Universidade Estadual de Campinas
Instituto de Computação

Lucas Ismaily Bezerra Freitas

Path partition problems in digraphs

Problemas de partição de caminhos em digrafos

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Resumo

Seja D um digrafo. Um subconjunto S de $V(D)$ é um *conjunto estável* se todo par de vértices em S é não adjacente em D. Uma coleção de caminhos disjuntos $\mathcal{P}$ de D é uma *partição em caminhos* de D, se todo vértice em $V(D)$ pertence a exatamente um caminho de $\mathcal{P}$. Dizemos que um conjunto estável S e uma partição em caminhos $\mathcal{P}$ são *ortogonais* se todo caminho de $\mathcal{P}$ contém exatamente um vértice de S. Um digrafo D satisfaz a *α-propriedade* se para todo conjunto estável máximo S de D existe uma partição em caminhos $\mathcal{P}$ tal que S e $\mathcal{P}$ são ortogonais. Um digrafo D é *α-diperfeito* se todo subdigrafo induzido de D satisfaz a α-propriedade. Em 1982, Berge propôs uma caracterização para digrafos α-diperfeitos em termos da proibição de *circuitos ímpares anti-orientados* induzidos. Em 2018, Sambinelli, Silva e Lee propuseram uma conjectura semelhante. Um digrafo D satisfaz a *Begin-End-propriedade*, ou *BE-propriedade*, se para todo conjunto estável máximo S de D, existe uma partição em caminhos $\mathcal{P}$ tal que (i) S e $\mathcal{P}$ são ortogonais e (ii) para todo caminho $P \in \mathcal{P}$, o início ou o fim de P pertence a S. Um digrafo D é *BE-diperfeito* se todo subdigrafo induzido de D satisfaz a BE-propriedade. Sambinelli, Silva e Lee propuseram uma caracterização para digrafos BE-diperfeitos em termos da proibição de *circuitos ímpares bloqueantes* induzidos. Nesta tese, verificamos ambas as conjecturas para digrafos arco-local in-semicompletos, arco-local out-semicompletos, arco-local semicompletos, 3-anti-circulantes, 3-anti-digon-circulantes e quase-transitivos. Além disso, provamos alguns resultados parciais para digrafos 3-quase-transitivos, 4-transitivos, k-semi-simétricos e digrafos com número de estabilidade igual a dois. Também demonstramos alguns resultados estruturais para digrafos α-diperfeitos e BE-diperfeitos. Além disso, fornecemos uma decomposição para digrafos arbitrários arco-local (out) in-semicompletos e arco-local semicompletos. Mostramos que a estrutura desses digrafos é semelhante à dos digrafos diperfeitos. Ademais, fornecemos alguns resultados estruturais para digrafos 3-anti-digon-circulantes. Demonstramos que a estrutura desses digrafos é semelhante à dos digrafos completos e bipartidos completos.

Abstract

Let D be a digraph. A subset S of $V(D)$ is a *stable set* if every pair of vertices in S is non-adjacent in D. A collection of disjoint paths $\mathcal{P}$ of D is a *path partition* of D, if every vertex in $V(D)$ belongs to exactly one path of $\mathcal{P}$. We say that a stable set S and a path partition $\mathcal{P}$ are *orthogonal* if every path of $\mathcal{P}$ contains exactly one vertex of S. A digraph D satisfies the *α-property* if for every maximum stable set S of D, there exists a path partition $\mathcal{P}$ such that S and $\mathcal{P}$ are orthogonal. A digraph D is *α-diperfect* if every induced subdigraph of D satisfies the α-property. In 1982, Berge proposed a characterization for α-diperfect digraphs in terms of forbidden *anti-directed odd cycles*. In 2018, Sambinelli, Silva and Lee proposed a similar conjecture concerning BE-diperfect digraphs. A digraph D satisfies the *Begin-End-property*, or *BE-property*, if for every maximum stable set S of D, there exists a path partition $\mathcal{P}$ such that (i) S and $\mathcal{P}$ are orthogonal and (ii) for every path $P \in \mathcal{P}$, either the initial vertex or the terminal vertex of P lies in S. A digraph D is *BE-diperfect* if every induced subdigraph of D satisfies the BE-property. Sambinelli, Silva and Lee proposed a characterization for BE-diperfect digraphs in terms of forbidden *blocking odd cycles*. In this text, we verify both conjectures for arc-locally in-semicomplete digraphs, arc-locally out-semicomplete digraphs, arc-locally semicomplete digraphs, 3-anti-circulant digraphs, 3-anti-digon-circulant digraphs and quasi-transitive digraphs. Also, we show some partial results for 3-quasi-transitive digraphs, 4-transitive digraphs, k-semi-symmetric digraphs and digraphs with stability number two. We also prove some structural results for α-diperfect and BE-diperfect digraphs. Furthermore, we provide a decomposition for arbitrary arc-locally (out) in-semicomplete digraphs and for arbitrary arc-locally semicomplete digraphs. We show that the structure of these digraphs is very similar to that of diperfect digraphs. Moreover, we provide some structural results for 3-anti-digon-circulant digraphs. We show that the structure of these digraphs is similar to that of complete and complete bipartite digraphs.

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Chapter 1

Introduction

In this chapter we present the two conjectures that we study in this thesis and the results we have obtained for both problems. In order to make the reading of this introduction more fluid, we postpone the definition of some standard terms of graph theory to Chapter 2.

Some important results in graph theory characterize a certain class of graphs (or digraphs) in terms of forbidden induced subgraphs (subdigraphs). The most famous one is probably the Strong Perfect Graph Conjecture proposed by Berge [6] in 1961. A *clique* in a (di)graph is a subset of pairwise adjacent vertices. A graph G is *perfect* if the size of maximum clique of G equals to the minimum number of colors need to (properly) color the vertices of G, and this also holds for every induced subgraph of G. Berge showed that neither an odd cycle of length at least five nor its complement is perfect. So clearly no perfect graph can contain an odd cycle of length at least five or its complement as an induced subgraph. Berge conjectured that a graph G is perfect if and only if it contains neither an odd cycle of length at least five nor its complement as an induced subgraph. This problem was studied by many researchers throughout the years resulting in a vast literature on the subject. The problem was finally settled in 2006, when Chudnovsky, Robertson, Seymour and Thomas [10] proved Berge's conjecture, which became known as the Strong Perfect Graph Theorem.

Theorem 1.1 (Chudnovsky, Robertson, Seymour and Thomas, 2006)**.** *A graph G is perfect if and only if G contains neither an odd cycle of length at least five nor its complement as an induced subgraph.*

In this text we are concerned with two conjectures on digraphs which are somehow similar to Berge's conjecture on perfect graphs. These conjectures relate *path partitions* and *stable sets*. We need a few definitions in order to present both conjectures.

Let $D = (V(D), A(D))$ be a digraph. A subset S of $V(D)$ is a *stable set* if every pair of vertices in S is non-adjacent in D. The cardinality of a maximum stable set in D is called the *stability number* and is denoted by $\alpha(D)$. A collection of disjoint (directed) paths $\mathcal{P}$ of D is a *path partition* of D if every vertex in $V(D)$ belongs to exactly one path of $\mathcal{P}$. The cardinality of a minimum path partition of D is denoted by $\pi(D)$. We say that a stable set S and a path partition $\mathcal{P}$ are *orthogonal* if $|V(P) \cap S| = 1$ for every $P \in \mathcal{P}$. We say that D is *transitive* if for every triple of pairwise distinct vertices $v_1, v_2, v_3 \in V(D)$, it follows that if $v_1v_2, v_2v_3 \in A(D)$, then $v_1v_3 \in A(D)$; we say that D is *symmetric* if for every arc $uv \in A(D)$, we have $vu \in A(D)$; we say that D is *acyclic* if D contains no directed cycle. Moreover, we say that D is *diperfect* if its underlying graph is perfect.

In 1950, Dilworth [12] proved one of the first known results relating maximum stable sets and minimum path partitions in digraphs which we describe next.

Theorem 1.2 (Dilworth, 1950)**.** *Let D be a transitive acyclic digraph. Then, $\pi(D) = \alpha(D)$.*

Since the vertex set of a path in a transitive acyclic digraph is also a clique (see Figure 1.1), a maximum stable set may intersect at most one vertex of each path of a path partition of D. So the inequality $\pi(D) \geq \alpha(D)$ easily follows for transitive acyclic digraphs. Note that without the transitivity hypothesis the inequality is false, since if D is a path with three vertices, then $\pi(D) = 1$ and $\alpha(D) = 2$. On the other hand, the inequality $\pi(D) \leq \alpha(D)$ is not so obvious for a transitive digraph and it is the most interesting part of Theorem 1.2.

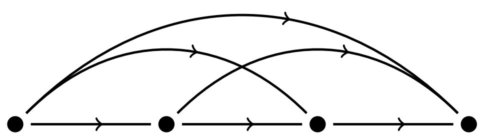


Figure 1.1: Example of subdigraph induced by the vertex set of a path in a transitive acyclic digraph.

In 1960, Gallai and Milgram [21] extended Dilworth's result and proved that the inequality $\pi(D) \leq \alpha(D)$ holds for arbitrary digraphs.

Theorem 1.3 (Gallai and Milgram, 1960)**.** *Let D be a digraph. Then, $\pi(D) \leq \alpha(D)$.*

In order to prove Theorem 1.3, Gallai and Milgram proved a stronger theorem which we present next.

Theorem 1.4 (Gallai and Milgram, 1960)**.** *Let D be a digraph. If $\mathcal{P}$ is a minimum path partition of D, then there exists a stable set S orthogonal to $\mathcal{P}$.*

Note that if $\mathcal{P}$ and S are orthogonal, then $\pi(D) = |\mathcal{P}| = |S| \leq \alpha(D)$ and Theorem 1.3 holds. Furthermore, Theorem 1.4 raises an interesting question:

Question 1. Let D be a digraph. If S is a maximum stable set S of D, then is there a path partition $\mathcal{P}$ orthogonal to S?

In general, the answer to Question 1 is no. For a counterexample, consider the maximum stable set $S = \{v_1, v_4\}$ in Figure 1.2. If a path partition contains v_1v_5 or v_1v_2, then the other vertices do not form a path, and hence, there exists no path partition orthogonal to S. However, the answer to Question 1 is true for some digraphs and this led Berge [7] to propose an interesting class of digraphs, called α-diperfect, which we define next.

Let D be a digraph and let S be a stable set of D. An *S-path partition* of D is a path partition $\mathcal{P}$ such that S and $\mathcal{P}$ are orthogonal (see Figure 1.3). We say that D satisfies

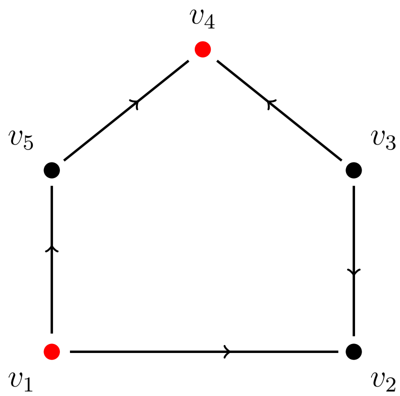


Figure 1.2: Counterexample for Question 1 with maximum stable set $S = \{v_1, v_4\}$.

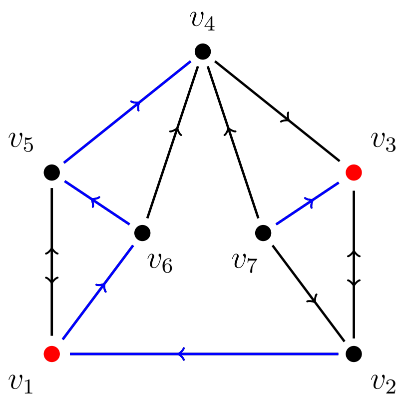


Figure 1.3: Example of a digraph with stable set $S = \{v_1, v_3\}$ and S-path partition $\mathcal{P} = \{v_2v_1v_6v_5v_4, v_7v_3\}$.

the *α-property* if for every maximum stable set S of D there exists an S-path partition of D, and we say that D is *α-diperfect* if every induced subdigraph of D satisfies the α-property. In Figure 1.4 we illustrate the α-property.

In [7], Berge proved that symmetric digraphs and diperfect digraphs are α-diperfect. For ease of reference, we state the following result.

Lemma 1.1 (Berge, 1982)**.** *Let D be a diperfect digraph. Then, D is α-diperfect.*

Berge also presented an important class of digraphs. A digraph C is an *anti-directed odd cycle* if (i) $C = x_1x_2 \ldots x_{2k+1}x_1$ is a non-oriented odd cycle, where $k \geq 2$, and (ii) each of the vertices $x_1, x_2, x_3, x_4, x_6, x_8, \ldots, x_{2k}$ is either a source or a sink (see Figure 1.5).

Berge [7] showed that anti-directed odd cycles do not satisfy the α-property, which led him to conjecture the following characterization of α-diperfect digraphs.

Conjecture 1.1 (Berge, 1982)**.** *A digraph D is α-diperfect if and only if D contains no anti-directed odd cycle as an induced subdigraph.*

Denote by $\mathfrak{B}$ the set of all digraphs which do not contain an induced anti-directed odd cycle. So Berge's conjecture can be stated as: D is α-diperfect if and only if D belongs to $\mathfrak{B}$.

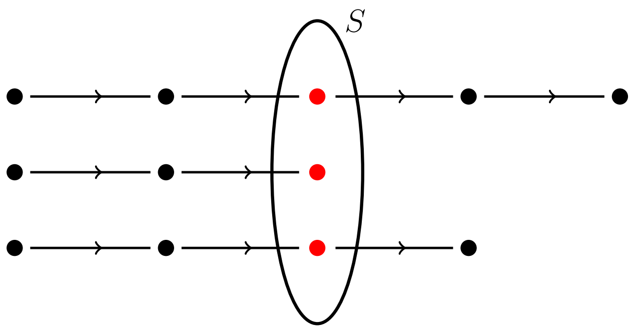


Figure 1.4: Illustration of the α-property.

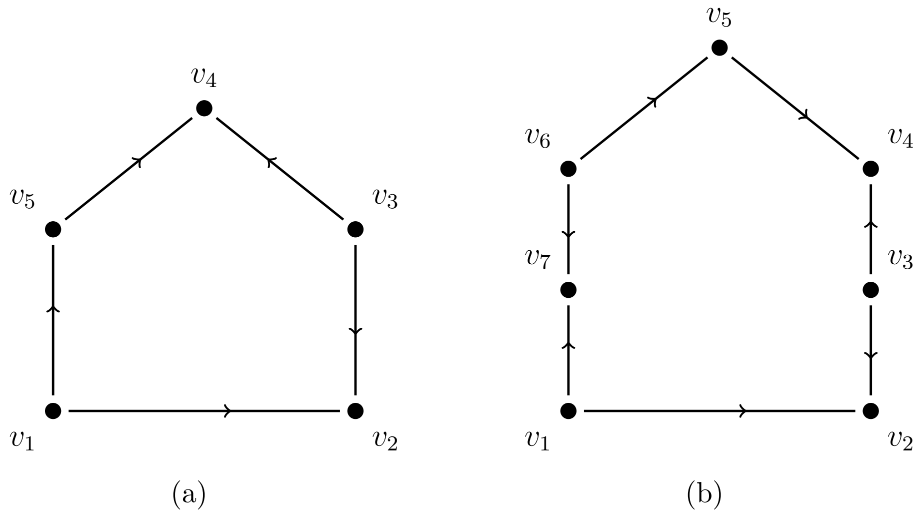


Figure 1.5: Examples of anti-directed odd cycles with length five and seven, respectively.

In the next three decades after Berge's paper, no results regarding this problem were published. In 2018, Sambinelli, Silva and Lee [28, 29] verified Conjecture 1.1 for locally in-semicomplete digraphs and digraphs whose underlying graph is series-parallel. In this thesis, we verified Conjecture 1.1 for several other classes of digraphs, which gave further support to the conjecture. Somewhat surprisingly, at the end of this project (beginning of 2022) we recently learned that Silva, Silva and Lee [31] found a counterexample to Conjecture 1.1. In fact, they showed that for every $k \geq 3$ there is exactly one non-α-diperfect digraph whose underlying graph is the complement of the odd cycle of length $2k + 1$. Let us refer to this family of digraphs as $\mathcal{T}$.

We note that every digraph in the classes for which we proved Conjecture 1.1 contains neither an anti-directed odd cycle nor a subdigraph whose underlying graph is the complement of an odd cycle of length at least five as an induced subdigraph. The proofs of these results are not trivial, but they do not seem to provide any insight on the role of the digraphs in $\mathcal{T}$ in a possible characterization of α-diperfect digraphs. An interesting problem would be to try to verify Conjecture 1.1 for some class of digraphs that may contain an induced subdigraph whose underlying graph is the complement of an odd cycle of length at least seven. This could lead us to a better understanding or a negative answer for the following question: if D is a digraph in $\mathfrak{B}$ that does not contain an induced subdigraph in $\mathcal{T}$, then is D an α-diperfect digraph? Note that a positive answer to this question would imply a characterization of α-diperfect digraphs in terms of forbidden subdigraphs, one of the main objectives of Berge in his seminal paper.

Now, let us present a similar conjecture to the one proposed by Berge. In an attempt to understand the structure of the α-diperfect digraphs, Sambinelli, Silva and Lee [28, 29] introduced the class of Begin-End-diperfect digraphs, or simply BE-diperfect digraphs, which we define next.

Let S be a maximum stable set of a digraph D. A path partition $\mathcal{P}$ is an *S_{BE}-path partition* of D if (i) $\mathcal{P}$ and S are orthogonal and (ii) every path of $\mathcal{P}$ starts or ends at some vertex in S (see Figure 1.6). We say that D satisfies the *BE-property* if for every maximum stable set S of D there exists an S_{BE}-path partition, and we say that D is *BE-diperfect* if every induced subdigraph of D satisfies the BE-property. In Figure 1.7 we illustrate the BE-property. Note that if D is BE-diperfect, then it is also α-diperfect, since every S_{BE}-path partition is also an S-path partition. So if D satisfies the BE-property, then D also satisfies the α-property. Since the BE-property is more restrictive than the α-property with respect to the type of orthogonal path partition we require, unsurprisingly the converse is not true. The smallest digraph which satisfies the α-property but not the BE-property is shown in Figure 1.8b (transitive triangle – see the next paragraph). In general any α-diperfect digraph which contains an induced transitive triangle is not BE-diperfect. It is also not hard to find α-diperfect digraphs which are not BE-diperfect but contains no induced transitive triangle (see the next paragraph).

Similarly to Berge's approach, Sambinelli, Silva and Lee presented an important class of digraphs. A digraph C is a *blocking odd cycle* if (i) $C = x_1x_2 \dots x_{2k+1}x_1$ is a non-oriented odd cycle, where $k \geq 1$, and (ii) x_1 is a source and x_2 is a sink (see Figure 1.8). In this case, we say that (x_1, x_2) is a *blocking pair* of C. Note that every anti-directed odd cycle is also a blocking odd cycle. For example, there could be a digon joining some pair x_i, x_{i+1} for some $i : 3 \leq i < 2k$. In the special case $k = 1$, we say that C is a *transitive triangle* (see Figure 1.8b). As the reader will see in the forthcoming chapters, forbidding transitive triangles impose some strong structure in the study of BE-diperfect digraphs. Sambinelli, Silva and Lee [28, 29] proved that blocking odd cycles do not satisfy the BE-property, which led them to conjecture the following characterization of BE-diperfect digraphs.

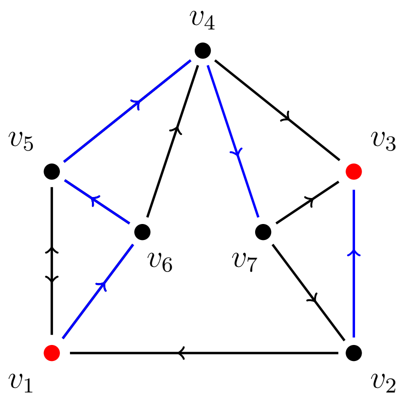


Figure 1.6: Example of an S_{BE}-path partition $\mathcal{P}$ of a digraph where $S = \{v_1, v_3\}$ and $\mathcal{P} = \{v_1v_6v_5v_4v_7, v_3v_2\}$.

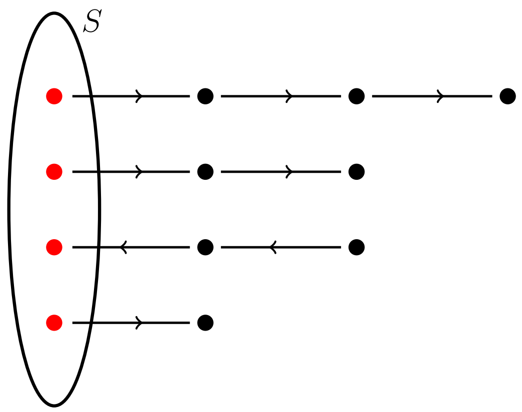


Figure 1.7: Illustration of the BE-property.

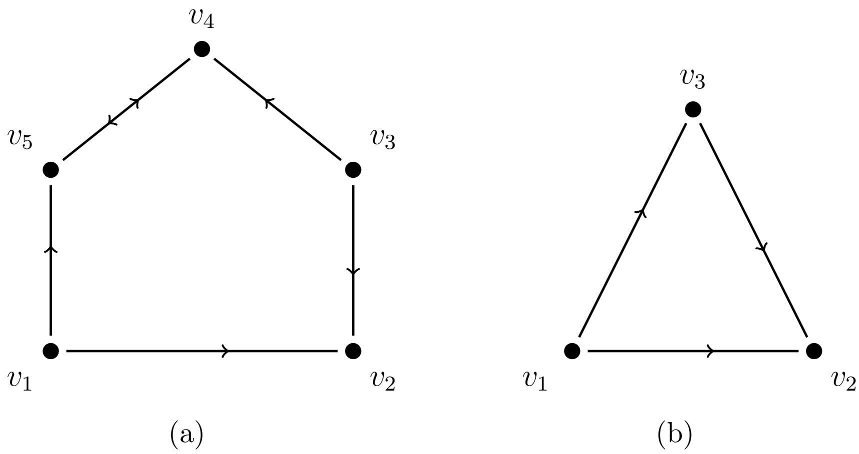


Figure 1.8: Examples of blocking odd cycles with length five and three, respectively. We also say that the digraph in (b) is a transitive triangle.

Conjecture 1.2 (Sambinelli, Silva and Lee, 2018)**.** *A digraph D is BE-diperfect if and only if D contains no blocking odd cycle as an induced subdigraph.*

Denote by $\mathfrak{D}$ the set of all digraphs which do not contain an induced blocking odd cycle. So Conjecture 1.2 can be stated as: D is BE-diperfect if and only if D belongs to $\mathfrak{D}$. Sambinelli, Silva and Lee [28, 29] verified Conjecture 1.2 for symmetric digraphs, locally in-semicomplete digraphs, diperfect digraphs and digraphs whose underlying graph are series-parallel. As we pointed out before, Silva, Silva and Lee [31] found the infinite family $\mathcal{T}$ of counterexamples to Conjecture 1.1. However, it is not hard to prove that every digraph in $\mathfrak{D}$ (specifically in this case, containing no induced transitive triangle) whose underlying graph is the complement of an odd cycle of length at least seven is BE-diperfect (see Chapter 6, Section 6.6). So Conjecture 1.2 remains open. We note that a diperfect digraph belongs to $\mathfrak{D}$ if it contains no induced transitive triangle. For ease of reference, we state the following result.

Lemma 1.2 (Sambinelli, Silva and Lee, 2018)**.** *Let D be a diperfect digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

Sambinelli, Silva and Lee [28, 29] also proved the following useful lemmas.

Lemma 1.3 (Sambinelli, Silva and Lee, 2018)**.** *Let D be a digraph. If $V(D)$ can be partitioned into $k \geq 2$ subsets $V_1, V_2, \ldots, V_k$ such that $D[V_i]$ satisfies the BE-property (resp., α-property) and $\alpha(D) = \sum_{i=1}^{k} \alpha(D[V_i])$, then D satisfies the BE-property (resp., α-property).*

Lemma 1.4 (Sambinelli, Silva and Lee, 2018)**.** *Let D be a digraph. If D has a clique cut, then $V(D)$ can be partitioned into two subsets V_1 and V_2 such that $\alpha(D) = \alpha(D[V_1]) + \alpha(D[V_2])$.*

Thus it follows from Lemmas 1.3 and 1.4 that if a digraph D is a minimal counterexample for Conjecture 1.1 or Conjecture 1.2, then D is connected and D does not contain a clique which is a vertex cut (of the underlying graph). Moreover, it follows from Lemmas 1.1 and 1.2 that D is also a non-diperfect digraph.

The rest of this text is organized as follows. In Chapter 2, we introduce the definitions and notation used. In Chapter 3, we present some structural results for BE-diperfect and α-diperfect digraphs. In Chapter 4, we characterize the structure of arbitrary arc-locally (out) in-semicomplete digraphs and arbitrary arc-locally semicomplete digraphs. We also verify both Conjecture 1.1 and Conjecture 1.2 for these classes of digraphs. In Chapter 5, we verify both Conjecture 1.1 and Conjecture 1.2 for 3-anti-circulant digraphs. We also present some structural results for 3-anti-digon-circulant digraphs. In Section 6, we present some partial results for several classes of digraphs. Finally, in Chapter 7, we present some final considerations.

Chapter 2

Definitions

In this chapter, we present the concepts of graph theory used in this text. The terminology used is standard, and for missing definitions we refer the reader to Bang-Jensen and Gutin's book [4] or to Bondy and Murty's book [8].

A *graph* G is an ordered triple $(V(G), E(G), \psi)$ where $V(G)$ is a finite set of elements called *vertices*, $E(G)$ is a finite set of elements, disjoint from $V(G)$, called *edges* and ψ is an *incidence function* that associates with each edge $e \in E(G)$ an unordered pair of (not necessarily distinct) vertices u and v of $V(G)$. We say that two vertices u and v are *adjacent* if there exists $e \in E(G)$ such that $\psi(e) = \{u, v\}$; otherwise, we say that u and v are *non-adjacent*. We also say that u and v are the *endvertices* of e and they are *incident* with e. Two edges are *adjacent* if they have at least one endvertex in common; otherwise, we say that they are *non-adjacent*. An edge e is a *loop* if $\psi(e) = \{v, v\}$ for some $v \in V(G)$. Two edges e and f are *multiple* if they have the same endvertices, that is, $\psi(e) = \{u, v\}$ and $\psi(f) = \{u, v\}$. We say that G is *simple* if G contains neither a loop nor multiple edges. In this text, we consider only simple graphs. Thus we generally leave the incidence function implicit, since a pair of distinct vertices uniquely define an edge in a simple graph. Henceforth, we denote a graph by $G = (V(G), E(G))$ and we use $e = uv$, or simply uv, instead of $\psi(e) = \{u, v\}$. In general, we write a graph meaning a simple graph, special cases being explained to the reader.

Let G be a graph and let X be a subset of $V(G)$. We define the *neighborhood* of X in G, denoted by $N(X)$, as the set of vertices in $V(G) - X$ that are adjacent to some vertex of X; when $X = \{v\}$, we simply write $N(v)$. Let v be a vertex of G. The *degree* of v in G, denoted by $d(v)$, is the cardinality of $N(v)$.

A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if every edge of $E(G)$ with both endvertices in $V(H)$ is in $E(H)$, then we say that H is *induced* by $X = V(H)$, and we write $H = G[X]$. We also denote $G[V(G)-X]$ by $G-X$; when $X = \{v\}$, we simply write $G-v$. We say that a subgraph H of G is *proper*, denoted by $H \subset G$, if $V(H) \subset V(G)$ or $E(H) \subset E(G)$. The *complement* of G, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$ and $E(\overline{G}) = \{uv : u \text{ and } v \text{ are non-adjacent in } G\}$.

A *path* P in a graph G is a sequence of distinct vertices $P = v_1 v_2 \ldots v_k$ such that for all v_i in P, $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq k-1$. Whenever it is appropriate, we treat P as being the subgraph of G with vertex set $V(P) = \{v_1, v_2, \ldots, v_k\}$ and edge set

$E(P) = \{v_i v_{i+1} : 1 \leq i \leq k-1\}$. We say that P *starts* at v_1 and *ends* at v_k. We also say that v_1, v_k are *endvertices* of P, v_1 is the *initial vertex* and v_k is the *terminal vertex* of P; to emphasize this fact we may write P as $v_1 P v_k$. We denote by $v_i P v_j$ a *subpath* of P where $1 \leq i \leq j \leq k$. We define the *length* of P as $k-1$. If $V(P) = V(G)$, then we say that P is a *hamiltonian path* and G is *traceable*. Let P and Q be two paths in G. If P ends at some vertex v and Q starts at some vertex u such that u and v are adjacent, then we denote by PQ the *concatenation* of P and Q. We use this notation only if PQ is a path.

A *cycle* C in graph G is a sequence of vertices $C = v_1 v_2 \dots v_k v_1$ such that $v_1 v_2 \dots v_k$ is a path, $v_k v_1 \in E(G)$ and $k \geq 3$. Sometimes, when convenient, we also treat C as being the subgraph of G with vertex set $V(C) = \{v_1, v_2, \dots, v_k\}$ and edge set $E(C) = \{v_i v_{i+1} : 1 \leq i \leq k\}$ where subscripts are taken modulo k. We define the *length* of C as k. If k is odd, then we say that C is an *odd cycle*. We denote by C_k the class of isomorphism of a cycle of length k. If $V(C) = V(G)$, then we say that C is a *hamiltonian cycle* and G is *hamiltonian*. If G contains at least one cycle, then the length of a longest cycle in G is called its *circumference*. Moreover, a *chord* of C is an edge in $E(G) - E(C)$ such that both endvertices lie on C.

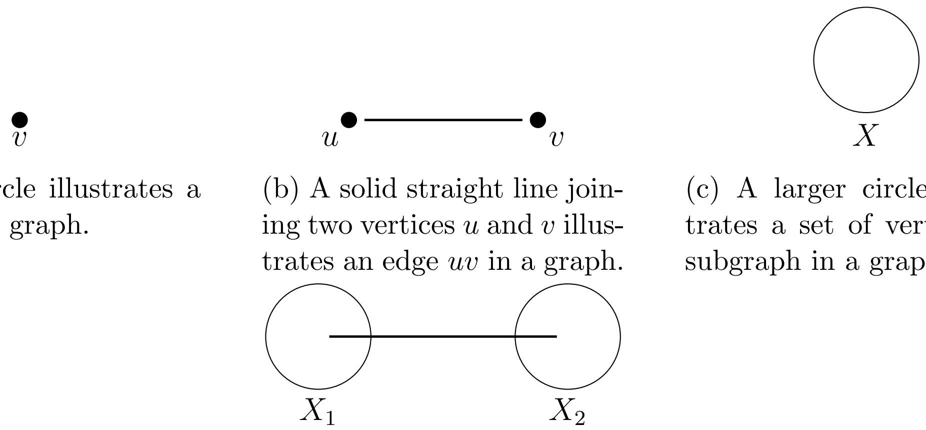
A set of vertices S of a graph G is a *stable set* if every pair of vertices in S is non-adjacent in G. We say that S is *maximum* if $|S| \geq |Z|$ for every stable set Z in G. The cardinality of a maximum stable set in G is called the *stability number* and is denoted by $\alpha(G)$. If $V(G)$ admits a bipartition into two stable sets, say X and Y, then we say that G is *bipartite*, denoted by $G[X, Y]$. Moreover, if every vertex in X is adjacent to every vertex in Y, then G is called a *complete bipartite graph*.

A *clique* in a graph G is a set of pairwise adjacent vertices of G. We say that a clique B is *maximum* if $|B| \geq |W|$ for every clique W in G. The *clique number* of G, denoted by $\omega(G)$, is the size of maximum clique of G. If $V(G)$ is a clique, then we say that G is *complete*. A *(proper) coloring* of G is a partition of $V(G)$ into stable sets $\{S_1, \dots, S_k\}$. A coloring $\mathcal{C}$ is *minimum* if $|\mathcal{C}| \leq |\mathcal{L}|$ for every coloring $\mathcal{L}$ of G. The *chromatic number* of G, denoted by $\chi(G)$, is the cardinality of a minimum coloring of G. We say that G is *perfect* if for every induced subgraph H of G, the equality $\omega(H) = \chi(H)$ holds.

Let G be a graph. We say that G is *connected* if for all pair of distinct vertices u and v, there exists a path in G with endvertices u and v; otherwise, we say that G is *disconnected*. If G is disconnected, then a *component* of G is a maximal induced subgraph of G which is connected. We denote by $c(G)$ the number of component of G. We say that a set of vertices $B \subset V(G)$ is a *vertex cut* if $c(G - B) > c(G)$. Moreover, if B is a clique, then we say that B is a *clique cut* of G.

A *matching* M in a graph G is a set of pairwise non-adjacent edges of G. We denote by $V(M)$ the set of vertices incident with the edges of M. We say that a vertex v is *covered* by M if $v \in V(M)$. We also say that a matching M *covers* $X \subseteq V(G)$ if $X \subseteq V(M)$. An *M-alternating path* P in G is a path whose edges are alternately in M and $E(G) - M$. If no endvertex of P is covered by M, then P is called an *M-augmenting path*. We say that M is *maximum* if $|M| \geq |N|$ for all matching N in G; and say that M is *perfect* if it covers $V(G)$.

In Figure 2.1, we show the representation we use to describe or draw certain structures in a graph.



(a) a full circle illustrates a vertex v in a graph.

(b) A solid straight line joining two vertices u and v illustrates an edge uv in a graph.

(c) A larger circle X illustrates a set of vertices or a subgraph in a graph.

(d) A solid straight line joining two large circles (or full circles) X_1 and X_2 illustrates that some vertex in X_1 and some vertex in X_2 are adjacent in a graph.

Figure 2.1: Notation used to describe the structure of a graph.

2.1 Directed graph

A *directed graph* or simply a *digraph* D is an ordered triple $(V(D), A(D), \psi)$ where $V(D)$ is a finite set of elements called *vertices*, $A(D)$ is a finite set of elements, disjoint from $V(D)$, called *arcs* and ψ is an *incidence function* that associates with each arc $a \in A(D)$ an ordered pair of (not necessarily distinct) vertices u and v of $V(D)$. Given two vertices u and v of $V(D)$, we say that u *dominates* v and that v is *dominated* by u, denoted by $u \to v$, if $\psi(a) = (u, v)$ for some $a \in A(D)$. We also say that u and v are the *endvertices* of a and they are *incident* with a. We say that u and v are *adjacent* if $u \to v$ or $v \to u$; otherwise, we say that u and v are *non-adjacent*. If $u \to v$ and $v \to u$, then we denote this by $u \leftrightarrow v$; we also say that $\{u, v\}$ is a *digon*. An arc a is a *loop* if $\psi(a) = (v, v)$ for some $v \in V(D)$. Two arcs a_1 and a_2 are *multiple* if they have the same endvertices in the same order, that is, $\psi(a_1) = (u, v)$ and $\psi(a_2) = (u, v)$. We say that D is *simple* if D contains neither a loop nor multiple arcs, but digons are allowed. In this text, we consider only simple digraphs. Thus we generally leave the incidence function implicit, since a pair of distinct vertices uniquely define an arc in a simple digraph. So we denote a digraph by $D = (V(D), A(D))$ and we use $a = uv$, or simply uv, instead of $\psi(a) = (u, v)$. Henceforth, we write a digraph meaning a simple digraph.

Let D be a digraph. We say that a vertex u is an *in-neighbor* (resp., *out-neighbor*) of a vertex v if $u \to v$ (resp., $v \to u$). Let X be a subset of $V(D)$. We denote by $N^-(X)$ (resp., $N^+(X)$) the set of vertices in $V(D) - X$ that are in-neighbors (resp., out-neighbors) of some vertex of X. We define the *neighborhood* of X as $N(X) = N^-(X) \cup N^+(X)$; when $X = \{v\}$, we write $N^-(v)$, $N^+(v)$ and $N(v)$. The *in-degree* (resp., *out-degree*) of X in

D, denoted by $d^-(X)$ (resp., $d^+(X)$), is the cardinality of $N^-(X)$ (resp., $N^+(X)$). The *degree* of X in D, denoted by $d(X)$, is the cardinality of $N(X)$; when $X = \{v\}$, we write $d^-(v)$, $d^+(v)$ and $d(v)$. We say that v is a *source* if $N^-(v) = \emptyset$ and a *sink* if $N^+(v) = \emptyset$.

We say that a digraph H is *inverse* of a digraph D if $V(H) = V(D)$ and $A(H) = \{uv : vu \in A(D)\}$. The *underlying graph* of D, denoted by $U(D)$, is the graph defined by $V(U(D)) = V(D)$ and $E(U(D)) = \{uv : u \text{ and } v \text{ are adjacent in } D\}$. Whenever it is appropriate, we may borrow terminology from graphs to digraphs. For instance, we say that a subset of arcs of a digraph D is a *matching* if its corresponding set of edges in $U(D)$ is a matching; we also say that D is *bipartite* if $U(D)$ is a bipartite graph. Moreover, we say that D is *diperfect* if $U(D)$ is perfect.

A *path* P in a digraph D is a sequence of distinct vertices $P = v_1v_2 \ldots v_k$ such that for all v_i in P, $v_iv_{i+1} \in A(D)$ for $1 \leq i \leq k-1$. Whenever it is appropriate, we treat P as being the subdigraph of D with vertex set $V(P) = \{v_1, v_2, \ldots, v_k\}$ and arc set $A(P) = \{v_iv_{i+1} : 1 \leq i \leq k-1\}$. We say that P *starts* at v_1 and *ends* at v_k. We also say that v_1, v_k are *endvertices* of P, v_1 is the *initial vertex* and v_k is the *terminal vertex* of P; to emphasize this fact we may write P as v_1Pv_k. Also, whenever it is convenient, we may omit the initial vertex or the terminal vertex of P as v_1P or Pv_k. We denote by v_iPv_j a *subpath* of P where $1 \leq i \leq j \leq k$. We define the *length* of P as $k-1$. We denote by $\overrightarrow{P_k}$ the class of isomorphism of a path of length $k-1$. If $V(P) = V(D)$, then we say that P is a *hamiltonian path*, and in this case, we say D is *traceable*. Let P and Q be two paths in D. If P ends at some vertex v and Q starts at some vertex u such that $v \to u$, then we denote by PQ the *concatenation* of P and Q. We use this notation only if PQ is a path.

Let D be a digraph. For disjoint subsets (or subdigraphs) X and Y of $V(D)$, we say that X *reaches* Y if there are $u \in X$ and $v \in Y$ such that there exists a path that starts at u and ends at v. The *distance* between $u \in V(D)$ and $v \in V(D)$, denoted by $\text{dist}(u, v)$, is the length of the shortest path which starts at u and ends at v. The distance between X and Y is defined by $\text{dist}(X, Y) = \min\{\text{dist}(u, v) : u \in X \text{ and } v \in Y\}$.

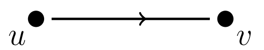
A *cycle* C in digraph D is a sequence of vertices $C = v_1v_2 \ldots v_kv_1$ such that $v_1v_2 \ldots v_k$ is a path, $v_kv_1 \in A(D)$ and $k \geq 2$. When convenient, we also treat C as being the subdigraph of D with vertex set $V(C) = \{v_1, v_2, \ldots, v_k\}$ and arc set $A(C) = \{v_iv_{i+1} : 1 \leq i \leq k\}$ where subscripts are taken modulo k. We define the *length* of C as k. If k is odd, then we say that C is an *odd cycle*. We denote by $\overrightarrow{C_k}$ the class of isomorphism of a cycle of length k. If $V(C) = V(D)$, then we say that C is a *hamiltonian cycle*, and in this case, we say that D is *hamiltonian*. We say that D is an *acyclic digraph* if D contains no cycles. We also say that C is a *non-oriented cycle* if C is not a cycle in D, but $U(C)$ is a cycle in $U(D)$. Moreover, a *chord* of C is an arc in $A(D) - A(C)$ such that the corresponding edge in $U(D)$ is a chord in $U(C)$. Note that by our definition, if $\{v_i, v_{i+1}\}$ is a digon, then $v_{i+1}v_i$ is not a chord in D.

Let D be a digraph. We say that D is *strong* if for each pair of vertices $u, v \in V(D)$, then there exists a path that starts at u and ends at v. A *strong component* of D is a maximal induced subdigraph of D which is strong. Let Q be a strong component of D. We denote by $\mathcal{K}^-(Q)$ (resp., $\mathcal{K}^+(Q)$) the set of strong components that reach (resp., are reached by) Q. We say that Q is an *initial strong component* (resp., *final strong component*) if there exists no vertex in $D - V(Q)$ that dominates (resp., is dominated by) some vertex of Q. For disjoint subsets (or subdigraphs) X and Y of $V(D)$, we say that X and Y are *adjacent* if some vertex of X and some vertex of Y are adjacent in D; $X \equiv Y$ means that every vertex in X is adjacent to every vertex in Y; $X \rightarrow Y$ means that every vertex in X dominates every vertex in Y; $X \Rightarrow Y$ means that there exists no arc from Y to X; and $X \mapsto Y$ means that both $X \rightarrow Y$ and $X \Rightarrow Y$ hold. When $X = \{x\}$ or $Y = \{y\}$, we simply write the element, as in $x \mapsto Y$ and $X \Rightarrow y$.

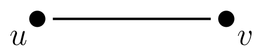
A digraph D is a *tournament* if for every pair of vertices $u, v \in V(D)$, it follows that either $u \mapsto v$ or $v \mapsto u$. We say that D is *semicomplete* if every pair of vertices of D are adjacent, and we say that D is complete if for every pair $u, v \in V(D)$, it follows that $u \leftrightarrow v$. A digraph D is *symmetric* if for every pair of adjacent vertices $u, v \in V(D)$, we have $u \leftrightarrow v$.

Let D be a digraph. We say that a collection of disjoint paths $\mathcal{P}$ of D is a *path partition* of D, if every vertex in $V(D)$ belongs to exactly one path of $\mathcal{P}$. We say that $\mathcal{P}$ is *minimum* if $|\mathcal{P}'| \geq |\mathcal{P}|$ for all path partition $\mathcal{P}'$ of D. The cardinality of a minimum path partition of D is denoted by $\pi(D)$. Let S be a stable set and $\mathcal{P}$ be a path partition of D, we say that S and $\mathcal{P}$ are *orthogonal* if $|V(P) \cap S| = 1$ for every $P \in \mathcal{P}$.

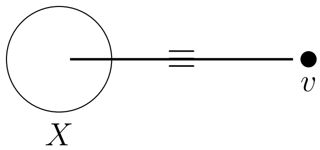
In Figure 2.2, we show the representation we use to describe or draw certain structures in a digraph. Similarly to graph, full circles and large circles illustrate vertices and a set of vertices (or a subdigraph), respectively.



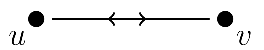
(a) A solid straight line with one arrow joining two vertices u and v illustrates an arc uv in a digraph.



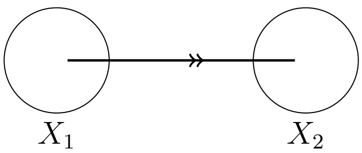
(b) A solid straight line joining two vertices u and v illustrates that u and v are adjacent in a digraph.



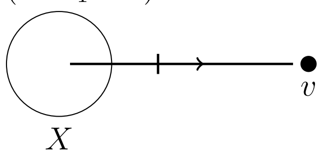
(c) A line with three solid straight lines joining large circles (or full circles) X_1 and X_2 illustrates that $X_1 \equiv X_2$ (or $X_1 \equiv v$).



(d) A solid straight line with two arrows in opposite directions joining two vertices u and v illustrates a digon $u \leftrightarrow v$.



(e) A line with two arrows in the same direction joining large circles (or full circles) X_1 and X_2 illustrates that $X_1 \Rightarrow X_2$.



(f) A line with one slash and one arrow in the same direction between large circles (or full circles) X_1 and X_2 illustrates that $X_1 \mapsto X_2$ (or $X_1 \mapsto v$).

Figure 2.2: Notation used to describe the structure of a digraph.

Chapter 3

Structural results

In this chapter, we compile some results presented in Freitas and Lee [15, 16]. This chapter contains a set of auxiliary results that are used in the forthcoming chapters. We acknowledge that this makes the reading of this chapter rather dry because the results are out of the context where they will be applied. In spite of these, we choose to group them in a single chapter for the following reasons: some results are used in more than one chapter and, it makes easier for a potential reader to find several general tools that can be applied to α-diperfect digraphs and BE-diperfect digraphs in other contexts. Besides, we note that the results presented here are extremely technical and it is relatively difficult to predict which ones will be most useful in the future. We have tried to group them thematically in each section to facilitate future reference.

This chapter is organized as follows. In Section 3.1, we present some structural results for arbitrary α-diperfect digraphs and arbitrary BE-diperfect digraphs. In Section 3.2, we show some auxiliary results for arbitrary digraphs. In Section 3.3, we prove structural results for BE-diperfect digraphs and α-diperfect digraphs related to matching theory. Finally, in Section 3.4, we show structural results for BE-diperfect and α-diperfect digraphs when they contain induced bipartite subdigraphs with some specific properties.

3.1 Structural results for arbitrary BE-diperfect and α-diperfect digraphs

In this section, we show some structural results for arbitrary α-diperfect digraphs and arbitrary BE-diperfect digraphs.

Let D be a digraph and let S be a maximum stable set of D. Recall that since every S_{BE}-path partition is also an S-path partition, it follows that if D satisfies the BE-property, then D also satisfies the α-property. Moreover, the *principle of directional duality* states that every structural result in a digraph has a companion structural result in its inverse digraph. Note that a digraph D is BE-diperfect (resp., α-diperfect) if and only if its inverse digraph is BE-diperfect (resp., α-diperfect). So we can use the principle of directional duality whenever it is convenient. This principle is very useful to fix an orientation for a given path or arc in a proof. In particular, some results of this section follows from principle of directional duality, and although it seems unnecessary, we choose to state them to facilitate the reading of the proofs of the following chapters.

Let us start with some nice structural lemmas.

Lemma 3.1. *Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set in D. Let $P = v_1v_2 \dots v_k$ be a path of D such that $V(P) \cap S = \emptyset$. If there exists a vertex u in $D - V(P)$ such that $u \notin S$, $N^+(u) \neq \emptyset$ and $N^+(u) \subseteq V(P)$, then D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Let v_i be a vertex such that $u \to v_i$ and i is minimum within $\{1, 2, \dots, k\}$. Let $P' = v_iPv_k$. Note that $N^+(u) \subseteq V(P')$. Let $D' = D - V(P')$. Note that u is a sink in D'. Since $V(P') \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let R be a path of $\mathcal{P}'$ such that $u \in V(R)$. Since u is a sink in D', it follows that R ends at u. Since $u \to v_i$, the collection $(\mathcal{P}' - R) \cup RP'$ is an S_{BE}-path partition of D. □

By the principle of directional duality, we have the following result.

Lemma 3.2. *Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set in D. Let $P = v_1v_2 \dots v_k$ be a path of D such that $V(P) \cap S = \emptyset$. If there exists a vertex u in $D - V(P)$ such that $u \notin S$, $N^-(u) \neq \emptyset$ and $N^-(u) \subseteq V(P)$, then D admits an S_{BE}-path partition (resp., S-path partition).* □

The next lemma is similar to Lemma 3.1, but here we have an arc u_1u_2 (instead of a vertex u) and a path P satisfying some technical conditions.

Lemma 3.3. *Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set in D. Let $P = v_1v_2 \dots v_k$ be a path of D such that $V(P) \cap S = \emptyset$. If there exists an arc u_1u_2 in $A(D)$ such that $u_1 \notin S$, $\{u_1, u_2\} \cap V(P) = \emptyset$, $v_k \to u_2$ and $N^+(u_1) \subseteq V(P) \cup u_2$, then D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Let v_i be a vertex such that $u_1 \to v_i$ and i is minimum within $\{1, 2, \dots, k\}$. Let $P' = v_iPv_k$. Note that $N^+(u_1) \subseteq V(P') \cup u_2$. Let $D' = D - V(P')$. Since $V(P') \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let R be a path of $\mathcal{P}'$ such that $u_1 \in V(R)$. If R ends at u_1, then since $u_1 \to v_i$, it follows that the collection $(\mathcal{P}' - R) \cup RP'$ is an S_{BE}-path partition of D. So we may assume that R does not end at u_1. Since $N^+(u_1) \subseteq V(P') \cup u_2$, it follows that u_1u_2 is an arc in R. Let w_1 and w_p be the endvertices of R. Let $R_1 = w_1Ru_1$ and let $R_2 = u_2Rw_p$. Since $u_1 \to v_i$ and $v_k \to u_2$, the collection $(\mathcal{P}' - R) \cup R_1P'R_2$ is an S_{BE}-path partition of D. □

By the principle of directional duality, we have the following result.

Lemma 3.4. *Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set in D. Let $P = v_1v_2 \dots v_k$ be a path of D such that $V(P) \cap S = \emptyset$. If there exists an arc u_1u_2 in $A(D)$ such that $u_2 \notin S$, $\{u_1, u_2\} \cap V(P) = \emptyset$, $u_1 \to v_1$ and $N^-(u_2) \subseteq V(P) \cup u_1$, then D admits an S_{BE}-path partition (resp., S-path partition).* □

Next, we show that if D contains a special partition of its vertex set, then it admits an S_{BE}-path partition (resp., S-path partition).

Lemma 3.5. *Let D be a digraph such that every proper induced subdigraph of D satisfies the BE-property (resp.,α-property). Let S be a maximum stable set of D. If $V(D)$ admits a partition (V_1, V_2, V_3) such that $V_1 \mapsto V_2 \mapsto V_3$, $D[V_2]$ is hamiltonian, $|V_2| \geq 2$ and $|V_2 \cap S| \leq 1$, then D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Let $k = |V_2|$. Let $C = v_1v_2 \ldots v_k$ be a hamiltonian cycle in $D[V_2]$. Let B be a subset of $V_2 - S$ with cardinality $k-1$ (note that B exists because $|V_2| \geq 2$ and $|V_2 \cap S| \leq 1$). Without loss of generality, we may assume that v_k is the vertex in $V_2 - B$. Let $D' = D - B$. Since $B \cap S = \emptyset$, S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let P be a path of $\mathcal{P}'$ such that $v_k \in V(P)$. First, suppose that P does not start at v_k. Let w be the vertex in P that immediately precedes v_k. Let $P_1 = Pw$ and let $P_2 = v_kP$. Since $V_1 \mapsto V_2 \mapsto V_3$ and $V(D') \cap V_2 = v_k$, it follows that w is in V_1. Let $R = v_1v_2 \ldots v_{k-1}$. Since $V_1 \mapsto V_2$, it follows that $w \to v_1$. Since $v_{k-1} \to v_k$, we conclude that the collection $(\mathcal{P}' - P) \cup P_1RP_2$ is an S_{BE}-path partition of D. So we may assume that P starts at v_k. Let w be the vertex in P that immediately follows v_k. Let $P_1 = v_k$ and let $P_2 = wP$. Let $R = v_1v_2 \ldots v_{k-1}$. Since $V_2 \mapsto V_3$, it follows that $v_{k-1} \to w$. Since $v_k \to v_1$, we conclude that the collection $(\mathcal{P}' - P) \cup P_1RP_2$ is an S_{BE}-path partition of D. □

Since the above lemmas consider BE-diperfect digraphs in the hypothesis, they hold for α-diperfect digraphs as well. The following lemmas are specific for α-diperfect digraphs.

Lemma 3.6. *Let D be a digraph such that every proper induced subdigraph of D satisfies the α-property. Let S be a maximum stable set of D. Let v_1v_2 be an arc of $A(D)$. Then,*

(i) if $v_1 \notin S$ and $N^-(v_2) = \{v_1\}$, then D admits an S-path partition,

(ii) if $v_2 \notin S$ and $N^+(v_1) = \{v_2\}$, then D admits an S-path partition.

Proof. By the principle of directional duality, it suffices to prove (i). Let $D' = D - v_1$. Since $v_1 \notin S$, S is a maximum stable set in D'. By hypothesis, D' is α-diperfect. Let $\mathcal{P}'$ be an S-path partition of D'. Let P be a path of $\mathcal{P}'$ such that $v_2 \in V(P)$. Since $N^-(v_2) = \{v_1\}$, it follows that P starts at v_2. Since $v_1 \to v_2$, the collection $(\mathcal{P}' - P) \cup v_1P$ is an S-path partition of D. □

Lemma 3.7. *Let D be a digraph such that every proper induced subdigraph of D satisfies the α-property. Let S be a maximum stable set in D. Let $P = v_1v_2 \ldots v_k$, $k > 1$, be a path of D such that $(V(P) - v_1) \cap S = \emptyset$. If there exists a vertex u in $D - V(P)$ such that $v_k \to u$ and $N^-(u) \subseteq V(P)$, then D admits an S-path partition.*

Proof. Let $P' = v_2Pv_k$ be a subpath of P. Let $D' = D - V(P')$. Since $V(P') \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is α-diperfect. Let $\mathcal{P}'$ be an S-path partition of D'. Let R be a path of $\mathcal{P}'$ such that $u \in V(R)$. Since $N^-(u) \subseteq V(P)$, it follows that R starts at u or v_1u is an arc of R. If P starts at u, then since $v_k \to u$, it follows that the collection $(\mathcal{P}' - R) \cup P'R$ is an S-path partition of D. So suppose that v_1u

is an arc of P. Let w_1 and w_p be the endvertices of R. Let $R_1 = w_1Rv_1$ and $R_2 = uRw_p$ be the subpaths of R. Thus the collection $(\mathcal{P}' - R) \cup R_1P'R_2$ is an S-path partition of D. □

By the principle of directional duality, we have the following result.

Lemma 3.8. *Let D be a digraph such that every proper induced subdigraph of D satisfies the α-property. Let S be a maximum stable set in D. Let $P = v_1v_2 \dots v_k$ be a path of D such that $(V(P) - v_k) \cap S = \emptyset$. If there exists a vertex u in $D - V(P)$ such that $u \to v_1$ and $N^+(u) \subseteq V(P)$, then D admits an S-path partition.*

3.2 Structural results for arbitrary digraphs

In this section, we prove some auxiliary results for arbitrary digraphs. These results are essential in the next sections. In order to do this, we need the celebrated Hall's Theorem [24] and Berge's Theorem [5] about matching in graphs.

Theorem 3.1 (Hall, 1935)**.** *A bipartite graph $G := G[X, Y]$ has a matching covering X if and only if $|N(W)| \geq |W|$ for all $W \subseteq X$.*

Theorem 3.2 (Berge, 1957)**.** *A matching M in a graph G is maximum if and only if G has no M-augmenting path.*

Next, we prove a useful tool. Note that the following lemmas are about matchings in a graph, but by our definition, a subset of arcs M of a digraph D is a matching if its corresponding set of edges in $U(D)$ is a matching. So we can apply Lemmas 3.9, 3.10 and 3.11 in D considering $U(D)$.

Lemma 3.9. *Let $G := G[X, Y]$ be a connected bipartite graph such that $|X| \geq 1$ and $|Y| \geq 1$. If G has no matching covering X, then there exists a non-empty subset $X' \subseteq X$ such that $G[X' \cup N(X')]$ has a matching covering $N(X')$.*

Proof. Assume that there exists no matching covering X in G. By Theorem 3.1, there exists a subset W of X such that $|N(W)| < |W|$; choose such W as small as possible. Since G is connected, it follows that $N(W) \neq \emptyset$. By the choice of W, for every $X' \subset W$ (and hence, for $X' \subset X$), it follows that $|N(X')| \geq |X'|$. Let X' be a subset of W with the same size as $|N(W)|$. Since for every $X^* \subseteq X'$, it follows that $|N(X^*)| \geq |X^*|$, we conclude by Theorem 3.1 that the graph $G[X' \cup N(X')]$ has a matching covering X' (and hence, $N(W)$). □

In the proof of the next lemma we use the symbol $\oplus$ to denote the symmetric difference of two sets. So $X \oplus Y = (X - Y) \cup (Y - X)$.

Lemma 3.10. *Let $G := G[X, Y]$ be a bipartite graph which has a matching covering X. For every $Y' \subset Y$, there exists a matching M covering X such that the restriction of M to $G[X' \cup Y']$, where $X' = N(Y')$, is a maximum matching of $G[X' \cup Y']$.*

Proof. Let $Y' \subset Y$. Let $H := H[X', Y']$ be a bipartite subgraph corresponding to $G[X' \cup Y']$, where $X' = N(Y')$. Let M be a matching covering X such that $|M \cap E(H)|$ as maximum as possible. Let $M' = M \cap E(H)$. Towards a contradiction, suppose that M' is not maximum in H. By Theorem 3.2, there exists an M'-augmenting path uPv in H. Since P is odd, we may assume that $u \in Y'$ and $v \in X'$. Since M covers X, there exists $w \in Y - Y'$ such that $wv \in M$. Since $X' = N(Y')$, u is not covered by an edge of M. Thus $M^* = (M \oplus E(P)) - wv$ is a maximum matching covering X such that $|M^* \cap E(H)| > |M \cap E(H)|$, a contradiction. □

Now, we prove a simple and useful lemma.

Lemma 3.11. *Let S be a maximum stable set in a graph G. Let X be a stable set disjoint from S and let $Y = N(X) \cap S$. Then, there exists a matching between X and Y covering X.*

Proof. Let H be the bipartite subgraph of G such that $V(H) = X \cup Y$ and $E(H) = \{uv : u \in X, v \in Y \text{ and } u \text{ and } v \text{ are adjacent in } G\}$. Note that for every $W \subseteq X$, $N_H(W) = N(W)$. Towards a contradiction, assume that there exists no matching between X and Y covering X. By Theorem 3.1 applied to H, there exists a subset W of X such that $|N(W)| < |W|$. Since $X \cap S = \emptyset$, it follows that $(S - N(W)) \cup W$ is a stable set larger than S in G, a contradiction. □

3.3 Matching in BE-diperfect and α-diperfect digraphs

In this section and in the next one, we show some structural results for digraphs such that every proper subdigraph is BE-diperfect digraphs (resp., α-diperfect). We state each lemma for both cases, but we only prove the corresponding statement for BE-diperfect digraphs; the proof is nearly identical for the corresponding statement for α-diperfect digraphs.

Initially, we prove that if there exists no matching covering S in a digraph D between a maximum stable set S and $V(D) - S$, then D admits an S_{BE}-path partition.

Lemma 3.12. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set of D. If there exists no matching between S and $N(S)$ covering S, then D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Note that we may assume that $|S| \geq 1$ and $|N(S)| \geq 1$. Let B be the digraph obtained from D by removing all edges with both endvertices in $N(S)$. Note that B is a bipartite digraph with bipartition $(S, N(S))$. Let $H := H[X, Y]$ be a maximal connected subdigraph of B with $X \subseteq S$ and $Y \subseteq N(S)$. Note that $|X| \geq 1$ and $|Y| \geq 1$, because every vertex in X is adjacent to some vertex in $N(S)$ (D is connected and $X \subseteq S$ is stable). By Lemma 3.9 applied to H, there exists a non-empty subset $X' \subset X$ such that $H[X' \cup N(X')]$ has a matching M covering $N(X')$. Let $D' = D - N(X')$. Since $N(X') \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be a S_{BE}-path partition of D'. Since $V(D') \cap N(X') = \emptyset$, every vertex in X' is a path of $\mathcal{P}'$. Let $\mathcal{P}_M$ be the set of paths in D corresponding to the edges in M. Thus the collection $(\mathcal{P}' - (X' \cap V(M))) \cup \mathcal{P}_M$ is an S_{BE}-path partition of D. $\square$

Next we prove a more specific lemma.

Lemma 3.13. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set of D. Let $H := H[X, Y]$ be an induced bipartite subdigraph of D such that $N^-(X) = Y$, $Y \Rightarrow X$, $Y \cap S = \emptyset$ and $N^+(X \cap S) = \emptyset$. If there exists no matching between X and Y covering X, then D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Let $H' := H[X', Y']$ be a maximal induced connected bipartite subdigraph of H such that $X' \subseteq X$, $Y' \subseteq Y$ and there exists no matching between X' and Y' covering X'. Note that $|X'| \geq 1$ and $|Y'| \geq 1$, because there exists no matching between X and Y covering X in H. By Lemma 3.9 applied to H', there exists a non-empty subset $X^* \subseteq X'$ such that $D[X^* \cup N^-(X^*)]$ has a matching M covering $N^-(X^*)$. Let $Y^* = N^-(X^*)$ and let $D' = D - Y^*$. Since $Y^* \subseteq Y$ and $Y \cap S = \emptyset$, it follows that S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D. Since $N^-(X^*) = Y^*$ and $N^+(X^* \cap S) = \emptyset$, it follows that every vertex v in X^* is the initial vertex of a path of $\mathcal{P}'$ (if $v \notin S$) or v is itself a path of $\mathcal{P}'$ (if $v \in S$). Since $Y^* \Rightarrow X^*$, it is easy to see that using the edges in M we can add the vertices of Y^* to paths of $\mathcal{P}'$ that starts at some vertex in $V(M) \cap X^*$, obtaining an S_{BE}-path partition of D. $\square$

The next lemma is important and will be used extensively throughout this text. For instance, it is essential in the proof of both conjectures for 3-anti-circulant digraphs (see Chapter 5).

Lemma 3.14. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). If D has a stable set Z such that $|N(Z)| \leq |Z|$, then D satisfies the BE-property (resp., α-property).*

Proof. Let S be a maximum stable set of D. First, we prove that there exists a perfect matching between Z and $N(Z)$. Let $Y = N(Z)$. Then, $|Z - S| \leq |Y \cap S|$, otherwise $(S - (Y \cap S)) \cup (Z - S)$ would be a stable set larger than S in D. Since $|Z| \geq |Y|$, this implies that $|Z \cap S| \geq |Y - S|$. By Lemma 3.12, we may assume that there exists a matching between S and $N(S)$ covering S, and hence, there exists a matching M_1 between $Z \cap S$ and $Y - S$ covering $Z \cap S$. This implies that $|Z \cap S| = |Y - S|$. Since $|Z| \geq |Y|$ and $|Z - S| \leq |Y \cap S|$, it follows that $|Z - S| = |Y \cap S|$. By Lemma 3.11, there exists a matching M_2 between $Z - S$ and $Y \cap S$ covering $Z - S$. Thus, the matching $M = M_1 \cup M_2$ is a perfect matching between Z and Y. Let $\mathcal{P}_M$ be the set of paths in D corresponding to the arcs of M. Note that $\mathcal{P}_M$ and S are orthogonal. Let $S' = S - V(M)$ and let $D' = D - V(M)$. Let $k = |S \cap V(M)| = |Z|$ and note that $|S'| = |S| - k$. Towards a contradiction, suppose that S' is not a maximum stable set of D'. Let S^* be a maximum stable set of D'. Since $|S^*| > |S| - k$ and Z has no neighbor in D', it follows that $S^* \cup Z$ is a stable set larger than S in D, a contradiction. So S' is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S'_{BE}-path partition of D'. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE}-path partition of D. $\square$

The next theorem shows that a minimal counterexample to both Conjecture 1.1 and Conjecture 1.2 cannot have large stability number.

Theorem 3.3. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). If $\alpha(D) \geq \frac{|V(D)|}{2}$, then D satisfies the BE-property (resp., α-property).*

Proof. Let S a maximum stable set of D. Let $\overline{S} = V(D) - S$. By hypothesis, it follows that $|S| \geq |\overline{S}|$, and hence, the result follows from Lemma 3.14. □

3.4 Induced bipartite subdigraphs in BE-diperfect and α-diperfect digraphs

In this section, we show some structural results for BE-diperfect digraphs and α-diperfect digraphs when they contain bipartite subdigraphs with some specific properties. These results are used to verify Conjecture 1.1 and Conjecture 1.2 for arc-locally (out) in-semicomplete digraphs (see Chapter 4).

The first lemma states that if $V(D)$ contains three disjoint nonempty subsets U, X and Y satisfying certain conditions, then D satisfies the BE-property. In an attempt to make the lemma more clear, we illustrate the structure of these subsets in Figure 3.2. Recall that $X_1 \equiv X_2$ means that every vertex in X_1 is adjacent to every vertex in X_2.

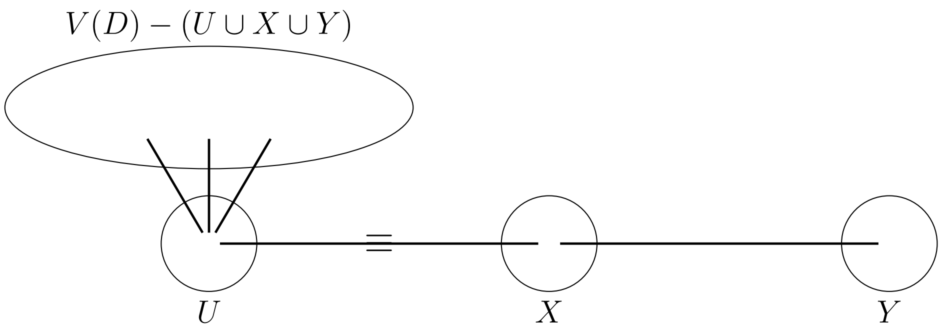


Figure 3.1: Illustration for Lemma 3.15. The subsets X and Y are stable, $N(Y) \subseteq X$, $N(X) \subseteq U \cup Y$ and every vertex in U is adjacent to every vertex in X.

Lemma 3.15. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). If $V(D)$ contains three disjoint nonempty subsets U, X, Y such that X and Y are stable, $N(Y) \subseteq X$, $N(X) \subseteq U \cup Y$ and every vertex in U is adjacent to every vertex in X, then D satisfies the BE-property (resp., α-property).*

Proof. Let S be a maximum stable set of D. Note that $N(Y \cap S) \subseteq X - S$ and $N(X \cap S) \cap Y \subseteq Y - S$. It follows from Lemma 3.11 that there exists a matching M_1 between $Y - S$ and $X \cap S$ covering $Y - S$. By Lemma 3.12, we may assume that there exists a matching between S and $N(S)$ covering S, and hence, there exists a matching M_2 between $Y \cap S$ and $X - S$ covering $Y \cap S$. Let $M = M_1 \cup M_2$, and note that M covers Y. Let

$D' = D - V(M)$ and let $S' = S - V(M)$. Let $k = |S \cap V(M)| = |Y| = |V(M) \cap X|$. Towards a contradiction, suppose that S' is not a maximum stable in D'. Let Z be a maximum stable set of D'. Thus, $|Z| > |S'| = |S| - k$. If $U \cap Z \neq \emptyset$, then since every vertex in U is adjacent a every vertex in X, it follows that $X \cap Z = \emptyset$. Since $N(Y) \cap U = \emptyset$ and Y is stable, the set $Z \cup Y$ is stable and larger than S in D, a contradiction. So we may assume that $U \cap Z = \emptyset$. Since $Y \cap Z = \emptyset$, $N(X) \subseteq U \cup Y$ and X is stable, it follows that $Z \cup (V(M) \cap X)$ is a stable set larger than S in D, a contradiction. So S' is a maximum stable in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S'_{BE}-path partition of D'. Let $\mathcal{P}_M$ be the set of paths in D corresponding to the arcs of M. Note that $\mathcal{P}_M$ and S are orthogonal. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE}-path partition of D. □

Similarly to the previous lemma, we illustrate the structure of specific induced bipartite subdigraph of Lemma 3.16 in Figure 3.2.

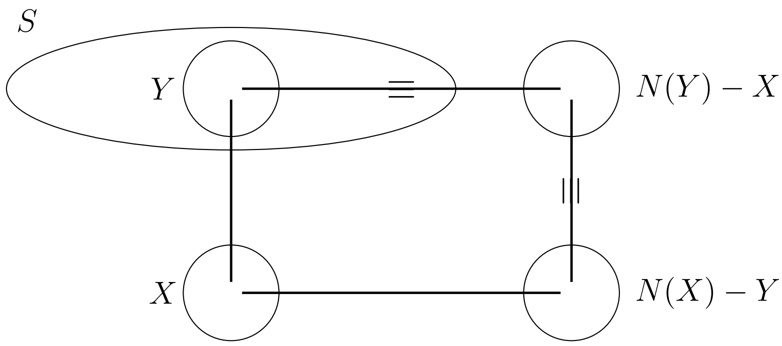


Figure 3.2: Illustration for Lemma 3.16. There exists no arc between X and $N(Y) - X$, and between Y and $N(X) - Y$; or equivalently, $N(X) \cap N(Y) = \emptyset$.

Lemma 3.16. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set of D. If D contains a connected induced bipartite subdigraph $H := H[X, Y]$ such that $Y \subseteq S$, $N(X) \cap S = Y$, $N(X) \cap N(Y) = \emptyset$ and $(N(Y) - X) \equiv N(X)$, then D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Note that $X \cap S = \emptyset$ because H is connected and $Y \subseteq S$. Since S is a maximum stable set and $N(X) \cap S = Y$, it follows from Lemma 3.11 that there exists a matching M between X and Y covering X. Let $D' = D - V(M)$ and let $S' = S - V(M)$. Note that $V(D') \cap X = \emptyset$. Towards a contradiction, suppose that S' is not a maximum stable set in D' and let Z be a maximum stable set in D'. Note that $|Z| > |S| - |V(M) \cap Y|$ and $|V(M) \cap Y| = |X|$. If $Z \cap (N(Y) - X) = \emptyset$, then since $V(D') \cap X = \emptyset$, it follows that $Z \cup (V(M) \cap Y)$ is a stable set in D larger than S, a contradiction. So we may assume that $Z \cap (N(Y) - X) \neq \emptyset$. Since every vertex in $N(Y) - X$ is adjacent to every vertex in $N(X)$ and $N(X) \cap N(Y) = \emptyset$, it follows that $N(X) \cap Z = \emptyset$. Thus $Z \cup X$ is a stable set in D larger than S in D, a contradiction. So S' is a maximum stable set in D'. Let $\mathcal{P}_M$ be the collection of paths corresponding to the arcs of M. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S'_{BE}-path partition of D'. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE}-path partition of D. □

Once again, we illustrate the structure of a digraph satisfying the properties stated in Lemma 3.17 in Figure 3.3. Recall that $X_1 \Rightarrow X_2$ means that there is no arc from X_2 to X_1 and $X_1 \mapsto X_2$ means that every vertex in X_1 dominates every vertex in X_2 and $X_1 \Rightarrow X_2$.

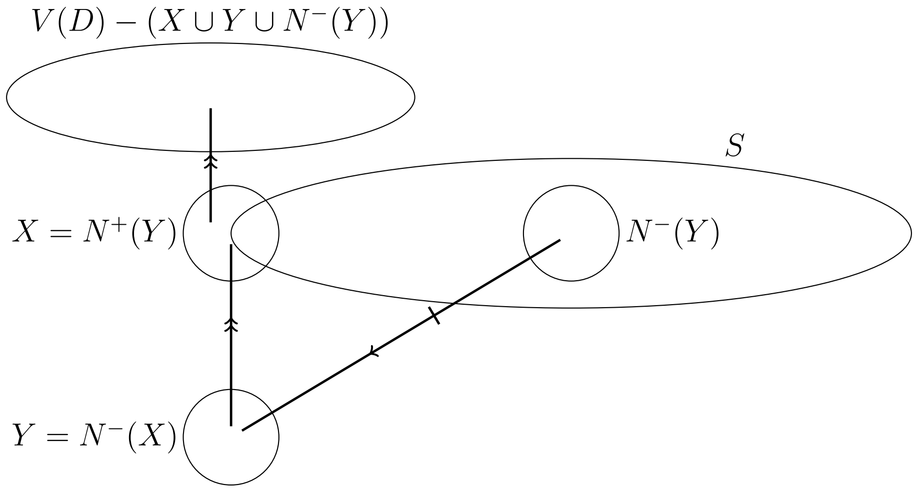


Figure 3.3: Illustration for Lemma 3.17.

Lemma 3.17. *Let D be a connected digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set of D. Let $H := H[X,Y]$ be an induced bipartite subdigraph of D such that $N^-(X) = Y$, $N^+(Y) = X$, $Y \Rightarrow X$, $Y \cap S = \emptyset$, $N^+(X \cap S) = \emptyset$, $N^-(Y) \subseteq S$ and $N^-(Y) \mapsto Y$. Then, D admits an S_{BE}-path partition (resp., S-path partition).*

Proof. Since $N^-(X) = Y$, $Y \Rightarrow X$, $Y \cap S = \emptyset$ and $N^+(X \cap S) = \emptyset$, we may assume by Lemma 3.13 that there exists a matching M between X and Y covering X. Suppose that $X \subseteq S$. Since $N^+(X \cap S) = \emptyset$ and $N^-(X) = Y$, it follows that $N(X) = Y$. Since $N^+(Y) = X$, $N(Y) = N^-(Y) \cup X$. Since $N^-(Y) \mapsto Y$, it follows from Lemma 3.15 applied to $N^-(Y)$, Y and X (in the roOnce again, we illustrate the structure of the next lemma in Figure 3.3. Recall that $X_1 \Rightarrow X_2$ means that there is no arc from X_2 to X_1 and $X_1 \mapsto X_2$ means that every vertex in X_1 dominates every vertex in X_2 and $X_1 \Rightarrow X_2$.les of U, X and Y, resp.) that D satisfies the BE-property. So we may assume that $X \not\subseteq S$.

Let $D' = D - (X - S)$. Since $(X - S) \cap S = \emptyset$, it follows that S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let $\mathcal{P}_M$ be the set of paths corresponding to the arcs of $M \cap E(D')$. First, suppose that $\mathcal{P}_M = \emptyset$. Since $\mathcal{P}_M = \emptyset$, it follows that $X \cap S = \emptyset$. Since $Y \cap S = \emptyset$ and $N^+(Y) = X$, it follows that every vertex in $V(M) \cap Y$ is the terminal vertex in some path of $\mathcal{P}'$. Since M covers X, it is easy to see that using the arcs of M, we can add the vertices of X to paths of $\mathcal{P}'$ that ends at some vertex in $V(M) \cap Y$, obtaining an S_{BE}-path partition of D. So we may assume that $\mathcal{P}_M \neq \emptyset$.

Let $\mathcal{P}_Y$ be the set of paths of $\mathcal{P}'$ that intersect Y. Since $N^-(Y) \subseteq S$, $X \cap V(D') \subseteq S$ and Y is a stable set, it follows that every path in $\mathcal{P}_Y$ has length one. Moreover, every $P \in \mathcal{P}_Y$ starts at some vertex of $N^-(Y)$ or ends at some vertex of $X \cap S$. Let $\mathcal{P}^* = (\mathcal{P}' - \mathcal{P}_Y) \cup \mathcal{P}_M$.

Note that every vertex of $X \cap S$ is the terminal vertex in some path in $\mathcal{P}^*$. Also, note that there might be some vertex of Y which does not belong to any path in $\mathcal{P}^*$. Since $\mathcal{P}'$ is an S_{BE}-path partition of D', every vertex in Y belongs to some path of $\mathcal{P}'$ and since every vertex in $X \cap S$ belongs to some path in $\mathcal{P}^*$, there are at least $|Y - V(M)|$ vertices in $N^-(Y)$ that do not belong to any path in $\mathcal{P}^*$. Since $N^-(Y) \mapsto Y$, we can add to $\mathcal{P}^*$ the path $u \to v$ where $v \in Y - V(M)$ and u is a vertex in $N^-(Y)$ that does not belong to any path of $\mathcal{P}$. Since M covers X, it easy to see that using the arcs of $M - (M \cap E(D'))$, we can add the vertices of $X - S$ to paths in $\mathcal{P}^*$ that ends at some vertex in $V(M) \cap Y$ that do not belong to any path in $\mathcal{P}^*$, obtaining an S_{BE}-path partition of D. This finishes the proof. □

Every lemma presented in this chapter is used in forthcoming chapters to verify Conjecture 1.1 and Conjecture 1.2 for some classes of digraphs. Also, Theorem 3.3 states that if a digraph D is a minimal counterexample to both conjectures, then $\alpha(D) < \frac{|V(D)|}{2}$. This result suggests that dealing with digraphs with small stability number may be the most difficult part of both conjectures. Furthermore, we believe that these results could help in proving both conjectures for other classes of digraphs.

Chapter 4

Arc-locally in-semicomplete digraphs

In this chapter, we compile the results presented in Freitas and Lee [14, 16]. We present a decomposition of the structure of arc-locally (out) in-semicomplete digraphs and arc-locally semicomplete digraphs. We show that the structure of these digraphs is very similar to diperfect digraphs. We also verify both Conjecture 1.1 and Conjecture 1.2 for these classes of digraphs.

In [2], Bang-Jensen introduced arc-locally (out) in-semicomplete and arc-locally semicomplete digraphs as a common generalization of both semicomplete and bipartite semicomplete digraphs. Let D be a digraph. We say that D is *arc-locally in-semicomplete* (resp., *arc-locally out-semicomplete*) if for every pair of adjacent vertices u and v, every in-neighbor (resp., out-neighbor) of u and every in-neighbor (resp., out-neighbor) of v either are adjacent or are the same vertex (see Figure 4.1). We say that D is *arc-locally semicomplete* if D is both arc-locally in-semicomplete and arc-locally out-semicomplete.

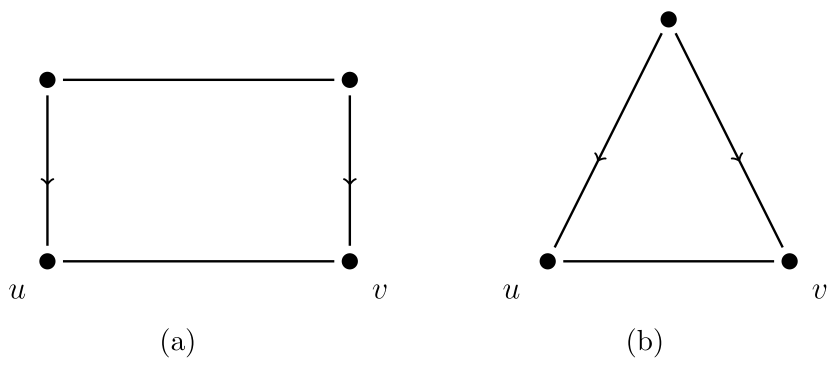


Figure 4.1: Examples of arc-locally in-semicomplete digraphs.

There are many results concerning these classes of digraphs [1, 3, 19, 34, 35]. In particular, Bang-Jensen [3] provided a characterization for strong arc-locally semicomplete digraphs, but Galeana-Sánchez and Goldfeder [19] and Wang and Wang [3] independently pointed out that one family of strong arc-locally semicomplete digraphs was missing. In [34], Wang and Wang provided a decomposition for strong arc-locally in-semicomplete digraphs. In [20], Galeana-Sánchez and Goldfeder extended the Bang-Jensen's results and characterized arbitrary arc-locally semicomplete digraphs. To the best of our knowledge, there is no characterization for arbitrary arc-locally (out) in-semicomplete digraphs. In this text, we extend the results of Wang and Wang [34] by presenting a decomposition for arbitrary arc-locally (out) in-semicomplete digraphs.

This chapter is organized as follows. In Section 4.1, we provide a decomposition for arbitrary arc-locally (out) in-semicomplete digraphs. In Section 4.2, we present a decomposition for arbitrary arc-locally semicomplete digraphs. In Section 4.3, we verify Conjecture 1.2 for arc-locally (out) in-semicomplete digraphs. Finally, in Section 4.4, we verify Conjecture 1.1 for arc-locally (out) in-semicomplete digraphs.

4.1 Decomposition for arbitrary arc-locally (out) in-semicomplete digraphs

In this section, we present some structural results for arbitrary arc-locally (out) in-semicomplete. In particular, we show that if D is a connected arc-locally (out) in-semicomplete digraph, then D is diperfect, or D admits a special partition of its vertices, or D has a clique cut. Since the inverse of an arc-locally in-semicomplete digraph is an arc-locally out-semicomplete digraph, for every statement regarding the former one, there is an equivalent one for the latter. So in this section we restrict ourselves to arc-locally in-semicomplete digraphs.

Let us start with a class of digraphs which is related to arc-locally in-semicomplete digraphs. Let C be a cycle of length $k \geq 2$ and let $X_1, X_2, \ldots, X_k$ be disjoint stable sets. The *extended cycle* $C := C[X_1, X_2, \ldots, X_k]$ is the digraph with vertex set $X_1 \cup X_2 \cup \cdots \cup X_k$ and arc set $\{x_i x_{i+1} : x_i \in X_i, x_{i+1} \in X_{i+1}, i \in \{1, 2, \ldots, k\}\}$, where subscripts are taken modulo k. Thus $X_1 \mapsto X_2 \mapsto \cdots \mapsto X_k \mapsto X_1$. Moreover, we say that the *length* of the extended cycle C is k (see Figure 4.2).

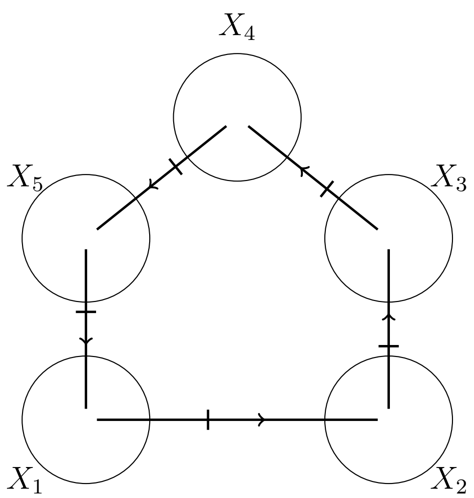


Figure 4.2: Illustration of an extended cycle.

For ease of reference, we state the following result presented in [34]. We have omitted the definition of a T-digraph, because it is a family of digraphs that does not play an important role in this context.

Theorem 4.1 (Wang and Wang, 2009)**.** *Let D be a strong arc-locally in-semicomplete digraph, then D is either a semicomplete digraph, a semicomplete bipartite digraph, an extended cycle or a T-digraph.*

Next, we prove a simple and useful lemma.

Lemma 4.1. *If D is an arc-locally in-semicomplete digraph, then D does not contain any induced non-oriented odd cycle of length at least five.*

Proof. Towards a contradiction, suppose that D contains an induced non-oriented odd cycle of length at least five. Let $P = u_1u_2 \ldots u_k$ be a maximum path in C. Since C is odd and has at least five vertices, then P has at least three vertices. Let w be the vertex of C distinct from u_{k-1} that dominates u_k. Since $u_{k-2} \to u_{k-1}$, $u_{k-1} \to u_k$, $w \to u_k$ and D is arc-locally in-semicomplete, we conclude that w and u_{k-2} must be adjacent, a contradiction. □

The next lemma states that not containing an induced odd cycle of length at least five is a necessary and sufficient condition for an arc-locally in-semicomplete digraph to be diperfect. Note that if a digraph D contains no induced odd cycle of length at least five, then D also contains no induced odd extended cycle of the same length.

Lemma 4.2. *Let D be an arc-locally in-semicomplete digraph. Then, D is diperfect if and only if D contains no induced odd cycle of length at least five.*

Proof. By Lemma 4.1, D does not contain any induced non-oriented odd cycle of length at least five. Thus it follows that every induced odd cycle of length at least five in $U(D)$ is also an induced odd cycle in D.

First, if D is diperfect, then the result follows from Theorem 1.1. To prove sufficiency, suppose that D is not diperfect. Since D contains no induced odd cycle of length at least five, it follows from Theorem 1.1 that $U(D)$ contains an induced complement of an odd cycle $U(C)$ of length at least five, denoted by $\overline{U(C)}$. Suppose that the vertices of C (and of $U(C)$) are labelled as $v_1, v_2, \ldots, v_{2k+1}$ so that the cycle in $\overline{U(C)}$ is $(v_1, v_2, \ldots, v_{2k+1}, v_1)$. Thus the non-adjacent vertices to v_i in $U(C)$ are v_{i-1} and v_{i+1}, where the indexes are taken modulo k. Since the complement of a C_5 is also a C_5, we may assume that C contains at least seven vertices. So consider the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ of $\overline{U(C)}$ (and hence, of $U(C)$). Recall that since D is arc-locally in-semicomplete, if x, y, u and v are distinct vertices such that u and v are adjacent, $x \to u$ and $y \to v$, then x and y must be adjacent in D or the same vertex.

The rest of the proof is divided into two cases, depending on whether $v_2 \to v_4$ or $v_4 \to v_2$.

Case 1. Assume that $v_2 \to v_4$. If $v_1 \to v_6$, then since v_4 and v_6 are adjacent, it follows that v_1 and v_2 are adjacent, a contradiction. So we may assume that $v_6 \to v_1$. If $v_3 \to v_7$, then since $v_2 \to v_4$ and, v_4 and v_7 are adjacent, it follows that v_2 and v_3 are adjacent in D, a contradiction. Thus $v_7 \to v_3$. Finally, since $v_7 \to v_3$, $v_6 \to v_1$ and v_1 and v_3 are adjacent, it follows that v_6 and v_7 are adjacent in D, a contradiction.

Case 2. Assume that $v_4 \to v_2$. If $v_3 \to v_6$, then since v_2 and v_6 are adjacent, it follows that v_3 and v_4 are adjacent, a contradiction. So we may assume that $v_6 \to v_3$. If $v_7 \to v_5$, since $v_6 \to v_3$ and, v_5 and v_3 are adjacent, it follows that v_7 and v_6 are adjacent in D, a contradiction. So $v_5 \to v_7$. Finally, since $v_5 \to v_7$, $v_4 \to v_2$ and v_2 and v_7 are adjacent, it follows that v_5 and v_4 are adjacent in D, a contradiction. This ends the proof. □

By Lemma 4.2, if D does not contain any induced odd cycle of length at least five, then it is easy to show that both the α-property and the BE-property are satisfied (see Chapter 1 or Section 4.5). Thus we prove next some properties of an arc-locally in-semicomplete digraph D when D has a strong component that induces an odd extended cycle of length at least five. To do this, we use the following auxiliary results.

Lemma 4.3 (Wang and Wang, 2011)**.** *Let D be an arc-locally in-semicomplete digraph and let H be a non-trivial strong subdigraph of D. For every $v \in V(D) - V(H)$, if there exists a path from v to H, then v and H are adjacent. In particular, if H is a strong component, then v dominates some vertex of H.*

Lemma 4.4 (Wang and Wang, 2011)**.** *Let D be an arc-locally in-semicomplete digraph and let K_1 and K_2 be two distinct non-trivial strong components of D with at least one arc from K_1 to K_2. Then either $K_1 \mapsto K_2$ or $D[V(K_1) \cup V(K_2)]$ is a bipartite digraph.*

Lemma 4.5 (Wang and Wang, 2011)**.** *Let D be an arc-locally in-semicomplete digraph and let Q be a non-trivial strong component of D. Let v be a vertex of $V(D) - V(Q)$ that dominates some vertex of Q. If $D[V(Q)]$ is non-bipartite, then $v \mapsto Q$.*

Recall that if Q is a strong component of a digraph D, then $\mathcal{K}^-(Q)$ (resp., $\mathcal{K}^+(Q)$) is the set of strong components that reach (resp., are reached by) Q in D.

Lemma 4.6. *Let D be a non-strong arc-locally in-semicomplete digraph. Let Q be a non-initial strong component of D that induces an odd extended cycle of length at least five. Let $W = \cup_{K \in \mathcal{K}^-(Q)} V(K)$. Then, each of the following holds:*

(i) every strong component in $\mathcal{K}^+(Q)$ is trivial,

(ii) $W \mapsto Q$,

(iii) $D[W]$ is a semicomplete digraph,

(iv) there exists a unique initial strong component that reaches Q in D.

Proof. Let $Q := Q[X_1, X_2, ..., X_{2k+1}]$ be a non-initial strong component that induces an odd extended cycle of length at least five of D.

(i) Towards a contradiction, suppose that there exists a non-trivial strong component K in $\mathcal{K}^+(Q)$. By definition of $\mathcal{K}^+(Q)$, there exists a path from some vertex of Q to some vertex of K. By Lemma 4.3, there must be some arc from Q to K. Note that $D[V(Q)]$ is a non-bipartite digraph. So it follows from Lemma 4.4 that $Q \mapsto K$. Let uv be an arc of K. Let $x_1 \in X_1$ and $x_3 \in X_3$ be vertices of Q. Since $x_1 \to u$, $x_3 \to v$ and D is arc-locally in-semicomplete, then x_1 and x_3 are adjacent, a contradiction to the fact that Q induces an extended cycle. Thus every strong component in $\mathcal{K}^+(Q)$ is trivial.

(ii) Let v be a vertex of W. By definition of $\mathcal{K}^-(Q)$ and W, there exists a path from v to Q. By Lemma 4.3, the vertex v dominates some vertex of Q. Since $D[V(Q)]$ is a non-bipartite digraph, it follows from Lemma 4.5 that $v \mapsto Q$.

(iii) Let u and v be two vertices in W. By (ii), $\{u, v\} \mapsto Q$. Let xy be an arc of Q. Since D is arc-locally in-semicomplete, $u \to x$ and $v \to y$, it follows that u and v are adjacent. Thus all vertices in W are adjacent, and hence, $D[W]$ is a semicomplete digraph.

(iv) Towards a contradiction, suppose that D contains two initial strong components that reach Q, say K_1 and K_2. By (iii), $D[W]$ is a semicomplete digraph. Since $V(K_1) \cup V(K_2) \subseteq W$, it follows that K_1 and K_2 are adjacent which is a contradiction. □

For the next lemma we need the following auxiliary result.

Lemma 4.7 (Wang and Wang, 2011)**.** *Let D be a connected non-strong arc-locally in-semicomplete digraph. If there is more than one initial strong component, then all initial strong components are trivial.*

Lemma 4.8. *Let D be an arc-locally in-semicomplete digraph. Let Q be a strong component that induces an odd extended cycle of length at least five of D. If Q is an initial strong component of D, then $V(D)$ admits a partition (V_1, V_2) such that $V_1 \Rightarrow V_2$, $V_1 = V(Q)$ and $D[V_2]$ is a bipartite digraph (V_2 could be empty).*

Proof. Let $Q := Q[X_1, X_2, ..., X_{2k+1}]$. Recall that $X_1 \mapsto X_2 \mapsto \cdots \mapsto X_{2k+1} \mapsto X_1$. If $V(D) = V(Q)$, then the result follows by taking the partition $(V(Q), \emptyset)$. So we may assume that $D - V(Q)$ is nonempty. In particular, D is non-strong. By Lemma 4.7, Q is the only initial strong component of D. Let $V_2 = V(D) - V(Q)$. Note that $V(Q) \Rightarrow V_2$. Now, we show that $D[V_2]$ is a bipartite digraph. By Lemma 4.6(i), every vertex of V_2 induces a trivial strong component, and hence, $D[V_2]$ is an acyclic digraph.

Claim 1. If a vertex u dominates a vertex v_1 of a transitive triangle T, then u is adjacent to a vertex v_2 distinct of v_1 in $V(T)$ such that $D[\{u, v_1, v_2\}]$ is a transitive triangle. In particular, if $u \in V(Q)$, then u dominates both v_1 and v_2.

Let $V(T) = \{v_1, v_2, v_3\}$. Assume that $u \to v_1$. If $v_2 \to v_3$ (resp., $v_3 \to v_2$), then u and v_2 (resp., u and v_3) are adjacent. Since $D[V_2]$ is an acyclic digraph, Q induces an odd extended cycle of length at least five, $V(Q) \Rightarrow V_2$ and $V(D) = V(Q) \cup V_2$, it follows that D contains no $\overrightarrow{C_3}$ as a subdigraph. Thus $D[\{u, v_1, v_2\}]$ (resp., $D[\{u, v_1, v_3\}]$) is a transitive triangle. Moreover, note that if $u \in V(Q)$, then u dominates both v_1 and v_2. This ends the proof of Claim 1.

Claim 2. There is no index $i \in \{1, 2, \ldots, 2k+1\}$, such that there are vertices $x_{i-1} \in X_{i-1}$, $x_i \in X_i$ and $v \in V_2$ for which $D[\{x_{i-1}, x_i, v\}]$ is a transitive triangle.

Since $V(Q) \Rightarrow V_2$, it follows that $x_{i-1} \mapsto \{x_i, v\}$. Let $x_{i-2} \in X_{i-2}$. Since $x_{i-2} \to x_{i-1}$, $x_i \to v$ and $x_{i-1} \to v$, we conclude that x_{i-2} and x_i are adjacent, a contradiction because Q induces an extended cycle of length at least five. This finishes the proof of Claim 2.

By Lemma 4.1, it follows that D (and hence, $D[V_2]$) contains no induced non-oriented odd cycle of length at least five. Since $D[V_2]$ is acyclic, to prove that $D[V_2]$ is a bipartite digraph, it suffices to show that $D[V_2]$ contains no transitive triangle as a subdigraph. Towards a contradiction, suppose that $D[V_2]$ contains a transitive triangle. Let T be a transitive triangle of $D[V_2]$ such that $dist(Q, T)$ is minimum. Let $V(T) = \{v_1, v_2, v_3\}$. Let $P = w_1 w_2 w_l \dots w_{l+1}$ be a minimum path from Q to T. Without loss of generality, assume that $w_{l+1} = v_1$. First, suppose that $l > 1$. By Claim 1, $D[\{w_l, v_1, v_j\}]$ for some $j \in \{2, 3\}$ is a transitive triangle which contradicts the choice of T. Thus it follows that $l = 1$, that is, there exists an arc from Q to T. Let $x_i \in X_i$ be a vertex of Q that dominates v_1 in $V(T)$. By Claim 1, $D[\{x_i, v_1, v_j\}]$ is a transitive triangle with $x_i \to \{v_1, v_j\}$ for some $j \in \{2, 3\}$. Let $x_{i-1} \in X_{i-1}$. By definition of extended cycle, $x_{i-1} \to x_i$. If $v_1 \to v_j$ (resp., $v_j \to v_1$), then since $x_i \to \{v_1, v_j\}$ and $x_{i-1} \to x_i$, it follows that $x_{i-1} \to v_1$ (resp., $x_{i-1} \to v_j$), a contradiction to Claim 2. Thus $D[V_2]$ is a bipartite digraph.

Since Q is the only initial strong component of D and $V(D) = V(Q) \cup V_2$, it follows that $(V(Q), V_2)$ is a partition of $V(D)$ such that $V(Q) \Rightarrow V_2$ and $D[V_2]$ is a bipartite digraph. This ends the proof. □

The next lemma is an analogue to Lemma 4.8 for the case in which D has a strong component Q that induces an odd extended cycle of length at least five but is not an initial strong component.

Lemma 4.9. *Let D be a connected non-strong arc-locally in-semicomplete digraph and let Q be a non-initial strong component of D that induces an odd extended cycle of length at least five. Then, D has a clique cut or $V(D)$ admits a partition $(V_1, V(Q), V_3)$, such that $D[V_1]$ is a semicomplete digraph, $V_1 \mapsto V(Q)$, $V_1 \Rightarrow V_3$, $V(Q) \Rightarrow V_3$ and $D[V_3]$ is a bipartite digraph (V_3 could be empty).*

Proof. Let $V_1 = \cup_{K \in \mathcal{K}^-(Q)} V(K)$ and let $V_3 = \cup_{K \in \mathcal{K}^+(Q)} V(K)$. Since Q is a non-initial strong component, it follows that V_1 is non-empty. By Lemma 4.6(iii), the digraph $D[V_1]$ is a semicomplete digraph. By Lemma 4.6(iv), there exists only one initial strong component K that dominates Q in D. Note that $V(K) \subseteq V_1$. Let $B = \{V_1 \cup V(Q) \cup V_3\}$. The rest of the proof is divided into two cases depending on whether $V(D) = B$ or $V(D) \neq B$.

Case 1. Assume that $V(D) = B$. Let $H = D - V_1$. Note that $V(H) = V(Q) \cup V_3$. By Lemma 4.7, Q is the unique initial strong component of H. By Lemma 4.8 applied to H, it follows that $V(Q) \Rightarrow V_3$ and $D[V_3]$ is a bipartite digraph. In D, it follows from Lemma 4.6(ii) that $V_1 \mapsto V(Q)$. By definition of $\mathcal{K}^-(Q)$ and $\mathcal{K}^+(Q)$, we conclude that $V_1 \Rightarrow V_3$. Thus $(V_1, V(Q), V_3)$ is a partition of $V(D)$ as described in the statement.

Case 2. Assume that $V(D) \neq B$. We show next that V_1 is a clique cut of D. First, we show that there exists no vertex v in $V(D) - B$ adjacent to $V(Q) \cup V_3$. Since $v \notin B$,

$V_1 \mapsto V(Q)$, $V_1 \Rightarrow V_3$ and $V(Q) \Rightarrow V_3$, it follows that v does not dominate and nor is dominated by any vertex in $V(Q)$, neither is dominated by any vertex in V_3. Thus it suffices to show that v does not dominate any vertex of V_3. Towards a contradiction, suppose that there exists $u \in V_3$ such that v dominates u. Choose u such that $dist(Q, u)$ is minimum. Let $P = w_1 w_2 w_l \dots u$ be a minimum path from Q to u. Suppose that $l > 1$. Let $w_1 \in V(Q)$, and hence, $\{w_2, ..., w_l, u\} \subseteq V_3$. Since $v \to u$, $w_l \to u$, $w_{l-1} \to w_l$ and D is arc-locally in-semicomplete, it follows that $v \to w_{l-1}$ which contradicts the choice of u or contradicts the fact that $v \notin B$ if $w_{l-1} \in V(Q)$. Thus we may assume that $w_1 \to u$ and $v \to u$. Since Q is a non-trivial strong component, let z be a vertex of $V(Q)$ that dominates w_1. Since $z \to w_1$, $v \to u$ and $w_1 \to u$, it follows that z and v are adjacent, a contradiction to the fact that $v \notin B$.

Since D is connected, B is a proper subset of D, $D[V_1]$ is a semicomplete digraph and there exists no vertex in $V(D) - B$ adjacent to $V(Q) \cup V_3$, we conclude that V_1 is a clique cut of D. This finishes the proof. □

For the main result of this section, we need the following auxiliary result.

Lemma 4.10 (Wang and Wang, 2009)**.** *Let D be a strong arc-locally in-semicomplete digraph. If D contains an induced cycle of length at least five, then D is an extended cycle.*

Theorem 4.2. *Let D be a connected arc-locally in-semicomplete digraph. Then,*

(i) D is a diperfect digraph, or

(ii) $V(D)$ admits a partition (V_1, V_2, V_3) such that $D[V_1]$ is a semicomplete digraph, $V_1 \mapsto V_2$, $V_1 \Rightarrow V_3$, $D[V_2]$ is an odd extended cycle of length at least five, $V_2 \Rightarrow V_3$ and $D[V_3]$ is a bipartite digraph (V_1 or V_3 could be empty), or

(iii) D has a clique cut.

Proof. If D contains no induced odd cycle of length at least five, then it follows from Lemma 4.2 that D is diperfect. So let C be an induced odd cycle of length at least five of D. Let Q be the strong component that contains C. By Lemma 4.10, Q induces an odd extended cycle of length at least five. First, suppose that Q is an initial strong component of D. Then, it follows from Lemma 4.8 that $V(D)$ admits a partition $(V_1, V(Q), V_3)$ such that V_1 is empty, $V(Q) \Rightarrow V_3$ and $D[V_3]$ is a bipartite digraph. So we may assume that Q is not an initial strong component of D. By Lemma 4.9, we conclude that D has a clique cut or $V(D)$ admits a partition $(V_1, V(Q), V_3)$ such that $D[V_1]$ is a semicomplete digraph, $V_1 \mapsto V(Q)$, $V_1 \Rightarrow V_3$, $V(Q) \Rightarrow V_3$ and $D[V_3]$ is a bipartite digraph. This ends the proof. □

Since the inverse of an arc-locally in-semicomplete digraph is an arc-locally out-semicomplete digraph, we have the following result.

Theorem 4.3. *Let D be a connected arc-locally out-semicomplete digraph. Then,*

(i) D is a diperfect digraph, or

(ii) $V(D)$ can be partitioned into (V_1, V_2, V_3) such that $D[V_1]$ is a semicomplete digraph, $V_2 \mapsto V_1$, $V_3 \Rightarrow V_1$, $D[V_2]$ is an odd extended cycle of length at least five, $V_3 \Rightarrow V_2$ and $D[V_3]$ is a bipartite digraph (V_1 or V_3 could be empty), or

(iii) D has a clique cut.

Next, we provide more structural properties of an arc-locally in-semicomplete digraph D for which $V(D)$ can be partitioned as described in Theorem 4.2(ii).

Lemma 4.11. *Let D be a connected arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). Then, the graph $U(D[V_2 \cup V_3])$ contains no cycle of length three.*

Proof. Let $Q := Q[X_1, X_2, ..., X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Since $U(D[V_3])$ is bipartite and $U(D[V(Q)])$ is an extended cycle of length at least five, it follows that both $U(D[V_3])$ and $U(D[V(Q)])$ do not contain a cycle of length three. Thus suppose that $U(D[V(Q) \cup V_3])$ contains a cycle T of length three. Note that $V(T) \cap V(Q) \neq \emptyset$ and $V(T) \cap V_3 \neq \emptyset$. Since $V(Q) \Rightarrow V_3$, it follows that T is a transitive triangle in D. Let $V(T) = \{x_1, x_2, x_3\}$. The rest of the proof is divided into two cases depending on the cardinality of $V(T) \cap V(Q)$.

Case 1. $|V(T) \cap V(Q)| = 2$. Let $x_1, x_2 \in V(Q)$ and let $x_3 \in V_3$. Without loss of generality, suppose that $x_1x_2 \in A(D)$, $x_1 \in X_1$ and $x_2 \in X_2$. Let $x_k \in X_k$ such that $x_k \to x_1$. Since $x_2 \to x_3$, x_1 and x_3 are adjacent, $x_k \to x_1$ and D is arc-locally in-semicomplete, it follows that x_k and x_2 are adjacent, a contradiction to the fact that $D[V(Q)]$ is an odd extended cycle of length at least five.

Case 2. $|V(T) \cap V(Q)| = 1$. Let $x_1 \in V_2$ and let $x_2, x_3 \in V_3$. Without loss of generality, suppose that $x_1 \in X_1$ and $x_2x_3 \in A(D)$. Let $x_k \in X_k$ such that $x_k \to x_1$. Since $x_2 \to x_3$, $x_1 \to x_3$, $x_k \to x_1$ and $V_2 \Rightarrow V_3$, it follows that $x_k \to x_2$. Thus $D[\{x_k, x_1, x_2\}]$ is a transitive triangle with $\{x_1, x_k\} \subset V(Q)$, and hence, the result follows from the previous case. □

Lemma 4.12. *Let D be an arc-locally in-semicomplete digraph. Let $H := H[X, Y]$ be an induced connected bipartite subdigraph of D such that $|X| \geq 1$, $|Y| \geq 1$ and $X \Rightarrow Y$. Let v be a vertex of $D - V(H)$ that dominates some vertex of X. If $v \Rightarrow X$, then $v \mapsto X$.*

Proof. Let u be a vertex in X such that $v \to u$. Note that we may assume that $|X| > 1$. Let w be a vertex in $X - u$. Since H is connected, $U(H)$ has a path $P = x_1y_1x_2y_2 \ldots x_{k-1}y_{k-1}x_k$ where $x_1 = u$ and $x_k = w$. Note that $x_i \in X$ and $y_i \in Y$. We prove by induction that v dominates every vertex x_i in P. The base case is trivial since v dominates $x_1 = u$. Suppose that v dominates x_{i-1}. Since $X \Rightarrow Y$, it follows that $x_{i-1} \to y_{i-1}$ and $x_i \to y_{i-1}$. Since D is arc-locally in-semicomplete, v and x_i are adjacent; but $v \Rightarrow X$, and hence, $v \to x_i$. So we conclude that v dominates w and thus $v \mapsto X$. □

For the next lemma, we need to define some sets. Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). Recall that $V_1 \mapsto V_2$, $(V_1 \cup V_2) \Rightarrow V_3$ and $D[V_2]$ is an odd extended cycle of length at least five. Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle corresponding to $D[V_2]$. Let $N_0 = V_2$ and for $d \geq 1$ denote by N_d the set of vertices that are at distance d from V_2. Note that $N_d \subseteq V_3$ for $d \geq 1$ because $V_1 \mapsto V_2$. For all $i \in \{1, 2, \ldots, k\}$, denote by R_i (resp., L_i) the subset of $N^+(X_i) \cap N_1$ consisting of those vertices that dominate (resp., are dominated by) some vertex in $N^+(X_{i+1}) \cap N_1$ (resp., $N^+(X_{i-1}) \cap N_1$). Moreover, let $I_i = (N^+(X_i) \cap N_1) - (L_i \cup R_i)$ and let $W_i = N^+(L_i \cup I_i \cup R_i) \cap N_2$. Note that $N^+(X_i) \cap N_1 = L_i \cup I_i \cup R_i$ (see Figure 4.3).

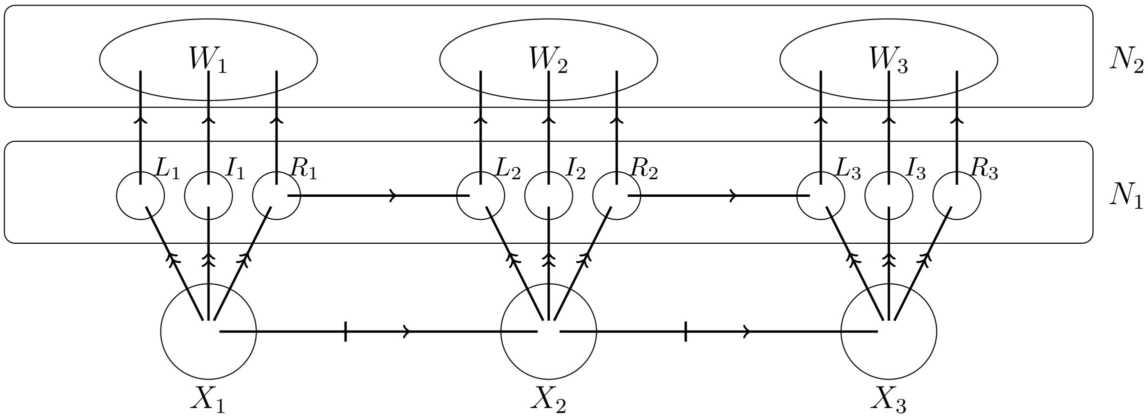


Figure 4.3: Illustration of sets L_i, I_i, R_i e W_i.

Lemma 4.13. *Let D be a connected arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Then, each one of the following holds:*

(i) N_d *is stable for all* $d \geq 2$,

(ii) there are no vertices $x_i \in X_i$, $x_j \in X_j$ *and* $y \in V_3$ *such that* $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$ *and* $\{x_i, x_j\} \to y$,

(iii) there are no vertices $u \in N^+(X_i) \cap N_1$, $v \in N^+(X_j) \cap N_1$ *such that* $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, X_i *and* X_j *are non-adjacent and* $u \to v$,

(iv) $N^+(X_i) \cap N_1 \Rightarrow N^+(X_{i+1}) \cap N_1$ *for all* $i \in \{1, 2, \ldots, k\}$,

(v) $N^-(N_d) \subseteq N_{d-1} \cup V_1$ *for all* $d \geq 1$,

(vi) the digraph $D[N_1]$ *contains no path of length two,*

(vii) $N^+(X_i) \cap N_1$ *is stable for all* $i \in \{1, 2, \ldots, k\}$,

(viii) the sets L_i, I_i *and* R_i *are pairwise disjoint,* $N^-(L_i) \subseteq R_{i-1} \cup X_i \cup V_1$, $N^-(I_i \cup R_i) \subseteq X_i \cup V_1$, $N^+(R_i) \subseteq W_i \cup L_{i+1}$, $N^+(L_i \cup I_i) \subseteq W_i$ *and* $X_i \mapsto R_i$ *for all* $i \in \{1, \ldots, k\}$,

(ix) $N^-(W_i) \subseteq L_i \cup I_i \cup R_i \cup V_1$ *for all* $i \in \{1, 2, \ldots, k\}$.

Proof. (i) First, we show that N_2 is stable. Towards a contradiction, suppose that there exists an arc uv with $\{u, v\} \subseteq N_2$. Let $x \in Q$ and $y \in N_1$ such that $x \to y$ and $y \to v$. Since D is arc-locally in-semicomplete, it follows that u and x are adjacent. Since $V_2 \Rightarrow V_3$, it follows that $x \to u$, a contradiction because $u \in N_2$. Thus N_2 is a stable set. Now, towards a contradiction, suppose that there exists d with $d > 2$ which N_d is not stable. Choose such d as small as possible. Let $u, v \in N_d$ such that $u \to v$. Let x, y be the vertices of N_{d-1} that such $x \to u$ and $y \to v$. Since $D[V_3]$ is bipartite, it follows that $x \neq y$. Since D is arc-locally in-semicomplete, it follows that x and y are adjacent, a contradiction to the choice of d. Thus N_d is stable for all $d \geq 2$.

(ii) Towards a contradiction, suppose that there are vertices $x_i \in X_i$, $x_j \in X_j$ and $y \in V_3$ such that $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$ and $\{x_i, x_j\} \to y$. Without loss of generality, assume that $i < j$. By Lemma 4.11, x_i and x_j are non-adjacent. So $X_{i-1} \neq X_j$ and $X_{j-1} \neq X_i$ where indices are taken modulo k. Let $x_{i-1} \in X_{i-1}$ and let $x_{j-1} \in X_{j-1}$. Since $x_{j-1} \to x_j$, $x_i \to y$, $x_j \to y$ and D is arc-locally in-semicomplete, it follows that $x_i \to x_{j-1}$, and hence, $i = j - 2$. Using the same argument but with the roles of X_i and X_j exchanged, we conclude that $x_j \to x_{i-1}$, a contradiction because $V(Q)$ induces an extended cycle of length at least five.

(iii) Towards a contradiction, suppose that there are vertices $u \in N^+(X_i) \cap N_1$, $v \in N^+(X_j) \cap N_1$ such that $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, X_i and X_j are non-adjacent and $u \to v$. Let x_i be a vertex in X_i that dominates u and let x_j be a vertex in X_j that dominates v. Since $uv \in A(D)$, $x_i \to u$, $x_j \to v$ and D is arc-locally in-semicomplete, it follows that x_i and x_j are adjacent, a contradiction because X_i and X_j are non-adjacent.

(iv) Towards a contradiction, suppose, without loss of generality, that there exists an arc $uv \in A(D)$ such that $u \in N^+(X_3) \cap N_1$ and $v \in N^+(X_2) \cap N_1$. Let x_3 be a vertex of X_3 that dominates u and let x_2 be a vertex of X_2 that dominates v. Let $x_1 \in X_1$. Since $x_2 v \in A(D)$, $u \to v$, $x_1 \to x_2$, $u \in V_3$, $V(Q) \Rightarrow V_3$ and D is arc-locally in-semicomplete, it follows that $x_1 \to u$ which contradicts (ii).

(v) Towards a contradiction, suppose that there exists a $d \geq 1$ such that $N^-(N_d) \not\subseteq N_{d-1} \cup V_1$. Choose such d as small as possible. Let uv be an arc in $A(D)$ such that $v \in N_d$, $u \in N_j$ and $j \neq d - 1$. By definition of N_d, it follows that $j > d$; otherwise, $v \notin N_d$. Let y be a vertex in N_{d-1} that dominates v and let x be a vertex of N_{d-2} that dominates y (if $d = 1$, then let $\{x, y\} \subseteq N_0 = V_2$). Since $u \to v$, $y \to v$, $x \to y$ and D is arc-locally in-semicomplete, it follows that x and u are adjacent. By definition of N_j, $u \to x$. Since $u \in V_3$ and $V_2 \Rightarrow V_3$, it follows that $x \notin N_0 = V_2$. So $x \in N_{d-2}$ has an in-neighbor $u \in N_j$ with $j \neq d - 3$, a contradiction to the choice of d. Thus $N^-(N_d) \subseteq N_{d-1} \cup V_1$ for all $d \geq 1$.

(vi) Towards a contradiction, suppose that there exists a path $P = u_1 u_2 u_3$ in $D[N_1]$. Let $x_i \in X_i$ be a vertex of Q that dominates u_3. Since $u_1 \to u_2$, $u_2 u_3 \in A(D)$, $x_i \to u_3$, D is arc-locally in-semicomplete and $V_2 \Rightarrow V_3$, it follows that $x_i \to u_1$. Let $x_{i-1} \in X_{i-1}$,

$x_{i-2} \in X_{i-2}$ and $x_{i-3} \in X_{i-3}$ be vertices of Q where indices are taken modulo k. So $x_{i-3} \to x_{i-2}$, $x_{i-2} \to x_{i-1}$ and $x_{i-1} \to x_i$. Since $x_{i-1} \to x_i$, $x_iu_3 \in A(D)$, $u_2 \to u_3$ and $V_2 \Rightarrow V_3$, it follows that $x_{i-1} \to u_2$. Analogously for x_{i-2}, u_1 and the arc $x_{i-1}u_2$, we conclude that $x_{i-2} \to u_1$. Since $x_i \to u_1$, we have a contradiction by (ii).

(vii) Towards a contradiction, suppose that there exists an arc u_1u_2 in $A(D)$ such that $\{u_1, u_2\} \subseteq N^+(X_i) \cap N_1$ for some i in $\{1, 2, \ldots, k\}$. Let v_1 and v_2 be vertices of X_i such that $v_1 \to u_1$ and $v_2 \to u_2$. By definition of extended cycle, X_i is stable. By Lemma 4.11, $v_1 \neq v_2$. Since $v_1 \to u_1$, $v_2 \to u_2$ and $u_1u_2 \in A(D)$, it follows that v_1 and v_2 are adjacent, a contradiction to the fact that X_i is a stable set.

(viii) By definition of I_i, it follows that I_i is disjoint from both L_i and R_i; also by (vi) it follows $L_i \cap R_i = \emptyset$ for all $i \in \{1, 2, \ldots, k\}$. Thus the sets L_i, I_i and R_i are pairwise disjoint.

Towards a contradiction, suppose that $N^-(L_i \cup I_i \cup R_i) \nsubseteq R_{i-1} \cup X_i \cup V_1$ for some $i \in \{1, 2, \ldots, k\}$. Let v be a vertex in $V(D) - (R_{i-1} \cup X_i \cup V_1)$ that dominates a vertex u in $L_i \cup I_i \cup R_i$. By (vii), $v \notin L_i \cup I_i \cup R_i$. By (ii), $v \notin V(Q)$. By (v), it follows that $v \notin N_d$ for all $d \geq 2$. Thus $v \in N^+(X_j) \cap N_1$ for some $j \neq i$. By (iv), $j \neq i + 1$ but this contradicts (iii). So $N^-(L_i \cup I_i \cup R_i) \subseteq R_{i-1} \cup X_i \cup V_1$. By (vii), $L_i \cup I_i \cup R_i$ is stable for all $i \in \{1, 2, \ldots, k\}$. So it follows from definition of L_i, I_i and R_i that $N^-(L_i) \subseteq R_{i-1} \cup X_i \cup V_1$ and $N^-(I_i \cup R_i) \subseteq X_i \cup V_1$.

By (iii), there exists no vertex in $L_i \cup I_i \cup R_i$ that dominates a vertex in $N^+(X_j) \cap N_1$ for $j \notin \{i-1, i+1\}$. By (iv) and $V_2 \Rightarrow V_3$, it follows that there exists no vertex in $L_i \cup I_i \cup R_i$ that dominates a vertex in $N^+(X_{i-1}) \cup V(Q)$. Since $V_1 \Rightarrow V_3$ and $N^+(X_i) \cap N_1$ is stable for all $i \in \{1, 2, \ldots, k\}$, we conclude we conclude that $N^+(L_i \cup I_i \cup R_i) \subseteq W_i \cup L_{i+1}$. By definition of L_i, I_i and R_i, it follows that $N^+(R_i) \subseteq W_i \cup L_{i+1}$ and $N^+(L_i \cup I_i) \subseteq W_i$.

Finally, let $x_i \in X_i$ and let $v \in R_i$; we want to show that $x_i \to v$. Let $w \in L_{i+1}$ such that $v \to w$. Let $x_{i+1} \in X_{i+1}$ such that $x_{i+1} \to w$. Since $x_i \to x_{i+1}$, $v \to w$, $x_{i+1} \to w$ and $V_2 \Rightarrow V_3$, it follows that $x_i \to v$. Thus $X_i \mapsto R_i$ for all $i \in \{1, 2, \ldots, k\}$.

(ix) Towards a contradiction, suppose that there exists $i \in \{1, 2, \ldots, k\}$ such that $N^-(W_i) \nsubseteq L_i \cup I_i \cup R_i \cup V_1$. Recall that, $W_i \subseteq N_2$. Let v be a vertex in $V(D) - (L_i \cup I_i \cup R_i \cup V_1)$ such that v dominates a vertex w in W_i. By (i), it follows that N_2 is stable, and hence, $v \notin N_2$. By definition of N_2, $v \notin V(Q)$. So it follows from (v) that $v \in N^+(X_j) \cap N_1$ for some $j \neq i$. Let x_j be a vertex in X_j such $x_j \to v$ and let u be a vertex in $L_i \cup I_i \cup R_i$ such that $u \to w$. Since $x_j \to v$, $v \to w$, $u \to w$ and $V(Q) \Rightarrow V_3$, it follows that $x_j \to u$. Let x_i be a vertex in X_i such that $x_i \to u$. So $\{x_i, x_j\} \to u$ which contradicts (ii). Thus $N^-(W_i) \subseteq L_i \cup I_i \cup R_i \cup V_1$ for all $i \in \{1, 2, \ldots, k\}$. □

4.2 Decomposition for arbitrary arc-locally semicomplete digraphs

In this section, we show that if D is a connected arc-locally semicomplete digraph, then D is either a diperfect digraph or an odd extended cycle of length at least five. Recall that a digraph D is arc-locally semicomplete if D is both arc-locally in-semicomplete and arc-locally out-semicomplete.

In [20], Galeana-Sánchez and Goldfeder presented a characterization of arbitrary connected arc-locally semicomplete digraphs. Since this result will not be used throughout the text and requires more technical definitions, we omit its statement. Here, we present another structural result for this class.

Let D be an arc-locally semicomplete digraph. Note that the inverse of D is also an arc-locally semicomplete digraph. So we can use the principle of directional duality whenever it is convenient. Moreover, every result valid for arc-locally in-semicomplete digraphs, also holds for arc-locally semicomplete digraphs, because they form a subclass of the former one.

The next lemma states that if a connected arc-locally semicomplete digraph D contains an induced odd extended cycle Q of length at least five, then $V(D) = V(Q)$.

Lemma 4.14. *Let D be a connected arc-locally semicomplete digraph. If D contains a strong component Q that induces an odd extended cycle of length at least five, then $V(D) = V(Q)$.*

Proof. Let $Q := Q[X_1, X_2, \ldots, X_{2k+1}]$. We show that $V(D) = V(Q)$. Suppose that there exists a vertex $u \in V(D) - V(Q)$ such that u is adjacent to Q. By the principle of directional duality, we may assume that u dominates some vertex in Q. Since $D[V(Q)]$ is a non-bipartite digraph, it follows from Lemma 4.5 that $u \mapsto Q$. Let $x_1 \in X_1$, $x_2 \in X_2$ and $x_3 \in X_3$. Since u and x_2 are adjacent, $u \to x_1$, $x_2 \to x_3$ and D is arc-locally out-semicomplete, it follows that x_1 and x_3 are adjacent, a contradiction to the fact that $V(Q)$ induces an extended cycle of length at least five. Since D is connected, this implies that $V(D) = V(Q)$. This ends the proof. □

Now, we are ready for the main result of this section.

Theorem 4.4. *Let D be a connected arc-locally semicomplete digraph. Then, D is either a diperfect digraph or an odd extended cycle of length at least five.*

Proof. If D contains no induced odd cycle of length at least five, then it follows from Lemma 4.2 that D is diperfect. Thus let C be an induced odd cycle of length at least five. Let Q be the strong component that contains C. By Lemma 4.10, Q induces an odd extended cycle of length at least five. Then, by Lemma 4.14 we conclude that $V(D) = V(Q)$, and hence, D is an odd extended cycle of length at least five. This finishes the proof. □

4.3 Begin-End conjecture

In this section, we prove that Conjecture 1.2 holds for arc-locally (out) in-semicomplete digraphs. Recall that $\mathfrak{D}$ denotes the set of all digraphs containing no induced blocking odd cycle.

Initially, we present an outline of the main proof. Let D be a connected arc-locally in-semicomplete digraph. Note that every induced subdigraph of D is also an arc-locally in-semicomplete digraph. Thus it suffices to show that D satisfies the BE-property. Recall that we may assume that D is non-diperfect and has no clique cut (see Lemmas 1.2, 1.3 and 1.4). So by Theorem 4.2(ii), $V(D)$ admits a partition (V_1, V_2, V_3) as described in the statement. First, we show that if $D \in \mathfrak{D}$, then $V_1 = \emptyset$. Next, we show that an extended cycle satisfies the BE-property. Finally, we show that if $V_3 \neq \emptyset$, then D satisfies the BE-property. This last case is divided into two subcases depending on whether there exists a vertex v in V_3 such that $\text{dist}(V_2, v) \geq 3$ or not.

Let us start with a simple lemma.

Lemma 4.15. *Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). If $D \in \mathfrak{D}$, then $V_1 = \emptyset$.*

Proof. Towards a contradiction, suppose that there exists v in V_1. Let xy be an arc of $D[V_2]$. Since $V_1 \mapsto V_2$ and $D[V_2]$ is an extended cycle, it follows that $D[\{v, x, y\}]$ is a transitive triangle, a contradiction to the fact that $D \in \mathfrak{D}$. □

The next lemma states that any extended cycle, even or odd, satisfies the BE-property.

Lemma 4.16. *Let D be an extended cycle. Then, D satisfies the BE-property.*

Proof. Let $D := D[X_1, X_2, \ldots, X_k]$. Let S be a maximum stable set of D. If k is even, then D is a bipartite digraph. Since a bipartite digraph is diperfect, the result follows from Lemma 1.2. So we may assume that k is odd. Note that for every X_i, it follows that $X_i \cap S = \emptyset$ or $X_i \subseteq S$ because $X_i \mapsto X_{i+1}$ for all $i \in \{1, 2, \ldots, k\}$. Also, if $X_i \cap S = X_i$, then $X_{i+1} \cap S = X_{i-1} \cap S = \emptyset$. Since k is odd, there exists some i such that $X_i \cap S = X_{i+1} \cap S = \emptyset$.

Without loss of generality, suppose that $X_1 \subseteq S$ and $X_2 \cap S = X_3 \cap S = \emptyset$. First, suppose that $|X_2| \leq |X_3|$. Let $D' = D - X_2$. Since $X_2 \cap S = \emptyset$, it follows that S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D. By definition of extended cycle, every vertex v of X_3 is a source in D', and hence, v is the initial vertex in some path of $\mathcal{P}'$. Since $X_2 \mapsto X_3$ and $|X_2| \leq |X_3|$, it is easy to see that we can add the vertices of X_2 to paths of $\mathcal{P}'$ that starts at some vertex in X_3, obtaining an S_{BE}-path partition of D. Note that if $|X_2| > |X_3|$, the proof proceeds similarly, but we use $D' = D - X_3$ instead of $D' = D - X_2$ and we add the vertices of X_3 to paths of $\mathcal{P}'$ that ends at some vertex in X_2. □

Now, we are ready to prove Conjecture 1.2 for arc-locally semicomplete digraphs.

Theorem 4.5. *Let D be an arc-locally semicomplete digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

Proof. Since every induced subdigraph of D is also an arc-locally semicomplete, it suffices to show that D satisfies the BE-property. Towards a contradiction, suppose the opposite and let D be a counterexample with the smallest number of vertices. Note that if D' is a proper induced subdigraph of D, then D' is an arc-locally semicomplete digraph, and hence, by the minimality of D, it follows that D' satisfies the BE-property. Thus D does not satisfy the BE-property. Moreover, note that by the minimality of D, we may assume that D is connected. Thus if follows from Theorem 4.4 that D is either a diperfect digraph or an odd extended cycle of length at least five. If D is diperfect, then it follows from Lemmas 1.2 that D satisfies the BE-property, a contradiction. Otherwise, we conclude by Lemma 4.16 that D satisfies the BE-property, a contradiction. This finishes the proof. □

Let D be an arc-locally in-semicomplete digraph. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Recall that $N_0 = V_2$ and N_d is the set of vertices that are at distance d from Q, R_i (resp., L_i) the subset of $N^+(X_i) \cap N_1$ consisting of those vertices that dominate (resp., are dominated by) some vertex in $N^+(X_{i+1}) \cap N_1$ (resp., $N^+(X_{i-1}) \cap N_1$). Moreover, $I_i = (N^+(X_i) \cap N_1) - (L_i \cup R_i)$ and $W_i = N^+(L_i \cup I_i \cup R_i) \cap N_2$.

Lemma 4.17. *Let D be a connected arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the BE-property. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). If $N_d = \emptyset$ for all $d \geq 3$ and $V_1 = \emptyset$, then D satisfies the BE-property.*

Proof. Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Let S be a maximum stable set of D. By hypothesis, $N^+(N_2) = \emptyset$ and $V_1 = \emptyset$. By Lemma 4.13(i) and (vii), it follows that W_i and $L_i \cup I_i \cup R_i$ are stable. Next, we prove some claims.

Claim 1. We may assume that $N^+(L_i) = \emptyset$ for all $i \in \{1, \ldots, k\}$.

Suppose that there exists some $i \in \{1, 2, \ldots, k\}$ such that $N^+(L_i) \neq \emptyset$. It follows from Lemma 4.13(viii) that $N^+(L_i) \subseteq W_i$. By Lemma 4.13(ix), $N^-(W_i) \subseteq L_i \cup I_i \cup R_i$. Let $H := H[X, Y]$ be a maximal induced connected bipartite subdigraph with arcs between L_i and $N^+(L_i)$. Let $X \subseteq L_i$ and let $Y \subseteq N^+(L_i) \subseteq W_i$. Since $Y \subseteq W_i$, it follows from Lemma 4.13(v) that $X \Rightarrow Y$. By Lemma 4.13(viii), the sets L_i, I_i and R_i are disjoint. Towards a contradiction, suppose that there exists a vertex $v \in I_i \cup R_i$ such that v dominates a vertex u in Y. Let $x \in X$ and $y \in R_{i-1}$ be vertices such that $x \to u$ and $y \to x$. Since $v \to u$, $y \to x$, $x \to u$ and D is arc-locally in-semicomplete, it follows that y and v are adjacent. So it follows from Lemma 4.13(iv) that $y \to v$, a contradiction to fact that $v \notin L_i$. Since H is maximal connected, $Y \subseteq W_i$ and $N^-(W_i) \subseteq L_i \cup I_i \cup R_i$, it follows that $N^-(Y) = X \subseteq L_i$. Let $U = N^-(X)$. By Lemma 4.13(viii), $U \subseteq R_{i-1} \cup X_i$. By Lemma 4.13(iv) and $V_2 \Rightarrow V_3$, it follows that $U \Rightarrow X$. By Lemma 4.12 applied to U and H, it follows that $U \mapsto X$. Since $N^+(Y) = \emptyset$, $N(Y) = X$. Since X and Y are stable, $N(Y) = X$, $N(X) = U \cup Y$ and every vertex in U is adjacent to every vertex in X, it follows from Lemma 3.15 applied to U, X and Y that D admits an S_{BE}-path partition.

So we may assume that $N^+(L_i) = \emptyset$ for all $i \in \{1, 2, \dots, k\}$. This ends the proof of Claim 1.

From now on, let $I_i^+ = N^-(W_i) \cap I_i$ for all $i \in \{1, 2, \dots, k\}$. So it follows from Lemma 4.13(viii) that $N^+(I_i - I_i^+) = \emptyset$. The Figure 4.4 illustrates the structure of D applying Claim 1 and Lemma 4.13(i)-(ix).

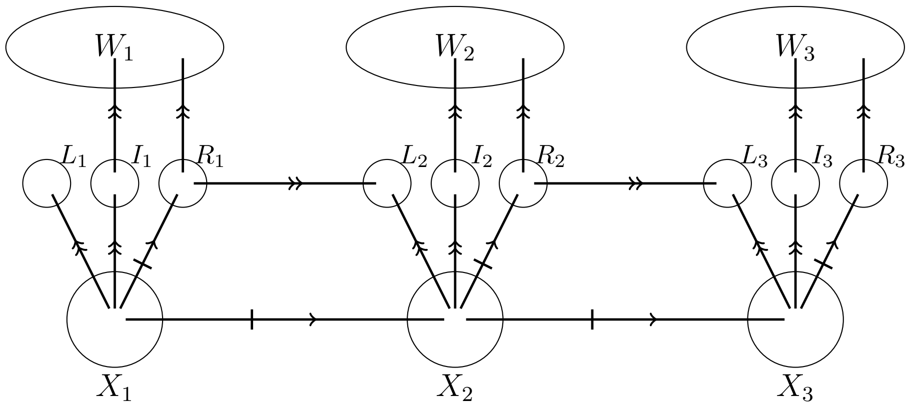


Figure 4.4: By Claim 1 and Lemma 4.13(i)-(ix) D has this structure: the sets L_i, I_i, R_i, W_i and X_i are stable, $L_i \cap I_i = \emptyset$, $I_i \cap R_i = \emptyset$, $L_i \cap R_i = \emptyset$, $N^-(W_i) \subseteq I_i^+ \cup R_i$, $N^-(L_i) \subseteq R_{i-1} \cup X_i$, $N^-(I_i \cup R_i) = X_2$, $N^+(L_i) = \emptyset$, $N^+(I_i^+) \subseteq W_i$, $N^+(I_i - I_i^+) = \emptyset$ and $N^+(R_i) \subseteq W_i \cup L_{i+1}$.

Claim 2. We may assume that $X_i \mapsto I_i^+ \cup R_i \cup X_{i+1}$ for all $i \in \{1, 2, \dots, k\}$.

Let i in $\{1, 2, \dots, k\}$. Since $V(Q) \Rightarrow V_3$ and Q is an extended cycle, it follows that $X_i \Rightarrow I_i^+ \cup R_i$ and $X_i \mapsto X_{i+1}$. By Lemma 4.13(viii), $X_i \mapsto R_i$. So it remains to show that $X_i \mapsto I_i^+$. Since $V_1 = \emptyset$, it follows from Lemma 4.13(ix) and by Claim 1 that $N^-(W_i) \subseteq I_i \cup R_i$. Let $H := H[X, Y]$ be a maximal induced connected bipartite subdigraph with arcs between $I_i^+ \cup R_i$ and W_i. Let $X \subseteq I_i^+ \cup R_i$ and let $Y \subseteq W_i$. Let $U = N^-(X)$. By Lemma 4.13(viii), $U \subseteq X_i$. Since $Y \subseteq W_i$, it follows from Lemma 4.13(v) that $X \Rightarrow Y$. Since $V(Q) \Rightarrow V_3$ and $X \Rightarrow Y$, it follows from Lemma 4.12 applied to U and H that $U \mapsto X$. Since $N^+(Y) = \emptyset$ and H is maximal connected, we conclude that $N(Y) = X$. Now, suppose that $X \subseteq I_i^+$. Since $N(X) = U \cup Y$ and $N(Y) = X$, it follows from Lemma 3.15 applied to U, X and Y that D admits an S_{BE}-path partition. So we may assume that $X \subseteq I_i^+ \cup R_i$ and $X \not\subseteq I_i^+$. Since $X_i \mapsto R_i$, it follows that $U = X_i$, and hence, $X_i \mapsto X$. Since H is arbitrary, it follows that $X_i \mapsto I_i^+$. So we may assume that $X_i \mapsto I_i^+ \cup R_i \cup X_{i+1}$ for all $i \in \{1, 2, \dots, k\}$. This ends the proof of Claim 2.

Claim 3. We may assume that if $S \cap X_i \neq \emptyset$, then $X_i \subseteq S$ for all $i \in \{1, 2, \dots, k\}$.

Suppose that there exists $i \in \{1, 2, \dots, k\}$ such that $X_i \cap S \neq \emptyset$ and $X_i \not\subseteq S$. Without loss of generality, assume that $i = 2$. By Claim 2, $X_2 \mapsto I_2^+ \cup R_2 \cup X_3$. Since $X_1 \mapsto X_2$, it follows that $(X_1 \cup I_2^+ \cup R_2 \cup X_3) \cap S = \emptyset$. Let $S_1 = S \cap (L_2 \cup (I_2 - I_2^+))$ and let $S_2 = S \cap W_1$. Since $X_2 - S \neq \emptyset$ and S is a maximum stable set, S_1 must be non-empty.

By Claim 1, $N^+(L_1) = N^+(L_2) = \emptyset$. By hypothesis, it follows that $N^+(W_1 \cup W_2) = \emptyset$ and $V_1 = \emptyset$, and hence, we conclude by Lemma 4.13(ix) that $N(W_1) \subseteq I_1 \cup R_1$. By definition of I_2^+ and by Lemma 4.13(viii), it follows that $N(I_2 - I_2^+) \subseteq X_2$ and $N(L_2) \subseteq R_1 \cup X_2$. So $N(S_1 \cup S_2) \subseteq I_1 \cup R_1 \cup X_2$. Since S is maximum and $(X_1 \cup X_3 \cup I_2^+ \cup R_2) \cap S = \emptyset$, we have that $|S_1 \cup S_2| \geq |N(S_1 \cup S_2)|$. Thus it follows from Lemma 3.14 applied to $S_1 \cup S_2$ that D satisfies the BE-property. So we may assume that if $S \cap X_i \neq \emptyset$, then $X_i \subseteq S$ for all $i \in \{1, 2, \dots, k\}$. This ends the proof of Claim 3.

Claim 4. We may assume that there exists no $i \in \{1, 2, \dots, k\}$ such that $(X_i \cup X_{i+1} \cup X_{i+2}) \cap S = \emptyset$, where subscripts are taken modulo k.

Without loss of generality, assume that $i = 1$. Since S is maximum, $(L_2 \cup I_2 \cup R_2) \cap S \neq \emptyset$. Let $S_1 = S \cap (L_2 \cup I_2 \cup R_2)$ and let $S_2 = S \cap W_1$. By Claim 1, $N^+(L_1) = N^+(L_2) = N^+(L_3) = \emptyset$. By hypothesis, $N^+(W_1 \cup W_2) = \emptyset$ and $V_1 = \emptyset$. So it follows from Lemma 4.13(ix) that $N(W_1) \subseteq I_1 \cup R_1$ and $N(W_2) \subseteq I_2 \cup R_2$. Also, we have by Lemma 4.13(viii) that $N(L_2 \cup I_2 \cup R_2) \subseteq R_1 \cup X_2 \cup W_2 \cup L_3$ and $N(I_1 \cup R_1) \subseteq X_1 \cup W_1 \cup L_2$. Thus $N(S_1 \cup S_2) \subseteq I_1 \cup R_1 \cup X_2 \cup W_2 \cup L_3$. Since S is maximum and $(X_1 \cup X_2 \cup X_3) \cap S = \emptyset$, we conclude that $|S_1 \cup S_2| \geq |N(S_1 \cup S_2)|$, and hence, by Lemma 3.14 applied to $S_1 \cup S_2$ it follows that D satisfies the BE-property. So we may assume that there exists no $i \in \{1, 2, \dots, k\}$ such that $(X_i \cup X_{i+1} \cup X_{i+2}) \cap S = \emptyset$. This ends the proof of Claim 4.

Since Q is an odd extended cycle, there exists a $i \in \{1, \dots, k\}$ such that $(X_i \cup X_{i+1}) \cap S = \emptyset$. Without loss of generality, assume that $(X_2 \cup X_3) \cap S = \emptyset$. By Claim 3 and 4, it follows that $X_1 \cup X_4 \subseteq S$. By Claim 1, $N^+(L_2) = \emptyset$. Since $X_1 \subseteq S$ and $(X_2 \cup X_3) \cap S = \emptyset$, we conclude that $(L_1 \cup I_1 \cup R_1) \cap S = \emptyset$ and $W_1 \cup L_2 \subseteq S$. The rest of the proof is divided into two cases depending on whether $R_2 \neq \emptyset$ or $R_2 = \emptyset$.

Case 1. $R_2 \neq \emptyset$. First, suppose that $(I_2^+ \cup R_2) \cap S \neq \emptyset$. Let $H := H[X, Y]$ be a maximal induced connected bipartite subdigraph with arcs between $(I_2^+ \cup R_2) \cap S$ and $W_2 \cup L_3$. Let $Y \subseteq (I_2^+ \cup R_2) \cap S$ and let $X \subseteq W_2 \cup L_3$. By hypothesis and by Claim 1, $N^+(W_2 \cup L_3) = \emptyset$. By Lemma 4.13(viii) and (ix), $N(X) \subseteq I_2^+ \cup R_2 \cup X_3$ and $N(Y) \subseteq X \cup X_2$. Note that $N(X) \cap N(Y) = \emptyset$. Since $X_3 \cap S = \emptyset$ and H is maximal connected, it follows that $N(X) \cap S = Y$. By Claim 2, $X_2 \mapsto (I_2^+ \cup R_2 \cup X_3)$, and hence, it follows that every vertex in $N(Y) - X$ is adjacent to every vertex in $N(X)$. Thus we conclude by Lemma 3.16 applied to H that D admits an S_{BE}-path partition. So we may assume that $(I_2^+ \cup R_2) \cap S = \emptyset$. Since $(X_2 \cup X_3) \cap S = \emptyset$, it follows that $W_2 \cup L_3 \subseteq S$, $I_2 - I_2^+ \subseteq S$ and $I_3 - I_3^+ \subseteq S$.

Now, let $H := H[X, Y]$ be a maximal induced connected bipartite subdigraph with arcs between $W_2 \cup L_3 \cup (I_3 - I_3^+)$ and $I_2^+ \cup R_2 \cup X_3$. Let $X \subseteq W_2 \cup L_3 \cup (I_3 - I_3^+)$ and let $Y \subseteq I_2^+ \cup R_2 \cup X_3$. By Lemma 4.13(viii) and (ix) and since H is maximal connected, we conclude that $Y = N(X)$. Note that $X \subseteq S$, and hence, $Y \cap S = \emptyset$. We may assume by Lemma 3.12 that there exists a matching between S and $N(S)$ covering S. Since $X \subseteq S$ and $Y = N(X)$, this implies that there exists a matching between X and Y covering X. We show next that exists a matching between X and Y covering X and

$Y \cap (I_2^+ \cup R_2)$. Let $Y' = Y \cap (I_2^+ \cup R_2)$ and let $X' = N^+(Y')$. Note that $X' \subseteq X \subseteq S$. Thus by Lemma 3.10 applied to $U(H)$ there exists a matching M between X and Y covering X such that the restriction of M on Y' and X' is a maximum matching. Since $(X_2 \cup I_2^+ \cup R_2) \cap S = \emptyset$, we conclude that $N(Y') \cap S = X'$. Thus by Lemma 3.11 there exists a matching between Y' and X' covering Y', and this implies that M covers Y'. So let M be a matching between X and Y covering $X \cup Y'$. Since H is maximal connected, it follows that $N(X) \cap (I_2^+ \cup R_2) = Y'$.

Let $D' = D - V(M)$ and $S' = S - X$. Since M covers X and Y', $V(D') \cap (X \cup Y') = \emptyset$. Towards a contradiction, suppose that S' is not a maximum stable set in D', and let Z be a maximum stable set in D'. So $|Z| > |S'| = |S| - |X| = |S| - |V(M) \cap Y|$. By Claim 2, $X_3 \mapsto I_3^+ \cup R_3 \cup X_4$. Since $X_2 \mapsto X_3$, if $Z \cap (X_2 \cup I_3^+ \cup R_3 \cup X_4) \neq \emptyset$, then $X_3 \cap Z = \emptyset$. Since $Y' \cap V(D') = \emptyset$ and $N(X) \cap (I_2^+ \cup R_2) = Y'$, we conclude that $Z \cup X$ is a stable set larger than S in D, a contradiction. So we may assume that $Z \cap (X_2 \cup I_3^+ \cup R_3 \cup X_4) = \emptyset$. Since $V(D') \cap X = \emptyset$, it follows that $Z \cup (V(M) \cap Y)$ is a stable set larger than S in D, a contradiction. So S' is a maximum stable set in D'. Let $\mathcal{P}_M$ be the set of paths in D corresponding to the arcs of M. By hypothesis, D' is DE-diperfect. Let $\mathcal{P}'$ be an S'_{BE}-path partition of D'. Thus the collection $\mathcal{P}' \cup \mathcal{P}_M$ is an S_{BE}-path partition of D.

Case 2. $R_2 = \emptyset$. First, we prove that $W_2 = \emptyset$. By Claim 2, $X_2 \mapsto I_2^+ \cup X_3$. By Claim 1, $N^+(L_2) = \emptyset$. Suppose that $W_2 \neq \emptyset$. Let $H := H[X, Y]$ be a maximal connected induced bipartite subdigraph with arcs between I_2^+ and W_2. Let $X \subseteq I_2^+$ and let $Y \subseteq W_2$. Since $N^+(W_2) = \emptyset$, $R_2 = \emptyset$ and H is maximal connected, we conclude that $N(Y) = X$ and $N(X) = X_2 \cup Y$. Since $X_2 \mapsto X$, it follows from Lemma 3.15 applied to X_2, X and Y that D admits an S_{BE}-path partition. So we may assume that $W_2 = \emptyset$. Since $X_1 \subseteq S$ and $(X_2 \cup X_3) \cap S = \emptyset$, it follows that $X_1 \cup W_1 \cup L_2 \cup I_2 \subseteq S$ and $(L_1 \cup I_1 \cup R_1) \cap S = \emptyset$.

Let $X := W_1 \cup L_2 \cup I_2 \cup X_3$ and let $Y = N^-(X)$. Note that $X \neq \emptyset$ because $X_3 \neq \emptyset$. By Lemma 4.13(viii) and (ix), it follows that $Y = I_1^+ \cup R_1 \cup X_2$ and $Y \Rightarrow X$. Let $H = D[X \cup Y]$ be an induced bipartite subdigraph of D. Note that X, Y is a bipartition of H and $N^+(Y) = X$. By Lemma 4.13(viii), $N^-(Y) = X_1$. So $N^-(Y) \subset S$. Since $N^+(W_1) = \emptyset$ and $W_2 = \emptyset$, we conclude that $N^+(X \cap S) = \emptyset$. By Claim 2, $X_1 \mapsto I_1^+ \cup R_1 \cup X_2$, and hence, $N^-(Y) \mapsto Y$. Since $X_1 \subseteq S$, $Y \cap S = \emptyset$. Thus since $N^-(X) = Y$, $N^+(Y) = X$, $Y \Rightarrow X$, $Y \cap S = \emptyset$, $N^+(X \cap S) = \emptyset$, $N^-(Y) \subset S$ and $N^-(Y) \mapsto Y$, it follows from Lemma 3.17 applied to H that D admits an S_{BE}-path partition. This finishes the proof. □

Lemma 4.18. *Let D be a connected arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the BE-property. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). If $N_d \neq \emptyset$ for some $d \geq 3$ and $V_1 = \emptyset$, then D satisfies the BE-property.*

Proof. Let $N_d \neq \emptyset$ such that d is maximum. By assumption $d \geq 3$. By Lemma 4.13(i), the sets N_d and N_{d-1} are stable. Since $V_1 = \emptyset$, it follows from Lemma 4.13(v) that $N^-(N_d) \subseteq N_{d-1}$, $N^-(N_{d-1}) \subseteq N_{d-2}$ and $N^-(N_{d-2}) \subseteq N_{d-3}$, this implies that $N_{d-2} \Rightarrow N_{d-1}$ and $N_{d-1} \Rightarrow N_d$. Also, by definition of N_d and since $V_2 \Rightarrow V_3$, we have that $N^+(N_{d-1}) = N_d$. Let $H := H[X, Y]$ be a maximal connected bipartite subdigraph with arcs between N_{d-1} and N_d. Let $X \subseteq N_{d-1}$ and let $Y \subseteq N_d$. Let $U = N^-(X)$. Since $U \Rightarrow X$ and $X \Rightarrow Y$, it follows from Lemma 4.12 applied to U and H that $U \mapsto X$. By the choice of N_d, $N^+(Y) = \emptyset$. Since H is maximal and connected, we conclude that $N(Y) = X$ and $N(X) = U \cup Y$. Since $U \mapsto X$, it follows from Lemma 3.15 applied to U, X and Y that D admits an S_{BE}-path partition. □

Now, we are ready for the main result of this section.

Theorem 4.6. *Let D be a connected arc-locally in-semicomplete digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

Proof. Since every induced subdigraph of D is also an arc-locally in-semicomplete digraph, it suffices to show that D satisfies the BE-property. If D is diperfect or D has a clique cut, then the result follows from Lemmas 1.2, 1.3 and 1.4. So we may assume that $V(D)$ can be partitioned into (V_1, V_2, V_3) as described in Theorem 4.2(ii). From Lemma 4.15, we get that $V_1 = \emptyset$. If $V_3 = \emptyset$, then the result follows from Lemma 4.16. Thus $V_3 \neq \emptyset$, and hence, we conclude by Lemmas 4.17 and 4.18 that D satisfies the BE-property. This finishes the proof. □

Let D be a connected arc-locally in-semicomplete digraph and let H be the inverse of D. Since D satisfies the BE-property if and only if H satisfies the BE-property, we have the following result.

Theorem 4.7. *Let D be a connected arc-locally out-semicomplete digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

□

4.4 Berge's conjecture

In this section, we prove that Conjecture 1.1 holds for arc-locally (out) in-semicomplete digraphs. Recall that we denote by $\mathfrak{B}$ the set of all digraphs containing no induced anti-directed odd cycle.

First, we present an outline of the main proof. Let D be a connected arc-locally in-semicomplete digraph. Since every induced subdigraph of D is also an arc-locally in-semicomplete digraph, it suffices to show that D satisfies the α-property. Recall that we may assume that D is non-diperfect and has no clique cut (see Lemmas 1.1, 1.3 and 1.4). So by Theorem 4.2(ii), $V(D)$ admits a partition (V_1, V_2, V_3) as described in the statement. So we show that if $V_1 = \emptyset$, then D satisfies the α-property. Next, we show that an extended cycle satisfies the α-property (it is analogous to the proof of Lemma 4.16). Finally, we show that if $V_1 \neq \emptyset$, then D satisfies the α-property.

Initially, we show that if $V_1 = \emptyset$, then D satisfies the α-property.

Lemma 4.19. *Let D be a connected arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the α-property. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). If $V_1 = \emptyset$, then D satisfies the α-property.*

Proof. Since $V_1 = \emptyset$, it follows from Lemma 4.11 that $U(D)$ contains no cycle of length three. So D contains no induced transitive triangle. Moreover, since every blocking odd cycle is also a non-oriented odd cycle, it follows from Lemma 4.1 that D contains no blocking odd cycle of length at least five as an induced subdigraph, and this implies that $D \in \mathfrak{D}$. Thus we conclude by Theorem 4.6 that D satisfies the BE-property, and hence, the α-property. □

The next lemma states that if a digraph D is an extended cycle, then D satisfies the α-property. We omit its proof since it is analogous to the proof of Lemma 4.16, but we use Lemma 1.1 instead of Lemma 1.2.

Lemma 4.20. *Let D be an extended cycle. If $D \in \mathfrak{B}$, then D satisfies the α-property.*

□

Now, we are ready to prove Conjecture 1.1 for arc-locally semicomplete digraphs. We also omit its proof since it is analogous to the proof of Theorem 4.5, but we use Lemmas 1.1 and 4.20 instead of Lemmas 1.2 and 4.16, respectively.

Theorem 4.8. *Let D be a connected arc-locally semicomplete digraph. If $D \in \mathfrak{B}$, then D is α-diperfect.*

□

Next, we prove that if (V_1, V_2, V_3) is a partition of $V(D)$ as described in Theorem 4.2(ii) and $V_1 \neq \emptyset$, then D satisfies the α-property.

Lemma 4.21. *Let D be a connected arc-locally in-semicomplete digraph such that every proper induced subdigraph of D satisfies the α-property. Let (V_1, V_2, V_3) be a partition of $V(D)$ as described in Theorem 4.2(ii). If $V_1 \neq \emptyset$, then D satisfies the α-property.*

Proof. Let S be a maximum stable set of D. The proof is divided into two cases depending on whether $S \cap V_1 = \emptyset$ or $S \cap V_1 \neq \emptyset$.

Case 1. $S \cap V_1 = \emptyset$. Let $D' = D - V_1$. Note that S is maximum in D'. By hypothesis, D' is α-diperfect. Let $\mathcal{P}'$ be an S-path partition of D'. Since $V_2 \Rightarrow V_3$, there exists a path xPy of $\mathcal{P}'$ such that x is in V_2. Since $D[V_1]$ is a semicomplete digraph, it follows that $D[V_1]$ is diperfect. By Lemma 1.1, $D[V_1]$ satisfies the α-property; since $\alpha(D[V_1]) = 1$, this implies that there exists a hamiltonian path $uP'v$ in $D[V_1]$. Since $V_1 \mapsto V_2$, we have $v \to x$. Thus the collection $(\mathcal{P}' - P) \cup P'P$ is an S-path partition of D.

Case 2. $S \cap V_1 \neq \emptyset$. Since $V_1 \mapsto V_2$, $S \cap V_2 = \emptyset$. Let $Q := Q[X_1, X_2, \ldots, X_k]$ be the odd extended cycle of length at least five corresponding to $D[V_2]$. Let $x_i \in X_i$ for all $i \in \{1, 2, \ldots, k\}$ and let $C = x_1 x_2 \ldots x_k x_1$ be a cycle of D. Let $D' = D - V(C)$. Since $V(C) \cap S = \emptyset$, S is maximum in D'. By hypothesis, D' is α-diperfect. Let $\mathcal{P}'$ be an S-path partition of D'. The rest of the proof is divided into two subcases depending on whether $V(Q) \neq V(C)$ or $V(Q) = V(C)$.

Case 2.1. $V(Q) \neq V(C)$. First, suppose that there exists a vertex $v_i \in X_i - x_i$ such that v_i is the initial (resp., terminal) vertex in some path P of $\mathcal{P}'$ for some $i \in \{1, 2, \ldots, k\}$. Let $x_i P' x_{i-1}$ (resp., $x_{i+1} P' x_i$) be a path in C containing $V(C)$. By definition of extended cycle, $x_{i-1} \to v_i$ (resp., $v_i \to x_{i+1}$). Thus the collection $(\mathcal{P}' - P) \cup P'P$ (resp., $(\mathcal{P}' - P) \cup PP'$) is an S-path partition of D. So we may assume that there exists no vertex v in $V(Q) - V(C)$ such that v is the initial vertex or the terminal vertex in some path of $\mathcal{P}'$. Thus there exists a vertex $v_i \in X_i - x_i$ such that v_i is an intermediate vertex in a path xPy of $\mathcal{P}'$ for some $i \in \{1, 2, \ldots, k\}$. Let w be the vertex of P that dominates v_i. Let $P_1 = xPw$ and $P_2 = v_iPy$ be the subpaths of P. Since x_i and v_i belong to the same X_i of Q and $V_2 \Rightarrow V_3$, it follows that $w \in V_1 \cup X_{i-1}$. Since $V_1 \cup X_{i-1} \mapsto X_i$, $w \to x_i$. By definition of extended cycle, $x_{i-1} \to v_i$. Let $x_i P' x_{i-1}$ be a path in C containing $V(C)$. Let $R = P_1 P' P_2$ be the path formed by inserting P' between P_1 and P_2. Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D.

Case 2.2. $V(Q) = V(C)$. Since $D' = D - V(C)$, $V(D') = V_1 \cup V_3$. Since $D[V_1]$ is a semicomplete digraph, $\alpha(D[V_1]) = 1$. Since $\alpha(D[V(Q)]) > 1$, $S \cap V_1 \neq \emptyset$ and S is a maximum stable set in D, it follows that $V_3 \neq \emptyset$. Recall that $N_0 = V_2$, N_d is the set of vertices that are at distance d from Q and $N_d \subseteq V_3$ for all $d \geq 1$. By Lemma 4.13(v), $N^-(N_1) \subseteq V(Q) \cup V_1$. Suppose there exists a vertex v in N_1 such that v is the initial vertex in some path P of $\mathcal{P}'$. Without loss of generality, assume that $x_1 \in V(C)$ dominates v in D. Let $x_2 P' x_1$ be a path in C containing $V(C)$. Thus the collection $(\mathcal{P}' - P) \cup P'P$ is an S-path partition of D. So we may assume that there exists no vertex v in N_1 such that v is the initial vertex in some path of $\mathcal{P}'$. Since $N^-(N_1) \subseteq V(Q) \cup V_1$ and $V_1 \Rightarrow V_3$, there exists a path P of $\mathcal{P}'$ such that P contains vertices $w \in V_1$ and $v \in N_1$ where $w \to v$. Let $P_1 = Pw$ and $P_2 = vP$ be the subpaths of P. Without loss of generality, assume that $x_1 \in V(C)$ dominates v in D. Let $x_2 P' x_1$ be a path in C containing $V(C)$. Since $V_1 \mapsto V_2$, $w \to x_2$. Let $R = P_1 P' P_2$ be the path formed by inserting P' between P_1 and P_2. Thus the collection $(\mathcal{P}' - P) \cup R$ is an S-path partition of D. This finishes the proof. □

Now, we are ready for the main result of this section.

Theorem 4.9. *Let D be a connected arc-locally in-semicomplete digraph. If $D \in \mathfrak{B}$, then D is α-diperfect.*

Proof. Since every induced subdigraph of D is also an arc-locally in-semicomplete digraph, it suffices to show that D satisfies the α-property. If D is diperfect or D has a clique cut, then the result follows from Lemmas 1.1, 1.3 and 1.4. So we may assume that $V(D)$ can be partitioned into (V_1, V_2, V_3) as described in Theorem 4.2(ii). If $V_1 = V_3 = \emptyset$, then we conclude by Lemma 4.20 that D satisfies the α-property. So $V_1 \cup V_3 \neq \emptyset$. If $V_1 = \emptyset$, then the result follows from Lemma 4.19; and if $V_1 \neq \emptyset$, then the result follows from Lemma 4.21. This ends the proof. □

Similarly to Theorem 4.7, we have the following result.

Theorem 4.10. *Let D be a connected arc-locally out-semicomplete digraph. If $D \in \mathfrak{B}$, then D is α-diperfect.*

□

The study of this class of arc-locally in-semicomplete digraphs shows the difficulty in proving both Conjectures 1.1 and 1.2. Note that with the results presented in Section 4.1, the structure of an arc-locally in-semicomplete digraph is well-defined and relatively simple, and even so, the proof for this class was quite challenging.

Moreover, it is reasonable to expect the Theorem 4.2 has future applications in other problems involving arc-locally in-semicomplete digraphs. In particular, we would like to point out that Theorem 4.2 was used by Silva, Silva and Lee [30] in the context of χ-diperfect digraphs (a class of digraphs introduced by Berge [7]). More specifically, they proved that every arc-locally in-semicomplete digraph is χ-diperfect (we omit the definition here).

Chapter 5

3-anti-circulant digraphs

In this chapter, we compile the results presented in Freitas and Lee [15], and it is organized as follows. In Section 5.1, we verify both Conjecture 1.1 and Conjecture 1.2 for 3-anti-circulant digraphs. In Section 5.2, we present some structural results for 3-anti-digon-circulant digraphs.

Let D be a digraph. We say that a set $\{v_1, v_2, v_3, v_4\} \subseteq V(D)$ is an *anti-P_4* if $v_1 \to v_2$, $v_3 \to v_2$ and $v_3 \to v_4$. Whenever it is convenient, we may write an anti-P_4 as $v_1 \to v_2 \leftarrow v_3 \to v_4$. Since every anti-directed odd cycle and every blocking odd cycle of length at least five contains an induced anti-P_4, it seems interesting to study digraphs that do not contain anti-P_4 as an induced subdigraph. Motivated by this observation, initially we decided to study the class of 3-anti-circulant digraphs which satisfy this property.

In [32], Wang characterized the structure of a strong 3-anti-circulant digraph admitting a partition into vertex-disjoint cycles and showed that the structure is very close to semicomplete and semicomplete bipartite digraphs. However, this characterization did not help in proving either conjecture. In order to obtain a better understanding of this class, we began to study a subclass of 3-anti-circulant digraphs that we called 3-anti-digon-circulant digraphs. We tried to obtain a characterization of this class; although we did not succeed in this task, we obtained nice structural results which were strong enough to settle both conjectures for this class. However, we realized that actually those results could be extended to settle both conjectures for 3-anti-circulant digraphs. Since we already had some interesting results for 3-anti-digon-circulant digraphs, we decided to include them in this chapter.

5.1 3-anti-circulant digraphs

In this section, we verify both Conjecture 1.1 and Conjecture 1.2 for 3-anti-circulant digraphs that were defined by Wang in [32]. Let D be a digraph. We say that D is 3-*anti-circulant* if for every anti-P_4 $v_1 \to v_2 \leftarrow v_3 \to v_4$, it follows that $v_4 \to v_1$ (see Figure 5.1a). Note that the inverse of D is also a 3-*anti-circulant* digraph. So we can use the principle of directional duality whenever it is convenient. Moreover, note that every 3-anti-circulant digraph belongs to $\mathfrak{B}$, and the only possible induced blocking odd cycle in a 3-anti-circulant digraph is a transitive triangle (see Figure 5.1b).

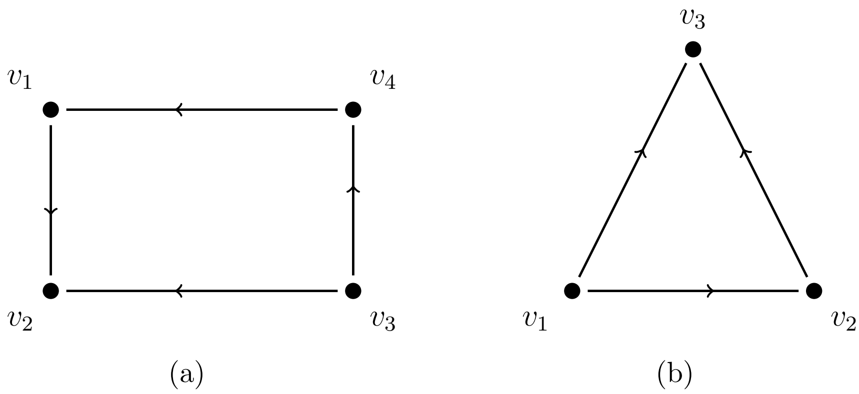


Figure 5.1: Examples of 3-anti-circulant digraphs.

Moreover, Wang also characterized the structure of a strong 3-anti-circulant digraph admitting a partition into vertex-disjoint cycles. However, this characterization does not help in proving either conjecture. Thus we use a different approach. First, we need the following definitions.

Let S be a maximum stable set of a digraph D. Denote by B^+ (resp., B^-) the subset of $V(D) - S$ such that $B \Rightarrow S$ (resp., $S \Rightarrow B$). Moreover, let $B^{\pm} = V(D) - (B^+ \cup B^- \cup S)$, that is, $B^{\pm}$ is a set of those vertices that both dominate and are dominated by some vertex in S (see Figure 5.2). Note that B^+, B^- and $B^{\pm}$ are pairwise disjoint and since S is a maximum stable set in D, it follows that $V(D) = S \cup B^+ \cup B^- \cup B^{\pm}$.

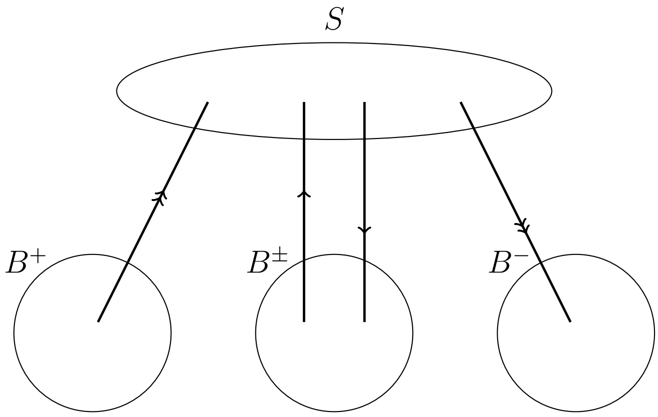


Figure 5.2: Illustration of B^+, $B^{\pm}$ and B^-.

Let us start with a simple and useful structural lemma.

Lemma 5.1. *Let D be a 3-anti-circulant digraph. Let S be a maximum stable set in D. Then, for every v in B^+ and for every u in B^-, it follows that $|N^-(v) \cap B^+| \leq 1$ and $|N^+(u) \cap B^-| \leq 1$.*

Proof. Note that by the principle of directional duality, it suffices to show that $|N^-(v) \cap B^+| \leq 1$. Towards a contradiction, suppose that $|N^-(v) \cap B^+| > 1$. So let v_1, v_2 be vertices in $N^-(v) \cap B^+$. By definition of B^+, there exists a vertex y in S such that $v_1 \to y$. Since

$v_2 \to v \leftarrow v_1 \to y$ and D is 3-anti-circulant, it follows that $y \to v_2$, a contradiction because $v_2 \in B^+$. Thus $|N^-(v) \cap B^+| \leq 1$ and $|N^+(u) \cap B^-| \leq 1$. □

5.1.1 Begin-End conjecture

In this subsection, we prove Conjecture 1.2 for 3-anti-circulant digraphs. Initially, we present an outline of the main proof. Let D be a 3-anti-circulant digraph and let S be a maximum stable set in D. Note that every induced subdigraph of D is also a 3-anti-circulant digraph. Thus it suffices to show that D satisfies the BE-property. First, we show that if $D \in \mathfrak{D}$, then there exists no arc connecting vertices of distinct sets in B^+, B^- and $B^\pm$. Next, we show that B^+, B^- and $B^\pm$ are stable. This implies that $|S| \geq |B^+ \cup B^- \cup B^\pm|$, and hence, it follows from Lemma 3.14 that D satisfies the BE-property.

In the next three lemmas we show that if $U(D)$ contains a cycle C of length three such that C contains a digon and $V(C) \cap S \neq \emptyset$, then D admits an S_{BE}-path partition.

Lemma 5.2. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. Let $\{v_1, v_2\}$ be a digon in $D - S$. If there exists a vertex v_3 in $V(D) - \{v_1, v_2\}$ such that $v_3 \in S$ and $D[\{v_1, v_2, v_3\}]$ contains a $\overrightarrow{C_3}$, then D admits an S_{BE}-path partition.*

Proof. With lost of generality, assume that $v_2 \to v_3$ and $v_3 \to v_1$. Let $D' = D - \{v_1, v_2\}$. Since $\{v_1, v_2\} \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let P be a path of $\mathcal{P}'$ such that $v_3 \in V(P)$. If $V(P) = \{v_3\}$, then the collection $(\mathcal{P}' - v_3) \cup v_1v_2v_3$ is an S_{BE}-path partition of D. So we may assume that $|V(P)| > 1$. By the principle of directional duality, we may assume that P starts at v_3. Let $P = v_3w_1w_2 \ldots w_k$. Next, we show by induction on k that $w_k \to v_1$ or $w_k \to v_2$ holds. First, suppose that $k = 1$. Since $v_2 \to v_1 \leftarrow v_3 \to w_1$ and D is 3-anti-circulant, it follows that $w_1 \to v_2$. Now, assume that $k > 1$. By induction hypothesis, $w_{i-1} \to v_1$ or $w_{i-1} \to v_2$ for some $i \in \{2, \ldots, k\}$. Since $v_1 \leftrightarrow v_2$ and $w_{i-1} \to w_i$, it follows that $w_i \to v_1$ or $w_i \to v_1$. Thus $w_k \to v_1$ or $w_k \to v_2$. Since $v_1 \leftrightarrow v_2$, the collection $(\mathcal{P}' - P) \cup Pv_1v_2$ or $(\mathcal{P}' - P) \cup Pv_2v_1$ is an S_{BE}-path partition of D. □

From now on, we prove some results for 3-anti-circulant digraphs that belong to $\mathfrak{D}$.

Lemma 5.3. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. Let $\{v_1, v_2\}$ be a digon in D. If $D \in \mathfrak{D}$ and there exists a vertex v_3 in $V(D) - \{v_1, v_2\}$ such that $\{v_1, v_2\} \to v_3$ and $\{v_1, v_2, v_3\} \cap S \neq \emptyset$, then D admits an S_{BE}-path partition.*

Proof. The proof is divided into two cases depending on whether $v_3 \in S$ or $v_3 \notin S$. First, we prove the following claim.

Claim 1. If there exists a vertex $v_4 \in V(D) - \{v_1, v_2, v_3\}$ such $v_4 \to v_3$, then $D[\{v_1, v_2, v_3\}]$ is a complete digraph.

Since $\{v_1, v_2\} \to v_3$, $v_1 \leftrightarrow v_2$ and D is 3-anti-circulant, it follows that $\{v_1, v_2\} \to v_4$. Since $v_2 \to v_4 \leftarrow v_1 \to v_3$, we conclude that $v_3 \to v_2$, and hence, $v_2 \leftrightarrow v_3$. Since $v_1 \to v_4 \leftarrow v_2 \to v_3$, it follows that $v_3 \to v_1$, and hence, $v_1 \leftrightarrow v_3$. Thus $D[\{v_1, v_2, v_3\}]$ is a complete digraph. This ends the proof of Claim 1.

Case 1. $v_3 \notin S$. If $N^-(v_3) \neq \{v_1, v_2\}$, then it follows from Claim 1 that $D[\{v_1, v_2, v_3\}]$ is complete, and hence, the result follows from Lemma 5.2. So $N^-(v_3) = \{v_1, v_2\}$. If $v_2 \in S$ (resp., $v_1 \in S$), then since $N^-(v_3) = \{v_1, v_2\}$ and $v_3 \notin S$, the result follows from Lemma 3.4 with $u_1 = v_1$ (resp., $u_1 = v_2$), $u_2 = v_3$ and $P = v_2$ (resp., $P = v_1$).

Case 2. $v_3 \in S$. Since $\{v_1, v_2\} \to v_3$, $\{v_1, v_2\} \cap S = \emptyset$. We may assume by Lemma 5.2 that $v_1 \mapsto v_3$ and $v_2 \mapsto v_3$. Thus it follows from Claim 1 that $N^-(v_3) = \{v_1, v_2\}$. First, suppose that there exists a vertex v_4 in $N^+(v_2) - \{v_1, v_3\}$. Since $v_1 \to v_3 \leftarrow v_2 \to v_4$, it follows that $v_4 \to v_1$. Since $v_4 \to v_1 \leftarrow v_2 \to v_3$, we conclude that $v_3 \to v_4$. Since $D \in \mathfrak{D}$, there exists at least one digon in $D[\{v_2, v_3, v_4\}]$; otherwise, $D[\{v_2, v_3, v_4\}]$ is an induced transitive triangle. Since $v_2 \mapsto v_3$ and $N^-(v_3) = \{v_1, v_2\}$, it follows that $v_2 \leftrightarrow v_4$. Thus the result follows from Lemma 5.2 applied to $D[\{v_2, v_3, v_4\}]$. So we may assume that $N^+(v_2) = \{v_1, v_3\}$. Let $P = v_1$. Since $v_2 \notin S$, $\{v_2, v_3\} \cap V(P) = \emptyset$, $v_2 \to v_1$, $v_1 \to v_3$ and $N^+(v_2) \subseteq V(P) \cup \{v_3\}$, the result follows from Lemma 3.3 with $u_1 = v_2$ and $u_2 = v_3$. This finishes the proof. □

By the principle of directional duality, we have the following result.

Lemma 5.4. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. Let $\{v_1, v_2\}$ be a digon in D. If $D \in \mathfrak{D}$ and there exists a vertex v_3 in $V(D) - \{v_1, v_2\}$ such that $v_3 \to \{v_1, v_2\}$ and $\{v_1, v_2, v_3\} \cap S \neq \emptyset$, then D admits an S_{BE}-path partition.*

The following lemma states that we may assume that for every transitive triangle T in $D \in \mathfrak{D}$, $V(T) \cap S = \emptyset$.

Lemma 5.5. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. If $D \in \mathfrak{D}$ and D contains a transitive triangle T such that $V(T) \cap S \neq \emptyset$, then D admits an S_{BE}-path partition.*

Proof. Let $V(T) = \{v_1, v_2, v_3\}$. Without loss of generality, assume that $v_1 \to v_2$ and $\{v_1, v_2\} \to v_3$. Since $D \in \mathfrak{D}$, there exists at least one digon in T; otherwise, T is an induced transitive triangle. If $v_1 \leftrightarrow v_2$ (resp., $v_2 \leftrightarrow v_3$), then the result follows from Lemma 5.3 (resp., Lemma 5.4). Thus $v_1 \leftrightarrow v_3$. If $v_2 \in S$, then the result follows from Lemma 5.2. So $\{v_1, v_3\} \cap S \neq \emptyset$. Without loss of generality, assume that $v_3 \in S$. We show next that $N^+(v_1) = \{v_2, v_3\}$. Suppose that there exists a vertex v_4 in $N^+(v_1) - \{v_2, v_3\}$. Since $v_2 \to v_3 \leftarrow v_1 \to v_4$ and D is 3-anti-circulant, we conclude that $v_4 \to v_2$. Also, since $v_4 \to v_2 \leftarrow v_1 \to v_3$, it follows that $v_3 \to v_4$. Thus the result follows from Lemma 5.3 applied to $D[\{v_1, v_3, v_4\}]$. So we may assume that $N^+(v_1) = \{v_2, v_3\}$. Let $P = v_2$. Since $v_1 \notin S$, $\{v_1, v_3\} \cap V(P) = \emptyset$, $v_1 \to v_2$, $v_2 \to v_3$ and $N^+(v_1) \subseteq V(P) \cup \{v_3\}$, the result follows from Lemma 3.3 with $u_1 = v_1$ and $u_2 = v_3$. This finishes the proof. □

The next lemma states that we may assume that $B^- \cup B^\pm \Rightarrow B^+$.

Lemma 5.6. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. If $D \in \mathfrak{D}$ and there are vertices $v_1 \in B^+$ and $v_2 \in B^- \cup B^\pm$ such that $v_1 \to v_2$, then D admits an S_{BE}-path partition.*

Proof. By definition of B^+, there exists a vertex y_1 in S such that $v_1 \to y_1$. By definition of $B^\pm$ and B^-, there exists a vertex y_2 in S such that $y_2 \to v_2$. Towards a contradiction, suppose that $y_1 \neq y_2$. Since $y_2 \to v_2 \leftarrow v_1 \to y_1$ and D is 3-anti-circulant, it follows that $y_2 \to y_1$, a contradiction because S is stable. So $y_1 = y_2$, and hence, the result follows from Lemma 5.5 applied to $D[\{v_1, v_2, y_1\}]$. □

By the principle of directional duality, we have the following result.

Lemma 5.7. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. If $D \in \mathfrak{D}$ and there are $v_1 \in B^+ \cup B^\pm$ and $v_2 \in B^-$ such that $v_1 \to v_2$, then D admits an S_{BE}-path partition.*

We show next that if $D \in \mathfrak{D}$, then we may assume that $B^\pm$ is a stable set.

Lemma 5.8. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. If $D \in \mathfrak{D}$ and $B^\pm$ is not stable, then D admits an S_{BE}-path partition.*

Proof. Let v_1, v_2 be adjacent vertices in $B^\pm$. Without loss of generality, assume that $v_1 \to v_2$. By definition of $B^\pm$, there are vertices y_1, y_2 in S such that $v_1 \to y_1$ and $y_2 \to v_2$. Since S is stable and D is 3-anti-circulant, it follows that $y_1 = y_2$, and hence, the result follows from Lemma 5.5 applied to $D[\{v_1, v_2, y_1\}]$. □

The next lemma states that if D contains an anti-P_4 disjoint from S, then D admits an S_{BE}-path partition.

Lemma 5.9. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. If $D \in \mathfrak{D}$ and D contains an anti-P_4 disjoint from S, then D admits an S_{BE}-path partition.*

Proof. Let $\{v_1, v_2, v_3, v_4\} \subseteq V(D)$ be an anti-P_4 in D such that $v_1 \to v_2 \leftarrow v_3 \to v_4$. Since D is 3-anti-circulant, we conclude that $v_4 \to v_1$. We show next that $v_2 \in B^+$ and $v_3 \in B^-$. Note that by the principle of directional duality, it suffices to show that $v_3 \in B^-$. Moreover, we may assume by Lemma 5.6 that $B^- \cup B^\pm \Rightarrow B^+$. Towards a contradiction, suppose that $v_3 \notin B^-$. Since $v_3 \notin S$, it follows that $v_3 \in B^+ \cup B^\pm$. If $v_3 \in B^+$, then since $v_3 \to \{v_2, v_4\}$ and $B^- \cup B^\pm \Rightarrow B^+$, we conclude that $\{v_2, v_4\} \subset B^+$. Since $v_4 \in B^+$, it follows that $v_1 \in B^+$, and hence, $|N^-(v_2) \cap B^+| > 1$, a contradiction by Lemma 5.1. If $v_3 \in B^\pm$, then since $v_3 \to v_4$, it follows from Lemma 5.8 that $v_4 \notin B^\pm$. By Lemma 5.7, $v_4 \notin B^-$. So $v_4 \in B^+$. Since $v_4 \in B^+$ and $B^- \cup B^\pm \Rightarrow B^+$, it follows that $v_1 \in B^+$. By definition of $B^\pm$, there exists a vertex y in S such that $v_3 \to y$. Since $v_1 \to v_2 \leftarrow v_3 \to y$, we conclude that $y \to v_1$, a contradiction because $v_1 \in B^+$. Thus $v_3 \in B^-$ and $v_2 \in B^+$.

Now, let $P_1 = v_4v_1v_2$ and let $P_2 = v_3v_4v_1$. Towards a contradiction, suppose that $N^+(v_3) \not\subseteq V(P_1)$ and $N^-(v_2) \not\subseteq V(P_2)$. So let w_1, w_2 vertices such that w_1 in $N^-(v_2) - V(P_2)$ and w_2 in $N^+(v_3) - V(P_1)$. First, suppose that $w_1 = w_2$. Since $D \in \mathfrak{D}$, there exists at least one digon in $D[\{v_2, v_3, w_1\}]$; otherwise, $D[\{v_2, v_3, w_1\}]$ is an induced transitive triangle. Since $v_2 \in B^+$ and $B^- \cup B^\pm \Rightarrow B^+$, we conclude that $w_1 \leftrightarrow v_3$, and since $v_3 \in B^-$, the result follows from Lemma 5.7. So we may assume that $w_1 \neq w_2$ (see Figure 5.3a).

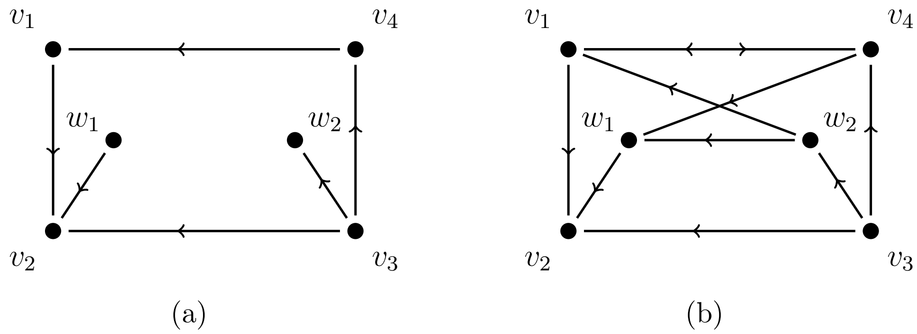


Figure 5.3: Illustration for the proof of Lemma 5.9.

Since $w_1 \rightarrow v_2 \leftarrow v_3 \rightarrow \{w_2, v_4\}$, we conclude that $\{w_2, v_4\} \rightarrow w_1$. Since $v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow w_2$, $w_2 \rightarrow v_1$. Also, since $v_4 \rightarrow w_1 \leftarrow w_2 \rightarrow v_1$, we conclude that $v_1 \rightarrow v_4$, and hence, $v_1 \leftrightarrow v_4$ (see Figure 5.3b). Since $v_3 \rightarrow v_4 \leftarrow v_1 \rightarrow v_2$, it follows that $v_2 \rightarrow v_3$, a contradiction because $v_2 \in B^+$, $v_3 \in B^-$ and $B^- \cup B^\pm \Rightarrow B^+$. Thus $N^+(v_3) \subseteq V(P_1)$ or $N^-(v_2) \subseteq V(P_2)$. Since $\{v_1, v_2, v_3, v_4\} \cap S = \emptyset$, the result follows from Lemma 3.1 with $u = v_3$ or by Lemma 3.2 with $u = v_2$. □

In the next lemmas, we show that if $D \in \mathfrak{D}$, then B^+ and B^- are stable. To do this, we show that there exists no arc v_1v_2 in D such that $v_1 \in B^+ \cup B^-$ and $v_2 \in B^\pm$.

Lemma 5.10. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set in D. If $D \in \mathfrak{D}$ and there are adjacent vertices v_1, v_2 in $V(D)$ such that $v_1 \in B^+ \cup B^-$ and $v_2 \in B^\pm$, then D admits an S_{BE}-path partition.*

Proof. By the principle of directional duality, we may assume that $v_1 \in B^+$. Also, we may assume by Lemma 5.6 that $B^- \cup B^\pm \Rightarrow B^+$. So $v_2 \mapsto v_1$. By definition of B^+, there exists a vertex y_1 in S such that $v_1 \mapsto y_1$. By definition of $B^\pm$, there exists a vertex y_2 in S such that $v_2 \rightarrow y_2$.

Claim 1. $N^-(v_1) \cap B^+ = \emptyset$.

Towards a contradiction, suppose that there exists $v_3 \in B^+$ such that $v_3 \rightarrow v_1$. Since $v_3 \rightarrow v_1 \leftarrow v_2 \rightarrow y_2$ and D is 3-anti-circulant, it follows that $y_2 \rightarrow v_3$, a contradiction by definition of B^+. Thus $N^-(v_1) \cap B^+ = \emptyset$. This finishes the proof of Claim 1.

If $N^-(v_1) = \{v_2\}$, then since $\{v_1, v_2\} \cap S = \emptyset$, the result follows from Lemma 3.2 with $P = v_2$ and $u = v_1$. So there exists a vertex v_3 in $N^-(v_1) - v_2$. By definition of B^+, $v_3 \notin S$. By Claim 1, $v_3 \in B^\pm \cup B^-$. The rest of proof is divided into two cases depending on whether $v_3 \in B^\pm$ or $v_3 \in B^-$.

Case 1. $v_3 \in B^\pm$. Recall that $v_2 \to y_2$ with $y_2 \in S$. Since $v_2 \in B^\pm$, we may assume by Lemma 5.8 that v_2 and v_3 are non-adjacent. By definition of $B^\pm$, there exists a vertex y_3 in S such that $v_3 \to y_3$. Towards a contradiction, suppose that $y_3 = y_2$. Since $v_3 \to y_2 \leftarrow v_2 \to v_1$, we conclude that $v_1 \to v_3$, a contradiction because $B^- \cup B^\pm \Rightarrow B^+$. So $y_3 \neq y_2$. Since $v_3 \to v_1 \leftarrow v_2 \to y_2$, $y_2 \to v_3$. Also, since $v_2 \to v_1 \leftarrow v_3 \to y_3$, $y_3 \to v_2$ (see Figure 5.4).

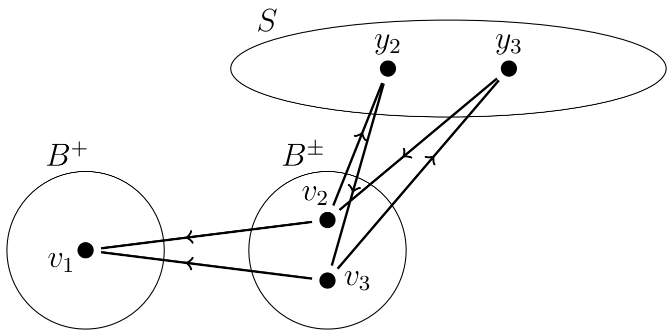


Figure 5.4: Illustration for the proof of Lemma 5.10.

Claim 2. $N^-(\{y_2, y_3\}) \cap (B^- \cup B^\pm) = \{v_2, v_3\}$.

By definition of B^-, $N^-(\{y_2, y_3\}) \cap B^- = \emptyset$. Towards a contradiction, suppose that there exists a vertex $v_4 \in B^\pm - \{v_2, v_3\}$ such that $v_4 \to y_i$ for some $i \in \{2, 3\}$. Since $v_4 \to y_i \leftarrow v_i \to v_1$, we conclude that $v_1 \to v_4$, a contradiction because $B^- \cup B^\pm \Rightarrow B^+$. So $N^-(\{y_2, y_3\}) \cap (B^- \cup B^\pm) = \{v_2, v_3\}$. This ends the proof of Claim 2.

Claim 3. $N^+(\{v_2, v_3\}) - S = \{v_1\}$.

Towards a contradiction, suppose that there exists $v_4 \in V(D) - (S \cup \{v_1\})$ such that $v_i \to v_4$ for some $i \in \{2, 3\}$. Then, $\{v_1, v_2, v_3, v_4\}$ is an anti-P_4 disjoint from S, and hence, the result follows from Lemma 5.9. So we may assume that $N^+(\{v_2, v_3\}) - S = \{v_1\}$. This finishes the proof of Claim 3.

Claim 4. If there exists a vertex v_4 in $V(D) - (S \cup \{v_1, v_2, v_3\})$ such that $v_4 \to v_i$ for some $i \in \{2, 3\}$, then $v_4 \in B^-$ and $N^+(v_4) = \{v_2, v_3\}$. Moreover, $N^-(\{v_2, v_3\}) - S = \{v_4\}$.

Without loss of generality, assume that $v_4 \to v_3$. Since $B^- \cup B^\pm \Rightarrow B^+$, $v_4 \notin B^+$. Since $\{v_2, v_3\} \subseteq B^\pm$, it follows from Lemma 5.8 that $v_4 \in B^-$ (see Figure 5.5).

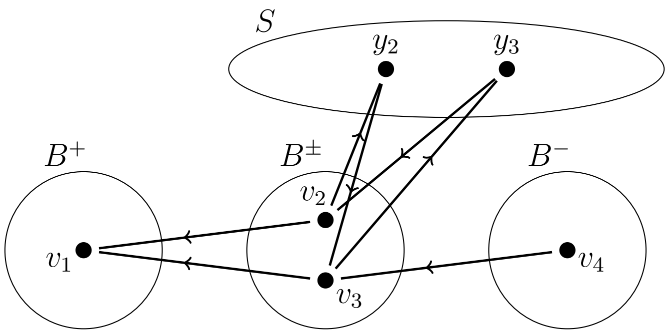


Figure 5.5: Illustration for the proof of Lemma 5.10.

By definition of B^-, $N^+(v_4) \cap S = \emptyset$. Now, we show that $N^+(v_4) \subseteq \{v_2, v_3\}$. First, suppose that $v_4 \to v_1$. Since $D \in \mathfrak{D}$, there exists at least in digon in $D[\{v_1, v_3, v_4\}]$; otherwise, $D[\{v_1, v_3, v_4\}]$ is an induced transitive triangle. Since $B^- \cup B^\pm \Rightarrow B^+$, $v_3 \leftrightarrow v_4$ which contradicts Claim 3. So $v_1 \notin N^+(v_4)$. Now, let v_5 be a vertex in $N^+(v_4) - \{v_2, v_3\}$. By definition of B^- and since $v_4 \in B^-$, it follows that $v_5 \notin S$. Since $y_2 \to v_3 \leftarrow v_4 \to v_5$, we conclude that $v_5 \to y_2$. Since $v_5 \to y_2 \leftarrow v_2 \to v_1$, we conclude that $v_1 \to v_5$. Thus since $\{v_1, v_3, v_4, v_5\} \cap S = \emptyset$ and $v_1 \to v_5 \leftarrow v_4 \to v_3$, the result follows from Lemma 5.9. So $N^+(v_4) \subseteq \{v_2, v_3\}$. If $N^+(v_4) = \{v_i\}$ for some $i \in \{2, 3\}$, then it follows from Lemma 3.1 with $P = v_i$ and $u = v_4$ that D admits an S_{BE}-path partition. Thus $N^+(v_4) = \{v_2, v_3\}$. Moreover, if $N^-(\{v_2, v_3\}) - S \supset \{v_4\}$, then D contains an anti-P_4 disjoint from S, and hence, the result follows from Lemma 5.9. Thus $N^-(\{v_2, v_3\}) - S = \{v_4\}$. This ends the proof of Claim 4.

Claim 5. If $N^-(\{v_2, v_3\}) - S \neq \emptyset$, then $N^-(v_1) = \{v_2, v_3\}$.

Let v_4 be a vertex in $N^-(\{v_2, v_3\}) - S$. it follows from Claim 4 that $N^+(v_4) = \{v_2, v_3\}$ and $N^-(\{v_2, v_3\}) - S = \{v_4\}$. Suppose that there exists a vertex v_5 in $N^-(v_1) - \{v_2, v_3\}$. By definition of B^+, $v_5 \notin S$. Since $v_5 \to v_1 \leftarrow v_2 \to y_2$, $y_2 \to v_5$. Also, since $v_4 \to v_3 \leftarrow y_2 \to v_5$, $v_5 \to v_4$. Since $\{v_1, v_3, v_4, v_5\} \cap S = \emptyset$ and $v_3 \to v_1 \leftarrow v_5 \to v_4$, it follows from Lemma 5.9 that D admits an S_{BE}-path partition. So we may assume that $N^-(v_1) = \{v_2, v_3\}$. This ends the proof of Claim 5.

The rest of proof is divided into two subcases depending on whether $N^-(\{v_2, v_3\}) - S \neq \emptyset$ or $N^-(\{v_2, v_3\}) - S = \emptyset$.

Subcase 1. $N^-(\{v_2, v_3\}) - S \neq \emptyset$. Let v_4 be a vertex in $N^-(\{v_2, v_3\}) - S$. it follows from Claim 4 that $N^+(v_4) = \{v_2, v_3\}$ and $N^-(\{v_2, v_3\}) - S = \{v_4\}$. By Claim 5, $N^-(v_1) = \{v_2, v_3\}$. Let $D' = D - \{v_2, v_3\}$. Note that v_1 is a source and v_4 is a sink in D'. Since $\{v_2, v_3\} \cap S = \emptyset$, S is a maximum stable set in D'. By hypothesis, D' is BE-perfect. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let P_1, P_2 be distinct paths of $\mathcal{P}'$ such that P_1 starts at v_1 and P_2 ends at v_4. Thus the collection $(\mathcal{P}' - \{P_1, P_2\}) \cup \{v_2 P_1, P_2 v_3\}$ is an S_{BE}-path partition of D.

Subcase 2. $N^-(\{v_2, v_3\}) - S = \emptyset$. By Claim 3, $N(\{v_2, v_3\}) - S = \{v_1\}$. Let $D' = D - v_1$. Since $v_1 \notin S$, S is a maximum stable set in D'. Let $\mathcal{P}'$ be an S_{BE}-path partition of D'. Let P_1 be a path of $\mathcal{P}'$ such that $v_2 \in V(P_1)$ and let P_2 be a path of $\mathcal{P}'$ such that $v_3 \in V(P_2)$. In D', $N(\{v_2, v_3\}) \subset S$. So it follows that both P_1 and P_2 have length one. If P_1 ends at v_2 or P_2 ends at v_3, then since $v_2 \to v_1$ and $v_3 \to v_1$, the collection $(\mathcal{P}' - P_1) \cup P_1 v_1$ or $(\mathcal{P}' - P_2) \cup P_2 v_1$ is an S_{BE}-path partition of D. Thus $P_1 = v_2 w_1$ and $P_2 = v_3 w_2$ with $w_1, w_2 \in S$. Since $\{v_2, v_3\} \to v_1$, $v_2 \to w_1$ and $v_3 \to w_2$, we conclude that $w_1 \to v_3$ and $w_2 \to v_2$. Thus the collection $(\mathcal{P}' - \{P_1, P_2\}) \cup \{w_2 v_2 v_1, w_1 v_2\}$ is an S_{BE}-path partition of D.

Case 2. $v_3 \in B^-$. By definition of B^-, $N^+(v_3) \cap S = \emptyset$. If there exists a vertex v_4 in $N^+(v_3) - \{v_1, v_2\}$, then since $\{v_1, v_2, v_3, v_4\} \cap S = \emptyset$ and $v_2 \to v_1 \leftarrow v_3 \to v_4$, the result follows from Lemma 5.9. Thus $N^+(v_3) \subseteq \{v_1, v_2\}$, and hence, since $\{v_1, v_2, v_3\} \cap S = \emptyset$, the result follows from Lemma 3.1 with $P = v_2 v_1$ and $u = v_3$. This ends the proof. $\square$

Now, we show that if $D \in \mathfrak{D}$, then we may assume that there exists no arc $v_1 v_2$ in D such that $v_1 \in B^+$ and $v_2 \in B^-$.

Lemma 5.11. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set of D. If $D \in \mathfrak{D}$ and there are adjacent vertices v_1, v_2 in $V(D)$ such that $v_1 \in B^+$ and $v_2 \in B^-$, then D admits an S_{BE}-path partition.*

Proof. We may assume by Lemma 5.6 that $B^- \cup B^{\pm} \Rightarrow B^+$. So $v_2 \mapsto v_1$. If $N^-(v_1) = \{v_2\}$, then since $\{v_1, v_2\} \cap S = \emptyset$, the result follows from Lemma 3.2 with $P = v_2$ and $u = v_1$. So there exists a vertex v_3 in $N^-(v_1) - v_2$. Since $v_1 \in B^+$, $v_3 \notin S$. Since $v_2 \in B^-$, $N^+(v_2) \cap S = \emptyset$. If there exists a vertex v_4 in $N^+(v_2) - \{v_1, v_3\}$, then since $\{v_1, v_2, v_3, v_4\} \cap S = \emptyset$ and $v_3 \to v_1 \leftarrow v_2 \to v_4$, the result follows from Lemma 5.9. So we may assume that $N^+(v_2) \subseteq \{v_1, v_3\}$. Since $\{v_1, v_2, v_3\} \cap S = \emptyset$, it follows from Lemma 3.1 with $P = v_3 v_1$ and $u = v_2$ that D admits an S_{BE}-path partition. This finishes the proof. $\square$

We show next that we may assume that $B^+ \cup B^-$ is a stable set.

Lemma 5.12. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the BE-property. Let S be a maximum stable set of D. If $D \in \mathfrak{D}$ and $B^+ \cup B^-$ is not a stable set, then D admits an S_{BE}-path partition.*

Proof. If there are adjacent vertices v_1, v_2 in $V(D)$ such that $v_1 \in B^+$ and $v_2 \in B^-$, then the result follows from Lemma 5.11. Let $v_1 v_2$ be an arc in $D[B^+ \cup B^-]$. By the principle of directional duality, we may assume that $\{v_1, v_2\} \subseteq B^+$. Towards a contradiction, suppose that $N^-(v_2) \supset \{v_1\}$. Let v_3 be a vertex in $N^-(v_2) - v_1$. By definition of B^+, $v_3 \notin S$. Moreover, we may assume by Lemmas 5.11 and 5.10 that $v_3 \in B^+$. By definition of B^+, let y be a vertex in S such that $v_1 \to y$. Since $v_3 \to v_2 \leftarrow v_1 \to y$, we conclude that $y \to v_3$, a contradiction by definition of B^+. Thus $N^-(v_2) = \{v_1\}$. Since $\{v_1, v_2\} \cap S = \emptyset$, it follows from Lemma 3.2 with $P = v_1$ and $u = v_2$ that D admits an S_{BE}-path partition. $\square$

Finally, we are ready for the main result of this subsection.

Theorem 5.1. *Let D be a 3-anti-circulant digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

Proof. Let S be a maximum stable set of D. Since every induced subdigraph of D is also a 3-anti-circulant digraph, it suffices to show that D satisfies the BE-property. Towards a contradiction, suppose the opposite and let D be a counterexample with the smallest number of vertices. Note that if D' is a proper induced subdigraph of D, then D' is a 3-anti-circulant digraph, and hence, by the minimality of D, it follows that D' satisfies the BE-property. Thus D does not satisfy the BE-property. it follows from Lemmas 5.8 and 5.12 that both $B^{\pm}$ and $B^{+} \cup B^{-}$ are stable. Thus it follows from Lemmas 5.10 and 5.11 that $B^{+} \cup B^{-} \cup B^{\pm}$ is stable. Since S is a maximum stable set of D, $|S| \geq |B^{+} \cup B^{-} \cup B^{\pm}|$. Thus we conclude by Lemma 3.14 that D satisfies the BE-property, a contradiction. This ends the proof. □

5.1.2 Berge's conjecture

In this subsection, we verify Conjecture 1.1 for 3-anti-circulant digraphs. Recall that every 3-anti-circulant digraph D belongs to $\mathfrak{B}$. This proof is divided into two cases depending on whether D contains an induced transitive triangle or not.

Initially, we prove that if D contains an induced transitive triangle, then D satisfies the α-property.

Lemma 5.13. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the α-property. If D contains an induced transitive triangle T, then D satisfies the α-property.*

Proof. Let S be a maximum stable set in D. Let $V(T) = \{v_1, v_2, v_3\}$. Without loss of generality, assume that $\{v_1, v_2\} \mapsto v_3$ and $v_1 \mapsto v_2$. First, we prove some claims.

Claim 1. $|N^{-}(v_3)| \leq 3$. Moreover, if there exists $v_4 \in N^{-}(v_3) - \{v_1, v_2\}$, then $v_4 \rightarrow v_1$ and $v_2 \rightarrow v_4$.

Towards a contradiction, suppose that there are distinct vertices v_4, v_5 in $N^{-}(v_3) - \{v_1, v_2\}$. Since $\{v_4, v_5\} \rightarrow v_3 \leftarrow v_1 \rightarrow v_2$ and D is 3-anti-circulant, it follows that $v_2 \rightarrow \{v_4, v_5\}$. Since $v_1 \rightarrow v_3 \leftarrow v_2 \rightarrow \{v_4, v_5\}$, we conclude that $\{v_4, v_5\} \rightarrow v_1$. Also, since $v_5 \rightarrow v_3 \leftarrow v_2 \rightarrow v_4$, it follows that $v_4 \rightarrow v_5$. Now, since $v_2 \rightarrow v_5 \leftarrow v_4 \rightarrow v_3$, it follows that $v_3 \rightarrow v_2$, and hence, $v_2 \leftrightarrow v_3$, a contradiction because $v_2 \mapsto v_3$. Thus $|N^{-}(v_3)| \leq 3$. Moreover, note that if there exists $v_4 \in N^{-}(v_3) - \{v_1, v_2\}$, then $v_4 \rightarrow v_1$ and $v_2 \rightarrow v_4$. This ends the proof of Claim 1.

Claim 2. $\{v_1, v_2\} \cap S \neq \emptyset$.

Suppose that $\{v_1, v_2\} \cap S = \emptyset$. First, suppose that there exists a vertex v_4 in $N^{-}(v_3) - \{v_1, v_2\}$. By Claim 1, it follows that $N^{-}(v_3) = \{v_1, v_2, v_4\}$, $v_4 \rightarrow v_1$ and $v_2 \rightarrow v_4$. Let $D' = D - \{v_1, v_2\}$. Since $\{v_1, v_2\} \cap S = \emptyset$, S is a maximum stable set in

D'. By hypothesis, D' is α-diperfect. Let $\mathcal{P}'$ be an S-path partition of D'. Let P be a path of $\mathcal{P}'$ such that $v_3 \in V(P)$. Since $N^-(v_3) = \{v_1, v_2, v_4\}$, it follows that P starts at v_3 or v_4v_3 is an arc of P. If P starts at v_3, then since $v_1 \to v_2$ and $v_2 \to v_3$, the collection $(\mathcal{P}' - P) \cup v_1v_2P$ is an S-path partition of D (note that if $N^-(v_3) = \{v_1, v_2\}$, then the result follows from previous argument). Thus v_4v_3 is an arc of P. Let w_1 and w_p be the endvertices of P. Let $P_1 = w_1Pv_4$ and $P_2 = v_3Pw_p$ be the subpaths of P. Since $v_4 \to v_1$, $v_1 \to v_2$ and $v_2 \to v_3$, the collection $(\mathcal{P}' - P) \cup P_1v_1v_2P_2$ is an S-path partition of D. So we may assume that $\{v_1, v_2\} \cap S \neq \emptyset$. This finishes the proof of Claim 2.

Claim 3. $\{v_2, v_3\} \cap S \neq \emptyset$.

By the principle of directional duality, the result follows from Claim 2. This ends the proof of Claim 3.

By Claim 2 and 3, $v_2 \in S$. First, suppose that there exists a vertex v_4 in $N^-(v_3) - \{v_1, v_2\}$. By Claim 1, it follows that $N^-(v_3) = \{v_1, v_2, v_4\}$, $v_4 \to v_1$ and $v_2 \to v_4$. Let $P = v_2v_4v_1$ and $u = v_3$. Since $(V(P) - v_2) \cap S = \emptyset$, $v_1 \to u$ and $N^-(u) \subseteq V(P)$, it follows from Lemma 3.7 that D admits an S-path partition. So we may assume that $N^-(v_3) = \{v_1, v_2\}$.

Now, suppose that $N^+(v_2) = \{v_3\}$. Since $v_3 \notin S$, the result follows from Lemma 3.6(ii). So we may assume that there exists a vertex w in $N^+(v_2) - \{v_1, v_3\}$. Since $v_1 \to v_3 \leftarrow v_2 \to w$, we conclude that $w \to v_1$. Let $P = v_2wv_1$ and let $u = v_3$. Since $(V(P) - v_2) \cap S = \emptyset$, $v_1 \to u$ and $N^-(u) \subset V(P)$, the result follows from Lemma 3.7. This finishes the proof. □

We show next that if D contains no induced transitive triangle, then D satisfies the α-property.

Lemma 5.14. *Let D be a 3-anti-circulant digraph such that every proper induced subdigraph of D satisfies the α-property. If D contains no induced transitive triangle, then D satisfies the α-property.*

Proof. Since every blocking odd cycle of length at least five contains an induced anti-P_4 and D is 3-anti-circulant, it follows that D contains no blocking odd cycle of length at least five. Moreover, D contains no induced transitive triangle, and this implies that D belongs to $\mathfrak{D}$. So by Theorem 5.1 D satisfies the BE-property, and hence, the α-property. □

Now, we prove the main result of this subsection.

Theorem 5.2. *Let D be a 3-anti-circulant digraph. Then, D is α-diperfect.*

Proof. Since every induced subdigraph of D is also a 3-anti-circulant digraph, it suffices to show that D satisfies the α-property. If D contains an induced transitive triangle, then the result follows from Lemma 5.13. Thus D contains no induced transitive triangle, and hence, the result follows from Lemma 5.14. This ends the proof. □

5.2 3-anti-digon-circulant digraphs

In this section we present some structural results for 3-anti-digon-circulant digraphs that we obtained in our research. Let D be a digraph. We say that D is 3-*anti-digon-circulant* if for every anti-P_4 $v_1 \to v_2 \leftarrow v_3 \to v_4$, we have $v_1 \leftrightarrow v_4$ (see Figure 5.6). We show that the structure of these digraphs is very close to complete and complete bipartite digraphs. Moreover, note that the inverse of a 3-anti-digon-circulant digraph is also a 3-anti-digon-circulant digraph. So we can use the principle of directional duality whenever it is convenient.

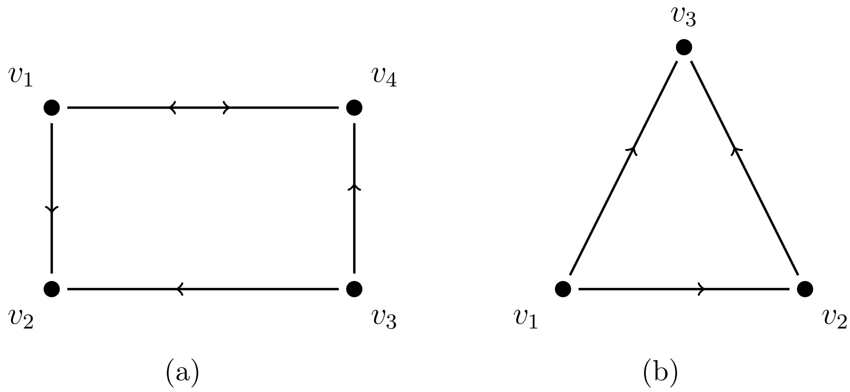


Figure 5.6: Examples of 3-anti-digon-circulant digraphs.

Let us start with a nice structural result.

Lemma 5.15. *Let D be a connected 3-anti-digon-circulant digraph and H be a strong subdigraph of D. If for every $u \in V(H)$, $d^-_H(u) \geq 2$ and $d^+_H(u) \geq 2$, then for every $v \in V(D) - V(H)$ there exists a vertex w in H such that $w \leftrightarrow v$.*

Proof. Let v be a vertex of $V(D) - V(H)$. Since D is connected, there exists a path between v and some vertex of H in $U(D)$. Let $P = u_1 u_2 \dots u_k v$ be a shortest path between a vertex u_1 in $V(H)$ and v in $U(D)$. Towards a contradiction, suppose that $k > 1$. By the principle of directional duality, we may assume that P starts at u_1 and ends at v in D. Since $d^+_H(u_1) \geq 2$, there exists x in H such $u_1 \to x$. Since $d^-_H(x) \geq 2$, there exists a vertex w in $H - u_1$ such $w \to x$. Since $w \to x \leftarrow u_1 \to u_2$ and D is 3-anti-digon-circulant, it follows that $w \leftrightarrow u_2$. Note that if $u_2 = v$, then we have a contradiction by minimality of P. So $k > 2$. Since $d^-_H(x) \geq 2$, there exists a vertex w_1 in H such that $w_1 \to w$. Since $w_1 \to w \leftarrow u_2 \to u_3$, it follows that $w_1 \leftrightarrow u_3$, which contradicts the minimality of P. Thus $k = 1$, and hence, v is adjacent to H. Using the same argument with v in the role of u_2, we conclude that $w \leftrightarrow v$ for some $w \in V(H)$. □

In the next lemmas, we show that the structure of D is very close to complete and complete bipartite digraph.

Lemma 5.16. *Let D be a connected 3-anti-digon-circulant digraph. If D contains an anti-P_4, then D is a complete digraph or a complete bipartite digraph.*

Proof. Let $\{v_1, v_2, v_3, v_4\} \subseteq V(D)$ be an anti-P_4 in D such that $v_1 \to v_2 \leftarrow v_3 \to v_4$. Since D is 3-anti-digon-circulant digraph, $v_1 \leftrightarrow v_4$. Since $v_1 \to v_4 \leftarrow v_3 \to v_2$, $v_1 \leftrightarrow v_2$. Similarly to v_2, v_3 and v_3, v_4, we have $v_2 \leftrightarrow v_3$ and $v_3 \leftrightarrow v_4$. Thus $v_i \leftrightarrow v_{i+1}$ for every $i \in \{1, 2, 3, 4\}$ where subscripts are taken modulo 4. Let $H = D[\{v_1, v_2, v_3, v_4\}]$. Moreover, from now on, let $i \in \{1, 2, 3, 4\}$ where subscripts are taken modulo 4.

Claim 1. For every $u \in V(D) - V(H)$, if u is adjacent to v_i, then $u \leftrightarrow v_i$ and $u \leftrightarrow v_{i+2}$.

By the principle of directional duality, we may assume that $u \to v_i$. Since $u \to v_i \leftarrow v_{i+1} \to v_{i+2}$, it follows that $u \leftrightarrow v_{i+2}$. Moreover, since $u \to v_{i+2} \leftarrow v_{i+1} \to v_i$, we conclude that $u \leftrightarrow v_i$. This ends the proof of Claim 1.

Claim 2. If there exists a vertex u in $V(D) - V(H)$ such that u is adjacent to both v_i and v_{i+1}, then D is a complete digraph.

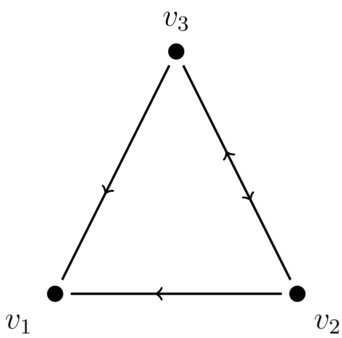
Without loss of generality, assume that $i = 1$. Since u is adjacent to both v_1 and v_2, it follows from Claim 1 that $u \leftrightarrow v_1$, $u \leftrightarrow v_2$, $u \leftrightarrow v_3$ and $u \leftrightarrow v_4$. Since $v_4 \to v_1 \leftarrow u \to v_2$, it follows that $v_2 \leftrightarrow v_4$. Since $v_3 \to v_4 \leftarrow u \to v_1$, $v_1 \leftrightarrow v_3$. So H is a complete digraph. If $V(D) = V(H)$, then the result follows. So $V(H) \subset V(D)$. Let x be a vertex in $V(D) - V(H)$. Note that $d^-_H(v) \geq 2$ and $d^+_H(v) \geq 2$ for every v in H. So it follows from Lemma 5.15 that x is adjacent to some vertex in H. By Claim 1, there are two vertices in H, say v_1 and v_3, such that $x \leftrightarrow v_1$ and $x \leftrightarrow v_3$. Since $x \to v_1 \leftarrow v_4 \to v_2$, we conclude that $x \leftrightarrow v_2$. By Claim 1, $x \leftrightarrow v_4$. Since x is arbitrary, it follows that for every vertex $x \in V(D) - V(H)$, we have $x \leftrightarrow v_1$, $x \leftrightarrow v_2$, $x \leftrightarrow v_3$, $x \leftrightarrow v_4$. Now, let x_1, x_2 be distinct vertices in $V(D) - V(H)$. Since $x_1 \to v_1 \leftarrow v_2 \to x_2$, we conclude that $x_1 \leftrightarrow x_2$. Thus it follows that D is a complete digraph. This finishes the proof of Claim 2.

Claim 3. If there are vertices v_i, v_{i+2} in H such that v_i and v_{i+2} are adjacent, then D is a complete digraph.

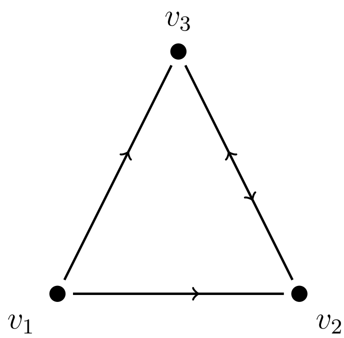
Without loss of generality, assume that $v_1 \to v_3$. Since $v_2 \to v_3 \leftarrow v_1 \to v_4$, it follows that $v_2 \leftrightarrow v_4$. Also, since $v_1 \to v_4 \leftarrow v_2 \to v_3$, we conclude that $v_1 \leftrightarrow v_3$. So H is a complete digraph. If $V(D) = V(H)$, then the result follows. So let x be a vertex in $V(D) - V(H)$. By Lemma 5.15, x is adjacent to H. By Claim 1, there are distinct vertices in H, say v_1 and v_3, such that $x \leftrightarrow v_1$ and $x \leftrightarrow v_3$. Since $x \to v_1 \leftarrow v_4 \to v_2$, we conclude that $x \leftrightarrow v_2$. Thus it follows from Claim 2 that D is complete. This finishes the proof of Claim 3.

If there are adjacent vertices v_i and v_{i+2} in H, then it follows from Claim 3 that D is a complete digraph. So we may assume that v_1 and v_3 (resp., v_2 and v_4) are non-adjacent in D. Let X be the subset of vertices in $V(D) - V(H)$ which are adjacent to $\{v_1, v_3\}$ and let Y be the subset of vertices in $V(D) - V(H)$ which are adjacent to $\{v_2, v_4\}$. By Lemma 5.15, $X \cup Y = V(D) - V(H)$. We show next that $(X \cup \{v_2, v_4\}, Y \cup \{v_1, v_3\})$ is a bipartition of D. By Claim 1, it follows that $x \leftrightarrow v_1$, $x \leftrightarrow v_3$, $y \leftrightarrow v_2$ and $y \leftrightarrow v_4$ for every $x \in X$ and for every $y \in Y$. If $X \cap Y \neq \emptyset$, then D is complete by Claim 2, a contradiction because because v_1 and v_3 are non-adjacent in D. So $X \cap Y = \emptyset$. Now, we show that X is stable. Towards a contradiction, suppose that there are vertices x_1, x_2 in X such that $x_1 \to x_2$. Since $v_3 \to x_2 \leftarrow x_1 \to v_1$, it follows that $v_1 \leftrightarrow v_3$, a contradiction. So X is stable, and analogously, Y is stable. Finally, we show next that for every $x \in X$ and for every $y \in Y$, $x \leftrightarrow y$. Let x be a vertex of X and let y be a vertex of Y. Since $x \to v_1 \leftarrow v_2 \to y$, it follows that $x \leftrightarrow y$. Thus D is a complete bipartite digraph with bipartition $(X \cup \{v_2, v_4\}, Y \cup \{v_1, v_3\})$. This ends the proof. □

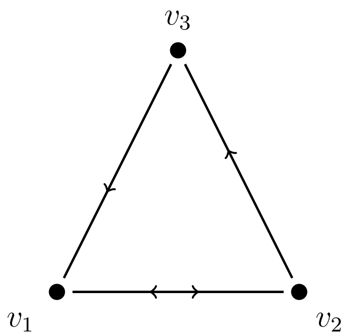
For the next lemmas, we need the following definitions. Let D be a digraph. First, suppose that D has order three. Let $V(D) = \{v_1, v_2, v_3\}$. We say that D is a TT^* digraph if $v_2 \mapsto v_1$, $v_3 \mapsto v_1$ and $v_2 \leftrightarrow v_3$ or if $v_1 \mapsto v_2$, $v_1 \mapsto v_3$ and $v_2 \leftrightarrow v_3$ (see Figure 5.7a and 5.7b). We say that D is a C_3^* digraph if $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \to v_1$ and there exists at least one digon in D (see Figure 5.7c). Now, suppose that D has order four. Let $V(D) = \{v_1, v_2, v_3, v_4\}$. We say that D is a F_4 digraph if $v_1 \to v_2$, $v_2 \to v_3$, $v_3 \to v_4$, $v_4 \to v_1$ and $v_1 \leftrightarrow v_4$ (see Figure 5.7d).



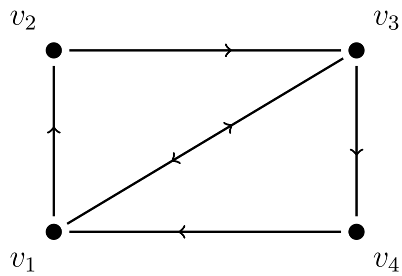
(a) TT^*



(b) TT^*



(c) C_3^*



(d) F_4

Figure 5.7: illustrations of TT^*, C_3^* and F_4 digraphs.

Lemma 5.17. *Let D be a connected 3-anti-digon-circulant digraph. If D contains a TT^*, then one of the following holds:*

(i) D is a complete digraph,

(ii) D is a complete bipartite digraph,

(iii) D is a $\overrightarrow{C}_3$ with at least two digons,

(iv) D *is a* TT^**, or*

(v) $D[\{v_1, v_2\}]$ *is an initial or a final strong component of* D *where* $\{v_1, v_2\}$ *is a digon in* TT^**.*

Proof. Let H be a TT^* in D. Let $V(H) = \{v_1, v_2, v_3\}$. Without loss of generality, suppose that $\{v_1, v_2\} \to v_3$ and $v_1 \leftrightarrow v_2$. If $V(D) = V(H)$, then D is a TT^* or a $\overrightarrow{C}_3$ with at least two digons. So we may assume that $V(H) \subset V(D)$. Since D is connected, there exists a vertex u in $V(D) - V(H)$ such that u is adjacent to some vertex in H. The rest of proof is divided into two cases depending on whether u dominates some vertex in H or some vertex in H dominates u.

Case 1. Suppose that u dominates a vertex v_i in H for some $i \in \{1, 2, 3\}$. If $u \to v_1$ (resp., $u \to v_3$), then since $u \to v_1 \leftarrow v_2 \to v_3$ (resp., $u \to v_1 \leftarrow v_2 \to v_3$), it follows that D contains an anti-P_4. Moreover, if $u \to v_2$, then since $u \to v_2 \leftarrow v_1 \to v_3$, we conclude that D contains also an anti-P_4. Thus it follows from Lemma 5.16 that D is a complete digraph or a complete bipartite digraph.

Case 2. Suppose that a vertex v_i in H dominates u for some $i \in \{1, 2, 3\}$. First, suppose that $N^-(\{v_1, v_2\}) \neq \emptyset$. So let w be a vertex in $N^-(\{v_1, v_2\})$. Suppose that $w \neq v_3$. Since $v_1 \leftrightarrow v_2$ and $\{v_1, v_2\} \to v_3$, it follows that D contains an anti-P_4, and hence, it follows from Lemma 5.16 that D is a complete digraph or a complete bipartite digraph. So we may assume that $w = v_3$. Since v_i in H dominates u for some $i \in \{1, 2, 3\}$ and there are two digons in H, it easy to see that D contains an anti-P_4, and hence, the result follows from Lemma 5.16. Thus $N^-(\{v_1, v_2\}) = \emptyset$, and this implies that $D[\{v_1, v_2\}]$ is an initial strong component of D. □

Now, we prove the main result of this section.

Lemma 5.18. *Let* D *be a connected* 3*-anti-digon-circulant digraph. If* D *contains a digon* $\{v_1, v_2\}$*, then one of the following holds:*

(i) D *is a complete digraph,*

(ii) D *is a complete bipartite digraph,*

(iii) D *is an* F_4*,*

(iv) D *is a* C_3^**,*

(v) D *is a* TT^**,*

(vi) $D[\{v_1, v_2\}]$ *is an initial or a final strong component of* D*, or*

(vii) D *has a clique cut.*

Proof. If $V(D) = \{v_1, v_2\}$, then D is complete, and the result follows. So we may assume that $|V(D)| \geq 3$. If $N(v_1) = \{v_2\}$ (resp., $N(v_2) = \{v_1\}$), then v_1 (resp. v_2) is a clique cut. Thus $|N(v_1)| \geq 2$ and $|N(v_2)| \geq 2$. If D contains an anti-P_4 as a subdigraph, then it follows from Lemma 5.16 that D is complete or complete bipartite. So we may assume D contains no anti-P_4 as a subdigraph.

Claim 1. If there exists a vertex v_3 in $V(D) - \{v_1, v_2\}$ such that $v_1 \leftrightarrow v_3$ or $v_2 \leftrightarrow v_3$, then D is complete, complete bipartite, a C_3^* or D has a clique cut.

With loss of generality, assume that $v_1 \leftrightarrow v_3$. Since $|N(v_2)| \geq 2$, there exists a vertex w in $N(v_2) - \{v_1\}$. Towards a contradiction, suppose that $v_3 \neq w$. Since $v_3 \leftrightarrow v_1$ and $v_1 \leftrightarrow v_2$, it easy to see that D contains an anti-P_4, a contradiction. Thus $v_3 = w$. If $V(D) = \{v_1, v_2, v_3\}$, then D is a C_3^* and the result follows. So $|V(D)| \geq 4$. Towards a contradiction, suppose that there exists a vertex w in $V(D) - \{v_1, v_2, v_3\}$ such that w and v_2 are adjacent. Since $v_1 \leftrightarrow v_2$ and $v_1 \leftrightarrow v_3$, it follows that D contains an anti-P_4, a contradiction. So $N(v_2) \subseteq \{v_1, v_3\}$. Since $|V(D)| \geq 4$ and $N(v_2) \subseteq \{v_1, v_3\}$, it follows that $\{v_1, v_3\}$ is a clique cut. This ends the proof of Claim 1.

Claim 2. If there are distinct vertices w_1, w_2 such that $w_1 \to v$ and $v \to w_2$, then D is complete, complete bipartite, a C_3^*, F_4 or D has a clique cut.

If $w_1 \leftrightarrow v$ or $v \leftrightarrow w_2$, then it follows from Claim 1 that D is complete, complete bipartite, a C_3^* or D has a clique cut. So we may assume that $w_1 \mapsto v$ and $v \mapsto w_2$. Since $|N(u)| \geq 2$, there exists a vertex x in $V(D) - v$ such that x is adjacent to u. If $x \notin \{w_1, w_2\}$, then D has an anti-P_4 because $w_1 \to v$, $v \to w_2$ and $u \leftrightarrow v$, a contradiction. So we may assume that $N(u) \subseteq \{v, w_1, w_2\}$. Note that this implies that $N(v) = \{w_1, u, w_2\}$.

First, suppose that $N(u) = \{w_1, v, w_2\}$. If $u \to w_2$, then since $w_1 \to v$ and $u \to v$, we have that D has an anti-P_4, a contradiction. So $w_2 \to u$. Similarly to u and w_1, we have $u \to w_1$. Towards a contradiction, assume that w_1 and w_2 are adjacent. If $w_2 \to w_1$, then since $w_2 \to u$ and $v \to u$, it follows that D has an anti-P_4, a contradiction. If $w_1 \to w_2$, then since $v \to w_1$ and $v \to u$, it follows that D has an anti-P_4, a contradiction. So we may assume that w_1 and w_2 are non-adjacent. Let $H = D[\{w_1, v, w_2, u\}]$. Note that H is a F_4. Towards a contradiction, assume that $V(H) \subset V(D)$. Since D is connected, let x be a vertex in $V(D) - V(H)$ such that x is adjacent to H. Since $N(u) = \{w_1, v, w_2\}$ and $N(v) = \{w_1, u, w_2\}$, it follows that x is adjacent to $\{w_1, w_2\}$. Since $u \leftrightarrow v$, $w_1 \to v$ and $v \to w_2$, it easy to see that D has an anti-P_4, a contradiction. So $V(D) = V(H)$, and hence, D is a F_4.

Now, suppose that $N(u) \subset \{w_1, v, w_2\}$. Since $|N(u)| \geq 2$, there exists a vertex x in $V(D) - v$ such x and u are adjacent. If $x \notin \{w_1, w_2\}$, then D has an anti-P_4 because $w_1 \to v$, $u \leftrightarrow v$ and $v \to w_2$. So $x \in \{w_1, w_2\}$. If u is adjacent to w_1, then $\{v, w_2\}$ is a clique cut; otherwise, if u is adjacent to w_2, then $\{w_1, v\}$ is a clique cut. This ends the proof of Claim 2.

Claim 3. If there exists a vertex w in $V(D) - \{u, v\}$ such that $u \to w$ and $w \to v$, then D is complete, complete bipartite, a C_3^* or D has a clique cut.

Towards a contradiction, assume that there exists a vertex x in $V(D) - \{u, v\}$ such that x and w are adjacent. By the principle of directional duality, we may assume that $x \to w$. Since $u \to w$ and $u \to v$, we have that D has an anti-P_4, a contradiction. So $N(w) = \{u, v\}$. If $V(D) \neq \{u, v, w\}$, then $\{u, v\}$ is a clique cut. Thus $V(D) = \{u, v, w\}$, and hence, D is a C_3^*. This finishes the proof of Claim 3.

By Claim 1, we may assume that there exists no vertex w in $V(D) - \{u, v\}$ such that $w_1 \leftrightarrow v$ or $w_1 \leftrightarrow u$. Since $|N(u)| \geq 2$, by the principle of directional duality, assume that there exists w in $N^+(u) - v$. By Claim 2, we may assume that $N^-(u) = \{v\}$. If $v \to w$, then $D[\{u, v, w\}]$ is a TT^*, and hence, it follows from Lemma 5.17 that D is a complete digraph, a complete bipartite digraph or a TT^*. If $w \to v$, then it follows from Claim 3 that D is complete, complete bipartite, a C_3^* or D has a clique cut. So w and v are non-adjacent. Since $|N(v)| \geq 2$, let x be a vertex in $N(v) - u$. If $x \to v$, then D has an anti-P_4 because $u \to v$ and $u \to w$. So $v \to x$. By Claim 2, we may assume that $N^-(v) = \{u\}$. Thus $N^-(\{u, v\}) = \emptyset$, and hence, $D[\{u, v\}]$ is an initial strong component of D. This finishes the proof. □

Note that with the results presented in this section, in order to obtain a complete characterization of a 3-anti-digon-circulant digraph D, it suffices to analyze the case where D contains no digon nor anti-P_4 as a subdigraph. We leave this as an open problem.

Furthermore, an interesting and natural followup would be analyzing digraphs which for every anti-P_4 $v_1 \to v_2 \leftarrow v_3 \to v_4$, it follows that v_1 and v_4 are adjacent. We believe this could be a challenging problem.

Finally, in context of this chapter, the most interesting problem would be characterizing digraphs which do not contain an anti-P_4 as an induced subdigraph. Such result would potentially give us some insight on how to deal with arbitrary digraphs in $\mathfrak{D}$. However, this problem is probably very difficult, since we do not even know a complete characterization of 3-anti-circulant digraphs, a special subclass of those digraphs.

Chapter 6

Other results

In this chapter we present some results for both Conjecture 1.1 and Conjecture 1.2 for several classes of digraphs. These results were obtained during our research but they do not fit into any of the other chapters. Let D be a digraph. We say that D is *k-quasi-transitive* (resp., k-transitive) if for every path uPv of length k, u and v are adjacent (resp., u dominates v) in D.

This chapter is organized as follows. In Section 6.1, we verify both conjectures for quasi-transitive digraphs. In Section 6.2, we present some results for 3-quasi-transitive digraphs. In Section 6.3, we verify both conjectures for strong 4-transitive digraphs and we provide some results for non-strong 4-transitive digraphs. In Section 6.4, we show that if a digraph D is a counterexample for Conjecture 1.1 that minimizes $|V(D)| + |A(D)|$, then D is not a 4-semi-symmetric digraph. We also show that if D is a counterexample for Conjecture 1.2 that minimizes $|V(D)|+|A(D)|$, then D is not a 3-semi-symmetric digraph. In Section 6.5, we provide some results for digraphs with stability number two. Finally, in Section 6.6, we show that if a digraph D belongs to $\mathfrak{D}$ and $U(D)$ is a complement of an odd cycle with length at least seven, then D is BE-diperfect.

6.1 Quasi-transitive digraphs

In this section, we prove that both Conjectures 1.1 and 1.2 are true for quasi-transitive digraphs. We say that a digraph D is *quasi-transitive* if D is 2-quasi-transitive. We also say that D is *transitive* if D is 2-transitive. In order to verify both conjectures for this class, we show that a quasi-transitive digraph D is diperfect, and hence, it follows from Lemmas 1.1 and 1.2 that D is α-diperfect and BE-diperfect, respectively.

It is well-known that transitive digraphs are diperfect. Moreover, Ghouila-Houri [23] proved the following theorem.

Theorem 6.1 (Ghouila-Houri, 1962)**.** *A graph G has a quasi-transitive orientation if and only if G has a transitive orientation.*

Thus it follows from Theorem 6.1 that every quasi-transitive digraph is also diperfect. Here, we present an alternative proof of this fact that relies on Strong Perfect Graph Theorem (Theorem 1.1). In order to do this, we need the next auxiliary result.

Lemma 6.1. *If D is a quasi-transitive digraph, then $U(D)$ contains no induced odd cycle of length at least five.*

Proof. Towards a contradiction, suppose that $U(D)$ contains an induced odd cycle C of length at least five. In D, let $P = u_1u_2 \dots u_k$ be a maximum path in C. Note that P has at least three vertices because C is odd and has at least five vertices. Since D is quasi-transitive, it follows that u_1 and u_3 must be adjacent in D, a contradiction. □

Theorem 6.2. *Let D be a digraph. If D is quasi-transitive, then D is diperfect.*

Proof. Towards a contradiction, suppose that D is not diperfect. By Lemma 6.2, $U(D)$ contains no induced odd cycle of length at least five. Thus it follows from Theorem 1.1 that $U(D)$ contains an induced complement, denoted by $\overline{U(C)}$, of an odd cycle $U(C)$ of length at least five. By definition of complement, two vertices are adjacent in $\overline{U(C)}$ (and in D) if and only if they are not consecutive in $U(C)$. Since the complement of a C_5 is also a C_5, we may assume that C contains at least seven vertices. In D, let $C = v_1v_2 \dots v_{2k+1}v_1$ where $k > 2$. By the principle of directional duality, we may assume that $v_1 \rightarrow v_6$.

Now, towards a contradiction, suppose that $v_6 \rightarrow v_2$. Since D is quasi-transitive, it follows that v_1 and v_2 are adjacent in D, a contradiction because $\overline{U(C)}$ is induced. So $v_2 \rightarrow v_6$. Similarly, we have $v_3 \rightarrow v_6$ and $v_4 \rightarrow v_6$. Again, towards a contradiction, suppose that $v_5 \rightarrow v_1$. Since $v_1 \rightarrow v_6$ and D is quasi-transitive, we conclude that v_5 and v_6 are adjacent in D, a contradiction. Thus $v_1 \rightarrow v_5$. Similarly, $v_1 \rightarrow v_4$. Now, if $v_7 \rightarrow v_4$, then since $v_4 \rightarrow v_6$, it follows that v_7 and v_6 are adjacent, a contradiction. So $v_4 \rightarrow v_7$. Similarly, $v_4 \rightarrow \{v_8, v_9, ..., v_{2k+1}\}$. Therefore, since $v_1 \rightarrow v_4$, $v_4 \rightarrow v_{2k+1}$ and D is quasi-transitive, we conclude that v_1 and v_{2k+1} are adjacent in D, a contradiction. This finishes the proof. □

With the results obtained for this class, we decided to study a natural generalization (3-quasi-transitive digraphs), which we present in the next section.

6.2 3-quasi-transitive digraphs

In this section, we present some results for 3-quasi-transitive digraphs. In [3], Bang-Jensen introduced the class of 3-quasi-transitive digraphs. Let D be a digraph. We say that D is 3-quasi-transitive if for every path of length three $P = v_1v_2v_3v_4$ in D, its endvertices v_1 and v_4 are adjacent. Here, we show that every strong 3-quasi-transitive digraph satisfies the BE-property (resp., α-property). We also present some results for non-strong 3-quasi-transitive digraphs. In particular, we show that if a non-strong 3-quasi-transitive digraph D contains a strong component Q such that $D[V(Q)]$ is non-bipartite, then D satisfies the BE-property (resp., α-property).

Since it was introduced by Bang-Jensen, several results for this class of digraphs have been presented in the literature [1, 13, 17, 18, 27, 35, 36]. In [17], Galeana-Sánchez, Goldfeder and Urrutia provided a characterization for strong 3-quasi-transitive digraphs. In [35], Wang and Wang proved the conjecture of Laborde, Payan and Xuong for arbitrary 3-quasi-transitive digraphs. This conjecture states that every digraph has a stable set intersecting every non-augmentable path (we do not define this concept here because we will not need it). In [1], Arroyo and Galeana-Sánchez proved the *path partition conjecture* for strong 3-quasi-transitive digraphs. This conjecture states that for every digraph D, and every choice of positive integers λ_1, λ_2 such that $\lambda_1 + \lambda_2$ is equals to the order of a longest path in D, there exists a partition of D in two vertex-disjoint subdigraphs D_1, D_2 such that the order of the longest path in D_i is at most λ_i for $i \in \{1, 2\}$.

First, let us define a class of digraphs which is related to 3-quasi-transitive digraphs. Let $n \geq 4$ be an integer. Let H_n be a digraph such that $V(H_n) = \{x_1, x_2, x_3 \ldots, x_n\}$ and $A(H_n) = \{x_1x_2, x_2x_3, x_3x_1\} \cup \{x_1x_i, x_ix_2 : \text{for all } i \in \{4, \ldots, n\}\}$ (see Figure 6.1).

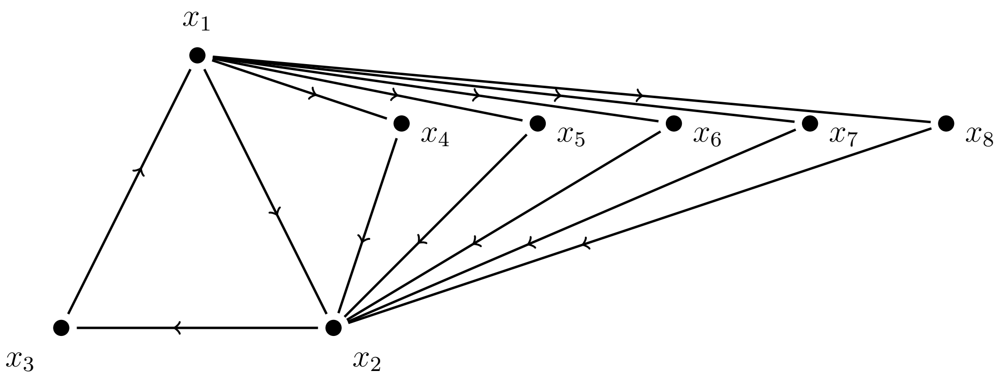


Figure 6.1: Illustration of a H_n with $n = 8$.

For ease of reference, we state the following result.

Theorem 6.3 (Galeana-Sánchez, Goldfeder and Urrutia, 2010)**.** *Let D be a strong 3-quasi-transitive digraph. Then D is either a semicomplete digraph, a semicomplete bipartite digraph or a H_n for some $n \geq 4$.*

Next we show that if a 3-quasi-transitive digraph D is strong, then D satisfies the BE-property (resp., α-property). Recall that we may assume that D is connected and D has no clique cut (see Lemmas 1.3 and 1.4).

Lemma 6.2. *Let D be a 3-quasi-transitive digraph. If D is strong and $D \in \mathfrak{D}$ (resp., $D \in \mathfrak{B}$), then D satisfies the BE-property (resp., α-property).*

Proof. By Theorem 6.3, D is either a semicomplete digraph, a semicomplete bipartite digraph or a H_n for some $n \geq 4$. If D is semicomplete or semicomplete bipartite, then D is diperfect, and hence, it follows from Lemma 1.2 (resp., Lemma 1.1) that D satisfies the BE-property (resp., α-property). So we may assume that D is a H_n for some $n \geq 4$. Let x_1, x_2, x_3 be vertices in D such that x_1x_2, x_2x_3, x_3x_1 are arcs in D. Since $n > 3$, it follows from definition of H_n that $\{x_1, x_2\}$ is a clique cut, a contradiction. □

Next we prove some results for both conjectures for a non-strong 3-quasi-transitive digraph D. In order to do this, we need the following two auxiliary results [9, 35].

Lemma 6.3 (Wang and Wang, 2011)**.** *Let D be a non-strong 3-quasi-transitive digraph. Let Q be a non-trivial strong component of D. Let $v \in V(D) - V(Q)$. Then,* *(i) if $D[V(Q)]$ is bipartite with bipartition (X, Y) and v dominates (resp., is dominated by) some vertex in X, then $v \mapsto X$ (resp., $X \mapsto v$),*

(ii) if $D[V(Q)]$ is non-bipartite and v dominates (resp., is dominated by) some vertex in Q, then $v \mapsto Q$ (resp., $Q \mapsto v$).

Lemma 6.4 (Camion, 1959)**.** *Let D be a semicomplete digraph. If D is strong, then D is hamiltonian.*

The next two lemmas state that if D contains a strong component Q such that $D[V(Q)]$ is non-bipartite, then D satisfies the BE-property and α-property.

Lemma 6.5. *Let D be a non-strong 3-quasi-transitive digraph such that every proper induced subdigraph of D satisfies the BE-property. If $D \in \mathfrak{D}$ and D contains a strong component Q such that $D[V(Q)]$ is non-bipartite, then D satisfies the BE-property.*

Proof. Let V_1 (resp., V_2) be a subset of $V(D) - V(Q)$ consisting of those vertices that dominate (resp., are dominated by) some vertex in Q. Since Q is a strong component of D, it follows that $V_1 \cap V_2 = \emptyset$. Since $D[V(Q)]$ is non-bipartite, it follows from Lemma 6.3(ii) that $V_1 \mapsto Q$ and $Q \mapsto V_2$. Let S be a maximum stable set of D. By Theorem 6.2, $D[V(Q)]$ is either a semicomplete digraph or H_n for some $n \geq 4$. By definition of H_n, every H_n contains at least one transitive triangle. Since $D \in \mathfrak{D}$, it follows that $D[V(Q)]$ is a semicomplete digraph. By Lemma 6.4, $D[V(Q)]$ is hamiltonian. Since $D[V(Q)]$ is semicomplete, $|S \cap V(Q)| \leq 1$. Since $V_1 \mapsto Q \mapsto V_2$, $D[V(Q)]$ is hamiltonian and $|S \cap V(Q)| \leq 1$, it follows from Lemma 3.5 that D admits an S_{BE}-path partition. □

Next, we show a version of Lemma 6.5 for α-diperfect digraphs.

Lemma 6.6. *Let D be a non-strong 3-quasi-transitive digraph such that every proper induced subdigraph of D satisfies the α-property. If $D \in \mathfrak{B}$ and D contains a strong component Q such that $D[V(Q)]$ is non-bipartite, then D satisfies the α-property.*

Proof. Similarly to Lemma 6.5, let V_1 (resp., V_2) be a subset of $V(D) - V(Q)$ consisting of those vertices that dominate (resp., are dominated by) some vertex in Q. Since Q is a strong component of D, it follows that $V_1 \cap V_2 = \emptyset$. Since $D[V(Q)]$ is non-bipartite, it follows from Lemma 6.3(ii) that $V_1 \mapsto Q$ and $Q \mapsto V_2$. Let S be a maximum stable set of D. By Theorem 6.2, $D[V(Q)]$ is either a semicomplete digraph or H_n for some $n \geq 4$. We omit the proof if $D[V(Q)]$ is a semicomplete digraph because it is analogous to the proof of Lemma 6.5. So we may assume that $D[V(Q)]$ is a H_n for some $n \geq 4$. Let $V(H_n) = \{x_1, x_2, x_3 \ldots, x_n\}$ and let $E(H_n) = \{x_1x_2, x_2x_3, x_3x_1\} \cup \{x_1x_i, x_ix_2 : \text{for all } i \in \{4, \ldots, n\}\}$. Let $C = x_1x_2x_3x_1$ be a cycle. Note that at least two vertices in C do not belong to S. So let x_i, x_j be vertices in C such that $\{x_i, x_j\} \cap S = \emptyset$ for some $i, j \in \{1, 2, 3\}$. Without loss of generality, assume that $x_i \to x_j$.

Let $D' = D - \{x_i, x_j\}$. Since $\{x_i, x_j\} \cap S = \emptyset$, S is maximum in D'. By hypothesis, D' is α-diperfect. Let $\mathcal{P}'$ be an S-path partition of D'. Let x_l be the vertex in $V(C) - \{x_i, x_j\}$. Since $x_i \to x_j$, it follows that $x_l \to x_i$ and $x_j \to x_l$. Let P be a path of $\mathcal{P}'$ such that $x_l \in V(P)$. First, suppose that P starts at x_l. Since $\{x_i, x_j\} \cap S = \emptyset$, $x_i \to x_j$ and

$x_j \to x_l$, it follows that the collection $(\mathcal{P}' - P) \cup x_i x_j P$ is an S-path partition of D. So we may assume that P does not start at x_l. Let w be the vertex in P such that $w \to x_l$. Let $P_1 = Pw$ and $P_2 = x_l P$ be the subpaths of P. Since $Q \mapsto V_2$ and $x_l \in V(Q)$, it follows that $w \in V_1 \cup V(Q)$. Suppose that $w \in V_1$. So $w \mapsto Q$. Since $w \to x_i$, $x_i \to x_j$ and $x_j \to x_l$, the collection $(\mathcal{P}' - P) \cup P_1 x_i x_j P_2$ is an S-path partition of D. So we may assume that $w \in Q$. If P ends at x_l, then the collection $(\mathcal{P}' - P) \cup P x_i x_j$ is an S-path partition of D. So let w_2 be the vertex in P such that $x_l \to w_2$. Since $V_1 \mapsto Q$ and $w \in Q$, it follows from definition of H_n that $w_2 \in V_2$. Since $Q \mapsto V_2$, $x_j \to w_2$. So the collection $(\mathcal{P}' - P) \cup P_1 x_l x_i x_j P_2 - x_l$ is an S-path partition of D. □

By Lemmas 6.5 and 6.6, if a 3-quasi-transitive digraph is a minimal counterexample for both conjectures, then D contains no strong component that induces a non-bipartite digraph. So we may assume that every strong component in D induces a bipartite digraph. We were not able to prove the conjectures under this scenario, but we obtained some interesting results which we describe in what follows. In order to do this, first we show some nice structural lemmas.

Lemma 6.7. *Let D be a non-strong 3-quasi-transitive digraph and let S be a maximum stable set of D. Let Q be a non-trivial strong component that induces a bipartite digraph with bipartition (X_1, X_2). If $S \cap X_1 \neq \emptyset$, then $X_1 \subseteq S$ and $X_2 \cap S = \emptyset$.*

Proof. By Theorem 6.3, $D[V(Q)]$ is a semicomplete bipartite digraph. Since $S \cap X_1 \neq \emptyset$, it follows that $X_2 \cap S = \emptyset$. We show next that $X_1 \subseteq S$. Towards a contradiction, suppose that there exists a vertex $v \in X_1$ such that $v \notin S$. It follows from Lemma 6.3(i) that $(N^-(X_1) - X_2) \mapsto X_1$ and $X_1 \mapsto (N^+(X_1) - X_2)$. Since $X_1 \cap S \neq \emptyset$ and $X_2 \cap S = \emptyset$ we conclude that $N(X_1) \cap S = \emptyset$. Thus $S \cup v$ is a stable set larger than S in D, a contradiction. □

We show next a nice structural result for a 3-quasi-transitive digraph in $\mathfrak{D}$.

Lemma 6.8. *Let D be a 3-quasi-transitive digraph. Let $U(C)$ be an odd cycle of length at least five in $U(D)$. If $D \in \mathfrak{D}$, then C contains a chord or a digon in D.*

Proof. If C is a cycle in D, then C contains a $\overrightarrow{P_4}$ because it has at least five vertices. Since D is 3-quasi-transitive, it follows that C contains a chord. So we may assume that C is a non-oriented odd cycle in D and C contains no $\overrightarrow{P_4}$. Since C is odd and C has length at least five, let $P = u_1u_2u_3$ be a maximum path in C. Since C has length at least five, let v_1, v_2 be two vertices in $V(C) - \{u_1, u_2, u_3\}$ such that v_1 is adjacent to u_1 and v_2 is adjacent to u_3. Since P is maximum in C, it follows that $u_1 \to v_1$ and $v_2 \to u_3$. Now, suppose that v_1 and v_2 are adjacent. If $v_2 \to v_1$, then C contains a chord or a digon; otherwise, C is a blocking odd cycle with (v_1, u_1) a blocking pair, a contradiction because $D \in \mathfrak{D}$. If $v_1 \to v_2$, then $P' = u_1v_1v_2u_3$ is a $\overrightarrow{P_4}$, a contradiction. So v_1 and v_2 are non-adjacent in D. Since C is odd, let v_3, v_4 be two vertices in $V(C) - \{u_1, u_2, u_3, v_1, v_2\}$ such that v_3 is adjacent to v_1 and v_4 is adjacent to v_2. If $v_3 \to v_1$, then C contains a chord or a digon; otherwise, C is a blocking odd cycle with (v_1, u_1) a blocking pair, a contradiction. So $v_1 \to v_3$. Similarly, we have $v_4 \to v_2$; otherwise, (u_3, v_2) is a blocking pair. Again, if $v_3 \to v_4$, then $P' = v_1v_3v_4v_2$ is a $\overrightarrow{P_4}$, a contradiction; and if $v_4 \to v_3$, then

C contains a chord or a digon. Thus v_3 and v_4 are non-adjacent. Since C is odd, let v_5, v_6 be two vertices in $V(C) - \{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ such that v_5 is adjacent to v_3 and v_6 is adjacent to v_4. Using analogous arguments to the previous ones, we have that $v_5 \to v_3$ and $v_4 \to v_6$. Note that v_5 and v_6 are non-adjacent. Moreover, note that if $v_i \to v_{i+2}$ where i is odd, then $v_{i+3} \to v_{i+1}$. Let $V(C) = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_k\}$. Since C is odd, we conclude that k is even. Let v_{k-1} and v_k. Since v_{k-1} and v_k are adjacent and $D \in \mathfrak{D}$, it follows that C contains a chord or a digon. □

Unfortunately, the the proof presented for Lemma 6.8 does not work for a digraph in $\mathfrak{B}$. The Lemma 6.8 helps to exemplify the difference between a blocking odd cycle and an anti-directed odd cycle. Note that in proof of lemma, the fact that C has no blocking pair was essential. However, anti-directed odd cycles may have a blocking pair which invalidates the argument. It is possible for C to have a blocking pair, to be induced, and still not to be an anti-directed odd cycle (see Figure 6.2).

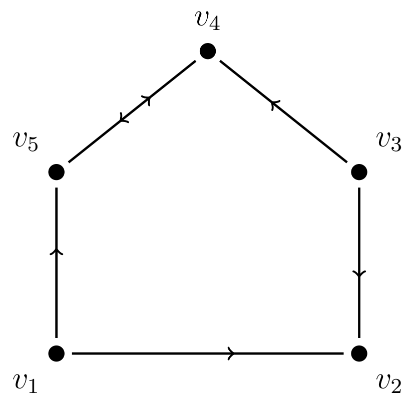


Figure 6.2: Example of an induced odd cycle that has a blocking pair but is not an anti-directed odd cycle.

Next, we show that if every strong component of a non-strong 3-quasi-transitive digraph D is trivial (equivalently, D is acyclic), then D is BE-diperfect.

Lemma 6.9. *Let D be a non-strong 3-quasi-transitive digraph such that every strong component of D induces a bipartite digraph. If $D \in \mathfrak{D}$ and D contains no digon, then D is BE-diperfect.*

Proof. Since D contains no digon, it follows from Lemma 6.8 that $U(D)$ contains no induced odd cycle of length at least five. We show next that D is a bipartite digraph. Towards a contradiction, suppose that D is bipartite and let C be an induced odd cycle of $U(D)$. Since every strong component of D induces a bipartite digraph, we conclude that C is non-oriented in D. By Lemma 6.8, C has length three. Since D contains no digon, it follows that C is an induced transitive triangle, a contradiction because $D \in \mathfrak{D}$. Thus D is bipartite, and hence, the result follows from Lemma 1.2. □

Now, we prove a simple, but useful lemma.

Lemma 6.10. *Let D be a non-strong 3-quasi-transitive digraph that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable* *set in D. Let Q be a non-trivial strong component of D such that $D[V(Q)]$ is a complete bipartite digraph. If Q is initial and $V(Q) \cap S = \emptyset$, then D admits an S_{BE}-path partition (resp., α-property).*

Proof. Let (X, Y) be the bipartition of $D[V(Q)]$. Without loss of generality, suppose that $|X| \leq |Y|$. Let $D' = D - X$. Since $V(Q) \cap S = \emptyset$, S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}$ be an S_{BE}-path partition of D. Since Q is an initial strong component of D, it follows that every vertex in Y is the initial vertex in some path of $\mathcal{P}$. Since $|X| \leq |Y|$ and $D[V(Q)]$ is a complete bipartite digraph, it is easy to see that using the arcs in Q, we can add the vertices of X to paths of $\mathcal{P}$ that start at some vertex in Y, obtaining an S_{BE}-path partition of D. □

By the principle of directional duality, we have the following result.

Lemma 6.11. *Let D be a non-strong 3-quasi-transitive digraph that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let S be a maximum stable set in D. Let Q be a non-trivial strong component of D such that $D[V(Q)]$ is a complete bipartite digraph. If Q is final and $V(Q) \cap S = \emptyset$, then D admits an S_{BE}-path partition (resp., α-property).*

The next lemma states that if a non-strong 3-quasi-transitive digraph D contains two non-trivial strong components Q_1, Q_2 such that $Q_1 \mapsto Q_2$ and $D[V(Q_i)]$ is bipartite for $i \in \{1, 2\}$, then D satisfies the BE-property.

Lemma 6.12. *Let D be a non-strong 3-quasi-transitive digraph that every proper induced subdigraph of D satisfies the BE-property. Let Q_1, Q_2 be two non-trivial strong components of D such that $D[V(Q_i)]$ is bipartite, for all $i \in \{1, 2\}$. If $Q_1 \mapsto Q_2$ and $D \in \mathfrak{D}$, then D satisfies the BE-property.*

Proof. Let (X_1, Y_1) and (X_2, Y_2) be the bipartition of $D[V(Q_1)]$ and $D[V(Q_2)]$, respectively. By Theorem 6.3, it follows that $D[V(Q_1)]$ and $D[V(Q_2)]$ are semicomplete bipartite digraphs. Since $Q_1 \mapsto Q_2$ and $D \in \mathfrak{D}$, we conclude that $D[V(Q_1)]$ and $D[V(Q_2)]$ are complete bipartite digraphs; otherwise, it easy to see that D contains an induced transitive triangle.

Claim 1. The strong component Q_1 is initial and Q_2 is final in D.

By the principle of directional duality, it suffices to show that Q_1 is initial. Towards a contradiction, suppose that Q_1 is non-initial. Let v be vertex in $V(D) - V(Q_1)$ such that v dominates a vertex x_1 in Q_1. Without loss of generality, suppose that $x_1 \in X_1$. Let x_2 be a vertex in X_2. Since $D[V(Q_1)]$ is complete and $Q_1 \mapsto Q_2$, there exists a $\overrightarrow{P_4}$ that starts at v and ends at x_2. Since D is 3-quasi-transitive and Q_1 and Q_2 are strong components of D, it follows that $v_1 \rightarrow x_2$. Thus $D[\{v, x_1, x_2\}]$ is an induced transitive triangle, a contradiction because $D \in \mathfrak{D}$. So Q_1 is initial and Q_2 is final in D. This finishes the proof of Claim 1.

Let S be a maximum stable set in D. Now, suppose that $V(Q) \cap S \neq \emptyset$. Since $Q_1 \mapsto Q_2, V(Q_2) \cap S = \emptyset$. By Claim 1, Q_2 is final in D. Since $D[V(Q_2)]$ is complete, it follows from Lemma 6.11 that D admits an S_{BE}. So we may assume that $V(Q) \cap S = \emptyset$, and hence, the result follows from Lemma 6.10. □

Similarly to Lemma 6.8, the proof presented for Lemma 6.12 does not work for a non-strong 3-quasi-transitive digraph D in $\mathfrak{B}$. In this case, the fact that D contains no induced transitive triangle is essential, and we need this fact to prove that each strong component of D induces a complete bipartite digraph. Without this argument, we cannot complete the proof for the version of D in $\mathfrak{B}$.

However, with the results presented in this section, the structure of the digraph that remains to verify both conjectures in this class is well-defined and relatively simple: every strong component of D induces a semicomplete bipartite digraph; if a strong component Q intersects a maximum stable set S, then the part of bipartition in Q that intersects S is entirely contained in S. Also, we may assume that there exists at least one digon in D, this implies that there exists at least one non-trivial strong component in D. Howsoever, it should be noted that even with a well-defined structure, verifying both conjectures for this class could be a challenging problem.

6.3 4-transitive digraphs

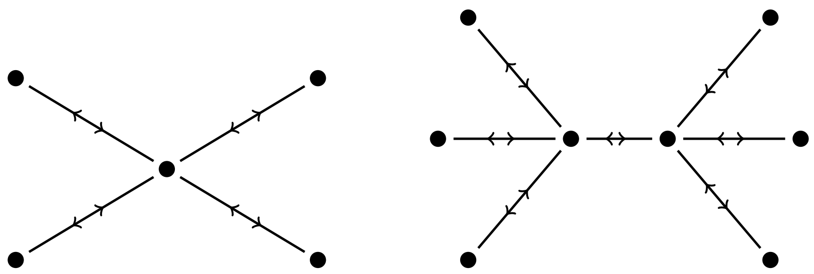
In this section, we present some results for 4-transitive digraphs. Recall that a digraph D is 4-transitive if for every path of length four $P = v_1v_2v_3v_4v_5$ in D, v_1 dominates v_5. Here, we show that a strong 4-transitive digraph satisfies the BE-property (resp., α-property). We also present some results for non-strong 4-transitive digraphs.

In [26], Hernández-Cruz and Galeana-Sánchez introduced 4-transitive digraphs. Since their introduction, several results have been presented in the literature for this class of digraphs [11, 22, 25, 33]. In [22], Garciá-Vázquez and Hernández-Cruz characterize 4-transitive digraphs having a 3-kernel and also 4-transitive digraphs having a 2-kernel (we omit this definition). Using the last one result, they verified the Laborde-Payan-Xuong conjecture (defined in Section 6.2) for 4-transitive digraphs. They also show that Seymour's Second Neighborhood Conjecture is true for 4-transitive digraphs. Seymour's Second Neighborhood conjecture states that every digraph has a vertex whose second out-neighborhood is at least as large as its first out-neighborhood.

Before we present the results of this section, we need to define some concepts. Let D be a digraph. We say that D is a *star* if $U(D)$ is a complete bipartite digraph with bipartition (X, Y) such that $|X| = 1$. We also say that the vertex in X is the *internal* vertex and the vertices in Y are *leaves* (see Figure 6.3a).

We say that D is a *double star* if $U(D)$ is obtained by joining the internal vertex of two vertex-disjoint stars (see Figure 6.3b). Recall that D is symmetric if for every pair of adjacent vertices $u, v \in V(D)$, we have $u \leftrightarrow v$.

In [25], Hernández-Cruz provided a characterization for strong 4-transitive digraphs.



(a) a symmetric star with four leaves

(b) a symmetric double star with six leaves

Figure 6.3: Example of a symmetric star and a symmetric double star.

Theorem 6.4 (Hernández-Cruz, 2013)**.** *Let D be a strong 4-transitive digraph. Then exactly one of the following possibilities holds:*

(i) D is a complete digraph,

(ii) D is an extended cycle of length 3,

(iii) D has circumference 3, an extended cycle of length 3 as a spanning subdigraph with cyclical partition $\{V_0, V_1, V_2\}$, at least one digon exists in D and for every digon $\{v_i, v_{i+1}\}$ in D, with $v_j \in V_j$ for $j \in \{i, i+1\}(mod\ 3)$, $|V_i| = 1$ or $|V_{i+1}| = 1$,

(iv) D has circumference 3, $U(D)$ is not 2-edge-connected and $\{S_1, S_2, \ldots, S_n\}$ are the vertex sets of the maximal 2-edge-connected subgraphs of $U(D)$, with $S_i = \{u_i\}$ for every $2 \leq i \leq n$ and such that $D[S_1]$ has an extended cycle of length 3 with cyclical partition $\{V_0, V_1, V_2\}$ as a spanning subdigraph. A vertex $v_0 \in V_0$ (without loss of generality) exists such that $v_0u_j, u_jv_0 \in A(D)$ for every $2 \leq j \leq n$. Also, $|V_0| = 1$ and $D[S_1]$ has the structure described in (i) or (ii), depending on the existence of digons,

(v) D is a symmetric digraph such that $U(D)$ is a C_5,

(vi) D is a symmetric star with at least 3 vertices,

(vii) D is a symmetric double star, or

(viii) D is a strong digraph with at most 4 vertices not included in the previous families.

In order to show that a strong 4-transitive digraph satisfies the BE-property, we need the following auxiliary result.

Lemma 6.13. *Let D be a digraph. If $U(D)$ has circumference 3, then D is diperfect.*

Proof. Towards a contradiction, suppose that D is non-diperfect. Since $U(D)$ has circumference 3, it follows from Strong Perfect Graph Theorem (Theorem 1.1) that $U(D)$ contains an induced complement, denoted by $\overline{U(C)}$, of an odd cycle $U(C)$ of length at least five. Since the complement of a C_5 is also a C_5, we have that $U(D)$ has circumference at least five, a contradiction. So D is diperfect. $\square$

The next lemma states that a strong 4-transitive digraph satisfies the BE-property.

Lemma 6.14. *Let D be a strong 4-transitive digraph. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

Proof. By Theorem 6.4, if D is a symmetric complete star or a symmetric complete double star, then D is bipartite, and hence, D is diperfect. Thus the result follows from Lemma 1.2. Also, if D is a complete digraph, an extended cycle of length 3 or a strong digraph of order less than or equal to 4, then D is diperfect, and the result follows. Moreover, if D has circumference 3, then it follows from Lemma 6.13 that D is diperfect, and the result follows. So we may assume that D is a symmetric digraph such that $U(D)$ is a C_5. Thus D is hamiltonian and the result follows easily. □

The next lemma is a version of Lemma 6.14 for α-diperfect digraphs, we omit its proof because it is analogous, but we use Lemma 1.1 instead of Lemma 1.2.

Lemma 6.15. *Let D be a strong 4-transitive digraph. If $D \in \mathfrak{B}$, then D is α-diperfect.* □

Now, we prove some results for a non-strong 4-transitive digraph. In order to do this, we need the following structural lemma. Recall that if Q is a strong component of a digraph D, then $\mathcal{K}^-(Q)$ (resp., $\mathcal{K}^+(Q)$) is the set of strong components that reach (resp., are reached by) Q in D.

Lemma 6.16. *Let D be a non-strong 4-transitive digraph. Let Q be a strong component Q with at least four vertices. If $D[V(Q)]$ is hamiltonian, then for every strong component K in $\mathcal{K}^-(Q)$, we have $K \mapsto Q$.*

Proof. Let u be a vertex in $K \in \mathcal{K}^-(Q)$. Note that there exists a path from u to Q in D. Let $P = uw_1w_2 \dots w_k$ be a minimum path from u to w_k such w_k in Q. We show next that $k = 1$. Towards a contradiction, suppose that $k > 1$. Since Q is a hamiltonian, let P' be a path in Q such that P' has length two, P' starts at w_k and ends at v. So $w_{k-2}w_{k-1}P'$ has length four (if $k = 2$, then $uw_{k-1}P'$). Since D is 4-transitive, it follows that $w_{k-2} \to v$ (if $k = 2$, then $u \to v$). Thus the path $uw_1 \dots w_{k-2}v$ contradicts the minimality of P. So we may assume that u dominates some vertex in Q. Now, we show that $u \mapsto Q$. Let $C = w_1w_2 \dots w_k$ be a hamiltonian cycle in Q where $k \geq 4$. Without loss of generality, suppose that $u \to w_1$. Since $k \geq 4$, let $P = uw_1w_2w_3w_4$ be a path of length four. Since D is 4-transitive, $u \to w_4$. Also, there exists a path uw_4Pw_i of length four in C, and hence, $u \to w_i$. Since C is a hamiltonian cycle, it easy to see that $u \to \{w_1, w_2, \dots, w_k\}$. Since Q is a strong component, we conclude that $u \mapsto Q$. Moreover, since u and K are arbitrary, the result follows. □

By the principle of directional duality, we have the following result.

Lemma 6.17. *Let D be a non-strong 4-transitive digraph. Let Q be a strong component Q with at least four vertices. If $D[V(Q)]$ is hamiltonian, then for every strong component K in $\mathcal{K}^+(Q)$, we have $Q \mapsto K$.*

The next lemma states that if a non-strong 4-transitive digraph D contains a strong component Q such that $D[V(Q)]$ is symmetric such that $U(D)$ is a C_5, then D satisfies the BE-property (resp., α-property).

Lemma 6.18. *Let D be a non-strong 4-transitive digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). If D contains a strong component Q such that $D[V(Q)]$ is symmetric and $U(D[V(Q)])$ is a C_5, then D satisfies the BE-property (resp., α-property).*

Proof. Let S be a maximum stable set in D. Let $V(Q) = \{v_1, v_2, v_3, v_4, v_5\}$. Since Q induces a symmetric digraph and $U(D[V(Q)])$ is a C_5, it follows that there exists at least one digon disjoint from S in Q. Without loss generality, suppose that $\{v_1, v_2\} \cap S = \emptyset$.

Now, let $D' = D - v_2$. Since $v_2 \notin S$, it follows that S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition. Let P be a path of $\mathcal{P}'$ such that $v_1 \in V(P)$. If P starts (resp., ends) at v_1, then since $v_1 \leftrightarrow v_2$, it follows that the collection $(\mathcal{P}' - P) \cup v_2P$ (resp., $(\mathcal{P}' - P) \cup Pv_2$) is an S_{BE}-path partition of D. So we may assume that P neither starts nor ends at v_1. Let w_1, w_2 be vertices in P such that $w_1 \to v_1$ and $v_1 \to w_2$. Since Q induces a symmetric digraph, $U(D[V(Q)])$ is a C_5 and $v_2 \notin V(D')$, it follows that $\{w_1, w_2\} \not\subseteq V(Q)$. By the principle of directional duality, we may assume that $w_2 \notin V(Q)$. Since $D[V(Q)]$ is hamiltonian, it follows from Lemma 6.17 that $Q \mapsto w_2$. So $v_2 \to w_2$. Let $P_1 = Pv_1$ and $P_2 = w_2P$ be the subpaths of P. Since $v_1 \leftrightarrow v_2$ and $v_2 \to w_2$, the collection $(\mathcal{P}' - P) \cup P_1v_2P_2$ is an S_{BE}-path partition of D. This ends the proof. □

Next, we show that if a non-strong 4-transitive digraph D contains an initial strong component Q such that $D[V(Q)]$ is a complete digraph with at least three vertices, then D satisfies the BE-property (resp., α-property).

Lemma 6.19. *Let D be a non-strong 4-transitive digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let Q be an initial strong component of D containing at least three vertices. If $D[V(Q)]$ is a complete digraph, then D satisfies the BE-property (resp., α-property).*

Proof. Let S be a maximum stable set in D. Since $D[V(Q)]$ is a complete digraph containing at least three vertices, there exists at least one digon in Q disjoint from S. So let $\{v_1, v_2\}$ be a digon in Q such that $\{v_1, v_2\} \cap S = \emptyset$.

Let $D' = D - v_2$. Since $v_2 \notin S$, it follows that S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition. Let P be a path of $\mathcal{P}'$ such that $v_1 \in V(P)$. If P starts at v_1, then since $v_1 \leftrightarrow v_2$, it follows that the collection $(\mathcal{P}' - P) \cup v_2P$ is an S_{BE}-path partition of D. So we may assume that P does not start at v_1. Let w be the vertex in P such that $w \to v_1$. Since Q is initial, we conclude that $w \in V(Q)$. Since $D[V(Q)]$ is a complete digraph, $w \leftrightarrow v_2$. Let $P_1 = Pw$ and $P_2 = v_1P$ be the subpaths of P. Thus the collection $(\mathcal{P}' - P) \cup P_1v_2P_2$ is an S_{BE}-path partition of D. □

By the principle of directional duality, we have the following result.

Lemma 6.20. *Let D be a non-strong 4-transitive digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). Let Q be a final strong component of D containing at least three vertices. If $D[V(Q)]$ is a complete digraph, then D satisfies the BE-property (resp., α-property).*

The next lemma states that if a non-strong 4-transitive digraph D contains a strong component Q such that $D[V(Q)]$ is a complete digraph with at least four vertices, then D satisfies the BE-property (resp., α-property).

Lemma 6.21. *Let D be a non-strong 4-transitive digraph such that every proper induced subdigraph of D satisfies the BE-property (resp., α-property). If D contains a strong component Q such that $D[V(Q)]$ is a complete digraph with at least four vertices, then D satisfies the BE-property (resp., α-property).*

Proof. Let S be a maximum stable set in D. Since Q induces a complete digraph, it follows that $|V(Q) \cap S| \leq 1$. Since Q contains at least four vertices, there exists a digon disjoint from S in Q. So let $\{v_1, v_2\}$ be a digon in Q such that $\{v_1, v_2\} \cap S = \emptyset$.

Let $D' = D - v_2$. Since $v_2 \notin S$, it follows that S is maximum in D'. By hypothesis, D' is BE-diperfect. Let $\mathcal{P}'$ be an S_{BE}-path partition. Let P be a path of $\mathcal{P}'$ such that $v_1 \in V(P)$. If P starts (resp., ends) at v_1, then since $v_1 \leftrightarrow v_2$, it follows that the collection $(\mathcal{P}' - P) \cup v_2P$ (resp., $(\mathcal{P}' - P) \cup Pv_2$) is an S_{BE}-path partition of D. So we may assume that P neither starts nor ends at v_1. Let w_1, w_2 be vertices in P such that $w_1 \to v_1$ and $v_1 \to w_2$. We show next that $w_1 \to v_2$. If $w_1 \in V(Q)$, then since $D[V(Q)]$ is a complete digraph, we have that $w_1 \leftrightarrow v_2$. So we may assume that $w_1 \notin V(Q)$. Since $D[V(Q)]$ is complete, it follows from Lemma 6.4 that $D[V(Q)]$ is hamiltonian. Since $w_1 \to v_1$, we conclude by Lemma 6.16 that $w_1 \mapsto Q$. So $w_1 \to v_2$. Now, let $P_1 = Pw_1$ and $P_2 = v_1P$ be the subpaths of P. Since $w_1 \to v_2$ and $v_2 \leftrightarrow v_1$, the collection $(\mathcal{P}' - P) \cup P_1v_2P_2$ is an S_{BE}-path partition of D. □

Unlike the previous lemmas, note that in Lemma 6.21 we need not suppose that the strong component Q is an initial (or final) strong component of D. This happens because we can use Lemma 6.16.

We believe it would be difficult to finish the proof of both conjectures for this class using Theorem 6.4. Perhaps a more promising approach would be to try a proof similar to the one we did for 3-anti-circulant digraphs (see Chapter 5, Section 5.1).

6.4 k-semi-symmetric digraphs

In [7], Berge showed that symmetric digraphs are α-diperfect. In [28], Sambinelli proved that 2-semi-symmetric digraphs and a certain 3-semi-symmetric digraph are BE-diperfect, which confirms both Conjecture 1.2 and Conjecture 1.1 for these classes of digraphs. In this section, we show that if a digraph D is a counterexample for Conjecture 1.1 that minimizes $|V(D)| + |A(D)|$, then D is not a 4-semi-symmetric digraph. We also show that if D is a counterexample for Conjecture 1.2 that minimizes $|V(D)|+|A(D)|$, then D is not a 3-semi-symmetric digraph.

First, we need the following definitions. Let D be a digraph. In order to make this text more fluid, we say that D is a $(V+A)$-*minimal counterexample* for both conjectures if D minimizes $|V(D)|+|A(D)|$. We say that an arc uv in $A(D)$ is *lonely* if $u \mapsto v$. We say that D is k-*semi-symmetric* if D contains at most k lonely arcs. In especial, the digraph D is *symmetric* if D is 0-semi-symmetric. We also say that a digon $\{u,v\}$ is an α-*good digon* (resp., *BE-good digon*) if both $D-uv$ and $D-vu$ do not contain an anti-directed odd cycle (resp., blocking odd cycle) as an induced subdigraph. Thus we have the following result.

Lemma 6.22. *Let D be a digraph such that D contains a BE-good digon $\{u,v\}$. If $D-uv$ satisfies the BE-property, then D satisfies the BE-property.*

Proof. Let S be a maximum stable set of D and let $D' = D - uv$. Since $u \leftrightarrow v$, it follows that S is a maximum stable set in D'. Since $u \leftrightarrow v$ is a BE-good digon, $D' \in \mathfrak{D}$. By hypothesis, D' admits an S_{BE}-path partition $\mathcal{P}$, and hence, it follows $\mathcal{P}$ is also an S_{BE}-path partition of D. □

We omit the proof of next lemma, since it is analogous to the proof of Lemma 6.22.

Lemma 6.23. *Let D be a digraph such that D contains an α-good digon $\{u,v\}$. If $D-uv$ satisfies the α-property, then D satisfies the α-property.*

We show next that if a digraph D is a $(V+A)$-minimal counterexample for Conjecture 1.1, then D is not a 4-semi-symmetric digraph.

Lemma 6.24. *Let D be a $(V+A)$-minimal counterexample for Conjecture 1.1. Then, D is not a 4-semi-symmetric digraph.*

Proof. Towards a contradiction, suppose that D is a 4-semi-symmetric digraph. Note that we may assume that D is connected. If D contains an α-good digon, then it follows from Lemma 6.23 that D satisfies the α-property, a contradiction because D is a $(V+A)$-minimal counterexample. So we may assume that D contains no α-good digon. If D contains no digon, then since D is 4-semi-symmetric, we conclude that $|A(D)| = 4$. So it follows from Theorem 1.1 that D is diperfect, and hence, we conclude by Lemma 1.1 that D satisfies the α-property, a contradiction. Thus let $\{u,v\}$ be a digon in D. Let $D' = D - uv$. Note that $\alpha(D) = \alpha(D')$. Since D contains no α-good digon, there exists at least one induced anti-directed odd cycle in D'. So let C be an induced anti-directed odd cycle of D'. By definition of an anti-directed odd cycle, it follows that C contains no digon, and this implies that C contains length five and C contains all four lonely arcs of D. Towards a contradiction, suppose that $V(D) \neq V(C)$. Since D is connected, there exists a vertex w in $V(D) - V(C)$ such that $w \leftrightarrow x$ for some $x \in V(C)$. Since all four lonely arcs are in C, it follows that $\{w,x\}$ is an α-good digon, a contradiction. So we may assume that $V(C) = V(D)$, and hence, D is isomorphic to the digraph in Figure 6.4.

Thus it easy to see that D satisfies the α-property, a contradiction. □

For the next lemma, we need the following auxiliary result.

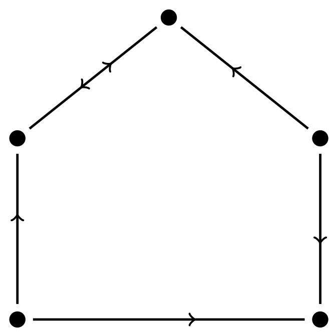


Figure 6.4: Illustration for the proof of Lemma 6.24.

Lemma 6.25 (Sambinelli, 2018)**.** *Let D be a 3-semi-symmetric digraph. If no pair of lonely arcs has a common endvertex, then D is BE-diperfect.*

The next lemma states that if a digraph D is a $(V + A)$-minimal counterexample for Conjecture 1.2, then D is not a 3-semi-symmetric digraph.

Lemma 6.26. *Let D be a $(V + A)$-minimal counterexample for Conjecture 1.2. If $D \in \mathfrak{D}$, then D is not a 3-semi-symmetric digraph.*

Proof. Towards a contradiction, suppose that D is a 3-semi-symmetric digraph. Since $D \in \mathfrak{D}$, it follows that D contains no blocking odd cycle as an induced subdigraph. Recall that we may assume that D is connected, non-diperfect and D has no clique cut. If $|V(D)| \leq 4$, then it follows from Theorem 1.1 that D is diperfect, a contradiction. So $|V(D)| \geq 5$. If D contains a BE-good digon, then it follows from Lemma 6.22 that D satisfies the BE-property, a contradiction because D is a $(V + A)$-minimal counterexample. So we may assume that D contains no BE-good digon. Also, it follows from Lemma 6.25 that there exists a pair of lonely arcs with a common endvertex. Let v_1, v_2, v_3 be endvertices of two lonely arcs with a common endvertex. Without loss generality, suppose that v_2 is a common endvertex. Since $|V(D)| \geq 5$ and D has no clique cut, it follows that there exists a vertex w_1 in $V(D) - \{v_1, v_2\}$ such that w_1 and v_3 are adjacent; otherwise, $\{v_1, v_2\}$ is a clique cut in D. Also, since $|V(D)| \geq 5$ and D is connected, let w_2 be a vertex in $V(D) - \{v_1, v_2, v_3, w_1\}$ such that w_2 is adjacent to some vertex in $\{v_1, v_2, v_3, w_1\}$. Thus we have two cases to deal with depending on whether $\{w_1, v_3\}$ is a digon or not.

Case 1. Assume that $\{v_1, w_1\}$ is a digon in D. If w_1 and v_2 are non-adjacent, then since every blocking odd cycle contains a blocking pair and two lonely arcs contains its endvertices in $\{v_1, v_2, v_3\}$, it follows that $\{w_1, v_3\}$ is a BE-good digon because we can form a $\overrightarrow{P_3}$ with the vertices $\{v_2, v_3, w_1\}$, and hence, D cannot contain a blocking odd cycle. So we conclude that the third lonely arcs has its endvertices in $\{v_2, w_1\}$. Since D is 3-semi-symmetric and all lonely arc have its endvertices in $\{v_1, v_2, v_3, w_1\}$, it follows that w_2 and some vertex in $\{v_1, v_2, v_3, w_1\}$ is a BE-good digon, a contradiction.

Case 2. Assume that every lonely arc contains its endvertices in $\{v_1, v_2, v_3, w_1\}$. Since $D \in \mathfrak{D}$, there exists no induced blocking odd cycle containing the vertices $\{v_1, v_2, v_3, w_1\}$.

Also, since D is 3-semi-symmetric, we conclude that w_2 and some vertex in $\{v_1, v_2, v_3, w_1\}$ is a digon. So we can form a $\overrightarrow{P_3}$ with w_2 and two vertices in $\{v_1, v_2, v_3, w_1\}$, and hence, we have a BE-good digon, a contradiction. □

We assume in Lemma 6.26 that D is not 3-semi-symmetric, and in Lemma 6.24 we assume that D is not 4-semi-symmetric. This helps, again, to exemplify the difference between a blocking odd cycle and an anti-directed odd cycle. Note that the fact that having a digon is suffices for a cycle to not be an anti-directed odd cycle is essential to proving Lemma 6.24. This does not happen with a blocking odd cycle because digons are allowed and possible in them. Also, it is essential to assume that $D \in \mathfrak{D}$, because a transitive triangle, for example, is 3-semi-symmetric. So we need to avoid these cases.

Moreover, we believe it is possible to use the same approach here to show that D is not 4-semi-symmetric, but the amount of cases to consider starts to increase considerably. This is understandable, since as the value of k increases, the problems become more like the general cases. Note that verifying both conjectures for an arbitrary k is exactly the same as proving the general case.

6.5 Digraphs with stability number two

In [28], Sambinelli proved some partial results for Conjecture 1.2 for digraphs with stability number two. In particular, Sambinelli showed that digraphs with stability number two contain at most five strong components. Also, Sambinelli proved that if a digraph D contains four or five strong components, then D satisfies the BE-property. Thus if we could verify the BE-property when D contains at most three strong components, then this would imply that Conjecture 1.2 holds for every digraph with stability number two.

In this section, we prove some results for this class of digraphs. In particular, we show that if D has no digon, then D satisfies the BE-property. Let us start with the following structural lemma.

Lemma 6.27. *Let D be a digraph such that $\alpha(D) = 2$. Let u, v be two vertices in D. If u and v are non-adjacent, then both $D[u \cup (N(u) - N(v))]$ and $D[v \cup (N(v) - N(u))]$ are semicomplete digraphs.*

Proof. Let $N_u = N(u) - N(v)$ and let $N_v = N(v) - N(u)$. Note that it suffices to show that $D[v \cup N_v]$ is a semicomplete digraph. Towards a contradiction, suppose that $D[v \cup N_v]$ is not a semicomplete digraph. Let w_1, w_2 be non-adjacent vertices in $D[v \cup N_v]$. By definition of N_v, it follows that $\{w_1, w_2, u\}$ is a stable set, which contradicts the fact that D has stability number two. So we conclude that both $D[v \cup N_v]$ and $D[u \cup N_u]$ are semicomplete digraphs. □

The next lemma states that if there are two non-adjacent vertices u, v in D such that there exists no vertex adjacent to both u and v, then D satisfies the BE-property.

Lemma 6.28. *Let D be a digraph such that $\alpha(D) = 2$. If $D \in \mathfrak{D}$ and there are two non-adjacent vertices u, v in D such that $N(u) \cap N(v) = \emptyset$, then D satisfies the BE-property.*

Proof. Since $N(u) \cap N(v) = \emptyset$, it follows from Lemma 6.27 that both $D[u \cup N(u)]$ and $D[v \cup N(v)]$ are semicomplete. Since $\alpha(D) = 2$, we conclude that $V(D) = \{u, v\} \cup N_u \cup N_v$. Thus $\alpha(D) = \alpha(D[u \cup N(u)]) + \alpha(D[v \cup N(v)])$, and hence, the result follows from Lemma 1.3. □

The next lemma is a version of Lemma 6.28 for α-property. We omit its proof because it is analogous.

Lemma 6.29. *Let D be a digraph such that $\alpha(D) = 2$. If $D \in \mathfrak{B}$ and there are two non-adjacent vertices u, v in D such that $N(u) \cap N(v) = \emptyset$, then D satisfies the α-property.* □

Next, we prove that if D has no digon, then D satisfies the BE-property. Recall that we may assume that D has no clique cut and D is non-diperfect. Moreover, recall that $X \equiv Y$ means that every vertex in X is adjacent to every vertex in Y.

Lemma 6.30. *Let D be a digraph such that $\alpha(D) = 2$. If $D \in \mathfrak{D}$ and D has no digon, then D satisfies the BE-property.*

Proof. Let $S = \{u, v\}$ be a maximum stable set in D. Let $N_u = N(u) - N(v)$, let $N_v = N(v) - N(u)$ and let $N_{uv} = N(v) \cap N(u)$. Note that the sets N_u, N_v and N_{uv} are pairwise disjoint. Since S is maximum, $V(D) = \{u, v\} \cup N_u \cup N_v \cup N_{uv}$. By Lemma 6.28, we may assume that $N_{uv} \neq \emptyset$.

Claim 1. The digraph $D[N_{uv}]$ has no $\overrightarrow{P_3}$ as a subdigraph.

Towards a contradiction, suppose that $D[N_{uv}]$ contains a path $P = x_1x_2x_3$ as a subdigraph. Since both x_1 and x_2 are adjacent to u and D has no digon, it follows that $D[\{u, x_1, x_2\}]$ is a $\overrightarrow{C_3}$; otherwise, $D[\{u, x_1, x_2\}]$ is an induced transitive triangle, a contradiction because $D \in \mathfrak{D}$. Since $x_1 \to x_2$, it follows that $u \to x_1 \to x_2 \to u$. By definition of N_{uv}, u and x_3 are adjacent. Since $x_2 \to \{u, x_3\}$ and D has no digon, it follows that $D[\{u, x_2, x_3\}]$ is an induced transitive triangle, a contradiction. So we may assume that $D[N_{uv}]$ has no $\overrightarrow{P_3}$ as a subdigraph. This finishes the proof of Claim 1.

Claim 2. The digraph $D[N_{uv}]$ is bipartite. Moreover, if (X, Y) is a bipartition of $D[N_{uv}]$, then $|X| \leq 2$ and $|Y| \leq 2$.

Towards a contradiction, suppose that $D[N_{uv}]$ is not a bipartite digraph. Let C be an odd cycle in $U(D)$. Since C is odd, we conclude that C contains a $\overrightarrow{P_3}$ as subdigraph, a contradiction by Claim 1. Moreover, let (X, Y) be a bipartition of $D[N_{uv}]$. Since $\alpha(D) = 2$, it follows $|X| \leq 2$ and $|Y| \leq 2$. This ends the proof of Claim 2.

Claim 3. We may assume that $|N_u| \leq 2$ and $|N_v| \leq 2$.

Note that it suffices to show that $|N_u| \leq 2$. Towards a contradiction, suppose that $|N_u| \geq 3$. Let w_1, w_2, w_3 be vertices in N_u. By Lemma 6.27, $D[u \cup N_u]$ is a semicomplete digraph. Since $D \in \mathfrak{D}$ and D has no digon, it follows that $D[\{w_1, w_2, w_3\}]$ is a $\overrightarrow{C_3}$; otherwise, $D[\{w_1, w_2, w_3\}]$ is an induced transitive triangle, a contradiction. Without loss of generality, assume that $w_1 \to w_2 \to w_3 \to w_1$. Since $w_1 \to w_2$ and u is adjacent to both w_1 and w_2, we conclude that $u \to w_1 \to w_2 \to u$. Since D has no digon, $w_2 \to \{u, w_3\}$ and w_3 and u are adjacent, it follows that $D[\{u, w_2, w_3\}]$ is an induced transitive triangle, a contradiction. So we may assume that $|N_u| \leq 2$ and $|N_v| \leq 2$. This ends the proof of Claim 3.

By Claim 2, $D[N_{uv}]$ is a bipartite digraph with bipartition (X, Y) such that $|X| \leq 2$ and $|Y| \leq 2$. The rest of proof is divided into two cases depending on whether if there exists an arc in $D[N_{uv}]$.

Case 1. There exists an arc x_1y_1 in $D[N_{uv}]$. Without loss of generality, assume that $x_1 \in X$ and $y_1 \in Y$. By definition of N_{uv}, $\{x_1, y_1\} \equiv \{u, v\}$. Since $x_1 \to y_1$, $D \in \mathfrak{D}$ and D has no digon, we conclude that $\{u, v\} \mapsto x_1$ and $y_1 \mapsto \{u, v\}$.

Case 1.1 Suppose that there are vertices $w_1 \in N_u$ and $w_2 \in N_v$ such that $w_1 \to u$ and $v \to w_2$ (see Figure 6.5).

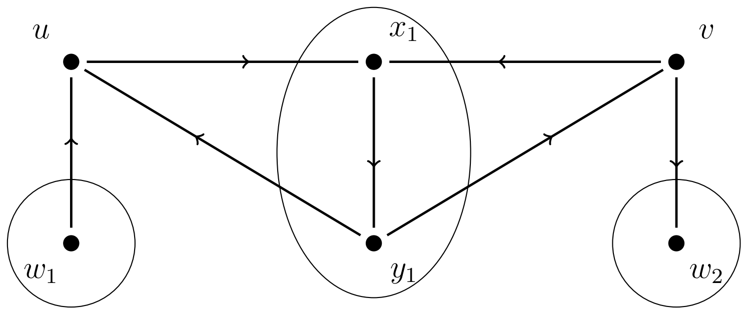


Figure 6.5: Illustration for the proof of Lemma 6.30.

We show next that $V(D) = \{u, x_1, w_1, v, y_1, w_2\}$. Since $D \in \mathfrak{D}$ and D has no digon, we conclude that w_1 and y_1 (resp., w_2 and x_1) are non-adjacent; otherwise, D contains an induced transitive triangle. Towards a contradiction, suppose that w_1 and x_1 are non-adjacent. Since $\alpha(D) = 2$, it follows that w_1 and w_2 are adjacent. So $D[\{w_1, u, x_1, v, w_2\}]$ is an induced blocking odd cycle with blocking pair $\{x_1, v\}$, a contradiction. So w_1 and x_1 are adjacent. Since $w_1 \to u \to x_1$, D has no digon and $D \in \mathfrak{D}$, we conclude that $x_1 \to w_1$. Similarly, we have $w_2 \to y_1$ (consider $D[\{w_1, u, y_1, v, w_2\}]$). Now, suppose that there exists a vertex w_3 in $N_u - w_1$. By Lemma 6.27, it follows that $D[u \cup N_u]$ is a semicomplete digraph. Since D contains no induced transitive triangle, we have $u \to w_3 \to w_1 \to u$. Thus w_3 and x_1 are non-adjacent; otherwise, $D[\{w_3, u, x_1\}]$ is an induced transitive triangle. Since $\alpha(D) = 2$ and w_2 and x_1 are non-adjacent, it follows that w_3 and w_2 are adjacent. So $D[\{w_3, u, x_1, v, w_2\}]$ is an induced blocking odd cycle with blocking pair $\{u, x_1\}$, a contradiction. So $N_u = \{w_1\}$, and similarly $N_v = \{w_2\}$.

We show next that $|X| = |Y| = 1$. Note that it suffices to show that $|X| = 1$. So suppose that there exists a vertex x_2 in $X - x_1$. Since $\alpha(D) = 2$, w_2 and x_1 are non-adjacent and $\{x_1, x_2\}$ is stable, we conclude that w_2 and x_2 are adjacent. Since D contains no induced transitive triangle and $v \to w_2$, it follows that $w_2 \to x_2 \to v$.

First, suppose that x_2 and y_1 are non-adjacent. Since w_1 and y_1 are non-adjacent and $\{y_1, x_2\}$ is stable, we conclude that w_1 and x_2 are adjacent. Since D contains no induced transitive triangle and $w_1 \to v$, it follows that $w_1 \to u \to x_2 \to w_1$. If there exists a vertex y_2 in $Y - y_1$, then since w_1 and y_1 are non-adjacent and $\{y_1, y_2\}$ is stable, it follows that w_1 and y_2 are adjacent. Since $w_1 \to u$ and D contains no induced transitive triangle, we conclude that $y_2 \to w_1 \to u \to y_2$. Since $u \to \{x_1, x_2\}$, $\alpha(D) = 2$, D has no digon and $D \in \mathfrak{D}$, we conclude that $D[\{u, x_1, x_2, y_2\}]$ contains an induced transitive triangle, a contradiction. So we may assume that $Y = \{y_1\}$. If w_1 and w_2 are non-adjacent, then $D[\{w_1, x_1, y_1, w_2, x_2\}]$ is an induced blocking odd cycle with blocking pair $\{x_1, y_1\}$, a contradiction. So w_1 and w_2 are adjacent, and since $w_2 \to x_2 \to w_1$, we conclude that $w_1 \to w_2$. Thus the collection $\{w_1w_2y_1v, ux_1\}$ is an S_{BE}-path partition of D.

So we may assume that x_2 and y_1 are adjacent. By Claim 1, we have $x_2 \to y_1$. Since $y_1 \to v$ and D contains no induced transitive triangle, we conclude $v \to x_2$. Since $\alpha(D) = 2$, w_2 must be adjacent to $\{x_1, x_2\}$, and hence, $D[\{w_2, x_1, x_2, v\}]$ contains an induced transitive triangle, a contradiction. Thus $|X| = |Y| = 1$. Since $N_u = \{w_1\}$ and $N_v = \{w_2\}$, we have $V(D) = \{u, x_1, w_1, v, y_1, w_2\}$. Since $\alpha(\{u, x_1, w_1\}) = \alpha(\{v, y_1, w_2\}) = 1$, it follows that $\alpha(D) = \alpha(\{u, x_1, w_1\}) + \alpha(\{v, y_1, w_2\})$. Thus the result follows from Lemma 1.3.

Case 1.2 For every vertex $w_1 \in N_u$ and every vertex $w_2 \in N_v$ we have $w_1 \to u$ and $v \to w_2$, or $u \to w_1$ and $w_2 \to v$. First, suppose that at least one between N_u and N_v is empty. Without loss of generality, suppose that $N_v = \emptyset$. Since $D[u \cup N_u]$ is semicomplete, $|N_u| \leq 2$, $|X| \leq 2$, $|Y| \leq 2$, $u \cup N_u$ and v are non-adjacent and $\{u, v\} \equiv X \cup Y$, it follows from Theorem 1.1 that D is diperfect, and the result follows.

So we may assume that both N_u and N_v are non-empty. Without loss of generality, we may assume that $w_1 \to u$ and $w_2 \to v$. Since D contains no induced transitive triangle, D has no digon and $D[u \cup N_u]$ and $D[v \cup N_v]$ are semicomplete digraphs, we conclude that $|N_u| \leq 1$ and $|N_v| \leq 1$ (see Figure 6.6). Since D contains no induced transitive triangle and D has no digon, it follows that w_1 and y_1 (resp., w_2 and y_1) are non-adjacent. Since $\alpha(D) = 2$, we conclude that w_1 and w_2 are adjacent. So $D[\{w_1, u, y_1, v, w_2\}]$ is an induced blocking odd cycle, a contradiction. This ends the proof of this case.

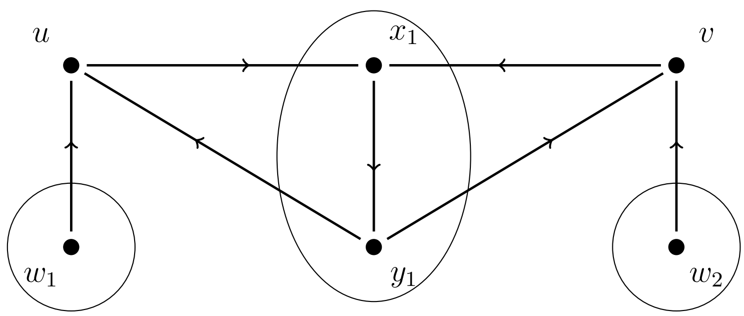


Figure 6.6: Illustration for the proof of Lemma 6.30.

Case 2. Suppose that N_{uv} is stable. Since $\alpha(D) = 2$, $|N_{uv}| \leq 2$. It follows from Lemma 6.27 that $D[u \cup N_u]$ and $D[v \cup N_v]$ are semicomplete digraphs. By Claim 3, we have that $|N_u| \leq 2$ and $|N_v| \leq 2$. If $|N_u| \leq 1$ and $|N_v| \leq 1$, then it is easy to check that D satisfies the BE-property. Without loss of generality, let w_1, w_2 be vertices in N_u such that $w_1 \to w_2$. Since $D[u \cup N_u]$ is semicomplete, $D \in \mathfrak{D}$ and D has no digon, it follows that $w_1 \to w_2 \to u \to w_1$. We show next that $|N_{uv}| = 1$. Towards a contradiction, suppose that

$|N_{uv}| = 2$. Let $N_{uv} = \{x_1, x_2\}$ (see Figure 6.7). Since D contains no induced transitive triangle and D has no digon, if $u \to \{x_1, x_2\}$ (resp., $\{x_1, x_2\} \to u$), then $\{w_1, x_1, x_2\}$ (resp., $\{w_2, x_1, x_2\}$) is a stable set larger than S in D, a contradiction. Therefore, without loss of generality, assume that $u \to x_1$ and $x_2 \to u$. Since D contains no induced transitive triangle, it follows that w_2 and x_2 (resp., w_1 and x_1) are non-adjacent. Thus we conclude that $x_1 \to w_2$ and $w_1 \to x_2$, and hence, $D[\{w_1, w_2, x_1, x_2, v\}]$ is an induced blocking odd cycle, a contradiction.

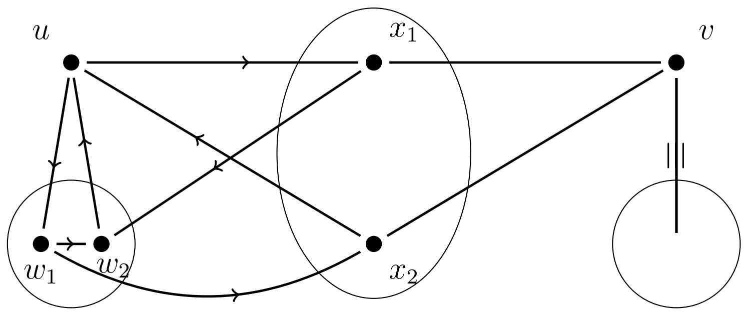


Figure 6.7: Illustration for the proof of Lemma 6.30.

So we may assume that $|N_{uv}| = 1$. Let $N_{uv} = \{x\}$. If $N_v = \emptyset$, then x is a clique cut, a contradiction. So let w_3 be a vertex in N_v. By principle of directional duality, suppose that $u \to x$. If N_v and N_u are non-adjacent, then x is a clique cut, a contradiction. So N_u and N_v are adjacent. Since D contains no induced transitive triangle and $u \to x$, it follows that w_1 and x are non-adjacent. We show next that $N_v = \{w_3\}$. Towards a contradiction, suppose that that $N_v = \{w_3, w_4\}$. Since $D[v \cup N_v]$ is semicomplete, D contains no induced transitive triangle and D has no digon, it follows that $D[\{v, w_3, w_4\}]$ is a $\overrightarrow{C_3}$. Without loss of generality, assume that $w_3 \to w_4 \to v \to w_3$. If $v \to x$, then x and w_3 are non-adjacent. Since $\alpha(D) = 2$ and w_1 and x are non-adjacent, we conclude that w_1 and w_3 are adjacent, and hence, $D[\{w_1, u, x, v, w_3\}]$ is an induced blocking odd cycle, a contradiction. Thus $x \to v$. Since D contains no induced transitive triangle, it follows that x and w_4 are non-adjacent. Similarly, it follows that w_1 and w_4 are adjacent, and this implies that $D[\{w_1, u, x, v, w_4\}]$ is an induced blocking odd cycle, a contradiction. So we may assume that $N_v = \{w_3\}$. Next, we show that x and w_3 are adjacent. Towards a contradiction, suppose that x and w_3 are non-adjacent. Since $\alpha(D) = 2$ and w_1 and x are non-adjacent, it follows that w_1 and w_3 are adjacent. If $w_3 \to w_1$, then $D[\{w_1, u, x, v, w_3\}]$ is an induced transitive triangle, a contradiction. So $w_1 \to w_3$. Since D contains no induced transitive triangle, we conclude that w_2 and w_3 are non-adjacent. Since $\alpha(D) = 2$ and w_3 and x are non-adjacent, we conclude that $x \to w_2$, and hence, $D[\{w_1, w_2, x, v, w_3\}]$ is an induced blocking odd cycle, a contradiction. Thus x and w_3 are adjacent, and this implies that $V(D)$ can be partitioned into $(\{u, w_1, w_2\}, \{v, x, w_3\})$ such that $\alpha(D) = \alpha(\{u, w_1, w_2\}) + \alpha(\{v, x, w_3\})$, and hence, the result follows from Lemma 1.3. This ends the proof. □

In Chapter 3, we showed some structural results for α-diperfect digraphs and BE-diperfect digraphs. In particular, the Theorem 3.3 suggests that dealing with digraph with small stability number may be the most difficult part of both conjectures. Therefore, it is not surprising that verifying them for this class of digraphs is a challenging problem; as Sambinelli [28] has pointed out, maybe the key to conclude the proof of both conjectures for this digraphs is to understand the case in which the digraph is strong.

6.6 Digraphs whose complement of the underlying graphs are odd cycles

In this section, we show that if a digraph D belongs to $\mathfrak{D}$ and $U(D)$ is a complement of an odd cycle of length at least five, then D is BE-diperfect.

Let D be a digraph. First, note that if $U(D)$ is a C_5, then its complement is also a C_5. So one may verify that if $D \in \mathfrak{D}$, then D is BE-diperfect. Thus we may assume that D contains at least seven vertices. Moreover, recall that we denote the complement of a graph G by $\overline{G}$.

Theorem 6.5. *Let D be a digraph such that $U(D)$ is a complement of an odd cycle of length at least seven. If $D \in \mathfrak{D}$, then D is BE-diperfect.*

Proof. Note that it suffices to show that D satisfies the BE-property. Let $\overline{U(D)}$ be an odd cycle of length at least seven. Suppose that the vertices of D (and of $U(D)$) are labelled as $v_1, v_2, \ldots, v_{2k+1}$ so that the cycle in $\overline{U(D)}$ is $(v_1, v_2, \ldots, v_{2k+1}, v_1)$. Thus the non-adjacent vertices to v_i in $U(D)$ are v_{i-1} and v_{i+1}, where the indexes are taken modulo k. Let S be a maximum stable set in D. Without loss of generality, suppose that $S = \{v_1, v_{2k+1}\}$. We show next that D admits an S_{BE}-path partition.

Let $B_1 = \{v_2, v_4, \ldots, v_{2k}\}$ and let $B_2 = \{v_3, v_5, \ldots, v_{2k-1}\}$. Note that $V(D) = \{v_1, v_{2k+1}\} \cup B_1 \cup B_2$. Since D contains at least seven vertices, we have that $|B_1| \geq 3$.

Claim 1. If there exists a hamiltonian path $w_1 P w_2$ in $D[B_1]$ such that $v_2 \notin \{w_1, w_2\}$, then D admits an S_{BE}-path partition.

By definition of complement, we conclude that $D[v_{2k+1} \cup B_2]$ is semicomplete, and hence, $D[v_{2k+1} \cup B_2]$ is diperfect. Since $D \in \mathfrak{D}$, it follows from Lemma 1.2 that $D[v_{2k+1} \cup B_2]$ satisfies the BE-property, and this implies that there exists a hamiltonian path P' in $D[v_{2k+1} \cup B_2]$ which v_{2k+1} is the initial (or the terminal) vertex of P'. On the other hand, since $v_2 \notin \{w_1, w_2\}$, it follows from definition of complement that v_1 is adjacent to both w_1 and w_2. If $v_1 \to w_1$ (resp., $w_2 \to v_1$), then $v_1 P$ (resp., $P v_1$) is a hamiltonian path in $D[v_1 \cup B_1]$, and hence, $\{P', v_1 P\}$ (resp., $\{P', P v_1\}$) is an S_{BE}-path partition of D. So we may assume that $w_1 \mapsto v_1$ and $v_1 \mapsto w_1$. Since w_1 and w_2 are adjacent and D contains no induced transitive triangle, it follows that $w_2 \to w_1$. Since $|B_1| \geq 3$, let u be the vertex in P that immediately succeeds w_1. Since $w_2 \to w_1$ and $w_1 \to v_1$, let $R = u P w_2 w_1 v_1$ be a hamiltonian path in $D[v_1 \cup B_1]$. Thus $\{P', R\}$ is an S_{BE}-path partition of D. This ends the proof of Claim 1.

Claim 2. If there exists a hamiltonian path w_1Pw_2 in $D[B_1]$ such that $v_{2k} \notin \{w_1, w_2\}$, then D admits an S_{BE}-path partition.

The proof of this claim is analogous to proof of Claim 1, but with the roles of v_1 and v_{2k+1} exchanged.

Since $D[B_1]$ is semicomplete and satisfies the BE-property, we conclude that for every vertex v in $D[B_1]$, there exists a hamiltonian path in $D[B_1]$ that starts (or ends) at v. Thus since $|B_1| \geq 3$, the result follows from Claim 1 or by Claim 2. This finishes the proof. □

Chapter 7

Concluding remarks

In this text, we presented some results for two conjectures related to maximum stable set and path partition of digraphs (Conjecture 1.1 and Conjecture 1.2).

In Chapter 3, we showed some structural results for α-diperfect digraphs and BE-diperfect digraphs. In Section 3.3, we proved structural results when a digraph D contains some special matchings. We also saw that minimal counterexamples to both Conjecture 1.1 and Conjecture 1.2 cannot have large stability number. In Section 3.4, we showed some structural results for BE-diperfect digraphs and α-diperfect digraphs when they contain some specific bipartite subdigraphs. The results presented in this chapter were of great help in obtaining results for the classes of digraphs that we studied. Furthermore, we believe that these results could help to obtain a proof for both Conjecture 1.1 and Conjecture 1.2 for several classes of digraphs, because they provide a different technique than the usual one, which consists of removing a certain structure of the digraph and somehow apply induction on the rest of the digraph.

In Chapter 4, we provided a decomposition for arbitrary arc-locally in-semicomplete digraphs, arbitrary arc-locally out-semicomplete digraphs and arbitrary arc-locally semicomplete digraphs. We also verified both Conjectures 1.1 and 1.2 for these classes of digraphs. In Section 4.1, we showed that if a digraph D is a connected arc-locally (out) in-semicomplete, then D is diperfect, or D admits a special partition of its vertices, or D has a clique cut. We also showed some structural results for arbitrary arc-locally (out) in-semicomplete. In particular, we proved that if (V_1, V_2, V_3) is a partition of $V(D)$ as described in Theorem 4.2(ii), then $U(D[V_2 \cup V_3])$ contains no cycle of length three. In Section 4.2, we provided another characterization for arbitrary arc-locally semicomplete digraphs. We showed that if a digraph D is a connected arc-locally semicomplete, then D is either a diperfect digraph or an odd extended cycle of length at least five. In our context, this decomposition was more useful and easier to use than one proved by Galeana-Sánchez and Goldfeder [20]. In Section 4.3, we verified Conjecture 1.2 for arc-locally (out) in-semicomplete digraphs and for arbitrary arc-locally semicomplete digraphs. In Section 4.4, we verified Conjecture 1.2 for the same classes of digraphs.

In Chapter 5, we studied 3-anti-circulant digraphs and 3-anti-digon-circulant digraphs. In Section 5.1, we verified both Conjectures 1.1 and 1.2 for 3-anti-circulant digraphs. In Section 5.2, we showed some structural results for 3-anti-digon-circulant digraphs. These digraphs do not contain anti-P_4 as an induced subdigraph, and hence, we believe that studying the structure of these digraphs should help towards obtaining a proof of both conjectures in the general case. Furthermore, the most interesting problem would be characterizing digraphs which do not contain an anti-P_4 as an induced subdigraph. Such result would potentially give us some insight on how to deal with arbitrary digraphs in $\mathfrak{D}$. However, this problem is probably very difficult, since we do not even know a complete characterization of 3-anti-circulant digraphs, a special subclass of those digraphs.

In Chapter 6, we showed some results for both conjectures for several classes of digraphs. In Section 6.1, we verified both conjectures for quasi-transitive digraphs. In order to do it, we showed an alternative proof for the fact that every quasi-transitive digraph is diperfect that relies on the Strong Perfect Graph Theorem (Theorem 1.1). In Section 6.2, we studied 3-quasi-transitive digraphs. We proved that every strong 3-quasi-transitive digraph satisfies the BE-property (resp., α-property). We also provided some results for non-strong 3-quasi-transitive digraphs. In particular, we show that if a non-strong 3-quasi-transitive digraph D contains a strong component Q such that $D[V(Q)]$ is non-bipartite, then D satisfies the BE-property (resp., α-property). In Section 6.3, we proved that every strong 4-transitive digraph satisfies the BE-property (resp., α-property). Also, we showed some results for non-strong 4-transitive digraphs. In particular, we proved that if a non-strong 4-transitive digraph D contains a strong component Q such that $D[V(Q)]$ is a symmetric C_5 or a complete digraph with at least four vertices, then D satisfies the BE-property (resp., α-property). These results could be useful towards obtaining a proof of these digraphs. Besides, we believe that an approach similar to that one used for 3-anti-circulant digraphs in Chapter 5 could be promissing. In Section 6.4, we showed that if a digraph D is a counterexample for Conjecture 1.1 that minimizes $|V(D)| + |A(D)|$, then D is not a 4-semi-symmetric digraph. We also proved that if D is a counterexample for Conjecture 1.2 that minimizes $|V(D)| + |A(D)|$, then D is not a 3-semi-symmetric digraph. We believe it is possible to use the same approach to show other values of k, however the amount of cases to consider starts to increase considerably. This seems natural, since as the value of k increases, the problems become more like the general cases. Note that verifying both conjectures for an arbitrary k is exactly the same as proving them in the general case. In Section 6.5, we proved that if a digraph D with stability number two has no digon, then D satisfies the BE-property. As Theorem 3.3 suggests and considering the hard-working proofs obtained by Sambinelli [28] for these digraphs, it could be difficult to verify both conjectures for these digraphs. However, we still consider it an interesting line to study in the near future. Finally, in Section 6.6, we proved that if a digraph D belongs to $\mathfrak{D}$ and $U(D)$ is a complement of an odd cycle of length at least five, then D is BE-diperfect.

We would like to remark that the results showed in Chapters 3, 4 and 5 were presented in [14], [16] and [15]. The paper [14] was submitted to the *Journal of Combinatorics* and it is under review. The paper [16] was submitted to the *Journal Graphs and Combinatorics* and it has now been accepted for publication. The paper [15] was submitted to the *Open Journal of Discrete Mathematics* and it also has now been accepted for publication.

Moreover, we compile the structural results in Table 7.1, the results obtained for Conjecture 1.1 in Table 7.2 and the results obtained for Conjecture 1.2 in Table 7.3. In Tables 7.2 and 7.3, complete results are marked by a green check symbol and partial results are marked by a slashed gray check symbol.

Table 7.1: Structural results.

Class of digraphs	Structural results
Arc-locally in-semicomplete digraphs	Theorem 4.2
Arc-locally out-semicomplete digraphs	Theorem 4.3
Arc-locally semicomplete digraphs	Theorem 4.4
3-anti-digon-circulant digraphs	Lemma 5.16, Lemma 5.17 and Theorem 5.18
3-quasi-transitive digraphs	Lemma 6.7 and Lemma 6.8
BE-diperfect digraphs	Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.12, Lemma 3.13, Lemma 3.14, Theorem 3.3, Lemma 3.15, Lemma 3.16 and Lemma 3.17
α-diperfect digraphs	Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4,Lemma 3.5, Lemma 3.6, Lemma 3.7, Lemma 3.8, Lemma 3.12, Lemma 3.13, Lemma 3.14, Theorem 3.3, Lemma 3.15, Lemma 3.16 and Lemma 3.17

Table 7.2: Results obtained for Conjecture 1.1. Complete results are marked by a green check symbol and partial results are denoted by a slashed gray check symbol.

Class of digraphs	Status	Results
Arc-locally in-semicomplete digraphs	✓	Theorem 4.9
Arc-locally out-semicomplete digraphs	✓	Theorem 4.10
Arc-locally semicomplete digraphs	✓	Theorem 4.8
3-anti-circulant digraphs	✓	Theorem 5.2
3-digon-anti-circulant digraphs	✓	Theorem 5.2
Quasi-transitive digraphs*	✓	Theorem 6.2
Strong 3-quasi-transitive digraphs	✓	Lemma 6.2
Strong 4-transitive digraphs	✓	Lemma 6.15
Arbitrary 3-quasi-transitive digraphs	✓̸	Lemma 6.6, Lemma 6.10 and Lemma 6.11
Arbitrary 4-transitive digraphs	✓̸	Lemma 6.18, Lemma 6.19, Lemma 6.20 and Lemma 6.21
4-semi-symmetric digraphs	✓̸	Lemma 6.24
With stability number two	✓̸	Lemma 6.29

*An alternative proof of result in [23].

Table 7.3: Results obtained for Conjecture 1.2. Complete results are marked by a green check symbol and partial results are denoted by a slashed gray check symbol.

Class of digraphs	Status	Results
Arc-locally in-semicomplete digraphs	✓	Theorem 4.6
Arc-locally out-semicomplete digraphs	✓	Theorem 4.7
Arc-locally semicomplete digraphs	✓	Theorem 4.5
3-anti-circulant digraphs	✓	Theorem 5.1
3-digon-anti-circulant digraphs	✓	Theorem 5.1
Quasi-transitive digraphs*	✓	Theorem 6.2
Strong 3-quasi-transitive digraphs	✓	Lemma 6.2
Strong 4-transitive digraphs	✓	Lemma 6.14
Complement of odd cycles	✓	Theorem 6.5
Arbitrary 3-quasi-transitive digraphs	✓̸	Lemma 6.5, Lemma 6.9, Lemma 6.10, Lemma 6.11 and Lemma 6.12
Arbitrary 4-transitive digraphs	✓̸	Lemma 6.18, Lemma 6.19, Lemma 6.20 and Lemma 6.21
3-semi-symmetric digraphs	✓̸	Lemma 6.26
With stability number two	✓̸	Lemma 6.28 and Lemma 6.30

*An alternative proof of result in [23].

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