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Decomposition of flows of diffeomorphisms: analytical and geometrical aspects

Decomposição de fluxos de difeomorfismos: alguns aspectos geométricos e analíticos

Campinas

2022

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Este trabalho corresponde à versão final da Tese defendida pelo aluno Lourival Rodrigues de Lima e orientada pelo Prof. Dr. Paulo Regis Caron Ruffino.

Campinas

2022

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

L934d
Lima, Lourival Rodrigues de, 1992-
Decomposition of flows of diffeomorphisms : analytical and geometrical aspects / Lourival Rodrigues de Lima. – Campinas, SP : [s.n.], 2022.

Orientador: Paulo Regis Caron Ruffino.
Coorientador: Xue-Mei Hairer.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Difeomorfismos. 2. Folheações (Matemática). 3. Semimartingala (Matemática). 4. Fluxo estocástico. 5. Integral de Young. I. Ruffino, Paulo Regis Caron, 1967-. II. Li, Xue-Mei. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações para Biblioteca Digital

Título em outro idioma: Decomposição de fluxos de difeomorfismos : alguns aspectos geométricos e analíticos
Palavras-chave em inglês:
Diffeomorphisms
Foliations (Mathematics)
Semimartingales (Mathematics)
Stochastic flow
Young integral
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Paulo Regis Caron Ruffino [Orientador]
Elbert Einstein Nehrer Macau
Fabiano Borges da Silva
Diego Sebastian Ledesma
Gabriel Ponce
Data de defesa: 10-08-2022
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0001-6826-3315
- Currículo Lattes do autor: http://lattes.cnpq.br/2058145161521443

Tese de Doutorado defendida em 10 de agosto de 2022 e aprovada

pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

In loving memory of my father, João Palma da Silva, which gave me the greatest gift anyone could give another person, he trusted me.

Acknowledgements

I am really grateful to have this opportunity to study and learn a bit about this fabulous universe of mathematics. I consider this thesis, up to now, as one of the most outstanding achievements in my career; in fact, it represents the end of a fruitful phase of my academic life. I offer my sincere appreciation for the learning opportunities provided by the State University of Campinas.

I sincerely thank my Ph.D. supervisor, Professor Paulo Ruffino, for his help, support, guidance, and encouragement during my graduate studies. I also thank my co-supervisor, Professor Xue-Mei Li, for her patience and suggestions for this work and for her hospitality during my time at Imperial College London.

I thank my beloved fiancee, Katherine Loor, for her moral and emotional support while completing this work. I also want to thank my family, particularly my mother Nilce and my sister Lidiane for the countless prayers and spiritual comfort along these years.

Thanks to all professors, friends, and colleagues who helped directly or indirectly.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001

" I have fought the good fight, I have finished the race, I have kept the faith. Now there is in store for me the crown of righteousness, which the Lord, the righteous Judge, will award to me on that day and not only to me, but also to all who have longed for his appearing. "

2 Timothy 4:7-8

Resumo

Seja M uma variedade compacta munida de um par de folheações complementares (vertical e horizontal). O objetivo desta tese é estudar decomposições de fluxos de difeomorfismos em um contexto de baixa regularidade. Provamos que dado um semimartingale Z_t (o qual pode ter infinitos saltos em intervalos compactos), então, até um tempo de parada τ, um fluxo de difeomorfismo em M dirigido por Z_t pode ser decomposto em um processo no grupo de Lie de difeomorfismos cujas trajetórias caminham ao longo das folhas horizontais composto com um processo no grupo de difeomorfismos cujas trajetórias caminham ao longo das folhas verticais. Equações para estes processos são determinadas. Os processos estocásticos com componentes de saltos são gerados por equações de Marcus (como em Kurtz, Pardoux and Protter, Annal. I.H.P., section B, 31 (1995)). Generalizamos ainda mais este contexto geométrico para quaisquer tipo de semimartingales. Mostramos também que esta decomposição também funciona para soluções de equações diferenciais de Young e exploramos alguns aspectos geométricos da integral de Young. No contexto de saltos, nossa técnica é baseada em uma extensão da fórmula de Itô-Ventzel-Kunita para processos com saltos. No contexto de integrais de Young, fazemos uma aplicação de uma fórmula de Itô-Ventzel-Kunita para caminhos α-Hölder Contínuos proposta por Castrequini e Catuogno (Chaos Solitons Fractals, 2022). Algumas obstruções geométricas e topológicas para decomposições também são consideradas.

Palavras-chave: Decomposição de fluxos, processos com saltos, integral de Young, integral de Marcus, fórmula de Itô-Ventzel-Kunita.

Abstract

Let M be a compact manifold equipped with a pair of complementary foliations, say horizontal and vertical. This thesis aims to study a decomposition of flows of diffeomorphisms in the low regularity context. Namely, we prove that given a general semimartingale Z_t (which can have an infinity number of jumps in compact intervals) up to a stopping time τ, a stochastic flow of local diffeomorphisms in M driven by Z_t can be decomposed into a process in the Lie group of diffeomorphisms which trajectories remain along the horizontal leaves composed with a process in the Lie group of diffeomorphisms which trajectories remain along the vertical leaves. SDEs of these processes are shown. The stochastic flows with jumps are generated by the classical Marcus equation (as in Kurtz, Pardoux and Protter, Annal. I.H.P., section B, 31 (1995)). We enlarge the scope of this geometric decomposition and consider flows driven by arbitrary semimartingales with jumps. We show that this decomposition also holds for solutions of Young differential equations exploring the geometry of Young integrals. In the jump context, our technique is based on our extension of the Itô-Ventzel-Kunita formula for stochastic flows, which may jump infinitely many times. In the Young integral context, we apply a Young Itô-Kunita formula for α-Hölder paths proved by Castrequini and Catuogno (Chaos Solitons Fractals, 2022). Geometrical and other topological obstructions for the decomposition are also considered, e.g., sufficient conditions for the existence of global decomposition for all $t \geqslant 0$.

Keywords: Decompositions of flows, jump processes, Young integral, Marcus integral, Itô-Ventzel-Kunita formula.

Contents

Introduction

In this thesis, we study some geometric aspects of decomposition of flows and dynamics generated by Marcus and Young differential equations. Motivated by the fact that in many kinds of dynamical systems, in order to obtain local or asymptotic parameters of the dynamics, one performs a befitting decomposition of the associated flow, our main example of application of this low regularity techniques in manifolds concerns a decomposition of the associated Marcus and Young flows. For good references related to Marcus integral and discontinuous noise, we recommend Marcus [45], Lévy [37], Protter [53], Kurtz et al. [33], Oksendal and Sulem [51], Applebaum [2], Hartmann and Pavlyukevich [26], [27], among many others. For works related to α-Hölder trajectories with $\alpha \in (1/2, 1]$ (our case) and Young integral, see e.g. the classical [65], or more recent Hairer and Friz [22], Gubinelli et al. [24], Lyons [38], Castrequini and Russo [13], Castrequini and Catuogno [15], Cong [19], Ruzmaikina [57], and others.

Generally, decomposition of flows appears in the literature related to distinct geometrical or analytical contexts. We mention few of them: given a system in a semi-simple Lie group, we get much information if we decompose the system into each component of the Iwasawa decomposition (see, e.g. in the stochastic context Malliavin and Malliavin [44]); given a stochastic flow in a Riemannian manifold, one can write this flow (up to some conditions) as a Markovian process in the group of isometries of the manifold composed with a process in the Lie group of diffeomorphisms which fix the initial condition and has derivatives at this point given by an upper triangular matrix, see Ming Liao [40], [41]. Also, given a flow in an m-dimensional manifold with a pair of complementary foliation (i.e., locally, the manifold and foliations are diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^{m-k}$), then locally, in time and space, a stochastic flow can be written as a composition of diffeomorphisms which preserve each of these foliations, see [47], [46]. We will make this last example more precise and explore its potential in the Marcus and Young integral context.

The decomposition of Marcus and Young flows is allowed thanks to Itô-Ventzel-Kunita type formulas in this low regularity context, Theorem 18 due to Castrequini and Catuogno [15] for flows generated by Young differential equations, and Theorem 11 for Marcus equation context. The framework where we apply those formulas is a pair of geometrical distributions (involutive, i.e., which generates a foliation or not). The main results in this framework establish the local decomposition of Marcus and Young flows of diffeomorphisms as one component given by a diffeomorphism generated by vector fields in one distribution and another component given by a diffeomorphism generated by the other distribution. Precise definitions are given in chapter 3. In this scenario of low regularity of trajectories, the geometric Young Itô Formula, Theorem 14, opens the possibility to many basic geometric constructions on this dynamics. These topics are explored in the next section, where we prove the existence of horizontal lifts on principal fiber bundles with an affine connection. In particular, considering a Riemannian manifold and its orthonormal bundle, parallel transport and covariant derivatives can be established along α-Hölder trajectories. Development and anti-development can also be constructed.

The thesis is organized as follows: in the first chapter, we recall basic properties and definitions of foliated spaces, Young integrals, and stochastic processes with jump components (general semimartingales), and we prove the relevant geometric results for later use. In chapter 2, we study the decomposition of stochastic flows defined over a Riemannian manifold M starting at an initial point $x_0 \in M$ and running exclusively along vertical concatenate with horizontal trajectories. We will study some geometrical and analytical conditions for the existence of decomposition of flows along the leaves of a foliated space. Some of these conditions can be intrinsically related to the manifold. In chapters 3 and 4, we prove the decomposition of Marcus and Young flows given complementary distributions. In chapter 5, we present examples. Initially, linear systems are treated with a pair of foliations given by affine parallel hyperplanes. We present conditions for the existence of global decomposition at any time in this context. The last example provides explicit calculations for decomposition of jump dynamics in the case of fiber bundles over homogeneous space $G \to M = G/H$ where G is a Lie group and $H < G$ is a closed subgroup. The last section of this work states some open problems related to the decomposition of flows and stochastic optimal control.

1 Preliminaries

1.1 Foliations

Given an n-dimensional smooth manifold M, a foliation $\mathcal{F}$ of dimension $1 \leqslant k < n$ in M is a partition of M into immersed connected submanifolds of dimension k, called the leaves of $\mathcal{F}$ with local foliated chart. More precisely, locally, $(M, \mathcal{F})$ is diffeomorphic to open sets of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, in such a way that the leaves have constant second coordinate. In fact, a foliation $(M, \mathcal{F})$ is identified with a foliated atlas which is coherent along the leaves in the following sense:

Definition 1. Let M be a smooth n-dimensional manifold. A (smooth) k-dimensional foliated atlas $\mathcal{A}$ of M is a maximal atlas on M which satisfies:

1) If $(U, \alpha) \in \mathcal{A}$, then $\alpha(U) = U_1 \times U_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ for U_1, U_2 open subsets of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively.

2) Given two local charts $(U, \alpha), (V, \beta) \in \mathcal{A}$, with $U \cap V \neq \varnothing$, then the change of coordinate map is given by $\alpha \circ \beta^{-1}(x, y) = (h_1(x, y), h_2(y))$, for some smooth maps h_1 and h_2 in the appropriate domain.

A foliated atlas $\mathcal{A}$ is said to be *regular* if it is locally finite and for any foliated chart $(U, \alpha) \in \mathcal{A}$, the closure of its domain $\bar{U}$ is a compact set contained in V, the domain of another foliated chart (V, β). The sets $\alpha^{-1}(B, \{y\}) \subset M$, for $(U, \alpha) \in \mathcal{A}, B \subset \mathbb{R}^k$ open set such that $(B, \{y\}) \subset \alpha(U)$ are called *plaques* of the atlas.

Consider the equivalence relation in M given by $x \sim y$ if and only if there exists a finite sequence of plaques $P_0, P_1, \dots, P_p$ with $x \in P_0$, $y \in P_p$ and $P_i \cap P_{i-1} \neq \varnothing$ for all $i = 1, \dots, p$. The equivalent classes here determine a one-to-one correspondence between regular foliated atlases and the leaves F of a foliated manifold (see e.g. [11, Thm. 1.2.18]). Given a point $p \in M$, the unique leaf of the foliation passing through p is denoted by $\mathcal{F}(p)$. The set $\mathcal{F}(S) = \cup_{p \in S} \mathcal{F}(p)$, for $S \subset M$, is called the *saturation of* S by $\mathcal{F}$.

Example 1. (Trivial foliation). Let M be an n-dimensional manifold. Then, $M \times \mathbb{R}^k$, for $k \in \mathbb{N}$, is a foliation. In fact, its leaves are $M \times \{p\}$, for $p \in \mathbb{R}^k$ and the leave space is $\mathbb{R}^k$. In general, given a submersion $g : M \to Q$, where Q is a k-dimensional manifold, the fibers of g can be seen as leaves of some foliation and Q as the leave space.

Example 2. (Fibre bundles). Let E, B and F be differentiable manifolds and consider a diferential map $\pi : E \to B$, a cover $\{U_i\}_{i\in I}$ of B and a family of diffeomorphisms Ψ_i, such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\Psi_i"] \arrow[d, "\pi"'] & U_i \times F \arrow[dl, "p_1"] \\ B & \end{tikzcd}$$

In the diagram, p_1 is the projection map in the first component of $U_i \times F$, for all $i \in I$. The group $\{\Psi_{ij} = \Psi_i \circ \Psi_j^{-1}\}_{U_i \cap U_j \neq \varnothing}$ is called the structure group of E. The sets $\pi^{-1}(b)$, $b \in B$ are called fibers and the fibers generate a foliation in the space E whose leaves are diffeomorphic to the connected components of F.

Example 3. (Foliations defined by submersions). Let M^m and N^n be Riemannian manifolds with dimensions m and n respectively. Consider $f : M^m \to N^n$ a smooth submersion. By the local form of submersion, it follows that for all $p \in M$, there exists local charts (U, φ) on M and (V, ψ) on N, such that $p \in U$, $q = f(p) \in N$ and

$$\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^{m-n} \times \mathbb{R}^n,$$

and

$$U_2 \subset V_1 = \psi(V),$$

with

$$\begin{aligned} \psi \circ f \circ \varphi^{-1} : U_1 \times U_2 &\to U_2 \\ (x, y) &\to y. \end{aligned}$$

Therefore, the chart (U, φ) define a foliation structure on the manifold M, where the leaves are generated by the connected components of $f^{-1}(c),\ c \in N$.

Example 4. (Non-singular vector fields). Let M be a Riemannian connected manifold and X be a non-singular vector fields on M. Consider the following ordinary differential equation:

$$\frac{dx}{dt} = X(x). \tag{1.1}$$

The solution curve of equation (1.1) is a leave of a 1-dimensional foliation for each initial condition $x_0 = y \in M$.

In some cases, we need the foliation to be well-behaved, i.e, we need it to be regular in the following sense:

i) The closure of each open set is a compact subset of some foliated chart.

ii) Each covering by coordinate systems is locally finite.

iii) The closure of each plaque of one chart intersects at most one plaque in each other chart.

In the sequel, we state two important definitions.

Definition 2. We say that the maximal atlas of $(M, \mathcal{H}, \mathcal{V})$ is transversely orientable (for the horizontal foliation) if for all change of coordinate map $y_i : \phi_2(U_1 \cap U_2) \longrightarrow \mathbb{R}$, with $\phi_1 \circ \phi_2^{-1} = (y_1, ..., y_n)$

$$\det \frac{\partial(y_{n-k+1}, ..., y_n)}{\partial(x_{n-k+1}, ..., x_n)} > 0. \tag{1.2}$$

Unless otherwise stated, we are going to assume that our foliation $(M, \mathcal{F})$ is tranversely orientable. This is not quite a restriction since if $\mathcal{F}$ is not transversely orientable, it can be lifted to a transversely-oriented foliation on a double covering of M, see e.g. [11, Prop. 3.5.1].

Definition 3. Let $\mathcal{H}$ and $\mathcal{V}$ be two complementary foliations on M. $\mathcal{A}$ is a biregular atlas on $(M, \mathcal{H}, \mathcal{V})$ if $\mathcal{A}$ is foliated and regular for $\mathcal{H}$ and $\mathcal{V}$ simultaneously. Namely, given two biregular coordinate systems (U, α_1) and (V, α_2), with $U \cap V \neq \emptyset$, then the change of coordinate map is given by $\alpha_1 \circ \alpha_2^{-1}(x, y) = (h_1(x), h_2(y))$, for some smooth maps h_1 and h_2 in the appropriate domain.

The following basic result is crucial on determining the topology of relevant sets we are going to introduce in chapter 3. It is a nontrivial result if one considers, for example, non-compact or dense leaves in a compact foliated space.

Proposition 1 (Uniform tranversality)**.** Consider $(M, \mathcal{H}, \mathcal{V})$, a manifold with complementary foliations $\mathcal{H}$ and $\mathcal{V}$. Fix a leaf F, say, in $\mathcal{H}$. Given two points $p, q \in F$, let $V_p, V_q \in \mathcal{V}$ be the vertical leaves passing thorough p and q respectively. Then, there exist open sets in the intrinsic topology $D_1 \subset V_p$, $D_2 \subset V_q$ with $p \in D_1$, $q \in D_2$ and a diffeomorphism $f : D_1 \to D_2$ such that $f(L \cap D_1) = L \cap D_2$ for every horizontal leaf L in $\mathcal{H}$.

Proof. Consider biregular charts $\varphi_p : U_p \to U_1 \times U_2$ and $\varphi_q : U_q \to \tilde{U}_1 \times \tilde{U}_2$ in a neighbourhood of p and q respectively, with $U_1, \tilde{U}_1 \subset \mathbb{R}^k$ and $U_2, \tilde{U}_2 \subset \mathbb{R}^{n-k}$. By the uniform transversality theorem, see e.g. [10, Thm. 3, Ch.III] there exist submanifolds N_1 and N_2, with $p \in N_1$ and $q \in N_2$ transverse to F and a diffeomorphism $\tilde{f} : N_1 \to N_2$ such that $\tilde{f}(L \cap N_1) = L \cap N_2$ for all horizontal leaf L .

To conclude the proof, we just have to show that N_1 and N_2 above can be chosen as open sets D_1 and D_2 of the vertical leaves V_p and V_q. Since N_1 is transverse to F at p, then the derivative at p of the non-linear projection $\psi_p := \varphi_p^{-1}(\{0\} \times \pi_2 \circ \varphi_p \circ i) : N_1 \to V_p$ is an isomorphism between the tangent spaces T_pN_1 and T_pV_p, where $i : N_1 \to M$ is the inclusion and $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is the projection. By the classical local inverse theorem, there exists an open set $\tilde{D}_1$ where the restriction of ψ_p is a diffeomorphism. By the same argument, we have that there exists an open set $\tilde{D}_2$ where the restriction of $\psi_q := \varphi_q^{-1}(\{0\} \times \pi_2 \circ \varphi_q \circ i)$ is also a diffeomorphism.

The diffeomorphism $f : D_1 \to D_2$ of the statement is given by $f = \psi_q \circ \tilde{f} \circ \psi_p^{-1}$ with $D_1 = \psi_p(\tilde{D}_1 \cap \tilde{f}^{-1}(\tilde{D}_2))$ and $D_2 = \psi_q(\tilde{f}(\tilde{D}_1) \cap \tilde{D}_2)$.

□

1.2 Distributions

Here we present another perspective of studying foliations which can be done via differentiable vector fields. These ideas are well known in the literature and can be found in Nomizu [49], Candel and Colon [11], Camacho and Lins Neto [10], among many others.

Definition 4. Let M be a Riemannian connected manifold. A distribution Δ, with dimension k, is a map that assigns each point $p \in M$, a subspace Δ_p. The distribution Δ is said to be differentiable if each point $p \in M$ has a neighbourhood U, such that for all $q \in U$, there exists a family of smooth vector fields $X_i(q)$, for $i = 1, \ldots, k$, which forms a basis for Δ_q.

In this context, we say that a differentiable vector field defined over U belongs to the distribution Δ, if $X(p) \in \Delta_p$, for all $p \in U$.

Definition 5. A differentiable distribution Δ is said to be involutive if for all local basis $X_1, \ldots, X_k$ of Δ, the brackets $[X_i, X_j]$ belong to Δ, i.e, the distribution is a Lie subalgebra of $\mathfrak{X}(M)$.

Usually in the literature, the manifold M is called an integral manifold of the distribution Δ if $\Delta_p = T_pM$, for all $p \in M$. The next theorem shows that foliations are sometimes generated by infinitesimal data, such as a smooth k dimensional distribution $\Delta \subset TM$. The involutiveness condition of a distribution is the classical result known as the Frobenius theorem.

Theorem 1 (Frobenius)**.** Let M be a connected Riemannian manifold and Δ be a distribution in TM. Then, there exists a foliation on M such that its leaves are integral manifolds of Δ, if and only if, Δ is involutive.

1.3 The α-Hölder space

Let E be a Banach space and $C([0,T],E)$ be the space of all continuous paths $x:[0,T]\to E$, with the norm:

$$||x||_\infty = \sup_{t\in[0,T]} |x_t|. \tag{1.3}$$

For $\alpha \in [0,1)$, we define the α-Hölder seminorm of a path x by:

$$||x||_\alpha = \sup \frac{|x_{st}|}{|t-s|^\alpha}, \tag{1.4}$$

where $x_{st} = x_t - x_s$. We denote by $C^\alpha([0,T],E)$ the space of all continuous paths, such that $||x||_\alpha < \infty$. In the sequel, we remark some well-known properties of the seminorm 1.4.

Remark 1.

1. If $0<\alpha<\beta\leqslant 1$, then $C^\beta([0,T],E)\subset C^\alpha([0,T],E)$. In fact, note that:
$$||x||_\alpha = \sup_{0\leqslant s,t\leqslant T} \frac{|x_{st}|}{|t-s|^\beta}|t-s|^{\beta-\alpha} \leqslant ||x||_\beta T^{\beta-\alpha}.$$
2. $C^\alpha([0,T],E)$ is a Banach space with the norm:
$$|x|_\alpha = |x_0| + ||x||_\alpha.$$
3. If $x\in C^\alpha([0,T],E)$, then x is a $\dfrac{1}{\alpha}$-variation path. Indeed, let π be a partition of the interval $[0,T]$. Then, for $p=\dfrac{1}{\alpha}$,
$$\sum_{t_i\in\pi} |x_{t_{i+1}} - x_{t_i}|^p \leqslant \sum_{t_i\in\pi} (||x||_\alpha |t_{i+1}-t_i|^\alpha)^p = ||x||_\alpha^p T^\alpha.$$
Hence,
$$|x|_p = \left(\sup_{\pi\in\mathcal{P}[0,T]} \sum_i |x_{t_{i+1}} - x_{t_i}|^p\right)^{\frac{1}{p}} \leqslant ||x||_\alpha T^\alpha.$$
4. **Lower semicontinuity and interpolation**. Let $x^n,\ x\in C^\alpha([0,T],E)$, such that $\lim_{n\to\infty} x^n = x$ pointwise. Then,
$$||x||_\alpha \leqslant \liminf ||x^n||_\alpha.$$
And
$$||x||_\alpha \leqslant ||x||_\beta^{\frac{\alpha}{\beta}} \left(\sup_{0\leqslant s,t\leqslant T} |x_{s,t}|\right)^{1-\frac{\alpha}{\beta}}.$$
It follows straightforward from inequalities:
$$\frac{|x_{s,t}|}{|t-s|^\alpha} = \liminf_n \frac{|x^n_{st}|}{|t-s|^\alpha} \leqslant \liminf_n ||x||_\alpha.$$
And
$$\frac{|x_{s,t}|}{|t-s|^\alpha} = \left(\frac{|x_{s,t}|}{|t-s|^\alpha}\right)^{\frac{\alpha}{\beta}} |x_{st}|^{1-\frac{\alpha}{\beta}}.$$

The next classical result states a condition for the possibility of approximating an α-Hölder path by a differentiable path in $C^\alpha([0,T],E)$. We denote by $C^{0,\alpha}([0,T],E)$ the closure of all differentiable paths in $C^\alpha([0,T],E)$.

Proposition 2. A path $x \in C^{0,\alpha}([0,T],E)$, if and only if,

$$\lim_{\delta\to 0}\sup_{|s-t|<\delta}\frac{|x_{st}|}{|t-s|^\alpha}=0.$$

For a proof, see Friz and Hairer [22].

Corollary 1. *Let* $0<\alpha<\beta<1$. *Then,*

$$C^\beta([0,T],E)\subset C^{0,\alpha}([0,T],E).$$

Proof. Take $x \in C^\beta([0,T],E)$ and consider a fixed $\delta>0$. Let $s,t\in[0,T]$, such that $|t-s|<\delta$. It follows that:

$$\frac{|x_{st}|}{|t-s|^\alpha}=\frac{|x_{st}|}{|t-s|^\beta}|t-s|^{\beta-\alpha}\leqslant |x|_\beta\delta^{\beta-\alpha}.$$

Hence,

$$\lim_{\delta\to 0}\sup_{|t-s|<\delta}\frac{|x_{st}|}{|t-s|^\alpha}=0.$$

By proposition (2), we conclude that $x\in C^{0,\alpha}([0,T],E)$.

□

Example 5. Let $x:[0,T]\to\mathbb{R}$ be a path defined by $x_t=t^\alpha$, for $\alpha\in(0,1)$. For $0\leqslant s,t\leqslant T$, we have that:

$$\frac{|x_{st}|}{|t-s|^\alpha}=\frac{|t^\alpha-s^\alpha|}{|t-s|^\alpha}=\frac{|1-\left(\frac{s}{t}\right)^\alpha|}{|\left(1-\frac{s}{t}\right)^\alpha|}\leqslant\frac{1-\frac{s}{t}}{1-\frac{s}{t}}=1.$$

Then, $||x||_\alpha\leqslant 1$ and $x\in C^\alpha([0,T],\mathbb{R})$. Moreover, we have that for all $t\in(0,T]$:

$$\frac{|x_{0t}|}{|t-0|^\alpha}=\frac{t^\alpha}{t^\alpha}=1.$$

Then,

$$\lim_{\delta\to 0}\sup_{|t-s|<\delta}\frac{|x_{st}|}{|t-s|^\alpha}\geqslant 1.$$

By proposition (2), $x\in C^{0,\alpha}([0,T],\mathbb{R})$.

Lemma 1 (Sewing Lemma)**.** Let E be a Banach space and $A:\Delta_T^2\to E$ a continuous functions. Where $\Delta_T^2=\{(a,b);\ 0\leqslant a\leqslant b\leqslant T\}$. For $0\leqslant s\leqslant u\leqslant t\leqslant T$, we set:

$$\delta A_{sut}:=A_{st}-A_{su}-A_{ut}.$$

We assume that there exist $\lambda > 0$ and $\epsilon > 0$, such that:

$$||\delta A_{sut}|| \leqslant \lambda |t-s|^{1+\epsilon}.$$

Then, there exists a continuous path $\sigma : [0,T] \to E$, with $\sigma_0 = 0$, and a constant $C = C(\epsilon) > 0$, such that:

$$||\sigma_t - \sigma_s - A_{st}|| \leqslant C\lambda |t-s|^{1+\epsilon},$$

for all $(s,t) \in \Delta_T^2$. Moreover, it holds that:

$$\lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} A_{uv} = \sigma_t - \sigma_s.$$

The next theorem states the existence of the called Young integral, for more details about the its convergence and properties, see the classical paper by Young in [65].

Theorem 2. Let E and F be two finite dimensional Banach spaces, take α, $\beta \in (0,1]$, such that $\alpha + \beta > 1$, $x \in C^{\alpha}([0,T], E)$ and $y \in C^{\beta}([0,T], \mathcal{L}(E,F))$. Then for all $t \in [0,t]$, there exists the following limit:

$$\int_0^t y_r dx_r = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} y_u x_{uv},$$

where $\pi \in \mathcal{P}[0,T]$. This limit is called the Young integral of y with respect to x. Moreover, it holds that for all $(s,t) \in \Delta_t^2$,

$$\left| \int_0^t y_r dx_r - y_s x_{st} \right| \leqslant K ||y||_{\beta} ||x||_{\alpha} |t-s|^{\alpha+\beta}.$$

Where K is a constant which depends only on $\alpha + \beta$.

Proof. set $A_{st} = y_s x_{st}$. Then,

$$\begin{aligned}
\delta A_{sut} &= A_{st} - A_{su} - A_{ut} \\
&= y_s x_{st} - y_s x_{su} - y_u x_{ut} \\
&= y_s (x_{st} - x_{su}) - y_u x_{ut} \\
&= y_s x_{ut} - y_u x_{ut} \\
&= -y_{su} x_{ut}.
\end{aligned}$$

Therefore,

$$|\delta A_{sut}| = |y_{su} x_{ut}| \leqslant ||y||_{\beta} ||x||_{\alpha} |t-s|^{\alpha+\beta}.$$

The existence of the Young integral follows directly from lemma (1).

□

The next classical theorem states the existence and uniqueness of solutions generate by Young differential equations. The proof follows via fixed point theorem. Again we refer the reader to [65], for more details.

Theorem 3. Let $x \in C^{\beta}([0,T],\mathbb{R}^d)$, with $\beta \in (1/2,1]$, $f \in C_b^2(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^d,\mathbb{R}^m))$ and $y \in \mathbb{R}^m$. Then, there exists a unique $y \in C^{\beta}([0,T],\mathbb{R}^m)$, such that:

$$y_t = y + \int_0^t f(y_s)dx_s,$$

for all $t \in [0,T]$.

The following theorem is an adaptation of H. Kunita result (see [32]) about composition of flows. In this context, the composition is considered for flows defined by Young differential equations.

Theorem 4. Let U, V and W be Banach spaces. Consider $x \in C^{\alpha}([0,T],V)$ and $h : [0,T] \times U \to \mathcal{L}(V,W)$ a differentiable map on U such that:

- $(t,x) \to Dh_t(x)$ is continuous.
- $h \in C(U, C^{\beta}([0,T],\mathcal{L}(V,W)))$, for $\frac{1}{2} < \beta \leqslant 1$.

Consider a map $g : [0,T] \times U \to W$, two times differentiable on U, such that the functions $(t,x) \to Dg_t(x)$ and $(t,x) \to D^2 g_t(x)$ are continuous. Assume that g satisfies:

$$g_t(x) = g_0(x) + \int_0^t h_s(x)dx_s.$$

Then, for any $u \in C^{\alpha}([0,T],U)$,

$$g_t(u_t) = g_0(u_0) + \int_0^t h_s(u_s)dx_s + \int_0^t D_x g_s(u_s)du_s.$$

Where the integral

$$\int_0^t D_x g_s(u_s)du_s \tag{1.5}$$

is understood in the Riemann-Stieltjes sense. If $Dg \in C(U, C^{\gamma}([0,T],\mathcal{L}(V,W)))$, for $\gamma \in (1/2,1]$, the integral (1.5) is a Young integral and $t \to g_t(u_t) \in C^{\alpha}(W)$.

Proof. see Castrequini and Catuogno [12, Thm. 3.1].

□

The next following corollaries are important tools which are going to be applied in the decomposition studied in Chapter 4. For a proof, see [12].

Corollary 2. Consider $x \in C^{\alpha}([0,T],V)$, $y \in C^{\alpha}([0,T],U)$, $f \in C^2(W,\mathcal{L}(V,W))$ and $g \in C^2(W,\mathcal{L}(U,W))$. Let η and ψ be solution maps associated with the YDE's $d\eta_t = f(\eta_t)dx_t$ and $d\psi_t = g(\psi_t)dy_t$ respectively. Then, the map $\varphi_t = \eta_t \circ \psi_t$ satisfies:

$$d\varphi_t = f(\varphi_t)dx_t + \eta_{t*}g(\varphi_t)d\psi_t.$$

Where $\eta_{s*}g := (D_x\eta_s \cdot g) \circ \eta_s^{-1}(x)$.

Corollary 3. Let u be a solution map associated with the YDE:

$$du_t = f(u_t)dx_t.$$

Then, the inverse map $t \to u_t^{-1}(z)$ satisfies the YDE:

$$dz_t = -Du_t(z_t)^{-1}f(u_t(z_t))dx_t.$$

With initial condition $z_0 = 1$.

1.4 Stochastic processes with jump components

For the reader's convenience, in this section we state the main results about stochastic processes with jump components. Those topics will be crucial in Chapter 3 where we propose a decomposition for diffusion generated by Marcus differential equations. For more details see e.g. Protter [53], Kurtz et al [33], among many others.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with a given filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant \infty}$. (By filtration we mean a family of σ-algebras $\mathcal{F}_t$ which increases as: $\mathcal{F}_s \subset \mathcal{F}_t$, if $s \leqslant t$). In this sense, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0 \leqslant t \leqslant \infty}, \mathbb{P})$ is said to **satisfy the usual hypotheses** if the filtration has the following properties:

- If $A \in \mathcal{F}$, with $\mathbb{P}(A) = 0$, then $A \in \mathcal{F}_0$.
- $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$, for all $0 \leqslant t < \infty$.

With this concept, a stochastic process x is said to be **adapted** if $x_t \in \mathcal{F}_t$, for each $t \geqslant 0$. From now on, we will always assume that the usual hypotheses hold.

Definition 6. A stochastic process x is called **càdlàg** if a.s. its trajectories are right continuous, with left limits. (càdlàg is actually an acronym from the french phrase: **continue à droite, limite à gauche**).

1.4.1 Lévy processes

Lévy processes were the first class of stochastic processes to be studied in a modern way back in the mid-1900s. It includes Brownian motions and Poisson processes as special cases, which is not actually expected since those are very different. Even though sample paths of Brownian motions are continuous and Poisson processes have discontinuous trajectories, there is one thing in common about these processes, both of them are Càdlág as do all Lévy processes.

Hence, in recent decades the study of Lévy processes as a whole class, rather than splitting up into individual cases, has become an attractive field of research which unifies all continuous and discontinuous processes theory. Of course there are many interesting books that deal with Lévy processes, see e.g. Khintchine [29], Applebaum [2], Lévy [37], Protter [53], among others.

Definition 7. An adapted process $Z_t = (Z_t)_{t \geqslant 0}$, with $Z_0 = 0$ a.s. is called a Lévy process if it satisfies the following properties:

1. Z has independent increments, i.e. $Z_t - Z_s$ does not depend on $\mathcal{F}_t$, for $t \geqslant 0$.
2. Z has stationary increments, i.e. $Z_t - Z_s \overset{d}{=} Z_{t-s}$, for $t \geqslant 0$.
3. Z is stochastically continuous, which means that for all ϵ, t, $s > 0$,

$$\lim_{t \to s} \mathbb{P}(|Z_t - Z_s| > \epsilon) = 0.$$

Example 6 (Brownian motion)**.** An adapted process $B = (B_t)_{t>0}$ is said to be a standard Brownian motion if it is a Lévy process such that:

1. $B_t \sim N(0, tI)$, for each $t \geqslant 0$.
2. B has continuous trajectories.

The Brownian motion is a Lévy process which has been intensively studied since early years of the twentieth century when it was introduced. It is important mentioning two basic properties: the Brownian motion is locally α-Hölder continuous for $\alpha \in (0, 1/2)$, i.e.

for all stopping time $T > 0$, and $\omega \in \Omega$, it follows that there exists a constant $K = K(t, \omega)$, such that:

$$|B_t(\omega) - B_s(\omega)| \leqslant K|t - s|^{\alpha},$$

for $t \geqslant 0$, and the sample paths $t \to B_t(\omega)$ are a.s. nowhere differentiable. For further examples and details of the proofs, we strongly recommend the classical Sato [61] and Revuz and Yor [56], among others.

Example 7 (The Poisson process). A **Poisson process** is a Lévy process N, such that $N_t \subset \mathbb{N} \cup \{0\}$, $t \geqslant 0$, and

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

For each $n = 0, 1, 2, \ldots$ where λ is called intensity or characteristic exponent of the Poisson process N. In this case, we use the notation: $N_t \sim \pi(\lambda t)$. Poisson processes are exhaustively studied and used in applications, see e.g. Kingman [30] and references therein.

Example 8 (The compound Poisson process). Consider a sequence of iid random variables Z_n, $n \in \mathbb{N}$ taking values in $\mathbb{R}^k$ and let N be a Poisson process as in example (7), that is independent of all Z_n. We define

$$Y_t := Z_1 + \ldots + Z_{N_t}.$$

For $t \geqslant 0$. The process Y is called **compound Poisson process**. Note that Y_t is a Lévy process (it is straightforward from Lévy definition and dominated convergence theorem). For $k = 1$ it is also a Poisson process.

Example 9 (Interlacing processes). Let Y and B be two independent compound Poisson process (as in example (8)) and Gaussian Lévy process (as in example 6) respectively. We define:

$$S_t = B_t + Y_t.$$

For all $t \geqslant 0$. Observe that S_t is a Lévy process (it follows directly from the definition, since B and Y are also Lévy processes). The next recursive formula shows that S have jumps of random size occurring at random times. Using the notation of examples (7) and (8), we have that for a sequence of stopping times $T_1 < T_2 < \ldots < T_n$:

$$S_t = \begin{cases} B_t, & \text{for } \ 0 \leqslant t < T_1 \\ B_{T_1} + Z_1, & \text{for } \ t = T_1 \\ S_{T_1} + B_t - B_{T_1}, & \text{for } \ T_1 < t < T_2 \\ S_{T_2} + Z_2, & \text{for } \ t = T_2. \end{cases}$$

And so on recursively.

Another Lévy property we want to remark is the infinitely divisibility.

Definition 8. A random variable Z_t has infinitely divisible distribution if for all $n \in \mathbb{N}$, there exists a sequence of iid variables $Y_1, Y_2, \ldots, Y_n$, such that:

$$Z_t \overset{d}{=} Y_1 + \ldots + Y_n.$$

Note that Lévy processes have infinitely division distribution. In fact, suppose that Z is a Lévy process, since it has stationary independent increments, for each $n \in \mathbb{N}$:

$$Z_t = Z_{\frac{t}{n}} + \left(Z_{\frac{2t}{n}} - Z_{\frac{t}{n}}\right) + \left(Z_{\frac{3t}{n}} - Z_{\frac{2t}{n}}\right) + \ldots + \left(Z_{\frac{nt}{n}} - Z_{\frac{(n-1)t}{n}}\right).$$

Where the increments are independent with the same distribution.

1.4.2 The jumps

Another characteristic of Lévy process is the instantaneous change of positions (jump). We formally define this using a very important process associated to a Lévy process Z called jump process, which is given by:

$$\Delta Z_t = Z_t - Z_{t^-},$$

for each $t \geqslant 0$, where Z_{t^-} is the left limit at time t. If $|\Delta Z_t| \leqslant C < \infty$ a.s, for a non-random constant C, we say that Z has bounded jumps.

Let Λ be a borel set in $\mathbb{R}$, such that $0 \in \bar{\Lambda}$, where $\bar{\Lambda}$ is the closure of Λ. We define the following random variables:

$$\begin{aligned} T_\Lambda^1 &= \inf\{t > 0;\ \Delta Z_t \in \Lambda\} \\ &\vdots \\ T_\Lambda^{n+1} &= \inf\{t > T_\Lambda^n;\ \Delta Z_t \in \Lambda\}. \end{aligned}$$

Note that the set $\{T_\Lambda^n \geqslant t\} \in \mathcal{F}_t$ (it follows by the fact that Z has càdlàg paths and $0 \notin \bar{A}$), thus T_Λ^n is a stopping time (more details can be found in Protter [53]). Moreover, by construction, it holds that $\lim_{n\to\infty} T_\Lambda^n = \infty$ a.s. We define the following process:

$$N_t^\Lambda = \sum_{0<s\leqslant t} 1_\Lambda(\Delta Z_s) = \sum_{n=1}^{\infty} 1_{\{T_\Lambda^n \leqslant t\}}.$$

The set functions $\Lambda \to N_t^\Lambda(\omega)$ and $\nu(\Lambda) = \mathbb{E}[N_1^\Lambda]$ define a σ-finite measure on $\mathbb{R}\backslash\{0\}$, see [Thm 35, [53], this measure is called Lévy measure of the Lévy process Z.

A commun difficulty in manipulating Lévy processes arises when:

$$\sum_{0 \leqslant s \leqslant t} |\Delta Z_s| = \infty \quad \text{a.s.}$$

Which is possible to occur. This problem in some cases can be solved using the fact that we always have:

$$\sum_{0 \leqslant s \leqslant t} |\Delta Z_s|^2 < \infty \ \ \text{a.s.}$$

This property and the next result will be crucial in Chapter 3.

Theorem 5. Let Z_t, $t \geqslant 0$ be a stochastic process. If Z_t is Càdlàg, then the set $S = \{t,\ \Delta Z_t \neq 0\}$ is at most countable.

Proof. For a proof, see [Thm 2.8.1, [2]].

□

1.4.3 The Lévy-Khintchine formula

The Lévy - Khintchine formula is one of the key results in the basic theory of Lévy processes which decomposes sample paths into continuous and jump parts. It gives an analytic expression for the characteristic function, which allows mathematicians to work with it in order to understand some probabilistic and/or geometric properties of Lévy processes, see Khintchine [29]. In the following we state the 1-dimensional version of Lévy - Khintchine formula.

Theorem 6 (Lévy - Khintchine formula)**.** Consider $a \in \mathbb{R}$, $b \in [0, \infty)$ and a measure ν on $\mathbb{R}\backslash\{0\}$, such that:

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, x^2)\nu(dx) < \infty.$$

For all $\lambda \in \mathbb{R}$, we define a function $h(\lambda)$ by:

$$h(\lambda) = ia\lambda + \frac{b\lambda^2}{2} + \int_{\mathbb{R}\backslash\{0\}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}\right) \nu(dx).$$

Then, there exists a unique Lévy process $Z = (Z_t)_{t \geqslant 0}$, which satisfies:

$$\mathbb{E}\left[e^{i\lambda Z_t}\right] = e^{-th(\lambda)}.$$

For all $t \geqslant 0$.

1.4.4 General semimartingales

We say that an adapted stochastic process Z is a semimartingale when it can be decomposed as:

$$Z_t = Z_0 + M_t + C_t,$$

where $M = (M_t)_{t\geqslant 0}$ is a local martingale (in the classical sense) and $C = (C_t)_{t\geqslant 0}$ is an adapted process with finite variation.

Theorem 7. Lévy processes are semimartingale

Proof. It follows by the Lévy - Itô decomposition, [Thm 2.4.11, [2]].

□

2 The existence of decomposition, geometrical and topological aspects.

Throughout this chapter, we study the decomposition of stochastic flows defined over a Riemannian manifold M starting at an initial point $x_0 \in M$ and running exclusively along vertical concatenate with horizontal trajectories. We are going to study some geometrical and analytical conditions for the existence of decomposition of flows along the leaves of a foliated space. Some of these conditions can be intrinsically related to the manifold, some examples are given in section 2.1.1, where is stated the concept of attainability index (which sometimes can be considered as a topological obstruction for the existence of this decomposition). We also state a technique to perform a decomposition of the form:

$$\varphi_t(x_0) = \eta^k_{t\vee s_k} \circ \psi^k_{t\vee s_k} \circ \ldots \circ \eta^2_{s_2} \circ \psi^2_{s_2} \circ \eta^1_{s_1} \circ \psi^1_{s_1}(x_0). \tag{2.1}$$

Where ψ^i and ξ^i are purely vertical and horizontal components respectively. This kind of decomposition is called of **alternate decomposition**.

2.1 Decomposition of diffeomorphisms in foliated spaces

The existence of the biregular atlas of the previous chapter is straightforward, see e.g. [11, Lemma 5.1.4]. Given an initial condition $x_0 \in M$, we take a local coordinate system $\alpha : U_{x_0} \subset M \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$. The product $\mathbb{R}^k \times \mathbb{R}^{n-k}$ can be seen as a canonical Cartesian k-dimensional pair of foliations on $\mathbb{R}^n$.

Remark 2. Given a diffeomorphism φ with initial condition x_0, it can be written as $\varphi = \big(\varphi^1(x,y), \varphi^2(x,y)\big)$, where x, $\varphi^1(x,y) \in \mathbb{R}^{n-k}$ and y, $\varphi^2(x,y) \in \mathbb{R}^k$. It follows directly by the inverse function theorem that there exists a unique (reducing the domain if necessary) decomposition $\phi = \eta \circ \psi$ in a neighbourhood of x_0, where η and ψ are horizontal and vertical preserving dffeomorphisms, if and only if

$$\det \frac{\partial \phi^2(x_0)}{\partial y} \neq 0. \tag{2.2}$$

Applying this characterization for a flow of diffeomorphism ϕ_t, one can guarantee the local existence of decomposition $\phi_t = \eta_t \circ \psi_t$ up to a stopping time τ, where η_t and ψ_t are flows of diffeomorphisms preserving horizontal and vertical components respectively and

$$\tau = \sup \left\{ t > 0; \det \frac{\partial \phi^2_s(x,y)}{\partial y} \neq 0 \ \text{ for all } \ 0 \leqslant s \leqslant t \right\}. \tag{2.3}$$

For more details see e.g. Melo et al [47]. In some cases, there exists some degree of compatibility of the vector fields with the complementary distributions in such a way that the decomposition presented above holds for all time t, see [46]. A basic but important example is a linear system on $\mathbb{R}^n \backslash \{0\}$ endowed with spherical and radial foliations, note that in this specific case, the system sends radial leaves into radial leaves, therefore the decomposition holds for all time. Another standard example in this context is the derivative flow $\varphi_{t*} : T_{x_0}M \longrightarrow T_{\varphi_t(x_0)}M$, for $x_0 \in M$, in the linear frame bundle $\pi : BM \longrightarrow M$. Note that φ_{t*} is an isomorphism between the fibres $\pi^{-1}(x_0)$ and $\pi^{-1}(\varphi_t(x_0))$, for all $t \geqslant 0$, therefore φ_{t*} has a decomposition $\varphi_{t*} = \eta_t \circ \psi_t$ for all time.

2.1.1 Attainability index and topological obstruction

In many interesting pairs of foliations, given an initial condition $x_0 \in M$, there exists a set of points which one cannot reach by a vertical trajectory concatenated with a horizontal path. Even if we are allowed to concatenate a number of alternating vertical and horizontal paths, see example (10). This topological restriction to accessibility represents also an obstruction for the decomposition of a dynamics given by a continuous family of diffeomorphisms φ_t which, say, send x_0 into a non-accessible point. This leads us to the following concept:

Definition 9. The k-attainable points from $x \in M$ with respect to the pair of foliation $(M, \mathcal{H}, \mathcal{V})$ is the composition of saturated sets

$$\mathcal{A}^k(x) = \underbrace{\cdots \mathcal{H}(\mathcal{V}(\mathcal{H}\mathcal{V}(x)))}_{2k \; times}. \tag{2.4}$$

In other words, we have k compositions of the pair of composed saturation $(\mathcal{H} \circ \mathcal{V})(\cdot)$.

Note that $\mathcal{A}^k(x)$ is horizontally saturated for all $k \in \mathbb{N}$ and for all $x \in M$. If a diffeomorphism $\phi_t(x)$ is decomposable (in the sense of (2.1)) in a neighbourhood of x, then $\phi_t(x) \in \mathcal{A}^k(x)$ for $k \in \mathbb{N}$ (the converse is not true: rotations of $\pi/2$ are counterexamples). Hence we can consider the non-k-attainability as an intrinsic obstruction to the decomposition of a diffeomorphism.

Proposition 3. Given a biregular foliated space $(M, \mathcal{V}, \mathcal{H})$, the attainable sets $\mathcal{A}^k(x)$ are open sets for all $x \in M$ and $k \in \mathbb{N}$.

Proof. Consider initially $k = 1$ and a point $y \in \mathcal{A}^1(x)$. By definition, there exists at least one point $z \in \mathcal{H}(y) \cap \mathcal{V}(x)$. By Proposition 1, there exists an open set $z \in D_1 \subset \mathcal{V}(x) = \mathcal{V}(z)$ which is sent diffeomorphically to an open set $y \in D_2 \subset \mathcal{V}(y)$ along the same horizontal leaves. Using a local biregular chart at y we conclude that the horizontal saturation of D_2 contains an open neighbourhood of y.

For $k \geqslant 2$ one just has to write

$$\mathcal{A}^k(x) = \bigcup_{y \in \mathcal{A}^{k-1}(x)} \mathcal{A}^1(y).$$

The result follows by induction.

□

Proposition 4. Given a biregular foliated space $(M, \mathcal{V}, \mathcal{H})$, if M is connected then $M = \cup_{k \in \mathbb{N}} \mathcal{A}^k(x)$ for all $x \in M$.

Proof. Indeed, we only have to prove that $\cup_{k \in \mathbb{N}} \mathcal{A}^k$ is a closed set. Suppose that there exists a point $x \in \partial \cup_{k \in \mathbb{N}} \mathcal{A}^k$. There exists a local biregular chart in a neighbourhood of x which is mapped in an open rectangle in $\mathbb{R}^r \times \mathbb{R}^{n-r}$. An infinite number of points of $\cup_{k \in \mathbb{N}} \mathcal{A}^k$ are also mapped in this open rectangle. Trivially, these points can also reach x with just one more step: vertical and horizontal trajectory. We conclude that $x \in \cup_{k \in \mathbb{N}} \mathcal{A}^k$ hence this set is open and closed in M.

□

It is particularly interesting when one can reach the whole manifold in a finite number of steps. This leads us to the following definition:

Definition 10. The index of attainability at $x \in M$ with respect to $(M, \mathcal{H}, \mathcal{V})$ is defined as the natural number

$$I_A(x, \mathcal{H}, \mathcal{V}) = min\{k \in \mathbb{N}; \mathcal{A}^k(x) = M\}, \tag{2.5}$$

when it exists. Otherwise we say that $I_A(x, \mathcal{H}, \mathcal{V}) = \infty$.

In other words, the attainability index of $x \in M$ is the maximal number of composition by horizontal and vertical foliations in such way that any point on the manifold is attainable from x. Decomposition of flows always open an interesting questions about the reversibility and commutativity of it. This motivate us to state the following definition.

Definition 11. The co-k-attainable set of $x \in M$ with respect to $(M, \mathcal{H}, \mathcal{V})$ is defined as

$$\mathcal{C}^k(x) = \underbrace{\mathcal{H} \circ \mathcal{V} \circ \mathcal{H} \circ \cdots \circ \mathcal{V}(x)}_{2k \ times} \cap \underbrace{\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} \circ \cdots \circ \mathcal{H}(x)}_{2k \ times}. \tag{2.6}$$

A point $y \in \mathcal{C}^k(x)$, if $y \in \mathcal{A}^k(x)$ and $x \in \mathcal{A}^k(y)$.

Since for each $x \in M$, $\mathcal{A}^k(x)$ is open and leaves of $\mathcal{H}$ are everywhere transverse to the leaves of $\mathcal{V}$, then points close enough to each other have the same k-attainable sets for all k. In particular, for each $x \in M$, there exists a neighbourhood U_0 of x, such that all points in U_0 have the same attainability index. Give $x \in M$ and for $1 \leqslant k \leqslant I_A(x, \mathcal{H}, \mathcal{V})$, there exists a natural chain

$$\mathcal{A}(x) \subsetneq \mathcal{A}^2(x) \subsetneq \mathcal{A}^3(x) \subsetneq \cdots \subsetneq \mathcal{A}^{I_A(x,\mathcal{H},\mathcal{V})}(x) = M. \tag{2.7}$$

Example 10. Consider $M = \mathbb{R}^{2*} = \mathbb{R}^2 \backslash \{(0,0)\}$ with the horizontal foliation given by $\mathcal{H} = \{(x,y) \in \mathbb{R}^{2*}; xy = \alpha, \text{ with } \alpha \in \mathbb{R}\}$ and let the vertical foliation $\mathcal{V}$ be the rotation by $\pi/4$ on the leaves of $\mathcal{H}$. $\mathcal{H}$ and $\mathcal{V}$ are given by grey and blue curves in the figure (1). For $p = (1,1)$, we have $\mathcal{A}(p) = \{(x,y) \in \mathbb{R}^{2*}; y + x > 0\}$, $\mathcal{A}^2(p) = M$. Then $I_A(p, \mathcal{H}, \mathcal{V}) = 2$, and

$$\begin{aligned} \mathcal{C}(p) &= \{(x,y) \in \mathbb{R}^{2*};\ x > 0, y > 0\} \\ \mathcal{C}^2(p) &= \{(x,y) \in \mathbb{R}^{2*};\ y > 0\}. \end{aligned}$$

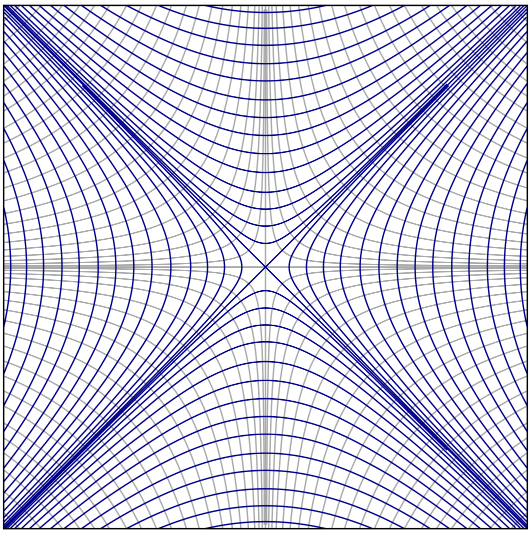

Figure 1 – Example of a biregular atlas with $I_A(p, \mathcal{H}, \mathcal{V}) = 2$

In many interesting cases the index of attainability is infinite, see the next example.

Example 11. For $M = \mathbb{R}^2 \backslash \{(0,0)\}$ with horizontal foliation $\mathcal{H} = \Big\{(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}; y = \Big|\dfrac{1}{\sin(x)}\Big| + c, \text{ for } c \in \mathbb{R}\backslash\{0\}\Big\} \cup \Big\{x = (2r+1)\dfrac{\pi}{2};\ r \in \mathbb{Z}\Big\}$ and vertical foliation given by $\mathcal{V} = \Big\{(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}; y = -\Big|\dfrac{1}{\sin(x)}\Big| + c, \text{ for } c \in \mathbb{R}\backslash\{0\}\Big\} \cup \Big\{x = \dfrac{2r\pi}{2};\ r \in \mathbb{Z}\Big\}$, see figure (2). In this case, note that for $p = (0,1)$:

$$\begin{aligned} \mathcal{A}(p) &= \Big\{(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}; -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}\Big\} \\ \mathcal{A}^2(p) &= \Big\{(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}; -\frac{3\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}\Big\}. \end{aligned}$$

In general,

$$\mathcal{A}^n(p) \quad = \quad \Big\{(x,y)\in\mathbb{R}^2\backslash\{(0,0)\}; -(2n-1)\frac{\pi}{2}\leqslant x\leqslant(2n-1)\frac{\pi}{2}\Big\}.$$

Thus,

$$\bigcup_{n=1}^{\infty}\mathcal{A}^n(p)=\mathbb{R}^2\backslash\{(0,0)\}.$$

Therefore, we conclude that $I_A(p,\mathcal{H},\mathcal{V})=\infty$. Moreover, $I_A(p,\mathcal{H},\mathcal{V})=\infty$ for all point in $(\mathbb{R}^2\backslash\{(0,0)\},\mathcal{H},\mathcal{V})$.

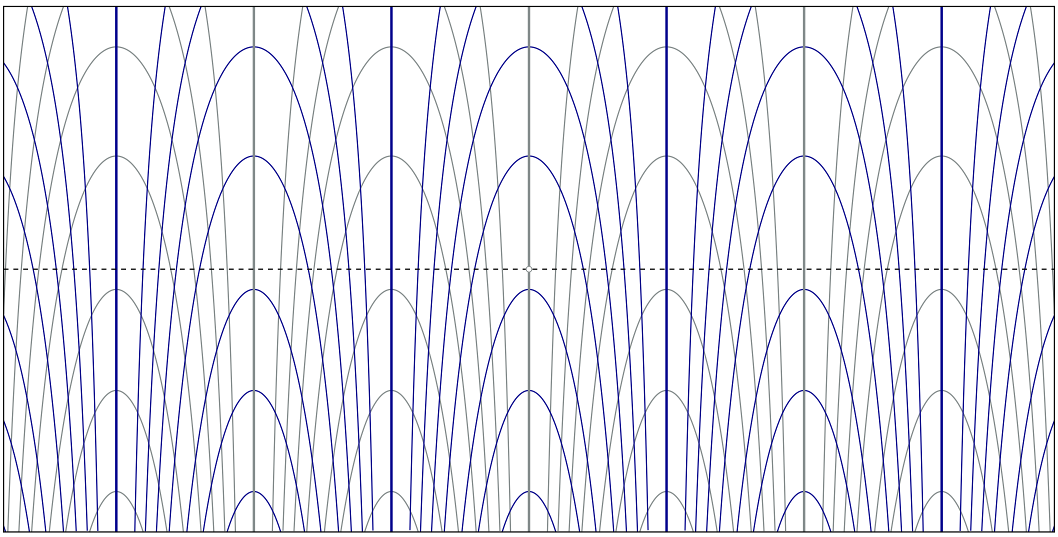

Figure 2 – A biregular atlas with $I_A(p,\mathcal{H},\mathcal{V})=\infty$ for all point p.

Proposition 5. Let M be a compact connected manifold. For each $x\in M$ and $k\in\mathbb{N}$,

i) If $\mathcal{A}^k(x)=\mathcal{C}^k(x)$, then $\mathcal{A}^k(x)=M$.

ii) $\mathcal{C}^{I_A(x,\mathcal{H},\mathcal{V})}(x)=\underbrace{\mathcal{V}\circ\mathcal{H}\circ\ldots\mathcal{V}\circ\mathcal{H}(x)}_{2k \text{ times}}$.

Proof. i) Let $y\in\partial\mathcal{A}^k$ which implies that $\mathcal{V}(y)\cap\mathcal{A}^k(x)$ is not empty, since $\partial\mathcal{A}^k$ is $\mathcal{H}$-saturated. Then, there exists $z\in\mathcal{V}(y)\cap\mathcal{A}^k(x)$. On the other hand, by the fact that $z\in\mathcal{A}^k(x)$, it follows that $z\in\mathcal{C}^k(x)$, therefore $\mathcal{V}(y)\cap\mathcal{H}(x)$ is not empty. Now, let γ be a horizontal curve starting at x and ending at the point $\omega\in\mathcal{V}(y)\cap\mathcal{H}(x)$. Either all vertical leaves of $\mathcal{V}(y)$ intersect $\partial\mathcal{A}^k$ or none of them intersects it, since $\partial\mathcal{A}^k$ is $\mathcal{H}$-saturated and M is compact. But $\mathcal{V}(\omega)=\mathcal{V}(y)$, hence $\mathcal{V}(x)$ intersects $\partial\mathcal{A}^k(x)$, thereby $\partial\mathcal{A}^k(x)\subset\mathcal{A}^k(x)$ and we conclude that $\mathcal{A}^k(x)=M$ since M is connected.

ii) By definition,

$$\begin{aligned}\mathcal{C}^{I_A(x.H,V)}(x) &= \mathcal{A}^{I_A(x.H,V)}(x) \cap \underbrace{\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} \circ \cdots \circ \mathcal{H}(x)}_{2k \text{ times}} \\ &= M \cap \underbrace{\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} \circ \cdots \circ \mathcal{H}(x)}_{2k \text{ times}} \\ &= \underbrace{\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} \circ \cdots \circ \mathcal{H}(x)}_{2k \text{ times}}.\end{aligned}$$

□

2.1.2 Alternate decomposition

In this section, we introduce a technique to rescue the decomposability of some flows of diffeomorphisms which in another way would not be possible to be performed: either by analytical or topological restrictions. It consists on stopping the decomposition close to the point where it no longer would exist and restart, from the identity, another couple of vertical-horizontal diffeomorphisms. In other words: just before the flow approaches a non-decomposable set in the group of diffeomorphisms, we restart the decomposition with a more convenient topological-analytical settings. This succession of dual decomposition, vertical composed with horizontal, represented by $(\mathcal{HV})$, will be called a cascade or alternate decomposition. So, typically, a cascade decomposition has the alternating structure $\mathcal{HV}\ldots\mathcal{HV}$. The last term on the left hand side being $\mathcal{H}$ or $\mathcal{V}$ is not relevant since ending with $\mathcal{V}$ means that the omitted $\mathcal{H}$ part is the identity.

Let $(\phi_t)_{t\in[0,a)}$ be a family of global diffeomorphisms on M, with $\phi_0 = Id$. Fix a point $p \in M$. Throughout this section, we consider two local biregular coordinate systems on possibly disjoint neighbourhoods: $\alpha_p : U_p \subset M \longrightarrow \mathbb{R}^{n-k} \times \mathbb{R}^k$ and $\alpha_{\phi_t(p)} : U_{\phi_t(p)} \subset M \longrightarrow \mathbb{R}^{n-k} \times \mathbb{R}^k$, where U_p and $U_{\phi_t(p)}$ are neighbourhoods of p and $\phi_t(p)$ respectively. With respect to these systems, one writes $\phi_t(x,y) = (\phi_t^1(x,y), \phi_t^2(x,y))$. We shall use the following notation: for $0 \leqslant u \leqslant v$, we set $\phi_{u,v} := \phi_v \circ \phi_u^{-1}$.

Definition 12. Let $(M, \mathcal{H}, \mathcal{V})$ be transversely orientable. The family of diffeomorphisms ϕ_t above is said to *preserve locally transverse orientation* at $p \in M$ along the interval $[0, s)$ if $\det \dfrac{\partial \phi_t^2(p)}{\partial y} > 0$, for all $t \in [0, s)$ and all elements α_p and $\alpha_{\phi_t(p)}$ in the atlas.

Preserving locally transversal orientation is a geometrical property which does not depend on the local biregular coordinate systems:

Lemma 2. Let $(M, \mathcal{H}, \mathcal{V})$ be transversely orientable. Preserving locally transversal orientation does not depend on the local biregular coordinate system.

Proof. Suppose we have two local biregular coordinate systems α_1 and α_2 in a neighbourhood of p. For a fixed time t in the interval of definition, suppose we have also two local biregular coordinate systems β_1 and β_2 in a neighbourhood of $\phi_t(p)$. By Definition 3 we have that the derivatives of the coordinate change $\alpha_1 \circ \alpha_2^{-1}$ and $\beta_1 \circ \beta_2^{-1}$ lie in the subgroup of matrices in $Gl(n, \mathbb{R})$ of the form

$$\begin{pmatrix} (*)_{k\times k} & 0 \\ 0 & (*) \end{pmatrix}_{n\times n},$$

with a positive minor of the lower right submatrix $(n-k)\times(n-k)$. Hence, we have that

$$\det \frac{\partial}{\partial y}\left[\beta_1 \circ \phi_t \circ \alpha_1^{-1}\right] > 0,$$

implies

$$\det \frac{\partial}{\partial y}\left[\beta_2 \circ \phi_t \circ \alpha_2^{-1}\right] > 0.$$

□

The next proposition, give us some properties of the dynamics on the leaves along components of a decomposable flows.

Proposition 6. Suppose ϕ_t is a flow of diffeomorphism which can be decomposed as

$$\phi_t(x) = \left(\eta^k_{t\vee s_{k-1}} \circ \psi^k_{t\vee s_{k-1}}\right) \circ \ldots \circ \left(\eta^2_{s_2} \circ \psi^2_{s_2}\right) \circ \left(\eta^1_{s_1} \circ \psi^1_{s_1}\right),$$

up to a time $\tau > 0$, where ψ^i and η^i are purely vertical and horizontal components respectively and $0 = s_0 < s_1 < s_2 < \ldots < s_r = s_{r+1} = \ldots = \tau$ is a non-decreasing sequence of times. Then, for $0 \leqslant t < \tau$, $\psi^i_t(x) \in \mathcal{V}(x) \cap \mathcal{H}\left(\eta^i_t \circ \psi^i_t(x)\right)$ and $\eta^i_t(y) \in \mathcal{H}(y) \cap \eta^i_t \circ \psi^i_t\left(\mathcal{V}(y)\right)$.

Proof. In fact, for the first statement, note that $\psi^i_t(x) \in \mathcal{V}(x)$, for $i = 1, \ldots, k$ and x in the appropriate domain. In addition, η_t preserves horizontal leaves for all $t < \tau$, then since $\mathcal{H}\left(\eta^i_t \circ \psi^i_t(x)\right) = \mathcal{H}\left(\psi^i_t(x)\right)$, it implies that $\psi^i_t(x) \in \mathcal{H}\left(\eta^i_t \circ \psi^i_t(x)\right)$. For the second statement, observe that $\eta^i_t(y) \in \mathcal{H}(y)$ and $y \in \mathcal{V}(y)$, therefore $\eta^i_t(y) \in \eta^i_t \circ \psi^i_t\left(\mathcal{V}(y)\right)$.

□

Before we state the main theorem of this chapter, consider an important theorem about decomposition of diffeomorphisms in a biregular atlas.

Theorem 8. Suppose that $(M, \mathcal{H}, \mathcal{V})$ is transversely orientable for the horizontal foliation. Then ϕ_t is globally decomposable (in the sense of remark (2)) for all $0 \leqslant t < a$, if and only if, it preserves transverse orientation.

Proof. For a proof, see [47, Thm. 2.5].

□

Theorem 9 (Alternate decomposition)**.** For any fixed $x \in M$, there exists an increasing sequence of times $0 = s_0 < s_1 < s_2 < \ldots < s_r = s_{r+1} = \ldots = a$, $(r \in \mathbb{N} \cup \{\infty\})$ such that for all $t \in [0, a)$, there exists a neighbourhood U_x of x, where we have the following foliated decomposition:

$$\phi_t(x) = \left(\eta^k_{s_{k-1},t} \circ \psi^k_{s_{k-1},t}\right) \circ \ldots \circ \left(\eta^2_{s_1,s_2} \circ \psi^2_{s_1,s_2}\right) \circ \left(\eta^1_{s_1} \circ \psi^1_{s_1}\right)(x),$$

for $t \in [s_{k-1}, s_k]$, with $\eta^k_{s_{k-1}} = Id$ and $\psi^k_{s_{k-1}} = Id$. Here η^j and ψ^j are horizontal and vertical diffeomorphisms respectively for all $j \in \mathbb{N}$.

Proof. For each $x_0 \in M$, we consider a neighbourhood U_{x_0} of x_0 and a coordinate system $\alpha_{x_0} : U_{x_0} \subset M \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$, with respect to it, we write $\phi_s(p) = (\phi^1_s(x, y), \phi^2_s(x, y))$, for $p \in U_{x_0}$, $s > 0$ and we define

$$s_1 = \inf\left\{s \in [0, a); \det \frac{\partial \phi^2_t}{\partial y} = 0\right\} - \epsilon_1,$$

for $\epsilon_1 > 0$ small enough. For $s < s_1$, $\phi_s(x_0)$ preserves transverse orientation. Applying a local version of Theorem 8, it follows that for all $x \in U_{x_0}$, $\phi_s(p)$ has a foliated decomposition

$$\phi_s(p) = \eta^1_s \circ \psi^1_s(p),$$

where η^1_s and ψ^1_s are horizontal and vertical flows of diffeomorphism respectively. If $s_1 = a$, the proof is done, otherwise, for $s > s_1$, we can write $\phi_s(x)$ as

$$\begin{aligned} \phi_s(p) &= \phi_{s_1,s} \circ \phi_{s_1}(p) \\ &= \phi_{s_1,s} \circ (\eta^1_{s_1} \circ \psi^1_{s_1})(p). \end{aligned}$$

Taking $u_1 = \eta^1_{s_1} \circ \psi^1_{s_1}(p)$ and considering u_1 as the initial value of diffeomorphism $\phi_{s_1,s}$, we set

$$s_2 = \inf\left\{s \in [s_1, a); \det \frac{\partial \phi^2_{s_1,s}}{\partial y}(u_1) = 0\right\} - \epsilon_2,$$

for a small $\epsilon_2 > 0$. For $s_1 < s < s_2$, we apply again theorem 8, and rewrite $\phi_s(p)$ as

$$\phi_s(p) = (\eta^2_{s_1,s} \circ \psi^2_{s_1,s}) \circ (\eta^1_{s_1} \circ \psi^1_{s_1})(p),$$

where η^2 and ψ^2 are horizontal and vertical diffeomorphisms respectively. If $s_2 = a$, the proof is done. For $s > s_2$, we rewrite $\phi_s(p)$ as

$$\begin{aligned} \phi_s(p) &= \phi_{s_2,s} \circ (\eta^2_{s_2,s} \circ \psi^2_{s_2,s}) \circ (\eta^1_{s_1,s_2} \circ \psi^1_{s_1,s_2})(p) \\ &= \eta^3_{s_2,s} \circ \psi^3_{s_2,s} \circ (\eta^2_{s_1,s_2} \circ \psi^2_{s_1,s_2}) \circ (\eta^1_{s_1} \circ \psi^1_{s_1})(p). \end{aligned}$$

Recursively, for all $p \in U_{x_0}$ and $s > s_{i-1}$, we take $u_i = (\eta^i_{s_i} \circ \psi^i_{s_i}) \circ \ldots \circ (\eta^2_{s_2} \circ \psi^2_{s_2}) \circ (\eta^1_{s_1} \circ \psi^1_{s_1})(p)$ to be the initial point of diffeomorphism $\phi_{s_{i-1},s}$ and we define

$$s_i = \inf \left\{ s \in [s_{i-1}, a); \det \frac{\partial \phi^2_{s_{i-1},s}}{\partial y}(u_{i-1}) = 0 \right\} - \epsilon_i.$$

For $\epsilon_i > 0$ small enough. Using the above construction, we rewrite $\phi_s(p)$ as

$$\phi_t(p) = \left(\eta^k_{s_{k-1},t} \circ \psi^k_{s_{k-1},t}\right) \circ \ldots \circ \left(\eta^2_{s_1,s_2} \circ \psi^2_{s_1,s_2}\right) \circ \left(\eta^1_{s_1} \circ \psi^1_{s_1}\right)(x),$$

for all $t \leqslant s_i$, where η^j and ψ^j are horizontal and vertical diffeomorphisms and $s_k = s_{k+1} = \ldots = a$.

□

Proposition 7. Suppose that $(M, \mathcal{H}, \mathcal{V})$ is transversely orientable for the horizontal foliation. If $\phi_s(p)$ approaches the boundary $\partial \mathcal{A}^k(p)$ of the k-attainable set, the determinant of $\dfrac{\partial \phi^2_s}{\partial y}(\phi_{s_{k-1}}(p))$ goes to zero.

Proof. The decomposition occurs in fact on $\mathcal{A}^1(\phi_{s_1}(p))$, hence if $\phi_s(p) = \phi_{s_{k-1},s}(p)$ approaches the boundary $\partial \mathcal{A}^k(p)$, then $\phi_s(p)$ necessarily approaches the boundary $\partial \mathcal{A}^1(\phi_{s_1}(p))$. Therefore, $\phi_s(p)$ is non-decomposable and by the analytical obstruction (2.2), it follows that $\det \dfrac{\partial \phi^2_{s_0}(x,y)}{\partial y} = 0$.

□

Example 12. Consider the following pure rotation system:

$$\dot{x}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_t, \tag{2.8}$$

whose solution flow is given by:

$$\phi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Note that this system is not decomposable (in the sense of Remark 2) for $t = \frac{\pi}{2}$, however, for all $\pi/4 \leqslant t \leqslant 3\pi/4$, ϕ_t has the following cascade decomposition:

$$\begin{aligned} \phi_t &= Rot\left(t - \frac{\pi}{4}\right) \circ Rot\left(\frac{\pi}{4}\right) \\ &= \begin{pmatrix} \sec(t-\pi/4) & -\tan(t-\pi/4) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin(t-\pi/4) & \cos(t-\pi/4) \end{pmatrix} \\ &\circ \begin{pmatrix} \sqrt{2}/2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}. \end{aligned}$$

The next example shows that Theorem 9 may hold even if $(M, \mathcal{H}, \mathcal{V})$ is not transversely orientable for the horizontal foliation.

Example 13. Let $M = [0,1]^3/\sim$, where $\sim$ is the identification of the following faces of the cube $[0,1]^3$:

$$(x, 0, z) \sim (1-x, 1, 1-z), \tag{2.9}$$

such that the section $\left(x, y, \dfrac{1}{2}\right) \cap [0,1]^3$ turns into a Mobius strip S. Note that M is a tubular neighbourhood of S. In this context, the horizontal and vertical foliations $\mathcal{H}$ and $\mathcal{V}$ are given by the image of the horizontal and vertical plaques respectively. It is worth mentioning that $(M \backslash S, \mathcal{H})$ is transversely orientable, but $(M, \mathcal{H})$ is not. Consider a complete family of diffeomorphisms given by $\phi_t(x, y, z) = (x, y+t, z)$. In this case, ϕ_t is a horizontal flow with respect to the pair of foliation $(\mathcal{H}, \mathcal{V})$, hence it can be decomposed as $\phi_t = \eta_t \circ \psi_t$, where $\eta_t = \phi_t$ and $\psi_t = Id$, for small $t > 0$. In the non-transversely orientable foliation case $(M, \mathcal{H})$, ϕ_t has an alternate decomposition $\phi_t(x_0) = \left(\eta_t^k \circ \psi_t^k\right) \circ \ldots \circ \left(\eta_{s_2}^2 \circ \psi_{s_2}^2\right) \circ \left(\eta_{s_1}^1 \circ \psi_{s_1}^1\right)(x_0)$, where $s_i \in \{(2k+1)2\pi, \text{with } k \in \mathbb{Z}\}$, for a local biregular coordinate system in a neighbourhood of an initial condition $x_0 \in S$. Each pair $\eta_t^j \circ \psi_t^j$ is given by the projection of the two reverting orientation diffeomorphisms $\eta_t^j(x, y, z) = (y, x, z)$ and $\psi_t^j(x, y, z) = (x, y, 1-z)$, since ϕ_t reverses the orientation of both vertical and horizontal components just before $t = s_i$. In the manifold $(M \backslash S, \mathcal{H})$, the decomposition is guaranteed by Theorem 8.

Corollary 4. Suppose that ϕ_t preserves transverse orientation over $\mathcal{A}^1(x)$ and $I_A(x, \mathcal{H}, \mathcal{V}) < \infty$ for all $x \in M$. Then there exists an increasing sequence of times $0 = s_0 < s_1 < s_2 < \ldots < s_r = s_{r+1} = \ldots = a$, $(r \in \mathbb{N} \cup \{\infty\})$ such that, locally, ϕ_t has an alternate decomposition for all $t \in [0, a)$ as

$$\phi_t(x) = \left(\eta_{s_{k-1},t}^k \circ \psi_{s_{k-1},t}^k\right) \circ \ldots \circ \left(\eta_{s_1,s_2}^2 \circ \psi_{s_1,s_2}^2\right) \circ \left(\eta_{s_1}^1 \circ \psi_{s_1}^1\right),$$

with a finite number of pairs $\left(\eta_{s_{i-1},s_i}^i \circ \psi_{s_{i-1},s_i}^i\right)$, where $\psi_{s_i}^i$, and $\eta_{s_i}^i$ are purely vertical and horizontal diffeomorphisms respectively and $k \leqslant \mathbf{I}_A(x, \mathcal{H}, \mathcal{V})$.

Proof. By Theorem 9, there exists a non-decreasing sequence of times $0 = s_0 < s_1 < s_2 < \ldots < s_r = s_{r+1} = \ldots = a$, such that ϕ_t has an alternate decomposition given by

$$\phi_t(x) = \left(\eta_{s_{k-1},t}^k \circ \psi_{s_{k-1},t}^k\right) \circ \ldots \circ \left(\eta_{s_1,s_2}^2 \circ \psi_{s_1,s_2}^2\right) \circ \left(\eta_{s_1}^1 \circ \psi_{s_1}^1\right)(x).$$

Since ϕ_t preserves transverse orientation over $\mathcal{A}^1(x)$ for all x, by Proposition 7, the subdeterminant $\dfrac{\partial \phi_s^2}{\partial y}(\phi_{s_{i-1}(p)})$ goes to zero just before it reaches $\mathcal{A}^1(\phi_{s_{i-1}}(p))$, it follows that the sequence $(s_i)_{i\in\mathbb{N}}$ must be finite.

□

The next theorem states that the alternate decomposition holds also for stochastic flows of diffeomorphisms. Consider the following Stratonovich Stochastic differential equation on the manifold M:

$$dx_t = \sum_{i=0}^{k} X_i(x_t) \circ dW_t^i, \tag{2.10}$$

with initial condition $x_0 \in M$, where $X_0, X_1, \ldots, X_k$ are smooth vector fields on M, $(W_t^1, \ldots, W_t^k)$ is a Brownian motion on $\mathbb{R}^k$, and $(W_t^0) = t$. We suppose that all this structure is well defined over an appropriate filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geqslant 0}, \mathbb{P})$. Let $\phi_t : \Omega \times M \longrightarrow M$ be the stochastic flow associated to the diffusion generated by equation (2.10). If we assume that the derivatives of the vector fields are bounded, then ϕ_t exists for all $t \geqslant 0$.

Theorem 10. There exists a non-decreasing sequence of stopping times $0 = t_0 < t_1 < t_2 < \ldots < t_r = t_{r+1} = \ldots = a$ such that, locally, $\phi_t(\omega, x)$ is alternately decomposable as

$$\phi_t(\omega, x) = \left(\eta^k_{t_{k-1},t} \circ \psi^k_{t_{k-1},t}\right) \circ \ldots \circ \left(\eta^2_{t_1,t_2} \circ \psi^2_{t_1,t_2}\right) \circ \left(\eta^1_{t_1} \circ \psi^1_{t_1}\right)(\omega, x), \tag{2.11}$$

where η^j and ψ^j are horizontal and vertical diffeomorphisms respectively for all $j \in \mathbb{N}$, $t \in [t_{k-1}, t_k]$ and $\omega \in \Omega$.

Proof. Theorem 9 guarantee that the alternate decomposition holds in the space of trajectories. Now consider the sequence of stopping times $(t_i)_{i\in\mathbb{N}}$, defined by

$$t_i(\omega, x) := \inf\left\{ t \in [t_{i-1}(\omega, x), a); \det \frac{\partial \phi^2_{t_{i-1},t}}{\partial y}(z) = 0 \right\} - \epsilon_i, \tag{2.12}$$

for $\epsilon_i > 0$ small enough, $\omega \in \Omega$, and an appropriated z in a neighbourhood of $\phi_{t_{i-1}}(x)$. By theorem 8 and the cocycle property for stochastic flows, it follows that

$$\begin{aligned} \phi_t(\omega, x) &= \phi_{t_k,t}(\theta_{t_k}(\omega), u_k) \circ \ldots \circ \phi_{t_1,t_2}(\theta_{t_1}(\omega), u_1) \circ \phi_{t_1}(\omega, x) \\ &= \left(\eta^k_{t_{k-1},t} \circ \psi^k_{t_{k-1},t}\right) \circ \ldots \circ \left(\eta^2_{t_1,t_2} \circ \psi^2_{t_1,t_2}\right) \circ \left(\eta^1_{t_1} \circ \psi^1_{t_1}\right)(\omega, x), \end{aligned}$$

where θ_t is the canonical shift operator on the probability space and η^j, ψ^j are horizontal and vertical stochastic flows of diffeomorphisms.

□

3 Decomposition of flows of diffeomorphisms with jump components

3.1 A generalization of Itô-Ventzel-Kunita formula

An interesting problem related to decomposition of stochastic flows of diffeomorphisms relates to discontinuous noise. In particular, we shall consider semimartingales with jumps such that the trajectories are càdlàg. The main result in this chapter, Theorem 11, is a generalization of Itô-Ventzel-Kunita formula for flows generated by the classical Marcus equation as in Kurtz, Pardoux and Protter [33]. We enlarge the scope of this formula allowing the noise to perform infinitely many jumps in compact intervals, it turns possible to use a big variety of noises which includes, for example, Lévy noise, see e.g. Applebaum [2], Protter [53], Oksendal and Sulem [51], among others.

3.1.1 Stratonovich SDE with jumps (SDEJ)

For the reader's convenience, we recall the main aspects and definitions of stratonovich SDEJ in the sense of Marcus equation. Let $Z = \{Z_t, t \geqslant 0\}$ be a k-dimensional semimartingale, with $Z_0 = 0$, and let $[Z, Z] = [Z^j, Z^m]$ be the covariation matrix which can be decomposed into $[Z, Z] = [Z, Z]^c + [Z, Z]^d$, where $[Z, Z]^c$ and $[Z, Z]^d$ represent the continuous and purely discontinuous parts respectively. In Kurtz, Pardoux and Protter [33], it was proposed the following: Let $Y \in C^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{R}^k, \mathbb{R}^d))$ with $Y = (Y^1, \dots, Y^k)$ k-vector fields in $\mathbb{R}^d$. Given an $\mathcal{F}_0$ initial condition x_0, the equation

$$x_t = x_0 + \int_0^t Y(x_s) \diamond dZ_s, \tag{3.1}$$

has a unique solution up to a stopping time τ. Here, the continuous part of the solution corresponds to the classical Stratonovich equations and the jump part are performed along ficticious time (jumps of ΔZ) along the deterministic flow of the corresponding vector field. See details in [33, Eq. (2.2)]. Moreover, although Marcus equation has many restrictions, a change of variables can be obtained in the following sense:

Proposition 8. If x_t is the solution of equation (3.1), then for any function $f \in C^2(\mathbb{R}^d)$:

$$f(x_t) = f(x_0) + \int_0^t f'(x_s) Y(x_s) \diamond dZ_s, \quad t \geqslant 0. \tag{3.2}$$

For a proof and precise definition, see [33, Prop. 4.2].

In a differentiable manifold, the natural extension is given by the following: let $X \in \mathcal{C}^\infty(M; \mathcal{L}(\mathbb{R}^k, TM))$, such that for each $x \in M$ the linear map $X(x)$ sends a vector $z \in \mathbb{R}^k$

into $X(x)z \in T_xM$. Assume that the vector field X is smooth on M and consider the equation

$$dx_t = X(x_t) \diamond dZ_t, \quad x(0) = x_0. \tag{3.3}$$

In this context, $x_t \in M$, $t \geqslant 0$ is a solution of equation (3.3) if for all $f \in C^2(M)$,

$$f(x_t) = f(x_0) + \int_0^t f'(x_s)X(x_s) \ \diamond dZ_s$$

in the sense of [33, Def. 4.1], if $X = (X_1, \dots, X^k)$:

$$\begin{aligned} f(x_t) &= \int_0^t df X^j(x_s)\, dZ_s^j + \frac{1}{2}\int_0^t \nabla^2 f(XdZ, XdZ)(x_s) \\ &+ \sum_{0<s\leqslant t} \left[f(\phi(X\Delta Z_s, x_{s^-})) - f(x_{s^-}) - Xf(x_{s^-})\Delta Z_s\right]. \end{aligned} \tag{3.4}$$

The first term on the right hand side of equation (3.4) is a standard Itô integral of the predictable process $dfX^j(x_s)$ with respect to the semimartingale Z_t. The second term is a Stieltjes integral of the Levi-civita connection applied in the derivative of the function f, with respect to the continuous part of the quadratic variation of Z_t. In the third term: $\phi(X\Delta Z_s, x_{s^-})$ indicates the solution at a fictitious time $t = 1$ of the ODE generated by the vector field $X\Delta Z_s$ and initial condition x_{s^-}. Thus, the jumps of this equation occurs in deterministic directions. It is worth mentioning that some regularities conditions over the linear map $X(x)$ and its derivatives guarantee the existence of a unique Stratonovich flow of diffeomorphisms φ, which is solution of equation (3.3). Moreover, for an embedded submanifold M in an Euclidean space, the support's theorem [33, Prop. 4.3] states that the solution still remains on the manifold after a jump. The next proposition is a change of variables formula for equation (3.1)

The next result shows an expression for equation (3.4) when it is written with respect to a coordinate system.

Proposition 9. Suppose that $x_t,\ t \geqslant 0$ is a solution of equation 3.3 and $\alpha : U \subset M \longrightarrow \mathbb{R}^n$ is a coordinate system of M in a neighbourhood of x_t. Then equation 3.4 can be written in $\mathbb{R}^n$ as

$$\begin{aligned} \tilde{x}_t &= \tilde{x}_0 + \int_0^t \tilde{X}(\tilde{x}_{s^-})dZ_s + \frac{1}{2}\int_0^t \tilde{X}'\tilde{X}(\tilde{x}_s)d[Z,Z]_s^c \\ &+ \sum_{0<s\leqslant t} \left[\tilde{\phi}(X\Delta Z_s, x_{s^-}) - \tilde{x}_{s^-} - \tilde{X}(\tilde{x}_{s^-})\Delta Z_s\right]. \end{aligned} \tag{3.5}$$

Where $\tilde{X}(\tilde{x}_t) = D\alpha(x_t)X(x_t)$ (the derivative of the coordinate system applied on the vector field X), $\tilde{x}_t = \alpha(x_t)$ and $\tilde{\phi}(X\Delta Z_s, x_{s^-}) = \alpha(\phi(X\Delta Z_s, x_{s^-}))$. Moreover, $\tilde{x}_t$ is a semimartigale for all coordinate system.

Proof. It follows straightforward by equation 3.4 and Proposition 8.

□

3.1.2 Itô-Ventzel-Kunita for Stratonovich SDEJs

In order to prove one of the main results in this chapter, which is an extension of the Itô- Ventzel- Kunita formula (infinite jump version), one needs to define the following integral that generalizes the classical Marcus integral (3.1). Precisely, let X and Y be two smooth vector fields on $\mathbb{R}^d$ and consider F_t and G_t, flows of diffeomorphisms generated by $dF_t = X(F_t) \diamond dZ_t$ and $dG_t = Y(G_t) \diamond dZ_t$, SDEJs in the sense of Marcus with respect to the same general semimartingale Z_t. We define the following integral:

$$\begin{aligned}\int_0^t (F_{s*}Y(G_s)) \diamond dZ_s \quad := & \quad \int_0^t (F_{s*}Y(G_{s^-}))dZ_s \\ + & \quad \frac{1}{2}\int_0^t (X'(Y(G_s)) + F_{s*}(Y'Y))d[Z,Z]_s^c \\ + & \quad \sum_{0\leqslant s\leqslant t} \Big\{\phi(X\Delta Z_s, F_{s^-}(\phi(Y\Delta Z_s, G_{s^-})) - \phi(X\Delta Z_s, F_{s^-}(G_{s^-})) \\ - & \quad (F_{s*}Y(G_s))\Delta Z_s\Big\},\end{aligned} \tag{3.6}$$

where the first term on the right hand side is the Itô integral of $F_{s*}Y(G_{s^-})$ with respect to Z_s. In the second term, note that $(X'(Y(G_s))+F_{s*}(Y'Y)) = dF_{s*}Y(G_s)(F_{s*}Y(G_s))$, so the second integral corresponds to the finite variation such that its continuous part satisfies the classical Itô - Ventzel - Kunita. On the last term, the expression $\phi(X\Delta Z_s, F_{s^-}(\phi(Y\Delta Z_s, G_{s^-}))$ has the following geometrical meaning: for a jump time $s \in [0,t]$, the flow G_r jumps at $r = s$ in the direction of solution ϕ with order ΔZ_s, then it is corrected by the flow $F_r : M \to M$, $r \in [0,t]$, then it jumps in the direction of X with order ΔZ_s. It is important to notice that the sum term is absolutely convergent, in fact, applying Taylor's theorem in the map $u \longrightarrow \phi(X\Delta Z_s, F_{s^-}(\phi(Y\Delta Z_s, G_{s^-}, u), 1)$, one gets

$$\begin{aligned}\phi(X\Delta_s Z, F_{s^-}(\phi(Y\Delta_s Z, G_{s^-}, 1), 1) \quad = & \quad \phi(X\Delta_s Z, F_{s^-}(G_{s^-})) + (F_{s*}Y(G_s))(F_s \circ G_s)\Delta Z_s \\ \\ + & \quad \frac{1}{2}(F_{s^-*}Y(G_{s^-}))'(F_{s*}Y(G_s))S(\theta_1,\theta_2)\Delta Z_s \Delta Z_s^t.\end{aligned}$$

Where $S(\theta_1,\theta_2) = \phi(X\Delta_s Z, F_{s^-}(\phi(Y\Delta_s Z, G_{s^-}, \theta_1), \theta_2)$, for $\theta_1, \theta_2 \in (0,1)$ which depends on (s,ω,x), with $x \in M$. Therefore,

$$\begin{aligned}&\sum_{0<s\leqslant t}\Big|\phi(X\Delta_s Z, F_{s^-}(\phi(Y\Delta_s Z, G_{s^-})) - \phi(X\Delta_s Z, F_{s^-}(G_{s^-})) - (F_{s*}Y(G_s))(F_s \circ G_s)\Delta_s Z\Big| \\ &\leqslant \sup_{0<s\leqslant t}\frac{1}{2}\Big|(X'(Y(F_{s^-})) + F_{s*}(Y'Y))f(\phi(X\Delta_s Z, F_{s^-}(\phi(Y\Delta_s Z, G_{s^-}, \theta_1), \theta_2))\Big| \sum_{0<s\leqslant t}|\Delta_s Z|^2 \\ &\leqslant K\sum_{0<s\leqslant t}|\Delta_s Z|^2.\end{aligned}$$

Which converges since $K(\omega)$ is finite and the sum of squares of the jumps of a general semimartingale is always finite a.s. The next theorem states an extension of Itô - Ventzel

- Kunita for general semimartingales. In this context, an infinite number of jumps may occur.

Theorem 11. *(Itô-Ventzel-Kunita for Stratonovich SDE, infinite jump version)* Suppose that F_s and G_s are solutions of SDEJs driven by the same general semimartingale Z_s and with respect to smooth vector fields X and Y on $\mathbb{R}^d$ respectively, for $s \in [0, a]$, Then:

$$F_s(G_s) = F_0(G_0) + \int_0^t X(F_s(G_s)) \diamond dZ_s + \int_0^t F_{s*}(Y(G_s)) \diamond dZ_s \tag{3.7}$$

Proof. It is known that formula (3.7) holds if Z_t is continuous for each $t \in [0, a]$, see e.g. Kunita [33, Thm. 8.3]. Moreover, in Melo et al [46] it was proven that if Z has a finite number of jumps on compact intervals, then formula (3.7) still holds. We are interested in proving this Theorem in the context where the semimartingale Z_t may jump infinitely many times. In this case the problem arises when the set of jump times have some accumulation points. We are going to overcome this problem by splitting the set of jump times of Z_t, into two disjoint subsets, $A = A(\epsilon, t)$ and $B = B(\epsilon, t)$, such that A has a finite number of elements and $\sum_{s \in B} (\Delta Z_s)^2 < \epsilon$, where $A \cup B$ exhausts the jump set of Z_t. Let

$$Z_t^A = Z_t - \sum_{s \leqslant t, s \in B} \Delta Z.$$

Consider F_t^A and G_t^A, solutions of equations $dF_t^A = X(F_t^A) \diamond dZ_t^A$ and $dG_t^A = Y(G_t^A) \diamond dZ_t^A$, respectively. Since formula (3.7) holds for all $s \in A$, it follows that

$$F_s^A(G_s^A) = F_0(G_0) + \int_0^t X(F_s^A(G_s^A)) \diamond dZ_s + \int_0^t F_{s*}^A(Y(G_s^A)) \diamond dZ_s^A. \tag{3.8}$$

Note that if s is a jump time on the interval $[0, t]$, then $s \in A$ for ϵ small enough. Therefore, the solutions F_t^A and G_t^A converge to solutions F_t and G_t a.s respectively, moreover, we have:

$$\begin{aligned}
\Big|\Big| \int_0^t X(F_t^A(G_t^A)) \diamond dZ_s^A - \int_0^t X(F_s(G_s)) \diamond dZ_s \Big|\Big| \;&\leqslant\; \sum_{s \in B} \Big|\Big| \Big\{ \phi(X \Delta Z_s, F_{s^-}(G_{s^-})) - F_{s^-}(G_{s^-}) \\
&-\; X(F_{s^-}(G_{s^-})) \Delta Z_s \Big\} \Big|\Big| \\
&\leqslant\; \sum_{s \in B} \Big|\Big| \Delta Z_t \Big|\Big|^2 \leqslant \epsilon.
\end{aligned}$$

Similarly,

$$\Big|\Big| \int_0^t F_{s*}^A(Y(G_s^A)) \diamond dZ_s^A - \int_0^t F_{s*}(Y(G_s)) \diamond dZ_s \Big|\Big| \leqslant \sum_{s \in B} \Big|\Big| \Delta Z_t \Big|\Big|^2 \leqslant \epsilon.$$

Therefore, when ϵ goes to zero, formula (3.8) converges to formula (3.7) a.s.

□

The next corollary states a Leibniz formula for Stratonovich SDEJs. The proof follows directly from Theorem (11). It will be used basically in the next section in order to compute explicit expressions for the components of a decomposition.

Corollary 5. *(Leibniz formula)* Let F_t and G_t be flows generated by Stratonovich SDEJs with respect to the same general semimartingale Z_t. Then

$$\diamond d(F \circ G)_t = \diamond d(F_t) \circ G_t + (F_t)_* \circ \diamond dG_t. \tag{3.9}$$

By Proposition (8) and local coordinate arguments we can easily extend all results in this section for a Riemannian manifold.

3.2 Decomposition of flows of diffeomorphism generated by SDEJs

Let $\mathrm{Diff}(M)$ be the infinite dimensional Lie group of smooth diffeomorphisms of a compact connected manifold M. The Lie algebra associated to $\mathrm{Diff}(M)$ is the infinite dimensional space of smooth vector fields on M, see e.g. Neeb [48], Omori [52], among others. The exponential map $\exp\{tY\} \in \mathrm{Diff}(M)$ is the associated flow of diffeomorphisms generated by the smooth vector field Y. In this context, given an element $\varphi \in \mathrm{Diff}(M)$ the derivative of the right translation is given by $R_{\varphi *}Y = Y(\varphi)$ for any smooth vector Y. The derivative of left translation $L_{\varphi *}Y = D\varphi(Y)$, and $\mathrm{Ad}(\varphi)Y = \varphi_*(Y(\varphi^{-1}))$.

Interesting problems arise when one decomposes a (flow of) diffeomorphism $\varphi \in \mathrm{Diff}(M)$, into composition of convenient prescribed components. This kind of decomposition appears in the literature, for example, in Bismut [8], Kunita [32] and many others. In particular, it is also relevant when each component of the decomposition belongs to prescribed subgroups of $\mathrm{Diff}(M)$, see e.g Melo et al [46], Catuogno et al [15], Iwasawa and non-linear Iwasawa decomposition [18], Ming Liao [40] among many others.

Suppose that locally M is endowed with a pair of regular differentiable distributions: i.e., every point $x \in M$ has a neighbourhood U and differentiable mappings $\Delta^1 : U \to Gr_k(M)$ and $\Delta^2 : U \to Gr_{m-k}(M)$ respectively, where

$$Gr_p(M) = \bigcup_{x \in M} Gr_p(T_x M)$$

is the Grasmannian bundle of p-dimensional subspaces over M, with $1 \leqslant p \leqslant m$. We assume that Δ^1 and Δ^2 are complementary in the sense that $\Delta^1(x) \oplus \Delta^2(x) = T_x M$, for all $x \in U$. With this notation we define the subgroup of $\mathrm{Diff}(M)$ which is generated by a certain distribution Δ by:

$$\mathrm{Diff}(\Delta, M) = \mathrm{cl}\big\{\exp(t_1 X_1)\ldots\exp(t_n X_n),\ \text{with } X_i \in \Delta, t_i \in \mathbb{R}, \forall n \in \mathbb{N}\big\}.$$

Note that if a distribution Δ is involutive, then each element of the group $\mathrm{Diff}(\Delta, M)$ preserves the leaves of the corresponding foliation.

Definition 13. We say that an element $\eta \in \mathrm{Diff}(M)$ preserves transversality of Δ^1 and Δ^2 in a neighbourhood $U \subset M$ if $\eta_* \Delta^2\left(\eta^{-1}(p)\right) \cap \Delta^1(p) = \{0\}$, for all $p \in U$.

3.2.1 Geometric set up

In the Lie group of diffeomorphisms $\mathrm{Diff}(M)$, the dynamics of the stochastic flow φ_t, which is solution of the stratonovich SDEJ (3.3), is written as the following right invariant SDEJ:

$$d\varphi_t = R_{\varphi_t^*} X \diamond dZ_t. \tag{3.10}$$

Using the same notation as in equation (3.3), we can write:

$$X\, dZ_t = \sum_{i=1}^{d} X_i \diamond dZ_t^i.$$

The Lie group $\mathrm{Diff}(\Delta, M)$ contains two important Lie subgroups: the group of all purely horizontal diffeomorphisms denoted by $\mathrm{Diff}(\Delta^1, M)$ and the group of all purely vertical diffeomorphisms denoted by $\mathrm{Diff}(\Delta^2, M)$. Locally, the intersection of these subgroups is the identity and each element of these groups preserves the leave of the corresponding foliation. The main result of this chapter consists of a decomposition of a flow of diffeomorphisms $\varphi_t \in \mathrm{Diff}(\Delta, M)$ into two components $F_t \in \mathrm{Diff}(\Delta^1, M)$ and $G_t \in \mathrm{Diff}(\Delta^2, M)$.

3.2.2 The existence of the decomposition

In the next result, we assume the condition of transversality preservation along $\mathrm{Diff}(\Delta^1, M)$ for the distributions Δ^1 and Δ^2. The next theorem states a decomposition of flows for the continuous case.

Theorem 12. Given a continuous stochastic flow $\varphi \in \mathrm{Diff}(\Delta, M)$, up to a stopping time, there exists a factorization $\varphi_t = F_t \circ G_t$, where F_t is a continuous diffusion on $\mathrm{Diff}(\Delta^1, M)$ and G_t is a continuous process in $\mathrm{Diff}(\Delta^2, M)$.

For a proof, see [15, Thm. 2.2]. A straightforward extension of this theorem for processes with jump components was proven in [46, Prop. 1] (considering that the process jumps just a finite number of times on compact intervals), this extension is easily proved if one consider that the flow of diffeomorphisms always jumps to a decomposable diffeomorphism. This assumption cannot be removed, in fact, consider the pure rotation system $dx_t = Ax_t \diamond dZ_t$, such that Z_t is a general semimartingale. The flow of this system is given by:

$$\varphi_t = \begin{pmatrix} \cos Z_t & -\sin Z_t \\ \sin Z_t & \cos Z_t \end{pmatrix}.$$

Locally, the flow φ_t can be decomposed by

$$\varphi_t = \begin{pmatrix} \sec(Z_t - \pi/4) & -\tan(Z_t - \pi/4) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin(Z_t - \pi/4) & \cos(Z_t - \pi/4) \end{pmatrix}.$$

Note that if t_0 is a jump time such that the process jumps to $Z_{t_0} \in \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$, then the above decomposition will no longer exist. The next theorem is the main result of this chapter.

Theorem 13. The stochastic flow of local diffeomorphisms φ_t can be decomposed (locally, up to a stopping time) as

$$\varphi_t = F_t \circ G_t,$$

where F_t is solution of an (autonomous) SDEJ in $\mathrm{Diff}(\Delta^1, M)$ and G_t is a process in $\mathrm{Diff}(\Delta^2, M)$. The decomposition is unique.

Proof. Let $\widetilde{X}_i$ be an element in the Lie algebra of the group $\mathrm{Diff}(\Delta^1, M)$, given by:

$$\widetilde{X}_i(x) = X_i(x) - V_i(x) \ \in \Delta^1.$$

Where V_i is the unique vector such that $\widetilde{X}_i$ is horizontal in T_xM. We define the component F_t as the solution of the Marcus equation in $\mathrm{Diff}(\Delta^1, M)$ given by:

$$dF_t = R_{F_{t*}}\widetilde{X}_0 dt + \sum_{i=1}^{m} R_{F_{t*}}\widetilde{X}_i \diamond dZ_t^i.$$

The second component is defined as $G_t = F_t^{-1} \circ \varphi_t$. In order to find a Marcus equation whose solution flow is G_t, one needs to apply Theorem 11 (Itô-Ventzel-Kunita for general semimartingales). Hence,

$$dG_t = F_{t*}^{-1} \diamond d\varphi_t + \diamond dF_t^{-1}(\varphi_t). \tag{3.11}$$

For each $F_t \in \mathrm{Diff}(\Delta^1, M)$. On the other hand, note that:

$$dF_t^{-1} = -L_{F_{t*}^{-1}}\widetilde{X}_0 dt - \sum_{i=1}^{m} L_{F_{t*}^{-1}}\widetilde{X}_i \diamond dZ_t^i. \tag{3.12}$$

By (3.11) and (3.12), it follows that:

$$dG_t = \sum_{i=0}^{m} \mathrm{Ad}(G_t^{-1})(V_i(G_t))(F_t) \diamond dZ_t^i.$$

It is important mentioning that $\mathrm{Ad}(F_t^{-1})(V_i)(x) \in \Delta^2(x)$, for all $x \in M$. Then, $G_t \in \mathrm{Diff}(\Delta^2, M)$.

$\square$

4 Geometry of young integral: decomposition of α-Hölder continuous paths

In this chapter we study geometric aspects of dynamics generated by Young differential equations (YDE) driven by α-Hölder trajectories with $\alpha \in (1/2, 1]$. More precisely, given a smooth manifold M, we focus on geometrical properties of equations of the type:

$$dx_t = X(x_t)\, dz_t, \tag{4.1}$$

with initial condition $x_0 \in M$ at $t = 0$, where $x \to X(x) \in \mathcal{L}(\mathbb{R}^d, T_x M)$ is a smooth assignment of d vector fields on M and $z \in C^\alpha([0,T], \mathbb{R}^d)$ is an α-Hölder continuous trajectory in $\mathbb{R}^d$. We say that a path $x : [0,T] \to M$ is a solution of equation (4.1) if for all test function $f \in C^\infty(M; \mathbb{R})$ we have that

$$f(x_t) = f(x_0) + \int_0^t Xf(x_s)\, dz_s, \tag{4.2}$$

where Xf is a short term for $\sum Df(x)X(x)e_i$, with e_i's the elements of the canonical basis of $\mathbb{R}^d$. The last term of equation (4.2) is an integral in the Young sense, see e.g. the classical [65], or more recent Hairer and Friz [22], Gubinelli et al. [24], Lyons [38], Castrequini and Russo [13], Castrequini and Catuogno [15], Cong [19], Ruzmaikina [57], among many others. We also encourage the readers to check our submitted paper [14], which includes the majority of results in this chapter.

4.1 Some geometric aspects of Young integral

In this section, we study some geometric aspects of dynamics generated by Young differential equations (YDE) driven by α-Hölder trajectories with $\alpha \in (1/2, 1)$. We present a number of properties and geometrical constructions on this low regularity context: Young Itô geometrical formula, horizontal lift in principal fibre bundles, parallel transport, covariant derivative, development and anti-development, among others.

4.1.1 Young differential equation on manifolds

We recall that for a general metric space (M, d), a curve $\sigma : [0,T] \to M$ is α-Hölder continuous, with $\alpha > 0$ if there exists a constant $C > 0$, such that

$$d(\sigma(t), \sigma(s)) \leqslant C|t-s|^\alpha, \tag{4.3}$$

for all $s, t \in [0, T]$. This concept extends naturally to a Riemaniann manifold, since it carries the well known induced metric $d(x, y)$ given by

$$d(x,y) = \inf \left\{ \int_0^1 ||\gamma'(t)||dt;\ \gamma : [0,1] \to M \text{ differentiable such that } \gamma(0) = x,\ \gamma(1) = y \right\}.$$

See e.g. [20] among many other classical books. Hence, naturally, α-Hölder paths are also well defined in Riemannian manifolds. Most of the classical analytic results on this regularity theory also holds for α-Hölder paths in a Riemannian manifold. For instance, composition of a differentiable function with an α-Hölder trajectory is also an α-Hölder path. Particularly, in a geometrical context, for readers convenience we prove the following

Proposition 10. Let M and N be Riemannian manifolds, $\dim N \geqslant 1$. A path $\sigma : [0, T] \to M$ is α-Hölder continuous on M if and only if, for all differentiable map $f : M \to N$, the path $f(\sigma(t))$ is α-Hölder continuous on N.

Proof. There are many interesting ways to prove this result. Here, we use an embedding argument. Initially consider that N is an Euclidean space $\mathbb{R}^n$ and take $\sigma(t)$ an α-Hölder trajectory on M. There exists an isometric embedding $i : M \to \mathbb{R}^d$ for a sufficiently large integer d (Nash theorem). For sake of notation we write $\sigma_t := \sigma(t)$.

Since $\|i(x) - i(y)\|_{\mathbb{R}^d} \leqslant d_M(x, y)$ for all $x, y \in M$, we have the following inequalities:

$$\|i(\sigma_t) - i(\sigma_s)\|_{\mathbb{R}^d} \ \leqslant\ d_M(\sigma_t, \sigma_s) \ \leqslant\ C|t-s|^\alpha,$$

which implies that $i(\sigma_t)$ is α-Hölder in $\mathbb{R}^d$. Now, for any differentiable function $f : M \to \mathbb{R}^n$, use the fact that it can be extended to a differentiable function $\bar{f} : U \to \mathbb{R}^n$ defined in a tubular neighbourhood U of $i \circ \sigma([0, T])$ in $\mathbb{R}^N$. Hence, $f(\sigma_t) = \bar{f}(i(\sigma_t))$. Since Hölder regularity is preserved by differentiable functions on Euclidean spaces, $f(\sigma_t)$ is α-Hölder continuous in $\mathbb{R}^n$. Mind that, in fact, in the compact set $i(\sigma_t)$ the metrics d_M, and ℓ_2 in $\mathbb{R}^d$ are uniformly equivalents, see Lemma 2.2 [35]. Hence, ℓ_2-norm Hölder regularity in $\mathbb{R}^d$ is equivalent to Hölder regularity on (M, d_M).

For a general Riemannian manifold N and a differentiable map $f : M \to N$, consider another isometric embedding $i' : N \to \mathbb{R}^{d'}$ for an integer d' sufficiently large. Then, the last paragraph shows that $i' \circ f(\sigma)$ is α-Hölder in $\mathbb{R}^{d'}$. From Lemma 2.2 [35] we have that there exists a positive constant C_1 such that

$$d_N(f(\sigma_t), f(\sigma_s)) \leqslant C_1 \|i(f(\sigma_t)) - i(f(\sigma_t))\|_{\mathbb{R}^{d'}} \leqslant C_2 |t-s|^\alpha,$$

for a positive constant C_2, which shows that $f(\sigma_t)$ is α-Hölder continuous in N.

Conversely, suppose that $f(\sigma_t) \in N$ is α-Hölder for all differentiable function $f : M \to N$. Denote the projections of $i(\sigma_t) \in \mathbb{R}^d$ by $\sigma_t^j := p_j \circ i(\sigma_t)$ for each $1 \leqslant j \leqslant d$. Let $\varphi : V \to$

$W \subset N$ be a local parametrization for N, with V an open set in an Euclidean space. There exist another local parametrization obtained from the previous one, just enlarging the domain by homothety, if necessary, which we call again by $\varphi : \tilde{V} \to W \subset N$ such that the set $\{(x, 0, \ldots, 0); x = \sigma^j(t) \text{ for some } t \in [0, T]\} \subset \tilde{V}$ for all $1 \leqslant j \leqslant d$. Consider the differentiable functions $f_j : M \to N$ given by $f_j(x) := \varphi(p_j(i(x)), 0, \ldots, 0))$. Then $f_j(\sigma_t) := \varphi(\sigma_t^j, 0, \ldots, 0))$ is α-Hölder by hypothesis. By metric equivalence in compact sets in the domain of the local parametrization, we have that σ_t^j is α-Hölder for all $1 \leqslant j \leqslant d$. We conclude that $\sigma(t) \in M$ is α-Hölder continuous on M.

□

Before we show conditions for existence and uniqueness of solutions for equation (4.1), we state the main geometric theorem that is a version of Itô's formula for α-Hölder continuous paths. We start with the definition of the Young integral of a real 1-form:

Definition 14 (Integration of real 1-forms)**.** Let N be an n-dimensional differentiable manifold with $\bigwedge^1(N)$ the space of real 1-forms. Consider $\beta \in \bigwedge^1(N)$ and a chart $(U, (y_1, \ldots, y_n))$ in N such that

$$\beta = \sum_{i=1}^{n} \beta_i \, dy^i. \tag{4.4}$$

The integral of β along an α-Hölder path $x : [0, T] \to N$ is defined by

$$\int_0^T \beta(x_t) \, dx_t = \sum_{i=1}^{n} \int_0^T \beta_i dx_t^i, \tag{4.5}$$

where the above integrals are Riemann-Stieltjes integral of β_i with respect to the i-th coordinate of the path x_t. Among others properties, this integration is independent of the local chart, see e.g, Abraham, Marsden and Ratiu [1] and Ikeda and Manabe [28].

The integration of real 1-forms above allows one to integrate many tensor fields in a manifold. In particular, if $F : M \to \mathbb{R}^d$ is a smooth function, the integration

$$\int_0^t DF(x_s) \, dx_s$$

makes sense, looking at each coordinate of $\mathbb{R}^d$. Also, if $F : M \to N$, with N another differentiable manifold, then we define

$$\int_0^t DF(x_s) \, dx_s := \int_0^t D\phi \circ F(x_s) \, dx_s, \tag{4.6}$$

where, ϕ is a local chart of N. Standard rough path calculus in Euclidean space (in particular, substitution formula) guarantees that this definition is independent of the local chart. Next Theorem is the basic property of the the α-Hölder calculus we are treating in this chapter.

Theorem 14 (Young Itô Formula)**.** Let M and N be Riemannian manifolds. Consider $x \in \mathcal{C}^\alpha([0,T], M)$ and a smooth function $F : M \to N$. Then

$$dF(x_t) = DF(x_s)\ dx_s. \tag{4.7}$$

Remark: We highlight that formula (4.7) above means that if β is a 1-form in N then

$$\int_0^t \beta\ dF(x_s) = \int_0^t (dF(x_s))^* \beta\ dx_s. \tag{4.8}$$

In particular, if N is an Euclidean space:

$$F(x_t) = F(x_0) + \int_0^t DF(x_s) dx_s. \tag{4.9}$$

Proof. Initially we prove the result for an Euclidean space $N = \mathbb{R}^d$. We use again the embedding argument from Nash's theorem: there exists a sufficiently large $p \in \mathbb{N}$ such that M can be isometrically embedded into $\mathbb{R}^{m+p}$. Abusing notation, we have $x \in \mathcal{C}^\alpha([0,T], \mathbb{R}^{m+p})$, F is defined in a tubular neighbourhood of the image of M and $DF(x) \in L(\mathbb{R}^{m+p}, \mathbb{R}^d)$. By Taylor's formula in Euclidean space,

$$F(x_t) - F(x_s) = DF(x_s) \cdot (x_t - x_s) + R(x_s, x_t),$$

with

$$R(x_s, x_t) = \int_0^1 (1-u) \operatorname{Hess}(F)(x_s + u(x_t - x_s))(x_t - x_s, x_t - x_s)\ du.$$

Since F is smooth,

$$\|R(x_s, x_t)\| \leqslant C \|x_t - x_s\|^2 \leqslant C' |t-s|^{2\alpha}.$$

Let $\pi = \{s_i\}$ be a partition of $[0,T]$. Then

$$F(x_t) - F(x_0) = \sum_i F(x_{s_{i+1}}) - F(x_{s_i}) = \sum_i DF(x_{s_i}) \cdot (x_{s_{i+1}} - x_{s_i}) + \sum_i R(x_{s_i}, x_{s_{i+1}}). \tag{4.10}$$

We have that

$$\sum_i \|R(x_{s_i}, x_{s_{i+1}})\| \leqslant C' \sum_i |s_{i+1} - s_i|^{2\alpha} \leqslant C' T \sup_i |s_{i+1} - s_i|^{2\alpha - 1}.$$

Thus, since $\alpha > 1/2$ we have that

$$\lim_{|\pi| \to 0} \sum_i \|R(x_{s_i}, x_{s_{i+1}})\| = 0.$$

Take the limit $|\pi| \to 0$ in equation (4.10) and the definition of Stieltjes (Young) integral to finish the proof in this context.

In general, when N is a Riemannian manifold, consider a local chart ϕ. The previous calculations hold with $\phi \circ F$ whose integration is independent of the coordinate system.

□

Note that multidimensional forms of Itô formula above, integration by parts, etc can be obtained from formula (4.7) considering the manifold M above as appropriate product spaces. We proceed to prove a Theorem of existence and uniqueness of solution for equation (4.1).

Theorem 15. Given an initial condition $x_0 \in M$, there exists a unique maximal solution of the Young differential equation (4.1) such that $x(0) = x_0$. Moreover, there exists a flow of (local) diffeomorphisms associated to the solutions.

Proof. A simple way to proof the result for local solutions is based on the existence and uniqueness results in the Euclidean space. In fact, given the initial condition x_0, let (U, Ψ) be a chart on M with $x_0 \in U$. Let $\tilde{X} := D\Psi(X(\Psi^{-1}(p)))$ be the induced vector field in the image of Ψ. The Young differential equation $dy_t = \tilde{X}(y_t)dz_t$ has a unique solution local solution y_t with $y_0 = \Psi(x_0)$. See e.g. Lejay [36], Caruana, Lyons and Thierry [43], Li and Lyons [38], Friz and Hairer [22] and references therein. Take $x_t = \Psi^{-1}(y_t) \subset U$. We claim that x_t is a solution of equation (4.1). In fact, consider a test function $f \in C^\infty(M)$. By Theorem 14, it follows that

$$\begin{aligned}
f(\Psi^{-1}(y_s)) &= f(\Psi^{-1}(y_0)) + \int_0^t D(f \circ \Psi^{-1})(y_s)dy_s \\
&= f(\Psi^{-1}(y_0)) + \int_0^t Df \Psi_*^{-1} \Psi_* X(y_s)dz_s \\
&= f(x_0) + \int_0^t Xf(x_s)dz_s.
\end{aligned}$$

Moreover, the solution x_t does not depend on the choice of local coordinate. In fact, let (V, Φ) be another chart on M, with $x_t \in U \cap V$ and let z_t be the solution of the Young differential equation $dw_t = \Phi_* X(w_t)dz_t$. Then

$$\begin{aligned}
dy_t &= \Psi_* X(y_t)dz_t \\
&= \Psi_* \Phi_*^{-1} \Phi_* X(z_t)dz_t \\
&= \Psi_* \Phi_*^{-1} dw_t.
\end{aligned}$$

Hence $y_t = \Psi\Phi^{-1}(w_t)$ and therefore $\Phi^{-1}(w_t) = \Psi^{-1}(y_t) = x_t$. A maximal solution is obtained in the classical way by extending a local solution up to its explosion time. The existence of local flow of (local) diffeomorphisms is also concluded from the Euclidean case using the same local chart argument.

□

4.1.2 Horizontal lifts

Let $\{P, M, G, \pi\}$ be a principal fibre bundle with base M, structure group G and total space P. In this case M is a smooth, connected and paracompact manifold. The projection

π is taken as $\pi : P \to M$. The group G acts freely on P on the right by the action $R_g : P \to P$ defined by $R_g(u) = ug$, for $u \in P$ and $g \in G$. Let $\mathfrak{g}$ be the Lie algebra of G, then an element $A \in \mathfrak{g}$ generates the exponential $\{\exp tA,\ t \in \mathbb{R}\}$, which induces a vector field on P by

$$A^*u = \frac{d}{dt} R_{\exp(tA)} u \,\Big|_{t=0},$$

If $\Gamma^\infty(TP)$ is the section of all smooth vector fields on P, then the map $A \to A^*$, from $\mathfrak{g}$ into $\Gamma^\infty(TP)$ is a Lie algebra homomorphism. For more details, see e.g. Shigekawa [62], the classical Kobayashi and Nomizu [31] among many others. The tangent space TP has a naturally defined subspace called the vertical tangent bundle VTP given by $VT_uP := \ker d\pi_u$ for all $u \in P$. Note that $A^*u \in VT_uP$ for all $A \in \mathfrak{g}$.

A connection in the principal fibre bundle is an assignment of a horizontal subspace HT_uP of T_uP which is the kernel of a $\mathfrak{g}$-valued 1-form ω in P with the following properties:

(i) (well-behaved vertically) $\omega dR_g = \mathrm{Ad}(g^{-1})\omega$, for all $g \in G$. Here the linear map $\mathrm{Ad}(g^{-1}) : \mathfrak{g} \to \mathfrak{g}$ is the derivative at the identity of the adjoint $\mathrm{Ad}(g^{-1}) : G \to G$ defined by $\mathrm{Ad}(g^{-1})a = g^{-1}ag$.

(ii) (vertical calibration) $\omega(A^*) = A$, where A^* is a vector field on VTP.

Such 1-form ω is called a connection form in the principal fibre bundle $\{P, M, G, \pi\}$. Moreover, ω defines the horizontal tangent bundle HTP given by $HT_uP = \ker \omega_u$. Hence, for all $u \in P$, the tangent space T_uP splits into $HT_uP \oplus VT_uP$ and $dR_g(HT_uP) = HT_{ug}P$.

Now we have the geometric set up to define the horizontal lift of α-Hölder continuous paths.

Definition 15. Let $x : [0, T] \to M$ be an α-Hölder continuous path. Consider $u \in P$, with $\pi(u) = x_0$. The horizontal lift of x_t starting at u is a path $\widetilde{x} : [0, T] \longrightarrow P$ such that:

(i) $\widetilde{x}_0 = u$.

(ii) $\pi(\widetilde{x}_t) = x_t$ for all $t \in [0, T]$.

(iii) $\displaystyle\int_0^t \omega\ d\widetilde{x}_s = 0$ for all $t \in [0, T]$.

Next result shows the existence and uniqueness of the horizontal lift for an α-Hölder continous path in a manifold. In the proof we apply the same technique used in Kobayashi and Nomizu [31] and in Shigekawa [62] where the existence and uniqueness of horizontal lift were proved in the context of C^1 paths and semimartingales respectively.

Theorem 16. Given an α-Hölder continuous path $x : [0,T] \to M$ and an element u in the fibre $\pi^{-1}(x_0)$, there exists (up to a explosion time) a unique horizontal lift $\tilde{x} : [0,T] \to P$ with $\tilde{x}_0 = u$.

Proof. Consider a local trivialization $\phi : \pi^{-1}(U) \to U \times G$ with $x_0 \in U$ and take the α-Hölder path $\nu_t = \phi^{-1}(x_t, e)$. If the horizontal lift of x_t exists at all, it has to be of the form $\tilde{x}_t = \nu_t a_t$, where $a_t \in G$ is an appropriate path which makes $\tilde{x}_t$ horizontal and $\nu_0\, a_0 = u$.

Let $\Psi : P \times G \to P$ be the right free action of G on P. Then, by Theorem 14 we have that

$$d\tilde{x}_t = \partial_1 \Psi(\nu_t, a_t)\ d\nu_t + \partial_2 \Psi(\nu_t, a_t)\ da_t.$$

Hence:

$$\begin{aligned} \int_0^t \omega\ d\,\tilde{x}_t &= \int_0^t (\partial_1 \Psi(\nu_t, a_t))^* \omega\ d\nu_t + \int_0^t (\partial_2 \Psi(\nu_t, a_t))^* \omega\ da_t. \\ &= \int_0^t R^*_{a_t} \omega\ d\nu_t + \int_0^t \theta\ da_t, \end{aligned} \tag{4.11}$$

by the vertical calibration of the connection ω, where θ is the canonical Cartan 1-form given by $\theta_g(dR_g A) = A$ for all $g \in G$ and $A \in \mathfrak{g}$. The lift $\tilde{x}_t$ is horizontal if and only if equation (4.11) vanishes for all $t \in [0,T]$, i.e. if and only if $\mathrm{Ad}(a_t^{-1})\omega\ d\nu_t = -\theta\ da_t$. Let $F_1, \ldots, F_n$ be a basis of the right invariant Lie algebra $\mathfrak{g}$. For all $t \in [0,T]$, there exist α-Hólder continuous real functions $\alpha_t^1, \ldots, \alpha_t^n$, such that:

$$\int_0^t \omega\ d\nu_s = \sum_{i=1}^n F_i \alpha_t^i. \tag{4.12}$$

Using this notation we have that a necessary and sufficient condition such that equation (4.11) vanishes is that

$$\int_0^t \omega\ d\,\tilde{x}_t = \sum_{i=1}^n \int_0^t \mathrm{Ad}(a_t^{-1}) F_i\ d\alpha^i + \int_0^t dR_{a_t^{-1}}\ da_t = 0,$$

for all $t \in [0,T]$, i.e., trajectory a_t has to satisfy

$$da_t = -\sum_{i=1}^n dR_{a_t}\ \mathrm{Ad}(a_t^{-1}) F_i\ d\alpha_t^i,$$

with initial condition a_0. There exists a unique solution by Theorem 15, hence there exists a unique horizontal lift $\tilde{x}_t$ up to a explosion. Mind that at the border of the local trivialization, one can extend further the solution applying again the same construction above. The maximal solution covers the whole interval $[0,T]$ (by compactness) if there is no explosion in the fibre.

Note that for initial element in the fibre $a_0 g$, the horizontal lift is given by $a_t g$.

□

In the sequel, we show another proof for Theorem 16. In this alternative proof, we write the canonical 1-form in terms of its local coordinates.

Proof. Let $\widetilde{a}_t$ be the unique solution of the following Young differential equation in G:

$$d\widetilde{a}_t = \sum_{i=1}^{n} F_i(\widetilde{a}_t) d\alpha_t^i, \tag{4.13}$$

where $F_i(\widetilde{a}_t) = dR_{\widetilde{a}_t}(F_i))$ Take $a_t = \widetilde{a}_t^{-1}$. Note that a_t is a α-Hölder path and satisfies the following equation:

$$da_t = \sum_{i=1}^{n} \int_0^t Ad(a_t^{-1}) F_i(a_t) d\alpha_t^i. \tag{4.14}$$

See Castrequini and Catuogno [12]. We aim to prove that $\widetilde{x}_t = \nu_t a_t$ is a horizontal lift of x_t. In fact, note that $\widetilde{x}_t \in C^{\min\{\alpha,\beta\}}$, and by construction, $\pi(\widetilde{x}) = x_t$. We just need to prove that $\displaystyle\int_{\widetilde{x}[0,T]} \omega = 0$. Consider the action $\psi : P \times G \longrightarrow P$, given by $\psi(x, y) = xy$ and let $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ be local coordinates in G and P respectively. Let $a_t^i = x^i(a_t)$, $\nu_t^i = y^i(\nu_t)$ and $\omega = \displaystyle\sum_{i=1}^{n} \omega_i dy^i$. Applying the Itô's formula for α-Hölder paths (theorem 14), it follows that:

$$\widetilde{x}_t^i = \widetilde{x}_0^i + \sum_{i=1}^{m} \int_0^t \frac{\partial \psi^i}{\partial a_s^j}(\nu_s^1, \ldots, \nu_s^n, a_s^1, \ldots, a_s^m) da_s^i + \sum_{i=1}^{n} \int_0^t \frac{\partial \psi^i}{\partial \nu_s^l}(\nu_s^1, \ldots, \nu_s^n, a_s^1, \ldots, a_s^m) d\nu_s^i.$$

By the definition of integrals of 1-forms, remark 14, the connection 1-form ω can be written as:

$$\begin{aligned}
\omega(d\widetilde{x}_t) &= \sum_{i=1}^{n} \omega_i(\widetilde{x}_t) d\widetilde{x}_t \\
&= \sum_{i=1}^{m} \sum_{i=1}^{n} \omega_i(\widetilde{x}_t) \frac{\partial \psi^i}{\partial a_s^j}(\nu_s^1, \ldots, \nu_s^n, a_s^1, \ldots, a_s^m) da_s^i \\
&+ \sum_{i=1}^{m} \sum_{i=1}^{n} \omega_i(\widetilde{x}_t) \frac{\partial \psi^i}{\partial \nu_s^l}(\nu_s^1, \ldots, \nu_s^n, a_s^1, \ldots, a_s^m) d\nu_s^i.
\end{aligned}$$

We define:

$$\begin{aligned}
(\psi_{\nu_t}^* \omega)_{a_t} &:= \sum_{i=1}^{m} \sum_{i=1}^{n} \omega_i(\widetilde{x}_t) \frac{\partial \psi^i}{\partial a_s^j} da_s^i \\
({}_{a_t}\psi^* \omega)_{\nu_t} &:= \sum_{i=1}^{m} \sum_{i=1}^{n} \omega_i(\widetilde{x}_t) \frac{\partial \psi^i}{\partial \nu_s^l} d\nu_s^i.
\end{aligned}$$

Hence,

$$\omega d\widetilde{x}_t = ({}_{a_t}\psi^*\omega)d\nu_t + (\psi^*_{\nu_t}\omega)da_t. \tag{4.15}$$

Note that $\psi^*\omega$ is the pull-back of the 1-form by the action ψ and ${}_{a_t}\psi^*$ is the differential of the right translation R_{a_t}. We are going to use that fact that $R^*_{a_t}\omega = Ad(a_t^{-1})\omega$ and $\psi^*_{\nu_t}\omega = \theta$, where θ is the canonical 1-form on G defined by $\theta_g(A(g)) = A$ for $g \in G$ and $A \in \mathfrak{g}$, see e.g [31, chapter 9]. Thus we have that:

$$\begin{aligned}
\int_0^t \omega d\widetilde{x}_t &= \int_0^t ({}_{a_t}\psi^*\omega)d\nu_t + \int_0^t (\psi^*_{\nu_t}\omega)da_t \\
&= \int_0^t R^*_{a_t}\omega d\nu_t + \int_0^t \theta da_t \\
&= \sum_{i=1}^n \int_0^t Ad(a_t^{-1})F_i d\alpha^i - \sum_{i=1}^n \int_0^t Ad(a_t^{-1})F_i d\alpha^i \\
&= 0.
\end{aligned}$$

Now, suppose that the horizontal lift is not unique. In this case, let $\widetilde{x}_t$ and $\widetilde{z}_t$ be horizontal lifts of x_t, with $\widetilde{x}_0 = \widetilde{z}_0$ and $\pi(\widetilde{x}_t) = \pi(\widetilde{z}_t) = x_t$, for all $t \in [0,T]$. Thus, there exists $u_t \in G$, with $u_0 = e$, such that $\widetilde{x}_t = \widetilde{z}_t u_t$, for all $t \in [0,T]$. It is easy to show that u_t is α-Hölder. Applying equation (4.15), we have:

$$\int_0^t \omega d\widetilde{x}_s = \int_0^t \theta du_t + \int_0^t Ad(u_s^{-1})\omega d\widetilde{z}_t = 0.$$

Since $\int_0^t Ad(u_s^{-1})\omega d\widetilde{z} = 0$, it follows that $\int_0^t \theta du_t = 0$ for all $t \in [0,T]$. Therefore, u_t must be the constant $u_t = e$, then we conclude that $\widetilde{x}_t = \widetilde{z}_t$, for all $t \in [0,T]$.

□

Besides the dynamics and the principal fiber bundle approach presented so far (which are basic to the next Sections), this low regularity Itô-Young calculus of Theorem 14 allows one to develop further geometrical properties. We mention the following three classical geometric aspects:

A. Parallel Transport and covariant derivative: Given a smooth manifold M, consider the frame bundle $BM \to M$ of basis $u : \mathbb{R}^n \to T_pM$, with $p \in M$, with the structure group $G = Gl(n, \mathbb{R})$. Last Theorem applied in this context establishes a parallel transport along α-Hölder path $x_t \in M$. In fact, given a horizontal lift u_t, the parallel transport of a vector $v \in T_{x_0}M$ is obtained by

$$/\!/_t\, v = u_t \circ u_0^{-1}(v) \in T_{x(t)}M.$$

It does not depend on the choice of the horizontal lift. Moreover, if we take the orthonormal frame bundle $OM \to M$ of basis orthonormal basis given by linear isometries $u : \mathbb{R}^n \to$

T_xM, with $x \in M$, with the structure group $G = O(n, \mathbb{R})$, the parallel transport is also an isometry.

Covariant derivative can now be defined along an α-Hölder path $x_t \in M$. Given a differentiable vector field Y, we have that its covariant derivative along $x(t)$ is given by:

$$DY(x_t) = /\!/_t \; d \; /\!/_t^{-1} \; Y(x_t).$$

where the differentials are interpreted in the sense of Young (Definition 14).

B. Development and anti-development: Let M be an m-dimensional Riemannian manifold, and consider an α-Hölder continuous path $x : [0, T] \to \mathbb{R}^m$. Take the horizontal operator $H : OM \times \mathbb{R}^m \to HTOM$ where $H(u, v)$ is the horizontal lift of $u(v) \in T_{\pi(u)}M$ up to HT_uOM. The development of x_t on M with initial orthonormal frame u_0 is obtained from u_t, the solution of the YDE:

$$d\,u_t = H(u_t, dx_t),$$

i.e. $\pi(u_t)$ is the development of $x(t)$ on M (rolling without slipping, with initial "contact plane" given by u_0). On the other hand, the anti-development of an α-Hölder continuous path $x : [0, T] \to M$ is described using its horizontal lift $\tilde{x}_t$ (Theorem 16) with initial condition $\tilde{x}_0$:

$$y_t = \int_0^t \tilde{x}_s^{-1} \, d\,x_s.$$

Note that, as expected, y_t depends on the choice of $\tilde{x}_0$. Compare this approach with the classical Brownian motion approach by Eells and Elworthy [21], and the isotropic Lévy processes approach in Applebaum and Estrade [3], among many others.

C. Continuous α-Hölder paths in M are solutions of Young differential equations: As established before, solutions of Young equations driven by α-Hölder paths on a manifold are also α-Hölder continuous paths. Reciprocally, every α-Hölder continuous paths on M is a solution of a Young differential equation (YDE) driven by an α-Hölder functions. In fact, take an embedding $i : M \to \mathbb{R}^{m+p}$ of M into a sufficiently large dimensional Euclidean space. Let U be a tubular neighbourhood with $\pi : U \to i(M)$ a projection of U into $i(M)$. Given an α-Hölder path y_t on M, let $z_t = i(y_t)$. Then z_t is an α-Hölder trajectory in $\mathbb{R}^{m+p}$. Consider the YDE in $i(M)$:

$$dx_t = D\pi(x_t)\, dz_t.$$

Then z_t is the solution of this YDE with initial condition $x_0 = z_0$: just check that the YDE is the differential version of the identity $z_t = \pi(z_t)$, according to Young Itô formula of Theorem 14. If the projection π is orthogonal, as in Elworthy [21] then the vector fields are gradients of the embedding. In general, the dynamics of other trajectories starting at $x_0 \neq y_0$ depends on the embedding and on the projection. This is an interesting topic to be studied further.

4.2 Decomposition of flow generated by Young differential equation

In the Lie group notation, a solution flow φ_t of an YDE is written as the solution of a right invariant Young differential equation in the Lie group of diffeomorphisms Diff(M):

$$d\varphi_t = R_{\varphi_t *} X \, dz_t. \tag{4.16}$$

Here we abuse notation in the sense that (using the same notation as in equation 4.1) one can write

$$X \, dZ_t = \sum_{i=1}^{d} X_i \, dz_t^i,$$

where $X_j = X(e_j)$ with e_j the elements of the canonical basis. Hence, equation (4.16) have to be interpret as

$$d\varphi_t = \sum_{i=1}^{d} R_{\varphi_t *} X_i \, dz_t^i.$$

In this section, we explore the Young calculus to proof the existence of a geometrical decomposition of flows generated by α-Hölder systems φ_t given by equation (4.1).

In particular, in this Section we focus on the subgroups Diff(Δ^1, M) and Diff(Δ^2, M). The main result of this paper (Theorem 18) establishes a local decomposition of the solution flow φ_t into two components: a curve (solution of an autonomous YDE) in Diff(Δ^1, M) composed with a non-autonomous path in Diff(Δ^2, M).

By continuity, for any pair of complementary distributions, there always exists a neighbourhood of the identity $1d \in$ Diff(M) where all elements in this neighbourhood preserve transversality. Moreover, if the distribution Δ^1 is involutive then all elements in Diff(Δ^1, M) preserves tranversality of Δ^1 and Δ^2: in fact, the derivative η_* above is a linear isomorphism which sends tangent spaces of the associated foliation to tangent spaces in the same leaf. In the sequence, we state an extended scope of the Itô-Kunita formula (see [32]) in the geometrical Young calculus.

Theorem 17 (Young Itô-Kunita formula)**.** Let X, $Y \in C^2(M, \mathcal{L}(\mathbb{R}^d, TM))$ and $z \in C^\alpha([0,T], \mathbb{R}^d])$ and suppose that η_t and ψ_t are solutions maps associated to the Young differential equations $d\eta_t = X(\eta_t)dz_t$ and $d\psi_t = Y(\psi_t)dz_t$ respectively. Then, $\varphi_t = \eta_t \circ \psi_t$ is the solution map associated with the Young differential equation

$$d\varphi_t = X(\varphi_t)dz_t + \mathrm{Ad}(\eta_t)Y(\varphi_t)dz_t. \tag{4.17}$$

For a proof in this low regularity context, see Castrequini and Catuogno [12, Thm. 4.1]. Next Corollary shows that the inverse of the solution flow of an YDE is also α-Hölder continuous.

Corollary 6. If η_t is the solution flow the Young differential equation on M

$$dx_t = X(x_t)\, dz_t, \tag{4.18}$$

then, the inverse map η_t^{-1} is the solution of the Young differential equation on M

$$du_t = -D\eta_t^{-1}(u_t)X(\eta_t(u_t))dz_t. \tag{4.19}$$

Proof. In fact, just apply expressions (4.18) and (4.19) into equation (4.17).

□

For a constructive proof of last Corollary see [12, Thm. 4.2]. Next Theorem states the main result of this section:

Theorem 18 (Decomposition of flows of YDE)**.** Up to a life time $\tau \in [0, T]$, the solution flow φ_t can be locally decomposed as

$$\varphi_t = \eta_t \circ \psi_t,$$

where η_t is solution of an (autonomous) Young differential equation in $\mathrm{Diff}(\Delta^1, M)$ and ψ_t is a path in $\mathrm{Diff}(\Delta^2, M)$.

Proof. Given $p \in M$, take $\eta \in \mathrm{Diff}(\Delta^1, M)$ sufficiently close to the identity such that it preserves tranversality, i.e. $\mathrm{Ad}(\eta_t)\Delta^2$ and Δ^1 are complementary. The tangent vector(s) $X(p)$ can be decomposed uniquely as

$$X(p) = h(p) + V(\eta_t, p), \tag{4.20}$$

where $h(p) \in \Delta^1(p)$ and $V(\eta_t, p) \in \mathrm{Ad}(\eta_t)\Delta^2(p)$, for all $p \in M$. We take the first component η_t as the solution map of the following Young differential equation in $\mathrm{Diff}(\Delta^1, M)$:

$$d\eta_t = R_{\eta_{t*}}\, h\, dz_t, \tag{4.21}$$

with initial condition $\eta_0 = 1d$, the identity. Even though the equation above is described in terms of a right translation, it is not a right invariant equation since h in general depends on η_t. We obtain the second component of decomposition of φ_t using that $\psi_t = \eta_t^{-1} \circ \varphi_t$. Applying Corollary 6, it follows that:

$$d\eta_t^{-1} = -L_{\eta_{t*}^{-1}}\, h\, dz_t,$$

where $L_{\eta_{t*}^{-1}}$ is the derivative of the left translation at the identity by η_t^{-1}. Finally, we find a equation for ψ_t by applying Theorem 17:

$$\begin{aligned} d\psi_t &= (\eta_t^{-1} h\, \eta_t\, \psi_t - \eta_t^{-1} X\, \eta_t\, \psi_t)\, dz_t \\ &= \mathrm{Ad}(\eta_t^{-1}) V(\eta_t)\; dz_t. \end{aligned} \tag{4.22}$$

Note that $V(\eta, p)$ does not necessarily belong to Δ^2. Still, $d\psi_t \in \Delta^2$ since $d\psi_t \in \mathrm{Ad}(\eta^{-1})\mathrm{Ad}(\eta)\Delta^2 = \Delta^2$. Then ψ_t is the Δ^2-component of φ_t.

□

Corollary 7. If the distributions Δ^1 and Δ^2 are integrable, then the decomposition of Theorem 18 is unique.

Proof. In fact, in this case $\mathrm{Diff}(\Delta^1, M) \cap \mathrm{Diff}(\Delta^2, M) = \{1d\}$.

□

5 Examples and open problems

In this chapter, we consider the same geometric structure used in chapter 4, i.e, a principal fibre bundle $\{P, M, G, \pi\}$, with base M, structure group G and total space P. Our main goal is to apply the decomposition which was proposed in Theorem 18 in general fibre bundles. Important notions such as 1-forms connection on fibre bundles, horizontal and vertical tangent bundles and others were discussed briefly in chapter 4, more details can be found for example in Kobayashi and Nomizu [31]. In the last section of this chapter, we also state some interesting open problems related to the decomposition of flows and stochastic optimal control.

5.1 Linear systems

Consider an Euclidean space $\mathbb{R}^n$, with a pair of complementary foliations given by the trivial Cartesian product $\mathbb{R}^k \times \mathbb{R}^\ell$, with $k + \ell = n$. More precisely, the horizontal foliation $\mathcal{F}_H$ is given by parallel leaves generated by affine translations $x + (\mathbb{R}^k \times \{0\})$, with $x \in \mathbb{R}^n$. Analogously, the vertical foliation $\mathcal{F}_V$ is given by parallel vertical leaves $x + (\{0\} \times \mathbb{R}^\ell)$, for all $x \in \mathbb{R}^n$. We consider the linear Young differential equation:

$$dx_t = A\, x_t\; dz_t, \tag{5.1}$$

with $x_0 \in \mathbb{R}^n$ and z_t an α-Hölder continuous trajectory in the real line. The Young calculus presented in the previous section shows that the fundamental linear solution flow of (5.1) is the exponential

$$F_t = \exp\{A(z_t - z_0)\}. \tag{5.2}$$

$$A = \begin{pmatrix} \begin{pmatrix} A_1 \end{pmatrix}_{k\times k} & \begin{pmatrix} A_2 \end{pmatrix} \\ \begin{pmatrix} A_3 \end{pmatrix} & \begin{pmatrix} A_4 \end{pmatrix}_{\ell\times\ell} \end{pmatrix}$$

The decomposition we are interested here is

$$F_t = \eta_t \circ \psi_t$$

such that $\eta_t \in \mathrm{Diff}(\Delta^1, M)$ and $\psi_t \in \mathrm{Diff}(\Delta^2, M)$. In general η_t and ψ_t does not have to be linear, even in quite symmetric situations. For example, if the pair of foliations in $\mathbb{R}^n \backslash \{0\}$ are given by radial and spherical coordinates, the components of the decomposition are not necessarily linear: in fact, the linear radial diffeomorphisms is reduced to a one dimensional group of uniform contractions and expansions $\lambda 1d$, with $\lambda > 0$, which, obviously, is not big enough to perform the decomposition. For the Cartesian pair of foliation $\mathbb{R}^k \times \mathbb{R}^\ell$

considered in this section, we do have that η_t and ψ_t are linear. In fact, in coordinates, write

$$F_t = \begin{pmatrix} \Big(F_1(t)\Big)_{k\times k} & \Big(F_2(t)\Big)_{k\times \ell} \\ & \\ \Big(F_3(t)\Big)_{\ell\times k} & \Big(F_4(t)\Big)_{\ell\times \ell} \end{pmatrix}.$$

Since η_t does not change the last ℓ coordinates the diffeomorphisms ψ must satisfies

$$\psi_t = \begin{pmatrix} \Big(1d\Big)_{k\times k} & 0 \\ & \\ F_3(t) & F_4(t) \end{pmatrix}.$$

Hence diffeomorphisms ψ_t and η_t, when exist, are global and linear.

A simple example: A system which illustrates not only these formulae, but also the lifetime of the decomposition is the pure rotation in $\mathbb{R}^2$ given by

$$dx_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_t\, dz_t,$$

whose decomposition of flow can be easily calculated as:

$$\begin{pmatrix} \cos z_t & -\sin z_t \\ \sin z_t & \cos z_t \end{pmatrix} = \begin{pmatrix} \sec z_t & -\tan z_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin z_t & \cos z_t \end{pmatrix}. \tag{5.3}$$

Note that if $z_t \in \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$, then the decomposition (5.3) no longer exists at the corresponding time t, i.e. we have explosion of the solutions of equations (4.21) or (4.22).

□

Back to the general linear case, the components of the decomposition in fact lie in the Lie group:

$$\psi_t \in G_V = \left\{ g \in Gl(n, \mathbb{R}); g = \begin{pmatrix} \Big(1d\Big)_{k\times k} & 0 \\ g_3 & \Big(g_4\Big)_{\ell\times \ell} \end{pmatrix} \right\}$$

whose Lie algebra is given by the vector space generated by

$$\begin{pmatrix} \Big(0\Big)_{k\times k} & 0 \\ & \\ \Big(*\Big) & \Big(*\Big)_{\ell\times \ell} \end{pmatrix},$$

where $(*)$ means nonzero matrices of the appropriate dimension. Analogously for the horizontal component:

$$\eta_t \in G_H = \left\{ g \in Gl(n, \mathbb{R}); g = \begin{pmatrix} \Big(g_1\Big)_{k\times k} & g_2 \\ 0 & \Big(1d\Big)_{\ell\times \ell} \end{pmatrix} \right\}$$

whose Lie algebra is given by the vector space generated by

$$\begin{pmatrix} \begin{pmatrix} * \end{pmatrix}_{k\times k} & \begin{pmatrix} * \end{pmatrix}_{k\times \ell} \\ & \\ 0 & \begin{pmatrix} 0 \end{pmatrix}_{\ell\times \ell} \end{pmatrix}.$$

Using the properties of the Young integral, we find the differential equations for the constituents submatrices g_1, g_2 and g_3, g_4 of η_t and ψ_t respectively. Let $\pi_2 : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the projection on the second subspace. From formula (4.20) we have that

$$V(\eta, \cdot) = \eta \circ \pi_2 \circ A(\cdot).$$

In fact, it is enough to check that $V(\eta, \cdot)$ is in the image of the vertical component by η and that $\pi_2 V(\eta, \cdot) = \pi_2 A(\cdot)$. From this formula, equations (4.21) and (4.22) we find the autonomous equation:

$$d\eta_t = (1d - \eta_t \circ \pi_2) A\, \eta_t\ dz_t,$$

and the well expected nonautonomous vertical diffeomorphisms:

$$d\psi_t = \pi_2\, A\, \eta_t \circ \psi_t\ dz_t.$$

Rewriting each constituent submatrices we find:

$$\begin{aligned} dg_1(t) &= \big[A_1\ g_1(t) - g_2(t)\ A_3\ g_1(t)\big]\, dz_t && (5.4)\\ dg_2(t) &= \big[A_1 g_2(t) + A_2 - g_2(t)\, A_4 - g_2\, A_3\, g_2(t)\big]\, dz_t, && (5.5)\\ dg_3(t) &= \big[A_3\, g_1 + A_3\, g_2\, g_3 + A_4 g_3\big]\, dz_t && (5.6)\\ dg_4(t) &= \big[A_3\, g_2\, g_4 + A_4\, g_4\big]\, dz_t. && (5.7) \end{aligned}$$

Explosion in the solutions of the equations of g_1 and g_2 can appear if A_3 is not zero (see example of equation (5.3), where $A_3 = [1]$). Otherwise, if $A_3 = 0$ then there exists the decomposition for all time $t \geqslant 0$. Using this feature, and the Jordan canonical form we can extend the scope of the decomposition in the next Proposition. Before that, let us fix a notation. Given two complementary subspaces $E_1 \oplus E_2 = \mathbb{R}^n$, let us denote by $\mathcal{F}(E_1)$ and $\mathcal{F}(E_2)$ the corresponding pair of complementary parallel foliations in $\mathbb{R}^n$.

Proposition 11. Consider a Young linear system in $\mathbb{R}^n$

$$dx_t = A\, x_t\ dz_t. \tag{5.8}$$

If dimension $n > 2$, then there exist a pair of parallel foliations $\mathcal{F}(E_1)$, $\mathcal{F}(E_2)$ generated by complementary subspaces E_1 and E_2 such that the decomposition of the flow of equation (5.8) exists for all time $t \in [0, T]$, i.e. there is no explosion time of the decomposition. Dimension of E_1 can be chosen as a number of the form $(a + 2b)$ where $a = 0, 1, \ldots, r = \#\{\text{real eigenvalues with multiplicities}\}$, and $b = 0, 1, \ldots, (n - r)/2$.

Proof. Let $A = P\,JP^{-1}$ be the canonical real Jordan form of A, with the choice of bases P such that the nilpotent component has, if necessary, 1's and identities I_2's above the diagonal. The change of coordinates $y = P\,x$ establishes the conjugate Young system:

$$dy_t = J\,y_t\;dz_t.$$

If $n > 2$, it is possible to write

$$J = \begin{pmatrix} \begin{pmatrix} J_1 \end{pmatrix}_{k\times k} & \begin{pmatrix} J_2 \end{pmatrix} \\ \begin{pmatrix} J_3 \end{pmatrix} & \begin{pmatrix} J_4 \end{pmatrix}_{\ell\times\ell} \end{pmatrix}$$

with $k = a + 2b$ and its complementary $\ell = n - k$, such that the submatrix $(J_3)_{\ell\times k} = 0$. The number a represents the number of real eingenvalues in the block J_3 and b represents the number of pairs of conjugate nonreal eigenvalues in this block. Hence, equations (5.7) guarantee the there is no explosion in the decomposition of y_t. By conjugacy, there is also no explosion in the decomposition of the linear fundamental solution F_t of (5.8) along the foliations generated by $E_1 = P\,(\mathbb{R}^k \times \{0\})$ and $E_2 = P\,(\{0\} \times \mathbb{R}^l)$. This proves the proposition.

□

Using the notation in the proof of last proposition, the decomposition of $F_t = \eta_t \circ \psi_t$ above are such that η_t lies in the group $P\,G_H\,P^{-1}$ and ψ_t lies in $P\,G_V\,P^{-1}$.

5.2 Principal fibre bundles over homogeneous spaces

Let G be a connected Lie group with a closed subgroup H and denote by $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras of right invariant vector fields, respectively. The group G acts on H by left translation gH, for all $g \in G$ and the orbits generate the homogeneous space $M := G/H$, see e.g. [31]. We have a principal fibre bundle given by the canonical projection $\pi : G \to M$. Given an element $A \in \mathfrak{g}$ consider the right invariant YDE:

$$d\,g_t = Ag_t\;dZ_t. \tag{5.9}$$

As it was done in chapter 4, here, we consider a connection ω in the principal fibre bundle $\pi : G \to M$. In this example we construct our decomposition of flow according to the vertical subspaces (involutive) and the horizontal subspace established by this connection. The solution flow (global in G up to lifetime of Z_t) is given by left action:

$$\varphi_t(x) = g_t x,$$

where $g_t = \exp\{A\,Z_t\}$. In this example the distributions Δ^1 and Δ^2 in the tangent space TG are given by the horizontal subspaces with respect to the connection ω and the tangent to the fibres gH (involutive). In order to decompose the flow φ_t as in Theorem 18, one has to identify the vector fields V and h as in equation (4.20) in the proof of the Theorem, i.e.:

$$Ax := h + V(\eta, x).$$

Elements $\eta \in \mathrm{Diff}(\Delta^1, G)$ can be written pointwise (with respect to $x \in G$) as a left action of elements of G at x. This action preserves the vertical component, i.e. $g_*\Delta^2 = \Delta^2$ for all $g \in G$. Hence, vector field V above is independent of η and one can easily calculate:

$$V(x) = \omega(Ax)^* \qquad \text{and} \qquad h = Ag - \omega(Ax)^*.$$

By equations (4.21) and (4.22) we have that each component of the decomposition $\varphi_t(\cdot) = \eta_t \circ \psi_t(\cdot)$ are given by:

$$d\,\eta_t = R_{\eta_t *}(A\eta_t(\cdot) - \omega(A\eta_t(\cdot))^*) \tag{5.10}$$

and

$$d\,\psi_t = \mathrm{Ad}(\eta_t)\ \omega(A\eta_t(\cdot))^*. \tag{5.11}$$

Let denote by $g_t^{H,x} \in G$ the α-Hölder curve in G such that $g_t^{H,x}x$ is the horizontal lift of $\pi(g_t\,x)$ starting at x, i.e. $g_t^{H,x}x$ is horizontal and $g_t^{H,x}x = g_t\,x\,v_t$ for some $v_t \in H$. With this notation, fixing the action at a point $x \in G$, the equations above reduce to well known finite dimensional equations (in G). This is the content of the following

Proposition 12. Consider the decomposition $\varphi_t(\cdot) = \eta_t \circ \psi_t(\cdot)$ of the solution flow of equation (5.9) according to horizontal and vertical distribution of the fibre bundle in the sense of Theorem 18. Then, at each point $x \in G$, the first component can be written as the left action:

$$\eta_t(x) = g_t^{H,x}x,$$

and the second component can be written as the right action:

$$\psi_t(x) = x\,h_t$$

where $h_t = x^{-1}\,(g_t^{H,x})^{-1}\,g_t\,x$.

Proof. The proof of the first equation follows straightforward when one applies equations (5.10) at a fixed inicial condition $x \in G$: it is the horizontal lift of $\pi(g_t x)$, cf. Theorem 16, using Itô formula 14. Regarding the second equation of the statement, one sees that $(x^{-1}\,(g_t^{H,x})^{-1}\,g_t\,x) \in H$ by definition of the horizontal lift: $g_t^{H,x}x = g_t\,x\,v_t$ for some $v_t \in H$. One checks that it solves (5.11) at a fixed point x.

□

5.2.1 Trivial fibre bundles

As a particular case, consider a trivial principal fibre bundle $\pi : G \times H \to H$ with strutural group H, where G and H are connected Lie groups. The trivial connection is given by $\omega_{(x,y)}(g'_t, h'_t) = y^{-1}h'_t \in \mathfrak{h}$. Consider a right invariant YDE in $G \times H$:

$$d(x_t, y_t) = (A \times B)\ (x_t, y_t)\, dz_t$$

where $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$, the Lie algebras of G and H respectively, with an initial condition (x_0, y_0). Since the connection in this case is invariant by left action of $G \times \{1d\}$, the factor $g_t^{H,x} \in G \times H$ of Proposition 12 does not depend on (x, y). One recovers the trivial components of the decomposition. In fact we get a global decomposition where the first component is given by the left action:

$$\eta_t(\cdot, \cdot) = (\exp(Az_t), 1d)(\cdot, \cdot).$$

And the second (vertical) component is given in terms of the right action:

$$\psi_t(\cdot, \cdot) = (\cdot, \cdot)(1d, h_t)$$

where $h_t = y^{-1}\ \exp(Bz_t)\, y$, according to Proposition 12.

5.3 Jump dynamics on reductive homogeneous spaces

We say that the homogeneous space M is reductive if the Lie algebra $\mathfrak{g}$ contains a subspace $\mathfrak{n}$, such that $\mathrm{Ad}(H)(\mathfrak{n}) \subset \mathfrak{n}$ and $\mathfrak{g}$ can be written as the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$. It is worth mentioning that each fibre $\pi^{-1}(x)$ is diffeomorphic to H. A similar decomposition was considered by Li [39] in the context of standard Brownian motion.

Corollary 8. For $S \in C_b^2(G, \mathcal{L}(\mathbb{R}^d, \mathfrak{h}))$ and $Y \in C_b^2(G, \mathcal{L}(\mathbb{R}^d, \mathfrak{n}))$. Let Z_t be a general semimartingale and ψ_t and η_t be solutions associated with the Marcus differential equations $d\psi_t = S^*(\psi_t) \diamond dZ_t$ and $d\eta_t = Y^*(\eta) \diamond dZ_t$. Then,

$$d(\eta_t\psi_t) = R_{\psi_t^*} d\eta_t + (L_{\psi_t^{-1}} d\psi_t)^*(\eta_t\psi_t). \tag{5.12}$$

Proof. By Corollary 5, it follows that

$$\begin{aligned} d(\eta_t\psi_t) &= R_{\psi_t *} Y^*(\eta_t) dZ_t + L_{(\eta_t\psi_t)*} L_{\psi^{-1}*} S^*(\eta_t\psi_t) dZ_t \\ &= R_{\psi_t *} d\eta_t + (L_{\psi^{-1}} d\psi)^*(\eta_t\psi_t). \end{aligned}$$

□

Let $\varphi_t, t < T$, be the flow of diffeomorphism of the following canonical Marcus stochastic differential equation,

$$d\varphi_t = W^*(\varphi_t) \diamond dZ_t. \tag{5.13}$$

Where W is a element of the Lie algebra $\mathfrak{g}$.

In the next theorem, we find explicit Marcus differential equations for the vertical and horizontal components of the solution φ_t. Let ω be the canonical connection 1-form on the principal bundle (P, G, H, π, M). As defined in chapter 4, $\omega(X) = 0$ for all vector field $X \in \mathfrak{n}$, and $\omega(A^*) = A$ if $A \in \mathfrak{h}$. We suppose that the Lie algebra $\mathfrak{g}$ is reductive, therefore it can be written as the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$. Thus, the vector field W can be decomposed into $W^*(g) = h^*(g) + V^*(g)$, where $h^*(g) \in \mathfrak{h}$ and $V^*(g) \in \mathfrak{n}$.

Theorem 19. The solution flow φ_t can be decomposed into $\varphi_t = \eta_t \psi_t$, such that the components η_t and ψ_t satisfies the following system of Marcus differential equations:

$$\begin{aligned} d\psi_t &= V^*(\psi_t) \diamond dZ_t^i, &(5.14)\\ d\eta_t &= (Ad(\psi_t)h)^*(\eta_t) \diamond dZ_t. &(5.15)\end{aligned}$$

Proof. Note that φ_t is a diffeomorphism for all $t > 0$, than it sends each fibre in another fibre, hence φ_t is decomposable for all $t < T$, see e.g. [46, Corollary 2], thus the solution flow φ_t can be rewritten as $\varphi_t = \eta_t \circ \psi_t$, where η_t and ψ_t are horizontal and vertical semimartingales respectively up to a stop time T. By Corollary 8, we have that:

$$d\varphi_t = R_{\psi_t} d\eta_t + \left(L_{\psi_t^{-1}} d\psi_t\right)^* (\varphi_t). \tag{5.16}$$

Applying the 1-form ω at $d\varphi_t$:

$$\omega(d\varphi_t) = \omega\left(L_{\psi_t^{-1}} d\psi_t\right)^* (\varphi_t) = L_{\psi_t^{-1}} d\psi_t.$$

Hence,

$$\begin{aligned} L_{\psi_t^{-1}} d\psi_t &= \omega\left(W^*(\varphi_t) \diamond dZ_t^i\right)\\ &= \omega\left(h^*(\varphi_t) \diamond dZ_t + V^*(\varphi_t) \diamond dZ_t\right)\\ &= V \diamond dZ_t.\end{aligned}$$

Therefore,

$$d\psi_t = V^*(\psi_t) \diamond dZ_t.$$

Using the identity $\psi_t \circ \psi_t^{-1} = 1$ and Corollary 8, it follows that:

$$d\psi_t^{-1} = -R_{\psi_t^{-1}} V \diamond dZ_t.$$

Applying Corollary 8 once again, for $\eta_t = \varphi_t \circ \psi_t^{-1}$, we have:

$$\begin{aligned}
d\eta_t &= R_{\psi_t^{-1}} d\varphi_t + L_{\varphi_t} d\psi_t^{-1} \\
&= R_{\psi_t^{-1}} \left(W^*(\varphi_t) \diamond dZ_t \right) - L_{\varphi_t} \left(R_{\psi_t^{-1}} V \diamond dZ_t \right) \\
&= \left[R_{\psi_t^{-1}} L_{\varphi_t} W - R_{\psi_t^{-1}} L_{\varphi_t} V \right] \diamond dZ_t \\
&= \left[R_{\psi_t^{-1}} L_{\eta_t} L_{\psi_t} h \right] \diamond dZ_t \\
&= L_{\eta_t} \mathrm{Ad}(\psi_t) h \diamond dZ_t = \left(\mathrm{Ad}(\psi_t) h \right)^* (\eta_t) \diamond dZ_t.
\end{aligned}$$

□

Let $\nu_t = \pi(\varphi_t)$, we want to compute a Marcus differential equation for ν_t. Using the fact that $d\pi(V^*(\varphi)) = 0$, it follows that:

$$\begin{aligned}
d\nu_t = d\pi(d\varphi_t) &= d\pi \left(W(\varphi_t) \right) \diamond dZ_t \\
&= d\pi \left(h^*(\varphi_t) + V^*(\varphi_t) \right) \diamond dZ_t \\
&= d\pi h^*(\varphi_t) \diamond dZ_t.
\end{aligned}$$

Therefore

$$d\nu_t = \bar{L}_{\eta_t *} \bar{L}_{\psi_t *} d\pi(h) \diamond dZ_t, \tag{5.17}$$

where $\bar{L}_a$ is the left translation on the base space M, for $a \in G$. Then, $\pi \circ L_a = \bar{L}_a \circ \pi$.

Proposition 13. The process $\eta_t, t < T$ satisfies the equation (5.15), if, and only if, it is a horizontal lift of ν_t.

Proof. Suppose that η_t is a solution flow of equation (5.15). Since $\omega_{\eta_t}(d\eta_t) = 0$, and $\psi_0 \in H$, taking $x_t = \pi(\eta_t)$, it holds that:

$$\begin{aligned}
dx_t &= d\pi \left(\mathrm{Ad}(\psi_t^{-1}) h \right)^* (\eta_t) \diamond dZ_t \\
&= d\pi \left(R_{\psi_t^{-1}} L_{\psi_t} h \right)^* (\eta_t) \diamond dZ_t \\
&= \bar{L}_{\eta_t} \bar{L}_{\psi_t} d\pi h \diamond dZ_t,
\end{aligned}$$

therefore, x_t satisfies equation (5.17), by uniqueness of solution of Marcus differential equations, we conclude that $x_t = \pi(\eta_t)$.

On the other hand, suppose that η_t is a horizontal lift of ν_t up to a stopping time T. Since φ_t is a solution of equation (5.13) and $\pi(\varphi_t) = \pi(\eta_t)$, then φ_t and η_t belong to the same fibre for all $t < T$. Therefore, there exists $C_t \in G$, such that $\eta_t C_t = \varphi_t$, for $t < T$. By Corollary 8:

$$d\eta_t = R_{C_t^{-1}} d\varphi_t + \left(L_{C_t} dC_t^{-1} \right)^* (\eta_t).$$

We rewrite the above expression by:

$$\begin{aligned} d\eta_t &= R_{C_t^{-1}}\left(h^*(\varphi_t) + V^*(\varphi_t)\right) \diamond dZ_t + \left(L_{C_t} dC_t^{-1}\right)^* (\eta_t) \\ &= R_{C_t^{-1}} h^*(\varphi_t) \diamond dZ_t + R_{C_t^{-1}} V^*(\varphi_t) \diamond dZ_t + \left(L_{C_t} dC_t^{-1}\right)^* (\eta_t) \\ &= \left(\mathrm{Ad}(C_t)h\right)^* (\eta_t) \diamond dZ_t + \left(\mathrm{Ad}(C_t)V\right)^* (\eta_t) \diamond dZ_t + \left(L_{C_t} dC_t^{-1}\right)^* (\eta_t). \end{aligned} \tag{5.18}$$

Now, applying the connection 1-form ω to the expression (5.18):

$$\begin{aligned} 0 &= \omega_{\eta_t}\left(R_{C_t^{-1}} V^*(\varphi_t)\right) \diamond dZ_t + L_{C_t} dC_t^{-1} \\ &= \omega_{\eta_t}\left(\left(\mathrm{Ad}(C_t)(V)\right)^*(\eta_t)\right) \diamond dZ_t + L_{C_t} dC_t^{-1} \\ &= \mathrm{Ad}(C_t)V \diamond dZ_t + L_{C_t} dC_t^{-1}. \end{aligned} \tag{5.19}$$

Here we used the fact that $\omega(d\eta_t) = 0$ and $\mathrm{Ad}(C_t)(h) \in \mathfrak{n}$.

From expression (5.19), it holds that:

$$dC_t^{-1} = -R_{C_t^{-1}} V \diamond dZ_t.$$

Hence,

$$dC_t = V^*(C_t) \diamond dZ_t.$$

Then, C_t is a solution of equation (5.14). By expression (5.19),

$$(L_{C_t} dC_t^{-1})^*(\eta_t) = -(\mathrm{Ad}(C_t)V)^*(\eta_t) \diamond dZ_t. \tag{5.20}$$

Finally, combining expressions (5.18) and (5.20), it follows that:

$$d\eta_t = (\mathrm{Ad}(C_t)h)^*(\eta_t) \diamond dZ_t.$$

□

5.4 Open problems

This section discusses some open problems related to the decomposition of flows. Note that in many cases we are dealing with low regularity problems which comes from a geometric and probabilistic context, since the decomposition's trajectories usually have low regularities besides a probabilistic struture.

5.4.1 Dynamics of càdlàg and α-Hölder continuous paths

Recently, the literature has shown a great interest in the study of differential equations driven by *rough path* and by α-Hölder continuous path with $\alpha \in (0, 1]$. Of course, the level of difficulty also depends on the sub-interval that includes the parameter α. See e.g. Lyons et al [43], Friz and Hairer [22] among many others. Recently, in Castrequini and Catuogno

[12], it was proposed a Itô-Ventzel formula in the Young integral context ([65], [19], [13], [57]) which motivated our submitted paper [14] (the chapter 4 of this thesis). An interesting problem would be work with α-Hölder trajectories with jump components for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Using those trajectories we are going to be able to generalize Theorem 11 and develop some basic geometric properties in this context of low regularities and jumps such as: horizontal lifts, parallel transport, covariant derivative, development and anti-development, etc. Note that here, the Marcus jumps described in chapter 3, are actually geodesic jumps and we definitely need to explore it. Another problem is to understand the construction of Lévy processes on manifolds using the development technique (see page 55) in the same sense of Applebaum and Estrade [3]. The idea is to use just the geometric properties of principal fibre bundles in order to construct a Lévy process which is not necessarily isotropic on a manifold. This technique is used by Elworthy in [21] in order to construct the trajectories of Brownian motion on manifolds.

5.4.2 Rough paths

Let E be a vector space. An α-rough path, for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ is a pair $\mathbf{X} = (X, \mathbb{X})$, such that $X : [0,T] \to E$ is an α-Hölder continuous and $\mathbb{X} : [0,T]^2 \to E \otimes E$ is 2α-Hölder continuous path, such that the following relation (Chen relation) holds:

$$\mathbb{X}_{st} = \mathbb{X}_{su} + \mathbb{X}_{ut} + X_{su} \otimes X_{ut}. \tag{5.21}$$

Where, $\mathbb{X}_{su} = \mathbb{X}_u - \mathbb{X}_s$, for $0 \leqslant s \leqslant u \leqslant t \leqslant T$. Consider the following equation:

$$dx_t = F(x_t) d\mathbf{X}_t. \tag{5.22}$$

Which has the following integral form:

$$x_t = x_0 + \int_0^t F(x_r) d\mathbf{X}_r.$$

Where $F = (F_1, \dots, F_d)$ is a smooth vector field and the above integral must be undestood in the rough sense, see [22]. This type of equation was widely studied on manifold context, see e.g. Bailleul [6], however, an interesting question still needs to be answered: is it possible to perform a decomposition in the solutions of equation (5.22) in the same sense of Theorem 18? In this context, it is reasonable to expect that if φ_t is a solution of equation (5.22), then it can be decomposed as $\varphi_t = \eta_t \circ \psi_t$, where η_t and ψ_t are diffeomorphisms satisfying:

$$\begin{aligned} d\eta_t &= H(\eta_t)\, d\mathbf{X}_t, \\ d\psi_t &= \left(\eta_{t*} V(\eta^{-1})\right) (\psi_t)\, d\mathbf{X}_t. \end{aligned} \tag{5.23}$$

Of course, in order to prove this result, we need to develop some tools such as: change of variables and some Itô-Ventzel adapted formula for rough paths.

5.4.3 An averaging principle for systems driven by fBM

Basically, an averaging principle refers to a interlacing between two dynamics in such way that one of them is, somehow, slower and it is affected by the other. In other words, the problem consists of studying the possibility, by some topology, of approximating those two dynamics. This problem appears in many works such as Arnold [4], Sanders et al [59], Sebastian Ledesma and Fabiano Silva [35], Gargate and Ruffino [23], Li [39], among others. In the sequel we do a brief description of the problem: given a Riemannian foliated manifold $(M, \mathcal{F})$ and let U be a open subset of M, the idea is considering diffeomorphisms of the form $\phi : U \to L_{x_0} \times V$, where $V \subset \mathbb{R}^d$ is an open connected subset which contains the origin and L_{x_0} is the leaf passing through a point $x_0 \in M$. Consider the following stochastic differential equation:

$$dx_t = X_0(x_t)dt + \sum_{k=1}^{d} X_k(x_t) \circ dB_H^k(t). \tag{5.24}$$

Where X_k, for $k = 1, \ldots, d$ is a smooth vector field on M and $B_H(t) = (B_H^1(t), \ldots, B_H^d(t))$ is a d-dimensional fraction Brownian motion (fBM). Let K be another smooth vector field on M, if we make a small perturbation of order $\epsilon > 0$ in the direction of K, then we can rewrite the system (5.24) as:

$$dy_t^\epsilon = X_0(y_t^\epsilon)dt + \sum_{k=1}^{d} X_k(y_t^\epsilon) \circ dB_H^k(t) + \epsilon K(y_t^\epsilon)dt. \tag{5.25}$$

If x_t and y_t^ϵ are solutions of systems (5.24) and (5.25) respectively, the idea is to explore the convergence of the following expression:

$$\left[\mathbb{E}\left(\sup_{s \leqslant t \wedge \tau^\epsilon} |f(y_s^\epsilon) - f(x_s)|^p\right)\right]^{\frac{1}{p}}.$$

Where τ^ϵ is a stopping time of the process y_t^ϵ and $f : M \to \mathbb{R}$ is a Lipschitz and continuous function. We believe that the projection of y_t^ϵ into the subset V converges for the solution of a deterministic equation when $\epsilon \to 0$, we are looking forward to find good estimates for the rate of convergence (part of the ergodic estimates will be based on Hairer [25]).

5.4.4 ϵ - Optimal stochastic control for non-Markovian systems (in Lie groups)

Control theory plays a major role in most applications of differential equation in any physical system. This theory is crucial when one can control – either constant in time or time-dependent, one or more parameters of a system, say, with conditions like: temperature, pressure, concentration of substances, investments, humidity, position and velocity of autonomous vehicles, satellites, electromagnetic parameters, action with vaccination in a population etc, just to mention few of them. Around the last few decades of the 20th century, emboldened by the well development of deterministic control theory and the constant improving of stochastic analysis and stochastic dynamics, the theory of stochastic control started to develop rapidly thanks also to a countless number of relevant application. As for the deterministic control theory, among hundreds of excellent introductory literature, we mention e.g. Colonius and Kliemann [17], Bullo and Lewis [9], Bacciotti [5], Tan [63], Ren and Tan [55], Zhang and Zhuo [66] and references therein; yet, for stochastic analysis, dynamics and control, among a list of excellent introductory texts, see e.g. Arnold [4], Oksendal [51], Protter [53], Leão et al [34], Zhou [67], Saporito [60], Qiu [54], Nutz [50] and references therein. Control theory in stochastic systems is fascinating in the sense that although the outcome is random and unpredictable, nevertheless in many cases, its law as a random variable can be controlled. It means that many useful properties and tools can be applied in order to optimise the chance that the outcomes are favourable.

Besides the pure geometrical motivation on extending the control settings to Lie groups, we point out some other applied importance: 1) this theory encapsulates the multiplicative approach, in the sense that the products here can be considered as the usual product of square matrices; 2) Lie groups are the appropriate frame to work with linear systems: in many of these cases G is the group (or a closed subgroup) of positive determinant $n \times n$-matrices $Gl^+(n, \mathbb{R})$; 3) this approach also includes the framework for any homogeneous space via quotient by closed subgroups, e.g. n-dimensional spheres, torus, Grasmannian, projective spaces, hyperboloid model and many others. In all these cases, the properties of the Lie group theory are important tools in the analysis and interpretation of the dynamics: decompositions (Iwasawa, polar, eingenvectors etc), invariance by translations, adjoints, geometrical structures of fibre bundles, connections, local diffeomorphism with to the corresponding Lie algebra, and many more. A vast and wide-ranging literature are available from applied to more theoretical approach. Just to mention few of them, see e.g. from the classical [16], the well known [64], the more introductory [7] or the more recent [58], including all references therein.

Let G be a connected Lie group with its corresponding Lie algebra $\mathfrak{g}$, identified with the tangent space T_eG, where e is the identity of G. The dynamics of the controlled trajectories $X^u(t) \in G$, with initial condition $X^u(0) = e$ in this context is described by the right invariant vector fields:

$$dX^u(t) = d(R_{X^u(t)})_e \; \alpha(t, X^u_t, u(t))\, dt + d(R_{X^u(t)})_e \; \sigma(t, X^u_t, u(t))\, dB_H(t), \tag{5.26}$$

where α and β are $\mathfrak{g}$-valued functions, $d(R_{X^u(t)})_e : T_eG \to T_{X^u(t)}G$ is the derivative at the identity of the right translation $R_{X^u}(t) : g \mapsto gX^u(t)$. The functions α and β depend on time t, the past trajectory $s \mapsto X^u(s)$ with $s \in [0, t]$ and the control function (bounded measurable) u. If the Lie group G is a subgroup of matrices, equation (5.26) can be written with a much more familiar notation

$$dX^u(t) = \alpha(t, X^u_t, u(t)) \cdot X^u(t)\, dt + \sigma(t, X^u_t, u(t)) \cdot X^u(t)\; dB_H(t),$$

where $\cdot$ stands for the usual product of square matrices.

Let $\xi : C_T \to \mathbb{R}$ be a Borel functional, where C_T is the set of continuous trajectories in G. Hence, for a certain final time $T > 0$ one is looking for the optimal performance given by

$$\sup_{u \in U^T} \mathbb{E}\left[\xi(X^u)\right]. \tag{5.27}$$

The idea here is to create an algorithm that will return good estimates for expression (5.27). We expect that for all $\epsilon > 0$ it is possible, numerically, estimate a stochastic control u^*, such that:

$$\mathbb{E}[\xi(X^{u^*})] > \sup_{u \in U^T} \mathbb{E}[\xi(X^u)] - \epsilon. \tag{5.28}$$

This idea of studying a stochastic optimal control on Lie groups was motivated by our recent paper [42].

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