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Symmetric spaces on the adjoint orbit

Espaços simétricos na órbita adjunta

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Resumo

Estudamos os espaços simétricos são variedades flag $\mathbb{F}_\Theta$. Mostramos que os fibrados cotangentes dessas variedades flag também são espaços simétricos. Estudamos propriedades dos espaços simétricos duais de $\mathbb{F}_\Theta$ como subvariedades da órbita adjunta $\mathrm{Ad}(G) \cdot H_\Theta$ para o grupo de Lie complexo simples G. Evidenciamos que as variedades simétricas Flag são de três tipos e os descrevemos com maior detalhe.

Palavras-chave: Órbita adjunta; variedade flag; espaço simétrico; fibrado cotangente.

Abstract

We study the symmetric spaces which are also flag manifolds $\mathbb{F}_\Theta$. We show that the cotangent bundles of these flags manifolds are also symmetric spaces. We study properties of the dual symmetric spaces of $\mathbb{F}_\Theta$ as submanifolds of the adjoint orbit $\mathrm{Ad}(G) \cdot H_\Theta$ for the complex simple Lie group G. We evidence that the flag symmetric manifolds are of three types and we describe them in detail.

Keywords: Adjoint orbit; flag manifold; symmetric space; cotangent bundle.

List of symbols

Π	Roots of an indicated complex Lie algebra.
G^σ	The set of σ fixed points in G: $\{g \in G : \sigma(g) = g\}$.
$\mathbb{F}_\Theta$	Flag manifold determined by the subset Θ of simple roots.
$\mathcal{K}(\cdot,\cdot)$	The Cartan-Killing form.
$\mathcal{S}$	The submanifold $\exp\sqrt{-1}\mathfrak{m}$ of the cotangent bundle of the flag.
C_g	Automorphism of a Lie group G, conjugation of the element $g \in G$.
$\langle \exp \mathfrak{h} \rangle$	The unique connected Lie subgroup $\{e^{Y_1} \cdots e^{Y_s} : s \geqslant 0, Y_i \in \mathfrak{h}\}$ of the Lie group G (indicated in the context) with Lie algebra $\mathfrak{h}$.
$G \cdot X$	The orbit of the group G in the element X. After define the action of a group, this notation means the adjoint orbit of the group G in an element X of the Lie algebra of G.
$g \cdot X$	$\mathrm{Ad}(g)X$, for an element g of a Lie group G and an element X of the Lie algebra of G.
b_Θ	Origin of the flag manifold $\mathbb{F}_\Theta$.
$\mathbb{C}^*$	Lie algebra of the multipliative Lie group $\mathbb{C}\backslash\{0\}$.
$\mathrm{Ad}_{\mathfrak{g}}(H)$	Let H be a Lie subgroup of G and let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathrm{Ad}_{\mathfrak{g}}(H) = \{\mathrm{Ad}(h) : \mathfrak{g} \to \mathfrak{g},\ h \in H\}$.
$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$	Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) = \{\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g},\ X \in \mathfrak{h}\}$.

Contents

Introduction

The symmetric spaces have been well studied by several authors such as Berger, Kobayashi, Nomizu, Helgason, among others. Kobayashi and Nomizu define the symmetric spaces as triples (G, H, σ) consisting of a connected Lie group G, a closed subgroup H and an involutive automorphism σ of G such that $G_0^\sigma \subset H \subset G^\sigma$, where $G^\sigma = \{g \in G : \sigma(g) = g\}$ and G_0^σ is the identity (connected) component of G^σ. If (G, H, σ) is a symmetric space, then G/H is said to be an affine symmetric manifold. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively, then the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ has the canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. If $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ is compact, then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is called an orthogonal symmetric Lie algebra. In [3], is proved that the orthogonal symmetric Lie algebras are of compact and non-compact type. Moreover they are of four classes. The types of symmetric spaces are related to their geometric properties as follows

Theorem 0.0.1. *[3] Let (G, H, σ) be a symmetric space with $\mathrm{Ad}_G(H)$ compact and let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be its orthogonal symmetric Lie algebra. Take any $G-$invariant Riemannian metric on G/H. Then we have.*

(1) If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of compact type, then G/H is a compact Riemannian symmetric space with non-negative sectional curvature and positive-defined Ricci tensor;

(2) If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of non-compact type, then G/H is a simply connected non-compact Riemannian symmetric space with non-positive sectional curvature and negative-definite Ricci tensor and is diffeomorphic to a Euclidean space.

In [4], Helgason shows a table with Riemannian symmetric spaces. Some of them are: the Grassmannian $SU(p+q)/S(U(p) \times U(q))$ of complex subspaces of $\mathbb{C}^n$, the space $Sp(n)/U(n)$ of complex structures on $\mathbb{H}^n$ compatible with the inner product and the space $SO(2n)/U(n)$ of orthogonal complex structures on $\mathbb{R}^{2n}$. All these examples have orthogonal symmetric Lie algebras.

The examples above are also described as complex flag manifolds of classical Lie groups in [11]. Alekseevsky and Arvanitoyeorgos shows a list of all complex flag manifolds in [11].

In [7], Berger says that the affine symmetric space G/H (resp. symmetric Lie algebra) is said to be $\mathbb{C}$-symmetric if there exists some complex structure of the vector space $\mathfrak{m}$, invariant by the linear representation $(\mathrm{Ad}(H), \mathfrak{m})$ [resp. $(\mathrm{ad}(\mathfrak{h}), \mathfrak{m})$]. On the other hand, the affine symmetric space G/H (resp. the symmetric Lie algebra) is said to be semi-Kahler if it is $\mathbb{C}$-symmetric and for a fixed complex structure of $\mathfrak{m}$, the representation $(\mathrm{Ad}(H), \mathfrak{m})$

[resp. (ad($\mathfrak{h}$),$\mathfrak{m}$)] leaves invariant an Hermitian form of $\mathfrak{m}$, non-degenerate. Berger proved the following proposition

Proposition 0.0.2. [7] Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra with $\mathfrak{g}$ simple. The symmetric Lie algebra is semi-Kahler if and only if the isotropy subalgebra $\mathfrak{h}$ contains the Lie algebra $\mathbb{R}$.

If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is a symmetric Lie algebra, we call $(\mathfrak{g}, \mathfrak{h})$ by a symmetric Lie pair. Berger gives a table of all symmetric Lie pairs indicating which are semi-Kahler. Some symmetric semi-Kahler Lie pairs are: $(\mathfrak{sl}(p+q, \mathbb{C}), \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(q, \mathbb{C}) \oplus \mathbb{C}^*)$, $(\mathfrak{sp}(n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}^*)$ and $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}^*)$. Note that in those examples, the isotropy subalgebras contain the complex vector space $\mathbb{C}$, then they contain $\mathbb{R}$ as in Proposition 0.0.2.

The examples of semi-Kahler symmetric pairs are the cotangent bundle of the examples of Riemannian symmetric spaces. This assertion shall be proved in this Thesis.

On the other hand, the adjoint orbit of a complex simple Lie group G has several realization indicated in [1]. One of those realizations is as the cotangent bundle of a complex flag manifold.

Recall that in [15] is indicated that every complex flag manifold is an homogeneous space U/K where U is a compact real form of a complex Lie group and $K \subset U$ is a connected Lie subgroup of U. By 0.0.3, an adjoint orbit $\mathrm{Ad}(G)X$ for a characteristic element X in $\mathfrak{g}$ is a vector bundle over a complex flag manifold U/K, where U is the compact (connected) real form of G. The adjoint orbit of a complex simple Lie group G is contained in the Lie algebra $\mathfrak{g}$, this allows inherit the required structures from the Lie algebra.

Theorem 0.0.3. *[1] The adjoint orbit $\mathcal{O}(H_\Theta) = Ad(G) \cdot H_\Theta \approx G/Z_\Theta$ of the characteristic element H_Θ is a C^∞ vector bundle over $\mathbb{F}_\Theta$ that is isomorphic to the cotangent bundle $T^*\mathbb{F}_\Theta$. Moreover, we can write down a diffeomorphism $\iota : Ad(G) \cdot H_\Theta \to T^*\mathbb{F}_\Theta$ such that*

(1) ι is equivariant with respect to the (adjoint) action of U, that is, for all $u \in U$,

$$\iota \circ \mathrm{Ad}(u) = \tilde{u} \circ \iota,$$

where $\tilde{u}$ is the lifting to $T^*\mathbb{F}_\Theta$ (via the differential) of the action of u on $\mathbb{F}_\Theta$; and

(2) the pullback of the canonical simplectic form on $T^*\mathbb{F}_\Theta$ by ι is the (real) Kirillov-Kostant-Souriaux form on the orbit.

On the other hand, considere a symmetric Lie algebra $(\mathfrak{u}, \mathfrak{k}, \sigma)$. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition. If we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the complexifications of $\mathfrak{u}$ and $\mathfrak{k}$, respectively, and by σ^c the involutive automorphism of $\mathfrak{g}$ induced by σ. We set

$$\mathfrak{u}^* := \mathfrak{k} + \sqrt{-1}\mathfrak{m}. \tag{0.0.1}$$

If we set $\sigma^* = \sigma^c \mid \mathfrak{u}^*$, then we obtain a symmetric subalgebra $(\mathfrak{u}^*, \mathfrak{k}, \sigma^*)$ of $(\mathfrak{g}, \mathfrak{h}, \sigma^c)$, which is called the **dual of** $(\mathfrak{u}, \mathfrak{k}, \sigma)$.

In this work, the flag manifolds U/K are affine symmetric manifolds and the adjoint orbits of G contains the affine symmetric manifold U^*/K, where U^* is the connected Lie subgroup of G with Lie algebra $\mathfrak{u}^*$. Again, we can inherit the required structures from the Lie algebra $\mathfrak{g}$ to U^*/K seen as a submanifold of the adjoint orbit.

The goals of this Thesis are

- To prove that if a complex flag manifold U/K is a symmetric space, then the cotangent bundle of U/K is a symmetric space.
- To determine an automorphism σ of G such that (U, K, σ) and (G, G^σ, σ) are symmetric spaces with G/G^σ being a realization of the cotangent bundle of U/K.
- To find Lagrangian submanifolds of the adjoint orbit G/G^σ.

This Thesis focuses in three topics, symmetric spaces, complex flag manifolds and adjoint orbits. Chapter 1 is dedicated to introducing the fundamental concepts that will be used throughout this work. For instance to talk about flag manifolds and adjoint orbits the preliminaries come mostly from the San Martin's books [6] and [5]. We define simple real and complex Lie algebras and real forms, Lie groups, actions of a Lie group, roots spaces, Cartan subalgebra, Cartan-Killing form and Hermitian 2-forms generated by it, homogeneous spaces and Riemannian and symplectic manifolds. Finally, we write a section about symmetric spaces using the references [3], [2] of Kobayashi and Nomizu.

Chapter 2 starts indicating all Riemannian symmetric spaces (in Table A.3, referenced from [4]) that are also complex flag manifolds (in Table A.2, referenced from [15]) and notices that they are the homogeneous spaces $SU(p+q)/S(U(p) \times U(q))$, $Sp(n)/U(n)$ and $SO(2n)/U(n)$ of classes A, C and D, respectively. These manifolds are called **flag symmetric spaces**. Since each of these symmetric spaces are flag manifolds, there exists a compact Lie group U for each case acting on them. The Lie algebra of each U is a compact real form $\mathfrak{u}$ of a complex Lie algebra $\mathfrak{g}$. In [11] is proved that a complex flag manifold U/K is determined by a subset of simple roots Θ of $\mathfrak{g}$, i.e. there exists an element H_Θ in a fixed Cartan subalgebra of $\mathfrak{u}$, such that $\alpha(H_\Theta) = 0$ for all $\alpha \in \Theta$ and $K = Z_U(H_\Theta)$. Thus, we use the notation $\mathbb{F}_\Theta := U/U_\Theta$ for the complex flag manifolds, where $U_\Theta = Z_U(H_\Theta)$.

We state the theorem 2.0.1, it says that there exists an H_Θ as above such that if $\sigma := C_{\exp H_\Theta}$, then (U, U^σ, σ) and (G, G^σ, σ) are symmetric spaces, where U^σ and G^σ are the σ-fixed points of U and G, respectively. This theorem is not proved in Chapter 2, however it is proved throughout chapters 3, 4 and 5. Using the result 1.2.12, referenced from [1], Theorem 2.0.1 indicates that the cotangent bundle $T^*(\mathbb{F}_\Theta)$ of the flag symmetric spaces $\mathbb{F}_\Theta$ or the adjoint orbit $\mathrm{Ad}(G)H_\Theta$ are symmetric spaces.

The symmetric Lie algebra of a flag symmetric space $\mathbb{F}_\Theta$ is $(\mathfrak{u}, \mathfrak{u}_\Theta, \sigma)$, where we use also the notation σ for the automorphism $\mathrm{Ad}(\exp H_\Theta)$ of $\mathfrak{g}$. This symmetric Lie algebra has the canonical decomposition $\mathfrak{u} = \mathfrak{u}_\Theta + \mathfrak{m}$. Then we denote by $\mathcal{S}$ the subset $\mathrm{Ad}(\exp(\sqrt{-1}\mathfrak{m}))H_\Theta$ of the adjoint orbit $\mathrm{Ad}(G)H_\Theta$. One of the main results is Theorem 2.1.1, it says that $\mathcal{S}$ is a submanifold of the adjoint orbit $\mathrm{Ad}(G)H_\Theta$ and has a realization a the dual symmetric space of $\mathbb{F}_\Theta$. Furthermore, $\mathcal{S}$ is a Riemannian manifold diffeomorphic to a vector space. If a fiber in the cotangent bundle $T^*\mathbb{F}_\Theta$ intersects $\mathcal{S}$, then that fiber and $\mathcal{S}$ are transverse.

The examples 2.0.1 and 2.1.1 show the case of the real flag manifold $SO(2)/\{\pm I\} \cong S^1$, which is a symmetric space. The cotangent bundle of S^1 has a realization as the one-sheeted hyperboloid $Q = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ and it is also a symmetric space. The flag S^1 is the intersection of Q with the plane $z = 0$. Here we study the orbit of the exponential of symmetric zero-trace matrices in the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This set has a realization as the curve $\mathcal{C} = \{(0, y, z) \in \mathbb{R}^3 : y^2 - z^2 = 1,\ y > 0\}$. This curve is intuitively symmetric with respect to the axis Y, as we can see in Figure 1. In the complex case, the generalization of this orbit is the submanifold $\mathcal{S}$ for the complex case. In this example we can verify Theorem 2.1.1, replacing the submanifold $\mathcal{S}$ by the curve $\mathcal{C}$.

Furthermore, in general the intersection of the submanifold $\mathcal{S}$ with the fibers of the bundle $\mathrm{Ad}(G)H_\Theta \to \mathbb{F}_\Theta$ is either nule or a single point. Then the projection of the cotangent bundle restricted to the submanifold $\mathcal{S}$ is an injective map, since each fiber intersecting $\mathcal{S}$ does so in a single element. Then $\mathcal{S}$ is continuously deformed into a part of the sphere $\mathbb{F}_\Theta$.

We define symplectic form

$$\omega(X, Y) = \mathrm{Im}\mathcal{K}(X, \tau Y), \qquad X, Y \in \mathfrak{g},$$

where $\mathcal{K}(\cdot, \cdot)$ is the Cartan-Killing form and τ is the conjugation in $\mathfrak{g}$ with respect to the compact real form $\mathfrak{u}$. Since the adjoint orbit $\mathrm{Ad}(G)H_\Theta$ is contained in the Lie algebra $\mathfrak{g}$, the symplectic form ω induces a symplectic form in the adjoint orbit through the pullback of ω. The submanifold $\mathcal{S}$ and the flag symmetric space $\mathbb{F}_\Theta$ are Lagrangian submanifold of $\mathrm{Ad}(G)H_\Theta$ with respect to ω. However, the submanifold $\mathcal{S}$ is not a Lagrangian submanifold with respect to the KKS form 2.2.4.

The symmetric spaces are Riemannian considering the Cartan-Killing form as metric. On the adjoint orbit of G, this metric is indefinite and invariant with respect to the symmetries. However, in the flag manifold $\mathbb{F}_\Theta$ it is the opposite of a (definite) Riemannian structure and in the symmetric dual space it is an invariant metric by the non-compact Lie group of symmetries. The Cartan-Killing form is the only metric that defines the affine invariant connection with null torsion and curvature with zero covariant derivative. With that connection, the geodesics are one-parameter subgroups and we can search geodesics in the adjoint orbit that are projected on geodesics in the flag manifold $\mathbb{F}_\Theta$.

We evidence that if $g \in G$, $X \in \mathfrak{u}$ and $g \in G \mapsto u \in U$, then the projection of the geodesics $\{g \exp(tX) \cdot H_\Theta\}_{t\in\mathbb{R}}$ of the adjoint orbit $\mathrm{Ad}(G)H_\Theta$ over $\mathbb{F}$ are also geodesics $\{u \exp(tX) \cdot H_\Theta\}_{t\in\mathbb{R}}$ in the flag symmetric space $\mathbb{F}_\Theta$. Furthermore, if $g \in G$, $Y \in \mathfrak{n}_\Theta^+$ and $g \in G \mapsto u \in U$, then the projection of the geodesics $\{g \exp(tY) \cdot H_\Theta\}_{t\in\mathbb{R}}$ of the adjoint orbit $\mathrm{Ad}(G)H_\Theta$ over $\mathbb{F}$ are also a geodesic $\{u \cdot H_\Theta\}$, where $\mathfrak{n}_\Theta^+$ is a subalgebra of $\mathfrak{g}$, as in 1.2.9, isomorphic to the fiber of the cotangent bundle $T^*(\mathbb{F}_\Theta)$.

In the chapters 3, 4 and 5 we found the automorphisms σ of the symmetric spaces (U, U_Θ, σ), its dual symmetric space $(U^*, (U^*)^\sigma)$ and (G, G^σ). We look for an element H_Θ such that $g_\Theta := \exp(H_\Theta)$ and H_Θ has the same centralizer in G and U, i.e.

$$Z_G(H_\Theta) = Z_G(g_\Theta) \qquad Z_U(H_\Theta) = Z_U(g_\Theta). \tag{0.0.2}$$

The automorphism σ is given by C_{g_Θ} for symmetric spaces and for symmetric Lie algebras, we use the same notation σ to the automorphism $\mathrm{Ad}(g_\Theta)$.

Assuming that $p + q = l + 1$ and $p \leqslant q$, we get

$$\begin{array}{cccc}
G & Sl(l+1,\mathbb{C}) & Sp(l,\mathbb{C}) & SO(2l,\mathbb{C}) \\
\Theta & \Sigma\backslash\{\alpha_p\} & \Sigma\backslash\{\alpha_l\} & \Sigma\backslash\{\alpha_l\} \\
\mathbb{F}_\Theta & SU(p+q)/S(U_p \times U_q) & Sp(l)/U(l) & SO(2l)/U(l) \\
T^*\mathbb{F}_\Theta & G/S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C})) & G/Gl(l,\mathbb{C}) & G/Gl(l,\mathbb{C})
\end{array}$$

and H_Θ satisfying 0.0.2 for A_l, C_l and D_l cases are

$$\frac{\sqrt{-1}\pi}{l+1}\begin{pmatrix} qI_p & 0 \\ 0 & -pI_q \end{pmatrix}, \qquad \frac{\sqrt{-1}\pi}{2}\begin{pmatrix} \mathrm{Id}_l & 0 \\ 0 & -\mathrm{Id}_l \end{pmatrix}$$

$$\frac{\sqrt{-1}\pi}{2}\begin{pmatrix} \mathrm{Id}_l & 0 \\ 0 & -\mathrm{Id}_l \end{pmatrix},$$

respectively.

Finally, in these three chapters, we proved the Proposition 2.1.2 calculating the intersection $\mathcal{S} \cap (H_\Theta + \mathfrak{n}_\Theta^+)$ for each case.

1 Preliminaries

1.1 Introduction to Lie theory

In this section we shall use L. A. B. San Martin's books [6] and [5] as main references.

A Lie algebra is a vector space $\mathfrak{g}$ provided with a product (breacket or commutator) $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisying the properties:

1. The bracket $[\cdot,\cdot]$ is bilinear.
2. Antisymmetry, i.e. $[X, Y] = -[Y, X]$, for all $X, Y \in \mathfrak{g}$.
3. Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \qquad X, Y, Z \in \mathfrak{g}.$$

A subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra if it is closed under the bracket. In this case $\mathfrak{h}$ is also a Lie algebra.

A subspace $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if for all $Y \in \mathfrak{h}$, $X \in \mathfrak{g}$, $[X, Y] \in \mathfrak{h}$, i.e. $[\mathfrak{g}, \mathfrak{h}]$ =spam$\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{h}\} \subset \mathfrak{h}$.

An example of Lie algebra is the set $\mathfrak{gl}(n, \mathbb{R})$ consisting of real $n \times n$ matrices with bracket given by the commutator of matrices

$$[A, B] = AB - BA.$$

A Lie group is a group that is also a differentiable manifold such that the product operation

$$p : (g, h) \in G \times G \mapsto gh \in G$$

is differentiable.

Let G be a Lie group, a subgroup $H \subset G$ of G is a **Lie subgroup** of G if H is an immersed submanifold of G, such that the product $H \times H \to H$ is differentiable with respect to the structure of H.

Example 1.1.1. Some examples of Lie groups and subgroups are:

1. The group $\mathrm{Gl}(n, \mathbb{R})$ of invertible $n \times n$ matrices over $\mathbb{R}$. This group with the product of matrices is a Lie group.

2. If G is a Lie group, then any one-parameter subgroup $\{\exp(tX) : X \in \mathfrak{g}, t \in \mathbb{R}\}$ is a Lie subgroup. If $t \mapsto \exp(tX)$ is a closed curve, then we get an injective immersion $S^1 \to G$. Otherwise, the one-parameter group defines an injective immersion $\mathbb{R} \to G$.

3. Some linear groups that are Lie subgroups of $Gl(n, \mathbb{R})$ are:

   (a) The special linear group $\mathrm{Sl}(n, \mathbb{R}) = \{g \in \mathrm{Gl}(n, \mathbb{R}) : \det g = 1\}$,

   (b) The unitary group $U(n)$ of $n \times n$ complex matrices such that $\bar{g}^T g = g\bar{g}^T = 1$ is a embedding submanifold.

   (c) The real symplectic group $\mathrm{Sp}(n, \mathbb{R})$ consisting of the $2n \times 2n$ real matrices, such that $g^T J g = gJg^T = J$, where

$$J = \begin{pmatrix} 0 & -\mathrm{id}_{n\times n} \\ \mathrm{id}_{n\times n} & 0 \end{pmatrix}.$$

4. The vector spaces of finite dimension V over $\mathbb{R}$. There are abelian Lie groups with the operation $+$. In particular $(\mathbb{R}, +)$ is a Lie group. The multiplicative groups $\mathbb{R}\backslash\{0\}$ and $\mathbb{C}\backslash\{0\}$ are also Lie groups.

Given an element $g \in G$, the left and right translations $L_g : h \in G \mapsto gh \in G$ and $R_g : h \in G \mapsto hg \in G$ are diffeomorphisms.

Definition 1.1.1. *Let G be a Lie group. A vector field X in G is called*

- ***right invariant*** *if for every $g \in G$, $d(R_g)_h(X(h)) = X(hg)$, for all $h \in G$.*
- ***left invariant*** *if for every $g \in G$, $d(L_g)_h(X(h)) = X(gh)$, for all $h \in G$.*

The right or left invariant fields are determined by its values at the identity element. Hence, each element in the tangent space T_1G determines a unique right invariant field and a unique left invariant field. Denoting by Inv^l and Inv^r the sets of the left and right invariant fields respectively, they are Lie subalgebras of the Lie algebra of all vector fields in G.

Definition 1.1.2. *A **Lie algebra** of the Lie group G denoted by $\mathfrak{g}$ is one of the isomorphic Lie algebras Inv^r, Inv^l, $(T_1G, [\cdot,\cdot]_r)$ or $(T_1G, [\cdot,\cdot]_l)$. We consider the Lie algebra of the left invariant fields in G.*

Example 1.1.2. The right invariant fields in $\mathrm{Gl}(n, \mathbb{R})$ has the form $X_A(g) = Ag$, with A being a $n \times n$ matrix. The left invariant fields are $Y_A(g) = gA$. In local coordinates, the Lie bracket of two fields is given by

$$[X, Y] = dY(X) - dX(Y).$$

For a matrix A, we get

$$[X_A, X_B](g) = B(Ag) - A(Bg) = X_{BA-AB}.$$

On the other hand, the Lie bracket of left invariant fields is $[Y_A, Y_B] = Y_{AB-BA}$. Hence, the Lie algebras Inv^r and Inv^l can be identify with the space of $n \times n$ matrices. In Inv^r, the bracket is $[A, B] = BA - AB$ and in Inv^l the bracket is $[A, B] = AB - BA$.

Example 1.1.3. If G is a discrete Lie group, $\dim G = 0$ and hence $\mathfrak{g} = \{0\}$.

In [6], it is showed that the flow X_t of a (right or left) invariant vector field X in the Lie group G is

$$X_t(hg) = X_t(h)g \quad X \in \mathrm{Inv}^r.$$

Likewise,

$$gY_t(h) = Y_t(gh) \quad Y \in \mathrm{Inv}^l.$$

Proposition 1.1.3 ([5])**.** Some properties of (right or left) invariant vector fields are:

- A (right or left) invariant field is complete
- If $X \in \mathrm{Inv}^r$ then $X_{t+s}(1) = X_t(X_s(1)) = X_s(1)X_t(1) = X_t(1)X_s(1)$.
- If $Y \in \mathrm{Inv}^l$ then $Y_{t+s}(1) = Y_t(Y_s(1)) = Y_s(1)Y_t(1) = Y_t(1)Y_s(1)$.

Definition 1.1.4. *Let $X \in T_1G$. Then, $expX := (X^r)_{t=1}(1) = (X^l)_{t=1}(1)$. That defines a map exp:* $\mathfrak{g} \to G$*, where* $\mathfrak{g} = T_1G$ *is a Lie group of* G*.*

Proposition 1.1.5. [5] The following statements hold:

1. If $X \in \mathrm{Inv}^r$ then $X_t = \mathrm{L}_{\exp(tX)}$, i.e. $X_t(g) = e^{tX}g$.
2. If $X \in \mathrm{Inv}^e$ then $X_t = \mathrm{R}_{\exp(tX)}$, i.e. $X_t(g) = ge^{tX}$.

Example 1.1.4. The right invariant field in $\mathrm{Gl}(n, \mathbb{R})$ have the form $X(g) = Ag$, where A is a $n \times n$ matrix. The exponential of matrices coincides with the exponential map $\exp A = \displaystyle\sum_{k\geqslant 0} \frac{1}{k!}A^k$ in $Gl(n, \mathbb{R})$.

Proposition 1.1.6. [5] Let G be a connected Lie group and take $g \in G$. Then, there are $X_1, \cdots, X_s \in \mathfrak{g}$ such that

$$g = \exp(X_1) \cdots \exp(X_s).$$

Proposition 1.1.7. [5] Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, for any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists a unique connected subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.

Theorem 1.1.8. *[5] Every closed subgroup H of a Lie group G is a Lie subgroup. Its Lie algebra is*

$$\mathfrak{h}_H = \{X \in \mathfrak{g} : \forall t \in \mathbb{R},\ \exp tX \in H\}.$$

Theorem 1.1.9. *[5] (Third Lie theorem) Let $\mathfrak{g}$ be a real Lie algebra with $\dim\mathfrak{g} < \infty$. Then*

1. *there exists a unique (up to isomorphism) simply connected and connected Lie group $\tilde{G}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}$;*

2. *if G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $G \approx \tilde{G}(\mathfrak{g})/\Gamma$, where $\Gamma \subset \tilde{G}(\mathfrak{g})$ is a central discrete subgroup, i.e. Γ is contained in the center $Z(\tilde{G}(\mathfrak{g}))$ of $\tilde{G}(\mathfrak{g})$. In that case Γ is isomorphic to the fundamental group $\pi_1(G)$.*

The **third Lie theorem** says that if $\mathfrak{g}$ is a real Lie algebra with finite dimension, then there is a connected Lie group G with Lie algebra (isomorphic to) $\mathfrak{g}$.

We can define representations on a Lie algebra $\mathfrak{g}$ using the left and right translations in the Lie group G, this representations are related to each other via the exponential map. An element $g \in G$ defines the inner automorphism $C_g(x) := gxg^{-1}$ and this automorphism allow us to define the following representations

Definition 1.1.10. *The **adjoint representation** $Ad\colon G \to Gl(\mathfrak{g})$ of G in its Lie algebra $\mathfrak{g}$ is defined by*

$$\mathrm{Ad}(g) := d(C_g)_1. \tag{1.1.1}$$

Furthemore,

$$g\exp(X)g^{-1} = \exp(\mathrm{Ad}(g)X).$$

Definition 1.1.11. *Let $\mathfrak{g}$ be a Lie algebra. Its **adjoint representation** is the application $ad\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by*

$$\mathrm{ad}(X)(Y) := [X, Y].$$

Example 1.1.5. In $\mathrm{Gl}(n, \mathbb{R})$, the adjoint $\mathrm{Ad}(g)$ coincides with the conjugation C_g, i.e. if $A \in \mathfrak{gl}(n, \mathbb{R})$ and $g \in \mathrm{Gl}(n, \mathbb{R})$ then $\mathrm{Ad}(g)A = gAg^{-1}$.

For two subsets A and B of the Lie algebra $\mathfrak{g}$, we shall use the notation $[A, B]$ to indicate the subspace generated by $\{[X, Y] : X \in A, Y \in B\}$. We define, by induction, the following subspaces of $\mathfrak{g}$:

$$\mathfrak{g}^{(0)} \qquad \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \qquad \cdots \qquad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}].$$

This sequence of ideals is known as derivative serie of $\mathfrak{g}$ and its components are called derivative algebras of $\mathfrak{g}$.

Example 1.1.6. Let $\mathfrak{g}$ be an algebra of upper triangular matrices

$$\mathfrak{g} = \left\{ \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & * \end{pmatrix}_{n\times n} \right\}.$$

Then $\mathfrak{g}$ is the algebra of upper triangular matrices with zero diagonal. Hence $\mathfrak{g}^{(k)} = \{0\}$ if $k \geqslant k_0$ for some k_0 large enough.

Example 1.1.7. If $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, then $\mathfrak{g}' = \mathfrak{g}$ and hence, $\mathfrak{g}^{(k)} = \mathfrak{g}$ for all $k \geqslant 0$.

The descending central series of the Lie algebra $\mathfrak{g}$ is defined by induction as

$$\mathfrak{g}^1 = \mathfrak{g} \qquad \mathfrak{g}^2 = [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}' \quad \cdots \quad \mathfrak{g}^k = [\mathfrak{g},\mathfrak{g}^{k-1}].$$

Definition 1.1.12. *An algebra is solvable if some of its derivative algebras is zero, i.e.*

$$\mathfrak{g}^{(k_0)} = 0$$

for some $k_0 \geqslant 1$ (and hence $\mathfrak{g}^{(k)} = 0$ for all $k \geqslant k_0$).

Example 1.1.8. The algebras of upper triangular matrices are solvable.

Example 1.1.9. The algebras $\mathfrak{sl}(n)$ are not solvable since its derivative algebras coincide with themselves.

Definition 1.1.13. *A Lie algebra is called nilpotent if its descending central series vanishes at some order, i.e.*

$$\mathfrak{g}^{k_0} = \{0\}$$

for some $k_0 \geqslant 1$ (and hence, $\mathfrak{g}^k = 0$ for every $k \geqslant k_0$).

Example 1.1.10.

1. The abelian algebras are nilpotent.
2. The following matrices subalgebras are nilpotent

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times n} \right\} \qquad \mathfrak{g} = \left\{ \begin{pmatrix} a & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a \end{pmatrix}_{n\times n} \right\}$$

3. The algebra of upper triangular matrices is not nilpotent.

Proposition 1.1.14. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then, there exists in $\mathfrak{g}$ a unique solvable ideal $\mathfrak{r} \subset \mathfrak{g}$ containing each solvable ideals of $\mathfrak{g}$.

Definition 1.1.15. *The ideal* $\mathfrak{r}$ *of Proposition 1.1.14 is called a solvable radical (or just radical) of* $\mathfrak{g}$*. The radical of* $\mathfrak{g}$ *will be denoted by* $\mathfrak{r}(\mathfrak{g})$*.*

Example 1.1.11. $\mathfrak{g}$ is solvable if and only if $\mathfrak{r}(\mathfrak{g}) = \mathfrak{g}$.

Definition 1.1.16. *A Lie algebra* $\mathfrak{g}$ *is called semi-simple if* $\mathfrak{r}(\mathfrak{g}) = 0$ *(i.e.* $\mathfrak{g}$ *contains no soluble ideals in addition to* 0*).*

Definition 1.1.17. *A algebra* $\mathfrak{g}$ *is simple if*

1 The only ideals of $\mathfrak{g}$ *are* 0 *and* $\mathfrak{g}$ *and*

2 dim$\mathfrak{g} \neq 1$.

Every simple Lie algebra is semi-simple.

Example 1.1.12. The algebras $\mathfrak{sl}(n, \mathbb{K})$ are simple if $\mathbb{K}$ has not characteristic equal to two.

We define the **Cartan-Killing form** in the Lie algebra $\mathfrak{g}$ as the symmetric bilinear form in $\mathfrak{g}$ given by

$$\mathcal{K}(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)), \qquad X, Y \in \mathfrak{g}. \tag{1.1.2}$$

Proposition 1.1.18. [6] If ϕ is an automorphism of the Lie algebra $\mathfrak{g}$ and $X, Y, Z \in \mathfrak{g}$, then

(a) $\mathcal{K}(\phi X, \phi Y) = \mathcal{K}(X, Y)$ and

(b) $\mathcal{K}([X, Y], Z) + \mathcal{K}(Y, [X, Z]) = 0$.

Theorem 1.1.19. *The Cartan-Killing form of* $\mathfrak{g}$ *is non-degenerate if and only if* $\mathfrak{g}$ *is semi-simple.*

1.1.1 Semi-simple Lie algebras

This work is focused on studying semi-simple Lie algebras and how it can be decomposed by its adjoint representation. Hence, we must define a Cartan subalgebra.

Definition 1.1.20. *Let* $\mathfrak{g}$ *be a Lie algebra. A Cartan subalgebra of* $\mathfrak{g}$ *is a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *satisfying*

1. $\mathfrak{h}$ *is nilpotent and*

2. the normalizer of $\mathfrak{h}$ *in* $\mathfrak{g}$ *coincides with* $\mathfrak{h}$*. This condition is equivalent to*

2'. If $[X, \mathfrak{h}] \subset \mathfrak{h}$ *then* $X \in \mathfrak{h}$*.*

Example 1.1.13. For $\mathfrak{sl}(2)$, the Cartan subalgebra $\mathfrak{h}$ could be equal to

$$\left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\right\} \text{or} \left\{\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}\right\},$$

since both $\mathfrak{h}$ are abelian and if $X \in \mathfrak{g}$ then $[\mathfrak{h}, X] \subset \mathfrak{h}$ if and only if $X \in \mathfrak{h}$ in both of cases.

Theorem 1.1.21. *[6] In complex Lie algebras, the Cartan subalgebras are conjugate.*

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan-subalgebra of $\mathfrak{g}$. The algebra can be decomposed as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_k},$$

with $\alpha_1, \cdots, \alpha_k$ the non-zero weights of the adjoint representation of $\mathfrak{h}$ in $\mathfrak{g}$. These weights will be called **roots** of $\mathfrak{h}$ with respect to $\mathfrak{g}$ and the set that contains all of them will be denoted by Π. The spaces $\mathfrak{g}_{\alpha_i}$ shall be called **root spaces**. Since the field is algebraically closed, the representation of $\mathfrak{h}$ in each $\mathfrak{g}_{\alpha_i}$ is given by the matrices

$$\text{ad}(H) = \begin{pmatrix} \alpha_i(H) & & * \\ & \ddots & \\ & & \alpha_i(H) \end{pmatrix}$$

for each $H \in \mathfrak{h}$. Furthermore, $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_j}] \subset \mathfrak{g}_{\alpha_i+\alpha_j}$.

Lemma 1.1.22. *[6] Let α and β be two weights of $\mathfrak{h}$ (roots or nule weight). If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ then*

$$\mathcal{K}(X, Y) = 0,$$

unless $\beta = -\alpha$.

Corollary 1.1.23. *[6] The following assertions hold:*

1. *The restriction of $\mathcal{K}(\cdot, \cdot)$ to $\mathfrak{h}$ is non-degenerate.*

2. *If α is a root, then $-\alpha$ is also a root.*

3. *For all $X \in \mathfrak{g}_\alpha$, there exists $Y \in \mathfrak{g}_{-\alpha}$ such that $\mathcal{K}(X, Y) \neq 0$.*

Proposition 1.1.24. [6] For all $H \in \mathfrak{h}$ and every weight α, $\text{ad}(H)\mid_{\mathfrak{g}_\alpha} = \alpha(H)\text{id}$ and the linear transformations $\text{ad}(H)$, $H \in \mathfrak{h}$ are simultaneously diagonalizable.

Proposition 1.1.25. $\mathfrak{h}$ is Abelian.

The Cartan-Killing form can also be "defined" in the dual $\mathfrak{h}^*$ of $\mathfrak{h}$. Since $\mathcal{K}(\cdot,\cdot)$ is a bilinear form, it defines a map $\mathfrak{h} \to \mathfrak{h}^*$ by

$$H \mapsto \alpha_H(\cdot) = \mathcal{K}(H,\cdot).$$

Since the restriction of the Cartan-Kiling form to $\mathfrak{h}$ is non-degenerate, this map is an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$. For $\alpha \in \mathfrak{h}^*$, its inverse image shall be denoted by H_α, i.e. H_α is defined by

$$\mathcal{K}(H_\alpha, H) = \alpha(H), \quad \forall H \in \mathfrak{h}.$$

We can define a bilinear form in $\mathfrak{h}^*$, using the same notation of the Cartan-Killing form, by

$$\mathcal{K}(\alpha,\beta) = \mathcal{K}(H_\alpha, H_\beta) = \alpha(H_\beta) = \beta(H_\alpha)$$

if α and β are linear functionals in $\mathfrak{h}$. This is a symmetric, non-degenerate bilinear form in $\mathfrak{h}^*$. This will also be called to as the Cartan-Killing form.

By the isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ defined from the Cartan-Kiling form, the roots $\alpha \in \Pi$ define a finite number of special elements H_α in $\mathfrak{h}$. As the set of roots generates $\mathfrak{h}^*$, the set $\{H_\alpha : \alpha \in \Pi\}$ generates $\mathfrak{h}$.

Let $\{v_1, \cdots, v_l\}$ be an ordered basis of $\mathfrak{h}^*_{\mathbb{Q}}$. Let $v, w \in V$ written in coordinates as

$$\begin{aligned} v = &\quad a_1 v_1 + \cdots + a_l v_l \\ w = &\quad b_1 v_1 + \cdots + b_l v_l. \end{aligned}$$

Fixed the lexicographic order in $\mathfrak{h}^*_{\mathbb{Q}}$ with respect to this basis defined by $v \leqslant w$ if $v = w$ or if $a_i < b_i$, where i is the first index such that the coordinates of v and w are different.

Proposition 1.1.26. [6] The Cartan-Killing form restricted to $\mathfrak{h}_{\mathbb{Q}}$ (and $\mathfrak{h}^*_{\mathbb{Q}}$) is an inner product.

Definition 1.1.27. *A root $\alpha \in \Pi$ is **simple** -with respect to the fixed order- if*

(i) $\alpha > 0$

(ii) there are no $\beta, \gamma \in \Pi$ such that β and γ are positive and $\alpha = \beta + \gamma$.

The set of simple roots will be denoted by Σ.

Lemma 1.1.28. *[6] Let $\beta \in \Pi$ with $\beta > 0$. Then β is uniquely written as*

$$\beta = n_1\alpha_1 + \cdots + n_l\alpha_l$$

with $n_1, \cdots, n_l$ integers $\geqslant 0$. In particular Σ generates $\mathfrak{h}^*_{\mathbb{Q}}$.

Definition 1.1.29. *A subset $\Sigma = \{\alpha_1, \cdots, \alpha_l\}$ satisfying the two conditions:*

a) Σ *is a basis of* $\mathfrak{h}^*_{\mathbb{Q}}$ *and*

b) each root β *can be written as* $\beta = n_1\alpha_1 + \cdots + n_l\alpha$ *are integer coefficients and each of them has the same sign,*

is called a simple root system.

Fixing a simple root system (or a lexicographic order), we can define

$$\Pi^+ = \{\alpha \in \Pi : \alpha > 0\} \quad \Pi^- = -\Pi^+ = \{\alpha \in \Pi : \alpha < 0\}.$$

Let us define the sets

$$\mathfrak{n}^+ = \sum_{\alpha\in\Pi^+} \mathfrak{g}_\alpha \qquad \mathfrak{n}^- = \sum_{\alpha\in\Pi^-} \mathfrak{g}_\alpha.$$

Then

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \tag{1.1.3}$$

and $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are dual by the Cartan-Killing form, since $\mathcal{K}(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$ and $\mathcal{K}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\beta \neq -\alpha$. The algebra $\mathfrak{n}^+$ is nilpotent, since if $X \in \mathfrak{g}_\alpha$ then $\mathrm{ad}(X)^k \mathfrak{g}_\beta \subset \mathfrak{g}_{k\alpha+\beta}$, the same is true for $\mathfrak{n}^-$ which is isomorphic to $\mathfrak{n}^+$. Thus $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a subalgebra and since $\mathfrak{n}^+$ is an ideal of $\mathfrak{b}$, this subalgebra is solvable. The subalgebra $\mathfrak{b}$ is known as **Borel subalgebra**.

1.1.2 Semi-simple real Lie algebras

Let $\mathfrak{g}$ be a complex Lie algebra, a real form of $\mathfrak{g}$ is a real Lie algebra such that their complexification is $\mathfrak{g}$, they are of two types compact and non-compact real forms. Every complex algebra has a unique (up to isomorphism) compact real form and the structure of these real forms is completly described by the complex algebra. The description of a non-compact real form is made from the decomposition obtained intersecting it with the compact real form (Cartan decomposition).

Let V be a real vector space and $V_{\mathbb{C}}$ and its complexification. The elements of $V_{\mathbb{C}}$ can be expressed as $u + \sqrt{-1}v$, $u, v \in V$. Writing the elements of $V_{\mathbb{C}}$ as above, we can define the conjugation $\sigma : V_{\mathbb{C}} \to V_{\mathbb{C}}$

$$\sigma(u + \sqrt{-1}v) = u - \sqrt{-1}v.$$

This conjugation satisfies $\sigma^2 = 1$ and it is antilinear (or sesquilinear) in $V_{\mathbb{C}}$, i.e. σ is linear over the real vector space V and

$$\sigma(zw) = \bar{z}w \quad z \in \mathbb{C}, w \in V_{\mathbb{C}}.$$

It is clear that $V = \{w \in V_{\mathbb{C}} : \sigma(w) = w\}$.

Definition 1.1.30. *Let F be a complex vector space. A conjugation in F is an antilinear transformation σ satisfying $\sigma^2 = 1$.*

An invertible antilinear transformation σ of $\mathfrak{g}$ satisfying

$$[\sigma X, \sigma Y] = \sigma[X, Y] \tag{1.1.4}$$

is called an *anti-automorphism.* Given an anti-automorphism in $\mathfrak{g}$, the real algebra

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : \sigma(X) = X\}$$

has as complexification the complex Lie algebra $\mathfrak{g}$.

Definition 1.1.31. *Let $\mathfrak{g}$ be a complex algebra. A **real form** of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the realification $\mathfrak{g}^{\mathbb{R}}$, which is the subspace of fixed points of a conjugation satisfying 1.1.4. If that happens, $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$.*

Example 1.1.14. In $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{C})$, let σ be the map in $\mathfrak{g}$ given by $\sigma(A) = -\bar{A}^T$ for $A \in \mathfrak{g}$. Then σ is an automorphism. The Lie algebra of its fixed points is $\mathfrak{su}(n) = \{A \in \mathfrak{sl}(n, \mathbb{C}) : A = -\bar{A}^T\}$, and hence it is a real form of $\mathfrak{sl}(n, \mathbb{C})$.

Definition 1.1.32. *A Lie algebra over $\mathbb{R}$ is called compact if its Cartan-Killing form is negative defined.*

Theorem 1.1.33. *[6] Every semi-simple complex Lie algebra admits compact real forms. If $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are compact real forms of $\mathfrak{g}$, then there exists an automorphism of ϕ of $\mathfrak{g}$ such that $\phi(\mathfrak{u}_1) = \mathfrak{u}_2$ and hence the compact real forms are isomorphic to each other.*

Let $\mathfrak{g}$ be a semi-simple complex algebra and let $\mathfrak{h}$ be a Cartan subalgebra with roots set Π and let Σ be the simple roots set in Π. A *Weyl basis* of $\mathfrak{g}$ is a basis of $\mathfrak{g}$ consisting of H_α, $\alpha \in \Sigma$ and $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Pi$ satisfying

- $[X_\alpha, X_{-\alpha}] = H_\alpha$ and
- $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root and such that $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$, then $m_{\alpha,\beta}$ is real.

Given a Weyl basis, let $\mathfrak{u}$ be a real subspace generated by

$$\begin{aligned} &\sqrt{-1} H_\alpha, \\ &A_\alpha := X_\alpha - X_{-\alpha}, \\ &Z_\alpha := \sqrt{-1}(X_\alpha + X_{-\alpha}), \end{aligned} \tag{1.1.5}$$

with α in the set Π^+ of positive roots. Then $\mathfrak{u}$ is a compact real form.

The Cartan-Killing form of $\mathfrak{u}$ coincides with the restriction of the Cartan-Killing form of $\mathfrak{g}$, since $\mathfrak{u}$ is a real form. Furthermore, if two roots are not opposite, the corresponding root spaces are orthogonal. Using the notation above

$$\mathcal{K}(\sqrt{-1}H_\alpha, A_\beta) = \mathcal{K}(\sqrt{-1}H_\alpha, Z_\beta) = \mathcal{K}(A_\alpha, Z_\beta) = 0. \tag{1.1.6}$$

We denote by $\mathfrak{h}_\mathbb{R}$ the real subspace of $\mathfrak{h}$ generated by $\{H_\alpha : \alpha \in \Pi\}$. Then, the subspace $\sqrt{-1}\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{u}$.

Lemma 1.1.34. *[6] Let τ be a conjugation with respect to the compact real form $\mathfrak{u}$ of the complex algebra $\mathfrak{g}$. Then, the expression*

$$\mathcal{H}_\tau(X, Y) = -\mathcal{K}(X, \tau Y) \tag{1.1.7}$$

defines a hermitian form in $\mathfrak{g}$.

Theorem 1.1.35. *[6] Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\mathfrak{u}$ a compact real form of $\mathfrak{g}$. Let $\mathfrak{g}_0$ any real form of $\mathfrak{g}$ and let us denote by σ the corresponding conjugation. Then, there exists an inner automorphism ϕ of $\mathfrak{g}$ such that σ commutes with the conjugation with respect to the compact real form $\phi(\mathfrak{u})$.*

Corollary 1.1.36. *[6] Let $\mathfrak{u}_1$ and $\mathfrak{u}_2$ be compact real forms of $\mathfrak{g}$. Then, there exists an automorphism ϕ of $\mathfrak{g}$ such that $\phi(\mathfrak{u}_1) = \mathfrak{u}_2$.*

The bijection between the semi-simple complex and compact Lie algebras is also used to develop compact algebras. Actually, the compact real forms are expressed as

$$\mathfrak{u} = \sqrt{-1}\mathfrak{h}_\mathbb{R} \oplus \sum_\alpha \mathfrak{k}_\alpha, \tag{1.1.8}$$

where $\mathfrak{k}_\alpha$ is the space generated by A_α and Z_α. Finally, the compact real forms of the classical Lie algebras are:

1. $\mathfrak{su}(n)$ is a compact real form of $\mathfrak{sl}(n, \mathbb{C})$.
2. A compact real form of $\mathfrak{so}(n, \mathbb{C})$, $n \geqslant 3$, is $\mathfrak{so}(n, \mathbb{R})$.
3. A compact real form of $\mathfrak{sp}(n, \mathbb{C})$, is the subalgebra of anti-hermitian matrices in $\mathfrak{sp}(n, \mathbb{C})$. This algebra is denoted by $\mathfrak{sp}(n)$:

   $$\mathfrak{sp}(n) = \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{su}(2n) \tag{1.1.9}$$

   and its elements have the form

   $$\begin{pmatrix} A & -\bar{C} \\ C & \bar{A} \end{pmatrix} \tag{1.1.10}$$

   with A being a $n \times n$ anti-hermitian and C being symmetric.

Let $\mathfrak{g}_0$ be a non-compact real form of the semi-simple complex algebra $\mathfrak{g}$ and σ its conjugation. If $\mathfrak{u}$ is the compact real form of $\mathfrak{g}$ with conjugation τ. Then

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{s},$$

where

$$\mathfrak{k} = \mathfrak{g}_0 \cap \mathfrak{u}, \qquad \mathfrak{s} = \mathfrak{g}_0 \cap \sqrt{-1}\mathfrak{u}. \tag{1.1.11}$$

This decomposition is known as *Cartan decomposition* of $\mathfrak{g}_0$. The brackets of the elements in the Cartan decomposition satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \qquad [\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s} \qquad [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}. \tag{1.1.12}$$

Hence $\mathfrak{k}$ is a subalgebra and $\mathfrak{s}$ is invariant under the adjoint representation of $\mathfrak{k}$. The subalgebra $\mathfrak{k}$ is called the *compact component* of the Cartan decomposition.

The restriction of $\mathcal{H}_\tau$ to the real form $\mathfrak{g}_0$ is an inner product, since the Cartan-Killing form of $\mathfrak{g}_0$ is the restriction of the Cartan-Killing form of $\mathfrak{g}$.

Proposition 1.1.37. [6] Given a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{s}$, the involutive automorphism θ defined by $\theta(X) = X$ if $X \in \mathfrak{k}$ and $\theta(Y) = -Y$ if $Y \in \mathfrak{s}$ is such that the bilinear form

$$B_\theta(X, Y) := -\mathcal{K}(X, \theta Y) \tag{1.1.13}$$

is an inner product in $\mathfrak{g}_0$. Conversely, given an automorphism θ satisfying 1.1.13 is an inner product in $\mathfrak{g}$ and its eigenspaces determine a Cartan decomposition. The automorphism is called Cartan involution.

Example 1.1.15. Taking $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, a compact real form is $\mathfrak{u} = \mathfrak{su}(n)$ and if $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R})$, then $\mathfrak{g}_0 \cap \mathfrak{u}$ is the subalgebra of real anti-hermitian matrices, i.e. the subalgebra $\mathfrak{so}(n)$ of anti-symmetric matrices. In addition, $\mathfrak{g}_0 \cap \sqrt{-1}\mathfrak{u}$ is the subspace of real matrices X such that $\sqrt{-1}X$ is anti-hermitian, i.e. the subspace of symmetric matrices. Then

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n, \mathbb{R}) \oplus \mathfrak{s}$$

is a Cartan decomposition. The corresponding Cartan involution is $\theta(X) = -X^T$, since $\theta = 1$ in $\mathfrak{so}(n, \mathbb{R})$ and $\theta = -1$ in $\mathfrak{s}$.

The Cartan-Killing form $\mathcal{K}_{\mathfrak{g}_0}(X, Y)$ of $\mathfrak{g}_0$ is negative defined in $\mathfrak{k}$ and positive defined in $\mathfrak{s}$. If $X \in \mathfrak{k}$ and $Y \in \mathfrak{s}$, then $\mathcal{K}_{\mathfrak{g}_0}(X, Y) = 0$.

The Lie algebra $\mathfrak{k}$ is a maximal compact subalgebra in $\mathfrak{g}_0$.

Theorem 1.1.38. *[5] Let G_0 be a connected semi-simple Lie group and let $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition of its Lie algebra. Write*

$$K = \langle \exp \mathfrak{k} \rangle \quad \text{and} \quad S = \exp \mathfrak{s}, \tag{1.1.14}$$

where $\langle \exp \mathfrak{k} \rangle$ *is the unique Lie subgroup of* G *with Lie algebra* $\mathfrak{k}$ *and* S *is the image of* exp *in* $\mathfrak{s}$*. Then,*

- $G_0 = SK = KS$ *and all* $g \in G_0$ *is uniquely written as*

$$g = sk \quad \text{or} \quad g = ks, \quad k \in K, \ s \in S, \tag{1.1.15}$$

- S *is an embedding submanifold of* G_0*, diffeomorphic to* $\mathfrak{s}$ *by the embedding exp:*$\mathfrak{s} \to S$*,*

- *The functions* $K \times S \to G_0$*;* $(k,s) \mapsto ks$ *and* $(k,s) \mapsto sk$ *are diffeomorphisms.*

we can combine these tools of Lie theory with differential geometry for the study of symmetric spaces. Now let's detail some terms of differential geometry.

1.2 Foundations of differential geometry

In order to study the symmetric spaces, we must know about homogeneous spaces, for that we shall use bibliography of San Martin ([5]) and of Kobayashi and Nomizu ([2] and [3]).

A **left action** of a group G in a set X is a function that associates to $g \in G$ an application $a(g) : X \to X$ and satisfy the properties:

1. $a(1)(x) = x$, for every $x \in X$;

2. $a(gh) = a(g) \circ a(h)$.

A **right action** is defined likewise, replacing the second property by $a(gh) = a(h) \circ a(g)$. A left action can be expressed using the notations $g(x), g \cdot x$ or gx to say $a(g)x$.

Given $x \in X$, its **orbit** by G, denoted by $G \cdot x$ or Gx, is defined as the set

$$G \cdot x = \{gx \in X : g \in G\}.$$

The set G_x of the elements of G fixing x is called **isotropy subgroup** or **stabilizer** of x:

$$G_x = \{g \in G : gx = x\}. \tag{1.2.1}$$

It is a subgroup of G since $(gh)x = g(hx)$.

Definition 1.2.1. *Let* a *be an action of* G *in* X*.*

1. *The action is said to be* ***effective*** *if ker*$(a) = \{g \in G : a(g) = \mathrm{id}_X\} = \{1\}$*, where* 1 *is the identity element of* G*.*

2. *The action is said to be* ***free*** *if the isotropy subgroups are equal to* $\{1\}$.

3. *The action is said to be* ***transitive*** *if* X *is an orbit of* G, *i.e. for every* $x, y \in X$, *there exists* $g \in G$ *such that* $gx = y$.

An action of the Lie group G is a function $\phi : G \times M \to M$, $\phi(g, x) = gx$, such that the partial function $g \mapsto \phi_g$, $\phi_g(x) = \phi(g, x)$, is an homomorphism of G in the group of the invertible transformations of M. The action is differentiable if ϕ is a differentiable function.

Proposition 1.2.2. [6] Suppose that the action of G in X is transitive and take $x \in X$. Then, the map $\xi : gG_x \in G/G_x \mapsto gx \in X$ is a bijection between G/G_x and X. The map ξ_x is **equivariant**, i.e. $g\xi_x(g_1G_x) = \xi((gg_1)G_x), g, g_1 \in G$, that means ξ_x commutes with the actions of G in G/G_x and X, respectively. Furthermore, if $y = gx$ then $\xi_y = \xi_x \circ D_g$.

From the identification in the Proposition above, a quotient G/H is called a **homogeneous space**, as are called the sets where the groups act transitively. The point x chosen to establish the identification of between X and G/G_x is called of **origin** of **base** of the homogeneous space X.

Theorem 1.2.3. *[5] Let* G *be a Lie group and let* $H \subset G$ *be a closed subgroup. Then, there exists a differentiable structure in* G/H, *compatible with the quotient topology, that satisfies*

1. $\dim G/H = \dim G - \dim H$.

2. *The canonical projection* $\pi : G \to G/H$ *is a submersion.*

3. *The natural action* $a : (g, xH) \in G \times G/H \mapsto (gx)H \in G/H$ *is differentiable.*

4. *For each* $g \in G$, *the induced map* $g : G/H \to G/H$, $xH \mapsto gxH$ *is a diffeomorphism.*

The differentiable structure defined in the theorem above is called **quotient differentiable structure**.

Definition 1.2.4. *Let* $\mathfrak{g}$ *be a Lie algebra and let* M *be a* C^∞ *manifold . Denote by* $\Gamma(TM)$ *the Lie algebra of vector fields in* M *provided with the Lie bracket. An infinitesimal action of* $\mathfrak{g}$ *in* M *is an homomorphism of* $\mathfrak{g} \to \Gamma(TM)$.

A differential action of G in M induces an infinitesimal action of $\mathfrak{g}$ as follows: given $X \in \mathfrak{g}$ and $x \in M$, the curve in M defined by $t \mapsto e^{tX}x$ is differentiable. Its derivative at the origin

$$\tilde{X}(x) = \frac{d}{dt}(e^{tX}x) \mid_{t=0} = \frac{d}{dt}\phi_x(e^{tX}) \mid_{t=0} = (d\phi_x)_1(X) \tag{1.2.2}$$

is a tangent vector in $x \in M$. Hence, $x \in M \mapsto \tilde{X}(x) \in T_xM$ defines a vector field in M.

Proposition 1.2.5. [5] The function $X \in \mathfrak{g} \mapsto \tilde{X} \in \Gamma(TM)$ is an homomorphism if $\mathfrak{g}$ is the Lie algebra of right invariant vector fields in G.

As a particular case of infinitesimal action, consider the action of G in G/H, where H is a closed subgroup. In this case, if $x_0 = 1 \cdot H$ is the origin of G/H then $\phi_{x_0} : G \to M$, $g \mapsto g \cdot x_0 = gH$, is the canonical projection $\pi : G \to G/H$. Thus, the following description of $\tilde{X}$ holds

Proposition 1.2.6. [5] Let G be a Lie group and let H be a closed subgroup and denote by $\pi : G \to G/H$ the canonical projection. Let X be a right invariant field of G, i.e.

$$d\pi_g(X(g)) = \tilde{X}(\pi(g)), \qquad g \in G. \tag{1.2.3}$$

The corresponding differentiable action is $\epsilon : \mathfrak{g} \to \Gamma(TM)$, $\epsilon(X) = \tilde{X}$. For $x \in M = G/H$, we define the subspace

$$T_xM = \{\tilde{X}(x) : X \in \mathfrak{g}\}, \tag{1.2.4}$$

for each $x \in M$, since the action of G in M is transitive.

We can define an action on the quotient spaces. Let $H \subset G$ be a subgroup and denote by G/H the set of all cosets gH, $g \in G$. Then the map $(g, g_1H) \mapsto g(g_1H) = (gg_1)H$ defines a natural left action of G in G/H. The action of G in G/H is transitive. Every transitive action is in bijection with a quotient space of G.

In a homogeneous space $M = G/H$ (H closed), the invariant structures are given by their values at the origin $x_0 = 1 \cdot H$. Since $h(x_0) = x_0$ for every $h \in H$, then its possible to define a representation $\rho : H \to \mathrm{Gl}(T_{x_0}M)$ of H in the tangent space $T_{x_0}M$. This function is called **isotropy representation** of G/H.

1.2.1 Fibre bundles

A **principal bundle** $P(M, H)$ (often denoted by $P \to M$) consists of the total space P of the basis M, both topological spaces and of the structure group G. These spaces are related as follows:

1. The group H acts freely to the right in P by the action $R : (p, g) \mapsto pg$, $p \in P$, $g \in H$. (i.e. if $pg = p$ for some p, then $g = 1$.)

2. The space of the orbits of this action is M. This means that there exists a sobrejective function $\pi : P \to M$, such that the orbits of H are the sets $\pi^{-1}\{x\}, x \in M$.

3. P is locally trivial, i.e. for every $x \in M$, there exists a neighborhood U of x and a bijective function, named of local trivialization, $\Psi : \pi^{-1}(U) \to U \times H$, such that

$\Psi(p) = (\pi(p), \phi(p))$, where $\phi : \pi^{-1}(U) \to H$ is a function satisfying $\phi(pg) = \phi(p)g$ for every $p \in \pi^{-1}(U)$ and $g \in H$.

The bundle $P \to M$ is called topological bundle if the functions in the definition are continuous (and homeomorphisms when are bijective functions). The principal bundle is of class C^k, $k \geqslant 1$, if the spaces involved are differentiable manifolds of class C^k.

The fibers of the principal bundle are denoted by $P_x = \pi^{-1}\{x\}$, $x \in M$, or $P_p = \pi^{-1}\{\pi(p)\}$, $p \in P$.

Example 1.2.1. $G(G/H, H)$: Let G a Lie group and H a closed subgroup of G. The group H acts on G by the right translation. We then obtain a differentiable principal bundle $G(G/H, H)$ over the base manifold G/H with structure group H.

Example 1.2.2. Let M be a differentiable manifold and $TM = \bigcup_{x \in M} T_xM$ its tangent bundle. The **bundle of linear frames** of M is the set BM of all basis of TM. A linear frame $u \in BM$ is an ordered basis $\{f_1, \cdots, f_n\}$ of some tangent space T_xM, $x \in M$. In [3] is indicated that the general linear group $GL(n, \mathbb{R})$ acts on BM on the right and this action can be accordingly interpreted as follows. Consider $a = (a^i_j) \in GL(n, \mathbb{R})$ as a linear transformation on $\mathbb{R}^n$ which maps e_j into $\sum_i a^i_j e_i$. Then $ua : \mathbb{R}^n \to T_xM$ is the composite of the following mappings

$$\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_xM.$$

Since H acts on P on the right, we can assign to each element $A \in \mathfrak{h}$ a vector field A^* on P as follows. The action of the 1−parameter subgroup $a_t = \exp tA$ on P induces a vector field on P, which will be denoted by A^*. The vector field A^* is called the **fundamental vector field** corresponding to A, and

$$A \to (A^*)_u \tag{1.2.5}$$

is a linear isomorphism of $\mathfrak{h}$ onto H_u for each $u \in P$.

Let $P(M, H)$ be a principal fiber bundle and F a manifold on which H acts on the left: $(a, \xi) \in H \times F \to a\xi \in F$. Can be constructed a fiber bundle $E(M, F, H, P)$ **associated** with P with standard fiber F. On the product manifold $P \times F$, we let H act on the right as follows: an element $a \in H$ maps $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$, The quotient space of $P \times F$ by this group action is denoted by $E = P \times_H F$. A differentiable structure will be introduced in E later and at this moment E is only a set.

Example 1.2.3. Tangent bundle: Let $GL(n; \mathbb{R})$ act on $\mathbb{R}^n$ as above. The tangent bundle $T(M)$ over M is the bundle associated with BM with standard fibre $\mathbb{R}$. It can be easily shown that the fibre of TM over $x \in M$ may be considered as T_xM.

1.2.1.1 Connections

Let $P(M, H)$ be a principal fibre bundle over a manifold M with group H. For each $u \in P$, let T_uP be the tangent space of P at u and G_u the subspace of T_uP consisting of vectors tangent to the fibre through u. A **connection** Γ in P is an assignment of a subspace Q_u of T_uP to each $u \in P$ such that

(a) $T_uP = H_u \oplus Q_u$

(b) $Q_{ua} = (R_a)_* Q_u$ for every $u \in P$ and $a \in H$; where R_a is the transformation of P induced by $a \in H$, $R_a u = ua$;

(c) Q_u depends differentiably on u.

For each $X \in T_uP$, we define $\omega(X)$ to be the unique $A \in \mathfrak{h}$ such that $(A^*)_u$ is equal to the vertical component of X (component in H_u of T_uP). It is clear that $\omega(X) = 0$ if and only if X is horizontal (belongs to $Q_u \subset T_uP$). The form ω is called the **connection form** of the given connection Γ.

A connection in the bundle BM of linear frames P over M is called a **linear connection** of M. The canonical form θ of BM is the $\mathbb{R}^n$-valued 1-form on BM defined by

$$\theta(X) = u^{-1}(\pi(X)), \qquad X \in T_uBM, \tag{1.2.6}$$

This allows us define the *torsion form* ϑ of a linear connection Γ by

$$\vartheta = D\theta \quad \text{(exterior covariant differential of } \theta\text{)}.$$

Theorem 1.2.7. *[3] Let ω, ϑ be the connection form and the torsion form of a linear connection Γ of M. Then, the first structure equation indicates that:*

$$\vartheta(X, Y) = d\theta(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)],$$

where $X, Y \in T_uBM$ and $u \in BM$.

Before defining a special type of connection, we need to define a Riemannian metric and in addition to that, we define a Riemannian manifold.

Definition 1.2.8. *A Riemannian metric m in a differentiable manifold M is a correspondence that associates to each point $p \in M$ an inner product $m_p(\cdot, \cdot)$ (i.e. a symmetric bilinear positive defined form) in the tangent space T_pM. Furthermore, m is smooth in the sense that for any smooth vector fields X and Y, the function $p \mapsto m_p(X(p), Y(p))$ is smooth. A differential manifold with a given Riemannian metric is called **Riemannian manifold**.*

A Riemannian metric $m(\cdot,\cdot)$ in the manifold M is invariant by the action of the group G if the elements of the group are isometries of the metric, i.e. if for each $g \in G$ and $x \in M$, we have

$$m_{gx}(dg_x u, dg_x v) = m(u,v) \qquad u, v \in T_x M.$$

If $M = G/H$ is a homogeneous space, an invariant metric is absolutely determined by its value at the origin x_0, which is an inner product in $T_{x_0}G/H$ invariant by the isotropy representation.

A necessary condition to the existence of an inner product m_{x_0} is to the image $\rho(H)$ of H by the isotropy representation ρ be a subgroup of the orthogonal group and hence $\rho(H)$ has compact closure. In particular, if H is compact, then G/H admits invariant Riemannian metrics.

Let M be an n-dimensional Riemannian manifold with metric g and $O(M)$ the bundle of orthonormal frames over M. Every connection in $O(M)$ determines a connection in the bundle BM of linear frames (see [3] for details of the proof), that is, a linear connection of M. A linear connection of M is called a **metric connection** if it is thus determined by a connection in $O(M)$.

Among all posible metric connections, the most important is the **Riemannian connection** (sometimes called the **Levi-Civita connection**) which is given by the following theorem

Theorem 1.2.9. *Every Riemannian manifold admits a unique torsion-free metric connection.*

1.2.2 Complex flag manifolds

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and roots set Π. Let Σ be a simple root system and let $\{H_\alpha : \alpha \in \Sigma\} \cup \{X_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Pi\}$ be a fixed (complex) Weyl basis of $\mathfrak{g}$. The compact real form of $\mathfrak{g}$ is denoted by $\mathfrak{u}$.

An element $H_\Theta \in \mathfrak{h}$ is *characteristic* for $\Theta \subset \Sigma$ if $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\}$. A subset Θ defines a parabolic subalgebra $\mathfrak{p}_\Theta$ with parabolic subgroup P_Θ and a flag manifold $\mathbb{F}_\Theta = G/P_\Theta$. In fact, the parabolic subalgebra is $\mathfrak{p}_\Theta = \oplus_{\lambda \geqslant 0}\mathfrak{g}_\lambda$, where λ runs through the non-negative eigenvalues of $\mathrm{ad}(H_\Theta)$. Conversely, starting with $H_0 \in \mathfrak{h}$ we define $\Theta_{H_0} = \{\alpha \in \Sigma : \alpha(H_0) = 0\}$. For instance, $\mathbb{F}_{H_0} = \mathbb{F}_{\Theta_{h_0}}$.

For any subset $Q \subset \Pi$ of simples roots, we have the decomposition $Q = Q^s \cup Q^a$, $Q^s = Q \cap (-Q)$, $Q^a = Q \backslash Q^s$ and we can denote by $\mathfrak{g}(Q)$ the algebra

$$\mathfrak{g}(Q) := \langle [X_\alpha, X_{-\alpha}] = H_\alpha, \alpha \in Q^s \rangle + \sum_{\alpha \in Q} \mathbb{C} X_\alpha \tag{1.2.7}$$

generated by Q.

Fixing a subset $\Theta = \{\alpha_1, \cdots, \alpha_m\} \subset \Sigma$ of simple roots, the set of roots generated by Θ is denoted by $\langle\Theta\rangle$. We denote the intersection $\langle\Theta\rangle \cap \Pi^{\pm}$ by $\langle\Theta\rangle^{\pm}$. For each subset Θ we get the decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}_{\Theta}^{-} \oplus \mathfrak{z}_{\Theta} \oplus \mathfrak{n}_{\Theta}^{+}, \tag{1.2.8}$$

where

$$\mathfrak{n}_{\Theta}^{-} = \mathfrak{g}(\Pi^{-}\backslash\langle\Theta\rangle^{-})$$

and

$$\mathfrak{n}_{\Theta}^{+} = \mathfrak{g}(\Pi^{+}\backslash\langle\Theta\rangle^{+}) \tag{1.2.9}$$

are nilpotent subalgebras of $\mathfrak{g}$. Also,

$$\mathfrak{z}_{\Theta} = \mathfrak{h} + \mathfrak{g}(\Theta)$$

is a reductive subalgebra, with

$$\mathfrak{g}(\Theta) := \mathfrak{g}(\langle\Theta\rangle) = \langle H_{\alpha}\rangle_{\alpha\in\langle\Theta\rangle} + \sum_{\alpha\in\langle\Theta\rangle} \mathbb{C}X_{\alpha}.$$

Then

$$\mathfrak{z}_{\Theta} = \mathfrak{z}_{\mathfrak{g}}(H_{\Theta}) \tag{1.2.10}$$

A **parabolic** subalgebra $\mathfrak{p}$ of a complex Lie algebra $\mathfrak{g}$ is a subalgebra wich contains a Borel subalgebra $\mathfrak{b}$. In [11], Alekseevsky constructs a parabolic subalgebra from a subset $\Theta \subset \Sigma$ of simple roots of $\mathfrak{g}$. A parabolic subalgebra of $\mathfrak{g}$ has the form:

$$\mathfrak{p}_{\Theta} = \mathfrak{h} + \mathfrak{g}(\langle\Theta\rangle^{-} \cup \Pi^{+}) = \mathfrak{z}_{\Theta} + \mathfrak{n}_{\Theta}^{+} = \sum_{\alpha\in\langle\Theta\rangle^{-}} \mathfrak{g}_{\alpha} + \mathfrak{b}. \tag{1.2.11}$$

Now we define flag manifolds associated with any connected complex semi-simple Lie group G. A parabolic subgroup P_{Θ} is the normalizer of $\mathfrak{p}_{\Theta}$ in G, i.e. $P_{\Theta} = \{g \in G : \mathrm{Ad}(g)\mathfrak{p}_{\Theta} = \mathfrak{p}_{\Theta}\}$ and the Lie algebra of P_{Θ} is $\mathfrak{p}_{\Theta}$.

Definition 1.2.10. *A **flag manifold** of a complex semi-simple Lie group G is the quotient $M = G/P_{\Theta}$ of G by a parabolic subgroup P_{Θ}.*

In [11], Alekseevsky proved the following proposition about flag manifolds.

Proposition 1.2.11. [11] Let G be a Lie group with compact real form U and let P_{Θ} be a parabolic subgroup of G. Any flag manifold $M = G/P_{\Theta}$ can be identified with the quocient $U/Z_{\Theta}^{\tau} := U/U_{\Theta}$ of a semi-simple compact Lie group $U \subset G$ modulo $Z_{\Theta}^{\tau} = Z_U(t_0)$, $t_0 \in \mathfrak{u}$ and, hence, with an adjoint orbit $\mathrm{Ad}(U)(t_0)$.

Now we provide a sketch of the proof:

Proof. The decomposition in 1.2.8 induces a decomposition of some open subset of G (connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$) into product of correspondent subgroups:

$$G_{\text{reg}} = N_\Theta^- \cdot Z_\Theta \cdot N_\Theta^+$$

with $Z_\Theta \cap N_\Theta^\pm = N_\Theta^+ \cap N_\Theta^- = \{e\}$.

The parabolic subgroup is

$$P_\Theta = Z_\Theta \cdot N_\Theta^+$$

This decomposition shows that the nilpotent subgroup N_Θ^- acts transitively on the open dense subset $M_{\text{reg}} = G_{\text{reg}}/P_\Theta$ of the flag manifold $M = G/P_\Theta$. Hence, any complex coordinates on N_Θ^- define local complex coordinates on M.

Let τ be the involution on $\mathfrak{g}$ with respect to the real compact form $\mathfrak{u}$ (the same notation is used for the Lie groups). A maximal compact subgroup $G^\tau =: U$ of G acts on M transitively. For a fixed $x \in M$, the isotropy subgroup in U will be denoted by U_x.

$$\mathfrak{u}_x = \mathfrak{u} \cap \mathfrak{p}_\Theta = \mathfrak{p}_\Theta^\tau = \{X \in \mathfrak{p}_\Theta : \tau X = X\}$$

Since the subalgebra $\mathfrak{z}_\Theta = \mathfrak{h} + \mathfrak{g}(\Theta)$ is invariant under the involution τ and

$$\tau(\mathfrak{n}_\Theta^+) = \mathfrak{n}_\Theta^-,$$

since $\tau(X_\alpha) = -X_{-\alpha}$ for all $\alpha \in \Pi$. Then

$$\mathfrak{u}_\Theta := \mathfrak{p}_\Theta^\tau = \mathfrak{z}_\Theta^\tau = \mathfrak{h}^\tau + (\mathfrak{g}(\Theta))^\tau \tag{1.2.12}$$

and is a compact form of the subalgebra $\mathfrak{z}_\Theta$. Therefore $\mathfrak{u}_\Theta = \mathfrak{z}_\Theta \cap \mathfrak{u}$ is the centralizer of a comutative subalgebra $\mathfrak{h}_\Theta := \mathfrak{h} \cap \mathfrak{u}_\Theta$. On the other hand, $\mathfrak{h}_\Theta$ generates a torus T in the compact connected semi-simple Lie group U. The centralizer of a torus in a connected compact semi-simple Lie group is connected and there exists an element t_0 in $\sqrt{-1}\mathfrak{h}_\mathbb{R}$ such that

$$U_\Theta = Z_U(\mathfrak{h}_\Theta) = Z_U(T) = Z_U(t_0). \tag{1.2.13}$$

□

In the Table A.2, we found all complex flag manifolds classified in [15]. We denote the flag manifold G/P_Θ by $\mathbb{F}_\Theta$.

In [1] the cotangent bundle of the complex flag manifold $\mathbb{F}_\Theta$ is realized by the adjoint orbit $G \cdot H_\Theta :=\text{Ad}(G)H_\Theta$ (the realization of the cotangent bundle is with the coadjoint orbit, however in the Example 1.3.1 this orbit is identify with the adjoint orbit) and is also realized by the homogeneous space G/Z_Θ, where Z_Θ is the centralizer of H_Θ in G by the adjoint representation.

Theorem 1.2.12. *[1] The adjoint orbit $\mathcal{O}(H_\Theta) = Ad(G) \cdot H_\Theta \approx G/Z_\Theta$ of the characteristic element H_Θ is a C^∞ vector bundle over $\mathbb{F}_\Theta$ that is isomorphic to the cotangent bundle $T^*\mathbb{F}_\Theta$. Moreover, we can write down a diffeomorphism $\iota : Ad(G) \cdot H_\Theta \to T^*\mathbb{F}_\Theta$ such that*

(1) ι is equivariant with respect to the actions of U, that is, for all $u \in U$,

$$\iota \circ \mathrm{Ad}(u) = \tilde{u} \circ \iota,$$

where $\tilde{u}$ is the lifting to $T^*\mathbb{F}_\Theta$ (via the differential) of the action of u on $\mathbb{F}_\Theta$; and

(2) the pullback of the canonical simplectic form on $T^*\mathbb{F}_\Theta$ by ι is the (real) Kirillov-Kostant-Souriaux form on the orbit.

The projection $\pi : G \cdot H_\Theta \to \mathbb{F}_\Theta$ is obtained via the action of G. The canonical fibration $gZ_\Theta \in G/Z_\Theta \mapsto gP_\Theta \in \mathbb{F}_\Theta$ has the fiber P_Θ/Z_Θ. In terms of the adjoint action the fiber is $\mathrm{Ad}(P_\Theta) \cdot H_\Theta$, which is the affine subspace $H_\Theta + \mathfrak{n}_\Theta^+$ (see [1]).

In [1] it is also described the isomorphism of the adjoint orbit $G \cdot H_\Theta$ with the cotangent bundle $T^*\mathbb{F}_\Theta$. The tangent space $T_{b_\Theta}\mathbb{F}_\Theta$ of the flag manifold $\mathbb{F}_\Theta$ at the origin b_Θ can be identified with $\mathfrak{n}_\Theta^-$ and the isotropy representation $U_\Theta \to \mathrm{Gl}(T_{b_\Theta}\mathbb{F}_\Theta)$ becomes the restriction of the adjoint representation. The subspace $\mathfrak{n}_\Theta^+$ is isomorphic to the dual $(\mathfrak{n}_\Theta^-)^*$ of $\mathfrak{n}_\Theta^-$ via the Cartan-Killing form $\mathcal{K}(\cdot,\cdot)$ of $\mathfrak{g}$. Thus, the map $X \in \mathfrak{n}_\Theta^+ \mapsto \mathcal{K}(X,\cdot) \in (\mathfrak{n}_\Theta^-)^*$ is an isomorphism.

1.3 Symplectic manifolds

In [1], each element of the Lie algebra $\mathfrak{g}$ is associated with a Hamiltonian vector field on $T^*\mathbb{F}_\Theta$ and they use the fact that $\mathfrak{g}$ is semi-simple to interchange the representations coadjoint and adjoint via the Cartan-Killing form and thus define the moment map $T^*\mathbb{F}_\Theta \to \mathfrak{g}$ of the Hamiltonian action of G on the cotangent bundle. That moment map is a diffeomorphism transforming the canonical symplectic form of $T^*\mathbb{F}_\Theta$ in the Kirillov-Kostant-Souriaux form on the adjoint orbit $\mathrm{Ad}(G)H_\Theta$. We also study the cotangent bundle of a flag manifold as a symplectic manifold. Some definitions in [12] and [16] to understand the symplectic manifolds are:

Definition 1.3.1. *A **symplectic form** ω in a vector space V ($dimV < \infty$) is a non-degenerate asymmetric bilinear form, i.e. for every $x \in V$, there exists $y \in V$ such that $\omega(x, y) \neq 0$.*

A **symplectic form** in a differentiable manifold M is a closed differential $2-$form ω ($d\omega = 0$) and non-degenerate.

A manifold M provided of a symplectic form is called **symplectic manifold**.

Definition 1.3.2. *Let (M,ω) be a $2n$-dimensional symplectic manifold. A submanifold Y of M is a* ***Lagrangian submanifold*** *if, at each $p \in Y$, $\omega_p \mid_{T_pY}= 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$.*

1.3.1 Coadjoint representation

The coadjoint representation $\mathrm{Ad}^*(g)$ of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is defined by $\mathrm{Ad}^*(g)\alpha = \alpha\circ\mathrm{Ad}(g^{-1})$, $g \in G$ and $\alpha \in \mathfrak{g}^*$. Its infinitesimal representation $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $\mathrm{ad}^*(X)\alpha = -\alpha\circ\mathrm{ad}(X)$. The group G acts on $\mathfrak{g}^*$ by the representation Ad^*. For this action, the induced vector fields $\tilde{X}$, $X \in \mathfrak{g}$, are given by $\tilde{X} =\mathrm{ad}^*(X)$.

For $\alpha \in \mathfrak{g}^*$, the coadjoint orbit $\mathrm{Ad}^*(G)\alpha$ is identified with the homogeneous space G/Z_α, where Z_α is the closed subgroup

$$Z_\alpha = \{g \in G : \alpha \circ \mathrm{Ad}(g^{-1}) = \alpha\}.$$

The Lie algebra $\mathfrak{z}_\alpha$ of Z_α is given by $\mathfrak{z}_\alpha = \{X \in \mathfrak{g} : \alpha \circ \mathrm{ad}(X) = 0\}$. The tangent space of $\mathrm{Ad}^*(G)\alpha$ at α is given by

$$T_\alpha(\mathrm{Ad}^*(G)\alpha) = \{\mathrm{ad}^*(X)\alpha : X \in \mathfrak{g}\}.$$

The symplectic form of **Kirillov-Kostant-Souriaux** Ω (KKS) in the coadjoint orbit $\mathrm{Ad}^*(G)\alpha$ is given by the expression

$$\Omega_\alpha(\tilde{X}(\alpha), \tilde{Y}(\alpha)) = \alpha[X, Y] \qquad X, Y \in \mathfrak{g}. \tag{1.3.1}$$

In [5] is proved that Ω_α is an invariant symplectic form in $\mathrm{Ad}^*(G)\alpha = G/Z_\alpha$. In addition, the form Ω at $\beta =\mathrm{Ad}^*(g)\alpha = \mathrm{Ad}^*(G)\alpha$ is given by

$$\begin{aligned}
\Omega_\beta(\tilde{X}(\beta), \tilde{Y}(\beta)) =&\Omega_\alpha(dg_\beta^{-1}\tilde{X}(\beta), dg_\beta^{-1}\tilde{Y}(\beta))\\
=&\Omega_\alpha(\widetilde{\mathrm{Ad}(g^{-1})X}(\alpha), \widetilde{\mathrm{Ad}(g^{-1})Y}(\alpha))\\
=&\alpha[\mathrm{Ad}(g^{-1})X, \mathrm{Ad}(g^{-1})Y]\\
=&\alpha \circ \mathrm{Ad}(g^{-1})[X, Y].
\end{aligned}$$

This means

$$\Omega_\beta(\tilde{X}(\beta), \tilde{Y}(\beta)) = \beta[X, Y]$$

has the same expression used to define Ω_α.

Example 1.3.1. [5] The Cartan-Killing form $\mathcal{K}(\cdot,\cdot)$ of a Lie algebra $\mathfrak{g}$ of the semi-simple group G is non-degenerate and define an isomorphism $W : \mathfrak{g} \to \mathfrak{g}^*$ by $W(X)(\cdot) = \mathcal{K}(X, \cdot)$. This isomorphism exchanges the adjoint and coadjoint representations, i.e. $W\mathrm{Ad}(g) =\mathrm{Ad}^*(g)W$ for every $g \in G$, since $\mathcal{K}$ is Ad-invariant. Hence W applies diffeomorphically the adjoint orbits in the coadjoint orbits, this allows transform the KKS symplectic forms Ω in the symplectic forms $W^*\Omega$ in the adjoint orbits. In the semi-simple case, the adjoint representation behaves like the coadjoint representation.

As a comment, we indicate a Poisson structure associated with Lie algebras. If $\mathfrak{g}$ is any Lie algebra, its dual $\mathfrak{g}^*$ carries the *Lie-Poisson structure*

$$\{F, G\}(\mu) = \langle \mu, [\frac{\delta F}{\delta u}, \frac{\delta G}{\delta \mu}] \rangle.$$

Here, $\frac{\delta F}{\delta u}$ and $\frac{\delta G}{\delta \mu}$ are the differentials of F and G considered as maps into $\mathfrak{g}$ rather than $\mathfrak{g}^{**}$, and $\langle , \rangle$ is the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$. If $X_1, \cdots, X_n$ form a basis for $\mathfrak{g}$ and $x_1, \cdots, x_n$ are the corresponding coordinate functions on $\mathfrak{g}^*$, the basic bracket relations are

$$\{x_i, x_j\} = \sum_k c_{ijk} x_k,$$

where the c_{ijk}'s are the structure constants of $\mathfrak{g}$. Conversely, any Poisson structure of the form as the last equation arises in this way from a Lie algebra. For the Lie-Poisson structure on $\mathfrak{g}$, the orbits of the coadjoint representation of a connected Lie group whose Lie algebra is $\mathfrak{g}$ has symplectic structure defined by Kirillov-Kostant-Souriau.

1.4 Symmetric spaces

In this section we shall use Kobayashi and Nomizu's books [3] and [2] as main references.

Let M be a $n-$dimensional manifold with an affine connection. The symmetry s_x at a point $x \in M$ is a diffeomorphism of a neighborhood U onto itself which sends $\exp X, X \in T_x M$, into $\exp(-X)$. Since the symmetry at x defined in one neighborhood U of x and the symmetry at x defined in another neighborhood V of x coincide in $U \cap V$, we can legitimately speak of the *symmetry* at x. If $\{x^1, \cdots, x^n\}$ is a normal coordinate system with origin at x, then s_x sends $(x^1, \cdots, x^n)$ into $(-x^1, \cdots, -x^n)$. The differential of s_x at x is equal to $-I_x$, where I_x is the identity transformation of $T_x M$. The symmetry s_x is involutive in the sense that $s_x \circ s_x$ is the identity transformation of a neighborhood of x. If s_x is an affine transformation for every $x \in M$, then M is said to be **affine locally symmetric**.

A manifold M with an affine connection is said to be **affine symmetric** if, for each $x \in M$, the symmetry s_x can be extended to a global affine transformation of M.

Theorem 1.4.1. *[3] A complete, simply connected, affine locally symmetric space is affine symmetric.*

Theorem 1.4.2. *[3] On every affine symmetric space, the group of affine transformations is transitive.*

Kobayashi and Nomizu, in [2], proved that the group of affine transformations $\mathfrak{U}(M)$ of M is known to be a Lie group. Let G denote the identity component of the $\mathfrak{U}(M)$. The identity component of a Lie group acting transitively on a manifold M is itself transitive on M, an affine symmetric space M may be written as a homogeneous space G/H.

Theorem 1.4.3. *[3] Let G be the largest connected group of affine transformations of an affine symmetric space M and H the isotropy subgroup of G at a fixed point o of M so that $M = G/H$. Let s_o be the symmetry of M at o and σ the automorphism of G defined by*

$$\sigma(g) = s_o \circ g \circ s_o^{-1} \quad for\ g \in G. \tag{1.4.1}$$

Let G_σ be the closed subgroup of G consisting of elements fixed by σ. Then H lies between G_σ and the identity component of G_σ.

From Theorem 1.4.3, Kobayashi and Nomizu suggests the following definiton. A *symmetric space* is a triple (G, H, σ) consisting of a connected Lie group G, a closed subgroup H of G and an involutive automorphism σ of G such that H lies between G_σ and the identity component of G_σ, where G_σ denotes the closed subgroup of G consisting of all elements left fixed by σ.

We define now an infinitesimal version of a symmetric space. A *symmetric Lie algebra* is a triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ consisting of a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and an involutive automorphism σ of $\mathfrak{g}$ such that $\mathfrak{h}$ consists of all elements of $\mathfrak{g}$ which are left fixed by σ.

Every symmetric space (G, H, σ) gives rise to a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ in a natural manner; $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively, and the automorphism σ of $\mathfrak{g}$ is the one induced by the automorphism σ of G by the automorphism σ of G. Conversely, if $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is a symmetric Lie algebra and if G is connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, then the automorphism σ of $\mathfrak{g}$ induces an automorphism σ of G and, for any subgroup H lying between G_σ and the identity component of G_σ, the triple is a symmetric space. The pair (G, H) is called a *symmetric pair.*

A triple (G', H', σ') is called a *symmetric subspace* of a symmetric space (G, H, σ) if G' is a Lie subgroup of G invariant by σ, if $H' = G' \cap H$ and if σ' is the restriction of σ to G'.

Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra. Since σ is involutive, its eigenvalues as a linear transformation of $\mathfrak{g}$ are 1 and -1 and $\mathfrak{h}$ is the eigenspace for 1. Let $\mathfrak{m}$ be the eigenspace of -1. The decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \tag{1.4.2}$$

is called the **canonical decomposition** of $(\mathfrak{g}, \mathfrak{h}, \sigma)$.

Proposition 1.4.4. [3] If $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the canonical decomposition of a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$, then

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{1.4.3}$$

Proposition 1.4.5. [3] Let (G, H, σ) be a symmetric space and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ its symmetric Lie algebra. If $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the canonical decomposition of $(\mathfrak{g}, \mathfrak{h}, \sigma)$, then

$$\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}. \tag{1.4.4}$$

The following theorems guarantee the existence of an invariant affine connection over the homogeneous symmetric space G/H.

Theorem 1.4.6 (Correspondence)**.** *[3] There is a one-to-one correspondence between the set of G-invariant connections in G (connection in the bundle $G(M, H)$ which is invariant by the left translations of G) and the set of linear mappings $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{h}$ such that*

$$\Lambda_{\mathfrak{m}}(Ad(h)X) = Ad(h)(\Lambda_{\mathfrak{m}}(X)), \; X \in \mathfrak{m}, h \in H, \tag{1.4.5}$$

The correspondence is given by $\Lambda : \mathfrak{g} \to \mathfrak{g}$

$$\Lambda(X) = \begin{cases} X, & X \in \mathfrak{h}, \\ \Lambda_{\mathfrak{m}}(X), & X \in \mathfrak{m} \end{cases} \tag{1.4.6}$$

To a G-invariant connection in G with connection form ω there corresponds the linear mapping defined by

$$\Lambda(X) = \omega_{u_0}(\tilde{X}) \qquad X \in \mathfrak{g},$$

where $\tilde{X}$ denotes the natural lift to G of a vector field $X \in \mathfrak{g}$ of $M = G/H$ and $u_0 \in G$ fixed.

The invariant connection corresponding to $\Lambda_{\mathfrak{m}} = 0$ is called **canonical connection** of (G, H, σ) or G/H

Theorem 1.4.7. *[natural torsion-free connection] [3] Every reductive homogeneous space $M = G/H$ admits a* ***unique*** *torsion-free G-invariant affine connection having the same geodesics as the canonical connection. It is defined by*

$$\Lambda_{\mathfrak{m}}(X)(Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}. \tag{1.4.7}$$

The invariant connection defined in the Theorem 1.4.7 shall called by the **natural torsion-free connection** on G/H (with respect to the decompostion $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$). If (G, H, σ) is a symmetric space, then the homogeneous space G/H is reductive with respect to the canonical decomposition, then $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and the canonical connection coincides with the natural torsion-free connection.

Theorem 1.4.8. *[3] Let (G, H, σ) be a symmetric space. The canonical connection is the only affine connection on $M = G/H$ which is <u>invariant by the symmetries</u> of M.*

The canonical connection makes it possible to find geodesics in symmetric spaces.

Theorem 1.4.9. *[3] With respect to the canonical connection of a symmetric space (G, H, σ), the homogeneous space $M = G/H$ with origin o is a (complete) affine symmetric space with symmetries s_x and possesses the following properties:*

(1) For each $X \in \mathfrak{m}$, define the curves $f_t = \exp tX$ in G and $x_t = f_t o$ in $M = G/H$. Then the curve x_t is a geodesic. Conversely, every geodesic in M starting from o is of the form $f_t \cdot o = \exp tX \cdot o$ for some $X \in \mathfrak{m}$.

(2) The canonical connection is complete.

In [2], an *indefinite Riemannian metric* is defined as a symmetric covariant tensor field m of degree 2 which is non-degenerate at each $x \in M$, that is, $m(X, Y) = 0$ for all $Y \in T_x(M)$ implies $X = 0$. The difference with the Riemannian metric defined in the section 1.2.8 is that an indefinite Riemannian metric associates $p \in M$ to non-degenerate bilinear symmetric $2-$form $m_p(\cdot, \cdot)$ in T_pM instead of an inner product.

Example 1.4.1. An indefinite Riemannian metric on a non-compact, semi-simple Lie group is given by the Cartan-Killing form.

Theorem 1.4.10. *[3] [Indefinite Riemannian metric] Let (G, H, σ) be a symmetric space with G semi-simple and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition. Then*

(1) the restriction of the Cartan-Killing form $\mathcal{K}(\cdot, \cdot)$ of $\mathfrak{g}$ to $\mathfrak{m}$ defines a G-invariant (indefinite) Riemannian metric on G/H and

(2) this (indefinite) Riemannian metric induces the canonical connection on G/H.

In Chapter IV of [2] there is a theorem saying that every Riemannian manifold admits a unique metric connection with vanishing torsion and that connection is called **Riemannian connection** (sometimes called the Levi-Civita connection). Since the canonical connection is also torsion-free by Theorem 1.4.7, it must be the Riemannian connection.

A homogeneous space $M = G/H$ with a G-invariant indefinite Riemannian metric is said to be **naturally reductive** if it admits an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying the condition

$$B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y) = 0, \qquad \forall X, Y, Z \in \mathfrak{m} \tag{1.4.8}$$

where $B(\cdot, \cdot)$ is an $\mathrm{Ad}(H)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$. Therefore, we can say every symmetric space is naturally reductive since $[X, Y]_{\mathfrak{m}} = 0$ for all $X, Y \in \mathfrak{m}$.

Then the affine symmetric space G/H, with G semi-simple, is naturally reductive with respect to the $G-$invariant indefinite Riemannian metric defined by the Cartan-Kililng form. Furthermore, in [3] is proved that the indefinite Riemannian metric on $\mathfrak{m}$ is a $\mathrm{Ad}(H)$-invariant, non-degenerate symmetric bilinear form that satisfies

$$\mathcal{K}([Z,X],Y)_o + \mathcal{K}(X,[Z,Y])_o = 0, \quad X, Y \in \mathfrak{m}, Z \in \mathfrak{h}, \tag{1.4.9}$$

where o is the origin of G/H. This metric is invariant by both right and left translations.

1.4.1 Riemannian symmetric spaces

A Riemannian manifold M is said to be *Riemannian locally (or globally) symmetric* if it is affine locally (or globally) symmetric with respect to the Riemannian connection.

Theorem 1.4.11. *[3] Let M be a Riemannian symmetric space, G the largest connected group of isometries of M and H the isotropy subgroup of G at a point o of M. Let s_o be the symmetry of M at o and σ the involutive automorphism of G defined by $\sigma(g) = s_o \circ g \circ s_o^{-1}$ for $g \in G$. Let G_σ be the closed subgroup of G consisting of elements fixed by σ. Then*

1. *G is transitive on M so that $M = G/H$;*
2. *H is compact and lies between G_σ and the identity component of G_σ.*

If (G, H, σ) is a symmetric space with compact $\mathrm{Ad}_{\mathfrak{g}}(H)$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition. Since $\mathfrak{h}$ and $\mathfrak{m}$ are invariant by $\mathrm{Ad}_{\mathfrak{g}}(H)$ (see Proposition 1.4.5) and since $\mathrm{Ad}_{\mathfrak{g}}(H)$ is compact, $\mathfrak{g}$ admits an $\mathrm{Ad}_{\mathfrak{g}}(H)$-invariant inner product with respect to which $\mathfrak{h}$ and $\mathfrak{m}$ are perpendicular to each other. This inner product restricted to $\mathfrak{m}$ induces a G-invariant Riemannian metric on G/H. By Theorem 1.4.10, any G-invariant Riemannian metric on G/H defines the canonical connection of G/H. Hence G/H is a Riemannian symmetric space.

1.4.1.1 Orthogonal symmetric Lie algebras

Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra. Consider the Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ of linear endomorphisms of $\mathfrak{g}$ consisting of $\mathrm{ad}X$ where $X \in \mathfrak{h}$. If the connected Lie group of linear transformations of $\mathfrak{g}$ generated by $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ is compact, then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is called an **orthogonal symmetric Lie algebra**. If (G, H, σ) is a symmetric space such that H has a finite number of connected components and if $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is its symmetric Lie algebra, then $\mathrm{Ad}_{\mathfrak{g}}(H)$ is compact if and only if $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is an orthogonal symmetric Lie algebra.

Proposition 1.4.12. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an orthogonal symmetric Lie algebra with $\mathfrak{g}$ simple. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition. Then

1. $\mathrm{ad}\mathfrak{h}$ is irreducible on $\mathfrak{m}$.
2. The Cartan-Killing form $\mathcal{K}$ of $\mathfrak{g}$ is (negative or positive) defined on $\mathfrak{m}$.

A symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is said to be *effective* if $\mathfrak{h}$ contains no non-zero ideal of $\mathfrak{g}$.

An effective symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is **irreducible** if $\mathrm{ad}([\mathfrak{m}, \mathfrak{m}])$ is irreducible on $\mathfrak{m}$. If $\mathfrak{g}$ is simple, then the symmetric Lie algebra is irreducible.

In general, an orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ with $\mathfrak{g}$ semi-simple is said to be of **compact type** or **non-compact type** according as the Cartan-Kiling form $\mathcal{K}$ of $\mathfrak{g}$ is negative-defined or positive-defined on $\mathfrak{m}$.

1.4.2 The dual symmetric space

Consider an arbitrary (real) symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ with canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

If we denote by $\mathfrak{g}^c$ and $\mathfrak{h}^c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and by σ^c the involutive automorphism of $\mathfrak{g}^c$ induced by σ, then $(\mathfrak{g}^c, \mathfrak{h}^c, \sigma^c)$ is a symmetric Lie algebra and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is a (real) symmetric subalgebra of $(\mathfrak{g}^c, \mathfrak{h}^c, \sigma^c)$. We set

$$\mathfrak{g}^* = \mathfrak{h} + \sqrt{-1}\mathfrak{m}. \tag{1.4.10}$$

If we set $\sigma^* = \sigma^c \mid \mathfrak{g}^*$, then we obtain a symmetric subalgebra $(\mathfrak{g}^*, \mathfrak{h}, \sigma^*)$ of $(\mathfrak{g}^c, \mathfrak{h}^c, \sigma^c)$, which is called the **dual of** $(\mathfrak{g}, \mathfrak{h}, \sigma)$.

Theorem 1.4.13. *[3] The irreducible orthogonal symmetric Lie algebras of compact type are divided into the following two classes:*

(I) $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *where* $\mathfrak{g}$ *is a simple Lie algebra of compact type;*

(II) $(\mathfrak{g}+\mathfrak{g}, \Delta\mathfrak{g}, \sigma)$ *where* $\mathfrak{g}$ *is a simple Lie algebra of compact type,* σ *maps* $(X, Y) \in \mathfrak{g}+\mathfrak{g}$ *into* $(Y, X) \in \mathfrak{g}+\mathfrak{g}$ *and* $\Delta\mathfrak{g}$ *is the diagonal of* $\mathfrak{g}+\mathfrak{g}$.

The irreducible orthogonal symmetric Lie algebras of non-compact type are divided into the following two classes:

(III) $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *where* $\mathfrak{g}$ *is a simple Lie algebra of non-compact type which does not admit a compatible complex structure;*

(IV) $(\mathfrak{g}^c, \mathfrak{g}, \sigma)$ *where* $\mathfrak{g}$ *is a simple Lie algebra of compact type,* $\mathfrak{g}^c$ *denotes the complexification of* $\mathfrak{g}$*, and* σ *is the complex conjugation in* $\mathfrak{g}^c$ *with respect to* $\mathfrak{g}$.

An orthogonal symmetric Lie algebras of type (I) is the dual of an orthogonal symmetric Lie algebra of type (III). An orthogonal symmetric Lie algebra of type (II) is the dual of an orthogonal symmetric Lie algebra of type (IV).

The types of symmetric spaces are related to their geometric properties as follows.

Theorem 1.4.14. *[3] Let (G, H, σ) be a symmetric space with $\mathrm{Ad}_G(H)$ compact and let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be its orthogonal symmetric Lie algebra. Take any G-invariant Riemannian metric on G/H. Then we have.*

(1) If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of compact type, then G/H is a compact Riemannian symmetric space with non-negative sectional curvature and positive-defined Ricci tensor;

(2) If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of non-compact type, then G/H is a simply connected non-compact Riemannian symmetric space with non-positive sectional curvature and negative-definite Ricci tensor and is diffeomorphic to a Euclidean space.

The Riemannian symmetric manifolds with orthogonal Lie algebras of compact type of class (I) and of non-compact type of class (III) are evidence in the Table A.3.

2 Symetric spaces on the adjoint orbit

Let $\mathfrak{g}$ be a complex simple Lie algebra with compact real form $\mathfrak{u}$ and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let U/K be a complex flag manifold and assume that the Lie algebra of U is $\mathfrak{u}$ and the Lie algebra of K is $\mathfrak{k}$. Comparing the tables A.2 and A.3, we can see that the symmetric spaces which are complex flag manifolds are $SU(p+q)/S(U(p)\times U(q))$, $Sp(l)/U(l)$ and $SO(2l)/U(l)$ of types A,C and D respectively. These flag manifolds are Riemannian symmetric manifolds and $(\mathfrak{u}, \mathfrak{k}, \sigma)$ is an orthogonal Lie algebra of compact type and class (I).

We call **flag symmetric spaces** to the symmetric spaces which are also complex flag manifolds. We shall see these spaces as homogeneous spaces with the canonical connection, which is also an invariant connection.

Throughout chapters $3, 4$ and 5, we shall prove Theorem 2.0.1, i.e. that the cotangent bundle of a flag symmetric space is a symmetric space.

Theorem 2.0.1. *Let $\mathbb{F}_\Theta$ be a flag symmetric space identified with the homogeneous space*

$$\mathbb{F}_\Theta = U/U_\Theta, \tag{2.0.1}$$

as in Proposition 1.2.11 . Then, there exists an element H_Θ in a fixed Cartan subalgebra of the compact Lie algebra $\mathfrak{u}$ such that for $\sigma := C_{\exp H_\Theta}$,

$$U_\Theta = \{u \in U : C_{\exp H_\Theta}(u) = u\}. \tag{2.0.2}$$

Moreover, let G be the complex Lie group such that U is its compact real form. Then, the cotangent bundle of $\mathbb{F}_\Theta$ has a realization as a symmetric space.

If we use also the notation σ for $\mathrm{Ad}_{\mathfrak{u}}(\exp H_\Theta)$ and $\mathrm{Ad}_{\mathfrak{g}}(\exp H_\Theta)$, then $(\mathfrak{u}, \mathfrak{u}_\Theta, \sigma)$ and $(\mathfrak{g}, \mathfrak{z}_{\mathfrak{g}}(H_\Theta), \sigma)$ are symmetric Lie algebras.

This theorem shows that one element H_Θ in a Cartan subalgebra determines the automorphism σ for two symmetric spaces and then for two symmetric Lie algebras.

In this chapter we shall use the following notations: Let $\mathfrak{g}$ be a complex simple Lie algebra with compact real form $\mathfrak{u}$. Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$ and U the connected compact Lie subgroup of G with Lie algebra $\mathfrak{u}$. The flag symmetric spaces shall be denoted by $\mathbb{F} = U/U_\Theta$, as in Proposition 1.2.11. The σ-fixed points set in $\mathfrak{g}$ is denoted by

$$\mathfrak{g}^\sigma = \mathfrak{z}_\Theta = \mathfrak{z}_{\mathfrak{g}}(H_\Theta). \tag{2.0.3}$$

In addition, we shall use the notation used in the section 1.2.2 for the flag symmetric spaces.

Moreover, the tangent space of the homogeneous space G/G^σ at the origin can be identify by the (direct) sum

$$\mathfrak{m}_G := \mathfrak{n}_\Theta^- + \mathfrak{n}_\Theta^+$$

Hence, the canonical decomposition of the complex Lie algebra is

$$\mathfrak{g} = \mathfrak{z}_\Theta + \mathfrak{m}_G. \tag{2.0.4}$$

Since $G^\sigma = Z_\Theta$, the adjoint orbit of G in H_Θ is G/G^σ (see 1.2.12) and its tangent space at H_Θ is isomorphic to the vector space $\mathfrak{n}_\Theta^- + \mathfrak{n}_\Theta^+$.

Analyzing the flag symmetric space U/U_Θ from Theorem 2.0.1, we have that $U_\Theta = Z_U(H_\Theta)$ and by (1.2.13) and (1.2.12), the compact real form of $\mathfrak{g}$ has the canonical decomposition:

$$\mathfrak{u} = \mathfrak{u}_\Theta + \mathfrak{m}, \tag{2.0.5}$$

where

$$\mathfrak{m} = \sum_{\alpha \in \Pi^+ \setminus \langle \Theta \rangle^+} \mathfrak{u}_\alpha, \tag{2.0.6}$$

with $\mathfrak{u}_\alpha = \mathrm{Span}_{\mathbb{R}}(A_\alpha, Z_\alpha)$ as in 1.1.5, is isomorphic to the tangent space of the flag in the origin. Then,

$$\mathfrak{m}_G = \mathfrak{m}^{\mathbb{C}} \tag{2.0.7}$$

where we identify $\mathfrak{m}_G$ with the complexification $\mathfrak{m}^{\mathbb{C}}$ of the tangent space $T_{b_\Theta}\mathbb{F}_\Theta$ of $\mathbb{F}_\Theta$ at the point $b_\Theta := eU_\Theta$. Then $\mathfrak{m}_G^\tau = \{X \in \mathfrak{g}, \tau X = X\}$ is identified with $T_{b_\Theta}\mathbb{F}_\Theta$ and the isotropy representation of U_Θ on $(T_{b_\Theta}\mathbb{F}_\Theta)^{\mathbb{C}}$ can be identified with the restriction of the adjoint representation $\mathrm{Ad}_{\mathfrak{g}}(U_\Theta)$ to $\mathfrak{m}_G$.

In the next chapters, let us characterize the flag symmetric spaces. They are only of three classes. In the Chapters 3, 4 and 5 is proved that the involutive inner automorphism σ mentioned in Theorem 2.0.1 is a generalized Cartan involution (extension of a Cartan involution from a noncompact real form to a complex Lie algebra) extended from a noncompact real form of $\mathfrak{g}$, denoted by $\mathfrak{g}_0$, with Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{s}, \tag{2.0.8}$$

such that $\mathfrak{k} = \mathfrak{u}_\Theta$. By 1.1.11 and $\mathfrak{g} = \mathfrak{g}_0 + \sqrt{-1}\mathfrak{g}_0$, we get $\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{s}$ and the vector subspace $\mathfrak{m}$ is isomorphic to the symmetric part of $\mathfrak{u}$ and we get the identification

$$\mathfrak{m} \cong \sqrt{-1}\mathfrak{s}. \tag{2.0.9}$$

Hence,

$$\mathfrak{g}^\sigma = \mathfrak{k} + \sqrt{-1}\mathfrak{k}, \tag{2.0.10}$$

since σ is the Cartan involution on $\mathfrak{g}_0$. The canonical decomposition of $(\mathfrak{g}, \mathfrak{g}^\sigma, \sigma)$ is

$$\mathfrak{g} = (\mathfrak{k} + \sqrt{-1}\mathfrak{k}) + (\mathfrak{s} + \sqrt{-1}\mathfrak{s}) \tag{2.0.11}$$

and the canonical decomposition of $(\mathfrak{u}, \mathfrak{u}^\sigma, \sigma)$ is:

$$\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{s} = \mathfrak{u}_\Theta + \sqrt{-1}\mathfrak{s} \tag{2.0.12}$$

Remark 1. A realization of the adjoint orbit in [1] is in Theorem 1.2.12. The adjoint orbit is realized as the associated vector bundle $U \times_\rho \mathfrak{n}_\Theta^+$ with principal bundle $U \to U_\Theta$, where ρ is the adjoint representation restricted to U_Θ on $\mathfrak{n}_\Theta^+$. Additionally, $G \cdot H_\Theta = \bigcup_{u \in U} \mathrm{Ad}(u)(H_\Theta + \mathfrak{n}_\Theta^+) \subset \mathfrak{g}$.

Since Lie groups here are all semi-simple, the Cartan Killng form $\mathcal{K}(\cdot,\cdot)$ is a non-degenerate bilinear form. Since it is invariant by automorphisms, then it is $\mathrm{Ad}(H)$ invariant, for H the isotropy subgroup both for the symmetric flag spaces and for the adjoint orbits. In the case of symmetric flag spaces, the Cartan Killing form is negative definite, then $-\mathcal{K}(\cdot,\cdot)$ generates a U-invariant Riemannian metric on $\mathbb{F}_\Theta = U/U_\Theta$ and for the cotangent bundle $T^*\mathbb{F}_\Theta \cong G/Z_\Theta$ there no exists an $\mathrm{Ad}(Z_\Theta)$-invariant (definite) metric, because of Z_Θ is not a group of isometries.

The theory studied here is a generalization of the examples 2.0.1 and 2.1.1. We shall show the case of an unidimensional real flag $SO(2)/\{\pm \mathrm{Id}\}$ which is a symmetric space and its dual symmetric space contained in the bidimensional adjoint orbit diffeomorphic to the cotangent bundle of $SO(2)/\{\pm \mathrm{Id}\}$.

Example 2.0.1. Let us consider the base of $\mathfrak{sl}(2,\mathbb{R})$:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.0.13}$$

The compact Lie group

$$SO(2) = \left\{ \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} : s \in \mathbb{R} \right\}$$

determines the maximal flag manifold

$$\begin{aligned} SO(2) \cdot B = & \left\{ \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \cdot B = \sin(2s)A + \cos(2s)B + 0C : s \in \mathbb{R} \right\} \\ = & S^1 \end{aligned} \tag{2.0.14}$$

contained in the real vector space $\mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2)$ =span$\{A, B\}$. This flag manifold is a symmetric space since is the 1-sphere S^1.

The adjoint orbit $\mathrm{Sl}(2,\mathbb{R})\cdot B := \mathrm{Ad}(\mathrm{Sl}(2,\mathbb{R}))B$ is the set of matrices $xA+yB+zC$ for $x,y,z\in\mathbb{R}$, with eigenvalues ± 1, i.e.

$$\det\begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.0.15}$$

Then, the adjoint orbit $\mathrm{Sl}(2,\mathbb{R})\cdot B$ is the one-sheeted hyperboloid

$$x^2+y^2-z^2=1. \tag{2.0.16}$$

This manifold has a realization as the homogeneous space $\mathrm{Sl}(2,\mathbb{R})/\mathrm{S}(\mathrm{Gl}(1,\mathbb{R})\times\mathrm{Gl}(1,\mathbb{R}))$ $\cong\mathrm{Sl}(2,\mathbb{R})/\mathbb{R}^*$, with $\mathbb{R}^*=\mathbb{R}\backslash\{0\}$.

The cotangent bundle of S^1 is also the union of fibers $T^*S^1 = \bigcup\limits_{k\in SO(2)} k\cdot(B+\mathfrak{n}^+)$, with $\mathfrak{n}^+$ being the Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ defined by the upper triangular matrices. For each element

$$k=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2), \quad t\in\mathbb{R}, \tag{2.0.17}$$

the fiber $k\cdot(B+\mathfrak{n}^+)$ is the one-dimensional affine vector space

$$\begin{aligned}
\mathrm{Ad}(k)(B+\mathfrak{n}^+) :=& \left\{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\cdot\left[B+\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}\right] : r\in\mathbb{R}\right\} \\
=& \left\{\begin{pmatrix} \cos(2t)-\frac{r}{2}\sin(2t) & -\sin(2t)+r(\cos t)^2 \\ -\sin(2t)-r(\sin t)^2 & -\cos(2t)-\frac{r}{2}\sin(2t) \end{pmatrix} : r\in\mathbb{R}\right\} \\
=&\{(-\sin(2t)+\frac{r}{2}\cos(2t))A \\
&+(\cos(2t)+\frac{r}{2}\sin(2t))B+\frac{r}{2}C : r\in\mathbb{R}\}.
\end{aligned} \tag{2.0.18}$$

A vector equation defining each fiber above is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix} + \frac{r}{2}\begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 1 \end{pmatrix} \quad r\in\mathbb{R}. \tag{2.0.19}$$

Example 2.0.2. Now we show a case of low dimension of the symmetric spaces studied in Chapter 3. Consider the Lie group $G=Sl(2,\mathbb{C})$, with compact real form $U=SU(2)$ and take the element in the Lie algebra $\mathfrak{u}=\mathfrak{su}(2)$

$$H_\Theta = \frac{\sqrt{-1}\pi}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This element defines the maximal flag manifold $\mathbb{F}_\Theta = U/U_\Theta$ for $\Theta=\varnothing$, where U_Θ is the centralizer of H_Θ in U. The Lie group

$$SU(2)=\left\{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2+|\beta|^2=1, \alpha,\beta\in\mathbb{C}\right\} \tag{2.0.20}$$

is diffeomorphic to the 3−sphere S^3 and $Z_U(H_\Theta)$ is the subgroup

$$\begin{aligned} S(U(1)\times U(1)) = & \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : |\alpha|^2 = 1, \alpha \in \mathbb{C} \right\} \\ = & \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi] \right\} \end{aligned}$$

diffeomorphic to the unidimensional sphere S^1.

Then the flag manifold $\mathbb{F}_\Theta$ is a symmetric space and is the Riemann sphere

$$SU(2)/S(U(1)\times U(1)) = S^3/S^1 = \mathbb{C}P^1 \cong S^2.$$

The cotangent bundle of $SU(2)/S(U(1)\times U(1))$ has a realization as the homogeneous space

$$G/Z_G(H_\Theta) = Sl(2,\mathbb{C})/S(GL(1,\mathbb{C})\times GL(1,\mathbb{C}))$$

with

$$S(GL(1,\mathbb{C})\times GL(1,\mathbb{C})) = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^* \right\} \cong \mathbb{C}^* = \mathbb{C}\backslash\{0\}.$$

In Chapter 3 we prove that in particular, the cotangent bundle of the 2−sphere is a symmetric space, since the element H_Θ is a particular case of the matrix in the equation 3.0.2. Then the homogeneous space $T^*(S^2) = Sl(2,\mathbb{C})/\mathbb{C}^*$ is symmetric.

Furthermore we know another realization of the cotangent bundle of S^2 as the adjoint orbit of $G = Sl(2,\mathbb{C})$ in H_Θ

$$\begin{aligned} G\cdot H_\Theta &= \left\{ \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} H_\Theta \begin{pmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{pmatrix} : z_1z_4 - z_2z_3 = 1, z_1, z_2, z_3 \in \mathbb{C} \right\} \\ &= \left\{ \frac{\sqrt{-1}\pi}{2} \begin{pmatrix} 1+2z_2z_3 & -2z_1z_2 \\ 2z_3z_4 & -1-2z_3z_2 \end{pmatrix} : z_1z_4 - z_2z_3 = 1, z_1, z_2, z_3 \in \mathbb{C} \right\}. \end{aligned}$$

Remark 2. In the three cases of flag symmetric spaces, we have symmetric Lie algebras, in the cotangent bundle, of the type $(\mathfrak{g}, \mathfrak{z}_\Theta, \sigma)$ with the following properties :

1. $\mathfrak{g}$ is simple and is a vector direct sum $\mathfrak{n}_\Theta^- + \mathfrak{z}_\Theta + \mathfrak{n}_\Theta^+$ with the relations

$$\begin{aligned} [\mathfrak{z}_\Theta, \mathfrak{z}_\Theta] \subset \mathfrak{z}_\Theta, \quad [\mathfrak{z}_\Theta, \mathfrak{n}_\Theta^-] \subset \mathfrak{n}_\Theta^-, \quad [\mathfrak{z}_\Theta, \mathfrak{n}_\Theta^+] \subset \mathfrak{n}_\Theta^+, \\ [\mathfrak{n}_\Theta^-, \mathfrak{n}_\Theta^+] \subset \mathfrak{z}_\Theta, \quad [\mathfrak{n}_\Theta^-, \mathfrak{n}_\Theta^+] = 0, \quad [\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^+] = 0; \end{aligned} \tag{2.0.21}$$

2. The canonical decomposition $\mathfrak{g} = \mathfrak{z}_\Theta \oplus \mathfrak{m}_G$ is given by the direct sum

$$\mathfrak{m}_G = \mathfrak{n}_\Theta^- + \mathfrak{n}_\Theta^+; \tag{2.0.22}$$

3. With respect to the Cartan-Killing form $\mathcal{K}$ of $\mathfrak{g}$, the subspaces $\mathfrak{n}_\Theta^\pm$ are dual to each other and, moreover,

$$\mathcal{K}(\mathfrak{n}_\Theta^-, \mathfrak{n}_\Theta^+) = 0 \quad \text{and} \quad \mathcal{K}(\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^+) = 0 \tag{2.0.23}$$

2.1 The dual symmetric space

We study the dual symmetric spaces of the flag symmetric spaces $(\mathfrak{u}, \mathfrak{u}_\Theta, \sigma)$. The dual symmetric Lie algebra is the direct sum

$$\mathfrak{u}^* = \mathfrak{u}_\Theta + \sqrt{-1}\mathfrak{m}, \tag{2.1.1}$$

with $\sigma^* = \mathrm{Ad}(e^{H_\Theta}) \mid_{\mathfrak{u}^*}$. In the expressions 3.0.15, 4.0.33 and 5.0.22, we shall show that $\mathfrak{u}^*$ is isomorphic to a classical real Lie algebra and the decomposition above is the Cartan decomposition of $\mathfrak{u}^*$ with Cartan involution given by σ^*

Let U^* be the non-compact connected Lie subgroup with Lie algebra $\mathfrak{u}^*$. This Lie group is a real form of the complex Lie group G. In this section, we study the set

$$\mathcal{S} := \mathrm{Ad}(\mathrm{e}^{\sqrt{-1}\mathfrak{m}})H_\Theta \subset G \cdot H_\Theta := \mathrm{Ad}(G)H_\Theta. \tag{2.1.2}$$

The theorems 2.1.1 and 2.1.3 are the main results of this section.

Theorem 2.1.1. *Let $\mathbb{F}_\Theta = U/U_\Theta$ be a flag symmetric space with symmetric Lie algebra*

$$\mathfrak{u} = \mathfrak{u}_\Theta + \mathfrak{m}. \tag{2.1.3}$$

Then

(i) $\mathcal{S}$ is a submanifold of the adjoint orbit $G \cdot H_\Theta$, with G being the complex Lie group with compact real form U. Furthermore,

$$\mathcal{S} \cong U^*/U_\Theta, \tag{2.1.4}$$

where U^*/U_Θ is the dual symmetric space of $\mathbb{F}_\Theta$.

(ii) The submanifold $\mathcal{S}$ is diffeomorphic to a vector space.

(iii) This homogeneous space is a Riemannian manifold with Riemannian structure defined by the Cartan-Killing form.

(iv) Let $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ be a fiber in the cotangent bundle of $\mathbb{F}_\Theta$ intersecting the submanifold $\mathcal{S}$, then both spaces are transverse.

Proof. (i) The quocient space U^*/U_Θ is an affine symmetric space by definition of dual symmetric space. Since U^* is a subgroup of G and $U_\Theta = U \cap Z_\Theta$, then $(U^*, U_\Theta, \sigma^*)$ is a symmetric subspace of (G, Z_Θ, σ), with $\sigma^* = \sigma \mid_{U^*}$. By a theorem in [3], U/U_Θ is a totally geodesic submanifold of G/Z_Θ (with respect to the canonical connection of G/Z_Θ) and the canonical connnection of G/Z_Θ restricted to U/U_Θ coincides with the canonical connection of G/Z_Θ.

The theorem 1.1.38 can be applied in the non-compact connected Lie group U^*. Then, U^*/U_Θ is diffeomorphic to $\exp(\sqrt{-1}\mathfrak{m})$, which is simply connected. Furthermore, the adjoint orbit

$$\mathrm{Ad}(U^*)H_\Theta = \mathrm{Ad}(e^{\sqrt{-1}\mathfrak{m}})\mathrm{Ad}(U_\Theta)H_\Theta = \mathrm{Ad}(e^{\sqrt{-1}\mathfrak{m}})H_\Theta =: \mathcal{S} \tag{2.1.5}$$

is diffeomorphic to U^*/U_Θ. Hence, $\mathcal{S}$ is diffeomorphic to $\exp(\sqrt{-1}\mathfrak{m})$, and thus it is diffeomorphic to the vector space $\sqrt{-1}\mathfrak{m}$.

(ii) Proved in the demonstration of (i).

(iii) The tangent space of the dual symmetric space U^*/U_Θ at the origin is isomorphic to $\sqrt{-1}\mathfrak{m}$ and the Cartan-Killing form in $\sqrt{-1}\mathfrak{m}$ is $\mathcal{K}(\sqrt{-1}X, \sqrt{-1}Y) = -\mathcal{K}(X,Y) =: \langle X, Y\rangle$ for $X, Y \in \mathfrak{m}$. Since $\mathfrak{m} \subset \mathfrak{u}$, then the Cartan-Killing form in $\sqrt{-1}\mathfrak{m}$ is positive defined. The homogeneous space U^*/U_Θ admits invariant metrics since U_Θ is compact. A natural metric in U^*/U_Θ is that given by the restriction of the Cartan-Killing form to $\sqrt{-1}\mathfrak{m}$. Since $\mathfrak{u}^*$ is simple, then the adjoint representation in $\sqrt{-1}\mathfrak{m}$ is irreducible and the invariant metric is essentially unique, defined by the Cartan-Killing form.

(iv) Every fiber has the form $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+) = u \cdot (N_\Theta^+ \cdot H_\Theta)$, for $u \in U$. To say that $\mathcal{S}$ is transverse to $H_\Theta + \mathfrak{n}_\Theta^+$ is equivalent to proving that for every $x \in \mathcal{S} \cap u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ must be satisfied

$$T_x(\mathcal{S}) + T_x(u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)) = T_x(G \cdot H_\Theta), \tag{2.1.6}$$

for each $u \in U$ such that $\mathcal{S} \cap u \cdot (H_\Theta + \mathfrak{n}_\Theta^+) \neq \varnothing$.

The Lie algebra $\mathfrak{g}$ can be seen as the set of left invariant fields. From the equation

$$\frac{d}{dt}(\mathrm{Ad}(e^{tX})(Y)) \mid_{t=0} = [X, Y](1), \qquad X, Y \in \mathfrak{g}, \tag{2.1.7}$$

the tangent space of $\mathcal{S}$ at $e^A \cdot H_\Theta = u \cdot (H_\Theta + X) = ue^Z \cdot H_\Theta$, with $A \in \sqrt{-1}\mathfrak{m}$, $X, Z \in \mathfrak{n}_\Theta^+$, $u \in U$ and $e^Z \cdot H_\Theta = H_\Theta + X$, is $[e^A \cdot H_\Theta, \sqrt{-1}\mathfrak{m}]$ and the tangent space to the fiber $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ at $e^A \cdot H_\Theta$ is $[e^A \cdot H_\Theta, \mathfrak{n}_\Theta^+]$. The sum of the two tangent spaces is

$$[e^A \cdot H_\Theta, \sqrt{-1}\mathfrak{m}] + [e^A \cdot H_\Theta, \mathfrak{n}_\Theta^+] = [e^A \cdot H_\Theta, \sqrt{-1}\mathfrak{m} + \mathfrak{n}_\Theta^+] = [e^A \cdot H_\Theta, \mathfrak{m}_G], \tag{2.1.8}$$

the tangent space to the adjoint orbit $G \cdot H_\Theta$ at $e^A \cdot H_\Theta$.

□

Proposition 2.1.2. The intersection of $\mathcal{S}$ with the fiber $H_\Theta + \mathfrak{n}_\Theta^+$ is the point set $\{H_\Theta\}$.

Proof. We shall compute the intersection of the submanifold $\mathcal{S}$ with the fiber $H_\Theta + \mathfrak{n}_\Theta^+$ in the three cases of flag symmetric spaces. The results will be evidenced in 3.0.21, 4.0.40 and 5.0.27. □

The symmetry $\tilde{\sigma}$ in the submanifold $\mathcal{S}$ is given by

$$\begin{array}{rlcl}\tilde{\sigma}: & \mathcal{S} & \to & \mathcal{S} \\ & e^A \cdot H_\Theta & \mapsto & \mathrm{C}_{e^{H_\Theta}}(e^A) \cdot H_\Theta,\end{array} \tag{2.1.9}$$

for $A \in \sqrt{-1}\mathfrak{m}$. Then $\tilde{\sigma}(e^A \cdot H_\Theta) = \exp(\mathrm{Ad}(e^{H_\Theta})A) \cdot H_\Theta = \exp(\sigma(A)) \cdot H_\Theta = \exp(-A) \cdot H_\Theta$, since $(\mathfrak{u}^*, \mathfrak{u}_\Theta, \sigma)$, $\sigma =\mathrm{Ad}(e^{H_\Theta})$ is a symmetric Lie algebra. Recall that we have the same notation σ for the automorphism in the Lie group and algebra.

Theorem 2.1.3. *Suppose that, for some $u \in U$, the fiber $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ intersects the submanifold $\mathcal{S}$. Then, the fiber $\sigma(u) \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ intersects $\mathcal{S}$.*

Proof. Each element $e^A \cdot H_\Theta$ of $\mathcal{S}$ is an element of some fiber of the cotangent bundle and has the form $u \cdot (H_\Theta + X) = ue^Z \cdot H_\Theta$, for some $u \in U$ and $Z, X \in \mathfrak{n}_\Theta^+$, since $\mathcal{S} \subset G \cdot H_\Theta$.

$$\begin{aligned}\tilde{\sigma}(e^A \cdot H_\Theta) =&\tilde{\sigma}(ue^Z \cdot H_\Theta)\\ e^{-A} \cdot H_\Theta =&\sigma(u)\sigma(e^Z) \cdot H_\Theta\\ =&\sigma(u)e^{-Z} \cdot H_\Theta, \quad Z \in \mathfrak{m}_G\\ \in&\sigma(u) \cdot (H_\Theta + \mathfrak{n}_\Theta^+).\end{aligned} \tag{2.1.10}$$

□

However, not every fiber intersects $\mathcal{S}$. The following theorem indicates us some fibers which does not intersect $\mathcal{S}$.

Theorem 2.1.4. *The fiber $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ does not intersect $\mathcal{S}$ if $u \cdot \mathfrak{n}_\Theta^+ \subset \mathfrak{u}^*$ and $u \cdot H_\Theta \notin \mathfrak{u}_\Theta$.*

Proof. Let $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ a fiber such that $u \cdot \mathfrak{n}_\Theta^+ \subset \mathfrak{u}^*$ and $u \cdot H_\Theta \notin \mathfrak{u}_\Theta$. Suppose that $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathcal{S} \neq \varnothing$. The fiber is cointained in $u \cdot H_\Theta + \mathfrak{u}^*$. If $u \cdot (H_\Theta + X) \in \mathcal{S} \subset \mathfrak{u}^*$, with $X \in \mathfrak{n}_\Theta^+$, then $u \cdot H_\Theta \in \mathfrak{u}^* \cap \mathfrak{u} = \mathfrak{u}_\Theta$ and this is a contradiction. Hence

$$(u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)) \cap \mathcal{S} = \varnothing. \tag{2.1.11}$$

□

Theorem 2.1.5. *Let $u \cdot (H_\Theta + X) \in \mathcal{S} \subset T^*\mathbb{F}_\Theta$ be an element that satisfies $u \cdot \mathfrak{n}_\Theta^+ \cap \mathfrak{u}^* = \{0\}$. Then, the intersection of the submanifold $\mathcal{S}$ with the fiber $u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$ is $\{u \cdot (H_\Theta + X)\}$.*

Proof. Consider an element $u \cdot (H_\Theta + Y)$ in the intersection $\mathcal{S} \cap u \cdot (H_\Theta + \mathfrak{n}_\Theta^+)$. Then $u \cdot (H_\Theta + Y) - u \cdot (H_\Theta + X) = u \cdot (Y - X) \in u \cdot \mathfrak{n}_\Theta^+ \cap \mathfrak{u}^* = \{0\}$, since $\mathcal{S} \subset \mathfrak{u}^*$. Hence $X = Y$. □

The projection $\pi : u\cdot(H_\Theta + X) \in T^*\mathbb{F}_\Theta \mapsto u\cdot H_\Theta \in \mathbb{F}_\Theta$, for $u \in U$, $X \in \mathfrak{n}_\Theta^+$; restricts to the submanifold $\mathcal{S}$ is an homeomorphism over $\pi(\mathcal{S})$. The adjoint orbit $G\cdot H_\Theta$ is a set of matrices in $\mathfrak{g}$ with imaginary eigenvalues, since H_Θ belongs to the the Cartan subalgebra of $\mathfrak{u}$. The elements in $\sqrt{-1}\mathfrak{u}$ do not intersect the adjoint orbit since $\mathrm{ad}(g\cdot H_\Theta) = \mathrm{Ad}(g)\circ\mathrm{ad}(H_\Theta)\circ\mathrm{Ad}(g^{-1})$, for each $g \in G$, has imaginary eigenvalues and the matrices in $\mathrm{ad}(\sqrt{-1}\mathfrak{u})$ has real eigenvalues.

Example 2.1.1. This example is the continuation of Example 2.0.1.

The symmetric part of the Cartan decomposition of $\mathfrak{sl}(2,\mathbb{R})$ is generated by the matrices A and B (see 2.0.13). Thus, the submanifold $\mathcal{S}$ is represented by

$$\begin{aligned}\mathrm{Ad}(e^{tA})B =&: \exp(tA)\cdot B\\ &= \left\{\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}\cdot B = 0A + \cosh(2t)B - \sinh(2t)C : t \in \mathbb{R}\right\}\\ &= \{(0,y,z) \in \mathbb{R}^3 : y^2 - z^2 = 1,\ y > 0\}\end{aligned} \tag{2.1.12}$$

and it is the intersection of the hyperboloid $x^2 + y^2 - z^2 = 1$ with the semi-plane $\{x = 0, y > 0\}$ contained in the vector space $\mathrm{span}_{\mathbb{R}}\{B, C\}$. Here the submanifold $\mathcal{S}$ is symmetric with respect to the abelian vector subspace of $\mathfrak{sl}(2,\mathbb{R})$ generated by B. Hence we have three symmetric spaces: The hyperboloid in 2.0.16, the 1-sphere and a branch of the hyperbola 2.1.12 contained in the plane YZ.

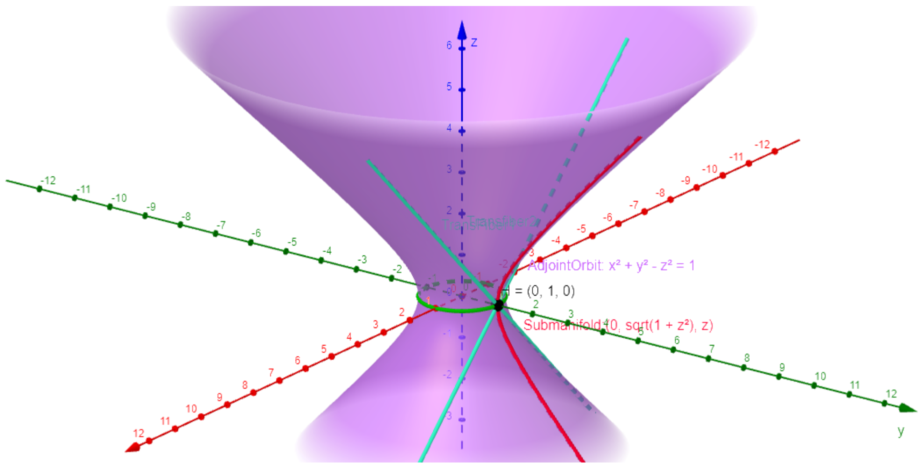

Figure 1 – Real adjoint orbit.

The intersection of $\mathcal{S}$ with the fiber $k\cdot(B+\mathfrak{n})$ is the set of the vectors $xA + yB + zC$ in 2.0.18, such that $x = 0$ and $y > 0$. These conditions are equivalent to the system

$$\begin{cases} -2\sin(2t) + r\cos(2t) = 0\\ 2\cos(2t) + r\sin(2t) > 0\end{cases},\ t \in [0,\pi]. \tag{2.1.13}$$

This system has solution only for $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$. Thus, the only fibers intersecting $\mathcal{S}$ are $k\cdot(B+\mathfrak{n})$ with k as in 2.0.17 and $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$. The intersection is the only vector satisfying the equation 2.0.19 with $r = 2\tan(2t)$. Furthermore, note that when k is equal to

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ or } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{2.1.14}$$

then $k \cdot B$ is equal to $-A$ and A, respectively. Hence, the fibers passing by $\pm A$ do not intersect the submanifold $\mathcal{S}$.

We can see this results in the Figure 2. All the lines in this figure are fibers and the sky blue lines are the fibers passing by $\pm A$. We can see how the fibers passing by the points in the (green) semicircle of radius 1 contained in $\{(x, y, 0) : x \in \mathbb{R}, y > 0\}$ intersect the submanifold $\mathcal{S}$ represented by the red curve.

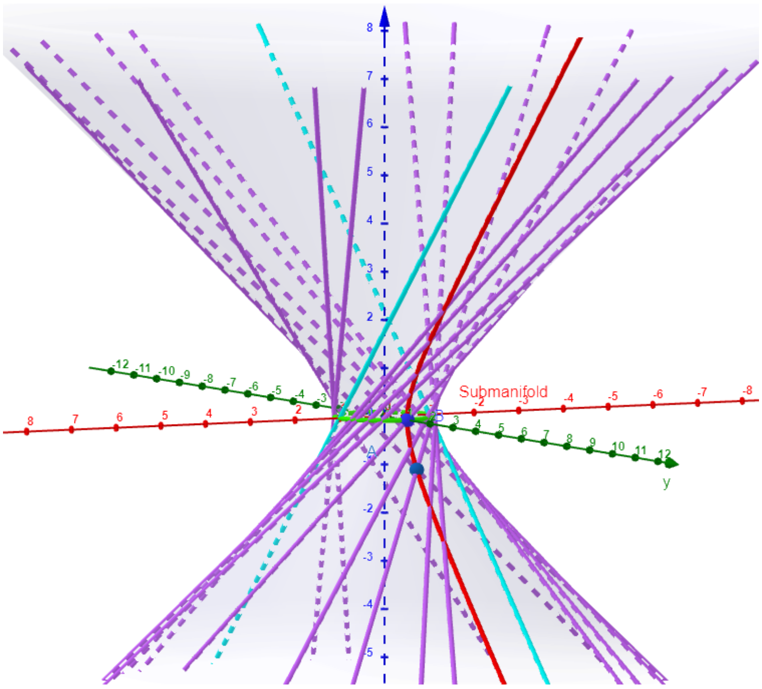

Figure 2 – Fibers of $T^*(S^1)$ and the submanifold $\mathcal{S}$.

Example 2.1.2. This example is a continuation of Example 2.0.2.

The manifold $\mathcal{S} :=(\exp\mathfrak{s}) \cdot H_\Theta$, with $\mathfrak{s} = \sqrt{-1}\mathfrak{m} = \left\{\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} : b \in \mathbb{C}\right\}$, is

$$\mathcal{S} = e^{\sqrt{-1}\mathfrak{m}} \cdot H_\Theta = \left\{ \frac{\sqrt{-1}\pi}{2} \begin{pmatrix} \cosh(2|b|) & -\frac{b}{|b|}\sinh(2|b|) \\ \frac{\bar{b}}{|b|}\sinh(2|b|) & -\cosh(2|b|) \end{pmatrix} : b \in \mathbb{C} \right\} \tag{2.1.15}$$

The diagonal entries of the matrices in $\mathcal{S}$ are complex numbers with real part equal to zero.

The fibers of the cotangent bundle are $u \cdot (H_\Theta + \mathfrak{n}^+)$, with $u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$ and each fiber is the set

$$\left\{ \begin{pmatrix} \frac{\sqrt{-1}\pi}{2}(2|\alpha|^2 - 1) + x\alpha\bar{\beta} & -\sqrt{-1}\pi\alpha\beta + x\alpha^2 \\ -\sqrt{-1}\pi\bar{\alpha}\bar{\beta} - x\bar{\beta}^2 & -\frac{\sqrt{-1}\pi}{2}(2|\alpha|^2 - 1) - x\alpha\bar{\beta} \end{pmatrix} : x \in \mathbb{C} \right\}. \tag{2.1.16}$$

To find out which fibers intersect $\mathcal{S}$, we are looking for elements in $u \cdot (H_\Theta + \mathfrak{n}^+)$ such that for each α and β as above, we must find if there exists $x \in \mathbb{C}$ satisfying the following conditions:

$$\bar{x}\bar{\alpha}\beta = -x\alpha\bar{\beta} \tag{2.1.17}$$

$$\sqrt{-1}\pi\bar{\alpha}\bar{\beta} + \bar{x}\bar{\alpha}^2 = -\sqrt{-1}\pi\bar{\alpha}\bar{\beta} - x\bar{\beta}^2 \tag{2.1.18}$$

$$\frac{\pi}{2}(2|\alpha|^2 - 1) + \operatorname{Im}(x\alpha\bar{\beta}) \geqslant \frac{\pi}{2} \tag{2.1.19}$$

The first and second conditions come from the form of matrices in $\mathfrak{su}(1,1)$ and the third condition is because, from 2.1.15, the imaginary part of the first element in the diagonal of matrices in $\mathcal{S}$ is greater than $\frac{\pi}{2}$.

If $\beta = 0$, then $u \in S(U(1) \times U(1))$ and $\alpha \neq 0$, thus the fiber is $(H_\Theta + \mathfrak{n}^+)$. The unique $x \in \mathbb{C}$ that satisfies the three conditions above is $x = 0$. Hence, the intersection $(H_\Theta + \mathfrak{n}^+) \cap \mathcal{S}$ is $\{H_\Theta\}$.

If $\alpha = 0$, then $|\beta| = 1$ and the fiber is $(H_\Theta + \mathfrak{n}^-)$. The equation 2.1.18 says us that $x = 0$ and putting $x = 0$ in 2.1.19, we have a contradiction. Hence, the intersection $(H_\Theta + \mathfrak{n}^-) \cap \mathcal{S}$ is empty.

If $\alpha \neq 0 \neq \beta$, from 2.1.17 and 2.1.18, we have the new condition

$$2\sqrt{-1}\pi\bar{\alpha}\beta = x(2|\alpha|^2 - 1). \tag{2.1.20}$$

Here, we have two cases:

a) If $2|\alpha|^2 - 1 = 0$, then $2\sqrt{-1}\pi\bar{\alpha}\beta = 0$. This is a contradiction, since $\bar{\alpha}\beta \neq 0$. Hence, the fibers $u \cdot (H_\Theta + \mathfrak{n}^+)$ with

$$u \in \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 = |\beta|^2 = \frac{1}{2}, \alpha, \beta \in \mathbb{C} \right\}, \tag{2.1.21}$$

has empty intersection with the manifold $\mathcal{S}$.

b) If $2|\alpha|^2 - 1 \neq 0$, from 2.1.17 and 2.1.18, we find the complex number

$$x = \frac{2\pi\sqrt{-1}}{2|\alpha|^2 - 1}\bar{\alpha}\beta. \tag{2.1.22}$$

The condition 2.1.19, with x as above, becomes in the following condition

$$\frac{1}{2} < |\alpha|^2 < 1. \tag{2.1.23}$$

Hence, $(u \cdot (H_\Theta + \mathfrak{n}^+)) \cap \mathcal{S}$ is not empty if

$$u \in \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \frac{1}{2} < |\alpha|^2 \leqslant 1, |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C} \right\}. \tag{2.1.24}$$

Furthermore,

$$(u \cdot (H_\Theta + \mathfrak{n}^+)) \cap \mathcal{S} = \left\{ \frac{\sqrt{-1}\pi}{2\,|\alpha|^2 - 1} \begin{pmatrix} 1 & \alpha\beta \\ -\bar{\alpha}\bar{\beta} & -1 \end{pmatrix} \right\}. \tag{2.1.25}$$

Considering the basis for $\mathfrak{u}$:

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \tag{2.1.26}$$

the matrices in $u \cdot H_\Theta$, with u as in 2.1.24, have the decomposition

$$\pi \mathrm{Im}(\alpha\beta) e_1 + \frac{\pi}{2}(2\,|\alpha|^2 - 1)e_2 - \pi \mathrm{Re}(\alpha\beta)e_3, \tag{2.1.27}$$

with the second component greater than zero and

$$\pi^2[\mathrm{Im}(\alpha\beta)]^2 + \frac{\pi^2}{4}(2\,|\alpha|^2 - 1)^2 + \pi^2[\mathrm{Re}(\alpha\beta)]^2 = \frac{\pi^2}{4}. \tag{2.1.28}$$

This is, $u \cdot H_\Theta$, with u as in 2.1.24, is the semi-sphere $\{(a, b, c) \in \mathfrak{su}(2) : a^2 + b^2 + c^2 = \frac{\pi^2}{4},\ b > 0\}$. We can conclude that $\mathcal{S}$ is projected into the flag as the semi-sphere, through the fibers.

Since the manifold $\mathcal{S}$ is contained in

$$\mathfrak{su}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} : \beta \in \mathbb{C}, \bar{\alpha} = -\alpha \right\}.$$

We can express each of its elments as a combination of the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, C = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}. \tag{2.1.29}$$

In that base, the manifold $\mathcal{S}$ is the part of the two-sheeted hyperboloid $x^2 + z^2 - y^2 = -\frac{\pi^2}{4}$ with positive Y-component:

$$\mathcal{S} = \left\{ \frac{\pi}{2} \sin t \sinh(2r) \cdot A + \frac{\pi}{2} \cosh(2r) \cdot B - \frac{\pi}{2} \cos t \sinh(2r) \cdot C : t \in [0, 2\pi], r \geqslant 0 \right\}. \tag{2.1.30}$$

Note that, the set of matrices $xA + yB + zC \in \mathfrak{su}(1,1)$ with determinant equal to $\det(H_\Theta) = \frac{\pi^2}{4}$ is the hyperboloid $x^2 + z^2 - y^2 = -\frac{\pi^2}{4}$ and has $|\mathcal{W}|$ connected components, where $\mathcal{W}$ is the Weyl group of $\mathfrak{sl}(2,\mathbb{R})$. Also in this example, the flag is maximal and H_Θ is a regular element. Let $w_0 \in \mathcal{W}$ be the principal involution, then $w_0 \cdot H_\Theta = -H_\Theta$ and the adjoint orbit $\mathrm{Ad}(SU(1,1))(-H_\Theta) = -\mathrm{Ad}(SU(1,1))H_\Theta$ is the sheet of the hyperboloid $x^2 + z^2 - y^2 = -\frac{\pi^2}{4}$ such that $y < 0$. Thus,

$$\begin{aligned}\{M \in \mathfrak{sl}(2,\mathbb{R}) : \det(M) = \frac{\pi^2}{4}\} =& \{(x,y,z) \in \mathbb{R}^3 : x^2 + z^2 - y^2 = -\frac{\pi^2}{4}\} \\ =& \mathrm{Ad}(\mathrm{SU}(1,1))H_\Theta \cup \mathrm{Ad}(\mathrm{SU}(1,1))w_0 H_\Theta\end{aligned}$$

2.2 Symplectic form

Let us define the 2−form in $\mathfrak{g}$,

$$\mathcal{H}_\tau(X,Y) = -\mathcal{K}(X,\tau Y) \qquad X,Y \in \mathfrak{g}. \tag{2.2.1}$$

It is an hermitian form in $\mathfrak{g}$ and then

$$\omega(X,Y) := -\mathrm{Im}\mathcal{H}_\tau(X,Y) \qquad X,Y \in \mathfrak{g} \tag{2.2.2}$$

is a symplectic form in $\mathfrak{g}$. In [8], it was proved that this 2-form induces a symplectic form in the adjoint orbit through the pullback of ω of the inclusion $G \cdot H_\Theta \hookrightarrow \mathfrak{g}$ and with $T_{g\cdot H_\Theta} G \cdot H_\Theta = [\mathfrak{g}, g \cdot H_\Theta]$, for each $g \in G$.

The flag symmetric space $\mathbb{F}_\Theta$ can be considered as the adjoint orbit $\mathrm{Ad}(U) \cdot H_\Theta$ of the compact Lie group U, contained in $\mathfrak{u}$. Furthermore, the conjugation τ restricts to $\mathfrak{u}$ is the identity function. Then, the symplectic form above in $\mathrm{Ad}(U) \cdot H_\Theta$ is

$$\omega(X,Y) = -\mathrm{Im}\mathcal{K}(X,Y) \qquad X,Y \in \mathfrak{u}. \tag{2.2.3}$$

The Cartan-Killing form is a symmetric 2−form and ω is anti-symmetric as well as the Lie algebra $\mathfrak{u}$ is real. Hence, the symplectic form in the submanifold $\mathbb{F}_\Theta$ is zero.

Now let us see who is the restriction of the symplectic form above to the submanifold $\exp(\sqrt{-1}\mathfrak{m}) \cdot H_\Theta = U^* \cdot H_\Theta$ of $G \cdot H_\Theta$. This submanifold is contained in $\mathfrak{u}^*$ and each tangent space $T_{v\cdot H_\Theta} U^* \cdot H_\Theta = [\mathfrak{u}^*, v \cdot H_\Theta]$, for each $v \in \exp(\sqrt{-1}\mathfrak{m})$, is also contained in $\mathfrak{u}^*$. The real Lie algebra $\mathfrak{u}^*$ is invariant by the conjugation τ. The Cartan-Killing form restricted to $\mathfrak{u}^*$ coincides with the Cartan-Killing form of $\mathfrak{u}^*$, since it is a real form. Then, the symplectic form restricted to $\mathfrak{u}^*$ is equal to zero.

Theorem 2.2.1. *The flag symmetric space $\mathbb{F}_\Theta = U/U_\Theta$ and its dual symmetric space $\mathcal{S}$ are Lagrangian submanifolds of the adjoint orbit $G \cdot H_\Theta$ with the symplectic form defined in 2.2.2.*

On the other hand, the Kirillov-Kostant-Souriau symplectic form (KKS) restricted to $\mathcal{S}$ is

$$\Omega_\xi([X,\xi],[Y,\xi]) = \mathcal{K}(\xi,[X,Y]), \qquad X,Y \in \mathfrak{u}^*, \xi \in \mathcal{S}. \tag{2.2.4}$$

If $\xi = H_\Theta$ and $X = \sqrt{-1}A_\alpha, Y = \sqrt{-1}Z_\alpha$, for some $\alpha \in \Pi^+ \backslash \langle\Theta\rangle^+$, then

$$\begin{aligned}
\Omega_{H_\Theta}([\sqrt{-1}A_\alpha, H_\Theta],[\sqrt{-1}Z_\alpha, H_\Theta]) =& \mathcal{K}(H_\Theta,[\sqrt{-1}A_\alpha,\sqrt{-1}Z_\alpha])\\
=&\mathcal{K}(H_\Theta,-[A_\alpha,Z_\alpha])\\
=&\mathcal{K}(H_\Theta,2\sqrt{-1}H_\alpha)\\
=&2\sqrt{-1}\alpha(H_\Theta)\\
\neq&0.
\end{aligned}$$

Hence, the submanifold $\mathcal{S}$ is not Lagrangian with respect to the KKS form.

2.3 Geodesics

The tangent vector space to the adjoint orbit at the origin can be expressed in the following ways (direct sums):

$$\mathfrak{m}_G = \mathfrak{m} + \sqrt{-1}\mathfrak{m} = \mathfrak{m} + \mathfrak{n}_\Theta^+. \tag{2.3.1}$$

Hence, every $X \in \mathfrak{m}_G$ have the decompositions

$$X = X_m + X_s, \qquad X_m \in \mathfrak{m},\ X_s \in \sqrt{-1}\mathfrak{m} \tag{2.3.2}$$

$$X = X_m + X_f, \qquad X_m \in \mathfrak{m},\ X_f \in \mathfrak{n}_\Theta^+. \tag{2.3.3}$$

With a canonical connection given by the Cartan-Killing form over G/Z_Θ, the maximal geodesics are the curves

$$\gamma(t) = \pi_G(g\exp(tX)), \quad g \in G, \quad X \in \mathfrak{m}_G \tag{2.3.4}$$

where $\pi_G : G \to G/Z_\Theta$ is the canonical projection defined by $g \in G \mapsto g \cdot o$, with $o = 1 \cdot Z_\Theta$. The maximal geodesics in $G/P_\Theta = U/U_\Theta$ are the curves

$$\gamma_U(t) = \pi_U(u\exp(tA)), \quad u \in U, \quad A \in \mathfrak{m} \cong T_{b_\Theta}\mathbb{F}_\Theta \tag{2.3.5}$$

where $\pi_U : U \to U/U_\Theta$ is the canonical projection defined by $u \in U \mapsto u \cdot b_\Theta$.

Furthermore, we can define another projection

$$\begin{array}{cccccc}
\pi : & G/Z_\Theta & \to & G/P_\Theta & \to & U/U_\Theta \\
 & gZ_\Theta & \mapsto & gP_\Theta & \mapsto & uU_\Theta
\end{array} \tag{2.3.6}$$

The following question arises: Is the projection of a geodesic in G/Z_Θ a geodesic in U/U_Θ? Here we analyze a case where the answer is yes and for this, we will focus on the study of geodesics that pass through the origin.

From the decomposition in (2.3.3), if $[X_\mathfrak{m}, X_f] = 0$, then $\exp(tX) = \exp(tX_m)\exp(tX_f)$ and

$$\pi(\exp(tX)\cdot o) = \exp(tX_\mathfrak{m})\cdot b_\Theta \tag{2.3.7}$$

since $X_m \in \mathfrak{m}$ and $X_f \in \mathfrak{n}_\Theta^+$.

When $X_f = 0$, we get the geodesic $\gamma(t) = g\exp(tX_\mathfrak{m})\cdot o$ is a horizontal curve in the vector bundle $G/Z_\Theta \to U/U_\Theta$ and the projection in U/U_Θ is the geodesic curve $\gamma_U(t) = u\exp(tX_\mathfrak{m})\cdot b_\Theta$, $\pi_{GU}(g) = u$ is the projection of G in U, and we have also the case when $X_\mathfrak{m} = 0$, then the geodesic is in the fiber $\mathfrak{n}_\Theta^+$ and the projection of this curve on U/U_Θ is $\gamma_U(t) = b_\Theta$.

From the realization $T^*\mathbb{F}_\Theta = G/Z_\Theta$, the elements in $g\cdot o \in G/Z_\Theta$ can be viewed as $uv\cdot H_\Theta = u\cdot(H_\Theta + X) \in T^*\mathbb{F}_\Theta$, with $u \in U, v \in N_\Theta^+$ such that $g = uv$ and $v\cdot H_\Theta = H_\Theta + X$. Thus, the projection will be $u\cdot(H_\Theta + X) \mapsto u\cdot H_\Theta$. Hence, for $\{u_t\} \subset U$ and $\{H_\Theta + X_t = e^{W_t}\cdot H_\Theta\} \subset N_\Theta^+\cdot H_\Theta$, we get

$$\gamma(t) = e^{tZ}\cdot H_\Theta = u_t e^{W_t}\cdot H_\Theta = u_t\cdot(H_\Theta + X_t). \tag{2.3.8}$$

Then, the projection of the geodesic is $\pi(\gamma(t)) = u_t\cdot H_\Theta$. When $Z \in \mathfrak{m}$, the geodesic $\gamma(t) = ge^{tZ}\cdot o$ is a horizontal curve in the vector bundle for $g \in G$ and its projection on the flag is the geodesic $\gamma_U(t) = ue^{tZ}\cdot b_\Theta$, with $\pi_{GU}(g) = u \in U$. Moreover, when $Z \in \mathfrak{n}_\Theta^+$, then

$$\gamma(t) = g\exp(tZ)\cdot o \xrightarrow{\pi} \gamma_U(t) = u\cdot b_\Theta$$

the geodesic is into the fiber $u\cdot(H_\Theta + \mathfrak{n}_\Theta^+)$ of the cotangent bundle and its vertical.

3 Case A_l

Let G be the Lie group $Sl(n, \mathbb{C})$ and let U be the compact Lie group $SU(n)$, for $n = l + 1$, with Lie algebras $\mathfrak{g}$ and $\mathfrak{u}$, respectively. The corresponding flag symmetric space is the set of Grassmannian of complex $p-$dimensional subspaces of $\mathbb{C}^{p+q}$

$$\mathbb{F}_\Theta = SU(p+q)/S(U(p) \times U(q)), \qquad n = p+q,\ p \leqslant q,$$

with $\Theta = \Sigma\backslash\{\alpha_p\} = \{\alpha_1 = \lambda_1 - \lambda_2, \cdots, \alpha_{p-1} = \lambda_{p-1} - \lambda_p, \alpha_{p+1} = \lambda_{p+1} - \lambda_{p+2}, \cdots, \alpha_l = \lambda_l - \lambda_{l+1}\}$ and each λ_i is given by

$$\lambda_i : \mathrm{diag}\{a_1, \cdots, a_{l+1}\} \mapsto a_i. \tag{3.0.1}$$

The compact real form of $\mathfrak{g}$ is $\mathfrak{u} = \mathfrak{su}(n)$ and an element in $\mathfrak{u}$ that defines the flag $\mathbb{F}_\Theta$ is the matrix

$$H_\Theta := \frac{i\pi}{n}\begin{pmatrix} qI_p & 0 \\ 0 & -pI_q \end{pmatrix} \in \mathfrak{su}(n) \subset \mathfrak{sl}(n, \mathbb{C}). \tag{3.0.2}$$

It belongs to the Cartan subalgebra of $\mathfrak{su}(n)$ and its exponential is the matrix

$$g_\Theta := exp(H_\Theta) = \begin{pmatrix} e^{i\frac{\pi q}{n}} I_p & 0 \\ 0 & e^{-i\frac{p\pi}{n}} I_q \end{pmatrix}. \tag{3.0.3}$$

Since $p + q = n$, then $e^{i\frac{\pi}{n}(p+q)} = -1$. So, we have $e^{i\pi\frac{q}{n}} \cdot e^{i\pi\frac{p}{n}} = -1$. Therefore,

$$e^{i\pi\frac{q}{n}} = -e^{-i\pi\frac{p}{n}} \tag{3.0.4}$$

and then

$$\begin{aligned} g_\Theta = & \begin{pmatrix} e^{i\pi\frac{q}{n}} I_p & 0 \\ 0 & -e^{i\pi\frac{q}{n}} I_q \end{pmatrix} \\ = & \ e^{i\pi\frac{q}{n}} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \\ := & \ e^{i\pi\frac{q}{n}} I_{p,q} \in SU(n) \end{aligned}$$

where $I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

Furthermore, $g_\Theta^2 = (e^{i\pi\frac{q}{n}})^2 \cdot I_{p,q}^2 = e^{2i\pi\frac{q}{n}} \cdot I_n$. This verifies that $(g_\Theta^2)^n = e^{2\pi q i} \cdot I_n = I_n$, i.e., $g_\Theta^2 \in Z(SU(n))$, since g_Θ^2 is a nth root of unity.

We shall use the same notation σ for both the symmetric automorphism for Lie groups and algebras. Seeing that $I_{p,q}^{-1} = I_{p,q}$, the inner automorphism

$$\begin{array}{rcl} \sigma: \; SU(n) & \rightarrow & SU(n) \\ h & \mapsto & C_{g_\Theta}(h) = g_\Theta h g_\Theta^{-1} = e^{i\pi\frac{q}{n}} I_{p,q} h e^{-i\pi\frac{q}{n}} I_{p,q}^{-1} = I_{p,q} h I_{p,q} \end{array}$$

is an involution, because of $C_{g_\Theta}^2(h) = I_{p,q}^2 h I_{p,q}^2 = h$, for all $h \in SU(n)$. The fixed points set of C_{g_Θ} in $SU(n)$ is:

$$\begin{aligned} & \{h \in SU(n) : I_{p,q} h I_{p,q} = h\} \\ =& \{h \in SU(n) : I_{p,q} = h^{-1} I_{p,q} h\} = Z_{SU(n)}(I_{p,q}) \\ =& \left\{ \begin{pmatrix} a_{p\times p} & b_{p\times q} \\ c_{q\times p} & d_{q\times q} \end{pmatrix} \in SU(n) : \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \\ =& \left\{ \begin{pmatrix} a_{p\times p} & 0 \\ 0 & d_{q\times q} \end{pmatrix} \in SU(n) \right\} \\ =& S(U(p) \times U(q)). \end{aligned}$$

Hence, the flag manifold $SU(n)/S(U(p) \times U(q))$ is a symmetric space. The tangent space to the flag symmetric at the origin is isomorphic to

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & Z \\ -\bar{Z}^T & 0 \end{pmatrix} : Z_{p\times q} \, \text{complex matrix} \right\} \tag{3.0.5}$$

and the symmetric pair is $(\mathfrak{su}(n), \mathfrak{u}_\Theta)$, where the subalgebra $\mathfrak{u}_\Theta$ consists of all elements in $\mathfrak{su}(n)$ which are fixed by C_{g_Θ} and is the Lie algebra of $S(U(p) \times U(q))$

$$\mathfrak{u}_\Theta = \left\{ \begin{pmatrix} A_{p\times p} & 0 \\ 0 & B_{q\times q} \end{pmatrix} : \operatorname{tr}(A) + \operatorname{tr}(B) = 0, A \in \mathfrak{u}(p),\; B \in \mathfrak{u}(q) \right\}. \tag{3.0.6}$$

In order to study the symmetry of the cotangent bundle of the flag symmetric space $\mathbb{F}_\Theta$, we study the centralizer of H_Θ in the Lie group G,

$$\begin{aligned} Z_{Sl(n,\mathbb{C})}(H_\Theta) =& \{h \in Sl(n,\mathbb{C}) : Ad(h) H_\Theta = H_\Theta\} \\ =& \left\{ h = \begin{pmatrix} a_{p\times p} & b \\ c & d_{q\times q} \end{pmatrix} \in Sl(n,\mathbb{C}) : h \begin{pmatrix} qI_p & 0 \\ 0 & -pI_q \end{pmatrix} = \begin{pmatrix} qI_p & 0 \\ 0 & -pI_q \end{pmatrix} h \right\} \\ =& \left\{ h \in Sl(n,\mathbb{C}) : \begin{pmatrix} qa & -pb \\ qc & -pd \end{pmatrix} = \begin{pmatrix} qa & qb \\ -pc & -pd \end{pmatrix} \right\} \\ =& \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Sl(n,\mathbb{C}) : a_{p\times p}, b_{q\times q} \right\} \\ =& S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C})) \end{aligned} \tag{3.0.7}$$

and note that the centralizer of the matrix $I_{p,q}$ in the group $\mathrm{Sl}(n,\mathbb{C})$ is

$$
\begin{aligned}
Z_{Sl(n,\mathbb{C})}(I_{p,q}) =& \{h = \begin{pmatrix} a_{p\times p} & b \\ c & d_{q\times q} \end{pmatrix} \in Sl(n,\mathbb{C}) : h \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} h\} \\
=& \{h \in Sl(n,\mathbb{C}) : \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}\} \\
=& \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Sl(n,\mathbb{C}) : a_{p\times p}, b_{q\times q}\} \\
=& S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C})) \\
=& Z_{Sl(n,\mathbb{C})}(H_\Theta)
\end{aligned}
$$

We can extend the inner automorphism $\sigma = C_{g_\Theta}$ to the complex Lie group $Sl(n,\mathbb{C})$:

$$
\begin{array}{rccl}
C_{g_\Theta} : & Sl(n,\mathbb{C}) & \to & Sl(n,\mathbb{C}) \\
 & h & \mapsto & ghg^{-1} = I_{p,q} h I_{p,q}
\end{array}
\tag{3.0.8}
$$

and we saw that it is an involution and its fixed points set is $Z_{Sl(n,\mathbb{C})}(I_{p,q}) = Z_{Sl(n,\mathbb{C})}(H_\Theta) = S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C}))$.

Hence the cotangent bundle of $\mathbb{F}_\Theta$, as homogeneous space,

$$
Sl(n,\mathbb{C})/S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C}))
$$

is a symmetric space associated to the symmetric pair $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(q,\mathbb{C}) \oplus \mathbb{C}^*)$. In fact, let $A \in \mathfrak{s}(\mathfrak{gl}(p,\mathbb{C}) \times \mathfrak{gl}(q,\mathbb{C}))$, then

$$
A = \begin{pmatrix} D_{p\times p} & 0 \\ 0 & B_{q\times q} \end{pmatrix}.
$$

(i) If $A \in \mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(q,\mathbb{C})$, then

$$
A = \left[A + \begin{pmatrix} -I_p & 0 \\ 0 & \frac{p}{q} I_q \end{pmatrix} \right] + \begin{pmatrix} I_p & 0 \\ 0 & -\frac{p}{q} I_q \end{pmatrix} \mapsto A' + \frac{p}{q}.
$$

(ii) If $A \notin \mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(q,\mathbb{C})$, then

$$
A = \left[A + \mathrm{tr}(B) \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right] + \mathrm{tr}(B) \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \mapsto A' + \mathrm{tr}(B).
$$

In both of cases, A' belongs to $\mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(q,\mathbb{C}) \oplus \mathbb{C}^*$. Hence $\mathfrak{s}(\mathfrak{gl}(p,\mathbb{C}) \times \mathfrak{gl}(q,\mathbb{C})) \subset \mathfrak{sl}(p,\mathbb{C}) \oplus \mathfrak{sl}(q,\mathbb{C}) \oplus \mathbb{C}^*$. The other containment is straightforward. Thus, this symmetric

Lie algebra is in Table A.3. Furthermore, the tangent space of that symmetric space at the origin is isomorphic to

$$\begin{aligned}\mathfrak{m}_G =&\mathfrak{m} + \sqrt{-1}\mathfrak{m}\\ \cong&\mathfrak{n}_\Theta^- + \mathfrak{n}_\Theta^+\\ =&\left\{\begin{pmatrix} 0 & A\\ B & 0\end{pmatrix} : A_{p\times q},\ B_{q\times p} \text{ complex matrices}\right\}\end{aligned} \tag{3.0.9}$$

where the elements in $\mathfrak{n}_\Theta^-$ are the matrices in 3.0.9 with $A = 0$, $\mathfrak{n}_\Theta^+$ is the set 3.0.9 with $B = 0$ and $\mathfrak{m}$ is the set in 3.0.5.

Using the results in [1], we get

$$G \cdot H_\Theta = G/Z_\Theta = T^*(U/U \cap Z_\Theta) = T^*(U/U_\Theta)$$

Then

$$Sl(n,\mathbb{C})/S(Gl(p,\mathbb{C}) \times Gl(q,\mathbb{C})) = T^*(SU(n)/S(U(p) \times U(q)))$$

Thus, the cotangent bundle of the flag symmetric manifold $SU(n)/S(U(p) \times U(q))$ is also a symmetric space with the same involutive automorphism $\sigma = C_{g_\Theta}$.

Note that the inner automorphism can be lifted to the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, obtaining the function

$$\begin{array}{rccl}\sigma = \mathrm{Ad}(g_\Theta): & \mathfrak{sl}(n,\mathbb{C}) & \to & \mathfrak{sl}(n,\mathbb{C})\\ & Z & \mapsto & g_\Theta Z g_\Theta^{-1} = I_{p,q} Z I_{p,q}.\end{array} \tag{3.0.10}$$

Denote by $\mathfrak{g}_0$ the non compact real form $\mathfrak{su}(p,q)$ of $\mathfrak{g}$, which elements are the $n \times n$ complex matrices satisfying

$$I_{p,q}X + \bar{X}^T I_{p,q} = 0, \qquad I_{p,q} = \begin{pmatrix} 1_p & 0\\ 0 & -1_q\end{pmatrix}. \tag{3.0.11}$$

This Lie algebra is

$$\mathfrak{su}(p,q) = \left\{\begin{pmatrix} A_{p\times p} & B_{p\times q}\\ \bar{B}^T_{q\times p} & C_{q\times q}\end{pmatrix} : A \in \mathfrak{u}(p),\ C \in \mathfrak{u}(q),\ \mathrm{tr}(A + C) = 0\right\}, \tag{3.0.12}$$

and has the Cartan decomposition $\mathfrak{su}(p,q) = \mathfrak{k} + \mathfrak{s}$ with $\mathfrak{k} = \mathfrak{u}_\Theta$ as in (3.0.6) and

$$\mathfrak{s} = \left\{\begin{pmatrix} 0 & B\\ \bar{B}^T & 0\end{pmatrix} : B_{p\times q} \text{ complex matrix}\right\}. \tag{3.0.13}$$

The involution $\mathrm{Ad}(\exp(H_\Theta))$ in $\mathfrak{sl}(n,\mathbb{C})$,

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} \mapsto \begin{pmatrix} A & -B\\ -C & D\end{pmatrix} \tag{3.0.14}$$

restricts to $\mathfrak{su}(p,q)$ is also an inner automorphism since $H_\Theta \in \mathfrak{u}_\Theta = \mathfrak{k}$ and coincides with the Cartan involution $Z \in \mathfrak{su}(p,q) \mapsto -\bar{Z}^T \in \mathfrak{su}(p,q)$.

From the definition of $\mathfrak{m}$ in (3.0.5), we have that the vector space $\sqrt{-1}\mathfrak{m}$ coincides with $\mathfrak{s}$ and therefore, the dual symmetric algebra of $(\mathfrak{su}(n),\mathfrak{k},\sigma)$ is

$$\mathfrak{su}(n)^* = \mathfrak{su}(p,q) = \mathfrak{k} + \mathfrak{s}. \tag{3.0.15}$$

In this way, as studied in the section 2.1, the dual symmetric space $SU(p,q)/S(U(p)\times U(q))$ of $SU(p+q)/S(U(p)\times U(q))$ is a submanifold of the adjoint orbit $Sl(n,\mathbb{C})\cdot H_\Theta$ (this H_Θ is from the A_l case).

The dual symmetric space $SU(p,q)/S(U(p)\times U(q))$ is diffeomorphic to $\mathcal{S} = \exp \mathfrak{s}\cdot H_\Theta$. To prove Proposition 2.1.2, let us calculate the intersection $\mathcal{S}\cap(H_\Theta + \mathfrak{n}_\Theta^+)$. Initially, note that

$$H_\Theta \in (H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathrm{Ad}(S)\cdot H_\Theta \subset \mathfrak{su}(p,q)\cap(H_\Theta + \mathfrak{n}_\Theta^+), \tag{3.0.16}$$

where S is the exponential of the vector space $\mathfrak{s}$. To calculate the intersection $\mathfrak{su}(p,q)\cap(H_\Theta + \mathfrak{n}_\Theta^+)$, we describe the elements of the fiber passing through the origin H_Θ:

$$\begin{pmatrix} \frac{\sqrt{-1}\pi}{n} qI_p & Z_{p\times q} \\ 0 & -\frac{\sqrt{-1}\pi}{n} pI_q \end{pmatrix}, \tag{3.0.17}$$

with $Z \in \mathfrak{n}_\Theta^+$. These matrices belong to $\mathfrak{su}(p,q)$ if and only if they satisfy the expression

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}\begin{pmatrix} \frac{\sqrt{-1}\pi}{n} qI_p & Z \\ 0 & -\frac{\sqrt{-1}\pi}{n} pI_q \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{-1}\pi}{n} qI_p & 0 \\ \bar{Z}^T & \frac{\sqrt{-1}\pi}{n} qI_p \end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} = \mathbf{0}$$

which is equivalent to

$$\begin{pmatrix} 0 & Z \\ \bar{Z}^T & 0 \end{pmatrix} = \mathbf{0} \Longleftrightarrow Z = \mathbf{0}. \tag{3.0.18}$$

It means that the intersection $\mathfrak{su}(p,q)\cap(H_\Theta + \mathfrak{n}_\Theta^+) \subset \{H_\Theta\}$. As from (4.0.36), we conclude that

$$\mathfrak{su}(p,q)\cap(H_\Theta + \mathfrak{n}_\Theta^+) = \{H_\Theta\}. \tag{3.0.19}$$

Thus,

$$H_\Theta \in ((H_\Theta + \mathfrak{n}_\Theta^+)\cap\mathcal{S}) \subset (H_\Theta + \mathfrak{n}_\Theta^+)\cap\mathfrak{su}(p,q)^*(l) = \{H_\Theta\}. \tag{3.0.20}$$

Hence,

$$(H_\Theta + \mathfrak{n}_\Theta^+)\cap\mathcal{S} = \{H_\Theta\}. \tag{3.0.21}$$

However, the fiber $-H_\Theta + \mathfrak{n}_\Theta^-$ is

$$\begin{pmatrix} \frac{-\sqrt{-1}\pi}{n} qI_p & 0 \\ Z_{p\times q} & \frac{\sqrt{-1}\pi}{n} pI_q \end{pmatrix} \tag{3.0.22}$$

The element H_Θ is the linear combination $H_\Theta = 2\sqrt{-1}\pi q[H_{\alpha_1} + 2H_{\alpha_2} + \cdots + pH_{\alpha_p}] + 2\sqrt{-1}\pi p[(q-1)H_{\alpha_{p+1}} + (q-2)H_{\alpha_{p+2}} + \cdots + 2H_{\alpha_{p+q-2}} + H_{\alpha_l}]$ and

$$\mathcal{K}(\sqrt{-1}H_{\alpha_p}, H_\Theta) = -\pi.$$

4 Case C_l

Let G be the Lie group $Sp(l,\mathbb{C})$ and U be the compact Lie group $Sp(l) = Sp(l,\mathbb{C}) \cap SU(2l)$, with Lie algebras $\mathfrak{g}$ and $\mathfrak{u}$, respectively. The corresponding flag symmetric manifold is the space of complex structures on $\mathbb{H}^l$ compatible with the inner product

$$\mathbb{F}_\Theta = Sp(l)/U(l)$$

with $\Theta = \{\lambda_1 - \lambda_2, \cdots, \lambda_{l-1} - \lambda_l\}$ and each λ_i is given by

$$\lambda_i : \operatorname{diag}\{a_1, \cdots, a_l\} \mapsto a_i. \tag{4.0.1}$$

The Lie algebra $\mathfrak{u} := \mathfrak{sp}(l) = \mathfrak{sl}(l,\mathbb{C}) \cap \mathfrak{su}(2l)$ is the set

$$\mathfrak{sp}(l) = \left\{ \begin{pmatrix} a_{l\times l} & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a \in \mathfrak{u}(l), b: \text{ symmetric complex matrix} \right\} \tag{4.0.2}$$

which is a compact real form of the Lie algebra of G, denoted by $\mathfrak{g} := \mathfrak{sp}(l,\mathbb{C}) = \{A \in M(2l,\mathbb{C}) : AJ + JA^T = 0\}$, where $J = \begin{pmatrix} 0 & -Id_l \\ Id_l & 0 \end{pmatrix}$ with $J^2 = -Id_{2l}$. The elements of $\mathfrak{g}$ are $2l \times 2l$ complex matrices which can be represented in blocks as

$$\begin{pmatrix} a_{l\times l} & b \\ c & -a^T \end{pmatrix}, \qquad b, c: \text{ symmetric.} \tag{4.0.3}$$

Since a, b, c are complex matrices, the matrix above is quaternionic. Furthermore, the Lie group G is the set

$$Sp(l,\mathbb{C}) = \left\{ M \in M_{2l\times 2l}(\mathbb{C}) : M^T \Omega M = \Omega \right\}, \quad \Omega = \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \tag{4.0.4}$$

$$= \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in M_{l\times l}(\mathbb{C}), \det(M) = 1 \right\} \tag{4.0.5}$$

and

$$\begin{aligned} -C^T A + A^T C &= 0, \\ -C^T B + A^T D &= I_l, \\ -D^T B + B^T D &= 0. \end{aligned} \tag{4.0.6}$$

An element in the Cartan subalgebra $\sqrt{-1}\mathfrak{h}_\mathbb{R}$ of the compact real form $\mathfrak{u}$ that defines the symmetric flag manifold is

$$H_\Theta := \frac{\pi}{2} \begin{pmatrix} \sqrt{-1} Id_l & 0 \\ 0 & -\sqrt{-1} Id_l \end{pmatrix} \in \mathfrak{sp}(l) \subset \mathfrak{sp}(l,\mathbb{C}) \tag{4.0.7}$$

and its exponential is the matrix

$$g_\Theta := e^{H_\Theta} = \begin{pmatrix} \sqrt{-1}Id_l & 0 \\ 0 & -\sqrt{-1}Id_l \end{pmatrix} \tag{4.0.8}$$

Since $H_\Theta = \frac{\pi}{2} g_\Theta$, then the centralizer of H_Θ in G,

$$Z_\Theta = Z_G(g_\Theta). \tag{4.0.9}$$

and since

$$g_\Theta^2 = -Id_{2l} \in Z(U) \subset Z(G), \tag{4.0.10}$$

then the inner automorphism

$$\begin{array}{rccc} \sigma = C_{g_\Theta} : & Sp(l,\mathbb{C}) & \to & Sp(l,\mathbb{C}) \\ & h & \mapsto & C_{g_\Theta}(h) = g_\Theta h g_\Theta^{-1} \end{array} \tag{4.0.11}$$

is an involution ($\sigma^2(h) = g_\Theta^2 h g_\Theta^{-2} = h g_\Theta^2 g_\Theta^{-2} = h$) and its fixed points set is:

$$\begin{aligned} Z_G(g_\Theta) = Z_\Theta &= \left\{ h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(l,\mathbb{C}) : a,b,c,d \in M(l,\mathbb{C}), g_\Theta h = h g_\Theta \right\} \\ &= \left\{ h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(l,\mathbb{C}) : b = c = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Sp(l,\mathbb{C}) \right\} \end{aligned}$$

But also,

$$\begin{aligned} & \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Sp(l,\mathbb{C}) \\ \Leftrightarrow & \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^T \begin{pmatrix} 0 & Id_l \\ -Id_l & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & Id_l \\ -Id_l & 0 \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} 0 & a^T d \\ -d^T a & d \end{pmatrix} = \begin{pmatrix} 0 & Id_l \\ -Id_l & 0 \end{pmatrix} \\ \Leftrightarrow & d = (a^T)^{-1}. \end{aligned} \tag{4.0.12}$$

Then

$$\begin{aligned} Z_\Theta = & \left\{ \begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1} \end{pmatrix} \in Sp(l,\mathbb{C}) \right\} \\ \cong & \quad Gl(l,\mathbb{C}) \end{aligned} \tag{4.0.13}$$

Thus,

$$G/Z_G(g_\Theta) = Sp(l,\mathbb{C})/Gl(l,\mathbb{C}) \tag{4.0.14}$$

is a symmetric space with symmetric pair $(\mathfrak{sp}(l,\mathbb{C}), \mathfrak{sl}(l,\mathbb{C}) \oplus \mathbb{C}^*)$. In fact, let $A \in \mathfrak{gl}(n,\mathbb{C})$.

(i) If $A \in \mathfrak{sl}(n,\mathbb{C})$, then

$$A = \left[A + \begin{pmatrix} -1 & 0 \\ 0 & \dfrac{1}{n-1}I_{n-1}\end{pmatrix}\right] + \begin{pmatrix} 1 & 0 \\ 0 & -\dfrac{1}{n-1}I_{n-1}\end{pmatrix} \mapsto A' + \frac{1}{1-n} \in \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}^*.$$

(ii) If $A \notin \mathfrak{sl}(n,\mathbb{C})$, then $A = \left[A - \dfrac{tr(A)}{n}I_n\right] + \dfrac{tr(A)}{n}I_n \mapsto \left[A - \dfrac{tr(A)}{n}I_n\right] + \dfrac{tr(A)}{n} \in \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}^*$.

Hence $\mathfrak{gl}(n,\mathbb{C}) \subset \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}^*$. The other containment is straightforward. Hence, this symmetric Lie algebra is in Table A.3. By adding the results in [1], the cotangent bundle

$$G/Z_\Theta = G \cdot H_\Theta = T^*(U/U_\Theta) \tag{4.0.15}$$

is a symmetric space. Since $U_\Theta = Z_\Theta \cap Sp(l)$ and using the fact that $Sp(l) = Sp(l,\mathbb{C}) \cap SU(2l)$, we have that

$$\begin{aligned} U_\Theta &= \left\{\begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1}\end{pmatrix} \in SU(2l) : a_{l\times l} \text{ complexa}\right\} \\ &= \left\{\begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1}\end{pmatrix} \cdot \begin{pmatrix} \bar{a}^T & 0 \\ 0 & \bar{a}^{-1}\end{pmatrix} = Id_{2l}\right\} \subset \mathrm{Sp}(l,\mathbb{C}) \\ &= \left\{\begin{pmatrix} a & 0 \\ 0 & (a^T)^{-1}\end{pmatrix} \in SU(2l) : a_{l\times l} \in U(l)\right\} \\ &= U(l). \end{aligned} \tag{4.0.16}$$

Hence, the cotangent bundle of $Sp(l)/U(l)$:

$$T^*(Sp(l)/U(l)) = Sp(l,\mathbb{C})/Gl(l,\mathbb{C}) \tag{4.0.17}$$

is a symmetric space.

The Lie algebra $\mathfrak{g} = \mathfrak{sp}(l,\mathbb{C})$ has the real normal form $\mathfrak{g}_0 = \mathfrak{sp}(l,\mathbb{R})$

$$\mathfrak{g}_0 = \left\{\begin{pmatrix} a_{l\times l} & b \\ c & -a^T\end{pmatrix} \mid b, c \text{ symmetrics}\right\} \tag{4.0.18}$$

which has the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{s}$$

where

$$\mathfrak{k} = \left\{\begin{pmatrix} a_{l\times l} & -b \\ b & a\end{pmatrix} \mid a \in \mathfrak{o}(l), b : \text{symmetric}\right\} \cong \mathfrak{u}(l) \tag{4.0.19}$$

$$\mathfrak{s} = \left\{ \begin{pmatrix} a_{l\times l} & b \\ b & -a \end{pmatrix} : a, b \,\text{symmetrics} \right\}. \tag{4.0.20}$$

The lift of the automorphism in (4.0.11) to the Lie algebra restricted to $\mathfrak{g}_0$ is

$$\begin{array}{rccc} \sigma = \mathrm{Ad}(g_\Theta): & \mathfrak{sp}(l,\mathbb{R}) & \to & \mathfrak{sp}(l,\mathbb{R}) \\ & X & \mapsto & \mathrm{Ad}(g_\Theta)X = g_\Theta X g_\Theta^{-1}. \end{array} \tag{4.0.21}$$

It does not coincide with the Cartan involution $\theta(X) = -X^T$. However, we have the symmetric pairs $(\mathfrak{sp}(l), \mathfrak{u}(l))$ and $(\mathfrak{sp}(l,\mathbb{C}), \mathfrak{sl}(l,\mathbb{C}) \oplus \mathbb{C})$, since the Lie algebra of Z_Θ is $\mathfrak{gl}(l,\mathbb{C}) \cong \mathfrak{sl}(l,\mathbb{C}) \oplus \mathbb{C}$.

The tangent bundle of the cotangent bundle of $\mathbb{F}_\Theta$, as a homogeneous space, at the origin is isomorphic to

$$\mathfrak{m}_G = \mathfrak{n}_\Theta^- + \mathfrak{n}_\Theta^+ \tag{4.0.22}$$

$$= \left\{ \begin{pmatrix} 0 & Y_{l\times l} \\ X_{l\times l} & 0 \end{pmatrix} : X, Y \,\text{symmetric complex matrices} \right\} \tag{4.0.23}$$

$$= \mathfrak{m} + \sqrt{-1}\mathfrak{m} \tag{4.0.24}$$

where $\mathfrak{n}_\Theta^\pm$ is the set of matrices in (4.0.23) with $X = 0$ and $Y = 0$, respectively. The tangent bundle of the flag symmetric manifold at the origin is isomorphic to

$$\mathfrak{m} \cong \sum_{\alpha \in \Pi^+ \setminus \langle\Theta\rangle^+} \mathfrak{u}_\alpha \tag{4.0.25}$$

$$= \left\{ \begin{pmatrix} 0 & B \\ -\bar{B} & 0 \end{pmatrix} : B_{l\times l} \,\text{symmetric complex} \right\} \tag{4.0.26}$$

The subspace $\mathfrak{u}_\alpha$ =span$_\mathbb{R}\{A_\alpha, Z_\alpha\}$ is generated by $A_\alpha = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ -E_{ij} - Eji & 0 \end{pmatrix}$ and $Z_\alpha = \begin{pmatrix} 0 & \sqrt{-1}(E_{ij} + E_{ji}) \\ \sqrt{-1}(E_{ij} + E_{ji}) & 0 \end{pmatrix}$ with $\alpha \in \Pi^+ \setminus \langle\Theta\rangle^+ = \{\lambda_i + \lambda_j, 1 \leqslant i, j \leqslant l\}$, for $E_{i,j} = \delta_i^j Id$.

The involution σ for the symmetric Lie algebra $(\mathfrak{sp}(l,\mathbb{C}), \mathfrak{gl}(l,\mathbb{C}), \sigma)$ is the following automorphism, with $H_\Theta \in \mathfrak{sp}(l)$ as in 4.0.7,

$$\mathrm{Ad}(e^{H_\Theta}) : \begin{pmatrix} A_{l\times l} & B \\ C & -A^T \end{pmatrix} \in \mathfrak{sp}(l,\mathbb{C}) \mapsto \begin{pmatrix} A & -B \\ -C & -A^T \end{pmatrix}. \tag{4.0.27}$$

The involution above is also an automorphism of $\mathfrak{sp}(l)$ and this symmetric Lie algebra has the canonical decomposition

$$\mathfrak{sp}(l) = \mathfrak{u}_\Theta + \mathfrak{m}, \tag{4.0.28}$$

with $\mathfrak{m}$ as in (4.0.26) and $\mathfrak{u}_\Theta$ being the subalgebra such that $\sigma(W) = W$, for all $W \in \mathfrak{u}_\Theta$

$$\mathfrak{u}_\Theta = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : A \in \mathfrak{u}(l) \right\} \cong \mathfrak{u}(l). \tag{4.0.29}$$

Then the dual symmetric Lie algebra of $(\mathfrak{sp}(l), \mathfrak{u}(l), \sigma)$ must have the canonical decomposition

$$\mathfrak{sp}(l)^* = \mathfrak{u}_\Theta + \sqrt{-1}\mathfrak{m}. \tag{4.0.30}$$

The set $\sqrt{-1}\mathfrak{m}$ is the vector space satisfying $\sigma(W) = -W$, for all $W \in \sqrt{-1}\mathfrak{m}$, it is

$$\left\{ \begin{pmatrix} 0 & B_{l\times l} \\ \bar{B} & 0 \end{pmatrix} : A \text{ symmetric complex matrix} \right\} \tag{4.0.31}$$

and then

$$\mathfrak{sp}(l)^* = \left\{ \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} : A \in \mathfrak{u}(l), B \text{ symmetric complex matrix} \right\}. \tag{4.0.32}$$

Let us define the following function

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{sp}(l)^* \mapsto \begin{pmatrix} \mathrm{Re}(A) + \mathrm{Re}(B) & \mathrm{Im}(B) - \mathrm{Im}(A) \\ \mathrm{Im}(A) + \mathrm{Im}(B) & \mathrm{Re}(A) - \mathrm{Re}(B) \end{pmatrix} \in \mathfrak{sp}(l, \mathbb{R}). \tag{4.0.33}$$

It is an isomorphism of Lie algebras since $A \in \mathfrak{u}(l)$ and B is symmetric. The function above is known as the Cayley transform. Furthermore,

$$\mathfrak{u}_\Theta \cong \mathfrak{k} \cong \mathfrak{u}(l), \qquad \sqrt{-1}\mathfrak{m} \cong \mathfrak{s}, \tag{4.0.34}$$

where $\mathfrak{sp}(l) = \mathfrak{k} + \mathfrak{s}$ is the Cartan decomposition indicated in 4.0.19 and 4.0.20. Hence, the dual symmetric Lie algebra is isomorphic to $(\mathfrak{sp}(l, \mathbb{R}), \mathfrak{k}, \theta)$, with θ being the Cartan involution $\theta(X) = \Omega X \Omega^{-1}$, where Ω is as in 4.0.4.

The function defined in 4.0.33 transforms $H_\Theta \in \mathfrak{sp}(l)$ in

$$\hat{H}_\Theta := \frac{\sqrt{-1}\pi}{2} \begin{pmatrix} 0 & -I_{l\times l} \\ I_{l\times l} & 0 \end{pmatrix} \in \mathfrak{sp}(l, \mathbb{R}). \tag{4.0.35}$$

The dual symmetric space $Sp(l)^*/U_\Theta \cong Sp(l, \mathbb{R})/U(l)$ is diffeomorphic to $\mathcal{S} = \exp \sqrt{-1}\mathfrak{m} \cdot H_\Theta \cong \exp \mathfrak{s} \cdot \hat{H}_\Theta$. To prove Proposition 2.1.2, let us calculate the intersection $\mathcal{S} \cap (H_\Theta + \mathfrak{n}_\Theta^+)$. Initially, note that

$$H_\Theta \in (H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathcal{S} \subset \mathfrak{sp}(l)^* \cap (H_\Theta + \mathfrak{n}_\Theta^+). \tag{4.0.36}$$

To calculate the intersection $\mathfrak{sp}(l)^* \cap (H_\Theta + \mathfrak{n}_\Theta^+)$, we describe the elements of the fiber passing through the origin H_Θ:

$$\left\{ \begin{pmatrix} \frac{\sqrt{-1}\pi}{2} I_{l\times l} & Z \\ 0 & -\frac{\sqrt{-1}\pi}{2} I_{l\times l} \end{pmatrix} : Z \in M_{l\times l}(\mathbb{C}) \right\}. \tag{4.0.37}$$

with $Z \in \mathfrak{n}_\Theta^+$. These matrices belong to $\mathfrak{sp}(l)^*$ if and only if they belong to the set in 4.0.32, i.e. Z is a symmetric complex matrix and $-\bar{Z} = 0$. It means that the intersection $\mathfrak{sp}(l)^* \cap (H_\Theta + \mathfrak{n}_\Theta^+) \subset \{H_\Theta\}$. As from (4.0.36), we conclude that

$$\mathfrak{sp}(l)^* \cap (H_\Theta + \mathfrak{n}_\Theta^+) = \{H_\Theta\}. \tag{4.0.38}$$

Thus,

$$H_\Theta \in ((H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathcal{S}) \subset (H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathfrak{sp}(l)^* = \{H_\Theta\}. \tag{4.0.39}$$

Hence,

$$(H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathcal{S} = \{H_\Theta\}. \tag{4.0.40}$$

The element H_Θ is the linear combination $2\pi(l+1)\sqrt{-1}[H_{\lambda_1-\lambda_2} + 2H_{\lambda_2-\lambda_3} + \cdots + (l-1)H_{\lambda_{l-1}-\lambda_l}] + l(l+1)\pi\sqrt{-1}H_{2\lambda_l}$ and

$$\mathcal{K}(\sqrt{-1}H_{2\lambda_l}, H_\Theta) = -4\pi(l+1).$$

5 Case D_l

Let G be the Lie group $SO(2l, \mathbb{C})$ and U be the compact Lie subgroup $SO(2l)$, with Lie algebras $\mathfrak{g} := \mathfrak{so}(2l, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -b^T & d \end{pmatrix} \in \mathfrak{gl}(2l, \mathbb{C}) : a_{l\times l}, d_{l\times l} \, \text{skew} - \text{symmetric} \right\}$ and $\mathfrak{u} := \mathfrak{so}(2l)$ respectively. The corresponding symmetric flag manifold is the space of orthogonal complex structures on $\mathbb{R}^{2l}$

$$\mathbb{F}_\Theta = SO(2l)/U(l)$$

with $\Theta = \{\lambda_1 - \lambda_2, \cdots, \lambda_{l-1} - \lambda_l\}$ and $\Sigma = \Theta \cup \{\lambda_{l-1} + \lambda_l\}$ where each λ_i was defined in 4.0.1.

As $\mathfrak{so}(2l, \mathbb{C})$ is defined over an algebraically closed field; if two not degenerated quadratic forms are equivalents, then the algebras of anti-symmetric matrices with respect to the quadratic forms are both isomorphic.

Let be suppose that

$$F = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}, \qquad f = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{-1} I_l & I_l \\ I_l & \sqrt{-1} I_l \end{pmatrix}. \tag{5.0.1}$$

Then

$$F = f^T I_{2l} f \tag{5.0.2}$$

and hence, the Lie algebra is isomorphic to

$$\mathfrak{g}_1 = \{A \in \mathfrak{sl}(2l, \mathbb{C}) : AF + FA^T = 0\} \tag{5.0.3}$$

Thus, $A \in \mathfrak{so}(2l, \mathbb{C})$ if and only if $fAf^{-1} \in \mathfrak{g}_1$. In other words $f\mathfrak{so}(2l, \mathbb{C})f^{-1} = \mathfrak{g}_1$ and the algebras $\mathfrak{so}(2l, \mathbb{C})$ and $\mathfrak{g}_1$ are isomorphic. To facilitate the process, we will use the Lie algebra $\mathfrak{g}_1$ instead of $\mathfrak{so}(2l)$. We shall denote by G_1 the connected Lie group with Lie algebra $\mathfrak{g}_1$.

In [6], a Cartan subalgebra of $\mathfrak{g}_1$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} : \Lambda_{l\times l} \text{ diagonal matrix} \right\}. \tag{5.0.4}$$

An element in the Cartan subalgebra $\sqrt{-1}\mathfrak{h}_\mathbb{R}$ of the compact real form $f\mathfrak{u}f^{-1} = f\mathfrak{so}(2l)f^{-1}$ that defines the symmetric flag manifold $\mathbb{F}_\Theta$ is

$$\hat{H}_\Theta := \frac{\pi}{2}\sqrt{-1} \begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{l\times l} \end{pmatrix} \in \sqrt{-1}\mathfrak{h}_\mathbb{R} \subset \mathfrak{g}_1 \tag{5.0.5}$$

and

$$g_\Theta := exp(\hat{H}_\Theta) = \sqrt{-1}\begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{l\times l} \end{pmatrix}. \tag{5.0.6}$$

Note that

$$g_\Theta^2 = -Id_{2l} \in Z(G_1) \tag{5.0.7}$$

Thus, the inner automorphism

$$\begin{array}{rccc} \sigma = C_{g_\Theta}: & G_1 & \to & G_1 \\ & A & \mapsto & C_{g_\Theta}(A) = g_\Theta A g_\Theta^{-1} \end{array}$$

is an involution, by 5.0.7, because of $(C_{g_\Theta})^2(A) = g_\Theta^2 A(g_\Theta^{-1})^2 = A$. Let's calculate the fixed points set, which is the centralizer of g_Θ in G_1. Since $H_\Theta = \frac{\pi}{2} g_\Theta$, then $Z_{G_1}(H_\Theta) = Z_{G_1}(g_\Theta)$, in fact

$$\begin{aligned} Z_{G_1}(g_\Theta) =& \{A \in G_1 : C_{g_\Theta}(A) = A\} \\ =& \{A \in G_1 : g_\Theta A g_\Theta^{-1} = A\} \\ =& \{A \in G_1 : H_\Theta A = \frac{\pi}{2} A g_\Theta\} \\ =& \{\begin{pmatrix} x_{l\times l} & y \\ z & w_{l\times l} \end{pmatrix} \in G_1 : H_\Theta A = \frac{\pi}{2}\sqrt{-1}\begin{pmatrix} x & -y \\ z & -w \end{pmatrix}\} \\ =& \{A = \begin{pmatrix} x_{l\times l} & y \\ z & w_{l\times l} \end{pmatrix} \in G_1 : H_\Theta A = \begin{pmatrix} \frac{\sqrt{-1}\pi}{2} x & -\frac{\sqrt{-1}\pi}{2} y \\ \frac{\sqrt{-1}\pi}{2} z & -\frac{\sqrt{-1}\pi}{2} w \end{pmatrix}\} \\ =& \{A = \begin{pmatrix} x_{l\times l} & y \\ z & w_{l\times l} \end{pmatrix} \in G_1 : H_\Theta A = A H_\Theta\} \\ =& Z_{G_1}(H_\Theta). \end{aligned} \tag{5.0.8}$$

On the other hand, the set of fixed points of $\sigma =$Ad(g_Θ) (which coincides with 4.0.27) in $\mathfrak{g}_1$ is

$$\begin{aligned} & \left\{A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \in \mathfrak{g}_1 : a,b,c,d \in M(l,\mathbb{C}), g_\Theta A g_\Theta^{-1} = A\right\} \\ =& \left\{A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \in \mathfrak{g}_1 : b = 0 = c\right\} \\ =& \{\begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix} \in \mathfrak{sl}(2l,\mathbb{C})\} \\ \cong& \mathfrak{gl}(l,\mathbb{C}). \end{aligned} \tag{5.0.9}$$

Using (5.0.8) and (5.0.9), we can see that

$$G/Z_\Theta \cong SO(2l,\mathbb{C})/Gl(l,\mathbb{C})$$

is a symmetric space. Its corresponding symmetric pair is $(\mathfrak{so}(2l,\mathbb{C}),\mathfrak{sl}(l,\mathbb{C})\oplus\mathbb{C}^*)$. In fact, the prove of this is in 4. Hence, this symmetric Lie algebra is in Table A.3. By adding the results in [1], the cotangent bundle

$$G/Z_\Theta = G\cdot H_\Theta = T^*(U/U_\Theta)$$

is a symmetric space. In order to find out the set U_Θ. Since $U = SO(2l)$, let us use the matrix $H_\Theta = f^{-1}\hat{H}_\Theta f \in \mathfrak{so}(2l)$. This matrix is

$$H_\Theta := \frac{\pi}{2}\begin{pmatrix} 0 & Id_l \\ -Id_l & 0 \end{pmatrix} \in \mathfrak{so}(2l,\mathbb{R}) \subset \mathfrak{so}(2l,\mathbb{C}). \tag{5.0.10}$$

Then

$$\begin{aligned} U_\Theta &= \left\{ u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2l) : a,b \in M_{l\times l}(\mathbb{R}), uH_\Theta = H_\Theta u \right\} \\ &= \left\{ \begin{pmatrix} a_{l\times l} & -b \\ b & a \end{pmatrix} \in Sl(2l,\mathbb{R}) : aa^T + bb^T = I_{l\times l}, ba^T = ab^T \right\} \\ &\cong \{a+ib \in Gl(l,\mathbb{C}) : (a+ib)(a+ib)^* = Id\} \\ &= U(l) \end{aligned} \tag{5.0.11}$$

Hence, the cotangent bundle of $SO(2l)/U(l)$:

$$T^*(SO(2l)/U(l)) \cong SO_{\mathbb{C}}(2l)/Gl(l,\mathbb{C}) \tag{5.0.12}$$

is also a symmetric space.

The tangent space of the cotangent bundle of $\mathbb{F}_\Theta$ at the origin is isomorphic to

$$\begin{aligned} \mathfrak{m}_G &= \mathfrak{n}_\Theta^- + \mathfrak{n}_\Theta^+ \\ &= \mathfrak{m} + \sqrt{-1}\mathfrak{m} \\ &\cong \left\{ \begin{pmatrix} 0 & A_{l\times l} \\ B_{l\times l} & 0 \end{pmatrix} : A, B \text{ anti}-\text{symmetric complex} \right\} \end{aligned} \tag{5.0.13}$$

where $\mathfrak{n}_\Theta^\pm$ are isomorphic to the subalgebras of matrices in 5.0.13 with $X = 0$ and $Y = 0$, respectively. The tangent space of the flag symmetric manifold at the origin b_Θ is isomorphic to

$$\mathfrak{m} \cong \sum_{\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{u}_\alpha \tag{5.0.14}$$

$$= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A, B \in \mathfrak{so}(l) \right\}. \tag{5.0.15}$$

Another real form of $\mathfrak{so}(2l,\mathbb{C})$ is $\mathfrak{g}_0 := \mathfrak{so}^*(2l) := \mathfrak{sl}(l,\mathbb{H}) \cap \mathfrak{so}(2l,\mathbb{C})$ and its elements are $2l \times 2l$ complex matrices of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix}, \qquad Z_1^T = -Z_1,\ \bar{Z_2}^T = Z_2, \tag{5.0.16}$$

where Z_1 and Z_2 are $l \times l$ complex matrices.

The involution σ in $\mathfrak{g}_0$ coincides with the Cartan involution and then, the Lie algebra $\mathfrak{g}_0$ has the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{s}$ with $\mathfrak{k} = \mathfrak{u}_\Theta \cong \mathfrak{u}(l)$ is the Lie algebra of U_Θ and $\mathfrak{s} = \sqrt{-1}\mathfrak{m}$.

In order to prove Proposition 2.1.2 let us study the intersection of the submanifold $\mathcal{S} = \exp\mathfrak{s} \cdot H_\Theta$ with the fiber passing through the origin. The dual symmetric space $SO(2l)^*/S(U(p) \times U(q))$ is diffeomorphic to $\mathcal{S}$ and the dual symmetric Lie algebra of $(\mathfrak{so}(2l)^*, \mathfrak{u}_\Theta, \sigma)$, with $\sigma =$Ad$(\exp(H_\Theta))$ (5.0.10) being the function

$$\sigma : \begin{pmatrix} A & B_{l\times l} \\ -B^T & D \end{pmatrix} \in \mathfrak{so}(2l)^* \mapsto \begin{pmatrix} D & B^T \\ -B & A \end{pmatrix} \in \mathfrak{so}(2l)^*, \tag{5.0.17}$$

is

$$\mathfrak{so}(2l)^* = \mathfrak{u}(l) + \sqrt{-1}\mathfrak{m}, \tag{5.0.18}$$

with $\mathfrak{m}$ as in (5.0.15) and $\mathfrak{u}_\Theta$ as the subalgebra isomorphic to $\mathfrak{u}(l)$:

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{pmatrix} : Z_1 \in \mathfrak{so}(l), Z_2 \text{ symmetric matrix} \right\} = \mathfrak{so}^*(2l) \cap \mathfrak{so}(2l) \tag{5.0.19}$$

Then

$$\sqrt{-1}\mathfrak{m} = \left\{ \begin{pmatrix} \sqrt{-1}A & \sqrt{-1}B \\ \sqrt{-1}B & -\sqrt{-1}A \end{pmatrix} : A, B \in \mathfrak{so}(l) \right\} = \mathfrak{so}^*(2n) \cap \sqrt{-1}\mathfrak{so}(2l). \tag{5.0.20}$$

Since $H_\Theta \in \mathfrak{so}^*(2l) = \mathfrak{sl}(l,\mathbb{H}) \cap \mathfrak{so}(2l,\mathbb{C})$, then σ is an automorphism of $\mathfrak{so}^*(2l)$ and coincides with σ^*. Hence, the dual symmetric Lie algebra is

$$\mathfrak{so}^*(2l) = \mathfrak{so}(2l)^\sigma + \sqrt{-1}\mathfrak{m} \tag{5.0.21}$$

$$= \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} : Z_1 \in \mathfrak{so}(l,\mathbb{C}), \bar{Z}_2^T = Z_2 \right\}. \tag{5.0.22}$$

Let us consider the isomorphism of $\mathfrak{so}(2n,\mathbb{C})$ with the algebra in 5.0.3. Thus, the fiber via this isomorphism is the affine vector space $\widetilde{H_\Theta + \mathfrak{n}_\Theta^+}$ defined by

$$\hat{H}_\Theta + f\mathfrak{n}_\Theta^+ f^{-1} = \left\{ \begin{pmatrix} \frac{\sqrt{-1}\pi}{2} I_l & Y \\ 0 & -\frac{\sqrt{-1}\pi}{2} I_l \end{pmatrix} : Y \in \mathfrak{so}(l,\mathbb{C}) \right\}. \tag{5.0.23}$$

Since $f\mathfrak{so}(2n,\mathbb{C})f^{-1} = \mathfrak{g}_1$, with f as in (5.0.1) and $\mathfrak{g}_1$ in (5.0.3), we can back to the original fiber, calculating $f^{-1}Mf$, for every $M \in H_\Theta + \mathfrak{n}_\Theta^+$, where

$$f^{-1} = \begin{pmatrix} -\dfrac{\sqrt{-1}}{\sqrt{2}}I_l & \dfrac{1}{\sqrt{2}}I_l \\ \dfrac{1}{\sqrt{2}}I_l & -\dfrac{\sqrt{-1}}{\sqrt{2}}I_l \end{pmatrix}.$$

Then, $H_\Theta + \mathfrak{n}_\Theta^+ =$

$$\left\{ f^{-1}\begin{pmatrix} \dfrac{\sqrt{-1}\pi}{2}I_l & Y \\ 0 & -\dfrac{\sqrt{-1}\pi}{2}I_l \end{pmatrix} f = \begin{pmatrix} -\dfrac{\sqrt{-1}}{2}Y & \dfrac{\pi}{2}I_l + \dfrac{1}{2}Y \\ -\dfrac{\pi}{2}I_l + \dfrac{1}{2}Y & \dfrac{\sqrt{-1}}{2}Y \end{pmatrix} : Y \in \mathfrak{so}(l,\mathbb{C}) \right\}. \tag{5.0.24}$$

Thus, the elements of the fiber has the form

$$\begin{pmatrix} \dfrac{1}{2}\mathrm{Im}(Y) & \dfrac{\pi}{2}I_l + \dfrac{1}{2}\mathrm{Re}(Y) \\ -\dfrac{\pi}{2}I_l + \dfrac{1}{2}\mathrm{Re}(Y) & -\dfrac{1}{2}\mathrm{Im}(Y) \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \dfrac{-1}{2}\mathrm{Re}(Y) & \dfrac{1}{2}\mathrm{Im}(Y) \\ \dfrac{1}{2}\mathrm{Im}(Y) & \dfrac{1}{2}\mathrm{Re}(Y) \end{pmatrix}, \tag{5.0.25}$$

with $Y \in \mathfrak{so}(l,\mathbb{C})$. Furthermore, the dual symmetric Lie algebra $\mathfrak{so}(2l)^*$, is the set of matrices

$$\begin{pmatrix} A+\sqrt{-1}C & B+\sqrt{-1}D \\ -B+\sqrt{-1}D & A-\sqrt{-1}C \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} + \sqrt{-1}\begin{pmatrix} C & D \\ D & -C \end{pmatrix}, \tag{5.0.26}$$

with $A, C, D \in \mathfrak{so}(l,\mathbb{R})$ and B being a real symmetric matrix.

Looking at the real part of matrices in (5.0.25) and comparing with the real part of matrices in (5.0.26), we can conclude that a matrix in the fiber belongs to $\mathfrak{so}(2l)^*$ if and only if $\mathrm{Im}(Y)$ and $\mathrm{Re}(Y)$ are equal to the null matrix, and then $Y = 0$. Hence

$$(H_\Theta + \mathfrak{n}_\Theta^+) \cap \mathfrak{so}(2l)^* = \{H_\Theta\}. \tag{5.0.27}$$

The element $\hat{H}_\Theta$ is the linear combination $2\pi(l-1)\sqrt{-1}[H_{\lambda_1-\lambda_2} + 2H_{\lambda_2-\lambda_3} + \cdots + (l-2)H_{\lambda_{l-2}-\lambda_{l-1}}] + (l-1)\pi\sqrt{-1}[(l-2)H_{\lambda_{l-1}-\lambda_l} + lH_{\lambda_{l-1}+\lambda_l}]$ and

$$\mathcal{K}(\sqrt{-1}H_{\lambda_{l-1}+\lambda_l}, \hat{H}_\Theta) = -4\pi(l-1).$$

6 Conclusions and open questions

The cotangent bundle of a flag symmetric space $\mathbb{F}_\Theta = U/U_\Theta$ is an affine symmetric manifold. There exists an element $H_\Theta \in \sqrt{-1}\mathfrak{h}_\mathbb{R}$ in the Cartan subalgebra of $\mathfrak{u}$ such that $g_\Theta := \exp H_\Theta$ satisfies $Z_U(H_\Theta) = Z_U(g_\Theta)$, $U_\Theta = Z_U(H_\Theta)$ and (U, U_Θ, σ) and $(G, Z_G(H_\Theta), \sigma)$ are symmetric spaces for $\sigma = C_{g_\Theta}$.

The projection $T^*\mathbb{F}_\Theta \to \mathbb{F}_\Theta$ induces an injective map from the submanifold $\mathcal{S}$ to the flag symmetric space $\mathbb{F}_\Theta$. The submanifold $\mathcal{S}$ is a Lagrangian submanifold of the cotangent bundle $T^*\mathbb{F}_\Theta$ with the symplectic form $\mathrm{Im}\mathcal{K}(\cdot, \tau\cdot)$ given by the Cartan-Killing form $\mathcal{K}$ and the conjugation τ with respect to the compact real form $\mathfrak{u}$.

Considering the canonical connection defined by the Cartan-Killing form, the geodesics are one-parameter curves. We asked ourselves the question whether the projection of a geodesic in the cotangent bundle was a geodesic on the flag manifold, however that question was only answered for geodesics on fibers and on the zero section, which are trivial cases, therefore the question is open.

Another realization of the adjoint orbit indicated in [1] is that of submanifold of a product of flag manifolds, the cotangent bundle is the orbit of the diagonal action of the group G passing through the origin in the product of the flag manifold $\mathbb{F}_\Theta$ with its dual flag. This orbit is dense in the compact manifold. An interesting problem would be to see what the $\mathcal{S}$ manifold is like inside in that realization, in addition to its geodesics, which are uniparametric curves, and its projection on the flag manifold $\mathbb{F}_\Theta$.

In Tojo's paper [10] it is shown that every flag manifold U/U_Θ is a k-symmetric space. A natural question is: Does the automorphism of order k on U extended to G also make the cotangent fibration of the flag manifold a k-symmetric space, and will the automorphism σ be inner? In fact, the problem of whether the cotangent bundle of any flag manifold is k-symmetric was an inspiration for this Thesis and allows us to continue to deepen our study.

Bibliography

[1] E. Gasparim, L. Grama and L. A. B. San Martin, *Adjoint orbits of semi-simple Lie groups and Lagrangian Submanifolds*, EMS , (2014).

[2] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 1, No. 2, New York, London, (1963).

[3] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol II. (1969), Interscience Publishers.

[4] S. Helgason, *Differential geometry and symmetric spaces*, Vol. 341. American Mathematical Soc., (2001).

[5] L. A. B. San Martin, *Grupos de lie*, (2012).

[6] L. A. B. San Martin, *Algebras de Lie*, Unicamp, (2010).

[7] M. Berger, *"Les espaces symétriques noncompacts*," Annales scientifiques de l'École Normale Supérieure, Vol. 74, No. 2. (1957).

[8] E. Gasparim, L. Grama, and L. A. B. San Martin, *Symplectic Lefschetz fibrations on adjoint orbits*, Forum Mathematicum. Vol. 28. No. 5. De Gruyter, (2016).

[9] A. L. Besse, *Einstein manifolds*, Springer Science and Business Media, (2007).

[10] K. Tojo, *Kähler C-spaces and k-symmetric spaces*, Osaka Journal of Mathematics, v. 34, n. 4, p. 803-820, (1997).

[11] D.V. Alekseevsky, *Flag manifolds*, Sbornik Radova, v. 11, p. 3-35, (1997).

[12] Abraham, Ralph, and Jerrold E. Marsden, *Foundations of mechanics*, No. 364, American Mathematical Soc., (2008).

[13] J. Eschenburg, *Lecture Notes on Symmetric Spaces*, Minicurso, (1997), (site: : https://www.researchgate.net/publication/241529230)

[14] O. Kowalski, *Generalized Symmetric Spaces*, Lect. notes in Math. Berlin, Springer, 805 (1980).

[15] D. Alekseevsky, and A. Arvanitoyeorgos, *Riemannian flag manifolds with homogeneous geodesics*, Transactions of the American Mathematical Society 359.8 (2007): 3769-3789.

[16] A. Da Silva, and A. Cannas Da Silva. *Lectures on symplectic geometry.* Vol. 3575. Berlin: Springer, (2008).

[17] V. Guillemin, and A. Pollack. *Differential topology.* Vol. 370. American Mathematical Soc., (2010).

[18] A. Weinstein. *Poisson structures and Lie algebras.* Astérisque, hors série, p. 421-434, (1985).

A Appendix

A.1 Classical semi-simple Lie algebras

This list of classical Lie algebras with its real forms are in chapter 12 of [6].

	Complex algebra	Compact real form	Type	Non-compact real forms
A_l, $l \geqslant 1$ $n = l+1$	$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n)$	AI	$\mathfrak{sl}(n, \mathbb{R})$
			AII	$\mathfrak{sl}(n/2, \mathbb{H})$, n even
			AIII	$\mathfrak{su}(p, q)$, $p \leqslant q$, $p+q=n$
B_l, l$\geqslant$ 2	$\mathfrak{so}(2l{+}1, \mathbb{C})$	$\mathfrak{so}(2l+1)$	BI	$\mathfrak{so}(p, q)$, $p \leqslant q$, $p+q = 2l+1$
C_l, $l \geqslant 3$	$\mathfrak{sp}(l, \mathbb{C})$	$\mathfrak{sp}(l) =$ $\mathfrak{sp}(l, \mathbb{C}) \cap \mathfrak{su}(2l)$	CI	$\mathfrak{sp}(l, \mathbb{R})$
			CII	$\mathfrak{sp}(p, q)$, $p+q=l$
D_l, $l \geqslant 4$	$\mathfrak{so}(2l, \mathbb{C})$	$\mathfrak{so}(2l)$	DI	$\mathfrak{so}(p, q)$, $p \leqslant q$, $p+q=2l$
			DIII	$\mathfrak{sl}(l, \mathbb{H}) \cap \mathfrak{so}(2l, \mathbb{C})$

A.2 Complex flag manifolds

The complex flag manifolds of the classical Lie groups are classified in [15]. Here, consider $n = n_1 + \cdots + n_s$, $n_1 \geqslant \cdots \geqslant n_s \geqslant 1$ and $l = l_1 + \cdots + l_k + m$, $l_1 \geqslant \cdots \geqslant l_k \geqslant 1$, $k, m \geqslant 0$.

	Flag manifold
A	$SU(n)/S(U(n_1) \times \cdots \times U(n_s))$
B	$SO(2l+1)/U(l_1) \times \cdots \times U(l_k) \times SO(2m+1)$
C	$Sp(l)/U(l_1) \times \cdots \times U(l_k) \times Sp(m)$
D	$SO(2l)/U(l_1) \times \cdots \times U(l_k) \times SO(2m)$

A.3 Symmetric spaces

In [4] we found the following table, which is a characterization of irreducible Riemannian symmetric spaces of type I and III (see the theorems 1.4.13 and 1.4.14), with $p+q=n$. The symmetric spaces in red and blue are the flag symmetric spaces and its dual symmetric spaces respectively.

	Compact symmetric space	Non-compact symmetric type
AI	$SU(n)/SO(n)$	$Sl(n,\mathbb{R})/SO(n)$
AII	$SU(2n)/Sp(n)$	$SU^*(2n)/Sp(n)$
AIII	$SU(p+q)/S(U(p)\times U(q))$	$SU(p,q)/S(U(p)\times U(q))$
BDI	$SO(p+q)/SO(p)\times SO(q)$	$SO_o(p,q)/SO(p)\times SO(q)$
DIII	$SO(2n)/U(n)$	$SO^*(2n)/U(n)$
CI	$Sp(n)/U(n)$	$Sp(n,\mathbb{R})/U(n)$
CII	$Sp(p+q)/Sp(p)\times Sp(q)$	$Sp(p,q)/Sp(p)\times Sp(q)$

In [7], Berger shows a list of all symmetric Lie algebras $(\mathfrak{g},\mathfrak{h},\sigma)$ that we will show here. The symmetric Lie algebras showed in chapters 3, 4 and 5 are in pink in the following tables.

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{su}^*(2n)$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(p,\mathbb{C})+\mathfrak{sl}(n-p,\mathbb{C})+\mathbb{C}^*$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(p,n-p)$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(p,\mathbb{R})+\mathfrak{sl}(n-p,\mathbb{R})+\mathfrak{so}(2)$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(p,n-p)$
$\mathfrak{sl}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{sl}(2n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})+\mathfrak{so}(2)$
$\mathfrak{su}(n,\mathbb{R})$	$\mathfrak{so}(n)$
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$
$\mathfrak{su}(2n)$	$\mathfrak{sp}(n)$
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p)+\mathfrak{su}(q)+\mathfrak{so}(2)$
$\mathfrak{su}^*(2n)$	$\mathfrak{su}^*(2p)+\mathfrak{su}(2n-2p)+\mathfrak{so}(2)$
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(p,n-p)$
$\mathfrak{su}^*(2n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{su}^*(2n)$	$\mathfrak{sl}(n,\mathbb{C})+\mathfrak{so}(2)$

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{su}(p, n-p)$	$\mathfrak{su}(k, h)+\mathfrak{su}(p-k, n-k-h)+\mathfrak{so}(2)$
$\mathfrak{su}(p, n-p)$	$\mathfrak{so}(p, n-p)$
$\mathfrak{su}(2p, 2n-2p)$	$\mathfrak{sp}(p, n-p)$
$\mathfrak{su}(n, n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{su}(n, n)$	$\mathfrak{sp}(n, \mathbb{R})$
$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) + \mathfrak{so}(2)$
$\mathfrak{so}^*(2n)$	$\mathfrak{su}(n) + \mathfrak{so}(2)$
$\mathfrak{so}(2n)$	$\mathfrak{su}(n) + \mathfrak{so}(2)$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathbb{C}^*$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(p, n-p)$	$\mathfrak{so}(p) + \mathfrak{so}(n-p)$
$\mathfrak{so}(n)$	$\mathfrak{so}(p) + \mathfrak{so}(n-p)$
$\mathfrak{so}(2, n-2)$	$\mathfrak{so}(n-2) + \mathfrak{so}(2)$
$\mathfrak{so}(n)$	$\mathfrak{so}(n-2) + \mathfrak{so}(2)$
$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{so}(p, \mathbb{C}) + \mathfrak{so}(n-p, \mathbb{C})$
$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{so}(n-2) + \mathbb{C}^*$
$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{so}(p, n-p)$
$\mathfrak{so}^*(2n, \mathbb{C})$	$\mathfrak{so}^*(2p) + \mathfrak{so}^*(2n-2p)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n-2) + \mathfrak{so}(2)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n, \mathbb{C})$
$\mathfrak{so}^*(2n)$	$\mathfrak{su}(p, n-p) + \mathfrak{so}(2)$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) + \mathfrak{so}(2)$
$\mathfrak{so}(p, n-p)$	$\mathfrak{so}(k, h) + \mathfrak{so}(p-k, n-h-p)$
$\mathfrak{so}(p, n-p)$	$\mathfrak{so}(p-2, n-p) + \mathfrak{so}(2)$
$\mathfrak{so}(p, n-p)$	$\mathfrak{so}(p-2, n-p) + \mathfrak{so}(2)$
$\mathfrak{so}(2p, 2n-2p)$	$\mathfrak{su}(p, n-p) + \mathfrak{so}(2)$
$\mathfrak{so}(n, n)$	$\mathfrak{sl}(n, \mathbb{R}) + \mathfrak{so}(2)$
$\mathfrak{so}(n, n)$	$\mathfrak{so}(n, \mathbb{C})$

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(n)+\mathfrak{so}(2)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})+\mathbb{C}^*$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{sp}(q,n-q)$	$\mathfrak{sp}(q)+\mathfrak{sp}(n-q)$
$\mathfrak{sp}(n)$	$\mathfrak{sp}(q)+\mathfrak{sp}(n-q)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(q,\mathbb{C})+\mathfrak{sp}(n-q,\mathbb{C})$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(q,n-q)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(q,\mathbb{R})+\mathfrak{sp}(n-q,\mathbb{R})$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(q,n-q)+\mathfrak{so}(2)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(q,n-q)+\mathfrak{so}(2)$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{sp}(i,n-i)$	$\mathfrak{su}(k,h)+\mathfrak{su}(i-k,n-k-h)$
$\mathfrak{sp}(p,n-p)$	$\mathfrak{su}(p,n-p)+\mathfrak{so}(2)$
$\mathfrak{sp}(n,n)$	$\mathfrak{su}^*(2n)+\mathfrak{so}(2)$
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$