mathbb{P}^3)$. Nosso resultado principal é a construção de uma componente irredutível de $\mathcal{N}(r,n)$ de dimensão $2n + rn - r^2 + 1$ quando $r < n$. Além disso, esta é a única componente quando $n \leqslant 10$.

Palavras chaves: espaços de módulos, esquema Quot, estabilidade, quasitrival.

Abstract

A torsion free sheaf E on $\mathbb{P}^d$ is called *quasitrivial* if $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ and $\dim(E^{\vee\vee}/E) = 0$. While such sheaves are always μ-semistable, they may not be semistable. We study the Gieseker–Maruyama moduli space $\mathcal{N}(r,n)$ of rank r semistable quasitrivial sheaves on $\mathbb{P}^3$ with $h^0(E^{\vee\vee}/E) = n$ via the Quot scheme of points $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$. We show that $\mathcal{N}(r,n)$ is empty when $r > n$, while $\mathcal{N}(n,n)$ has no stable points and is isomorphic to the symmetric product $\mathrm{Sym}^n(\mathbb{P}^3)$. Our main result is the construction of an irreducible component of $\mathcal{N}(r,n)$ of dimension $2n + rn - r^2 + 1$ when $r < n$. Furthermore, this is the only irreducible component when $n \leqslant 10$.

Keywords: moduli spaces, quot scheme, stability, quasitrivial.

Abstrait

Une faisceau E sans torsion sur $\mathbb{P}^d$ est dite *quasitriviale* si $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ et $dim(E^{\vee\vee}/E) = 0$. Bien que de telles faisceaux soient toujours μ-semistables, elles peuvent ne pas être semi-stables. Nous étudions l'espace des modules de Gieseker–Maruyama $\mathcal{N}(r,n)$ de rang r des faisceaux quasitrivaux semi-stables sur $\mathbb{P}^3$ avec $h^0(E^{\vee\vee}/E) = n$ via le schéma de Quot des points $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$. Nous montrons que $\mathcal{N}(r,n)$ est vide lorsque $r > n$, tandis que $\mathcal{N}(n,n)$ n'a pas de points stables et est isomorphe au produit symétrique $\mathrm{Sym}^n(\mathbb{P}^3)$. Notre résultat principal est la construction d'une composante irréductible de $\mathcal{N}(r,n)$ de dimension $2n + rn - r^2 + 1$ lorsque $r < n$. De plus, c'est la seule composante irréductible lorsque $n \leqslant 10$.

Mots clés: espaces de modules, schéma Quot, stabilité, quasitrivalité.

Contents

Introduction

Let $\mathcal{M}(r, c_1, c_3, c_3)$ denote the Gieseker–Maruyama moduli space of semistable rank r sheaves on $\mathbb{P}^3$ with the first, second and third Chern classes equal to c_1, c_2 and c_3, respectively. Maruyama proved in [18] that the space $\mathcal{M}(r, c_1, c_3, c_3)$ is a projective scheme. However, the geometry of such a scheme remains largely unknown, despite the efforts of many authors in the past four decades, and questions about connectedness, irreducibility, the number of irreducible components, and so on, remain open.

When $r = 1$ and $c_1 = 0$ (which can always be achieved after twisting by an appropriate line bundle), one gets that $\mathcal{M}(1, 0, c_2, c_3)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^{d,g}(\mathbb{P}^3)$ of 1-dimensional schemes of degree $d = -c_2$ and genus $g = c_3 - 2c_2$ [15, Lemma B.5.6], which is known to always be connected [7]. Not much is known in general when $r \geqslant 2$, though:

1. $\mathcal{M}(2, c_1, c_2, c_3)$ is irreducible for $c_3 = c_2^2 - c_2 + 2$ when $c_1 = 0$, or $c_3 = c_2^2$ when $c_1 = -1$, see [24, Theorem 1.1] and the references therein;
2. $\mathcal{M}(2, 0, 2, c_3)$ has 2 irreducible components when $c_3 = 2$ and it has 3 irreducible components when $c_3 = 0$ [13, Section 6].
3. $\mathcal{M}(2, -1, 2, c_3)$ has 2 irreducible components when $c_3 = 2$ and it has 4 irreducible components when $c_3 = 0$ [1, Main Theorem 3].

Moreover, the moduli spaces in items (2) and (3) are connected. For higher values of c_2, one can check that the number of irreducible components of $\mathcal{M}(2, c_1, c_2, 0)$ grows with c_2, see [5, Proposition 3.6]; it is not known whether $\mathcal{M}(2, c_1, c_2, c_3)$ is always connected.

The goal of this work is to explore a somewhat exotic case, namely

$$\mathcal{M}(r, 0, 0, -2n) =: \mathcal{N}(r, n),$$

whose points correspond to *quasitrivial rank r sheaves*, that is, semistable rank r sheaves E on $\mathbb{P}^3$ such that $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ and $\dim(E^{\vee\vee}/E) = 0$; this nomenclature is borrowed from Artamkin [2]. The motivation comes from its close relationship, described in the body of the paper, between $\mathcal{N}(r, n)$ and the Hilbert and Quot schemes of points in $\mathbb{P}^3$. Moreover, even though the main focus of this paper is the moduli space of semistable quasitrivial sheaves, we also provide some results regarding μ-semistable quasitrivial sheaves.

First, we study μ-semistable sheaves E on $\mathbb{P}^d$ with $\mathrm{rk}(E) \geqslant 1$ and $c_1(E) = c_2(E) = 0$, and show that they are always extensions of ideal sheaves of subschemes of $\mathbb{P}^d$ of codimension at least 3, see Theorem 2.8 below. In addition, we prove that the moduli space of such sheaves is a GIT quotient of a Quot scheme $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$, where u is a polynomial of degree less than or equal to $d-3$, see Theorem 2.12.

We then focus on the case $d = 3$, for which we can get more concrete and precise results. Here is the main result of this work.

Main Theorem.

1. $\mathcal{N}(r,n)$ is empty whenever $r > n$ or $n < 0$.
2. $\mathcal{N}(n,n)$ is isomorphic to $\mathrm{Sym}^n(\mathbb{P}^3)$.
3. If $r < n$, then $\mathcal{N}(r,n)$ has an irreducible component of dimension $2n + rn - r^2 + 1$. Moreover, if $n \leqslant 10$, $\mathcal{N}(r,n)$ is irreducible.

The bound on n comes from the fact that the variety $\mathcal{C}(n)$ of triples of $n \times n$ commuting matrices is known to be irreducible precisely for $n \leqslant 10$; in fact, our conclusion is that $\mathcal{N}(r,n)$ is irreducible whenever $\mathcal{C}(n)$ is.

The thesis is organized as follows. We assume that the reader is familiar with Hartshorne's book [8]. The first chapter is devoted to some preliminaries that are not covered in Hartshorne's book and that we are going to use several times in the course of the work. The first section treats torsion free and reflexive sheaves, as well as their stability. In the second section, we briefly introduce the general theory to spectral sequences, with more focus on the local-to-global spectral sequence. In Section 3 we recall the theory of moduli spaces in general, and specialize this theory to the Quot scheme and the moduli space of semistable sheaves. The last section of the first chapter is dedicated to the study of Families of extensions, where we do a walk-through to Lange's paper [16], recalling the main results that we are going to use.

In Chapter 2, we start by studying reflexive sheaves on $\mathbb{P}^d$ with vanishing Chern classes in Section 2.1; we prove a key technical result about the triviality of these sheaves, which is subsequently used in the following sections. In Section 2.2 we begin exploring the relation between semistable sheaves and 3-codimensional quotients of the trivial sheaf on $\mathbb{P}^d$, and extensions of a sheaf of ideals. In Section 2.3 we give a criterion that tells when a torsion free sheaf coming arising as the kernel of an element of $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ is (semi)stable, and explain the relation between sheaf (semi)stability and the GIT-stability with respect to the natural action of GL_r on $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$.

We then start Chapter 3 by restricting the results obtained so far to $\mathbb{P}^3$. Here is where we establish items (1) and (2) of the Main Theorem. Here is where we construct an irreducible component for $\mathcal{N}(r,n)$. We begin constructing an irreducible component for $\mathcal{N}(2,n)$, which will serve as an induction step for the construction of an irreducible component for $\mathcal{N}(r,n)$ done in the final section of this chapter.

We work over the complex numbers $\mathbb{C}$. Every cohomology is taken over $\mathbb{P}^3$ as well as the Ext sheaves and Ext groups unless otherwise stated, that is, $H^i(F) = H^i(\mathbb{P}^3, F)$, etc. For a torsion free sheaf E, (semi)stable always means Gieseker (semi)stability, while μ-(semi)stability refers to stability in the sense of Mumford–Takemoto. As usual, we denote by the lower capital letters the dimension of the respective cohomology or Ext group: $h^i(F) = \dim H^i(F)$ and $\mathrm{ext}^i(F, G) = \dim \mathrm{Ext}^i(F, G)$.

Introdução

Seja $\mathcal{M}(r, c_1, c_3, c_3)$ o espaço de módulos de Gieseker–Maruyama de feixes semiestáveis de posto r em $\mathbb{P}^3$ com primeira, segunda e terceira classes de Chern iguais à c_1, c_2 e c_3, respectivamente. Maruyama provou em [18] que o espaço $\mathcal{M}(r, c_1, c_3, c_3)$ é um esquema projetivo. No entanto, a geometria desse esquema continua muito desconhecida, apesar do esforço de diversos autores nas últimas quatro décadas, e questões sobre conexidade, irredutibilidade, número de componentes irredutíveis, e assim por diante, continuam abertas.

Quando $r = 1$ e $c_1 = 0$ (o que é sempre possível depois de torcer por um fibrado de linha apropriado), temos que $\mathcal{M}(1, 0, c_2, c_3)$ é isomorfo ao esquema de Hilbert $\mathrm{Hilb}^{d,g}(\mathbb{P}^3)$ de esquemas de dimensão 1, grau $d = -c_2$ e genus $g = c_3 - 2c_2$ [15, Lemma B.5.6], o que é conhecido por ser sempre conexo [7]. Em geral, nada é muito conhecido para $r \geqslant 2$, temos:

1. $\mathcal{M}(2, c_1, c_2, c_3)$ é irredutível para $c_3 = c_2^2 - c_2 + 2$ quando $c_1 = 0$, ou $c_3 = c_2^2$ quando $c_1 = -1$, veja [24, Theorem 1.1] e as referências contidas no mesmo;
2. $\mathcal{M}(2, 0, 2, c_3)$ tem 2 componentes irredutíveis quando $c_3 = 2$ e tem 3 componentes irredutíveis quando $c_3 = 0$ [13, Section 6].
3. $\mathcal{M}(2, -1, 2, c_3)$ tem 2 componentes irredutíveis quando $c_3 = 2$ e tem 4 componentes irredutíveis quando $c_3 = 0$ [1, Main Theorem 3].

Além disso, os espaços de módulos nos itens (2) e (3) são conexos. Para valores maiores de c_2, é possível mostrar que o número de componentes irredutíveis de $\mathcal{M}(2, c_1, c_2, 0)$ cresce com c_2, veja [5, Proposition 3.6]; não é conhecido se $\mathcal{M}(2, c_1, c_2, c_3)$ é sempre conexo.

O Objetivo deste trabalho é explorar um caso exótico, isto é, $\mathcal{M}(r, 0, 0, -2n) =: \mathcal{N}(r, n)$, cujo os pontos correspondem à feixes *quasitriviais* de posto r e $\dim(E^{\vee\vee}/E) = 0$; essa nomenclatura foi emprestada de Artamkin [2]. A motivação vem por sua relação próxima, descrita no corpo do texto, entre $\mathcal{N}(r, n)$ e o esquema de Hilbert e o esquema Quot de pontos em $\mathbb{P}^3$. Além disso, apesar de que o foco principal é o espaço de módulos de feixes quasitriviais semiestáveis, nós também mostramos diversos resultados sobre feixes quasitriviais μ-semiestáveis.

Primeiro, estudamos feixes μ-semiestáveis E em $\mathbb{P}^d$ com $\mathrm{rk}(E) \geqslant 1$ e $c_1(E) = c_2(E) = 0$, e mostramos que eles sempre são extensão de ideais of subesquemas de $\mathbb{P}^d$ de codimensão pelo menos 3, veja o Teorema 2.8 abaixo. Adicionalmente, provamos que o espaço de módulos de tais feixes é o quociente GIT do esquema Quot $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$, em que u é um polinômio de grau menor ou igual à $d-3$, veja o Teorema 2.12.

Depois focamos no caso em que $d = 3$, em que focamos em resultados mais concretos e precisos. O principal resultado deste trabalho é o seguinte.

Teorema Principal.

1. $\mathcal{N}(r,n)$ é não vazio sempre que $r > n$ ou $n < 0$.
2. $\mathcal{N}(n,n)$ é isomorfo à $\mathrm{Sym}^n(\mathbb{P}^3)$.
3. Se $r < n$, então $\mathcal{N}(r,n)$ tem uma componente irredutível de dimensão $2n+rn-r^2+1$. Além disso, se $n \leqslant 10$, $\mathcal{N}(r,n)$ é irredutível.

A cota superior de n vêm do fato de que a variedade $\mathcal{C}(n)$ de triplas de matrizes $n \times n$ comutantes é conhecida por ser irredutível precisamente para $n \leqslant 10$; in fact, our conclusion is that $\mathcal{N}(r,n)$ é irredutível sempre que $\mathcal{C}(n)$ o é.

A tese está organizada da seguinte forma. Assumimos que o leitor está familiarizado com o livro de Hartshorne [8]. O primeiro capítulo é destinado a algumas preliminares que não são abordadas no livro de Hartshorne e que usaremos várias vezes no decorrer do trabalho. A primeira seção trata de feixes reflexivos e sem torção, bem como sua estabilidade. Na segunda seção, apresentamos brevemente a teoria geral das sequências espectrais, com mais foco na sequência espectral local-global. Na Seção 3, relembramos a teoria dos espaços de módulos em geral, e especializamos essa teoria para o esquema de Quot e o espaço de módulos de feixes semistáveis. A última seção do primeiro capítulo é dedicada ao estudo de Famílias de extensões, onde fazemos um walk-through ao artigo de Lange [16], relembrando os principais resultados que iremos utilizar.

No Capítulo 2, começamos estudando feixes reflexivos em $\mathbb{P}^d$ com o anulamento da primeira e segunda classes de Chern na Seção 2.1; provamos um resultado técnico fundamental sobre a trivialidade desses feixes, que é posteriormente usado nas seções a seguir. Na seção 2.2, começamos a explorar a relação entre feixes semestáveis e quocientes 3-codimensionais do feixe trivial em $\mathbb{P}^d$ e extensões de feixes de ideais. Na seção 2.3, damos um critério que informa quando um feixe livre de torção surge como o núcleo de um elemento de $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{oplusr}, u)$ é (semi)estável, e explica a relação entre a (semi)estabilidade do feixe e a estabilidade GIT com respeito à ação natural de GL_r em $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$.

Em seguida, iniciamos o Capítulo 3 restringindo os resultados obtidos até agora a $\mathbb{P}^3$. Aqui é onde estabelecemos os itens (1) e (2) do Teorema Principal. O Capítulo 4 é dedicado à construção de um componente irredutível para $\mathcal{N}(r,n)$. Começando a construir um componente irredutível para $\mathcal{N}(2,n)$, que servirá como uma etapa de indução para a construção de um componente irredutível para $\mathcal{N}(r,n)$ feito na seção final deste capítulo.

Nós trabalhamos sobre os números complexos $\mathbb{C}$. Toda cohomologia é tomada sobre $\mathbb{P}^3$ assim como os feixes Ext e grupos Ext, a menos que o contrário seja explicitado, isto é, $H^i(F) = H^i(\mathbb{P}^3, F)$, etc. Para um feixe sem torção E, (semi)estabilidade sempre significa a (semi)estabilidade de Gieseker, enquanto μ-(semi)estabilidade se refere a estabilidade no sentido de Mumford–Takemoto. Como de usual, vamos denotas pelas letras minúsculas a dimensão do respectivo grupo de cohomologia ou grupo Ext: $h^i(F) = \dim H^i(F)$ e $\mathrm{ext}^i(F, G) = \dim \mathrm{Ext}^i(F, G)$.

Introduction

Soit $\mathcal{M}(r, c_1, c_3, c_3)$ l'espace des modules de Gieseker–Maruyama des faisceaux de rang r semistable sur $\mathbb{P}^3$ avec les première, deuxième et troisième classes de Chern égales à c_1, c_2 et c_3, respectivement. Maruyama a prouvé dans [18] que l'espace $\mathcal{M}(r, c_1, c_3, c_3)$ est un schéma projectif. Cependant, la géométrie d'un tel schéma reste largement inconnue, malgré les efforts de nombreux auteurs au cours des quatre dernières décennies, et les questions sur la connexité, l'irréductibilité, le nombre de composants irréductibles, etc., restent ouvertes.

Lorsque $r = 1$ et $c_1 = 0$ (ce qui peut toujours être obtenu après torsion par un faisceau de lignes approprié), on obtient que $\mathcal{M}(1, 0, c_2, c_3)$ est isomorphe au schéma de Hilbert $\mathrm{Hilb}^{d,g}(\mathbb{P}^3)$ de schémas à 1 dimension de degré $d = -c_2$ et de genre $g = c_3 - 2c_2$ [15, Lemme B.5.6], qui est connu pour être toujours connecté [7]. On ne sait pas grand-chose en général quand $r \geqslant 2$, cependant:

1. $\mathcal{M}(2, c_1, c_2, c_3)$ est irréductible pour $c_3 = c_2^2 - c_2 + 2$ lorsque $c_1 = 0$, ou $c_3 = c_2^2$ lorsque $c_1 = -1$, voir [24, Théorème 1.1] et les références qu'il contient;
2. $\mathcal{M}(2, 0, 2, c_3)$ a 2 composantes irréductibles lorsque $c_3 = 2$ et il a 3 composantes irréductibles lorsque $c_3 = 0$ [13, Section 6].
3. $\mathcal{M}(2, -1, 2, c_3)$ a 2 composantes irréductibles lorsque $c_3 = 2$ et il a 4 composantes irréductibles lorsque $c_3 = 0$ [1, Main Theorem 3].

De plus, les espaces de modules dans les éléments (2) et (3) sont connectés. Pour des valeurs plus élevées de c_2, on peut vérifier que le nombre de composantes irréductibles de $\mathcal{M}(2, c_1, c_2, 0)$ croît avec c_2, voir [5, Proposition 3.6]; on ne sait pas si $\mathcal{M}(2, c_1, c_2, c_3)$ est toujours connecté.

Le but de cet article est d'explorer un cas quelque peu exotique, à savoir $\mathcal{M}(r, 0, 0, -2n) =: \mathcal{N}(r, n)$, dont les points correspondent au faisceaux *quasirivial* rang r, soit le semi-stable rang r faisceau E sur $\mathbb{P}^3$ tel que $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ et $\dim(E^{\vee\vee}/E) = 0$; cette nomenclature est empruntée à Artamkin [2]. La motivation vient de sa relation étroite, décrite dans le corps de l'article, entre $\mathcal{N}(r, n)$ et les schémas de points de Hilbert et Quot dans $\mathbb{P}^3$. De plus, même si l'objectif principal de cet article est l'espace des modules des faisceaux quasitrivaux semistables, nous fournissons également quelques résultats concernant les faisceaux quasitrivaux μ-semistables.

Premièrement, nous étudions les faisceaux μ-semistables E sur $\mathbb{P}^d$ avec $\mathrm{rk}(E) \geqslant 1$ et $c_1(E) = c_2(E) = 0$, et montrons que ce sont toujours des extensions de faisceaux idéaux de sous-schémas de $\mathbb{P}^d$ de codimension au moins 3, voir le théorème 2.8 ci-dessous. De plus, nous montrons que l'espace des modules de tels faisceaux est un quotient GIT d'un schéma de Quot $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$, où u est un polynôme de degré inférieur ou égal à $d-3$, voir le théorème 2.12.

On se concentre alors sur le cas $d = 3$, pour lequel on peut obtenir des résultats plus concrets et précis. Plus précisément, voici le résultat principal de cet article.

Théorème PrincipalI.

1. $\mathcal{N}(r,n)$ est vide chaque fois que $r > n$ ou $n < 0$.
2. $\mathcal{N}(n,n)$ est isomorphe à $\mathrm{Sym}^n(\mathbb{P}^3)$.
3. Si $r < n$, alors $\mathcal{N}(r,n)$ a une composante irréductible de dimension $2n + rn - r^2 + 1$. De plus, si $n \leqslant 10$, $\mathcal{N}(r,n)$ est irréductible.

La borne sur n vient du fait que la variété $\mathcal{C}(n)$ de triplets de $n \times n$ matrices de commutation est connue pour être irréductible précisément pour $n \leqslant 10$; en fait, notre conclusion est que $\mathcal{N}(r,n)$ est irréductible chaque fois que $\mathcal{C}(n)$ l'est.

La thèse est organisée comme suit. Nous supposons que le lecteur est familier avec le livre de Hartshorne [8]. Le premier chapitre est destiné à quelques préliminaires qui ne sont pas abordés dans le livre de Hartshorne et que nous allons utiliser plusieurs fois au cours de l'ouvrage. La première section traite des faisceaux sans torsion et réfléchissantes, ainsi que de leur stabilité. Dans la deuxième section, nous présentons brièvement la théorie générale des séquences spectrales, en nous concentrant davantage sur la séquence spectrale du local au global. Dans la section 3, nous rappelons la théorie des espaces de modules en général, et spécialisons cette théorie au schéma de Quot et à l'espace de modules des faisceaux semi-stables. La dernière section du premier chapitre est consacrée à l'étude des Familles d'extensions, où nous faisons un tour d'horizon de l'article de Lange [16], rappelant les principaux résultats que nous allons utiliser.

Au chapitre 2, nous commençons par étudier les faisceaux réflexifs sur $\mathbb{P}^d$ avec des classes de Chern évanouissantes dans la section 2.1; nous prouvons un résultat technique clé sur la trivialité de ces faisceaux, qui est ensuite utilisé dans les sections suivantes. Dans la section 2.2 nous commençons à explorer la relation entre les faisceaux semi-stables et les quotients codimensionnels à 3 du faisceau trivial sur $\mathbb{P}^d$, et les extensions du faisceau d'idéaux. Dans la section 2.3 nous donnons un critère qui indique quand une gerbe sans torsion apparaît comme le noyau d'un élément de $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ est la (semi)stabilité, et explique la relation entre la (semi)stabilité du faisceau et la stabilité GIT par rapport à l'action naturelle de GL_r sur $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$.

Nous commençons ensuite le chapitre 3 en restreignant les résultats obtenus jusqu'à présent à $\mathbb{P}^3$. C'est ici que nous établissons les éléments (1) et (2) du théorème principal. Le chapitre 4 est dédié à la construction d'une composante irréductible pour $\mathcal{N}(r,n)$. Nous commençons à construire une composante irréductible pour $\mathcal{N}(2,n)$, qui servira d'étape d'induction pour la construction d'une composante irréductible pour $\mathcal{N}(r,n)$ faite dans la section finale de ce chapitre.

On travaille sur les nombres complexes $\mathbb{C}$. Chaque cohomologuey est repris $\mathbb{P}^3$ ainsi que les faisceaux Ext et les groupes Ext sauf indication contraire, c'est-à-dire $H^i(F) = H^i(\mathbb{P}^3, F)$, etc. Pour une faisceau sans torsion E, (semi)stable signifie toujours (semi)stabilité de Gieseker, tandis que μ-(semi)stabilité fait référence à la stabilité au sens de Mumford–Takemoto. Comme d'habitude, nous désignons par les lettres majuscules inférieures la dimension de la cohomologie ou du groupe Ext respectif : $h^i(F) = \dim H^i(F)$ et $\mathrm{ext}^i(F,G) = \ dim\,\mathrm{Ext}^i(F,G)$.

1 Preliminary

We assume the reader is familiar with the basics of algebraic geometry, that is, Hartshorne's book from Chapters 1 to 3. In this chapter we cover some the notions not contained in that, but in which we are going to use in this work. Of course, the reader is encouraged to read the original references for more details. We begin studying torsion free and reflexive sheaves and their stability.

1.1 Stability of sheaves

In this section we recall some useful definitions and results concerning torsion free sheaves and the stability of sheaves, which plays an important role in constructing the moduli spaces that we are going to deal in this work. The reader can check [12, 21] for more details.

1.1.1 Torsion free and reflexive sheaves

Let X be a smooth projective variety.

Definition 1.1. A coherent sheaf F over X is *torsion free* if every stalk F_x is a torsion free $\mathcal{O}_{X,x}$-module; that is $fa = 0$ for $f \in \mathcal{O}_{X,x}$, $a \in F_x$ implies $a = 0$ or $f = 0$.

As an example of torsion free sheaves, we have that vector bundles (locally free sheaves) are torsion free and subsheaves of torsion free sheaves are again torsion free.

Now given a coherent sheaf F we have the canonical morphism from F to its double dual $F^{\vee\vee}$

$$\mu : F \to F^{\vee\vee},$$

and one can see that F is torsion free if, and only if, μ is a monomorphism. In fact, when we say torsion free sheaf, we usually keep in mind the equivalent property above.

Definition 1.2. A coherent sheaf F over X is said to be reflexive if $\mu : F \to F^{\vee\vee}$ is an isomorphism.

In particular, reflexive sheaves are torsion free.

Proposition 1.3. The dual of any coherent sheaf is reflexive.

Proof. [21, Corollary 1.2] □

Lemma 1.4. A reflexive sheaf of rank 1 is a line bundle.

Proof. [21, II, Lemma 1.1.15]. □

We end this section with a result about the singularity set of torsion free and reflexive sheaves.

Definition 1.5. Let F be a coherent sheaf on a scheme X. We define the *singularity set* of F by

$$S(F) = \{x \in X \mid F_x \text{ is not free over } \mathcal{O}_{X,x}\}.$$

Proposition 1.6. The singularity set of a torsion free sheaf is at least 2-codimensional. The singularity set of a reflexive sheaf is at least 3-codimensional.

Proof. [21, II, Corollary 1.1.8 and Lemma 1.1.10]. □

1.1.2 μ-stability

Let E be a torsion free coherent sheaf of rank r on $\mathbb{P}^n$. Define the slope of E as

$$\mu(E) := \frac{c_1(E)}{\mathrm{rk}(E)}.$$

Definition 1.7. Let E be a torsion free coherent sheaf on $\mathbb{P}^n$. We say that E is μ-(semi)stable if for every coherent subsheaf $0 \neq F \subset E$

$$\mu(F) < (\leqslant)\mu(E).$$

Theorem 1.8. Let E be a torsion free sheaf over $\mathbb{P}^n$. The following statements are equivalent:

1. E is (semi)stable.

2. $\mu(F) < (\leqslant)\mu(E)$ for all coherent subsheaves $F \subset E$ with $0 < \mathrm{rk}\, F < \mathrm{rk}\, E$ whose quotient E/F is torsion free.

3. $\mu(Q) > (\geqslant)\mu(E)$ for all torsion free quotients $E \twoheadrightarrow Q$ with $0 < \mathrm{rk}\, Q < \mathrm{rk}\, E$.

Proof. [21, II, Theorem 1.2.2]. □

In view of the above theorem, we will use the equivalent statement that is more suitable according to each situation.

Lemma 1.9.

1. Line bundles are μ-stable.

2. The sum $E_1 \oplus E_2$ of two μ-semistable sheaves is μ-semistable if, and only if, $\mu(E_1) = \mu(E_2)$.

3. E is μ-(semi)stable if, and only if, $E^\vee$ is.

4. E is μ-(semi)stable if, and only if, $E(k)$ is.

Proof. [21, II, Lemma 1.2.4] □

1.1.3 Gieseker stability

Let E be a coherent torsion free sheaf over $\mathbb{P}^n$ and let $P_E(t)$ be its Hilbert Polynomial. We define the reduced Hilbert polynomial of E as

$$p_E(t) := \frac{P_E(t)}{\operatorname{rk}(E)}.$$

Definition 1.10. We say that E is (semi)stable if for every coherent subsheaf $F \subset E$ with $0 < \operatorname{rk} F < \operatorname{rk} E$ we have

$$p_F(t) < (\leqslant) p_E(t).$$

We also have a similar theorem as we did with the μ-stability.

Theorem 1.11. Let E be a torsion free sheaf over $\mathbb{P}^n$. The following statements are equivalent.

1. E is (semi)stable.

2. For every coherent subsheaf $F \subset E$, $0 < \operatorname{rk} F < \operatorname{rk} E$, with torsion free quotient E/F
$$p_F(t) < (\leqslant) p_E(t).$$

3. For every torsion free quotient $E \twoheadrightarrow Q$, $0 < \operatorname{rk} Q < \operatorname{rk} E$,
$$p_Q(t) > (\geqslant) p_E(t).$$

Lemma 1.12. μ-stable torsion free coherent sheaves over $\mathbb{P}^n$ are also stable. Semistable sheaves over $\mathbb{P}^n$ are also μ-semistable.

Proof. [21, II, Lemma 1.2.12]. □

Lemma 1.13. Stable sheaves E over $\mathbb{P}^n$ are simple, that is, $\operatorname{End}(E) \cong \mathbb{C}$.

Proof. [21, Theorem 1.2.9]. □

1.2 Spectral Sequences

In this section, we recall the basics of spectral sequences and, in the end, we define the local-to-global spectral sequence, which we use several times in this work. The reader can check [19] for more about spectral sequences. We begin with an informal motivation.

Let us say that we want to compute H^* where H^* is a graded vector object, which, for simplicity, we will assume is a graded vector space. Suppose also that H^* is filtered, that is, we have

$$H^* \supset \cdots \supset F^n H^* \supset F^{n+1} H^* \supset \cdots \supset \{0\}.$$

For example, let $H^n = 0$ for $n < 0$, then

$$F^p H^* = \oplus_{n \geqslant p} H^n$$

gives a filtration for H^*.

Definition 1.14. Given H^* and F^* as above we define the associated graded vector space as

$$E_0^p(H^*) := \frac{F^p H^*}{F^{p+1} H^*}.$$

In good cases, we can recover H^* up to isomorphism by taking

$$H^* \cong \oplus_{p=0}^{\infty} E_0^p(H^*).$$

Note that in our previous example one has that

$$E_0^p(H^*) = \frac{F^p H^*}{F^{p+1} H^*} = \frac{\oplus_{n \geqslant p} H^n}{\oplus_{n \geqslant p+1} H^n} = H^p,$$

which implies that $H^* \cong \oplus_{p=0}^{\infty} E_0^p(H^*)$.

Thus we can consider as a first approximation to H^* the associated graded vector space to a filtration.

Remark 1.15. We can make $E_0^p(H^*)$ to a bigraded object: define $F^p H^r := F^p H^* \cap H^r$ and take

$$E_0^{p,q} = \frac{F^p H^{p+q}}{F^{p+1} H^{H^{p+q}}}.$$

We say that the index p is the filtration degree and the index q is the complementary degree.

Note that $E^p(H^*) = \oplus_{q=0}^{\infty} E_0^{p,q}$ and $H^r = \oplus_{p+q=r} E_0^{p,q}$.

Now we can define the first notion of a spectral sequence.

Definition 1.16. A (first quadrant, cohomological) spectral sequence, is a sequence of differential bigraded vector spaces, that is, for $r = 1, 2, \dots$ and $p, q > 0$, we have a vector space $E_r^{p,q}$. Furthermore, each bigraded vector space $E_r^{*,*}$ is equipped with a linear mapping

$$d_r : E_r^{*,*} \to E_r^{*,*},$$

which is a differential of bidegree $(r, 1-r)$, meaning that $d_r \circ d_r = 0$ and

$$d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}.$$

Finally, for all $r \geqslant 1$, $E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r)$, that is,

$$E_{r+1}^{p,q} = \frac{\ker(E_r^{p,q} \to E_r^{p+r,q+1-r})}{\operatorname{Im}(E_r^{p-r,q+r-1} \to E_r^{p,q})}.$$

We usually call the objects E_r as the r-th page of the sequence.

Remark 1.17. Let $E_r^{p,q}$ be a spectral sequence. If $r > \max(p, q+1)$, then $d_r = 0$ and, in this case, $E_{r+1}^{p,q} = E_r^{p,q}$. Which also implies that $E_r^{p,q} = E_{r+k}^{p,q}$ for $k \geqslant 0$. We denote this vector space by $E_\infty^{p,q}$.

Definition 1.18. Let $(E_r^{*,*}, d_r)$ be a spectral sequence. We say that $(E_r^{*,*}, d_r)$ converges to a graded vector space H^* if H^* has a filtration F^* and

$$E_\infty^{p,q} = E_0^{p,q}(H^*) = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

So our initial objective is approximated if we can find a spectral sequence converging to H^*.

Definition 1.19. We say that a spectral sequence $(E_r^{*,*}, d_r)$ collapse at the N-th page if $d_r = 0$ for $r > N$.

Remark 1.20. Note that if $(E_r^{*,*}, d_r)$ collapse at N, then

$$E_N^{*,*} \cong E_{N+1}^{*,*} \cong \cdots \cong E_\infty^{*,*}.$$

Spectral sequences can arise from various forms, for example: filtered differential modules, exact couples, double complex or Grothendieck's theorem. We will use the latter one to construct the local-to-global spectral sequence.

Let A, B and C be abelian categories and let $F : A \to B$ and $G : B \to C$ functors. Grothendieck's theorem relates the derivated functors of F and G with the derivated functor of $G \circ F$.

Theorem 1.21 (Grothendieck)**.** Suppose that F and G are covariant functors, G is left exact and F takes injectives to G-acyclic objects. Then there is a spectral sequence with

$$E_2^{p,q} \cong (R^p G)(R^q F(a))$$

and converging to $R^*(G \circ F)(a)$ for $a \in A$.

Now we can define the local-to-global spectral sequence. For a scheme X and a coherent sheaf $\mathcal{G}$ on X we take $G = H^0(X, -)$ and $F = \mathcal{H}om(-, \mathcal{G})$. So Grothendieck's theorem says that we have a spectral sequence with second page

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))$$

converging to $\operatorname{Ext}^*(\mathcal{F}, \mathcal{G})$. This is called the local-to-global spectral sequence.

Assume that for a given $\mathcal{F}$ and $\mathcal{G}$, the spectral sequence collapses at the second page. Then, from Remark 1.15, we have

$$\operatorname{Ext}^r(\mathcal{F}, \mathcal{G}) = \oplus_{p+q=r} H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})).$$

To illustrate how to work with this spectral sequence, we will explicit the details of a formula contained in [13] about torsion free sheaves with 0-dimensional singularities on $\mathbb{P}^3$.

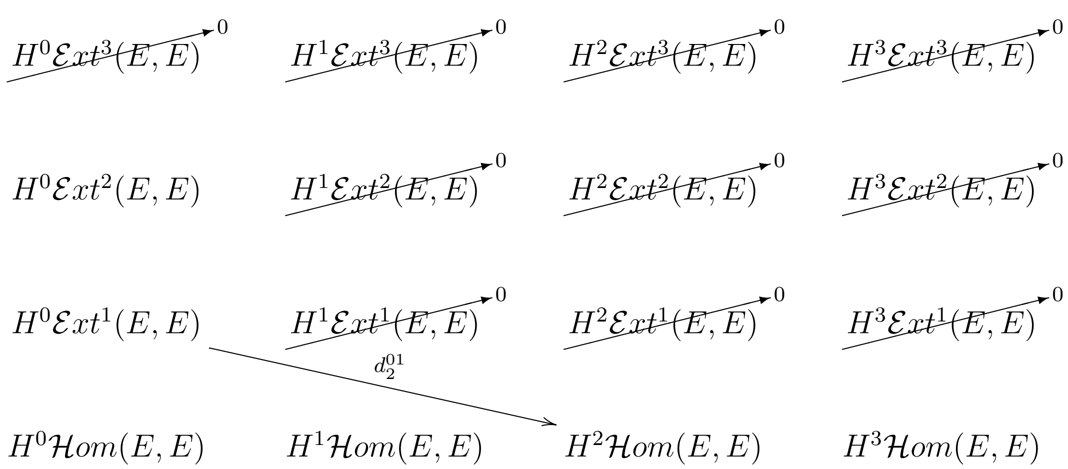
Example 1.22. Let E be a torsion free sheaf with 0-dimensional singularities on $\mathbb{P}^3$, that is, we have the following exact sequence

$$0 \to E \to E^{\vee\vee} \to Q_E \to 0,$$

where $\dim Q_E = 0$. Then

$$\sum_{j=0}^{3} (-1)^j \operatorname{ext}^j(E, E) = \chi(\mathcal{H}om(E, E)) - h^0 \mathcal{E}xt^1(E, E) + h^0 \mathcal{E}xt^2(E, E).$$

First thing to note is that $\mathcal{E}xt^1(E, E)$ and $\mathcal{E}xt^2(E, E)$ are 0-dimensional sheaves, while $\mathcal{E}xt^3(E, E) = 0$. Now the second page E_2 of the local-to-global spectral sequence for the pair (E, E) is given by:



The third page E_3 will be now given by

$$\begin{array}{cccc}
H^0\mathcal{E}xt^2(E,E) & & & \\
\ker d_2^{01} & & & \\
H^0\mathcal{H}om(E,E) & H^1\mathcal{H}om(E,E) & \operatorname{coker} d_2^{01} & H^3\mathcal{H}om(E,E)
\end{array}
\qquad d_3^{02}\colon H^0\mathcal{E}xt^2(E,E) \to H^3\mathcal{H}om(E,E)$$

Finally, the fourth page E_4 will be the following.

$$\begin{array}{cccc}
\ker d_3^{02} & & & \\
\ker d_2^{01} & & & \\
H^0\mathcal{H}om(E,E) & H^1\mathcal{H}om(E,E) & \operatorname{coker} d_2^{01} & \operatorname{coker} d_3^{02}
\end{array}$$

Note now that $d_4 = 0$, so the spectral sequence collapse on page 4. By Remark 1.15, we can write

- $\operatorname{Ext}^1(E,E) = H^1\mathcal{H}om(E,E) \oplus \ker d_2^{01}$.
- $\operatorname{Ext}^2(E,E) = \ker d_3^{02} \oplus \operatorname{coker} d_2^{01}$.
- $\operatorname{Ext}^3(E,E) = \operatorname{coker} d_3^{02}$.

Analysing the map d_2^{01} via the exact sequence

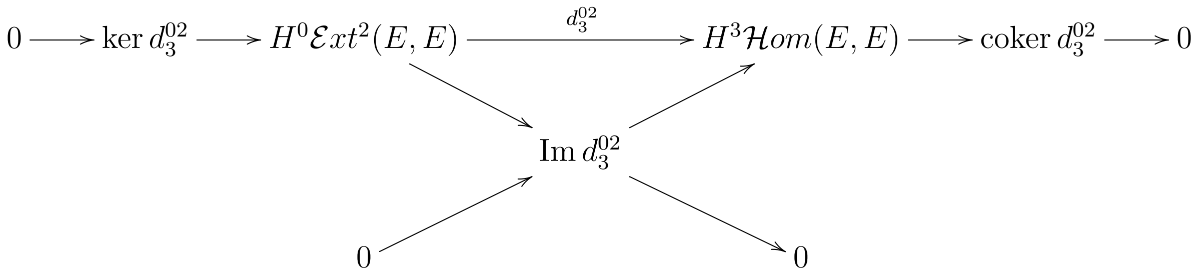
$$0 \longrightarrow \ker d_2^{01} \longrightarrow H^0\mathcal{E}xt^1(E,E) \xrightarrow{\quad d_2^{01} \quad} H^2\mathcal{H}om(E,E) \longrightarrow \operatorname{coker} d_2^{01} \longrightarrow 0$$

$$H^0\mathcal{E}xt^1(E,E) \twoheadrightarrow \operatorname{Im} d_2^{01} \hookrightarrow H^2\mathcal{H}om(E,E), \qquad 0 \to \operatorname{Im} d_2^{01} \to 0$$

we have that

- $\dim \ker d_2^{01} = h^0\mathcal{E}xt^1(E,E) - \dim \operatorname{Im} d_2^{01}$.
- $\dim \operatorname{coker} d_2^{01} = h^2\mathcal{E}xt^1(E,E) - \dim \operatorname{Im} d_2^{01}$.

Doing the same analysis to the map d_2^{02} with sequence



we have

- $\dim\ker d_3^{02} = h^0\mathcal{E}xt^2(E,E) - \dim \operatorname{Im} d_3^{02}$.
- $\dim\operatorname{coker} d_3^{02} = h^3\mathcal{H}om(E,E) - \dim \operatorname{Im} d_3^{02}$.

Finally, comparing the above equations we get

$$\sum_{j=0}^{3}(-1)^j \operatorname{ext}^j(E,E) = \chi(\mathcal{H}om(E,E)) - h^0\mathcal{E}xt^1(E,E) + h^0\mathcal{E}xt^2(E,E).$$

1.3 Moduli spaces

We briefly introduce to the general theory of moduli spaces, then introduce the two moduli spaces that we are concerned with in this work.

1.3.1 General theory

A moduli problem is essentially a classification problem: we want to classify certain geometric objects up to some notion of equivalence. For example, if we want to classify vector bundles on a fixed variety up to isomorphism or closed subschemes of a given scheme. We begin with the notion of a naive moduli problem. Following Hoskins' lecture notes [11].

Definition 1.23. A *naive moduli problem* is a collection $\mathcal{A}$ of objects (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Definition 1.24. Let $(\mathcal{A},\sim)$ be a naive moduli problem. Then an extended moduli problem is given by

1. Sets $\mathcal{A}_S$ of families over S and an equivalence relation $\sim_S$ on $\mathcal{A}_S$, for all schemes S,
2. Pullback maps $f^* : \mathcal{A}_S \to \mathcal{T}$, for every morphism of schemes $T \to S$, satisfying the following properties:

   a) $(\mathcal{A}_{\operatorname{Spec} k}, \sim_{\operatorname{Spec} k}) = (\mathcal{A},\sim)$;

b) for the identity $\mathrm{id} : S \to S$ and any family $\mathcal{F}$ over S, we have $\mathrm{id}^* \mathcal{F} = \mathcal{F}$;

c) for a morphism $f : T \to S$ and equivalent families $\mathcal{F} \sim_S \mathcal{G}$ over S, we have $f^*\mathcal{F} \sim_T f^*\mathcal{G}$;

d) for morphisms $f : T \to S$ and $g : S \to R$, and a family $\mathcal{F}$ over R, we have an equivalence $(g \circ f)^*\mathcal{F} \sim_T f^*g^*\mathcal{F}$.

For a family $\mathcal{F}$ over S and a point $s : \operatorname{Spec} k \to S$, we write $\mathcal{F}_s := s^*\mathcal{F}$ to denote the corresponding family over $\operatorname{Spec} k$.

Lemma 1.25. A moduli problem defines a functor $\mathcal{M}$ given by

$$\mathcal{M}(S) := \{\text{families over } S\}/\sim_S$$
$$\mathcal{M}(f : T \to S) = f^* : \mathcal{M}(S) \to \mathcal{M}(T).$$

We will often refer to a moduli problem simply by its moduli functor. A good answer to a moduli problem would be a scheme that represents our given moduli functor, that is the notion of fine moduli space.

Definition 1.26. Let $\mathcal{M} : Sch^o \to Set$ be a moduli functor. A scheme M is a fine moduli space for $\mathcal{M}$ if it represents $\mathcal{M}$.

The above definition says that if M is a fine moduli space for $\mathcal{M}$, then there is a natural isomorphism $\eta : \mathcal{M} \to \operatorname{Hom}(-, M)$. Hence, for every scheme S, we have a bijection

$$\eta_S : \mathcal{M}(S) \longleftrightarrow \operatorname{Hom}(S, M)$$

In particular, if $S = \operatorname{Spec} k$, then the k-points of M are in bijection with the set $\mathcal{A}/\sim$. Furthermore, these bijections are compatible with morphisms $T \to S$, that is, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\eta_S} & \operatorname{Hom}(S, M) \\ {\scriptstyle \mathcal{M}(f)} \downarrow & & \downarrow {\scriptstyle \hom(-,M)(f)} \\ \mathcal{M}(T) & \xrightarrow[\eta_T]{} & \operatorname{Hom}(T, M) \end{array}$$

The natural isomorphism $\eta : \mathcal{M} \to \operatorname{Hom}(-, M)$ determines an element $\mathcal{U} = \eta_M^{-1}(\mathrm{id}_M)$; that is, $\mathcal{U}$ is a family over M (up to equivalence).

Definition 1.27. Let M be a fine moduli space for $\mathcal{M}$. The family $\mathcal{U} \in \mathcal{M}(M)$ corresponding to the identity morphism on M is called the universal family.

This family is called the universal family, as any family $\mathcal{F}$ over a scheme S (up to equivalence) corresponds to a morphism $f : S \to M$ and, moreover, as the families $f^*\mathcal{U}$ and $\mathcal{F}$ correspond to the same morphism $\mathrm{id}_M \circ f = f$, we have

$$f^*\mathcal{U} \sim_S \mathcal{F};$$

that is, any family is equivalent to a family obtained by pulling back the universal family.

Not every moduli functor has a fine moduli space; thus we arrive at a weaker answer: a coarse moduli space.

Definition 1.28. A coarse moduli space for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of functors $\eta : \mathcal{M} \to h_M$ such that

1. $\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective.
2. For a scheme N and natural transformation $\nu : \mathcal{M} \to h_N$, there exists a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$, where $h_f : h_M \to h_N$ is the corresponding natural transformation.

1.3.2 Quot scheme

The Quot scheme is a fine moduli space that generalises the Grassmannian in the sense that it parametrizes quotients of a fixed sheaf. It is an important technical tool in many branches of algebraic geometry, for example, in the construction of many moduli spaces as we will see the example of the moduli space of semistable sheaves.

Let Y be a projective scheme and $\mathcal{F}$ be a fixed coherent sheaf on X. Then one can consider the moduli problem of classifying quotients of $\mathcal{F}$, that is, we consider surjective sheaf morphisms $q : \mathcal{F} \to Q$ up to the equivalence relation

$$(q : \mathcal{F} \to Q) \sim (q' : \mathcal{F} \to Q') \Leftrightarrow \ker q = \ker q'.$$

Equivalently, $(q : \mathcal{F} \to Q) \sim (q' : \mathcal{F} \to Q')$ if there is a sheaf isomorphism $\Phi : Q \to Q'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{q} & Q \\ & \searrow^{q'} & \downarrow \Phi \\ & & Q'. \end{array}$$

Thus we get a naive moduli problem. The next definition gives the extended moduli problem.

Definition 1.29. Let $\mathcal{F}$ be a coherent sheaf over Y. Then for any scheme S, we let $\mathcal{F}_S := \pi_Y^*\mathcal{F}$ denote the pullback of $\mathcal{F}$ to $Y \times S$ via the projection $\pi_Y : Y \times S \to Y$. A *family of quotients* of $\mathcal{F}$ over a scheme S is a surjective $\mathcal{O}_{Y\times S}$-linear morphism of sheaves over $Y \times S$

$$q_S : \mathcal{F}_S \to Q,$$

such that Q is flat over S. Two families $q_S : \mathcal{F}_S \to Q$ and $q'_S : \mathcal{F}_S \to Q'$ are equivalent if $\ker q_S = \ker q'_S$. As flatness is preserved by base change, we can pullback families, then we let

$$\mathcal{Q}uot_Y(\mathcal{F}) : Sch \to Set$$

denote the associated moduli functor.

Definition 1.30. For a fixed ample line bundle L on Y, we have a decomposition

$$\mathcal{Q}uot_Y(\mathcal{F}) = \cup_{P\in\mathbb{Q}[t]}\mathcal{Q}uot_Y^{P,L}(\mathcal{F})$$

into Hilbert polynomials P taken with respect to L.

Next Theorem shows the existence of a fine moduli space for the $\mathcal{Q}uot$ functor.

Theorem 1.31. Let Y be a projective scheme and L be an ample invertible sheaf on Y. Then for any coherent sheaf $\mathcal{F}$ over Y and any polynomial P, the functor $\mathcal{Q}uot_Y^{P,L}(\mathcal{F})$ is represented by a projective scheme $\mathrm{Quot}_Y^{P,L}(F)$.

When $F = \mathcal{O}_Y$ then the Quot scheme, turns to be the Hilbert scheme, parametrizing closed subschemes of Y .

1.3.3 Semistable sheaves

A moduli space of semistable sheaves is a scheme that is in some sense in natural bijection to equivalence classes of semistable sheaves on some fixed polarized projective scheme (X, H). As we will see, the expression natural bijection will become the notion of coarse moduli space and the equivalence will turn out to the S-equivalence.

This moduli space can be constructed as a quotient of a certain Quot scheme by a natural group action. Although we will not do this construction with full details here, this section contains the basics definitions and results that we are going to use in the course of the next chapters. We begin recalling some definitions important to defining the moduli functor.

Definition 1.32. Let E be a semistable sheaf of dimension d. A Jordan-Holder filtration of E is a filtration

$$0 = E_0 \subset E_0 \subset \cdots \subset E_l = E,$$

such that the factors $gr_i(E) = {}^{E_i}/_{E_{i-1}}$ are stable with reduced Hilbert polynomial $p(E)$.

Proposition 1.33. Jordan-Holder filtration always exist. The graded object $gr(E) := \oplus_i gr_i(E)$ does not depend on the choice of the Jordan-Holder filtration.

Proof. [12, Proposition 1.5.2]. □

Definition 1.34. Two semistable sheaves E_1 and E_2 with the same Hilbert polynomial are called S-equivalent if $gr(E_1) \cong gr(E_2)$.

Definition 1.35. A semistable sheaf E is called polystable if E is the direct sum of stable sheaves.

Now we can define the moduli functor.

Let $(X, \mathcal{O}_X(1))$ be a polarized projective scheme over an algebraically closed field k. For a fixed polynomial $P \in \mathbb{Q}[z]$ define a functor

$$\mathcal{M}' : (Sch/k)^o \to Sets$$

as follows. If S is an object in Sch/k, let $\mathcal{M}'(S)$ be the set of isomorphism classes of S-flat families of semistable sheaves on X with Hilbert polynomial P. And if $f : S' \to S$ is a morphism in Sch/k, let $\mathcal{M}'(f)$ be the map obtained by pulling-back sheaves via $f_X = f \times \mathrm{id}_X$:

$$\begin{array}{rccc} \mathcal{M}'(f): & \mathcal{M}'(S) & \to & \mathcal{M}'(S') \\ & [F] & \to & [f_X^* F]. \end{array}$$

If $F \in \mathcal{M}'(S)$ is an S-flat family of semistable sheaves, and if L is an arbitrary line bundle on S, then $F \otimes p^* L$ is also an S-flat family, and the fibers F_s and $(F \otimes p^* L)_s = F_s \otimes_{k(s)} L(s)$ are isomorphic for each point $s \in S$. It is therefore reasonable to consider the quotient functor $\mathcal{M} = \mathcal{M}' / \sim$, where $\sim$ is the equivalence relation:

$$F \sim F' \Leftrightarrow F \cong F' \otimes p^* L$$

for $F, F' \in \mathcal{M}'(S)$ and for some $L \in \mathrm{Pic}(S)$. Note that this definition depends on the polarization and we will write $\mathcal{M}_{\mathcal{O}_X(1)}(P)$ if we want to emphasize this fact.

We will see that there is always a projective coarse moduli space for this moduli functor (see Theorem 1.37). In general, however, there is no hope that $\mathcal{M}$ can be represented, i.e. to exist a fine moduli space for $\mathcal{M}$ as the next lemma shows.

Lemma 1.36. Suppose M is a coarse moduli space for $\mathcal{M}$. Then S-equivalent sheaves correspond to identical closed points in M. In particular, if there is a properly semistable sheaf F, (i.e. semistable but not stable), then there is no fine moduli space for $\mathcal{M}$.

Proof. [12, Lemma 4.1.2]. □

Next Theorem shows the existence of a coarse moduli space for the functor $\mathcal{M}$.

Theorem 1.37. There is a projective scheme $M_{\mathcal{O}_X(1)}(P)$ such that it is a coarse moduli space for the moduli functor $\mathcal{M}_{\mathcal{O}_X(1)}(P)$. Closed points in M are in bijection with S-equivalence classes of semistable sheaves with Hilbert polynomial P.

Proof. [12, Theorem 4.3.4]. □

1.4 Families of extensions

In this section, we review the results contained in [16] that we are going to use in the course of this work. We begin by recalling the definition of the relative-Ext sheaves following Birkar's notes [3].

Definition 1.38. Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ be the category of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules respectively. Let F be a $\mathcal{O}_X$-module. We define the functor $\mathcal{E}xt^p_f(F, -)$ to be the right derived functors of the left exact functor $f_*\mathcal{H}om_{\mathcal{O}_X}(F, -) : \mathfrak{M}(X) \to \mathfrak{M}(Y)$.

We now note that from the above definition we can derive several types of cohomologies by considering special cases.

Remark 1.39.

1. If Y is just a point, then $\mathcal{E}xt^p_f(F, G)$ is the usual Ext group $\mathrm{Ext}^p_{\mathcal{O}_X}(F, G)$, that is, $\mathcal{E}xt^p_{\mathcal{O}_X}(F, G) = \mathrm{Ext}^p_{\mathcal{O}_X}(F, G)$ is the right derived functors of the left exact functor $\mathrm{Hom}_{\mathcal{O}_X}(F, -)$. Moreover, note that the usual cohomology functor $H^p(X, -) \cong \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_X, -)$ because we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, -) \cong H^0(X, -)$.

2. If f is the identity, then instead of $\mathcal{E}xt^p_f(F, G)$ we write $\mathcal{E}xt^p_{\mathcal{O}_X}(F, G)$. That is, $\mathcal{E}xt^p_{\mathcal{O}_X}(F, -)$ are the right derived functors of the left exact functor $\mathcal{H}om_{\mathcal{O}_X}(F, -)$, the usual Ext sheaves. In particular, $\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{O}_X, G) \cong G$ and $\mathcal{E}xt^p_{\mathcal{O}_X} = 0$ if $p > 0$ because the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, -) \cong -$ is exact and so its right derived functors are trivial.

3. Since $f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, -) = f_*(-)$, we have $R^p f_*(-) = \mathcal{E}xt^p_f(\mathcal{O}_X, -)$. The functors $R^p f_*(-)$ are the right derived functors of the left exact functor f_*, the usually called higher direct images defined in [8].

Theorem 1.40. The sheaf $\mathcal{E}xt^p_f(F, G)$ is the sheaf associated to the presheaf

$$U \mapsto \mathrm{Ext}^p_{\mathcal{O}_{f^{-1}U}}(F|_{f^{-1}U}, G|_{f^{-1}U})$$

on Y. In particular, for any open subset $W \subseteq Y$, we have

$$\mathcal{E}xt_f^p(F,G)|_W \cong \mathcal{E}xt_f^p(F|_{f^{-1}W}, G|_{f^{-1}W}).$$

Proof. [3, Theorem 1.1.3]. □

Theorem 1.41. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. Then, for any $\mathcal{O}_X$-module G, we get a long exact sequence

$$\cdots \to \mathcal{E}xt_f^p(F'',G) \to \mathcal{E}xt_f^p(F,G) \to \mathcal{E}xt_f^p(F',G) \to \mathcal{E}xt_f^{p+1}(F'',G) \to \cdots$$

We can also generalize the local-to-global spectral sequence defined in Section 1.2 to its relative version as follows. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces and let $h = g \circ f$. Then we have a commutative diagram of functors

$$\begin{array}{ccc} \mathfrak{M}(X) & \xrightarrow{\alpha} & \mathfrak{M}(Y) \\ & \searrow^{\gamma} & \downarrow{\scriptstyle\beta} \\ & & \mathfrak{M}(Z) \end{array}$$

in which α is the functor $f_*\mathcal{H}om_{\mathcal{O}_X}(F,-)$, β is the functor g_* and γ is the functor $h_*\mathcal{H}om_{\mathcal{O}_X}(F,-)$.

Theorem 1.42. For any $\mathcal{O}_X$-module G, there is a spectral sequence

$$E_2^{p,q} = R^p g_* \mathcal{E}xt_f^p(F,G) \Longrightarrow \mathcal{E}xt_h^{p+q}(F,G)$$

When Z is just a point and f is the identity map, then we recover the spectral sequence described in Section 1.2.

Now let X be a projective variety over a field k and let F and G coherent sheaves on X. We know that the vector space $\mathrm{Ext}_X^1(F,G)$ parametrizes the extensions of F by G over X and there is an universal extension of p_1^*F by p_1^*G on $X \times \mathrm{Ext}_X^1(F,G)$ such that for every k-rational point $v \in \mathrm{Ext}_X^1(F,G)$ its restriction to $X \times \{v\}$ is just the extension represented by v, where $p_1 : X \times \mathrm{Ext}_X^1(F,G) \to X$ is the projection. Moreover, $\mathbb{P} = \mathbb{P}(\mathrm{Ext}_X^1(F,G))$ parametrizes the classes of nonsplitting extensions of F and G on X modulo k^* and there is a universal family of extensions of p_1^*F by $p_1^*G \otimes p_2^*\mathcal{O}_{\mathbb{P}}(1)$ on $X \times \mathbb{P}$ such that for every k-rational point p of $\mathbb{P}$ its restriction to $X \times \{p\}$ represents just the class of extensions given by p.

The rest of the section aims to generalize these results to the relative case, which will be used later in this work. For this, we follow [16].

Let $f : X \to Y$ be a flat projective morphism of noetherian schemes and let F and G be coherent $\mathcal{O}_X$-modules, flat over Y. We want to know how does the group $\mathrm{Ext}^i_{X_y}(F_y, G_y)$ varies as a function of $y \in Y$. To do this, we will find some relation between these groups and the i-th relative Ext-sheaf $\mathcal{E}xt^i_f(F, G)$, which turns out to be a generalization of Grothendieck's base change theory for cohomology.

Take a fixed projective embedding of X over Y, whence the invertible sheaves $\mathcal{O}_X(n)$ are defined, and consider a morphism $u : Y' \to Y$ of noetherian schemes. Let $X' = Y' \times_Y X$ and p_i denote the i-th projection.

Lemma 1.43. There is an integer $N(G)$ such that $\mathcal{E}xt^i_{p_2}(\mathcal{O}_{X'}(-n), G \otimes_Y M) = 0$ for all $n \geqslant N(G)$, $i \geqslant 1$ and quasi-coherent $\mathcal{O}_{Y'}$-modules M.

Proof. [16, Lemma 1.1]. □

Now take a locally free resolution $J_\bullet \to F \to 0$ with $J_j := \mathcal{O}_X(-k_j)^{l_j}$ and $k_j \geqslant N(G)$, and consider the complex $L^\bullet$ with $L^j := f_*\mathcal{H}om_{\mathcal{O}_X}(J_j, G) = f_*G(k_j)^l_j$.

Corollary 1.44.

1. $L^\bullet$ is a complex of coherent locally free $\mathcal{O}_Y$-modules.
2. For every quasi-coeherent $\mathcal{O}_{Y'}$-module M, there is a canonical isomorphism
$$p_2 * \mathcal{H}om_{\mathcal{O}_{X'}}(p_1^* J_j, G \otimes_Y M) \to L^j \otimes_Y M.$$
3. For every quasi-coherent $\mathcal{O}_{Y'}$-module M and every $i \geqslant 0$ there is a canonical isomorphism
$$\mathcal{E}xt^i_{p_2}(p_1^* F, G \otimes_Y M) \to H^i(L^\bullet \otimes_Y M).$$

Proof. [16, Corollary 1.2] □

For evert quasi-coherent $\mathcal{O}_Y$-module M, define
$$C^i(M) := H^i(L^\bullet \otimes_Y M).$$
Then we have the canonical homomorphism
$$C^i : \mathcal{O}_Y \otimes_{Y'} M \to C^i(M).$$

Using Corollary 1.44 it yields, for every $u : Y' \to Y$ of noetherian schemes, the base change homomorphism
$$\tau^i(u) : u^*\mathcal{E}xt^i_f(F, G) \to \mathcal{E}xt^i_{p_2}(p_1^* F, p_1^* G).$$

Using a locally free resolution of F and the ordinary base change theorem for the flat morphism $u : Y' \to Y$ we get the following proposition.

Proposition 1.45. For every flat morphism $u : Y' \to Y$ of noetherian schemes and every $i \geqslant 0$ the base change homomorphism $\tau^i(u)$ is an isomorphism.

Restricting to an open affine set in Y, the base change property applies directly in our situation to give the following base change theorem for relative Ext-sheaves.

Theorem 1.46. Let $y \in Y$ be a point and assume the base change morphism $\tau^i(y)$: $\mathcal{E}xt^i_f(F,G) \otimes_Y k(y) \to \mathcal{E}xt^i_{X_y}(F_y, G_y)$ to be surjective. Then

1. there is a neighbourhood U of y such that $\tau^i(y')$ is an isomorphism for all $y' \in U$;
2. $\tau^{i-1}(y)$ is surjective if and only if $\mathcal{E}xt^i_f(F,G)$ is locally free in a neighbourhood of y.

If $\tau^i(y)$ is an isomorphism for all $y \in Y$ we say that $\mathcal{E}xt^i_f(F,G)$ *commutes with base change*. This implies that $\tau^i(u)$ is an isomorphism for all u as above.

1.4.1 Universal Families of Extensions

Now we construct a family of extensions that parametrizes all extensions of F_y by G_y on X_y for all points $y \in Y$. After we turn to the construction of a family of extensions that parametrizes all classes of nonsplitting extensions of F_y by G_y on X_y modulo the equivalence relation identifying families which differ by a nonzero constant. To do this we will assume for this subsection that $\mathcal{E}xt^0_f(F,G)$ and $\mathcal{E}xt^1_f(F,G)$ commute with base change.

First, note that by Theorem 1.46, $\mathcal{E}xt^1_f(F,G)$ is locally free on Y. For every morphism $g : S \to Y$, consider the pullback diagram

$$\begin{array}{ccc} X_S & \xrightarrow{q_S} & X \\ {\scriptstyle p_S}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow[g]{} & Y \end{array} \qquad (1.1)$$

and define

$$E(S) := H^0(S, \mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)).$$

If $\alpha : S' \to S$ is a morphism over Y, define a map $E(\alpha) : E(S) \to E(S')$ by composing

$$H^0(S, \mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)) \to H^0(S', \alpha^*\mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)) \to H^0(S', \mathcal{E}xt^1_{p_S}(q_{S'}^*F, q_{S'}^*G)),$$

where the last map is given by $\tau^1(\alpha)$.

Since $\mathcal{E}xt^1_f(F,G)$ commutes with base change, so does $\mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)$ and E is a contravariant functor from the category of noetherian Y-schemes to the category of sets.

Proposition 1.47. Suppose $\mathcal{E}xt^1(F,G)$ commutes with base change for $i = 0, 1$. Then the functor E is representable by the vector bundle $V = V(\mathcal{E}xt^1_f(F,G)^*)$ over Y associated to the locally free sheaf $\mathcal{E}xt^1_f(F,G)^*$.

Proof. [16, Proposition 3.1] □

Now for the construction of the family for nonsplitting extensions, we need the projection formula for relative Ext-sheaves.

Lemma 1.48. For every coherent locally free $\mathcal{O}_Y$-module M and every $i \geqslant 0$ there is a canonical isomorphism

$$\mathcal{E}xt^i_f(F,G) \otimes_Y M \to \mathcal{E}xt^i_f(F, G \otimes f^*M).$$

For $g : S \to Y$ we can consider de pullback diagram (1.1) and we define

$$PE(S) := \text{ set of invertible quotients of } \mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)^*.$$

If $\alpha : S' \to S$ is a morphism over Y and $\mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G) \to L \to 0$ is an invertible quotient, then $\mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)$ is locally free and commutes with base change, and hence

$$\mathcal{E}xt^1_{p_{S'}}(q_{S'}^*F, q_{S'}^*G)^* = \alpha^*\mathcal{E}xt^1_{p_S}(q_S^*F, q_S^*G)^* \to \alpha^*L \to 0$$

is an invertible quotient of $\mathcal{E}xt^1_{p_{S'}}(q_{S'}^*F, q_{S'}^*G)^*$. Therefore we get a map $PE(\alpha) : PE(S) \to PE(S')$ and altogether a contravariant functor from the category of noetherian Y-schemes to the category of sets.

Proposition 1.49. Suppose $\mathcal{E}xt^i_f(F,G)$ commutes with base change for $i = 0, 1$. Then the functor PE is representable by the projective bundle $P = P(\mathcal{E}xt^1_f(F,G)^*)$ associated to the locally free sheaf $\mathcal{E}xt^1_f(F,G)^*$.

Corollary 1.50. Suppose Y is reduced and $\mathcal{E}xt^i_f(F,G)$ commutes with base change for $i = 0, 1$. Then there is a family of extensions $(e_p)_{p \in P}$ of q_P^*F by $q_P^*G \otimes p_P^*\mathcal{O}_P(1)$ over $P = P(\mathcal{E}xt^1_f(F,G)^*)$ which is universal on the category of reduced noetherian Y-schemes for the classes of families of nonsplitting extensions of q_S^*F by $q_S^*G \otimes p_S^*L$ over S with arbitrary $L \in \mathrm{Pic}(S)$ modulo the canonical operation of $H^0(S, \mathcal{O}_S^*)$.

Corollary 1.51. Suppose $\mathcal{E}xt^0_f(F,G) = 0$ and $\mathcal{E}xt^1_f(F,G)$ commutes with base change. Then there is an extension (e_P)

$$q_P^*G \otimes p_P^*\mathcal{O}_P(1) \to E_P \to q_P^*F \to 0$$

on X_P, $P = P(\mathcal{E}xt^1_f(F,G)^*)$, which is universal on the category of noetherian Y-schemes for the classes of extensions of q_S^*F by $q_S^*G \otimes p_S^*L$ on X_S with arbitrary $L \in \mathrm{Pic}(S)$, which split nowhere over S, modulo the canonical operation of $H^0(S, \mathcal{O}_{S^*})$.

Remark 1.52. If X is a projective variety over a field k, every coherent sheaf F or G is flat over k and, by Proposition 1.45, $\mathcal{E}xt^1_f(F,G)$ commutes with base change for $i = 0, 1$. Therefore we get the universal family of extensions over $P(\mathrm{Ext}^1_X(F,G)^*)$ mentioned in the beginning of this subsection as special case of Corollary 1.50.

2 Torsion free sheaves with $c_1 = c_2 = 0$

2.1 Semistable reflexive sheaves with vanishing Chern classes

A particular case of a result due to Simpson, see [25, Theorem 2], establishes that if F be a μ-semistable reflexive sheaf on a smooth projective variety X with $c_1(F) = c_2(F) = 0$, then F an extension of μ-stable locally free sheaves with vanishing Chern classes.

When X is a projective space, we can prove the following refinement, which will be very relevant to achieve the goals of this work. From now on, stability and μ-stability of sheaves on $X = \mathbb{P}^d$ are measured with respect to the hyperplane divisor.

Lemma 2.1. Let F be a μ-semistable reflexive sheaf of rank r on $\mathbb{P}^d$ with $d \geqslant 3$. If $c_1(F) = c_2(F) = 0$, then F is isomorphic to $\mathcal{O}_{\mathbb{P}^d}^{\oplus r}$.

Proof. The first step is to show that the only μ-stable locally free sheaf with vanishing Chern classes on $\mathbb{P}^d$ is $\mathcal{O}_{\mathbb{P}^d}$. Indeed, let G be a μ-stable locally free sheaf with vanishing Chern classes on $\mathbb{P}^d$ and take a 2-dimensional linear subspace $\wp \subset \mathbb{P}^d$; by [17, Theorem 3.1], $G|_\wp$ is a μ-stable locally free sheaf with $c_1(G|_\wp) = c_2(G|_\wp) = 0$. Following [17, proof of Proposition 8.2], we have that $\hom(G|_\wp, G|_\wp) = 1$, $\operatorname{ext}^2(G|_\wp, G|_\wp) = \hom(G|_\wp, G|_\wp(-3)) = 0$ and

$$\chi(G|_\wp, G|_\wp) = r^2 - \Delta(G|_\wp) = 1 - \operatorname{ext}^1(G|_\wp, G|_\wp) \leqslant 1.$$

But $\Delta(G|_\wp) = 0$ since $F|_\wp$ has vanishing Chern classes, thus $r = 1$ and $G|_\wp \cong \mathcal{O}_\wp$. We then conclude that $F \cong \mathcal{O}_{\mathbb{P}^d}$, as desired.

Since extensions of trivial bundles on $\mathbb{P}^d$ are always trivial, Simpson's result implies that $F \cong \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$. □

In particular, we note that every μ-semistable reflexive sheaf F on $\mathbb{P}^d$ with $c_1(F) = c_2(F) = 0$ is also semistable. As we will see below, this is no longer true if one considers *torsion free* sheaves with vanishing first and second Chern classes.

Theorem 2.2. Let X be a smooth projective variety such that every μ-stable reflexive sheaf with $c_1 = c_2 = 0$ is a line bundle. If E is a semistable reflexive sheaf with $c_1 = c_2 = 0$, then its Jordan-Holder filtration has factors in $\operatorname{Pic}^0(X)$.

Proof. We argue by induction on $\operatorname{rk}(E) = r$. If E is a rank 2 semistable reflexive sheaf with $c_1 = c_2 = 0$, then by Simpson's result, E must be an extension of μ-stable locally free sheaves with vanishing Chern classes, that is, we can write E in the following short exact sequence

$$0 \to L \to E \to L' \to 0,$$

where L and L' are in $\mathrm{Pic}^0(X)$. Thus we can take

$$0 \subset L \subset E$$

as the desired Jordan-Holder filtration.

Now suppose the result valid for rank less then r and let E be a semistable reflexive sheaf with $c_1 = c_2 = 0$. Again, by Simpson's result, there exists a filtration of E

$$0 = G_0 \subset G_1 \subset \cdots \subset G_k \subset E,$$

such that each quotient is a μ-stable locally free sheaf with vanishing Chern classes. Let $F := G_k$ and $F' := E/G_k$ and consider the following exact sequence

$$0 \to F \to E \to F' \to 0.$$

Since F' is a μ-stable locally free with vanishing Chern classes, the hypothesis implies that $F' \in \mathrm{Pic}^0(X)$. Now we can apply the induction hypothesis to F which has rank $r - 1$, and we get a Jordan-Holder filtration of F

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F$$

whose factors are in $\mathrm{Pic}^0(X)$. Finally, we take

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F \subset E,$$

and this is a Jordan-Holder filtration for E which satisfies our requirements. □

The hypothesis that every μ-stable reflexive sheaf with $c_1 = c_2 = 0$ is a line bundle is valid whenever the fundamental group is abelian, this is true by Corlette-Simpson correspondence [25, Corollary 1.3]. Examples of such varieties are Fano varieties, rational surfaces, abelian varieties, K3 surfaces, products of the previous ones, quotients of simply connected varieties by finite abelian groups (e.g., Enriques surfaces).

The following theorem is essentially the same Theorem 2.2 for surfaces with stronger assumptions, where we make the proof by hand, without relying on Simpson's results.

Theorem 2.3. Let X be a smooth projective surface with $K_X \cdot H \leqslant 0$ and satisfying $\chi(\mathcal{O}_X) = 1$. If E is a semistable reflexive sheaf (with respect to H) with $c_1 = c_2 = 0$, then its Jordan-Holder filtration has factors in $\mathrm{Pic}^0(X)$.

Proof. Let E be a semistable reflexive sheaf with $c_1 = c_2 = 0$ and let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = E,$$

be a Jordan–Holder filtration of E. We need to show that every quotient F_l/F_{l-1} is in $\mathrm{Pic}^0(X)$. We will prove that every stable sheaf F with $c_1 = c_2 = 0$ on X is a line bundle, so the statement will follow.

For a stable rank r sheaf F on a smooth projective surface Hirzebruch-Riemann-Roch formula [12, p. 103] gives us

$$\chi(\mathcal{O}_X) - \chi(F,F) = \Delta(F) - (r^2-1)\cdot\chi(\mathcal{O}_X), \tag{2.1}$$

where $\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2$ and $\chi(F,F) = \sum_{i=0}^{2}(-1)^i \operatorname{ext}^i(F,F)$.

Let us assume first that $K_X \cdot H < 0$. So in this case, if F is stable, then $\hom(F,F) = 1$ and $\operatorname{ext}^2(F,F) = \hom(F, F\otimes\omega_X)$ which is 0 because $K_X \cdot H < 0$ so $p_F > p_{F\otimes\omega_X}$ [12, Proposition 1.2.7]. Since $c_1 = c_2 = 0$ implies that $\Delta(F) = 0$, equation (2.1) gives us

$$\begin{aligned}
&\chi(F,F) = \chi(\mathcal{O}_X)r^2 - \Delta(F)\\
\Rightarrow\ & r^2 = \chi(F,F)\\
\Rightarrow\ & r^2 = 1 - \operatorname{ext}^1(F,F)\\
\Rightarrow\ & r^2 \leqslant 1.
\end{aligned}$$

Therefore the rank of a stable sheaf F must be equal to 1. In this case, since F is torsion free, we have the canonical monomorphism $F \hookrightarrow F^{\vee\vee}$, which gives us the following short exact sequence

$$0 \to F \to F^{\vee\vee} \to Q \to 0 \tag{2.2}$$

with $\dim Q = 0$. It follows that $F^{\vee\vee}$ is a rank 1 reflexive sheaf, thus a line bundle. By sequence (2.2), $c_1(F^{\vee\vee}) = 0$. Let L^{-1} be the inverse of $F^{\vee\vee}$ and twist sequence (2.2) to obtain

$$0 \to F\otimes L^{-1} \to \mathcal{O}_X \to Q \to 0.$$

So $F\otimes L^{-1}$ is some sheaf o ideals I_Z with $\dim Z = 0$. Therefore $F = I_Z \otimes L$ with $L \in \mathrm{Pic}^0(X)$. But $c_2(I_Z(L)) = \mathrm{length}(Z) = 0$, that is, Z is empty which implies that F is a line bundle as desired.

If $K_X \cdot H = 0$, then we still have $\hom(F,F) = 1$, but now $\operatorname{ext}^2(F,F) = \hom(F, F\otimes\omega_X) \leqslant 1$. If $\hom(F, F\otimes\omega_X) = 0$ we are in the previous case. If $\hom(F, F\otimes\omega_X) = 1$, then equation (2.1) gives us

$$r^2 \leqslant \frac{2}{\chi(\mathcal{O}_X)} = 2.$$

Then $r = 1$ and the proof goes exactly like the above case again. □

2.2 Quot scheme and extensions of ideals

Let us now turn our attention to general μ-semistable sheaves (not necessarily reflexive) on $\mathbb{P}^d$, starting by understanding how they are related to 3-codimensional quotients of the trivial sheaf.

Let (φ, Q) be an element of the Quot scheme $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ with $u \in \mathbb{Q}[t]$ such that $\deg(u) \leqslant d-3$, that is, $\varphi\colon \mathcal{O}_{\mathbb{P}^d}^{\oplus r} \to Q$ is an epimorphism onto a sheaf Q with Hilbert polynomial $P_Q(t) = u(t)$. Now let $E := \ker\varphi$. In that case, we have a short exact sequence

$$0 \to E \to \mathcal{O}_{\mathbb{P}^d}^{\oplus r} \to Q \to 0. \tag{2.3}$$

As E is a subsheaf of a locally free sheaf, E is a torsion free sheaf of rank r.

Now let us calculate the Hilbert polynomial and the Chern classes of E. By sequence (2.3) and additivity of the Hilbert polynomial, we have that

$$\begin{aligned} P_E(t) &= P_{\mathcal{O}_{\mathbb{P}^d}^{\oplus r}}(t) - P_Q(t) \\ &= r \cdot \binom{t+d}{d} - u(t). \end{aligned}$$

Note that the first and second Chern classes of Q are zero as well as the first and second Chern classes of $\mathcal{O}_{\mathbb{P}^d}^{\oplus r}$, so it follows, from the multiplicative property of Chern classes, that the first and second Chern classes of E are also equal to zero.

Proposition 2.4. Given $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ with $r > 1$ and $u(t) \in \mathbb{Q}[t]$ such that $\deg(u) \leqslant d-3$, then the sheaf $E := \ker\varphi$ is strictly μ-semistable.

Proof. Applying $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^d})$ to sequence (2.3) we have

$$0 \to Q^\vee \to (\mathcal{O}_{\mathbb{P}^d}^{\oplus r})^\vee \to E^\vee \to \mathcal{E}xt^1(Q, \mathcal{O}_{\mathbb{P}^d}).$$

Since $\mathrm{codim}(Q) \geqslant d-3$, we have that $Q^\vee = 0 = \mathcal{E}xt^1(Q, \mathcal{O}_{\mathbb{P}^d})$ by [12, Proposition 1.1.6], so that $E^\vee \cong (\mathcal{O}_{\mathbb{P}^d}^{\oplus r})^\vee \cong \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$. Therefore E is μ-semistable by Proposition 1.9. □

Next, we establish a converse to the previous claim.

Proposition 2.5. Let E be a μ-semistable sheaf of rank r on $\mathbb{P}^d$ with $c_1(E) = c_2(E) = 0$. Then there is $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ with $\deg(u) \leqslant d-3$ such that $E \cong \ker\varphi$.

Proof. Since E is a torsion free sheaf, we have a canonical monomorphism $E \hookrightarrow E^{\vee\vee}$; let Q_E denote its cokernel. Then we have a short exact sequence

$$0 \to E \to E^{\vee\vee} \to Q_E \to 0. \tag{2.4}$$

We claim that $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$. First, since E is torsion free, we have $\dim Q_E \leqslant d-2$ by Proposition 1.6. It follows that we can write the Hilbert polynomial of Q_E as $P_{Q_E}(t) = at^{d-2} + u(t)$ for some $u \in \mathbb{Q}[t]$ with $\deg(u) \leqslant d-3$. Hence, by sequence (2.4),

$$\begin{aligned} P_{E^{\vee\vee}}(t) &= P_E(t) + P_{Q_E}(t) \\ &= r \cdot \binom{t+d}{d} - n + at^{d-2} + u(t). \end{aligned}$$

In this case we have $c_1(E^{\vee\vee}) = c_1(E) = 0$ and $c_2(E^{\vee\vee}) = -a$; by hypothesis, $E^{\vee\vee}$ is a μ-semistable reflexive sheaf of rank r.

Therefore, by the Bogomolov inequality, we have that $c_1(E^{\vee\vee})^2 - 4c_2(E^{\vee\vee}) \leqslant 0$. Thus $c_2(E^{\vee\vee}) \geqslant 0$ which implies that $a \leqslant 0$. But a, being the leading coefficient of the Hilbert polynomial of Q_E, must be greater or equal to 0. Therefore $a = 0$.

We conclude that $E^{\vee\vee}$ is a μ-semistable reflexive sheaf of rank r in $\mathbb{P}^3$ with $c_1(E^{\vee\vee}) = c_2(E^{\vee\vee}) = 0$. Therefore, by Lemma 2.1, $E^{\vee\vee} \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ and $P_{Q_E}(t) = u(t)$ as we desired. □

Remark 2.6. Following the proof of Proposition 2.5, we can see that if E is a μ-semistable sheaf of rank r on $\mathbb{P}^3$ with Chern classes $c_1(E) = c_2(E) = 0$, then $c_3(E) \leqslant 0$. In other words, there are no μ-semistable sheaves with Chern classes $c_1 = c_2 = 0$ and $c_3 > 0$ on $\mathbb{P}^3$.

Clearly, the ideal sheaf I_Z of a subscheme $Z \subset \mathbb{P}^d$ of codimension at least 3 is a μ-stable sheaf with Chern classes $c_1 = c_2 = 0$. However, such μ-stable sheaves do not occur in higher rank.

Corollary 2.7. There are no μ-stable sheaves E of rank $r \geqslant 2$ on $\mathbb{P}^d$ with Chern classes $c_1(E) = c_2(E) = 0$.

Proof. If E a μ-stable torsion free sheaf of rank $r \geqslant 2$ with Chern classes $c_1(E) = c_2(E) = 0$ and $c_d(E) = -2n$, then by Proposition 2.5, we have that $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$. On the other hand, Proposition 1.9 implies that $E^{\vee\vee}$ is μ-stable, giving a contradiction. □

Since a μ-semistable sheaf of rank $r \geqslant 2$ with Chern classes $c_1 = c_2 = 0$ is not μ-stable, it is natural to consider its μ-Jordan–Hölder filtration, namely a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k = E \tag{2.5}$$

such that the factors $G_i := F_i/F_{i-1}$ of the filtration are μ-stable sheaves with $\mu(G_i) = 0$.

Theorem 2.8. If E is a μ-semistable sheaf on $\mathbb{P}^d$ with $\mathrm{rk}(E) \geqslant 1$ $c_1(E) = c_2(E) = 0$, then E is an extension of ideal sheaves of subschemes of $\mathbb{P}^d$ of codimension at least 3.

Note that this statement can be regarded as a generalization of [25, Theorem 2] for torsion free sheaves on $\mathbb{P}^d$.

Proof. We show that each factor of the Jordan–Hölder filtration of E is the ideal sheaf of a subscheme in $\mathbb{P}^d$ of codimension at least 3. We argue by induction on the length of the filtration, denoted by k in display (2.5).

If $k = 1$ then E is actually μ-stable, and Corollary 2.7 implies that $\mathrm{rk}(E) = 1$, so in fact E is the ideal sheaf of a 3-codimensional subscheme of $\mathbb{P}^d$.

Assume that $k > 1$, and consider the epimorphism $\eta_k : E \twoheadrightarrow G_k$, so that $F_{k-1} = \ker \eta_k$. We then obtain a diagram of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{k-1} & \longrightarrow & F_{k-1}^{\vee\vee} & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & E & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{\oplus r} & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \\
 & & \downarrow \eta_k & & \downarrow \eta_k^{\vee\vee} & & \downarrow & & \\
0 & \longrightarrow & G_k & \longrightarrow & G_k^{\vee\vee} & \longrightarrow & Q'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that $G_k^{\vee\vee}$ is a μ-stable reflexive sheaf with $c_1(G_k^{\vee\vee}) = 0$. In addition, the fact that $G_k^{\vee\vee}$ is the quotient of the semistable sheaf $\mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ implies that $c_2(G_k^{\vee\vee}) = 0$, so the proof of Lemma 2.1 implies that $G_k^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}$; since $\mathrm{codim}(Q'') = 3$, we get that G_k is the ideal sheaf of a 3-codimensional scheme.

It also follows that F_{k-1} is a μ-semistable sheaf of rank $r - 1$ and Chern classes $c_1(F_{k-1}) = c_2(F_{k-1}) = 0$. By induction hypothesis, the factors of the Jordan–Hölder filtration of F_{k-1}, which coincide with G_i for $i < k$ (the factors of the Jordan–Hölder filtration of E), are ideal sheaves of a 3-codimensional subschemes of $\mathbb{P}^3$. □

In particular, as a consequence of the previous Theorem, if E is a μ-semistable rank 2 torsion free sheaf with $c_1(E) = c_2(E) = 0$, then we can write E as extension in the following way:

$$0 \to I_Z \to E \to I_{Z'} \to 0 \tag{2.6}$$

where $Z, Z' \subset \mathbb{P}^d$ are subschemes of codimension at least 3.

More generally, we can use the Jordan–Hölder filtration for semistability to obtain the following statement.

Lemma 2.9. Let E be a semistable sheaf on with $c_1(E) = c_2(E) = 0$ on $\mathbb{P}^d$. There are $(\psi, Q_F) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus s}, v(t))$ and $(\psi', Q_G) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus(r-s)}, P_E - v)$ for some polynomial $v \in \mathbb{Q}[t]$ with $\deg(v) \leqslant d - 3$ such that E can be written as an extension

$$0 \to F \to E \to G \to 0,$$

with $F = \ker \psi$ being semistable and $G = \ker \psi'$ being stable.

Proof. Since E is semistable, by [12, Proposition 1.5.2], we have a Jordan-Holder filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E,$$

that is, each E_i is semistable with reduced Hilbert polynomial p_E and each quotient E_i/E_{i-1} is stable also with reduced Hilbert polynomial p_E, for $i > 0$.

Now we can take $F = E_{l-1}$, so $G = E/E_{l-1}$. By the above, F is semistable and G is stable. Finally, note that the Hilbert polynomial of G is $P_G(t) = P_E(t) \cdot \dfrac{\mathrm{rk}(G)}{\mathrm{rk}(E)}$, thus G must be of the desired form. Similarly, the same thing must happen to F as desired. $\square$

2.3 Moduli spaces of semistable sheaves on $\mathbb{P}^d$ with $c_1 = c_2 = 0$

Consider first the following action of $\mathrm{GL}_r \simeq \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r})$ on $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$:

$$g \cdot (\varphi, Q) := (\varphi \circ g, Q). \tag{2.7}$$

Note that $g \cdot (\varphi, Q)$ is clearly in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$ again. Propositions 2.4 and 2.5 lead to the following theorem characterizing the set of isomorphism classes of μ-semistable sheaves.

Theorem 2.10. There is a bijection between the set of isomorphism classes of μ-semistable sheaves E of rank r with Chern classes $c_1(E) = c_2(E) = 0$ on $\mathbb{P}^d$, and the set of orbits $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)/\mathrm{GL}_r$ for $u \in \mathbb{Q}[t]$ given by $u(t) := r \cdot \binom{t+d}{d} - P_E(t)$

Proof. Let (φ, Q), (φ', Q') be points in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ lying in the same GL_r-orbit, that is, there is some $g \in \mathrm{GL}_r$ such that $(\varphi, Q) = g \cdot (\varphi', Q') = (\varphi' \circ g, Q')$. Hence, there is an isomorphism $f : Q \to Q'$ such that $f \circ \varphi = \varphi' \circ g$. Let $E = \ker \varphi$ and $E' = \ker \varphi'$ be the corresponding μ-semistable sheaves given by Proposition 2.4. Thus we can consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \mathcal{O}_{\mathbb{P}^d}^{\oplus r} & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & E' & \longrightarrow & \mathcal{O}_{\mathbb{P}^d}^{\oplus r} & \xrightarrow[\varphi']{} & Q' & \longrightarrow & 0. \end{array} \tag{2.8}$$

Since g and f are isomorphisms, the snake lemma implies that $h : E \to E'$ is also an isomorphism as desired.

Now let E and E' two isomorphic μ-semistable sheaves on $\mathbb{P}^d$ with Chern classes $c_1 = c_2 = 0$ and let $h : E \to E'$ be an isomorphism. By Proposition 2.5, let (φ, Q) and (φ', Q') the corresponding elements in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, u)$. Since $E^{\vee\vee} \simeq (E')^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$, we obtain an induced morphism $h^{\vee\vee} : \mathcal{O}_{\mathbb{P}^d}^{\oplus r} \to \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$ which is an isomorphism, since h is. We then construct a commutative diagram like the one in display (2.8), with $g = h^{\vee\vee}$. In particular, we get an isomorphism $f : Q \to Q'$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^d}^{\oplus r} & \xrightarrow{\varphi} & Q \\ & \searrow^{\varphi\circ g} & \downarrow f \\ & & Q', \end{array}$$

that is, $(\varphi, Q) = (\varphi' \circ g, Q')$ in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$. Therefore $(\varphi, Q) = g \cdot (\varphi', Q')$. □

Proposition 2.5 implies that every semistable sheaf E on $\mathbb{P}^d$ with $c_1 = c_2 = 0$ can be realized as $\ker \varphi$ for some $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ for $u \in \mathbb{Q}[t]$ with $\deg(u) \leqslant d-3$. However, the converse is not true; Theorem 2.11 below provides a characterization of those $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, u)$ for which $\ker \varphi$ is semistable.

Theorem 2.11. Let (φ, Q) in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ for $u \in \mathbb{Q}[t]$ with $\deg(u) \leqslant d-3$, and let $E = \ker \varphi$. Then E is not *(semi)stable* if, and only if, there is a torsion free sheaf $F \hookrightarrow E$ with $F = \ker \psi$ for some (ψ, Q_F) in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus s}, v)$ satisfying $0 < s < r$ and

$$v < \quad (\leqslant) \frac{s \cdot u}{r}.$$

Proof. Let us analyse torsion free subsheaves $F \hookrightarrow E$ with $\mathrm{rk}\, F = s$ and $0 < s < \mathrm{rk}\, E$ such that the quotient $E/F = G$ is torsion free. Since F and G are torsion free we can form a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F & \to & F^{\vee\vee} & \to & Q_F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E & \to & \mathcal{O}_{\mathbb{P}^d}^{\oplus r} & \to & Q & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G & \to & G^{\vee\vee} & \to & Q_G & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & . \end{array}$$

By Proposition 2.4, E is μ-semistable, thus $c_1(F) \leqslant c_1(E) = 0$. Now we begin examining the reduced Hilbert polynomials of E and F. For a polynomial $p(t)$, we write $[p]_i$ for the coefficient of the t^i term. Note that $[p_E]_d = [p_F]_d$. By the Riemann–Roch Theorem, we can write $[P_F]_{d-1} = \frac{1}{2}c_1(F) + s$ and $[P_E]_{d-1} = r$. It follows that

$$[p_F]_{d-1} = \frac{c_1(F)}{2 \cdot s} + 1 \text{ and } [p_E]_{d-1} = 1.$$

If $c_1(F) < 0$, then F does not destabilize E, so we can assume $c_1(F) = 0$.

Note that Q_F being a subsheaf of Q, has codimension 3, thus $c_1(Q_F) = c_2(Q_F) = 0$. By the multiplicative property of Chern classes on short exact sequences, we have $c_1(F) = c_1(F^{\vee\vee})$ and $c_2(F) = c_2(F^{\vee\vee})$. Again by the Riemann–Roch Theorem, one can check that $[P_{F^{\vee\vee}}]_{d-2} = -2c_2(F) + \frac{11}{6}s$ and $[P_{\mathcal{O}_{\mathbb{P}^3}^{\oplus r}}]_{d-2} = \frac{11}{6}r$. Thus,

$$[p_{F^{\vee\vee}}]_{d-2} = -\frac{2c_2(F)}{s} + \frac{11}{6} \text{ and } [p_{\mathcal{O}_{\mathbb{P}^3}^{\oplus r}}]_{d-2} = \frac{11}{6}.$$

Now, if $c_2(F) > 0$, then F does not destabilize E. If $c_2(F) < 0$, then we would have $p_{F^{\vee\vee}} > p_{\mathcal{O}_{\mathbb{P}^3}^{\oplus r}}$ so $\mathrm{Hom}(F^{\vee\vee}, \mathcal{O}_{\mathbb{P}^3}^{\oplus r}) = 0$ by [12, Proposition 1.2.7], and this cannot happen. Therefore we can assume $c_2(F) = 0$ and, in this case, by Lemma 2.1, $F^{\vee\vee} \cong \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$. That is, $F = \ker \psi$, for ψ in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus s}, v)$, where v is a polynomial with degree $\leqslant d - 3$.

Finally, we have $p_E - p_F = \frac{v}{s} - \frac{u}{r}$. It follows that, $p_E < \ (\leqslant)\ p_F$ if, and only if,

$$v < \ (\leqslant)\ \frac{u \cdot s}{r}.$$

□

Let (φ, Q) be an element in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ and let $E = \ker \varphi$. We can also relate the (semi)stability of E given in the above theorem with GIT-stability applied to the GL_r-action on $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ given by (2.7). To do this, we use the results in [12, Lemma 4.4.5]. Essentially, let V be a finite dimensional $\mathbb{C}$-vector space; a closed point (φ, Q) in $\mathrm{Quot}(V \otimes \mathcal{O}_{\mathbb{P}^3}, u)$ is GIT-(semi)stable if, and only if, for every non-trivial proper linear subspace $V' \subset V$ and the induced subsheaf $Q' := \varphi(V' \otimes \mathcal{O}_{\mathbb{P}^3}) \subset Q$, the following inequality holds:

$$P_{Q'} > \ (\geqslant)\ \frac{\dim(V') \cdot P_Q}{\dim(V)}. \tag{2.9}$$

Theorem 2.12. Let (φ, Q) be an element in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ and let $E = \ker \varphi$. Then (φ, Q) is GIT-(semi)stable with respect to the GL_r action in display (2.7) if, and only if, E is (semi)stable.

Proof. Note that the existence of a subspace $V' \subset V$ satisfying (2.9), means that $\ker(V' \otimes \mathcal{O}_{\mathbb{P}^d} \to \varphi(V' \otimes \mathcal{O}_{\mathbb{P}^d}))$ is a torsion free sheaf F and $F \hookrightarrow E$. In this case we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & V' \otimes \mathcal{O}_{\mathbb{P}^d} & \longrightarrow & \varphi(V' \otimes \mathcal{O}_{\mathbb{P}^d}) =: Q_F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & V \otimes \mathcal{O}_{\mathbb{P}^d} & \xrightarrow{\varphi} & Q & \longrightarrow & 0. \end{array} \tag{2.10}$$

So (2.9) translates to

$$P_{Q_F} < (\leqslant) \frac{\dim V' \cdot P_Q}{\dim V} = \frac{\operatorname{rk} F \cdot P_Q}{\operatorname{rk} E},$$

which is the inequality given in Theorem 2.11.

On the other hand, if F is a torsion free sheaf $F = \ker \psi$ with $F \hookrightarrow E$ as in Theorem 2.11, we can choose V' to agree with the composition $F \hookrightarrow E \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}^d}$ and obtain the diagram (2.10) again such that the inequality (2.9) is satisfied.

Therefore, (φ, Q) is GIT-(semi)stable if, and only if, $E = \ker \varphi$ is (semi)stable. □

As an immediate consequence, we can write the moduli space $\mathcal{M}_d^P$ of semistable sheaves on $\mathbb{P}^d$ with Hilbert polynomial given by $P(t) = r \cdot \binom{t+d}{d} - u(t)$ for a given polynomial $u \in \mathbb{Q}[t]$ with $\deg(u) \leqslant d-3$ as a GIT-quotient.

Corollary 2.13. $\mathcal{M}_d^P$ is the GIT-quotient of $\operatorname{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ by GL_r.

Remark 2.14. Cazzaniga and Ricolfi recently provided in [4] a characterization of the affine Quot scheme $\operatorname{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$ as a moduli space of framed torsion free sheaves on $\mathbb{P}^d$ with Chern character $(r, 0, \ldots, 0, -n)$. We observe that this moduli space is different from the moduli spaces $\mathcal{M}_d^P$ described above, when $u(t)$ is a constant polynomial, in two ways: first, we do not considers framings; second, a framed torsion free sheaf need not be semistable.

Corollary 2.13 leads us to believe that

$$\dim \mathcal{M}_d^P = \dim \operatorname{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u) - r^2 + 1.$$

Indeed, let take $(Q, \varphi) \in \operatorname{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ such that $E := \ker \varphi$ is stable. Applying the functor $\operatorname{Hom}(E, \cdot)$ to the exact sequence (2.3), we obtain

$$0 \to \operatorname{Hom}(E, E) \to \operatorname{Hom}(E, \mathcal{O}_{\mathbb{P}^d}^{\oplus r}) \to \operatorname{Hom}(E, Q) \to \operatorname{Ext}^1(E, E) \to 0$$

since

$$\operatorname{Ext}^1(E, \mathcal{O}_{\mathbb{P}^d}^{\oplus r}) \simeq H^{n-1}(E(-n-1))^{\oplus r} = 0$$

since $H^{n-2}(Q) = H^{n-1}(Q) = 0$ because $\operatorname{codim} Q \geqslant 3$. Moreover, we have that

$$\operatorname{Hom}(E, \mathcal{O}_{\mathbb{P}^d}^{\oplus r}) = H^0(E^\vee)^{\oplus r};$$

since $E^\vee \simeq \mathcal{O}_{\mathbb{P}^d}^{\oplus r}$, we get that $\hom(E, \mathcal{O}_{\mathbb{P}^d}^{\oplus r}) = r^2$. In addition, the stability of E guarantees that $\hom(E, E) = 1$, thus

$$\begin{aligned} \dim T_E \mathcal{M}_d^P &= \mathrm{ext}^1(E, E) = \hom(E, Q) - r^2 + 1 \\ &= \dim T_{(\varphi, Q)} \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u) - r^2 + 1, \end{aligned}$$

where $T_E \mathcal{M}_d^P$ and $T_{(\varphi,Q)} \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$ denote the Zariski tangent spaces of $\mathcal{M}_d^P$ and $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, u)$, respectively.

3 Moduli spaces of semistable quasitrivial sheaves on $\mathbb{P}^3$

We are finally ready to focus on the main character of this paper, namely the Gieseker–Maruyama moduli space

$$\mathcal{N}(r,n) := \mathcal{M}(r,0,0,-2n) \quad \text{with} \quad r,n \geqslant 1$$

of semistable quasitrivial sheaves of rank r on $\mathbb{P}^3$ and Chern classes $c_1(E) = c_2(E) = 0$, and $c_3(E)) = -2n$; note that the Hilbert polynomial of such a sheaf is given by $P_E(t) = r \cdot \binom{t+3}{3} - n$, while Remark 2.6 implies that it is enough to consider $n > 0$. Let $\mathcal{N}^{\mathrm{st}}(r,n)$ denote the (open, possibly empty) subset consisting of stable sheaves.

In this chapter we will show that $\mathcal{N}(r,n)$ is irreducible for $n \leqslant 10$, and we will construct an irreducible component of $\mathcal{N}(r,n)$. First, we will construct an irreducible component for $\mathcal{N}(2,n)$, and later this will serve as an induction step to prove the general result.

The first observation is that $\mathcal{N}(1,n)$ coincides with the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}^3$ with length n. Therefore we will focus on $r \geqslant 2$, and our initial task is to check whether $\mathcal{N}(r,n)$ is non-empty. To do this, we begin by translating how the results established in the previous sections translate to this particular case.

Every sheaf $E \in \mathcal{N}(r,n)$ is also μ-semistable, therefore Proposition 2.5 implies that E is $\ker\varphi$ for some $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ for $n \in \mathbb{N}$. For the converse, Proposition 2.4 says that every $\ker\varphi$ is μ-semistable for $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$. Lemma 2.9 allows us to express E as an extension

$$0 \to F \to E \to G \to 0,$$

with $F \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus s}, k)$ semistable and $G \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r-s}, n-k)$ stable, for some $k \in \{1,\ldots,n-1\}$ and $s \in \{1,\ldots,r-1\}$. Moreover, when $r = 2$ this implies that E is an extension of sheaf of ideals

$$0 \to I_Z \to E \to I_{Z'} \to 0$$

where Z and Z' are 0-dimensional subschemes of $\mathbb{P}^3$ with $h^0(\mathcal{O}_Z) + h^0(\mathcal{O}_{Z'}) = n$.

The criterion obtained in Theorem 2.11 applied to our case translates into the following claim: $E = \ker\varphi$ for $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ is (semi)stable if, and only if, there is a torsion free sheaf $F \hookrightarrow E$ with $F = \ker\psi$ for some (ψ, Q_F) in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus s}, k)$ satisfying $0 < s < r$ and

$$k < \;\; (\leqslant) \;\; \frac{s \cdot n}{r}.$$

For the rank 2 case, this criterion reduces to check whether there is a sheaf of ideals $I_Z \hookrightarrow E$ satisfying

$$k < \quad (\leqslant) \ \frac{n}{2}$$

where $k = h^0(\mathcal{O}_Z)$.

Finally, Corollary 2.13 yields that $\mathcal{N}(r,n)$ is the GIT–quotient of $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ by GL_r.

Proposition 3.1. $\mathcal{N}(r,n) = \varnothing$ for $r > n$.

Proof. Let E be a semistable sheaf with $c_1(E) = c_2(E) = 0$. By Proposition 2.5, we can write E as $\ker\varphi$ for (φ, Q) in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$. We must have $H^0(E) = 0$, since otherwise we would have a section $\mathcal{O}_{\mathbb{P}^3} \hookrightarrow E$, and then $p_{\mathcal{O}_{\mathbb{P}^3}} > p_E$, destabilizing E, but we are assuming E to be semistable. Now taking cohomologies on the exact sequence $0 \to E \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \to Q \to 0$ we have an injective map

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}) \to H^0(Q)$$

which forces $r \leqslant h^0(Q) = n$. Therefore, $\mathcal{N}(r,n) = \varnothing$ for $r > n$. □

The next lemma will help us treat the case where $r = n > 1$ to study $\mathcal{N}(n,n)$.

Lemma 3.2. Let E be a semistable rank n torsion free sheaf with $c_1(E) = c_2(E) = 0$ and $c_3(E) = -2n$. Then E is stable if, and only if, $n = 1$.

Proof. If $n = 1$, then $E = I_p$ for some $p \in \mathbb{P}^3$, therefore stable.

Now assume $n > 1$ and E stable. By Lemma 2.9, we can write E as extension of G and F with F semistable and G stable. Moreover F is the kernel of some map $\mathcal{O}_{\mathbb{P}^3}^{\oplus s} \to Q_F \to 0$ with $h^0(Q_F) = k$ and G is the kernel of $\mathcal{O}_{\mathbb{P}^3}^{\oplus (r-s)} \to Q_F \to 0$ with $h^0(Q_G) = n - k$. Since E is stable, by Theorem 2.11, we must have

$$k > \frac{n \cdot s}{r} = s.$$

In this case, $n - k < r - s$ and, by Proposition 3.1, $\mathcal{N}(r-s, n-k) = \varnothing$. That is, G cannot be stable, providing a contradiction. □

Proposition 3.3. If $n > 1$, then $\mathcal{N}^{\mathrm{st}}(n,n) = \varnothing$, and $\mathcal{N}(n,n) = \mathrm{Sym}^n(\mathbb{P}^3)$.

Proof. First part of the statement is given by Lemma 3.2. To do the last part we will show that every $E \in \mathcal{N}(n,n)$ is S-equivalent to a sum of ideals $\oplus_{i=1}^n I_{p_i}$.

By Lemma 2.9, we can write E as extension of $F = \ker\psi$ and $G = \ker\psi'$ for $(\psi, Q_F) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus s}, k)$ and $(\psi', Q_G) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus (r-s)}, n-k)$. Since E is semistable, by Theorem 2.11, we must have $k \geqslant \dfrac{s \cdot n}{r} = s$. But if $k > s$, then $n-k < r-s$ and this cannot happen, because G is stable. It follows that $s = k$, which implies that $r - s = n - k$ as well.

That is, if $E \in \mathcal{N}(n,n)$, then we can write E as extension of F and G with $F \in \mathcal{N}(k,k)$ and $G \in \mathcal{N}^{st}(n-k,n-k)$. Moreover, by Lemma 3.2, we must have $F \in \mathcal{N}(n-1,n-1)$ and $G = I_p$ for some $p \in \mathbb{P}^3$.

By the argument above, write $0 \to F_1 \to E \to I_{p_1} \to 0$ with $F_1 \in \mathcal{N}(n-1,n-1)$. Now note that we can do the same with F_1 and write $0 \to F_2 \to F_1 \to I_{p_2} \to 0$ for $F_2 \in \mathcal{N}(n-2,n-2)$, and so on. Going through this induction process, we find a filtration

$$0 = F_n \subset F_{n-1} \subset \cdots \subset F_2 \subset F_1 \subset F_0 = E, \tag{3.1}$$

where each factor F_i/F_{i+1} is the ideal sheaf of a point $I_{p_{i+1}}$, for $i \in \{0,\ldots,n-1\}$. Since each factor of this filtration is stable and with the same reduced Hilbert polynomial, we conclude that (3.1) is a Jordan–Holder filtration for E and $gr(E) = \oplus_{i=1}^n I_{p_i}$. Therefore E is S-equivalent to $\oplus_{i=1}^n I_{p_i}$. Note that the points p_i need not be distinct, and the graded object associated to the Jordan–Holder filtration is only unique up to ordering; we therefore obtain the desired ismorphism with the symmetric product $\mathrm{Sym}^n(\mathbb{P}^3)$. □

We will later show that $\mathcal{N}^{\mathrm{st}}(r,n) \neq \varnothing$ when $r < n$, see Chapter 3 below. For now, we shift our attention to the irreducibility of $\mathcal{N}(r,n)$.

Our starting point is the fact that the affine Quot scheme $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^3}^{\oplus r}, n)$ is irreducible for $n \leqslant 10$, see [9]. In order to see that the same holds for the projective Quot scheme $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$, we will use the following technical lemma.

Lemma 3.4. Let X be the union of two irreducible subschemes A and B, and let C be the intersection of A and B. Assume C is open in A and B. Then X is irreducible

Proof. The closure of A in X is the same as the closure of C in X. Indeed, a function vanishing on C vanishes on A because C is open in A. Also, the closure of B in X is the same as the closure of C in X. Since $X = A \cup B$, we conclude that X is the closure of C in X. But C is an open subset of an irreducible scheme A; thus X is irreducible. □

As a simple consequence, we have:

Lemma 3.5. If $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$ is irreducible, then $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$ is irreducible.

Proof. Fix coordinates $[x_0 : x_1 : \ldots : x_m]$ for $\mathbb{P}^d$ and let $H_i = \{x_i = 0\}$ so $A_i := \mathbb{P}^d \backslash H_i \cong \mathbb{A}^d$. For $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$, if $\mathrm{Supp}(Q) \cap H_i = \varnothing$, then we can restrict φ to A_i and we obtain an element of $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$. Since we are assuming $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$ to be irreducible, we can write $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$ as union of irreducible subschemes such that the two-by-two intersection is open inside those two. Therefore, by induction and the previous lemma, we get the statement. □

In particular, we conclude that $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ is irreducible for $n \leqslant 10$. Theorem 2.11 implies that $\mathcal{N}(r,n)$ is the GIT-quotient of $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ by GL_r. Thus, whenever $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ is irreducible, $\mathcal{N}(r,n)$ is also irreducible. We have therefore established the following claim.

Corollary 3.6. $\mathcal{N}(r,n)$ is irreducible for $n \leqslant 10$.

We will construct an explicit irreducible component of $\mathcal{N}(r,n)$ in Theorem 3.17 (case $r = 2$) and in Theorem 3.23 (case $r \geqslant 2$) below. When $n \leqslant 10$, Corollary 3.6 tells us that this is the only component, and provides an explicitly description of $\mathcal{N}(r,n)$.

Remark 3.7. Let $\mathcal{C}(d,n)$ the variety consisting of d-tuples of $n \times n$ commuting matrices. In [14], the authors classified the components of $\mathcal{C}(d,n)$ for $n \leqslant 7$, which gives a classification of the components of $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$ for $n \leqslant 7$, and in [9, Proposition 6.1] the authors proved that the number of irreducible components of the affine quotient scheme $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^3}^{\oplus r}, n)$ is always smaller than or equal to the number of irreducible components of $\mathcal{C}(3,n)$ (regardless of the value of r). Therefore, we can conclude that $\mathcal{N}(r,n)$ is irreducible whenever $\mathcal{C}(3,n)$ is.

Determining whether $\mathcal{C}(3,n)$ is irreducible is an interesting open problem. Currently, $\mathcal{C}(n)$ is known to be irreducible for $n \leqslant 10$ but reducible for $n \geqslant 29$, see [6, 10, 23, 22, 26, 27]. Although not yet published, it is also claimed that $\mathcal{C}(3,n)$ is irreducible for $n = 11$, see [10, p. 27].

3.1 Irreducible component of $\mathcal{N}(2,n)$

The main goal of this section is to show that $\mathcal{N}^{\mathrm{st}}(2,n)$ is nonempty and construct an irreducible component of $\mathcal{N}(2,n)$ for each $n \geqslant 3$. First, let us analyze the case of $\mathcal{N}(2,3)$ which will motivate the notion of unbalanced sheaves later in this chapter. Next lemma ensures stability when E is an extension in the most "balanced" way.

Lemma 3.8. Let n be a positive odd integer and let $E \in \mathrm{Ext}^1(I_{Z'}, I_Z)$ where Z and Z' are 0-dimensional subschemes of $\mathbb{P}^3$ of length $\dfrac{n+1}{2}$ and $\dfrac{n-1}{2}$, respectively. Then E is stable if, and only if, E is not the trivial extension $I_Z \oplus I_{Z'}$.

Proof. Clearly, if $E = I_Z \oplus I_{Z'}$, then E is not stable. Thus we get the "if" part of the lemma. Now assume E is not stable, that is, there is an injective map $I_Y \hookrightarrow E$ with $h^0(\mathcal{O}_Y) \leqslant \dfrac{n-1}{2}$. Let us consider two separate cases: when $h^0(\mathcal{O}_Y) < \dfrac{n-1}{2}$ and when $h^0(\mathcal{O}_Y) = \dfrac{n-1}{2}$.

In the first case, since $h^0(I_{Z'}) = \dfrac{n-1}{2}$ and $h^0(I_Y) < \dfrac{n-1}{2}$, it follows that $\mathrm{Hom}(I_Y, I_{Z'}) = 0$. Thus, as we did in the theorem above, we can consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & I_Y & \xrightarrow{\text{id}} & I_Y & & & & \\ & & \downarrow & & \downarrow & \searrow^{0} & & & \\ 0 & \longrightarrow & I_Z & \longrightarrow & E & \longrightarrow & I_{Z'} & \longrightarrow & 0. \end{array}$$

Again, as $p_{I_Y} > p_{I_Z}$, $\mathrm{Hom}(I_Y, I_Z) = 0$ and then the map $I_Y \to E$ is zero. A contradiction. Therefore, first case does not happen.

Now assume $h^0(\mathcal{O}_Y) = \dfrac{n-1}{2} = h^0(\mathcal{O}_{Z'})$. In this case, $p_{I_Y} = p_{I_{Z'}}$ and then the composition $I_Y \to E \to I_{Z'}$ must be an isomorphism. Hence the sequence

$$0 \to I_Z \to E \to I_{Z'} \to 0$$

splits, and $E = I_Z \oplus I_{Z'}$ like we wanted. □

The previous lemma tell us that if n is odd and E is a nonsplit extension

$$0 \to I_Z \to E \to I_{Z'} \to 0,$$

in which we are in the most balanced case, that is, $h^0(\mathcal{O}_Z) = \dfrac{n+1}{2}$ and $h^0(\mathcal{O}_{Z'}) = \dfrac{n-1}{2}$, then E is always stable.

Proposition 3.9. If E is an element of $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}, I_Z)$ for $Z \in \mathrm{Hilb}^n(\mathbb{P}^3)$ with $n > 1$, then E can be written as an element of $\mathrm{Ext}^1(I_{Y'}, I_Y)$ for some 0-dimensional subschemes Y, Y' of $\mathbb{P}^3$.

Proof. Consider the short exact sequence

$$0 \to I_Z \to E \to \mathcal{O}_{\mathbb{P}^3} \to 0 \tag{3.2}$$

Now let $p \in Z$ and let $f : I_Z \to I_p$ be the induced injective morphism. Applying $\mathrm{Hom}(-, I_p)$ to sequence (3.2) we get that

$$0 \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}, I_p) \to \mathrm{Hom}(E, I_p) \to \mathrm{Hom}(I_Z, I_p) \to \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}, I_p).$$

Note that $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}, I_p) = H^0(I_p) = 0$ and $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}, I_p) = H^1(I_p) = 0$. Hence

$$\mathrm{Hom}(E, I_p) \cong \mathrm{Hom}(I_Z, I_p)$$

and we can lift f and consider the following diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & I_Z & \longrightarrow & E & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & I_p & \xrightarrow[\text{id}]{} & I_p. & & & & \end{array}$$

Thus, by the snake lemma, we have the exact sequence

$$0 \to K \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{Z\backslash\{p\}} \to C \to 0,$$

where $K = \ker(E \to I_p)$ and $C = \operatorname{coker}(E \to I_p)$. The image of $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{Z\backslash\{p\}}$ is some $\mathcal{O}_Y$ for $Y \subseteq Z\backslash\{p\}$. It follows that $K = I_Y$ and, therefore we can write

$$0 \to I_Y \to E \to I_{Y'} \to 0$$

with Y, Y' 0-dimensional subschemes of $\mathbb{P}^3$. □

By Theorem 2.11 and Theorem 2.8, a stable sheaf with Chern classes $c_1 = c_2 = 0$ and $c_3 = -6$ must be an element of $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}, I_Z)$ for some $Z \in \operatorname{Hilb}^3(\mathbb{P}^3)$ or $\operatorname{Ext}^1(I_p, I_Z)$ for some $Z \in \operatorname{Hilb}^2(\mathbb{P}^3)$. By Proposition 3.9, the first case does not happen, thus we have to analyze just sheaves coming from $\operatorname{Ext}^1(I_p, I_Z)$. Note that this is the most balanced case for our case of $n = 3$, hence by Lemma 3.8, any nontrivial element in $\operatorname{Ext}^1(I_p, I_Z)$ is indeed stable.

Lemma 3.10. Let Z, Z' be 0-dimensional subschemes of $\mathbb{P}^3$ with $Z \cap Z' = \varnothing$. Then $\mathcal{E}xt^1(I_{Z'}, I_Z) = 0$ and

(i) $\operatorname{Ext}^1(I_{Z'}, I_Z) = H^1(\mathcal{H}om(I_{Z'}, I_Z)) = H^1(I_Z)$,

(ii) $\operatorname{Ext}^2(I_{Z'}, I_Z) = H^0(\mathcal{E}xt^2(I_{Z'}, I_Z)) = H^0(\mathcal{E}xt^3(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})) = H^0(\mathcal{O}_{Z'})$

(iii) $\operatorname{Ext}^3(I_{Z'}, I_Z) = 0$.

Proof. Let Z, Z' be 0-dimensional subschemes of $\mathbb{P}^3$ with $Z \cap Z' = \varnothing$ and consider the short exact sequence

$$0 \to I_Z \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0. \tag{3.3}$$

Note that $\mathcal{E}xt^i(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3}) = 0$ for $i \leqslant 2$ by [12, Proposition 1.1.6], and, since $Z \cap Z' = \varnothing$, we have $\mathcal{E}xt^j(\mathcal{O}_{Z'}, \mathcal{O}_Z) = 0$ for every j. It follows that if we apply $\mathcal{H}om(\mathcal{O}_{Z'}, -)$ to (3.3) we get $\mathcal{E}xt^i(\mathcal{O}_{Z'}, I_Z) = 0$ for $0 \leqslant i \leqslant 2$ and $\mathcal{E}xt^3(\mathcal{O}_{Z'}, I_Z) = \mathcal{E}xt^3(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})$.

Now observe that $\mathcal{H}om(\mathcal{O}_{\mathbb{P}^3}, I_Z) = I_Z$ and $\mathcal{E}xt^i(\mathcal{O}_{\mathbb{P}^3}, I_Z) = 0$ and apply $\mathcal{H}om(-, I_Z)$ to sequence

$$0 \to I_{Z'} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{Z'} \to 0, \tag{3.4}$$

so we have $\mathcal{H}om(I_{Z'}, I_Z) = I_Z$, $\mathcal{E}xt^2(I_{Z'}, I_Z) = \mathcal{E}xt^3(\mathcal{O}_{Z'}, I_Z) = \mathcal{E}xt^3(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})$ and $\mathcal{E}xt^1(I_{Z'}, I_Z) = \mathcal{E}xt^3(I_{Z'}, I_Z) = 0$.

Using the above isomorphisms and vanishings, the local-to-global spectral sequence turns as follows

$$
\begin{array}{cccc}
\cancelto{0}{H^0(\mathcal{E}xt^3(I_{Z'}, I_Z))} & \cancelto{0}{H^1(\mathcal{E}xt^3(I_{Z'}, I_Z))} & \cancelto{0}{H^2(\mathcal{E}xt^3(I_{Z'}, I_Z))} & \cancelto{0}{H^3(\mathcal{E}xt^3(I_{Z'}, I_Z))} \\
H^0(\mathcal{E}xt^2(I_{Z'}, I_Z)) & \cancelto{0}{H^1(\mathcal{E}xt^2(I_{Z'}, I_Z))} & \cancelto{0}{H^2(\mathcal{E}xt^2(I_{Z'}, I_Z))} & \cancelto{0}{H^3(\mathcal{E}xt^2(I_{Z'}, I_Z))} \\
\cancelto{0}{H^0(\mathcal{E}xt^1(I_{Z'}, I_Z))} & \cancelto{0}{H^1(\mathcal{E}xt^1(I_{Z'}, I_Z))} & \cancelto{0}{H^2(\mathcal{E}xt^1(I_{Z'}, I_Z))} & \cancelto{0}{H^3(\mathcal{E}xt^1(I_{Z'}, I_Z))} \\
H^0(\mathcal{H}om(I_{Z'}, I_Z)) & H^1(\mathcal{H}om(I_{Z'}, I_Z)) & \cancelto{0}{H^2(\mathcal{H}om(I_{Z'}, I_Z))} & H^3(\mathcal{H}om(I_{Z'}, I_Z)).
\end{array}
$$

Therefore, the spectral sequence collapses in the second page and we have the following.

(i) $\operatorname{Ext}^1(I_{Z'}, I_Z) = H^1(\mathcal{H}om(I_{Z'}, I_Z)) = H^1(I_Z)$;

(ii) $\operatorname{Ext}^2(I_{Z'}, I_Z) = H^0(\mathcal{E}xt^2(I_{Z'}, I_Z)) = H^0(\mathcal{E}xt^3(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})) = H^0(\mathcal{O}_{Z'})$;

(iii) $\operatorname{Ext}^3(I_{Z'}, I_Z) = H^3(I_Z) = 0$;

where the equality $H^0(\mathcal{E}xt^3(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})) = H^0(\mathcal{O}_{Z'})$ follows from Serre's duality and the local-to-global spectral sequence applied to the pair $(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})$, in which it collapses in the second page because the sheaves $\mathcal{E}xt^i(\mathcal{O}_{Z'}, \mathcal{O}_{\mathbb{P}^3})$ are 0-dimensional sheaf, so all the higher cohomology groups vanishes. □

Now to construct an irreducible component of $\mathcal{N}(2,3)$ we start studying $\operatorname{Ext}^1(I_p, I_Z)$ with the case where all points are distinct from each other, that is, $E \in \operatorname{Ext}^1(I_p, I_{q_1,q_2})$. By Lemma 3.10, we have that $\operatorname{ext}^1(I_p, I_{q_1,q_2}) = 1$, that is, we have a unique extension of I_p and I_{q_1,q_2} up to scalar multiplication.

Consider $S = \operatorname{Sym}^3(\mathbb{P}^3)\backslash\Delta$, where Δ denotes the diagonals of $\operatorname{Sym}^3(\mathbb{P}^3)$ to ensure that we choose 3 distinct points. In this case, as we seen above, S can be regarded as a flat family of stable (by Lemma 3.8) sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = c_2 = 0$ and $c_3 = -6$. Thus we get a morphism $S \to \mathcal{N}(2,3)$ and we denote by $\overline{S}$ the closure of the image of S in $\mathcal{N}(2,3)$. Theorem 3.14 bellow ensures that $\overline{S}$ is an irreducible component of $\mathcal{N}(2,3)$.

The problem with using "balanced" sheaves to construct an irreducible component for $n > 3$ is that the dimension of the family obtained in a similar construction is not enough to fill a irreducible component. We will see that the family of extensions that will give us an irreducible component will be nonsplit extensions of the following form

$$0 \to I_Z \to E \to I_q \to 0.$$

This motivates the following definition.

A rank 2 torsion free sheaf E is said to be *unbalanced* if it satisfies a non-split exact sequence of the form

$$0 \to I_Z \to E \to I_q \to 0. \tag{3.5}$$

where Z is a 0-dimensional scheme. Note that unbalanced sheaves are μ-semistable, since they are obtained as an extension of μ-stable sheaves with the same slope. Proposition 2.5 then implies that an unbalanced sheaf E satisfies the exact sequence

$$0 \to E \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to Q \to 0,$$

and one can check that $\dim Q = 0$, thus E is quasitrivial; note that $c_3(E) = -2(h^0(\mathcal{O}_Z)+1)$.

Note that Lemma 3.10 yields $\operatorname{Ext}^1(I_q, I_Z) \neq 0$ when $q \notin Z$ and $h^0(\mathcal{O}_Z) > 1$, thus showing that unbalanced quasitrivial sheaves E do exist when $c_3(E) \leqslant -4$

Lemma 3.11. Let E be a μ-semistable quasitrivial sheaf of rank r on $\mathbb{P}^3$. If $\operatorname{Hom}(E, I_q) \neq 0$, then $q \in \operatorname{Supp}(E^{\vee\vee}/E)$.

Proof. Let E be a quasitrivial sheaf of rank r, so that $E^\vee \simeq E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$; set $Q := E^{\vee\vee}/E$. Take a point $q \in \mathbb{P}^3$ and apply the functor $\operatorname{Hom}(\cdot, I_q)$ to the exact sequence

$$0 \to E \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \to Q \to 0$$

we conclude that $\operatorname{Hom}(E, I_q) \simeq \operatorname{Ext}^1(Q, I_p)$, since $h^0(I_q) = h^1(I_q) = 0$. Since $\dim Q = 0$, the spectral sequence of local to global Ext's yields the first of the following isomorphisms

$$\operatorname{Ext}^1(Q, I_p) \simeq H^0(\mathcal{E}xt^1(Q, I_p)) \simeq H^0(\mathcal{E}xt^2(Q, \mathcal{O}_p)),$$

with the second identification being a consequence of $\mathcal{E}xt^1(Q, I_p) \simeq \mathcal{E}xt^2(Q, \mathcal{O}_p)$. Thus if $\operatorname{Hom}(E, I_q) \neq 0$, then $\mathcal{E}xt^2(Q, \mathcal{O}_p) \neq 0$, which implies that $p \in \operatorname{Supp}(Q)$. □

Next, we argue for the existence of stable unbalanced sheaves.

Proposition 3.12. For each pair (q, Z) consisting of a point q and a reduced 0-dimensional scheme Z not containing q with $h^0(\mathcal{O}_Z) \geqslant 2$, there exists a stable unbalanced sheaf $E \in \operatorname{Ext}^1(I_q, I_Z)$ such that for every $p \in Z$ there is an epimorphism $\varepsilon : E \to I_p$ with $\ker \varepsilon \simeq I_{Z'}$ where $Z' = \big(Z \backslash \{p\}\big) \cup \{q\}$. In particular, $\mathcal{N}^{st}(2, n) \neq \varnothing$ for every $n \geqslant 3$.

Proof. Let $\tilde{Z} = \{p_1, \cdots, p_n\}$ be a reduced 0-dimensional scheme, and set

$$Z_j := \{p_1, \cdots, p_n\} \backslash \{p_j\}.$$

Consider the morphism $\varphi : \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to \mathcal{O}_{\tilde{Z}}$ given by the choice of vectors

$$\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \in \mathbb{C}^2 \quad \text{for} \quad i = 1, \dots, n \quad \text{with} \quad \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \neq \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \quad \text{for all} \quad i \neq j$$

and let $E = \ker\varphi$. Fix $j \in \{1, \ldots, n\}$ and choose $(a, b) \in \mathbb{C}^2$ such that

$$(a, b) \in (\mathbb{C} \cdot (\alpha_j, \beta_j))^\perp \text{ and } (a, b) \notin (\mathbb{C} \cdot (\alpha_i, \beta_i))^\perp, \forall i \neq j,$$

where $\mathbb{C} \cdot (\alpha_i, \beta_i)$ is the $\mathbb{C}$-vector space generated by (α_i, β_i) and $V^\perp$ is the orthogonal complement of a subspace $V \subset \mathbb{C}^2$.

The choice of such vector (a, b) gives a section of $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$ such that the image of the composition

$$\mathcal{O}_{\mathbb{P}^3} \overset{(a,b)}{\to} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to \mathcal{O}_{\tilde{Z}}$$

is precisely $\mathcal{O}_{Z_j}$, leading to the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_{Z_j} & \to & \mathcal{O}_{\mathbb{P}^3} & \to & \mathcal{O}_{Z_j} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E & \to & \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} & \to & \mathcal{O}_{\tilde{Z}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_{p_j} & \to & \mathcal{O}_{\mathbb{P}^3} & \to & \mathcal{O}_{p_j} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

That is, $E \in \operatorname{Ext}^1(I_{p_j}, I_{Z_j})$ for all $j = 1, \ldots, n$. We claim that E is stable.

Indeed, if E is not stable, by Theorem 2.11, then there is a sheaf of ideal $I_Y \hookrightarrow E$ such that $h^0(\mathcal{O}_Y) \leqslant \dfrac{n}{2}$. Let $p_k \in \tilde{Z}$ such that $p_k \notin Y$ and consider E as an element of $\operatorname{Ext}^1(I_{p_k}, I_{Z_k})$. Hence we have the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & I_Y & = & I_Y & & & & \\
 & & \downarrow & & \downarrow & \overset{0}{\searrow} & & & \\
0 & \to & I_{Z_k} & \to & E & \to & I_{p_k} & \to & 0
\end{array}$$

The composition $I_Y \to E \to I_{p_k}$ is zero by the choice of $p_k \notin Y$. Thus the monomorphism $I_Y \hookrightarrow E$ must factor through I_{Z_k}; but is also zero since $h^0(\mathcal{O}_Z) \leqslant \dfrac{n}{2} < n - 1$. Hence the monomorphism $I_Z \hookrightarrow E$ is zero, which is impossible. □

Having established the existence of stable rank 2 quasitrivial sheaves, we now provide a characterization of strictly semistable quasitrivial sheaves.

Proposition 3.13. A rank 2 quasitrivial sheaf E on $\mathbb{P}^3$ is strictly semistable if and only if it is an extension of ideal sheaves I_Z and $I_{Z'}$, where Z and Z' are 0-dimensional subschemes of the same length.

Proof. If E is a strictly semistable quasitrivial sheaf of rank 2, then the fact that E can be expressed and an extension of ideal sheaves I_Z and $I_{Z'}$ with $h^0(\mathcal{O}_Z) = h^0(\mathcal{O}'_Z)$ is a consequence of Theorem 2.8.

Conversely, we argue that every extension of $I_{Z'}$ and I_Z is semistable. Let $E \in \operatorname{Ext}^1(I_{Z'}, I_Z)$ with $h^0(\mathcal{O}_Z) = h^0(\mathcal{O}'_Z) = l$ and assume that E is not semistable, that is, there exists an injective morphism $I_Y \to E$; since $p_{I_Y} > p_E$, we conclude that $\dim Y = 0$ and $h^0(\mathcal{O}_Y) < l$. In that case, we have the inequality between the reduced Hilbert polynomials $p_{I_Y} > p_{I_{Z'}}$ and, it follows that, $\operatorname{Hom}(I_Y, I_{Z'}) = 0$. Hence we can form the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I_Y & \xrightarrow{\text{id}} & I_Y & & & & \\
 & & \downarrow & & \downarrow & \searrow^{0} & & & \\
0 & \longrightarrow & I_Z & \longrightarrow & E & \longrightarrow & I_{Z'} & \longrightarrow & 0.
\end{array}$$

Again, $p_{I_Y} > p_{I_Z}$ implies that $\operatorname{Hom}(I_Y, I_Z) = 0$. Hence the map $I_Y \to I_Z$ in the diagram is zero. Since the diagram commutes, the map $I_Y \to E$ is also zero, thus providing a contradiction.

Therefore, any extension of ideals with the same length is semistable as desired. □

It follows that $\mathcal{N}(2, 2k+1) = \mathcal{N}^{st}(2, 2k+1)$ for every $k \geqslant 1$; furthermore, there is a bijection between the set of S-equivalence classes of strictly semistable sheaves within $\mathcal{N}(2, 2k)$ and the symmetric product $\operatorname{Sym}^2\left(\operatorname{Hilb}^k(\mathbb{P}^3)\right)$.

The next step is to describe the Zariski tangent space $T_E\mathcal{N}(r, n) \simeq \operatorname{Ext}^1(E, E)$ when E is a stable quasitrivial sheaf.

Lemma 3.14. Let $E \in \mathcal{N}^{\text{st}}(r, n)$. If $Q = E^{\vee\vee}/E$ satisfies the following condition

$$\operatorname{ext}^1(Q, Q) - \hom(Q, Q) = 2n,$$

then $\operatorname{ext}^1(E, E) = 2n + rn - r^2 + 1$. In particular, if E is a stable rank 2 unbalanced sheaf with $c_3(E) = -2n$, then $\operatorname{ext}^1(E, E) = 4n - 3$.

Proof. If E is stable, then $h^0(E) = 0$ and $\hom(E, E) = 1$. The long exact sequence of cohomology induced the exact sequence

$$0 \to E \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \to Q \to 0, \tag{3.6}$$

yields $h^1(E) = h^0(Q) - r \cdot h^0(\mathcal{O}_{\mathbb{P}^3}) = n - r$.

Applying the functor $\mathrm{Hom}(Q,-)$ to (3.6) we obtain:

$$\mathrm{Hom}(Q,Q) \cong \mathrm{Ext}^1(Q,E) \quad \text{and} \quad \mathrm{Ext}^1(Q,Q) \cong \mathrm{Ext}^2(Q,E).$$

Next, applying the functor $\mathrm{Hom}(-,E)$ to (3.6) and we get the following exact sequence

$$0 \to \mathrm{Hom}(E,E) \to \mathrm{Ext}^1(Q,E) \to H^1(E)^{\oplus r} \to \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(Q,E) \to 0. \qquad (3.7)$$

It follows that

$$\begin{aligned} \mathrm{ext}^1(E,E) &= 1 + r(n-r) - \mathrm{hom}(Q,Q) + \mathrm{ext}^1(Q,Q) \\ &= rn - r^2 + 1 + 2n, \end{aligned}$$

as desired. When E is a rank 2 unbalanced sheaf, then, as we observed in the paragraph just before Lemma 3.10, the quotient sheaf Q is the structure sheaf of a reduced 0-dimensional scheme, which fulfills the hypothesis in the first part of the lemma. □

Returning to the case $r = 2$, Lemma 3.14 indicates that we should construct an irreducible family of stable rank 2 quasitrivial sheaves on $\mathbb{P}^3$ of dimension $4n - 3$.

In order to construct the family of stable quasitrivial sheaves we are looking for, we will recall certain general results regarding relative Ext sheaves to fix our notation, see Section 1.4 or [16, 3] for more details. Let $f : X \to Y$ be a morphism of schemes, and let $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ be the category of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules respectively. Let $F \in \mathfrak{M}(X)$. We define $\mathcal{E}xt^p_f(F,-)$ to be the right derived functors of the left exact functor $f_*\mathcal{H}om_{\mathcal{O}_X}(F,-) : \mathfrak{M}(X) \to \mathfrak{M}(Y)$.

Let $f : X \to Y$ a flat projective morphism of noetherian schemes and F, G coherent $\mathcal{O}_X$-modules, flat over Y. For every $u : Y' \to Y$ of noetherian schemes, we have the base change morphism

$$\tau^i(u) : u^*\mathcal{E}xt^i_f(F,G) \to \mathcal{E}xt^i_{p_2}(p_1^*F, p_1^*G),$$

where p_1 and p_2 are the projections in the following diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y. \end{array}$$

If $y \in Y$ and $u : \mathrm{Spec}\, k(y) \to Y$ is the respective map, we denote the base change morphism by

$$\tau^i(y) : \mathcal{E}xt^i_f(F,G) \otimes_Y k(y) \to \mathrm{Ext}^i_{X_y}(F_y, G_y).$$

The following result due to Lange will be used several times; see [16, Theorem 1.4].

Theorem 3.15. Let $y \in Y$ be a point and assume the base change morphism $\tau^i(y) : \mathcal{E}xt^i_f(F, G) \otimes_Y k(y) \to \mathrm{Ext}^i_{X_y}(F_y, G_y)$ to be surjective. Then

1. there is a neighbourhood U of y such that $\tau^i(y')$ is an isomorphism for all $y' \in U$;

2. $\tau^{i-1}(y)$ is surjective if and only if $\mathcal{E}xt^i_f(F, G)$ is locally free in a neighbourhood of y.

Our next lemma is the crucial technical fact to be explored in the desired irreducible family of quasitrivial sheaves.

Lemma 3.16. Let $f : X \to Y$ be a projective morphism of noetherian schemes with Y reduced, and let F and G be coherent sheaves on X flat over Y. If $\mathrm{Ext}^3_{X_y}(F_y, G_y) = 0$ and the dimension of $\mathrm{Ext}^i_{X_y}(F_y, G_y)$ is constant for $i = 1, 2$ for all $y \in Y$, then the base change morphism $\tau^1(y)$ is an isomorphism for every $y \in Y$ and $\mathcal{E}xt^1_f(F, G)$ is locally free.

Proof. Since $\mathrm{Ext}^3_{X_y}(F_y, G_y) = 0$, the base change morphism is $\tau^i(y)$ is trivially surjective. By the first item of Theorem 3.15, we have that the sheaf $\mathcal{E}xt^3_f(F, G)$ is zero and item (ii) shows that $\tau^2(y)$ is surjective. Applying Theorem 3.15 again for $i = 2$, we get that $\tau^2(y)$ is an isomorphism. Moreover, [20, Lemma 1, p. 51] implies that $\mathcal{E}xt^2_f(F, G)$ is local free. Finally, applying Theorem 3.15 again for $i = 1$, we obtain that $\tau^1(y)$ is an isomorphism and, since $\mathrm{ext}^1_{X_y}(F_y, G_y)$ is constant, [20, Lemma 1, p. 51] implies that $\mathcal{E}xt^1_f(F, G)$ is locally free. □

We are finally in position to construct the family of quasitrival sheaves we are looking for; let $\mathcal{H}^i$ the universal sheaf for the Hilbert scheme of i points on $\mathbb{P}^3 \times \mathrm{Hilb}^i(\mathbb{P}^3)$. Consider the following diagram

$$\begin{array}{ccccc} & & \mathrm{Hilb}^1(\mathbb{P}^3) \times \mathrm{Hilb}^{n-1}(\mathbb{P}^3) & & \\ & & \uparrow f & & \\ & & \mathbb{P}^3 \times \mathrm{Hilb}^1(\mathbb{P}^3) \times \mathrm{Hilb}^{n-1}(\mathbb{P}^3) & & \\ & \swarrow p_1 & & p_2 \searrow & \\ \mathbb{P}^3 \times \mathrm{Hilb}^1(\mathbb{P}^3) & & & & \mathbb{P}^3 \times \mathrm{Hilb}^{n-1}(\mathbb{P}^3), \end{array}$$

Let

$$U := \{(q, Z) \in \mathrm{Hilb}^1(\mathbb{P}^3) \times \mathrm{Hilb}^{n-1}(\mathbb{P}^3) \mid q \notin Z\ ,\ \ Z = Z_{\mathrm{red}}\} \overset{\text{open}}{\subset} \mathrm{Hilb}^1(\mathbb{P}^3) \times \mathrm{Hilb}^{n-1}(\mathbb{P}^3),$$

and set $X := \mathbb{P}^3 \times U$ with $\pi : X \to U$ being the canonical projection. Define $\mathcal{E}^i := \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1})$. Let (q, Z) be a point in U and let $\tau^i(q, Z)$ be the corresponding base change morphism, that is,

$$\tau^i(q, Z) : \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) \otimes k(q, Z) \to \mathrm{Ext}^i_{\mathbb{P}^3}(I_q, I_Z).$$

Proposition 3.10 and Lemma 3.16 shows us that $\mathcal{E}^1$ is locally free on U. Note that, by Theorem 3.15, $\tau^0(q, Z)$ is an isomorphism, that is, for every $(q, z) \in U$ we have

$$\tau^0(q, Z) : \mathcal{E}xt^0_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) \otimes k(q, Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{P}^3}(I_q, I_Z).$$

However, $\mathrm{Hom}_{\mathbb{P}^3}(I_q, I_Z) = 0$ for every pair $(q, Z) \in U$, which implies that

$$\mathcal{E}xt^0_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) = 0$$

as well. In this case, by [16, Corollary 4.5], there is an universal extension $\mathcal{H}$ on $\mathbb{P}^3 \times H$ with $H := \mathbb{P}((\mathcal{E}^1)^\vee)$ such that for every $h \in H$, the restriction $E_h := \mathcal{H}|_{\mathbb{P}^3 \times \{h\}}$ is a nonsplit extension of two sheaf of ideals of 0-dimensional subschemes of $\mathbb{P}^3$ of the following form

$$0 \to I_Z \to E \to I_q \to 0. \tag{3.8}$$

In other words, every member E_h of the family $\mathcal{H}$ satisfies the exact sequence in display (3.8), and therefore is an unbalanced sheaf; since stability is an open condition, Proposition 3.12 guarantees that there is an open subset $H' \subset H$ whose projection $H' \to U$ is surjective and such that E_h is stable for every $h \in H'$. Therefore, $\mathcal{H}|_{H'}$ is a family of stable rank 2 quasitrivial sheaves parametrized by the scheme H', whose dimension can be easily computed as follows

$$\dim H' = \dim U + \mathrm{ext}^1(I_q, I_Z) - 1 = 3 + 3(n-1) + n - 3 = 4n - 3.$$

Moreover, Lemma 3.14 yields $\mathrm{ext}^1(E, E) = 4n - 3$.

Theorem 3.17. For every $n \geqslant 3$, $\mathcal{N}(2, n)$ contains an irreducible component of dimension $4n - 3$.

Proof. Now $\mathcal{N}(2, n)$ is a coarse moduli space, so our family $\mathcal{H}$ on $X \times H'$ gives us a modular morphism $\Psi : H' \to \mathcal{N}(2, n)$ whose image is precisely the subset of stable unbalanced sheaves. However, as we have seen in Proposition 3.12, the representation of an unbalanced sheaf as an extension of the ideal sheaf of a point by the ideal sheaf of a 0-dimensional scheme is not unique, meaning that the morphism Ψ is not injective. Nonetheless, we argue that it is a quasi-finite map.

Indeed, note that the Lemma 3.11 shows that an unbalanced sheaf can be represented as an extension of an ideal sheaf of a point by an ideal sheaf of a reduced 0-dimensional scheme in at most n different ways. In other words, if $E \in \mathrm{Im}\,\Phi \subset \mathcal{N}(2, n)$, then $\Phi^{-1}(E)$ consists of at most n different points.

This means that the dimension of image of Ψ in $\mathcal{N}(2, n)$ is equal to the dimension of H. Since every $E \in \mathrm{Im}\,\Psi$ satisfies

$$\dim T_E\mathcal{N}(2, n) = \mathrm{ext}^1(E, E) = 4n - 3 = \dim \mathrm{Im}\,\Psi$$

we conclude that the closure of $\mathrm{Im}\,\Psi$ within $\mathcal{N}(2, n)$ is an irreducible component of $\mathcal{N}(2, n)$, as desired. □

3.2 Irreducible component of $\mathcal{N}(r,n)$

In this section we will construct an irreducible component of $\mathcal{N}^{\mathrm{st}}(r,n)$ whenever $r < n$, where the previous section will serve as an induction step.

We begin by generalizing the notion of unbalanced sheaf introduced in the previous section to quasitrivial sheaves of any rank $r \geqslant 2$.

A quasitrivial sheaf E of rank r is called *unbalanced* if it admits an epimorphism $E \to I_q$ to an ideal sheaf of a point $q \in \mathbb{P}^3$, and it does not admit a morphism $I_q \to E$ such that the composition $E \to I_q \to E$ is the identity. Note that if E is unbalanced, then one can consider the kernel of $E \to I_q$ and the following exact sequence

$$0 \to F \to E \to I_q \to 0,$$

with F being a μ-semistable quasitrivial sheaf of rank $r-1$ and $c_3(F) = -2(n-1)$. Moreover, the second condition in the definition implies that the above sequence does not split. Therefore, the present definition generalizes to quasitirival sheaves the notion of unbalanced rank 2 torsion free sheaves present at the beginning of Section 3.1.

Next lemma guarantees the existence of stable unbalanced sheaves.

Lemma 3.18. Let $r < n$ be positive integers. There exists a stable unbalanced sheaf $E \in \mathcal{N}(r,n)$ such that E can be written as an extension of the following form

$$0 \to F \to E \to I_p \to 0,$$

with $F \in \mathcal{N}(r-1,n-1)^{st}$ and $\{p\} \cup \operatorname{Supp} Q_F = \{p_1,\ldots,p_n\}$ distinct points in $\mathbb{P}^3$.

Proof. Let $p_1,\ldots,p_n$ be n points in $\mathbb{P}^3$ distinct with each other. Let $(\varphi, \mathcal{O}_{p_1\cup\ldots\cup p_n})$ be an element in $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$. Note that φ is defined by the choice of n vectors in $\mathbb{C}^r$: $\alpha_i = (\alpha_{1,i},\ldots,\alpha_{n,i})$ for $i = 1,\ldots,r$. Choose α_i's such that $\alpha_i \neq \alpha_j$ for $i \neq j$, defining a matrix

$$\alpha_{n\times r} = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,r} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,r} \end{bmatrix}.$$

Now, we can choose a matrix

$$A_{r\times r-1} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,r-1} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,r-1} \end{bmatrix}$$

such that the product $\alpha \cdot A$ has the row i equal zero for $i = 1,\ldots,n$. The choice of A in this way gives a map $\mathcal{O}_{\mathbb{P}^3}^{\oplus r-1} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r}$ such that the composition $\mathcal{O}_{\mathbb{P}^3}^{\oplus r-1} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \xrightarrow{\varphi} \mathcal{O}_{p_1\cup\ldots\cup p_n}$

vanishes at $\mathcal{O}_{p_i}$. So we can form a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & \mathcal{O}_{\mathbb{P}^3}^{\oplus r-1} & \to & \mathcal{O}_{p_1\cup\ldots\cup\hat{p_i}\cup\ldots\cup p_n} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E & \to & \mathcal{O}_{\mathbb{P}^3}^{\oplus r} & \to & \mathcal{O}_{p_1\cup\ldots\cup p_n} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_{p_i} & \to & \mathcal{O}_{\mathbb{P}^3} & \to & \mathcal{O}_{p_i} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

That is, for every choice of $1 \leqslant i \leqslant n$, we can write E as extension of F and I_{p_i} with $\operatorname{Supp}(Q_F) = \{p_1, \ldots, \hat{p_i}, \ldots, p_n\}$.

We claim that E must be stable. Indeed if E is not stable, that is, there is $F' \hookrightarrow E$ such that

$$h^0 F' < \frac{\operatorname{rk} F' \cdot n}{r}, \tag{3.9}$$

by Theorem 2.11. By the above construction, we can write E as extension of F and I_{p_i} such that $p_i \notin \operatorname{Supp} Q_{F'} = \varnothing$. By Lemma 3.20, $\operatorname{Hom}(F', I_{p_i}) = 0$, so we have the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F' & = & F' & \overset{0}{\dashrightarrow} & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F & \to & E & \to & I_{p_i} & \to & 0.
\end{array}$$

Now we have $h^0(Q_F) = n - 1$, and note that

$$\frac{h^0 Q_F}{\operatorname{rk} F} = \frac{n-1}{r-1} = \frac{n-r+r-1}{r-1} = \frac{n-r}{r-1} + 1 > \frac{n-r}{r} + 1 = \frac{n}{r}.$$

By (3.9), it follows that,

$$h^0 F' < \frac{\operatorname{rk}(F') \cdot n}{r} < \frac{\operatorname{rk}(F') \cdot h^0(Q_F)}{\operatorname{rk}(F)}.$$

But since F is stable, this cannot happen. Therefore E must be stable. □

Corollary 3.19. *If $r < n$, then $\mathcal{N}(r,n)^{st} \neq \varnothing$.*

Lemma 3.20. *Let $(\varphi, Q_F) \in \operatorname{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}, n)$ and $(\psi, Q_G) \in \operatorname{Quot}(\mathcal{O}_{\mathbb{P}^3}^{\oplus s}, k)$ such that the support of Q_F and Q_G does not intersect. Let $F = \ker \varphi$ and $G = \ker \psi$. Then*

$$\operatorname{ext}^1(G, F) = \operatorname{rk}(G)(h^0(F) + h^0(Q_F) - \operatorname{rk}(F)),$$

and

$$\hom(G,F) = \operatorname{rk} G \cdot h^0(F).$$

Proof. We have two short exact sequences:

$$0 \to F \to \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \to Q_F \to 0 \tag{3.10}$$

and

$$0 \to G \to \mathcal{O}_{\mathbb{P}^3}^{\oplus s} \to Q_F \to 0. \tag{3.11}$$

Apply $\operatorname{Hom}(Q_G, -)$ to sequence (3.10) and we get

$$\begin{aligned} 0 \to \operatorname{Hom}(Q_G, F) \to \operatorname{Hom}(Q_G, \mathcal{O}_{\mathbb{P}^3}^{\oplus r}) \to \operatorname{Hom}(Q_G, Q_F) \\ \to \operatorname{Ext}^1(Q_G, F) \to \operatorname{Ext}^1(Q_G, \mathcal{O}_{\mathbb{P}^3}^{\oplus r}) \to \operatorname{Ext}^1(Q_G, Q_F). \end{aligned} \tag{3.12}$$

Note that, by the local-to-global spectral sequence and the hypothesis that $\operatorname{Supp}(Q_F) \cap \operatorname{Supp}(Q_G) = \varnothing$, it follows that $\operatorname{Ext}^i(Q_G, Q_F) = 0$ for $i \geqslant 0$. By Serre's duality,

$$\operatorname{Ext}^i(Q_G, \mathcal{O}_{\mathbb{P}^3}^{\oplus r}) \cong H^{3-i}(Q_G)^{\oplus r}.$$

Since $\dim Q_G = 0$, we have $\operatorname{Ext}^i(Q_G, \mathcal{O}_{\mathbb{P}^3}^{\oplus r}) = 0$ for $i \leqslant 2$. Thus, by (3.12), $\operatorname{Ext}^i(Q_G, F) = 0$ for $i \geqslant 0$.

Now apply $\operatorname{Hom}(-, F)$ to (3.11) to get

$$\begin{aligned} 0 \to \operatorname{Hom}(Q_G, F) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^{\oplus s}, F) \to \operatorname{Hom}(G, F) \\ \to \operatorname{Ext}^1(Q_G, F) \to \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}^{\oplus s}, F) \to \operatorname{Ext}^1(G, F) \to \operatorname{Ext}^2(Q_G, F). \end{aligned} \tag{3.13}$$

By the above, (3.13) give us

$$\operatorname{Hom}(G, F) \cong H^0(F)^{\oplus s} \text{ and } \operatorname{Ext}^1(G, F) \cong H^1(F)^{\oplus s}.$$

Finally, taking cohomologies on (3.10),

$$0 \to H^0(F) \to \overbrace{H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus r})}^{\cong \mathbb{C}^r} \to H^0(Q_F) \to H^1(F) \to \overbrace{H^1(\mathcal{O}_{\mathbb{P}^3}^{\oplus s})}^{0}.$$

Taking dimensions:

$$h^1 F = h^0 F + h^0 Q_F - \operatorname{rk} F,$$

and note that the same holds for G. Summing up everything,

$$\begin{aligned} \operatorname{ext}^1(G, F) &= \operatorname{rk} G \cdot h^1(F) \\ &= \operatorname{rk}(G)(h^0(F) + h^0(Q_F) - \operatorname{rk}(F)). \end{aligned}$$

□

By Lemma 3.14, we must find a flat family of stable sheaves on $\mathcal{N}(r,n)$ with dimension $2n+rn-r^2+1$. For the rank 2 case, our component was given, essentially, by $\mathbb{P}(\mathrm{Ext}^1(I_p,I_Z))$, for $p\in\mathbb{P}^3$ and $Z=\{p_1,\dots,p_{n-1}\}$ all distinct from each other. By Lemma 2.9, we can write $E\in\mathcal{N}(r,n)$ as extension of $F\in\mathcal{N}(s,k)$ and $G\in\mathcal{N}^{st}(r-s,n-k)$. Hence, we can try to find a combination of $[s,k]$ such that the dimension of $\mathbb{P}(\mathrm{Ext}^1(G,F))=2n+rn-r^2+1$.

Suppose $\mathcal{N}(s,k)$ has an irreducible component $H_{s,k}$ of dimension $2k+sk-s^2+1$. In this case the dimension of this family would be given by:

$$\overbrace{2k+sk-s^2+1}^{H_{s,k}}+\overbrace{2(n-k)+(r-s)(n-k)-(r-s)^2+1}^{H_{r-s,n-k}}+\overbrace{(r-s)(n-k)}^{\mathrm{Ext}^1\,G,F}-1. \quad (3.14)$$

Moreover, we could assume F and G to be stable since stability is an open property.

Now, in order to this to work we need to find which values $[s,k]$ satisfies (3.14) equal to $2n+rn-r^2+1$. Solving this in terms of $[s,k]$ we find

$$[s,k]=[s,n-r+s],$$

that is, for each choice of s, we take $k=n-r+s$ and we would have H_s to have dimension $2n+rn-r^2+1$.

Now observe that by choosing $1\leqslant s\leqslant r$ so $k=n-r+s$, that is, F is in $\mathcal{N}(s,n-r+s)$ and, in this case, G is in $\mathcal{N}(r-s,n-(n-r+s))=\mathcal{N}(r-s,r-s)$. But since we are assuming F and G to be stable, by Lemma 3.2, the only case that we need to consider is when $r-s$ is equal to 1, hence $G=I_p$ for $p\in\mathbb{P}^3$. We now make this precise following the idea used for the rank 2.

We will argue by induction. The case $r=2$ is already done, so suppose we have a family $\mathcal{H}_{r,n}$ that gives us an irreducible component $H_{r,n}$ of $\mathcal{N}(r,n)$ with dimension $2n+rn-r^2+1$, that is, $\mathcal{H}_{r,n}$ is sheaf on $\mathbb{P}^3\times H_{r,n}$ such that for every $h\in H_{r,n}$, $\mathcal{H}_{r,n}|_h\in\mathcal{N}(r,n)^{st}$.

Proposition 3.21. Let F and G be torsion free sheaves on $\mathbb{P}^3$ defined by the sequences

$$0\to F\to\mathcal{O}_{\mathbb{P}^3}^{\oplus r}\to Q_F\to 0 \quad (3.15)$$

and

$$0\to G\to\mathcal{O}_{\mathbb{P}^3}^{\oplus s}\to Q_G\to 0, \quad (3.16)$$

where Q_F and Q_G are supported on 0-dimensional subschemes of $\mathbb{P}^3$. If $\mathrm{Supp}\,Q_F\cap\mathrm{Supp}\,Q_G=\varnothing$, then

- $\mathrm{Ext}^1(F,G)=\mathrm{rk}(F)\cdot H^1(G)$,

- $\operatorname{Ext}^2(F,G) = \operatorname{rk}(G)\cdot H^0(Q_F)$
- $\operatorname{Ext}^3(F,G) = 0.$

Proof. First, taking cohomologies on the sequences defining F and G we obtain that $H^2(F(t)) = H^2(G(t)) = 0$ for all $t \in \mathbb{Z}$. Now, apply $\operatorname{Hom}(Q_F, -)$ to sequence (3.16) and we obtain $\operatorname{Ext}^i(Q_F,G) = 0$ for $i = 0,1,2$ and $\operatorname{Ext}^3(Q_F,G) \cong \operatorname{rk} G \cdot H^0 Q_F$. Apply $\operatorname{Hom}(-,G)$ to sequence (3.15) and we get $\operatorname{Ext}^3(F,G) = 0$, $\operatorname{Ext}^2(F,G) = \operatorname{rk} G\cdot H^0(Q_G)$ and $\operatorname{Ext}^1(F,G) = \operatorname{rk} F\cdot H^1(G)$, as we wanted. □

Remark 3.22. Let F be a torsion free sheaf on $\mathbb{P}^3$ defined by sequence (3.15). Note that if F is stable, then $H^0(F) = 0$. Therefore, taking cohomologies on sequence (3.15), we get that $H^0(Q_F) = H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus r}) \oplus H^1(F)$, that is

$$h^1(F) = h^0(Q_F) - \operatorname{rk}(F). \tag{3.17}$$

Consider the following diagram

$$\begin{array}{ccccc} & & \operatorname{Hilb}^1 \times H_{r-1,n-1} & & \\ & & \uparrow f & & \\ & & \mathbb{P}^3 \times \operatorname{Hilb}^1 \times H_{r-1,n-1} & & \\ & \swarrow p_1 & & \searrow p_2 & \\ \mathbb{P}^3 \times \operatorname{Hilb}^1 & & & & \mathbb{P}^3 \times H_{r-1,n-1}, \end{array}$$

where $H_{r-1,n-1}$ denotes the irreducible component given by the induction hypothesis on $\mathcal{N}(r-1,n-1)$.

Define

$$U := \{(y,F) \in \operatorname{Hilb}^1 \times H_{r-1,n-1} \mid y \notin \operatorname{Supp} Q_F,\quad \operatorname{Supp} Q_F = (\operatorname{Supp} Q_F)_{\mathrm{red}}\}$$

as an open subset of $\operatorname{Hilb}^1 \times H_{r-1,n-1}$ and let $X := \mathbb{P}^3 \times Y$ with $\pi : X \to U$ the projection.

Let $\mathcal{E}^i := \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}_{r-1,n-1})$. By Proposition 3.21 and Lemma 3.16, $\mathcal{E}^1$ is a locally free on $\operatorname{Hilb}^1 \times H_{r-1,n-1}$ sheaf whose fibres over a point $(y,F) \in \mathbb{P}^3 \times \mathcal{N}(r-1,n-1)^{st}$ is $\operatorname{Ext}^1(I_y,F)$ for every $y \notin \operatorname{Supp} Q_F$.

Note that $\operatorname{Hom}_{\mathbb{P}^3}(I_p,F) = 0$ for every $F \in \mathcal{N}(r-1,n-1)^{\mathrm{st}}$. So, by Theorem 3.15, $\mathcal{E}^0 = 0$ and [16, Corollary 4.2] implies that there is an universal extension $\mathcal{H}$ on $X \times H$ with $H = \mathbb{P}((\mathcal{E}^1)^*)$ such that for every $h \in H$, the restriction $\mathcal{H}|_h$ is a nonsplit extension the following form

$$0 \to F \to E \to I_y \to 0. \tag{3.18}$$

In other words, every member E_h of the family $\mathcal{H}$ satisfies the exact sequence in display (3.18), and therefore is an unbalanced sheaf; since stability is an open condition, Proposition 3.18 guarantees that there is an open subset $H' \subset H$ whose projection $H' \to U$ is surjective and such that E_h is stable for every $h \in H'$. Therefore, $\mathcal{H}|_{H'}$ is a family of stable rank r quasitrivial sheaves parametrized by the scheme H', whose dimension can be computed as follows

$$\begin{aligned}
\dim H' &= \dim U + \mathrm{ext}^1(I_p, F) - 1 \\
&= 3 + 2(n-1) + (r-1)(n-1) - (r-1)^2 + 1 + (n-1-(r-1)) - 1 \\
&= 2n + rn - r^2 + 1.
\end{aligned}$$

Note that Lemma 3.14 implies $\mathrm{ext}^1(E, E) = 2n + rn - r^2 + 1$.

Theorem 3.23. Let $r < n$ be positive integers. $\mathcal{N}(r, n)$ contains an irreducible component of dimension $2n + rn - r^2 + 1$.

Proof. $\mathcal{N}(r, n)$ is a coarse moduli space, so our family $\mathcal{H}$ on $X \times H'$ gives us a modular morphism $\Psi : H' \to \mathcal{N}(r, n)$ whose image is precisely the subset of stable unbalanced sheaves. However, as we have seen in Proposition 3.18, the representation of an unbalanced sheaf as an extension of the ideal sheaf of a point by a quasitrivial sheaf supported in a 0-dimensional scheme is not unique, meaning that the morphism Ψ is not injective. Nonetheless, we argue that it is a quasi-finite map.

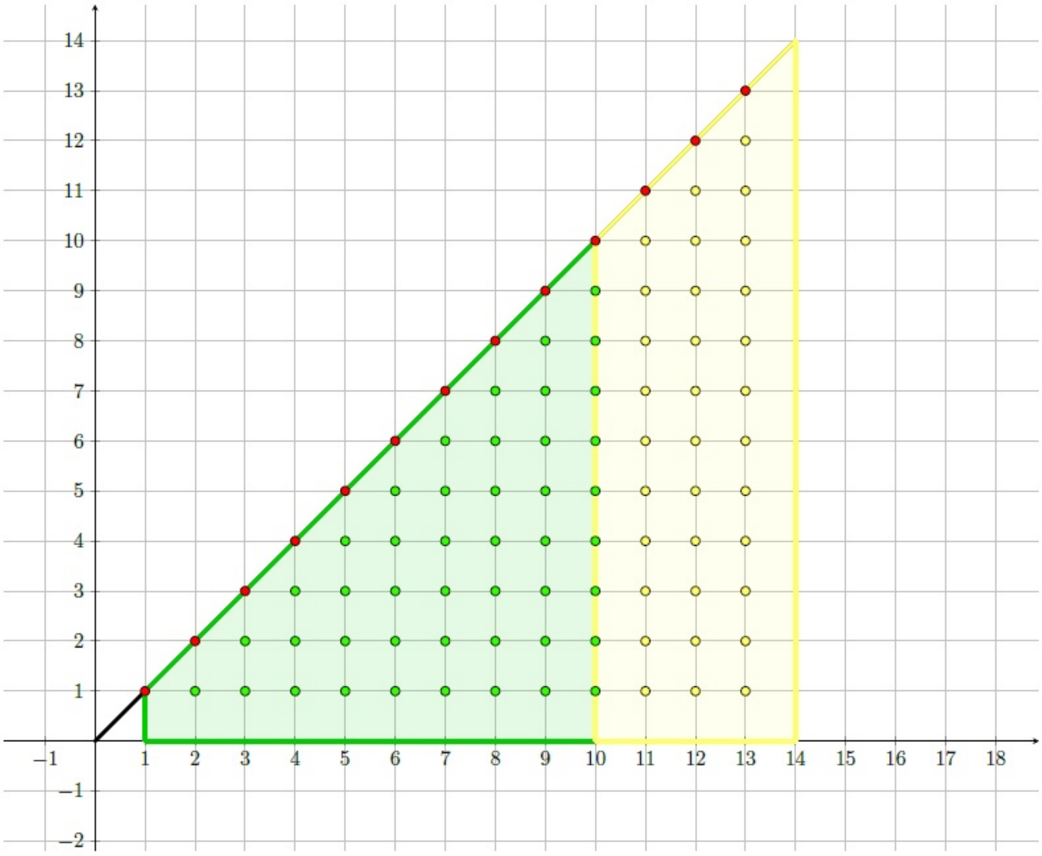
Indeed, note that the Lemma 3.11 shows that an unbalanced sheaf can be represented as an extension as in (3.18) in at most n different ways. In other words, if $E \in \mathrm{Im}\,\Phi \subset \mathcal{N}(r, n)$, then $\Phi^{-1}(E)$ consists of at most n different points.

This means that the dimension of image of Ψ in $\mathcal{N}(r, n)$ is equal to the dimension of H. Since every $E \in \mathrm{Im}\,\Psi$ satisfies

$$\dim T_E \mathcal{N}(r, n) = \mathrm{ext}^1(E, E) = 2n + rn - r^2 + 1 = \dim \mathrm{Im}\,\Psi$$

we conclude that the closure of $\mathrm{Im}\,\Psi$ within $\mathcal{N}(r, n)$ is an irreducible component of $\mathcal{N}(r, n)$, as desired. □

We end this chapter with a graph that summarizes our results.



Here $\mathcal{N}(r, n)$ is represented by the points in the picture where n is running along x-axis and r along y-axis. Red dots are the points where $r = n$ and we have the bijection with $\text{Sym}^n(\mathbb{P}^3)$. Green dots are the points where we know that $\mathcal{N}(r, n)$ is irreducible, therefore our component constructed in Chapter 3 gives a description to $\mathcal{N}(r, n)$. Yellow dots are the points are we have an irreducible component, but we do not know whether it is the only one. As we have seen in Chapter 3, we should only care about points bellow the line $r = n$, because $\mathcal{N}(r, n) = \varnothing$ whenever $r > n$.

Remark 3.24. We have restricted ourselves to $\mathbb{P}^3$ from Chapter 3 onwards because we believe that this is the most interesting case, deserving special attention; as explained in Remark 3.7, it is on dimension 3 that we have irreducibility results for the Quot schemes of points. However, most of the results contained in these sections can be generalized to higher dimensional projective spaces with little effort. Indeed, one can check that the moduli space of quasitrivial sheaves on $\mathbb{P}^d$, namely the moduli space of semistable sheaves

$$\mathcal{M}_d^P \quad \text{for } P(t) = r \cdot \binom{t+d}{d} - n,$$

admits an irreducible component of dimension

$$(r - 1 + d)n - r^2 + 1$$

when $r < n$, and whose generic point corresponds to a stable unbalanced sheaf.

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