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Cohomology of Maximal Flag Manifolds

Cohomologia de Variedades Bandeira Maximais

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"It is by logic that we prove,
but by intuition that we discover."
(Henri Poincaré)

Resumo

Em 1973, Bernstein, Gel'fand e Gel'fand publicaram um artigo sobre a homologia dos espaços G/P, que chamaremos de variedades bandeira. Lá eles mostram qual é a relação entre a homologia gerada pela decomposição celular de Schubert e a cohomologia usando as classes características de Chern dada por fibrados de retas complexas. Este trabalho apresenta técnicas alternativas de se obter essa mesma relação em variedades bandeira maximais, porém utilizando o homomorfismo de Chern-Weil, que associa polinômios P-invariantes à formas diferenciais fechadas, e portanto representantes em $H^*(G/P;\mathbb{R})$. Como se sabe, cada célula de Schubert está associada a um elemento w do grupo de Weyl. O que fizemos foi construir formas volumes invariantes em tais células que *não* dependem da decomposição minimal de w. Utilizando propriedades geométricas das células de Schubert—fibrações de esfera sobre células menores—demonstramos a dualidade entre células e polinômios, um problema que se reduziu à integração sobre esferas. Tais técnicas permitem inclusive a obtenção de resultados análogos para variedades bandeira quaterniônicas. Explorando mais afundo essas variedades quaterniônicas, apresentamos formas explícitas das classes de Pontryagin e por último, como tais classes representam uma obstrução para existência de certos referenciais na variedade bandeira, provamos que ali não existe uma estrutura hipercomplexa invariante pela ação do grupo estrutural.

Palavras-chave: Variedades bandeira. Homologia. Cohomologia. Teoria de Chern-Weil. Teoria de Lie. Células de Schubert. Álgebra de polinômios invariantes. Decomposição de Bruhat.

Abstract

In 1973, Bernstein, Gel'fand and Gel'fand published a paper about the homology of the spaces G/P, which we call flag manifolds. There they showed what is the relationship between the homology generated by the Schubert cellular decomposition and the cohomology using Chern characteristic classes of complex line bundles. This work presents alternative techniques aiming to prove the same relations in maximal flag manifolds, but using instead the Chern-Weil homomorphism, which takes P-invariant polynomials to *closed* differential forms, that is, representants in $H^*(G/P; \mathbb{R})$. It is known that each Schubert cell is indexed by an element w of the Weyl group. We build invariant volume forms on each cell which does *not* depend on the minimal decomposition of w. Making use of geometrical properties of the Schubert cells—sphere fibrations over smaller cells—we have managed to demonstrate the duality between cells and polynomials, thus reducing the problem to that of integration over spheres. Such techniques allowed us to go even further and prove analogous results for the quaternionic maximal manifold. We deeper explore the quaternionic manifolds and show explicit formulas for the Pontryagin classes. Lastly, since such characteristic classes provides an obstruction for the existence of certain frames on the manifold, we proved that there is no hypercomplex structure which is also invariant by the structure group action.

Keywords: Flag manifolds. Homology. Cohomology. Chern-Weil theory. Lie theory. Schubert cells. Invariant polynomial algebra. Bruhat decomposition.

Contents

Introduction

This is a study on the homology and cohomology of a specific family of homogeneous spaces, the so called *flag manifolds*. A first definition is given by seeing these manifolds as collections of all chain of subspaces of a given vector space, the chains satisfying a given form, a.k.a the flag *signature*. Using some basic linear algebra, a differentiable structure of these sets is stablished by considering them as a quotient G/B_Θ.[1] The case when $G = \mathrm{Sl}_n(\mathbb{C})$ is done in Example 1.2 and Example 2.1. The most known examples are the projective spaces of a vector space and, more generally, the Grassmannians. Therefore, the methods used to compute the homology and cohomology of the flag manifolds $\mathbb{F}_\Theta = G/B_\Theta$ are several.

For the cohomology, one method that we use, in particular, is by associating *G-invariant polynomials* to *closed differential forms* on $\mathbb{F}_\Theta$ via the *Chern-Weil homomorphism* (Theorem 1.2), therefore inducing *characteristic classes* on the flag, which are widely known topological invariants (CHERN, 1946). To exemplify their importance, the non-annihilation of these characteristic classes establishes an obstruction to the existence of certain frames on the manifold. Theorem 3.6 is an example of a result in this direction.

For the homology, remember that for G a semisimple Lie group, one has the *Bruhat decomposition* $G = \bigcup BwB$, where the union is taken over the elements w of the *Weyl group* $\mathcal{W}$ (KNAPP, 2013). By letting B_Θ act on G from the right, the corresponding flag manifold $\mathbb{F}_\Theta$ inherits a similar decomposition into orbits. Fix a point $x_0 \in \mathbb{F}_\Theta$. The closure of the orbit Bwx_0 is what defines a *Schubert cell* $\mathcal{S}_w$ (see section 2.2). These cells allows us to see $\mathbb{F}_\Theta$ as a CW-complex (RABELO; MARTIN, 2019), therefore it contains information about the homology. Let d_k denote the number of cells $\mathcal{S}_w$ such that $k = 2\,l(w)$. By Corollary 2.1 (about the dimension), one immediately sees that

$$H_k(\mathbb{F};\mathbb{R}) = \begin{cases} \mathbb{R}^{d_k} & \text{if } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Aware of the natural pairing between homology and cohomology, one is naturally led to the question:

What is the relationship between the Schubert cellular structure on $\mathbb{F}_\Theta = G/B_\Theta$ and the polynomial algebra of the structure group B_Θ?

[1] Unlike other more traditional studies on flag manifolds, I'll use the letter B (instead of P) to stand for a parabolic subgroup. The reason for this is that the main object of study in this work is the *maximal* flag manifold $\mathbb{F} = G/B$, where B is a *Borelian* subgroup. Also, the letter P is reserved for principal bundles.

A trio of mathematicians—Bernstein, Gel'fand and Gel'fand (BGG from now on)—answered this question in a paper published in 1973 going by the tittle *Schubert Cells and Cohomology of the Spaces G/P* (BERNSTEIN; GEL'FAND; GEL'FAND, 1973). To each Schubert cell $\mathcal{S}_w$, they've showed a formula for the corresponding invariant polynomial P_w, starting from the Schubert cell associated to the maximal length $w_0 \in \mathcal{W}$ and going down in length recursively. The main tool in the mathematical apparatus used in this formula consisted of the "derivation" operator $A_\gamma : \mathrm{Inv}(G) \to \mathrm{Inv}(G)$ defined by

$$A_\gamma p = \frac{p - w_\gamma(p)}{\gamma},$$

where γ is a root of $\mathfrak{g}$ and p is an invariant polynomial. By considering the many ways an element $w \in \mathcal{W}$ can be decomposed, they proved the theorem stated below (for your convenience).

Theorem 0.1. *Let $w \in \mathcal{W}$ with length $l(w) = k$. Let $\gamma_1, \ldots, \gamma_k \in \mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $D(\cdot, \cdot)$ be the natural pairing operator between homology and cohomology. Then*

$$D_w(\mathcal{S}_w, \gamma_1 \cdots \gamma_k) = \sum \gamma_1(H_{\alpha_1}) \cdots \gamma_k(H_{\alpha_k}) \tag{1}$$

where the sum is taken over all sequences $(\alpha_1, \ldots, \alpha_k)$ of roots such that $w = r_{\alpha_1} \cdots r_{\alpha_k}$.

In the paper by BGG, they describe the operator D in terms of the A_γ, which leads formulas of *combinatoric nature*. In chapter 2, we actually prove a similar result, but using other perhaps more geometrical and elementary techniques. To each cell $\mathcal{S}_w$, we first build an invariant volume form described in terms of the characteristic classes of $G \to \mathbb{F}$. Then we integrate it over $\mathcal{S}_w$ leading us to a similar formula, like Equation 1. Because each $\mathcal{S}_w$ is related to a downward sequence of sphere fibrations, we do this integration fundamentally by comparing the metric induced by such volume forms and the *Euclidean metric*, giving us the explicit integral

$$\beta_1(H_{\beta_1}) \cdots \beta_k(H_{\beta_k}) = \frac{1}{(4\pi)^k} \int_{\mathcal{S}_w} (\beta_1 \cdots \beta_k)(\Omega).$$

Notice that this formula does *not* depend on the choice of a minimal decomposition of w. This is the content of Theorem 2.2. The above formula is generalized to any Chern-Weil class $f(\Omega)$—see Theorem 2.3. The advantages of our method is that we have obtained similar results even for non-complex flag manifolds, like the maximal flag of $\mathfrak{sl}_n(\mathbb{H})$. See Theorem 3.2.

Continuing, BGG used Equation 1 to investigate representations of the Weyl group in the invariant polynomial algebra and cohomology $H^*(G/P)$. By realizing the Weyl group as a subgroup of G as in Equation 2.1, there is a right-action of $\mathcal{W}$ in $\mathbb{F}$, therefore $\mathcal{W}$ represents itself in the cohomology by considering the pullback operator. The authors proved that such representation is actually equivalent to the *regular representation* of the Weyl group. In this work, we prove an analogous result in the case of the quaternionic maximal flag. See section 3.2.

The structure of the thesis is organized as follows. In chapter 1, the basics of the Chern-Weil theory are established alongside notation. It follows very closely what is done in (KOBAYASHI; NOMIZU, 1963; KOBAYASHI; NOMIZU, 1996). I've decided to put it first (instead of adding it as an appendix) because I've explored important examples in the middle of it. Now, chapter 2 is where the real work begins. It talks about flag manifolds from the Lie-theoretic point of view, stating (with proofs) fundamental facts about such spaces; it also explores the Schubert cells in great depth. The chapter ends with the main results about the maximal flag varieties of a complex semisimple Lie group, realizing the duality between elements of $\mathcal{W}$ and invariant polynomials by integration. The contents of chapter 3 shows how the techniques developed in the previous chapter can be used to prove analogous results to the quaternionic case. Last, but not least, it indicates how the Weyl group acts on the polynomial algebra and cohomology and proves equivalences of these representations and the (right) regular representation. In the appendix, I've put some basic facts about the quaternion linear algebra, some examples that were too big or inconvenient and some useful theorems.

I hope you enjoy!

1 Chern-Weil Theory

A fundamental problem in topology is to find out whether two given spaces are (topologically) equivalent, *i.e.*, if there exists a homeomorphism between them. To answer this question, one way is to take a deep dive into the realm of algebraic topology, a field where all the modern tools are. In this text, we give special attention to the homology and cohomology theory: the (co)homology of equivalent spaces are equal, one of the so called topological invariants. Therefore, if the (co)homology of two given spaces differ, then they can not be equivalent. Theoretically, this is all it takes, but actually compute the homology and cohomology might not be straightforward.

The aim of this chapter is mostly to introduce preliminary concepts used in the following chapters. This includes our main tool which is the *Chern-Weil homomorphism* defined in Theorem 1.2. This homomorphism allows us to compute characteristic classes (elements in the homology, essentially) of natural principal bundles over the flag manifolds. Everything involved in this construction is what we call *Chern-Weil theory*. This introductory chapter follows very closely what is done in (KOBAYASHI; NOMIZU, 1963) and (KOBAYASHI; NOMIZU, 1996), so most of the proofs are skipped. Instead, we take this opportunity to give examples and develop the main concepts.

1.1 Principal Bundles

Let M be a differentiable manifold and G a Lie group. A (differentiable) principal bundle over M with group G is a space P and an action $P \times G \to P$ satisfying

1. G acts freely on P from the right;
2. M is the quotient space of P by the equivalence relation induced by G, *i.e.*, $M = P/G$ and the canonical projection $\pi : P \to M$ is differentiable;
3. P is locally trivial.

A principal bundle will be denoted by (P, M, G) or simply by P. The set P is called the principal space, G is the structure group, M is the base space and π is simply the projection. For each $x \in M$, the set $\pi^{-1}(x)$ is called the fiber over x. Each fiber is a closed submanifold of P and is diffeomorphic to G.

Example 1.1. Let G act on $M \times G$ by right multiplication on the G factor. Then $P = M \times G$ is a principal bundle, the so called *trivial bundle*. Whether any given principal bundle $P(M, G)$ is trivial or not is is a fundamental question in algebraic topology. Most of the theory developed in this thesis actually provides a toolbox to answer this; Another class of principal bundles that we are going to explore are the *quotient bundles* $(G, H, G/H)$. Let H be a closed subgroup of G. Naturally, H acts on G from the right, so G can be seen as a principal bundle over G/H with structure group H. □

Remark 1.1. Most of the objects defined in this thesis have some sort of invariant property, allowing us to almost completely ignore the local triviality of a principal bundle, even though it is necessary. The fact that one is able to reduce any global discussion to an open neighborhood of a point—and by extension, to the study of the Lie algebra— is a common feature in the study of Lie groups.

In what follows, the *most important example* in this text is presented.

Example 1.2. A *flag* in $\mathbb{C}^n$ is a sequence

$$x = (V_1, V_2, \dots, V_{n-1})$$

where each V_k is a subspace of $\mathbb{C}^n$ of (complex) dimension k; and $V_k \subseteq V_{k+1}$. The set of all such flags is called *the maximal complex flag manifold* and is denoted by $\mathbb{F}$. Now, consider the distinguished flag

$$x_0 = (E_1, E_2 \dots, E_{n-1}) \in \mathbb{F}$$

where each E_k is the subspace of $\mathbb{C}^n$ spanned by the basic vectors $\{e_j\}_{j=1}^k$. The simple Lie group $\mathrm{Sl}_n(\mathbb{C})$ acts naturally from the left on $\mathbb{F}$

$$g \cdot x = (gE_1, gE_2, \dots, gE_{n-1}).$$

This action is easily seen to be transitive and the stabilizer of x_0 is a Borel subgroup B of $\mathrm{Sl}_n(\mathbb{C})$, that is, B is the subgroup of determinant one upper triangular matrices. A well-known result about orbit and stabilizers[1] provides the diffeomorphism

$$\mathbb{F} \approx \mathrm{Sl}_n(\mathbb{C})/B.$$

In other words, $\mathrm{Sl}_n(\mathbb{C})$ is a principal bundle over $\mathbb{F}$ with structure group B. □

A *homomorphism* of principal bundles (P, M, G) and (P', M', G') is a pair consisting of a differentiable map $f' : P' \to P$ and a homomorphism $f'' : G' \to G$ denoted by the same letter f such that $f(pg) = f(p)f(g)$. One says that P' is a *subbundle* of P if f' is an imbedding and f'' is injective.

Let (P, M, G) be a principal bundle and take G' to be any subgroup of G. The structure group G is said to be reducible to G' if P admits a subbundle (P', M, G'). In this case, the subbundle P' is a *reduction* of P to the structure group G'; and G is said to be *reducible* to G'.

[1] This is a generalization of proposition 2.33 on (SANMARTIN, 2021)

Example 1.3. Continuing the last example, consider the Iwasawa decomposition of the Lie group $G = \mathrm{Sl}_n(\mathbb{C})$ into KAN where $K = \mathrm{SU}_n$, A is the subgroup of diagonal matrices with determinant one and N is the subgroup of upper diagonal matrices with '1' in the diagonal entries. Notice that $B = AN$, so K is a subbundle of G with structure group T, the diagonal subgroup in K, also known as the *maximal torus*. This leads us to another representation of the maximal flag as a quotient, namely $\mathbb{F} \approx \mathrm{SU}_n/T$. □

We dedicate the rest of this section to present some results that are used to show that in a quaternionic maximal flag, no G-invariant quaternionic structure will be hypercomplex. See section 3.4.

Let F be a manifold on which G acts from the left. Then G acts on $P \times F$ from the right by letting $(p, f) \cdot g = (pg, g^{-1}f)$. Denote by $E = P \times_G F$ the quotient group by this action with coset denoted by $[p, f]$. It is known that E has a differentiable structure such that the projection $\pi_E : E \to M$ given by $\pi_E[p, f] = \pi(p)$ is differentiable and $E \to M$ is a (general) bundle with fiber F. More precisely, $E(M, F, G, P)$ is said to be the *fiber bundle E over M with fiber F associated to the principal bundle* (P, M, G).

Example 1.4. Let H be a closed subgroup of G. Then H acts naturally on the quotient space G/H from the left. Therefore, $E = G \times_H G/H$ is a fiber bundle with fiber G/H associated to the principal bundle $(G, G/H, H)$.

Remark 1.2. A similar result is true when one considers a more general principal bundle P, *i.e.*, the associated bundle E is diffeomorphic to the quotient P/H. See Proposition 5.5 of Chapter I in (KOBAYASHI; NOMIZU, 1963).

A *cross section* of a vector bundle $E \to M$ is a mapping $\sigma : M \to E$ such that $\pi \circ \sigma$ is the identity on M. Notice that local sections always exist, since P is locally trivial. We finish this section with the following important result giving a criteria for when a reduction is possible.

Proposition 1.1. *The structure group G of (P, M, G) is reducible to a closed subgroup H if, and only if, the associated bundle $E(M, G/H, G, P)$ admits a cross section $\sigma : M \to E \approx P/H$.* □

1.2 Connection theory

Let (P, M, G) be a principal bundle with projection $\pi : P \to M$. By definition, the map π is differentiable. A tangent vector $X \in T_pP$ is said to be *vertical* if $X \in \ker d\pi_p$. Let $\mathcal{V}_p$ denote the kernel of π and $v : T_pP \to \mathcal{V}_p$ is the vertical part mapping. Although in any principal bundle, there is the notion of a vertical vector field, there is no standard way to say that a vector is "horizontal". The notion of horizontality is introduced by endowing a principal bundle with a connection. Formally, a *connection* on a principal bundle P is a *distribution* $\mathcal{H}$ of subspaces in the tangent bundle TP such that at each point $p \in P$, one has the direct sum

$$T_pP = \mathcal{H}_p \oplus \mathcal{V}_p.$$

We also require that this distribution is equivariant by the G-action, *i.e.*, if $R_g : P \to P$ denotes the right action by an element $g \in G$, then $\mathcal{H}$ must satisfy

$$\mathcal{H}_{p \cdot g} = (R_g)_*(\mathcal{H}_p)$$

where $(R_g)_*$ denotes the differential.

Associated to every connection is the *connection form*, denoted by ω. It is the unique linear functional defined on P and taking values on the Lie algebra $\mathfrak{g}$ such that for a vector field X, $\omega_p X = A$ where $A \in \mathfrak{g}$ is such that

$$\left.\frac{d}{dt}\right|_{t=0} (p \cdot e^{tA}) = v(X_p). \tag{1.1}$$

The correspondence between connections (as distributions) and connection forms are 1:1, so on we may refer to them unambiguously simply as a *connection*.

The expression on the left-hand side of Equation 1.1 is called the *fundamental vector field corresponding to* $A \in \mathfrak{g}$ and is denoted by A^*, that is, at a point p

$$A_p^* = \left.\frac{d}{dt}\right|_{t=0} (p \cdot e^{tA}).$$

In short, the connection ω provides a way to measure the $\mathfrak{g}$-change of a vector field X by taking as a reference the exponential flows on P among different fibers.

Example 1.5. Let $G = \mathrm{Sl}_n(\mathbb{C})$ and $K = \mathrm{SU}_n$ be the compact real form. Let $(K, \mathbb{F}, T)$ be the principal bundle over the maximal flag manifold $\mathbb{F}$ as shown in Example 1.3. Let $x_0 \in K$ denote the origin. Then $T_{x_0}K = \mathfrak{su}_n$ which has a decomposition in terms of root spaces $\mathfrak{g}_\alpha$ given by

$$\mathfrak{su}_n = \mathfrak{t} \oplus \sum_{\alpha > 0} \mathfrak{u}_\alpha$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{su}_n$ and $\mathfrak{t}$ is the Lie algebra of T. One particularly useful example of an *invariant connection* on K is constructed by first defining the horizontal space at the origin to be the sum

$$\mathcal{H}_{x_0} = \sum_{\alpha > 0} \mathfrak{u}_\alpha.$$

Then define $\mathcal{H}_g = (L_g)_* \mathcal{H}_{x_0}$, where L_g is left multiplication by $g \in K$. Notice that $L_g \circ \pi = \pi \circ L_g$, so one has the direct sum $T_gK = \mathcal{H}_g \oplus \mathcal{V}_g$. Now, for any $t \in T$

$$(R_t)_* \mathcal{H}_{x_0} = (L_t)_* \circ \mathrm{Ad}(t^{-1}) \mathcal{H}_{x_0} = (L_t)_* \mathcal{H}_{x_0} = \mathcal{H}_t$$

because H_{x_0} is invariant by $\mathrm{Ad}(T)$. Since L_g commutes with R_t for all $t \in T$ and $g \in K$, one immediately obtains the equivariance condition by the T-action. □

Remark 1.3. A connection on a principal bundle allows one to speak of horizontal curves and liftings, which leads to the notion of parallel transport (in (KOBAYASHI; NOMIZU, 1963), the authors call it *parallel displacement*). It is known that a parallel transport induces an isomorphism between the fibers (essentially, the group G) over the starting and ending points. Considering it on closed loops, one arrives at the notion of *holonomy groups* $\Phi(x)$ which, roughly speaking, is defined as the set of G-isomorphisms derived from the parallel transports among all closed loops at a base point $x \in M$. See Chapter II, Sections 3 and 4 for more on the subject; Section 8 is devoted to the proof of a theorem on holonomy by Ambrose and Singer.

Back to the general theory. Let (P, M, G) be a principal bundle with a given connection form ω. We introduce now the notion of a curvature form on P, the covariant derivative of a connection form. The curvature form is related to the more geometrical notion of a curvature on vector bundles[2]. Precisely, the *curvature form* Ω is a differential form of degree 2 defined on P taking values on $\mathfrak{g}$. It is defined as the covariant derivative of the curvature form ω, that is, if we let $h : TP \to \mathcal{H}$ be the projection onto the horizontal part, then

$$\Omega(X, Y) = d\omega(hX, hY).$$

By definition, if X is vertical, then $\Omega(X, Y) = 0$.

Just as in the case of vector bundles, the connection form measures the non-commutativity of the Lie bracket. Equation 1.2 is known as the *structure equation*.

Theorem 1.1. *Let ω be a connection form and Ω be its curvature form. Then*

$$d\omega(X, Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y) \tag{1.2}$$

for $X, Y \in TP$.

A special case of the above theorem is when all vector fields involved are horizontal. It simplifies the structure equation, since ω is a vertical form. See Chapter II, section V of (KOBAYASHI; NOMIZU, 1963).

Corollary 1.1. *Let X and Y be horizontal vector fields on P. Then*

$$\omega([X, Y]) = -2\,\Omega(X, Y).$$

We go back to the *most important example* in this text and provide an explicit formula for the curvature. It is used in section 3.3 to give an explicit set of generators for the cohomology of the quaternionic maximal flag.

[2] The relationship between these two is presented in (SPIVAK, 1999), Addendum 3 of Chapter 8.

Example 1.6. Let us expand on Example 1.5, where $U = \mathrm{SU}_n$ and an invariant connection was constructed using

$$\mathcal{H}_{x_0} = \sum_{\alpha>0} \mathfrak{u}_\alpha.$$

The associated connection form ω is a left-invariant differential form, meaning that for $u, v \in U$, one has

$$(L_u^*\omega)_v(X_v) = \omega_{L_u(v)}(d(L_u)_v(X_v)) = \omega_v(X_v),$$

or simply

$$L_u^*\omega = \omega \quad \forall\, u \in U.$$

So, for a left-invariant vector field induced by $X \in \mathfrak{m}$, one has for any $u \in U$ that

$$\omega_u(X_u) = \omega_e(X_e) = v(X)$$

where v is the projection $T_eU \to \mathcal{V}_e = \ker d\pi_e$. Since the connection is left-invariant, the induced vector field X of a horizontal X_e is also horizontal (and invariant). Apply Corollary 1.1 to get the *nice formula*

$$\Omega(X, Y) = -\frac{1}{2}v([X, Y]). \tag{1.3}$$

□

1.3 Chern-Weil Homomorphism

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A *form* of degree k on G is a symmetric multilinear map

$$f : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{(k \text{ factors})} \to \mathbb{R}.$$

If f and g are forms of degree k and l, respectively, define their product to be the form fg of degree $k + l$ given by the formula

$$\begin{aligned} &fg(X_1, \ldots, X_{k+l}) \\ &= \frac{1}{(k+l)!}\sum_\sigma f(X_{\sigma(1)}, \ldots, X_{\sigma(k)})g(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}) \end{aligned} \tag{1.4}$$

where σ runs through the permutations of $\{1, 2, \ldots, k + l\}$. Clearly, one has $fg = gf$. Denote by $\mathrm{Sym}_k(G)$ the set of all such forms of degree k and put

$$\mathrm{Sym}(G) = \sum_k \mathrm{Sym}_k(G).$$

Then $\mathrm{Sym}(G)$ has the structure of a commutative (graded) algebra over $\mathbb{R}$. This algebra is called *the polynomial algebra of* G.

Remark 1.4. There are two reasons for the name "polynomial algebra": 1) I already use "form" in "differential forms", and 2) the polynomial algebra is isomorphic to an actual set of polynomials. Such isomorphism is explicitly stated in (KOBAYASHI; NOMIZU, 1996), Section 2 of Chapter 12.

A form $f \in \mathrm{Sym}(G)$ is said to be $\mathrm{Ad}(G)$-*invariant* if $f \circ \mathrm{Ad}(g) = f$ for all $g \in G$. Here the composition is being taken entry-wise. Denote by $\mathrm{Inv}(G)$ the set of all $\mathrm{Ad}(G)$-invariant forms. This is a (commutative) subalgebra of $\mathrm{Sym}(G)$. Naturally, one defines $\mathrm{Inv}_k(G) = \mathrm{Inv}(G) \cap \mathrm{Sym}_k(G)$.

Let (P, M, G) be a principal bundle with connection form ω and curvature form Ω. Take $f \in \mathrm{Inv}(G)$. Then $f(\Omega)$ is the differential $2k$-form defined on P by

$$\begin{aligned} &f(\Omega)(X_1, \dots, X_{2k}) \\ &\quad = \frac{1}{(2k)!} \sum_{\sigma} \epsilon_\sigma f(\Omega(X_{\sigma(1)}, X_{\sigma(2)}), \dots, \Omega(X_{\sigma(2k-1)}, X_{\sigma(2k)})) \end{aligned} \tag{1.5}$$

where ϵ_σ denotes the sign of the permutation σ on $\{1, 2, \dots, 2k\}$. Sometimes the sum in Equation 1.5 is referred as the *σ-alternating sum of* $f(\Omega(X_1, X_2), \dots, \Omega(X_{2k-1}, X_{2k}))$.

A basic fact of the Chern-Weil theory is that any differential form $\xi = f(\Omega)$ is projectable in M in the sense that there exists a differential form η on M such that $\pi^*(\eta) = \xi$ where π^* is the pullback of differential forms.

The following result summarizes the constructions and results concerning the Chern-Weil homomorphism.

Theorem 1.2 (Chern-Weil Theorem)**.** *Let (P, M, G) be a differentiable principal bundle with projection $\pi : P \to M$. Endow P with a connection form ω and let Ω be its curvature.*

Let $f \in \mathrm{Inv}(G)$. Then there are the following results:

1. *The differential $2k$-form $f(\Omega)$ projects (uniquely) onto a differential $2k$-form on M. The projected form is also denoted by $f(\Omega)$;*

2. *The differential form $f(\Omega)$ on M is closed. Its de Rham cohomology class $[f(\Omega)] \in H^{2k}(M; \mathbb{R})$ is independent of the choice of the connection ω on P. This cohomology class is denoted simply by $[f] = [f(\Omega)]$;*

3. *The map*
$$\begin{array}{rccc} [\cdot]: & \mathrm{Inv}(G) & \to & H^*(M; \mathbb{R}) \\ & f & \mapsto & [f] \end{array}$$
is an algebra homomorphism.

The map $[\cdot]$ is called the *Chern-Weil homomorphism* and the cohomology classes $[f]$ are the Chern-Weil (characteristic) classes of the principal bundle $\pi : P \to M$.

If G is a complex (real) semisimple Lie group, the Chern-Weil classes of a principal bundle (P, M, G) are actually Chern classes (Pontryagin classes) in the standard way, that is, they are characteristic classes of complex (real) vector bundles —see Section 3 of Chapter 12 in (KOBAYASHI; NOMIZU, 1996). Simply calling such classes "Chern-Weil classes" makes sense because we deal with both of them: Chern classes on chapter 2 and Pontryagin classes on chapter 3.

A book by Loring Tu ((TU, 2017)) provides a nice alternative reference for this topic. For more on characteristic classes, see (MILNOR; STASHEFF, 1974) and (HATCHER, 2002). A somewhat didactic presentation on this subject is found on the lecture notes by Michael Uscher called "Vector Bundles" (see (USCHER, 2012)).

2 Complex Flag Manifolds

In this chapter, we formally introduce the flag manifolds as a quotient of a semisimple Lie group by a parabolic subgroup. The flag $\mathbb{F}$ has a CW-complex structure given by the Schubert cells. So, in order to study the cohomology of $\mathbb{F}$, we give full attention to these cells and their properties, which are explored in section 2.2 and section 2.3. Our aim is to construct invariant volume forms on each of those cells and integrate them to obtain a BGG-like formula[1] for the case when $\mathbb{F}$ is maximal. The techniques used here are different from those of BGG and can be applied to non-complex flags. We did it for the quaternionic maximal flag of $\mathfrak{sl}_n(\mathbb{H})$ (see chapter 3).

2.1 Flag Manifolds

The flag manifolds of a complex semisimple Lie group G are the main object of study in this text. They are generally defined as the quotient of G by some parabolic group B_Θ. By definition, each flag manifold is naturally associated to a principal bundle $G \to G/B_\Theta$ with structure group B_Θ, as was done in Example 1.1. The most important example to us is given in Example 1.2, *i.e.*, $\mathbb{F} = \mathrm{Sl}_n(\mathbb{C})/B$, where B is the minimal parabolic subgroup of G—the *Borelian* subgroup (see below).

To start with, let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by Π the associated set of roots. Then $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha.$$

For a choice of a simple system of roots $\Sigma \subseteq \Pi$, let Π^+ denote the set of positive roots. The *Borel subgroup* B is the connected subgroup of G with Lie algebra

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha.$$

For a subset $\Theta \subseteq \Sigma$, let $\langle \Theta \rangle$ be the subset of Π^+ generated by the roots $\alpha \in \Theta$. Define the *parabolic subgroup* $B_\Theta = \exp \mathfrak{b}_\Theta$ as the connected subgroup of G with Lie algebra

$$\mathfrak{b}_\Theta = \mathfrak{b} \oplus \sum_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_{-\alpha}.$$

The *flag manifold* $\mathbb{F}_\Theta$ is defined as the quotient G/B_Θ. If Θ is the empty set, then $B_\Theta = B$ and the corresponding flag manifold is called the *maximal flag manifold* of G. In this case, one simply writes $\mathbb{F}$.

[1] This is Equation 1.

Example 2.1. Just as in Example 1.2, the *partial flag manifolds* of $G = \mathrm{Sl}_n(\mathbb{C})$ can be seen as a set of nested subspaces of $\mathbb{C}^n$. For instance, the complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ is a partial flag manifold with parabolic subgroup B_Θ given by $\Theta = \{\alpha_k\}$. A partial flag in $\mathbb{C}^n$ is obtained from the maximal flag by forgetting some of the subspaces. Which subspaces are forgotten is indicated by the *flag signature*, that is, a sequence of integers $(d_1, \dots, d_m)$ such that $d_1 < d_2 < \cdots < d_m < n$. When $\mathbb{F}$ is maximal, $d_m = m$ for all $0 < m < n$ (equivalently, $m = n$). So generally speaking, a flag manifold (partial or maximal) in $\mathbb{C}^n$ of signature $(d_1, \dots, d_m)$ is a sequence of nested subspaces

$$x = (V_1, \dots, V_m)$$

where $\dim V_k = d_k$, $1 < k_j < n$ and $V_j \subset V_{j+1}$. Denote by $\mathbb{F}_\Theta$ the set of all such flags in $\mathbb{C}^n$ (Θ is to be determined). It is known, by linear algebra, that G acts transitively on $\mathbb{F}_\Theta$ with isotropy given by some subgroup, say B_Θ. Then

$$\mathbb{F}_\Theta = G/B_\Theta.$$

As the notation already suggests, B_Θ is the integral subgroup of the Lie algebra $\mathfrak{b}_\Theta$ for some set $\Theta \subset \Sigma$. To find out what Θ is this, let

$$x_0^\Theta = (E_{d_1}, E_{d_2}, \dots, E_{d_m}) \in \mathbb{F}_\Theta$$

where E_k is the subspace of $\mathbb{C}^n$ spanned by the set of canonical vectors $\{e_j\}_{j=1}^k$. As noted, G acts transitively on $\mathbb{F}_\Theta$, so we turn our focus to the isotropy subgroup B_Θ, that is, the set of all $g \in G$ such that $g \cdot x_0^\Theta = x_0^\Theta$. Notice that $g \in B_\Theta$ if and only if $g \cdot E_k = E_k$ for each k in the signature. For E_{d_1} to be fixed by g, one needs to send the canonical vectors $\{e_1, e_2, \dots, e_{d_1}\}$ inside the space generated by themselves, or more specifically, the image cannot involve a linear combination containing any element e_k for $k > d_1$ (this is the condition). Take the matrix-block representation of

$$g = \begin{bmatrix} g_{d_1} & ** \\ * & g_{n-d1} \end{bmatrix}.$$

The submatrix g_m is of size $m \times m$. The condition on the linear combination is equivalent to the "$*$" block of size $(n - d_1) \times d_1$ being null, which is equivalent to $g \cdot E_{d_1} = E_{d_1}$. By continuing this process for $E_{d_2}, E_{d_3}, \dots, E_{d_m}$, the final form of the matrix g is

$$g = \begin{bmatrix} g_{d_1} & * & \cdots & * \\ 0 & g_{d_2-d_1} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{n-d_m} \end{bmatrix}.$$

The root system of $\mathfrak{sl}_n(\mathbb{C})$ is of type A. Each root α_{jk} is decomposed as a sum of simple roots

$$\alpha_{jk} = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{k-1}$$

where α_l is short for the simple root $\alpha_{l;l+1}$. Thus one immediately[2] sees that the isotropy B_Θ corresponds to

$$\Theta = \Sigma \backslash \{\alpha_{d_1}, \ldots, \alpha_{d_m}\}.$$

So, starting from the maximal signature "forgetting" an entry d_j is the same as adding the respective simple root to the isotropy. The smaller flag, the larger the isotropy. □

Other examples of interesting flag manifolds are obtained when one endows $\mathbb{C}^n$ with a non-degenerate bilinear form φ, *e.g.,* the cannonical inner product. Take G be the group of automorphisms of $\mathbb{C}^n$ preserving such form. Then $G \cdot x_0$ is the set of complete flags $(V_1, \ldots, V_{n-1})$ such that V_k and V_{n-k} are mutually φ-orthogonal.

2.2 Schubert Cellular Decomposition

Each flag manifold G/B_Θ has a special cellular decomposition, the so called *Schubert cell decomposition* and each cell is called a *Schubert cell*, denoted by $\mathcal{S}_w$. They come from the *Bruhat decomposition*.

In order to introduce the Bruhat decomposition, first consider the more general group of invertible matrices $G = \mathrm{Gl}_n(\mathbb{C})$. This example provides a good motivation for the definitions below. It is known that any $g \in G$ can be written as a product of the form bwb', where b and b' are both upper triangular matrices and w is a permutation matrix. Therefore, one writes G as the union of double cosets

$$G = \bigcup_{w \in \mathcal{W}} BwB$$

where B is the set of all upper triangular matrices in G and $\mathcal{W}$ is the set of all permutations in n elements represented as $n \times n$ matrices. In the quotient space G/B, the double cosets becomes "N-orbits" (explained below), so

$$G/B = \bigcup_{w \in \mathcal{W}} Bw \cdot x_0,$$

where $x_0 = 1 \cdot B$ is the coset class corresponding to the neutral element '1' in G. Each orbit $Bw \cdot x_0$ is what one calls a Bruhat cell, all indexed by the group $\mathcal{W}$. The closure of an orbit is what defines a Schubert cell.

Now we go back to the theory of semisimple Lie groups. Let G be a complex semisimple Lie group and let U denote its *compact real form*. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{h}$ a Cartan subalgebra with root system Π and Σ the set of simple roots. The *Weyl group* is first defined as the finite group generated by the reflections

$$r_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

[2] The the Dynkin diagram of $\mathfrak{sl}_n(\mathbb{C})$ is the simplest—see Chapter 8 of (SANMARTIN, 1999).

with respect to the simple roots $\alpha \in \Sigma$ where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{h}^*$ defined by the Cartan-Killing form. As an example, take the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ with the standard Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. Then the Weyl group $\mathcal{W}$ is the permutation group in n elements that acts in $\mathfrak{h}$ by changing the order of the diagonal entries. In $\mathfrak{h}^*$ the action is similarly obtained by transposition.

The Weyl group $\mathcal{W}$ is isomorphic to the quotient $\mathcal{W} = \mathrm{Norm}/T$ where[3]

$$\mathrm{Norm} = \mathrm{Norm}_U(\mathfrak{h}) = \{ u \in U \ : \ \mathrm{Ad}(u)\mathfrak{h} \subset \mathfrak{h} \} \tag{2.1}$$

is the normalizer and $T \subset U$ is the maximal torus. The *Bruhat decomposition* for G is[4]

$$G = \bigcup_{w \in \mathcal{W}} BwB. \tag{2.2}$$

This union is disjoint up to T, that is, $BwB \cap Bw'B \neq \emptyset$ if and only if $wT = w'T$ and in this case $BwB = Bw'B$. Hence the set of components BwB is in bijection with the Weyl group $\mathcal{W} = \mathrm{Norm}/T$.

The double coset decomposition of G factors down to the maximal flag $\mathbb{F} = G/B$. As seen in the section 2.1, the Lie algebra of B is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Set $H = \exp \mathfrak{h}$ and $N = \exp \mathfrak{n}^+$. Then $B = HN = NH$. Let $x_0 = 1 \cdot B$ be the origin of this coset space and for $w \in \mathrm{Norm}$, put $x_w = w \cdot x_0$. The double coset space BwB is projected onto $N \cdot x_w$. In fact, $BwB \cdot x_0 = Bw \cdot x_0 = B \cdot x_w$. Because $B = NH$ and $H \cdot x_0 = x_0$, one has $B \cdot x_w = N \cdot x_w$ (w normalizes H). The orbits $\mathcal{B}_w = N \cdot x_w$ are the *Bruhat cells* in $\mathbb{F}$. In light of Equation 2.2, they exhaust the N-orbits and are in bijection with the Weyl group $\mathcal{W}$.

More generally, in a partial flag manifold $\mathbb{F}_\Theta = G/B_\Theta$ the *Bruhat cells* are the N-orbits $\mathcal{B}_w^\Theta = N \cdot x_w^\Theta$ where $x_0^\Theta = 1 \cdot B_\Theta$ is the origin and $x_w^\Theta = w \cdot x_0^\Theta$ with $w \in \mathrm{Norm}$. The same way the cells are indexed by the Weyl group. The difference now is that the indexing is not exact since it may happen that $\mathcal{B}_w^\Theta = \mathcal{B}_{w'}^\Theta$ with $w \neq w'$.

Definition 2.1. A Schubert cell is denoted by $\mathcal{S}_w^\Theta \subseteq \mathbb{F}_\Theta$, indexed by $w \in \mathcal{W}$; and is defined as the closure of the Bruhat cell $\mathcal{B}_w^\Theta = N \cdot w x_0^\Theta$.

Remark 2.1. Notice that $\mathcal{B}_w^\Theta = N \cdot x_w^\Theta$ is a submanifold of $\mathbb{F}_\Theta$ since it is an orbit of a Lie subgroup (it can be proved that any $\mathcal{B}_w^\Theta$ is diffeomorphic to a Lie subgroup of N and hence to an Euclidean space). A Schubert cell $\mathcal{S}_w^\Theta$, on the other hand, is not in general a submanifold. It contains the corresponding Bruhat cell $\mathcal{B}_w^\Theta$ as an open and dense subset, which enables several computations to be worked out in $\mathcal{S}_w^\Theta$ as it were a submanifold. In section 2.4, integration of volume forms on $\mathcal{S}_w^\Theta$ is done using this open and dense property of the Bruhat cells.

[3] The isomorphism is given by Theorem 4.54 of Chapter IV in (KNAPP, 2013).
[4] This is Theorem 7.40 of Chapter VII in (KNAPP, 2013).

Example 2.2. Let $G = \mathrm{Sl}_n(\mathbb{C})$ and Lie algebra $\mathfrak{g}$ with the standard set of simple roots $\alpha_{j;\,j+1} = \lambda_j - \lambda_{j+1}$ and take $\mathbb{F} = G/B$ to be the maximal flag manifold. Choose $\alpha = \alpha_{1;\,2}$ and let $w = r_\alpha$ be a simple reflection. Then x_0 corresponds to the distinguished flag in Example 1.2 and w acts on the canonical vectors e_j via permutation of the indices. So the Bruhat cell $\mathcal{B}_w$ is the N-orbit through the flag

$$x_w = (V_1, E_2, E_3, \ldots, E_{n-1}) \in \mathcal{F}$$

where V_1 is the one-dimensional subspace of $\mathbb{C}^n$ spanned by e_2. Each $g \in N$ is an upper triangular matrix with '1' in each entry of the diagonal. By letting N act on x_w, one sees that

$$\mathcal{B}_w = \{\, (V_1, E_2, E_3, \ldots, E_{n-1}) \,:\, E_1 \neq V_1 \,\}$$

which is homeomorphic to $\mathbb{C}\mathrm{P}^1 \backslash \{x_0\}$, where

$$x_0 = (E_1, E_2, \ldots, E_{n-1}).$$

Its closure, that is, the Schubert cell $\mathcal{S}_w$ *is* homeomorphic $\mathbb{C}\mathrm{P}^1$, which in turn is a two-dimensional sphere. This is a feature of the Schubert cells (see Proposition 2.3 and Corollary 2.2). □

Local properties of the cells

In what follows, local properties of $\mathcal{B}_w$ (and $\mathcal{S}_w$) of $\mathbb{F}$, mainly, are described in terms of the indexer element $w \in \mathcal{W}$.

To write the *tangent space* of $\mathcal{B}_w$ at x_w in terms of the root spaces, first take a representative $u \in$ Norm of $w \in \mathcal{W}$. Set

$$\mathfrak{n}^{w^{-1}} = \mathrm{Ad}(u^{-1})\mathfrak{n}^+ = \mathrm{Ad}(u^{-1}) \left(\sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \right) = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{w^{-1}\alpha},$$

where $w^{-1}\alpha = \alpha \circ w$ and $N^{w^{-1}} = u^{-1}Nu = \exp \mathfrak{n}^{w^{-1}}$. Then

$$\mathcal{B}_w = N \cdot ux_0 = u(u^{-1}Nu) \cdot x_0 = u(N^{w^{-1}} \cdot x_0).$$

Since $N \cdot x_0 = x_0$, it follows that $N^{w^{-1}} \cdot x_0 = \left(N^{w^{-1}} \cap N^- \right) \cdot x_0$. The root space decomposition shows that $N^{w^{-1}} \cap N^- = \exp \left(\mathfrak{n}^{w^{-1}} \cap \mathfrak{n}^- \right)$ where

$$\mathfrak{n}^{w^{-1}} \cap \mathfrak{n}^- = \sum_\beta \mathfrak{g}_\beta$$

with the sum extended over the set $w^{-1}\Pi^+ \cap \Pi^-$ of roots $\beta < 0$ such that $\beta = w^{-1}\alpha$ with $\alpha > 0$.

Turning back to the N-orbit through x_w, it follows that

$$N \cdot x_w = u\left((N^{w^{-1}} \cap N^-) \cdot x_0\right) = u(N^{w^{-1}} \cap N^-)u^{-1} \cdot x_w.$$

Hence the Bruhat cell $\mathcal{B}_w = N \cdot x_w$ is the orbit by x_w of the subgroup

$$u(N^{w^{-1}} \cap N^-)u^{-1} = \exp \mathrm{Ad}(u)(\mathfrak{n}^{w^{-1}} \cap \mathfrak{n}^-) = \exp\left(\sum_{\alpha \in \Pi_w} \mathfrak{g}_{-\alpha}\right)$$

where

$$\Pi_w = w(w^{-1}\Pi^+ \cap \Pi^-) = \Pi^+ \cap w(\Pi^-) \tag{2.3}$$

is the set of positive roots that are mapped into negative roots by w^{-1}.

To write tangent vectors we use the notation $X \cdot x$ to stand for the value at x of the vector field induced by $X \in \mathfrak{g}$, namely

$$X \cdot x = \left.\frac{d}{dt}\right|_{t=0} (e^{tX} \cdot x).$$

More generally, if $\mathfrak{v}$ is a vector subspace of the Lie algebra $\mathfrak{g}$ and x is an element of a flag manifold $\mathbb{F}_\Theta$, then

$$\mathfrak{v} \cdot x = \{X \cdot x \in T_x\mathbb{F}_\Theta : X \in \mathfrak{v}\}.$$

Now the tangent space $T_{x_w}\mathcal{B}_w$ of the Bruhat cell $\mathcal{B}_w$ at its origin x_w is the tangent space of the $u(N^{w^{-1}} \cap N^-)u^{-1}$-orbit, that is,

$$T_{x_w}\mathcal{B}_w = \sum_{\alpha \in \Pi_w} \mathfrak{g}_{-\alpha} \cdot x_w$$

with $\Pi_w = \Pi^+ \cap w(\Pi^-)$ as in Equation 2.3. This is the tangent space $T_{x_w}\mathcal{S}_w$ to the Schubert cell as well.

The expression for the tangent space will be used afterwards for the computation of the integrals of Chern-Weil forms in the Schubert cells.

The tangent space $T_{x_w}\mathcal{B}_w = T_{x_w}\mathcal{S}_w$ can be written in terms of subspace of the compact real form $\mathfrak{u}$. To get it for a root $\alpha > 0$ put $\mathfrak{u}_\alpha = \mathfrak{u} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. If $\alpha \in \Pi_w$ then $\mathfrak{g}_{-\alpha} \cdot x_w = \mathfrak{u}_\alpha \cdot x_w$ so that

$$T_{x_w}\mathcal{B}_w = T_{x_w}\mathcal{S}_w = \sum_{\alpha \in \Pi_w} \mathfrak{u}_\alpha \cdot x_w$$

We summarize the above discussion in the following proposition.

Proposition 2.1. *The tangent space of $\mathcal{S}_w$ at the origin x_w coincides with the tangent space of $\mathcal{B}_w = N \cdot x_w$. It is given by*

$$T_{x_w}\mathcal{B}_w = \sum_{\alpha \in \Pi_w} \mathfrak{g}_{-\alpha} \cdot x_w. \tag{2.4}$$

In terms of the compact real form U, let $\mathfrak{u}_\alpha = \mathfrak{u}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Then

$$T_{x_w}\mathcal{B}_w = T_{x_w}\mathcal{S}_w = \sum_{\alpha \in \Pi_w} \mathfrak{u}_\alpha \cdot x_w. \tag{2.5}$$

To write a basis of $T_{x_w}\mathcal{B}_w = T_{x_w}\mathcal{S}_w$, we recall the concept of a *Weyl basis* for $\mathfrak{g}$, which is a set of vectors $X_\alpha \in \mathfrak{g}_\alpha$ satisfying the properties

1. $[X_\alpha, X_{-\alpha}] = H_\alpha$;
2. $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root.
3. The constant $m_{\alpha,\beta} \in \mathbb{R}$ satisfies $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$.

For each root $\alpha \in \Pi^+$, the vectors

$$S_\alpha = i(X_\alpha + X_{-\alpha}) \quad \text{and} \quad A_\alpha = X_\alpha - X_{-\alpha} \tag{2.6}$$

forms a real basis of $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u}$. This leads to

Corollary 2.1. *The set*

$$B_w = \{\, S_\alpha, A_\alpha \ : \ \alpha \in \Pi_w \,\} \tag{2.7}$$

is a real basis of $T_{x_w}(S_w)$, so the real dimension of $\mathcal{S}_w$ is $2\,l(w)$. □

Remark 2.2. Take a reduced expression $w = r_{\alpha_1} \cdots r_{\alpha_k}$ with respect to the simple roots. It is well known that

$$\Pi_w = \{\, \alpha_1, r_{\alpha_1}\alpha_2, \ldots, r_{\alpha_1} \cdots r_{\alpha_{k-1}}\alpha_k \}$$

and that Π_w is independent of the chosen reduced expression. See (SANMARTIN, 1999; VARADARAJAN, 1974).

We take this opportunity to state another property of such vectors which is used later in the text. This is found in (SANMARTIN, 1999).

Proposition 2.2. *The vectors defined in Equation 2.6 satisfies the bracket identities:*

$$\begin{aligned} [A_\alpha, A_\beta] &= m_{\alpha,\beta} A_{\alpha+\beta} + m_{-\alpha,\beta} A_{\alpha-\beta}, \\ [S_\alpha, S_\beta] &= -m_{\alpha,\beta} A_{\alpha+\beta} - m_{\alpha,-\beta} A_{\alpha-\beta}, \\ [A_\alpha, S_\beta] &= m_{\alpha,\beta} S_{\alpha+\beta} + m_{\alpha,-\beta} S_{\alpha-\beta} \end{aligned} \tag{2.8}$$

and

$$[A_\alpha, S_\alpha] = 2iH_\alpha.$$

Alternative definitions of the Schubert cells

There is an alternative way to define the Schubert cells in the maximal flag that does not require mentioning the underlying Bruhat cells. This is done via fibrations over a selection of partial flag manifolds. In order to do that, let $w \in \mathcal{W}$ and let

$$w = r_{\alpha_1} \cdots r_{\alpha_k}$$

be a reduced decomposition with respect to the simple roots. Put $\Theta_j = \{\alpha_j\}$ and write $\mathbb{F}_j = \mathbb{F}_{\Theta_j} = G/B_{\Theta_j}$. Let $\pi_j = \pi_{\Theta_j}$ be the canonical projection $\mathbb{F} \to \mathbb{F}_j$ of the maximal flag onto the partial flag. In terms of flags of $\mathbb{C}^n$, this projection acts by forgetting some entries of each maximal flag $x = (V_1, V_2, \ldots, V_{n-1})$, therefore changing the signature of the flag. Then, define the *fiber exhausting maps* γ_j by setting

$$\gamma_j(A) = \pi_j^{-1}(\pi_j(A)), \quad \text{for } A \subseteq \mathbb{F}.$$

Such maps are used below to (re)define the Schubert cells. For instance, the Schubert cell in Example 2.2 is recovered by noticing that

$$\mathcal{S}_w = \gamma_{\alpha_{12}}(\{x_0\}).$$

Definition 2.2. Let $w \in \mathcal{W}$ with reduced decomposition $w = r_{\alpha_1} \cdots r_{\alpha_k}$ with respect to the simple roots. Then the Schubert cell $\mathcal{S}_w$ is given by

$$\mathcal{S}_w = \gamma_k \circ \cdots \circ \gamma_1(\{x_0\}).$$

The equivalence between Definition 2.1 and Definition 2.2 is the statement of Theorem 5.3 in (SANMARTIN, 1998).[5] We use the second one to provide a description of S_w as product orbit. Start with a unit length $w_1 = r_{\alpha_1}$, it is true that $\mathcal{S}_w = \gamma_1(\{x_0\}) = B_{\Theta_1} \cdot x_0$. For $w_2 = r_{\alpha_1} r_{\alpha_2}$, one has

$$\gamma_2\gamma_1(\{x_0\}) = \gamma_2\Big(\bigcup_{g \in B_{\Theta_1}} g \cdot x_0\Big) = \bigcup_{g \in B_{\Theta_1}} \Big(g \cdot \gamma_2(\{x_0\})\Big).$$

The last equality follows by the equivariance of γ_j, since π_j is equivariant. Continuing, notice that $\gamma_2(\{x_0\}) = B_{\Theta_2} \cdot x_0$, so

$$\gamma_2\gamma_1(\{x_0\}) = \bigcup_{g \in B_{\Theta_1}} \Big(gB_{\Theta_2} \cdot x_0\Big) = B_{\Theta_1} B_{\Theta_2} \cdot x_0.$$

Following by induction, one is lead to the general formula

$$\mathcal{S}_w = B_{\Theta_1} \cdots B_{\Theta_k} \cdot x_0,$$

where $w = r_{\alpha_1} \cdots r_{\alpha_k}$. The same is true when one considers the compact real form K.

Proposition 2.3. *Let $w = r_{\alpha_1} \cdots r_{\alpha_k}$ be a reduced expression with respect to the simple roots. Let $K_\Theta = K \cap B_\Theta$, where K is the compact real form of G. Then*

$$\mathcal{S}_w = K_{\Theta_1} \cdots K_{\Theta_k} \cdot x_0.$$

Proof. This follows almost immediately from the fact the $B_{\Theta_j} \cdot x_0 = K_{\Theta_j} \cdot x_0$ —by the Langlands decomposition $B_{\Theta_j} = K_{\Theta_j} AN$ and the fact the $AN \cdot x_0 = x_0$. □

[5] In section 4, in the appendix, we explore an example of Schubert cell using both definitions.

We end this section with a proposition that illustrates the nice behavior of the Schubert cells under the canonical projection $\mathbb{F}_\Theta \to \mathbb{F}_{\Theta'}$ and a minimal decomposition of $w \in \mathcal{W}$.

Proposition 2.4. *Let $w = r_{\alpha_1} \cdots r_{\alpha_k}$ be a minimal decomposition with respect to the simple roots. Let π be the restriction of $\pi_{\{\alpha_k\}} : \mathbb{F} \to \mathbb{F}_k$ to $\mathcal{S}_w$ and put $w' = r_{\alpha_1} \cdots r_{\alpha_{k-1}}$. Then*

1. *the projection $\pi : \mathcal{S}_w \to \mathcal{S}_w^k$ is an equivariant sphere fibration.*
2. *$\pi(\mathcal{S}_w) = \pi(\mathcal{S}_{w'})$.*
3. *$\pi(\mathcal{S}_{w'}) \approx \mathcal{S}_w^k$.*

Proof. item 1 Let $x_0^k = \pi(x_0)$. By Definition 2.2, x_0^k is a flag of $\mathbb{C}^n$ that is almost complete. The only entry that is missing is the subspace corresponding to the root α_k. So the fiber over x_0^k is diffeomorphic to a $\mathbb{C}P^1 \approx S^2$, because the fiber is the set of all j-dimensional vector subspaces containing a fixed $(j-1)$-dimensional which is also inside a $(j+1)$-dimensional vector subspace of $\mathbb{C}^n$; item 2 follows immediately by Definition 2.2

$$\pi(\mathcal{S}_w) = \pi \circ \pi^{-1} \circ \pi(\mathcal{S}_w) = \pi(\mathcal{S}_{w'})$$

because $\pi \circ \pi^{-1}$ is the identity on $\mathbb{F}_k$. □

2.3 Parametrization of the Schubert Cells

Our aim is to express the dual relationship between homology and cohomology of $\mathbb{F}$ by integrating representatives of the Chern-Weil classes—closed differential forms—over the Schubert cells which are known to be a basis of the homology. To do that, first one needs a suitable parametrization of such cells.

Since we are dealing with semisimple complex Lie groups, the simplest case comes from the maximal flag of $\mathfrak{sl}_2(\mathbb{C})$, which has a single positive root, call it α. The Weyl group is $\{\pm 1\}$ and therefore, the only non-trivial Schubert cell $\mathcal{S}_w$ is the whole flag $\mathbb{F}$. The non-trivial element $w = r_\alpha$ is represented in SU_2 by the matrix

$$r = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

By what has been discussed previously, we know that $\mathcal{S}_w = \mathrm{SU}_2 \cdot x_0$. The proposition below provides a useful map ψ of the two-dimensional ball $B^2 \subset \mathbb{C}$ into SU_2. This map, composed with the canonical projection $\mathrm{SU}_2 \to \mathrm{SU}_2/M$ gives a suitable parametrization of the cell, since $\mathrm{SU}_2/M \approx \mathrm{SU}_2 \cdot x_0$. The closed ball B^2 below is identified with $S^1 \times [0, \pi/2]$ in a way that $(S^1 \times \{0\})$ is taken into the center and $S^1 \times \{\pi/2\}$ is taken into the boundary ∂B^2.

Proposition 2.5. *There is a parametrization $\psi : B^2 \to \mathrm{SU}_2$ such that*

1. *The center of B^2 is taken into r_α, i.e., $\psi(S^1 \times \{0\}) = r$ where r is a representative of r_α in SU_2.*
2. *The boundary ∂B^2, namely $S^1 \times \{\pi/2\}$, is taken into M, so*
$$\psi(\partial B^2) \cdot x_0 = x_0.$$
3. *The mapping $B^2 \backslash \partial B^2 \to \mathcal{B}_w$, $(z,t) \mapsto \psi(z,t) \cdot x_0$ is a diffeomorphism.*

Proof. The idea is to use the exponential map $\exp : \mathfrak{su}_2 \to \mathrm{SU}_2$. Notice that

$$\mathfrak{su}_2 = \mathfrak{m} \oplus \mathfrak{u}_\alpha$$

where $\mathfrak{m}$ is the diagonal part. Consider the inclusion $S^1 \times [0, \pi/2] \hookrightarrow \mathfrak{u}_\alpha$ given by

$$S^1 \times [0,\pi/2] = \left\{ \begin{bmatrix} 0 & -t\bar{z} \\ tz & 0 \end{bmatrix} : |z| = 1,\, t \in [0,\pi/2] \right\}.$$

Let ϕ denote the restriction to B^2 of the exponential map considering such identification. Then

$$\phi(z,t) = \begin{bmatrix} \cos t & -\bar{z}\sin t \\ z \sin t & \cos t \end{bmatrix}. \tag{2.9}$$

Notice that ϕ is not the desired mapping, since $\phi(0) = \mathrm{Id} \neq r$. Therefore, let $\psi(z,t) = r\phi(z,t)$, *i.e.*,

$$\psi(z,t) = \begin{bmatrix} z \sin t & \cos t \\ -\cos t & \bar{z}\sin t \end{bmatrix}$$

so $\psi(0) = r$; and at the boundary one has

$$\psi(z, \pi/2) = \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \in M.$$

This proves items item 1 and item 2.

The mapping in item 3 is the composition of ψ with the projection $\mathrm{SU}_2 \to \mathrm{SU}_2/M$ identified with the orbit through x_0. In fact, x_0 is the one dimensional subspace of $\mathbb{C}^2$ spanned by the canonical vector e_1, denoted by $\langle e_1 \rangle$. Now, the corresponding Bruhat cell is defined as $\mathcal{B}_w = N \cdot rx_0$. On one hand, the subgroup N is given by

$$N = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{C} \right\}$$

so

$$N \cdot rx_0 = \left\langle \begin{bmatrix} z \\ 1 \end{bmatrix} \right\rangle$$

for all $z \in \mathbb{C}$. Here, $\langle v \rangle$ indicates the one-dimensional subspace generated by $v \in \mathbb{C}^2$. On the other hand,

$$\psi(z,t) \cdot x_0 = \left\langle \begin{bmatrix} z \sin t \\ \cos t \end{bmatrix} \right\rangle.$$

If $t \in [0, \pi/2)$, then $\cos t \neq 0$ and

$$\psi(z,t) \cdot x_0 = \left\langle \begin{bmatrix} z \tan t \\ 1 \end{bmatrix} \right\rangle \quad z \in S^1.$$

This shows the diffeomorphism between $B^2 \backslash \partial B^2$ and $\mathcal{B}_w$. □

At this point, we take advantage of the notation established to emphasize the fact the a minimal dimensional Schubert cell is homeomorphic to S^2.

Corollary 2.2. *A minimal dimensional Schubert cell $\mathcal{S}_w$ is topologically equivalent to a two-dimensional sphere S^2.*

The proof below is only a sketch and the details to show the bijection are left out, since we will need the mapping $\mathcal{S}_w \mapsto S^2$ only.

Proof. Consider $\psi(z,t) \cdot x_0$; or in terms of the corresponding matrix multiplication

$$\begin{bmatrix} z \sin t & \cos t \\ -\cos t & \bar{z} \sin t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z \sin t \\ -\cos t \end{bmatrix}$$

where $z = e^{i\theta} \in S^1$, so one can write $z = \cos\theta + i \sin\theta$. Identify (as sets) $\mathbb{C}$ and $\mathbb{R}^2$. Then

$$\mathcal{S}_w \ni \begin{bmatrix} e^{i\theta} \sin t \\ -\cos t \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta \sin t \\ \sin\theta \sin t \\ -\cos t \end{bmatrix} \in S^2 \subseteq \mathbb{R}^3. \tag{2.10}$$

By letting $(\theta, t) \in [0, \pi] \times [-\pi/2, \pi/2]$, one obtains the usual spherical coordinates on the sphere. □

Now we go back to general complex flag $\mathbb{F}$, which is a flag of a complex semisimple Lie group G with compact real form given by U. Denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{u}$. Let Σ be the set of simple roots and take $w = r_\alpha$ to be a simple reflection, $\alpha \in \Sigma$. Denote by $\mathfrak{u}(\alpha)$ the Lie algebra generated by $\mathfrak{u}_\alpha$. It is known that $\mathfrak{u}(\alpha) \approx \mathfrak{su}_2$. Denote by $U(\alpha) \approx \mathrm{SU}_2$ the corresponding analytical subgroup with Lie algebra $\mathfrak{u}(\alpha)$. By Proposition 2.3,

$$\mathcal{S}_w = U(\alpha) \cdot x_0.$$

Then by Proposition 2.5, there is a map $\psi_\alpha : B^2 \to U(\alpha)$ satisfying those stated properties.

Remark 2.3. One can only do this for simple roots, since for a non-simple root β, the orbit $Nr_\beta \cdot x_0$ is *not* contained in $U(\beta) \cdot x_0$.

The above discussion is generalized easily to Schubert cells of higher dimensions, in light of Proposition 2.3.

Theorem 2.1. *Let $\mathcal{S}_w$ be a Schubert cell in $\mathbb{F}$. Choose a minimal decomposition $w = r_{\alpha_1} \cdots r_{\alpha_k}$ and let $\psi_{\alpha_j} : B^2 \to U(\alpha)$ be defined as before. Define the map*

$$\begin{array}{cccc} \Psi : & B^{2k} & \to & \mathcal{S}_w \\ & (u_1, \dots, u_k) & \mapsto & \psi_{\alpha_1}(u_1) \cdots \psi_{\alpha_k}(u_k) \cdot x_0 \end{array}$$

where u_j is of the form (z_j, t_j), $j = 1, \dots, k$. Then Ψ is a parametrization of the Schubert cell $\mathcal{S}_w$ in terms of the minimal decomposition for $w \in \mathcal{W}$.

Proof. As noted, this follows immediately from the previous discussion on the minimal Schubert cells and we are able to write $\mathcal{S}_w$ as a product orbit given by Proposition 2.3. □

Remark 2.4. Note that each $\psi_{\alpha_j}(u_j)$ is a $n \times n$ matrix belonging to $U(\alpha) \subseteq U$, the compact real form of G.

2.4 Homology and the Main Theorem

The maximal flag manifold $\mathbb{F}$ of a complex semisimple Lie group G is a CW-complex with cellular structure consisting of the Schubert cells. From Corollary 2.1, each cell is even-dimensional. This implies the homology chain complex is zero on odd dimensions and the corresponding homology is freely generated by the Schubert cells, since each boundary map is null (HATCHER, 2002). The aim of this section is to prove Theorem 2.2, which presents invariant volume forms for each Schubert cell and shows a formula for when they are integrated over the corresponding cell. Later we generalize and prove Theorem 2.3. A result of this kind was first proven in (BERNSTEIN; GEL'FAND; GEL'FAND, 1973). In this chapter, we do it using somewhat simpler techniques exploiting various geometrical properties of the Schubert cells and a lot of Lie theory. As a bonus, the same techniques are used to prove Theorem 3.2, the quaternionic version of Theorem 2.2 (a non-complex flag).

Recall that for $w = r_\alpha$ ($\alpha \in \Sigma$), the tangent space of $\mathcal{S}_w$ at the origin is canonically identified with the root space $\mathfrak{u}_\alpha$. It was shown that $\mathcal{S}_w$ is topologically equivalent to a two-dimensional sphere. Theorem 1.2 asserts that the differential 2-form $\alpha(\Omega)$ is invariant. Since its degree coincides with the real dimension of $\mathcal{S}_w$ (see the remark after Corollary 2.1), it must be a volume form. This is the content of the following lemma.

Lemma 2.1. *Let $\mathcal{S}_w$ be the Schubert cell associated to the simple reflection $w = r_\alpha$. Then $\alpha(\Omega)$ is an invariant volume form on $\mathcal{S}_w$. Even more: if δ is the invariant metric on $\mathcal{S}_w$ which induces the volume form $\alpha(\Omega)$, then $(\mathcal{S}_w, \delta)$ is a sphere with radius $r = \sqrt{\alpha(H_\alpha)}$.*

Proof. The differential form $\alpha(\Omega)$ is a Chern-Weil form, hence invariant; and its degree equals the dimension of the cell $\mathcal{S}_w$ so $\alpha(\Omega)$ is a volume form if and only if it is non-zero when evaluated at any base of $T_{x_w}(\mathcal{S}_w) = \mathfrak{u}_\alpha \cdot x_w$. An suitable basis is $B_w = \{S_\alpha, A_\alpha\}$ (see Corollary 2.1). Their Lie bracket is $[A_\alpha, S_\alpha] = 2iH_\alpha \in \mathfrak{m}$, hence $\Omega_{x_w}(S_\alpha, A_\alpha) = iH_\alpha$ (this is a consequence of Equation 1.3 in Example 1.6). By definition (see Equation 1.5),

$$\alpha(\Omega)_{x_w}(S_\alpha, A_\alpha) = \alpha\left(\Omega_{x_w}(S_\alpha, A_\alpha)\right) = \alpha(H_\alpha) \neq 0.$$

The imaginary unit was dropped, since $\mathfrak{m} \approx i\mathfrak{m}$. So $\alpha(\Omega)$ is an invariant volume form on $\mathcal{S}_w$.

Now we prove the second part of the lemma. The local computation can be done using the open cell $\mathcal{B}_w$ instead of its closure $\mathcal{S}_w$. Recall from Proposition 2.5 that $\mathcal{B}_w$ is diffeomorphic to $B^2 \backslash \partial B^2$ which in turn can be mapped into an open cell of S^2. One such mapping is given by (see proof of Corollary 2.2)

$$\begin{array}{cccc} f: & \mathcal{B}_w & \to & S^2 \\ & \phi(e^{i\theta}, t) \cdot x_w & \mapsto & (\cos t, \cos\theta \sin t, \sin\theta \sin t) \end{array}$$

by letting $(\theta, t) \in [0, 2\pi) \times (-\pi/2, \pi/2)$. Here, the map ϕ is the same as in Equation 2.9. Notice that the diffeomorphism $[0, 2\pi) \times (-\pi/2, \pi/2) \to \mathcal{B}_w$ composed with f provides a local chart of S^2.

There is an invariant metric, say δ, defined on $\mathcal{S}_w$ for which the corresponding volume form is precisely $\alpha(\Omega)$. Now, on the sphere S^2, one has the Riemannian metric ϵ induced by the Euclidean metric on $\mathbb{R}^3$. Let ν denote the (also invariant) volume form on S^2 induced by ϵ. Up to a sign, the metrics δ and ϵ have the same group of isometries, *i.e.*, $\mathrm{SO}_3 \approx \mathrm{SU}_2/\{\pm\mathrm{Id}\}$. This allows one to compare both metrics via the map f. There is a constant $c > 0$ such that $\alpha(\Omega) = c\, f^*\nu$, where f^* is the pullback of differential forms. Then, to find the radius of $\mathcal{S}_w$, one only need to find this constant c. In fact, for any two-dimensional sphere of radius R, their *surface area* is given by the formula $4\pi R^2$. Let R_δ denote the *Chern-Weilian* radius of $\mathcal{S}_w$ with respect to the metric delta, in contrast to the Euclidean radius $R_\epsilon = 1$ of S^2 with volume $= 4\pi$. Then

$$4\pi R_\delta^2 = \int_{\mathcal{S}_w} \alpha(\Omega) = c \int_{S^2} \nu = c\,(4\pi) \tag{2.11}$$

leading to $R_\delta = \sqrt{c}$.

The rest of the proof is dedicated to find the value of c. Let $I = [0, 2\pi) \times (-\pi/2, \pi/2)$ and take two curves γ_A and γ_S on I such that

$$\phi \circ \gamma_A(0) \cdot x_w = \phi \circ \gamma_S(0) \cdot x_w = x_w$$

satisfying

$$\left.\frac{d}{dt}\right|_{t=0} \phi \circ \gamma_A(t) \cdot x_w = A_\alpha \qquad \text{and} \qquad \left.\frac{d}{dt}\right|_{t=0} \phi \circ \gamma_S(t) \cdot x_w = S_\alpha.$$

We choose $\gamma_A(t) = (\pi, t)$ and $\gamma_S(t) = (\pi/2, t)$, so

$$\phi \circ \gamma_A(t) = e^{tA_\alpha} \quad \text{and} \quad \phi \circ \gamma_S(t) = e^{tS_\alpha}.$$

A simple computation shows that the corresponding curves in S^2 satisfy

$$\left.\frac{d}{dt}\right|_{t=0} f\left(\phi \circ \gamma_A(t) \cdot x_w\right) = -e_2 \qquad \text{and} \qquad \left.\frac{d}{dt}\right|_{t=0} f\left(\phi \circ \gamma_S(t) \cdot x_w\right) = e_3.$$

Therefore,

$$\alpha(H_\alpha) = \alpha(\Omega)_{x_w}(S_\alpha, A_\alpha) = c\,(f^*\nu)_{x_w}(S_\alpha, A_\alpha) = c\,\nu(e_3, -e_2) = c$$

so $R_\delta = \sqrt{\alpha(H_\alpha)}$. □

Remark 2.5. An alternative way to calculate the radius of such cells is achieved by noticing first that the curve $a(t) = \exp(tA_\alpha) \cdot x_0$ is a great circle in $\mathcal{S}_w$ and one loop is taken when t runs throughout the interval $[0, \pi]$. Its perimeter is given by $2\pi r$, so

$$2\pi r = \int_0^\pi \|a'(t)\| dt = \|A_\alpha\|\pi. \tag{2.12}$$

The norm in the equation above is with respect to the metric δ on $\mathcal{S}_w$ which induces the volume form $\alpha(\Omega)$. Let $\varphi : \mathrm{SU}_2 \to \mathrm{SO}_3$ be the 2 to 1 homomorphism given by $\varphi(g) \cdot v = gvg^{-1}$, where $g \in \mathrm{SU}_2 \approx \mathrm{Sp}_1$ (the unit quaternions) and $v \in \mathbb{R}^3 \approx \operatorname{Im}\mathbb{H}$. The map φ induces a well-defined isomorphism $\Phi : \mathcal{S}_w \to S^2 \in \mathbb{R}^3$, $g \cdot x_0 \mapsto \varphi(g) \cdot e_1$ where e_1 is the canonical vector of $\mathbb{R}^3$. A direct computation shows that $d\Phi_{x_0}(A_\alpha) = 2e_2$ and $d\Phi_{x_0}(S_\alpha) = 2e_3$, both having Euclidean norm $= 2$. If ϵ is the pullback by Φ of the Euclidean metric, then ϵ is also invariant. Indeed, $\mathrm{SO}_3 = \varphi(\mathrm{SU}_2)$ is composed of Euclidean isometries. Again, there is a multiplying constant $c = \alpha(H_\alpha)$ such that $\delta = c\,\epsilon$. This leads to

$$\|A_\alpha\| = \delta(A_\alpha, A_\alpha)^{\frac{1}{2}} = \sqrt{c}\,\epsilon(A_\alpha, A_\alpha)^{\frac{1}{2}} = 2\sqrt{c}.$$

By Equation 2.12, we have that $r = \sqrt{c}$.

Now we move on to the next step, which is to build an invariant volume form associated to each Schubert cell, including those of larger dimensions.

Lemma 2.2. *Let $w = r_{\alpha_1} \cdots r_{\alpha_k}$ be a minimal decomposition with respect to the simple roots α_j. Let $\Pi_w = \{\beta_1, \ldots, \beta_k\}$ where $\beta_1 = \alpha_1$ and $\beta_j = r_{\alpha_1} \cdots r_{\alpha_{j-1}}(\alpha_j)$. Define*

$$\beta_w = \beta_1 \wedge \cdots \wedge \beta_k. \tag{2.13}$$

Then $\beta_w(\Omega)$ is an invariant volume form on $\mathcal{S}_w$.

Proof. The proof is by induction on the length of w. The case when $l(w) = 1$ is part of the statement of Lemma 2.1.

Let $w' = r_{\alpha_1} \cdots r_{\alpha_{k-1}}$ and suppose $\beta_{w'} = (\beta_1 \cdots \beta_{k-1})(\Omega)$ is an invariant volume form on $\mathcal{S}_{w'}$. Let $\pi : \mathcal{S}_w \to \mathcal{S}_{w'}$ be the sphere fibration associated to w (check Proposition 2.4). Recall that the tangent space of $\mathcal{S}_w$ at x_0 is isomorphic to $\sum_{\alpha \in \Pi_w} \mathfrak{u}_\alpha$. This means that the tangent space to the fiber $\pi^{-1}(x_0)$ at x_0 is the complement of $T_{x_0}\mathcal{S}_{w'}$ in $T_{x_0}\mathcal{S}_w$, *i.e.*,

$$T_{x_0}\left(\pi^{-1}(x_0)\right) \approx \mathfrak{u}_{\beta_k}.$$

From the proof of *Lemma* 2.1, $\beta_k(\Omega)$ *is* an invariant volume form on the fibers $\approx S^2$. By equivariance, the pullback $\pi^*(\beta_{w'})$ is also invariant on $\mathcal{S}_w$. Therefore, the wedge product

$$\beta_w(\Omega) = \pi^*(\beta_{w'}) \wedge \beta_k(\Omega) \tag{2.14}$$

is a volume form for $\mathcal{S}_w$, since it is a non-zero top form. The differential form $\beta_w(\Omega)$ is also invariant, since it is a product of two invariant forms. □

Remark 2.6. The volume form stated in Lemma 2.2 is slightly inaccurate. The precise version is the one given in Equation 2.14. Let

$$w^{(j)} = r_{\alpha_1} \cdots r_{\alpha_{k-j}} \qquad j = 0, \ldots, k-1,$$

and $\pi_j = \pi_j^w$ be the canonical sphere fibration

$$\pi_j : \mathcal{S}_{w^{(j-1)}} \to \mathcal{S}_{w^{(j)}}.$$

The invariant form on Equation 2.14 when fully expanded is given by

$$\begin{aligned}\beta_{w^{(0)}}(\Omega) &= \beta_k(\Omega) \wedge \pi_1^*(\beta_{w^{(1)}})(\Omega) \\ &= \beta_k(\Omega) \wedge \pi_1^*(\beta_{k-1}(\Omega)) \wedge (\pi_2 \circ \pi_1)^*(\beta_{w^{(2)}}(\Omega)) \\ &\vdots \\ &= \beta_k(\Omega) \wedge \pi_1^*(\beta_{k-1}(\Omega)) \wedge \cdots \wedge (\pi_{k-1} \circ \cdots \circ \pi_1)^*(\beta_1(\Omega)).\end{aligned} \tag{2.15}$$

Notice that these differential forms commute without changing sign, since they are all of even degree.

Theorem 2.2. *Let $\mathbb{F} = U/M$ be a complex flag manifold with Schubert cell decomposition $\mathbb{F} = \bigcup_{w \in \mathcal{W}} \mathcal{S}_w$. Let $w = r_{\alpha_1} \cdots r_{\alpha_k}$ be a minimal decomposition with respect to the simple roots α_j. Let $\Pi_w = \{\beta_1, \ldots, \beta_k\}$ where $\beta_1 = \alpha_1$ and $\beta_j = r_{\alpha_1} \cdots r_{\alpha_{j-1}}(\alpha_j)$. Denote by H_β the element in $\mathfrak{m}$ such that $\beta(H) = \langle H, H_\beta \rangle$ with respect to the Cartan-Killing form. Then*

$$\beta_1(H_{\beta_1}) \cdots \beta_k(H_{\beta_k}) = \frac{1}{(4\pi)^k} \int_{\mathcal{S}_w} (\beta_1 \cdots \beta_k)(\Omega). \tag{2.16}$$

This does not depend on the minimal decomposition of w.

Proof. Let $w' = r_{\alpha_1} \cdots r_{\alpha_{k-1}}$ and $\pi : \mathcal{S}_w \to \mathcal{S}_{w'}$ be the sphere fibration — here we are following the notation as in the proof of Lemma 2.2. By Fubini's theorem (see Theorem .1 in the appendix)

$$\int_{\mathcal{S}_w} \beta_w(\Omega) = \int_{\mathcal{S}_{w'}} \left(\int_{\pi^{-1}(\{x\})} \beta_k(\Omega) \right) \beta_{w'}(\Omega). \tag{2.17}$$

where the integral between parentheses is to be interpreted as a function on $\mathcal{S}_{w'}$

$$x \mapsto \int_{\pi^{-1}(\{x\})} \beta_k(\Omega). \tag{2.18}$$

Because π is an equivariant fibration and the differential form is invariant, the function described in Equation 2.18 is constant equals $4\pi\, \beta_k(H_{\beta_k})$ — see the proof of Lemma 2.1. So the right-hand side of Equation 2.17 becomes

$$4\pi\, \beta_k(H_{\beta_k}) \int_{\mathcal{S}_{w'}} \beta_{w'}(\Omega). \tag{2.19}$$

Notice the integral on Equation 2.19 is similar to the integral on the left-hand side of Equation 2.17 so we can repeatedly apply Fubini's theorem. This leads to

$$\int_{\mathcal{S}_w} \beta_w(\Omega) = (4\pi)^k\, \beta_1(H_{\beta_1}) \cdots \beta_n(H_{\beta_k}).$$

This formula is independent of the chosen minimal decomposition, because Π_w does not depend on this choice. See Corollary 2.1. □

A similar results holds when one deals with the quaternionic maximal flag manifold instead—the one corresponding to the group $\mathrm{Sl}_n(\mathbb{H})$. It is an analogue of what happens to the complex flag of $\mathrm{Sl}_n(\mathbb{C})$, essentially switching $\mathbb{C}$ by $\mathbb{H}$ in the proofs. See the next chapter for more.

Surjectivity of the homomorphism

The Chern-Weil homomorphism $[\cdot]$ is, in particular, a linear transformation of $\mathrm{Inv}(M)$ into $H^*(\mathbb{F};\mathbb{R})$, both seen as a vector space over $\mathbb{R}$. The dual vector space of $H^*(\mathbb{F};\mathbb{R})$ is identified with the homology $H_*(\mathbb{F})$, which has $\mathcal{S}_w$ as a basis. The Schubert cell $\mathcal{S}_w : H^*(\mathbb{F};\mathbb{R}) \to \mathbb{R}$ is defined as the linear functional

$$(\mathcal{S}_w)([f]) = \int_{\mathcal{S}_w} f$$

where f is a closed differential form on $\mathbb{F}$, therefore a representative of a class in $H^*(\mathbb{F};\mathbb{R})$. Notice that $\mathcal{S}_w$ (as a manifold) has no boundary, so the integral of any exact form is zero (by Stoke's Theorem). This shows that the linear functional $\mathcal{S}_w$ is well-defined.

The *dual homomorphism* $[\cdot]^*$ is a linear transformation from the dual of $H^*(\mathbb{F};\mathbb{R}) \approx H_*(\mathbb{F})$ into $\mathrm{Inv}(M)^*$, the dual algebra of invariant polynomials. It is defined by the equation

$$[\mathcal{S}_w]^*(f) = \int_{\mathcal{S}_w} f(\Omega)$$

Here $f(\Omega)$ is a representative of $[f]$ for some curvature Ω on $U \to \mathbb{F}$. The Chern-Weil homomorphism is surjective if the dual $[\cdot]^*$ is injective. This is a general fact, but we only prove here the direction we are going to make use of, all this exploring the fact that $H^*(\mathbb{F};\mathbb{R})$ is finite-dimensional.

Proposition 2.6. *Let $T : U \to V$ be a linear transformation of vector spaces where V is finite dimensional. Let $T^* : U^* \to V^*$ denote the dual transformation. If T^* is injective, then T is surjective.*

Proof. The annihilator of a subset $S \subseteq V$ is the *vector subspace* $S^0 \subseteq V^*$ defined by $S^0 = \{ f \in V^* \mid f(s) = 0 \,\forall\, s \in S \}$. If $S^0 = \{0\}$ (the null-functional), then $S = V$. In fact, let $S^\perp$ be the orthogonal complement of S according to some inner product $\langle \cdot, \cdot \rangle$ on V (since V is finite dimensional). Take any $s \in S^\perp$. Then its dual $s^* \in V^*$, defined by $s^*(v) = \langle s, v \rangle$, is an element of $S^0 = \{0\}$. Then s^* is the null-functional on V^* and this can only be true if $s = 0$. To finish the proof, take $S = T(U)$ and notice that $(T(U))^0 = \ker T^* = \{0\}$ by hypothesis, so $T(U) = S = V$. □

The rest of this section is dedicated to prove the surjectivity of $[\cdot]$ by showing that $[\cdot]^*$ is injective. The first step is to generalize what is done in Theorem 2.2. Suppose there is a general formula of the form

$$\langle f, f_w \rangle = \frac{1}{c} \int_{\mathcal{S}_w} f$$

where $f_w = \beta_1 \cdots \beta_k$ is the polynomial given in Theorem 2.2, $\langle \cdot, \cdot \rangle$ is a inner product on $\mathrm{Inv}(M)$, $c \neq 0$ is a normalization constant and f is *any* polinomial. In terms of dualization, this formula is simply

$$[\mathcal{S}_w]^*(f) = c\langle f, f_w \rangle.$$

If such formula is true, then the kernel of $[\cdot]^*$ must contain only the trivial Schubert cell—the origin x_0—so $[\cdot]^*$ is injective.

The linear functionals λ_j generates the algebra $\mathrm{Inv}(M)$. This is so because M is a cartesian product of circles S^1 and the Lie algebra of S^1 is unidimensional, meaning that *any* non-zero linear functional on $\mathbb{R}$ does the job.[6]

[6] See "The Invariant Polynomial Algebra" (sub)section in the next chapter for a clearer explanation.

Lemma 2.3. *Let α be a simple root and consider $w = r_\alpha \in \mathcal{W}$. Take $f \in \mathrm{Inv}(M)$ of degree 1, that is, a linear functional on $\mathfrak{m}$. Then*

$$\int_{\mathcal{S}_w} f(\Omega) = 4\pi f(H_\alpha).$$

Proof. This proof goes in the same spirit of Lemma 2.1. Let S_α and A_α form the basis for $\mathfrak{u}_\alpha \approx T_{x_0}(\mathcal{S}_w)$. Then

$$f(\Omega)_{x_0}(S_\alpha, A_\alpha) = f(H_\alpha).$$

The 1/2 of Chern-Weil is canceled with the factor 2 of Equation 2.8. The rest follows by integration over S_w exactly as is done in the proof of Lemma 2.1. □

Theorem 2.3. *Let $w = r_{\alpha_1} \cdots r_{\alpha_k}$ be a minimal decomposition with respect to the simple roots α_j. Let $\Pi_w = \{\beta_1, \dots, \beta_k\}$ where $\beta_1 = \alpha_1$ and $\beta_j = r_{\alpha_1} \cdots r_{\alpha_{j-1}}(\alpha_j)$. Denote by H_β the element in $\mathfrak{m}$ such that $\beta(H) = \langle H, H_\beta \rangle$ with respect to the Cartan-Killing form. Then for any invariant polynomial $f \in \mathrm{Inv}(M)$ of degree k, one has*

$$f(H_{\beta_1}, \dots, H_{\beta_k}) = \frac{1}{(4\pi)^k} \int_{\mathcal{S}_w} f(\Omega) \tag{2.20}$$

This does not depend on the minimal decomposition of w.

Proof. First we prove for f a monomial, that is, f can be written as a product

$$f = \lambda_{i_1} \cdots \lambda_{i_k}.$$

To integrate $f(\Omega)$ over $\mathcal{S}_w$, we use Theorem .1 (Fubini), just as it was done in the proof of Theorem 2.2, but with a minor twist by writting such integral as

$$\int_{\mathcal{S}_w} f(\Omega) = \frac{1}{k} \sum_{j=1}^{k} \Big(\int_{\mathcal{S}_{w'}} \left(\int_{\mathfrak{u}_{\beta_k}} \lambda_j(\Omega) \right) \lambda_1 \cdots \widehat{\lambda_j} \cdots \lambda_k(\Omega) \Big) \tag{2.21}$$

where $w' = r_{\alpha_1} \cdots r_{\alpha_{k-1}}$ and $\widehat{\lambda_j}$ means that this factor is skipped. Equation 2.21 makes sense, since the $\lambda_j(\Omega)$ commutes between each other. Then all integrals in the summand above are equal.

The innermost integral on Equation 2.21 is a constant function on the basis $\mathcal{S}_{w'}$, but the notation needs some explanation. The domain of integration is set to $\mathfrak{u}_{\beta_k}$ to mean that this integral is to be computed over the fibers of $\mathcal{S}_w \to \mathcal{S}_{w'}$ which has tangent plane $\approx \mathfrak{u}_{\beta_k}$. By Lemma 2.3, this integral if $4\pi\lambda_j(H_{\beta_k})$. Now we apply induction on the index k. Suppose Equation 2.20 is true for monomials of degree $k-1$, then continuing the expansion

$$\begin{aligned}
\int_{\mathcal{S}_w} f(\Omega) &= \frac{4\pi}{k} \sum_{j=1}^{k} \Big(\lambda_j(H_{\beta_k}) \int_{\mathcal{S}_{w'}} \lambda_1 \cdots \widehat{\lambda_j(\Omega)} \cdots \lambda_k(\Omega) \Big) \\
&= \frac{(4\pi)^k}{k} \sum_{j=1}^{k} \Big(\lambda_j(H_{\beta_k}) \cdot \lambda_1 \cdots \widehat{\lambda_j(\Omega)} \cdots \lambda_k(H_{\beta_1}, \dots, \widehat{H_{\beta_j}}, \dots, H_{\beta_k}) \Big)
\end{aligned}$$

Notice that each λ_* appears applied to each H_{β_*}, so that if you expand the definition (Equation 1.4) of the product of the λ_i and carefully collect terms, you arrive at the first equality below.

$$\begin{aligned}\frac{1}{(4\pi)^k}\int_{\mathcal{S}_w} f(\Omega) = \frac{1}{k!}\sum_\sigma \lambda_1(H_{\beta_{\sigma(1)}})\cdots\lambda_k(H_{\beta_{\sigma(k)}}) \\ = \lambda_1\cdots\lambda_k(H_{\beta_1},\ldots,H_{\beta_k}) \\ = f(H_{\beta_1},\ldots,H_{\beta_k}).\end{aligned}$$

Let $g \in \mathrm{Inv}(M)$ be any form of degree k. The λ_* generates the invariant polynomial algebra, so g can be written as a linear combination of monomials like f above. Since $[\cdot]^*$ is linear, Equation 2.20 follows. This does not depend on the chosen minimal decomposition of w, since Π_w does not, meaning that the β_* that appears above are always the same. □

Corollary 2.3. *The homomorphism* $[\cdot]$ *is surjective.*

Proof. Suppose $[\mathcal{S}_w]^*$ is in the null-functional on $\mathrm{Inv}(M)$. Then, for any polynomial f

$$0 = [\mathcal{S}_w]^*(f) = \int_{\mathcal{S}_w} f(\Omega).$$

In light of Theorem 2.3, for every *non-trivial* Schubert cell $\mathcal{S}_{w'}$ one can find a volume form $\beta_{w'}(\Omega)$ so $[\mathcal{S}_{w'}]^*(\beta_{w'}(\Omega)) \neq 0$. Therefore, S_w is the trivial cell x_0. This shows that the homomorphism $[\cdot]^*$ is injective. By Proposition 2.6, the homomorphism $[\cdot]$ is surjective. □

3 Quaternionic Flag Manifolds

The techniques presented in chapter 2 have the advantage that they can also be applied to the quaternionic flag manifolds, since in this case the homology is non-trivial only in the dimensions multiples of 4. The analogue of the main theorem in last chapter is now Theorem 3.2. Even though there are many similarities in the complex case, it is where they differ that makes them so interesting and worth it.

We start with section 3.1 which contains the definition of a quaternionic flag. Most of what was done in chapter 2 can be applied here and most of the proofs can be copier almost-verbatim, only swapping $\mathbb{C}$ for $\mathbb{H}$, 2 for 4 here and there and so on. In those cases, the proofs is skipped. When it is not, the differences are only highlighted. I believe this will make the reading lighter.

In section 3.2, we go deeper on the action of the Weyl group both on the homology and cohomology; and also on the invariant polynomial algebra. The aim of this section is to give the kernel of the Chern-Weil homomorphism in the quaternionic case, which I believe is new stuff. This is a known result in the complex case (BERNSTEIN; GEL'FAND; GEL'FAND, 1973).

Now in section 3.3, we show how to represent the Chern-Weil forms $[\kappa]$ in terms of volume forms on each rootspace. Chronologically speaking, this was our first result on the matter. We explore some ideas involving various projections of the maximal flag into smaller ones, clarifying the graded algebra nature of the cohomology of $\mathbb{F}$ in light of the Leray-Hirsch theorem (put in the appendix).

Lastly, in section 3.4, we provide an application of the Chern-Weil classes $[\kappa]$ by looking at them as an obstruction to the existence of certain frames on $\mathbb{F}$. This goes in the direction of what was done in Conegundes' PhD thesis (CONEGUNDES, 2019), but here we apply it to the quaternionic maximal flag using the tools developed so far.

3.1 Quaternionic Flags

Start with the quaternionic Lie algebra. It is defined as

$$\mathfrak{sl}_n(\mathbb{H}) = \{\, X \in \mathrm{Mat}_n(\mathbb{H}) \,:\, (\mathrm{Re}(\mathrm{Tr}))(X) = 0 \,\}\,.$$

This algebra has Cartan decomposition

$$\mathfrak{sl}_n(\mathbb{H}) = \mathfrak{u} \oplus \mathfrak{s}$$

where $\mathfrak{s}$ is the set of Hermitian matrices in $\mathfrak{sl}_n(\mathbb{H})$ and

$$\mathfrak{u} = \mathfrak{sp}_n = \left\{\, X \in \mathrm{Mat}_n(\mathbb{H}) \,:\, X + \bar{X}^* = 0 \,\right\}$$

is the set of anti-Hermitian matrices (X^* is the conjugate transpose). A maximal abelian subalgebra in $\mathfrak{s}$ is

$$\mathfrak{a} = \left\{ \operatorname{diag}(a_1, \dots, a_n) \in \mathfrak{sl}_n(\mathbb{H}) \;:\; a_r \in \mathbb{R}, \sum_r a_r = 0 \right\}.$$

A root system is given by the linear functionals $\alpha_{rs} : \mathfrak{a} \to \mathbb{R}$ defined by

$$\alpha_{rs}(H) = a_r - a_s, \quad 1 \leqslant r, s \leqslant n$$

and our choice of simple root set is

$$\Sigma = \{\alpha_{12}, \dots, \alpha_{n-1;n}\}.$$

Therefore, the root spaces of $(\mathfrak{sl}_n(\mathbb{H}), \mathfrak{a})$ are the sets $\mathfrak{g}_{\alpha_{rs}}$ consisting of matrices having zero entries in every position, except (possibly) in the (r, s)-entry — the quaternionic analogues of $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{C})$.

Let $\mathfrak{n} = \mathfrak{n}^+$ be the sum of all positive root spaces. The Iwasawa decomposition reads as

$$\mathfrak{sl}_n(\mathbb{H}) = \mathfrak{u} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

and a choice of minimal parabolic algebra is given by

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

where $\mathfrak{m} = \mathfrak{sp}_1 \oplus \cdots \oplus \mathfrak{sp}_1$ is the diagonal part of $\mathfrak{u}$.

Before we turn the discussion to the group level, let $\Phi : \mathrm{Mat}_n(\mathbb{H}) \to \mathrm{Mat}_{2n}(\mathbb{C})$ be the injective algebra homomorphism defined by the rule

$$\Phi(A + jB) = \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix}$$

where $A, B \in \mathrm{Mat}_n(\mathbb{C})$. Then our choice of Lie group corresponding to $\mathfrak{sl}_n(\mathbb{H})$ is given by exponentiation. Explicitly,

$$\mathrm{Sl}_n(\mathbb{H}) = \{ g \in \mathrm{Mat}_n(\mathbb{H}) \;:\; \det(\Phi(g)) = 1 \}. \tag{3.1}$$

The compact part is

$$U = \mathrm{Sp}_n = \{ g \in \mathrm{Mat}_n(\mathbb{H}) \;:\; gg^* = g^*g = \mathrm{Id} \}$$

where g^* is the conjugate transpose of g. For the rest of the Lie groups, let $M = \exp \mathfrak{m}$ and the same for A and N, so the minimal parabolic subgroup is $B = MAN$. Notice that $M = \mathrm{Sp}_1 \times \cdots \times \mathrm{Sp}_1$ (n times) is *not* abelian — unlike the complex and real cases. Also notice that $\mathfrak{sp}_1$ is the set of purely imaginary quaternions with $\exp \mathfrak{sp}_1 = \mathrm{Sp}_1$, the set of *unit* quaternions isomorphic to the sphere S^3.

Remark 3.1. In Equation 3.1, the necessity of applying Φ comes from the fact that for a quaternionic matrix, the usual determinant is *not* well-defined. This is a consequence of the non-commutativity of the quaternion algebra $\mathbb{H}$.

Definition 3.1. The quaternionic maximal flag manifold is given by

$$\mathbb{F} = G/B$$

where $G = \mathrm{Sl}_n(\mathbb{H})$ and B is the minimal parabolic subgroup.

Remark 3.2. The associated principal bundle $G \to \mathbb{F}$ is reducible to $U \to \mathbb{F}$ with structural group M, allowing one to write the compact version of the previous definition, *i.e.*,

$$\mathbb{F} = U/M.$$

The Schubert cell decomposition is indexed by the Weyl group, which in this case is the same as the permutation group of the sequence $(1, \ldots, n)$. They're denoted by $\mathcal{S}_w$, where $w \in \mathcal{W}$. The only difference between this case and the complex case is that now the Schubert cells $\mathcal{S}_w$ are $4k$-dimensional, where $k = l(w)$. For instance, the lowest dimensional orbit is actually derived from the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$, while in the complex case we obtain it (in essence) from the classical fibration $S^1 \hookrightarrow S^3 \to S^2$. Every result in the Schubert cells section applies here with clear and easy adaptions.

The Invariant Polynomial Algebra

In the following, we present the generators of the algebra $\mathrm{Inv}(\mathrm{Sp}_1)$ and $\mathrm{Inv}(M)$. Let Ad denote the adjoint representation of Sp_1

$$\mathrm{Ad} : \mathrm{Sp}_1 \to \mathrm{Gl}(\mathfrak{sp}_1) \quad \text{given by} \quad \mathrm{Ad}(g)(X) = gXg^{-1};$$

and denote by $\kappa(\cdot, \cdot)$ the Cartan-Killing form of $\mathfrak{sp}_1$ which is isomorphic to the imaginary part of $\mathbb{H}$. Up to a multiplicative constant, κ is the only Ad-invariant symmetric bilinear form on $\mathfrak{sp}_1$, so $\mathrm{Inv}(\mathrm{Sp}_1)$ is generated by κ. This is case $n = 1$ in Theorem 2.8 of Chapter 12 in (KOBAYASHI; NOMIZU, 1996).

Now, consider $\mathfrak{m} = \mathfrak{sp}_1 \oplus \cdots \oplus \mathfrak{sp}_1$ and denote by κ_s the Cartan-Killing form on the s-th entry of $\mathfrak{m}$. Because $M = \mathrm{Sp}_1 \times \cdots \times \mathrm{Sp}_1$ is a product group, the algebra $\mathrm{Inv}(M)$ is then generated by the set $\{\kappa_1, \ldots, \kappa_n\}$, *i.e.*, each form in $\mathrm{Inv}(M)$ is a linear combination of monomials of type

$$\kappa_1^{m_1} \cdots \kappa_n^{m_n}, \qquad m_1, \ldots, m_s \geqslant 0.$$

Although there are some differences, the bilinear forms κ_s plays the same role as the roots α_j do in the complex case: they are the polynomials used to construct invariant volume forms on the Schubert cells. See the next section to check some applications exclusive to quaternionic case.

Now, the rest of this section is devoted to make similar constructions as done in chapter 2. We provide explicit invariant forms $[\kappa]$ on each of the Schubert cells with useful properties. Comparing the complex and quaternionic, the $[\kappa]$ play a similar role as the $[\alpha]$, but the proofs involves permutations of length 4, leading into slightly more complex argumentation. See the proof below.

Lemma 3.1. *Let $\mathbb{F} = \mathrm{Sp}_2/(\mathrm{Sp}_1 \times \mathrm{Sp}_1)$ be the maximal flag of $\mathfrak{sl}_2(\mathbb{H})$. There is only one positive root $\alpha = \alpha_{12}$ and $\mathcal{W} = \{\pm \mathrm{Id}\}$, therefore $\mathcal{S}_w = \mathbb{F}$. Let κ_1 and κ_2 be the generators of $\mathrm{Inv}(\mathrm{Sp}_1 \times \mathrm{Sp}_1)$. Choose any connection form with curvature Ω. Then $\kappa_1(\Omega)$ and $\kappa_2(\Omega)$ are both invariant volume forms on $\mathcal{S}_w$.*

Proof. Let $\kappa = \kappa_1$. Unlike the complex case, it is *not* immediate that $\kappa(\Omega)$ is a volume form. Again, one only needs to provide a basis for $T_{x_0}\mathbb{F} \approx \mathfrak{u}_\alpha = \mathfrak{u} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ such that $\kappa(\Omega)$ evaluated on that basis in non-zero. Let $p \in \mathbb{H}$. A general element in $\mathfrak{u}_\alpha$ is given by the matrix

$$A_p = \begin{bmatrix} 0 & p \\ -\bar{p} & 0 \end{bmatrix}.$$

Then, for quaternions p and q, one has

$$[A_p, A_q] = \begin{bmatrix} -p\bar{q} + q\bar{p} & 0 \\ 0 & -\bar{p}q + \bar{q}p \end{bmatrix} = (-2) \begin{bmatrix} \mathrm{Im}(p\bar{q}) & 0 \\ 0 & \mathrm{Im}(\bar{p}q) \end{bmatrix}.$$

A real basis for $\mathfrak{u}_\alpha$ is given by $\{A_1, A_i, A_j, A_k\}$. We wish to compute $\kappa(\Omega)$ at this basis, but due to combinatoric aspect in the definition of the Chern-Weil form, in order to simplify, rename its elements as

$$X_1 = A_1, \quad X_2 = A_i, \quad X_3 = A_j \quad \text{and} \quad X_4 = A_k.$$

Now, at the origin x_0 one has

$$\begin{aligned} \kappa(\Omega)_{x_0}(X_1, &X_2, X_3, X_4) \\ &= \frac{1}{4!} \sum_\sigma \epsilon_\sigma \kappa\big(\Omega_{x_0}(X_{\sigma(1)}, X_{\sigma(2)}), \Omega_{x_0}(X_{\sigma(3)}, X_{\sigma(4)})\big). \end{aligned}$$

Notice that

$$\begin{aligned} \kappa\big(\Omega_{x_0}(X_{\sigma(1)}, &X_{\sigma(2)}), \Omega_{x_0}(X_{\sigma(3)}, X_{\sigma(4)})\big) \\ &= \epsilon_\sigma \kappa\big(\Omega_{x_0}(X_1, X_2), \Omega_{x_0}(X_3, X_4)\big) \end{aligned}$$

for all $\sigma \in \Gamma$, the group generated by the permutations $\{(12), (34), (13)(24)\}$[1]. This group has 8 elements, and therefore has 3 (right) lateral classes represented by Γ, $\Gamma(13)$ and

[1] This is so because of the symmetry and anti-symmetry of κ and Ω, respectively.

$\Gamma(14)$. Let $\tau \in \{\mathrm{Id}, (13), (14)\}$. Then,

$$\sum_{\sigma \in \Gamma\tau} \epsilon_\sigma \kappa\big(\Omega_{x_0}(X_{\sigma(1)}, X_{\sigma(2)}), \Omega_{x_0}(X_{\sigma(3)}, X_{\sigma(4)})\big) = 8\epsilon_\tau \kappa\big(\Omega_{x_0}(X_{\tau(1)}, X_{\tau(2)}), \Omega_{x_0}(X_{\tau(3)}, X_{\tau(4)})\big).$$

Therefore

$$\begin{aligned} \kappa(\Omega)_{x_0}(X_1, X_2, X_3, X_4) &= \frac{1}{4!}\Bigg(8\kappa\Big(\Omega_{x_0}(X_1, X_2), \Omega_{x_0}(X_3, X_4)\Big) \\ &+ 8\epsilon_{(13)}\kappa\Big(\Omega_{x_0}(X_3, X_2), \Omega_{x_0}(X_1, X_4)\Big) \\ &+ 8\epsilon_{(14)}\kappa\Big(\Omega_{x_0}(X_4, X_2), \Omega_{x_0}(X_3, X_1)\Big)\Bigg). \end{aligned} \tag{3.2}$$

Remember, the cannonical connection has curvature given by the projection of the bracket onto the Lie algebra of the isotropy (see Example 1.6). Let

$$E_{rr} = \mathrm{diag}(0, \ldots, 1, \ldots, 0)$$

with 1 on the r-th diagonal entry. Then each combination of $\Omega(A_p, A_q)$ for $p, q \in \{1, i, j, k\}$ is computed below.

$$\begin{aligned} \Omega(A_1, A_i) &= 2iE_{11} - 2iE_{22} \\ \Omega(A_j, A_k) &= 2iE_{11} + 2iE_{22} \\ \Omega(A_j, A_i) &= -2kE_{11} - 2kE_{22} \\ \Omega(A_1, A_k) &= 2kE_{11} - 2kE_{22} \\ \Omega(A_k, A_i) &= 2jE_{11} + 2jE_{22} \\ \Omega(A_j, A_1) &= -2jE_{11} + 2jE_{22} \end{aligned} \tag{3.3}$$

Therefore

$$\begin{aligned} \kappa\big(\Omega_{x_0}(X_1, X_2), \Omega_{x_0}(X_3, X_4)\big) &= 4 \\ \kappa\big((\Omega_{x_0}(X_3, X_2), \Omega_{x_0}(X_1, X_4)\big) &= -4 \\ \kappa\big((\Omega_{x_0}(X_4, X_2), \Omega_{x_0}(X_3, X_1)\big) &= -4 \end{aligned} \tag{3.4}$$

and by direct substitution in Equation 3.2, we have

$$\kappa(\Omega)_{x_0}(X_1, X_2, X_3, X_4) = \frac{8}{4!}(4 + 4 + 4) = 4 \neq 0. \tag{3.5}$$

Since $\kappa(\Omega)$ is a differential form of degree 4 on a basic cell $\mathcal{S}_w$ which has 4 dimensions, $\kappa(\Omega) = \kappa_1(\Omega)$ must be a volume form. An analogous computation shows that

$$\kappa_2(\Omega)_{x_0}(X_1, X_2, X_3, X_4) = -4 \neq 0.$$

The reason for the change in sign is that κ_2 is the inner product on the *second diagonal entry* (of opposite sign). In fact, when $p = 1$ and $q = i, j, k$, we actually have

$$\mathrm{Im}(p\bar{q}) = -\mathrm{Im}(\bar{p}q).$$

Thus each one of Equation 3.4 changes in sign. □

Remark 3.3. Let $\nu = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, where x_j are the coordinates with respect to the base of $\mathfrak{u}_\alpha$ consisting of the X_j above. Then

$$\kappa_1(\Omega)_{x_0} = 4\,\nu = -\kappa_2(\Omega)_{x_0}.$$

The next step is to look at the lowest dimensional non-trivial Schubert cells inside a flag of $\mathfrak{sl}_n(\mathbb{H})$.

Theorem 3.1. *Let $\mathbb{F}$ be the maximal flag of $\mathfrak{sl}_n(\mathbb{H})$ $(n \geqslant 2)$. Let $\mathcal{S}_w \subseteq \mathbb{F}$ be the Schubert cell associated to the simple reflection $w = r_{\alpha_s}$. Then $\kappa_s(\Omega)$ and $\kappa_{s+1}(\Omega)$ are both invariant volume forms on $\mathcal{S}_w$.*

Proof. Let $\alpha = \alpha_s = \lambda_s - \lambda_{s+1}$ be a simple root and r_α be the associated simple reflection. By definition,

$$\mathcal{S}_w = U(\alpha) \cdot x_0$$

where $U(\alpha)$ is the connected subgroup of $U = \mathrm{Sp}_n$ with Lie algebra

$$\mathfrak{u}(\alpha) = \mathfrak{m}(\alpha) \oplus \mathfrak{u}_\alpha$$

where $\mathfrak{m}(\alpha) = \mathfrak{sp}_1 \oplus \mathfrak{sp}_1$ is the sum of terms corresponding to the s-th and $(s+1)$-th entries of $\mathfrak{m} = \mathfrak{sp}_1 \oplus \cdots \oplus \mathfrak{sp}_1$. The Lie algebra $\mathfrak{u}(\alpha)$ is isomorphic to $\mathfrak{sp}_2$. Also, at the group level, we have $U(\alpha) \approx \mathrm{Sp}_2$. Therefore, all the computations done in the proof of the Lemma 3.1 apply. In other words, if we let $\{X_1, X_2, X_3, X_4\}$ be the corresponding basis for $T_{x_0}(\mathcal{S}_w) \approx \mathfrak{u}_\alpha$, then

$$\kappa_s(\Omega)(X_1, X_2, X_3, X_4) = 4$$

and

$$\kappa_{s+1}(\Omega)(X_1, X_2, X_3, X_4) = -4.$$

□

Parametrization of the Schubert cells

To compute the volumes of the minimal Schubert cells —or equivalently, their radii[2]— one must first describe an adequate parametrization of Sp_2. The essence of this

[2] This is an immediate consequence of known formulas for the volume(R) of even-dimensional spheres of radius R.

parametrization is analogous to the complex case, but we make a few notes anyway. In Proposition 2.5, we defined a map ϕ by restricting the exponential map to the image of the closed ball in the Lie algebra $\mathfrak{u}$. In the quaternionic case, the same technique holds, changing the unit complex $z \in S^1$ by a unitary quaternion $p \in S^3 \subset \mathbb{H}$. Such quaternion can be written as exponential e^q where $q \in \mathrm{Im}(\mathbb{H})$, so q plays the same role as $i\theta$ in $z = e^{i\theta}$. Even more is true: one can write

$$e^q = \cos(|q|) + \frac{q}{|q|}\sin(|q|).$$

The next proposition is the quaternionic analogue of Proposition 2.5. Here B^4 is the closed ball in $\mathbb{H}$ identified with $S^3 \times [0, \pi/2]$ where $S^3 \times \{0\}$ is taken into the center.

Proposition 3.1. *There is a parametrization $\psi : B^4 \to \mathrm{Sp}_2$ such that*

1. *The center of B^4 is taken into r_α, i.e., $\psi(S^3 \times \{0\}) = r$ where r is a representative of r_α in Sp_2.*

2. *The boundary ∂B^4, namely $S^3 \times \{\pi/2\}$, is taken into M, so*

$$\psi(\partial B^4) \cdot x_0 = x_0.$$

3. *The mapping $B^4 \backslash \partial B^4 \to \mathcal{B}_w$, $(z, t) \mapsto \psi(z, t) \cdot x_0$ is a diffeomorphism.*

Explicitly, the quaternionic version of the mapping is

$$\phi(p, t) = \begin{bmatrix} \cos t & -\bar{p}\sin t \\ p\sin t & \cos t \end{bmatrix}$$

and $\psi = r_\alpha \phi$.

Now, let $x_w = w x_0$ be the origin of the Bruhat cell $\mathcal{B}_w = N \cdot x_w$. Let

$$\begin{array}{cccc} f: & \mathcal{B}_w & \to & S^4 \subset \mathbb{R}^5 \\ & \phi(e^q, t) \cdot x_w & \mapsto & (\cos t, e^q \sin t) \end{array}$$

Here $(q, t) \in A \times [-\pi/2, \pi/2]$ where $A \subset \mathrm{Im} \approx \mathbb{R}^3$ is a region around 0 just big enough to have $e^A = S^3$. Let q be fixed and consider the curve $\gamma_q(t) = (\cos t, e^q \sin t)$ (the exponential is taken here to ensure that $\gamma_q(t)$ belongs to S^4). Then $\gamma_q'(0) = (0, e^q)$, so by letting $q \in \{0, i, j, k\}$, the corresponding curves are such that their tangent vector forms a basis given by the canonical vector $\{e_2, e_3, e_4, e_5\}$.

Let ν to be the Euclidean volume form, which is invariant since $\mathrm{SU}_2 \approx \mathrm{SO}_3$ up to a sign. By similar arguments, there is a constant $c > 0$ such that $\kappa_1(\Omega) = c f^* \nu$. By evaluation of such forms on the basis, on one hand by Equation 3.5, we have that $c = 4$. Therefore, the *volume* of $\mathcal{S}_w = S^4$ is

$$\int_{\mathcal{S}_w} \kappa(\Omega) = 4 \int_{S^4} \nu = 4\,\mathrm{vol}(S^4). \tag{3.6}$$

where the volume $\mathrm{vol}(S^4)$ of an Euclidean sphere of unitary radius is $8\pi^2/3$.

Now we generalize the construction of invariant volume forms for larger Schubert cells and compute their volume arriving at an (quaternionic) analogue version of Theorem 2.2.

Theorem 3.2. *Let $\mathbb{F}$ be a quaternionic flag manifold with Schubert cell decomposition $\mathbb{F} = \bigcup_{w \in \mathcal{W}} \mathcal{S}_w$. Let $w = r_{\alpha_1} \cdots r_{\alpha_s}$ be a minimal decomposition with respect to the simple roots. Define*

$$\kappa_w = \kappa_1 \wedge \cdots \wedge \kappa_s \in \mathrm{Inv}^{2s}(M).$$

Then $\kappa_w(\Omega)$ is an invariant volume form on $\mathcal{S}_w$ and

$$\int_{\mathcal{S}_w} \kappa_w(\Omega) = (\mathrm{vol}(S^4))^s.$$

Proof. Let $w' = r_{\alpha_1} \cdots r_{\alpha_{s-1}}$ and $\pi : \mathcal{S}_w \to \mathcal{S}_{w'}$ be the sphere fibration. By Fubini's theorem

$$\int_{\mathcal{S}_w} \kappa_w(\Omega) = \int_{\mathcal{S}_{w'}} \left(\int_{\pi^{-1}(\{x\})} \kappa_s(\Omega) \right) \kappa_{w'}(\Omega). \tag{3.7}$$

where the integral between parentheses is to be interpreted as a function on $\mathcal{S}_{w'}$

$$x \mapsto \int_{\pi^{-1}(\{x\})} \kappa_s(\Omega). \tag{3.8}$$

Because π is an equivariant fibration and the differential form is invariant, the function described in Equation 2.18 is constant, say c'. The fiber on which this integral is evaluated is homeomorphic to a sphere S^4, so we repeat the arguments of equation Equation 3.6 to see that c' is simply the volume of S^4 according to the volume form κ_s, that is, $c' = 4\,\mathrm{vol}(S^4)$. So the right-hand side of Equation 3.7 becomes

$$4\,\mathrm{vol}(S^4) \int_{\mathcal{S}_{w'}} \kappa_{w'}(\Omega). \tag{3.9}$$

Notice the integral on Equation 3.9 is similar to the integral on the left-hand side of Equation 3.7 so we can repeatedly apply Fubini's theorem. This leads to

$$\int_{\mathcal{S}_w} \kappa_w(\Omega) = (4\,\mathrm{vol}(S^4))^s.$$

□

Surjectivity of the homomorphism

The rest of this section is dovoted to prove that the Chern-Weil homomorphism, in the quaternionic case, is also surjective. The techniques involved are analogous to those used in the proof of Corollary 2.3. In later sections, we recover the same result using a very different path, but still as a consequence of the Chern-Weil classes.

Let $f \in \mathrm{Inv}(M)$ be a symmetric form of degree 2. There are real constants c^r such that $f = \sum_r c^r \kappa_r$. Let Ω be the cannonical choice of curvature on the principal bundle $U \to \mathbb{F}$. Since $f(\Omega)$ is an invariant differential form, to integrate it over a cell $\mathcal{S}_w$ with $w = r_\alpha$ ($\alpha \in \Sigma$), one needs only to check its value at a basis of $T_{x_0}(\mathcal{S}_w)$. With this idea in mind, in the next paragraphs we generalize the computations for κ at the origin extending those formulas for any f of degree 2. For that, let $\alpha = \alpha_{rs}$ be any root of $\mathfrak{sl}_n(\mathbb{H})$. Let $p \in \mathbb{H}$. A general element in $\mathfrak{u}_{\alpha_{rs}} \approx \mathbb{H}$ is written as $A_p = pE_{rr} - \bar{p}E_{ss}$, so A_1, A_i, A_j and A_k together form a basis for $\mathfrak{u}_\alpha$. Just as we did in Equation 3.2, one also has at the origin x_0 that

$$
\begin{aligned}
&f(\Omega)_{x_0}(A_1, A_i, A_j, A_k) \\
&\quad = \frac{1}{4!}\Bigg(8f\Big(\Omega_{x_0}(A_1, A_i), \Omega_{x_0}(A_j, A_k)\Big) \\
&\qquad\quad + 8\epsilon_{(13)} f\Big(\Omega_{x_0}(A_j, A_i), \Omega_{x_0}(A_1, A_k)\Big) \\
&\qquad\qquad + 8\epsilon_{(14)} f\Big(\Omega_{x_0}(A_k, A_i), \Omega_{x_0}(A_j, A_1)\Big)\Bigg).
\end{aligned}
\tag{3.10}
$$

For any rootspace $\mathfrak{u}_\alpha$ with $\alpha = \alpha_{rs}$, Equation 3.3 generalizes to Equation 3.11.

$$
\begin{aligned}
\Omega(A_1, A_i) &= 2iE_{rr} - 2iE_{ss} \\
\Omega(A_j, A_k) &= 2iE_{rr} + 2iE_{ss} \\
\Omega(A_j, A_i) &= -2kE_{rr} - 2kE_{ss} \\
\Omega(A_1, A_k) &= 2kE_{rr} - 2kE_{ss} \\
\Omega(A_k, A_i) &= 2jE_{rr} + 2jE_{ss} \\
\Omega(A_j, A_1) &= -2jE_{rr} + 2jE_{ss}.
\end{aligned}
\tag{3.11}
$$

Remember that each κ_r is an inner product on the r-th entry of $\mathfrak{m}$; also κ_r does not mix the diagonal entries, that is, if $s \neq r$ then $\kappa_r(pE_{ss}, X) = 0$ for all $p \in \mathbb{H}$ and all $X \in \mathfrak{m}$. Because $f = \sum_r c_r \kappa_r$, the same is true for f. So, using Equation 3.11 and this observation, the first term on the right-hand side of Equation 3.10 is

$$\begin{aligned} f\big(\Omega_{x_0}(A_1, A_i), \Omega_{x_0}(A_j, A_k)\big) &= f(2iE_{rr} - 2iE_{ss}, 2iE_{rr} + 2iE_{ss}) \\ &= 4\big(f(iE_{rr}, iE_{rr}) - f(iE_{ss}, iE_{ss})\big) \\ &= 4\big(c_r\kappa_r(iE_{rr}, iE_{rr}) - c_s\kappa_s(iE_{ss}, iE_{ss})\big). \end{aligned}$$

Each κ_r is an Euclidean inner product on the r-th diagonal entry of $\mathfrak{m}$, so

$$c_r\kappa_r(iE_{rr}, iE_{rr}) - c_s\kappa_s(iE_{ss}, iE_{ss}) = c_r - c_s.$$

The computation of the other permutations is analogous. In short,

$$\begin{aligned} f\big(\Omega_{x_0}(A_1, A_i), \Omega_{x_0}(A_j, A_k)\big) &= 4(c_r - c_s), \\ f\big((\Omega_{x_0}(A_j, A_i), \Omega_{x_0}(A_1, A_k)\big) &= -4(c_r - c_s), \\ f\big(\Omega_{x_0}(A_k, A_i), \Omega_{x_0}(A_j, A_1)\big) &= -4(c_r - c_s). \end{aligned}$$

Finally, substitute those in Equation 3.10 to obtain

$$f(\Omega)_{x_0}(A_1, A_i, A_j, A_k) = 4(c_r - c_s). \tag{3.12}$$

In an attempt to do the analogous computations in the context of Theorem 2.3, it is desired to rewrite Equation 3.12 in terms of the symmetric bilinear form f. In order to do that, first define an inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Inv}(M)$. This algebra is generated by $\{\kappa_r\}_{r=1}^n$, so it is non-trivial only in the even-dimensional gradings $\mathrm{Inv}_{2r}(M)$. On the first grading, for instance, define the inner product by $\langle\kappa_r, \kappa_s\rangle = \delta_r^s$ (the "zero or one"-delta). The second grading, as a vector space, has basis $\{\, \kappa_r\kappa_s \,:\, 1 \leqslant r \leqslant s \leqslant n \,\}$, so define

$$\langle \kappa_r\kappa_s \,,\, \kappa_u\kappa_v\rangle = \begin{cases} 1, & (r,s) = (u,v); \\ 0, & \text{otherwise.} \end{cases}$$

I believe it is clear how to define it for higher gradings. Note that by this definition, the gradings $\mathrm{Inv}_{2r}(M)$ are orthogonal. Then, by letting $\Omega = (1/2)[\cdot,\cdot]$ (so the constant '4' is canceled), Equation 3.12 is rewritten as

$$f(\Omega)_{x_0}(A_1, A_i, A_j, A_k) = \langle f, \kappa_r - \kappa_s\rangle. \tag{3.13}$$

Notation 3.1. Consider the following notation for the rest of this section:

- For $\alpha = \lambda_r - \lambda_s$, let $\kappa_\alpha = \kappa_r - \kappa_s$;
- Let $\langle f, \kappa_\alpha\rangle = f(\kappa_\alpha)$;

- For f and g of degrees r and s, respectively, define

$$\begin{aligned} & fg(\kappa_1, \dots, \kappa_{r+s}) \\ & \quad = \frac{1}{(r+s)!} \sum_{\sigma} f(\kappa_{\sigma(1)}, \dots, \kappa_{\sigma(r)}) g(\kappa_{\sigma(r+1)}, \dots, \kappa_{\sigma(r+s)}). \end{aligned} \tag{3.14}$$

The next lemma is straightforward.

Lemma 3.2. *Let $w = r_\alpha$ be a simple reflection, i.e., $\alpha \in \Sigma$. Let $f \in \mathrm{Inv}_2(M)$ be written as $f = \sum_r c_r \kappa_r$. Then*

$$\frac{1}{\mathrm{vol}(S^4)} \int_{\mathcal{S}_w} f(\Omega) = f(\kappa_\alpha). \tag{3.15}$$

Proof. The Chern-Weil class is invariant and we are integrating it over a Schubert cell that is a 4-dimensional sphere. We follow the idea explored in Lemma 2.1, which is to compare the differential form $f(\Omega)$ to the Euclidean volume form on S^4. From the calculations done in the mentioned lemma, the constant factor is precisely $f(\Omega)_{x_0}(A_1, A_i, A_j, A_k) = f(\kappa_\alpha)$, so

$$\int_{\mathcal{S}_w} f(\Omega) = f(\kappa_\alpha) \int_{S^4} \nu = \mathrm{vol}(S^4) f(\kappa_\alpha)$$

where ν is the Euclidean volume form and $\mathrm{vol}(S^4)$ is the Euclidean volume of S^4. □

Theorem 3.3. *Let $\mathbb{F}$ be a quaternionic flag manifold with Schubert cell decomposition $\mathbb{F} = \bigcup_{w \in \mathcal{W}} \mathcal{S}_w$. Let $w = r_{\alpha_1} \cdots r_{\alpha_s}$ be a minimal decomposition with respect to the simple roots. Let $\Pi_w = \{\beta_1, \dots, \beta_s\}$ and let $f \in \mathrm{Inv}(M)$ of degree $2s$. Then*

$$\left(\frac{1}{\mathrm{vol}(S^4)} \right)^s \int_{\mathcal{S}_w} f(\Omega) = f(\kappa_{\beta_1}, \dots, \kappa_{\beta_s}). \tag{3.16}$$

Sketch of proof. We only do a sketch here, because it is essentially the same computation done in the proof of Theorem 2.3. The difficult part, perhaps, is already done in Lemma 3.2. For instance, if f is an invariant polynomial of degree 4, then it can be written as

$$f = \sum_{i_1, i_2 = 1}^{n} a_{i_1 i_2} \kappa_{i_1} \kappa_{i_2}$$

with the symmetric coefficients. Let $w = r_{\alpha_1} r_{\alpha_2}$ and $w' = r_{\alpha_1}$, so $\Pi_w = \{\beta_1, \beta_2\}$ and $\Pi_{w'} = \{\beta_1\}$. Then

$$\int_{\mathcal{S}_w} f(\Omega) = \sum_{i_1 i_2 = 1}^{n} a_{i_1, i_2} \int_{\mathcal{S}_w} \kappa_{i_1} \kappa_{i_2}(\Omega). \tag{3.17}$$

Now, for one term, we have

$$\begin{aligned} \int_{\mathcal{S}_w} \kappa_1 \kappa_2(\Omega) &= \frac{1}{2} \Big(\int_{\mathcal{S}_{w'}} \kappa_1(\Omega) \Big) \Big(\int_{\mathfrak{u}_{\beta_2}} \kappa_2(\Omega) \Big) + \frac{1}{2} \Big(\int_{\mathcal{S}_{w'}} \kappa_2(\Omega) \Big) \Big(\int_{\mathfrak{u}_{\beta_2}} \kappa_1(\Omega) \Big) \\ &= \big(\mathrm{vol}(S^4) \big)^2 \left(\frac{\kappa_1(\kappa_{\beta_1}) \kappa_2(\kappa_{\beta_2}) + \kappa_2(\kappa_{\beta_1}) \kappa_1(\kappa_{\beta_2})}{2} \right) \\ &= \big(\mathrm{vol}(S^4) \big)^2 \kappa_1 \kappa_2(\kappa_{\beta_1}, \kappa_{\beta_2}). \end{aligned}$$

Substitute this in Equation 3.17 and use Equation 3.14 to get

$$\int_{\mathcal{S}_w} f(\Omega) = \left(\mathrm{vol}(S^4)\right)^2 f(\kappa_{\beta_1}, \kappa_{\beta_2}).$$

The computation for a polynomial f of arbitrary degree follows very closely what was done in the proof of Theorem 2.3. □

So we obtain the same result as we did at the end of chapter 2. We skip the proof, since it is analogous to the complex case.

Corollary 3.1. *The homomorphism* $[\cdot]$ *is surjective.* □

3.2 Representations of the Weyl group

In this section, two representations of the Weyl group $\mathcal{W}$ of $\mathfrak{sl}_n(\mathbb{H})$ are explored in order to determine the kernel of the Chern-Weil homomorphism. The spaces of such representations are the domain and codomain of the Chern-Weil homomorphism $[\cdot]$: $\mathrm{Inv}(M) \to H^*(\mathbb{F}; \mathbb{R})$. Both of them are related to the regular representation of $\mathcal{W}$.

The Weyl group of $\mathfrak{sl}_n(\mathbb{H})$ is the quotient $\mathcal{W} = \mathrm{Norm}_U(\mathfrak{a})/M$ where

$$\mathrm{Norm}_U(\mathfrak{a}) = \{\, g \in U \,:\, \mathrm{Ad}(g)(\mathfrak{a}) \subset \mathfrak{a} \,\}$$

is the normalizer and $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{s}$. This subgroup coincides with $\mathrm{Norm}_U(M)$. Because M (and $\mathfrak{a}$) are both sets of diagonal matrices, the group $\mathcal{W}$ is identified with the permutation matrix group — just as it happens for $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{C})$.

Let $\mathbb{F} = U/M$ be the quaternionic maximal flag manifold. The Weyl group acts on $\mathbb{F}$ from the right by $R_w(xM) = xM\bar{w} = x\bar{w}M$ where $\bar{w} \in \mathrm{Norm}_U(M)$ is a representative for $w \in \mathcal{W}$. From now on, whenever there is no risk of confusion, we use the same symbol w for both. Notice that each R_w is a homeomorphism of $\mathbb{F}$ and R_w has fixed points if and only if $w = 1$, in which case every point is fixed.

The right action of $\mathcal{W}$ on $\mathbb{F}$ induces a representation of $\mathcal{W}$ into the cohomology ring $H^*(\mathbb{F}; \mathbb{R})$ given by R_w^*. Notice that the *left* action by U induces the trivial representation on $H^*(\mathbb{F}; \mathbb{R})$, since U is connected — each homeomorphism L_g is homotopic to the identity.

The Lefschetz fixed point theorem asserts that if a homeomorphism ϕ of a topological space has no fixed point, then its *Lefschetz number* $\mathcal{L}(\phi)$ is zero. The Lefschetz number is defined as

$$\mathcal{L}(\phi) = \sum_{l \geqslant 0} (-1)^l \operatorname{tr}_l(\phi^*)$$

where tr_l is the trace of ϕ^* on the l-th cohomology $H^l(\mathbb{F}; \mathbb{R})$.

Proposition 3.2. *The representation of* $\mathcal{W}$ *in* $H^*(\mathbb{F};\mathbb{R})$ *is equivalent to the regular (right) representation.*

Proof. The cohomology $H^l(\mathbb{F})$ is non-trivial if and only if l is even. Therefore,

$$\mathcal{L}(R_w) = \sum_{l\geqslant 0}(-1)^l \operatorname{tr}_l(R_w^*) = \sum_{l\geqslant 0}\operatorname{tr}_l(R_w^*).$$

The last sum is the character $\chi(w)$ of the representation of $\mathcal{W}$ in $H^*(\mathbb{F};\mathbb{R})$. If $w \neq 1$, then $\mathcal{L}(R_w) = 0$. On the other hand,

$$\chi(1) = \operatorname{tr}(\mathrm{Id}) = \dim H^*(\mathbb{F};\mathbb{R}) = |\mathcal{W}|.$$

The last equation follows from the fact that the elements of $\mathcal{W}$ are in bijection with the Schubert cells and those forms a basis for the homology. Therefore

$$\chi(w) = \begin{cases} 0 & \text{if } w \neq 1 \\ |\mathcal{W}| = \dim H^*(\mathbb{F};\mathbb{R}) & \text{if } w = 1. \end{cases}$$

This is precisely the character of the regular representation, so they are equivalent. □

The Weyl group also represents itself in the space $\mathrm{Inv}(M)$, *i.e.*, in the space generated by the monomials

$$\kappa_1^{m_1}\cdots\kappa_n^{m_n}, \qquad m_1,\ldots,m_s \geqslant 0$$

by permutation of the indices on each monomial and extending it linearly. Each $w \in \mathcal{W}$ corresponds to a permutation, so

$$w\cdot(\kappa_1^{m_1}\cdots\kappa_n^{m_n}) = \kappa_{w(1)}^{m_1}\cdots\kappa_{w(n)}^{m_n}.$$

In what follows, we will verify that $[\cdot] : \mathrm{Inv}(M) \to H^*(\mathbb{F};\mathbb{R})$ is equivariant by the actions of $\mathcal{W}$ on $\mathrm{Inv}(M)$ and $H^*(\mathbb{F};\mathbb{R})$. First we prove a lemma.

Lemma 3.3. *Let* ω *and* Ω *be the connection and curvature forms on* $U \to U/M$*. If* $\bar{w} \in \mathrm{Norm}_U(M)$*, then* $R_{\bar{w}}^*\omega = \mathrm{Ad}(\bar{w}^{-1})\omega$ *and* $R_{\bar{w}}^*\Omega = \mathrm{Ad}(\bar{w}^{-1})\Omega$*.*

Proof. The Lie algebras of M and $\mathrm{Norm}_U(M)$ are both equal to $\mathfrak{m}$ with complement $\mathfrak{c} = \sum_{\alpha>0}\mathfrak{u}_\alpha$. Not only for $U \to U/M$, the complement $\mathfrak{c}$ also defines a connection and curvature on the principal bundle $U \to U/\mathrm{Norm}_U(M)$ which are equal to the one defined on $U \to U/M$. The statement in the lemma follows, since such equalities $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ and $R_g^*\Omega = \mathrm{Ad}(g^{-1})\Omega$ are valid in every principal fiber bundle (see (KOBAYASHI; NOMIZU, 1963), proposition 1.1 of section 1, chapter II; and section 5 of chapter II). In particular, it is true for $\bar{w} \in \mathrm{Norm}_U(M)$. □

Proposition 3.3. *Let $f \in \mathrm{Inv}(M)$. Take $\bar{w} \in \mathrm{Norm}_U(M)$ to be a representative of $w \in \mathcal{W}$. Then $[w \cdot f] = R^*_{\bar{w}}[f]$. In other words, $[\cdot] : \mathrm{Inv}(M) \to H^*(\mathbb{F};\mathbb{R})$ is equivariant by the actions of $\mathcal{W}$.*

Proof. Choose ω *any* curvature form on U and let Ω be the corresponding curvature form. By definition of pullback

$$R^*_{\bar{w}}(f(\Omega))(X_1, \dots, X_{2k}) = f(\Omega)((R_{\bar{w}})_* X_1, \dots, (R_{\bar{w}})_* X_{2k}). \tag{3.18}$$

By Equation 1.5, the right-hand side of Equation 3.18 is the σ-alternating sum of

$$f(\Omega\left((R_{\bar{w}})_* X_1, (R_{\bar{w}})_* X_2\right), \dots, \Omega\left((R_{\bar{w}})_* X_{2k-1}, (R_{\bar{w}})_* X_{2k}\right))$$

which equals to

$$f\left(\mathrm{Ad}(\bar{w}^{-1})\Omega(X_1, X_2), \dots, \mathrm{Ad}(\bar{w}^{-1})\Omega(X_{2k-1}, X_{2k})\right) \\ = (wf)(\Omega(X_1, X_2), \dots, \Omega(X_{2k-1}, X_{2k})). \tag{3.19}$$

This equation follows because $\mathrm{Ad}(\bar{w})$ acts as permutation of the entries in the diagonal matrix $\Omega(X_r, X_s)$. The σ-alternating sum of the right-hand side of Equation 3.19 is $wf(\Omega)(X_1, \dots, X_{2k})$. Notice this is all done at the level of the principal space U. One still needs to project it down to $H^*(\mathbb{F};\mathbb{R})$. This follows from the fact that $R_w \circ \pi = \pi \circ R_{\bar{w}}$, where $\pi : U \to U/M$. If we denote the projected form by $\bar{f}(\Omega)$, then $\pi^*(\bar{f}(\Omega)) = f(\Omega)$. □

Corollary 3.2. *Let $\mathcal{I}$ denote the ideal of $\mathrm{Inv}(M)$ generated by the $\mathcal{W}$-invariant forms. Then $\mathcal{I} \subseteq \ker[\cdot]$.*

Proof. Let $f \in \mathrm{Inv}_k(M)$, $k > 0$, and suppose that f is $\mathcal{W}$-invariant. The subspace of $H^k(\mathbb{F};\mathbb{R})$ generated by the form $[f]$ is also invariant, since $[\cdot]$ is equivariant. The representation of $\mathcal{W}$ in $H^*(\mathbb{F};\mathbb{R})$ is equivalent to the regular representation by Proposition 3.2. The regular representation has only one-dimensional invariant subspace, that is, if $V = V(\mathcal{W})$ is the freely generated vector space with basis $\mathcal{W}$, then the unique invariant one-dimensional subspace is the one generated by the element $v = \sum_{w \in \mathcal{W}} w$ (to see this, remember that $\mathcal{W}$ is the group of all permutations in n elements!). On $H^*(\mathbb{F};\mathbb{R})$, the only $\mathcal{W}$-invariant one-dimensional subspace is then H^0, so the subspace generated by $[f]$ is *not* one-dimensional, that is, $[f] = 0$. Because $[\cdot]$ is a ring homomorphism, it follows that the ideal $\mathcal{I}$ is also inside $\ker[\cdot]$. □

The representation of the Weyl group in the quotient space $\mathrm{Inv}(M)/\ker[\cdot]$ is also equivalent to the regular representation[3], just as the representation on $H^*(\mathbb{F};\mathbb{R})$. Since the $\mathcal{W}$-actions are equivariant with respect to $[\cdot] : \mathrm{Inv}(M) \to H^*(\mathbb{F};\mathbb{R})$, we must have that $\mathcal{I}$ is the whole kernel.

[3] Chevalley's Theorem. See section 3.6 in (HUMPHREYS, 1990)

Corollary 3.3. *The ideal generated by the $\mathcal{W}$-invariant polynomials is the kernel of the Chern-Weil homomorphism, that is,* $\ker[\cdot] = \mathcal{I}$. □

Remark 3.4. The Schubert cells provide a basis for the homology of $\mathbb{F}$ and its dual is a basis for the cohomology $H^*(\mathbb{F};\mathbb{R})$. Since the representation spaces $H^*(\mathbb{F};\mathbb{R})$ and $\mathrm{Inv}(M)/\ker[\cdot]$ are isomorphic, such dual basis provides a natural basis for the invariant polynomial algebra (excluding the kernel).

3.3 Cohomology of the Flag

The computations done in the proof of Lemma 3.1 can be generalized to matrices of $\mathfrak{sl}_n(\mathbb{H})$. Let $\alpha = \alpha_{rs} = \lambda_r - \lambda_s$. Let E_{rs} denote the $n \times n$ matrix consisting of 1 in the (r, s)-entry and zero everywhere else. Then a basis for $\mathfrak{u}_\alpha$ is given by $\{A_1^\alpha, A_i^\alpha, A_j^\alpha, A_k^\alpha\}$, where

$$A_p^\alpha = pE_{rs} - \bar{p}E_{sr}, \quad p \in \mathbb{H}.$$

The bracket behaves in a similar fashion. For $p, q \in \mathbb{H}$,

$$[A_p^\alpha, A_q^\alpha] = (-2)\Big(\mathrm{Im}(p\bar{q})E_{rr} + \mathrm{Im}(\bar{p}q)E_{ss}\Big). \tag{3.20}$$

So, for $1 \leqslant t \leqslant n$, one might have many roots $\alpha = \alpha_{rs}$ such that

$$\kappa_t\left([A_1^\alpha, A_i^\alpha], [A_j^\alpha, A_k^\alpha]\right) \neq 0. \tag{3.21}$$

Example 3.1. Let $\mathbb{F}$ be the maximal flag of $\mathfrak{sl}_3(\mathbb{H})$. The rootspaces are those $\mathfrak{u}_\beta$ with $\beta \in \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}$. Because $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha\pm\beta}$ (whichever sign provides a root), $\Omega(\mathfrak{u}_\alpha, \mathfrak{u}_\beta) = 0$ (remember that Ω_{x_0} is projection onto $\mathfrak{m}$). Therefore, the only chance of having $\kappa_t(\Omega)_{x_0} \neq 0$ is if its entries belongs all to a single rootspace. To find out which rootspaces are these, simply take a closer look at Equation 3.21: the brackets must have non-zero coefficient at E_{tt}, so the roots α_{rs} for which the restriction of κ_t is non-zero are

$$\{\, \alpha_{rs} \in \Pi \;:\; r = l \text{ or } s = l \,\}\,.$$

The computations done previously reveals that

$$\kappa_t(\Omega)\Big(A_1^{\alpha_{rs}}, A_i^{\alpha_{rs}}, A_j^{\alpha_{rs}}, A_k^{\alpha_{rs}}\Big) = \pm 4,$$

being positive if $t = r$ or negative if $t = s$ (see Equation 3.20). □

We summarize the above discussion in the following proposition.

Theorem 3.4. *Let* $\mathbb{F} = U/M$ *be a maximal flag of* $\mathfrak{sl}_n(\mathbb{H})$*, where* $U = \mathrm{Sp}_n$ *and* $M = \mathrm{Sp}_1 \times \cdots \times \mathrm{Sp}_1$ *with corresponding linear algebras* $\mathfrak{u}$ *and* $\mathfrak{m}$*. For* $\alpha = \alpha_{rs} \in \Pi$*, let* $\mathfrak{u}_\alpha$ *be the* *corresponding rootspace and* $\nu_\alpha = \nu_{rs}$ *the volume form in* $\mathfrak{u}_\alpha$ *given by the coordinates with tangent vectors* $\{A_1^\alpha, A_i^\alpha, A_j^\alpha, A_k^\alpha\}$*. Then for* $1 \leqslant t \leqslant n$ *and* $\alpha = \alpha_{rs}$*,*

$$\kappa_t(\Omega)_{x_0}(A_1^\alpha, A_i^\alpha, A_j^\alpha, A_k^\alpha) = \begin{cases} 0 & r \neq t \neq s \\ -4 & r = t < s \\ 4 & r < t = s \end{cases}.$$

With respect to the volume forms ν_α*,*

$$\kappa_t(\Omega)_{x_0} = -4\Big(\nu_{1;t} + \cdots + \nu_{t-1;t}\Big) + 4\Big(\nu_{t;t+1} + \cdots + \nu_{t;n}\Big) \tag{3.22}$$

The following matrix indicates the coefficient on each volume form ν_{rs} *that appears in the composition of* $\kappa_t(\Omega)_{x_0}$*.*

$$\begin{bmatrix} (1,1) & & & -4 & & & \\ & \ddots & & \vdots & & & \\ & & \ddots & -4 & & & \\ & & & (t,t) & +4 & \cdots & +4 \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & (n,n) \end{bmatrix} \tag{3.23}$$

Remark 3.5. The volume forms ν_α need a little bit of explanation. They are first defined at a root space $\mathfrak{u}_\alpha$, as stated in the theorem. Then it is easily extended to the whole tangent space at the origin, since $\mathfrak{u}_\alpha \subset T_{x_0}(\mathbb{F})$. The volume form ν_α is invariant by $M = \mathrm{Sp}_1 \times \cdots \times \mathrm{Sp}_1$, so left translation to other points of $\mathbb{F} = U/M$ is well-defined. This translation does not alter volume or changes orientation, since for $m \in M$, $\det m = 1$ and M is connected.

There is a natural pairing between homology and closed differential forms: to each cohomology class $[f]$, there corresponds a homomorphism on the chains of $\mathbb{F}$ defined by the integral of f over said chain. Consider the set $\{\mathcal{S}_\alpha : \alpha \in \Sigma\}$. As we have seen, this set is a basis of $H_4(\mathbb{F})$ due to the homology being non-trivial only in even dimensions. Theorem 3.4 can be used to find the dual basis in $H^4(\mathbb{F};\mathbb{R})$. In fact, let $\alpha_r = \lambda_r - \lambda_{r+1}$ denote the simple roots (for now) and see $[\kappa]$ as an element in the dual $(H_4(\mathbb{F}))^* \approx H^4(\mathbb{F};\mathbb{R})$ (De Rham's Theorem). By Theorem 3.4, the restriction of $[\kappa_r]$ is a volume form with a certain orientation or it is null, all depending on Equation 3.22. Therefore, one sees immediately that $[\kappa_1](S_{\alpha_j})$ is $\neq 0$ if and only if $j = 1$. By the same reasoning, $[\kappa_1 + \kappa_2](S_{\alpha_j})$ is $\neq 0$ if and only if $j = 2$ and so on. This proves the proposition below.

Proposition 3.4. *Up to a normalization constant, the set*

$$\{[\kappa_1], [\kappa_1 + \kappa_2], \dots, [\kappa_1 + \cdots + \kappa_{n-1}]\}$$

is the dual basis of $\{\, \mathcal{S}_\alpha \,:\, \alpha \in \Sigma \,\} \subset H_4(\mathbb{F})$. □

Let $\Sigma = \{\alpha_1, \dots, \alpha_l\}$ be a system of simple roots of type A_l. The associated maximal flag of $\mathfrak{sl}_n(\mathbb{H})$ is denoted by $\mathbb{F}$ ($n = l + 1$). Consider the subsets $\Theta = \Sigma\backslash\{\alpha_1\}$ and $\Theta_1 = \Sigma\backslash\{\alpha_1, \alpha_l\}$. Their associated flag manifolds are $\mathbb{F}_\Theta = \mathbb{H}P^{n-1}$, the quaternionic projective space, and $\mathbb{F}_{\Theta_1}$, the set of all quaternionic flags of signature (V_1, V_{n-1}).

To each of these flag manifolds, there is a natural principal bundle with total space $U = \mathrm{Sp}_n$. Their respective structural groups are denoted by M_Θ and M_{Θ_1}, and their Lie algebras will be following the same notational pattern. Explicitly, those structure groups are subgroups of Sp_n consisting of block diagonal matrices

$$M_\Theta = \mathrm{Sp}_1 \times \mathrm{Sp}_{n-1} \qquad \text{and} \qquad M_{\Theta_1} = \mathrm{Sp}_1 \times \mathrm{Sp}_{n-2} \times \mathrm{Sp}_1. \tag{3.24}$$

Let $\kappa_1 \in \mathrm{Inv}(\mathfrak{m}_\Theta)$ be the symmetric 2-form which is the inner product on the first factor of $\mathfrak{m}_\Theta$ and zero on the others. Naturally, κ_1 is also a symmetric form on $\mathfrak{m}_{\Theta_1}$ by restriction of its domain. Because $M_{\Theta_1} \subseteq M_\Theta$, this restricted form, still denoted by κ_1, belongs to $\mathrm{Inv}(\mathfrak{m}_{\Theta_1})$. In general, there is the inclusion $\mathrm{Inv}(\mathfrak{m}_\Theta) \subseteq \mathrm{Inv}(\mathfrak{m}_{\Theta_1})$.

Lemma 3.4. *Let* $\pi : \mathbb{F}_{\Theta_1} \to \mathbb{F}_\Theta$ *be the canonical projection. Choose the canonical connection forms on each flag manifold and denote by* Ω_Θ *and* Ω_{Θ_1} *the associated curvatures. Let* κ_1 *be the symmetric form associated to the first factor of both* M_Θ *and* M_{Θ_1}*. Then*

$$\kappa_1(\Omega_{\Theta_1}) = \pi^*\kappa_1(\Omega_\Theta). \tag{3.25}$$

Proof. Take arbitrary horizontal vectors $X_1, \dots, X_4$ on the tangent space $T(\mathrm{Sp}_n)$ with respect to the chosen connection in $\mathrm{Sp}_n \to \mathbb{F}_\Theta$. Let $X'_r \in T(\mathbb{F}_{\Theta_1})$ be a vector field such that $\pi^*(X'_r) = X_r$. By definition,

$$\pi^*\kappa_1(\Omega_\Theta)(X'_1, \dots, X'_4) = \kappa_1(\Omega_\Theta)(X_1, \dots, X_4).$$

The right-hand side of the above equation is defined as the σ-alternating sum of

$$\kappa_1(\Omega_\Theta(X_1, X_2), \Omega_\Theta(X_3, X_4)) = \kappa_1\Big([X_1, X_2]_{\mathfrak{m}_\Theta}, [X_3, X_4]_{\mathfrak{m}_\Theta}\Big).$$

Since κ_1 is the inner product on the (first) factor $\imath$ of both $\mathfrak{m}_\Theta$ and $\mathfrak{m}_{\Theta_1}$, the brackets in the last equation are only counted for their E_{11} component. Therefore, projection onto $\mathfrak{m}_{\Theta_1}$ or $\mathfrak{m}_\Theta$ are interchangeable, because we will be applying κ_1 right after. So

$$\kappa_1\Big([X_1, X_2]_{\mathfrak{m}_\Theta}, [X_3, X_4]_{\mathfrak{m}_\Theta}\Big) = \kappa_1\Big([X_1, X_2]_{\mathfrak{m}_{\Theta_1}}, [X_3, X_4]_{\mathfrak{m}_{\Theta_1}}\Big).$$

On the other hand, $\kappa_1(\Omega_{\Theta_1})(X_1', \ldots, X_4')$ is the σ-alternating sum of

$$\kappa_1(\Omega_{\Theta_1}(X_1', X_2'), \Omega_{\Theta_1}(X_3', X_4')) = \kappa_1\Big([X_1', X_2']_{\mathfrak{m}_{\Theta_1}}, [X_3', X_4']_{\mathfrak{m}_{\Theta_1}}\Big).$$

Now, the only way to have $\kappa_1\Big([X_1', X_2']_{\mathfrak{m}_{\Theta_1}}, [X_3', X_4']_{\mathfrak{m}_{\Theta_1}}\Big) \neq 0$ is if each $X_r' \in \mathfrak{u}_\beta$, where $\beta = \lambda_1 - \lambda_t$, for any $t > 1$ (the projection onto $\mathfrak{m}_{\theta_1}$ of a bracket involving any other X' will null first coordinate). In this case,

$$[X_r', X_s']_{\mathfrak{m}_{\Theta_1}} = [X_r, X_s]_{\mathfrak{m}_{\Theta_1}}.$$

This shows that $\kappa_1(\Omega_{\Theta_1})$ and $\pi^*\kappa_1(\Omega_\Theta)$ have the same non-zero σ-alternating sum. □

Example 3.2. We now use the projections of the maximal flag onto the smaller partial flags presented above to compute the homology of the maximal flag of $\mathfrak{sl}_3(\mathbb{H})$. In fact, let $\Sigma = \{\alpha_1, \alpha_2\}$ and define $\Theta = \Sigma \backslash \{\alpha_2\} = \{\alpha_1\}$. The maximal flag is the set of all sequences (V_1, V_2) of nested quaternionic subspaces of $\mathbb{H}^3$, then $\mathbb{F}_\Theta$ is the set $\left\{ V_2 \subset \mathbb{H}^3 \,:\, E_1 \subset V_2 \right\}$ (remember Example 2.1). Here E_1 is the quaternionic line generated by the canonical vector $e_1 \in \mathbb{H}^3$. Let $\pi : \mathbb{F} \to \mathbb{F}_\Theta$ denote the projection $(V_1, V_2) \mapsto V_2$. Then the fiber over $E_2 \approx \mathbb{H}^2$ is the set of all V_1 inside E_2, that is, the fiber is the quaternionic projective line $\mathbb{H}P^1$. Notice that $\mathbb{F}_\Theta$ is also diffeomorphic to $\mathbb{H}P^1$, since all V_2 contains the line E_1. Then again, in essence an application of Theorem .2, one has that $\{\kappa_d(\Omega), \kappa_3(\Omega)\}$ ($d = 1$ or 2) generates $H^*(\mathbb{F}; \mathbb{R})$. □

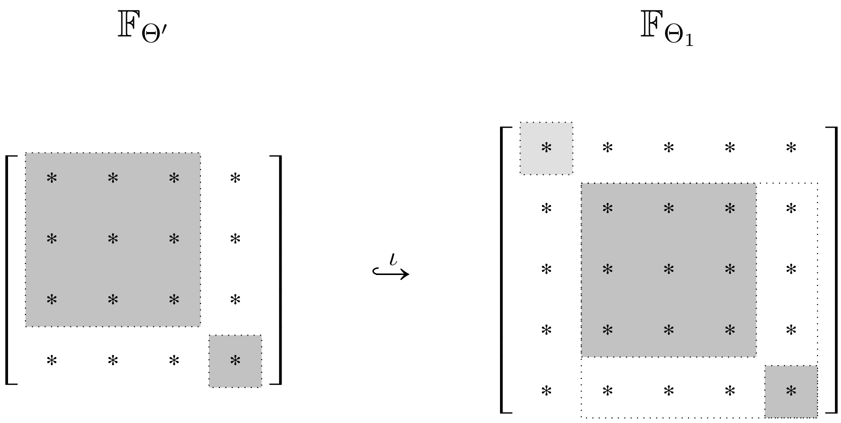
Let $\mathcal{F}$ be the fiber of the principal bundle $\mathbb{F}_{\Theta_1} \to \mathbb{F}_\Theta$, which consists of all subspaces V_{n-1} containing a fixed V_1. Then $\mathcal{F}$ is diffeomorphic to the set of flags of type V_{n-2} inside $\mathbb{H}^{n-1}$. If we write $\Sigma' = \{\alpha_2, \ldots, \alpha_l\}$, then $\mathcal{F} = \mathbb{F}_{\Theta'}$ where $\Theta' = \Sigma' \backslash \{\alpha_l\}$. It is known that this flag is isomorphic to the projective space $\mathbb{H}P^{n-2}$, so its cohomology has a single generator—see (HATCHER, 2002) or (BAKER, 1999).

Lemma 3.5. *Let $\Theta_1 = \Sigma \backslash \{\alpha_1, \alpha_l\}$. Then for $n > 2$, the cohomology $H^*(\mathbb{F}_{\Theta_1}; \mathbb{R})$ is generated by the characteristic classes in $\{\kappa_1^*(\Omega_{\Theta_1}), \kappa_n^*(\Omega_{\Theta_1})\}$.*

Proof. The formula for $\kappa_r(\Omega)$ on Theorem 3.4 allows us to obtain a similar formula specifically for $\kappa_1(\Omega_{\Theta_1})$ and $\kappa_n(\Omega_{\Theta_1})$, since they are the same at both smaller and maximal flag. Therefore

$$\kappa_n(\Omega_{\Theta_1}) = -\nu_{1;n} - \nu_{2;n} - \cdots - \nu_{n-1;n} \qquad \text{(at the origin)}.$$

Let $\iota : \mathbb{F}_{\Theta'} \hookrightarrow \mathbb{F}_{\Theta_1}$ denote the inclusion. In terms of the tangent plane at the origin—which is a sum of rootspaces— the inclusion ι is visually represented below for the case $n = 5$.



The rootspaces in tangent plane are represented by the stars with white background, the complement is the Lie algebra of the isotropy.

Because the pullback ι^* is simply a restriction, one has

$$\iota^*\kappa_n(\Omega_{\Theta_1}) = -\nu_{2;n} - \cdots - \nu_{n-1;n}.$$

The form $\iota^*\kappa_n^*(\Omega_{\Theta_1})$ is non-zero, because it is the sum of volume forms on the roots spaces (and one cannot mix those). Therefore, it is a generator of $H^*(\mathcal{F};\mathbb{R})$ and the Leray-Hirsch condition is met. By Theorem .2, the cohomology of $\mathbb{F}_{\Theta_1}$ is generated by $\{\pi^*\kappa_1(\Omega_\Theta), \kappa_n(\Omega_{\Theta_1})\}$. In light of Lemma 3.4, this set equals

$$\{\kappa_1(\Omega_{\Theta_1}), \kappa_n(\Omega_{\Theta_1})\}.$$

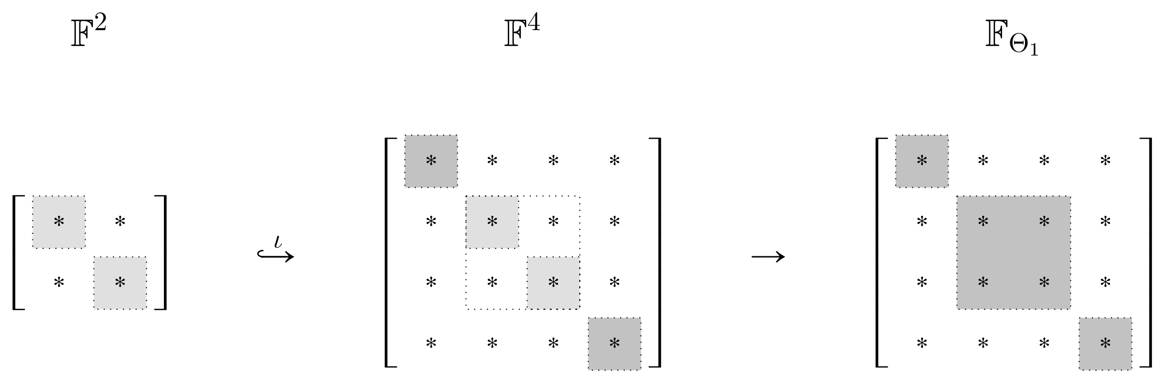
□

Remark 3.6. The dimension of the fiber is lesser than the dimension of the total space, so the smallest k such that $\kappa_n^*(\Omega_\Theta)^k = 0$ is greater than the smallest l such $\iota^*\kappa_n^*(\Omega_\Theta)^l = 0$.

Our aim is to show that the set of all $\kappa(\Omega)$ forms a set of generators of the cohomology $H^*(\mathbb{F};\mathbb{R})$. The idea is to reduce the maximal flag by removing the first and last roots by letting $\Theta = \Sigma\backslash\{\alpha_1, \alpha_l\}$ (we call this the *"comendo pelas beirada"*-technique). The projection $\mathbb{F} \to \mathbb{F}_\Theta$ is a bundle with fiber diffeomorphic to the smaller maximal flag of $\mathfrak{sl}_{n-2}(\mathbb{H})$. By the Leray-Hirsch argument, the $H^*(\mathbb{F};\mathbb{R})$ is generated by generating sets for $H^*(\mathcal{F})$ and $H^*(\mathbb{F}_\Theta)$ —the latter being given by Lemma 3.5. This process is illustrated in Example 3.3.

Notation 3.2 (Temporary notation)**.** $\mathbb{F}_\Theta^n$ stands for the flag manifold of $\mathfrak{sl}_n(\mathbb{H})$.

Example 3.3. Let $\Sigma = \{\alpha_1, \alpha_2, \alpha_3\}$. Consider $\Theta_1 = \Sigma\backslash\{\alpha_1, \alpha_3\} = \{\alpha_2\}$. Then we have the fibration $\mathbb{F}^2 \hookrightarrow \mathbb{F}^4 \to \mathbb{F}_{\Theta_1}$.



By Lemma 3.5, $H^*(\mathbb{F}_{\Theta_1};\mathbb{R})$ is generated by $\{\kappa_1(\Omega), \kappa_4(\Omega)\}$ and by Lemma 3.1, $H^*(\mathbb{F}^2;\mathbb{R})$ is generated by either $\iota^*\kappa_2(\Omega)$ or $\iota^*\kappa_3(\Omega)$, as indicated by the matrix representation of the fibration above. The Leray-Hirsch condition is met, then $H^*(\mathbb{F}^4;\mathbb{R})$ is generated by $\{\kappa_1(\Omega), \kappa_d(\Omega), \kappa_4(\Omega)\}$, $d = 2$ or 3.

The case $n = 5$ is slightly different from the case $n = 4$ above, since the fibers in this case is diffeomorphic to the maximal flag $\mathbb{F}^3$. Its cohomology is covered in Example 3.2. So, a generating set for $\mathbb{F}^5$ is $\{\kappa_1(\Omega), \kappa_d(\Omega), \kappa_4(\Omega), \kappa_5(\Omega)\}$. □

Remark 3.7. We will choose the smallest d from now on.

Theorem 3.5. *Let $\mathbb{F}^n$ denote the maximal flag manifold of $\mathfrak{sl}_n(\mathbb{H})$. Choose $\{\kappa_1, \dots, \kappa_n\}$ to be the standard set of generators of* $\mathrm{Inv}(M)$. *For any curvature 2-form Ω, $H^*(\mathbb{F}^n;\mathbb{R})$ has*

$$\{\kappa_1(\Omega), \kappa_2(\Omega), \dots, \kappa_n(\Omega)\}\backslash\{\kappa_{m+1}(\Omega)\} \tag{3.26}$$

as a set of generators, where m is such that either $n = 2m$ or $n = 2m + 1$.

Proof of Theorem 3.5. The proof is done via (reversed) induction on n which depends on its parity, so regard n as equals to $2m$ or $2m + 1$. Let $\mathbb{F}^n$ be the maximal flag of $\mathfrak{sl}_n(\mathbb{H})$. Let $\Sigma = \{\alpha_1, \dots, \alpha_l\}$ $(l = n - 1)$ be the associated system of simple roots and define

$$\Theta_k = \Sigma\backslash\{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{l-k}, \dots, \alpha_{l-1}, \alpha_l\}, \quad 0 \leqslant k \leqslant m - 1.$$

So Θ_{m-1} has either one root in it ($n = 2m$) or two roots left ($n = 2m + 1$). There is a natural projection $\pi : \mathbb{F}^n \to \mathbb{F}_{\Theta_1}$ whose fibers are diffeomorphic to a smaller maximal flag manifold $\mathbb{F}^{n-2}$. This induces a fibration

$$\mathbb{F}^{n-2} \hookrightarrow \mathbb{F}^n \to \mathbb{F}_{\Theta_1}. \tag{3.27}$$

Suppose the fibration in (3.27) satisfies the Leray-Hirsch condition. Then

$$H^*(\mathbb{F}^n;\mathbb{R}) = H^*(\mathbb{F}^{n-2};\mathbb{R}) \otimes H^*(\mathbb{F}^n_{\Theta_1};\mathbb{R}). \tag{3.28}$$

The flag manifold $\mathbb{F}^{n-2}$ can be seen as the flag associated to the root system $\{\alpha_2, \ldots, \alpha_{l-1}\} = \Theta_1$ (*"comendo pelas beiradas"*) which is of type A. Therefore, by repeating the process, its cohomology is

$$H^*(\mathbb{F}^{n-2}; \mathbb{R}) = H^*(\mathbb{F}^{n-4}; \mathbb{R}) \otimes H^*(\mathbb{F}^{n-2}_{\Theta_2}; \mathbb{R}). \tag{3.29}$$

Substitute Equation 3.29 in Equation 3.28 to arrive at

$$H^*(\mathbb{F}^{n}; \mathbb{R}) = H^*(\mathbb{F}^{n-4}; \mathbb{R}) \otimes H^*(\mathbb{F}^{n-2}_{\Theta_2}; \mathbb{R}) \otimes H^*(\mathbb{F}^{n}_{\Theta_1}; \mathbb{R}).$$

Continue all the way down to the smallest non-trivial flag $\mathbb{F}^{n-2(m-1)}$ (*i.e.* $\mathbb{F}^2$ or $\mathbb{F}^3$) to obtain

$$H^*(\mathbb{F}^{n}; \mathbb{R}) = H^*(\mathbb{F}^{n-2(m-1)}; \mathbb{R}) \otimes H^*(\mathbb{F}^{n-2(m-2)}_{\Theta_{m-1}}; \mathbb{R}) \otimes \cdots H^*(\mathbb{F}^{n}_{\Theta_1}; \mathbb{R}). \tag{3.30}$$

Each factor in Equation 3.30 of the form $H^*(\mathbb{F}^{n-2(r-1)}_{\Theta_r}; \mathbb{R})$ has $\{\kappa_r(\Omega), \kappa_{n-r+1}(\Omega)\}$ as generating set, by Lemma 3.5. Collecting all closed forms, one has

$$\bigcup_{1 \leqslant r \leqslant m-1} \{\kappa_r(\Omega), \kappa_{n-r+1}(\Omega)\}.$$

As for the cohomology of the remaining factor $\mathbb{F}^d$, $d = 2, 3$, we have that a generating set is either $\{\kappa_m(\Omega)\}$ or $\{\kappa_m(\Omega), \kappa_{m+2}(\Omega)\}$. In any case, putting everything together, we arrive at

$$\{\kappa_1(\Omega), \kappa_2(\Omega), \ldots, \kappa_n(\Omega)\} \backslash \{\kappa_{m+1}(\Omega)\}$$

as the desired generating set for $H^*(\mathbb{F}; \mathbb{R})$. □

Remark 3.8. The "left out" $[\kappa_{m+1}]$ on equation Equation 3.26 might as well be included in the set of generators, since it can be written as a linear combination of the others. In fact,

$$[\kappa_1(\Omega) + \kappa_2(\Omega) + \cdots + \kappa_n(\Omega)] = 0$$

since it invariant by the Weyl group, so $\kappa_{m+1}(\Omega)$ is the negative sum of the others.

The above theorem, together with Proposition 3.4, guarantees that the image of the Chern-Weil homomorphism contains the dual basis given by the Schubert cellular structure of $\mathbb{F}$.

Corollary 3.4. *The Chern-Weil homomorphism* $[\cdot] : \mathrm{Inv}(M) \to H^*(\mathbb{F}; \mathbb{R})$ *is surjective.*

Proof. By Proposition 3.4, the Chern-Weil class $[\kappa_r]$ is in the dual basis of Schubert cells. The same $[\kappa_r]$ is also in the generating set of the algebra $H^*(\mathbb{F}; \mathbb{R})$, by Theorem 3.5. So each closed differential form $f \in H^*(\mathbb{F}; \mathbb{R})$ can be written as sum of products of the $[\kappa_r]$. □

Remark 3.9. The (reversed) induction on n in the proof of Theorem 3.5 is due to the Leray-Hirsch, which we know can be applied only retroactively starting from $\mathbb{F}^d$.

Remark 3.10. Putting together Corollary 3.3 and Corollary 3.4, we arrive at a complete description of $H^*(\mathbb{F}; \mathbb{R})$ in terms of the invariant polynomial algebra.

3.4 Quaternionic Structures

The aim of this section is to present a nice application of the Chern-Weil classes description given by Equation 3.22 to show that on the maximal flag manifold $\mathbb{F}$, no G-invariant quaternionic structure (explained below) will be hypercomplex.

Let M be a smooth manifold of real dimension $4n$. The set of all linear isomorphisms $p : \mathbb{R}^{4n} \to T_xM$ $(x \in M)$ forms the principal bundle $\mathrm{Fr}(TM)$, called the *frame bundle*, with structure group $\mathrm{Gl}_{4n}(\mathbb{R})$. Another important bundle is $\mathrm{End}(TM)$, the *endomorphism vector bundle*, where each fiber $\mathrm{End}_x(TM)$ over $x \in M$ is the set of all endomorphisms of T_xM.

A *quaternionic structure* on M is a subbundle $\mathcal{Q} \subseteq \mathrm{End}(TM)$ where each fiber is isomorphic to the quaternion algebra $\mathbb{H}$. Locally, there is a pair of anti-commuting complex structures (I, J) defined on an open set of M induced by the quaternions i and j.

Let $\mathbb{H}^n \approx \mathbb{R}^{4n}$ be the canonical right $\mathbb{H}$-vector space. For each $q \in \mathbb{H}$, define the transformation (denoted by the same symbol)

$$q(v) = vq, \qquad v \in \mathbb{H}^n$$

and let Sp_1 be the subset of those transformations corresponding to unitary quaternions. The group

$$\mathrm{Gl}_n(\mathbb{H}) = \{\, g \in \mathrm{Gl}_{4n}(\mathbb{R}) \;:\; gq = qg \;\; \forall\, q \in \mathbb{H} \,\}$$

is the centralizer of $\mathbb{H}$ (seen as transformations of $\mathbb{H}^n$) in $\mathrm{Gl}_{4n}(\mathbb{R})$ and it represents the set of all $\mathbb{H}$-linear transformations of $\mathbb{H}^n$.

Associated to a quaternionic structure $\mathcal{Q}$, there is a principal subbundle $P \subseteq \mathrm{Fr}(TM)$ with structure group

$$\mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H}) = \{\, qg \;:\; q \in \mathrm{Sp}_1 \text{ and } g \in \mathrm{Gl}_n(\mathbb{H}) \,\}$$

whose fiber P_x over $x \in M$ is

$$P_x = \left\{\, p \in \mathrm{Fr}M_x \;:\; pqp^{-1} \in \mathcal{Q}_xM \;\; \forall q \in \mathbb{H} \,\right\}.$$

Conversely, given a $\mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$-bundle P there corresponds a quaternionic structure $\mathcal{Q}$ with fibers given by

$$\mathcal{Q}_xM = p\mathbb{H}p^{-1}$$

where p is *any* element of P_x. Thus we will also call the principal bundle P a quaternionic structure.

When the base manifold is homogeneous, i.e., $M = G/H$, then G acts naturally on M by the map $g : x \mapsto gx$ and the differential (also denoted by g) induces a left action on $\mathrm{End}(TM)$ by composition.

We will say that a *quaternionic structure $\mathcal{Q}$ is G-invariant* if

$$g\mathcal{Q}_x g^{-1} = \mathcal{Q}_{gx}$$

for arbitrary x and g. In terms of the principal bundle P, this is the same as having $g(P_x) \subseteq P_{gx}$ with g being composed on the left.

Given a general bundle $E \to M$ with fiber F, one often asks whether E is trivial or not, i.e., if there is a bundle isomorphism $E \approx M \times F$. One way to tackle this question is to look for the characteristic classes of E. As we have seen, for a vector bundle, its characteristic classes are generated by the Chern-Weil classes. See Chapter 12, Section 3 (or 4) of (KOBAYASHI; NOMIZU, 1996).

Let $\mathcal{Q} \subseteq \mathrm{End}(TM)$ be a quaternionic structure on M. If $\mathcal{Q}$ is trivial, that is, $\mathcal{Q} \approx M \times \mathbb{H}$, then there is a pair of anti-commuting complex structures (I, J) *globally defined*. We call this a *hypercomplex structure* on M.

Take P to be the $\mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$-principal bundle associated to a quaternionic structure $\mathcal{Q}$. If $\mathcal{Q}$ is trivial, then the structure group of P is reducible to $\mathrm{Gl}_n(\mathbb{H})$. In fact, let

$$R_x = \{\, p \in P_x \;:\; pq = qp \;\; \forall q \in \mathbb{H} \,\}$$

where on the right-hand side we are identifying q and $(x, q) \in \mathcal{Q}_x$. Then $R = \bigcup_{x \in M} R_x$ is a $\mathrm{Gl}_n(\mathbb{H})$-subbundle of P. For any $a \in \mathrm{Gl}_n(\mathbb{H})$ and $v \in \mathbb{H}^n$

$$pa(vq) = p(a(vq)) = p((av)q) = (p(av))q = (pa(v))q$$

which shows that $pa \in R_x$ and for p and r in the same fiber, define $a = p^{-1}r$ which is an element of $\mathrm{Gl}_n(\mathbb{H})$ such that $pa = r$.

On the other hand, given a $\mathrm{Gl}_n(\mathbb{H})$-reduction R of $\mathrm{Fr}(TM)$, one arrives at a natural pair (I, J) of sections for $\mathrm{Fr}(TM)$: define $I_{gx_0} = gI_0g^{-1}$ and similarly for J. This discussion is summarized in the next proposition.

Proposition 3.5. *A quaternionic structure P on M is hypercomplex if and only the structure group $\mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$ is reducible to the subgroup $\mathrm{Gl}_n(\mathbb{H})$.*

From now on, the base manifold considered is homogeneous ($M = G/H$) and every quaternionic structure on it will be G-invariant.

Let x_0 be the origin of M, i.e., the class corresponding to the coset eH. The isotropy representation is given by

$$\rho : H \to \mathrm{Gl}(T_{x_0}M).$$

Proposition 3.6. *If ρ is injective, then G immerses itself as a subbundle of P with structure group $\rho(H) \subseteq \mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$.*

Proof. Let $\beta_0 \in P_{x_0}$. Then the orbit $G\beta_0 \approx G/G_{\beta_0}$. Since P is G-invariant, $G_{\beta_0} \subseteq H$. Moreover, $h\beta_0 = \beta_0$ if, and only if, h acts as the identity. Since ρ is injective, we must have $h = e$ (neutral element). Therefore $G_{\beta_0} \approx G$. □

Example 3.4. Let $M = G/H = \mathrm{Sp}_2/\mathrm{Sp}_1 \times \mathrm{Sp}_1$. This is the maximal flag manifold of $\mathfrak{sl}_2(\mathbb{H})$. The tangent space at the origin is identified with the quaternion algebra $\mathbb{H}$. The isotropy representation is the adjoint representation given by

$$\mathrm{Ad} : \mathrm{Sp}_1 \times \mathrm{Sp}_1 \to \mathrm{Gl}(\mathbb{H}) \quad \text{with } \mathrm{Ad}(p, q)v = qv\bar{p}.$$

Denote by H_1 the first Sp_1-factor of H. Then H_1 is a normal subgroup of H and induces a quaternionic structure $\mathcal{Q}_0$ on $T_{x_0}M$ (as a vector space) by the pair

$$I_{x_0} = \mathrm{Ad}(i, 1) \quad \text{and} \quad J_{x_0} = \mathrm{Ad}(j, 1).$$

Since H_1 is normal in H, let

$$\mathcal{Q}_{gx_0} = g\mathcal{Q}_0 g^{-1}.$$

This is well-defined and induces a G-invariant quaternionic structure on M. Notice that we could define another quaternionic structure on M by taking the second factor of H. □

Remark 3.11. In the example above, the isotropy representation is *not* injective, since both $\pm h \in \mathrm{Sp}_1 \times \mathrm{Sp}_1$ have the same effect on $\mathbb{H}$. However, this is not a problem since the Lie algebras of both G and $G/\{\pm \mathrm{Id}\}$ are equal resulting in no changes in the Chern-Weil classes.

Now we give the relationship between the various characteristic classes defined so far. A general result from the theory of principal bundles is given below. This is Proposition 1.1, but restated with the adapted notation.

Proposition 3.7. *The structure group $\mathcal{G}$ of P is reducible to a closed subgroup $\mathcal{H}$ if and only if the associated bundle $E = P \times_{\mathcal{G}} \mathcal{G}/\mathcal{H}$ admits a cross section.*

In our case, the groups being $\mathcal{G} = \mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$ and $\mathcal{H} = \mathrm{Gl}_n(\mathbb{H})$, the associated bundle E is identified with $P/\mathcal{H}$ which in turn is a Sp_1-principal bundle over M (this is not true in general). *So the existence of a cross section mentioned in the proposition is equivalent to $P/\mathcal{H}$ being trivial.*

The aim of this section is to relate the various Chern-Weil classes appearing in the principal bundles G, P and E. See Diagram 1 for a visual reference.

Let $f_0 \in \mathrm{Inv}(\mathrm{Sp}_1)$. Denote by $f \in \mathrm{Inv}(\mathrm{Sp}_1 \oplus \mathrm{Gl}_n(\mathbb{H}))$ the trivial extension and by $f' \in \mathrm{Inv}(\rho(H))$ the restriction of f. Here the representation is seen as $\rho : \mathfrak{h} \to \mathfrak{sp}_1 \oplus \mathfrak{gl}_n(\mathbb{H})$.

Consider G as a subbundle of P with structure group $\rho(H) \approx H$. Choose a connection ω' on G with curvature Ω'. Let ω be a curvature on P which extends ω' and Ω its curvature form. Then for any $f \in \mathrm{Inv}(\mathrm{Sp}_1 \oplus \mathrm{Gl}_n(\mathbb{H}))$, the restriction of $f(\Omega)$ to G is equal to $f'(\Omega')$, therefore their projection on M are also the same, that is,

$$[f(\Omega)] = [f'(\Omega')] \in H^*(M;\mathbb{R}). \quad (3.31)$$

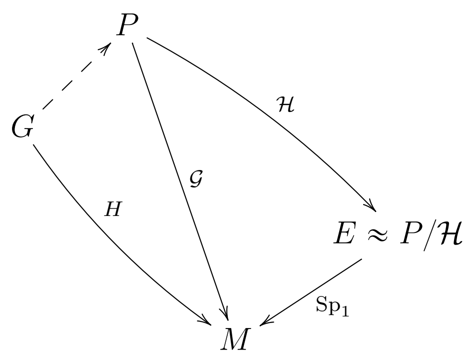


Diagram 1

Now, let ω be a connection in P and $p : \mathfrak{sp}_1 \oplus \mathfrak{gl}_n(\mathbb{H}) \to \mathfrak{sp}_1$ the projection so that the 1-form $p \circ \omega$ takes values in $\mathfrak{sp}_1$.

Proposition 3.8. *The 1-form $p \circ \omega$ projects to a (unique) differential 1-form ω_0 on E, i.e., if $\pi_0 : P \to E$, then $\pi_0^*(\omega_0) = p \circ \omega$.*

Proof. We have to show that $p \circ \omega$ is vertical and right-invariant by $\mathrm{Gl}_n(\mathbb{H})$.

We need to consider only the fundamental vector fields A^* in P where $A \in \mathfrak{gl}_n(\mathbb{H}) \subseteq \mathfrak{sp}_1 \oplus \mathfrak{gl}_n(\mathbb{H})$. Notice that the verticality mentioned here is with respect to the principal bundle $P \to E$. One of the defining properties of a connection states that $\omega(A^*) = A$. Then $p \circ \omega(A^*) = p(A) = 0$.

For the right-invariance, let $a \in \mathrm{Gl}_n(\mathbb{H})$. Then

$$R_a^*(p \circ \omega) = p \circ (R_a^*\omega) = p \circ \mathrm{Ad}(a^{-1}) \cdot \omega.$$

Notice that $p \circ \mathrm{Ad} = p$ for $a \in \mathrm{Gl}_n(\mathbb{H})$. Thus $p \circ \mathrm{Ad}(a^{-1}) \cdot \omega = p \circ \omega$. □

Proposition 3.9. *The 1-form ω_0 is a connection on $E \to M$. Moreover, the associated curvature Ω_0 satisfies $\pi_0^*(\Omega_0) = p \circ \Omega$.*

Proof. Let A^* be a fundamental vector field for $A \in \mathfrak{sp}_1$. The inclusion $\mathfrak{sp}_1 \hookrightarrow \mathfrak{sp}_1 \oplus \mathfrak{gl}_n(\mathbb{H})$ allows us to take the horizontal lift in P to also be the field A^*. Thus

$$\omega_0(A^*) = p \circ \omega(A^*) = p(A) = A.$$

For the second defining property of a connection, let $a \in \mathrm{Sp}_1$. Then for any field X on E, we have

$$R_a^*\omega_0(X) = \omega_0(R_a X) = p \circ \omega(\widetilde{R_a X}),$$

where $\tilde{X}$ is the horizontal lift of the vector X.

Notice that $\pi_0 \circ R_a = R_a \circ \pi_0$ for any $a \in \mathrm{Sp}_1$. Then $\widetilde{R_a X} = R_a \widetilde{X}$ and we have

$$p \circ \omega(R_a \widetilde{X}) = p \circ \mathrm{Ad}(a^{-1}) \cdot \omega \widetilde{X}.$$

For $a \in \mathrm{Sp}_1$, $\mathrm{Ad}(a)$ commutes with p, so $R_a^* \omega_0 = \mathrm{Ad}(a^{-1}) \cdot \omega_0$. This is the second defining property of a connection.

To prove the equation regarding the curvature forms, simply take the covariant differential on both sides of the equation $p \circ \omega = \pi_0^*(\omega_0)$. □

Corollary 3.5. *Let* $f_0 \in \mathrm{Inv}(\mathrm{Sp}_1)$ *and* $f \in \mathrm{Inv}(\mathrm{Sp}_1 \times \mathrm{Gl}_n(\mathbb{H}))$ *the trivial extension. Then*

$$f_0(\Omega_0) = f(\Omega) \quad \text{in } M. \tag{3.32}$$

Proof. For a vector field X on M, denote its horizontal lift to P by $\widetilde{X}$, i.e., if $\pi : P \to M$, then $d\pi(\widetilde{X}) = X$. Notice that $d\pi_0(\widetilde{X})$ is a lift of X to E, because the diagram mentioned earlier is commutative. Denote temporarily by $f(\Omega)^*$ the projection on M of the differential form $f(\Omega)$ on P. So one has

$$\begin{aligned} f_0(\Omega_0)^*(X_1, \ldots, X_{2k}) &= f_0(\Omega_0)(d\pi \widetilde{X}_1, \ldots, d\pi \widetilde{X}_{2k}) \\ &= f_0(\pi_0^* \Omega_0)(\widetilde{X}_1, \ldots, \widetilde{X}_{2k}) \\ &= f_0(p \circ \Omega)(\widetilde{X}_1, \ldots, \widetilde{X}_{2k}), \end{aligned}$$

where on the last equality we used Proposition 3.9. Now, by construction, the form f is zero on $\mathfrak{gl}_n(\mathbb{H})$, therefore $f = f_0 \circ p$ (composition being entry-wise). Then $f_0(p \circ \Omega) = f(\Omega)$. □

Remark 3.12. The Chern-Weil forms $f_0(\Omega_0)^*$ are defined using only P, i.e., without any mention to the bundle E.

Combining Equation 3.31 and Equation 3.32, we arrive at

$$f'(\Omega')^* = f_0(\Omega_0)^*. \tag{3.33}$$

By Equation 3.33, if any form $f'(\Omega') \neq 0$, then E won't be trivial. By Proposition 3.7, the structure group $\mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$ is not reducible to $\mathrm{Gl}_n(\mathbb{H})$ and by Proposition 3.5, the given quaternionic structure is not hypercomplex. Thus *the Chern-Weil forms of G constitutes an obstruction* for to the given quaternionic structure to be hypercomplex.

We finally arrive at the last step, that is, when $M = \mathbb{F}$ the maximal flag manifold of $\mathfrak{sl}_n(\mathbb{H})$, that is, $G = \mathrm{Sp}_n$ and $H = \mathrm{Sp}_1 \times \cdots \times \mathrm{Sp}_1$ (n-times). Throughout the section, we denote by H_k the k-th factor of the group H. The subgroup H_k is normal in H for $k = 1, \ldots, n$. We are going to choose $\{\kappa_1, \ldots, \kappa_n\}$ as a set of generators of the cohomology $H^*(\mathbb{F}; \mathbb{R})$.

Example 3.5 (continuation from Example 3.4)**.** Consider the quaternionic structure determined by H_1 in $\mathbb{F} = \mathrm{Sp}_2/\mathrm{Sp}_1 \times \mathrm{Sp}_1$. Let $\kappa_0 \in \mathrm{InvSp}_1$ be our preferred choice of generator. Following the notation of the previous section, the form $\kappa' \in \mathrm{Inv}(\rho(\mathfrak{h}))$ is actually our κ_1 which is non-zero. Therefore, by the discussion in the previous subsection, the quaternionic structure determined by H_1 (or H_2 for that matter) is not hypercomplex.

Our aim in this section is to show, inspired by this example, that every G-invariant quaternionic structure $\mathcal{Q}$ on $\mathbb{F}$ is *not* trivial, i.e., hypercomplex.

Remark 3.13. The discussion so far begs the question: is every G-invariant quaternionic structure on $\mathbb{F}$ determined by a factor H_l for some $l \in \{1, \ldots, n\}$? We do not know yet, but this is intimately related on *how* the isotropy H intersects the structure group $\mathrm{Sp}_1 \subseteq \mathrm{Sp}_1\mathrm{Gl}_n(\mathbb{H})$. The answer is probably in this intersection.

In general, for $\kappa_0 \in \mathrm{Inv}(\mathrm{Sp}_1)$, there exists real numbers a_l $(l = 1, \ldots, n)$ such that

$$\kappa' = a_1\kappa_1 + \cdots + a_n\kappa_n.$$

Notice that κ_l is zero when evaluated outside $\mathfrak{h}_l \approx \rho(\mathfrak{h}_l)$. Therefore, the restriction of κ' to any $\mathfrak{h}_l$ is exactly $a_l\kappa_l$. Now, by looking at the dimension of the Lie algebras and keeping in mind that ρ is injective, we conclude that there is exactly one index, say l, such that the intersection between $\rho(\mathfrak{h}_l)$ and the factor $\mathfrak{sp}_1 \subseteq \mathfrak{sp}_1 \oplus \mathfrak{gl}_n(\mathbb{H})$ is non-zero. The coefficient a_l is non-zero by construction ($\kappa_0 \neq 0$). Thus we must have $\kappa' = a_l\kappa_l$ and the corresponding Chern-Weil class must be $\neq 0$. This proves the final theorem below.

Theorem 3.6. *Any G-invariant quaternionic structure on the maximal flag of $\mathfrak{sl}_n(\mathbb{H})$ is not hypercomplex.*

In section 4 of the Appendix, there are two charts: one visually represents how one goes back and forth from the principal bundle P to the vector bundle $\mathcal{Q}$ (fiber-wise) and the other is an attempt to summarize the various extensions and restrictions done using the several curvature forms and polynomials.

4 Final Comments

The paper (BERNSTEIN; GEL'FAND; GEL'FAND, 1973) has inspiring results: the integration formula for general spaces G/P and the representations of the Weyl group. What goes in the essence of their work is how they see the Bruhat/Schubert cells by immerging the whole flag inside a big enough projective space, so they see $\mathbb{F}$ as a projective variety.[1] For them, the Schubert cells appears as intersection between G/P and some projective planes. Another aspect that differs from what was explored in here is the use of the operator A_γ. We have interpreted it as a derivative operator with respect to some root seen as a variable. For $w \in \mathcal{W}$ of greatest length, $\mathcal{S}_w$ is the whole space, so in this case is easy to find a Chern-Weil class which is in the dual basis for the Schubert cells. Then, with this volume form as starting point, they descend to the lower dimensional Schubert cells, finding the Chern-Weil classes recursively with the aid of the operator A_γ.

So what did we do differently?

Inspired by those, we proved the same integration formula in the case of a maximal flag manifold. Not only for complex ones (Theorem 2.2), but also for quaternionic flags (Theorem 3.2). We believe the techniques developed so far sheds *even more* light on the subject by arriving at the same results while using a different approach, a different *way* of expressing mathematics. Considering $\mathbb{F}$ as a reductive homogeneous space on which $\mathcal{W}$ acts, while also carefully choosing generators of the invariant polynomial algebra, we have set ourselves directly at the *heart* of the beast, that is, a closed differential form of maximum degree that is *not* based on the minimal decomposition of the corresponding Weyl group element. From there, we climbed down the sphere-fibrations like a mountaineer does with its ropes: one step at a time, checking carefully each nail and knot. From top to bottom; and there lied the *soul* of said beast: the minimal dimensional (but non-trivial) cell, which is also the most beautiful: a round two-dimensional sphere of radius the size of a root! And there, dear reader, we can integrate just as our calculus teacher taught us. Isn't that beautiful?

The last paragraph is the part that inspired me the most, but there is more: we have also managed to recover other results, such as the surjectivity and the kernel of the Chern-Weil homomorphism, and see the Weyl group action on cohomology as the regular representation. And a few more and minor things here and there.

[1] In the Appendix section 4, we show how to do it for $\mathrm{Gr}_k(\mathbb{C}^n)$. For the whole flag one needs to take in consideration tensorial representations.

Expectation from the future and the elephant in the room.

In terms of results and statements, not everything is new. But the path taken to get there was our own and, during the walk, new stones were collected. Stones with different shapes that could spark new ideas.

Now we address said elephant. A beast crazier than a maximal flag is the partial flag manifold. The way things were done in the quaternionic maximal flag, we relied heavily on the way the tangent plane and the rootspaces are put together. Precisely speaking, we have seen how the curvature form is basically projection of the Lie bracket onto the Lie algebra of the isotropy group. A few computations were greatly simplified by the simple fact that $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha\pm\beta}$, so Ω is annihilated when the entries comes from different rootspaces. This does *not* happen in the partial flag manifold, because the isotropy is larger and the rootspace $\mathfrak{u}_{\alpha\pm\beta}$ can be included there, *i.e.*, Ω won't be necessarily annihilated anymore. Even more, the invariant polynomial algebra gets more complicated the larger the isotropy group. The κ_r does not suffice, because we know, in theory, the generators Sp_n for $n \geqslant 2$. But not all is lost, since one can naturally embed the partial Schubert cells inside the maximal flag manifold and pullback the Chern-Weil classes using this inclusion. This seems to be the natural next step.

Another direction one could follow is to look at the other real algebras, since the only one we took care of was $\mathfrak{sl}_n(\mathbb{H})$. Also, something that can be done is to try and find similar formulas for each of the complex semisimple Lie algebras in the spirit of the one in Theorem 3.4, using instead of the $[\kappa]$, a more suitable choose of functionals. For instance, in the case of $\mathfrak{sl}_n(\mathbb{C})$, such choice is given by the linear functionals $\lambda_j : \mathfrak{h} \to \mathbb{C}$. Beware: applying Chern-Weil is not always possible. For instance, in the case of $\mathfrak{sl}_n(\mathbb{R})$, the isotropy is discrete, so one cannot use Chern-Weil, since those have null curvature.

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Appendix

The Quaternion Algebra

The quaternion algebra, denoted by $\mathbb{H}$, is the algebra generated by the pair $\{i, j\}$ satisfying the relations $ij = -ji$ and $i^2 = -1 = j^2$. As a real vector space, $\mathbb{H}$ is identified with $\mathbb{R}^4$ by setting $\{1, i, j, k\}$ as the canonical orthonormal basis, where $k = ij$.

A generic element q in $\mathbb{H}$ is represented as a sum

$$q = a + ib + jc + kd \qquad \text{where } a, b, c, d \in \mathbb{R}.$$

Another useful representation of $\mathbb{H}$ arises when we write $q = (a + bi) + j(c - di)$ which is a "j-sum" of complex numbers, inducing the identification $\mathbb{H} \approx \mathbb{C} \oplus j\mathbb{C}$.

Similar to the complex numbers, we denote by $\bar{q}$ the conjugate of a quaternion. In terms of the representation, the conjugate is defined by $\bar{q} = a - bi - cj - dk$. Also $\|q\|^2 = q\bar{q} = \bar{q}q$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^4$.

Going further, we denote by $\mathbb{H}^n$ the (right) vector space with quaternions acting as scalars. Most of basic linear algebra results are also valid here. A fundamental difference occurs when one considers $\mathbb{H}$-linear transformations (the linearity on the right, of course). Let φ be a $\mathbb{H}$-linear transform of $\mathbb{H}^n$. Then its matrix representation actually multiplies the corresponding column vector *on the left*.

The space $\mathbb{H}^n$ is identified with $\mathbb{R}^{4n}$ via the morphism which takes

$$(a_1 + ib_1 + jc_1 + kd_1, \ldots, a_n + ib_n + jc_n + kd_n) \in \mathbb{H}^n$$

into

$$(a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n) \in \mathbb{R}^{4n}.$$

Notice that multiplication by i and j are real linear transformations, then by considering the previous identification, they induce the block-diagonal matrices $I_{4n} = \operatorname{diag}(I, \ldots, I)$ and $J_{4n} = \operatorname{diag}(J, \ldots, J)$ acting on the left where

$$I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

A quaternionic subspace then is a vector subspace that is invariant by $\mathbb{H}$-multiplication on the right. The canonical basis $\beta = \{e_1, \ldots, e_n\}$ is the one where $e_r = (0, \ldots, 1_r, \ldots, 0) \in \mathbb{H}^n$.

A quaternionic maximal flag in $\mathbb{H}^n$ is a sequence of nested quaternionic subspaces of the form

$$\{0\} \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{H}^n$$

where V_r has real dimension $4r$. The set of all such flags is called the maximal flag manifold and is denoted by $\mathbb{F}$.

The canonical maximal flag, usually denoted by x_0 is the flag

$$\{0\} \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq \mathbb{H}^n$$

where $E_r = \mathrm{Span}_{\mathbb{H}}\{e_1, \ldots, e_r\}$.

Vector Spaces over C and H

In this text $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ are seen as right vector spaces (or modules, if you will). This discussion serves to show that we can interchange between the matrix representations of $\mathfrak{sl}_n(\mathbb{H})$ when seen as quaternionic matrices or inside the set of complex matrices.

Let φ be the homomorphism between $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ given by

$$\mathbb{H}^n \ni q = z + jw = \begin{bmatrix} z_1 + jw_1 \\ \vdots \\ z_n + jw_n \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{C}^{2n}.$$

The last equation is notational. Let Φ be the homomorphism between $\mathfrak{gl}_n(\mathbb{H})$ and $\mathfrak{gl}_{2n}(\mathbb{C})$ given by

$$X = A + jB \mapsto \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix}.$$

Proposition .1. $\varphi(Xq) = \Phi(X)\varphi(q)$.

Proof. Using the fact that $jz = \bar{z}j$ for $z \in \mathbb{C}$, we have

$$Xq = (A + jB)(z + jw) = Az - \bar{B}w + j(\bar{A}w + Bz).$$

Then

$$\varphi(Xq) = \begin{bmatrix} Az - \bar{B}w \\ \bar{A}w + Bz \end{bmatrix}.$$

On the other hand, we have

$$\Phi(X)\varphi(q) = \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} Az - \bar{B}w \\ Bz + \bar{A}w \end{bmatrix} = \varphi(Xq).$$

□

The product by j (from the right) in $\mathbb{H}^n$ induces a antilinear isomorphism J on $\mathbb{C}^{2n}$ given by

$$J\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} -\bar{w} \\ \bar{z} \end{bmatrix}.$$

Considering this isomorphism, the Lie algebra $\mathfrak{gl}_n(\mathbb{H})$ is the (real) subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})$ composed by the elements X such that

$$JX\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = XJ\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) \quad \forall\ z, w \in \mathbb{C}^n.$$

The proof is easy and left to someone else. Define the Lie algebra $\mathfrak{sl}_n(\mathbb{H})$ as the set of matrices $X \in \mathfrak{gl}_n(\mathbb{H})$ having $\mathrm{ReTr}X = 0$.

The next step is to understand the homomorphism $\phi : \bigwedge^k \mathbb{H}^n \to \bigwedge^k \mathbb{C}^{2n}$ given by

$$\phi(q^1 \wedge \cdots \wedge q^k) = \varphi(q^1) \wedge \cdots \wedge \varphi(q^k).$$

The canonical action of $X \in \mathfrak{gl}_n(\mathbb{H})$ on $\bigwedge^k \mathbb{H}^n$ is given by

$$X \cdot q^1 \wedge \cdots \wedge q^k = \sum_{m=1}^{k} q^1 \wedge \cdots \wedge Xq^m \wedge \cdots \wedge q^k. \tag{1}$$

The action of $\mathfrak{gl}_{2n}(\mathbb{C})$ on $\bigwedge^k \mathbb{C}^{2n}$ is analogous.

Proposition .2. *Let φ, Φ and ϕ be the homomorphisms previously defined. Then*

$$\phi(X \cdot q^1 \wedge \cdots \wedge q^k) = \Phi(X) \cdot \phi(q^1 \wedge \cdots \wedge q^k).$$

The above proposition is still true when we consider the group level action of $\mathrm{Gl}_n(\mathbb{H})$ acting on $\bigwedge^k \mathbb{H}^n$ from the left by

$$g \cdot q^1 \wedge \cdots \wedge q^k = gq^1 \wedge \cdots \wedge gq^k. \tag{2}$$

Lastly, let $\tilde{J}$ be the antilinear isomorphism of $\bigwedge^k \mathbb{C}^{2n}$ given by

$$\tilde{J}(v^1 \wedge \cdots \wedge v^k) = Jv^1 \wedge \cdots \wedge Jv^k.$$

Proposition .3. $\tilde{J}^2 = (-1)^k \mathrm{Id}$ *and $\tilde{J}$ commutes with both canonical actions, at the level of the group or the Lie algebra.*

The existence of such antilinear isomorphism is of paramount importance when one wishes to understand the basics representations of $\mathfrak{sl}_n(\mathbb{H})$.

Corollary .1. *The representation $\bigwedge^k \mathbb{H}^n$ is quaternionic when k is odd and real otherwise.*

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ with the standard choice of a Cartan subalgebra $\mathfrak{h}$ as the set of diagonal matrices in $\mathfrak{g}$ with zero trace. Define the linear functionals $\lambda_j : \mathfrak{h} \to \mathbb{C}$ by $(a_1, \ldots, a_n) \mapsto a_j$. It is known that, for $1 \leqslant k \leqslant n-1$, the functionals $\lambda_1 + \cdots + \lambda_k$ forms the dual basis of $\{H_{\alpha_k} = E_{k;k} - E_{k+1;k+1}\}_{k=1}^{n-1}$ where each $H_{\alpha_k} \in \mathfrak{h}$ is defined by the relation

$$\alpha_k(\cdot) = \langle H_{\alpha_k}, \cdot \rangle.$$

Fix k in the range $1, \dots, n-1$. Set $W = \bigwedge^k \mathbb{C}^n$ to be the wedge representation (defined above) of degree k. By Equation 1, $\lambda = \lambda_1 + \cdots + \lambda_k$ is the highest weight of W. The corresponding highest vector is given by $v_0 = e_1 \wedge \cdots \wedge e_k$. The weight space W_λ is one-dimensional, generated by $v_0 = e_1 \wedge \cdots \wedge e_k$. This is clear, because for any $v = e_{i_1} \wedge \cdots \wedge e_{i_k} \in W$ and any $X = (x_{ij})_{i,j=1}^n \in \mathfrak{g}$, one has

$$X \cdot v = (x_{i_1 i_1} + \cdots + x_{i_k i_k})\, v.$$

Continuing, let $G = \mathrm{Sl}_n(\mathbb{C})$ act on W by means of Equation 2. Consider the orbit through $G \cdot v_0$. Each indecomposable element $v_1 \wedge \cdots \wedge v_k \neq 0$ of the orbit is associated to a k-subspace of $\mathbb{C}^n$ generated by the elements $v_1, \dots, v_k$. Clearly, vector multiples are associated to the same k-subspace. Denote by $P(W)$ the projectivization of W according the the natural complex structure of $\mathbb{C}^n$. Then $P(G \cdot v_0)$ is actually the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. In fact, let $g \in G$ such that $g \cdot v_0 = * \, v_0$, *i.e.*, $g \cdot v_0$ is a multiple of v_0. At the level of $P(W)$, this means that g fixes the associated k-subspace $[v_0]$. Using the argumentation done in Example 2.1, we see that the isotropy of $[v_0]$ is precisely the parabolic set B_Θ with $\Theta = \Sigma \backslash \{\alpha_k\}$.

Example of Schubert Cell

This section contains an explicit example of a Schubert cell using both definitions 2.1 and 2.2. First we do it using N-orbit, then using the γ fibrations.

Let $\mathbb{F}$ be a maximal flag of $\mathrm{Sl}_3(\mathbb{C})$ and consider the canonical choices for roots in $\mathfrak{g}$. Let $w = r_{\alpha_1} r_{\alpha_2}$. In matrix form,

$$w = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Let $x_0 = (E_1, E_2)$ be our choice of an origin in $\mathbb{F}$, where E_1 and E_2 are the canonical subspaces of dimensions 1 and 2 in $\mathbb{C}^3$.

Notation .1. For this section, consider the notation $V(v_1, \ldots, v_k)$ to denote the k-subspace of $\mathbb{C}^n$ that is generated by $\{v_1, \ldots, v_k\}$.

With the new notation, $x_0 = (V(e_1), V(e_1, e_2))$. Then

$$w \cdot x_0 = \Big(V(e_2), V(e_2, e_3) \Big).$$

The subgroup N is the set of matrices of the form

$$\begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{bmatrix}$$

where $\lambda_j \in \mathbb{C}$. With this notation, we compute explicitly the form of the Bruhat cell $\mathcal{B}_w = N \cdot wb_0$. It is given by the set of all flags of the form

$$\Big(V(ae_1 + e_2), V(ae_1 + e_2,\, be_1 + ce_2 + e_3) \Big).$$

Notice the $V(ae_1 + e_2)$ is always inside $V(e_1, e_2)$ and it can get arbitrarily close to $V(e_1)$; and $V(ae_1 + e_2,\, be_1 + ce_2 + e_3)$ can get arbitrarily close to $V(e_1, e_2)$. Just take a, b and c infinitely large. This means that $f = (V(e_1), V(e_1, e_2))$ is in the Schubert cell $\mathcal{S}_w$. In fact, f is the only flag in $\mathcal{S}_w \backslash \mathcal{B}_w$.

Onto the next definition. Let $\gamma_j = \pi_j^{-1} \circ \pi_j$ and $w' = r_{\alpha_1}$. Then

$$\mathcal{S}_w = \gamma_2 \circ \gamma_1 \{x_0\} = \gamma_2(\mathcal{S}_{w'}). \tag{3}$$

First, $\pi_1(x_0) = (V(e_2))$, that is, a flag consisting of a single subspaces $V = V(e_2)$ inside $\mathbb{C}^3$. Then

$$\pi_1^{-1}(V(e_2)) = \{\, (V_1, V(e_1, e_2)) \in \mathbb{F} \,:\, \dim_{\mathbb{C}} V_1 = 1 \,\} = \mathcal{S}_{w'}. \tag{4}$$

Now apply the second projection to get

$$\pi_2(\mathcal{S}_{w'}) = \{\, ((V_1)) \,:\, V_1 \subseteq V(e_1, e_2) \,\} = \mathcal{S}_{w'}^{\{\alpha_2\}}. \tag{5}$$

Finally pull it back to $\mathbb{F}$

$$\mathcal{S}_w = \pi_2^{-1}(\mathcal{S}_{w'}^{\{\alpha_2\}}) = \{\, (V_1, V_2) \in \mathbb{F} \,:\, V_1 \subset V(e_1, e_2) \,\} \,. \tag{6}$$

This illustrates how the first and second definitions of Schubert cells match. This example also shows how the sphere fibrations appears. In fact, the projection $\pi_2 : \mathcal{S}_w \to \mathcal{S}_{w'}^{\{\alpha_2\}}$ has fibers homeomorphic to $\mathbb{C}P^1 \approx S^2$. Consider the fibration over $V(e_1)$

$$\pi_2^{-1}(V(e_1)) = \{\, (V(e_1), V_2) \,:\, \dim V_2 = 2 \,\} \,.$$

Taking the quotient by $V(e_1)$, this set is homeomorphic to the space of all complex lines inside $\mathbb{C}^2$, which is $\mathbb{C}P^1$.

Useful Theorems

Below is a generalization of Fubini's theorem from calculus adapted to differentials form over a bundle. It states that if a differential form on top can be written as a wedge product of forms on the basis and fiber, then we can integrate such form by first integrating on the fibers, then on the basis — see (SULANKE; WINTGEN, 1972).

Theorem .1 (Fubini)**.** *Let $\varphi : M \to N$ where M, N are smooth manifolds of dimensions m, n respectively, with $m \geqslant n$. Let $\omega \in \Omega^{m-n}(M)$ and $\eta \in \Omega^n(N)$, and let $f : M \to \mathbb{R}$ be measurable (meaning, its superposition with any map is Lebesgue measurable). Assume the set of critical values of φ has measure zero in N (again, this means the image under any map has Lebesgue measure zero). If the m-form $f\omega \wedge \varphi^*\eta$ is integrable on M, then for almost all $x \in N$ the integral*

$$\int_{\varphi^{-1}(x)} f\omega$$

is well-defined and, moreover, when treated as a function of x and multiplied by η, it is integrable on N and

$$\int_M f\omega \wedge \varphi^*\eta = \int_N \left(\int_{\varphi^{-1}(x)} f\omega \right) \eta.$$

Next we state the Leray-Hirsh theorem, taken from (HATCHER, 2002). The *Leray-Hirsch condition* that we mention throughout the text is the combination of item 1 and item 2 below.

Theorem .2. *Let $p : E \to B$ be a fiber bundle with fiber F such that, for some commutative coefficient ring R:*

1. *$H^n(F; R)$ is a finitely generated free R-module for each n.*
2. *There exist classes $c_j \in H^{k_j}(E; R)$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F; R)$ in each fiber F, where $i : F \to E$ is the inclusion.*

Then the map $\Phi : H^*(B; R) \otimes_R H^*(F; R) \to H^*(E; R)$, $\sum_{ij} b_i \otimes i^*(c_j) \mapsto \sum_{ij} p^*(b_i) \cup c_j$ is an isomorphism of R-modules.

Charts

This section is an attempt to put in a single place all quaternionic structures and characteristic classes used in section 3.4, visually indicated by the charts below.

The following table indicates the relationship between the principal and vector bundles corresponding to quaternionic structures. The horizontal arrows indicates how to starting from one and define the other. The vertical arrows indicates only that the lower is the intended reduction from the upper. If such reduction is possible, then the starting quaternionic structure would be hypercomplex.

$$\begin{array}{ccc}
P_x = \{\, p \in \mathrm{Fr} M_x \;:\; p\mathbb{H}p^{-1} \subseteq \mathcal{Q}_x M \,\} & \longleftrightarrow & \mathcal{Q}_x = p\mathbb{H}p^{-1} \text{ (for any } p \in P_x) \\
\downarrow & & \downarrow \\
R_x = \{\, p \in \mathrm{Fr} M_x \;:\; pq = qp \,\} & \longleftrightarrow & \mathcal{Q}_x M = \mathrm{Span}\{I_x, J_x\}
\end{array}$$

The next table indicates how the various invariant polynomials κ and curvatures Ω defined in the bundles G, P or E are obtained one from another, either by extension, restriction or induction.

$$\begin{array}{ccccc}
G & \longrightarrow & P & \longrightarrow & E \\
\Omega' & \xrightarrow{\text{extended}} & \Omega & \xrightarrow{\text{induced}} & \Omega_0 \\
\kappa' & \xleftarrow{\text{restricted}} & \kappa & \xleftarrow{\text{extended}} & \kappa_0
\end{array}$$

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