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Regularity of foliations and rigidity for Anosov endomorphisms

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Resumo

Endomorfismos de Anosov são uma versão mais geral de mapas que exibem hiperbolicidade uniforme, contendo a já bem estudada classe dos difeomorfismos de Anosov e muitos outros exemplos. Esses endomorfismos são difeomorfismos locais apresentando taxas uniformes de contração e expansão ao longo de uma sequência de espaços tangente. Neste trabalho, exploramos alguns aspectos desses mapas, focando nossos esforços na compreensão de seu comportamento em superfícies, e na existência e regularidade de uma conjugação com um endomorfismo linear do toro.

Para melhor compreender a regularidade da conjugação, quando ela existe, estudamos as propriedades geométricas e mensuráveis das folheações estáveis e instáveis. A propriedade principal é a de *densidade uniformemente limitada (UBD)*, a versão uniformemente limitada de continuidade absoluta folha-a-folha para folheações, que provamos ser equivalente às holonomias terem jacobianos uniformemente limitados.

Finalmente, apresentamos no 2-toro uma caracterização da conjugação suave de endomorfismos de Anosov especiais com suas linearizações em termos da propriedade UBD, desde que o sistema *quase preserve volume* ao longo das folheações invariantes, condição que substitui a conservatividade para endomorfismos. Primeiro provamos que a condição de regularidade nas folheações e a condição de volume implicam que os expoentes de Lyapunov do endomorfismo são constantes e iguais aos da sua linearização. Então provamos que, se os expoentes de Lyapunov nos pontos periódicos correspondentes de dois endomorfismos de Anosov conjugados são iguais, então a conjugação é tão regular quanto os endomorfismos.

Palavras-chave: dinâmica suave. dinâmica hiperbólica. dinâmica não inversível. rigidez.

Abstract

Anosov endomorphisms are a more general version of maps displaying uniform hyperbolicity, containing the well studied class of Anosov diffeomorphisms and many other examples. These endomorphisms are local diffeomorphisms presenting uniform rates of contraction and expansion along a sequence of tangent spaces. In this work, we explore some aspects of these maps, focusing our efforts in comprehending their behavior on surfaces, and in the existence and regularity of a conjugacy with a linear toral endomorphism.

To better grasp the regularity of the conjugacy, when it does exist, we study geometric and measurable properties of the stable and unstable foliations. The main property is the *uniform bounded density (UBD) property*, the uniform bounded version of leafwise absolute continuity for foliations, which we prove to be equivalent to the holonomies having uniformly bounded Jacobians.

Finally, we give on the 2-torus a characterization of smooth conjugacy of special Anosov endomorphisms with their linearizations in terms of the UBD property, provided that the system *quasi preserves volume* along invariant foliations, a condition that replaces conservativeness for endomorphisms. We first prove that the regularity condition on the foliations and the volume condition imply that the Lyapunov exponents of the map are constant and equal to the ones of its linearization. Then we prove that, if the Lyapunov exponents on corresponding periodic points of two conjugate Anosov endomorphisms are equal, then the conjugacy is as regular as the maps.

Keywords: smooth dynamics. hyperbolic dynamics. non-invertible dynamics. rigidity.

Contents

Introduction

Along this introduction, we present some context for our field and the main concepts we work with, as a way to display how our results are related to the existing research in Dynamical Systems. Then we present the results obtained in this work and the structure of the text.

Uniform hyperbolicity

Nowadays, phenomena related to hyperbolicity are central in some fields of work in Dynamical Systems. To comprehend how the history of the field provided this development, it is key to highlight the role of stability when studying Dynamical Systems. The information on this section is mostly inspired by the Preface of [9].

The field of Dynamical Systems emerged at the end of the 19th century from the works of Henri Poincaré, who was the first to explore the behavior of orbits of a system modeling celestial mechanics from a qualitative point of view. Since this paradigm change, instead of studying systems of equations with analytic tools, many mathematicians started to study topological and measure-theoretic properties of most orbits to describe the general nature of the system.

Natural questions followed, mainly about stability, since it was already a important question for celestial mechanics. The concept of stability was first explored for states on the phase space, that is, given a system, it was desirable to know if a state of the system would resist small perturbations, for instance on the positions and velocities for mechanical systems.

In the 30s, Aleksandr Andronov and Lev Pontryagin [2] introduced another kind of stability, now regarding the stability of the system instead of the stability of its points or orbits. We say that a system is *structurally stable* if its topological behavior persists under small perturbations of the map on some set of functions. Both concepts of stability are vital to applications, since computations always incur errors, and if the phenomenon modeled with small errors is not similar to the original one, the models may not be good for predictions — for instance, as was noticed empirically by Edward Lorenz [35] when he discovered the famous attractor that carries his name.

At the middle of the 20th century, mathematicians were interested in characterize the systems with structural stability, with the belief that they were typical among all systems on a given manifold — mostly because Andronov and Pontryagin proved [2] that most flows on surfaces are structurally stable —, but Stephen Smale proved that it is not the case for higher dimensions or discrete-time dynamics [57].

Dmitri Anosov introduced — through the study of geodesic flows on negatively curved manifolds [3] — a class of examples that are structurally stable, the *uniformly hyperbolic diffeomorphisms* (also known as *Anosov diffeomorphisms*), that was formalized by Smale. Uniformly hyperbolic diffeomorphisms have a splitting of the tangent space of each point into a direction with uniform contraction and another with uniform expansion. Later, in 1970, Jacob Palis and Stephen Smale [47] conjectured that this splitting on the non-wandering set is fundamental, in fact, together with a transversality condition, it is equivalent to structural stability. This conjecture was proved by Ricardo Mañe [37].

Uniform hyperbolicity, however, is too restrictive. Indeed, in dimension 2 and 3 tori are the only manifolds which support Anosov diffeomorphisms [22], and any Anosov diffeomorphism on $\mathbb{T}^n$ is topologically conjugate (has the same topological dynamics) to a linear toral automorphism [39]. Additionally, uniform hyperbolicity is not generic, there are open sets of non-hiperbolic maps for any manifold.

Thenceforth, the field expanded to include weaker forms of hyperbolicity, such as *partial hyperbolicity*, *dominated splitting* and *non-uniform hyperbolicity*. See [9] for a complete view on the theory. Although not structurally stable, the interest in these systems began as they surged as examples with robust properties (meaning that all nearby systems have this property), such as the example given by [36] of a diffeomorphism with robust transitivity that is not uniformly hyperbolic, but is partially hyperbolic.

Endomorphisms

In his thesis [54], Michael Shub widen the work of his advisor, Stephen Smale, to non-invertible differentiable maps on manifolds. He proved the classical result that, on the circle $\mathbb{S}^1$, any C^1 endomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is topologically conjugate to $z \mapsto z^d$, where $d = \deg f$. This result is in fact a particular case of a more general one, for he introduced the concept of *expanding endomorphism* as a C^1 map on a closed manifold such that its derivative acts as a uniform expansion on the tangent bundle. Shub proved that any two homotopic expanding maps on a compact manifold are topologically conjugate, thus providing examples of structurally stable endomorphisms.

Despite his pioneer work about expanding maps on manifolds, Shub generalized in his thesis [54, Theorem 6] some of his results for Anosov endomorphisms (having both directions of expansion and contraction), including the structural stability, but he considered every Anosov endomorphism to have a global unstable bundle, which was proved independently by Feliks Przytycki [49] and by Ricardo Mañé and Charles Pugh [38] to be false.

This error in Michael Shub thesis was propagated in the work of John Franks [22], for instance [22, Corollary 1.3] affirms that unstable leaves on the universal cover are preserved under deck transformations, which is false. And in the remark after [22, Proposition 1.9] it is stated that the stable manifolds are dense, which is also false, and we detail it in Section 1.3. We do not know if [22, Proposition 1.9] is correct and the unstable foliations for Anosov endomorphisms are always dense.

The fact that Anosov endomorphisms that are not invertible or expanding maps are not structurally stable was proven by R. Mañé and C. Pugh [38] and F. Przytycki [49], were they also introduced several properties of these systems.

Rigidity

The term *rigidity* is often use in the theory of dynamical systems, and it is usually associated to the following idea

> *"Values of finitely many invariants determine the system either locally, i.e. in a certain neighborhood of a 'model', or globally within an a priori defined class of systems."* [8]

In [8] the authors work with a complementary concept, the one of *flexibility*, meaning that on a fixed class of systems and under general restrictions, a set of invariants can take arbitrary values. These two paradigms allow to better understand the general behavior of a class of dynamical systems.

In our case, as stated in Theorem E, this is the precise meaning of rigidity: we have that the invariants are the stable and unstable Lyapunov exponents, and they determine that the system is a smooth coordinate change of a linear one, provided it only has one unstable direction for each point. This is the particular case for which we know examples of, but the proof works for a possible more general setting, as we explain better at the end of this introduction.

The aforementioned result of [54] is a rigidity result, since the degree of a C^1 endomorphism of the circle determines its topological conjugacy class. These kind of results were largely studied for one dimensional endomorphisms. For instance, together with Dennis Sullivan, Shub [56] also proved that if f and g are C^r, $r \geqslant 2$, expansions on the circle that are absolutely continuously conjugate, then this conjugacy have the same regularity C^r. Thus, the degree of the endomorphism actually determines its smooth conjugacy class.

For C^r diffeomorphisms on the circle, the invariant to look at is the rotation number α: if the rotation number of f is irrational, then the map is topologically conjugate to a rotation by α [17]; and with a condition bounding the derivatives Df^n, we have that this conjugacy is C^r [31]. These results are part of the *smooth classification problem*, that consists in giving conditions under which two smooth dynamical systems that are topologically equivalent are in fact smooth equivalent, and they are under the umbrella of the rigidity study. To more information on the one dimensional case, see [19, 15].

For higher dimensions, we consider the smooth classification problem for uniformly hyperbolic systems. For Anosov diffeomorphisms on $\mathbb{T}^2$ and Anosov flows on $\mathbb{T}^3$, this problem was completed solved by Rafael de la Llave, José Manuel Marco and Roberto Moriyón in a serie of works [40, 16, 41, 18], where they prove that two Anosov diffeomorphisms are smooth conjugate if and only if the Lyapunov exponents for corresponding periodic points coincide. In $\mathbb{T}^3$, this problem is also addressed by Andrey Gogolev and Misha Guysinsky [24, 23]. De la Llave shows in [17] that this characterization fails to hold on manifolds of dimension greater than 3.

Theorem E then regards this problem for endomorphisms on $\mathbb{T}^2$, and it is more general than a previous result by Fernando Micena [42]. Micena treats as well conditions for smooth conjugacy for Anosov endomorphism on higher dimensional tori, by requiring more hypothesis for it to hold.

When studying dynamical systems, we often are interested simultaneously in different aspects of the systems, whether geometric, measurable or topological. More generally, rigidity can reefer to results in which a condition under one of this aspects implies a very specific behavior of the system under the same aspect or other. In this sense, we also prove for Anosov endomorphisms in $\mathbb{T}^2$ that a kind of measurable regularity of the stable and unstable foliations implies that the Lyapunov exponents are constant.

Regularity of foliations

As we mentioned above, one of the conditions that we require to obtain smooth conjugacy is a kind of measurable regularity of the stable and unstable foliations. More conventionally, this regularity is *absolute continuity*, and we have two approaches to it, one of then using *conditional measures*, and the other using *holonomies*.

For the precise definitions of the concepts here mentioned, see Section 1.2, here we recall the concepts briefly. Fubini's Theorem essentially says that, to integrate in $\mathbb{R}^n$ over a product of balls $I^k \times I^{n-k}$ with respect to the product measure $m_k \times m_{n-k}$, we can integrate first with respect to m_k on each set $I^k \times \{y\}$ for $y \in I^{n-k}$, and then integrate the result with respect to m_{n-k}. We want to generalize it by replacing $\{I^k \times \{y\}\}_y$ with a partition $\mathcal{P}$ of the space, considering any probability measure μ, and then getting a similar decomposition of the integrals. This decomposition is called *disintegration* of μ with respect to $\mathcal{P}$, and each measure μ_P, for $P \in \mathcal{P}$, is called a *conditional measure*.

We always have local conditional measures for continuous foliations, and if they are equivalent to the induced volume, we say that the foliation is *leafwise absolutely continuous*. The other extreme case is when they are atomic measures, in which case we say that the foliations is *atomic*. Even foliations arising from dynamical systems can present this behavior, see [48], for instance. For uniformly hyperbolic systems, however, the foliations are always leafwise absolutely continuous.

A holonomy for a foliation is a map between two transverse discs that takes one point in the first one and, by intersecting the leaf of this point with the second disc, obtain its image. Roughly speaking, a point navigates from one disk to the other along the leaves of the foliation. We say that a foliation is *transverse absolutely continuous* if its holonomies preserve null measure sets on the discs.

These two definitions of absolute continuity are related: the former is weaker than the latter, as we state precisely in Theorem 3. And the former implies the latter for "good transversals", as proved in the unpublished notes [50].

This regularity is not enough to guarantee smooth conjugacy. Even requiring that the foliation is $C^{1+\alpha}$ is not enough, since in $\mathbb{T}^2$ the foliations are $C^{1+\alpha}$, but there are conjugate systems that are not smoothly conjugate. To overcome this we require that, besides from being leafwise absolutely continuous, the conditional measures are uniformly equivalent to the volume on the leaves, which we call *UBD property*, as defined by Fernando Micena and Ali Tahzibi [43].

Theorem D is inspired by the following result.

Theorem 1. *[60] Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a C^∞ conservative partially hyperbolic diffeomorphism with hyperbolic linearization A. The center foliation has the UBD property if and only if f is C^∞ semiconjugate to A.*

Regularity conditions for the holonomies are frequently used in studying ergodic theory for systems with some kind of hyperbolicity. For instance, codimension one foliations for Anosov maps are C^1, and this plays a role in our rigidity results. Another example is given in [1, p. 105], where the condition (Y5) — introduced among other conditions to guarantee the existence of *Young structures* for nonuniformly hyperbolic dynamics— is that the holonomies have Jacobians with integrals uniformly bounded. The uniform bound for the Jacobian is specifically used in [13], together with other conditions, to obtain *Markov partitions* for uniformly hyperbolic invertible dynamical systems with singularities.

This motivates our study of the relation between holonomies with bounded Jacobians and the UBD property, as done in Theorems A and B.

Results and structure of the thesis

During Chapter 1, we present some necessary concepts to this work, such as the definitions and properties of foliations and of uniform hyperbolicity for local diffeomorphisms.

Chapter 2 is dedicated to explore the relation between uniform bounded absolute continuity with respect to the holonomies (transverse) and with respect to the disintegration (leafwise). The results that we prove about this relation are Theorems A and B. In Theorem A, similarly to the proof that transverse absolute continuity implies leafwise absolute continuity, we prove an uniformly bounded version.

Theorem A. *If $\mathcal{F}$ is a transversely absolutely continuous foliation with uniformly bounded Jacobians, then $\mathcal{F}$ has conditional measures along its leaves equivalent to the Lebesgue measure and with uniformly bounded densities.*

The proof of Theorem A is a direct consequence of the definitions. Its reciprocal, however, is more complex, and holds for "well behaved" traversals.

Theorem B. *If a foliation $\mathcal{F}$ has the UBD property, then for every transverse local foliation $\mathcal{T}$ which is transversely absolutely continuous with uniformly bounded Jacobians, the holonomy map $h_{\mathcal{F}} : \mathcal{T}_1 \to \mathcal{T}_2$ between almost every pair of $\mathcal{T}$-leaves is absolutely continuous with bounded Jacobian. The boundedness constant does not depend on T_1 and T_2.*

To prove Theorem B, we construct a family of transversely absolutely continuous functions $\{h_g\}_{g\in\mathbb{R}^k}$ with uniformly bounded Jacobian (where k is the dimension of $\mathcal{F}$) on a foliated box, in such way that every local leaf $\mathcal{T}_1$ of $\mathcal{T}$ is taken to any other local leaf $\mathcal{T}_2$ by a unique element of $\{h_g\}_{g\in\mathbb{R}^k}$, that act as an holonomy through $\mathcal{F}$-leaves. Moreover, restricted to $\mathcal{F}$-leaves, these functions are "translations by g" with uniformly bounded Jacobian. This implies that, for almost every pair of $\mathcal{T}$-leaves, h_g is an $\mathcal{F}$-holonomy and it is absolutely continuous with bounded Jacobian.

The main issue when we adapt the proof of [50, Lemma 2.6] to the uniformly bounded context is in Lemma 1, where we need, in order to complete the argument by absurd, to construct a small measurable "rectangle" where some inequalities hold. Other difficulties arise when defining the domain of each of the functions h_g, as we do it in such a way that they do not distort to much the lengths on $\mathcal{F}$-leaves.

The previous theorems further motivate the definition of foliations having the UBD property, in addition to the comments already made in this introduction. This property is one of the hypothesis on Theorem C, our rigidity theorem inspired by Theorem 1.

Working with preservation of volume for maps that are not invertible is more subtle, since a conservative endomorphism does not have constant Jacobian. To overcome the lack of conservativeness and its consequences, we introduce a technical hypothesis, that may be improved. For an f-invariant foliation, we say that f *quasi preserves densities along the foliation* — and we call it in this text Hypothesis (C) — if:

$$(C) : C^{-1} \leqslant \frac{d\hat{\lambda}^k_{f^k(x)}}{df^k_* \hat{\lambda}^0_x} \leqslant C,$$

for some uniform $C > 1$, where we fix a local leaf $\mathcal{W}$, and $\hat{\lambda}^k_x$ is the normalized volume on $f^k(\mathcal{W})$. Essentially, this hypothesis says that, by iterating with f, the densities of the induced volume are not distorted too much. That means, in the hyperbolic case, that the expansion/contraction seen on the leaf is "well distributed" along the leaf.

Theorem C. *[12] Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^∞ special Anosov endomorphism with quasi preservation of densities along its invariant foliations, and let A be its linearization. The stable and unstable foliations of f are absolutely continuous with uniformly bounded densities if and only if f is C^∞ conjugate to A.*

Theorem C is a consequence of Theorems D and E. The first one gives us that the regularity of foliations implies equality of the Lyapunov exponents.

Theorem D. *[12] Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^∞ Anosov endomorphism and A its linearization. If the stable and unstable foliations of f satisfy the Hypothesis (C) and are leafwise absolutely continuous with uniformly bounded densities, then $\lambda^\sigma_f \equiv \lambda^\sigma_A$ for $\sigma \in \{u, s\}$.*

In the above result, f need not to be special. Besides f does not have a global unstable foliation in general, since each point can have more than one unstable direction, so by "the unstable foliations of f", we are actually referring to a foliation on the universal cover of $\mathbb{T}^2$.

The following theorem is inspired by previous rigidity results for Anosov diffeomorphism on $\mathbb{T}^2$ and $\mathbb{T}^3$ [40, 16, 41, 18, 17, 24], and guarantees that, for two surface Anosov endomorphisms that are conjugate, if the Lyapunov exponents on corresponding periodic points coincide, then the conjugacy is as regular as the endomorphisms.

Theorem E. *[12] Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^k, $k \geqslant 2$, Anosov endomorphisms topologically conjugated by $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to Id. If the corresponding periodic points of f and g have the same Lyapunov exponents, the conjugacy h is C^k. In particular, if f and g are C^∞, h is also C^∞.*

In the case that f is special and $g = f_* = A$ is its linearization, Theorem E was firstly proven by F. Micena [42, Theorem 1.10], and in the case that f and g are not special, he proves a similar result with different hypotheses [42, Theorem 1.7]. He also explores conditions for regularity on higher dimensions with additional hypotheses. Theorem E has counterexamples in higher dimensions [17], in which more hypotheses are required for the invertible case [26, 25].

It remains to give examples of non-special Anosov endomorphisms on $\mathbb{T}^2$ that are topologically conjugate but not smoothly conjugate. If f and g are topologically conjugate, by lifting the conjugacy to the inverse limit space, f and g are inverse-limit conjugate. Then, a necessary condition is that f and g have the same linearization, since by Nobuo Aoki and Koichi Hiraide [4, Theorem 6.8.1] f and its linearization are inverse-limit conjugate. Additionally, the conjugacy must necessarily be a homeomorphism between $W^u_f(\tilde{x})$ and $W^u_g(\tilde{h}(\tilde{x}))$ for each of the unstable directions. Micena and Tahzibi [44] proved that, if f is not special, there is a residual subset $\mathcal{R} \in \mathbb{T}^2$ such that every $x \in \mathcal{R}$ has infinitely many unstable directions. This suggests the complexity of this problem.

Question 1. Under which conditions do we have a topological conjugacy between two non-special Anosov endomorphisms on $\mathbb{T}^n$ with the same linearization?

Remarks on the results and previous attempts

The results of this work are fruit of a constant (if somewhat slow) process of better understating the nature of Anosov endomorphisms on tori, either topologically, geometrically or measure theoretically.

The first problem that took most of our time was to understand the topological (semi)conjugacy between an Anosov endomorphism and a nearby perturbation, in particular when this perturbation is its linearization. Our first impression was that an Anosov endomorphism with infinitely many unstable manifolds for a given point could be semi conjugate to a linear one, with the semiconjugacy “collapsing” all unstable manifolds of the same point into only one for the linear model, based on the possibility that the conjugacy present on the universal cover or the natural extension could be projected when its inverse could not. We prove in Proposition 11 that it cannot occur.

When attempting to understand the nature of the possible conjugacies between two nearby Anosov endomorphisms, we tried to construct it with the shadowing property, as presented in Section 1.3.1 and similar to the proof for diffeomorphisms, but it turned out that is gives origin precisely to the conjugacy of their natural extensions, as already stated by [49].

We then proceeded to consider only the case on which a conjugacy does in fact exist. The natural hypothesis is suppose that the Anosov endomorphism f is special (only has one unstable direction for each point). We attempted to encompass the case in which f and g are not special but are topologically conjugate with h homotopic to identity, which was successful in Theorem E, although we do not know of examples that make the theorem not vacuous for this case, that is, examples that are conjugated by a homeomorphism that is not C^1.

Additionally, our first proof of Theorem D contained an error, as we did not required any preservation of volume and attempt to prove that f somehow preserved normalized induced volume on the leaves, which is true only for large segments inside each leaf, due to the quasi-isometry and the conjugacy with a linear foliation. To overcome this, we added a hypothesis on the foliations that we call *quasi preservation of densities*. We hope to refine this hypothesis in the nearby future.

At the time that we did not know of this error, our natural continuation of this work was to either explore the construction of examples, or to motivate the UBD property present on our hypotheses. Our first efforts was on this second topic, which gave origin to the Chapter 2.

Therefore, there is still much to comprehend and explore on the topics of this work, including the study of rigidity for Anosov endomorphisms on higher dimensional manifolds. We remark that, along the proofs of Theorems D and E, we make explicit use of the fact that the stable and unstable foliations are both one dimensional, thus the generalization to higher dimensions would require several different arguments, as well as additional hypotheses for dimension greater than three.

1 Preliminary concepts

This chapter is dedicated to a brief presentation of some concepts relevant to our results. Our main goal is to provide references to facilitate the reading of this text, with the aim of making it as self-contained as possible. We omit most of the proofs of the results here stated and provide references where they may be found.

During the first section, we present the general theory of dynamical systems, focusing on concepts that will be particularly relevant along the text. The second section contains an introduction to foliations, as well as some of their geometric and measurable properties. The last section presents the theory of uniform hyperbolic dynamical systems, including non-invertible ones.

1.1 General dynamical systems

Here, we briefly review the fundamental concepts in the general theory of Dynamical Systems and Ergodic Theory, introducing concepts such as topological dynamical system, measurable dynamical system, invariant sets, ergodicity, entropy, and so on. We also state some properties. For a thorough introduction, see [34, 52, 61] (there is also [46], the original Brazilian Portuguese version of [61]).

A *dynamical system* is a space X with some structure (metric space, measure space or Riemannian manifold, for instance) and an action f of a semigroup G on this space that preserves the structure (f is continuous, measurable or differentiable, for instance). Most commonly, the semigroup G is actually a group: $G = \mathbb{Z}$ or $G = \mathbb{R}$.

If $f : X \to X$ is a continuous map, its compositions define an $\mathbb{N}$-action, and in particular, if f is an invertible map, we can compose f^{-1} and we have a $\mathbb{Z}$-action. Reciprocally, a semigroup (group) action defines a (invertible) map $f(1, \cdot)$. So we say that an action is *invertible* if it is a group action.

We say that f is a *discrete time dynamical system* if $G = \mathbb{N}$ or $G = \mathbb{Z}$, and $f : G \times X \to X$ is denoted by $f^n : X \to X$ for each $n \in G$. In this case $f^0 = \mathrm{id}$ and $f^n \circ f^m = f^{n+m}$ for all $m, n \in G$.

Similarly, we say that ϕ is a *continuous time dynamical system* if $G = \mathbb{R}_+$ or $G = \mathbb{R}$, and $\phi : G \times X \to X$ is denoted by $\phi_t : X \to X$ for each $t \in G$. In this case $\phi_0 = \mathrm{id}$ and $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in G$.

In these two cases, the action of the semigroup is often intuitively related with the passage of time, since f^n or ϕ_t can represent the state of a physical system after n

observations or after t units of time have passed. Thus, a dynamical system is a model of some phenomenon over time, given a general law of movement or behavior.

A *topological dynamical system* is a continuous action on a topological space (often a metric space). A *measurable dynamical system* is a measurable action on a measure space (often a probability space that also has a topological structure). Henceforth, we suppose that X is a metric space and we deal exclusively with the discrete time case, with f being a continuous action of $\mathbb{Z}$ in the invertible case and an action of $\mathbb{N}$ in the non-invertible setting. Most of the definitions and properties for discrete time dynamical systems are applicable to continuous ones.

The general approach of the theory of dynamical systems is to understand the qualitative behavior of a system, that is, we do not want to compute the precise valor of f^n on some point, but to describe its behavior as n goes to infinity. The following sets are some of the tools used to grasp this general behavior:

- *Fixed points of* f: $\mathrm{Fix}(f) = \{p \in X : f(p) = p\}$;
- *Periodic points of* f: $\mathrm{Per}(f) = \{p \in M : \exists\ k \in \mathbb{N}$ such that $f^k(p) = p\}$, and the smallest such k is the *period* of p;
- ω*-limit of* $p \in X$: $\omega_f(p) = \{y \in X : \exists\ n_k \xrightarrow{k\to\infty} \infty$ such that $f^{n_k}(p) \xrightarrow{k\to\infty} y\}$;
- α*-limit of* $p \in X$ (if f is invertible):
$$\alpha_f(p) = \{y \in X : \exists\ n_k \xrightarrow{k\to\infty} -\infty \text{ such that } f^{n_k}(p) \xrightarrow{k\to\infty} y\};$$
- *Non-wandering set of* f: $p \in X$ is *non-wandering* if, given a neighborhood U of p, there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. Otherwise, the point is said to be a *wandering point*. We denote by $\Omega(f)$ the set of non-wandering points of f.

These sets, besides being relevant to the understanding of the system, also have the property of being somewhat preserved by the evolution of the system. To be precise, we introduce the concept of invariance.

Definition 1. We say that a set $A \subseteq X$ is f*-invariant* if $f^{-1}(A) = A$. Additionally, A is *positively invariant* if $f(A) \subset A$ and *negatively invariant* if $f^{-1}(A) \subset A$.

Evidently, every fixed point is periodic, and every periodic point is in its ω- and α-limits. Furthermore, $\omega_f(x)$ is closed and positively invariant and, if f is a homeomorphism, the limit sets $\omega_f(x)$ and $\alpha_f(x)$ are closed and invariant. Finally, $\omega_f(x) \subseteq \Omega(f)$ for all $x \in X$, $\Omega(f)$ is positively invariant, and if f is a homeomorphism then $\alpha_f(x) \subseteq \Omega(f)$ for all $x \in X$ and $\Omega(f)$ is invariant. For the proofs of these facts, we recommend [52, Chapter 2].

If two topological dynamical systems have "qualitatively" the same behavior, we want to say that they are somehow equivalent. To make this notion precise, we define topological conjugacy.

Definition 2. If X and Y are compact metric spaces, with $f : X \to X$ and $g : Y \to Y$ continuous maps, we say that f and g are *topologically conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

If $h : X \to Y$ is surjective, we say that f and g are *topologically semiconjugate*, or that g is a *topological factor* of f.

Another sense of equivalence can be considered if we take into account measure-theoretic aspects of the dynamics.

Definition 3. If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, with $f : X \to X$ and $g : Y \to Y$ measurable maps, we say that f and g are *ergodically equivalent* if there are $X' \subseteq X$ and $Y' \subseteq Y$, with $\mu(X \backslash X') = 0$ and $\nu(Y \backslash Y') = 0$, and a measurable bijection $h : X' \to Y'$ such that $h \circ f = g \circ h$.

If $h : X' \to Y'$ is surjective, we say that g is a *factor* of f.

These notions of equivalence are in fact equivalence relations, and maps in the same class are alike. For the topological conjugacy, two equivalent maps have corresponding fixed points, periodic orbits and dense orbits; the limit sets and non-wandering set are also preserved. As such, the topological behavior of the orbits is indistinguishable from one system to the other.

There are topological and measure theoretic notions of how much a function mix points on the space. We recall some of these definitions.

Definition 4. Given X a metric space, we say that a continuous map $f : X \to X$ is *topologically transitive* if, for all $U, V \subseteq X$ non-empty open sets, there is $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.

Often the above definition is taken to be equivalent to the existence of a dense orbit. Even though it is not always the case [20], this equivalence holds if X is separable, second category and does not have isolated points.

Definition 5. Given X a metric space, we say that a closed invariant set $\Lambda \subseteq X$ for a continuous map $f : X \to X$ is *minimal* if every orbit $\mathcal{O}_f(x) := \{f^n(x) \ : \ n \in \mathbb{N}\}$ of points $x \in \Lambda$ is dense in Λ. We say that f is a minimal dynamical system if X is a minimal set for f.

Minimal dynamical systems are transitive, but there are transitive systems that are not minimal, for instance, a transitive system with fixed points.

Definition 6. Given X a metric space, we say that a continuous map $f : X \to X$ is *topologically mixing* if, for all $U, V \subseteq X$ non-empty open sets, there is $k \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$ for all $k \geqslant n$.

Topological transitivity and mixture express that the system is somewhat "chaotic" — in the sense that any point can approximate any other point after some number of iterations — with the second property being stronger than the former. In the same fashion, we have concepts for measurable dynamical systems that express something similar, but from a measure-theoretic point of view. More precisely, we have the concepts of *ergodicity*, *weak mixture* and *mixture* defined as follows.

Definition 7. If $(X, \mathcal{B}, \mu)$ is a measure space and $f : X \to X$ a measurable map, we say that f *preserves* μ or that μ *is* f*-invariant* if

$$\mu(A) = \mu(f^{-1}(A)) \text{ for all } A \in \mathcal{B}.$$

In this case we say that $(X, \mathcal{B}, \mu, f)$ is a *measurable dynamical system.*

Definition 8. A measurable dynamical system $(X, \mathcal{B}, \mu, f)$ is:

- *ergodic* if every measurable invariant set A satisfies $\mu(A) = 0$ or $\mu(X \backslash A) = 0$;
- *weak mixing* if, for any $A, B \in \mathcal{B}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(f^{-i}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

- *mixing* if, for any $A, B \in \mathcal{B}$,

$$\lim_{n \to \infty} \mu(f^{-n}(A) \cap B) - \mu(A)\mu(B) = 0.$$

It is easy to see that for a system, being mixing implies being weak mixing, which itself in turn implies being ergodic [61].

Another important concept is the one of unique ergodicity, that may seem to be measure-theoretic, but it is actually topological. A dynamical system is *uniquely ergodic* if it admits only one invariant probability measure. In this case, this measure is necessarily ergodic [61]. An example of uniquely ergodic actions is given by transitive translations on compact and metrizable topological groups; the invariant measure in this case is the *Haar measure.*

Topological transitivity and minimality do not relate directly to the measure-theoretic concept of ergodicity, for there are measurable dynamical systems with topological structure that are minimal but not ergodic with respect to a "natural" measure (see [61, Theorem 6.2.2]) or ergodic for some measure μ but not transitive (a non-transitive continuous function on a compact metric space admits an ergodic measure), thus they can not be minimal.

However, for a measurable dynamical system $(X, \mathcal{B}, \mu, f)$, we have that if f is mixing, then its restriction to the support of μ is topologically mixing. A similar result holds for weak mixing systems, implying that they are *topologically weak mixing*[1] restricted to the support of μ. Also, a uniquely ergodic system is minimal restricted to the support of its unique invariant measure [61].

1.1.1 Stability

Most commonly in dynamical systems, we deal with the stability of points for a fixed system. More precisely, given a topological dynamical system $f : X \to X$, we say that $x \in X$ is *stable* for f if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \varepsilon$ for all $y \in X$ and $n \in \mathbb{N}$. That is, any point sufficiently close to x has its forward orbit as close as we want to the one of x. In contrast, a point x is said to be *unstable* if there are points as close as we want to x whose forward orbits do not follow the one of x.

We will deal in this work with other kind of stability, not for points under a fixed map, but for maps on a suitable space of functions.

For topological dynamical systems, we can perturb the map on a small neighborhood of an isolated periodic point to create many new periodic points, which obstructs topological conjugacy between the original map and its perturbation. To avoid this sensibility, we focus on differentiable dynamical systems and restrict the space in which the perturbations can take place. A *differentiable dynamical systems* is given by a Riemannian manifold M and a local diffeomorphism $f \in C^1(M, M)$. We imbue $C^1(M, M)$ with the C^1*-topology*, that, roughly speaking, is given by the notion that two maps are close when the supremum of the distance between their images on the same point and the supremum of the distance between their derivatives on the same point are both small. For more details, we refer to [32].

This allows us to formulate a concept of stability for C^1 maps: if their perturbations have the same "topological behavior", then they are stable.

Definition 9. A diffeomorphism $f : M \to M$ is *structurally stable* if there is a neighborhood U of f, with respect to the C^1 topology, such that, for every $g \in U$, f and g are topologically conjugate.

[1] The system $(X, \mathcal{B}, \mu, f)$ is topologically weak mixing if its Koopman operator $U_f : L^1(\mu) \to L^1(\mu)$ has no non-constant continuous eigenfunctions. Remember that the Koopman operator is given by $\phi \mapsto \phi \circ f$.

One cannot expect that the conjugacy between f and g is differentiable in general. A differentiable conjugacy means that the systems are equal up to a C^1 coordinate change. In particular, this implies that some properties have to be preserved under C^1 conjugacy. For instance, by applying the chain rule to compute the differential of $f^k(x) = h^{-1} \circ g^k \circ h(x)$, we have that

$$Df^k_x = Dh^{-1}_{g^k(h(x))} \circ Dg^k_{h(x)} \circ Dh_x,$$

for all $k \in \mathbb{Z}$. Taking x to be a periodic point with period p, then $g^p(h(x)) = h(x)$, $Dh^{-1}_{h(x)} = (D^h_x)^{-1}$ and

$$Df^p_x = (D^h_x)^{-1} \circ Dg^p_{h(x)} \circ Dh_x.$$

This means that a necessary condition to have C^1 conjugacy is that the derivatives of corresponding periodic points are conjugate matrices. We see in Section 1.3.3 that this is equivalent to the fact that corresponding periodic points have similar rates of exponential growth of their derivatives along some directions.

Thus, for differentiable dynamical systems, it makes sense to consider stability in terms of topological equivalence, since differentiable conjugacy is more subtle. See [34, Chapter 2] for a full discussion on this matter.

1.1.2 Natural extension

Dynamical systems are not always invertible, and most physical phenomena are not reversible, so it is relevant to study non-invertible systems. For certain dynamical aspects — such as unstable directions, as we see in Section 1.3 —, we need to analyze the past orbit of a point. Since every point has more than one preimage, there are several "choices of past", and we can make each one of these choices a point on a new space, defined as follows.

Definition 10. Let (X, d) be a compact metric space and $f : X \to X$ continuous. The *inverse limit space* (or *natural extension*) associated to the triple X, d and f is

- $\tilde{X} = \{\tilde{x} = (x_k) \in X^{\mathbb{Z}} : x_{k+1} = f(x_k),\ \forall k \in \mathbb{Z}\}$,
- $(\tilde{f}(\tilde{x}))_k = x_{k+1}$ $\forall k \in \mathbb{Z}$ and $\forall \tilde{x} \in \tilde{X}$,
- $\tilde{d}(\tilde{x}, \tilde{y}) = \sum_k \dfrac{d(x_k, y_k)}{2^{|k|}}$.

We have that $(\tilde{X}, \tilde{d})$ is a compact metric space and the shift map $\tilde{f}$ is continuous and invertible. Considering $\pi : \tilde{M} \to M$ the projection on the 0th coordinate, $\pi(\tilde{x}) = x_0$, then π is a continuous surjection. Therefore, every non-invertible topological dynamical system is a topological factor of an invertible topological dynamical system.

With a metric over $\tilde{M}$, we can define precisely the continuity of objects that depend on the orbit of a point, such as the invariant manifolds in Section 1.3.

By making use of the inverse limit space, it is possible to better comprehend non-invertible systems. For C^1 endomorphisms, it is the natural environment to look for structural stability [7, 6], and it provides means to explore the measure-theoretic properties of these systems [51].

1.2 Foliations

In this section we define some geometric and measurable properties of foliations, and establish the kind of foliation we work with: continuous foliations with C^1 leaves. Consider $B^k = B(0,1)$ the open ball in $\mathbb{R}^k$ centered at the origin and with radius one. Given a partition $\mathcal{P}$, we denote by $\mathcal{P}(x)$ the element $P \in \mathcal{P}$ containing x.

Definition 11. A *foliation* $\mathcal{F}$ of dimensrion k is a partition of M into C^1 k-submanifolds with, for every $x \in M$, local charts $\phi : B^k \times B^{m-k} \to U \ni x$ such that $\phi(B^k \times \{z\}) = \mathcal{F}(\phi(0,z)) \cap U$. Additionally, h is C^1 along B^k and continuous along B^{n-k}.

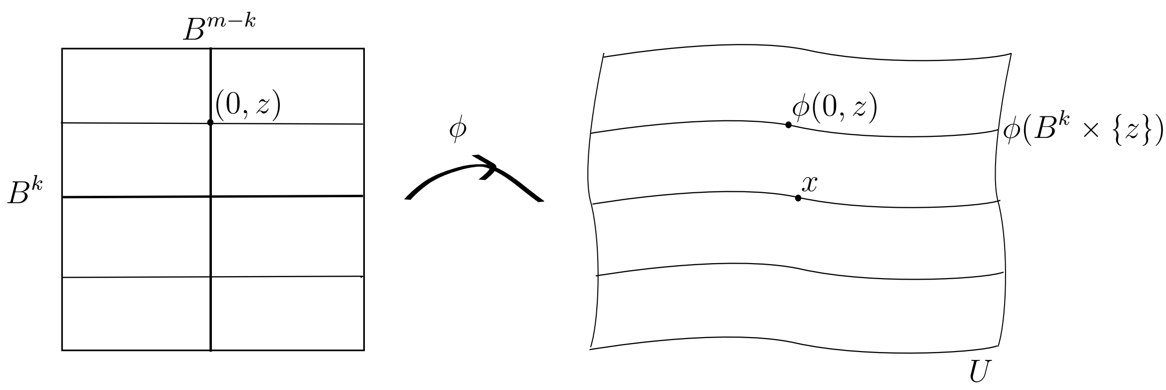

Figure 1 – A coordinate chart (U, ϕ) for the foliation around the point x.

In other words, a continuous foliation with C^1 leaves (or simply *foliation*) is a partition into C^1 submanifolds "stacked" continuously. The study of this kind of foliation arises naturally in dynamical systems, since systems with uniform hyperbolicity — uniform contraction and expansion along some directions in TM — provide foliations with this regularity, as we see in Section 1.3.

A *vector bundle* is a way to associate, for each point of a given *base space* B, a vector space in such way that this family of vector spaces vary continuously with respect to the base space. Focusing on the case that B is a compact and connected topological space, we define a vector bundle as a triple (E, π, B), where $\pi : E \to X$ is a continuous surjection and the *fiber* $E_x := \pi^{-1}(\{x\})$ is a k-dimensional vector space, with $k \in \mathbb{N}$ the *rank* of E. Additionally, there must hold the *compatibility condition*, meaning that over sufficiently small neighborhoods in B, the bundle $\pi^{-1}(U) \subseteq E$ is homeomorphic to $U \times \mathbb{R}^k$, and the homeomorphism $\phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ satisfies: $\pi \circ \phi(x, v) = x$ for all $x \in U$ and $v \in \mathbb{R}^k$; and $\phi(x, \cdot)$ is a linear isomorphism between $\mathbb{R}^k$ and E_x. We say that (U, π) is a *local trivialization* of E.

An example of vector bundle is given by *subbundles*, that is, W a family of linear subspaces $W_x \subseteq E_x$ that makes W a vector bundle over M with the projection $\pi\big|_W$. Another example of vector bundle of rank k is the tangent bundle TM over a differential k-manifold M. A subbundle of TM is called a *distribution*.

A distribution E is *integrable* if there is a foliation $\mathcal{F}$ with $T_x\mathcal{F}(x) = E_x$, that is, the leaves of $\mathcal{F}$ are tangent to E. Reciprocally, a k-dimensional foliation $\mathcal{F}$ provides an integrable distribution by considering $E_x = T_x\mathcal{F}(x)$ for each $x \in M$.

1.2.1 Quasi-isometry

A property frequently required for foliations in $\mathbb{R}^n$ when studying hyperbolic systems is the one of quasi-isometry. It means, roughly speaking, that at a large scale the foliation has lengths (along the leaf) between two points uniformly comparable with the Euclidean distance between them.

Definition 12. Given a foliation $\mathcal{F}$ of $\mathbb{R}^n$, with $d_{\mathcal{F}}$ the distance along the leaves, we say that $\mathcal{F}$ is *quasi-isometric* if there are constants $a, b > 0$ such that, for every $y \in \mathcal{F}(x)$,

$$d_{\mathcal{F}}(x, y) \leqslant a\|x - y\| + b.$$

In particular, if the foliation $\mathcal{F}$ is uniformly continuous, the above definition is equivalent to the existence of $Q > 0$ such that, for every $y \in \mathcal{F}(x)$,

$$d_{\mathcal{F}}(x, y) \leqslant Q\|x - y\|.$$

For invertible systems with uniform hyperbolicity, quasi-isometry is guaranteed by the existence of a conjugacy between the map and a linear one, as we see in Section 1.3.

For systems with partial hyperbolicity — systems whose tangent space splits into invariant directions E^u, E^s with contraction and expansion, and a direction E^c that is neither as expanded or contracted by the map —, quasi-isometry is the main ingredient to guarantee *dynamical coherence* [10], that is, the distributions $E^{uc} = E^u \oplus E^c$ and $E^{cs} = E^c \oplus E^s$ are integrable. The quasi-isometry, as a geometrical property of the foliations, also implies for partially hyperbolic systems with one-dimensional center bundle that the dynamics on the leaves can be topologically classified accordingly to the dynamics of a linear associated model [28].

Thus, quasi-isometry is a geometrical property of foliations that, when present for invariant foliations of a dynamical systems, provide geometric and topological consequences. We see in Theorems D and E that, together with measurable hypotheses, it also has consequences in the smooth classification of the hyperbolic dynamics on surfaces.

1.2.2 Conditional measures

Before we define measure theoretical ways to see how regular a foliation is, let us digress for a short while into necessary concepts of measure theory. For the basics of measure theory, see [53, Part Three], for instance.

Let $(X, \mathcal{A}, \mu)$ be a probability space, $\mathcal{P}$ a partition of X into measurable sets, and $\pi : M \to \mathcal{P}$ the projection that assigns for each point $x \in M$ the element of $\mathcal{P}$ containing x. We define a σ-algebra and a measure in $\mathcal{P}$ as follows: $\mathcal{Q} \subseteq \mathcal{P}$ is measurable if $\pi^{-1}(\mathcal{Q})$ is measurable in X, and $\hat{\mu}(\mathcal{Q}) = \pi_*\mu(\mathcal{Q}) = \mu(\pi^{-1}(\mathcal{Q}))$.

Thus, we have a way to measure collections of elements of a partition. If we define a measure along each atom of the partition, we can ask whether we can compose these two ways of measurement into the original measure of X. More specifically, we want to generalize Fubini's Theorem for the case in which we do not have a product space to begin with. This is achieved if there is a family of measures defined on each atom of the partition permitting this decomposition of the integral.

Definition 13. A family $\{\mu_P\}_{P\in\mathcal{P}}$ of probability measures on X is a *system of conditional measures* (or a *disintegration of* μ) with respect to a partition $\mathcal{P}$ if, for $\phi : M \to \mathbb{R}$ continuous

1. $P \mapsto \int \phi d\mu_P$ is measurable;
2. $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;
3. $\int \phi d\mu = \int_{\mathcal{P}} \int_P \phi d\mu_P d\hat{\mu}$.

Given a partition $\mathcal{P}$, if the σ-algebra $\mathcal{A}$ is countably generated and there is a system of conditional measures for μ with respect to $\mathcal{P}$, then it is unique with respect to $\hat{\mu}$. More precisely, we have the following result, which is proved in [61, Proposition 5.1.7], for instance.

Proposition 1. *If the σ-algebra $\mathcal{A}$ has a countable generator and $\{\mu_P\}$ and $\{\nu_P\}$ are disintegrations with respect to $\mathcal{P}$, then $\mu_P = \nu_P$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$.*

The Rokhlin disintegration theorem guarantees the existence of conditional measures for partitions that can be generated by a countable family of sets.

Definition 14. A partition $\mathcal{P}$ is a *measurable partition* (or *countably generated*) with respect to μ if there is $M_0 \in M$ with $\mu(M_0) = 1$ and a family $\{A_i\}_{i\in\mathbb{N}}$ of measurable sets such that, given $P \in \mathcal{P}$, there is $\{P_i\}_{i\in\mathbb{N}}$ with $P_i \in \{A_i, A_i^{\complement}\}$ such that $P = \bigcap_{i\in\mathbb{N}} P_i$ restricted to M_0.

Theorem 2 (Rokhlin disintegration)**.** *If X is a complete and separable metric space and $\mathcal{P}$ is a measurable partition, then the probability μ has a disintegration on a family of conditional measures μ_P.*

The Borel σ-algebra is always countably generated for a separable metric space X, and the disintegration given by Rokhlin disintegration theorem is unique for $\hat{\mu}$-almost every P. In particular, if the partition $\mathcal{P}$ is preserved by a measurable function $T : M \to M$ that also preserves μ, then T carries conditional measures into conditional measures, that is, $T_*\mu_P = \mu_{T(P)}$ for $\hat{\mu}$-almost every P, since $\{T_*\mu_P\}$ is a disintegration of μ with respect to $\mathcal{P}$.

In our context, we want to disintegrate the measure using a foliation to give the partition. We have the disintegration locally for small sets filled with leaves of the foliation. More precisely, a *foliated box* for a foliation $\mathcal{F}$ of dimension k on a complete Riemannian n-manifold M is given by a local leaf X and a local $(n-k)$-dimensional transversal Y to this leaf. The foliated box $\mathcal{B}$ is homeomorphic to $X \times Y$ and the map $\phi : X \times Y \to \mathcal{B}$ takes sets on the form $X \times \{y\}$ to local leaves $\mathcal{F}_{loc}(y)$. We identify $X \times Y$ with $\mathcal{B}$ and $X \times \{y\}$ with $\mathcal{F}_{loc}(y)$.

A foliated box has its local leaves as a measurable partition. Indeed, with the Riemannian structure inherited from M, Y is a separable metric space, having a countable base of open sets $\{Y_i\}_{i\in\mathbb{N}}$. The sets $A_i := \bigcup_{y\in Y_i} \mathcal{F}_{loc}(y)$ for $i \in \mathbb{N}$ form a countable generator for the partition, independent of the measure. Then, for foliated boxes, we can always consider a disintegration of any probability measure, by the Rokhlin disintegration theorem. The same does not necessarily hold for the whole partition, as it can not be countably generated.

1.2.3 Absolute continuity

We regard the regularity of a foliation not only with respect to its topological or differential properties, but with respect to metric properties of its leaves and the holonomy maps that move points through leaves of the foliation.

Definition 15. Given a foliation $\mathcal{F}$, we define the *holonomy* $h_{\mathcal{F}} : \Sigma_1 \to \Sigma_2$ between two local discs Σ_1 and Σ_2 transverse to $\mathcal{F}$ by $q \mapsto \mathcal{F}(q) \cap \Sigma_2$, where $\mathcal{F}(q)$ is the leaf of $\mathcal{F}$ containing q.

In this sense, we can use the holonomy to ascertain the regularity of a foliation. But first, let us recall the definition of an absolutely continuous measurable function.

Definition 16. Given $(M, \mathcal{A}, \mu)$ and $(N, \mathcal{B}, \nu)$ measure spaces and a measurable function $f : M \to N$. We say that f is *absolutely continuous* if, for each $A \in \mathcal{A}$, we have that $f(A) \in \mathcal{B}$ and

$$\mu(A) = 0 \implies \nu(f(A)) = 0.$$

Equivalently, f is absolutely continuous if there is a positive and measurable map $q : M \to \mathbb{R}$, called the *Jacobian* of f, such that, for each $A \in \mathcal{A}$

$$\nu(f(A)) = \int_A q(z) d\mu(z).$$

Definition 17. We say that a foliation $\mathcal{F}$ is *transversely absolutely continuous* if, given two smooth transversals T_1 and T_2, the holonomy $h_{\mathcal{F}}$ between them is absolutely continuous with respect to λ_{T_1} and λ_{T_2}, their respective Riemannian volumes.

Another perspective from which to consider this regularity is regarding conditional measures on the leaves as a partition.

Definition 18. We say that a foliation $\mathcal{F}$ is *leafwise absolutely continuous* if, given a foliated box, the measures $m_{\mathcal{F}(x)}$, given on each leaf by the disintegration of the volume m, are equivalent to the induced Lebesgue measure $\lambda_{\mathcal{F}(x)}$ on the same leaf.

This definition does not depend on the foliated box. Indeed, when foliated boxes overlap, the conditional measures on a given leaf are the same up to multiplication by a constant [5, Lemma 3.2]. Moreover, this definition is weaker than transversely absolute continuity.

Theorem 3 ([11])**.** *Transverse absolute continuity implies leafwise absolute continuity.*

In contrast to leafwise absolute continuity, we have other extreme behavior on foliations: if the conditional $\mu_{\mathcal{F}(x)}$ is a sum of Dirac measures for almost every leaf, we say that $\mathcal{F}(x)$ has *atomic disintegration* with respect to μ.

To understand how the leaves of a foliation with atomic disintegration behave under holonomies, we have the following.

Theorem 4 ([14])**.** *Let $\mathcal{F}$ be a foliation with mono-atomic disintegration. Then, for almost every pair of transversals, the holonomy between them takes a set of full Lebesgue measure to a set of zero Lebesgue measure.*

By "almost every pair of transversals" we mean the following: given $\mathcal{T}$ a smooth transverse foliation for $\mathcal{F}$, we consider $\mathcal{T}(x)$ and $\mathcal{T}(y)$ for $m \times m$-a. e. point $(x, y) \in M \times M$.

Theorem 4 is a direct consequence of the result below, first presented in [14], that we present here for completeness. Along the proof, since we only work with volume along $\mathcal{T}$-leaves, we denote $\lambda_{\mathcal{T}(x)}$ by λ_x.

Theorem 5 ([14])**.** *Let $\mathcal{F}$ be a foliation on a manifold M and $\mathcal{T}$ a transverse C^1 foliation. Assume that $\mathcal{F}$ has atomic disintegration with $k \in \mathbb{N}$ and a set $N \subseteq M$, $m(N) = 1$, such that the number of atoms in $N \cap \mathcal{F}(x)$ is smaller than k for each $x \in N$. Then, for $m \times m$-almost every $(x, y) \in N \times N$, the set $A_x \subseteq \mathcal{T}(x)$ given by*

$$A_x = \mathcal{T}(x) \cap \Lambda,$$

is such that $\lambda_x(A_x) = 1$ and $\lambda_y(A_{x,y}) = 0$, where Λ is the set of atoms for $\mathcal{F}$, $A_{x,y} = h_{\mathcal{F},y}(A_x)$ and $h_{\mathcal{F},y}$ is the holonomy between the leaves $\mathcal{T}(x)$ and $\mathcal{T}(y)$.

Proof. Considering

$$A_x = \mathcal{T}(x) \cap \Lambda,$$

we have that, for m-almost every point x, A_x has full λ_x-measure, since the set of atoms has full m-measure.

The proof of the theorem follows once we show that $\lambda_y(A_{x,y}) = 0$. By contradiction, assume that there is a set $W \subseteq N \times N$, with positive $m \times m$ measure, such that

$$\lambda_y(A_{x,y}) > 0, \text{ for all } (x, y) \in W.$$

By Fubini's theorem, there is a set $V \subset N$ such that $m(V) > 0$ and, for each $y \in V$, there is a set $H_y \subseteq N$ such that $m(H_y) > 0$ and, for each $x \in H_y$,

$$\lambda_y(A_{x,y}) > 0.$$

Without loss of generality, we may assume that, for $y \in V$, H_y is such that $\lambda_x(H_y \cap \mathcal{T}(x)) > 0$.

For each $y \in V$, the set H_y is uncountable, thus there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset H_y$ such that

$$\lambda_y(A_{x_i,y}) > 1/n, \text{ for some } n \in \mathbb{N}.$$

Note that for each $x_i \in H_y$, the set $A_{x_i} = \mathcal{T}(x_i) \cap \Lambda$ is formed by atoms from the leaves of $\mathcal{F}$.

Since $1/n < \lambda_y(A_{x_i,y}) \leqslant 1$, for all $i \in \mathbb{N}$, among the $n + 1$ sets

$$A_{x_1,y},\ A_{x_2,y},\ A_{x_3,y}, \ldots,\ A_{x_{n+1},y}$$

there exist two of them, that we denote by $A_{x_{n_1},y}$ and $A_{x_{n_2},y}$, such that their intersection $A^1_{x_{n_1},y} := A_{x_{n_1},y} \cap A_{x_{n_2},y}$ satisfies $\lambda_y(A^1_{x_{n_1},y}) > 1/n^2$. The leaves $\mathcal{T}(x_{n_1})$ and $\mathcal{T}(x_{n_2})$ are distinct, then there exist leaves of $\mathcal{F}$ with at least two atoms intersecting $A^1_{x_{n_1},y}$.

Analogously, among the $n+1$ sets

$$A_{x_{n+2},y},\ A_{x_{n+3},y},\ A_{x_{n+4},y},\ldots,\ A_{x_{2n+2},y}$$

there exist two of them with intersection, which we denote by $A^1_{x_{n_2},y}$, satisfying $\lambda_y(A^1_{x_{n_2},y}) > 1/n^2$. Additionally, there are leaves of $\mathcal{F}$ with at least two atoms intersecting $A^1_{x_{n_2},y}$. By keeping this process, we obtain a sequence of sets $(A^1_{x_{n_i},y})_{i\in\mathbb{N}}$, $\lambda_y(A^1_{x_{n_i},y}) > 1/n^2$, such that, for each $i \in \mathbb{N}$, there exist leaves of $\mathcal{F}$ with at least two atoms that intersect $A^1_{x_{n_i},y}$. Now, repeating this argument as before with the sequence $(A^1_{x_{n_i},y})_{i\in\mathbb{N}}$, we obtain a sequence $(A^2_{x_{n_i},y})_{i\in\mathbb{N}}$, $\lambda_y(A^2_{x_{n_i},y}) > 1/n^4$, such that, for each $i \in \mathbb{N}$, there exist leaves of $\mathcal{F}$ with at least 2^2 atoms intersecting $A^2_{x_{n_i},y}$. Therefore, repeating this process, we can always find leaves of $\mathcal{F}$ with an arbitrary amount of atoms, which contradicts the hypothesis of the theorem. □

Question 2. Is the converse statement of Theorem 4 valid? In other words, how do we characterize atomic disintegration in terms of holonomies?

1.2.4 Uniform boundedness for absolute continuity

We now introduce a uniformly bounded formulation of transverse absolute continuity and leafwise absolute continuity.

Definition 19. A leafwise absolutely continuous foliation $\mathcal{F}$ is said to have the *uniformly bounded density property* (*UBD property*) if there is $K > 1$ such that, for every foliated box $\mathcal{B}$, the disintegration $\{m^{\mathcal{B}}_{\mathcal{F}(x)}\}$ of volume normalized to $\mathcal{B}$ satisfies

$$K^{-1} \leqslant \frac{dm^{\mathcal{B}}_{\mathcal{F}(x)}}{d\lambda^{\mathcal{B}}_{\mathcal{F}(x)}} \leqslant K,$$

where $\lambda^{\mathcal{B}}_{\mathcal{F}(x)}$ is the normalized induced volume in the connected component of $\mathcal{F} \cap \mathcal{B}$ containing x.

The above definition is given by [43], where the authors use it to prove that the Lyapunov exponents are constant a. e. on a non-ergodic setting. Moreover, it is the natural hypothesis to characterize smooth conjugacy with a linear model for low dimensional Anosov maps [60, 12]. Indeed, in [60], it is shown that a smooth conservative partially hyperbolic diffeomorphism on $\mathbb{T}^3$ is smoothly conjugate to its linearization if, and only if, the center foliation has the UBD property. This can be seen as a sharp result, since it is given an example on [59] of a conservative partially hyperbolic diffeomorphism on $\mathbb{T}^3$ such that the center foliation is C^1 but the conjugacy map is not C^1.

We want to introduce a similar boundedness for transverse absolute continuity. However, the uniform constant can not exist unless we limit the angles between the transversals and the foliation. Take for instance $\mathcal{F}$ to be a one-dimensional foliation by horizontal lines in $[0,1]^2 \subseteq \mathcal{R}^2$. Let $\mathcal{T}_1$ be a vertical line and $\mathcal{T}_2$ a line with slope $\alpha \in (0,\pi)$, as in Figure 2. Given a subset $A \subseteq \mathcal{T}_1$, we have that $h_{\mathcal{F}}(A) \subseteq \mathcal{T}_2$ satisfies $\lambda_{\mathcal{T}_2}(h_{\mathcal{F}}(A)) = \dfrac{1}{\text{sen}(\alpha)}\lambda_{\mathcal{T}_1}(A)$. As α goes to 0, $\dfrac{1}{\text{sen}(\alpha)}$ goes to infinity, so there is no uniform bound for the Jacobian. But a foliation by lines is as regular as a foliation can be, and so we want to exclude this possibility.

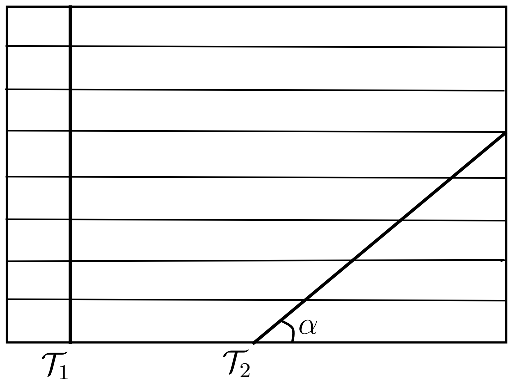

Figure 2 – We need the angle condition to avoid the case in which a small set in $\mathcal{T}_1$ is taken to a very large set in $\mathcal{T}_2$.

Definition 20. A foliation $\mathcal{F}$ is *transversely absolutely continuous with uniformly bounded Jacobians* if, for every $\beta > 0$ and any pair of C^1 transverse discs T_1 and T_2 with $\angle(T_x\mathcal{F}(x), T_xT_i) > \beta$ for all $x \in T_i$, $i \in \{1,2\}$, we have that the holonomy $h_{\mathcal{F}} : T_1 \to T_2$ is absolutely continuous with respect to λ_1 and λ_2 and its Jacobian $q_{\mathcal{F}}$ is uniformly bounded: there is $C_\beta > 1$ such that

$$\frac{1}{C_\beta} < q_{\mathcal{F}}(x) < C_\beta \text{ for } \lambda_1\text{-a. e. } x \in T_1.$$

The constant C_β is fixed for all leaves of a given transverse foliation $\mathcal{T}$ if M is compact. Additionally, it is also fixed locally for foliated boxes in any manifold.

Remark 1. The angle condition on the transverse discs above is satisfied, for instance, for $\mathcal{F} = \mathcal{F}^s$ the stable foliation for a uniformly hyperbolic diffeomorphism and $T_1 = \mathcal{F}^u(x)$ and $T_2 = \mathcal{F}^u(y)$, $y \notin \mathcal{F}^u(x)$, two distinct unstable leaves. A more general example is given by leaves of invariant foliations of a diffeomorphism with dominated splitting (see [9]).

1.3 Uniform hyperbolicity

This section is intended to introduce the main concepts, results and tools of hyperbolic dynamics — the field of dynamics interested in the study of maps that have a splitting of the tangent space of each point into a direction with uniform contraction and another with uniform expansion. We focus specifically in the case in which the hyperbolicity is uniform, that is, the rates of expansion and contraction are the same for each point. Such maps, if invertible, are called *Anosov diffeomorphisms*, as a tribute to Dmitri Anosov who introduced their study, as we mentioned at the introduction.

If the map is a local diffeomorphism but it is not invertible, several complications arise, but we can still define uniform hyperbolicity, and maps having this property for each point are called *Anosov endomorphisms*. The definition is analogous, as we state in the following paragraphs, but the splitting is not global, it is defined along a fixed orbit (equivalently, along points of the natural extension).

Most of the results here stated have the proofs omitted, but we reefer to their demonstration. For a more detailed approach to the invertible case, see [34, 52, 55, 9]. Additionally, if possible, we state the results on the more general setting of endomorphisms, highlighting the particularities of the invertible or expanding setting, cases that are included in the definition of uniform hyperbolic endomorphisms, see [49, 38, 51, 4].

Let M be a closed (finite dimensional, compact, connected and without boundary) C^∞ Riemannian manifold. The classical concept of a *uniformly hyperbolic diffeomorphism* (or *Anosov diffeomorphism*) over M is that there is a splitting $TM = E^u \oplus E^s$ of the tangent bundle that is DF-invariant and such that DF acts as a uniform contraction over E^s and a uniform expansion over E^u. There is a more general notion of a diffeomorphism with a *hyperbolic set*: there are $\Lambda \subseteq U \subseteq M$, with Λ closed and f-invariant and U open, such that $f : U \to M$ is regular and $f\big|_\Lambda$ is uniformly hyperbolic.

Let us define a hyperbolic set for a local C^1 diffeomorphism $f : U \subseteq M \to M$.

Definition 21 ([49])**.** Let $f : U \subseteq M \to M$ be a local C^1 diffeomorphism. The closed invariant set Λ is an *hyperbolic set* for f if, for all $\tilde{x} \in \tilde{\Lambda}$, there is, for all $i \in \mathbb{Z}$, a splitting $T_{x_i}M = E^u(x_i) \oplus E^s(x_i)$ such that

- $Df(x_i)E^u(x_i) = E^u(x_{i+1})$;
- $Df(x_i)E^s(x_i) = E^s(x_{i+1})$;
- there are constants $c > 0$ and $\lambda > 1$ such that, for a Riemannian metric on M,

  $||Df^n(x_i)v|| \geqslant c^{-1}\lambda^n||v||$, $\forall v \in E^u(x_i)$, $\forall i \in \mathbb{Z}$,

  $||Df^n(x_i)v|| \leqslant c\lambda^{-n}||v||$, $\forall v \in E^s(x_i)$, $\forall i \in \mathbb{Z}$.

$E^{s/u}(x_i)$ is called the *stable/unstable direction* for x_i. When it is needed to make explicit the choice of past orbit, we denote by $E^u(\tilde{x})$ the unstable direction at the point $\pi(\tilde{x}) = x_0$ with respect to the orbit $\tilde{x}$.

In particular, if $U = \Lambda = M$, we say that f is an *Anosov endomorphism*, or a *uniformly hyperbolic endomorphism*. This definition includes Anosov diffeomorphisms (if f is invertible) and expanding maps (if $E^u(x) = T_xM$ for each x). Throughout this work, however, we consider the case in which E^s is not trivial, excluding the expanding case, unless it is mentioned otherwise.

A point can have more than one unstable direction under an Anosov endomorphism, even though the stable direction is always unique. Indeed, we find in [49] an example in which a point has uncountable many unstable directions. Before we state this example, let us introduce the stable and unstable manifolds. To guarantee their existence and their properties, we state the *Hadamard–Perron Theorem*, which implies the *Stable Manifold Theorem*.

Definition 22. Considering $\lambda < \mu$, we say that a sequence $L_m : \mathbb{R}^n \to \mathbb{R}^n$, $m \in \mathbb{Z}$, of invertible linear maps admits a (λ, μ)-*splitting* if $\mathbb{R}^n = E^u_m \oplus E^s_m$ for every m, $L_m E^{s/u}_m = E^{s/u}_{m+1}$ and

$$\left\| L_m\big|_{E^s_m} \right\| \leqslant \lambda \quad \text{and} \quad \left\| L_m^{-1}\big|_{E^u_m} \right\| \leqslant \mu^{-1}.$$

We say that $\{L_m\}_m$ admits a *hyperbolic splitting* if $\lambda < 1 < \mu$.

Theorem 6 (Hadamard–Perron)**.** *Consider $\lambda < \mu$ and $r \geqslant 1$, and for all $m \in Z$, let $f_m : \mathbb{R}^n \to \mathbb{R}^n$ be a C^r diffeomorphism such that*

$$f_m(x, y) = (A_m x + \alpha_m(x, y), B_m y + \beta_m(x, y)),$$

for all $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, where $A_m : \mathbb{R}^k \to \mathbb{R}^k$ and $B_m : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ are linear maps with $|A_m^{-1}\| \leqslant \mu^{-1}$, $\|B_m\| \leqslant \lambda$, $\alpha_m(0) = 0$, $\beta_m(0) = 0$. Then, for $0 < \gamma < \min\left(1, \sqrt{\frac{\mu}{\lambda}} - 1\right)$, there is $\delta = \delta(\lambda, \mu, \gamma) > 0$ such that, if $\|\alpha_m\|_{C^1} < \delta$ and $\|\beta_m\|_{C^1} < \delta$ for all $m \in \mathbb{Z}$, then

1. *there is a unique family $\{W^u_m\}_m$ of C^1 k-dimensional manifolds*

$$W^u_m\{(x, \varphi^u_m(x) : x \in \mathbb{R}^k\} = \text{ graph } \varphi^u_m,$$

   where $\varphi^u_m : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and $\|D\varphi^u_m\| \leqslant \gamma$;

2. *there is a unique family $\{W^s_m\}_m$ of C^1 $(n-k)$-dimensional manifolds*

$$W^s_m\{(x, \varphi^s_m(x) : x \in \mathbb{R}^{n-k}\} = \text{ graph } \varphi^s_m,$$

   where $\varphi^u_m : \mathbb{R}^{n-k} \to \mathbb{R}^k$ and $\|D\varphi^s_m\| \leqslant \gamma$;

3. *these families are invariant: $f_m(W^s_m) = W^s_{m+1}$, $f_m(W^u_m) = W^u_{m+1}$;*

4.

$$\|f_m(z)\| < \underbrace{(1+\gamma)(\lambda+\delta(1+\gamma))}_{\lambda'}\|z\| \text{ for all } z \in W^s_m,$$

$$\text{and } \|f^{-1}_{m-1}(z)\| < \underbrace{\left(\frac{\mu}{1+\gamma}-\delta\right)^{-1}}_{(\mu')^{-1}}\|z\| \text{ for all } z \in W^u_m;$$

5. *If $\nu \in (\lambda', \mu')$ and $\|f_{m+l-1} \circ \cdots \circ f_m(z)\| \leqslant C\nu^l\|z\|$ for a $C > 0$ and all $l \geqslant 0$, then $z \in W^s_m$, analogously, if $\|f^{-1}_{m-l} \circ \cdots \circ f^{-1}_{m-1}(z)\| \leqslant C\nu^{-l}\|z\|$ for a $C > 0$ and all $l \geqslant 0$, then $z \in W^u_m$.*

Note that, for Anosov endomorphisms, there is not a global splitting of the tangent space; the splitting is along a given orbit $\tilde{x} = (x_i)$ on $\tilde{M}$. But $\{Df_{x_i}\}$ is a hyperbolic sequence, and one can apply the Hadamard–Perron theorem in the same way it is done for Anosov diffeomorphisms to prove that f has *local stable* and *local unstable manifolds*, denoted by $W^s_{f,R}(\tilde{x})$ and $W^s_{f,R}(\tilde{x})$, tangent to the stable and unstable directions.

Theorem 7 ([49])**.** *Let Λ be a hyperbolic set with a (λ, λ^{-1}) splitting for a C^1 local diffeomorphism $f : U \to M$. Then, for each $\tilde{x} \in \tilde{\Lambda}$ and each $i \in \mathbb{Z}$:*

1. *The sets*

$$W^s_{f,R}(x_i) := \{y \in M : \forall k \geqslant 0,\ d(f^k(y), f^k(x_i)) < R\} \text{ and}$$

$$W^u_{f,R}(x_i) = \{y \in M : \exists \tilde{y} \in \tilde{M} \text{ such that } \pi(\tilde{y}) = y \text{ and } \forall k \geqslant 0,\ d(y_{-k}, x_{i-k}) < R\}$$

are C^1 manifolds, called respectively local stable manifold *and* local unstable manifold *of f at the point x_i with respect to $\tilde{f}^i(\tilde{x})$. R is a small constant that is locally constant for f.*

2. $T_{x_i}W^{s/u}_{f,R}(x_i) = E^{s/u}_f(x_i)$.

3. *If $y, z \in W^s_{f,R}(x_i)$, then*

$$d(f^{k+1}(y), f^{k+1}(z)) \leqslant \frac{2+\lambda}{3} \cdot d(f^k(y), f^k(z)) \text{ and}$$

if $y, z \in W^u_{f,R}(x_i)$, then, for the corresponding $\tilde{y}, \tilde{z} \in \tilde{\Lambda}$,

$$d(y_{-k-1}, z_{-k-1}) \leqslant \frac{2+\lambda}{3} \cdot d(y_{-k}, z_{-k})$$

for $k \in \mathbb{Z}$.

Additionally, for $R > 0$ sufficiently small, these manifolds are characterized by

$$W^s_{f,R}(\tilde{x}) = \{y \in M : \forall k \geqslant 0,\ d(f^k(y), f^k(x_0)) < R\}$$

and

$$W^u_{f,R}(\tilde{x}) = \{y \in M : \exists \tilde{y} \in \tilde{M} \text{ such that } \pi(\tilde{y}) = y \text{ and } \forall k \geqslant 0,\, d(y_{-k}, x_{-k}) < R\}.$$

The *global stable/unstable manifolds* are

$$W^s_f(\tilde{x}) = \{y \in M : d(f^k(y), f^k(x_0)) \xrightarrow{k\to\infty} 0\}$$

and

$$W^u_f(\tilde{x}) = \{y \in M : \exists \tilde{y} \in \tilde{M} \text{ such that } \pi(\tilde{y}) = y \text{ and } d(y_{-k}, x_{-k}) \xrightarrow{k\to\infty} 0\}.$$

Moreover, these manifolds are as regular as f. The stable manifolds do not depend on the choice of past orbit for x_0, but the unstable ones do.

In the case that the unstable directions do not depend on $\tilde{x}$, that is, $E^u(\tilde{x}) = E^u(\tilde{y})$ for any $\tilde{x}, \tilde{y} \in \tilde{M}$ with $x_0 = y_0$, then we say that f is a *special Anosov endomorphism*. Hyperbolic toral endomorphisms are examples of special Anosov endomorphisms.

The fact that non-invertible Anosov endomorphisms, or expanding maps, are not structurally stable was proven by R. Mañé and C. Pugh [38] and independently by F. Przytycki [49] in the 1970's, when they introduced the concept of Anosov endomorphisms as we know today.

Since the stable and unstable manifolds are characterized topologically using the distance, they are preserved under conjugacies, that is, if $h \circ f = g \circ h$, then $h(W^{s/u}_f(x)) = W^{s/u}_g(h(x))$. Thus, the number of different unstable manifolds for x and $h(x)$ is the same under f and g, respectively. The following example illustrates this lack of structural stability, with a map f nearby a linear one with infinitely many unstable directions at a given point.

Example 1 ([49])**.** Consider

$$A = \begin{pmatrix} n & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & n \end{pmatrix},$$

that defines in $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ a special Anosov endomorphism $f_A([x]) = [Ax]$. If n is sufficiently large, then for every neighborhood U of A in the C^1 topology and for each $x \in \mathbb{T}^3$, there exists $f \in U$ such that x has infinitely many unstable directions for f.

This is a consequence of [49, Theorem 2.15], in which the strategy is to give labels to the pre-images of x, and perturb the map f_A on small neighborhoods depending on this label.

If $\sigma = \deg f_A = |\det A|$, then each point has σ pre-images, and there is $\tau > 0$ with, for every $z_1, z_2 \in \mathbb{T}^3$, if $f_A(z_1) = f_A(z_2)$ and $z_1 \neq z_2$, then $d(z_1, z_2) > 4\tau$. Without loss of generality, assume that x is not periodic. We denote the points $\bigcup_{n=0}^{\infty} f_A^{-n}(\{x\})$ by n-tuples $(\sigma_1, \ldots, \sigma_n)$ of integers $0 \leqslant \sigma_i < \sigma$, as such:

- At least $\sigma - 1$ points from $A^{-1}(\{x\})$ are outside the ball $B(x, 2\tau)$. We denote these points by $(0), (1), \dots (\sigma - 2)$ and the remaining one is $(\sigma - 1)$;
- If $(\sigma_1, \dots, \sigma_n)$ is a point in $A^{-n}(\{x\})$, the at least $\sigma - 1$ points from $A^{-1}(\{(\sigma_1, \dots, \sigma_n)\})$ are outside the ball $B(x, 2\tau)$. We denote these points by

$$(\sigma_1, \dots, \sigma_n, 0), (\sigma_1, \dots, \sigma_n, 1), \dots (\sigma_1, \dots, \sigma_n, \sigma - 2)$$

and the remaining one is $(\sigma_1, \dots, \sigma_n, \sigma - 1)$.

The orbits in $\pi^{-1}(x)$ are sequences $(\sigma_1, \sigma_2, \dots)$. For each $y = (\sigma_1, \dots, \sigma_n) \in \bigcup_{n=0}^{\infty} f_A^{-n}(\{x\})$ with $\sigma_n \neq \sigma - 1$, we define a neighborhood $U_{(\sigma_1, \dots, \sigma_n)}$ as a projection of a small box on $T_y\mathbb{T}^3$, in such way that the U_y are f_A-invariant and disjoint. We perturb f_A on these projected boxes to generate different unstable local leaves that, when iterated until they reach x, give origin to different unstable leaves for x. For the details of this perturbation, see [49].

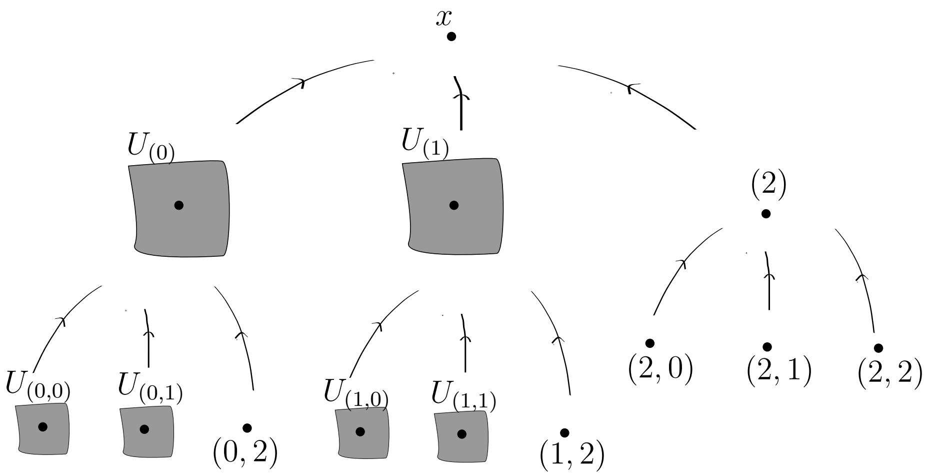

Figure 3 – If the degree of f_A is 3, then the gray sets illustrate where the perturbations are made.

Mañé and Pugh also proved the following proposition, which is very useful to generalize properties of Anosov diffeomorphisms to endomorphisms.

Proposition 2 ([38]). *Let $\overline{M}$ be the universal cover of M and $F : \overline{M} \to \overline{M}$ a lift for f. Then f is an Anosov endomorphism if and only if $F : \overline{M} \to \overline{M}$ is an Anosov diffeomorphism. Additionally, the stable bundle of F projects onto that of f.*

Usually, to study Anosov diffeomorphisms over a manifold M, we require M to be compact. In our case, even though the universal cover is not compact, since F is a lift for a map on a compact space, F carries some uniformity. This allows us to prove some results, that were originally stated for compact spaces, for the lifts proven in Proposition 2 to be Anosov diffeomorphisms.

Proposition 3 ([44])**.** *If* $f : \mathbb{T}^n \to \mathbb{T}^n$ *is a* $C^{1+\alpha}$ *Anosov endomorphism,* $\alpha > 0$*, and* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a lift for* f *to the universal cover, then there are* W_F^u *and* W_F^s *transversely absolutely continuous foliations tangent to* E_F^u *and* E_F^s*.*

The above proposition is presented in [44] as Lemma 4.1. The proof is the same one as in the compact case, since F projects on the torus, then its derivative is periodic with respect to compact fundamental domains. With the same argument, we can prove the following just as it is in [34, §19.1].

Proposition 4. *Let* $f : M \to M$ *be an Anosov endomorphism and* $F : \overline{M} \to \overline{M}$ *a lift for* f *to the universal cover. If the unstable distribution of* F *has codimension one, then it is* C^1*.*

If $\dim M = 2$, since F is invertible, then both the stable and unstable distribution are C^1. In general, these distributions are only Hölder continuous. This also implies that the stable and unstable holonomies are C^1.

Definition 23. Given a foliation $\mathcal{F}$, we define the *holonomy* $h_{\Sigma_1,\Sigma_2} : \Sigma_1 \to \Sigma_2$ between two local discs Σ_1 and Σ_2 transverse to $\mathcal{F}$ by $q \mapsto \mathcal{F}(q) \cap \Sigma_2$, where $\mathcal{F}(q)$ is the leaf of $\mathcal{F}$ containing q.

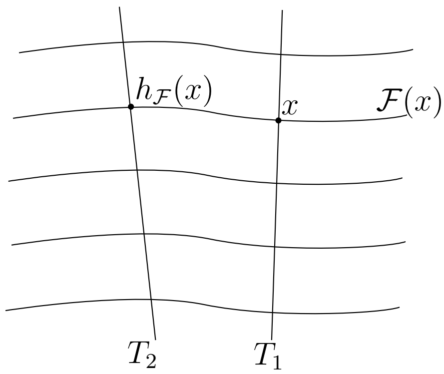

Figure 4 – The holonomy $h_{\mathcal{F}}$ moves $x \in T_1$ to T_2 along the leaf $\mathcal{F}(x)$.

That is, a holonomy moves the point q through its leaf on $\mathcal{F}$. For Anosov endomorphisms, we have transverse foliations on the universal cover, so the *stable holonomy* can have local unstable leaves as the discs. When there is no risk of ambiguity, we denote it simply by h^s. The same goes for the unstable holonomy.

An important feature of Anosov endomorphisms on tori is transitivity. The following theorem is a consequence of results in [4] for *topological Anosov maps*, which are continuous surjections with some kind of expansiveness and shadowing property, as we introduce in Section 1.3.1. Anosov endomorphisms are particular cases of topological Anosov maps that are differentiable.

Proposition 5 ([4]). *Every Anosov endomorphism on $\mathbb{T}^n$ is transitive.*

Proof. By [4, Theorem 8.3.5], every topological Anosov map f on $\mathbb{T}^n$ has $\Omega(f) = \mathbb{T}^n$, its nonwandering set is the whole manifold, which implies transitivity. □

In Section 1.2.1 we introduced quasi-isometry as a geometric property of foliations on $\mathbb{R}^n$, consisting of a uniform equivalence between the measure along the leaves and the Euclidean measure. If the endomorphism is close to is linearization, we have that

Proposition 6 ([44]). *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an Anosov endomorphism C^1-close to its linearization A. Then W^u_F and W^s_F are quasi-isometric.*

The above proposition is proved in [44] as Lemma 4.4. In the case that f is a special Anosov endomorphism, however, the quasi-isometry is guaranteed by the conjugacy between f and A: the stable and unstable foliations lift to foliations of $\mathbb{R}^n$, that are invariant under deck transformations, and together with the fact that H is uniformly bounded and with the global product structure, this allows us to bound the lengths properly. In particular, for dimension 2, $W^{u/s}_F$ is homeomorphic to a foliation by lines, therefore it is Reebless and, by the classification of foliations on compact surfaces given in [30, §4.3], $W^{u/s}_F$ is the suspension of a circle homeomorphism, thus being quasi-isometric (this is similar to the argument given in [29] to prove dynamical coherence for special partially hyperbolic endomorphisms on $\mathbb{T}^2$).

This argument does not apply for W^u_F if f is not special, since this foliation does not project to a foliation of $\mathbb{T}^n$, but the quasi-isometry of W^u_F follows from [27, Proposition 2.10] for partially hyperbolic endomorphisms on $\mathbb{T}^2$.

Concluding, for Anosov endomorphisms, the stable foliation is always quasi-isometric. We have quasi-isometry of the unstable manifolds for special Anosov endomorphisms, it only is guaranteed locally around the linear maps for the general non-special case, and on $\mathbb{T}^2$ it always holds. We still do not know if there are examples of Anosov endomorphisms on $\mathbb{T}^n$ with unstable foliations that fail to be quasi-isometric, or if it can be proved that Anosov endomorphisms always have quasi-isometric unstable manifolds.

1.3.1 Shadowing and c-expansivity

In this section, we prove that Anosov endomorphisms have two properties that, together, imply the structurally stability at the natural extension level. These are the *shadowing property* and *c-expansivity*. In the invertible case, this proves the structural stability.

A δ-*pseudo-orbit* is a sequence $\{x_i\}_{i\in\mathbb{Z}\cap[a,b]}$ of points of X, $a < b \in \mathbb{Z}$, such that $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z} \cap [a, b)$. If $a = -\infty$ and $b = \infty$, we say that the δ-pseudo-orbit is bi-infinite.

Essentially, pseudo orbits are orbits allowing a small error at each step, but on a non-cumulative fashion: after an iteration, we are allowed to have instead a point near the "right one", but then we iterate the "right one" and get a new neighborhood of allowed points.

We say that a δ-pseudo orbit is ε-*shadowed* by $x \in X$ if $d(f^i(x), x_i) \leqslant \varepsilon$ for $i \in \mathbb{Z} \cap [a, b]$ (with $a \geqslant 0$ if f is non-invertible). The map $f : X \to X$ has the *shadowing property* if, for $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit is ε-shadowed by a $x \in X$.

We can shadow a bi-infinite δ-pseudo-orbit for an endomorphism using points of the natural extension.

The shadowing property gives us that every "approximated" orbit can be replaced by a real orbit nearby. To prove that it is the case for Anosov endomorphisms, let us first establish the local product structure.

The following is a consequence of the stable manifold theorem [51, Theorem IV.2.3], [49, Theorems 2.3 and 2.6].

Theorem 8 (Local product structure for endomorphisms [51, 49])**.** *If Λ a hyperbolic set for the endomorphism $f : U \subseteq M \to M$, then, given $\varepsilon > 0$ small enough, there is $0 < \delta < \varepsilon$ such that*

1. *For $\tilde{y} \in \Lambda^f$ and $x_0 \in \Lambda$ with $d(x_0, y_0) < \delta$, we have that $W^u_\varepsilon(\tilde{y}) \pitchfork W^s_\varepsilon(x_0)$ is a single point $z_{x_0,\tilde{y}}$. See Figure 5.*
2. *If $U_\delta(\Lambda) = \{(x_0, \tilde{y} \in \Lambda \times \Lambda^f \ : \ d(x_0, y_0) < \delta\}$, then*
$$\begin{array}{rccc} [.,.]_{\varepsilon,\delta} : & U_\delta(\Lambda) & \to & \Lambda \\ & (x_0, \tilde{y}) & \mapsto & z_{x_0,\tilde{y}} \end{array}$$
is continuous.
3. *If Λ is a basic (maximal and transitive) subset for f, then for all $(x_0, \tilde{y}) \in U_\delta(\Lambda)$ there is a single point $\tilde{w} \in \Lambda^f$ such that $\pi(\tilde{w}) = z_{x_0,\tilde{y}}$ and $d(w_{-n}, y_{-n}) < \varepsilon$ for all $n \in \mathbb{Z}^+$.*

Remark 2. We need that Λ is an axiom A basic set only for the uniqueness of $\tilde{w}$. Its existence is given by the definition of local unstable manifold.

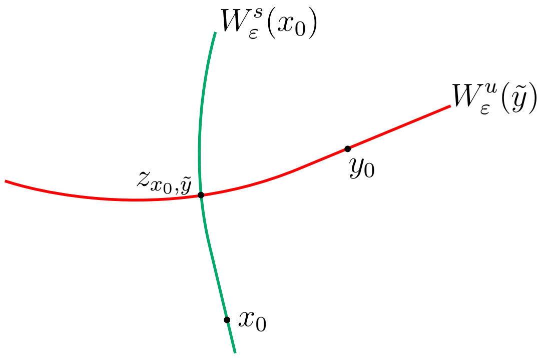

Figure 5 – The local product between x_0 and y_0 is obtained from the intersection of the stable manifold for x_0 and an unstable manifold for y_0. The unstable manifold depends on a choice of pre images for y_0, given by $\tilde{y}$.

In what follows, we prove the Shadowing Lemma for endomorphisms, which is already stated in [49, Corollary 1.14] and [4, Theorem 1.2.1], with our proof here being an adaptation of [55, Proposition 8.20] to endomorphisms. Remember that a bi-infinite pseudo-orbit for and endomorphism $f : M \to M$ is not shadowed by M, but by a point of the natural extension M^f.

Theorem 9 (Shadowing Lemma)**.** *Let Λ be a hyperbolic set for the C^r endomorphism $f : U \subseteq M \to M$, $r > 1$. If Λ has local product structure, then for any $\beta > 0$ there is $\alpha > 0$ such that all α-pseudo-orbit in Λ is β-shadowed.*

Proof. Consider in M the adapted metric, that is, a metric equivalent to the Riemannian metric such that, for $\lambda \in (0,1)$,

$$||Df_x(v)|| \leqslant \lambda ||v|| \text{ if } v \in E^s(x)$$

$$||Df_{x_0}(v)|| \geqslant \lambda^{-1} ||v|| \text{ if } v \in E^u(\tilde{x}).$$

Let $0 < \varepsilon < (1-\lambda)\beta$ be sufficiently small for the existence of local stable and unstable manifolds. Consider $0 < \eta = \dfrac{\varepsilon}{1-\lambda} < \beta$. By Theorem 8, there is $0 < \delta < \min\{\varepsilon, \beta - \eta\}$ such that

$$\begin{array}{rccc} [.,.]_{\varepsilon,2\delta} : & U_\delta(\Lambda) & \to & \Lambda \\ & (x_0, \tilde{y}) & \mapsto & z_{x_0,\tilde{y}} \end{array}$$

is continuous.

We choose $0 < \alpha < \delta$ such that, given $\tilde{w} \in \Lambda^f$ with $d(z_0, w_0) < \alpha$ we have that

$$[z_0, \pi^{-1}(W^s_{\lambda\delta}(w_0) \cap \Lambda)] \subset W^s_\delta(z_0).$$

Such α exists by the continuity of the local product. See Figure 6.

Given a finite α-pseudo-orbit $\underline{x} = \{x_0, \ldots, x_n\}$ in Λ, we define inductively $\tilde{y}_0, \ldots, \tilde{y}_n$ with

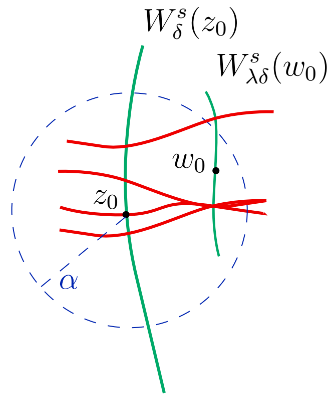

Figure 6 – We illustrate the choice of α in such way that the local unstable manifolds passing through the local stable manifold of w_0 intercept the local stable manifold of z_0 at a single point.

1. $y_0 := x_0 \in \Lambda$ and we fix a $\tilde{y}_0 \in \pi^{-1}(y_0)$.

2. $y_1 := [x_1, \tilde{f}(\tilde{y}_0)]$. This local product is well defined because

$$d(x_1, \pi(\tilde{f}(\tilde{y}_0))) = d(x_1, f(x_0)) < \alpha < \delta.$$

   Besides, from the definition of α and since $d(x_1, \pi(\tilde{f}(\tilde{y}_0))) < \alpha$, we have that $y_1 = [x_1, \tilde{f}(\tilde{y}_0)] \in W^s_\delta(x_1)$.

   We choose $\tilde{y}_1$ as a point of Λ^f such that $\pi(\tilde{y}_1) = y_1$ and $d((\tilde{y}_1)_{-n}, (\tilde{f}(\tilde{y}_0))_{-n}) < \varepsilon$ for all $n \in \mathbb{Z}^+$, that exists due to the definition of local stable manifold.

3. We define $y_k := [x_k, \tilde{f}(\tilde{y}_{k-1})]$ for $1 \leqslant k \leqslant n$, and $\tilde{y}_k$ as a point of Λ^f such that $\pi(\tilde{y}_k) = y_k$ and $d((\tilde{y}_k)_{-n}, (\tilde{f}(\tilde{y}_{k-1}))_{-n}) < \varepsilon$ for all $n \in \mathbb{Z}^+$.

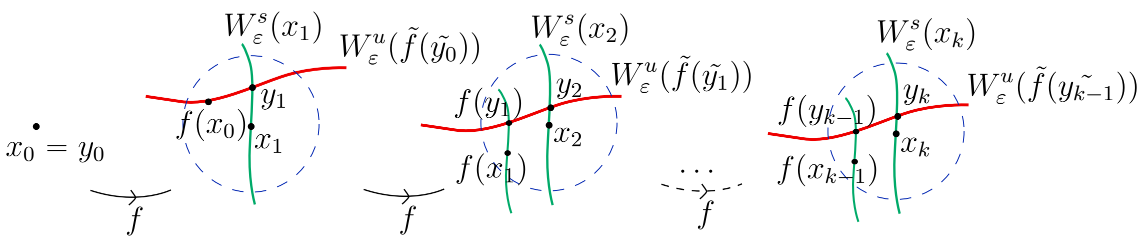

Figure 7 – We start with $x_0 = y_0$. At the next step, y_1 is defined as the intersection between the stable manifold of x_1 and the unstable manifold of $\tilde{f}(\tilde{y}_0)$. Thus, the define y_i inductively.

To check that y_k are well defined, we first prove by induction that $y_k \in W^s_\delta(x_k)$. Indeed, we saw that $y_1 \in W^s_\delta(x_1)$, so it remains to assume the induction hypothesis and conclude that it holds for k.

Suppose that $y_{k-1} \in W^s_\delta(x_{k-1})$. Then $f(y_{k-1}) \in W^s_{\lambda\delta}(f(x_{k-1}))$. Since

$$d(x_k, f(x_{k-1})) < \alpha,$$

by the definition of α, we have that

$$[x_k, \pi^{-1}(W^s_{\lambda\delta}(f(x_{k-1})) \cap \Lambda)] \subset W^s_\delta(x_k).$$

Therefore, $y_k = [x_k, \tilde{f}(\tilde{y}_{k-1})] \in W^s_\delta(x_k)$.

To conclude that $y_k = [x_k, \tilde{f}(\tilde{y}_{k-1})]$ is well defined, it suffices to prove that

$$d(x_k, \pi \circ \tilde{f}(\tilde{y}_{k-1})) = d(x_k, f(y_{k-1})) < 2\delta.$$

But $d(x_k, f(y_{k-1})) \leqslant d(x_k, f(x_{k-1})) + d(f(x_{k-1}), f(y_{k-1}))$. The first term is lesser than α, since $\underline{x}$ is an α-pseudo-orbit.

Since $y_{k-1} \in W^s_\delta(x_{k-1})$, then $f(y_{k-1}) \in W^s_{\lambda\delta}(f(x_{k-1}))$, thus

$$d(f(x_{k-1}), f(y_{k-1})) < \lambda\delta.$$

Therefore, $d(x_k, f(y_{k-1})) < \alpha + \lambda\delta < 2\delta$.

Now let us see how to use $\tilde{y}_i$ to find the shadowing orbit for the α-pseudo-orbit $\underline{x}$.

Since, by definition, $y_k \in W^u_\varepsilon(\tilde{f}(\tilde{y}_{k-1}))$, then, for $j \in Z^+$,

$$f^j(y_k) \in W^u_{\lambda^{-j}\varepsilon}(\tilde{f}^{j+1}(\tilde{y}_{k-1})) \tag{1.1}$$

and, with $\tilde{y}_k = (\dots, (\tilde{y}_k)_{-2}, (\tilde{y}_k)_{-1}, y_k, f(y_k), f^2(y_k), \dots)$, we have that

$$(\tilde{y}_k)_{-j} \in W^u_{\lambda^j\varepsilon}(\tilde{f}^{-(j-1)}(\tilde{y}_{k-1})). \tag{1.2}$$

Claim: $\tilde{y} := \tilde{f}^{-n}(\tilde{y}_n)$ β-shadows $\underline{x}$, that is, $y = \pi(\tilde{y}) = (\tilde{y}_n)_{-n}$ satisfies $d(f^i(y), x_i) < \beta$ for $i = 0, \dots, n$.

Indeed, we have that

$$d((\tilde{y}_k)_{-j}, y_{k-j}) < \theta_j := \sum_{i=1}^{j} \lambda^j \varepsilon, \tag{1.3}$$

for $k, j \in \mathbb{N}$ such that $0 \leqslant j \leqslant k \leqslant n$, that can be verified inductively over j by 1.2.

Remark 3. In [55], in the case that f is a diffeomorphism, we have that $(\tilde{y}_k)_{-j} \in W^u_{\theta_j}(y_{k-j})$, that does not hold in our case due to the existence of multiple unstable directions. However, the inequality 1.3 still holds. See Figure 8.

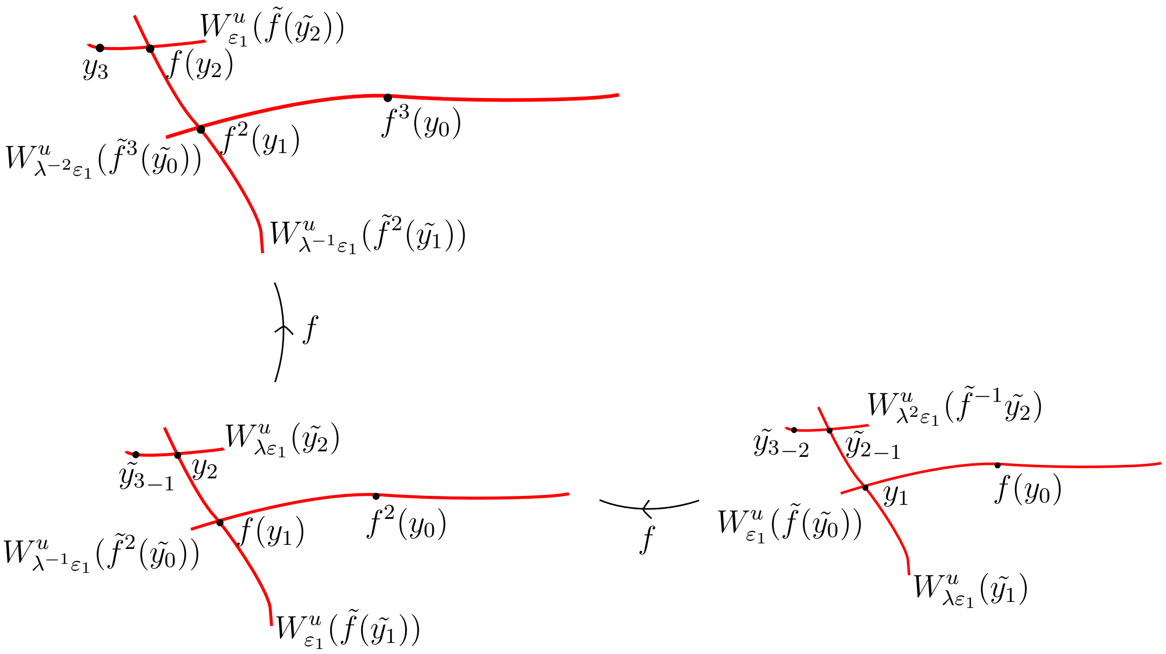

Figure 8 – The unstable leaf of $f(y_{k-1})$ is not necessarily the one of y_k, but we can still control their distances.

Thus, $d(y, x_0) = d((\tilde{y}_n)_{-n}, y_0) < \theta_n < \eta < \beta$ and

$$d(f(y), x_1) = d((\tilde{y}_n)_{-(n-1)}, x_1) \leqslant d((\tilde{y}_n)_{-(n-1)}, y_1) + d(y_1, x_1) < \eta + \delta < \beta$$

where the second term is lesser than δ since $y_k \in W^s_\delta(x_k)$ and the first one is lesser than η due to inequality 1.3.

In the same way, for the general case with $1 \leqslant i \leqslant n$,

$$d(f^i(y), x_i) = d((\tilde{y}_n)_{-(n-i)}, x_i) \leqslant d((\tilde{y}_n)_{-(n-i)}, y_i) + d(y_i, x_i) < \eta + \delta < \beta,$$

and this concludes the claim.

For any finite pseudo-orbit, the previous proof applies by reindexing the sequence to make the first term x_0.

If $\underline{x}$ is a bi-infinite α-pseudo-orbit,

$$\underline{x} = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots),$$

each finite segment $\underline{x}_{a,b}$ of $\underline{x}$ is β-shadowed by an orbit $\tilde{y}_{a,b}$. Let

$$\mathcal{A} := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \text{ such that } a < b - 1\},$$

then $\{\tilde{y}_{a,b} : (a, b) \in \mathcal{A}\}$ is countable in Λ^f compact, thus it has an accumulation point $\tilde{y}$, that β-shadows $\underline{x}$. □

To apply the Shadowing Lemma to construct a conjugacy between the natural extensions of nearby Anosov endomorphisms, we need the uniqueness of the shadowing for small β. In the invertible case, this is guaranteed by the expansivity, and here we generalize this concept.

Definition 24. If $f : X \to X$ is a continuous and surjective function the compact metric space X, we say that f is *c-expansive* if there is $\varepsilon > 0$ such that, if $\tilde{x}$ and $\tilde{y}$ are distinct points of the natural extension X^f, there is $i \in \mathbb{Z}$ such that $d(x_i, y_i) > \varepsilon$.

Equivalently, f is c-expansive if there is $\varepsilon > 0$ such that if $d(x_k, y_y) < \varepsilon$ for all $k \in \mathbb{Z}$, then $\tilde{x} = \tilde{y}$.

See [4] for the properties of c-expansive functions. For instance, the following result [4, Theorem 2.2.29] can be easily verified.

Proposition 7 ([4])**.** *$f : X \to X$ is c-expansive if and only if $\tilde{f} : X^f \to X^f$ is expansive at the natural extension.*

Other result of [4] is the following.

Proposition 8 ([4])**.** *If Λ is a hyperbolic set for the endomorphism $f : U \subseteq M \to M$, then f is c-expansive in Λ.*

Thus, we have c-expasivity for hyperbolic sets of endomorphisms, and taking β in Theorem 9 to be half the expansivity constant, we get uniqueness. The conjugacy between $\tilde{f}$ and $\tilde{g}$, the inverse limit of two nearby Anosov endomorphisms, is given as follows: if $d_{C^1}(f, g) < \alpha$, then any $\tilde{x} \in M^f$ is an α-pseudo orbit for g, thus it is β-shadowed by a unique $\tilde{y} \in M^g$; we define $\tilde{h}(\tilde{x}) = \tilde{y}$ that takes each pseudo orbit to its shadowing, and we can prove that $\tilde{h}$ is a conjugacy, as in the invertible case [52, Theorem 7.1].

The proofs of this section are valid without the differentiable structure. We say that a local homeomorphism $f : M \to M$ is an *topological Anosov map* if it is a continuous surjection that has the shadowing property and is c-expansive. Topological Anosov maps are stable on the natural extension [4, Theorem 6.8.1].

Having a conjugacy between the natural extensions of two nearby Anosov endomorphisms could cause us to think we can project it somehow, but if it projects, then f and g would be at least topologically semiconjugate. In the case that g is special and f is not, it cannot occur, as we show in Proposition 11.

1.3.2 Conjugacy with a linear model

Given an Anosov endomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$, its *linearization* $A : \mathbb{T}^n \to \mathbb{T}^n$ is the unique linear toral endomorphism homotopic to f, that is, $A = f_*$ extended from

$\mathbb{Z}^n$ to $\mathbb{T}^n$. Often the behavior of f is related to the one of A. In fact, if f is invertible or expansive, f and A are topologically conjugate. In the more general non-invertible setting, since a conjugacy should preserve stable and unstable manifolds, it does not exist if f is not special.

With the results of the previous subsection, we see that f and A are conjugate at the natural extension level [49, Theorem 1.20], but if this conjugacy projects to the original manifold $\mathbb{T}^n$ then f is special, a necessary condition to establish the conjugacy between f and A. We can approach the conjugacy using the universal cover, as was classically done for diffeomorphisms [21, 39].

The version of Theorem E given by F. Micena requires the Anosov endomorphism to be *strongly special* — that is, $W_f^s(x)$ is dense for each $x \in M$ — in order to guarantee the existence of conjugacy with its linearization. This relies on Proposition 9 stated below and given by Aoki and Hiraide in [4] for topological Anosov maps.

Proposition 9 ([4])**.** *If $f : \mathbb{T}^n \to \mathbb{T}^n$ is a strongly special Anosov endomorphism, then its linearization A is hyperbolic and f is topologically conjugate to A.*

We know that Anosov endomorphisms on tori are topologically transitive, and a classical open question is whether every Anosov diffeomorphism is transitive. For diffeomorphisms, transitivity is equivalent to the density of stable and unstable manifolds.

However even in the transitive case, general Anosov endomorphisms may have non-dense stable manifolds. For instance, consider the linear Anosov endomorphism on $\mathbb{T}^3$ induced by the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

It is easy to check that $\dim E^u = 2$, $\dim E^s = 1$, $A : \mathbb{T}^3 \to \mathbb{T}^3$ is transitive and $W_A^u(x)$ is dense in $\mathbb{T}^3$ for each x, but, if $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, $W_A^s(x)$ is restricted to $\mathbb{T}^2 \times \{x_3\}$, then it is not dense.

More recently, S. Moosavi and K. Tajbakhsh [45] proved, similarly to Aoki and Hiraide, that the hypothesis on the density of the stable set is not required, and extended the conjugacy result to topological Anosov maps on nil-manifolds. As a consequence, we have the following.

Proposition 10 ([58, 45])**.** *An Anosov endomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ is special if and only if it is conjugate to its linearization by a map $h : \mathbb{T}^n \to \mathbb{T}^n$ homotopic to Id.*

We saw in Example 1 that a small perturbation of a special Anosov endomorphism may be not special, and in particular, a perturbation can have uncountable many unstable directions on a given point. This is an obstruction to topological conjugacy. On the universal cover, however, we do have a conjugacy.

If $f : \mathbb{T}^n \to \mathbb{T}^n$ is an Anosov endomorphism, by [4, Theorem 8.2.1] there is a unique continuous surjection $H : \mathbb{R}^n \to \mathbb{R}^n$ on the universal cover with

- $A \circ H = H \circ F$;
- H is uniformly close to Id;
- H is uniformly continuous.

And, by [4, Proposition 8.4.2], the inverse H^{-1} exists and it is uniformly continuous, regardless of the distance between f and A. These two results hold for f topological Anosov map on the n-torus, with the c-expansivity playing a key hole to the injectivity of H as in Section 1.3.1.

A necessary condition for H to project to the torus is that f is special. In fact, we prove that even the existence of a semiconjugacy with the linearization on $\mathbb{T}^n$ implies that f is special.

Proposition 11 ([12]). *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an Anosov endomorphism and A its linearization. If A is a factor of f, then f is special.*

Proof. Supposing that A is a factor of f, then there is a continuous surjective map $h : \mathbb{T}^n \to \mathbb{T}^n$ with $h \circ f = A \circ h$. Considering H a lift of h to $\mathbb{R}^n$, we have that $H(x+a) = H(x)+Ba$ for each $x \in \mathbb{R}^n$ and $a \in \mathbb{Z}^n$, where $B : \mathbb{Z}^n \to \mathbb{Z}^n$. Let $F, A : \mathbb{R}^n \to \mathbb{R}^n$ be lifts of f and A.

For $x \sim y$, i.e., $y = x + a$ with $a \in \mathbb{Z}^n$, we prove that $p(W^u_F(x)) = p(W^u_F(y))$. Since H takes unstable leaves of F to unstable lines of A, then

$$H(W^u_F(x)) = W^u_A(H(x)) \text{ and}$$

$$H(W^u_F(y)) = H(W^u_F(x + a)) = W^u_A(H(x + a)) = W^u_A(H(x)) + Ba.$$

For any $z \in W^u_F(x)$, we have $H(z) \in W^u_A(H(x))$, then

$$H(z) + Ba \in W^u_A(H(x)) + Ba = H(W^u_F(y)).$$

By [4], H is invertible, then $H(z)+Ba = H(z+a)$ and, therefore, $z+a \in W^u_F(y)$. Therefore $p(W^u_F(x)) \subseteq p(W^u_F(y)))$, and the converse inclusion is analogous.

Thus, the set of unstable directions projected from the universal cover for each point in $\mathbb{T}^n$ is unitary. Since the set of all unstable directions for a point is the closure of the ones projected from the universal cover [44, Proposition 2.5], we conclude that f is special. □

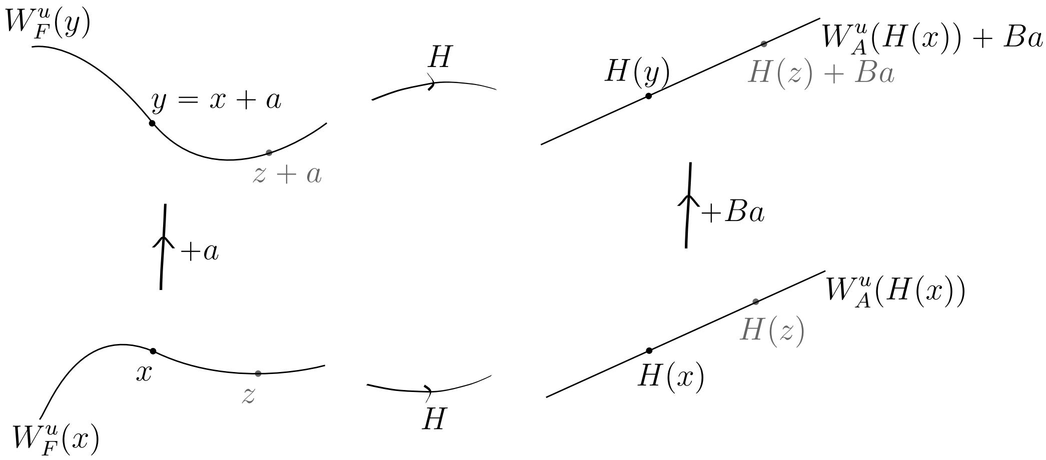

Figure 9 – For any point on $W^u_F(x)$, we show that its image under a deck transformation $+a$ is contained on $W^u_F(x + a)$.

In the conclusion of the previous result, we get that the invariance under deck transformations of the unstable leaves on the universal cover is equivalent to f being special, and the existence of a semiconjugacy on the torus would imply this invariance because H is invertible.

1.3.3 Lyapunov exponents

Given a differentiable dynamical system $f : M \to M$, Lyapunov exponents are quantities defined for almost every point that describe the rates of exponential growth along some directions of the tangent space T_xM. Their existence and properties are given by the *Multiplicative Ergodic Theorem*, as follows.

Theorem 10 ([51])**.** *For $f : M \to M$ a C^1 endomorphism, there is a f-invariant Borel subset $N \subseteq M$ that has full μ-measure for any f-invariant probability μ and*

1. *there is a measurable function $r : N \to \mathbb{Z}^+$ with $r \circ f = r$;*
2. *there are real numbers $+\infty > \lambda^1(x) > \lambda^2(x) > \cdots > \lambda^{r(x)}(x) \geqslant -\infty$ for all $x \in N$;*
3. *for all $x \in N$, there are linear subspaces of T_xM with*
$$V^{(0)}(x) = T_xM \supset V^{(1)} \supset \cdots \supset V^{(r(x))} = \{0\};$$
4. *for $x \in N$ and $1 \leqslant i \leqslant r(x)$, we have for all $v \in V^{(i-1)}(x) \backslash V^{(i)}(x)$ that*
$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x) \cdot v| = \lambda^i(x) \text{ and}$$
$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x) \cdot v| = \sum_{i=1}^{r(x)} \lambda^i(x) m_i(x),$$
where $m_i(x) = \dim V^{(i-1)}(x) - \dim V^{(i)}(x)$ for all $1 \leqslant i \leqslant r(x)$;

5. *$\lambda^i(x)$ is measurably defined on $\{x \in N \,:\, r(x) \geqslant i\}$ and f-invariant;*

6. *$V^{(i)}(x)$ is Df-invariant for all $1 \leqslant i \leqslant r(x)$.*

The numbers $\lambda^1(x), \cdots \lambda^{r(x)}(x)$ are called *Lyapunov exponents* of f at the point x and $m_i(x)$ are their *multiplicity*.

If (f, μ) is ergodic, then $r(x)$ and $\lambda^i(x)$, for all $1 \leqslant i \leqslant r(x)$, are constants μ-almost everywhere. Additionally, if (f, μ) satisfies the integrability condition $\log|\det Df(x)| \in L^1(\mu)$, then we have that

$$\int_M \log|\det Df(x)|\mu(x) = \int_N \sum_{i=1}^{r(x)} \lambda^i(x) m_i(x)\mu(x).$$

We saw in Section 1.1.1 that the conjugacy of derivatives of corresponding periodic points is a a necessary condition to have C^1 conjugacy. For our rigidity result, Theorem E, we require that corresponding periodic points have the same Lyapunov exponents. Let us see that these conditions are equivalent in the context of this work.

Consider $f, g : M \to M$ topologically conjugate endomorphisms, $h \circ f = g \circ h$. If $Df^p_x = B^{-1} \circ Dg^p_{h(x)} \circ B$ for all periodic x with $f^p(x) = x$, then the matrix B gives a change of basis between the $V^{(i)}$ directions of f and g, with the same Lyapunov exponents at the periodic points.

Reciprocally, if $Df(x)$ and $Dg(x)$ are diagonalizable for each $x \in M$ — that is the case if $M = \mathbb{T}^2$ and f and g are uniformly hyperbolic —, then

$$\lambda^i_f(x) = \lambda^i_g(h(x))$$

implies the conjugacy of derivatives of corresponding periodic points. Indeed, for all $n \in \mathbb{N}$, there are $k \in \mathbb{N}$ and $r \in \{0, \cdots, p-1\}$ such that $n = kp + r$. Then we can decompose the derivatives to compute the Lyapunov exponents as

$$Df^n(x) = D(f^k \circ D^{kp})(x) = Df^r(x)(Df^p(x))^k$$
$$\text{and, analogously, } Dg^n(h(x)) = Dg^r(h(x))(Dg^p(h(x)))^k.$$

Then we have that

$$\begin{aligned}
\lambda^i_f(x) &= \lim_{n\to\infty} \frac{1}{n} \log|Df^r(x)(Df^p(x))^k| \\
&= \lim_{k\to\infty} \frac{1}{kp+r} \log|Df^r(x)| + \lim_{k\to\infty} \frac{k}{kp+r} \log|Df^p(x)| \\
&= \frac{1}{p} \log|Df^p(x)|,
\end{aligned}$$

and $\lambda^i_g(h(x)) = \dfrac{1}{p} \log|Dg^p(h(x))|$ along one dimensional corresponding directions, that is, $|Df^p(x)| = |Dg^p(h(x))|$ along these directions and the matrices are conjugate.

2 Holonomies for foliations with extreme disintegration behavior

In this chapter, we explore the relation of two approaches to the regularity of a foliation with respect to the Lebesgue measure: by the regularity of the associated conditional measures (leafwise absolute continuity), or by the measurable regularity of the holonomy maps (transverse absolute continuity), as introduced in Sections 1.2.3 and 1.2.4.

As we have mentioned, the transverse absolute continuity is stronger than leafwise absolute continuity, and, in Theorem A, we prove an analogous result for their respective uniform formulations: transverse absolute continuity with uniformly bounded Jacobians is stronger than leafwise absolute continuity with uniformly bounded densities (and we call this *UBD property*).

Conversely, in the unpublished notes [50], the authors prove that leafwise absolute continuity implies transverse absolute continuity "almost everywhere". More precisely, they prove that, if a foliation is leafwise absolutely continuous, then for any transversely absolutely continuous transverse local foliation $\mathcal{T}$, we have that the $\mathcal{F}$-holonomies between almost every pair of $\mathcal{T}$-leaves are absolutely continuous. We also provide a proof of a similar result for foliations having the UBD property. In Theorem B, we prove that if a foliation has the UBD property, then for any transversely absolutely continuous local foliation $\mathcal{T}$ with uniformly bounded Jacobians and transverse to $\mathcal{F}$, we have that the $\mathcal{F}$-holonomies between almost every pair of $\mathcal{T}$-leaves are absolutely continuous with uniformly bounded Jacobians. This result answers a question left as a footnote in [43], when the authors define the UBD property.

These results also motivate the hypothesis that the stable and unstable foliations have the UBD property in Theorem D that we prove in the next chapter, since requiring that the Jacobian for the holonomies are uniformly bounded is common when studying hyperbolic dynamics.

Another extreme behavior for a foliation — in contrast to the regularity given by absolute continuity — is atomic disintegration, that is, when the conditionals are sums of Dirac measures for almost every leaf. In this case, if almost every leaf has at most k atoms, we prove in Theorem 5 that almost every holonomy should take a full Lebesgue measure set to a null one.

2.1 Proof of Theorem A

Consider $\mathcal{T}$ a transverse local foliation to $\mathcal{F}$. Take $\mathcal{T}$ to be transversely absolutely continuous with uniformly bounded Jacobians and with the UBD property. There is a foliation coordinate chart (U, h) for $\mathcal{F}$ such that U is a foliated box for both $\mathcal{F}$ and $\mathcal{T}$. Hereafter, all the computations take place on U, and we omit it from the notation.

Let m be the normalized volume on U, $\lambda_{\mathcal{F}(x)}$ the induced normalized volume on $\mathcal{F}(x)$ and $m_{\mathcal{F}}(x)$ the conditional of the volume on $\mathcal{F}(x)$.

Since $\mathcal{T}$ has the UBD property, then the densities $\delta_y = \dfrac{dm_{\mathcal{T}(y)}}{d\lambda_{\mathcal{T}(y)}}$ are uniformly bounded for each leaf $\mathcal{T}(y)$.

For each measurable $A \subseteq U$ we have, using the conditionals, that

$$m(A) = \int_{\mathcal{F}(x)} \int_{\mathcal{T}(y)} \chi_A(z)\delta_y(z)d\lambda_{\mathcal{T}(y)}(z)d\lambda_{\mathcal{F}(x)}(y). \tag{2.1}$$

Fixed $x \in U$, consider the holonomy $h_{\mathcal{F},y} : \mathcal{T}(x) \to \mathcal{T}(y)$ that takes $\mathcal{T}(x)$ to $\mathcal{T}(y)$ through the leaves of $\mathcal{F}$ and $q_{\mathcal{F},y}$ its Jacobian. Then

$$\int_{\mathcal{T}(y)} \chi_A(z)\delta_y(z)d\lambda_{\mathcal{T}(y)}(z) = \int_{\mathcal{T}(x)} \chi_A(h_{\mathcal{F},y}(s))q_{\mathcal{F},y}(s)\delta_y(h_{\mathcal{F},y}(s))d\lambda_{\mathcal{T}(x)}(s), \tag{2.2}$$

since

$$\int_{\mathcal{T}(x)} \chi_C(z)q_{\mathcal{F},y}(z)d\lambda_{\mathcal{T}(x)} = \int_{\mathcal{T}(y)} \chi_{h_{\mathcal{F},y}(C)}(s)d\lambda_{\mathcal{T}(y)}(s)$$

by the definition of Jacobian, and $\chi_{f(C)} = \chi_C \circ f^{-1}$.

By replacing (2.2) in the equation (2.1) and switching the order of integration, we have that

$$m(A) = \int_{\mathcal{T}(x)} \int_{\mathcal{F}(x)} \chi_A(h_{\mathcal{F},y}(s))q_{\mathcal{F},y}(s)\delta_y(h_{\mathcal{F},y}(s))d\lambda_{\mathcal{F}(x)}(y)d\lambda_{\mathcal{T}(x)}(s). \tag{2.3}$$

Consider the holonomy $h_{\mathcal{T},s} : \mathcal{F}(x) \to \mathcal{F}(s)$ that takes $\mathcal{F}(x)$ to $\mathcal{F}(s)$ through the leaves of $\mathcal{T}$ and $q_{\mathcal{T},s}$ its Jacobian. Consider the change of variables $r = h_{\mathcal{F},y}(s)$, $y = h_{\mathcal{T},x}(r) = h^{-1}_{\mathcal{T},s}(r)$. Then

$$\int_{\mathcal{F}(x)} \chi_A(h_{\mathcal{F},y}(s))q_{\mathcal{F},y}(s)\delta_y(h_{\mathcal{F},y}(s))d\lambda_{\mathcal{F}(x)}(y) = \int_{\mathcal{F}(s)} \chi_A(r)q_{\mathcal{F},y}(s)\delta_y(r)q^{-1}_{\mathcal{T},s}(r)d\lambda_{\mathcal{F}(s)}(r), \tag{2.4}$$

therefore

$$m(A) = \int_{\mathcal{T}(x)} \int_{\mathcal{F}(s)} \chi_A(r)\delta_y(r)q^{-1}_{\mathcal{T},s}(r)q_{\mathcal{F},y}(s)d\lambda_{\mathcal{F}(s)}(r)d\lambda_{\mathcal{T}(x)}(s). \tag{2.5}$$

Thus, $\delta_s(r) := \delta_y(r)q^{-1}_{\mathcal{T},s}(r)q_{\mathcal{F},y}(s)$ is the density $\dfrac{dm_{\mathcal{F}(s)}}{d\lambda_{\mathcal{F}(s)}}$. The Jacobian $q^{-1}_{\mathcal{T},s}$ is uniformly bounded, since $\mathcal{T}$ is transversely absolutely continuous with uniformly bounded

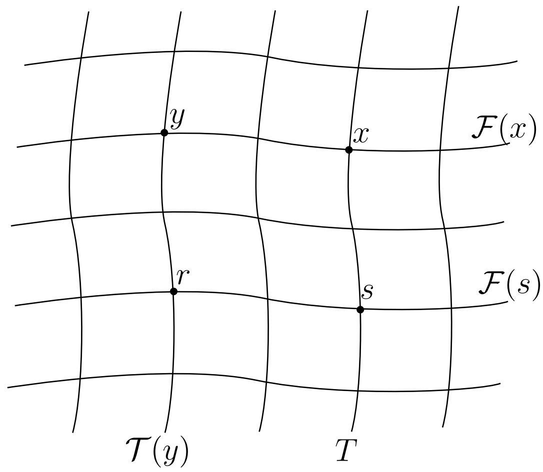

Figure 10 – We change the variables from y and s to r and s.

Jacobians, and $\mathcal{T}$ also has the UBD property, then $\delta_y(r)$ is also uniformly bounded. Finally, $q_{\mathcal{F},y}(s)$ is uniformly bounded by our hypothesis on $\mathcal{F}$, and we conclude that δ_s is uniformly bounded.

2.2 Proof of Theorem B

Let (U, h) be a foliation coordinate chart for $\mathcal{F}$ in such way that U is also a foliated box for $\mathcal{T}$. The strategy of this proof is to construct a family of functions that have uniformly bounded Jacobian, that restricted to $\mathcal{F}$-leaves are translations with uniformly bounded Jacobian, and that restricted to $\mathcal{T}$-leaves are holonomies along $\mathcal{F}$. Using this family of functions, we prove that $h_{\mathcal{F}}$ has uniformly bounded Jacobian for almost every pair of $\mathcal{T}$-leaves.

Fixed $F = \mathcal{F}(x)$ a leaf of $\mathcal{F}$ on U, there is a diffeomorphism $i : F \to \mathbb{R}^k$, which allows us to pullback to F the action by translations on $\mathbb{R}^k$. More precisely, if the action $a : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is defined by $g \cdot x = a(g, x) = g + x$, then the pullback action is given by

$$\begin{array}{rccc} \overline{a}: & \mathbb{R}^k \times F & \to & F \\ & (g, x) & \mapsto & i^{-1}(g \cdot i(x)). \end{array}$$

Having no risk of ambiguity, we denote $g \cdot x = \overline{a}(g, x)$ as the induced action on F.

For every $g \in \mathbb{R}^k$, since $a(g, \cdot)$ is smooth and preserves the Lebesgue measure on $\mathbb{R}^k$, then $\overline{a}(g, \cdot)$ is C^1 (it is as regular as the leaves of $\mathcal{F}$) and absolutely continuous. However, since i takes a bounded manifold F to $\mathbb{R}^k$, it takes sets of fixed volume on F to arbitrarily large sets on $\mathbb{R}^k$, therefore $\overline{a}(g, \cdot)$ does not have bounded Jacobian.

To overcome this, consider $V \subseteq U$ a compact foliated box for $\mathcal{F}$ and $\mathcal{T}$, and $\tilde{F} = F \cap V$. We only consider holonomies between leaves of $\mathcal{T}$ intercepting $\tilde{F}$, that is, between $\mathcal{T}(x)$ and $\mathcal{T}(x')$ with $x, x' \in \tilde{F}$. Then, consider $A \subseteq \mathbb{R}^k$ such that $g \in A$ implies that $\overline{a}(g, \cdot) : \tilde{F} \to F$ has image satisfying $\overline{a}(g, \tilde{F}) \cap \tilde{F} \neq \varnothing$. It is easy to see that A is closed and its interior is a non-empty neighborhood of 0 in $\mathbb{R}^k$. For $g \in A$, let $F_g := \overline{a}(-g, \overline{a}(g, \tilde{F}) \cap \tilde{F}) \neq \varnothing$ be the pre image of $\overline{a}(g, \tilde{F}) \cap \tilde{F}$ under $\overline{a}(g, \cdot)$. In other words, F_g is the subset of $\tilde{F}$ that remains inside $\tilde{F}$ after the translation by g. Then there exists a constant C_V such that, for all $g \in A$, $\overline{a}(g, \cdot)$ restricted to F_g has Jacobian $q_{\overline{a},g}$ satisfying

$$\frac{1}{C_V} < q_{\overline{a},g} < C_V,$$

since i restricted to $\tilde{F}$ and i^{-1} restricted to $d(\tilde{F})$ are C^1 functions over fixed compact sets.

Hereafter, all the leaves are local leaves with respect to the foliated box V. We can use the holonomies along the leaves of $\mathcal{F}$ to define, for each $g \in A$, a homeomorphism

$$h_g : V_g \to h_g(V_g) \subseteq V, \text{ where } V_g = \bigcup_{x \in F_g} \mathcal{T}(x).$$

We define h_g as follows: for $y \in V_g$, there is $x \in F_g$ such that $y \in \mathcal{T}(x)$. Considering the holonomy $h_{\mathcal{F}} : \mathcal{T}(x) \to \mathcal{T}(g \cdot x)$,

$$\begin{array}{rccc} h_g : & V_g & \to & h_g(V_g) \\ & y & \mapsto & h_{\mathcal{F}}(y). \end{array}$$

In other words, $h_g(y) = h_{\mathcal{T}}(g \cdot h_{\mathcal{T}}^{-1}(y))$, where $h_{\mathcal{T}} : F \to \mathcal{F}(y)$ is the holonomy along the leaves of $\mathcal{T}$. Also, by the construction of h_g, its restriction to $\mathcal{T}$-leaves is the holonomy $h_{\mathcal{F}} : \mathcal{T}(x) \to \mathcal{T}(g \cdot x)$.

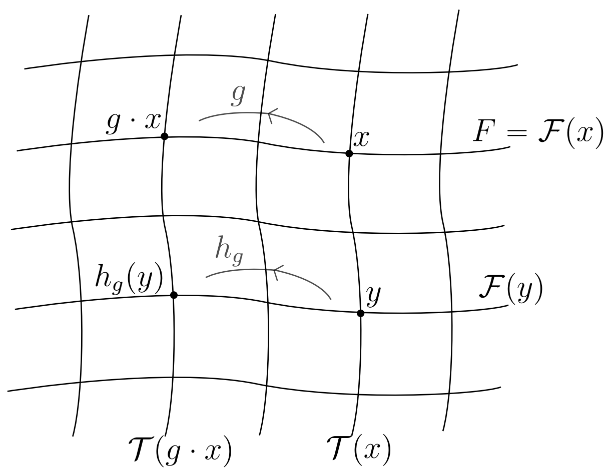

Figure 11 – We define h_g as a translation on $\mathcal{F}$-leaves with restrictions to $\mathcal{T}$-leaves being $\mathcal{F}$-holonomies.

We have that h_g restricted to $\mathcal{F}$-leaves is absolutely continuous with uniformly bounded Jacobian. Indeed, given a leaf $\mathcal{F}(y)$ and a measurable $I \subseteq \mathcal{F}(y) \cap V_g$, we have that $h_g(I) = h_{\mathcal{T}}(g \cdot h_{\mathcal{T}}^{-1}(I))$, with the holonomies $h_{\mathcal{T}}$ along the leaves of $\mathcal{T}$ having uniformly bounded Jacobian and $\overline{a}(g, \cdot)$ also being bounded with the constant C_V that does not depend on g.

Furthermore, if $z \in \mathcal{F}(y)$, then $h_g(z) \in \mathcal{F}(y)$. This means that h_g preserves a measure transverse to the foliation $\mathcal{F}$. But h_g restricted to $\mathcal{F}$-leaves is absolutely continuous with uniformly bounded Jacobian, thus $h_g : V_g \to h_g(V_g)$ is absolutely continuous with uniformly bounded Jacobian.

We have a map h_g that is absolutely continuous with uniformly bounded Jacobian for whole open sets in the manifold and along the foliation $\mathcal{F}$. But, restricted to $\mathcal{T}$-leaves, this map is precisely the $\mathcal{F}$-holonomies, which we want to prove that is absolutely continuous with uniformly bounded Jacobian along $\mathcal{T}$. Thus, the heart of the proof is the following lemma.

Lemma 1. *There exists a constant $C > 1$ such that, for each $g \in A$, there is a λ_{F_g}-full measure set $X_g \subseteq F_g$ such that for each $x \in X_g$ the holonomy $h_g|_{\mathcal{T}(x)} = h_{\mathcal{F}} : \mathcal{T}(x) \to \mathcal{T}(g \cdot x)$ is absolutely continuous and its Jacobian q satisfies $q \leqslant C$.*

Proof. Suppose that the lemma does not hold. Then for each $C > 1$ there exists $g \in A$ and $B = B_{C,g} \subseteq F_g$ with positive volume such that for each $x \in B$ we have that $h_g|_{\mathcal{T}(x)} : \mathcal{T}(x) \to \mathcal{T}(g \cdot x)$ does not have Jacobian q satisfying $q \leqslant C$. Thus, for each $x \in B$, there is $Z_{x,C} \subseteq \mathcal{T}(x)$ with positive volume such that $q(y) > C$ for all $y \in Z_{x,C}$.

Let z be a density point of $Z_{x,C}$ with respect to the volume measure $\lambda_{\mathcal{T}(x)}$ on the leaf. That is, if $B_{\mathcal{T}}(z, \delta)$ is the ball of center z and radius δ on the leaf and $Z_{x,C,\delta} := Z_{x,C} \cap B_{\mathcal{T}}(z, \delta)$, then

$$\lim_{\delta \to 0} \frac{\lambda_{\mathcal{T}(x)}(Z_{x,C,\delta})}{\lambda_{\mathcal{T}(x)}(B_{\mathcal{T}}(z, \delta))} = 1.$$

Since the foliation $\mathcal{T}$ is continuous, the holonomy $h_{\mathcal{F}}$ along $\mathcal{F}$ leaves is continuous and the Lebesgue volumes on $\mathcal{T}$-leaves vary continuously, then, for a sufficiently small interval $I \subseteq \mathcal{F}(z)$ containing z, we can estimate the volumes of the images of $Z_{x,C,\delta}$ under $\mathcal{F}$-holonomies and its images under h_g. More precisely, for all $w \in I$, considering $h_{\mathcal{F},w} : \mathcal{T}(x) \to \mathcal{T}(w)$, we have that $Z_{w,C,\delta} := h_{\mathcal{F},w}(Z_{x,C,\delta})$ satisfies

$$\frac{\delta}{2} \leqslant \lambda_{\mathcal{T}(w)}(Z_{w,C,\delta}) \leqslant \frac{3\delta}{2}.$$

Moreover, its image $h_g(Z_{w,C,\delta})$ satisfies

$$\lambda_{\mathcal{T}(g \cdot w)}(h_g(Z_{w,C,\delta})) \geqslant \frac{C\delta}{2},$$

since it is close to $h_g(Z_{x,C,\delta})$. Therefore, we have that

$$\lambda_{\mathcal{T}(g\cdot w)}(h_g(Z_{w,C,\delta})) > \frac{C}{4}\lambda_{\mathcal{T}(w)}(Z_{w,C,\delta}). \tag{2.6}$$

Fixed $C > 1$, a small $\delta > 0$ and I sufficiently small so that the above inequality holds, consider $Z := \bigcup_{w\in I} Z_{w,C,\delta}$. To see that Z is measurable, it suffices to "see" it as a product of I and $Z_{x,C,\delta}$. Indeed, consider

$$\begin{array}{rccc} \rho: & Z & \to & I \times Z_{x,C,\delta} \\ & p & \mapsto & (h_{\mathcal{T},z}(p), h_{\mathcal{F},z}(p)), \end{array}$$

where $h_{\mathcal{T},z} : \mathcal{F}(p) \to \mathcal{F}(z)$ is the holonomy along $\mathcal{T}$-leaves and $h_{\mathcal{F},z} : \mathcal{T}(p) \to \mathcal{T}(z)$ is the holonomy along $\mathcal{F}$-leaves. By our definition of Z and the fact that the holonomies are continuous, we have that ρ is a homeomorphism, and then Z is measurable.

Since $\mathcal{T}$ is transversely absolutely continuous by hypothesis, then Z has positive volume. We aim to conclude that $m(h_g(Z)) > \alpha C m(Z)$ for some fixed constat α, which leads to a contradiction to the fact that $h_g : V_g \to h_g(V_g)$ has uniformly bounded Jacobian.

Note that $h_g(Z) = h_g\left(\bigcup_{w\in I} Z_{w,C,\delta}\right) = \bigcup_{w\in I} h_g(Z_{w,C,\delta})$, then

$$\begin{aligned} m(h_g(Z)) &= \int_{h_g(I)} m_{\mathcal{T}(g\cdot w)}(h_g(Z_{w,C,\delta}))\, d\lambda_{\mathcal{F}} \\ &> \int_{h_g(I)} \frac{1}{K}\lambda_{\mathcal{T}(g\cdot w)}(h_g(Z_{w,C,\delta}))\, d\lambda_{\mathcal{F}} \\ &\overset{(2.6)}{\geqslant} \int_I \frac{C}{4K}\lambda_{\mathcal{T}(w)}(Z_{w,C,\delta})\, d\lambda_{\mathcal{F}} \\ &> \int_I \frac{C}{4K^2} m_{\mathcal{T}(w)}(Z_{w,C,\delta})\, d\lambda_{\mathcal{F}} = \frac{C}{4K^2}m(Z). \end{aligned}$$

where K is the constant from the UBD property. Since K does not depend on C or g, we have that for each $C > 1$, there is $g \in A$ such that h_g is not uniformly bounded, a contradiction. □

Analogously to Lemma 1, we prove that

Lemma 2. *There exists a constant $C > 1$ such that, for each $g \in A$, there is a λ_{F_g}-full measure set $Y_g \subseteq F_g$ such that for each $x \in Y_g$ the holonomy $h_g\big|_{\mathcal{T}(x)} = h_{\mathcal{F}} : \mathcal{T}(x) \to \mathcal{T}(g\cdot x)$ is absolutely continuous and its Jacobian q satisfies $\frac{1}{C} < q$.*

Now we proceed to prove that $h_{\mathcal{F}} : \mathcal{T}(x) \to \mathcal{T}(x')$ is absolutely continuous with uniformly bounded Jacobian for almost every $(x, x') \in \tilde{F} \times \tilde{F}$.

Consider

$$\tilde{F} \times \tilde{F} = \bigcup_{g \in A} \{(x, g \cdot x) : x \in F_g\},$$

as a partition of the product $\tilde{F} \times \tilde{F}$. This is indeed a partition since, for all $x, y \in \tilde{F}$, there exists a unique g such that $g \cdot x = y$. Moreover, $\tilde{F}$ is C^1, and each set $\mathcal{G}_g := \{(x, g \cdot x) : x \in F_g\}$ is also C^1. Therefore, the partition $\{\mathcal{G}_g\}_{g \in A}$ is in fact a foliation of $\tilde{F} \times \tilde{F}$ homeomorphic to an affine foliation of $\mathbb{R}^k \times \mathbb{R}^k$ by k-plaques restricted to the box $B(0,1) \times B(0,1)$, with $B(0,1)$ a unit ball in $\mathbb{R}^k$.

By the Lemmas 1 and 2, for each $g \in A$ there is a full volume set $Z_g = X_g \cap Y_g$ in F_g such that, for every $x \in Z_g$, the holonomy between $\mathcal{T}(x)$ and $\mathcal{T}(g \cdot x)$ is absolutely continuous with uniformly bounded Jacobian. Thus, for all

$$(x, x') \in \bigcup_{g \in A} \{(x, g \cdot x) : x \in Z_g\},$$

the holonomy $h_{\mathcal{F}} : \mathcal{T}(x) \to \mathcal{T}(x')$ is absolutely continuous with uniformly bounded Jacobian. But $Z_g \times g \cdot Z_g$ is a full measure subset of $\mathcal{G}_g$, then a Fubini argument guarantees that such pairs form a full measure set of $\tilde{F} \times \tilde{F}$.

3 Rigidity for Anosov Endomorphisms on surfaces

In this chapter, we prove two results to obtain as a corollary Theorem C, that characterizes, for Anosov endomorphisms on surfaces, the smooth conjugacy between an Anosov endomorphism and its linearization with the condition that the unstable and stable foliations have a uniform kind of leafwise absolute continuity. This condition is called UBD property, as defined in Section 1.2.4.

In the first result, Theorem D, we obtain a rigidity result relating the Lyapunov exponents with the regularity of the unstable and stable foliations. More precisely, given $f : \mathbb{T}^2 \to \mathbb{T}^2$ a smooth Anosov endomorphism, we prove that, if the stable and unstable foliations have the UBD property and a volume condition is satisfied, then the Lyapunov exponents of f are constant and equal to the ones of A. This volume condition replace the role of conservativity for the non-invertible case, and we introduce it with more details along the proof.

The second result, Theorem E, guarantee that, for two surface Anosov endomorphisms that are conjugate, if the Lyapunov exponents on corresponding periodic points coincide, then the conjugacy is as regular as the endomorphisms.

By joining Theorems D and E for f special with the volume condition and $g = A$, we have that the UBD property implies equality of Lyapunov exponents, which implies that h is as regular as f. Reciprocally, if the conjugacy h is C^∞, then a lift H of h takes unstable leaves of F to unstable lines of A, implying the UBD property on the unstable foliation of F. The same holds for the stable foliation, and Theorem C follows.

The results of this chapter were sent to publication and a pre-print is available at [12].

3.1 Proof of Theorem D

Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ to be a lift of f to $\mathbb{R}^n$ the universal cover, and $p : x \mapsto [x]$ to be the canonical projection from $\mathbb{R}^2$ to $\mathbb{T}^2$. By [4, Theorem 8.2.1] there is a conjugacy $H : \mathbb{R}^2 \to \mathbb{R}^2$ between F and A, with $d(H, Id) < \delta$. In particular, this gives us global product structure for the stable and unstable foliations of F.

Proposition 3 implies that the unstable and stable foliations W^u_F and W^s_F are transversely absolutely continuous, therefore being leafwise absolutely continuous. So, by requiring the Uniform Bounded Density property — promoting the leafwise absolute continuity to a uniform formulation — for both foliations, together with the volume condition given by Hypothesis (C), we prove that the exponents are constant at each point on $\mathbb{T}^2$. We proceed by proving that $\lambda_f^u \equiv \lambda_A^u$ and the proof for the stable exponents follows analogously by using F^{-1}.

We have that F has exactly one fixed point, because it is conjugate to A, and we can suppose that $F(0) = 0$. Consider $B := W_F^s(0)$ the stable leaf of 0 with respect to F. We have that B is F-invariant and the unstable leaves of F intersect B transversely at a unique point. Then we can define $p_F^u : \mathbb{R}^2 \to B$ as the projection taking each point z to $W_F^u(z) \cap B$. We can also fix an orientation on each leaf by choosing a connected component of $B \backslash \{0\}$ as positive. Assuming that F preserves the orientation on unstable manifolds, or working with F^2 instead, consider the *foliated strip*

$$\mathcal{B}_0 := \{y \in \mathbb{R}^2 : d^u(p_F^u(y), y) \leqslant \gamma_0,\ y \in W_F^{u,+}\},$$

where $\gamma_0 > 0$ is a constant that satisfies $p(\mathcal{B}_0) = \mathbb{T}^2$ and is greater than 4δ. We can visualize $\mathcal{B}_0$ as a strip "above" B whose projection covers the whole torus. Let $\mathcal{B}_k := F^k(\mathcal{B}_0)$ be the iterates of $\mathcal{B}_0$. Since B is F-invariant and it expands along the unstable leaves, $\mathcal{B}_{k-1} \subsetneq \mathcal{B}_k$.

Define $m^k := m\big|_{\mathcal{B}_k}$ to be the induced volume on $\mathcal{B}_k$. Even with the m^k being an infinite measure, we can still consider conditional probability measures on foliated boxes inside $\mathcal{B}_k$: m_x^k is the conditional probability measure in $W_k(x) := W_F^u(x) \cap \mathcal{B}_k$ for m-almost every point in $\mathcal{B}_k$. Additionally, this probability measure is unique up to multiplication by a constant for almost every leaf.

Remark 4. Given this setting, before we proceed with the proof, let us explain the Hypothesis (C). We want it to hold to guarantee that, as we iterate under F the conditionals on $\mathcal{B}_0$, obtaining $F_*^k m_{F^{-k}(x)}^0$, they remain uniformly equivalent to the normalized volume $\hat{\lambda}_x^k$ on W_k. This is the condition that allows us to compute the rates of exponential growth of f as being equal to the ones of A along this proof.

Denoting with $\sim$ the uniform equivalence between two measures, we already have that $m_x^k \sim \hat{\lambda}_x^k$ and $F_*^k m_{F^{-k}(x)}^0 \sim F_*^k \hat{\lambda}_{F^{-k}(x)}^0$ by the UBD property. So, to obtain $F_*^k m_{F^{-k}(x)}^0 \sim \hat{\lambda}_x^k$ as we wish, it suffices to have that $F_*^k m_{F^{-k}(x)}^0 \sim m_x^k$ or $F_*^k \hat{\lambda}_{F^{-k}(x)}^0 \sim \hat{\lambda}_x^k$, with the latter being Hypothesis (C), and the both being equivalent under the UBD property. Thus, in this case, Hypothesis (C) regards the relation between $F_*^k m_{F^{-k}(x)}^0$ and m_x^k. We have that

a) $dF_*^k m^0 = \underbrace{|\det DF^{-k}(\cdot)|}_{JF^{-k}(\cdot)} dm^k$ in $\mathcal{B}_k$;

b) $\{m_x^k\}_x$ is a disintegration of m^k;

c) $\{F_*^k m_{F^{-k}(x)}^0\}_x$ is a disintegration of $F_*^k m^0$.

Then, for any $\varphi : \mathcal{B}_k \to \mathbb{R}$, we have on one hand that

$$\int_{\mathcal{B}_k} \varphi dF_*^k m^0 \overset{c)}{=} \int_B \left(\int_{W_k(x)} \varphi(z) dF_*^k m^0_{F^{-k}(x)}(z) \right) dF_*^k \mu^0(x). \tag{3.1}$$

On the other hand, we have that

$$\int_{\mathcal{B}_k} \varphi dF_*^k m^0 \overset{a)}{=} \int_{\mathcal{B}_k} \varphi \; JF^{-k} dm^k \overset{b)}{=} \int_B \left(\int_{W_k(x)} \varphi(z) JF^{-k}(z) dm_x^k(z) \right) d\mu^k(x). \tag{3.2}$$

By comparing (3.1) and (3.2), we have that

$$dF_*^k m^0_{F^{-k}(x)}(z) \frac{dF_*^k \mu^0}{d\mu^k}(x) = JF^{-k}(z) dm_x^k(z) \tag{3.3}$$

for any $z \in W_k(x)$. Therefore, Hypothesis (C) is equivalent (under the UBD property) to the existence of $C > 1$ such that, for all $x \in \mathcal{B}_k$ and $z \in W_k(x)$

$$C^{-1} \leqslant JF^k(z) \frac{dF_*^k \mu^0}{d\mu^k}(x) \leqslant C. \tag{3.4}$$

It is easy to check that this condition is satisfied if f is linear, or more generally if JF is constant. The transverse measures μ_k and $F_*^k \mu^0$ given by the disintegration satisfy

$$\frac{dF_*^k \mu^0}{d\mu^k}(x) = \lim_{\varepsilon \to 0} \frac{F_*^k \mu^0(I_\varepsilon^B)}{\mu_k(I_\varepsilon^B)} = \lim_{\varepsilon \to 0} \frac{m \circ F^{-k}(A_\varepsilon)}{m(A_\varepsilon)},$$

where $I_\varepsilon^B \subseteq B$ is a ball with center x and radius ε on $B = W_F^s(0)$, and $A_\varepsilon = \bigcup_{y \in I_\varepsilon^B} W_k(y)$. Since

$$m(A_\varepsilon) = \int_{F^{-k}(A_\varepsilon)} JF^k dm,$$

then Hypothesis (C) tells us that the Jacobian JF^k behaves regularity on $W_k(x)$ independently of $k \in \mathbb{N}$, replacing the role of conservativeness. This concludes our remarks on this hypothesis. Whether this can be obtained from the other hypotheses or be refined, is open to exploration.

The idea of this proof is to construct measures on the local leaves of $\mathcal{B}_k$ with densities that decrease as k grows to ∞ with a rate equal to α the unstable Lyapunov exponent of the linearization A, and in such way that they are invariant under the dynamics up to multiplication to α. This allows us to conclude that the unstable Lyapunov exponents of F and A are the same. More specifically, we want to construct measures η_x such that $F_* \eta_x = \alpha^{-1} \eta_{F(x)}$ and uniformly equivalent to the volume induced on the global leaves.

Consider the measures η_x^k defined inductively as follows:

$$\eta_x^0 := m_x^0 \text{ and}$$
$$\eta_x^k := \alpha F_* \eta_{F^{-1}(x)}^{k-1} = \alpha^k F_*^k m_{F^{-k}(x)}^0.$$

Here α is the unstable eigenvalue of A. Since $|\alpha| > 1$, η_x^k is not a probability. Actually, we prove that it is comparable with λ_x, where λ_x is the induced volume along the global leaf $W_F^u(x)$. For this, we use the quasi-isometry and the existence of a conjugacy uniformly near to Id to show that the length of the unstable segments $W_k(x)$ grows uniformly with α as we apply F.

Lemma 3. *There is $K > 1$ such that, for each $k \in \mathbb{N}$ and $x \in \mathcal{B}_0$, we have that*

$$K^{-1} \leqslant \frac{\lambda_{F^k(x)}(W_k(F^k(x)))}{\alpha^k} \leqslant K.$$

Proof. The quasi-isometry implies that, to estimate $\lambda_{F^k(x)}(W_k(F^k(x)))$, it suffices to estimate $\|a_k - b_k\|$, with a_k, b_k being the extreme points of $W_k(F^k(x))$. But we have that $a_k = F^k(a_0)$ and $b_k = F^k(b_0)$, with a_0 and b_0 the extreme points of $W_0(x)$. Then

$$\begin{aligned}\|F^k(a_0) - F^k(b_0)\| &\leqslant \|F^k(a_0) - H \circ F^k(a_0)\| + \|H \circ F^k(a_0) - H \circ F^k(b_0)\| \\ &+ \|H \circ F^k(b_0) - F^k(b_0)\| \leqslant 2\delta + \|A^k \circ H(a_0) - A^k \circ H(b_0)\|,\end{aligned}$$

where $H(a_0), H(b_0)$ are on the same unstable line for A, and $\delta = d(Id, H) > 0$.

Since

$$\begin{aligned}\|A^k \circ H(a_0) - A^k \circ H(b_0)\| &= \alpha^k \|H(a_0) - H(b_0)\| \\ &\leqslant \alpha^k(\|a_0 - b_0\| + 2\delta) \leqslant \alpha^k(\lambda_x(W_0(x)) + 2\delta),\end{aligned}$$

the upper bound follows because $\lambda_x(W_0(x)) = \gamma_0$, and the lower bound is analogous. □

The multiplicative ergodic theorem for endomorphisms (see [51], for instance), gives us that the unstable Lyapunov exponent of f on a given point does not depend on the choice of orbit that defines the unstable direction, and then we can obtain it at Lebesgue almost every point in $\mathcal{B}_0$, since $p(\mathcal{B}_0) = \mathbb{T}^2$, p is a local isometry and the projection of unstable leaves of F are unstable leaves of f.

The next estimations are for points $x \in \mathcal{B}_0$ such that m_x^k is well defined for all $k \geqslant 0$. We then construct the measures η_x as the limit of measures η_x^k for a given $x \in \mathcal{B}_0$. For that, we need η_x^k to be well defined for every large enough k, which is only guaranteed for a full volume set on the foliated strip $\mathcal{B}_0$. Indeed, for all $k \geqslant 0$, there is a full volume set $A_k \subseteq \mathcal{B}_k$ such that m_x^k is well defined for each $x \in A_k$. Considering $A := \bigcap_{k \geqslant 0} A_k$ and $D := \bigcap_{n \in \mathbb{N}} F^n(A)$, we have that D is F-invariant, has full measure in $\mathcal{B}_0$ and m_x^k is well defined for each $x \in D$ and $k \geqslant 0$.

Now we construct η_x for $x \in D$, and later, using the density of D, we construct using the holonomies other measures to compute the Lyapunov exponents for all point in $\mathcal{B}_0$.

Lemma 4. *For m-almost every $x \in \mathcal{B}_0$ there is a measure η_x on $W^u_F(x)$ such that $F_*\eta_x = \alpha^{-1}\eta_{F(x)}$ and $\eta_x = \rho_x\lambda_x$ with ρ_x uniformly bounded.*

Proof. By the definition of η^k_x, $F_*\eta^k_x = \alpha^{-1}\eta^{k+1}_{F(x)}$. Thus, it suffices to show that the sequence $\{\eta^k_x\}_{k\geqslant 0}$ has an accumulation point η_x such that $\eta^{k+1}_{F(x)}$ converges under the same subsequence to $\eta_{F(x)}$. First, to guarantee the existence of accumulation points to the sequence $\{\eta^k_x\}_{k\geqslant 0}$, we show that η^k_x is uniformly equivalent to the induced volume on the leaf, which will also imply the uniform bound for ρ_x.

Essentially, we have uniform equivalence between the following measures

1. $m^k_x \sim \hat{\lambda}^k_x$ by the UBD property;
2. $m^k_x \sim F^k_* m^0_{F^{-k}(x)}$ by the Hypothesis (C);
3. $F^k_* m^0_{F^{-k}(x)} \sim \hat{\lambda}^k_x$ by items 1 and 2;
4. $\eta^k_x = \alpha^k F^k_* m^0_{F^{-k}(x)} \sim \alpha^k \hat{\lambda}^k_x$.

By Lemma 3, $\alpha^k\hat{\lambda}^k_x$ is uniformly equivalent to λ_x, the induced volume on $W^u_F(x)$. Let us now formalize these ideas.

Consider $\rho_k(x,.) : W_k(x) \to \mathbb{R}$ to be the density relating the probability m^k_x and the normalized volume $\hat{\lambda}^k_x := \dfrac{\lambda_x}{\lambda_x(W_k(x))}$ on the leaf segment $W_k(x)$, i. e., $dm^k_x(.) = \rho_k(x,.)d\hat{\lambda}^k_x(.)$. For all $k \in \mathbb{N}$, since the unstable foliation of F has the UBD property, $\rho_k \in [C^{-1}, C]$, then

$$\frac{C^{-1}}{\lambda_x(W_k(x))}d\lambda_x \leqslant dm^k_x \leqslant \frac{C}{\lambda_x(W_k(x))}d\lambda_x. \tag{3.5}$$

So, the density $\dfrac{dm^k_x}{d\hat{\lambda}^k_x}$ is bounded with C^{-1} and C. By the Hypothesis (C), the density $\dfrac{dm^k_x}{dF^k_* m^0_{F^{-k}(x)}}$ is also bounded with C^{-1} and C, thus

$$C^{-2} \leqslant \frac{dF^k_* m^0_{F^{-k}(x)}}{d\hat{\lambda}^k_x} \leqslant C^2.$$

By multiplying by α^k, we have that

$$C^{-2}\alpha^k d\hat{\lambda}^k_x \leqslant d\eta^k_x \leqslant C^2\alpha^k d\hat{\lambda}^k_x.$$

By Lemma 3,

$$K^{-1}d\lambda_x \leqslant \alpha^k d\hat{\lambda}_x^k = \alpha^k \frac{d\lambda_x}{\lambda(W_K(x))} \leqslant K d\lambda_x.$$

This implies that

$$C^{-2}K^{-1}d\lambda_x \leqslant d\eta_x^k \leqslant C^2 K d\lambda_x$$

then the sequence $\{\eta_x^k\}_{k\geqslant 0}$ have an accumulation point. Let k_i index a subsequence such that $\eta_x := \lim_i \eta_x^{k_i}$. The sequence $\{\eta_{F(x)}^{k_i+1}\}_i$ has an accumulation point for a subsequence with indices that also make $\{\eta_x^k\}_{k>j}$ converge. By a diagonal argument, we get η_x as desired. □

To compute the Lyapunov exponents, we would like to have the measures η_x to be defined at every point in $\mathcal{B}_0$. But we only have them on D, a full volume set. To overcome this, for any $z \in \mathcal{B}_0$, we define the measures $\mathfrak{m}_z$ and $\mathfrak{M}_z$ using the stable holonomy h^s, which carries points from a local unstable leaf of F to other by traveling on stable leaves. Consider $(z_n)_n$ to be a sequence of points in $\mathcal{B}_0$ such that η_{z_n} exists and $z_n \xrightarrow{n\to\infty} z$. Consider $\mathcal{I}_z$ to be the set of connected intervals on $W_F^u(z)$. Define for all $I \in \mathcal{I}_z$

$$\mathfrak{m}_z(I) := \liminf_n \frac{1}{n}\sum_{i=0}^{n-1} \eta_{z_i}(h^s(I)),$$

$$\mathfrak{M}_z(I) := \limsup_n \frac{1}{n}\sum_{i=0}^{n-1} \eta_{z_i}(h^s(I)).$$

In the following, we see that these measures are uniformly equivalent to λ_x, the volume induced on the global leaves, and that they are well behaved with respect to F, as the measures η_x are. There is a constant γ and $(z_n)_n$ such that

$$\gamma^{-1}\lambda_z(I) \leqslant \mathfrak{m}_z(I) \leqslant \mathfrak{M}_z(I) \leqslant \gamma\lambda_z(I), \tag{3.6}$$

for all $I \in \mathcal{I}_z$ small enough, with γ independent of the choice of z. Indeed, by Lemma 4,

$$\mathfrak{m}_z(I) = \liminf_n \frac{1}{n}\sum_{i=0}^{n-1} \rho_{z_i}\lambda_{z_i}(h^s(I)) \geqslant C^{-2}K^{-1}\liminf_n \frac{1}{n}\sum_{i=0}^{n-1} \lambda_{z_i}(h^s(I)).$$

The holonomies are C^1 by Proposition 4, and for I small enough $\lambda_{z_i}(h^s(I))$ is uniformly close to $\lambda_z(I)$. Thus $\mathfrak{m}_z(I) \geqslant \gamma^{-1}\lambda_z(I)$. $\mathfrak{M}_z(I) \leqslant \gamma\lambda_z(I)$ follows analogously.

For each $I \in \mathcal{I}_{F^k(z)}$ small enough

$$\begin{aligned} F_*^k\mathfrak{m}_z(I) &= \alpha^{-k}\mathfrak{m}_{F^k(z)}(I), \\ F_*^k\mathfrak{M}_z(I) &= \alpha^{-k}\mathfrak{M}_{F^k(z)}(I). \end{aligned} \tag{3.7}$$

Indeed, by Lemma 4 and the fact that the holonomies are F-invariant:

$$F_*^k\mathfrak{m}_z(I) = \mathfrak{m}_z(F^{-k}(I)) = \liminf_n \frac{1}{n}\sum_{i=0}^{n-1}\eta_{z_i}(h^s(F^{-k}(I))) = \liminf_n \frac{1}{n}\sum_{i=0}^{n-1}\eta_{z_i}(F^{-k}(h^s(I)))$$
$$= \liminf_n \frac{1}{n}\sum_{i=0}^{n-1}\alpha^{-k}\eta_{F^k(z_i)}(h^s(I)) = \alpha^{-k}\mathfrak{m}_{F^k(z)}(I).$$

And for $\mathfrak{M}$ it is analogous.

The inequality (3.6) and the equations (3.7) imply that the unstable Lyapunov exponent of F at every point is $\log(\alpha)$. Indeed, for $n \in \mathbb{N}$

$$F_*^n(\gamma^{-1}\lambda_z)(I) \overset{(3.6)}{\leqslant} F_*^n(\mathfrak{m}_z)(I) \overset{(3.7)}{=} \alpha^{-n}\mathfrak{m}_{F^n(z)}(I) \text{ and}$$
$$F_*^n(\gamma\lambda_z)(I) \overset{(3.6)}{\geqslant} F_*^n(\mathfrak{M}_z)(I) \overset{(3.7)}{=} \alpha^{-n}\mathfrak{M}_{F^n(z)}(I),$$

thus

$$\gamma^{-1}F_*^n(\lambda_z)(I) \leqslant \alpha^{-n}\mathfrak{m}_{F^n(z)}(I) \leqslant \alpha^{-n}\mathfrak{M}_{F^n(z)}(I) \leqslant \gamma F_*^n(\lambda_z)(I). \tag{3.8}$$

By dividing by $F_*^n(\lambda_z)(I)$ and decrease the interval I, we get the derivative of F^{-n} at the unstable direction. More precisely, if $x \in I$, consider $I_\varepsilon := \{y \in W_F^u(F^n(z)) : d^u(x,y) < \varepsilon\}$ the open ball around x on the unstable leaf of $F^n(z)$. Then, since by (3.6) $\mathfrak{m}_{F^n(z)}$ is uniformly bounded with respect to the volume measure on the leaf, then $\mathfrak{m}_{F^n(z)} = \tau_{F^n(z)}\lambda_{F^n(z)}$ with $\tau_{F^n(z)} \in [\beta^{-1}, \beta]$, and

$$\alpha^{-n}\frac{\int_{I_\varepsilon}\beta^{-1}d\lambda_{F^n(z)}(\xi)}{\lambda_z(F^{-n}(I_\varepsilon))} \leqslant \alpha^{-n}\frac{\mathfrak{m}_{F^n(z)}(I_\varepsilon)}{F_*^n(\lambda_z)(I_\varepsilon)}$$
$$= \alpha^{-n}\frac{\int_{I_\varepsilon}\tau_{F^n(z)}(\xi)d\lambda_{F^n(z)}(\xi)}{\lambda_z(F^{-n}(I_\varepsilon))} \leqslant \alpha^{-n}\frac{\int_{I_\varepsilon}\beta d\lambda_{F^n(z)}(\xi)}{\lambda_z(F^{-n}(I_\varepsilon))}.$$

Taking the limit as $\varepsilon \to 0$, we have that

$$\alpha^{-n}\beta^{-1}\left\|DF^{-n}\big|_{E_F^u}(x)\right\|^{-1} \leqslant \lim_{\varepsilon\to 0}\frac{\alpha^{-n}\mathfrak{m}_{F^n(z)}(I_\varepsilon)}{\lambda_z(F^{-n}(I_\varepsilon))} \leqslant \alpha^{-n}\beta\left\|DF^{-n}\big|_{E_F^u}(x)\right\|^{-1}.$$

Now, by applying $\frac{1}{n}\log$ and taking the limit as $n \to \infty$, we have

$$-\log(\alpha) - \lambda_{F^{-1}}^u(x) \leqslant 0 \leqslant -\log(\alpha) - \lambda_{F^{-1}}^u(x),$$

where in the central expression we use the inequality (3.8) to bound $\alpha^{-n}\dfrac{\mathfrak{m}_{F^n(z)}(I_\varepsilon)}{\lambda_z(F^{-n}(I_\varepsilon))}$.

Hence, $\lambda_F^u(x) \equiv \lambda_A^u$ for all $x \in \mathcal{B}_0$. Since $p : \mathbb{R}^2 \to \mathbb{T}^2$ is a local isometry, $\lambda_f^u(x) \equiv \lambda_A^u$ for all $x \in \mathbb{T}^2$. We conclude analogously that $\lambda_f^s(x) \equiv \lambda_A^s$ for all $x \in \mathbb{T}^2$.

3.2 Proof of Theorem E

We adapt the proof in [24], that uses Journé's Regularity Theorem (11), for the non-invertible case. Even with f and g not invertible, we can use local transverse foliations on the universal cover to apply Theorem 11. So we prove that the lift H of h restricted to some local unstable foliation is C^k. For the stable foliations, the proof is analogous. Then, H is C^k on a small foliated box on the covering space $\mathbb{R}^2$.

Theorem 11 ([33]). *Let M_j be a manifold, W_j^s, W_j^u continuous transverse foliations with uniformly smooth leaves ($j = 1, 2$) and $h : M_1 \to M_2$ a homeomorphism such that $h(W_1^\sigma) = W_2^\sigma$ ($\sigma = s, u$). If h restricted to the leaves of the foliations W_1^s and W_1^u is uniformly $C^{r+\alpha}$, with $r \in \mathbb{N}$ and $\alpha \in (0, 1)$, then h is $C^{r+\alpha}$.*

Consider $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ lifts for f and g, and $p : \mathbb{R}^2 \to \mathbb{T}^2$ the canonical projection. We have that F and G are Anosov diffeomorphisms and have stable and unstable foliations $W^s_{F/G}$ and $W^u_{F/G}$, which are quasi-isometric. For any $\xi \in \mathbb{T}^2$, we consider a foliated box $\mathcal{B}$ with respect to f containing ξ by fixing $\overline{\xi} \in p^{-1}(\xi)$ and projecting a small foliated box $\mathcal{B}_{\overline{\xi}}$ with respect to the foliation W_F^u containing $\overline{\xi}$, in such way that $p\big|_{\mathcal{B}_{\overline{\xi}}}$ is a bijection over its image $\mathcal{B}$. By doing so, we have that $p\big|_{\mathcal{B}_{\overline{\xi}}}$ is an isometry, which allows us to work either on $\mathcal{B}$ or $\mathcal{B}_{\overline{\xi}}$, since the regularity of h on $\mathcal{B}$ is the same as the regularity of H on $\mathcal{B}_{\overline{\xi}}$. On $\mathcal{B}$, the projected unstable foliation is transverse to W_f^s. We consider $H(\mathcal{B})$ the foliated box with respect to g obtained by applying H to $\mathcal{B}$.

By Proposition 2, the stable leaves on $\mathcal{B}$ are invariant under deck transformations, thus they do not depend on the choice of $\overline{\xi}$, and we denote them by $W^s_{\mathcal{B}}(x)$. However, the unstable leaves do depend on the choice of $\overline{\xi}$. More specifically, given $x \in \mathcal{B}$, the unstable leaf $W^u_{\mathcal{B}}(x)$ is a local unstable leaf with respect to the orbit $\tilde{x} = \{p(F^k(\overline{x}))\}_{k\in\mathbb{Z}}$, where $\overline{x} = p\big|_{\mathcal{B}_{\overline{\xi}}}^{-1}(x)$.

Along this proof, we use $\tilde{x}$ to refer to the orbit of $x \in \mathcal{B}$ given by the projection of the orbit of $\overline{x} = p\big|_{\mathcal{B}_{\overline{\xi}}}^{-1}(x)$.

Lemma 5. *h is uniformly Lipschitz along $W^u_{\mathcal{B}}$.*

Proof. For $x \in \mathcal{B}$ and $y \in W^u_{\mathcal{B}}(x)$, consider

$$\rho_f(x, y) := \prod_{n=1}^{\infty} \frac{D_F^u(F^{-n}(\overline{x}))}{D_F^u(F^{-n}(\overline{y}))}, \tag{3.9}$$

where $D_F^u(z) = \left| DF\big|_{E_F^u}(z) \right|$ and $\overline{x}$ and $\overline{y}$ are the lifts of x and y in $\mathcal{B}_{\overline{\xi}}$. We will use ρ_f to construct on $\mathcal{B}$ metrics on each leaf that behave "linearly" with respect to D_f^u, that is, that present the exact expansion of the derivative with respect to the unstable direction.

Since f is not necessarily special, then each point can have more than one unstable direction for f, and each lift of a point on $\mathbb{R}^2$ can have a different unstable direction for F, so the definition of ρ_f depends on the choice of $\mathcal{B}$. More specifically, $D^u_F(.)$ is well defined for points in $\mathcal{B}$, for we fix a lift of this point in $\mathbb{R}^2$ as the one in $\mathcal{B}_{\overline{\xi}}$, and $\rho_f(x,.) : W^u_{\mathcal{B}}(\tilde{x}) \to \mathbb{R}$ is well defined and Hölder continuous, since D^u_F is uniformly bounded and F^{-n} contracts uniformly. Moreover, given $x \in \mathcal{B}$, $D^u_F(.)$ is well defined for points on $W^u_f(\tilde{x})$, the projection of the global unstable leaf of $\overline{x}$ with respect to F. Indeed, p is a bijection between $W^u_F(x)$ and $W^u_f(\tilde{x})$, which makes the choice of direction in $\mathbb{R}^2$ to compute $D^u_F(\overline{y})$ unambiguous. Furthermore, since we have the orbits fixed as the ones projected from $\mathcal{B}_{\overline{\xi}}$, $D^u_F(.)$ is well defined for points belonging to iterates of these fixed global unstable leaves, that is, on $p(F^i(W^u_F(\overline{x})))$ for every $i \in \mathbb{Z}$, which are precisely $W^u_f(\tilde{f}^i(\tilde{x}))$ for each $i \in \mathbb{Z}$.

Hence, if $x \in \mathcal{B}$ and $y \in W^u_f(\tilde{x})$, we can compute $\rho_f(f^i(x), f^i(y))$, and

$$\rho_f(f(x), f(y)) = \frac{D^u_F(\overline{x})}{D^u_F(\overline{y})}\rho_f(x,y). \tag{3.10}$$

Moreover, $\rho_f(.,.)$ is the unique continuous function satisfying both (3.10) and $\rho_f(x,x) = 1$. Besides, note that, for all $K > 0$, there is $C > 0$ such that $d^u(x,y) < K$ implies $C^{-1} < \rho_f(x,y) < C$. We define $\rho_g(.,.)$ likewise by using the foliated box $H(\mathcal{B})$.

Fixed a $p \in \mathcal{B}$, consider $h_p : W^u_f(\tilde{p}) \to W^u_g(h(\tilde{p}))$. We aim to prove that h_p is Lipschitz with a constant independent of $\mathcal{B}$ and p, i. e., that there exists $K > 0$ such that $d^u_g(h_p(x), h_p(y)) \leqslant K d^u_f(x,y)$, with d^u the distance along the leaves. We actually do that for a equivalent metric along the leaves, defined using ρ_f as follows. Let λ_p be the induced volume on $W^u_f(\tilde{p})$. For $x, y \in W^u_f(\tilde{p})$,

$$\tilde{d}_f(x,y) := \int_x^y \rho_f(x,z)d\lambda_p(z) \tag{3.11}$$

is a metric in the leaf, and for all $K > 0$ there exists $C > 0$ such that $d^u(x,y) < K$ implies $C^{-1}\tilde{d}_f(x,y) < d^u_f(x,y) < C\tilde{d}_f(x,y)$, since $\rho_f(x,\cdot)$ is uniformly bounded along the leaf. Moreover, $\tilde{d}_f(f(x), f(y)) = D^u_F(\overline{x})\tilde{d}_f(x,y)$, and, inductively, for all $n \in \mathbb{N}$

$$\tilde{d}_f(f^n(x), f^n(y)) = \prod_{i=0}^{n-1} D^u_F(F^i(\overline{x}))\, \tilde{d}_f(x,y). \tag{3.12}$$

Additionally, $\tilde{d}_f$ is uniformly continuous: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all x, y, z, q with $y \in W^u_f(\tilde{x})$, $q \in W^u_f(\tilde{z})$, $z \in B(x,\delta)$, and $q \in B(y,\delta)$ we have that $|\tilde{d}_f(x,y) - \tilde{d}_f(z,q)| < \varepsilon$.

Thus, to prove that h_p is Lipschitz, it suffices to prove that, for an uniform K, $\tilde{d}_g(h_p(x), h_p(y)) < K\tilde{d}_f(x,y)$, where $\tilde{d}_g$ is defined similarly to $\tilde{d}_f$.

Since the conjugacy h is homotopic to Id and H is a lift for h, $H(x+m) = H(x)+m$ for all $x \in \mathbb{R}^2$ and $m \in \mathbb{Z}^2$, then there is $C > 0$ such that $\|H(x)-H(y)\| \leqslant C\|x-y\|$ for $\|x-y\| \geqslant 1$, where $\|.\|$ is the Euclidean norm on $\mathbb{R}^2$.

Using the fact that p is an isometry between unstable leaves and by the quasi-isometry of the unstable foliations W_F^u and W_G^u, the same inequality holds for the induced metric on unstable leaves, that is, there is $C > 0$ such that

$$d_g^u(h(x), h(y)) \leqslant C d_f^u(x, y) \text{ for } d_f^u(x, y) \geqslant 1. \tag{3.13}$$

So we already have the Lipschitz inequality for points far enough apart. For x and y close, we use the Livshitz Theorem, that holds for Anosov endomorphisms, as observed by F. Micena in [42].

Theorem 12 (Livshitz Theorem)**.** *Let M be a Riemannian manifold and $f : M \to M$ be a transitive Anosov endomorphism. If $\varphi_1, \varphi_2 : M \to \mathbb{R}$ are Hölder continuous and*

$$\prod_{i=1}^{n} \varphi_1(f^i(x)) = \prod_{i=1}^{n} \varphi_2(f^i(x)) \text{ for all } x \text{ such that } f^n(x) = x,$$

then there is a function $P : M \to \mathbb{R}$ such that $\dfrac{\varphi_1}{\varphi_2} = \dfrac{P \circ f}{P}$. P is Hölder continuous and it is unique up to a multiplicative constant.

We apply the above theorem for $\varphi_1(z) = D_F^u(\overline{z})$ and $\varphi_2(z) = D_G^u(H(\overline{z}))$. Remember that $\varphi_1(z)$ is only well defined for $z \in \mathcal{B}$, and for $z \in W_f^u(\tilde{f}^i(\tilde{x}))$ for each $i \in \mathbb{Z}$, where $x \in \mathcal{B}$. But, by the transitivity of f, there is a $x \in \mathcal{B}$ with dense orbit. Thus we can extend φ_1, for it is well defined for every point in the orbit of x. The same holds for φ_2, and both maps satisfy the hypothesis on periodic points due to the hypothesis on Lyapunov exponents. Therefore, it follows from the Livshitz Theorem that

$$\frac{\varphi_1(x)}{\varphi_2(x)} = \frac{D_F^u(\overline{x})}{D_G^u(H(\overline{x}))} = \frac{P(f(x))}{P(x)},$$

and, inductively,

$$\frac{P(f^n(x))}{P(x)} = \prod_{i=0}^{n-1} \frac{D_F^u(F^i(\overline{x}))}{D_G^u(H(F^i(\overline{x})))} \text{ for all } x \in \mathbb{T}^2 \text{ and } n \in \mathbb{N}. \tag{3.14}$$

For $x, y \in W_f^u(\tilde{p})$, consider $N \in \mathbb{N}$ to be the smallest n such that $d_f^u(f^n(x), f^n(y)) \geqslant 1$. Then $d_g^u(h(f^n(x)), h(f^n(y))) \leqslant C d_f^u(f^n(x), f^n(y))$.

The property (3.12) of the distance $\tilde{d}_f$ implies that

$$\tilde{d}_f(x, y) = \frac{\tilde{d}_f(f^N(x), f^N(y))}{\prod_{i=0}^{N-1} D_F^u(F^i(\overline{x}))},$$

and an analogous equality holds for $\tilde{d}_g$, thus

$$\frac{\tilde{d}_g(h(x), h(y))}{\tilde{d}_f(x, y)} = \prod_{i=0}^{N-1} \frac{D_F^u(F^i(\overline{x}))}{D_G^u(H(F^i(\overline{x})))} \frac{\tilde{d}_g(h(f^N(x)), h(f^N(y)))}{\tilde{d}_f(f^N(x), f^N(y))}.$$

The first term of this product is is bounded, since it is equal to $\dfrac{P(f^n(x))}{P(x)}$, and the second one is bounded by the Lipschitz constant given by the inequality (3.13), using the equivalent metric $\tilde{d}_f$. □

In fact, the above lemma proves that H is Lipschitz along $W^u_F(\overline{x})$ for all $\overline{x} \in \mathcal{B}_{\overline{\xi}}$, that is, h is Lipschitz along $W^u_f(\tilde{x})$ for x in $\mathcal{B}$. Then, h is differentiable along unstable leaves, *u-differentiable*, for almost every point with respect to the induced volume on the leaves. If h is u-differentiable for x, then it is for $f^k(x)$, $k \in \mathbb{N}$ $k \in \mathbb{Z}$, since f is C^k and the unstable leaves are f-invariant, and the same goes for H and $F^k(\overline{x})$.

Lemma 6. *If h is u-differentiable at $x \in W^u_B(p)$, then it is u-differentiable at every $y \in W^s_B(x)$.*

Proof. This proof is similar to the one of Step 1 on Lemma 5 in [24], which consists in estimate the u-derivative of a point using a nearby u-differentiable point.

For $y \in W^s_{\mathcal{B}}(x)$ and each $n \in \mathbb{N}$, we fix a $y_n \in W^u_F(F^n(\overline{y}))$ close to $F^n(\overline{y})$. Remember that $\tilde{d}_f$ and $\tilde{d}_g$ are uniformly continuous, P is Hölder continuous and H is Lipschitz. Thus, for each small $\varepsilon > 0$, there is $\delta > 0$ independent of n such that $\zeta \in B(F^n(\overline{y}), \delta)$ implies

$$|P(p(\zeta)) - P(f^n(y))| < \varepsilon \tag{3.15}$$

and there exists $q \in W^u_F(\zeta)$ with $\tilde{d}_f(\zeta, q) = \tilde{d}_f(F^n(\overline{y}), y_n)$, q has the same orientation as y_n, q belongs to a small neighborhood of y_n and

$$|\tilde{d}_g(H(F^n(\overline{y})), H(y_n)) - \tilde{d}_g(H(\zeta), H(q))| < \varepsilon. \tag{3.16}$$

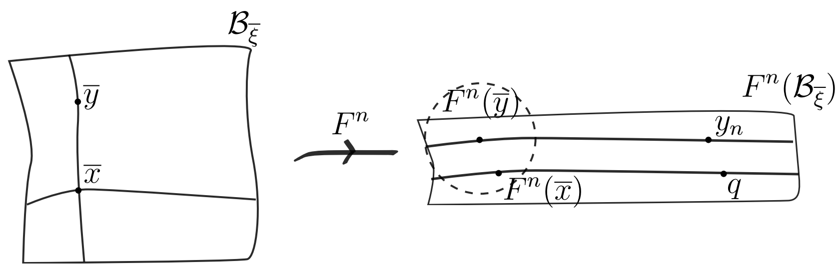

Figure 12 – We estimate the u-derivative of H at the point $F^n(\overline{y})$ using the one of H at the nearby point $F^n(\overline{x})$.

We have that $y \in W^s_{\mathcal{B}}(x)$, then $d(F^k(\overline{y}), F^k(\overline{x})) \xrightarrow{k\to\infty} 0$ and we can fix $n \in \mathbb{N}$ such that $F^n(\overline{x}) \in B(F^n(\overline{y}), \delta)$. Then there exists $q \in W^u_F(F^n(\overline{x}))$ that satisfies (3.16) (see

Figure 12). Consider $z \in W^u_F(\overline{x})$ being such that $F^n(z) = q$. Then, by taking n sufficiently large, we have that $\tilde{d}_f(\overline{x}, z)$ is small enough so that

$$\left| \frac{\tilde{d}_g(H(\overline{x}), H(z))}{\tilde{d}_f(\overline{x}, z)} - D^u_H(\overline{x}) \right| < \varepsilon. \tag{3.17}$$

Thus

$$\begin{aligned}
\tilde{d}_g(H(F^N(\overline{y})), H(y_N)) &\overset{(3.16)}{=} \varepsilon_1 + \tilde{d}_g(H(F^n(\overline{x})), H(F^n(z))) \\
&\overset{(3.12)}{=} \varepsilon_1 + \prod_{i=0}^{N-1} D^u_G(H(F^i(\overline{x})))\, \tilde{d}_g(H(\overline{x}), H(z)) \\
&\overset{(3.17)}{=} \varepsilon_1 + \prod_{i=0}^{N-1} D^u_G(H(F^i(\overline{x})))\, (D^u_H(\overline{x}) + \varepsilon_2)\tilde{d}_f(\overline{x}, z) \\
&\overset{(3.12)}{=} \varepsilon_1 + \prod_{i=0}^{N-1} D^u_G(H(F^i(\overline{x})))\, (D^u_H(\overline{x}) + \varepsilon_2) \frac{\tilde{d}_f(F^N(\overline{x}), F^N(z))}{\prod_{i=0}^{N-1} D^u_F(F^i(\overline{x}))} \\
&\underset{(3.14)}{\overset{(3.16)}{=}} \varepsilon_1 + \frac{P(x)}{P(f^N(x))}\, (D^u_H(\overline{x}) + \varepsilon_2)\tilde{d}_f(F^n(\overline{y}), y_n) \\
&\overset{(3.15)}{=} \varepsilon_1 + \frac{P(x)}{P(f^N(y)) + \varepsilon_3}\, (D^u_H(\overline{x}) + \varepsilon_2)\tilde{d}_f(F^n(\overline{y}), y_n),
\end{aligned}$$

with $|\varepsilon_i| < \varepsilon$, $i = 1, 2, 3$. As $\varepsilon \longrightarrow 0$, we have

$$\frac{\tilde{d}_g(H(F^N(\overline{y})), H(y_N))}{\tilde{d}_f(F^n(\overline{y}), y_N)} = \frac{P(x)}{P(f^N(y))} D^u_H(\overline{x}),$$

with the right-hand side not depending on y_N, then H is u-differentiable at $F^n(\overline{y})$. Hence, H is u-differentiable at $\overline{y}$ and h is u-differentiable at y. □

If $\mathcal{K} := \{x \in \mathcal{B} : h \text{ is u-differentiable at } x\}$ and

$$\mathcal{K}(p) := \{x \in W^u_{\mathcal{B}}(p) : h \text{ is u-differentiable at } x\},$$

then the above lemma implies that $\bigcup_{x \in \mathcal{K}(p)} W^s_{\mathcal{B}}(x) \subseteq \mathcal{K}$. But $W^u_{\mathcal{B}}(p)$ is transverse to the leafwise absolutely continuous foliation $W^s_{\mathcal{B}}$. Then, since $\mathcal{K}(p) \subseteq W^u_{\mathcal{B}}(p)$ has full volume, then $m\big|_{\mathcal{B}}(\mathcal{K}) = 1$. This implies that $\mathcal{K}$ is dense in $\mathcal{B}$.

During the proof of Lemma 6 we did not use the fact that $y \in W^s_{\mathcal{B}}(x)$, unless to guarantee that there is a u-differentiable point close to the orbit of y. Since the u-differentiable points form a dense set, we can use the same argument as in the lemma and estimate the u-derivative of any point with nearby u-differentiable points. In particular, if $x \in \mathcal{B}$ is a u-differentiable point with respect to h and it is sufficiently close to y, then

$$D^u_H(\overline{y}) = \frac{P(x)}{P(y)} D^u_H(\overline{x}).$$

Thus $\mathcal{K} = \mathcal{B}$, that is, h is u-differentiable for each point in $\mathcal{B}$ and D^u_H is $C^{1+\alpha}$.

In order promote the regularity of h to the one of f, let us see that

$$\rho_g(h(x), h(y)) = \frac{D^u_H(\overline{x})}{D^u_H(\overline{y})}\rho_f(x, y), \tag{3.18}$$

for $x, y \in W^u_{\mathcal{B}}(p)$. Considering

$$\tilde{\rho}_g(h(x), h(y)) := \frac{D^u_H(\overline{x})}{D^u_H(\overline{y})}\rho_f(x, y),$$

we have that $\tilde{\rho}_g$ satisfies (3.10) for g, then it is equal to ρ_g by uniqueness. Indeed,

$$\begin{aligned}
\tilde{\rho}_g(g(h(x)), g(h(y))) = \tilde{\rho}_g(h(f(x)), h(f(y))) &= \frac{D^u_H(F(\overline{x}))}{D^u_H(F(\overline{y}))}\rho_f(f(x), f(y)) \\
&= \frac{D^u_H(F(\overline{x}))}{D^u_H(F(\overline{y}))}\frac{D^u_F(\overline{x})}{D^u_F(\overline{y})}\rho_f(x, y) = \frac{D^u_{H\circ F}(\overline{x})}{D^u_{H\circ F}(\overline{y})}\rho_f(x, y) \\
&= \frac{D^u_{G\circ H}(\overline{x})}{D^u_{G\circ H}(\overline{y})}\rho_f(x, y) = \frac{D^u_G(H(\overline{x}))}{D^u_G(H(\overline{y}))}\frac{D^u_H(\overline{x})}{D^u_H(\overline{y})}\rho_f(x, y) \\
&= \frac{D^u_G(H(\overline{x}))}{D^u_G(H(\overline{y}))}\tilde{\rho}_g(h(x), h(y)).
\end{aligned}$$

With the same argument as [23, Lemma 2.4], we have that H is C^k: by [42, Lemma 3.8], ρ_g and ρ_f are C^{k-1}. Thus, the relation (3.18) implies that D^u_H is C^{k-1}, and H is C^k along W^u_F.

The proof for the stable direction is analogous, as we replace F^{-n} by F^n in the definition of ρ_f (3.9). Then we can apply Theorem 11 for H in $\mathcal{B}_{\overline{\xi}}$ and conclude that H is C^k.

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