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Specht property and rational Hilbert series for superalgebras with superinvolution and algebras with Hopf algebra action

Propriedade Specht e série de Hilbert racional para superálgebras com superinvolução e álgebras com ação de álgebra de Hopf

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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"There is a single light of science,
and to brighten it anywhere is to brighten it everywhere."
(Isaac Asimov)

Resumo

Um dos principais problemas na PI-teoria é provar a propriedade de Specht para uma determinada álgebra e provar a racionalidade da série de Hilbert da álgebra relativamente livre. Nesta tese de doutorado consideramos um corpo F de característica 0 e provamos a propriedade de Specht para a variedade de superalgebras com superinvolução finitamente gerada sobre F e para a variedade de álgebras H_m-módulo geradas pela álgebra $UT_2(F)$ de matrizes triangulares superiores 2×2 onde H_m é uma álgebra de Taft de dimensão m^2. Alem disso, provamos a racionalidade da série de Hilbert da PI-álgebra A sobre F tanto no caso A sendo uma superalgebra com superinvolução como no caso de uma álgebra de Hopf semisimples de dimensão finita agindo sobre A.

Palavras-chave: Identidades polinomiais, álgebras de Hopf, propriedade de Specht, série de Hilbert, álgebras H-módulo, superinvoluções, cocaracteres.

Abstract

One of the main problems in PI-theory is to prove the Specht property for a given algebra and the rationality of the Hilbert series of its relatively free algebra. In this doctoral thesis we consider a field F of characteristic 0 and we prove the Specht property for the variety of finitely generated superalgebras with superinvolution over F and for the variety of H_m-module algebras generated by the algebra $UT_2(F)$ of 2×2 upper triangular matrices, where H_m is a Taft's Hopf algebra of dimension m^2. Moreover, we prove the rationality of the Hilbert series of the PI-algebra A over F both in the case A is a superalgebra with superinvolution and when a finite dimensional semisimple Hopf algebra acts on A.

Keywords: Polynomial identities, Hopf algebras, Specht property, Hilbert series, H-module algebras, superinvolutions, cocharacters.

List of symbols

F	An arbitrary field, 20
$M_n(F)$	The set of $n \times n$ matrices over F, 20
$\mathrm{Hom}_F(V, V)$	The linear transformations from V to V, 20
Id	The identity map $\mathrm{Id}\colon V \to V$, 20
$\mathrm{End}_F(V)$	The algebra of endomorphism of V, 20
$F[x]$	Polynomials in x over F, 21
A/I	The factor algebra between an algebra A and an ideal $I \subseteq A$, 21
$V \otimes W$	Tensor product of vector spaces V and W, 21
$C\langle a_1, \dots, a_l\rangle$	Algebra over C generates by $a_i, \dots, a_l$, 22
$\mathrm{Ann}(M)$	The annihilator of the module M, 23
$J(A)$	The Jacobson radical of the algebra A, 23
$A = \bar{A} \bigoplus J$	The Wedderburn-Malcev decomposition of A, 24
$F\langle X\rangle$	The free algebra genereted by X, 24
$f = f(x_1, \dots, x_m)$	Polynomial in the variables $x_1, \dots, x_m$, 24
$f \equiv 0$	Polynomial identity, 25
$[x_1, x_2]$	The Lie commutator of x_1 and x_2, 25
$UT_n(F)$	The set of $n \times n$ upper triangular matrices over F, 25
s_n	The standar polynomial, 25
S_n	The symmetric group on $\{1, \dots, n\}$, 25
$\mathrm{Id}(A)$	The set of all polynomial identities of A, 25
$\langle S\rangle_T$	The T-ideal generated by the set S, 26
$A_1 \sim_{PI} A_2$	The PI-equivalence between A_1 and A_2, 26
$\mathrm{var}(S)$	Variety determined by a set S, 26

$\mathbb{Z}_2$	The group $\mathbb{Z}/2\mathbb{Z}$, 29
$V = \bigoplus_{n\geqslant 0} V^{(n)}$	The grading of the vector space V, 29
$F[x_1, \ldots, x_m]$	Polynomials in $x_1, \ldots, x_m$ over F, 29
$T(V)$	The tensor algebra of the vector space V, 30
$\dim(W)$	The dimension of a vector space W, 30
∞	Infinity, 30
$\mathrm{Hilb}(V, t)$	The Hilbert series of V, 30
G	An arbitrary group, 32
$A^{(g)}$	The homogeneous component of A of degree g, 32
e_{pq}	The matrix unit whose cell (p, q) is 1, 32
F_n	The free algebra $F\langle x_1, \ldots, x_n\rangle$ of rank $n \geqslant 1$, 33
Δ	The Vandermonde matrix, 34
$\tilde{s}_n$	The symmetric polynomial, 35
$\binom{d}{i}$	The binomial coefficient $\binom{d}{i}$, 35
$F\langle X\rangle^{gr}$	The free G-graded algebra generated by X, 36
$\mathrm{Id}^{gr}(A)$	The set of graded identities, 36
$\lvert a\rvert$	The homogeneous degree of a, 37
$*$	The superinvolution $*\colon A \to A$, 37
A_i^+	The set of symmetric elements $A_i^+ = \{a \in A_i : a^* = a\}$, 37
A_i^-	The set of skew elements $A_i^- = \{a \in A_i : a^* = -a\}$, 37
$F\langle Y \cup Z, *\rangle$	The free $*$-algebra, 37
y_i^+	A symmetric variable of even degree, 37
y_i^-	A skew variable of even degree, 37
z_i^+	A symmetric variable of odd degree, 37
z_i^-	A skew variable of odd degree, 37
$\mathrm{Id}^*(A)$	The set of $*$-identities, 38

$A \sim_{T_2^*} B$	The T_2^*-equivalence between two $*$-algebras A and B, 38
$\langle f_1, \ldots, f_n \rangle_{T_2^*}$	The T_2^*-ideal generated by the $*$-polynomials $f_1, \ldots, f_n$,38
P_n^*	The space of multilinear $*$-polynomials, 38
trp	The transpose superinvolution, 39
osp	The orthosympletic supertinvolution, 39
A^{sop}	The opposite superalgebra of A, 40
ex	The exchange superinvolution, 40
Δ	The comultiplication, 41
ε	The counit, 41
τ	The twist map, 41
FS	The F-vector space with basis S, 41
$\sum c_1 \otimes c_2$	The sigma notation for $\Delta(c)$, 42
$\mathrm{Ker}(f)$	The Kernel of f, 43
$\mathrm{Im}(f)$	The image of f, 43
FG	The group algebra, 43
$\mathfrak{g}$	An arbitrary Lie algebra, 44
$U(\mathfrak{g})$	The universal enveloping algebra of $\mathfrak{g}$, 44
$\mathcal{O}(F^2)$	The quantum plane $\mathcal{O}(F^2) = F\langle x, y \mid xy = qyx \rangle$, 44
H_4	The Sweedler's Hopf algebra, 46
$H_{m^2}(\xi)$	The Taft's Hopf algebra of dimension m^2, 46
$\int_H^l$	The space of left integrals in H, 48
$\int_H^r$	The space of right integrals in H, 48
$F^H\langle X \rangle$	The Free H-module algebra generated by X, 51
$\mathrm{Id}^H(W)$	The set of all H-identities of W, 52
$W_1 \sim_{T^H} W_2$	The T^H-equivalence between two H-module algebras W_1, W_2, 52

$\mathrm{var}^H(S)$	The variety of H-module algeras determined by S, 52
$GL(V)$	The general linear group of V, 53
χ_ϕ	The character of the representation ϕ, 56
$\lambda \vdash n$	A partition λ of n, 56
D_λ	The Young diagram of $\lambda \vdash n$, 56
T_λ	A Young tableau of the Young diagram D_λ, 57
$M_{d_\lambda}(F)$	The irreducible S_n-submodule of FS_n associated to $\lambda \vdash n$, 57
χ_λ	The character of $M_{d_\lambda}(F)$,57
R_{T_λ}	The row stabilizer of T_λ, 58
C_{T_λ}	The column stabilizer of T_λ, 58
e_{T_λ}	The essential idempotent associated to T_λ,58
P_n	The vector space of all multilinear polynomials of degree n, 59
$P_n(A)$	The subspace of $F\langle X\rangle/\,\mathrm{Id}(A)$ constituted by multilinear polynomials in the first n variables, 59
$\chi_n(A)$	The cocharacter of the algebra A, 59
$c_n(A)$	The codimension of the algebra A, 60
$l_n(A)$	The colength of the algebra A, 60
$c_{n,i}$	The i-th Capelli polynomial of degree $2n$, 65
$\mathrm{Ind}^*(\Gamma)$	The $*$-index of Γ, 65
$K(\Gamma)$	The Kemer set of Γ, 66
$n(A)$	The nilpotency index of $J(A)$, 67
$\mathrm{Par}^*(A)$	The 3-tuple $(d(B_0), d(B_1), n(A)-1)$, 67
$\bar{F}\langle Y \cup Z, *\rangle$	Free $*$-algebra on a set of finite variables, 90
UT_2	The algebra of 2×2 upper triangular matrices, 107
$\mathrm{ad}_\alpha(y)$	A inner derivation, 107
$F\langle Y \cup Z\rangle$	The free superalgebra on $Y \cup Z$, 108

$F\langle Y \cup Z \mid D_2 \rangle$	The free superalgebra on $Y \cup Z$ with action of D_2,109
$P_n^{\mathbb{Z}_2, D_2}$	Vector space of multilinear $\mathbb{Z}_2$-D_2-polynomials of degree n, 110
$\mathbb{Z}_2 \wr S_n$	Wreath product of Z_2 and S_n, 110
$\chi_{\lambda,\mu}$	The irreducible $\mathbb{Z}_2 \wr S_n$-character, 110
$\leqslant$	Binary relation, 116
$\overline{B}$	Closure of $B \subseteq A$, 116

Contents

Introduction

The theory of polynomial identities (PI-theory) plays an important role in ring theory and has been the subject of study of many algebraists for the past 70 years. In the early papers [17] and [62] the authors studied non commutative polynomials vanishing on an algebra, but the systematic study of algebras with polynomial identities only started in 1948 with Kaplansky's work [37]. In that paper it was proved that any primitive algebra satisfying a polynomial identity is a finite dimensional simple algebra over its center, suggesting an important finiteness condition for an algebra. A polynomial identity of an algebra A is a polynomial in non-commuting indeterminates vanishing under all evaluations in A and the algebras having at least one nontrivial polynomial identity (i.e., a nonzero polynomial) are called PI-algebras.

One of the main problems concerning the qualitative approach in PI-theory is determining the polynomial identities of specific algebras and studying the properties of the varieties they define. Although the most classic results in this area deal with polynomial identities for associative algebras over fields of characteristic 0, in the recent decades different classes of algebras with additional structure, such as group graded algebras or algebras with involution, have been studied in the context of PI-theory.

In this work we focus our attention on H-module algebras, where H is a finite dimensional Hopf algebra and on the setting of superalgebras with superinvolution. In particular, we study the so-called Hilbert series and the Specht property for these class of PI-algebras.

We outline briefly what the Specht problem is: given a variety of algebras (associative, Lie, Jordan, graded, etc.) one can ask whether or not any of its subvarieties is finitely generated ([57]). In the language of T-ideals (the ideals of polynomial identities of a given algebra), the Specht problem can be formulated as follows: given any algebra A, is it true that any T-ideal containing the T-ideal of A is finitely generated (or based) as a T-ideal? If we restrict our attention to the associative environment, the Specht problem was solved positively in [39] and [40] by Kemer provided the ground field of the algebras therein is of characteristic 0. His proof is based on deep structure theory of the T-ideals which has given a new impetus to the subject. Further generalizations of Kemer's result are due to Sviridova [59] (PI-algebras graded by a finite abelian group), Aljadeff and Kanel-Belov [2] (PI-algebras graded by a finite group), and Karasik [38] (PI-algebras that are module algebras under the action of a finite dimensional semisimple Hopf algebra).

On the other hand, Hilbert polynomials, Hilbert series or Hilbert-Poincaré series of graded (in a classical meaning) algebras are strongly related notions which attracted several mathematicians in the last century. The Hilbert series of an algebra represents a crucial algebraic tool in computational algebraic geometry, as it is the easiest known way for computing the dimension and the degree of an algebraic variety defined by explicit polynomial equations. We recall that the question of whether the Hilbert series of an algebra is the Taylor expansion of a rational function is fundamental in the commutative setting because of its relations with other invariants related to the growth of an algebra such as the Gelfand-Kirillov dimension or the Krull dimension of algebras.

Let F be a field and consider a graded (in a classical meaning) F-algebra A with finite generating set X. If we denote by $A^{(n)}$ the F-vector space generated by the monomials of degree n in the elements from X, then the **Hilbert series** of A is

$$\mathrm{Hilb}(A, t) = \sum_{n=0}^{\infty} \dim A^{(n)} t^n.$$

If A is a finitely generated (affine) commutative algebra, then the **Hilbert-Serre Theorem** says that the Hilbert series of A is rational ([6]). Nevertheless, such a theorem is not true in the case the algebra is non-commutative and, in this context, we cite the work [5] by Anick where the author showed a famous counterexample. Anyway, there is a big class of non-commutative affine algebras whose Hilbert series is rational. In case of relatively free algebras, that is, algebras isomorphic to the quotient of a finitely generated free algebra by a T-ideal, it is known that their Hilbert series is a rational function ([9]).

The analog of Hilbert-Serre Theorem for relatively free algebras carried a lot of results in PI-theory and we would like to cite among them the paper [12] by Berele and Regev in which the authors showed the exact asymptotic behaviour of the codimension sequence of a PI-algebra satisfying the Capelli identity. The analog of Hilbert-Serre Theorem also holds for classes of relatively free algebras with additional structure, such as the class of finitely generated G-graded relatively free algebras, where G is a finite group and the underlying graded T-ideal is the ideal of G-graded polynomial identities of a G-graded algebra satisfying an ordinary polynomial identity (Aljadeff and Kanel-Belov in [3]). We emphasize however that the rationality of the Hilbert series is not a corollary of representability since there are examples of representable algebras which have a transcendental (so non-rational) Hilbert series (see for instance, [36, Example 11.3.8]).

In this work, we present a proof of the Hilbert-Serre Theorem in the case of relatively free algebras of H-module algebras where H is a finite dimensional semisimple Hopf algebra (Theorem 3.2.6) and in the case of relatively free algebras of superalgebras with superinvolution (Theorem 2.5.15). In both cases we have to assume that the algebra satisfies an ordinary polynomial identity.

Superalgebras with superinvolution are a natural generalization of algebras with involution and they play a prominent role in the setting of Lie and Jordan algebras (see, for instance, [35, 51]). In recent years, such a kind of algebras has been extensively studied by several mathematicians. In particular the importance of such algebras has been highlighted in 2017 by Aljadeff, Giambruno and Karasik. In [1], they showed that any algebra with involution has the same identities of the Grassmann envelope of a finite dimensional superalgebras with superinvolution. The last result is a generalization of a classical result in PI-theory due to Kemer, known as the Representability Theorem.

We would like to point out that the structure of H-module algebra generalizes several notions such as gradings by finite abelian groups and involutions whereas superalgebras with superinvolution cannot be seen as H-module algebras.

In this work we also present a positive solution to the Specht problem in the case of finitely generated superalgebras with superinvolutions satifying an ordinary polynomial (Theorem 2.4.5) and in the case of H-module algebras generated by the algebra $UT_2(F)$ of 2×2 upper triangular matrices over a field of characteristic 0 containing a primitive m-th root of unit and where $H = H_m$ is a Taft's Hopf algebra of dimension m^2 (Theorem 4.3.8). As far as we know this is the first result in the literature toward Specht property of varieties of algebras under the action of a Taft's Hopf algebra.

This Ph.D. thesis is divided in four chapters. Chapter 1 is a review of the background on PI-theory. We introduce the basic definitions and we give an account of the main results of the structure theory of PI-algebras. The content is focused on showing the concepts necessary to understand the theoretical framework of the next chapters.

In Chapter 2 we deal with superalgebras with superinvolutions. We show an explicit form of the so-called *Kemer polynomials* which are crucial in the proof of the rationality of the Hilbert series of any relatively free algebra. We introduce the *Kemer index* for these algebras and finally we give a positive solution to the Specht problem and the rationality of the Hilbert series in this setting.

Chapter 3 is devoted to the proof of the Hilbert-Serre Theorem in the context of H-module algebras, where H is a finite dimensional semisimple Hopf algebra. If we specialize H with the dual algebra of the group algebra FG, where G is a finite abelian group, we get the notion of G-graded algebra and we would have a result analogous to that obtained by Aljadeff and Kanel-Belov in [3] for abelian groups.

Finally, in Chapter 4, we deal with the algebra $UT_2(F)$ of 2×2 upper triangular matrices with an action of a Taft's algebra H_m. We give a complete description of the space of its multilinear H_m-identities in the language of Young diagrams through the representation theory of the hyperoctahedral group. We finally prove that the variety of H_m-module algebras generated by $UT_2(F)$ has the Specht property.

Part of this work has been published in [13] and [14].

1 Preliminaries

In this chapter we will give the tools we are going to use to understand the Kemer's theory for superalgebras with superinvolution and for algebras with an Hopf action, and the Hilbert series of their relatively free algebra. As we only work with associative algebras with unity in this work, whenever we talk about algebra, we will be considering them associative with unity.

1.1 Algebras with polynomial identities

1.1.1 Basic properties of algebras

Let F be a field of any characteristic. We start with the basic definition of algebra.

Definition 1.1.1. *A vector space A is called an associative F-**algebra** if A is equipped with a function $(a,b) \mapsto ab$ from $A \times A$ to A, called **multiplication**, satisfying the following axioms:*

A1) $a(bc) = (ab)c$ *for all* $a, b, c \in A$.

A2) $a(b+c) = ab + ac$ *for all* $a, b, c \in A$.

A3) $(a+b)c = ac + bc$ *for all* $a, b, c \in A$.

A4) $\alpha(ab) = (\alpha a)b = a(\alpha b)$ *for all* $a, b, c \in A$, $\alpha \in F$.

An associative F-algebra A is called **unitary** if there exists an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

Example 1.1.2. *Some examples of (associative unitary) F-algebras:*

1. *Any field F is an algebra over F.*

2. *$M_n(F)$ the set of all $n \times n$ matrices with entries from F with the usual multiplication of matrices. Here 1 is the identity matrix I_n.*

3. *If V is any vector space over F, then $Hom_F(V,V)$ becomes an associative algebra over F when we define the product of two linear transformations T_1 and T_2 to be their composite $T_1 \circ T_2$. Here 1 is the identity map $\mathrm{Id}\colon V \to V$. Linear transformations from V to V are called **endomorphisms** of V. The algebra $End_F(V) = Hom_F(V,V)$ is called the **algebra of endomorphisms** of V.*

4. *Let x be a variable, then $F[x]$, the vector space of polynomials in one variable is an algebra with the usual product of polynomials.*

Definition 1.1.3. *A subspace S of an algebra A is called **subalgebra** if $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. The subalgebra I is called **left ideal** if $AI \subseteq I$. Similarly, I is called **right ideal** if $IA \subseteq I$. The subalgebra I is called **two-sided ideal** (or simply **ideal**) if I is both left and right ideal. An ideal I of A is called **proper** if $I \neq A$.*

An ideal I of an algebra A is said to be a **nilpotent ideal** if there exists a natural number k such that $I^k = 0$.

Definition 1.1.4. *Let A and B be F-algebras. The F-linear map $\varphi\colon A \to B$ is called **homomorphism of algebras** if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.*

Remark 1.1.5. *Let A be an algebra, I a ideal of A and $p\colon A \to A/I$ the canonical projection of F-vector spaces. Then there exists a unique algebra structure on A/I (called the **factor algebra**) such that p is a homomorphism of algebras.*

Remark 1.1.6. *Let A be a non-unitary associative F-algebra and let the direct sum of vector spaces $A_1 = F \oplus A$ be equipped with multiplication*

$$(\alpha_1 + a_1)(\alpha_2 + a_2) = \alpha_1\alpha_2 + (\alpha_1 a_2 + \alpha_2 a_1 + a_1 a_2),$$

*with $\alpha_1, \alpha_2 \in F$, $a_1, a_2 \in A$. Then A_1 is a unitary algebra. (we say that A_1 is obtained from A by **formal adjoint of unity**).*

Definition 1.1.7. *Let V and W be vector spaces over F with bases $\{v_i \mid i \in I\}$ and $\{w_j \mid j \in J\}$, respectively. The **tensor product** $V \otimes W$ of V and W is the vector space over F with basis $\{v_i \otimes v_j \mid i \in I, j \in J\}$.*

The tensor product $V \otimes W$ induces a bilinear map $\varphi\colon V \times W \to V \otimes W$ such that $(v, w) \mapsto v \otimes x$ is characterized by the following universal property: if $\phi\colon V \times W \to Z$ is any bilinear map from the cartesian product $V \times W$ to any vector space Z, then there exists a unique linear map $T\colon V \otimes W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} V \times V \arrow[r, "\varphi"] \arrow[dr, "\phi"] & V \otimes W \arrow[d, "T"] \\ & Z \end{tikzcd}$$

If V and W are algebras, then $V \otimes W$ is also an algebra with multiplication given by

$$(v_1 \otimes w_1)(v_2 \otimes w_2) = v_1 v_2 \otimes w_1 w_2, \qquad \text{for all } v_1, v_2 \in V, w_1, w_2 \in W.$$

Our main interest arises in the following important class of algebras:

Definition 1.1.8. *An algebra A is* ***affine*** *or* ***finitely generated*** *over a commutative ring C (or a field) if there exists a finite set of elements $a_1, \dots, a_l$ of A such that every element of A can be expressed as a C-linear combination of products in $a_1, \dots, a_l$. In this case we write $A = C\langle a_1, \dots, a_l\rangle$.*

In most cases, we shall be considering affine algebras over a field F, so unless specified otherwise, "affine" will mean "affine over a field".

Definition 1.1.9. *Let A be an algebra. An element $e \in A$ is* ***idempotent*** *if $e^2 = 2$. The* ***trivial idempotents*** *are 0 and 1. Two idempotents $e_1, e_2 \in A$ are* ***orthogonal*** *if $e_1e_2 = e_2e_1 = 0$*

Given a nontrivial idempotent $e \in A$. The **Peirce decomposition** of A is

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

More generally, if $e_1, \dots, e_n$ are mutually orthogonal idempotents with sum 1, then A can be decomposed as:

$$A = \bigoplus e_iAe_j \quad \text{for } 1 \leqslant i, j \leqslant n.$$

For algebras without 1, the Peirce decomposition is as follows: given any idempotent element e of A, define formally $e'a$ as $a - ea$, ae' as $a - ae$, and $e'ae'$ as $(e'a)e'$ for all $a \in A$. Now

$$A = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'.$$

More generally, if $e_1, \dots, e_n$ are mutually orthogonal idempotents, take $e = \sum_{i=1}^{t} e_i$, and thus, A can be decomposed as:

$$A = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'.$$

1.1.2 The Wedderburn-Malcev Theorem

Essential in this work will be the notion of simple subalgebras, semisimple subalgebras, and Jacobson radical of an algebra. In this section we introduce these concepts from the notion of modules over a ring R or, in short, an R-module and we will finish with the Wedderburn-Malcev Theorem which states that an algebra can be decomposed into its radical part and its semisimple part.

Definition 1.1.10. *The additive abelian group M is said to be an (left)* ***R-module*** *if there is a mapping $R \times M \to M$ sending (r, m) to rm such that:*

- $(r + q)m = rm + qm,$

- $r(m + n) = rm + rn$,
- $(rq)m = r(qm)$,

for all $r, q \in R$ *and* $m, n \in M$.

If R has unit element 1, and if $1m = m$ for all $m \in M$, we then describe M to be a unitary R-module.

Let A be an F-algebra. We can consider the notion of module over the F-algebra A simply taking $R = A$ in the definition above. In this case, we say that M is an (left) A-module.

Definition 1.1.11. *Let A be an F-algebra, and let M be an A-module. The **annihilator** of M, denoted* $\mathrm{Ann}(M)$ *is the set of all elements $a \in A$ such that, for all $m \in M$, $am = 0$. In set notation,*

$$\mathrm{Ann}(M) = \{a \in A \mid am = 0 \quad \forall m \in M\}.$$

Lemma 1.1.12. $\mathrm{Ann}(M)$ *is a two-sided ideal of A.*

Proof. Let $a \in A$ and $b \in \mathrm{Ann}(M)$. For any $m \in M$ we have $(ba)m = b(am)$, since $am \in M$ then $b(am) = 0$, which implies $\mathrm{Ann}(M)$ is a right ideal of A. On the other hand, $(ab)m = a(bm) = a(0) = 0$ implies $\mathrm{Ann}(M)$ is a left ideal of A □

An A-module M is called **irreducible** if $AM \neq 0$ and if the only submodules of M are $\{0\}$ and M. The A-module M is called **completely reducible** if it is isomorphic to a direct sum of irreducible modules.

Definition 1.1.13. *Let A be an algebra. The **Jacobson radical** $J(A)$ of A, is the set of all elements of A which annihilate all the irreducible A-modules. A finite-dimensional algebra A is said to be **semisimple** if $J(A) = \{0\}$.*

Note that $J(A) = \bigcap \mathrm{Ann}(M)$, where this intersection runs over all irreducible A-modules M. Since the $\mathrm{Ann}(M)$ are two-sided ideals of A, we see that $J(A)$ is a two-sided ideal of A. Moreover, $J(A)$ contains all nilpotent ideals, and if A is finite-dimensional, $J(A)$ itself is a nilpotent ideal.

Definition 1.1.14. *An algebra A is called **simple** if it has no proper ideals and $A^2 = \{ab \mid a, b \in A\} \neq \{0\}$.*

The following proposition characterizes semisimple algebras.

Proposition 1.1.15. *A finite dimensional algebra is semisimple if and only if it can be written as a direct sum of simple algebras.*

Finally, we recall a famous classical result by Wedderburn and Malcev:

Theorem 1.1.16 (Wedderburn-Malcev)**.** *For any finite dimensional algebra A over an algebraically closed field, there is a vector space isomorphism*

$$A \cong \bar{A} \oplus J$$

where $J = J(A)$ is nilpotent and $\bar{A}$ is a semisimple subalgebra of A isomorphic to A/J. Furthermore, if there is another decomposition $A = B \oplus J$, then there is an invertible $a \in A$ such that $B = a\bar{A}a^{-1}$.

Proof. See, for instance, [30, Theorem 3.4.3]. □

Remark 1.1.17. *Suppose A is a finite dimensional algebra without 1. Consider the Wedderburn-Malcev decomposition $A = \bar{A} \oplus J(A)$. The semisimplicity of $\bar{A}$ implies $\bar{A}$ has a unit element e which is idempotent in A. Adjoint a unit element 1 to A as in Remark 1.1.6, and note that $\hat{1} = (1, 0)$ is the multiplicative unit of $A_1 = F \oplus A$. Define $e' = \hat{1} - e$. Now we embed our Pierce decomposition of A into*

$$A_1 = eA_1e \oplus eA_1e' \oplus e'A_1e \oplus e'A_1e'.$$

More generally, if $e_1, \dots, e_n$ are mutually orthogonal idempotents of A, take $e = \sum_{i=1}^{t} e_i$. Again, define $e_0 = \hat{1} - e \in A_1$ and

$$A_1 = (e + e_0)A_1(e + e_0) = \bigoplus_{i,j=0}^{n} e_iA_1e_j.$$

1.1.3 Free algebras and polynomial identities

Let F be a field and $X = \{x_1, x_2, \dots\}$ a countable set of variables. The algebra $F\langle X\rangle$ whose basis consists of all the words in the alphabet X (including the empty word 1) and multiplication defined by juxtaposition of words, is the **associative unitary free algebra** (or simply, **free algebra**) generated by X over F. Each word is called **monomial** and the elements of $F\langle X\rangle$ are called **polynomials** in the non-commuting variables X. If $f \in F\langle X\rangle$ we will write $f = f(x_1, \dots, x_m)$ to indicate that $x_1, \dots, x_m \in X$ are the only indeterminates appearing in f. The cardinality of X is called the **rank** of $F\langle X\rangle$.

The algebra $F\langle X\rangle$ is defined by the following universal property: if $g\colon X \to A$ is a map from X to an unitary F-algebra A, then there exists a unique homomorphism of algebras $\alpha\colon F\langle X\rangle \to A$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "g"'] & F\langle X\rangle \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

here, $i\colon X \to F\langle X\rangle$ is the inclusion map.

Definition 1.1.18. *Let A be an F-algebra and $f = f(x_1, \dots, x_m) \in F\langle X\rangle$ a polynomial, then we say that f is a **polynomial identity for** A if $f(a_1, \dots a_m) = 0$ for all $a_1, \dots, a_m \in A$, and we write $f \equiv 0$.*

Consequently, $f \in F\langle X\rangle$ is a polynomial identity for A if and only if f is in the kernel of all homomorphisms $F\langle X\rangle \to A$.

Definition 1.1.19. *If an algebra A satisfies a nontrivial polynomial identity $f \equiv 0$ (i.e., f is a nonzero element of $F\langle X\rangle$), we say that A is a **PI-algebra**.*

Example 1.1.20. *If A is a commutative algebra, then A is a PI-algebra, since it satisfies the identity $[x_1, x_2] \equiv 0$ where $[x_1, x_2] = x_1x_2 - x_2x_1$ is called the **Lie commutator** of x_1 and x_2.*

Example 1.1.21. *An associative algebra without unit A is said to be a **nilpotent algebra** if there exists some integer $n \geqslant 0$ such that $a_1a_2\cdots a_n = 0$ for all $a_1, a_2, \dots, a_n \in A$. Clearly A is a PI-algebra, because it satisfies the polynomial identity $x_1 \cdots x_n \equiv 0$.*

Example 1.1.22. *Let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices over F. Then $UT_n(F)$ is a PI-algebra since it safisfies the identity*

$$[x_1, x_2][x_3, x_4]\cdots[x_{2n-1}, x_{2n}] \equiv 0.$$

Example 1.1.23. *Let A be a finite dimensional associative algebra and let $n > \dim A$. Then A satisfies the **standard polynomial** of degree n*

$$s_n(x_1. \dots, x_n) = \sum_{\sigma\in S_n} (\operatorname{sgn}\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)},$$

where S_n is the symmetric group of order n.

1.1.4 T-ideals and varieties

We now turn to a general description of the set of identities of an algebra and its varieties.

Definition 1.1.24. *An ideal I of $F\langle X\rangle$ is called **T-ideal** if $\varphi(I) \subseteq I$ for all endomorphism φ of $F\langle X\rangle$.*

Let A be an F-algebra, we define

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\},$$

the set of all the polynomial identities of A. Then $\mathrm{Id}(A)$ is a T-ideal. Indeed, since each endomorphism ϕ of $F\langle X\rangle$ is defined by the image of X, i.e., by $\phi(x_i) = g_i \in F\langle X\rangle$, then $f(g_1, \dots, g_n) \in \mathrm{Id}(A)$ for all $f \in \mathrm{Id}(A)$.

Remark 1.1.25. *Each T-ideal of $F\langle X\rangle$ is the set of polynomial identities of some algebra. In fact, if I is a T-ideal, then $I = \mathrm{Id}(F\langle X\rangle/I)$. Indeed, for all $g_1, \ldots, g_n \in F\langle X\rangle$,*

$$\begin{aligned} f \in \mathrm{Id}(F\langle X\rangle/I) &\Leftrightarrow f(g_1 + I, \ldots, g_n + I) = I \\ &\Leftrightarrow f(g_1, \ldots, g_n) + I = I \\ &\Leftrightarrow f(g_1, \ldots, g_n) \in I \\ &\Leftrightarrow f \in I. \end{aligned}$$

Let S be a set of polynomials in $F\langle X\rangle$. The **T-ideal generated** by S in $F\langle X\rangle$ is the smallest T-ideal of $F\langle X\rangle$ containing S, and it is denoted by $\langle S\rangle_T$. The following proposition characterizes $\langle S\rangle_T$.

Proposition 1.1.26. *Let $S = \{f_i \mid i \in I\}$ be a set of polynomials. Then the T-ideal generated by S is the set*

$$\langle S\rangle_T = \left\{ \sum_{i \in J \subset I} \alpha_i u_i f_i(g_{i1}, \ldots, g_{in_i}) v_i \mid \alpha_i \in F, u_i, g_{ij}, v_i \in F\langle X\rangle, f_i \in S, J \text{ finite} \right\}.$$

Proof. Clearly, the right side of the previous expression is a T-ideal containing S. To see that it is the smallest T-ideal containing S, suppose K is a T-ideal of $F\langle X\rangle$ containing S, then $u_i f_i(g_{i1}, \ldots, g_{in_i}) v_i \in K$ for each i because f_i belong to $S \subset K$, K is an ideal and K is invariant under all endomorphism and the proof is complete. □

Definition 1.1.27. *Let S be a set of polynomials in $F\langle X\rangle$ and $f \in F\langle X\rangle$. We say that f is a **consequence** of the polynomials in S if $f \in \langle S\rangle_T$, the T-ideal generated by the set S.*

Definition 1.1.28. *Two sets of polynomials are said to be **equivalent** if they generate the same T-ideal.*

Definition 1.1.29. *Two algebras A_1, A_2 are called **PI-equivalent** if $\mathrm{Id}(A_1) = \mathrm{Id}(A_2)$; in this case we write $A_1 \sim_{PI} A_2$.*

Definition 1.1.30. *Given a non-empty set $S \subseteq F\langle X\rangle$, the class of all associative algebras A such that $f \equiv 0$ on A for all $f \in S$ is called the **variety** $\mathcal{V} = \mathrm{var}(S)$ **determined** by S. A variety $\mathcal{V} = \mathrm{var}(S)$ is non-trivial if $S \neq \{0\}$.*

For example, the class of commutative algebras forms a variety of algebras, because each commutative algebra satisfies the polynomial identity $[x_1, x_2] = x_1x_2 - x_2x_1$.

Definition 1.1.31. *Let $\mathcal{V}$ be a variety, $A \in \mathcal{V}$ an algebra and $Y \subseteq A$ a subset of A. We say that A is **relatively free on** Y (with respect to $\mathcal{V}$) if for any function $g\colon Y \to B$*

from Y to $B \in \mathcal{V}$, there exists a unique homomorphism of algebras $\beta\colon A \to B$ such that the following diagram commutes, where i is the inclusion map $i\colon Y \to A$.

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "g"'] & A \arrow[d, "\beta"] \\ & B \end{tikzcd}$$

There is a one-to-one correspondence between T-ideals of $F\langle X\rangle$ and varieties of algebras. Let $\mathcal{V}$ be a variety with corresponding ideal $\mathrm{Id}(\mathcal{V}) \subseteq F\langle X\rangle$. Then $F\langle X\rangle/\,\mathrm{Id}(\mathcal{V})$ is a relatively free algebra on the set $\bar{X} = \{x + \mathrm{Id}(\mathcal{V}) \mid x \in X\}$. Moreover, any two relatively free algebras with respect to $\mathcal{V}$ of the same rank are isomorphic ([22, Proposition 2.2.5]).

We close this section with a theorem of Birkhoff, which gives the properties characterizing the varieties.

Theorem 1.1.32 (Birkhoff)**.** *A non-empty class of algebras $\mathcal{V}$ is a variety if and only if $\mathcal{V}$ is closed under taking Cartesian sums, subalgebras and factor algebras.*

Proof. See for instance [22, Theorem 2.3.2]. □

1.1.5 The Specht problem

One of the most interesting questions about T-ideals is whether the generating set S of a T-ideal can be reduced to a finite set, which generates the same T-ideal. We can see this question as the analogue in non-commutative algebra of the Hilbert's Basis Theorem for commutative algebras which states that every algebraic variety can be defined by a finite set of commutative polynomials. In order to formally establish the problem, we begin with the following definition.

Definition 1.1.33. *A variety of algebras $\mathcal{V}$ is called **finitely based** if $\mathcal{V}$ can be determinded by a finite set of polynomial identities (from $F\langle X\rangle$). If $\mathcal{V}$ cannot be determined by a finite set of identities, it is called **infinitely based**. If all subvarieties of $\mathcal{V}$, including $\mathcal{V}$ itself, are finitely based, $\mathcal{V}$ satisfies the **Specht property**.*

The following problem was posed by Specht in 1950 for associative algebras over a field of characteristic 0. Now it is known as the **Specht problem** ([57]).

Problem 1.1.34. *Is every variety of associative algebras finitely based?*

In 1987 Kemer gave a positive solution for the Specht problem for associative algebras over a field of characteristic 0.

Theorem 1.1.35. *[40, Theorem 1] Every variety of associative algebras over a field of characteristic* 0 *is finitely based.*

To achieve this, Kemer developed a powerful technique which is known as *Kemer's Theory*. This technique is contained mostly in his monograph ([41]). The key step in Kemer's Theory is the Representability Theorem for affine PI-algebras. In the paper [4] the authors provide a much more detailed proof of this theorem.

Definition 1.1.36. *An algebra W is* ***PI-representable*** *if $W \sim_{PI} A$ for some algebra A which is finite dimensional over some field.*

Theorem 1.1.37 (Kemer's Representability Theorem)**.** *Let W be an affine PI-algebra over a field F of characteristic zero. Then W is PI-representable.*

Kemer's theory, besides being quite technical and sophisticated, contains a remarkable number of new ideas opening new avenues of research in the study of varieties of algebras. We outline the main steps of the proof of Kemer's theorem.

Step 1. Show that there exists a finite dimensional algebra A with $\mathrm{Id}(A) \subseteq \Gamma = \mathrm{Id}(W)$.

Step 2. Definition of $\mathrm{Ind}(\Gamma) = (\alpha, s)$, the Kemer index of any T-ideal Γ which contains the T-ideal of a finite dimensional algebra A.

Step 3. Definition of Kemer polynomials of a T-ideal Γ. These are extremal polynomials which are not in Γ whose alternation realize the Kemer index $\mathrm{Ind}(\Gamma)$.

Step 4. Construction of basic algebras, in which the parameters α and s of $\mathrm{Ind}(\Gamma)$ coincide respectively with the integers $\dim \bar{A}$ and $n_A - 1$, where $\bar{A}$ is the semisimple part of A and n_A is the nilpotency index of the Jacobson radical of A.

Step 5. From the connection between the parameters of the Kemer index of any basic algebra A and its geometrical properties (namely $\mathrm{Ind}(\Gamma) = (\dim \bar{A}, n_A - 1)$), we obtain the Phoenix property of Kemer polynomials of A.

Step 6. Find a finite dimensional algebra B with $\mathrm{Id}(A) \subseteq \mathrm{Id}(B) \subseteq \Gamma$ such that $\mathrm{Id}(B)$ and Γ have the same Kemer index and have the same Kemer polynomials.

Step 7. Construction of the representable algebra $B_{(\alpha,s)}$ over F with $\mathrm{Id}(B_{\alpha,s)}) \supseteq \Gamma$ and such that all Kemer polynomials of Γ are non-identities of $B_{(\alpha,s)}$.

Step 8. Consider $\Gamma' = \Gamma + S$ where S is the T-ideal generated by all Kemer polynomials of Γ.

Step 9. Show that all polynomials of S are non-identities of $B_{(\alpha,s)}$. From that one concludes that $\Gamma = \mathrm{Id}(A' + B_{(\alpha,s)})$ where A' is over a field extension L of F with $\Gamma' = \mathrm{Id}(A')$.

Most of these steps will be emulated in Chapter 2 to develop Kemer's Theory for superalgebras with superinvolutions.

1.2 Graded algebras

In this section we study algebras which are graded by a finite group. We consider the reduction of arbitrary polynomial identities to polynomial identities of special form: homogeneous, multilinear and symmetric. We introduce the notion of free graded algebras and the tensor algebra and define the involution map in graded algebras, in particular, the so called superinvolution for algebras graded by the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

1.2.1 Graded vector spaces

We will start with the classical notion of graded vector spaces.

Definition 1.2.1. *Let V be a F-vector space. We say that V is **graded** if it is a direct sum of its subspaces $V^{(n)}$, $n \geqslant 0$, i.e.*

$$V = \bigoplus_{n\geqslant 0} V^{(n)}.$$

The subspaces $V^{(n)}$ are called the **homogeneous components** of degree n of V. The subspace W of the graded vector space $V = \oplus_{n\geqslant 0} V^{(n)}$ is a **graded subspace** if $W = \oplus_{n\geqslant 0}(W \cap V^{(n)})$. In this case, the factor space V/W can also be naturally graded.

Example 1.2.2. *The polynomial algebra $F[x_1, \dots, x_m]$ is graded assuming that the homogeneous polynomials of degree n (in the usual sense) are the homogeneous elements of degree n.*

Remark 1.2.3. *If A, B are graded vector spaces, then $A \otimes B$ can be graded via*

$$(A \otimes B)^{(n)} = \bigoplus_{j+k=n} A^{(j)} \otimes B^{(k)}.$$

Analogously we can define the notion of grading for algebras: Let A be an F-algebra. We say that A is **graded algebra** if it is a direct sum of its subalgebras $A^{(n)}$, $n \geqslant 0$, i.e.

$$A = \bigoplus_{n\geqslant 0} A^{(n)}.$$

and if $A^{(n)}A^{(m)} \subseteq A^{(n+m)}$ for each $n, m \geqslant 0$.

We present an example of graded algebra: the tensor algebra. We show first the categorical definition.

Definition 1.2.4. *Let V be a F-vector space. A **tensor algebra** of V is a pair (X, i), where X is an F-algebra and $i\colon V \to X$ is a F-linear map, such that the following universal property is satisfied: for any F-algebra A, and any F-linear map $f\colon V \to A$, there exists a unique homomorphism of algebras $\phi\colon X \to A$ such that $\phi i = f$, that is, the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & X \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

We present its construction. Denote by $T^0(V) = F$, $T^1(V) = V$, and for $n \geqslant 2$ by $T^n(V) = V \otimes V \otimes \cdots \otimes V$, the tensor product of n copies of the vector space V. Define

$$T(V) = \bigoplus_{n \geqslant 0} T^n(V),$$

and $i\colon V \to T(V)$ by $i(v) = v \in T^1(V)$ for any $v \in V$. On $T(V)$ we define the multiplication as follows: if $x = v_1 \otimes \cdots \otimes v_n \in T^n(V)$ and $y = w_1 \otimes \cdots \otimes w_m \in T^m(V)$, then define the product

$$x \cdot y = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m \in T^{n+m}(V).$$

The multiplication of two arbitrary elements from $T(V)$ is obtained by extending the above formula by linearity. In this way $T(V)$ becomes an associative unitary F-algebra with identity element $1 \in T^0(V)$, and the pair $(T(V), i)$ is a tensor algebra of V.

The tensor algebra $A = T(V)$ has a natural grading by setting $A = \oplus_{n \geqslant 0} A^{(n)}$ where $A^{(n)} = T^n(V)$ for all $n \geqslant 0$.

Remark 1.2.5. *Let V be a F-vector space with a countable basis $\{v_1, v_2, \dots\}$. The tensor algebra $T(V)$ is just the free associative algebra (defined in page 24) generated by $\{v_1, v_2, \dots\}$ over F.*

Now, we present the notion of Hilbert series for vector spaces. We denote by $\dim(W)$ the dimension of an arbitrary F-vector space W.

Definition 1.2.6. *Let $V = \oplus_{n \geqslant 0} V^{(n)}$ be a graded vector space and let $\dim V^{(n)} < \infty$ for all $n \geqslant 0$. The formal power series*

$$\mathrm{Hilb}(V, t) = \sum_{n \geqslant 0} \dim V^{(n)} t^n,$$

*is called the **Hilbert series** of V.*

For a function $f(t)$, we make the usual convention that $\mathrm{Hilb}(V,t) = f(t)$ if the series $\mathrm{Hilb}(V,t)$ converges in some neighbourhood of 0 and the function $\mathrm{Hilb}(V,t)$ and $f(t)$ are equal here.

Example 1.2.7. *Let $V = F[x]$ be the polynomial algebra in one indeterminate x. Then $V^{(n)} = Fx^n$, so each $\dim V^{(n)} = 1$. Thus,*

$$\mathrm{Hilb}(F[x],t) = \sum_{n\geqslant 0} t^n = \frac{1}{1-t}.$$

Example 1.2.8. *Let $V = F\langle x_1,\dots,x_m\rangle$ be the free algebra. There are m^n words of length n, so $\dim V^{(n)} = m^n$. Thus,*

$$\mathrm{Hilb}(F\langle x_1,\dots,x_m\rangle,t) = 1 + mt + m^2t^2 + \cdots = \frac{1}{1-mt}.$$

Proposition 1.2.9. *Let A, B be graded vector spaces. Then*

1. $\mathrm{Hilb}(A\oplus B,t) = \mathrm{Hilb}(A,t) + \mathrm{Hilb}(B,t)$;
2. $\mathrm{Hilb}(A\otimes B,t) = \mathrm{Hilb}(A,t)\mathrm{Hilb}(B,t)$;
3. $\mathrm{Hilb}(A,t) = \mathrm{Hilb}(A/B,t) + \mathrm{Hilb}(B,t)$, *if B is a subspace of A.*

Proof. (1) Since $\dim(A\oplus B) = \dim A + \dim B$, we have

$$\begin{aligned}\mathrm{Hilb}(A\oplus B,t) &= \sum_{n\geqslant 0}\dim((A\oplus B)^{(n)})t^n\\ &= \sum_{n\geqslant 0}\dim(A^{(n)}\oplus B^{(n)})t^n\\ &= \sum_{n\geqslant 0}\dim A^{(n)}t^n + \dim B^{(n)}t^n\\ &= \mathrm{Hilb}(A,t) + \mathrm{Hilb}(B,t).\end{aligned}$$

(2) Since $\dim(A\otimes B) = \dim A\dim B$, and by Remark 1.2.3 we have

$$\begin{aligned}\mathrm{Hilb}(A\otimes B,t) &= \sum_{n\geqslant 0}\dim((A\otimes B)^{(n)})t^n\\ &= \sum_{n\geqslant 0}\dim(\bigoplus_{j+k=n} A^{(j)}\otimes B^{(k)})t^n\\ &= \sum_{n\geqslant 0}\sum_{j+k=n}\dim A^{(j)}\dim B^{(k)}t^n\\ &= \sum_{j\geqslant 0}\dim A^{(j)}t^j\sum_{k\geqslant 0}\dim B^{(k)}t^k\\ &= \mathrm{Hilb}(A,t)\mathrm{Hilb}(B,t).\end{aligned}$$

(3) Since $\dim B + \dim(A/B) = \dim A$, the result follows. □

Example 1.2.10. $V = F[x_1,\dots,x_m] = \otimes_{i=1}^m F[x_i]$. *Then by the previous proposition and Example 1.2.7,*

$$\mathrm{Hilb}(V,t) = \mathrm{Hilb}(\otimes_{i=1}^m F[x_i],t) = \mathrm{Hilb}(F[x_1],t)\cdots\mathrm{Hilb}(F[x_m],t) = \frac{1}{(1-t)^m}.$$

1.2.2 Group-graded algebras

In this section we study the algebras graded by a group G.

Definition 1.2.11. *Let A be an algebra over a field F and let G be a group, we say that A is G-**graded algebra** if A can be written as the direct sum of subspaces $A = \bigoplus_{g\in G} A^{(g)}$ such that for all $g, h \in G$, $A^{(g)}A^{(h)} \subseteq A^{(gh)}$.*

The subspaces $A^{(g)}$ are called **homogeneous components of** A. Consequently, an element $a \in A$ is homogeneous of **degree** g if $a \in A^{(g)}$.

Given $a \in A$, we can write $a = \sum_{g\in G} a_g$ (uniquely) where $a_g \in A^{(g)}$, so any element can be written uniquely as a sum of homogeneous elements. In particular, $(a+b)_g = a_g + b_g$ for all $a, b \in A$ and all $g \in G$. Write e for the identity element of G. Then $A^{(e)}$ is always a subalgebra of A.

Example 1.2.12. *Any algebra A can be graded by a group G by setting $A = A^{(e)}$ and $A^{(g)} = 0$ for any $g \neq e$. This grading is called trivial.*

Example 1.2.13. *Let $A = M_2(F)$ and $G = \mathbb{Z}_2$. If we set $A^{(0)} = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid a, d \in F\}$ and $A^{(1)} = \{\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \mid b, c \in F\}$, then A is $\mathbb{Z}_2$-graded algebra.*

Example 1.2.14. *If $A = M_n(F)$ (the algebra of $n \times n$ matrices with entries of F) and G is a group, let $(g_1, \dots, g_n)$ be a n-tuple of elements of G, then A is G-graded by an **elementary grading** if we set $A^{(g)} = Span_F\{e_{pq} \mid \|e_{pq}\| = g\}$, where e_{pq} are the **matrix units** (i.e, matrices whose entries are all 0 except in the cell (p, q) whose value is 1), $\|e_{pq}\| = g_q g_p^{-1}$ and $A = \oplus_{g\in G} A^{(g)}$. The previous example is a particular case, taking the couple $(0, 1)$ of $G = \mathbb{Z}_2$.*

Definition 1.2.15. *Let A and B be G-graded algebras. A function $g \colon A \to B$ is called G-**graded homomorphism** if g is a homomorphism of algebras and $g(A^{(g)}) \subseteq B^{(g)}$ for all $g \in G$.*

Definition 1.2.16. *An ideal I of A is a **graded ideal** if I is graded as a subalgebra of A.*

Thus, I is a graded ideal of A if and only if $I = \sum_{g\in G} I^{(g)}$ where $I^{(g)} = I \cap A^{(g)}$, i.e., each element of I is a sum of homogeneous elements of I. It is easy to see that an ideal is graded if and only if it is generated by homogeneous elements.

Remark 1.2.17. *If I is a graded ideal of A, then $A/I = \bigoplus_{g\in G} A^{(g)}/I^{(g)}$ is G-graded as an algebra, where $(a + I)_g = a_g + I^{(g)}$.*

1.2.3 Homogeneous and multilinear polynomials

Let $F_n = F\langle x_1, \ldots, x_n\rangle$ be the free algebra of rank $n \geqslant 1$ over F. We can write

$$F_n = F_n^{(0)} \oplus F_n^{(1)} \oplus F_n^{(2)} \oplus \cdots$$

where $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k, for $k \geqslant 0$. Since $F_n^{(i)} F_n^{(j)} \subseteq F_n^{(i+j)}$, for all $i, j \geqslant 0$, then F_n is graded by the degree or that it has a structure of graded algebra. Then the $F_n^{(i)}$'s are the homogeneous components of F_n.

Definition 1.2.18. *A polynomial f is **linear in the variable** x_i if x_i occurs with degree 1 in every monomial of f. A polynomial which is linear in each of its variables is called **multilinear**.*

For a multilinear polynomial, we can write

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $\alpha_\sigma \in F$ and S_n is the symmetric group on $\{1, \ldots, n\}$. Moreover, if $f(x_1, \ldots, x_n)$ is a linear polynomial in one variable, say x_1, then

$$f(\sum \alpha_i y_i, x_2, \ldots, x_n) = \sum \alpha_i f(y_i, x_2, \ldots, x_n),$$

for every $\alpha_i \in F, y_i \in F\langle X\rangle$.

Proposition 1.2.19. *Let A be an algebra and*

$$f(x_1, \ldots, x_m) = \sum_{i=0}^{n} f_i \in F\langle X\rangle,$$

where f_i is the homogeneous component of f of degree i in x_1. If the base field F contains more than n elements (e.g. F is infinite), then if $f \equiv 0$ is a polynomial identity for the algebra A, then every homogeneous component f_i, $i = 0, 1, \ldots, n$ is still a polynomial identity for A.

Proof. Choose $n+1$ different elements $\alpha_0, \alpha_1, \ldots, \alpha_n$ of F. Since $\mathrm{Id}(A)$ is a T-ideal, for every $j = 0, \ldots, n$,

$$f(\alpha_j x_1, \ldots, x_m) \in \mathrm{Id}(A)$$

and therefore, for each $j = 0, \ldots, n$,

$$f(\alpha_j x_1, x_2, \ldots, x_m) = \sum_{i=0}^{n} f_i(\alpha_j x_1, x_2, \ldots, x_m) = \sum_{i=0}^{n} \alpha_j^i f_i(x_1, x_2, \ldots, x_m) \in \mathrm{Id}(A). \quad (1.1)$$

Consider the **Vandermonde matrix**

$$\Delta = \begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^n \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^n \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^n \end{pmatrix}.$$

If for every $a_1, \ldots, a_m \in A$ we write $\bar{f}_i := f_i(a_1, \ldots, a_m)$ for $i = 0, \ldots, n$, then Equation (1.1) says that

$$\Delta \begin{pmatrix} \bar{f}_0 \\ \vdots \\ \bar{f}_n \end{pmatrix} = 0. \tag{1.2}$$

It is known that the determinant of the Vandermonde matrix is $\det(\Delta) = \prod_{i<j} (\alpha_j - \alpha_i)$, and it is different from 0, then the homogeneous system (1.2) only has the trivial solution $\bar{f}_0 = 0, \ldots, \bar{f}_n = 0$. Thus $f_0 \equiv 0, \ldots, f_n \equiv 0$ are identities of A, i.e. the polynomial identities $f_i \equiv 0$ are consequences of $f \equiv 0$. □

In the proof of the next theorem we shall use the so-called **process of multilinearization** and can be described briefly as follows:

Suppose the polynomial $f(x_1, \ldots, x_m)$ has degree $n > 1$ in the variable x_1. Define the **partial linearization**

$$h(y_1, y_2, x_2, \ldots, x_m) := f(y_1 + y_2, x_2, \ldots, x_m) - f(y_1, x_2, \ldots, x_m) - f(y_2, x_2, \ldots, x_m).$$

Notice that h is still a polynomial identity for A when $f \in \mathrm{Id}(A)$.

In the situation in which x_1 does not appear in each monomial of f, we can define g $= f(0, x_2, \ldots, x_m)$. If $f \in \mathrm{Id}(A)$, then $g \in \mathrm{Id}(A)$. Thus $f - g \in \mathrm{Id}(A)$, so we can replace f by $f - g$ and thereby assume that any indeterminate appearing in f appears in each monomial of f as desired.

Let $n > 1$ the degree of x_1 in f. Iterating the partial linearization procedure $n - 1$ times (each time introducing a new intedeterminate y_i) yields an n-linear polynomial $\bar{f}(y_1, \ldots, y_n, x_2, \ldots, x_m)$. For each monomial in f we now have $n!$ monomials in $\bar{f}$. Thus. when f is homogeneous in x_1, we have

$$\bar{f}(x_1, \cdots, x_1, x_2, \ldots, x_m) = n! f.$$

We call $\bar{f}$ the **linearization** of f in x_1. In characteristic 0 we can recover f from $\bar{f}$.

Repeating the linearization process for each indeterminate appearing in f yields a multilinear polynomial, called the **complete linearization** of f.

Example 1.2.20. *The multilinearization of the polynomial x^n is called the* ***symmetric polynomial***

$$\tilde{s}_n = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Theorem 1.2.21. *If the characteristic of the base field F is zero, then every non-zero polynomial $f \in F\langle X\rangle$ is equivalent to a finite set of multilinear polynomials.*

Proof. By Proposition 1.2.19 we may assume that $f(x_1, \ldots, x_m)$ is homogeneous in each of its variables. Let $\deg_{x_1} f = d$. We write

$$f(y_1 + y_2, x_2, \ldots, x_m) = \sum_{i=0}^{d} f_i(y_1, y_2, x_2, \ldots, x_m),$$

where f_i is the homogeneous component of degree i in y_1. Hence $f_i \in V$, for each $i \in \{0, 1, \ldots, d\}$. Since $\deg_{y_j} < d$, $i = 0, 1, \ldots, d-1$, $j = 1, 2$, we apply inductive arguments and obtain a set of multilinear consequences of $f = 0$. Also, since the characteristic of F is zero, the binomial coefficient $\binom{d}{i}$ is different from 0, then

$$f_i(y_1, y_1, x_2, \ldots, x_m) = \binom{d}{i} f(y_1, x_2, \ldots, x_m),$$

and this implies that the multilinear identities are equivalent to $f = 0$. □

Corollary 1.2.22. *If the characteristic of the base field F is zero, each T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains.*

Actually, we conclude that in order to study the polynomial identities of a algebra over a field F of characteristic zero, we just need to find the multilinear polynomial identities.

1.2.4 Free graded algebras

Let $F\langle X\rangle$ be the free algebra over F on a countable set X and let G be a finite group. We write X in the form

$$X = \bigcup_{g \in G} X^{(g)},$$

where $X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$ are disjoint sets. The indeterminates from $X^{(g)}$ are said to be of homogeneous degree g. The **homogeneous degree** (or **G-degree**) of a monomial $x_{i_1}^{(g_1)} \cdots x_{i_t}^{(g_t)} \in F\langle X\rangle$ is defined to be $g_1 g_2 \cdots g_t$, as opposed to its **total degree**, which is defined to be t. Denote by $F\langle X\rangle^{(g)}$ the subspace of the algebra $F\langle X\rangle$ generated by all the monomials having homogeneous degree g. Notice that $F\langle X\rangle^{(g)} F\langle X\rangle^{(h)} \subseteq F\langle X\rangle^{(gh)}$ for every $g, h \in G$. It follows that

$$F\langle X\rangle = \bigoplus_{g \in G} F\langle X\rangle^{(g)}$$

is a G-grading on $F\langle X\rangle$. We denote by $F^G\langle X\rangle$ the algebra $F\langle X\rangle$ endowed with this grading.

Definition 1.2.23. *$F^G\langle X\rangle$ is called the **free G-graded algebra** of countable rank over F.*

The algebra $F^G\langle X\rangle$ has the following universal property: if $f\colon X \to A$ is any map from X to any G-graded algebra A such that $f(X^{(g)}) \subseteq A^{(g)}$ for each $g \in G$, then there exists a unique homomorphism of G-graded algebras $\alpha\colon F^G\langle X\rangle \to A$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F^G\langle X\rangle \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

where $i\colon X \to F^G\langle X\rangle$ is the inclusion map.

Given A a G-graded algebra and $f = f(x_1^{(g_1)}, \dots, x_m^{(g_m)}) \in F^G\langle X\rangle$ a graded polynomial, we say that f is a **graded identity for** A if $f(a_1^{(g_1)}, \dots a_m^{(g_m)}) = 0$ for all $a_1^{(g_1)} \in A^{(g_1)}, \dots, a_m^{(g_m)} \in A^{(g_m)}$, and we write $f \equiv 0$.

Consequently, $f \in F^G\langle X\rangle$ is a graded identity for A if and only if f is in the kernel of all graded homomorphisms $F^G\langle X\rangle \to A$.

Definition 1.2.24. $\mathrm{Id}^G(A) = \{f \in F^G\langle X\rangle \mid f \equiv 0$ *on* $A\}$ *is called the **ideal of graded identities** of A.*

$\mathrm{Id}^G(A)$ is a two-side ideal of the free G-graded algebra $F^G\langle X\rangle$. Moreover, $\mathrm{Id}^G(A)$ is a T-ideal of G-graded identities of A. As in the classical case, the T-ideals of G-graded identities are generated by multilinear polynomials.

In [2], the authors proved the G-grading version of the Representability Theorem (Theorem 1.1.37) and the Specht problem (Theorem 1.1.35). We present below the affine case.

Let W be a finitely generated associative PI-algebra over a field F of characteristic zero. Assume $W = \sum_{g\in G} W^{(g)}$ is G-graded where G is a finite group.

Theorem 1.2.25. *[2, Theorem 1.1] There exists a field extension K of F and a finite dimensional G-graded algebra A over K such that* $\mathrm{Id}^G(W) = \mathrm{Id}^G(A)$ *in* $F^G\langle X\rangle$.

Theorem 1.2.26. *[2, Theorem 1.2]* $\mathrm{Id}^G(W)$ *is finitely generated as a T-ideal in* $F^G\langle X\rangle$.

1.2.5 Superalgebras and superinvolutions

We shall call **superalgebra** any $\mathbb{Z}_2$-graded (associative) algebra. In this case $A = A^{(0)} \oplus A^{(1)}$ and the subspaces $A^{(0)}$ and $A^{(1)}$ are called the **even** and the **odd component** of A respectively and their elements are called homogeneous of degree zero (**even elements**) and of degree one (**odd elements**), respectively. If a is a homogeneous element we shall write $\deg(a)$ or $|a|$ to indicate its homogeneous degree.

In what follows, the field F is supposed to be of characteristic zero.

Notation 1.2.27. *From now on, for simplicity, we denote the subspaces $A^{(0)}$ and $A^{(1)}$ by A_0 and A_1 respectively.*

Recall from Definition 1.2.15, if $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are two superalgebras, then $\varphi : A \to B$ is a graded homomorphism if $\varphi(A_i) \subseteq B_i$, $i = 0, 1$.

Definition 1.2.28. *A* ***superinvolution*** *on a superalgebra $A = A_0 \oplus A_1$ is a graded map $* : A \to A$ such that:*

1. $(a^*)^* = a$, *for all* $a \in A$,
2. $(ab)^* = (-1)^{|a||b|} b^* a^*$, *for any homogeneous elements* $a, b \in A_0 \cup A_1$.

Since the characteristic of F is zero, we can write

$$A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-,$$

where for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^* = a\}$ and $A_i^- = \{a \in A_i \mid a^* = -a\}$ denote the sets of **symmetric** and **skew elements** of A_i, respectively.

We shall refer to a superalgebra with superinvolution simply as a **$*$-algebra**.

The free algebra with superinvolution (called the **free $*$-algebra**), denoted by $F\langle Y \cup Z, *\rangle$, is generated by symmetric and skew elements of even and odd degree. We write

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree. In order to simplify the notation, sometimes we denote by y any even variable, by z any odd variable and by x an arbitrary variable. The elements of $F\langle Y \cup Z, *\rangle$ are called **$*$-polynomials**.

Definition 1.2.29. *A $*$-polynomial $f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_t^+, z_1^-, \ldots, z_s^-)$ in $F\langle Y \cup Z, *\rangle$ is a $*$-identity of the $*$-algebra $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, and we write $f \equiv 0$,* *if, for all* $u_1^+, \dots, u_n^+ \in A_0^+$, $u_1^-, \dots, u_m^- \in A_0^-$, $v_1^+, \dots, v_t^+ \in A_1^+$ *and* $v_1^-, \dots, v_s^- \in A_1^-$, *we have*

$$f(u_1^+, \dots, u_n^+, u_1^-, \dots, u_m^-, v_1^+, \dots, v_t^+, v_1^-, \dots, v_s^-) = 0.$$

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, *\rangle : f \equiv 0 \text{ on } A\}$ the $\mathbf{T}_2^*$**-ideal** of $*$-identities of A, i.e. $\mathrm{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, *\rangle$ invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra $F\langle Y \cup Z\rangle$ commuting with the superinvolution $*$.

Given two $*$-algebras A and B, we say that A is $\mathbf{T}_2^*$**-equivalent** to B, and we write $A \sim_{T_2^*} B$, in case $\mathrm{Id}^*(A) = \mathrm{Id}^*(B)$. Moreover, we denote by $\langle f_1, \dots, f_n\rangle_{T_2^*}$ the T_2^*-ideal generated by the $*$-polynomials $f_1, \dots, f_n \in F\langle Y \cup Z, *\rangle$.

Because we are in characteristic 0, as in the ordinary and the graded case, it is easily seen that every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by

$$P_n^* = \mathrm{span}_F\left\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \left\{y_i^+, y_i^-, z_i^+, z_i^-\right\},\ i = 1, \dots, n\right\}$$

the space of multilinear $*$-polynomials of degree n in $y_1^+, y_1^-, z_1^+, z_1^-, \dots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial with degree 1) the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geqslant 1$.

Definition 1.2.30. *An ideal I of a $*$-algebra A is a* ***$*$-ideal*** *of A if it is a graded ideal and $I^* = I$. The $*$-algebra A is a* ***simple $*$-algebra*** *if $A^2 \neq 0$ and A has no non-trivial $*$-ideals.*

The Wedderburn-Malcev analog for $*$-algebras was proved in [23, Theorem 4.1].

Theorem 1.2.31. *Let A be a finite dimensional $*$-algebra over a field F of characteristic* 0. *Then there exists a semisimple $*$-subalgebra B such that*

$$A = B \oplus J(A)$$

as vector spaces and $J(A)$ is a $*$-ideal of A. Moreover $B \cong A_1 \times \cdots \times A_q$, where $A_1, \dots, A_q$ are simple $*$-algebras.

Of course, if $A = B \oplus J(A)$ with B semisimple $*$-subalgebra, the Wedderburn-Malcev decomposition enables us to consider **semisimple** and **radical** (or nilpotent) substitutions. More precisely, since in order to check whether a given multilinear $*$-polynomial is an identity of A it is sufficient to evaluate the variables in any spanning set of even/skew homogeneous elements, we may take a basis consisting of even/skew homogeneous elements of B or of $J(A)$. We refer to such evaluations as semisimple or radical evaluations, respectively. Moreover, the semisimple substitutions may be taken from $*$-simple components. This kind of evaluations, i.e., the ones from the set

$$\bigcup_{i=1}^{q} A_i \cup J(A),$$

are called **elementary** . In what follows, whenever we evaluate a polynomial on a finite dimensional $*$-algebra, we shall only consider elementary evaluations.

1.2.6 Finite dimensional simple $*$-algebras

We shall present the classification of the finite dimensional simple $*$-algebras over an algebraically closed field F of characteristic zero.

Definition 1.2.32. *Let A and B be two superalgebras endowed with superinvolutions $*$ and $\star$, respectively, then $(A,*)$ and $(B,\star)$ are isomorphic, as $*$-algebras, if there exists an isomorphism of superalgebras $\psi : A \to B$ such that $\psi(x^*) = \psi(x)^\star$, for all $x \in A$.*

If $n = k + h$, the matrix algebra $M_n(F)$ becomes a superalgebra, denoted by $M_{k,h}(F)$, endowed with the grading

$$(M_{k,h}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F),\ T \in M_h(F) \right\},$$
$$(M_{k,h}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

In [50], Racine proved that, up to isomorphism and if the field F is algebraically closed and of characteristic different from 2, it is possible to define on $M_{k,h}(F)$ only the following superinvolutions.

1. The **transpose** superinvolution, denoted by *trp* and defined for $h = k$ by

   $$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix},$$

   where t is the usual transpose.

2. The **orthosymplectic** superinvolution *osp* defined when $h = 2l$ is even by

   $$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} X^t & Z^tQ \\ QY^t & -QT^tQ \end{pmatrix},$$

   where $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and I_k, I_l are the $k \times k$, $l \times l$ identity matrices, respectively.

Furthermore, if A is a superalgebra, the **opposite superalgebra** A^{sop} is the superalgebra with the same graded vector space structure of A and product given on homogeneous elements $a, b \in A^{sop}$ by

$$a \circ b = (-1)^{|a||b|} ba.$$

The direct sum $R = A \oplus A^{sop}$ is a superalgebra with $R_0 = A_0 \oplus A_0^{sop}$ and $R_1 = A_1 \oplus A_1^{sop}$. Given $x, y \in R$, $x = (a, b) = (a_0 + a_1, b_0 + b_1)$, $y = (a', b') = (a'_0 + a'_1, b'_0 + b'_1)$, where $a_0, a'_0 \in A_0$, $a_1, a'_1 \in A_1$, $b_0, b'_0 \in A_0^{sop}$ and $b_1, b'_1 \in A_1^{sop}$, the product in R is given by

$$(a_0+a_1, b_0+b_1)\cdot(a'_0+a'_1, b'_0+b'_1) = (a_0a'_0+a_1a'_1+a_0a'_1+a_1a'_0, b'_0b_0-b'_1b_1+b'_0b_1+b'_1b_0). \quad (1.3)$$

Moreover R is a $*$-algebra since it is endowed with the **exchange** superinvolution ex defined by:

$$(a, b)^{ex} = (b, a)\,.$$

For example, if we consider the superalgebra $Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$, with $c^2 = 1$, then $Q(n) \oplus Q(n)^{sop}$ is a $*$-algebra with exchange superinvolution.

The following result gives the classification of the finite dimensional simple $*$-algebras (see [7, 31, 50]).

Theorem 1.2.33. *Let A be a finite dimensional simple $*$-algebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic (as a $*$-algebra) to one of the following:*

1. *$M_{k,h}(F)$ with the orthosymplectic or the transpose superinvolution,*

2. *$M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$ with the exchange superinvolution,*

3. *$Q(n) \oplus Q(n)^{sop}$ with the exchange superinvolution.*

Remark 1.2.34. *In Theorem 1.2.33, the $*$-algebra A has always an identity element that is symmetric of homogeneous degree 0.*

Proof. Let I be the identity matrix of $M_n(F)$. If $A \cong M_{k,h}(F)$, $n = k + h$, then I is the identity of A. Suppose that $A \cong M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$ or $A \cong Q(n) \oplus Q(n)^{sop}$, then the pair (I, I) is the identity of A. Finally it is not difficult to see that the identity of A is a symmetric even element. □

We conclude this section with the following result announced in [1, Theorem 1].

Theorem 1.2.35. *Let F be an algebraically closed field of characteristic zero. Let $\mathcal{V}$ be a variety generated by a finitely generated $*$-algebra A over F, satisfying an ordinary non-trivial identity. Then $\mathcal{V} = \mathrm{var}^*(B)$, for some finite dimensional $*$-algebra B over F.*

1.3 Hopf algebras

In this section we will introduce the Hopf Algebra environment and explore basic ideas of bialgebras and modules, taking as reference [47] and [49]. We remand to the books [49, 47, 52, 60] for further information about Hopf algebras.

1.3.1 Algebras and coalgebras

Definition 1.1.1 of F-algebra is equivalent to say that an (associative unitary) F-**algebra** is a vector space A over F with two F-linear maps, the multiplication $m\colon A\otimes A \to A$ and the unit $u\colon F \to A$, so the following diagrams are commutative:

a) associativity

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$

b) unit

$$\begin{tikzcd} & A\otimes A \arrow[dd, "m"] & \\ F\otimes A \arrow[ur, "u\otimes id"] \arrow[dr] & & A\otimes F \arrow[ul, "id\otimes u"'] \arrow[dl] \\ & A & \end{tikzcd}$$

We now “dualize” the notion of algebra.

Definition 1.3.1. *An F-**coalgebra** (with unit) is a vector space C over F with two F-linear maps, the **comultiplication** $\Delta\colon C \to C\otimes C$ and the **counit** $\varepsilon\colon C \to F$, such that the following diagrams are commutative:*

a) coassociativity

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "\Delta\otimes id"] \\ C\otimes C \arrow[r, "id\otimes\Delta"] & C\otimes C\otimes C \end{tikzcd}$$

b) counit

$$\begin{tikzcd} & C \arrow[dl, "1\otimes"'] \arrow[dr, "\otimes 1"] \arrow[dd, "\Delta"] & \\ F\otimes C & & C\otimes F \\ & C\otimes C \arrow[ul, "\varepsilon\otimes id"] \arrow[ur, "id\otimes\varepsilon"'] & \end{tikzcd}$$

The two upper maps in Definition 1.3.1(b) are given by $c \mapsto 1\otimes c$ and $c \mapsto c\otimes 1$ for any $c \in C$. We say C is **cocommutative** if $\tau\circ\Delta = \Delta$, where $\tau\colon C\otimes C \to C\otimes C$ is the **twist map**, defined by $\tau(c_1\otimes c_2) = c_2\otimes c_1$ for all $c_1, c_2 \in C$.

Example 1.3.2. *Let S be a nonempty set. Denote by FS the F-vector space with basis S. Then FS is a coalgebra with comultiplication Δ and counit ε defined by $\Delta(s) = s\otimes s$ and $\varepsilon(s) = 1$ for any $s \in S$, indeed,*

$$(\mathrm{Id}\otimes\Delta)\Delta(s) = (\mathrm{Id}\otimes\Delta)(s\otimes s) = s\otimes s\otimes s = (\Delta\otimes\mathrm{Id})(s\otimes s) = (\Delta\otimes\mathrm{Id})\Delta(s),$$

$$(\varepsilon\otimes\mathrm{Id})\Delta(s) = (\varepsilon\otimes\mathrm{Id})(s\otimes s) = 1\otimes s,$$

$$(\mathrm{Id}\otimes\varepsilon)\Delta(s) = (\mathrm{Id}\otimes\varepsilon)(s\otimes s) = s\otimes 1.$$

Note that this coalgebra is cocommutative. This example shows that any vector space can be endowed with a F-coalgebra structure.

Example 1.3.3. *Let C be an F-vector space with basis $\{s, c\}$. Then C is a coalgebra with comultiplication Δ and counit ε defined by*

$$\begin{aligned}\Delta(s) &= s \otimes c + c \otimes s,\\ \Delta(c) &= c \otimes c - s \otimes s,\\ \varepsilon(s) &= 0,\\ \varepsilon(c) &= 1.\end{aligned}$$

Indeed, we have

$$(\mathrm{Id} \otimes \Delta)\Delta(s) = s \otimes c \otimes c + c \otimes s \otimes c + c \otimes c \otimes s - s \otimes s \otimes s = (\Delta \otimes \mathrm{Id})\Delta(s),$$

$$(\mathrm{Id} \otimes \Delta)\Delta(c) = c \otimes c \otimes c - s \otimes s \otimes c - s \otimes c \otimes s - c \otimes s \otimes s = (\Delta \otimes \mathrm{Id})\Delta(c),$$

$$\begin{aligned}(\varepsilon \otimes \mathrm{Id})\Delta(s) &= (\varepsilon \otimes \mathrm{Id})(s \otimes c + c \otimes s) = 0 \otimes c + 1 \otimes s = 1 \otimes s,\\ (\mathrm{Id} \otimes \varepsilon)\Delta(s) &= (\mathrm{Id} \otimes \varepsilon)(s \otimes c + c \otimes s) = s \otimes 1 + c \otimes 0 = s \otimes 1,\\ (\varepsilon \otimes \mathrm{Id})\Delta(c) &= (\varepsilon \otimes \mathrm{Id})(c \otimes c - s \otimes s) = 1 \otimes c - 0 \otimes s = 1 \otimes c,\\ (\mathrm{Id} \otimes \varepsilon)\Delta(c) &= (\mathrm{Id} \otimes \varepsilon)(c \otimes c - s \otimes s) = c \otimes 1 - s \otimes 0 = c \otimes 1.\end{aligned}$$

Note that this coalgebra is cocommutative.

Notation 1.3.4. *The **sigma notation** for Δ is given as follows: for any $c \in C$, we write*

$$\Delta c = \sum c_1 \otimes c_2.$$

With the usual summation conventions we should have written

$$\Delta c = \sum_{i=1,n} c_{i1} \otimes c_{i2}.$$

The sigma notation supresses the index "i". The subscripts "1" and "2" are symbolic, and do not indicate particular elements of C.

Using the sigma notation, the counit diagram in Definition 1.3.1.b) can be expressed as

$$c = \sum \varepsilon(c_1)c_2 = \sum c_1\varepsilon(c_2). \tag{1.4}$$

Definition 1.3.5. *Let C and D be coalgebras with comultiplications Δ_C and Δ_D, and counits ε_C and ε_D, respectively.*

1. *A map $f : C \to D$ is a **homomorphism of coalgebras** if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and if $\varepsilon_C = \varepsilon_D \circ f$.*

2. *A subspace $I \subseteq C$ is a* ***coideal*** *if $\Delta_C I \subseteq I \otimes C + C \otimes I$ and if $\varepsilon_C(I) = 0$.*

Remark 1.3.6. *Let C be a coalgebra, I a coideal of C and $p: C \to C/I$ the canonical projection of F-vector spaces. Then:*

1. *There exists a unique coalgebra structure on C/I (called the* ***factor coalgebra****) such that p is a homomorphism of coalgebras.*
2. *If $f: C \to D$ is a homomorphism of coalgebras with $I \subseteq Ker(f)$, then there exists a unique homomorphism of coalgebras $\bar{f}: C/I \to D$ such that $\bar{f}p = f$. So, there exists a canonical isomorphism of coalgebras between $C/Ker(f)$ and $Im(f)$.*

1.3.2 Bialgebras

Let H be a F-vector space which is simultaneously endowed with an algebra structure (H, m, u) and a coalgebra structure (H, Δ, ε). The following definition establishes the situation in which the two structures are compatible.

Definition 1.3.7. *A 5-tuple $(H, m, u, \Delta, \varepsilon)$ is called* ***bialgebra*** *if (H, m, u) is an algebra, (H, Δ, ε) is a coalgebra and m and u are homomorphism of coalgebras (or, equivalently, Δ and ε are homomorphism of algebras).*

Remark 1.3.8. *The compatibility relations between the operations of a bialgebra H give us the following conditions for all $h, g \in H$:*

$$\begin{aligned}
\Delta(hg) &= \sum h_1 g_1 \otimes h_2 g_2, \\
\varepsilon(hg) &= \varepsilon(h)\varepsilon(g), \\
\Delta(1) &= 1 \otimes 1, \\
\varepsilon(1) &= 1.
\end{aligned}$$

We say that a **bialgebra morphism** is an F-linear map $f: H \to H'$ that is both a homomorphism of algebras and a homomorphism of coalgebras. A subspace $I \subseteq H$ is a **biideal** if it is both an ideal (in the underlying algebra of H) and a coideal (in the underlying coalgebra of H). Then the structures of factor algebra and the factor coalgebra define the bialgebra H/I.

Example 1.3.9. *Let G be a (multiplicative) group and let $H = FG$ be its* ***group algebra****. We recall that FG is an F-vector space with basis $\{g \mid g \in G\}$ and its elements are of the form $\sum_{g \in G} \alpha_g g$ with $(\alpha_g)_{g \in G}$ a family of elements from F having only a finite number of nonzero elements . The multiplication and unit are defined by $m(h \otimes g) = hg$ and $u(1) = 1_H = e$ and extended by linearity, where e is the identity element of G.*

On the group algebra H we also have a coalgebra structure (as in Example 1.3.2), in which $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, for all $g \in G$. Note that

$$\Delta(hg) = hg \otimes hg = (h \otimes h)(g \otimes g) = \Delta(h)\Delta(g)$$

and

$$\varepsilon(hg) = 1 = \varepsilon(h)\varepsilon(g).$$

This shows that Δ and ε are homomorphism of algebras and H is a bialgebra.

Example 1.3.10. *Let $\mathfrak{g}$ be a Lie algebra over F and let $U(\mathfrak{g})$ be its universal enveloping algebra. Then $U(\mathfrak{g})$ is a bialgebra with $\Delta(x) = x \otimes 1 + 1 \otimes 1$, and $\varepsilon(x) = 0$, for all $x \in \mathfrak{g}$.*

Example 1.3.11. *Take $0 \neq q \in F$ and let $H = \mathcal{O}(F^2) = F\langle x, y \mid xy = qyx\rangle$, which is called the* ***quantum plane****. H has a bialgebra structure given by setting $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes 1 + x \otimes y$, $\varepsilon(x) = 1$ and $\varepsilon(y) = 0$.*

Example 1.3.12. *Assume that the characteristic of F is different from 2. Let H be the algebra given by generators and relations as follows: H is generated as an F-algebra by c and x satisfying the relations*

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx.$$

Then H has dimension 4 as F-vector space with basis $\{1, c, x, cx\}$. The coalgebra structure is induced by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c \otimes x + x \otimes 1,$$

$$\varepsilon(c) = 1, \quad \varepsilon(x) = 0.$$

In this way, H becomes a bialgebra. Note that H is neither commutative nor cocommutative.

1.3.3 Hopf algebras

Definition 1.3.13. *A bialgebra $(H, m, u, \Delta, \varepsilon)$ is called* ***Hopf algebra*** *if there exists a F-linear map $S\colon H \to H$ (called* ***antipode****) such that the following diagram commutes:*

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S\otimes \mathrm{Id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & F \arrow[rr, "u"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{Id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

This property can also be expressed as

$$\sum (Sh_1)h_2 = \varepsilon(h)1_H = \sum h_1(Sh_2) \text{ for all } h \in H, \tag{1.5}$$

where 1_H is the identity element of H.

Definition 1.3.14. *Let C be a coalgebra and A an algebra. Then $Hom_F(C, A)$ becomes an algebra under the* ***convolution product***

$$(f * g)(c) = \sum f(c_1)g(c_2),$$

for all $f, g \in Hom_F(C, A)$, $c \in C$.

Notice that the unit element of $\mathrm{Hom}_F(C, A)$ is $u\varepsilon$, since

$$(f * u\varepsilon)(c) = \sum f(c_1)u\varepsilon(c_2) = \sum f(c_1)\varepsilon(c_2)1 = f(c).$$

Similarly, $(u\varepsilon) * f = f$.

Let H be an Hopf algebra with antipode $S\colon H \to H$. Then S is the inverse of the identity map $I\colon H \to H$ with respect to the convolution product in $\mathrm{Hom}_F(H, H)$.

We can define **morphisms of Hopf algebras** which are simply bialgebra morphisms, since the bialgebra morphisms preserve antipodes, that is, if $f\colon H \to B$ is a bialgebra morphism between two Hopf algebras H and B with antipodes S_H and S_B respectively, then $S_B f = f S_H$ [49, Proposition 4.2.5].

Let H be a Hopf algebra, and I a **Hopf ideal** of H, i.e. I is an ideal of the algebra H, a coideal of the coalgebra H, and $S(I) \subseteq I$, where S is the antipode of H. Then on the factor space H/I we can attach a natural structure of Hopf algebra. When this structure is settled up, the canonical projection $p\colon H \to H/I$ is a morphism of Hopf algebras.

Proposition 1.3.15. *Let H be a Hopf algebra with antipode S. Then:*

i) $S(hg) = S(g)S(h)$ *for any* $g, h \in H$.

ii) $S(1) = 1$.

iii) $\Delta(S(h)) = \sum S(h_2) \otimes S(h_1)$.

iv) $\varepsilon(S(h)) = \varepsilon(h)$.

Proof. See [49, Proposition 4.2.6]. $\square$

Example 1.3.16. *Let G be a group and $H = FG$ the bialgebra defined in Example 1.3.9. Then H has a Hopf algebra structure with antipode S defined by $S(g) = g^{-1}$ for all $g \in G$. Indeed,*

$$\sum (Sg_1)g_2 = S(g)g = g^{-1}g = 1_H = \varepsilon(g)1_H$$

and

$$\sum g_1(Sg_2) = gS(g) = gg^{-1} = 1_H = \varepsilon(g)1_H.$$

Example 1.3.17. *The universal enveloping algebra $H = U(\mathfrak{g})$ is a Hopf algebra with antipode S defined by $S(x) = -x$ for each $x \in \mathfrak{g}$. Indeed,*

$$\sum (Sx_1)x_2 = s(x)1 + s(1)x = -x + x = 0 = \varepsilon(x)1_H$$

and

$$\sum x_1(Sx_2) = xS(1) + 1S(x) = x - x = 0 = \varepsilon(x)1_H$$

Example 1.3.18. *Let $\mathcal{O}(f^2)$ be the quantum plane defined in Example 1.3.11. Consider the bialgebra $H = \mathcal{O}(f^2)[x^{-1}]$ where $\Delta(x^{-1}) = x^{-1} \otimes x^{-1}$ and $\varepsilon(x^{-1}) = x^{-1}$. Then H has a Hopf algebra structure with antipode S defined by $S(x) = x^{-1}$, $S(x^{-1}) = x$ and $S(y) = -x^{-1}y$.*

Example 1.3.19. *Consider the bialgebra $H_4 = F\langle 1, c, x, cx \mid c^2 = 1, x^2 = 0, xc = -cx\rangle$ defined in Example 1.3.12. Then H_4 becomes a Hopf algebra with antipode S define by $S(c) = c^{-1} = c$ and $S(x) = -cx$, indeed,*

$$\sum (Sc_1)c_2 = s(c)c = c^2 = 1 = \varepsilon(c)1 = cS(c) = \sum c_1(Sc_2)$$

and

$$\begin{aligned}\sum (Sx_1)x_2 = s(c)x + s(x)1 = cx - cx = 0 = \varepsilon(x)1 \\ = -c^2x + x = cS(x) + xS(1) = \sum x_1(Sx_2).\end{aligned}$$

*This Hopf algebra is known as **Sweedler's 4-dimensional Hopf algebra**.*

The next example is a generalization of the Sweedler's Hopf algebras.

Example 1.3.20. *Let F be a field containing a m-th root of the unit ξ for some positive integer m. Let $H_{m^2}(\xi)$ be the bialgebra defined by generators c and x with relations*

$$c^m = 1, \quad x^m = 0, \quad xc = \xi cx.$$

The coalgebra structure is given by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c \otimes x + x \otimes 1,$$

$$\varepsilon(c) = 1, \quad \varepsilon(x) = 0.$$

*As a F-vector space, $H_{m^2}(\xi)$ has dimension m^2 with basis $\{c^i x^j \mid 0 \leqslant i, j, \leqslant m-1\}$. The bialgebra $H_{m^2}(\xi)$ becomes a Hopf algebra if we define the antipode as $S(c) = c^{-1}$ and $S(x) = -c^{-1}x$. This Hopf algebra is known as the **Taft's Hopf algebra** of dimension m^2.*

1.3.4 Modules and comodules

We begin this section by defining modules over an algebra using only morphism and diagrams. Then by dualization we obtain the notion of a comodule over a coalgebra.

Definition 1.3.21. *Let A be an F-algebra, a (left) A-**module** is F-vector space X with a F-linear map $\gamma\colon X \otimes A \to X$ such that the following diagrams commute:*

$$\begin{tikzcd} X \otimes A \otimes A \arrow[r, "\mathrm{Id} \otimes m"] \arrow[d, "\gamma \otimes \mathrm{Id}"'] & X \otimes A \arrow[d, "\gamma"] \\ X \otimes A \arrow[r, "\gamma"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \otimes F \arrow[r, "\mathrm{Id} \otimes u"] \arrow[dr, "\textit{scalar mult.}"'] & X \otimes A \arrow[d, "\gamma"] \\ & X \end{tikzcd}$$

Definition 1.3.22. *Let C be a F-coalgebra, a (right) C-**comodule** is a F-vector space M with a F-linear map $\rho\colon M \to M \otimes C$ such that the following diagrams commute:*

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "\mathrm{Id} \otimes \Delta"] \\ M \otimes C \arrow[r, "\rho \otimes \mathrm{Id}"'] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\otimes 1"'] & M \otimes C \arrow[d, "\mathrm{Id} \otimes \varepsilon"] \\ & M \otimes F \end{tikzcd}$$

Notation 1.3.23. *For any element $m \in M$, we write $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$ where $m_0 \in M$ and $m_1 \in C$.*

Example 1.3.24. *Let (C, Δ, ε) be a coalgebra. Take $M = C$ with $\rho = \Delta$. Then, (M, ρ) is a right C-comodule.*

Example 1.3.25. *Let (C, Δ, ε) be a coalgebra and let V be a F-vector space. Then $M = V \otimes C$ is a right C-comodule with $\rho = I \otimes \Delta$. Thus $\rho(v \otimes c) = \sum v \otimes c_1 \otimes c_2$.*

Example 1.3.26. *Let $C = FG$ with coalgebra structure of the Example 1.3.9. Let M be a G-graded module, i.e. $M = \bigoplus_{g \in G} M_g$ where $(M_g)_{g \in G}$ is a family of F-vector spaces. Then M is a right C-comodule with $\rho(m_g) = m_g \otimes g$ for any $g \in G$ and $m_g \in M_g$.*

We define the morphisms of comodules dualizing the corresponding definition of morphisms of modules. We use commutative diagrams for this purpose.

Definition 1.3.27. *Let A be an F-algebra, and let (X, γ), (Y, ν) be two left A-modules. The F-linear map $f\colon X \to Y$ is a **morphism of A-modules** if the following diagram* *commutes:*

$$\begin{tikzcd} A \otimes X \arrow[r, "\mathrm{Id}\otimes f"] \arrow[d, "\gamma"'] & A \otimes Y \arrow[d, "\nu"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Definition 1.3.28. *Let C be a F-coalgebra, and let (M, ρ), (N, ϕ) be two right C-comodules. The F-linear map $g\colon M \to N$ is a **morphism of C-comodules** if the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\rho"'] & N \arrow[d, "\phi"] \\ M \otimes C \arrow[r, "g\otimes id"'] & N \otimes C \end{tikzcd}$$

1.3.5 Semisimple Hopf algebras

In this section, consider H to be a finite dimensional Hopf algebra. We will study the relationship between integrals, semisimplicity and finite dimensional Hopf algebras.

Definition 1.3.29. *A **left integral in** H is an element $t \in H$ such that $ht = \varepsilon(h)t$, for all $h \in H$. Similarly, a **right integral in** H is an element $t' \in H$ such that $t'h = \varepsilon(h)t'$, for all $h \in H$.*

Denote by $\int_H^l$ the F-space of left integrals in H, and by $\int_H^r$ the F-space of right integrals in H.

Example 1.3.30. *Let $H = FG$. The element $t = \sum_{g\in G} g$ is a left and right integral in H. Indeed, if $h \in G$,*

$$ht = h\left(\sum_{g\in G} g\right) = \sum_{g\in G} g = t = \varepsilon(g)t.$$

If $h = \sum_{g'\in N} \alpha_{g'}g'$ where N is a subgroup of G and $\alpha_{g'} \in F$ for all $g' \in N$,

$$ht = \left(\sum_{g'\in N} \alpha_{g'}g'\right) t = \sum_{g'\in N} \alpha_{g'}t = \varepsilon(h)t.$$

This implies that t is a left integral in H. Similarly, t is a right integral in H.

Example 1.3.31. *Let $H = H_2$ be the Sweedler's Hopf algebra. Note that*

$$c(x + cx) = cx + x = \varepsilon(c)(x + cx),$$

$$x(x+cx) = xcx = 0 = \varepsilon(x)(x+cx).$$

Then $x+cx$ is a left integral in H. On the other hand, note that

$$(x-cx)c = xc - cxc = x - xc = \varepsilon(c)(x-cx),$$

$$(x-cx)x = 0 = \varepsilon(x)(x-cx).$$

Then $x-cx$ is a right integral in H. Moreover $\int_H^l = F(x+cx)$ and $\int_H^r = F(x-cx)$.

Definition 1.3.32. *An Hopf algebra H is called **semisimple** if every (left) H-module is completely reducible.*

A necessary and sufficient condition to determine whether a Hopf algebra is semisimple is given by the following version of the Maschke's theorem for Hopf algebras due to Larson and Sweedler.

Theorem 1.3.33. *[47, Theorem 2.2.1]. Let H be any finite dimensional Hopf algebra. Then the following conditions are equivalent:*

- *H is semisimple;*
- $\varepsilon\left(\int_H^l\right) \neq 0$;
- $\varepsilon\left(\int_H^r\right) \neq 0$.

Example 1.3.34. *Let H_4 the Sweedler's Hopf algebra. By Example 1.3.31, we have $\varepsilon\left(\int_H^l\right) = \varepsilon(F(x+cx)) = 0$, and $\varepsilon\left(\int_H^r\right) = \varepsilon(F(x-cx)) = 0$. So H_4 is not semisimple.*

In general, the Taft algebra $H_{m^2}(\xi)$ is not semisimple.

1.3.6 H-module algebras

In this section we study actions of a Hopf algebra H on an F-algebra A.

Definition 1.3.35. *An F-algebra A is a (left) H**-module algebra** if the following conditions hold:*

MA1) *A is a left H-module (with action of $h \in H$ on $a \in A$ denoted by $h \cdot a$.*

MA2) $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$, *for all $h \in H$ and $a, b \in A$.*

MA3) $h \cdot 1_A = \varepsilon(h) 1_A$ *for all $h \in H$.*

Remark 1.3.36. *Some authors omit (MA3) in the definition of H-module algebra since (MA3) can be obtained from (MA2), indeed,*

$$\begin{aligned}
h \cdot 1_A &= (h \cdot 1_A)1_A \\
&= ((\sum h_1\varepsilon(h_2)) \cdot 1_A)1_A && \text{By equation 1.4} \\
&= \sum (h_1 \cdot 1_A)(\varepsilon(h_2) \cdot 1_A) \\
&= \sum (h_1 \cdot 1_A)((h_2S(h_3)) \cdot 1_A) && \text{By equation 1.5} \\
&= \sum (h_1 \cdot 1_A)(h_2 \cdot (S(h_3) \cdot 1_A)) && \text{By associativity of scalars from } H \\
&= \sum h_1 \cdot (1_A(S(h_2) \cdot 1_A)) && \text{By (MA2)} \\
&= \sum h_1 \cdot (S(h_2) \cdot 1_A) \\
&= (\sum h_1S(h_2)) \cdot 1_A && \text{By associativity of scalars from } H \\
&= \varepsilon(h)1_A. && \text{By equation 1.5}
\end{aligned}$$

Example 1.3.37. *For any Hopf algebra H and any F-algebra A, we have a structure of H-module algebra given by the trivial action $h \cdot a = \varepsilon(h)a$ for all $h \in H$ and $a \in A$.*

Example 1.3.38. *Let A be an H-module algebra with $H = FG$ where G is a group. Since $\Delta(g) = g \otimes g$ for every $g \in G$ then $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for every $a, b \in A$, and thus g acts as an endomorphism of A. Moreover, g acts as an automorphism since $g^{-1}g = 1$. Thus, we have a homomorphism of groups $G \to Aut_F(A)$. Conversely if G is a group acting as automorphism on any F-algebra A, then A is a FG-module algebra.*

Example 1.3.39. *Any Hopf algebra H acts on itself by the **adjoint action**, defined by*

$$h \cdot l = (\operatorname{ad} h)l = \sum h_1lS(h_2), \qquad \text{for all } h, l \in H.$$

To see this, it is sufficient to prove (MA2). For any $h, l, m \in H$, we have

$$\begin{aligned}
h \cdot (ml) &= \sum h_1lmS(h_2) \\
&= \sum h_1\varepsilon(h_2)lmS(h_2) && \text{By equation 1.4} \\
&= \sum h_1l\varepsilon(h_2)mS(h_2) \\
&= \sum h_1lS(h_2)h_3mS(h_4) && \text{By equation 1.5} \\
&= \sum ((\operatorname{ad} h_1)l)((\operatorname{ad} h_2)m) \\
&= \sum (h_1 \cdot l)(h_2 \cdot m).
\end{aligned}$$

As a particular case, if $H = FG$ then $(\operatorname{ad} x)y = xyx^{-1}$, $x, y \in G$, and if $H = U(\mathfrak{g})$, then $(\operatorname{ad} x)h = xh - hx, x \in \mathfrak{g}, h \in H$.

Example 1.3.40. *Let H_4 the Sweedler's Hopf algebra (Example 1.3.19). Consider the algebra $M_2(F)$ of matrices 2×2 over F and the H_4-action on $M_2(F)$ induced by*

$$c \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{pmatrix} \qquad \text{and} \qquad x \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} -a_3 & a_1 - a_4 \\ 0 & -a_3 \end{pmatrix}.$$

To see that $M_2(F)$ is an H_4-module algebra we will prove (MA2): Let $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$ be elements of $M_2(F)$. Then,

$$\begin{aligned} c \cdot (AB) &= c \cdot \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right] = c \cdot \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 & -a_1b_2 - a_2b_4 \\ -a_3b_1 - a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & -b_2 \\ -b_3 & b_4 \end{pmatrix} = (c \cdot A)(c \cdot B). \end{aligned}$$

and

$$\begin{aligned} x \cdot (AB) &= x \cdot \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right] = x \cdot \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \\ &= \begin{pmatrix} -a_3b_1 - a_4b_3 & a_1b_1 + a_2b_3 - a_3b_2 - a_4b_4 \\ 0 & -a_3b_1 - a_4b_3 \end{pmatrix} \\ &= \begin{pmatrix} -a_1b_3 & a_1b_1 - a_1b_4 + a_2b_3 \\ a_3b_3 & -a_3b_1 + a_3b_4 - a_4b_3 \end{pmatrix} + \begin{pmatrix} -a_3b_1 + a_1b_3 - a_4b_3 & -a_3b_2 + a_1b_4 - a_4b_4 \\ -a_3b_3 & -a_3b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{pmatrix}\begin{pmatrix} -b_3 & b_1 - b_4 \\ 0 & -b_3 \end{pmatrix} + \begin{pmatrix} -a_3 & a_1 - a_4 \\ 0 & -a_3 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= (c \cdot A)(x \cdot B) + (x \cdot A)(1 \cdot B). \end{aligned}$$

Definition 1.3.41. *Let H be an Hopf algebra and let A and B be two H-module algebras. We say that a homomorphism of algebras $\phi\colon A \to B$ is a **homomorphism of H-module algebras** or, simply, an **H-homomorphism** if ϕ is a morphism of H-modules, i.e. $\phi(h \cdot a) = h \cdot \phi(a)$ for every $h \in H$ and $a \in A$.*

1.3.7 Free H-module algebra

Given H an m-dimensional Hopf algebra over a field F, let $X = \{x_1, x_2, \dots\}$ be a set of non-commutative variables and consider the vector space $V = \mathrm{Span}_F\{x_1, x_2, \dots\} \otimes_F H$. The **free H-module algebra** generated by X, denoted by $F^H\langle X\rangle$ is the tensor algebra over V, that is,

$$F^H\langle X\rangle = T(V) = \bigoplus_{n \geqslant 0} T^n(V) = \bigoplus_{n \geqslant 0} T^n(\mathrm{Span}_F\{x_1, x_2, \dots\} \otimes_F H).$$

An element of $F^H\langle X\rangle$ is called H**-polynomial**. We prefer the notation

$$x_{i_1}^{h_1}x_{i_2}^{h_2}\cdots x_{i_n}^{h_n} := (x_{i_1}\otimes h_1)\otimes(x_{i_2}\otimes h_2)\otimes\cdots\otimes(x_{i_n}\otimes h_n).$$

Suppose $\{b_1,\ldots,b_m\}$ is a basis for H (basis as vector space), then $F^H\langle X\rangle$ is isomorphic to the free algebra over F with free formal (non-commutative) generators x^{b_j}, where $j\in\{1,\ldots,m\}$ and $x\in X$. Define in $F^H\langle X\rangle$ the structure of a left H-module algebra by

$$h(x_{i_1}^{h_1}x_{i_2}^{h_2}\cdots x_{i_n}^{h_n}) = x_{i_1}^{h_{(1)}h_1}x_{i_2}^{h_{(2)}h_2}\cdots x_{i_n}^{h_{(n)}h_n}$$

where $h_{(1)}\otimes h_{(2)}\otimes\cdots\otimes h_{(n)}$ is the image of $h\in H$ under the comultiplication Δ of H applied $(n-1)$ times.

The H-module algebra $F^H\langle X\rangle$ has the following universal property: if $\alpha\colon X\to W$ is any map from X to any H-module algebra W, then there exists a unique H-homomorphism $\beta\colon F^H\langle X\rangle\to W$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\alpha"'] & F^H\langle X\rangle \arrow[d, "\beta"] \\ & W \end{tikzcd}$$

where $i\colon X\to F^H\langle X\rangle$ is the function $i(x) = x^{1_H}\in F^H\langle X\rangle$ for all $x\in X$.

Given any H-module algebra W, we say that $f\in F^H\langle X\rangle$ is an H**-identity** of W if for every H-homomorphism $\phi\colon F^H\langle X\rangle\to W$ the polynomial f is in the kernel of ϕ. In other words, $f(x_1,x_2,\ldots,x_n)\in F^H\langle X\rangle$ is a H-identity of W if and only if $f(w_1,w_2,\ldots,w_n)=0$ for all $w_1,w_2,\ldots,w_n\in W$. The set $\mathrm{Id}^H(W)$ of all H-identities of W is a ideal of $F^H\langle X\rangle$ and is invariant under all H-endomorphisms of $F^H\langle X\rangle$. The ideals with this property are called T^H**-ideals**. If I is an T^H-ideal of $F^H\langle X\rangle$ then $\mathrm{Id}^H(F^H\langle X\rangle/I)=I$. Two H-module algebras W_1 and W_2 are said to be T^H**-equivalent**, and we write $W_1\sim_{T^H} W_2$), if $\mathrm{Id}^H(W_1)=\mathrm{Id}^H(W_2)$.

Given a non-empty set $S\subseteq F^H\langle X\rangle$, the class $\mathrm{var}^H(S)$ of all H-module algebras W such that f is an H-identity for W for all $f\in S$ is called the **variety determined** or **generated** by S. Similarly, given an H-module algebra W, the variety of H-module algebras generated by W, denoted by $\mathrm{var}^H(W)$, is the class of all H-module algebras satisfying the H-identities of W. Hence we say that $A\in\mathrm{var}^H(W)$ if and only if $\mathrm{Id}^H(W)\subseteq\mathrm{Id}^H(A)$.

1.4 Representation Theory

In this section we deal with the method of representation theory of groups in the study of PI-algebras. We will gather basic results on finite-dimensional representations and representations of the symmetric group. Finally, we will exploit the permutation action of the symmetric group S_n on the space of multilinear polynomials in n variables. For more detailed background see for example the books by Steinberg [58], James and Kerber [34], and Sagan [54].

Even if most of the results in this section hold over a field of arbitrary characteristic, we shall assume, throughout the section, that F is an algebraically closed field of characteristic 0.

1.4.1 Representation of finite groups

Let V be a vector space. The **general linear group** $GL(V)$ of V is the group

$$GL(V) := \{A \in \mathrm{End}(V) \mid A \text{ is invertible}\}.$$

If $\dim V = m < \infty$, fixing a basis of V, we identify the group $GL(V)$ with the group $GL_m(F)$ of invertible $m \times m$ matrices with entries from F.

Definition 1.4.1. *A representation of a group G is a homomorphism $\phi\colon G \to GL(V)$ for some finite-dimensional vector space V. The dimension of V is called the* ***degree of*** *ϕ. We usually write ϕ_g for $\phi(g)$ and $\phi_g(v)$, or simply $\phi_g v$, for the action of ϕ_g on $v \in V$.*

Example 1.4.2. *The* ***trivial representation*** *of a group G is the homomorphism $\phi\colon G \to F\backslash\{0\}$ given by $\phi(g) = 1$ for all $g \in G$.*

Example 1.4.3. *$\phi\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}\backslash\{0\}$ defined by $\phi([m]) = e^{2\pi i m/n}$ is a representation.*

Example 1.4.4. *Let S_n be the symmetric group of order n. Define $\phi\colon S_n \to GL_n(F)$ on the standard basis by $\phi_\sigma(e_i) = e_{\sigma(i)}$. So, for instance, when $n = 3$ we have*

$$\phi_{(1\,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \phi_{(1\,2\,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Definition 1.4.5. *Two representations $\phi\colon G \to GL(V)$ and $\varphi\colon G \to GL(W)$ are said to be* ***equivalent*** *if there exists an isomorphism $T\colon V \to W$ such that $\varphi g = T\phi g T^{-1}$ for all $g \in G$, i.e., $\varphi_g T = T\phi_g$ for all $g \in G$. In this case, we write $\phi \sim \varphi$. In pictures, we have that the following diagram commutes.*

$$\begin{tikzcd} V \arrow[r, "\phi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\varphi_g"] & W \end{tikzcd}$$

Definition 1.4.6. *Suppose that representations $\phi^{(1)}\colon G \to GL(V_1)$ and $\phi^{(2)}\colon G \to GL(V_2)$ are given. Then their* ***direct sum*** *$\phi = \phi^{(1)} \oplus \phi^{(2)}\colon G \to GL(V_1 \oplus V_2)$ is given by*

$$\phi_g(v_1, v_2) = (\phi_g^{(1)}(v_1), \phi_g^{(2)}(v_2)).$$

Their ***tensor product*** *$\phi = \phi^{(1)} \otimes \phi^{(2)}\colon G \to GL(V_1 \otimes V_2)$ is given by*

$$\phi_g(v_1 \otimes v_2) = \phi_g^{(1)}(v_1) \otimes \phi_g^{(2)}(v_2).$$

Example 1.4.7. *If $n > 1$, then the representation $\phi\colon G \to GL_n(K)$ given by $\phi_g = I$ all $g \in G$ is not equivalent to the trivial representation; rather, it is equivalent to the direct sum of n copies of the trivial representation.*

Definition 1.4.8. *Let $\phi\colon G \to GL(V)$ be a representation. If W is a G-invariant subspace of V, i.e., $\phi_g(W) = W$, then the representation $\varphi\colon G \to GL(W)$ defined by $\varphi_g(w) = \phi_g(w)$ for all $g \in G$ and $w \in W \subseteq V$ is called a* ***subrepresentation*** *of the representation $\phi\colon G \to GL(V)$. The subrepresentation φ is* ***proper*** *if $W \neq \{0\}$ and $W \neq V$.*

Example 1.4.9. *Let $\phi\colon S_n \to GL_n(F)$ be the representation given in example 1.4.4. Notice that $W = F(e_1 + \cdots + e_n)$ is a S_n-invariant subspace of $V = F^n$, in fact, since σ is a permutation and addition is commutative, we have that $\phi_\sigma(e_1+\cdots+e_n) = e_{\sigma(1)}+\cdots+e_{\sigma(n)} = e_1 + \cdots + e_n$. Moreover, if $n > 1$, the subrepresentation $\varphi\colon G \to GL(W)$, given by $\varphi_g(w) = \phi_g(w)$ is proper.*

Definition 1.4.10. *A representation $\phi\colon G \to GL(V)$ is said to be* ***irreducible*** *if it has no proper subrepresentations. ϕ is said to be* ***completely reducible*** *if it is a direct sum of finitely many irreducible representations.*

If FG is the group algebra of G over F and ϕ is a representation of G on V, this representation induces a homomorphis of algebras $\phi'\colon FG \to \mathrm{End}_F(V)$ given by

$$\phi'(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g \phi(g),$$

such that $\phi'(1_G) = 1$.

Theorem 1.4.11 (Maschke)**.** *Every finite dimensional representation of a finite group G is completely reducible. Then the group algebra FG is isomorphic to a direct sum of matrix algebras,*

$$FG \cong M_{d_1}(F) \oplus \cdots \oplus M_{d_r}(F).$$

A representation of a group G uniquely determines a finite dimensional G-module in the following way: if $\phi\colon G \to GL(V)$ is a representation, V becomes a (left) G-module by defining $gv = \phi_g(v)$ for all $g \in G$, $v \in V$. On the other hand, if M is a

G-module which is finite dimensional as a vector space over F, then $\phi\colon G \to GL(M)$, such that $\phi_g(m) = gm$, for $g \in G$, $m \in M$, defines a representation of G on M.

Let us introduce some notation. If V is a vector space, ϕ a representation and $m > 0$, then we set

$$mV = \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}} \quad \text{and} \quad m\phi = \underbrace{\phi \oplus \cdots \oplus \phi}_{n \text{ times}}.$$

Let $\phi^{(1)}, \ldots, \phi^{(s)}$ be a complete list of irreducible representations of G, up to equivalence (by Maschke Theorem). Let ρ a representation of G. If $\rho \sim m_1\phi^{(1)} \oplus \cdots \oplus m_s\phi^{(s)}$, then m_i is called the **multiplicity** of $\phi^{(i)}$ in ρ.

The **regular representation** of G is the homomorphism $L\colon G \to GL(FG)$ defined by

$$L_g(\sum_{h\in G} \alpha_h h) = \sum_{h\in G} \alpha_h gh.$$

Considering FG as a left G-module, we always assume that G acts on FG in this way, as a group of left translations. The subrepresentations of L correspond to left ideals of the group algebra FG and the irreducible representations correspond to minimal left ideals of FG.

By Maschke Theorem 1.4.11 the regular representation has the following decomposition

$$FG \cong n_1V_1 \oplus \cdots \oplus n_kV_k,$$

where $n_iV_i = V_i \oplus \cdots \oplus V_i$ (n_i times), n_i is the multiplicity of V_i in FG, and $V_i \cong \sum_{l=1}^{n_i} Fe_{li}$ is a minimal left ideal of $M_{n_i}(F)$. Notice that n_i is the degree of the representation V_i.

Proposition 1.4.12. *[58, Theorem 4.4.4] Every irreducible representation of G (up to equivalence) appears in the regular representation of G with multiplicity equal to its degree.*

Recall from Definition 1.1.9 that an element e in an algebra A is called idempotent if $e^2 = e$. A nonzero left (resp. right) ideal I of an algebra A is called **minimal ideal** if it contains no other nonzero left (resp. right) ideal. An element in A is called **minimal** if it generates a minimal one-side ideal of A.

Proposition 1.4.13. *If M is an irreducible representation of G, then $M \cong V_i$, a minimal left ideal of $M_{n_i}(F)$, for some $i \in \{1, \ldots, k\}$. Hence there exists a minimal idempotent $e \in FG$ such that $M \cong FGe$.*

Now, we will define the characters of the representations of a group G, which describe the multiplicities of irreducible representations for FG. Let $\operatorname{tr}\colon \operatorname{End}(V) \to F$ be the trace function on $\operatorname{End}(V)$.

Definition 1.4.14. *Let $\phi\colon G \to GL(V)$ be a representation of G. Then the map $\chi_\phi\colon G \to F$ such that $\chi_\phi(g) = tr(\phi_g)$ is called the* ***character*** *of ϕ. The character of an irreducible representation is called an irreducible character.*

Let $C(G)$ be the vector space of class functions on G, (that is, all the functions $f\colon G \to F$ such that $f(g) = f(hgh^{-1})$ for all $g, h \in G$). It is easy to show that the character belongs to $C(G)$. One can define an inner product $\langle\ ,\ \rangle$ on $C(G)$ by setting

$$\langle \chi, \psi \rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1}), \qquad \chi, \psi \in C(G).$$

Proposition 1.4.15. *[29, Proposition 2.1.10] Let $\phi^{(1)}, \ldots, \phi^{(s)}$ be a complete list (up to equivalence) of irreducible representations of G, with characters $\chi_1, \ldots, \chi_s$, respectively. Let ρ be a representation of G and write $\rho \sim m_1\phi^{(1)} \oplus \cdots \oplus m_s\phi^{(s)}$. Then:*

1. $\chi_\rho = \displaystyle\sum_{i=1}^{k} m_i\chi_i$.
2. $\langle \chi_\rho, \chi_i \rangle = m_i$, *for all i.*
3. $\langle \chi_\rho, \chi_\rho \rangle = \displaystyle\sum_{i=1}^{k} m_i^2$.
4. $\langle \chi_\rho, \chi_\rho \rangle = 1$ *if and only if ρ is irreducible.*
5. $\chi_\rho = \chi_{\rho'}$ *if and only if $\rho \sim \rho'$, where ρ' is another representation of G.*

Thus, the group algebra FG descomposes as $FG = \oplus_{i=1}^{k} e_iFG$, with $e_iFG \cong M_{n_i}(F)$ and

$$e_i = \frac{\chi_i(1)}{|G|}\sum_{g\in G}\chi_i(g^{-1})g$$

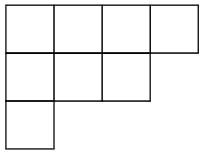
is a minimal central idempotent of FG.

1.4.2 Representations of the symmetric group

Definition 1.4.16. *Let $n \geqslant 1$ be an integer. A* ***partition*** *λ of n is a finite sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ in decreasing order (i.e. $\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0$) such that $\displaystyle\sum_{i=1}^{r}\lambda_i = n$. In this case we write $\lambda \vdash n$ or $|\lambda| = n$.*

Definition 1.4.17. *The* ***Young diagram*** *D_λ of a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ can be formally defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leqslant j \leqslant \lambda_i$, $i = 1, \ldots, r$.*

Graphically, we draw the diagrams replacing the points by square boxes such that the first coordinate i (the row index) increases from top to the bottom and the second coordinate j (the column index) increases from left to right. For instance, for $\lambda = (4, 3, 1)$, the corresponding Young diagram is given by



For a partition $\lambda \vdash n$ we shall denote by $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ the **conjugate partition** of λ, where λ'_j is the length of the j-th column of D_λ.

Definition 1.4.18. *Let $\lambda \vdash n$. A* ***Young tableau*** *T_λ of the Young diagram D_λ is a filling of the boxes of D_λ with the integers $1, 2, \ldots, n$. The tableau T_λ is* ***standard*** *if the integers in each row and in each column of T_λ increase from left to right and from top to bottom, respectively.*

Example 1.4.19. *Let $n = 3$. For $\lambda = (2, 1)$, the standard tableaux for λ are:*

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Proposition 1.4.20. *[29, Proposition 2.2.2] Let $n \geqslant 1$. There is a one-to-one correspondence between irreducible S_n-characters and partitions of n. Let $\{\chi_\lambda \mid \lambda \vdash n\}$ be a complete set of irreducible characters of S_n and $d_\lambda = \chi_\lambda(1)$ be the degree of χ_λ, $\lambda \vdash n$. Then*

$$FS_n \cong \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F),$$

where $I_\lambda = e_\lambda FS_n \cong M_{d_\lambda}(F)$ is the minimal two-sided ideal of FS_n corresponding to $\lambda \vdash n$, and $e_\lambda = \sum\limits_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ is the essential central idempotent, which is up to a scalar, the unit element of I_λ.

We denote by $M(\lambda)$ the irreducible S_n-module related with the partition $\lambda \vdash n$. The degrees of the irreducible representations of S_n can be obtained in two ways.

Theorem 1.4.21. *[22, Theorem 12.2.12] Let $\lambda \vdash n$.*

1. *The dimension d_λ of the irreducible S_n-module $M(\lambda)$ is equal to the number of standard Young tableaux T_λ.*

2. *(The Hook Formula)*

$$d_\lambda = \frac{n!}{\prod\limits_{(i,j)\in D_\lambda} (\lambda_i + \lambda'_j - i - j + 1)}.$$

Example 1.4.22. *Let $n = 5$. For $\lambda = (3,2)$, the dimension $d_{(3,2)}$ of the irreducible S_5-module $M(3,2)$ is:*

$$d_{(3,2)} = \frac{5!}{4\cdot 3\cdot 1\cdot 2\cdot 1} = 5.$$

The five standard tableaux for $\lambda = (3,2)$ are:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}$$

Now we describe a complete set of minimal left ideals of FS_n. Let $\lambda \vdash n$ be a partition and T_λ be a Young tableau. The **row stabilizer** R_{T_λ} of T_λ is the subgroup of S_n which maps every element of $\{1,\dots,n\}$ into an element standing in the same row in T_λ. The **column stabilizer** C_{T_λ} of T_λ is the subgroup of S_n which maps every element of $\{1,\dots,n\}$ into an element standing in the same column in T_λ. Clearly $R_{T_\lambda} \cap C_{T_\lambda} = \{1\}$.

The following theorem describes the irreducible representations of the symmetric group.

Theorem 1.4.23. *[22, Theorem 12.2.7] Let $\lambda \vdash n$ be a partition and T_λ be a Young tableau. Consider the element of the group algebra FS_n*

$$e_{T_\lambda} = \sum_{\substack{\sigma\in R_{T_\lambda}\\ \tau\in C_{T_\lambda}}} (\operatorname{sgn}\tau)\sigma\tau.$$

1. *$e_{T_\lambda}^2 = a e_{T_\lambda}$, where $a = \dfrac{n!}{d_\lambda}$, i.e., e_{T_λ} is an essential minimal idempotent of FS_n and generates a minimal left ideal of FS_n (i.e. an irreducible S_n-module).*

2. *If T_λ and $T^\star_\lambda$ are Young tableaux of the same partition $\lambda \vdash n$, then $FS_n e_{T_\lambda} \cong FS_n e_{T^\star_\lambda}$ as S_n-modules; moreover $\sigma e_{T_\lambda}\sigma^{-1} = e_{T^\star_\lambda}$ for some $\sigma \in S_n$.*

3. *If μ is another partition of n, then $FS_n e_{T_\mu} \ncong FS_n e_{T_\lambda}$.*

4. *Every irreducible S_n-module is isomorphic to $FS_n e_{T_\lambda}$ for some partition $\lambda \vdash n$.*

Example 1.4.24. *Let $n = 3$. Given the Young tableau $T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}$ of the partition $\lambda = (2,1)$. Then*

$$e_{T_{(2,1)}} = (1 + (1\,2))(1 - (1\,3)) = 1 + (1\,2) - (1\ 3) - (1\ 3\ 2)$$

is an essential idempotent of FS_3 which generates a two dimensional irreducible S_3-module.

1.4.3 S_n-actions on multilinear polynomials

In this section we introduce an action of the symmetric group S_n on the space of multilinear polynomials in n fixed variables.

Let A be a PI-algebra over a field of characteristic 0. By Corollary 1.2.22 $\mathrm{Id}(A)$ is determined by multilinear polynomials. We denote by P_n the vector space of all polynomials in $F\langle X\rangle$ which are multilinear of degree n. Clearly, P_n is of dimension $n!$ and we can write

$$P_n = \mathrm{Span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}.$$

Consider the map

$$\psi \colon FS_n \to P_n,$$
$$\sum_{\sigma\in S_n} \alpha_\sigma \sigma \mapsto \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

it is clear that ψ is a linear isomorphism. This isomorphism turns P_n into a left S_n-module with the action

$$\sigma(\sum \alpha_i x_{i_1} \cdots x_{i_n}) = \sum \alpha_i x_{\sigma(i_1)} \cdots x_{\sigma(i_n)}, \qquad \sigma \in S_n, \quad \alpha_i \in F, \quad x_{i_1} \cdots x_{i_n} \in P_n.$$

The meaning of the left S_n-action on a polynomial $f(x_1, \ldots, x_n) \in P_n$, for $\sigma \in S_n$ is

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

that is, of permuting the variables according to σ.

Since T-ideals are invariant under all endomorphisms, in particular, under substitution of variables, we obtain that T-ideals are invariant under all permutations of the variables. Thus, $P_n \cap \mathrm{Id}(A)$ is a left S_n-submodule of P_n. Hence

$$P_n(A) := \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

has an induced structure of left S_n-module. $P_n(A)$ is the subspace of the relatively free algebra $F\langle X\rangle/\mathrm{Id}(A)$ constituted by multilinear polynomials in the first n variables.

Definition 1.4.25. *Let A be a PI-algebra and let $P_n(A) = \dfrac{P_n}{P_n \cap \mathrm{Id}(A)}$, $n = 0, 1, 2, \ldots$. The S_n-character*

$$\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda(A)\chi_\lambda \tag{1.6}$$

*is called the **n-cocharacter** of the polynomial identities of the algebra A. The sequence*

$$\chi_n(A), \qquad n = 0, 1, 2, \ldots,$$

*is called the **cocharacter sequence** of A. The non-negative integer*

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

*is called the n-**th codimension** of A. The non-negative integer*

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)$$

*is called the n-**th colength** of A. Finally, the **PI-exponent** of A is defined by*

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}.$$

Since the n-th codimension $c_n(A)$ of the PI-algebra A is equal to the dimension of the S_n-module $P_n(A)$, we obtain inmediately that $c_n(A)$ is equal to evaluation of $\chi_n(A)$ on the identity permutation, i.e., $c_n(A) = \chi_n(A)(1)$. Moreover, $l_n(A)$ counts the number of irreducible S_n modules appearing in the decomposition of $P_n(A)$.

Example 1.4.26. *Let A be a commutative non-nilpotent algebra. Since the T-ideal of A coincides with the commutator ideal of $F\langle X\rangle$, the relatively free algebra $F\langle X\rangle/\operatorname{Id}(A)$ is isomorphic to the polynomial algebra $F[X]$ in infinitely many commuting variables. Hence $P_n(A) = \operatorname{Span}\{x_1 \cdots x_n\}$ and $\sigma(x_1 \cdots x_n) = x_1 \cdots x_n$ for all $\sigma \in S_n$, that is, $P_n(A)$ is the trivial module of S_n. Therefore for all n, $\chi_n(A) = \chi_n$, $c_n(A) = 1$, $l_n(A) = 1$ and* $\exp(A) = 1$.

An useful tool in computing the n-th cocharacter is the following.

Theorem 1.4.27. *[29, Theorem 2.1] Let A be a PI-algebra with n-th cocharacter $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda(A)\chi_\lambda$. For a fixed partition $\lambda \vdash n$, we have that $m_\lambda = 0$ if and only if for any Young tableau T_λ and for any polynomial $f \in P_n$, the algebra satisfies the identity $e_{T_\lambda} f \equiv 0$.*

1.4.4 Representations of general linear groups

In this section we will deal with the polynomial representations of general linear groups. The main application of representation theory of $GL_m(F)$ in the context of PI-algebras is the theorem of Drensky and Berele, which gives that any result on multilinear polynomial identities obtained in the language of representation of symmetric group is equivalent to a corresponding result on homogeneous polynomial identities obtained in the language of representation of the general linear group.

Denote by V_m the m-dimensional vector space with basis $\{x_1, \ldots, x_m\}$ over a field F and denote by $F\langle V_m\rangle$ the free associative algebra freely generated by $x_1, \ldots, x_m$, that is, $F\langle V_m\rangle = F\langle x_1, \ldots, x_m\rangle$.

The canonical representation $\varphi\colon GL_m(F) \to GL(V_m)$ of the general linear group $GL_m(F)$ on V_m is given by

$$\varphi_g(x_i) = \sum_{p=1}^{m} \alpha_{pi} x_p, \quad i = 1, \ldots, m,$$

where $g = (\alpha_{pq}) \in GL_m(F)$.

We shall build a representation $\phi\colon GL_m(F) \to GL(F\langle V_m\rangle)$ by extending the representation φ diagonally on $F\langle V_m\rangle$ by

$$\phi_g(x_{i_1}\cdots x_{i_n}) = \varphi_g(x_{i_1})\cdots\varphi_g(x_{i_n}), \quad g \in GL_m(F), x_{i_1},\dots,x_{i_n} \in F\langle V_m\rangle.$$

This turns $F\langle V_m\rangle$ in a left $GL_m(F)$-module which is a direct sum of its submodules $(F\langle V_m\rangle)^{(n)}$, $n = 0,1,2,\dots$, where $(F\langle V_m\rangle)^{(n)}$ is the homogeneous component of degree n of $F\langle V_m\rangle$. Moreover, if $f(x_1,\dots,x_m)$ belongs to the T-ideal U of $F\langle X\rangle$, and $g \in GL_m(F)$, then

$$\phi_g(f(x_1,\dots,x_m)) = f(\phi_g(x_1),\dots,\phi_g(x_m)) \in U,$$

and $U \cap F\langle V_m\rangle$ is $GL_m(F)$-invariant. Then the vector spaces $U \cap F\langle V_m\rangle$ and $U \cap (F\langle V_m\rangle)^{(n)}$ are submodules of $F\langle V_m\rangle$.

Definition 1.4.28. *Let $\phi\colon GL_m(F) \to GL_s(F)$ be a finite dimensional representation of the general linear group $GL_m(F)$. The representation ϕ is **polynomial** if the entries $(\phi_g)_{pq}$ of the $s \times s$ matrix ϕ_g are polynomials of the entries a_{kl} of g for $g \in GL_m(F)$, $k,l = 1,\dots,m$, $p,q = 1,\dots,s$. The polynomial representation ϕ is **homogeneous** of degree d if the polynomials $(\phi_g)_{pq}$ are homogeneous of degree d.*

The polynomials representations of $GL_m(F)$ have many properties similar to those of the representations of finite groups.

Theorem 1.4.29. *Every polynomial representation of $GL_m(F)$ is a direct sum of irreducible homogeneous polynomial subrepresentations. Moreover, every irreducible homogeneous polynomial representation of $GL_m(F)$ of degree $n \geqslant 0$ is isomorphic to a submodule of $(F\langle V_m\rangle)^{(n)}$.*

The irreducible homogeneous polynomial representations of degree n of $GL_m(F)$ are described by partitions of n in not more than m parts and Young diagrams.

Theorem 1.4.30. *[22, Theorem 12.4.4]*

1. *The pairwise nonisomorphic irreducible homogeneous polynomial $GL_m(F)$-representations of degree $n \geqslant 0$ are in 1-1 correspondence with the partitions $\lambda = (\lambda_1,\dots,\lambda_m)$ of n. We denote by $W_m(\lambda)$ the irreducible $GL_m(F)$-module related to λ.*

2. *Let $\lambda = (\lambda_1,\dots,\lambda_m) \vdash n$. The $GL_m(F)$-module $W_m(\lambda)$ is isomorphic to a submodule of $(F\langle V_m\rangle)^{(n)}$. Moreover, $(F\langle V_m\rangle)^{(n)}$ has the decomposition*

$$(F\langle V_m\rangle)^{(n)} \cong \sum d_\lambda W_m(\lambda),$$

where d_λ is the dimension of the irredubible S_n-module $M(\lambda)$ and the summations runs on all partitions $\lambda \vdash n$ in not more than m parts.

Now we introduce a right action of S_n on $(F\langle V_m\rangle)^{(n)}$ by

$$(x_{i_1}\cdots x_{i_n})\tau^{-1} = x_{i_{\tau(1)}}\cdots x_{i_{\tau(n)}}, \quad x_{i_1}\cdots x_{i_n}\in (F\langle V_m\rangle)^{(n)}, \quad \tau\in S_n.$$

Pay attention that the left action S_n on P_n defined in the previous section is an action on the variables and now S_n acts on the position of the variables.

Example 1.4.31. *Let $n=3$ and $f(x_1,x_2,x_3)=x_3x_1x_2$.. Then $\tau=(12)$ acts from the left as follows:*

$$\tau(x_3x_1x_2)=\tau(f(x_1,x_2,x_3))=f(x_{\tau(1)},x_{\tau(2)},x_{\tau(3)})=f(x_2,x_1,x_3)=x_3x_2x_1.$$

On the other hand, we have that $f(x_1,x_2,x_3)=x_3x_1x_2=x_{i_1}x_{i_2}x_{i_3}$ with $i_1=3, i_2=1, i_3=2$. Then $\tau=(12)$ acts from the right as follows:

$$(x_3x_1x_2)\tau = x_{i_{\tau(1)}}x_{i_{\tau(2)}}x_{i_{\tau(3)}} = x_{i_2}x_{i_1}x_{i_3} = x_1x_3x_2.$$

Let $\lambda=(\lambda_1,\dots,\lambda_m)$ be a partition of n in not more than m parts and let $q_1,\dots,q_k$ be the lengths of the columns of the Young diagram D_λ (that is, $q_j=\lambda'_j$ and $k=\lambda_1$). Fix $q=q_1$ and let $s_\lambda=s_\lambda(x_1,\dots,x_q)$ be the polynomial of $F\langle V_m\rangle$ defined as follows:

$$s_\lambda(x_1,\dots,x_q)=\prod_{j=1}^{k} s_{q_j}(x_1,\dots,x_{q_j}),$$

where $s_p(x_1,\dots,x_p)$ is the standard polynomial (see Example 1.1.23).

Example 1.4.32. *1. If $\lambda=(n)$, then $s_\lambda=s_{(n)}(x_1)=\prod_{j=1}^{n}s_1(x_1)=x_1^n$.*

2. If $\lambda=(1^n)$, then $s_\lambda=s_{(1^n)}(x_1,\dots,x_n)=s_n(x_1,\dots,x_n)=\sum_{\sigma\in S_n}(\operatorname{sgn}\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}$.

The following theorem is the analog of Theorem 1.4.23 for representations of $GL_m(F)$.

Theorem 1.4.33. *[22, Theorem 12.4.12] Let $\lambda=(\lambda_1,\dots,\lambda_m)$ be a partition of n in not more than m parts and let $(F\langle V_m\rangle)^{(n)}$ be the homogeneous component of degree n in $F\langle V_m\rangle$.*

1. *The element $s_\lambda(x_1,\dots,x_q)$, defined above, generates an irreducible $GL_m(F)$-submodule of $(F\langle V_m\rangle)^{(n)}$ isomorphic to $W_m(\lambda)$.*
2. *Every $W_m(\lambda)\subseteq (F\langle V_m\rangle)^{(n)}$ is generated by a nonzero element*

$$w_\lambda(x_1,\dots,x_q)=s_\lambda(x_1,\dots,x_q)\sum_{\sigma\in S_n}\alpha_\sigma\sigma, \quad \alpha_\sigma\in F.$$

The element $w_\lambda(x_1,\dots,x_q)$ is unique up to a multiplicative constant and it is called the ***highest weight vector*** *of $W_m(\lambda)$.*

Lemma 1.4.34. *If f and g are homogeneous polynomials of degree n in $F\langle V_m\rangle$, then the polynomial identity $g \equiv 0$ is a consequence of $f \equiv 0$ if and only if g belongs to the $GL_m(F)$-module generated by f.*

Proof. See Exercise 12.4.17 in [22]. $\square$

There is a close relationship between the irreducible polynomial representations of $GL_m(F)$ and the irreducible representations of the symmetric group S_n.

Proposition 1.4.35. *Let $m \geqslant n$, $\lambda \vdash n$ and let $W_m(\lambda) \subseteq F\langle V_m\rangle$. The set $M = W_m(\lambda) \cap P_n$ of all multilinear elements in $W_m(\lambda)$ is an S_n-submodule of P_n isomorphic to $M(\lambda)$. Every submodule $M(\lambda)$ can be obtained in this way.*

Proof. See, for instance, Proposition 12.4.18 in [22]. $\square$

The following theorem, due to Drensky [19] and Berele [11], establishes that any result on multilinear polynomial identities obtained in the language of representations of the symmetric group is equivalent to a corresponding result on homogeneous polynomial identities obtained in the language of representations of the general linear group.

Theorem 1.4.36. *Let A be a PI-algebra and let*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

be the cocharacter sequence of the T-ideal of A. Then, for any m, the relatively free algebra $F\langle V_m\rangle/\operatorname{Id}(A)$ is isomorphic as a $GL_m(F)$-module to the direct sum

$$\sum_{n \geqslant 0} \sum_{\lambda \vdash n} m_\lambda(A) W_m(\lambda),$$

with the same multiplicities $m_\lambda(A)$ as in the cocharacter sequence (Equation 1.6) and $W_m(\lambda) = 0$ if λ is a partition in more than m parts. On the other hand, if $m \geqslant n$ and

$$F\langle V_m\rangle/\operatorname{Id}(A) \cong \sum_{\lambda \vdash n} n_\lambda(A) W_m(\lambda),$$

for some $n_\lambda(A)$, then

$$\chi_n(A) = \sum_{\lambda \vdash n} n_\lambda(A)\chi_\lambda.$$

2 Superalgebras with superinvolution

The purpose of this chapter is to give a positive answer to the Specht's problem in the setting of finitely generated superalgebras with superinvolution ($*$-algebras). More precisely, if W is a finitely generated $*$-algebra over a field F of characteristic 0 satisfying an ordinary identity, we shall find a finite generating set for the T_2^*-ideal of identities $\mathrm{Id}^*(W)$. We recall that this result was announced in [1]. Here we shall give an explicit construction of Kemer's polynomials that are the key ingredient in solving the Specht's problem (Theorem 2.4.5). Finally, we present a proof of the Hilbert-Serre Theorem in the case of relatively free algebras of superalgebras with superinvolution (Theorem 2.5.15).

2.1 Kemer points and Kemer polynomials

Let Γ be a T_2^*-ideal. Recall that, since F is a field of characteristic zero, Γ is generated by multilinear $*$-polynomials. Let X be a set of variables. We can write

$$X = X_0^+ \cup X_0^- \cup X_1^+ \cup X_1^-,$$

where X_0^+ is the subset of symmetric even variables (the y_i^+'s), X_0^- is the subset of skew even variables (the y_i^-'s), X_1^+ is the subset of symmetric odd variables (the z_i^+'s) and X_1^- is the subset of skew odd variables (the z_i^-'s). Let S_0 and S_1 be subsets of $Y = X_0^+ \cup X_0^-$ and $Z = X_1^+ \cup X_1^-$ respectively, and let $R_0 = Y \backslash S_0$, $R_1 = Z \backslash S_1$. Of course, if $S_i = \{x_1, \dots, x_m\}$, then the variables x_j's are of homogeneous degree i and symmetric or skew.

Definition 2.1.1. *Let $f = f(X)$ be a multilinear $*$-polynomial. We say that f is* ***alternating*** *in $S_i = \{x_1, \dots, x_m\}$, $i \in \{0, 1\}$, if there exists a multilinear $*$-polynomial $h(S_i, R_i) := h(x_1, \dots, x_m, R_i)$ such that*

$$f(X) = \sum_{\sigma \in S_m} (-1)^\sigma h(x_{\sigma(1)}, \dots, x_{\sigma(m)}, R_i).$$

If $S_1, \dots, S_p$, are p disjoint sets of variables of X (belonging to Y or Z), we say that $f(X)$ is **alternating** in $S_{i_1}, \dots, S_{i_p}$, if it is alternating in each of them.

Now we will consider polynomials which alternate in 2ν disjoint sets of the form S_i, $i = 0, 1$.

Definition 2.1.2. *Let $f = f(X)$ be a multilinear $*$-polynomial alternating in $S_{i_1}, \dots, S_{i_{2\nu}}$. If all the sets $S_{i_1}, \dots, S_{i_{2\nu}}$ belonging to the same set (Y or Z) have the same cardinality (say $d_i, i \in \{0, 1\}$), then we will say that*

$$f(X) \text{ is } \nu\text{-fold } (d_0, d_1)\text{-alternating.}$$

In order to define the $*$-index of a T_2^*-ideal Γ we need the notion of **i-th Capelli polynomial**, $i \in \{0,1\}$. Let $X_{n,i} = \{x_1, \ldots, x_n\}$ be a set of n variables of homogeneous degree $i \in \{0,1\}$ and let $W = \{w_1, \ldots, w_{n+1}\}$ be a set of $n+1$ ungraded variables. The i-th Capelli polynomial $c_{n,i}$ of degree $2n+1$ is the polynomial obtained by alternating the set of variables $x_1, \ldots, x_n$ in the monomial $w_1 x_1 w_2 \cdots x_n w_{n+1}$. Hence

$$c_{n,i} = \sum_{\sigma \in S_n} (-1)^\sigma w_1 x_{\sigma(1)} w_2 \cdots x_{\sigma(n)} w_{n+1}.$$

Clearly $c_{n,i}$, $i \in \{0,1\}$ is a multilinear $*$-polynomial alternating in $\{x_1 \ldots, x_n\}$.

Lemma 2.1.3. *For any $i \in \{0,1\}$ there exists an integer n_i such that the T_2^*-ideal Γ contains $c_{n_i,i}$.*

Proof. Let A be a finite dimensional $*$-algebra such that $\mathrm{Id}^*(A) \subseteq \Gamma$ (such algebra exists by Theorem 1.2.35). We consider the decomposition $A = A_0 \oplus A_1$ and we take $n_i = \dim A_i + 1$, $i \in \{0,1\}$. It is clear that $c_{n_i,i} \in \mathrm{Id}^*(A)$ and the proof is complete. □

As a consequence, we get the following result.

Corollary 2.1.4. *If $f = f(X)$ is a multilinear $*$-polynomial alternating on a set S_i of cardinality n_i, then $f \in \Gamma$. Consequently, there exists an integer M_i which bounds (from above) the cardinality of the alternating homogeneous sets in any $*$-polynomial h which is not in Γ.*

Let Γ denote the T_2^*-ideal of a finitely generated $*$-algebra. Now we are in a position to define the $*$-index $\mathrm{Ind}^*(\Gamma)$ of Γ. Here we want to highligh that in [2] Aljadeff and Belov introduced the analogous object in the setting of G-graded algebras, where G is a finite group.

$\mathrm{Ind}^*(\Gamma)$ will consist of a finite set of points (α, s) in the lattice $L = \mathbb{N}^2 \times (\mathbb{N} \cup \infty)$. Given $\alpha = (\alpha_0, \alpha_1)$, $\beta = (\beta_0, \beta_1) \in \mathbb{N}^2$, we put $\alpha \preceq \beta$ if and only if $\alpha_i \leqslant \beta_i$, for $i = 0,1$. This gives a partial order in $\mathbb{N}^2$. As a consequence, we obtain a partial order on L. Given $(\alpha, s), (\beta, s') \in L$, we write $(\alpha, s) \preceq (\beta, s')$ if and only if either

1) $\alpha \prec \beta$, or

2) $\alpha = \beta$ and $s \leqslant s'$ (notice that $s < \infty$ for every $s \in \mathbb{N}$).

We first determine the set $\mathrm{Ind}^*(\Gamma)_0$, namely the projection of $\mathrm{Ind}^*(\Gamma)$ into $\mathbb{N}^2$.

Definition 2.1.5. *A point $\alpha = (\alpha_0, \alpha_1)$ is in $\mathrm{Ind}^*(\Gamma)_0$ if for any integer ν there exists a multilinear $*$-polynomial outside Γ with ν alternating homogeneous sets (of degree i) of cardinality α_i for every $i = 0,1$.*

Lemma 2.1.6. *The following facts hold:*

1. *The set* $\mathrm{Ind}^*(\Gamma)_0$ *is bounded (finite).*

2. *If* $\alpha \in \mathrm{Ind}^*(\Gamma)_0$*, then* $\alpha' \preceq \alpha$ *is also in* $\mathrm{Ind}^*(\Gamma)_0$*.*

Proof. The first statement follows since $\Gamma \supseteq \mathrm{Id}^*(A)$, for some finite dimensional $*$-algebra A at light of Theorem 1.2.35. The second one is a consequence of the definition of $\mathrm{Ind}^*(\Gamma)_0$. □

Definition 2.1.7. *A point* $\alpha \in \mathrm{Ind}^*(\Gamma)_0$ *is* ***extremal*** *if for any* $\beta \in \mathrm{Ind}^*(\Gamma)_0$*,* $\beta \succeq \alpha$ *implies* $\beta = \alpha$*.*

We denote by $E_0(\Gamma)$ the set of all extremal points in $\mathrm{Ind}^*(\Gamma)_0$.

For any point $\alpha = (\alpha_0, \alpha_1) \in E_0(\Gamma)$ and every integer ν, consider the set $\Omega_{\alpha,\nu}$ of all ν-fold alternating polynomials in homogeneous sets of cardinality α_i, where $i = 0, 1$, that are not in Γ. Given $f \in \Omega_{\alpha,\nu}$, we consider the number $s_\Gamma(\alpha, \nu, f)$ of alternating homogeneous sets of disjoint variables, of cardinality $\alpha_i + 1$, $i = 0, 1$. The set of integers $\{s_\Gamma(\alpha, \nu, f)\}_{f \in \Omega_{\alpha,\nu}}$ is bounded. We define $s_\Gamma(\alpha, \nu) = \max\{s_\Gamma(\alpha, \nu, f)\}_{f \in \Omega_{\alpha,\nu}}$. The sequence $s_\Gamma(\alpha, \nu)$ is monotonically decreasing as a function of ν. As a consequence, there exists an integer $\mu = \mu(\Gamma, \alpha)$ for which the sequence stabilizes, that is for $\nu \geqslant \mu$, the sequence $s_\Gamma(\alpha, \nu)$ is constant. We let $s(\alpha) = \lim_{\nu \to \infty} s_\Gamma(\alpha, \nu) = s_\Gamma(\alpha, \mu)$. At this point the integer μ depends on α. However, since the set $E_0(\Gamma)$ is finite by Lemma 2.1.6, we take μ to be the maximum of all μ's considered above. Keeping in mind the definition of μ, we have the following definitions.

Definition 2.1.8. *The* ***$*$-index*** $\mathrm{Ind}^*(\Gamma)$ *of* Γ *is the set of points* $(\alpha, s) \in L$ *such that* $\alpha \in \mathrm{Ind}^*(\Gamma)_0$ *and* $s = s_\Gamma(\alpha)$ *if* $\alpha \in E_0(\Gamma)$ *or* $s = \infty$ *otherwise.*

Definition 2.1.9. *Given a* T_2^**-ideal* Γ *containing the* $*$*-identities of a finite dimensional* $*$*-algebra* A*, we let the* ***Kemer set*** *of* Γ*, denoted* $K(\Gamma)$*, be the set of points* (α, s) *in* $\mathrm{Ind}^*(\Gamma)$*, where* α *is extremal. We refer to the elements of* $K(\Gamma)$ *as the* ***Kemer points*** *of* Γ*.*

The next remark follows immediately.

Remark 2.1.10. *Let* $\Gamma_1 \supseteq \Gamma_2$ *be two* T_2^**-ideals containing* $\mathrm{Id}^*(A)$*, where* A *is a finite dimensional* $*$*-algebra. Then:*

1. $\mathrm{Ind}^*(\Gamma_1) \subseteq \mathrm{Ind}^*(\Gamma_2)$*.*

2. *For every* $(\alpha, s) \in K(\Gamma_1)$ *there is a Kemer point* $(\beta, s') \in K(\Gamma_2)$ *such that* $(\alpha, s) \preceq (\beta, s')$*.*

We are now ready to define Kemer polynomials for a T_2^*-ideal Γ.

Definition 2.1.11. *Let (α, s) be a Kemer point of Γ. A $*$-polynomial f is said to be a **Kemer $*$-polynomial** for the point (α, s) if $f \notin \Gamma$ and it has at least ν-folds of alternating homogeneous sets (of degree i) of cardinality α_i (small sets), where $i = 0, 1$, and s homogeneous sets of disjoint variables μ (of some homogeneous degree) of cardinality $\alpha_i + 1$ (big sets). A $*$-polynomial f is **Kemer** for Γ if it is Kemer for a Kemer point of Γ.*

If we choose a Kemer point (α, s), then α is extremal. Because of this, we get the next result.

Remark 2.1.12. *A $*$-polynomial f cannot be Kemer simultaneously for different Kemer points of Γ.*

2.2 Decomposition in basic $*$-algebras

In this section we shall introduce the so-called basic $*$-algebras and we shall prove that every finitely generated $*$-algebra satisfying an ordinary non-trivial identity is T_2^*-equivalent to the direct product of finitely many basic $*$-algebras.

First, let A be a finite dimensional $*$-algebra and consider its Wedderburn-Malcev decomposition:

$$A = B + J(A).$$

The semisimple part B is a $*$-algebra too and so we can consider its decomposition in symmetric and skew spaces of homogeneous degree 0 and 1, respectively:

$$B = B_0 \oplus B_1 = B_0^+ \oplus B_0^- \oplus B_1^+ \oplus B_1^-.$$

We use the following notation:

- $d(B_i) = \dim_F B_i$, $i \in \{0, 1\}$,
- $n(A)$ is the nilpotency index of $J(A)$.

We write $\mathrm{Par}^*(A)$ to indicate the 3-tuple $(d(B_0), d(B_1), n(A) - 1) \in \mathbb{N}^2 \times \mathbb{N}$.

Proposition 2.2.1. *If $(\alpha, s) = (\alpha_0, \alpha_1, s)$ is a Kemer point of A, then $(\alpha, s) \preceq \mathrm{Par}^*(A)$.*

Proof. Suppose, by contradiction, that this does not happen. Hence, $\alpha_i > d(B_i)$ for some $i = 0, 1$, or $\alpha_i = d(B_i)$ in any case and $s > n(A) - 1$. We shall see that both these possibilities cannot occur. First recall that, since (α, s) is a Kemer point of A, then there exist multilinear $*$-polynomials f which are non-identities of A with arbitrary many alternating homogeneous sets of cardinality α_i, $i = 0, 1$.

1. Suppose $\alpha_i > d(B_i)$, for some $i = 0, 1$.
   We have that in each such alternating set there must be at least one radical substitution in any non-zero evaluation of a polynomial f. This implies that we cannot have more than $n(A) - 1$ alternating homogeneous sets of cardinality α_j, contradicting our previous statement.

2. Suppose $\alpha_i = d(B_i)$ in any case and $s > n(A) - 1$.
   This means that we have s alternating sets (of a certain homogeneous degree) of cardinality $\alpha_i + 1 = d(B_i) + 1$, for some $i = 0, 1$. Again this means that f will vanish if we evaluate any of these sets by semisimple elements. It follows that in each one of these s sets at least one of the evaluations is radical. Since $s > n(A) - 1$, the polynomial f vanishes on such evaluations as well and hence it is a $*$-identity of A. We reach a contradiction in this case too, and this complete the proof.

□

In order to establish a precise relation between Kemer points of a finite dimensional $*$-algebra A and its structure we need to find appropriate finite dimensional algebras which will serve as minimal models for a given Kemer point. We start with the decomposition of a finite dimensional $*$-algebra into the product of subdirectly irreducible components.

Definition 2.2.2. *A finite-dimensional $*$-algebra A is said to be **subdirectly irreducible** if there are no non-trivial $*$-ideals I and J of A such that $I \cap J = (0)$.*

Lemma 2.2.3. *Let A be a finite dimensional $*$-algebra over F. Then A is T_2^*-equivalent to a direct product $C_1 \times \cdots \times C_n$ of finite dimensional subdirectly irreducible $*$-algebras. Furthermore for every $i = 1, \ldots, n$, $\dim_F(C_i) \leqslant \dim_F(A)$ and the number of $*$-simple components in C_i is bounded by the number of such components in A.*

Proof. If A is subdirectly irreducible there is nothing to prove. If A is not subdirectly irreducible, then there exist non-trivial $*$-ideals I and J of A such that $I \cap J = (0)$. It is clear that A/I (and at the same way A/J) is a $*$-algebra with superinvolution $\bar{*}\colon A/I \to A/I$ induced from the superinvolution $*$ of A by $\bar{*}(a + I) = a^* + I$, for any $a \in A$. Moreover, it is easy to prove that A is T_2^*-equivalent to $A/I \times A/J$. This completes the first part of the proof.

The second one follows by induction by taking into account the fact that $\dim_F(A/I)$ and $\dim_F(A/J)$ are strictly smaller than $\dim_F A$. □

By Theorem 1.2.31, a $*$-algebra A can be decomposed as

$$A = B + J \cong A_1 \times \cdots \times A_q + J,$$

where J is the Jacobson radical of the algebra (a $*$-ideal) and $A_1, \ldots, A_q$ are simple $*$-algebras.

Definition 2.2.4. *Let A be a finite-dimensional $*$-algebra and let $f = f(x_1, \ldots, x_m)$ be a multilinear $*$-polynomial. We say that A is* ***full with respect to*** *f, if there exists a non-vanishing evaluation of f such that every $*$-simple component is represented (among the semisimple substitutions) in any substitution. In other words, if $f(\bar{x}_1, \ldots, \bar{x}_m) \neq 0$ for suitable substitutions $\bar{x}_i$ such that*

$$A_k \cap \{\bar{x}_1, \ldots, \bar{x}_m\} \neq, \qquad \textit{for all } 1 \leqslant k \leqslant q.$$

A finite dimensional $*$-algebra A is ***full*** *if it is full with respect to some multilinear $*$-polynomial f.*

Lemma 2.2.5. *A finite dimensional $*$-algebra A is full if and only if we have a permutation π of $\{1, \ldots, q\}$ such that*

$$A_{\pi(1)} J A_{\pi(2)} J \cdots J A_{\pi(q)} \neq 0.$$

Proof. If $A_{\pi(1)} J A_{\pi(2)} \cdots J A_{\pi(q)} \neq 0$, then A is full with respect to the monomial $x_1 \cdots x_{2q-1}$. On the other hand, suppose A is full with respect to a monomial f. Since the substitutions pass through each component and f is non-vanishing in A, these substitutions should be connected by radical substitutions and the proof is complete. □

For $i = 1, \ldots, q$, let e_i denote the identity element of A_i and consider the decomposition

$$A \cong \bigoplus_{i,j=1}^{q} e_i A e_j.$$

By the previous Lemma, whenever $i_1, \ldots, i_q$ are distinct, it follows that if A is not full,

$$e_{i_1} A e_{i_2} \cdots e_{i_{q-1}} A e_{i_q} = e_{i_1} J e_{i_2} \cdots e_{i_{q-1}} J e_{i_q} = 0. \tag{2.1}$$

Remark 2.2.6. *Let A be a $*$-algebra over a field F of characteristic zero and let $\bar{F}$ be the algebraic closure of F. Then $\bar{A} = A \otimes_F \bar{F}$ is a $*$-algebra with superinvolution $(a \otimes \alpha)^{\bar{*}} = a^* \otimes \alpha$. We have that*

- $\dim_F A = \dim_{\bar{F}} \bar{A}$,
- $\mathrm{Id}^*(A) = \mathrm{Id}^*(\bar{A})$, *viewed as $*$-algebras over F,*

We wish to show that any finite dimensional algebra may be decomposed (up to T_2^*-equivalence) into the direct product of full algebras. Algebras without an identity element are treated separately.

Lemma 2.2.7. *Let A be a $*$-algebra subdirectly irreducible and not full.*

1. *If A has an identity element then it is T_2^*-equivalent to a direct product of finite-dimensional $*$-algebras, each having fewer $*$-simple components.*

2. *If A has no identity element then it is T_2^*-equivalent to a direct product of finite-dimensional $*$-algebras, each having either fewer $*$-simple components than A or else it has an identity element and the same number of $*$-simple components as A.*

Proof. Suppose first that A has unit. By Equation (2.1)

$$e_{i_1}Ae_{i_2}\cdots e_{i_{q-1}}Ae_{i_q} = e_{i_1}Je_{i_2}\cdots e_{i_{q-1}}Je_{i_q} = 0.$$

Let us consider the commutative algebra $R = F[\lambda_1, \ldots, \lambda_q]/I$, where I is the ideal generated by $\lambda_i^2 - \lambda_i$ and $\lambda_1 \cdots \lambda_q$. We denote by $\tilde{e}_i$ the image of λ_i in R. It is clear that $\tilde{e}_i^2 = \tilde{e}_i$ and $\tilde{e}_1 \cdots \tilde{e}_q = 0$. The algebra $A \otimes_F R$ is a $*$-algebra with superinvolution $\bar{*}$ induced via the superinvolution $*$ of A as in Remark 2.2.6. Let $\tilde{A}$ be the $*$-subalgebra generated by all $e_iAe_j \otimes \tilde{e}_i\tilde{e}_j$, for every $1 \leqslant i, j \leqslant q$. We claim that $A \sim_{T_2^*} \tilde{A}$. Clearly $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(A \otimes_F R) \subseteq \mathrm{Id}^*(\tilde{A})$. Hence it suffices to prove that any non $*$-identity f of A is also a non-identity of $\tilde{A}$. Clearly, we may assume that f is multilinear. Evaluating f on A it suffices to consider maps of the form $x_l^{\pm} \mapsto v_l^{i_l,\pm}$, where $x \in \{y, z\}$ and $i_l \in \{0, 1\}$ (symmetric or skew elements of homogeneous degree 0 or 1) and $v_l^{i_l,\pm} \in e_{j_k}Ae_{j_{k+1}}$, for some k. In order to have $v_1^{i_1,\pm} \cdots v_n^{i_n,\pm} \neq 0$, the set of indices $\{j_k\}$ must contain at most $q - 1$ distinct elements, so $e_{j_1} \cdots e_{j_n} \neq 0$. Then

$$f(v_1^{i_1,\pm} \otimes \tilde{e}_{i_1}, \ldots, v_n^{i_n,\pm} \otimes \tilde{e}_{i_n}) = f(v_1^{i_1,\pm}, \ldots, v_n^{i_n,\pm}) \otimes \tilde{e}_{i_1} \cdots \tilde{e}_{i_n} \neq 0.$$

Hence f is not in $\mathrm{Id}^*(\tilde{A})$ and this proves the claim.

In order to complete the proof we need to show that $\tilde{A}$ can be decomposed into a direct product of $*$-algebras, each having fewer $*$-simple components. Let $I_j = \langle e_j \otimes \tilde{e}_j, e_j^* \otimes \tilde{e}_j \rangle$ be a $*$-ideal of $\tilde{A}$. Hence

$$\bigcap_{j=1}^{q} I_j = (1 \otimes \tilde{e}_1 \cdots 1 \otimes \tilde{e}_q)\left(\bigcap_{j=1}^{q} I_j\right) = (1 \otimes \tilde{e}_1 \cdots \tilde{e}_q)\left(\bigcap_{j=1}^{q} I_j\right) = (0).$$

It follows that $\tilde{A}$ is subdirectly reducible to the direct product of $\tilde{A}/I_j$. Furthermore, each component $\tilde{A}/I_j$ has less than q $*$-simple components since we eliminated the idempotent corresponding to the j-th $*$-simple component. This completes the proof of the first part of the lemma.

Consider now the case in which the algebra A has no identity element. In the notation of Remark 1.1.17, let $e_0 = \hat{1} - (e_1 + \cdots + e_q)$; we consider the decomposition

$$A \cong \bigoplus_{i,j=0}^{q} e_iAe_j$$

and we carry on as in the first case but with $q+1$ idempotents, variables, and so on. As above, A/I_j will have less than q $*$-simple components if $1 \leqslant j \leqslant q$ whereas A/I_0 will have an identity element and exactly q $*$-simple components. The proof now is complete. □

By putting together Lemmas 2.2.3 and 2.2.7 we get the following result.

Corollary 2.2.8. *Every finite dimensional $*$-algebra A is T_2^*-equivalent to a direct product of full, subdirectly irreducible finite dimensional $*$-algebras.*

Remark 2.2.9. *In the decomposition above, the nilpotency index of the components in the direct product is bounded by the nilpotency index of A.*

In the following definition we introduce the so-called minimal algebras.

Definition 2.2.10. *We say that a finite dimensional $*$-algebra A is* ***minimal*** *if* $\mathrm{Par}^*(A)$ *is minimal (with respect to the partial order defined before) among all finite dimensional $*$-algebras which are T_2^*-equivalent to A.*

Definition 2.2.11. *A finite dimensional $*$-algebra A is said to be* ***basic*** *if it is minimal, full and subdirectly irreducible.*

As a consequence of the results and definitions of this section we obtain the following theorem.

Theorem 2.2.12. *Every finite dimensional $*$-algebra A is T_2^*-equivalent to the direct product of finitely many basic $*$-algebras.*

Combining this result with Theorem 1.2.35 we obtain the following corollary.

Corollary 2.2.13. *Every finitely generated $*$-algebra W satisfying an ordinary non-trivial identity is T_2^*-equivalent to the direct product of finitely many basic $*$-algebras.*

2.3 Kemer's lemmas

The task in this section is to show that any basic $*$-algebra A has a Kemer set which consists of a unique point $(\alpha, s) = \mathrm{Par}^*(A)$. To achieve this, throughout the section we show a constructive way to obtain the appropriate Kemer $*$-polynomials that will comprise the backbone for the rest of the work on superalgebras with superinvolution. We start with some preliminaries in the framework of finite dimensional simple $*$-algebras.

Let $A = (a_{ij})$ be an $n \times n$ matrix. For $j = 2, \ldots, n$, the **j-th hook** of A is the set of elements:

$$\{a_{1j}, a_{2j}, \ldots, a_{jj}, a_{j1}, a_{j2}, \ldots, a_{jj-1}\}.$$

Remark 2.3.1. *There exists a product of the matrix units e_{ij}, $i, j \in \{1, \ldots, n\}$, with value e_{11}.*

Proof. Let us consider the matrix $E = (e_{ij}) \in M_n(F)$ and let

$$e_{1j}, e_{2j}, \ldots, e_{jj}, e_{j1}, e_{j2}, \ldots, e_{jj-1}$$

be the elements in the j-th hook of E. We have

$$e_{1j}e_{j2}e_{2j}e_{j3}e_{3j}\cdots e_{jj-1}e_{j-1j}e_{jj}e_{j1} = e_{11}.$$

For any $j = 2, \ldots, n$, we denote by H_j the previous product of matrix units. The proof now works because

$$e_{11}H_2H_3\cdots H_n = e_{11}.$$

□

Now let us consider the $*$-algebra $M_{k,k}(F)$ with the transpose superinvolution *trp*. Notice that:

$$(M_{k,k}(F),\ trp)_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} |X \in M_k(F) \right\},$$

$$(M_{k,k}(F),\ trp)_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} |X \in M_k(F) \right\},$$

$$(M_{k,k}(F),\ trp)_1^+ = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} |Y = -Y^t,\ Z = Z^t,\ Y, Z \in M_k(F) \right\},$$

$$(M_{k,k}(F),\ trp)_1^- = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} |Y = Y^t,\ Z = -Z^t,\ Y, Z \in M_k(F) \right\}.$$

The following elements form a **$*$-basis** (i.e., basis as a vector space with homogeneous symmetric or skew elements) of $(M_{k,k}(F), trp)$:

- $\{e_{i,j} + e_{k+j,k+i}\}$, $i, j = 1, \ldots, k$.
- $\{e_{i,j} - e_{k+j,k+i}\}$, $i, j = 1, \ldots, k$.
- $\{e_{i,k+j} - e_{j,k+i},\quad e_{k+i,j} + e_{k+j,i},\quad e_{k+l,l}\}$, $1 \leqslant i < j \leqslant k$ and $l = 1, \ldots, k$.
- $\{e_{i,k+j} + e_{j,k+i},\quad e_{k+i,j} - e_{k+j,i},\quad e_{l,k+l}\}$, $1 \leqslant i < j \leqslant k$ and $l = 1, \ldots, k$.

Lemma 2.3.2. *There exists a product of the above $*$-basis elements with value e_{11}.*

Proof. Let us consider the matrix units e_{pq}, $p, q = 1, \ldots, 2k$. It is easy to see that in the above $*$-basis there is at least one element in which e_{pq} appears with a plus, for any $p, q \in \{1, \ldots, 2k\}$. When there are two elements of this kind, we make the following choice: we fix the element of the $*$-basis corresponding to e_{pq} to be that one in which in the second part of the element appears a minus. We shall denote by $\bar{e}_{pq}$ the element of the $*$-basis corresponding to e_{pq}. For instance, $e_{1,1}$ appears in the $*$-basis both in $e_{1,1} + e_{k+1,k+1}$ and $e_{1,1} - e_{k+1,k+1}$. Hence $\bar{e}_{1,1} = e_{1,1} - e_{k+1,k+1}$. In this way we are sure that $\bar{e}_{k+1,k+1} = e_{1,1} + e_{k+1,k+1}$ (notice that $e_{k+1,k+1}$ appears with a plus, as desired).

Now we construct the following $2k \times 2k$ matrix E: in the entry (p, q) we put the element of the $*$-basis $\bar{e}_{pq}$. As in Remark 2.3.1, we denote by H_j the product of the elements in the j-th hook of the matrix E, $j = 2, \ldots, 2k$. Moreover, in any element of the $*$-basis of the form $e_{ab} \pm e_{cd}$, we have $a \neq c$. Hence, as desired, we get

$$\bar{e}_{11} H_2 \cdots H_{2k} = e_{11}. \tag{2.2}$$

□

Let us consider the monomial $M = w_1 \cdots w_{4k^2}$, where each variable w_i has a certain homogeneous degree and it is symmetric or skew according to the corresponding element in the product in (2.2).

If we border each matrix $\bar{e}_{i,j}$ in the product (2.2) with idempotents $e_{i,i}$ and $e_{j,j}$, then we can consider the monomial obtained by M by surrounding each variable with a variable of homogeneous degree 0:

$$M' = y_1 w_1 y_2 w_2 \cdots y_{4k^2} w_{4k^2} y_{4k^2+1}.$$

Clearly, the monomial M' has the property that there exists an evaluation φ such that $\varphi(M') = e_{11}$. Moreover, we have

$$e_{j,j} = \begin{cases} \dfrac{(e_{i,i} + e_{k+i,k+i}) + (e_{i,i} - e_{k+i,k+i})}{2}, & \text{if } 1 \leqslant i = j \leqslant k, \\ \dfrac{(e_{i,i} + e_{k+i,k+i}) - (e_{i,i} - e_{k+i,k+i})}{2}, & \text{if } j = k+i,\ 1 \leqslant i \leqslant k. \end{cases} \tag{2.3}$$

Thus we can write each bordering element $e_{i,i}$ in terms of the $*$-basis elements. In this way, we can replace each variable y_i in the monomial M' by $(y_i^+ + y_i^-)/2$ or $(y_i^+ - y_i^-)/2$ according to (2.3), where y_i^+ is a symmetric variable of zero degree and y_i^- is a skew variable of degree 0. Denote by P this $*$-polynomial. Then we have the next result that is a consequence of Lemma 2.3.2.

Lemma 2.3.3. *Consider the $*$-polynomial*

$$P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y_{4k^2}^+ \pm y_{4k^2}^-}{2} w_{4k^2} \frac{y_{4k^2+1}^+ \pm y_{4k^2+1}^-}{2}$$

defined above. Then there exists an evaluation φ of P such that $\varphi(P) = e_{11}$.

Now let us consider $A = (M_{k,2l}(F),\ osp)$ be the $*$-algebra of $(k+2l)\times(k+2l)$ matrices endowed with the orthosymplectic superinvolution. Recall we have the following:

$$
\begin{aligned}
A_0^+ &= \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = X^t,\ T = -QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},\\
A_0^- &= \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = -X^t,\ T = QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},\\
A_1^+ &= \left\{ \begin{pmatrix} 0 & Z^tQ \\ Z & 0 \end{pmatrix} \mid Z \text{ is a } 2l \times k \text{ matrix} \right\},\\
A_1^- &= \left\{ \begin{pmatrix} 0 & -Z^tQ \\ Z & 0 \end{pmatrix} \mid Z \text{ is a } 2l \times k \text{ matrix} \right\}.
\end{aligned}
$$

It is easy to see that the following sets $\mathcal{B}_0^+, \mathcal{B}_0^-, \mathcal{B}_1^+, \mathcal{B}_1^-$ form a $*$-basis of $A_0^+, A_0^-, A_1^+, A_1^-$ respectively:

$$
\mathcal{B}_0^+ = \left\{ \begin{array}{cc}
e_{i.i} & 1 \leqslant i \leqslant k, \\
e_{i,j} + e_{j,i} & 1 \leqslant i < j \leqslant k, \\
e_{k+i,k+j} + e_{k+l+j,k+l+i} & 1 \leqslant i, j \leqslant l, \\
e_{k+i,k+l+j} - e_{k+j,k+l+i} & 1 \leqslant i < j \leqslant l, \\
e_{k+l+i,k+j} - e_{k+l+j,k+i} & 1 \leqslant i < j \leqslant l
\end{array} \right\}
$$

$$
\mathcal{B}_0^- = \left\{ \begin{array}{cc}
e_{i,j} - e_{j,i} & 1 \leqslant i < j \leqslant k, \\
e_{k+i,k+j} - e_{k+l+j,k+l+i} & 1 \leqslant i, j \leqslant l, \\
e_{k+i,k+l+j} + e_{k+j,k+l+i} & 1 \leqslant i < j \leqslant l, \\
e_{k+l+i,k+j} + e_{k+l+j,k+i} & 1 \leqslant i < j \leqslant l, \\
e_{k+i,k+l+i} & 1 \leqslant i \leqslant l, \\
e_{k+l+i,k+i} & 1 \leqslant i \leqslant l
\end{array} \right\}
$$

$$
\mathcal{B}_1^+ = \left\{ \begin{array}{cc}
e_{i,k+j} - e_{k+l+j,i} & 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l, \\
e_{i,k+l+j} + e_{k+j,i} & 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l
\end{array} \right\}
$$

$$
\mathcal{B}_1^- = \left\{ \begin{array}{cc}
e_{i,k+j} + e_{k+l+j,i} & 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l, \\
e_{i,k+l+j} - e_{k+j,i} & 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l
\end{array} \right\}.
$$

With a similar construction to that of Lemma 2.3.2, it is not difficult to show that there exists a product of the above $*$-basis elements with value e_{11}. In this way, we get an analog of Lemma 2.3.3.

Lemma 2.3.4. *Let $(M_{k,2l}, osp)$ be the $*$-algebra of $(k+2l)\times(k+2l)$ matrices endowed with the orthosymplectic superinvolution. Then there exists an evaluation φ of the $*$-polynomial*

$$
P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y_{(k+2l)^2}^+ \pm y_{(k+2l)^2}^-}{2} w_{(k+2l)^2} \frac{y_{(k+2l)^2+1}^+ \pm y_{(k+2l)^2+1}^-}{2}
$$

in a $*$-basis of (M_{k2l}, osp) such that $\varphi(P) = e_{11}$.

Now let us focus our attention on the $*$-algebra $M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$ endowed with the exchange superinvolution. A $*$-basis of such an algebra is the following:

$$B = \{(e_{ij}, e_{ij}), (e_{ij}, -e_{ij})\}_{i,j=1,\dots,k+h}.$$

We construct the following two matrices: A^+ is the matrix having in the entry (i,j) the element (e_{ij}, e_{ij}) whereas A^- is the matrix having in the entry (i,j) the element $(e_{ij}, -e_{ij})$. Now let H_j^+ (H_j^-, resp.) be the product of the elements in the j-th hook of the matrix A^+ (A^-, resp.). By taking into account the multiplication rule in equation (1.3), we get

$$Q = (e_{11}, e_{11})H_2^+ \cdots H_{k+h}^+ = (e_{11}, 0) \qquad \text{and} \qquad Q' = (e_{11}, -e_{11})H_2^- \cdots H_{k+h}^- = (e_{11}, 0).$$

If we consider the same product Q' but in the opposite direction, i.e. we start with the last element and we finish with the first one of Q' (we denote such a new product by Q^*), we get

$$Q^* = \begin{cases} (0, e_{11}) & \text{if } k+h+kh \text{ is even,} \\ (0, -e_{11}) & \text{if } k+h+kh \text{ is odd.} \end{cases}$$

Lemma 2.3.5. *Let us consider the monomials $M = w_1 \cdots w_{(k+h)^2}$ and $M^* = u_1 \cdots u_{(k+h)^2}$, where each variable w_i and u_i has a certain homogeneous degree and it is symmetric or skew according to the corresponding element in the products Q and Q^* above, respectively. Then we can consider the monomials obtained by M and M^*, respectively, by surrounding each variable with a symmetric even variable:*

$$M' = y_1^+ w_1 y_2^+ w_2 \cdots y_{(k+h)^2}^+ w_{(k+h)^2} y_{(k+h)^2+1}^+$$

$$(M^*)' = \mathfrak{y}_1^+ u_1 \mathfrak{y}_2^+ u_2 \cdots \mathfrak{y}_{(k+h)^2}^+ u_{(k+h)^2} \mathfrak{y}_{(k+h)^2+1}^+.$$

Consider the evaluation φ of the $*$-polynomial $f = M' \pm (M^*)'$ ($+$ if $k+h+kh$ is even, $-$ otherwise) :

1. *each variable w_i (u_i, resp.) is evaluated in the corresponding element of Q (Q^*, resp.),*
2. *each variable y_j^+, $\mathfrak{y}_j^+$ is evaluated in the suitable idempotent element $(e_{l_j l_j}, e_{l_j l_j}) \in M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$.*

We get that

$$\varphi(f) = (e_{11}, e_{11}).$$

Finally, let us focus our attention to the $*$-algebra $Q(n) \oplus Q(n)^{sop}$ endowed with the exchange superinvolution. A $*$-basis of such an algebra is the following:

$$B = \{(e_{ij}, e_{ij}), (e_{ij}, -e_{ij}), (ce_{ij}, ce_{ij}), (ce_{ij}, -ce_{ij})\}_{i,j=1,\dots,n}.$$

We construct the following four matrices: A_0^+ is the matrix having in the entry (i,j) the element (e_{ij}, e_{ij}), A_0^- is the matrix having in the entry (i,j) the element $(e_{ij}, -e_{ij})$, A_c^+ is the matrix having in the entry (i,j) the element (ce_{ij}, ce_{ij}), A_c^- is the matrix having in the entry (i,j) the element $(ce_{ij}, -ce_{ij})$. Now, let $H_j^{0,+}$, $H_j^{0,-}$, $H_j^{c,+}$, $H_j^{c,-}$ be the product of the elements in the j-th hook of the matrices A_0^+, A_0^-, A_c^+ and A_c^-, respectively. Consider the following products:

$$Q = (e_{11}, e_{11})H_2^{0,+} \cdots H_n^{0,+}(ce_{11}, ce_{11})H_2^{c,+} \cdots H_n^{c,+},$$

$$Q' = (e_{11}, -e_{11})H_2^{0,-} \cdots H_n^{0,-}(ce_{11}, -ce_{11})H_2^{c,-} \cdots H_n^{c,-}.$$

Hence we have

$$Q = Q' = \begin{cases} (e_{11}, 0) & \text{if } n \text{ is even,} \\ (ce_{11}, 0) & \text{if } n \text{ is odd.} \end{cases}$$

If we consider the same product Q' but in the opposite direction, i.e., we start with the last element and we finish with the first one of Q' (we denote such a new product by Q^*), we get

$$Q^* = \begin{cases} (0, e_{11}) & \text{if } n \text{ is even,} \\ (0, -ce_{11}) & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 2.3.6. *Let us consider the monomials $M = w_1 \cdots w_{2n^2}$ and $M^* = u_1 \cdots u_{2n^2}$, where each variable w_i and u_i has a certain homogeneous degree and it is symmetric or skew according to the corresponding element in the products Q and Q^* above, respectively. Then we can consider the monomials obtained by M and M^*, respectively, by surrounding each variable with a symmetric even variable:*

$$M' = y_1^+ w_1 y_2^+ w_2 \cdots y_{2n^2}^+ w_{2n^2} y_{2n^2+1}^+, \qquad \text{and} \qquad (M^*)' = \mathfrak{y}_1^+ u_1 \mathfrak{y}_2^+ u_2 \cdots \mathfrak{y}_{2n^2}^+ u_{2n^2} \mathfrak{y}_{2n^2+1}^+.$$

Finally we construct the following $*$-polynomial f:

$$f = \begin{cases} M' + (M^*)' & \text{if } n \text{ is even,} \\ M'z^+ - (M^*)'z^+ & \text{if } n \text{ is odd.} \end{cases}$$

We consider the following evaluation φ:

1. *each variable w_i (u_i, resp.) is evaluated in the corresponding element of Q (Q^*, resp.),*

2. *each variable y_j^+ is evaluated in the suitable idempotent element $(e_{l_j l_j}, e_{l_j l_j}) \in Q(n) \oplus Q(n)^{sop}$,*

3. *the variable z^+ is evaluated in the element (ce_{11}, ce_{11}).*

We get that

$$\varphi(f) = (e_{11}, e_{11}).$$

Remark 2.3.7. *In all the results above we have considered monomial or polynomial with value e_{11} or (e_{11}, e_{11}). Of course it is possible to obtain the same result for any e_{ii} or (e_{ii}, e_{ii}).*

The following result is the $*$-algebra version of Kemer's First Lemma.

Lemma 2.3.8. *Let $A = B + J$ be a finite dimensional $*$-algebra, subdirectly irreducible and full with respect to a polynomial f. Then for any integer ν there exists a non-identity f' of A in the T_2^*-ideal generated by f with ν-folds (d_0, d_1)-alternating, where $d_i = \dim B_i$ for $i \in \{0, 1\}$.*

Proof. Consider the Wedderburn-Malcev decomposition $A = B + J = A_1 \times \cdots \times A_q + J$, where the A_i's are $*$-simple algebras (Theorem 1.2.31). Since A is full, there is a multilinear $*$-polynomial $f(x_1^{\pm}, \ldots, x_q^{\pm}, w_1, \ldots, w_p)$ (where $x \in \{y, z\}$ and $w_1, \ldots, w_p$ are variables disjoint from $\{x_1^{\pm}, \ldots, x_q^{\pm}\}$) which does not vanish under an elementary evaluation of the form $x_j^{\pm} = v_j^{i_j,\pm} \in A_j$, $j = 1, \ldots, q$, $i_j \in \{0, 1\}$, and the variables w_j's get elementary values in A.

Now we consider the polynomial obtained from f by multiplying on the left each one of the variables $\{x_1^{\pm}, \ldots, x_q^{\pm}\}$ by symmetric variables of even degree $y_1^+, \ldots, y_q^+$ respectively. Clearly such a polynomial is a non-identity since the variables y_j^+'s may be evaluated on the identity elements 1_{A_j} of A_j. By Remark 1.2.34, we may write the identity element of A_j as $1_{A_j} = e_{1,1} + \cdots + e_{n_j,n_j}$ or $1_{A_j} = (e_{1,1}, e_{1,1}) + \cdots + (e_{n_j,n_j}, e_{n_j,n_j})$. Applying linearity there exists a non-zero evaluation where the variables $y_1^+, \ldots, y_q^+$ take values of the form e_{i_j,i_j} or $(e_{i_j,i_j}, e_{i_j,i_j})$, with $1 \leqslant i_j \leqslant n_j$, $j = 1, \ldots, q$.

Now we replace each variable $y_1^+, \ldots, y_q^+$ by $*$-polynomials $Y_1 \ldots, Y_q$ such that:

- Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating, $j = 1, \ldots, q$,
- Y_j takes the value e_{i_j,i_j} or $(e_{i_j,i_j}, e_{i_j,i_j})$, $j = 1, \ldots, q$.

In the construction of the $*$-polynomials Y_j we have to consider 4 distinct cases.

Case 1.1: $A_j \cong M_{k,k}(F)$ with the transpose superinvolution *trp*.

Fix $1 \leqslant i_j \leqslant k + h$ and consider the $*$-polynomial P constructed in Lemma 2.3.3:

$$P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y_{4k^2}^+ \pm y_{4k^2}^-}{2} w_{4k^2} \frac{y_{4k^2+1}^+ \pm y_{4k^2+1}^-}{2}.$$

We refer to the variables w_i's as **designated variables** . Next we consider the product of ν $*$-polynomials P (with distinct variables). We denote the long $*$-polynomial obtained in this way by P_ν. Finally, we construct the $*$-polynomial Y_j by alternating separately the variables of even/odd degree in each set of designated variables w_i of P_ν. Clearly the $*$-polynomial Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating. We only need to show that Y_ν takes the value e_{i_j,i_j}, so that it will be a non-identity of A_j.

By Lemma 2.3.3 and Remark 2.3.7 there exists a suitable evaluation φ of P such that $\varphi(P) = e_{i_j,i_j}$. We consider the following evaluation for Y_j: for each polynomial P (with distinct variables) we consider the corresponding evaluation φ giving out the value e_{i_j,i_j}.

Notice that the monomials of Y_j assuming a non-zero value under this evaluation are those corresponding to permutations that only transpose the variables corresponding to elements of type $e_{i_1,j_1} + e_{i_2,j_2}$ and $e_{i_1,j_1} - e_{i_2,j_2}$. Moreover, it is not difficult to see that each of these monomials takes the value e_{i_j,i_j} (considering it with the sign). In conclusion, the evaluation of Y_j is a scalar multiple of e_{i_j,i_j} and since char$F = 0$ we are done.

Case 1.2: $A_j \cong M_{k,h}(F)$ with the orthosymplectic superinvolution.

This case can be treated as the previous one. We just need to consider the $*$-polynomial

$$P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y^+_{(k+2l)^2} \pm y^-_{(k+2l)^2}}{2} w_{(k+2l)^2} \frac{y^+_{(k+2l)^2+1} \pm y^-_{(k+2l)^2+1}}{2}$$

constructed in Lemma 2.3.4 and then define the $*$-polynomial Y_j as before. Such a polynomial is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating and assume the value e_{i_j,i_j} as desired.

Case 2: $A_j \cong M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$.

Fix $1 \leqslant i_j \leqslant k+h$ and consider the $*$-polynomial f (remark it is not multilinear) constructed in Lemma 2.3.5:

$$\begin{aligned} f &= M' \pm M'^* \\ &= y_1^+ w_1 y_2^+ w_2 \cdots y^+_{(k+h)^2} w_{(k+h)^2} y^+_{(k+h)^2+1} \pm \mathfrak{y}_1^+ u_1 \mathfrak{y}_2^+ u_2 \cdots \mathfrak{y}^+_{(k+h)^2} u_{(k+h)^2} \mathfrak{y}^+_{(k+h)^2+1}. \end{aligned}$$

We consider the product of ν $*$-polynomials f (with distinct variables) and we denote the long $*$-polynomial obtained in this way by P_ν. Finally, we construct the $*$-polynomial Y_j by alternating separately the variables of even/odd degree in each set of designated variables w_i of P_ν. Clearly the $*$-polynomial Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating. We need to show that Y_ν is a non-identity of A_j.

By Lemma 2.3.5 and Remark 2.3.7 there exists a suitable evaluation φ of f such that $\varphi(f) = (e_{i_j,i_j}, e_{i_j,i_j})$. Notice that the permutation that only transposes the variables corresponding to elements of the type $(e_{i_1,j_1}, e_{i_2,j_2})$ and $(e_{i_1,j_1}, -e_{i_2,j_2})$ does not vanish in the evaluation φ: in fact, the evaluations in this kind of permutations are equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. Transpositions of other types vanish (in the above evaluation) because the bordering elements are different. Therefore, the evaluation of a permutation obtained from an even number of transpositions is equal to $(e_{i_j,i_j}, e_{i_j,i_j})$ and the evaluation of a permutation obtained from an odd number of transpositions is equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. In conclusion the evaluation φ of Y_j is a scalar multiple of $(e_{i_j,i_j}, e_{i_j,i_j}) - (e_{i_j,i_j}, -e_{i_j,i_j})$.

Case 3: $A_j \cong Q(n) \oplus Q(n)^{sop}$.

Fix $1 \leqslant i_j \leqslant n$ and consider the $*$-polynomial f defined in Lemma 2.3.6. Notice that the polynomial f is not multilinear. We consider the product of ν $*$-polynomials f (with distinct variables) and we denote the long $*$-polynomial obtained in this way by P_ν. Then we construct the $*$-polynomial Y_j by alternating separately the variables of even/odd degree in each set of designated variables w_i of P_ν. Clearly the $*$-polynomial Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating. We need to show Y_ν is a non-identity of A_j.

By Lemma 2.3.6, there exists a suitable evaluation φ of f such that $\varphi(f) = (e_{i_j,i_j}, e_{i_j,i_j})$. Notice that the permutation that only transposes the variables corresponding to elements of the type $(e_{i_1,j_1}, e_{i_2,j_2})$ and $(e_{i_1,j_1}, -e_{i_2,j_2})$ or of the type $(ce_{i_1,j_1}, ce_{i_2,j_2})$ and $(ce_{i_1,j_1}, -ce_{i_2,j_2})$ does not vanish in the evaluation φ: in fact, the evaluations in this kind of permutations are equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. Moreover, transpositions of other types vanish (in the above evaluation) because the bordering elements are different. Therefore the evaluation of a permutation obtained from an even number of transpositions is equal to $(e_{i_j,i_j}, e_{i_j,i_j})$ and the evaluation of a permutation obtained from an odd number of transpositions is equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. In this way the evaluation φ of Y_j is a scalar multiple of $(e_{i_j,i_j}, e_{i_j,i_j}) - (e_{i_j,i_j}, -e_{i_j,i_j})$.

In order to complete the proof we construct a $*$-polynomial f' by alternating the (symmetric/skew of a certain homogeneous degree) sets which come from different Y_j's. Clearly $f' \notin Id^*(A)$ and f' has ν-folds (d_0, d_1)-alternating as desired and the proof follows. □

Proposition 2.3.9. *Let A be a finite dimensional $*$-algebra, full and subdirectly irreducible. Then there is an extremal point α in $E_0(A)$ with $\alpha = (d(A)_0), d(A)_1)$. In particular, this extremal point is unique.*

Proof. The existence follows immediately by Lemma 2.3.8. The uniqueness is a consequence of Proposition 2.2.1. □

The last goal of this section is to give the analog of Kemer's Lemma 2 in the setting of $*$-algebras. In order to reach this goal we need some definitions and preliminary results. Recall that, if A is a $*$-algebra, then by using the Wedderburn-Malcev decomposition, we can write $A = B + J$, where B is the semisimple part and J is the Jacobson radical of A, which is a nilpotent $*$-ideal ($n(A)$ is its nilpotency index).

Lemma 2.3.10. *If (α, s) is a Kemer point of a finite dimensional $*$-algebra A, then $s \leqslant n(A) - 1$.*

Proof. By the definition of the parameter s we know that for arbitrary large ν there exist multilinear $*$-polynomials, not in $\mathrm{Id}^*(A)$, being ν-folds alternating on homogeneous (small) sets of cardinality $d(A)_i$ and s (big) sets of cardinality $d(A)_i + 1$, for each $i \in \{0, 1\}$. It follows that an alternating homogeneous set of cardinality $d(A)_i + 1$ in a non-identity polynomial must have at least one radical evaluation. Consequently we cannot have more than $n(A) - 1$ of such alternating sets and we are done. □

The next construction (see [36, Remark 6.10.1]) will enable us to take some "control" on the nilpotency index of the radical of a finite dimensional $*$-algebra.

Let $B = \bar{B} + J$ be any finite-dimensional $*$-algebra and let $B' = \bar{B} \cdot F\langle X, *\rangle$ be the $*$-algebra of $*$-polynomials in the variables $X = \{x_{i_1}^{\dagger_1}, \ldots, x_{i_m}^{\dagger_m}\}$ with coefficients in $\bar{B}$, the semisimple component of B, where $i_j \in \{0, 1\}$ and $\dagger_j \in \{+, -\}$, for $j = 1, \ldots, m$. The number of homogeneous symmetric (skew) variables that we take is at least the dimension of the homogeneous symmetric (skew) component of $J(B)$. The superinvolution in B' is induced by

$$(b \cdot x)^* = (-1)^{|b||x|} x^* b^*,$$

where $b \in \bar{B}$ and x is a variable in X. Observe that any element of B' is represented by a sum of elements of the form $b_1 f_1 b_2 f_2 \cdots b_k f_k b_{k+1}$, where $b_1, \ldots, b_{k+1} \in \bar{B}$ and $f_1, \ldots, f_k \in F\langle X, *\rangle$.

Let I_1 be the $*$-ideal of B' generated by all the evaluations of the $*$-polynomials of $\mathrm{Id}^*(B)$ on B' and let I_2 be the $*$-ideal of B' generated by the variables $\{x_{i_j}^{\dagger_j}\}_{j=1}^m$. For any $u > 1$, define $\hat{B}_u = B'/(I_1 + I_2^u)$.

Proposition 2.3.11. *The following statements hold:*

1. $\mathrm{Id}^*(\hat{B}_u) = \mathrm{Id}^*(B)$, *whenever $u \geqslant n(B)$ (the nilpotency index of B). In particular $\hat{B}_u$ and B have the same Kemer points.*
2. *$\hat{B}_u$ is finite dimensional.*
3. *The nilpotency index of $\hat{B}_u$ is u.*

Proof. (1) By definition of $\hat{B}_u$, $\mathrm{Id}^*(\hat{B}_u) \supseteq \mathrm{Id}^*(B)$. On the other hand, by the fact that the number of symmetric (skew) homogeneous variables that we take is at least the dimension of the symmetric (skew) homogeneous component of $J(B)$, we can construct a surjective map $\phi \colon B' \to B$ such that the variables $\{x_{i_j}^{\dagger_j}\}$ are mapped onto a spanning set of $J(B)$ and $\bar{B}$ is mapped isomorphically. Indeed, $\phi(b \cdot 1_X) = b$, where 1_X represents the empty word in $F\langle X, *\rangle$ and $b \in \bar{B}$. This map is a homomorphism of superalgebras with superinvolution. The $*$-ideal I_1 consists of all evaluations of $\mathrm{Id}^*(B)$ on B' and hence is contained in $\ker(\phi)$. Also the $*$-ideal I_2^u is contained in $\mathrm{Ker}(\phi)$ since $u \geqslant n(B)$ and $\phi(x_{i_j}^{\dagger_j}) \in J(B)$. By the universal property, there exists a surjective homomorphism of superalgebras with superinvolution $\hat{B}_u \to B$. Hence, $\mathrm{Id}^*(\hat{B}_u) \subseteq \mathrm{Id}^*(B)$ and we are done.

(2) Notice that any element in $\hat{B}_u$ is represented by a sum of elements of the form $b_1 w_1 b_2 w_2 \cdots b_l w_l b_{l+1}$, where $l < u$, $b_k \in \bar{B}$ and $w_k \in \{x_{i_j}^{\dagger_j}\}$ for $k = 1, \ldots, l$. Then $\hat{B}_u$ is of course finite dimensional.

(3) Notice that I_2 generates a radical ideal in $\hat{B}_u$ and since $B'/I_2 \cong \bar{B}$ we have that

$$\hat{B}_u/I_2 \cong B'/(I_1 + I_2^u + I_2) = B'/(I_1 + I_2) \cong (B'/I_2)/I_1 \cong \bar{B}/I_1 = \bar{B}.$$

We see that I_2 generates the radical of $\hat{B}_u$ and therefore its nilpotency index is bounded by u. □

Definition 2.3.12. *Let f be a multilinear $*$-polynomial which is not in* $\mathrm{Id}^*(A)$. *We say that f has the **property** K if f vanishes on every evaluation with less than $n(A) - 1$ radical substitutions. We say that a finite-dimensional $*$-algebra A has the **property** K if it satisfies the property with respect to some multilinear $*$-polynomial which is a non-identity of A.*

Proposition 2.3.13. *Let A be a finite dimensional $*$-algebra which is minimal (in the sense of Definition 2.2.10). Then A has the property K.*

Proof. Assume A has not the property K. This means that any multilinear $*$-polynomial which vanishes on less than $n(A) - 1$ radical evaluations is in $\mathrm{Id}^*(A)$. Consider the algebra $\hat{A}_u$ (from the proposition above). We claim that, for $u = n(A) - 1$, $\hat{A}_u$ is T_2^*-equivalent to A. Once this is accomplished, we would have that the nilpotency index of $\hat{A}_u$ is $n(A) - 1$, a contradiction to the minimality of A.

By construction we have $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(\hat{A}_u)$. For the converse take a $*$-polynomial f which is not in $\mathrm{Id}^*(A)$. Then by assumption, there is a non-zero evaluation $\tilde{f}$ of f on A with less than $n(A) - 1$ radical substitutions (say k). Following this evaluation we refer to the variables of f that get semisimple (radical) values as semisimple (radical) variables, respectively. Let $X = \{x_{i_1}, \ldots, x_{i_m}\}$ be a set of variables. Consider the evaluation $\hat{f}$ of f on $A' = \bar{A} \cdot F\langle X, *\rangle$, where semisimple variables are evaluated as in $\tilde{f}$ whereas the radical variables are evaluated on $\{x_{i_j}\}$, respecting the surjective map $\phi\colon A' \to A$. Our aim is to show that $\tilde{f} \notin I_1 + I_2^u$ because, in this case, we would have $f \notin \mathrm{Id}^*(\hat{A}_u)$ and this will complete the proof.

To show $\tilde{f} \notin I_1 + I_2^u$, notice that f is not in I_1 by definition. Moreover, an element of A' is in I_1 if and only if each one of its multihomogeneous components in the variables $\{x_{i_j}\}$ is in I_1. But by construction $\tilde{f}$ is multihomogeneous of degree $k < n(A) - 1$

in the variables $\{x_{i_j}\}$ whereas any element of $I_2^u \subseteq A'$ is the sum of multihomogeneous elements of degree $\geqslant n(A) - 1$. We therefore have that $\tilde{f} \in I_1 + I_2^u$ if and only if $\tilde{f} \in I_1$ and we are done. $\square$

Let A be a basic $*$-algebra. By Proposition 2.3.13 we have A satisfies the property K with respect to a non-identity f. Moreover, we have A is full with respect to a non-identity h. Our goal now is showing A is full and has property K with respect to the same $*$-polynomial.

Now we give the definition of Phoenix property.

Definition 2.3.14. *Let Γ be a T_2^*-ideal. Let P be any property which may be satisfied by $*$-polynomials (e.g. being Kemer). We say that P is Γ-**Phoenix** (or in short **Phoenix**) if given a polynomial f having P which is not in Γ and any $f' \in \langle f \rangle_{T_2^*}$, the T_2^*-ideal generated by f, which is not in Γ as well, there exists a polynomial $f'' \in \langle f' \rangle_{T_2^*}$ which is not in Γ and satisfies P. We say that P is **strictly Phoenix** if f' itself satisfies P.*

The next lemma shows that property K and the property of being full are "preserved" in a T_2^*-ideal.

Lemma 2.3.15. *Let A be a finite dimensional $*$-algebra over F.*

1. *The property of a non-identity of A of being ν-folds alternating on homogeneous sets of cardinality $d(A)_i$, $i = 0, 1$, is Phoenix.*
2. *Property K is strictly Phoenix.*

Proof. (1) Let f be a non-identity which is ν-fold alternating on homogeneous sets of cardinality $d(A)_i$, $i \in \{0, 1\}$ (in particular A is full with respect to f). We want to show that if $f' \in \langle f \rangle$ is a non-identity in the T_2^*-ideal generated by f, then there exists a non-identity $f'' \in \langle f' \rangle$ which is ν-fold alternating on homogeneous sets of cardinality $d(A)_i$. In view of Lemma 2.3.8, it is sufficient to show that A is full with respect to f'. Remark that, for each $i \in \{0, 1\}$, in at least one alternating set S_i, the evaluations of the corresponding variables must consist of semisimple elements of A in any non-zero evaluation of the $*$-polynomial. This is clear if f' is in the ideal (rather than in the T_2^*-ideal) generated by f. Therefore, we assume that f' is obtained from f by substituting variables x_i's by monomials Z_i's. Clearly, if one of the evaluations in any of the variables of Z_i is radical, then the value of Z_i is radical. Hence in any non-zero evaluation of f' there is an alternating set Δ_i of cardinality $d(A)_i$ in f such that the variables in monomials of f' (corresponding to the variables in Δ_i) assume only semisimple values. Furthermore, each $*$-simple component must be represented in these evaluations: in fact, otherwise we would have a $*$-simple component not represented in the evaluations of the Δ_i's and this is impossible. In conclusion we get that A is full with respect to f'.

(2) If $f' \in \langle f \rangle$ is a non-identity and has less than $n(A) - 1$ radical evaluations, then the same is true for f and hence f' vanishes. □

Finally, the following lemma can be proved following word by word the proof of [2, Proposition 6.6].

Lemma 2.3.16. *Let A be a finite-dimensional $*$-algebra, which is full, subdirectly irreducible and satisfying the property K. Let f be a non-identity which is ν-folds alternating on homogeneous sets of cardinality $d(A)_i$, $i \in \{0, 1\}$ and let h be a $*$-polynomial with respect to which A has the property K. Then there is a non-identity in $\langle f \rangle \cap \langle h \rangle$. Consequently there exists a non-identity $\hat{f}$ which is ν-folds alternating on homogeneous sets of cardinality $d(A)_i$, $i = 0, 1$, and with respect to which A has the property K.*

We are in a position to prove the $*$-algebra version of Kemer's Lemma 2.

Lemma 2.3.17. *Let $A = B + J$ be a finite dimensional basic $*$-algebra. Then for any integer ν there exists a multilinear non-identity f which is ν-folds alternating on homogeneous sets of cardinality $d(B_i) = \dim_F(B_i)$, $i = 0, 1$, and $n(A) - 1$ sets of homogeneous variables of cardinality $d(B_i) + 1$, $i = 0, 1$.*

Proof. By Lemma 2.3.16, there exists a multilinear non-identity f with respect to which A is full and has property K. Let us fix a non-zero evaluation $x_i \mapsto \hat{x}_i$ realizing the "full" property. Notice that by Lemma 2.3.10, f cannot have more than $n(A) - 1$ radical evaluations, and by property K, f cannot have less than $n(A) - 1$ radical evaluation. Thus, f has precisely $n(A) - 1$ radical substitutions whereas the remaining variables only take semisimple values. Let us denote by $w_1, \ldots, w_{n(A)-1}$ the variables taking radical values (in the evaluation above) and by $\hat{w}_1, \ldots, \hat{w}_{n(A)-1}$ their corresponding values.

Suppose further $B \cong A_1 \times \cdots \times A_q$ (A_i are $*$-simple algebras). We will consider four distinct cases corresponding to whether $q = 1$ or $q > 1$ and whether A has or does not have an identity element.

Case 1: A has an identity element and $q > 1$.

Choose a monomial M in f which does not vanish upon the evaluation above. By multilinearity of f, the monomial M is full (i.e. visits every $*$-simple component of A). Notice that the variables of M which get semisimple evaluations from different $*$-simple components must be separated by radical variables. Next, we may assume that the evaluation of any radical variable w_i is of the form $1_{A_{j(i)}} \hat{w}_i 1_{A_{\tilde{j}(i)}}$, $i = 1, \ldots, n(A) - 1$, where 1_{A_j} is the identity element of the $*$-simple component A_j. Notice that the evaluation remains full.

Consider the radical evaluations which are bordered by pairs of elements $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$, where $j(i) \neq \tilde{j}(i)$ (i.e. they belong to different $*$-simple components). Then, since M is full, every $*$-simple component is represented by one of the elements in those pairs.

For $t = 1, \ldots, q$, we fix a variable w_{r_t} whose radical value is $1_{A_{j(r_t)}} \hat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$, where

1. $j(r_t) \neq \tilde{j}(r_t)$ (i.e. different $*$-simple components),
2. one of the elements $1_{A_{j(r_t)}}$, $1_{A_{\tilde{j}(r_t)}}$ is the identity element of A_t.

We replace now the variables w_{r_t}, $t = 1, \ldots, q$, by the product $y_{r_t} w_{r_t}$ or $w_{r_t} \tilde{y}_{r_t}$ (according to the position of the bordering), where the variables y_{r_t}'s and $\tilde{y}_{r_t}$'s are symmetric variables of even degree. Clearly, by evaluating the variable y_{r_t} by $1_{A_{j(r_t)}}$ (or the variable $\tilde{y}_{r_t}$ by $1_{A_{\tilde{j}(r_t)}}$) the value of the $*$-polynomial remains the same and we obtain a non-identity.

Remember that by Remark 1.2.34, we may write the identity element of A_j as $1_{A_j} = e^j_{1,1} + \cdots + e^j_{n_j,n_j}$ or $1_{A_j} = (e_{1,1}, e_{1,1})^j + \cdots + (e_{n_j,n_j}, e_{n_j,n_j})^j$. Thus applying linearity, each $\hat{w}_i$ may be bordered by elements of the form $e^{j(i)}_{k_{j(i)},k_{j(i)}}$ or $(e_{k_{j(i)},k_{j(i)}}, e_{k_{j(i)},k_{j(i)}})^{j(i)}$ with $1 \leqslant k_{j(i)} \leqslant n_{j(i)}$. As in the proof of Lemma 2.3.8 we can insert in the y_{r_t}'s suitable $*$-polynomials and obtain a $*$-polynomial which is ν-folds alternating on homogeneous sets of cardinality $\dim_F(B_i)$, $i \in \{0, 1\}$.

Consider the variables with radical evaluations which are bordered by variables with evaluations from different $*$-simple components (these include the variables which are bordered by the y_{r_t}). Let χ_i be such a variable of a certain homogeneous degree (according to $i \in \{0, 1\}$). We attach it to a (small) alternating homogeneous set S_i (according with i). We claim that if we alternate this set (of cardinality $d(A)_i + 1$) we obtain a non-identity. Indeed, any non-trivial permutation of χ_i with one of the variables of S_i, keeping the evaluation above, will yield a zero value since the idempotents values in the framed variables of each variable of S_i belong to the same $*$-simple component whereas the pair of idempotents $1_{A_{j(\chi)}} \hat{\chi}_i 1_{A_{\tilde{j}(\chi)}}$ belong to different $*$-simple components. At this point we have constructed the desired number of small sets and some of the big sets.

Now, we need to attach the radical variables w_i whose evaluation is $1_{A_{j(i)}} \hat{w}_i 1_{A_{\tilde{j}(i)}}$ where $j(i) = \tilde{j}(i)$ (i.e. the same $*$-simple component) to some small set S_i. We claim that if we alternate this set (of cardinality $d(A_i) + 1$) we obtain a non-identity. Indeed, any non-trivial permutation represents an evaluation with fewer radical evaluations in the original polynomial which must vanish by property K. This completes the proof in this case.

Case 2: A has an identity element and $q = 1$.

We start with a non-identity f which satisfies property K. Clearly we may multiply f by a symmetric homogeneous variable x_0^+ of even degree and get a non-identity (since x_0^+ may be evaluated by 1). Again by Lemma 2.3.8 we may replace x_0^+ by a polynomial h which is ν-folds alternating on homogeneous sets of cardinality $d(A_i)$. Consider the polynomial hf. We attach the radical variables of f to some of the small sets in h. Any non-trivial permutation vanishes because f satisfies property K. This completes the proof in this case.

Case 3: A has no identity element and $q > 1$.

In the notation of Remark 1.1.17, let $e_0 = \hat{1} - 1_{A_1} - 1_{A_2} - \cdots - 1_{A_q}$ and include e_0 to the set of elements which border the radical values $\hat{w}_j$. A similar argument shows that also here every $*$-simple component $(A_1, \ldots, A_q)$ is represented in one of the bordering pairs $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$ where the pairs are different (the point is that one of these pairs may be e_0). Now we complete the proof exactly as in Case 1.

Case 4: A has no identity element and $q = 1$.

For simplicity we write $e_1 = 1_{A_1}$ and $e_0 = \hat{1} - e_1$. Let $f(x_{i_1}, \ldots, x_{i_n})$ be a non-identity of A satisfying property K and let $f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})$ be a non-zero evaluation for which A is full. If $e_1 f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n}) \neq 0$ or $f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n}) e_1 \neq 0$, we proceed as in Case 2. To treat the remaining case we may assume that

$$e_0 f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n}) e_0 \neq 0.$$

By linearity, each one of the radical values $\hat{w}$ may be bordered by one of the pairs $\{(e_0, e_0), (e_0, e_1), (e_1, e_0), (e_1, e_1)\}$. Hence, if we replace the evaluation $\hat{w}$ of w by the corresponding element $e_i \hat{w} e_j$, $i, j = 0, 1$, we get a non-zero value.

Now, if one of the radical values (say $\hat{w}_0$) in $f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})$ allows a surrounding by the pair (e_0, e_1) (and remains non-zero), then replacing w_0 by $w_0 y$ yields a non-identity (since we may evaluate y by e_1). Invoking Lemma 2.3.8, we may replace the variable y by a polynomial h with ν-folds alternating (small) homogeneous sets of variables of cardinality $\dim_F(B)_i = \dim_F(A_1)_i$ for every $i \in \{0, 1\}$. Then we attach the radical variable w_0 to a suitable small set. Clearly, the value of any non-trivial permutation of w_0 with any element of the small set is zero since the borderings are different. Similarly, attaching radical variables w whose radical value is $e_i \hat{w} e_j$ where $i \neq j$, to small sets yields zero for any non-trivial permutation and hence the value of the polynomial remains non-zero. The remaining possible values of radical variables are either $e_0 \hat{w} e_0$ or $e_1 \hat{w} e_1$. Notice that since semisimple values can be bordered only by the pair (e_1, e_1), any alternation of the radical variables whose radical value is $e_0 \hat{w} e_0$ with elements of a small set vanishes and again the value of the polynomial remains unchanged. Finally (in order to complete this case, namely where the radical variable w_0 is bordered by the pair (e_0, e_1)) we attach the remaining radical variables (whose values are bordered by (e_1, e_1)) to suitable small sets in h. Here, the value of any non-trivial permutation of w_0 with elements of the small set is zero because of property K (as in Case 1). This settles this case. Obviously, the same holds if the bordering pair of $\hat{w}_0$ above is (e_1, e_0).

The outcome is that we may assume that all radical values may be bordered by either (e_0, e_0) or (e_1, e_1). Under this assumption, notice that all pairs that border radical values are equal, that is are all (e_0, e_0) or all (e_1, e_1). Indeed, if we have of both kinds, we must have a radical value which is bordered by a mixed pair since the semisimple variables can be bordered only by the pair (e_1, e_1) (and in particular they cannot be bordered by mixed pairs). This of course contradicts our assumption.

A similar argument shows that we cannot have radical variables w with values $e_0\hat{w}e_0$ since semisimple values can be bordered only by (e_1, e_1) and this will force the existence of a radical value bordered by mixed idempotents.

The remaining case is the case where all values (radical and semisimple) are bordered by the pair (e_1, e_1) and this contradicts the assumption $e_0 f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})e_0 \neq 0$. This completes the proof of the lemma. □

Remark 2.3.18. *Any non-zero evaluation of such f must consist only of semisimple evaluations in the ν-folds and each one of the big sets must have exactly one radical evaluation.*

Corollary 2.3.19. *If A is a finite dimensional basic $*$-algebra, then its Kemer set consists of precisely one point $(\alpha, s) = \mathrm{Par}^*(A)$.*

2.4 Specht's problem for finitely generated $*$-algebras

Let W be a finitely generated $*$-algebra over F satisfying an ordinary non-trivial identity. The goal of this section is to find a finite generating set for the T_2^*-ideal $\mathrm{Id}^*(W)$. By Theorem 1.2.35 (and by Remark 2.2.6), there exists a field extension $\bar{F}$ of F and a finite dimensional $*$-algebra A such that

$$\mathrm{Id}^*(W) = \mathrm{Id}^*(A).$$

Let $m = \dim_{\bar{F}} A$. Then clearly W satisfies the (ordinary) Capelli identity c_{m+1} on $2(m+1)$ variables, or equivalently, the finite set of $*$-identities $c_{m+1,i}$ which follow from c_{m+1} by setting its variables to be of homogeneous degree 0 or 1.

Now, observe that any T_2^*-ideal of $*$-identities is generated by at most a countable number of $*$-identities (indeed, for each n the space of multilinear $*$-identities of degree n is finite dimensional). Hence we may take a sequence of $*$-identities $f_1, \ldots, f_n, \ldots$, which generate $\mathrm{Id}^*(W)$.

Let Γ_1 be the T_2^*-ideal generated by the polynomials $c_{m+1,i} \cup \{f_1\}$, …, Γ_n be the T_2^*-ideal generated by the polynomials $c_{m+1,i} \cup \{f_1, \ldots, f_n\}$, and so on. Clearly, since the set $c_{m+1,i}$ is finite, in order to prove the finite generation of $\mathrm{Id}^*(W)$, it is sufficient to show that the ascending chain of graded T_2^*-ideals $\Gamma_1 \subseteq \cdots \subseteq \Gamma_n \subseteq \cdots$ stabilizes.

Now, for each n, the T_2^*-ideal Γ_n corresponds to a finitely generated $*$-algebra (see [24, Theorem 5.2]). Hence, invoking Theorem 1.2.35, we may replace each Γ_n by $\mathrm{Id}^*(A_n)$, where A_n is a finite dimensional $*$-algebra over a suitable field extension K_n of F. Clearly, extending the coefficients to a sufficiently large field K, we may assume that all algebras A_n are finite dimensional over an algebraically closed field K.

Our goal is to show that the sequence $\mathrm{Id}^*(A_1) \subseteq \cdots \subseteq \mathrm{Id}^*(A_n) \subseteq \cdots$ stabilizes in $F\langle Y \cup Z, *\rangle$ or equivalently in $K\langle Y \cup Z, *\rangle$.

Consider the Kemer sets of the algebras A_n, $n \geqslant 1$. Since the sequence of ideals is increasing, the corresponding Kemer sets are monotonically decreasing (recall that this means that for any Kemer point (α, s) of A_{i+1} there is a Kemer point (α', s') of A_i with $(\alpha, s) \preceq (\alpha', s')$). Furthermore, since these sets are finite, there is a subsequence $\{A_{i_j}\}$ whose Kemer points (denoted by E) coincide. Clearly it is sufficient to show that the subsequence $\{\mathrm{Id}^*(A_{i_j})\}$ stabilizes and so, in order to simplify notation, we replace our original sequence $\{\mathrm{Id}^*(A_i)\}$ by the subsequence.

Choose a Kemer point (α, s) in E. By Theorem 2.2.12, for any i, we may replace the algebra A_i by a direct product of basic algebras

$$A'_{i,1} \times \cdots \times A'_{i,u_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}},$$

where the $A'_{i,j}$'s correspond to the Kemer point (α, s) and the $\widehat{A_{i,l}}$'s have Kemer index $\neq (\alpha, s)$ (notice that their index may or may not be in E).

Let A be a basic $*$-algebra corresponding to the Kemer point (α, s). Let $A = B + J(A)$ be the Wedderburn-Malcev decomposition of A into the semisimple and radical components. As shown in Section 2.3, we have that $\alpha_i = \dim(B_i)$, for every $i = 0, 1$. Hence, in particular, the dimension of B is determined by α.

By considering the $*$-algebras presented in Theorem 1.2.33, the following result is obvious.

Proposition 2.4.1. *The number of isomorphism classes of semisimple $*$-algebras of a given dimension is finite.*

Immediately, we get the following corollary.

Corollary 2.4.2. *The number of structures on the semisimple components of all basic $*$-algebras which correspond to the Kemer point (α, s) is finite.*

It follows that by passing to a subsequence $\{i_s\}$ we may assume that all basic algebras that appear in the decompositions above and correspond to the Kemer point

(α, s) have $*$-isomorphic semisimple components (which we denote by C) and have the same nilpotency index s.

Let us now consider the $*$-algebras

$$\widehat{C}_i = \frac{C \cdot K\langle \bar{X} \rangle}{I_i + J},$$

where

- $\bar{X}$ is a set of $*$-variables of cardinality $4(s-1)$,
- $C \cdot K\langle \bar{X} \rangle$ is the algebra of $*$-polynomials in the variables of $\bar{X}$ and coefficients in C,
- I_i is the ideal generated by all evaluations of $\mathrm{Id}^*(A_i)$ on $C \cdot K\langle \bar{X} \rangle$,
- J is the ideal generated by all words in $C \cdot K\langle \bar{X} \rangle$ with s variables from $\bar{X}$.

Proposition 2.4.3. *The following facts hold:*

1. *The ideal generated by variables from $\bar{X}$ is nilpotent.*
2. *For any i, the algebra $\widehat{C}_i$ is finite dimensional.*
3. *For any i, $\mathrm{Id}^*(A_i) = \mathrm{Id}^*(\widehat{C}_i \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}})$.*

Proof. (1) By definition of J, the number of variables appearing in a non-zero monomial of the $*$-algebra $\widehat{C}_i$ is bounded by $s-1$, then such an ideal is nilpotent.

(2) Consider a typical non-zero monomial of the $*$-algebra $\widehat{C}_i$. It has the form

$$a_{t_1} x_{t_1} a_{t_2} x_{t_2} \cdots a_{t_r} x_{t_r} a_{t_{r+a}}.$$

Since the set of variables $\bar{X}$ is finite and the index r is bounded by $s-1$, we have that the number of different configurations of these monomials is finite. Between these variables we have the elements a_{t_j}, $j = 1, \ldots, r+1$, which are taken from the finite-dimensional $*$-algebra C. Therefore the $*$-algebra $\widehat{C}_i$ is finite-dimensional.

(3) Since $\mathrm{Id}^*(A'_{i,1} \times \cdots \times A'_{i,u_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) = \mathrm{Id}^*(A_i)$, then $\mathrm{Id}^*(\widehat{A_{i,j}}) \supseteq \mathrm{Id}^*(A_i)$ for $j = 1, \ldots, r_i$. Also, from the definition of I_i we have that $\mathrm{Id}^*(\widehat{C}_i) \supseteq \mathrm{Id}^*(A_i)$ and so $\mathrm{Id}^*(\widehat{C}_i \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) \supseteq \mathrm{Id}^*(A_i)$. On the other hand, first let us show that $\mathrm{Id}^*(\widehat{C}_i) \subseteq \mathrm{Id}^*(A'_{i,j})$ for every $j = 1, \ldots, u_i$. This implies that $\mathrm{Id}^*(\widehat{C}_i) \subseteq \mathrm{Id}^*(A'_{i,1} \times \cdots \times A'_{i,u_i})$ and therefore $\mathrm{Id}^*(\widehat{C}_i \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) \subseteq \mathrm{Id}^*(A'_{i,1} \times \cdots \times A'_{i,u_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) = \mathrm{Id}^*(A_i)$.

To see $\mathrm{Id}^*(\widehat{C}_i) \subseteq \mathrm{Id}^*(A'_{i,j})$ let us take $f = f(x_{i_1}, \ldots, x_{i_t})$ a multilinear $*$-polynomial which is a non-identity of $A'_{i,j}$ and show that f is in fact a non-identity of $\widehat{C}_i$ (the variables x_{i_j} are homogeneous of degree zero or one). Fix a non-vanishing evaluation of f in $A'_{i,j}$ where $x_{j_1} = d_1, \ldots, x_{j_k} = d_k$ ($k \leqslant s-1$) are the variables with the corresponding radical evaluations and $x_{q_1} = c_1, \dots, x_{q_l} = c_l$ are the other variables with their semisimple evaluations. Consider the following homomorphism of $*$-algebras

$$\phi\colon C \cdot K\langle \bar{X} \rangle \to A'_{i,j}$$

where C is mapped isomorphically and a subset of k variables $\{\bar{x}_1, \dots, \bar{x}_k\}$ of $\bar{X}$ (with appropriate $\mathbb{Z}_2$-grading) are mapped onto the set $\{d_1, \dots, d_k\}$. The other variables from $\bar{X}$ are mapped to zero.

Notice that $(I_i + J) \subseteq \mathrm{Ker}(\phi)$ and hence we obtain a homomorphism of $*$-algebras $\bar{\phi}\colon \widehat{C}_i \to A'_{i,j}$. By construction, the evaluation of the $*$-polynomial $f(x_{i_1}, \dots, x_{i_t})$ on $\widehat{C}_i$, where $x_{q_1} = c_1, \dots, x_{q_l} = c_l$ and $x_{j_1} = \bar{x}_1, \dots, x_{j_k} = \bar{x}_k$, is non-zero and the result follows. □

The following lemma shows how to replace (for a subsequence of indices i_k) the direct product of the basic algebras corresponding to the Kemer point (α, s), $A'_{i,1} \times \cdots A'_{i,u_i}$, by a certain $*$-algebra U such that, for all i:

$$\mathrm{Id}^*(A_i) = \mathrm{Id}^*(U \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}).$$

Lemma 2.4.4. *We may replace the algebra $A'_{i,1} \times \cdots A'_{i,u_i}$ by a certain $*$-algebra U as above.*

Proof. At light of the Proposition 2.4.3 we are in the following situation. We have a sequence of T_2^*-ideals

$$\mathrm{Id}^*(\widehat{C}_1 \times \widehat{A_{1,1}} \times \cdots \times \widehat{A_{1,r_1}}) \subseteq \cdots \subseteq \mathrm{Id}^*(\widehat{C}_i \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) \\ \subseteq \mathrm{Id}^*(\widehat{C}_{i+1} \times \widehat{A_{i+1,1}} \times \cdots \times \widehat{A_{i+1,r_{i+1}}}) \subseteq \cdots .$$

In order to complete the construction of the algebra U we will show that in fact that, by passing to a subsequence, all $\widehat{C}_i$ are $*$-isomorphic. Indeed, since $\mathrm{Id}^*(A_i) \subseteq \mathrm{Id}^*(A_{i+1})$, we have a surjective map φ from $\widehat{C}_i$ to $\widehat{C}_{i+1}$. Since the $*$-algebras $\widehat{C}_i$'s are finite dimensional the result follows. □

At light of Lemma 2.4.4, we can continue as follows. Replace the sequence of indices $\{i\}$ by the subsequence $\{i_k\}$. Clearly, it is sufficient to show that the subsequence of T_2^*-ideals $\{\mathrm{Id}^*(A_{i_k})\}$ stabilizes.

Let I be the T_2^*-ideal generated by Kemer polynomials of U which correspond to the Kemer point (α, s). Notice that the polynomials in I are identities of the basic algebras $\widehat{A_{i,l}}$'s. It follows that the Kemer sets of the T_2^*-ideals $\{(\mathrm{Id}^*(A_i) + I)\}$ do not contain the point (α, s) and so they are strictly smaller. By induction we obtain that the following sequence of T_2^*-ideals stabilizes:

$$(\mathrm{Id}^*(A_1) + I) \subseteq (\mathrm{Id}^*(A_2) + I) \subseteq \cdots .$$

For any i, we have that:

1. $\mathrm{Id}^*(A_i) = \mathrm{Id}^*(U \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}})$.
2. $I \subseteq \mathrm{Id}^*(\widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}})$.

It follows that, for any i, j,

$$I \cap \mathrm{Id}^*(A_i) = I \cap \mathrm{Id}^*(A_j).$$

Combining the last statements we get the Specht property for $*$-algebras:

Theorem 2.4.5. *Let W be a finitely generated $*$-algebra. Then $\mathrm{Id}^*(W)$ is finitely generated, as a T_2^*-ideal.*

2.5 Rationality of the Hilbert series of relatively free $*$-algebras

Let $F\langle Y \cup Z, *\rangle$ be the free $*$-algebra on the set of countable variables $y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots$. In what follows we shall denote by $\bar{F}\langle Y \cup Z, *\rangle$ the free $*$-algebra on the set of finite variables $Y = \{y_1^+, \ldots, y_p^+, y_1^-, \ldots, y_q^-\}$ and $Z = \{z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_s^-\}$. Consider a T_2^*-ideal I in $\bar{F}\langle Y \cup Z, *\rangle$ containing at least an ordinary non-trivial identity and let $\bar{F}\langle Y \cup Z, *\rangle/I$ be the corresponding relatively free $*$-algebra.

Remark 2.5.1. *Since $\mathrm{Id}^*(\bar{F}\langle Y \cup Z, *\rangle/I) = I$, the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/I$ is PI, i.e. it contains an ordinary non-trivial identity.*

Let Ω_n be the (finite) set of monomials of degree n in the variables of $Y \cup Z$ and let c_n be the dimension of the F-subspace of $\bar{F}\langle Y \cup Z, *\rangle/I$ spanned by the monomials of Ω_n.

Definition 2.5.2. *The **Hilbert series** of $\bar{F}\langle Y \cup Z, *\rangle/I$ is given by*

$$\mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/I, t) = \sum_n c_n t^n. \tag{2.4}$$

The purpose of this section is to prove that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/I$ is a rational function. We wish to point out that giving a positive solution to the problem of the rationality of the Hilbert series for the relatively free algebra of a given algebra A has important applications to other growth invariants of A (see for instance [3, 5, 8, 9, 20, 21, 42, 43, 55]).

For the reader's convenience, we start by recalling some well-known facts of classical PI-theory (see [36, Chapther 2] and [10]).

Definition 2.5.3. *Let W be an finitely generated PI-algebra over F and let $a_1, \dots a_s$ be a set of generators of W. For a fixed positive integer m, consider B to be the (finite) set of all words in $a_1, \dots a_s$ of length $\leqslant m$. We say that W has a* ***Shirshov base of length*** m ***and of height*** h *if W is spanned (over F) by elements of the form $b_1^{k_1} \cdots b_l^{k_l}$, where $b_i \in B$ and $l \leqslant h$.*

Moreover, we say that the set B is an **essential Shirshov base of** W (of length m and of height h) if there exists a finite set D such that the elements of the form $d_{i_1} b_{i_1}^{k_1} d_{i_2} \cdots d_{i_l} b_{i_l}^{k_l} d_{i_{l+1}}$ span W, where $d_{i_j} \in D$, $b_{i_j} \in B$ and $l \leqslant h$.

Theorem 2.5.4. *Let W be a finitely generated PI-algebra over F satisfying a multilinear identity of degree m. Then W has a Shirshov base of length m and of height h, where h depends only on m and on the number of generators of W.*

Let us recall the following definitions.

Definition 2.5.5. *Let A be a commutative ring and C a subring of A. An element $b \in A$ is* ***integral over*** C *if there is a monic polynomial $f(x) \in C[x]$ such that $f(b) = 0$.*

Definition 2.5.6. *Let C be a ring and A be a left C-module. We say that A is a* ***finite module over*** C *if there exists $a_1, \dots, a_s \in A$ such that for any $x \in A$, there exists $c_1, \dots, c_s \in C$ with $x = c_1 a_1 + \cdots + c_s a_s$.*

The following result was proved in [2, Theorem 7.9].

Theorem 2.5.7. *Let C be a commutative algebra over F and let $A = C\langle a_1, \dots a_s \rangle$ be an affine algebra over C (see Definition 1.1.8). If A has an essential Shirshov base (in particular, if A has a Shirshov base) whose elements are integral over C, then A is a finite module over C.*

The following proposition is a classical result.

Proposition 2.5.8. *Any finite module M over a commutative affine algebra A has a rational Hilbert series.*

Finally we recall the following result given in [36, Theorem I, page 42].

Theorem 2.5.9. *Let $A \subseteq M_n(F)$ be an algebra and let V be a d-dimensional subalgebra of $M_n(F)$ with an F-basis $a_1, \dots, a_d$ of elements of A. Given an F-linear transformation $T \colon V \to V$, let $\lambda^d + \sum_{i=1}^{d} (-1)^i \gamma_i \lambda^{d-i}$ be the characteristic polynomial of T. For any polynomial*

$f(x_1, \ldots, x_d, Y)$ which is alternating in the variables $x_1, \ldots, x_d$, and where Y is a set of variables disjoint from $\{x_1, \ldots, x_d\}$, the following equation holds:

$$\gamma_i f(a_1, \ldots, a_d, \hat{Y}) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i \in \{0,1\}}} f(T^{k_1}(a_1), \ldots, T^{k_d}(a_d), \hat{Y}) \tag{2.5}$$

where $\hat{Y}$ is any evaluation of the variables in Y.

Now we focus our attention to $*$-algebras.

Proposition 2.5.10. *Let $A = A_0 \oplus A_1$ be an affine $*$-algebra satisfying an ordinary non-trivial identity. Then A has an essential Shirshov base of elements of A_0.*

Proof. Since A is a $\mathbb{Z}_2$-graded affine algebra, the result follows from [2, Proposition 7.10]. □

Our next goal is to prove the following lemma.

Lemma 2.5.11. *Let S be a set of multilinear $*$-polynomials in $F\langle Y \cup Z, *\rangle$ and let I be the T_2^*-ideal generated by S. Given a $*$-algebra W, we consider $\mathcal{S}$, $\mathcal{I}$ to be the sets of all evaluations on W of the polynomials of S and I, respectively. Then $\mathcal{I} = \langle \mathcal{S} \rangle$ (the $*$-ideal generated by $\mathcal{S}$).*

Proof. In order to prove the lemma, we start by showing that $\mathcal{I}$ is a $*$-ideal of W. Let $a, b \in \mathcal{I}$ and consider the $*$-polynomials p_a and p_b in I with evaluations a and b, respectively. Since I is invariant under all the endomorphism of $F\langle Y \cup Z, *\rangle$ commuting with the superinvolution $*$, we may change variables and assume that p_a and p_b have disjoint sets of variables. Then we get $a + b$ as an evaluation of the $*$-polynomial $p_a + p_b$ and so it follows that $a + b \in \mathcal{I}$. Now let $c \in W$. We may take a variable x disjoint from the variables of p_a and so we get ca and ac as evaluations of xp_a and $p_a x$, respectively. Hence ca and ac belong to $\mathcal{I}$. So far we have proved that $\mathcal{I}$ is an ideal. In order to prove that $\mathcal{I}$ is a graded ideal, we have to show that $\mathcal{I} = (\mathcal{I} \cap W_0) \oplus (\mathcal{I} \cap W_1)$, where W_0 and W_1 are the homogeneous components of W. Now let $a = w_0 + w_1 \in \mathcal{I}$, $a_0 \in W_0$ and $a_1 \in W_1$. Hence there exists a $*$-polynomial $p_a \in I$ with evaluation a. Since I is a graded ideal, we have that $p_a = (p_a)_0 + (p_a)_1$, with $(p_a)_0 \in F\langle Y \cup Z, *\rangle_0$ and $(p_a)_1 \in F\langle Y \cup Z, *\rangle_1$ (the homogeneous components of $F\langle Y \cup Z, *\rangle$). Clearly $(p_a)_i$ takes value w_i, $i = 0, 1$. In conclusion $w_i \in \mathcal{I}$, $i = 0, 1$ and we are done. Finally, let $a \in \mathcal{I}$ and consider the $*$-polynomial $p_a \in I$ with evaluation a. Since I is $*$-invariant, we have that $p_a^* \in I$ (and also $-p_a^* \in I$). It is not difficult to see that one of these polynomials takes value a^*. Therefore $a^* \in \mathcal{I}$ and this implies that $\mathcal{I}$ is a $*$-ideal of W.

In order to complete the proof, it remains to show that $\mathcal{I} = \langle \mathcal{S} \rangle$. Since $S \subseteq I$, then $\mathcal{S} \subseteq \mathcal{I}$ and so $\langle \mathcal{S} \rangle \subseteq \mathcal{I}$. On the other hand, consider the $*$-algebra $\bar{W} = W/\langle \mathcal{S} \rangle$. Since the $*$-polynomials of S are identities of $\bar{W}$, then $I \subseteq \mathrm{Id}(\bar{W})$. Therefore, all evaluations of I on W are contained in $\langle \mathcal{S} \rangle$, that is, $\mathcal{I} \subseteq \langle \mathcal{S} \rangle$. $\square$

Remark 2.5.12. *Let K be a T_2^*-ideal of $F\langle Y \cup Z, * \rangle$ and let $f \in F\langle Y \cup Z, * \rangle$ be a $*$-polynomial such that $f \notin K$. Let J the T_2^*-ideal generated by f and K. Taking $S = K \cup \{f\}$ and $W = F\langle Y \cup Z, * \rangle / K$ in the previous lemma, we have that J/K is the $*$-ideal of $F\langle Y \cup Z, * \rangle / K$ generated by all the evaluations on $F\langle Y \cup Z, * \rangle / K$ of the polynomial f.*

In order to prove the main result of this section we need the following technical results.

Lemma 2.5.13. *Let K and J be T_2^*-ideals of $\bar{F}\langle Y \cup Z, * \rangle$ such that $K \subseteq J$. Then the following holds:*

$$\mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / K, t) = \mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / J, t) + \mathrm{Hilb}(J/K, t).$$

Proof. Let I be an ideal of an ordinary algebra A. It is well-known that A may be decomposed into the direct sum $(A/I) \oplus I$ (decomposition as vector spaces). Moreover, the ordinary Hilbert series of algebras satisfies the following relation (Proposition 1.2.9):

$$\mathrm{Hilb}(A, t) = \mathrm{Hilb}(A/I, t) + \mathrm{Hilb}(I, t).$$

Since K is a $*$-ideal of $\bar{F}\langle Y \cup Z, * \rangle$, then K is $*$-ideal of J (here J becomes a $*$-algebra with the operation of $\bar{F}\langle Y \cup Z, * \rangle$ restricted to J). Moreover, since J is a $*$-ideal of $\bar{F}\langle Y \cup Z, * \rangle$, we have that J/K is a $*$-ideal of the $*$-algebra $\bar{F}\langle Y \cup Z, * \rangle / K$. Both $\bar{F}\langle Y \cup Z, * \rangle / J$ and J/K are $*$-algebras. Taking $A = \bar{F}\langle Y \cup Z, * \rangle / K$ and $I = J/K$ we get the decomposition

$$\frac{\bar{F}\langle Y \cup Z, * \rangle / K}{J/K} \oplus \frac{J}{K} \cong \frac{\bar{F}\langle Y \cup Z, * \rangle}{J} \oplus \frac{J}{K}.$$

Now the proof is complete since we have:

$$\mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / K, t) = \mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / J, t) + \mathrm{Hilb}(J/K, t).$$

$\square$

Lemma 2.5.14. *Let I' and I'' be T_2^*-ideals of $\bar{F}\langle Y \cup Z, * \rangle$. Then the following holds:*

$$\begin{aligned} \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, * \rangle}{I' \cap I''}, t\right) =& \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, * \rangle}{I'}, t\right) + \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, * \rangle}{I''}, t\right) \\ & - \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, * \rangle}{I' + I''}, t\right). \end{aligned}$$

Proof. Taking $J = I' + I''$ and $K = I''$ in the previous lemma, we have:

$$\begin{aligned} \mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / I'', t) &= \mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / (I' + I''), t) + \mathrm{Hilb}(I' + I''/I'', t) \\ &= \mathrm{Hilb}(\bar{F}\langle Y \cup Z, * \rangle / (I' + I''), t) + \mathrm{Hilb}(I'/(I' \cap I''), t). \end{aligned}$$

Now we complete the proof by using again the previous lemma with $J = I'$ and $K = I' \cap I''$:

$$\begin{aligned}
\mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I' \cap I''}, t\right) =& \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I'}, t\right) + \mathrm{Hilb}\left(\frac{I'}{I' \cap I''}, t\right)\\
=& \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I'}, t\right) + \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I''}, t\right)\\
&- \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I' + I''}, t\right).
\end{aligned}$$

□

We have the key ingredients to prove the main result of this section, namely, the Hilbert-Serre Theorem for $*$-algebras.

Theorem 2.5.15. *Let $\bar{F}\langle Y \cup Z, *\rangle$ be the free $*$-algebra on the set of finite variables $Y = \{y_1^+, \ldots, y_p^+, y_1^-, \ldots, y_q^-\}$ and $Z = \{z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_s^-\}$, where F is an algebraically closed field of characteristic zero. If I is a T_2^*-ideal of $\bar{F}\langle Y \cup Z, *\rangle$ containing at least one ordinary non-trivial identity, then the Hilbert series of the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/I$ is rational.*

Proof. Suppose that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/I$ is non-rational. By the Specht's property for $*$-algebras (Theorem 2.4.5) there exists a T_2^*-ideal K of $\bar{F}\langle Y \cup Z, *\rangle$ containing an ordinary non-trivial identity and that it is maximal among T_2^*-ideals containing ordinary non-trivial identities and having non-rational Hilbert series of this relatively free $*$-algebra (i.e., of $\bar{F}\langle Y \cup Z, *\rangle/K$). Indeed, if there is no such an ideal, then we get an infinite ascending chain of T_2^*-ideals containing an ordinary non-trivial identity that does not stabilize and this contradicts the fact that the union of the T_2^*-ideals is finitely generated.

The maximality of K implies that the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/K$ is T_2^*-equivalent to a single basic $*$-algebra A. Indeed, assuming the converse, by Corollary 2.2.13, we get that

$$\bar{F}\langle Y \cup Z, *\rangle/K \sim_{T_2^*} A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are basic $*$-algebras, $m \geqslant 2$ and $\mathrm{Id}^*(A_i) \nsubseteq \mathrm{Id}^*(A_j)$, $1 \leqslant i, j \leqslant m$ with $i \neq j$. Thus

$$\mathrm{Id}^*(\bar{F}\langle Y \cup Z, *\rangle/K) = \mathrm{Id}^*(A_1 \oplus \cdots \oplus A_m) = \bigcap_{i=1}^{m} \mathrm{Id}^*(A_i).$$

For every $i \in \{1, \ldots, m\}$, clearly $\mathrm{Id}^*(\bar{F}\langle Y \cup Z, *\rangle/K) \subsetneq \mathrm{Id}^*(A_i)$. Let I_i be the evaluation on $\bar{F}\langle Y \cup Z, *\rangle$ of the T_2^*-ideal $\mathrm{Id}^*(A_i)$, $1 \leqslant i \leqslant m$. Then I_i properly contains K and their intersection is K. By the maximality of K, the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/I_i$ is rational for every i and by Lemma 2.5.14 we obtain that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/K$ is rational, a contradiction. Hence, $m = 1$ and so $\bar{F}\langle Y \cup Z, *\rangle/K$ is T_2^*-equivalent to a single basic $*$-algebra A.

Let f be a Kemer $*$-polynomial of the basic $*$-algebra A (see Theorem 2.3.17) and let J be the T_2^*-ideal generated by f and K. Since f is not a $*$-identity of A, then it is not a $*$-identity of $\bar{F}\langle Y \cup Z, *\rangle/K$ and so $K \subsetneq J$. By the maximality of K, the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/J$ is rational.

Our next goal is to show that the Hilbert series of J/K is a rational function.

Consider the decomposition $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ and let $\{\alpha_1^+, \ldots, \alpha_k^+\}$, $\{\alpha_1^-, \ldots, \alpha_l^-\}$, $\{\beta_1^+, \ldots, \beta_m^+\}$, $\{\beta_1^-, \ldots, \beta_n^-\}$ be F-bases of $A_0^+, A_0^-, A_1^+, A_1^-$, respectively. Let

$$\Lambda = \{\lambda_{ij}^+\}_{\substack{1\leqslant i\leqslant p\\1\leqslant j\leqslant k}} \cup \{\lambda_{ij}^-\}_{\substack{1\leqslant i\leqslant q\\1\leqslant j\leqslant l}} \cup \{\mu_{ij}^+\}_{\substack{1\leqslant i\leqslant r\\1\leqslant j\leqslant m}} \cup \{\mu_{ij}^-\}_{\substack{1\leqslant i\leqslant s\\1\leqslant i\leqslant n}}$$

be a set of commuting indeterminates which centralize with the elements of A. Now we consider the F-algebra $F\Lambda$. It is not difficult to see that the F-algebra $A \otimes_F F\Lambda$ is a $*$-algebra. The $\mathbb{Z}_2$-grading in A induces a $\mathbb{Z}_2$-grading in $A \otimes_F F\Lambda$:

$$A \otimes_F F\Lambda = (A_0 \otimes_F F\Lambda) \oplus (A_1 \otimes_F F\Lambda)\,.$$

The superinvolution $\bar{*}$ in $A \otimes_F F\Lambda$ is given by $(a \otimes P)^{\bar{*}} = a^* \otimes P$, with $a \in A$, $P \in F\Lambda$ and $*$ the superinvolution defined on A. Indeed, $((a \otimes P)^*)^* = (a^*)^* \otimes P = a \otimes P$ and $((a \otimes P)(a' \otimes P'))^* = (aa' \otimes PP')^* = (aa')^* \otimes PP' = (-1)^{|a||a'|}(a')^* a^* \otimes PP' = (-1)^{|a||a'|}(a')^* a^* \otimes P'P = (-1)^{|a||a'|}((a')^* \otimes P')(a^* \otimes P) = (-1)^{|a\otimes P||a'\otimes P'|}(a' \otimes P')^*(a \otimes P)^*$ for any $a' \in A$ and $P' \in F\Lambda$.

Consider the map $\varphi\colon \bar{F}\langle Y \cup Z, *\rangle/K \to A \otimes_F F\Lambda$, induced by

$$y_i^+ \longmapsto \sum_{j=1}^{k} \alpha_j^+ \otimes \lambda_{ij}^+, \qquad y_i^- \longmapsto \sum_{j=1}^{l} \alpha_j^- \otimes \lambda_{ij}^-, \qquad z_i^+ \longmapsto \sum_{j=1}^{m} \beta_j^+ \otimes \mu_{ij}^+, \qquad z_i^- \longmapsto \sum_{j=1}^{n} \beta_j^- \otimes \mu_{ij}^-.$$

Clearly φ is a well-defined homomorphism of $*$-algebras. Indeed, given a $*$-polynomial

$$g(y_1^+, \ldots, y_p^+, y_1^-, \ldots, y_q^-, z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_s^-) \in K,$$

then $g \in \mathrm{Id}^H(A)$. Hence, g vanishes on all evaluations of the basis of A. Thus,

$$\begin{aligned}
&\varphi(g(y_1^+, \ldots, y_p^+, y_1^-, \ldots, y_q^-, z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_s^-))\\
&\quad = g(\varphi(y_1^+), \ldots, \varphi(y_p^+), \varphi(y_1^-), \ldots, \varphi(y_q^-), \varphi(z_1^+), \ldots, \varphi(z_r^+), \varphi(z_1^-), \ldots, \varphi(z_s^-))\\
&\qquad = g(\sum_{j=1}^{k} \alpha_j^+ \otimes \lambda_{1j}^+, \ldots, \sum_{j=1}^{k} \alpha_j^+ \otimes \lambda_{pj}^+, \sum_{j=1}^{l} \alpha_j^- \otimes \lambda_{1j}^-, \ldots, \sum_{j=1}^{l} \alpha_j^- \otimes \lambda_{qj}^-,\\
&\qquad\quad \sum_{j=1}^{m} \beta_j^+ \otimes \mu_{1j}^+, \ldots, \sum_{j=1}^{m} \beta_j^+ \otimes \lambda_{rj}^+, \sum_{j=1}^{n} \beta_j^- \otimes \mu_{1j}^-, \ldots, \sum_{j=1}^{n} \beta_j^- \otimes \lambda_{sj}^-)\\
&\qquad\qquad = \sum_{\Delta} g(\alpha_{i_1}^+, \ldots, \alpha_{i_p}^+, \alpha_{j_1}^-, \ldots, \alpha_{j_q}^-, \beta_{k_1}^+, \ldots, \beta_{k_r}^+, \beta_{l_1}^-, \ldots, \beta_{l_s}^s) \otimes Q = 0
\end{aligned}$$

with $\Delta = (\bar{i}, \bar{j}, \bar{k}, \bar{l})$ where $\bar{i} = (i_1, \ldots, i_p) \in \{1, \ldots, k\}^p, \bar{j} = (j_1, \ldots, j_q) \in \{1, \ldots, l\}^q, \bar{k} = (k_1, \ldots, k_r) \in \{1, \ldots, m\}^r, \bar{l} = (l_1, \ldots, l_s) \in \{1, \ldots, n\}^s$ and Q is some polynomial in $F\Lambda$.

This shows that φ is well defined. By definition φ is $\mathbb{Z}_2$-graded homomorphism. It is easy to check that $\varphi(x^*) = \varphi(x)^*$ for any variable x in $Y \cup Z$. This prove that φ is $*$-homomorphism.

By definition, we have that φ is also injective. Hence $\mathcal{A} := \mathrm{Im}(\varphi)$ is isomorphic (as $*$-algebras) to $\bar{F}\langle Y \cup Z, *\rangle/K$. Thus, we can see $\bar{F}\langle Y \cup Z, *\rangle/K$ as a $*$-subalgebra of $A \otimes_F F\Lambda$.

Consider the following decompositions as $\mathbb{Z}_2$-graded algebras: $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $A = A_0 \oplus A_1$. Moreover, let $\bar{\mathcal{A}}_0$ and $\bar{A}_0$ be the semisimple parts of $\mathcal{A}_0$ and A_0, respectively. We can embed (embedding of $\mathbb{Z}_2$-graded algebras) $\bar{\mathcal{A}}_0$ into $\mathrm{End}_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda) \cong M_d(F\Lambda)$, where $d = \dim(\bar{A}_0)$, via the regular left $\bar{\mathcal{A}}_0$-action on $\bar{A}_0 \otimes_F F\Lambda$. Notice that each semisimple element $\bar{a} \in \bar{A}_0$ satisfies a Cayley-Hamilton identity (characteristic polynomial of $\bar{a}$) of degree d.

By Remark 2.5.1 we get $\bar{F}\langle Y \cup Z, *\rangle/K$ is a PI-algebra. Hence Proposition 2.5.10 applies and we get that $\bar{F}\langle Y \cup Z, *\rangle/K$ has an essential Shirshov base. As a consequence, $\mathcal{A}$ has an essential Shirshov base of elements of $\mathcal{A}_0$. Moreover, we may choose generators of $\mathcal{A}_0$ such that the corresponding essential Shirshov base is $\mathcal{B} = \bar{\mathcal{B}} \cup \mathcal{B}_J$ (disjoint union), where $\bar{\mathcal{B}} \subseteq \bar{\mathcal{A}}_0$ and $\mathcal{B}_J \subseteq J(\mathcal{A}_0)$ (the radical part of $\mathcal{A}_0$). Since $J(A) \otimes_F F\Lambda$ is nilpotent, the elements of $\mathcal{B}_J \subseteq J(\mathcal{A}_0) \subseteq J(A) \otimes_F F\Lambda$ are integrals over F.

In view of the embedding $\bar{\mathcal{A}}_0 \hookrightarrow \mathrm{End}_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda)$, each element of $\bar{\mathcal{B}}$ satisfies a characteristic polynomial of degree d with coefficients in $F\Lambda$. Let C be the F-subalgebra of $F\Lambda$ generated by these coefficients. Since $\mathcal{A}$ has unit, we may consider $\mathcal{B}$ having unit and therefore C has it too. Since the essential Shirshov base is finite, C is an affine commutative F-algebra and therefore a Noetherian F-algebra.

Consider the $*$-algebra $\mathcal{A}_C := C[\mathcal{A}]$. Notice that the elements of the essential Shirshov base of $\mathcal{A}$ are integral over C because the elements of $\mathcal{B}_J$ are integral over F and we may see F as the F-subspace spanned by the unit 1_C of C. On the other hand, given an element of $\bar{\mathcal{B}}$, by the Cayley-Hamilton Theorem, this satisfies its characteristic polynomial with coefficients in $F\Lambda$. But, by construction, these coefficients belong to C and so the elements of $\bar{\mathcal{B}}$ are integral over C. Thus, by Theorem 2.5.7, $\mathcal{A}_C$ is a finite module over C. By Proposition 2.5.8 we obtain that $\mathcal{A}_C$ has a rational Hilbert series.

We come back now to the study of the $*$-ideal J/K of the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/K$. We denote by $\mathcal{J}$ the image through φ of J/K. By Lemma 2.5.11 and Remark 2.5.12, $\mathcal{J}$ is the $*$-ideal of $\mathcal{A}$ generated by all the evaluations on $\mathcal{A}$ of the Kemer $*$-polynomial f.

Now, we want to show that $\mathcal{J}$ is a C-submodule of $\mathcal{A}_C$, that is $\mathcal{J}$ is closed under the multiplication of the coefficients of the characteristic polynomials of the elements in $\mathcal{B}$. Given an element $b_0 \in \bar{\mathcal{B}}$ and its characteristic polynomial $\lambda^d + \sum_{i=1}^{d}(-1)^i\gamma_i\lambda^{d-i}$, it is sufficient to show that for the Kemer $*$-polynomial $f(X_d, Y)$, where X_d and Y are disjoint sets of variables and X_d has exactly d variables of degree zero, we have $\gamma_i f(\hat{X}_d, \hat{Y}) \in \mathcal{J}$, for every $i \in \{1, \ldots, d\}$, where $\hat{X}_d = \{\hat{x}_1, \ldots, \hat{x}_d\}$ and $\hat{Y}$ denote any evaluation by elements of $\mathcal{A}$. Since $d = \dim_F(\bar{A}_0) = \dim_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda)$ and $J(A) \otimes_F F\Lambda$ has the same nilpotency index of $J(A)$, we have that $A \otimes_F F\Lambda$ has the same Kemer index of A. Hence Remark 2.3.18 implies that the $\hat{x}_i$'s can only assume semisimple values in $\bar{\mathcal{A}}_0 \subseteq \bar{A}_0 \otimes_F F\Lambda$, for $1 \leqslant i \leqslant d$. Denote these values by $a_1, \ldots, a_d$. Since f is alternating in the set of variables X_d, the value $f(a_1, \ldots, a_d, \hat{Y})$ is zero unless the elements $a_1, \ldots, a_d$ are linearly independent over $F\Lambda$. In this case, since $d = \dim_F(\bar{A}_0) = \dim_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda)$, the set $\{a_1, \ldots, a_d\}$ would be a linear basis of $\bar{A}_0 \otimes_F F\Lambda$ over $F\Lambda$. Finally, since we may see $b_0 \in \bar{\mathcal{B}}$ as an element of $\mathrm{End}_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda) \cong M_d(F\Lambda)$, we use Lemma 2.5.9 and conclude that

$$\gamma_i f(\hat{X}, \hat{Y}) = \gamma_i f(a_1, \ldots, a_d, \hat{Y}) = \sum_{\substack{k_1+\cdots+k_d=i\\ k_i\in\{0,1\}}} f((b_0)^{k_1}(a_1), \ldots, (b_0)^{k_d}(a_d), \hat{Y}) \in \mathcal{J}.$$

Since C is Noetherian, $\mathcal{J}$ is a finitely generated C-module and so, by Proposition 2.5.8, $\mathcal{J}$ has a rational Hilbert series. Since $\mathcal{A} = 1_C \cdot \mathcal{A} \subseteq \mathcal{A}_C$, we have that $\mathcal{J}$ is a common ideal of $\mathcal{A}$ and $\mathcal{A}_C$. We conclude that J/K has a rational Hilbert series.

So far we have proved that $\bar{F}\langle Y \cup Z, *\rangle/J$ and J/K have rational Hilbert series. Now, by applying Lemma 2.5.13, we get that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/K$ is rational, which is a contradiction. The contradiction arised from the assumption the Hilbert series of the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/I$ is not a rational function. The proof is complete. $\square$

3 H-module algebras

The purpose of this chapter is to give a proof of the Hilbert-Serre Theorem in the case of relatively free algebras of H-module algebras satisfying an ordinary polynomial identity, where H is a finite dimensional semisimple Hopf algebra over a field F of characteristic zero (Theorem 3.2.6).

Throughout this chapter H will denote a finite dimensional semisimple Hopf algebra over a field F of characteristic zero. We refer to the Section 1.3 for basic definitions, examples of Hopf algebras and H-module algebras.

3.1 Specht's problem for H-module algebras

In this section we shall introduce some definitions and present several results concerning the theory of Specht in the setting of H-module algebras. We refer the reader to the paper [38] by Karasik for more details.

Let W be a H-module algebra. Recall that $\mathrm{Id}(W)$ is the T-ideal of $F\langle X\rangle$ consisting of all ordinary identities of W and $\mathrm{Id}^H(W)$ is the T^H-ideal of $F^H\langle X\rangle$ consisting of all H-identities of W. Notice that the ordinary identities of W are H-identities of W taking the identification $F\langle X\rangle \cong F\langle X\rangle \otimes_F 1_H \subseteq F^H\langle X\rangle$. Thus $\mathrm{Id}(W) \subseteq \mathrm{Id}^H(W)$. On the other hand, W does not necessarily have ordinary identities, even if it has H-identities. This is the case, for example, of the free non-commutative algebra W with H-action given by $1_H w = w$ and $hw = 0$, for all $w \in W$ and $h \in H$, $h \neq 1_H$.

Since the field F is of characteristic zero, every T^H-ideal is generated by multilinear H-polynomials, i.e. H-polynomials $f(x_1,\dots,x_n) \in F^H\langle X\rangle$ such that

$$f(x_1,\dots,x_{i-1},\alpha x_i + y, x_{i+1},\dots,x_n) = \alpha f(x_1,\dots,x_n) + f(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n),$$

for every $i \in \{1,\dots,n\}$ and $\alpha \in F$.

Now we recall the following results whose proofs can be found in [38, Theorem 4.1].

Theorem 3.1.1. *Let W be an affine (i.e., finitely generated) H-module algebra satisfying an ordinary non-trivial identity. Then there exists a finite dimensional H-module algebra A such that $\mathrm{Id}^H(A) \subseteq \mathrm{Id}^H(W)$.*

Remark 3.1.2. *Let I be a T^H-ideal of $F^H\langle X\rangle$ containing an ordinary non-trivial identity. Since $\mathrm{Id}^H(F^H\langle X\rangle/I) = I$, the relatively free H-module algebra $F^H\langle X\rangle/I$ contains an ordinary non-trivial identity.*

The following result is the Representability Theorem for H-module algebras due to Karasik in [38].

Theorem 3.1.3. *Let W be a finitely generated H-module F-algebra, where F is a field containing $\mathbb{C}$, satisfying an ordinary non-trivial identity. Then there exists a field extension K of F and a finite dimensional H-module algebra A over K such that $W \sim_{T^H} A$ (notation as on pag. 52).*

Definition 3.1.4. *Let $f(x_1, \ldots, x_n, Y) \in F^H\langle X\rangle$ be a multilinear H-polynomial, where Y is a set of variables disjoint from $x_1, \ldots, x_n$. We say f is **alternating** in $\{x_1, \ldots, x_n\}$ if there exists a multilinear H-polynomial $h(x_1, \ldots, x_n, Y)$ such that*

$$f(X) = \sum_{\sigma \in S_n} (-1)^\sigma h(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, Y).$$

Given an H-module algebra W, we say that W satisfies a Capelli identity of rank m if every H-polynomial $f(x_1, \ldots, x_n, Y)$ alternating on $x_1, \ldots, x_n$ is in $\mathrm{Id}^H(W)$. By Theorem 3.1.1, an affine H-module algebra satisfying an ordinary non-trivial identity satisfies a Capelli identity.

Definition 3.1.5. *Let W be an H-module algebra satisfying an ordinary non-trivial identity. The H**-Kemer index** of W is the ordered pair $(\beta(W), \gamma(W)) \in \mathbb{N} \times \mathbb{N}$, where*

- *$\beta(W)$ is the maximal integer such that, for every μ, there exists a multilinear H-polynomial $f = f(X_1, \ldots, X_\mu, Y) \notin \mathrm{Id}^H(W)$ which is alternating with respect to the sets $X_1, \ldots, X_\mu$, which are all of cardinality $\beta(W)$,*
- *$\gamma(W)$ is the maximal integer such that, for every μ, there exists a multilinear H-polynomial $g = g(X_1, \ldots, X_\mu, X'_1, \ldots, X'_{\gamma(W)}, Y) \notin \mathrm{Id}^H(W)$ which is alternating with respect to the sets $X_1, \ldots, X_\mu$, $X'_1, \ldots, X'_{\gamma(W)}$, where $X_1, \ldots, X_\mu$ are of cardinality $\beta(W)$ and $X'_1, \ldots, X'_{\gamma(W)}$ are of cardinality $\beta(W) + 1$.*

*The polynomials g are called H**-Kemer polynomials of rank** μ.*

Let W be a finite dimensional H-module algebra and let $J = J(W)$ be the Jacobson radical of W. In [44] it was proved that J is H-invariant and so W/J is semisimple. By the Wedderburn-Malcev Theorem, W may be decomposed as $W = \bar{W} + J$ (decomposition as vector spaces), where $\bar{W}$ is a semisimple H-module subalgebra of W which is H-isomorphic to W/J. Let d_W be the dimension of $\bar{W}$ and let n_W be the nilpotency index of J. Denote by $\mathrm{Par}(W) = (d_W, n_W - 1)$ the *parameter* of W.

A finite dimensional H-module algebra A is called H**-basic** if there are no finite dimensional H-module algebras $B_1, \ldots, B_s$ such that $\mathrm{Par}(B_i) < \mathrm{Par}(A)$, $i \in \{1, \ldots, s\}$, and $A \sim_{T^H} B_1 \times \cdots \times B_s$. By induction on $\mathrm{Par}(W)$, every finite dimensional H-module algebra W is T^H equivalent to a finite product of H-basic algebras (see Remark 5.8 of [38]).

Kemer's Lemmas 1 and 2 for H-module algebras are given in [38, Lemmas 5.12 and 6.6]. They imply that, if A is H-basic, then $(d_A, n_A - 1) = (\beta(A), \gamma(A))$ ([38, Corollary 6.9]). It follows that A has an H-Kemer polynomial f having, say at least, n_A alternating sets of variables of cardinality d_A and a total of $n_A - 1$ alternating sets of variables of cardinality $d_A + 1$.

In particular, Kemer's Lemma 2 implies the following remark (see [38, Remark 6.8]).

Remark 3.1.6. *Any non-zero evaluation of f must consist only of semisimple evaluations in the sets of variables of cardinality d_A.*

Let W be an affine H-module algebra over a field F cointaining $\mathbb{C}$, satisfying an ordinary non-trivial identity. As a consequence of the Representability Theorem for H-modules algebras (Theorem 3.1.3), W is T^H-equivalent to a finite product of H-basic algebras $A_1, \ldots, A_m$ over a field extension K of F. Notice that, since $\mathrm{Id}^H(A_1 \oplus \cdots \oplus A_m) = \cap_{i=1}^m \mathrm{Id}^H(A_i)$, we may assume that $\mathrm{Id}^H(A_i) \nsubseteq \mathrm{Id}^H(A_j)$, for every $1 \leqslant i, j \leqslant m$ with $i \neq j$. By passing to the algebraic closure of K, we may assume that the H-basic algebras A_i are finite dimensional over the same field F.

The following theorem is the Specht property for H-module algebras (see[38, Theorem 1.4]).

Theorem 3.1.7. *Let W be an affine H-module algebra satisfying an ordinary non-trivial identity. If $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of T^H-ideals of W containing an ordinary non-trivial identity, then the chain stabilizes.*

3.2 Rationality of the Hilbert series of relatively free H-module algebras

Let H be a Hopf algebra over F with basis $\{b_1, \ldots, b_m\}$. We denote by $F^H\langle X_r \rangle$ the free H-module algebra on the set of finite variables $X_r = \{x_1, \ldots, x_r\}$. Given a T^H-ideal I in $F^H\langle X_r \rangle$, then $F^H\langle X_r \rangle / I$ is the corresponding relatively free H-module algebra. Write Ω_n to denote the (finite) set of monomials of degree n on the variables $x_i^{b_j}$, $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, r\}$. If c_n is the dimension of the F-subspace of $F^H\langle X_r \rangle / I$ spanned by the monomials of Ω_n, then the **Hilbert series of** $F^H\langle X_r \rangle / I$ with respect to the generators $\{x_i^{b_j}\}_{ij}$ is defined by

$$\mathrm{Hilb}(F^H\langle X_r \rangle / I, t) = \sum_n c_n t^n.$$

Given any T^H-ideal I in $F^H\langle X_r\rangle$, it is convenient to view I as the evaluation on $F^H\langle X_r\rangle$ of a T^H-ideal $\mathcal{I}$ of the free H-module algebra $F^H\langle X\rangle$. As already mentioned in the previous section, every T^H-ideal is generated by multilinear H-polynomials. Unfortunately, passing from $\mathcal{I}$ to I (by evaluation) the multilinearity condition could no longer be true.

The main goal of this section is to show that, in case H is a semisimple Hopf F-algebra, then the Hilbert series of $F^H\langle X_r\rangle/I$ is a rational function. Let W be an H-module algebra and consider the T^H-ideal of the identities $\mathrm{Id}^H(W)$ satisfied by W. Since $\mathrm{char}F = 0$, we have $\mathrm{Id}^H(W) = \mathrm{Id}^H(W \otimes_F \bar{F})$, where $\bar{F}$ is the algebraic closure of F. This means that the ideal of identities of $W_{\bar{F}}$ over $\bar{F}$ is the span (over $\bar{F}$) of the T^H-ideal of identities of W over F. Therefore the Hilbert series remains the same when passing to the algebraic closure of F. From now on, we assume that $F = \bar{F}$.

We start by proving the following technical result.

Lemma 3.2.1. *Let $A \subseteq M_n(F)$ be an algebra which is a H-module and let V be a d-dimensional subalgebra of $M_n(F)$ with an F-basis $a_1, \ldots, a_d$ of elements of A. Given an F-linear transformation $T\colon V \to V$, let $\lambda^d + \sum_{i=1}^{d}(-1)^i\gamma_i\lambda^{d-i}$ be its characteristic polynomial. Then for any multilinear H-polynomial $f(x_1, \ldots, x_d)$ which is alternating in the variables $x_1, \ldots, x_d$, the following equation holds:*

$$\gamma_i f(a_1, \ldots, a_d) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i\in\{0,1\}}} f(T^{k_1}(a_1), \ldots, T^{k_d}(a_d)).$$

Proof. We first show that the following equation holds:

$$\det(T)f(a_1, \ldots, a_d) = f(T(a_1), \ldots, T(a_d)).$$

Suppose that $T(a_j) = \sum_{i=1}^{d} c_{ij}a_i$, with $c_{ij} \in F$, $1 \leqslant i, j \leqslant d$. Since the H-action is linear, then $h \cdot (T(a_j)) = \sum_{i=1}^{d} c_{ij}ha_i$, with $h \in H$. Also, since $f(x_1, \ldots, x_d)$ is an alternating multilinear H-polynomial and T is an F-linear transformation, we get that

$$\begin{aligned} f(T(a_1), \ldots, T(a_d)) &= f\left(\sum_{i=1}^{d} c_{i1}a_i, \ldots, \sum_{i=1}^{d} c_{id}a_i\right) \\ &= \sum_{\sigma\in S_d} c_{\sigma(1),1}\cdots c_{\sigma(d),d}f(a_{\sigma(1)}, \ldots, a_{\sigma(d)}) \\ &= \sum_{\sigma\in S_d} (-1)^\sigma c_{\sigma(1),1}\cdots c_{\sigma(d),d}f(a_1, \ldots, a_d) \\ &= \det(T)f(a_1, \ldots, a_d). \end{aligned}$$

Here S_d is the symmetric group of order d.

Using the F-linear transformation $\lambda I_d - T$ in place of T, we get:

$$\det(\lambda I_d - T)f(a_1, \dots, a_d) = f((\lambda I_d - T)(a_1), \dots, (\lambda I_d - T)(a_d)).$$

Now we remark that

$$\begin{aligned}
f((\lambda I_d &- T)(a_1), \dots, (\lambda I_d - T)(a_d)) \\
&= f(\lambda a_1 - T(a_1), \dots, \lambda a_d - T(a_d)) \\
&= \lambda^d f(a_1, \dots, a_d) - \lambda^{d-1} \sum_{k_1+\cdots+k_d=1} f(T^{k_1}(a_1), \dots, T^{k_d}(a_d)) \\
&\quad + \lambda^{d-2} \sum_{k_1+\cdots+k_d=2} f(T^{k_1}(a_1), \dots, T^{k_d}(a_d)) - \cdots \\
&\quad + (-1)^d \lambda^0 f(T(a_1), \dots, T(a_d)),
\end{aligned}$$

with $k_i \in \{0, 1\}$ for all $i \in \{1, \dots, d\}$. On the other hand,

$$\det(\lambda I_d - T) = \lambda^d + \sum_{i=1}^{d} (-1)^i \gamma_i \lambda^{d-i},$$

the characteristic polynomial of T with coefficients $\gamma_i \in F$, $1 \leqslant i \leqslant d$. In conclusion we get

$$\gamma_i f(a_1, \dots, a_d) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i \in \{0,1\}}} f(T^{k_1}(a_1), \dots, T^{k_d}(a_d)).$$

□

Lemma 3.2.2. *Let S be a set of H-polynomials in $F^H\langle X\rangle$ and let I be the T^H-ideal generated by S. Given an H-module algebra W, consider $\mathcal{S}, \mathcal{I}$ to be the sets of all evaluations on W of the polynomials of S and I, respectively. Then $\mathcal{I} = \langle \mathcal{S} \rangle$ (the ideal generated by $\mathcal{S}$).*

Proof. Given $a, b \in \mathcal{I}$ and given polynomials p_a and p_b in I with evaluations a and b respectively, then by the T^H-property of I (i.e. I is invariant under all H-endomorphism of $F^H\langle X\rangle$), we may change variables and assume that p_a and p_b have disjoint sets of variables. Then we may get $a + b$ as an evaluation of the polynomial $p_a + p_b$, so $a + b \in \mathcal{I}$. If $c \in W$, we may take a variable x which is not in p_a and get ca and ac as evaluations of xp_a and $p_a x$ respectively, so ca and ac belong to $\mathcal{I}$. If $h \in H$, then $hp_a \in I$ which implies $ha \in \mathcal{I}$. Thus, $\mathcal{I}$ is ideal of W.
Now, we show $\mathcal{I} = \langle \mathcal{S} \rangle$. Since $S \subseteq I$, then $\mathcal{S} \subseteq \mathcal{I}$ and $\langle \mathcal{S} \rangle \subseteq \mathcal{I}$. On the other hand, consider the H-module algebra $\bar{W} = W/\langle \mathcal{S} \rangle$. Since the polynomials of S are identities of $\bar{W}$ then $I \subseteq \mathrm{Id}(\bar{W})$. Therefore, all evaluations of I on W are contained in $\langle \mathcal{S} \rangle$, that is, $\mathcal{I} \subseteq \langle \mathcal{S} \rangle$. □

Remark 3.2.3. *Let K be a T^H-ideal of $F^H\langle X\rangle$ and let $f \in F^H\langle X\rangle$ be an H-polynomial such that $f \notin K$. Let J be the T^H-ideal generated by f and K. Taking $S = K \cup \{f\}$ and $W = F^H\langle X\rangle/K$ in the previous lemma, we have J/K is the ideal of $F^H\langle X\rangle/K$ generated by all evaluations on $F^H\langle X\rangle/K$ of the polynomial f.*

Let J be a T^H-ideal of $F^H\langle X_r\rangle$. In particular, J is an ideal of $F^H\langle X_r\rangle$ (as F-algebra). Then J becomes an H-module algebra with the operations of $F^H\langle X_r\rangle$ restricted to J.

The following lemmas can be proved by using the same arguments employed in the corresponding results of Section 2.5 (Lemmas 2.5.13 and 2.5.14).

Lemma 3.2.4. *Let K and J be T^H-ideals of $F^H\langle X_r\rangle$ such that $K \subseteq J$. Then the following holds:*

$$\mathrm{Hilb}(F^H\langle X_r\rangle/K, t) = \mathrm{Hilb}(F^H\langle X_r\rangle/J, t) + \mathrm{Hilb}(J/K, t).$$

Lemma 3.2.5. *Let I' and I'' be T^H-ideals of $F^H\langle X_r\rangle$. Then the following holds:*

$$\mathrm{Hilb}(F^H\langle X_r\rangle/(I' \cap I''), t)$$

$$= \mathrm{Hilb}(F^H\langle X_r\rangle/I', t) + \mathrm{Hilb}(F^H\langle X_r\rangle/I'', t) - \mathrm{Hilb}(F^H\langle X_r\rangle/(I' + I''), t).$$

Finally we are in a position to prove the main theorem of this chapther, namely, the Hilbert-Serre Theorem for H-module algebras.

Theorem 3.2.6. *Let $F^H\langle X_r\rangle$ be the free H-module algebra on the set of variables $X_r = \{x_1, \cdots, x_r\}$, where H is a finite dimensional semisimple Hopf algebra and F is a field of characteristic zero. If I is a T^H-ideal of $F^H\langle X_r\rangle$ containing at least one ordinary non-trivial identity, then the Hilbert series of the relatively free H-module algebra $F^H\langle X_r\rangle/I$ is rational.*

Proof. The proof is very similar to the one given for the analogous result in the setting of $*$-algebras (Theorem 2.5.15). For this reason we will give here just a sketch of it.

Suppose that the Hilbert series of $F^H\langle X_r\rangle/I$ is non-rational. By the Specht's property for H-module algebras (Theorem 3.1.7) there exists a T^H-ideal K of $F^H\langle X_r\rangle$ containing an ordinary non-trivial identity and that it is maximal among T^H-ideals containing ordinary non-trivial identities and having non-rational Hilbert series of their relatively free H-module algebra.

The maximality of K implies that $F^H\langle X_r\rangle/K$ is T^H-equivalent to a single H-basic H-module algebra A. To this end we just need to use Theorem 3.1.3 and Lemma 3.2.5.

Now let f be a H-Kemer polynomial of the H-basic H-module algebra A and let J be the T^H-ideal generated by f and K. Since f is not an H-identity of A, then f is not an H-identity of $F^H\langle X_r\rangle/K$, and hence, $K \subsetneq J$. By the maximality of K, the Hilbert series of $F^H\langle X_r\rangle/J$ is rational.

In order to complete the proof we need to show that the Hilbert series of J/K is a rational function. In fact, once this is accomplished, we will have that $F^H\langle X_r\rangle/J$ and

J/K have rational Hilbert series. Then by Lemma 3.2.4, the Hilbert series of $F^H\langle X_r\rangle/K$ is rational, which is a contradiction. The contradiction arises from having assumed that the Hilbert series of the relatively free H-module algebra $F^H\langle X_r\rangle/I$ is not a rational function.

From now on, our only goal is to prove that the Hilbert series of J/K is a rational function.

Suppose that $\{\alpha_1,\dots,\alpha_l\}$ is an F-basis of A and let $\Lambda = \{\lambda_{ij}\colon 1\leqslant i\leqslant r, 1\leqslant j\leqslant l\}$ be a set of commuting indeterminates centralizing with the elements of A. Consider the F-algebra $F\Lambda$ endowed with a formal H-action. We prefer the notation $\lambda_{ij}^h := h\cdot\lambda_{ij}$ to denote the formal action of some $h\in H$ in each λ_{ij}. The H-module structure is given by

$$h\cdot(\lambda_{i_1j_1}^{h_1}\lambda_{i_2j_2}^{h_2}\cdots\lambda_{i_nj_n}^{h_n}) = \lambda_{i_1j_1}^{h_{(1)}h_1}\lambda_{i_2j_2}^{h_{(2)}h_2}\cdots\lambda_{i_nj_n}^{h(n)h_n} \quad\text{and}\quad h\cdot 1_\Lambda = 1_\Lambda,$$

where 1_Λ is the unit element of $F\Lambda$.

Consider the algebra $A\otimes_F F\Lambda$ and define the action of H in $A\otimes_F F\Lambda$ for F-basic elements: if H_0 is an F-basis of H and Λ_0 is an F-basis of $F\Lambda$ then $h(\alpha\otimes P) = h\alpha\otimes hP$, where $h\in H_0$, $P\in\Lambda_0$ and $\alpha\in\{\alpha_1,\dots,\alpha_l\}$. If we extend the H-action linearly in H and $A\otimes_F F\Lambda$ we obtain a structure of H-module. Notice that, for any $a_1,a_2\in A$, $P_1,P_2\in F\Lambda$ and $h\in H$, we have that:

$$\begin{aligned} h((a_1\otimes P_1)(a_2\otimes P_2)) &= h(a_1a_2\otimes P_1P_2) = h(a_1a_2)\otimes h(P_1P_2)\\ &= h_{(1)}a_1h_{(2)}a_2\otimes h_{(1)}P_1h_{(2)}P_2\\ &= h_{(1)}(a_1\otimes P_1)h_{(2)}(a_2\otimes P_2),\end{aligned}$$

$$h(1_A\otimes 1_\Lambda) = h(1_A)\otimes h(1_\Lambda) = \varepsilon(h)1_A\otimes 1_\Lambda = \varepsilon(h)(1_A\otimes 1_\Lambda).$$

This shows that $A\otimes_F F\Lambda$ is an H-module algebra. Now, consider the H-homomorphism $\varphi\colon F^H\langle X_r\rangle/K\to A\otimes_F F\Lambda$, induced, for any $h\in H$, by

$$x_i^h \longmapsto \sum_{j=1}^{l} h(\alpha_j)\otimes\lambda_{ij}^h.$$

Given a multilinear H-polynomial $g(x_1,\dots,x_r)\in K$, then $g\in\mathrm{Id}^H(A)$. Hence, g vanishes on all evaluations of the basis $\{\alpha_1,\dots,\alpha_l\}$ of A. Thus,

$$\begin{aligned}\varphi(g(x_1,\dots,x_r)) = g(\varphi(x_1),\dots\varphi(x_r)) &= g(\sum_{j=1}^{l}\alpha_j\otimes\lambda_{1j},\dots,\sum_{j=1}^{l}\alpha_j\otimes\lambda_{rj})\\ &= \sum_{\bar{i}} g(\alpha_{i_1},\dots,\alpha_{i_r})\otimes Q = 0\end{aligned}$$

with $\bar{i} = (i_1,\dots,i_r)\in\{1,\dots,l\}^r$ and Q is some polynomial in $F\Lambda$. This shows that φ is well defined H-homomorphism. It is not difficult to see that φ is injective. Hence we get that $\mathcal{A} := \mathrm{Im}(\varphi)$ is H-isomorphic (isomorphic as H-module algebras) to $F^H\langle X_r\rangle/K$. Thus, we can see $F^H\langle X_r\rangle/K$ as a subalgebra of $A\otimes_F F\Lambda$.

Let $\bar{A}$ be the H-invariant semisimple part of A. We can embed (embedding of F-algebras) $\bar{A}$ into $\mathrm{End}_F(\bar{A}) \cong M_d(F)$, where $d = \dim(\bar{A})$, via the regular left $\bar{A}$-action on $\bar{A}$. This induce an embedding $\bar{A} \otimes_F F\Lambda$ into $\mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$ via the regular action. Notice that each semisimple element $\bar{a} \in \bar{A}$ satisfies a Cayley-Hamilton identity (characteristic polynomial of $\bar{a}$) of degree d.

Since A may be decomposed into the direct sum $\bar{A} \oplus J(A)$ where $J(A)$ is the Jacobson radical of A, we may decompose $\mathcal{A}$ into the direct sum $\bar{\mathcal{A}} \oplus \mathcal{A}_J$ where $\bar{\mathcal{A}} \subseteq \bar{A} \otimes_F F\Lambda \hookrightarrow \mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$ and $\mathcal{A}_J \subseteq J(A) \otimes_F F\Lambda$. We shall call $\bar{\mathcal{A}}$ the semisimple part of $\mathcal{A}$ and $\mathcal{A}_J$ the radical part of $\mathcal{A}$.

Remark 3.1.2 implies that $F^H\langle X_r\rangle/K$ is a PI-algebra. By Theorem 2.5.4, $F^H\langle X_r\rangle/K$ has a Shirshov base, then $\mathcal{A}$ has a Shirshov base. Moreover, we may choose generators of $\mathcal{A}$ such that the corresponding Shirshov base is $\mathcal{B} = \bar{\mathcal{B}} \cup \mathcal{B}_J$ (disjoint union), where $\bar{\mathcal{B}} \subseteq \bar{\mathcal{A}}$ and $\mathcal{B}_J \subseteq \mathcal{A}_J$. In fact, if we choose generators $b_1, \ldots, b_s$ of $\mathcal{A}$ either from $\bar{\mathcal{A}}$ or $\mathcal{A}_J$, a basic element $b_{i_1} b_{i_2} \cdots b_{i_t}$ belongs to $\bar{\mathcal{B}}$ if and only if $b_{i_j} \in \bar{\mathcal{A}}$ for all $j \in \{1, \ldots, t\}$. Since $J(A) \otimes_F F\Lambda$ is nilpotent, the elements of $\mathcal{B}_J$ are integrals over F.

In view of the embedding $\bar{\mathcal{A}} \hookrightarrow \mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$, each element of $\bar{\mathcal{B}}$ satisfies a characteristic polynomial of degree d with coefficients in $F\Lambda$. Let C the F-subalgebra of $F\Lambda$ generated by these coefficients. Since $\mathcal{A}$ has unit, we may consider $\mathcal{B}$ having unit, and therefore C has unit. Since the Shirshov base is finite, C is an affine commutative F-algebra and therefore a Noetherian F-algebra.

Consider the H-module C-algebra $\mathcal{A}_C := C[\mathcal{A}]$. Notice that the elements of the Shirshov base of $\mathcal{A}$ are integrals over C because the elements of $\mathcal{B}_J$ are integrals over F and we may see F as the F-subspace spanned by the unit 1_C of C. On the other hand, given an element of $\bar{\mathcal{B}}$, by the Cayley-Hamilton Theorem this satisfies its characteristic polynomial with coefficients in $F\Lambda$. But, by construction, these coefficients belongs to C, then the elements of $\bar{\mathcal{B}}$ are integral over C. Thus, by Theorem 2.5.7 $\mathcal{A}_C$ is a finite module over C. Then $\mathcal{A}_C$ has a rational Hilbert series by Proposition 2.5.8.

We come back now to the study of the ideal J/K of the relatively free H-module algebra $F^H\langle X_r\rangle/K$. We denote by $\mathcal{J}$ the image by φ of J/K. By Lemma 3.2.2 and Remark 3.2.3, $\mathcal{J}$ is the ideal of $\mathcal{A}$ generated by all the evaluations on $\mathcal{A}$ of the H-Kemer polynomial f. We will show that $\mathcal{J}$ is a C-submodule of $\mathcal{A}_C$, that is, we show that $\mathcal{J}$ is closed under the multiplication of the coefficients of the characteristic polynomials of the elements in $\mathcal{B}$. So, given an element $b_0 \in \bar{\mathcal{B}}$ and $\lambda^d + \sum_{i=1}^{d}(-1)^i \gamma_i \lambda^{d-i}$ its characteristic polynomial, it is sufficient to show that for the H-Kemer polynomial $f(X_d, Y)$, where X_d and Y are sets of disjoint variables and X_d has d elements, we have $\gamma_i f(\hat{X}_d, \hat{Y}) \in \mathcal{J}$, where $\hat{X}_d = \{\hat{x}_1, \ldots, \hat{x}_d\}$ and $\bar{Y}$ denote an evaluation of elements of $\mathcal{A}$.

In view of the embedding $\mathcal{A} \subseteq A \otimes_F F\Lambda \subseteq (\bar{A} \otimes_F F\Lambda) \oplus (J(A) \otimes_F F\Lambda)$, an element $v \in \hat{X}_d \cup \hat{Y}$ can be written as $v = \bar{v} + v_J$ where $\bar{v} \in \bar{A} \otimes_F F\Lambda$ and $v_J \in J(A) \otimes_F F\Lambda$. Since $d = \dim_F(\bar{A}) = \dim_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$ and $J(A) \otimes_F F\Lambda$ has the same nilpotency index as $J(A)$, then $A \otimes_F F\Lambda$ has the same H-Kemer index as A. If we denote by a_i the semisimple part of $\hat{x}_i$ and by c_i the radical part of $\hat{x}_i$ for $1 \leqslant i \leqslant d$, Remark 3.1.6 implies $f(\hat{X}_d, \hat{Y}) = f(a_1, \ldots, a_d, \hat{Y})$. Since f is alternating in the set of variables X_d, the value $f(a_1, \ldots, a_d, \hat{Y})$ is zero unless the elements $a_1, \ldots, a_d$ are linearly independent over $F\Lambda$ and since $d = \dim_F(\bar{A}) = \dim_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$, the set $\{a_1, \ldots, a_d\}$ is a linear basis of $\bar{A} \otimes_F F\Lambda$ over $F\Lambda$.

Since we may see $b_0 \in \bar{\mathcal{B}}$ as an element of $\mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$, by Lemma 3.2.1 we get that

$$\gamma_i f(\hat{X}, \hat{Y}) = \gamma_i f(a_1, \ldots, a_d, \hat{Y}) = \sum_{\substack{k_1 + \cdots + k_d = i \\ k_i \in \{0,1\}}} f((b_0)^{k_1}(a_1), \ldots, (b_0)^{k_d}(a_d), \hat{Y}) \in \mathcal{J}.$$

Since C is Noetherian, $\mathcal{J}$ is a finitely generated C-module as well and again by Proposition 2.5.8, $\mathcal{J}$ has a rational Hilbert series. Since $\mathcal{A} = 1_C \cdot \mathcal{A} \subseteq \mathcal{A}_C$, we have that $\mathcal{J}$ is a common ideal of $\mathcal{A}$ and $\mathcal{A}_C$. We conclude that J/K has a rational Hilbert series. $\square$

4 H_m-module algebra UT_2

In this chapter we study the Specht property for the variety of H_m-module algebras generated by the algebra UT_2 of 2×2 upper triangular matrices over a field of characteristic 0 containing a primitive m-th root of unit and where H_m denotes a Taft's Hopf algebra of dimension m^2. We would like to point out that we cannot use Karasik's result (Theorem 3.1.7) in order to establish whether or not our variety satisfies the Specht property because although the Hopf algebra H_m is finite dimensional, it is not semisimple (see Example 1.3.34).

Hereby we would like to highlight the role of UT_2 in the theory of PI-algebras. In [53] Regev proved the codimension sequence of any associative PI-algebra is exponentially bounded. Later Kemer in [41] showed such codimensions are either polynomially bounded or grow exponentially. Moreover, Giambruno and Zaicev in a famous couple of paper (see [27] and [28]) computed the exponential rate of growth of a PI-algebra and proved that it is a non-negative integer. By a well known Kemer's result [39] we get the variety of algebras generated by UT_2 is a variety of almost polynomial growth, i.e., it has exponential growth but every proper subvariety has polynomial growth. An analogous result was found by Valenti in [61] for varieties of algebras graded by a finite group and by Mishchenko and Valenti in [46] for varieties of algebras with involution. We would also like to cite the paper [25] by Giambruno and Rizzo toward differential identities: there the authors prove that UT_2 under the action of its algebra of derivation does not generate a variety of almost polynomial growth and they construct a subvariety of almost polynomial growth. Notice that the variety of H_m-module algebras generated by UT_2 is not of almost polynomial growth too as showed by Centrone and Yasumura in [16].

4.1 The action of H_m on the algebra UT_2

Let UT_2 be the algebra of 2×2 upper triangular matrices over the field F and let G be group. A detailed description of the G-graded identities satisfied by the algebra UT_2 when the characteristic of F is 0 is given in [61]. In particular, in [61, Theorem 1] the author shows that, up to isomorphism, there is only one non-trivial grading. So any G-grading on UT_2 is actually a $\mathbb{Z}_2$-grading.

Definition 4.1.1. *Given an algebra A over a field F, a α-**derivation** is an F-linear map $\delta\colon A \to A$ such that for every $a, b \in A$ we have $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, where α is a suitable endomorphism of A. The α-derivation δ is called **inner** if there exists an element $y \in A$ so that $\delta(a) = ya - \alpha(a)y$ and we write $ad_\alpha(y)$ instead of δ.*

Taft's algebras were introduced in Example 1.3.20. From now on, F is a field of characteristic zero containing a primitive m-th root of the unit. We shall denote by H_m the m-th Taft's Hopf algebra over F.

Consider a H_m-action on UT_2.

Theorem 4.1.2. *[16, Theorem 11] The H_m-action on UT_2 is completely determined by a choice of an automorphism α of UT_2 or order m, and an inner α-derivation by an element $y \in UT_2$ such that $\alpha(y) = \gamma^{-1}y$, and $ad_\alpha(y)^m = 0$.*

Equivalently, the H_m-action on UT_2 is completely determined by a choice of a $\mathbb{Z}_m$ grading on UT_2 and a homogeneous element $d \in UT_2$ of homogeneous degree γ^{-1} such that $ad_\alpha(y)^m = 0$.

Then there exist three "structures" of H_m-module algebra on UT_2 (see [16, page 738]):

i) The trivial grading (so d acts trivially): in this case, $\mathrm{Id}^{H_m}(UT_2)$ is merely the ideal of ordinary polynomial identities of UT_2, which was calculated by Malcev in [45].

ii) The canonical $\mathbb{Z}_2$-grading and d acts trivially: in this case, $\mathrm{Id}^{H_m}(UT_2)$ coincides with the ideal of $\mathbb{Z}_2$-graded polynomial identities of UT_2 which was originally calculated by Valenti in [61] and generalized by Di Vincenzo, Koshlukov and Valenti in [18].

iii) The canonical $\mathbb{Z}_2$-grading and d acts non-trivially. In this case, perforce $d = \mathrm{ad}_\alpha(ae_{12})$, for some $0 \neq a \in F$, that is, if $A = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \in UT_2$, then

$$A^d = \mathrm{ad}_\alpha(ae_{12})\left(\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}\right) = \begin{pmatrix} 0 & a(x_{22} - x_{11}) \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

The Specht property for (i) and (ii) are particular cases of the Specht property for ordinary PI-algebras [40] and G-graded PI-algebras [2], respectively. Therefore, we will study the case (iii). Thus, from now on, an H_m-action on UT_2 means the canonical $\mathbb{Z}_2$-grading on UT_2 with a non-trivial action of d on UT_2. This forces us to see an action of H_m on the algebra UT_2 as an action of H_2 on UT_2. It is worth recalling in [32] the author gives an explicit description of the simple algebras that are module algebra under the action of a Sweedler's algebra that is a Taft's algebra of dimension 4.

Let $F\langle X\rangle$ be the free associative algebra over the countable set $X = \{x_1, x_2, \dots\}$. If we write $X = Y \cup Z$ where $Y = \{y_1, y_2, \dots\}$ is the countable set of variables of degree zero and $Z = \{z_1, z_2, \dots\}$ is the countable set of variables of degree one, and $Y \cap Z = \emptyset$, then $F\langle Y \cup Z\rangle$ has a natural structure of **free superalgebra on** $Y \cup Z$.

We recall from Section 1.2.4 that a graded polynomial $f(y_1, \ldots, y_t, z_1, \ldots, z_s) \in F\langle Y \cup Z\rangle$ is a graded identity of a superalgebra $A = A_0 \oplus A_1$, and we write $f \equiv 0$, if, for all $a_1, \ldots, a_t \in A_0$, $b_1, \ldots, b_s \in A_1$,, we have $f(a_1, \ldots, a_t, b_1, \ldots, b_s) = 0$. $\mathrm{Id}^{gr}(A)$ denote the ideal of graded identities of A. Notice that $\mathrm{Id}^{gr}(A)$ is a T_2-ideal of $F\langle Y \cup Z\rangle$, i.e., an ideal that is invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra $F\langle Y \cup Z\rangle$. Since the characteristic of F is zero, it is well known that $\mathrm{Id}^{gr}(A)$ is completely determined by its multilinear graded polynomials.

Now, we construct $F\langle Y \cup Z \mid D_2\rangle$ the free superalgebra on $X = Y \cup Z$ with action of $D_2 = F\langle d \mid d^2 = 0\rangle$ as follows. The algebra $F\langle Y \cup Z \mid D_2\rangle$ is the algebra freely generated by the set $\{x^{d_1} = d_1(x) \mid x \in Y \text{ or } x \in Z, d_1 \in D_2\}$. We let D_2 act on $F\langle Y \cup Z \mid D_2\rangle$ by requiring that if $d_1, d_2 \in D_2$, then $(x^{d_1})^{d_2} = x^{d_1 d_2}$, and then by extending this action on all of $F\langle Y \cup Z \mid D_2\rangle$ as follows: if v, w are monomials, then define $(vw)^d = v^d w + (-1)^{\deg(v)} v w^d$ and then extend this action by linearity to all of $F\langle Y \cup Z \mid D_2\rangle$. The elements of $F\langle Y \cup Z \mid D_2\rangle$ are called **$\mathbb{Z}_2$-D_2-polynomials**.

The algebra $F\langle Y \cup Z \mid D_2\rangle$ has the following universal property: Given any superalgebra $A = A_0 \oplus A_1$ with D_2-action, any set theorical map $\varphi\colon Y \cup Z \to A$ such that $\varphi(Y) \subseteq A_0$ and $\varphi(Z) \subseteq A_1$, extends uniquely to a homomorphism of superalgebras $\bar{\varphi}\colon F\langle Y \cup Z \mid D_2\rangle \to A$ such that $\bar{\varphi}(f_1^d) = \bar{\varphi}(f)_1^d$, for any $f \in F\langle Y \cup Z \mid D_2\rangle$, $d_1 \in D_2$.

If we let Φ be the set of all such homomorphisms, then $\mathrm{Id}^{\mathbb{Z}_2, D_2}(A) = \cap_{\bar{\phi}\in\Phi} \ker \bar{\phi}$ is the **ideal of $\mathbb{Z}_2$-D_2-polynomials identities** of A. This means that a $\mathbb{Z}_2$-D_2-polynomial $f(y_1, \ldots, y_s, z_1, \ldots, z_t) \in F\langle Y \cup Z \mid D_2\rangle$ is a $\mathbb{Z}_2$-D_2-identity for A if for all $a_1, \ldots, a_s \in A_0$ and $b_1, \ldots, b_t \in A_1$, $f(a_1, \ldots, a_s, b_1, \ldots, b_t) = 0$. We write $f \equiv 0$ on A, in this case.

Definition 4.1.3. *$\mathrm{Id}^{\mathbb{Z}_2, D_2}(A) = \{f \in F\langle Y \cup Z \mid D_2\rangle \mid f \equiv 0 \text{ on } A\}$ is the ideal of $\mathbb{Z}_2$-D_2-polynomial identities of A.*

Proposition 4.1.4 ([16] Proposition 14)**.**

$$\mathrm{Id}^{H_2}(A) = \mathrm{Id}^{\mathbb{Z}_2, D_2}(A),$$

and

$$F^{H_2}\langle X\rangle \cong F\langle Y \cup Z \mid D_2\rangle.$$

Theorem 4.1.5 (Theorem 17, [16])**.** *For each $j = 0, 1, \ldots, m-1$, let $\beta_j = \sum_{l=0}^{m-1} \gamma^{jl} c^l$, $y_i = x_i^{\beta_0}$ and $z_i = x_i^{\beta_1}$. Then the T^{H_m}-ideal of UT_2 is generated by the following polynomials*

$$[y_1, y_2],\ z_1 x^h z_2,\ z^d,\ x^{d^2},\ y_1^d x^h y_2^d,\ x^{\beta_j},$$

where $h \in H_m$ and $j = 2, \ldots, m-1$.

Let $P_n^{\mathbb{Z}_2,D_2}$ be the subspace of $F\langle Y \cup Z \mid D_2\rangle$ consisting of multilinear $\mathbb{Z}_2$-D_2-polynomials of degree n in $x_1,\dots,x_n$, i.e.,

$$P_n^{\mathbb{Z}_2,D_2} = \operatorname{span}_F\{x_{\sigma(1)}^{d_1}\cdots x_{\sigma(n)}^{d_n} \mid \sigma \in S_n, d_i \in D_2, x_i = y_i \text{ or } x_i = z_i, i = 1,\dots,n\}.$$

Recall that the wreath product of $\mathbb{Z}_2$ and S_n (called the **hyperoctahedral group**) is the group defined by

$$\mathbb{Z}_2 \wr S_n = \{(g_1,\dots,g_n;\sigma) \mid g_1,\dots,g_n \in \mathbb{Z}_2, \sigma \in S_n\}$$

with multiplication given by

$$(g_1,\dots,g_n;\sigma)(h_1,\dots,h_n;\tau) = (g_1h_{\sigma^{-1}(1)},\dots,g_nh_{\sigma^{-1}(n)};\sigma\tau).$$

Let $\mathbb{Z}_2 = \{1,c\}$. Then the space $P_n^{\mathbb{Z}_2,D_2}$ has a structure of left $\mathbb{Z}_2 \wr S_n$-module induced by defining for $(g_1,\dots,g_n;\sigma) \in \mathbb{Z}_2 \wr S_n$ and $f(x_1,\dots,x_n) \in P_n^{\mathbb{Z}_2,D_2}$ (see [30, Lemma 10.1.5]),

$$(g_1,\dots,g_n;\sigma)f(x_1,\dots,x_n) = f(x_{\sigma(1)}^{g_{\sigma(1)}^{-1}},\dots,x_{\sigma(n)}^{g_{\sigma(n)}^{-1}}),$$

where $y_{\sigma(i)}^c = y_{\sigma(i)}$ and $z_{\sigma(i)}^c = -z_{\sigma(i)}$.

Notice that the vector space $P_n^{\mathbb{Z}_2,D_2} \cap \mathrm{Id}^{\mathbb{Z}_2,D_2}(A)$ is invariant under this action, hence $P_n^{\mathbb{Z}_2,D_2}(A) := P_n^{\mathbb{Z}_2,D_2}/(P_n^{\mathbb{Z}_2,D_2} \cap \mathrm{Id}^{\mathbb{Z}_2,D_2}(A))$ is a left $\mathbb{Z}_2 \wr S_n$-module. Let $\chi_n^{\mathbb{Z}_2,D_2}(A)$ be its character. It is known (see for instance Section 10.4 of [30]) that there is a one-to-one correspondence between irreducible $\mathbb{Z}_2 \wr S_n$-character and pairs of partitions (λ,μ), where $\lambda \vdash r$, $\mu \vdash n-r$, for all $r = 0,1,\dots,n$. If $\chi_{\lambda,\mu}$ denotes the irreducible $\mathbb{Z}_2 \wr S_n$-character corresponding to (λ,μ) then we can write

$$\chi_n^{\mathbb{Z}_2,D_2}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $m_{\lambda,\mu} \geqslant 0$ are the corresponding multiplicities.

For fixed $r \in \{0,\dots,n\}$, let

$$\begin{aligned} P_{r,n-r} = \operatorname{span}_F\{x_{\sigma(1)}^{d_1}\cdots x_{\sigma(n)}^{d_n} \mid \sigma \in S_n, d_i \in D_2, x_i = y_i \text{ for } i = 1,\dots,r,\\ \text{and } x_i = z_i \text{ for } i = r+1,\dots,n\} \end{aligned}$$

be the subspace of multilinear $\mathbb{Z}_2$-D_2-polynomials in the variables $y_1,\dots,y_r$, $z_{r+1},\dots,z_n$. In order to study $P_n^{\mathbb{Z}_2,D_2}(A)$ it is enough to study

$$P_{r,n-r}(A) = \frac{P_{r,n-r}}{P_{r,n-r} \cap \mathrm{Id}^{\mathbb{Z}_2,D_2}(A)}$$

for all $r = 0,\dots,n$. If we let S_r acting on the variables $y_1,\dots,y_r$ and S_{n-r} acting on the variables $z_{r+1},\dots,z_n$, we obtain an action of $S_r \times S_{n-r}$ on $P_{r,n-r}$ and $P_{r,n-r}(A)$ becomes a left $S_r \times S_{n-r}$-module. Let $\chi_{r,n-r}(A)$ be its character. It is well known that the irreducible $S_r \times S_{n-r}$-characters are obtained by taking the outer tensor product of S_r and S_{n-r} irreducible characters, respectively. Then, we can write

$$\chi_{r,n-r}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu),$$

where χ_λ (respectively, χ_μ) denotes the irreducible S_r-character (respectively S_{n-r}-character) and $m_{\lambda,\mu} \geqslant 0$ are the corresponding multiplicities.

The relation between the character $\chi_n^H(A)$ and the character $\chi_{r,n-r}(A)$ for any H_m-module algebra A is given by

$$\chi_n^{\mathbb{Z}_2,D_2}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu} \quad \text{and} \quad \chi_{r,n-r}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu)$$

for all $r \leqslant n$. Moreover,

$$c_n^H(A) = \sum_{r=0}^{n} \binom{n}{r} \dim_F P_{r,n-r}(A).$$

Remark that since $[y_1, y_2]^d$ is an H_m-identity of UT_2, then we have the following equality modulo $\mathrm{Id}^{H_m}(UT_2)$

$$y_1^d y_2 - y_2^d y_1 = y_2 y_1^d - y_1 y_2^d. \tag{4.2}$$

Moreover, for every $n \geqslant 0$, a linear basis for the space $P_{n,0}(UT_2)$ is given by the following set of polynomials:

- $y_1 \cdots y_n$,
- $w_S := y_{i_1} \cdots y_{i_{k-1}} y_{i_k}^d y_{i_{k+1}} \cdots y_{i_n}$,

where S denotes the ordered k-tuple $(i_1, \ldots, i_k)$, $i_j \in \{1, \ldots, n\}$ and all the other indexes are ordered. This implies that the space $P_{n,0}(UT_2)$ has dimension $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

A linear basis for the space $P_{n-1,1}(UT_2)$ is given by the following set of polynomials:

- $u_S := y_{i_1} \cdots y_{i_{k-1}} z y_{i_{k+1}} \cdots y_{i_n}$,

where S denotes the ordered k-tuple $(i_1, \ldots, i_k)$, $i_j \in \{1, \ldots, n\}$ and all the other indexes are ordered. Since the number of polynomials u_S is given by $\displaystyle\sum_{k=0}^{n-1} \binom{n-1}{k}$, then the space $P_{n-1,1}(UT_2)$ has dimension 2^{n-1}. The spaces $P_{r,n-r}(UT_2)$ vanishes for $r = 0, 1, \ldots, n-2$. Therefore we obtain the following.

Proposition 4.1.6. *The n-th H_m-codimension of UT_2 is*

$$c_n^{H_m}(UT_2) = \sum_{r=0}^{n} \binom{n}{r} \dim_F P_{r,n-r}(UT_2) = n2^{n-1} + 2^n = (n+2)2^{n-1},$$

and the H_m PI-exponent of UT_2 is

$$\exp^{H_m}(UT_2) = \limsup_{n\to\infty} \sqrt[n]{c_n^{H_m}(UT_2)} = 2.$$

Example 4.1.7. *Let us calculate the space of multilinear polynomials for $n = 2, 3$.*

- *For $n = 2$:*

  $P_{2,0}(UT_2) = \mathrm{Span}_F\{y_1y_2, y_1^dy_2, y_2^dy_1, y_1y_2^d\}$,

  $P_{1,1}(UT_2) = \mathrm{Span}_F\{zy, yz\}$.

  Therefore,

  $P_n^{H_m}(UT_2) = \mathrm{Span}_F\{y_1y_2, y_1^dy_2, y_2^dy_1, y_1y_2^d, y_2z_1, y_1z_2, z_1y_2, z_2y_1\}$.

- *For $n = 3$:*

  $P_{3,0}(UT_2) = \mathrm{Span}_F\{y_1y_2y_3, y_1^dy_2y_3, y_2^dy_1y_3, y_3^dy_1y_2, y_1y_2^dy_3, y_1y_3^dy_2, y_2y_3^dy_1, y_1y_2y_3^d\}$,

  $P_{1,1}(UT_2) = \mathrm{Span}_F\{zy_1y_2, y_1zy_2, y_2zy_1, y_1y_2z\}$.

  Therefore,

  $P_n^{H_m}(UT_2) = \mathrm{Span}_F\{y_1y_2y_3, y_1^dy_2y_3, y_2^dy_1y_3, y_3^dy_1y_2, y_1y_2^dy_3, y_1y_3^dy_2, y_2y_3^dy_1, y_1y_2y_3^d,$
  $z_3y_1y_2, z_2y_1y_3, z_1y_2y_3, y_1z_3y_2, y_1z_2y_3, y_2z_1y_3, y_2z_3y_1, y_3z_2y_1, y_3z_1y_2, y_1y_2z_3, y_1y_3z_2,$
  $y_2y_3z_1\}$.

4.2 H_m-Cocharacters of UT_2

The goal of this section is giving a complete description of the H_m-cocharacter sequence of UT_2, where H_m is an m-th Taft's Hopf algebra.

The following lemma is well known (see, for instance, [30, Theorem 10.4.2]).

Lemma 4.2.1. *Let $\lambda \vdash r$, $\mu \vdash n-r$ and let $W_{\lambda,\mu}$ be a left irreducible $S_r \times S_{n-r}$-module. If T_λ is a tableau of λ and T_μ is a tableau of μ, then*

$$W_{\lambda,\mu} \cong F(S_r \times S_{n-r})e_{T_\lambda}e_{T_\mu}.$$

For a partition $\lambda \vdash n$ we denote by $h(\lambda)$ the height of the diagram associated to λ, that is, if $\lambda = (\lambda_1, \dots, \lambda_k)$, then $h(\lambda) = k$.

We can now write the explicit decomposition of the n-th H_m-cocharacter of UT_2 into irreducibles.

Remark 4.2.2. *In the proof of the following theorem, the notation* $y_{i_1}\cdots y_{i_{j-1}}y_{i_j}dy_{i_{j+1}}\cdots y_{i_m}$ *means* $y_{i_1}\cdots y_{i_{j-1}}(y_{i_j})^d y_{i_{j+1}}\cdots y_{i_m}$.

Theorem 4.2.3. *Let*

$$\chi_n^{\mathbb{Z}_2,D_2}(UT_2)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be the n-th H_m-cocharacter of the H-module algebra UT_2. Then

i) $m_{\lambda,\varnothing}=l+1$ *if* $\lambda=(k+l,k)$;

ii) $m_{\lambda,\mu}=q+1$ *if* $\lambda=(p+q,p)$, $\mu=(1)$;

iii) $m_{\lambda,\mu}=0$ *in all other cases.*

Proof. Let $A=UT_2$ and consider the canonical grading $A=A_0\oplus A_1$, where $A_0=\operatorname{span}\{e_{11},e_{22}\}$ and $A_1=\operatorname{span}\{e_{12}\}$. Since $\dim A_0=2$ and $\dim A_1=1$, any H_m-polynomial alternating on three even variables or in two odd variables vanishes on A; it follows that $m_{\lambda,\mu}=0$ if either $h(\lambda)\geqslant 3$ or $h(\mu)\geqslant 2$. By Proposition 4.1.5, $z_1xz_2\in \operatorname{Id}^{H_m}(A)$, then $m_{\lambda,\mu}=0$ whenever $|\mu|\geqslant 2$. So we have two cases left to study, namely $\mu=\varnothing$ or $\mu=(1)$, and (iii) is already proven.

First we consider the case $\mu=\varnothing$. Let $\lambda=(k+l,k)$, with $k\geqslant 0$, $l\geqslant 0$ and $2k+l=n$. For each $i=0,\dots,l$ let us consider the following tableau:

$$T_\lambda^{(i)}=\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline i+1 & i+2 & \cdots & i+k & 1 & 2 & \cdots & i & i+2k+2 & \cdots & n\\ \hline i+k+2 & i+k+3 & \dots & i+2k+1 \\ \cline{1-4}\end{array}.$$

We associate to $T_\lambda^{(i)}$ the H_m-polynomial

$$b_{k,l}^{(i)}(y_1,y_2)=\sum_{\sigma_1,\dots,\sigma_k\in S_2}(-1)^{\sigma_1}\cdots(-1)^{\sigma_k}y_1^i y_{\sigma_1(1)}\cdots y_{\sigma_k(1)}dy_{\sigma_1(2)}\cdots y_{\sigma_k(2)}y_1^{l-i}.$$

We shall prove the $l+1$ H_m-polynomials $b_{k,l}^{(i)}(y_1,y_2)$, $i=0,\dots,l$, are linearly independent over F modulo $\operatorname{Id}^{H_m}(A)$. For the sake of convenience, let us rewrite each polynomial $b_{k,l}^{(i)}(y_1,y_2)$, $i=0,\dots,l$, as

$$b_{k,l}^{(i)}(y_1,y_2)=y_1^i\underbrace{\overline{y_1}\cdots\widehat{y_1}\widetilde{y_1}d}_{k}\underbrace{\overline{y_2}\cdots\widehat{y_2}\widetilde{y_2}}_{k}y_1^{l-i},$$

where $-,\hat{},\sim$ mean alternation on the corresponding elements. Suppose by absurd $\sum_{i=0}^{l}\beta_i b_{k,l}^{(i)}(y_1,y_2)=0 \pmod{\operatorname{Id}^{H_m}(A)}$ and let $t=\max\{i\mid \beta_i\neq 0\}$. Then $\beta_t b_{k,l}^{(t)}(y_1,y_2)+$

$\sum_{i<t}\beta_i b_{k,l}^{(i)}(y_1,y_2)=0 \pmod{\mathrm{Id}^{H_m}(A)}$. If we consider the substitution $y_1=y_1+y_3$, we get

$$\begin{aligned}\beta_t(y_1+y_3)^t\overline{(y_1+y_3)}\cdots\widehat{(y_1+y_3)}\widetilde{(y_1+y_3)}d\bar{y}_2\cdots\widehat{y}_2\widetilde{y}_2(y_1+y_3)^{l-t}\\ +\sum_{i<t}\beta_i(y_1+y_3)^i\overline{(y_1+y_3)}\cdots\widehat{(y_1+y_3)}\widetilde{(y_1+y_3)}d\bar{y}_2\cdots\widehat{y}_2\widetilde{y}_2(y_1+y_3)^{l-i}\\ =0(\mathrm{mod}\ \ \mathrm{Id}^{H_m}(A)).\quad(4.3)\end{aligned}$$

Let us consider the homogeneous component of degree $t+k$ in y_1 and of degree $l-t$ in y_3. Considering the substitution $y_1=e_{11}$ and $y_2=y_3=e_{22}$, then, by Equation (4.1) we get $y_1^d=-ae_{12}$ and we obtain $(-\beta_t a)e_{12}=0$, which implies $\beta_t=0$, a contradiction. Hence the H_m-polynomials $b_{k,l}^{(i)}(y_1,y_2)$, $i=0,\dots,l$, are linearly independent $(\mathrm{mod}\ \mathrm{Id}^{H_m}(A))$.

Notice that, for all i, $e_{T_\lambda^{(i)}}(y_1,\dots,y_n)$ is the complete linearization of the H_m-polynomial $b_{k,l}^{(i)}(y_1,y_2)$. It follows that the H_m-polynomials $e_{T_\lambda^{(i)}}$, $i=0,\dots,l$, are linearly independent $(\mathrm{mod}\ \mathrm{Id}^{H_m}(A))$ and this implies that $m_{\lambda,\mu}\geqslant l+1$.

We want to prove the multiplicities are exactly $l+1$. For, let T_λ be any tableau and $e_{T_\lambda}(y_1,\dots,y_n)$ the corresponding H_m-polynomial. If $e_{T_\lambda}\notin \mathrm{Id}^{H_m}(A)$, then any two alternating variables in e_{T_λ} must lie on different sides of the elements of type y_i^d. Since e_{T_λ} is a linear combination $(\mathrm{mod}\ \mathrm{Id}^{H_m}(A))$ of H_m-polynomials, each alternating on k pairs of y_i's, we get e_{T_λ} is a linear combination of the H_m-polynomials $e_{T_\lambda^{(i)}}$, $i=0,\dots,l$. Hence $m_{\lambda,\mu}=l+1$ and this proves item (i) of the sentence.

We only need to study the case $\mu=(1)$. Let $\lambda=(p+q,p)$, with $p\geqslant 0$, $q\geqslant 0$ and $2p+q=n-1$. This case can be proved following word by word the last part of the proof of Theorem 3 of [61], where the H_m-polynomials

$$a_{p,q}^{(i)}(y_1,y_2,z)=\sum_{\sigma_1,\dots,\sigma_p\in S_2}(-1)^{\sigma_1}\cdots(-1)^{\sigma_p}y_1^i y_{\sigma_1(1)}\cdots y_{\sigma_p(1)}zy_{\sigma_1(2)}\cdots y_{\sigma_p(2)}y_1^{q-i},$$

with $i=0,1,\dots,q$, are the highest weight vectors corresponding to λ. As above,

$$a_{p,q}^{(i)}(y_1,y_2,z)=y_1^i\,\underbrace{\overline{y_1}\cdots\widetilde{y_1}}_{p}\,z\,\underbrace{\overline{y_2}\cdots\widetilde{y_2}}_{p}\,y_1^{q-i}.$$

This proves (ii) and the proof is complete. □

Recall that in characteristic zero, any result on multilinear polynomial identities obtained in the language of representations of the symmetric group is equivalent to a corresponding result on homogeneous polynomial identities obtained in the language of representations of the general linear group (Theorem 1.4.36).

Notice that the H_m-polynomial $b_{k,l}^{(i)}$ is obtained from the essential idempotent corresponding to the tableau $T_\lambda^{(i)}$ by identifying all the elements in each row of λ. Therefore, the H_m-polynomial $b_{k,l}^{(i)}$ is a highest weight vector, according to the representation theory of GL_n (see Section 1.4.4). We recall that the complete linearization of a highest weight vector associated to an irreducible GL_n-module generates an irreducible S_n-module.

Corollary 4.2.4. *The highest weight vectors whose characters appear with non-zero multiplicity in the decomposition of $\chi_n^{\mathbb{Z}_2,D_2}(UT_2)$ are linear combinations of H_m-polynomials of the form:*

1.

$$b_{k,l}^{(i)}(y_1, y_2) = y_1^i \underbrace{\overline{y_1} \cdots \widehat{y_1}\widetilde{y_1}}_{k}{}^{d} \underbrace{\overline{y_2} \cdots \widehat{y_2}\widetilde{y_2}}_{k} y_1^{l-i}, \qquad i = 0, 1, \ldots, l,$$

where $2k + l = n$; and

2.

$$a_{p,q}^{(i)}(y_1, y_2, z) = y_1^i \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{p}\, z \underbrace{\overline{y_2} \ldots \widetilde{y_2}}_{p} y_1^{q-i}, \qquad i = 0, 1, \ldots, q,$$

where $2p + q + 1 = n$.

If $\chi_n^{\mathbb{Z}_2,D_2}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} \chi_{\lambda,\mu}$ is the decomposition of the $\mathbb{Z}_2 \wr S_n$-character of A, then one defines the n-th $\mathbb{Z}_2 \wr S_n$-*colength* of A as

$$l_n^{\mathbb{Z}_2,D_2}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}.$$

By Theorem 4.2.3 we immediately get the following.

Corollary 4.2.5. *For all $n \geqslant 1$,*

$$l_n^{\mathbb{Z}_2,D_2}(UT_2) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} = \frac{n^2 + 3n + 2}{2}.$$

4.3 Specht property for the H_m-module algebra UT_2

In this section we prove that the variety of H_m-module algebras generated by the H_m-module algebra UT_2 has the Specht property. We recall the definition of Specht property in the language of T^H-ideals of H-module algebras (compare with Definition 1.1.33).

Definition 4.3.1. *Let W be an H-module algebra. We say that $\mathrm{Id}^H(W)$ has the* ***Specht property*** *if any T^H-ideal I such that $I \supseteq \mathrm{Id}^H(W)$, has a finite basis, that is, I is finitely generated as a T^H-ideal. We say that the variety $\mathcal{V}$ has the Specht property if the corresponding T^H-ideal has the Specht property.*

We recall that a binary relation $\leqslant$ on a set A is a **quasi-order** if $\leqslant$ is reflexive and transitive, i.e., (i) $a \leqslant a$ for all $a \in A$, and (ii) $a \leqslant b$ and $b \leqslant c$ imply $a \leqslant c$, with $a, b, c \in A$. If B is a subset of a quasi-ordered set A, the **closure** of B, written $\overline{B}$, is defined as

$$\overline{B} = \{a \in A \mid \text{ exists } b \in B \text{ such that } b \leqslant a\}.$$

We say that the quasi-ordered set A has the **finite basis property** (**f.b.p.**) if for any subset B of A, there exists a finite subset B_0 of A such that $B_0 \subseteq B \subseteq \overline{B_0}$. Every well-ordered set has **f.b.p.** (because every non-empty subset is the closure of a single element). In particular, the set $\mathbb{N}$ of natural numbers with standard ordering has **f.b.p.**. However, $\mathbb{Z}$ the set of integers has not **f.b.p.**. The following theorem gives a equivalent definition for **f.b.p.**.

Theorem 4.3.2. *[33, theorem 2.1] The following conditions on a quasi-ordered set A are equivalent.*

1. *If B is any subset of A, there is a finite set B_0 such that $B_0 \subseteq B \subseteq \overline{B_0}$;*
2. *There exists neither an infinite strictly descending sequence in A nor an infinite one of mutually incomparable elements of A.*

Let $A_1, A_2, \ldots, A_n$ be quasi-ordered sets. The cartesian product $A_1 \times A_2 \times \cdots \times A_n$ ordened by $(a_1, a_2, \ldots, a_n) \leqslant (b_1, b_2, \ldots, b_n)$ if and only if $a_i \leqslant b_i$ for all $i \in \{1, 2, \ldots, n\}$ is a quasi-ordered set.

The following theorems will be useful in the sequel.

Theorem 4.3.3. *[33, Theorem 2.3] Let $A_1, A_2, \ldots, A_n$ be quasi-ordered sets satisfying* **f.b.p.***, so their cartesian product satisfies* **f.b.p.**.

Theorem 4.3.4. *Let $A_1, A_2, \ldots, A_n$ be quasi-ordered sets satisfying* **f.b.p.***, so the disjoint union $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ endowed with the quasi-order $a \leqslant b$ if and only if $a, b \in A_i$ and $a \leqslant_{A_i} b$ for some $i \in \{1, \ldots, n\}$ satisfies* **f.b.p.**.

The free H-module algebra $F^H\langle X\rangle$ is a quasi-ordered set if we define for $f, g \in F^H\langle X\rangle$,

$$f \leqslant g \text{ if and only if } g \in \langle f\rangle_{T^H},$$

where $\langle f\rangle_{T^H}$ denotes the T^H-ideal generated by f.

If I is a T^H-ideal of $F^H\langle X\rangle$, the quasi-order on $F^H\langle X\rangle$ is inherited by $\dfrac{F^H\langle X\rangle}{I}$.

Remark 4.3.5. *Let M be a subset of $F^H\langle X\rangle$. Then $\overline{M} \subseteq \langle M\rangle_{T^H}$ by definition. On the other hand, since $M \subseteq \overline{M}$ we have that $\langle \overline{M}\rangle_{T^H} = \langle M\rangle_{T^H}$*

Let A be an H-module algebra such that $\mathrm{Id}^H(A)$ is finitely generated. A strategy to give a positive solution to the Specht problem for $\mathrm{Id}^H(A)$ is:

S1) Find a set of polynomials $M \subseteq F^H\langle X\rangle/\mathrm{Id}^H(A)$, not necessarily finite, such that for every T^H-ideal I of $F^H\langle X\rangle/\mathrm{Id}^H(A)$,

$$I = \langle M'\rangle_{T^H} \text{ for some } M' \subseteq M.$$

S2) Show that $(M, \leqslant)$ satisfies **f.b.p.** where $\leqslant$ is the quasi-order $f \leqslant g$ if and only if g is a consequence of f in $F^H\langle X\rangle/\mathrm{Id}^H(A)$.

Then, in the light of Lemma 1.4.34, a natural set M satisties step (S1) is the set of highest weight vectors generating irreducible modules whose characters appear with non-zero multiplicity in the decomposition of the cocharacter of the H-module algebra A. The step (S2) is to show that these highest weight vectors satisfy **f.b.p.** with the quasi-order inherited by $F^H\langle X\rangle/\mathrm{Id}^H(A)$. In this way the Specht property is proved because if I is a T^H-ideal of $F^H\langle X\rangle/\mathrm{Id}^H(A)$, then by (S1) there exists $M' \subseteq M$ such that $I = \langle M'\rangle_{T^H}$ and by (S2) there exists a finite set $M_0 \subset M'$ such that $M_0 \subset M' \subset \bar{M}_0$. Thus by Remark 4.3.5,

$$I = \langle M'\rangle_{T^{H_m}} = \langle\overline{M_0}\rangle_{T^{H_m}} = \langle M_0\rangle_{T^{H_m}}.$$

This strategy has been used to prove the Specht property in different algebras environments (see for instance [26, 48, 56]). Problems arise in the strategy when the multiplicities of the irreducible characters are greater than 1, and according to the Theorem 4.2.3, the H_m-module algebra UT_2 has multiplicities greater than 1. So we will approach it in a different way.

We shall consider $\dfrac{F^{H_m}\langle X\rangle}{\mathrm{Id}^{H_m}(UT_2)}$ as a quasi-ordered set. Hence, if $f, g \in F^{H_m}\langle X\rangle$, we define

$$f \leqslant g \text{ if and only if } g \in \langle\{f\} \cup \mathrm{Id}^{H_m}(UT_2)\rangle_{T^{H_m}}.$$

In this case we say that g is a consequence of f modulo $\mathrm{Id}^{H_m}(UT_2)$ or simply that g is a consequence of f.

Let M be the set of all the highest weight vectors corresponding to the cocharacters appearing with non-zero multiplicities in $\chi_n^{\mathbb{Z}_2,D_2}(UT_2)$. By Corollary 4.2.4, the highest weight vectors lying in M are a linear combination of H_m-polynomials of the form:

(1) $y_1^i \underbrace{\overline{y_1}\cdots\widetilde{y}_1}_{p}\, z\, \underbrace{\overline{y_2}\dots\widetilde{y}_2}_{p}\, y_1^{q-i}$,

(2) $y_1^i \underbrace{\overline{y_1}\cdots\widehat{y}_1\widetilde{y}_1{}^{d}}_{k}\underbrace{\overline{y_2}\cdots\widehat{y}_2\widetilde{y}_2}_{k}\, y_1^{l-i}$.

Let us denote by $\mathcal{B}_1$ the set of H_m-polynomials of the form (1) and $\mathcal{B}_2$ the set of H_m-polynomials of the form (2). For $i = 1, 2$, we define the quasi-order $\leqslant$ in $\mathcal{B}_i$ by $f \leqslant g$ if and only if g is a consequence of f, where $f, g \in \mathcal{B}_i$. We consider the following sets which are in one-to-one correspondence with the highest weight vectors of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively:

$$\begin{aligned} B_1 &= \{(i, q-i, p) \mid 0 \leqslant i \leqslant q\} \\ &= \{(i,j,p)\} = \mathbb{N}^3; \\ B_2 &= \{(i, l-i, k) \mid 0 \leqslant i \leqslant l\} \\ &= \{(i,j,k)\} = \mathbb{N}^3. \end{aligned}$$

By Theorem 4.3.3, B_1 and B_2 have **f.b.p.** with the natural quasi-order of $\mathbb{N}^3$. We shall show that the quasi-order $\leqslant$ in B_1 and B_2 induces the quasi-order $\leqslant$ in $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively.

Lemma 4.3.6. *We have*

1. $y_1^i \underbrace{\overline{y_1}\cdots\widetilde{y_1}}_{p}\, z\, \underbrace{\overline{y_2}\dots\widetilde{y_2}}_{p}\, y_1^j \leqslant y_1^{i'} \underbrace{\overline{y_1}\cdots\widetilde{y_1}}_{p'}\, z\, \underbrace{\overline{y_2}\dots\widetilde{y_2}}_{p'}\, y_1^{j'}$ *where* $(i,j,p) \leqslant (i',j',p')$*;*

2. $y_1^i \underbrace{\overline{y_1}\cdots\widehat{y_1}\widetilde{y_1}^{\,d}}_{k}\, \underbrace{\overline{y_2}\cdots\widehat{y_2}\widetilde{y_2}}_{k}\, y_1^j \leqslant y_1^{i'} \underbrace{\overline{y_1}\cdots\widehat{y_1}\widetilde{y_1}^{\,d}}_{k'}\, \underbrace{\overline{y_2}\cdots\widehat{y_2}\widetilde{y_2}}_{k'}\, y_1^{j'}$ *where* $(i,j,k) \leqslant (i',j',k')$*.*

Proof. By transitivity of the quasi-order, in order to prove (1) we prove that

(i) $(i,j,p) \leqslant (i',j,p)$ implies $a_{i,j,p} \leqslant a_{i',j,p}$;

(ii) $(i,j,p) \leqslant (i,j',p)$ implies $a_{i,j,p} \leqslant a_{i,j',p}$;

(iii) $(i,j,p) \leqslant (i,j,p')$ implies $a_{i,j,p} \leqslant a_{i,j,p'}$,

where i, i', j, j', p, p' are integers and

$$a_{i,j,p} = a_{i,j,p}(y_1, y_2, z) = y_1^i \underbrace{\overline{y_1}\cdots\widetilde{y_1}}_{p}\, z\, \underbrace{\overline{y_2}\dots\widetilde{y_2}}_{p}\, y_1^j. \tag{4.4}$$

The statements (i) and (ii) follow from the fact that $a_{i,j,k} \equiv y_1^{i'-i} a_{i,j,k} \mod (\mathrm{Id}^H(UT_2))$ and $a_{i,j,k} \equiv a_{i,j,k} y_1^{j'-j} \mod (\mathrm{Id}^H(UT_2))$ respectively. In order to prove the statement (iii), without loss of generality, we may suppose $p' = p+1$. The general statement will follows by a standard induction argument.

Notice that $a_{i,j,p}$ is a linear combination (mod $(\mathrm{Id}^H(UT_2))$) of the polynomials:

$$y_1^{i+t} y_2^{p-t} z y_1^{p-t} y_2^t y_1^j, \qquad t = 0, 1, \dots, p.$$

Thus, if we multiply by appropriate variables y's to the right or to the left of these polynomials, we obtain that for all $t = 0, 1, \ldots, p$,

$$y_1^{i+t} y_2^{p-t} z y_1^{p-t} y_2^t y_1^j \leqslant y_1^{i+t} y_2^{(p+1)-t} z y_1^{(p+1)-t} y_2^t y_1^j,$$

and therefore $a_{i,j,p} \leqslant a_{i,j,p'}$.

The proof of (2) is analogous. $\square$

Lemma 4.3.7. *The sets $\mathcal{B}_1$ and $\mathcal{B}_2$ with the quasi-order given above satisfy the* **f.b.p.**.

Proof. Let $\mathcal{B}_1'$ be a subset of $\mathcal{B}_1$ and B_1' a subset of $B_1 = \{(i, q-i, p) \mid a_{p,q}^{(i)} \in \mathcal{B}_1\}$ corresponding to $\mathcal{B}_1'$, i.e., $B_1' = \{(i, q-i, p) \mid a_{p,q}^{(i)} \in \mathcal{B}_1'\}$. Since $B_1' \subseteq B_1 \subseteq \mathbb{N}^3$, and, by Theorem 4.3.3, $\mathbb{N}^3$ has **f.b.p.**, we have there is a finite set $B_1^0 \subseteq B_1'$ such that $B_1^0 \subseteq B_1' \subseteq \overline{B_1^0}$. Consider $\mathcal{B}_1^0 = \{a_{p,q}^{(i)} \mid (i, q-i, p) \in B_1^0\} \subseteq \mathcal{B}_1'$ and $a_{p,q}^{(i)} \in \mathcal{B}_1'$. This implies $(i, q-i, p) \in B_1' \subseteq \overline{B_1^0}$; therefore there is $(i_0, q_0 - i_0, p_0) \in B_1^0$, where $(i_0, q_0 - i_0, p_0) \leqslant (i, q-i, p)$. By the previous lemma, $a_{p_0,q_0}^{(i_0)} \leqslant a_{p,q}^{(i)}$, where $a_{p_0,q_0}^{(i_0)} \in \mathcal{B}_1^0$. Thus $a_{p,q}^{(i)} \in \overline{\mathcal{B}_1^0}$ and consequently $\mathcal{B}_1^0 \subseteq \mathcal{B}_1' \subseteq \overline{\mathcal{B}_1^0}$, where $\mathcal{B}_1^0$ is a finite set. This shows $(\mathcal{B}_1, \leqslant)$ satisfies **f.b.p.**.

The proof for the set $(\mathcal{B}_2, \leqslant)$ is analogous and we are done. $\square$

We already have the key ingredients to prove the main result of this chapter. We want to highlight we are going to use the algorithm described in full details in the paper [15].

Theorem 4.3.8. *$var^{H_m}(UT_2)$ has the Specht property.*

Proof. If $I = \mathrm{Id}^{H_m}(UT_2)$, then Theorem 4.1.5 ensures us that I is finitely generated. So let us suppose $I \supsetneq \mathrm{Id}^{H_m}(UT_2)$. Let M be the set of highest weight vectors corresponding to cocharacters appearing with non-zero multiplicities in $\chi_n^{\mathbb{Z}_2, D_2}(UT_2)$, $n \geqslant 0$; hence, $F^{H_m}\langle X\rangle$ is generated by M modulo $\mathrm{Id}^{H_m}(UT_2)$. Since $F^{H_m}\langle X\rangle \supseteq I \supsetneq \mathrm{Id}^{H_m}(UT_2)$, there exists $M' \subseteq M$ such that I is generated by M' modulo $\mathrm{Id}^{H_m}(UT_2)$. We will show that $(M, \leqslant)$ satisfies **f.b.p.**, where $\leqslant$ is the quasi-order given by the consequence, i.e., $f \leqslant g$ if and only if g is a consequence of f in $F^{H_m}\langle X\rangle / \mathrm{Id}^{H_m}(UT_2)$.

A highest weight vector of degree n in M is a linear combination of H_m-polynomials of the form $a_{p,q}^{(i)}$, $i = 0, \ldots, q$ and p, q fixed such that $2p + q + 1 = n$, or H_m-polynomials of the form $b_{k,l}^{(i)}$, $i = 0, \ldots, l$ and k, l fixed such that $2k + l = n$ because they correspond to different modules. Thus $M = \mathcal{S}_1 \sqcup \mathcal{S}_2$, where $\mathcal{S}_1$ is the set of highest weight vectors associated to $\mathcal{B}_1$ and $\mathcal{S}_2$ is the set of highest weight vectors associated to $\mathcal{B}_2$. Then, by Theorem 4.3.4, it suffices to show that the sets $\mathcal{S}_i$ satisfy **f.b.p.**, where $f \leqslant g$ if and only if g is a consequence of f, where $f, g \in \mathcal{S}_i$ for $i = 1, 2$.

Consider the set $\mathcal{S}_1$. A highest weight vector of degree n in $\mathcal{S}_1$ is of the form $\sum_{i=0}^{q} \alpha_i a_{p,q}^{(i)}$. Define the leading term of this highest weight vector as the element $a_{p,q}^{(i_0)}$, where

$i_0 = \min\{i \mid \alpha_i \neq 0\}$. Notice that $\mathcal{B}_1$ can be seen as the set of all the leading terms of the set $\mathcal{S}$ and, by Lemma 4.3.7, $(\mathcal{B}_1, \leqslant)$ satisfies **f.b.p.**. Hence, $\mathcal{B}_1$ has a finite subset $\mathcal{B}_1^0$ such that every element in $\mathcal{B}_1$ is bigger than some element of $\mathcal{B}_1^0$. Let $\mathcal{S}_1^0 \subseteq \mathcal{S}_1$ be the finite subset with leading terms in $\mathcal{B}_1^0$.

Let

$$h_1 = \sum_{i=0}^{q} \alpha_i a_{p,q}^{(i)} \in \mathcal{S}_1^0 \quad \text{and} \quad h_2 = \sum_{j=0}^{q'} \beta_j a_{p',q'}^{(j)} \in \mathcal{S}_1$$

be two highest weight vectors with leading terms $a_{p,q}^{(i_0)}, a_{p',q'}^{(j_0)}$ respectively, and such that $a_{p,q}^{(i_0)} \leqslant a_{p',q'}^{(j_0)}$. Then,

$$a_{p',q'}^{(j_0)} \equiv y_1^{j_0-i_0} \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{p'-p} a_{p,q}^{(i_0)} \underbrace{\overline{y_2} \cdots \widetilde{y_2}}_{p'-p} y_1^{q'-j_0-q+i_0} \quad (\text{mod } \mathrm{Id}^{H_m}(UT_2)).$$

where $-, \sim$ mean alternation on the corresponding elements. At light of this, we consider the highest weight vector

$$h := \sum_{i=0}^{q} \alpha_i y_1^{j_0-i_0} \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{p'-p} a_{p,q}^{(i)} \underbrace{\overline{y_2} \cdots \widetilde{y_2}}_{p'-p} y_1^{q'-j_0-q+i_0},$$

which is a consequence of h_1 and its leading term is exactly $a_{p',q'}^{(j_0)}$. Therefore the leading term of

$$h_2 - \frac{\beta_{j_0}}{\alpha_{i_0}} h,$$

is smaller than the leading term of h_2 and by inductive arguments is a consequence of $\mathcal{S}_1^0$. This shows that $\mathcal{S}_1$ satisfies **f.b.p.**.

Similarly, $\mathcal{S}_2$ satisfies **f.b.p.** too.

Finally, since I is generated by M' modulo $\mathrm{Id}^{H_m}(UT_2)$ and $(M, \leqslant)$ satisfies **f.b.p.**, then there exists a finite set $M_0 \subseteq M' \subseteq M$ such that $M_0 \subseteq M' \subseteq \overline{M_0}$. By Remark 4.3.5,

$$I = \langle M' \rangle_{T^{H_m}} = \langle \overline{M_0} \rangle_{T^{H_m}} = \langle M_0 \rangle_{T^{H_m}},$$

that is, I is finitely generated as a T^{H_m}-ideal. $\square$

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Bibliography

[1] Aljadeff, E., Giambruno, A., and Karasik, Y. Polynomial identities with involution, superinvolutions and the Grassmann envelope. *Proc. Amer. Math. Soc. 145*, 5 (2017), 1843–1857.

[2] Aljadeff, E., and Kanel-Belov, A. Representability and Specht problem for **G**-graded algebras. *Adv. Math. 225* (2010), 2391–2428.

[3] Aljadeff, E., and Kanel-Belov, A. Hilbert series of PI relatively free **G**-graded algebras are rational functions. *Bull. Lond. Math. Soc. 44*, 3 (12 2012), 520–532.

[4] Aljadeff, E., Kanel-Belov, A., and Karasik, Y. Kemer's theorem for affine PI algebras over a field of characteristic zero. *J. Pure Appl. Algebra 220*, 8 (2016), 2771–2808.

[5] Anick, D. J. Non-commutative graded algebras and their Hilbert series. *J. Algebra 78*, 1 (1982), 120 – 140.

[6] Atiyah, M. F., and Macdonald, I. G. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., 1969.

[7] Bahturin, Y., Tvalavadze, M., and Tvalavadze, T. Group gradings on superinvolution simple superalgebras. *Linear Algebra Appl. 431*, 5-7 (2009), 1054–1069.

[8] Bell, J. The Hilbert series of prime PI rings. *Israel J. Math. 139* (2004), 1–10.

[9] Belov, A. Y. Rationality of Hilbert series with respect to free algebras. *Russian Math. Surveys 52* (1997), 394–395.

[10] Belov, A. Y., Borisenko, V. V., and Latychev, V. N. Monomial algebras. *J. Math. Sci. 87*, 3 (1997), 3463–3575.

[11] Berele, A. Homogeneous polynomial identities. *Israel J. Math. 42* (1982), 258–272.

[12] Berele, A., and Regev, A. Asymptotic behaviour of codimensions of PI algebras satisfying Capelli identities. *Trans. Amer. Math. Soc. 360*, 10 (2008), 5155–5172.

[13] Centrone, L., and Estrada, A. Specht property for the algebra of upper triangular matrices of size two with a Taft's algebra action. *Canad. Math. Bull.* (2022).

[14] Centrone, L., Estrada, A., and Ioppolo, A. On PI-algebras with additional structures: Rationality of Hilbert series and Specht's problem. *J. Algebra 592* (2022), 300–356.

[15] Centrone, L., Martino, F., and Souza, M. S. Specht property for some varieties of Jordan algebras of almost polynomial growth. *J. Algebra 521* (2019), 137–165.

[16] Centrone, L., and Yasumura, F. Actions of Taft's algebras on finite dimensional algebras. *J. Algebra 560* (2020), 725–744.

[17] Dehn, M. Über die Grunddlagen der projektiven Geometrie und algemeine Zahlsysteme. *Math. Annalen 85* (1922), 184–194.

[18] Di Vincenzo, O., Koshlukov, P., and Valenti, A. Gradings on the algebra of upper triangular matrices and their graded identities. *J. Algebra 275*, 2 (2004), 550–566.

[19] Drensky, V. Representations of the symmetric group and varieties of linear algebras. *Algebra i Logika 20* (1981), 282–290. (Russian). Traslation: *Math. USSR Sb* 43 (1981), 85-101.

[20] Drensky, V. Codimensions of T-ideals and Hilbert series of relatively free algebras. *J. Algebra 91* (1984), 1–17.

[21] Drensky, V. On the Hilbert series of relatively free algebras. *Comm. Algebra 12* (1984), 2335–2347.

[22] Drensky, V. *Free Algebras and PI-Algebras.* Graduate Course in Algebra. Springer-Verlag, 2000.

[23] Giambruno, A., Ioppolo, A., and La Mattina, A. Varieties of algebras with superinvolution of almost polynomial growth. *Algebr. Represent. Theory 19*, 3 (2016), 599–611.

[24] Giambruno, A., Ioppolo, A., and La Mattina, D. Superalgebras with involution or superinvolution and almost polynomial growth of the codimensions. *Algebr. Represent. Theory 22*, 4 (2019), 961–976.

[25] Giambruno, A., and Rizzo, C. Differential identities, 2×2 upper triangular matrices and varieties of almost polynomial growth. *J. Pure Appl. Algebra 223*, 4 (2019), 1710–1727.

[26] Giambruno, A., and Souza, M. S. Graded polynomial identities and Specht property of the Lie algebra $\mathfrak{sl}_2$. *J. Algebra 389* (2013), 6–22.

[27] Giambruno, A., and Zaicev, M. On codimension growth of finitely generated associative algebras. *Adv. Math. 140*, 2 (1998), 145–155.

[28] Giambruno, A., and Zaicev, M. Exponential codimension growth of PI algebras: an exact estimate. *Adv. Math. 142*, 2 (1999), 221–243.

[29] Giambruno, A., and Zaicev, M. Codimension growth and minimal superalgebras. *Trans. Amer. Math. Soc. 355* (12 2003), 5091–5117.

[30] Giambruno, A., and Zaicev, M. *Polynomial identities and asymptotic methods*, vol. 122 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005.

[31] Gomez-Ambrosi, C., and Shestakov, I. P. On the Lie structure of the skew-elements of a simple superalgebra with involution. *J. Algebra 208*, 1 (1998), 43–71.

[32] Gordienko, A. S. Algebras simple with respect to a Sweedler's algebra action. *J. Pure Appl. Algebra 219*, 8 (2015), 3279–3291.

[33] Higman, G. Ordering by divisibility in abstract algebras. *Proc. Lond. Math. Soc. 3*, 2 (1952), 326–336.

[34] James, G., and Kerber, A. *The Representation Theory of the Symmetric Group*, vol. 16 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley, 1981.

[35] Kac, V. Lie superalgebras. *Adv. Math. 26*, 1 (1977), 8–96.

[36] Kanel-Belov, A., Karasik, Y., and Rowen, L. H. *Computational aspects of polynomial identities*, 2 ed., vol. 1 of *Monogrpths and research notes in mathematics.* Taylor & Francis Group, 2016.

[37] Kaplansky, I. Rings with a polynomial identity, revisted. *Bull. Amer. Math. Soc. 54* (1948), 496–500.

[38] Karasik, Y. Kemer's theory for H-module algebras with application to the PI exponent. *J. Algebra 457* (2016), 194–227.

[39] Kemer, A. R. Varieties of finite rank. *Proc. 15-th All the Union Algebraic Conf. 2* (1979), 73. (Russian).

[40] Kemer, A. R. Finite basis property of identities of associative algebras. *Algebra i Logika 26*, 5 (1987), 597–641. (Russian).

[41] Kemer, A. R. *Ideals of identities of associative algebras.* No. 87 in AMS Translations of Mathematical Monograph. AMS, Providence, RI, 1988.

[42] Kobayashi, Y. Another graded algebra with a nonrational Hilbert series. *Proc. Amer. Math. Soc. 81*, 1 (1981), 19–22.

[43] Koshlukov, P. Rational Hilbert series of relatively free Jordan algebras. *J. Algebra 121*, 2 (1989), 301–309.

[44] Linchenko, V., Montgomery, S., and Small, L. W. Stable Jacobson radicals and semiprime smash products. *Bull. Lond. Math. Soc. 37*, 6 (2005), 860–872.

[45] Malcev, Y. A basis for the identities of the algebra of upper triangular matrices. *Algebra i Logika 10* (1971), 393–400. (Russian).

[46] Mishchenko, S., and Valenti, A. A star-variety with almost polynomial growth. *J. Algebra 223*, 1 (2000), 66–84.

[47] Montgomery, S. *Hopf algebras and their actions on rings.* No. 82 in Regional Conference Series in Mathematics. American Mathematical Society, 1993.

[48] Morais, P., and Souza, M. The algebra of 2×2 upper triangular matrices as a commutative algebra: Gradings, graded polynomial identities and Specht property. *J. Algebra 593* (2022), 217–234.

[49] Născălescu, S., Năstăsescu, C., and Raianu, Ş. *Hopf algebras: an introduction.* Marcel Dekker, 2001.

[50] Racine, M. L. Primitive superalgebras with superinvolution. *J. Algebra 206*, 2 (1998), 588–614.

[51] Racine, M. L., and Zelmanov, E. I. Simple Jordan superalgebras with semisimple even part. *J. Algebra 270*, 2 (2003), 374–444.

[52] Radford, D. E. *Hopf Algebras.* No. 49 in Series on Knots and Everything. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ., 2012.

[53] Regev, A. Existence of identities in $A \otimes B$. *Israel J. Math. 11* (1972), 131–152.

[54] Sagan, B. E. *The Symmetric Group. Representations, combinatorial algorithms, and symmetric functions.* No. 203 in Graduate text in mathematics. Springer-Verlag, 2001.

[55] Shearer, J. A graded algebra with a non-rational Hilbert series. *J. Algebra 62*, 1 (1980), 228–231.

[56] Silva, D., and Souza, M. Specht property for the 2-graded identities of the Jordan algebra of a bilinear form. *Comm. Algebra 45*, 4 (2017), 1618–1626.

[57] Specht, W. Gesetze in Ringen. *I, Math Z. 52* (1950), 557–589.

[58] Steinberg, B. *Representation Theory of Finite Groups. An introduction approach.* Universitext. Springer, 2012.

[59] Sviridova, I. Identities of PI-algebras graded by a finite abelian group. *Comm. Algebra 39*, 9 (2011), 3462–3490.

[60] Sweedler, M. E. *Hopf Algebras.* Mathematics Lecture Note Series. W. A. Benjamin, New York, 1969.

[61] Valenti, A. The graded identities of upper triangular matrices of size two. *J. Pure Appl. Algebra 172* (2002), 325–335.

[62] Wagner, W. Über die Grundlagen der projektiven Geometrie und algemeine Zahlsysteme. *Math. Annalen 113* (1937), 528–564.