UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

LAURA ESTIVALEZ FRANCO DA SILVA

Combinatorial Aspects of Affine Weyl Groups and Orbits of Dominant Weights

Aspectos Combinatórios de Grupos de Weyl Afins e Órbitas de Pesos Dominantes

Campinas

2022

Laura Estivalez Franco da Silva

Combinatorial Aspects of Affine Weyl Groups and Orbits of Dominant Weights

Aspectos Combinatórios de Grupos de Weyl Afins e Órbitas de Pesos Dominantes

Dissertação apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestra em Matemática.

Dissertation presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Mathematics.

Orientador: Adriano Adrega de Moura

Este trabalho corresponde à versão final da Dissertação defendida pela aluna Laura Estivalez Franco da Silva e orientada pelo Prof. Dr. Adriano Adrega de Moura.

Campinas

2022

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

Si38c Silva, Laura Estivalez Franco da, 1996-
Combinatorial aspects of affine Weyl groups and orbits of dominant weights / Laura Estivalez Franco da Silva. – Campinas, SP : [s.n.], 2022.

Orientador: Adriano Adrega de Moura.
Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Análise combinatória. 2. Grupos de Coxeter. 3. Álgebras de Kac-Moody. 4. Grupos de Weyl. 5. Órbitas. I. Moura, Adriano Adrega de, 1975-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Aspectos combinatórios de grupos de Weyl afins e órbitas de pesos dominantes
Palavras-chave em inglês:
Combinatorial analysis
Coxeter groups
Kac-Moody algebras
Weyl groups
Orbits
Área de concentração: Matemática
Titulação: Mestra em Matemática
Banca examinadora:
Adriano Adrega de Moura [Orientador]
Dijana Jakelic
Iryna Kashuba
Data de defesa: 08-03-2022
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0003-0313-5296
- Currículo Lattes do autor: http://lattes.cnpq.br/5327441763643919

Dissertação de Mestrado defendida em 08 de março de 2022 e aprovada pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). ADRIANO ADREGA DE MOURA

Prof(a). Dr(a). DIJANA JAKELIC

Prof(a). Dr(a). IRYNA KASHUBA

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Agradecimentos

Gostaria de agradecer a todos que me apoiaram, tornando este trabalho possível.

Ao meu orientador por ter me proposto um tema de estudo tão bonito, interessante e instigante. Também gostaria de agradecê-lo por ser um orientador tão atencioso e presente, sempre me ajudando a desemaranhar as ideias ao apontar correções e comentários profundos sobre o trabalho. Isto me proporcionou, para além do conhecimento matemático, reflexões sobre as estruturas de demonstrações matemáticas e sobre acessibilidade do texto ao leitor.

À bolsa de estudos que forneceu o suporte financeiro. O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001.

Ao Grupo Elza, institucionalizado no IMECC, do qual fui representante de pós-graduação, por dar visibilidade a mulheres em STEM e lutar para tornar este um espaço mais inclusivo, atrativo e agradável para que meninas e mulheres jovens possam inspirar-se e decidirem tornar-se pesquisadoras.

Aos meus professores do mestrado e de graduação na UFSM, em especial às minhas ex-orientadoras, Daiana e Saradia, por terem me apresentado a Álgebra e me inspirado a iniciar a vida acadêmica.

À minha família, por terem acreditado em mim e apoiado minhas escolhas acadêmicas, mas em especial à minha madrasta, que sempre me encorajou a não ter fronteiras e perseguir meus sonhos, mas faleceu no ano passado, sem ter a chance de ver isto se concretizando.

Aos meus amigos e colegas de casa por usarem máscara, por todo café e por todas as risadas, que amenizaram o pesadelo de ser brasileira durante a pandemia de coronavírus, com um ambiente seguro, alegre e confortável para pensar e produzir matemática.

Por último, mas não menos importante, gostaria de agradecer aos meus amigos e colegas de casa animais, Rocky, Becky e Ravi, por todo o suporte emocional e pelo lembrete de esticar as pernas entre um cálculo e outro, ao pausar as contas para mimá-los.

Resumo

O objetivo desta dissertação é apresentar alguns aspectos combinatórios de grupos de Weyl afins, motivados por suas conexões com a teoria de representações de álgebras de Kac-Moody afins. Mais especificamente, estamos interessados em ferramentas combinatórias que possam simplificar o cômputo de certas órbitas de pesos dominantes de um grupo de Weyl afim $\widehat{W}$. Em particular, é conveniente trabalhar com um conjunto mínimo de expressões para os elementos de $\widehat{W}$. Neste sentido, introduzimos o conceito de extratos completamente longos à esquerda (cle) e mostramos que todo elemento possui um único tal extrato. Quando $\widehat{W}$ tem tipo $\widehat{A}_n$, $n \geq 2$, descrevemos uma expressão reduzida específica para cada componente dos extratos cle. Isso acaba por estabelecer uma correspondência dos extratos cle com certos pares de sequências finitas monótonas de inteiros não-negativos, chamadas sequência esquerda e sequência direita. Estas sequências dão origem a um grafo orientado, chamado grafo de extratificação, o qual codifica uma família de fórmulas para expressões de todos os elementos de $\widehat{W}$. Conjecturamos que este é um conjunto mínimo de expressões e, utilizando propriedades de grupos de Coxeter, mostramos que este é de fato o caso para expressões onde figuram até duas vezes um certo elemento distinguido. Por fim, demonstramos a conjectura para $n = 2$ usando formas de permutações e de alcovas para os elementos de $\widehat{W}$ e, então, calculamos as órbitas desejadas.

Palavras-chave: Grupos de Weyl afins. Caminhos de alcovas. Grupos de Coxeter. Combinatória. Expressões reduzidas. Órbitas.

Abstract

The goal of this dissertation is to present some combinatorial aspects of affine Weyl groups, motivated by its connection with representation theory of affine Kac-Moody algebras. More specifically, we are interested in combinatoric tools which might simplify the computation certain orbits of dominant weights of an affine Weyl group $\widehat{W}$. In particular, it is convenient to work with a minimal set of expressions for the elements of $\widehat{W}$. In this sense, we introduce the concept of fully left-long (fll) extract and show that every element has such an extract. When $\widehat{W}$ has type $\widehat{A}_n$, $n \geq 2$, we describe a specific reduced expression for each component of an fll extract. This establishes a correspondence of fll extracts with certain pairs of finite monotonous sequences of non-negative integers, called left and right sequences. These sequences give rise to a directed graph, called the extract graph, which encodes a family of formulae for expressions for all elements of $\widehat{W}$. We conjecture that this is a minimal set of expressions and show that that is indeed the case for expressions where a certain distinguished element appears at most two times. Finally, we prove the conjecture for $n = 2$, using permutation and alcove forms for the elements of $\widehat{W}$ and, then, compute the desired orbit.

Keywords: Affine Weyl groups. Alcove paths. Coxeter groups. Combinatorics. Reduced expressions. Orbits.

Contents

Introduction

The theory of representations of Lie algebras emerged at the end of nineteenth century in the context of the study of differential equations from the point of view of its symmetry groups. Besides being an interesting research area by itself, it emerged as a powerful tool for several areas of mathematics and physics. The structure of a simple Lie algebra can be codified by a matrix with integer entries, which is called the associated Cartan matrix. Later, Victor Kac and Robert Moody generalized the concept of Cartan matrix and associated Lie algebras to them. In particular, we are interested in the affine Kac-Moody algebras, which are the algebraic structures used in many areas of physics, as conformal field theory, string theory, and integrable systems of statistical mechanics.

The relation between certain areas of physics and Lie theory has become stronger since the development of quantum groups four decades ago. Quantum groups may be realized as one-parameter deformations of the universal enveloping algebras of the Kac-Moody algebras. Quantum groups became even more interesting once they established unexpected connections between distinct fields. In particular, we mention the construction of topological invariants that made possible the classification theorems on knot theory, advances in the study of representation theory of algebraic groups over positive characteristic, and several results related to number theory. These results have strong combinatorial aspects, including the main motivation for the concept of Cluster algebras, which enabled revolutionary progress in many areas of mathematics at the beginning of this millennium.

The purpose of this dissertation is to study combinatorial aspects of Coxeter groups, in particular, the subclass of affine Weyl groups, seeking to apply this knowledge in representation theory of affine Kac-Moody algebras and current algebras, on which combinatorial aspects are prominent. Our main motivation arises from results presented in (JAKELIĆ; MOURA, 2018) for the case of the algebra of type $\widehat{A}_1$ and whose extension to higher rank requires a better understanding of the characterization of certain elements of the corresponding affine Weyl group and, hence, of the underlying combinatorics. Since the main part of this text will not address the underlying Lie theoretic background behind our main focus of study, let us dedicate the next few paragraphs to provide further explanation of the wider context and the motivation behind the problem being investigated in this thesis.

The main result of (JAKELIĆ; MOURA, 2018) explores the connection between two kinds of multiplicity problems, establishing a formula for computing one of them in terms of the other. On one hand, there is the problem of computing the so-called outer multiplicities for tensor products of simple modules in the category of integrable weight modules for an affine Kac-Moody algebra. This is a classical problem which is relevant for mathematical physics and number theory and whose various answers are very rich in combinatorial aspects. There are algorithmic answers such as the ones given in (KING; WELSH, 1991; LITTELMANN, 1994; OKADO; SCHILLING; SHIMOZONO, 2003) and more direct answers leading to proofs of Rogers-Ramanujan-type identities as well as partition identities (see (FEINGOLD, 1981; LEPOWSKY; WILSON, 1982; MISRA; WILSON, 2013; MISRA; WILSON, 2014) and references therein).

On the other hand, the study of multiplicities in Demazure flags for graded modules for current algebras has risen to prominence in recent years motivated by the study of certain structural aspects of representations of quantum affine algebras, specially after a standing conjecture about the character of the so called local Weyl modules was settled in (NAOI, 2012) by using results on Demazure flags from (JOSEPH, 2003; JOSEPH, 2006). More precisely, it was proved in (NAOI, 2012) that, if the underlying finite-dimensional simple Lie algebra $\mathfrak{g}$ is simply laced, any level-ℓ Demazure module for the current algebra admits a Demazure flag of level $\ell + 1$. The main result of (JAKELIĆ; MOURA, 2018) was inspired by a reversed engineering of the proof of this result and establishes a connection with the problem of outer multiplicities discussed previously since the results of (JOSEPH, 2003; JOSEPH, 2006) are strongly related to that problem. It is interesting to remark that (JOSEPH, 2003; JOSEPH, 2006) make strong use of the context of quantum groups, global basis, and combinatorics of crystals.

The papers (BISWAL; CHARI; KUS, 2018; BISWAL et al., 2016; BISWAL et al., 2021; BISWAL; KUS, 2021; CHARI et al., 2014) are dedicated to studying multiplicities in Demazure flags in certain special cases obtaining various formulae in terms of generalizations of combinatorial objects such as MacDonald polynomials, Fibonacci sequences, and theta functions. In particular, in the case $\mathfrak{g} = \mathfrak{sl}_2$, the multiplicities in level-2 Demazure flags for level-1 Demazure modules were described in (CHARI et al., 2014) in terms of gaussian binomials. Combined with the main result of (JAKELIĆ; MOURA, 2018), this leads to an expression for outer multiplicities for tensor products of fundamental modules for the affine Kac-Moody algebra of type $\widehat{A}_1$ in terms of a finite sum of partitions with bounded parts. The same tensor products were studied in (MISRA; WILSON, 2014) for type $\widehat{A}_n$ by completely different methods (Demazure flags do not play a role) and the results were also expressed in terms of different kinds of partitions. Thus, a comparison of the two formulae in the $\widehat{A}_1$ case led to a proof of partition identities.

In addition to the result about Demazure flags from (CHARI et al., 2014), the other main ingredient used in (JAKELIĆ; MOURA, 2018) to obtain such expression for outer multiplicities was an explicit description of the elements of the affine Weyl group or, more importantly, the characterization of the elements of the orbit of a given dominant affine weight whose projections onto the weight lattice of $\mathfrak{g}$ is dominant. More precisely, let $\widehat{W}$ denote the affine Weyl group, $\widehat{P}$ the affine weight lattice, P the underlying finite type weight lattice so that $\widehat{P} = \mathbb{C}\Lambda_0 \oplus P \oplus \mathbb{C}\delta$, where Λ_0 is the highest weight of the basic representation of the affine Kac-Moody algebra and δ is the generator of imaginary roots, and let $\pi : \widehat{P} \to P$ be the associated projection. The computation in (JAKELIĆ; MOURA, 2018) relies on an explicit description of the set

$$\Gamma_\Lambda := \{w\Lambda : \pi(w\Lambda) \text{ is dominant}\}$$

for every dominant affine weight Λ. While such computations for type $\widehat{A}_1$ are straightforward, moving to higher rank they are much more challenging, even for type $\widehat{A}_2$. Thus, studying several combinatorial approaches seeking to find efficient ways of describing the above set was the motivating goal of the present work. Using the notion of alcove walks and computer based computations, an answer for type $\widehat{A}_2$ was partially given in (BURGER, 2017).

The action of an affine Weyl group on an Euclidean space gives rise to a hyperplane arrangement which partitions the complement of the hyperplanes into some open convex simplices called alcoves. These objects have a structural importance for the theory of affine Weyl groups. Since these groups act simply transitively on the set of alcoves, their elements are in a bijective correspondence with the alcoves determined by their action. On the other hand, a realization of affine Weyl groups as permutation groups was introduced by (LUSZTIG, 1983) in order to understand when certain irreducible representations of the Hecke algebra of an affine Weyl group correspond to a square integrable representation of a simple p-adic group. Different characterizations of affine Weyl groups of type $\widehat{A}$ were compared in (SHI, 1980), such as expressions in terms of Coxeter generators, permutation forms and alcove forms. Later, in (SHI, 1987), the author extended the description of the alcove forms for all types. Moreover, in (SHI, 1999), formulae for the transition between alcove and permutation forms for other types were obtained.

The classical Bertrand's Ballot Problem can be rephrased in the context of finite Weyl groups as the following question: how many ways are there to walk from the origin to an arbitrary point using positive unit steps in such a way that the walk remains in the region $x_1 \geq x_2 \geq \ldots \geq x_n$? In (GESSEL; ZEILBERGER, 1992), this question was generalized and answered for affine Weyl groups by counting the number of distinct k-step random walks that remain in the same Weyl chamber. In (GRABINER; MAGYAR, 1993), this generalization was shown to be equivalent to the problem of describing the decomposition of the k-th tensor power of certain representations of reductive Lie groups as a direct sum of simple modules. Later, when studying results related to Markov chains, (BIDIGARE; HANLON; ROCKMORE, 1999) and (BROWN; DIACONIS, 1998) described random walks in the set of facets of alcoves of affine Weyl groups. This description was recently used in (DEFOSSEUX, 2016) to present some characters as the eigenfunctions of the Dirichlet problem on such alcoves, while (LAM, 2015) used it to describe the possible shapes of randomly generated elements of an affine Weyl group, and (LECOUVEY; TARRAGO, 2020) used it for the description of certain affine Grasmannian elements.

Some ideas behind random alcove walks have been independently developed in the works of Littelman (LITTELMANN, 1994) and (LITTELMAN, 1995), which construct a path model in order to describe the characters of irreducible representations of complex semisimple Lie groups by counting some paths related to the corresponding semisimple Lie algebra, in particular, the so-called LS-paths. Prior to (LECOUVEY; TARRAGO, 2020), connections with the geometry of affine Grasmannian were made in (GAUSSENT; LITTELMAN, 2005), by replacing the path model by an equivalent gallery model with the concept of LS-gallery, a certain sequence of facets of alcoves of the corresponding affine Weyl group. This made it possible to describe the coefficients of Hall-Littlewood polynomials in (SCHWER, 2006) and, hence, the structure constants of the spherical Hecke algebra in a combinatorial way.

The definition of an alcove path as a sequence of alcoves was given in (LENART; POSTNIKOV, 2007). It was shown that the decomposition of the corresponding group in product of Coxeter generators is in bijective correspondence with alcove paths starting in the fundamental alcove. Alternatively, (RAM, 2006) and (PARKINSON; RAM, 2008) introduced alcove walks as some sequences of crossings from an alcove to an adjacent one and foldings of an alcove on itself. This definition gives rise to what (LENART; POSTNIKOV, 2007) calls an alcove path, but allowing also paths from the alcove on itself. This generalization clarified the relations between the path model with crystal theory, for instance. It also made it possible to relate affine Hecke algebras with the combinatorics of spherical functions on p-adic groups, the so-called Hall-Littlewood polynomials. Ram's theory of alcove walks was used to provide an alternative method to deal with presentations of affine Hecke algebras, which helped in the study of problems related to sheaves on affine flag manifolds. Later, (LENART, 2011) also defined alcove walks but in a different language and was able to generalize results about Hall-Littlewood polynomials that were previously known just for type $\widehat{A}$.

The definition of alcove walk used by (BURGER, 2017) coincides with the definition of alcove path given by (LENART, 2011). Thus, in this work, we chose to adopt the terminology of alcove path instead of alcove walk, which is already used for a generalized notion. For type $\widehat{A}_2$, the approach of (BURGER, 2017) consisted of describing specific alcove paths from the fundamental alcove to each fixed alcove, inducing a corresponding set of expressions for the elements of the group in terms of its Coxeter generators. Then, using the software SageMath, the author computed the corresponding expression of each element, determining 52 sets of conditions on elements in the orbit of affine dominant weights that, under the action of $\widehat{\mathcal{W}}$, project to anti-dominant weights of $\mathcal{W}$. An analysis of the conditions presented by (BURGER, 2017) made us find out that, at least in this case, the set of elements corresponding to alcoves contained in the fundamental Weyl chamber is always contained in the orbit of affine dominant weights that project to dominant weights.

Although the approach using alcove walks used in (BURGER, 2017) provides an interesting geometric intuition and the tables with computer based calculations from (BURGER, 2017) were fundamental for our initial understanding of the problem, the complexity of such analysis for higher rank grows so fast that, at this moment, we do not see how to handle it beyond affine rank 3. However, building up on this work, we sought a more conceptual manner of approaching the problem which, in particular, led to a much simpler answer for type $\widehat{A}_2$ without relying on computer based computations. From now on we describe the main features of our approach which is based on a specific choice of reduced expressions for the elements of $\widehat{\mathcal{W}}$. Alcove paths will not play a significant role in our strategies, but we intend to bring as much connections as we find.

First we will describe a presentation of the elements of an arbitrary affine Weyl group $\widehat{\mathcal{W}}$ on uniquely determined finite sequences of elements of $\mathcal{W}$ in Section 1.8. We will refer to each one of these sequences as the fully left-long (fll) extract of the corresponding element. We obtain that $\widehat{\mathcal{W}} = \mathcal{W}\mathcal{E}$, where $\mathcal{W}$ is the underlying finite Weyl group and $\mathcal{E}$ is the set of elements of $\widehat{\mathcal{W}}$ corresponding to alcoves contained in the fundamental Weyl chamber. In particular, if $\mathcal{E}$ projects dominant affine weights to dominant weights, then $\Gamma_\Lambda = \mathcal{E}\Lambda$. In Section 3.3, we prove that this occurs for type $\widehat{A}_2$. Our strategy consists of choosing some specific expressions for the elements of $\mathcal{E}$ (of an arbitrary rank) and then computing the respective orbit in rank 2.

More generally, in Section 2.3, we prove that, in type $\widehat{A}_n$, $n \geq 2$, each element is determined by a pair of monotonous finite sequences of non-negative integers satisfying certain conditions, which we call the left and right sequences. Conversely, the set of pairs of finite sequences satisfying such conditions gives rise to a directed graph which we call the extract graph of type $\widehat{A}_n$. Each oriented path in this graph corresponds to a formula for a family of expressions coming from certain left and right sequences. Thus, the set of such paths provides a class of reduced expressions for the elements of $\widehat{\mathcal{W}}$.

For type $\widehat{A}_2$, we prove that this set is minimal, that is, the elements corresponding to the set of expressions induced by the graph are all distinct. Using the alcove form of each one of these elements in Section 3.1, we prove that the expressions are all reduced, while the computation of the permutations forms in Section 3.2 shows that the elements are all distinct. Moreover, each one of these expressions correspond to the fll extract of the element. In Section 3.3, we compute the expression associated to each element of $\mathcal{E}$ applied to a dominant affine weight Λ, obtaining 2 families of formulae to describe the elements of Γ_Λ.

Even though in type $\widehat{A}_2$ there are only 6 maximal oriented paths in the extract graph, giving rise to 6 families of formulae used to compute Γ_Λ, in $\widehat{A}_3$ this number increases to 18. Thus, finding an efficient way of dealing with the associated formulae for higher rank remains a challenge. Nevertheless, since part of the formula in (JAKELIĆ; MOURA, 2018) is essentially controlled by a certain "asymptotic" behavior of such computations, it might happen that the combinatorics of these graphs are sufficient to obtain information in order to understand this asymptotic behavior. This is one of the directions we shall start working on next.

It is interesting to remark that the reason behind the fact that we have proved some of the results here only for type $\widehat{A}_2$, arises from a similar jump in the difficulty-level of the combinatorics we have been exploring here. However, as we were finalizing the text, we came across the recent preprint (AL HARBAT, 2021) which apparently proves the two conjectures for type $\widehat{A}$ made in Section 2.4. We added comments comparing (AL HARBAT, 2021) to the present work in Remark 2.4.5 and develop a few steps of the strategy utilized in (AL HARBAT, 2021) in Section 2.5. We regard the fact that one of the main questions addressed in this work was the topic of a recent preprint as a further indication of the relevance of the project that led to this dissertation.

This work is organized as follows. Chapter 1 contains most of the theoretical framework about Weyl groups and Coxeter groups. In Chapter 2, we present different presentations of the main example of the work, the affine Weyl group of type $\widehat{A}_n, n \geq 2$. Finally, in Chapter 3, we focus in the case $n = 2$, applying the combinatorial tools developed in the previous chapters, in order to compute Γ_Λ.

Throughout the work, we adopt the notations

$$\mathbb{Z}_{\geq i} = \{x \in \mathbb{Z} : x \geq i\} \quad \text{and} \quad I_{i,j} = \{x \in \mathbb{Z} : i \leq x \leq j\},$$

for $i, j \in \mathbb{Z}$.

Chapter 1

Weyl groups

The aim of this chapter is to introduce affine *Weyl groups*, which are special cases of *Coxeter groups*. Since we are most interested in combinatorial aspects of affine Weyl groups, we begin, in Section 1.1, by presenting some general combinatorial properties of Coxeter groups concerning *reduced expressions* for its elements, following (HUMPHREYS, 1990) and (BJÖRNER; BRENTI, 2005). These properties provide a special factorization of a Coxeter group as the end of a chain of inclusion of certain proper subgroups, which we present in Section 1.2 following (BJÖRNER; BRENTI, 2005) and (STEMBRIDGE, 1997). It has been proved that finite Coxeter groups are precisely finite reflection groups, a class of groups which contains the finite Weyl groups. In Section 1.3, we introduce finite Weyl groups under a geometric point of view, by seeing them as reflection groups acting on an Euclidean space, following (HUMPHREYS, 1990). These groups are related with representation of semisimple Lie algebras, as we briefly expose in Section 1.4, following (KAC, 1990). In Section 1.5, we present affine Weyl groups, which are infinite Coxeter groups, following (HUMPHREYS, 1990). Although there exist other classes of infinite Coxeter groups beyond affine Weyl groups, in this work we will focus on this particular class only. Affine reflections fix pointwise certain hyperplanes which not necessarily pass through the origin. The action of an affine Weyl group on the connected components of the intersection of all such hyperplanes is simply transitive, inducing a bijection between the components, called *alcoves*, and the elements of the group, which are the subject of Sections 1.6 and 1.7, following (HUMPHREYS, 1990), (LENART; POSTNIKOV, 2007), (SHI, 1987), and (SHI, 1999). Finally, in Section 1.8, we present a characterization of special reduced expressions of the elements of an affine Weyl group. Such expressions will be crucial in Section 2.3. As a rule, we omit all proofs contained in books here and, with a few exceptions, we present the proofs of the papers (SHI, 1987) and (SHI, 1999).

1.1 Coxeter Groups

Let $\mathcal{W}$ be a general group generated by a set S. Thus, each element $w \in \mathcal{W}$ can be written non uniquely as a product of generators $w = s_1 s_2 \dots s_k$, for some $s_i \in S$. If k is minimal among all such expressions for w, then k is called the *length* of w and denoted by $\ell(w) = k$. In this case, the expression $s_1 s_2 \dots s_k$ is called a *reduced expression* for w (with respect to S). By convention, $\ell(e) = 0$, where e is the identity element of $\mathcal{W}$. Given $w \in \mathcal{W}$, it will be convenient to consider the *left descent set* associated with w,

$$D(w) := \{s \in S : \ell(sw) < \ell(w)\}.$$

Analogously, one can define right descent sets, but in this work we will only deal with the left version.

Several properties of finite and affine Weyl groups can be deduced directly from the presentation $(\mathcal{W}, S)$, where S is a finite generating set subjected to the relations of the form

$$(ss')^{m(s,s')} = 1,$$

for some $m(s,s') \in \mathbb{Z}_{>0} \cup \{\infty\}$, with $m(s,s) = 1$, and $m(s,s') \geq 2$ for $s \neq s'$. The condition $m(s,s') = \infty$ means no relation of the form $(ss')^n = 1$, $n \in \mathbb{Z}_{>0}$, is imposed. An abstract group admitting this kind of presentation is called a *Coxeter group*.

The pair $(\mathcal{W}, S)$ is called a *Coxeter system* and the elements of S are the *Coxeter generators*. The cardinality of S is called the *rank* of the system. In particular, if $s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced expression for $w \in \mathcal{W}$, then

$$s_{i_j} \neq s_{i_{j+1}}, \quad \text{for all } 1 \leq j < l. \tag{1.1.1}$$

From the definition, we see that a Coxeter system is determined by a finite set S and a set $\{m(s,s') : s, s' \in S\} \subseteq \mathbb{Z}_{>0} \cup \{\infty\}$, which satisfies the conditions $m(s,s) = 1$ and $m(s,s') \geq 2$, if $s \neq s'$. Thus, the Coxeter system can be represented by its *Coxeter graph*, that is, the undirected graph whose vertex set is S and whose edges are the unordered pairs $\{s, s'\}$ such that $m(s,s') \geq 3$. Each edge is labeled by the corresponding number $m(s,s')$. As a simplifying convention, the edges for which $m(s,s') = 3$ have the label omitted. By definition, a Coxeter graph has no loops and $m(s,s') = 2$ if $s \neq s'$ are not connected by an edge.

A Coxeter system is called *irreducible* if its Coxeter graph is connected. It can be shown that each Coxeter group $\mathcal{W}$ is a direct product of certain subgroups $\mathcal{W}_i$ with generating set S_i, so that each pair $(\mathcal{W}_i, S_i)$ is irreducible (HUMPHREYS, 1990, Proposition 6.1).

The length function on a Coxeter group can be characterized by two important properties, which are called the *Exchange Property* and the *Deletion Property*. These properties characterize a Coxeter system in the sense that, if $\mathcal{W}$ is a group generated by the finite set S all of whose elements have order 2, then the pair $(\mathcal{W}, S)$ is a Coxeter system if and only if the Exchange Property or, equivalently, the Deletion Property is satisfied.

Proposition 1.1.1. (Exchange Property) (HUMPHREYS, 1990, Theorem 5.8) Let $w \in \mathcal{W}$, $s \in S$, and $s_1 s_2 \ldots s_k$ be any expression for w. If $s \in D(w)$, then $sw = s_1 \ldots \widehat{s_i} \ldots s_k$, for some $1 \leq i \leq k$. If the expression is reduced, then the index i is unique. □

Corollary 1.1.2. (BJÖRNER; BRENTI, 2005, Corollary 1.4.4) Let $w = s_1 \ldots s_k \in \mathcal{W}$ and $s \in D(w)$. Then, there exists $1 \leq i \leq k$ such that $s = u s_i u^{-1}$ with $u = s_1 s_2 \ldots s_{i-1}$. If the expression is reduced, then the index i is unique. □

Corollary 1.1.3. (BJÖRNER; BRENTI, 2005, Corollary 1.4.6) Let $w \in \mathcal{W}$ and $s \in S$. Then, $s \in D(w)$ if and only if some reduced expression for w begins with s. □

Corollary 1.1.4. Suppose $u, w \in \mathcal{W}$ satisfy $\ell(uw) = \ell(u) + \ell(w)$. Then, $D(u) \subseteq D(uw)$.

Proof. Let $s_1 \ldots s_m$ and $s_{m+1} \ldots s_l$ be reduced expressions for u and w, respectively so that $s_1 \ldots s_l$ is a reduced expression for uw. If $s \in D(u)$, we can assume $s_1 = s$ by Corollary 1.1.3. It follows that $suw = s_2 \ldots s_l$ and, hence,

$$\ell(suw) \leq l - 1 < l = \ell(uw),$$

showing $s \in D(uw)$. □

Proposition 1.1.5. (Deletion property) (BJÖRNER; BRENTI, 2005, Proposition 1.4.7) If $w = s_1 s_2 \ldots s_k$ and $\ell(w) < k$, then $w = s_1 \ldots \widehat{s_i} \ldots \widehat{s_j} \ldots s_k$, for some $1 \leq i < j \leq k$. □

Corollary 1.1.6. (BJÖRNER; BRENTI, 2005, Corollary 1.4.8)

(i) Any expression $w = s_1 s_2 \ldots s_k$ contains a reduced expression for w, obtainable by deleting an even number of Coxeter generators.

(ii) The set of Coxeter generators appearing in a reduced expression for w is independent of the chosen expression.

(iii) S is a minimal set of generators for $\mathcal{W}$. That is, no Coxeter generator can be expressed in terms of the others. □

The next theorem characterizes Coxeter groups in terms of the Exchange and the Deletion properties. For instance, assume that $\mathcal{W}$ is a general group with a set of generating S. The pair $(\mathcal{W}, S)$ is said to have the Exchange Property (Deletion Property) if Proposition 1.1.1 (respectively, Proposition 1.1.5) is true.

Theorem 1.1.7. (BJÖRNER; BRENTI, 2005, Theorem 1.5.1) Let W be a group and S a set of generators which have order 2 in W. Then the following are equivalent:

(i) (W, S) is Coxeter system;

(ii) (W, S) has the Exchange Property;

(iii) (W, S) has the Deletion Property. □

Let V be the real Euclidean space with basis $\{\alpha_s : s \in S\}$. Consider the symmetric bilinear form $(.,.)$ on V, given by

$$(\alpha_s, \alpha_{s'}) = -\cos\left(\lim_{t\to m(s,s')} \pi/t\right), \quad \text{for } s, s' \in S. \tag{1.1.2}$$

Then, it is possible to show that the form is positive definite precisely when W is finite (HUMPHREYS, 1990, Theorem 6.4). Finite Coxeter groups correspond to finite reflection groups. The groups for which the form is positive semi-definite are infinite and correspond to affine reflection groups. Weyl groups are special cases of reflections groups, which are the subject of Sections 1.3 and 1.5.

1.2 Parabolic Coxeter Subgroups

Given $X \subseteq S$, let $W_X = \langle X \rangle$ be the subgroup generated by X. It is well-known that (W_X, X) is also Coxeter system and that the length function on W_X relative to X coincides with the restriction of the length function on W to W_X. The subgroups of W of the form W_X, $X \subseteq S$, are called *parabolic* subgroups of W. Let also

$$W^X = \{w \in W : \ell(xw) > \ell(w)\ \forall\ x \in X\},$$

which is called the *set of shortest right coset representatives* of W relative to W_X because of the next proposition. For a proof of the proposition, see (BJÖRNER; BRENTI, 2005, Proposition 2.4.4) or (HUMPHREYS, 1990, Proposition 1.10(c)).

Proposition 1.2.1. For every $w \in W$, there exists a unique pair $(u, v) \in W_X \times W^X$ such that $w = uv$. Moreover, this pair satisfies $\ell(w) = \ell(u) + \ell(v)$ and v is the unique minimal length representative of the right coset $W_X w$. □

In particular, we have:

$$\ell(uv) = \ell(u) + \ell(v) \quad \text{for all} \quad (u, v) \in W_X \times W^X,\ X \subseteq S. \tag{1.2.1}$$

We shall use this later on with $X = S \setminus D(w)$ for elements w such that $\#D(w) = 1$. In that case we have $w \in W^X$.

Suppose $\mathbb{W} := \{e\} = W_0 \subset W_1 \subset \cdots \subset W_n = \mathcal{W}$ is a chain of inclusion of maximal proper parabolic subgroups and, for $1 \leq i \leq n$, let $\mathbb{W}_i$ be the set of shortest right coset representatives of W_i relative to W_{i-1}. It follows that every $w \in \mathcal{W}$ admits a unique factorization

$$w = w_1 w_2 \cdots w_n \quad \text{with} \quad w_i \in \mathbb{W}_i \quad \text{for all} \quad 1 \leq i \leq n$$

and, moreover,

$$\ell(w) = \sum_{i=1}^{n} \ell(w_i).$$

We shall refer to this factorization as the $\mathbb{W}$-factorization of w (it is referred to as canonical factorization in (STEMBRIDGE, 1997)). The element w_i of the factorization will be referred to as the i-th component of the factorization.

1.3 Finite Weyl Groups

From now on, we fix V a real Euclidean space endowed with a positive definite symmetric bilinear form $(\cdot,\cdot)$, $GL(V)$ denotes the linear group of V and $O(V)$ denotes the subgroup of orthogonal transformations.

A *reflection* is a diagonalizable element $s \in O(V)$, whose eigenvalues are 1 and -1, the latter with multiplicity 1. That is, s sends some nonzero vector α to its negative while it fixes pointwise the hyperplane orthogonal to α. We shall denote by s_α the reflection with such property. In other words,

$$s_\alpha \beta = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \quad \text{for all} \quad \beta \in V.$$

Given a finite set of reflections S, the subgroup $\mathcal{W}$ of $GL(V)$ generated by S is called a *finite reflection group*.

Let $\mathcal{W}$ be a finite reflection group. A subset $\Phi \subseteq V$ of nonzero vectors is said to be a *root system* for $\mathcal{W}$ if

$$\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}, \quad s_\alpha(\Phi) = \Phi \quad \text{for all} \quad \alpha \in \Phi, \quad \text{and} \quad \mathcal{W} = \langle s_\alpha : \alpha \in \Phi \rangle. \tag{1.3.1}$$

In that case, the elements of Φ are referred to as *roots*.

Fix a total ordering in V satisfying:

(i) if $\lambda \neq 0$, then $\lambda > 0 \Leftrightarrow -\lambda < 0$,

(ii) if $\lambda, \mu > 0$ and $c \in \mathbb{R}_{>0}$, then $\lambda + \mu > 0$ and $c\lambda > 0$.

For instance, the lexicographic ordering with respect to some fixed ordered basis of V. An element λ of V is said to be *positive* (*negative*) if $\lambda > 0$ (respectively, $\lambda < 0$). A subset

Φ^+ of Φ is called a *positive system* if it consists of all of the *positive roots* (with respect to the fixed ordering). Similarly, the subset $\Phi^- := -\Phi^+$ is called a *negative system*. Of course, $\Phi = \Phi^+ \cup \Phi^-$. A subset Δ of Φ^+ is called a *simple system* if Δ is a basis for the real vector space generated by Φ, so that each element of Φ has all of its coefficients relative to Δ all of the same sign. It is possible to show that every positive system in Φ contains a unique simple system, and that each simple system in Φ is contained in some unique positive system in Φ (HUMPHREYS, 1990, Theorem 1.3). Of course, each choice of total ordering in V leads to different positive and simple systems in Φ.

It can be shown that any two positive (simple) systems are conjugate under $\mathcal{W}$ (HUMPHREYS, 1990, Theorem 1.4, Theorem 1.8). The next proposition characterizes $\alpha \in \Delta$ as the unique positive root made negative by s_α.

Proposition 1.3.1. (HUMPHREYS, 1990, Proposition 1.4) Let Δ be a simple system with Φ^+ the corresponding positive system. If $\alpha \in \Delta$ and $\beta \in \Phi^+ \setminus \{\alpha\}$, then $s_\alpha \beta \in \Phi^+$. □

Fix Φ a root system for $\mathcal{W}$ and choose a positive system Φ^+ and corresponding simple system Δ. Set $S := \{s_\alpha : \alpha \in \Delta\}$. Then, it is possible to show that $(\mathcal{W}, S)$ is a Coxeter system (HUMPHREYS, 1990, Theorem 1.9), i.e., $\mathcal{W}$ is generated by S subjected only to relations of the form $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$, $\alpha, \beta \in \Delta$, where $m(\alpha, \beta)$ is the order of the element $s_\alpha s_\beta$ in $\mathcal{W}$. In this case, the Coxeter generators are called *simple reflections*. In particular, the simple system Δ can be chosen in such a way that the bilinear form in V coincides with (1.1.2) in the basis Δ (HUMPHREYS, 1990, Theorem 6.4). The possible graphs that can be realized as connected Coxeter graphs coming from finite reflection groups are shown in Figure 1.3.1. The subscript in each case is the rank n of the system. They are all distinct, except for small values of n, and classify up to isomorphism all irreducible pairs $(\mathcal{W}, S)$.

Associated with each $\alpha \in \Delta$, there exists a hyperplane $H_\alpha := \{\lambda \in V : (\lambda, \alpha) = 0\}$. Each hyperplane determines a semiplane $H_\alpha^+ := \{\lambda \in V : (\lambda, \alpha) > 0\}$. The intersection $\mathcal{C}_e$ of all H_α^+, $\alpha \in \Delta$, is called the *fundamental Weyl chamber* and the remaining *Weyl chambers* are $\mathcal{C}_w := w\mathcal{C}_e$, $w \in \mathcal{W}$. It is possible to show that $\mathcal{W}$ acts simply transitively on the set of Weyl chambers (HUMPHREYS, 1990, Proposition 1.12). In particular, the simple transitivity implies that there must exist a unique element $w_o \in \mathcal{W}$ sending Φ^+ to Φ^-. This element, called the *longest element* of $\mathcal{W}$ is the unique element with maximal length $\ell(w_o) = \#\Phi^+$ (HUMPHREYS, 1990, Section 1.8).

We are particularly interested in the special class of finite reflection groups called *Weyl groups*. Their associated root systems play an important role in the theory of semisimple Lie algebras. A root system Φ is called a *crystallographic root system* if for all $\alpha, \beta \in \Phi$, we have $s_\alpha \beta = \beta + k\alpha$, for some $k \in \mathbb{Z}$. The finite reflection group $\mathcal{W}$ associated with a crystallographic root system is called the Weyl group of the crystallographic

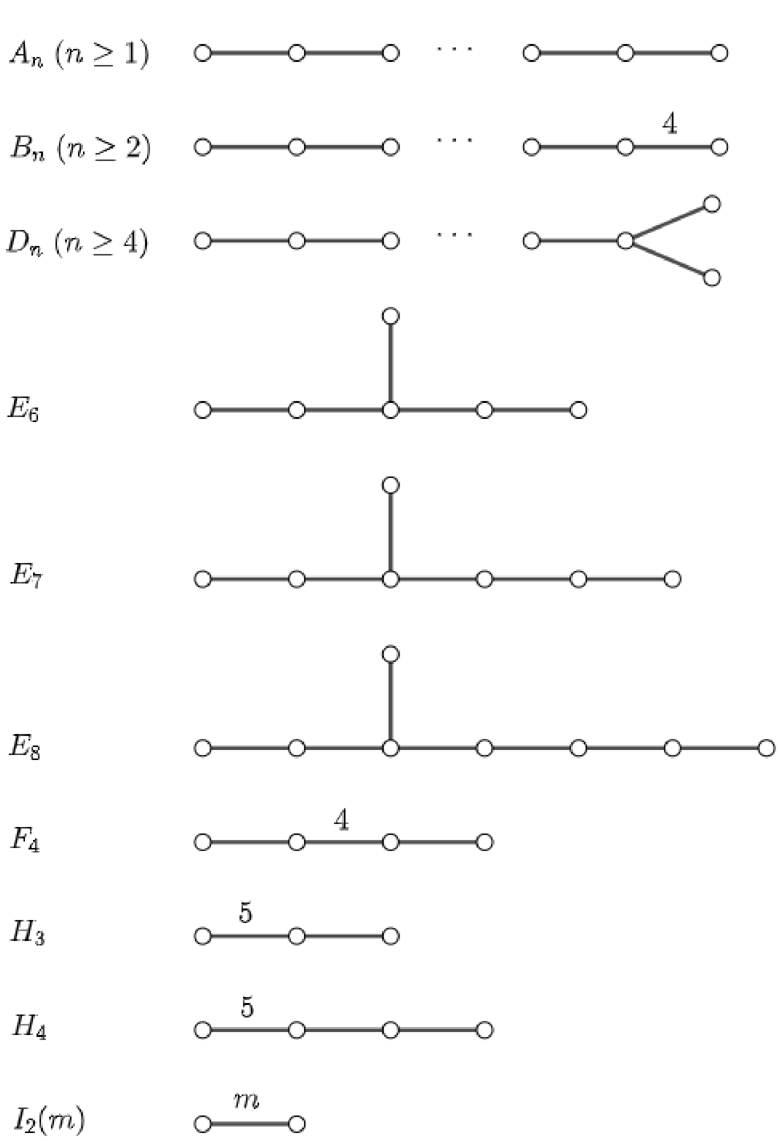


Figure 1.3.1 – Classification of finite reflection groups.

root system. In this case, W stabilizes some lattice in V.

Weyl groups of crystallographic root systems are precisely the reflections groups for which all $m(\alpha,\beta)$ lie in $\{2,3,4,6\}$. When W is irreducible at most two distinct root lengths are possible and the ratios of the squared lengths of *long roots* and *short roots* can only be 2 or 3. If just one root length occur, all roots are considered as short ones. Geometrically, since the ratios are constant, we can assume from now on, without loss of generality, all short roots have length 1, that is $|\alpha|^2 := (\alpha,\alpha) = 1$, for $\alpha \in \Delta$. Most finite reflection groups are Weyl groups, except for the classes H_3, H_4 and $I_2(m)$, for $m \notin \{2,3,4,6\}$ (HUMPHREYS, 1990, Proposition 2.8). Moreover, when W is irreducible, there must exist a unique *highest root* α_0, so that for all positive roots α, $\alpha_0 - \alpha$ is a sum of simple roots.

Finally, we present some W-stable lattices in V. The $\mathbb{Z}$-span Q of Φ in V is called the *root lattice* associated with Φ. For each $\alpha \in \Phi$, define $\alpha^\vee := \dfrac{2\alpha}{(\alpha,\alpha)}$, which is called the *coroot* associated with α. Note $(\alpha^\vee)^\vee = \alpha$, for all $\alpha \in \Phi$. The set $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$ of all coroots is also a crystallographic system in V, with simple system $\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta\}$, which is called the *dual root system*. The root lattice $Q^\vee$ associated with $\Phi^\vee$ is called the *coroot lattice* associated with Φ. In particular, if all roots are short, then we may consider $\alpha^\vee = 2\alpha$, for all $\alpha \in \Phi$. Since $s_\alpha = s_{\alpha^\vee}$ for all $\alpha \in \Phi$, it follows that the group $\mathcal{W}^\vee$ associated with $\Phi^\vee$ coincides with $\mathcal{W}$.

The *weight lattice* of Φ is

$$P := \{\lambda \in V : (\lambda, \alpha^\vee) \in \mathbb{Z}, \text{ for all } \alpha \in \Phi\}.$$

The latter is generated by the *fundamental weights* $\omega_1, \ldots, \omega_n$, defined as $(\omega_i, \alpha_j^\vee) = \delta_{ij}$, where n is the rank of Φ. Consider also the set of *dominant weights*

$$P^+ := \{\lambda \in P : (\lambda, \alpha) \geq 0, \text{ for all } \alpha \in \Phi\}.$$

Example 1.3.2. The symmetric group S_{n+1} is a Weyl group of type A_n, for $n \geq 1$. Indeed, it is well known that S_{n+1} can be generated by the set of transpositions $S = \{s_i : i \in I_{1,n}\}$, where $s_i = (i \quad i+1)$. Let $V = \mathbb{R}^{n+1}$ with standard basis $\{\varepsilon_1, \ldots, \varepsilon_{n+1}\}$. Make S_{n+1} act on V by permuting the subscripts of the fixed basis. Consider $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n$. Notice that

$$s_i\alpha_i = s_i(\varepsilon_i - \varepsilon_{i+1}) = \varepsilon_{i+1} - \varepsilon_i = -\alpha_i,$$

while s_i fixes pointwise the hyperplane $\left\{ \sum_{j \in I_{1,n+1}} a_j\varepsilon_j : a_i = a_{i+1} \right\}$, for $1 \leq i \leq n$. Thus, S_{n+1} acts on V as a finite reflection group of rank n with simple system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Since S_{n+1} satisfies relations of the form

$$|s_is_j| = \begin{cases} 3, & \text{if } j \in \{i-1, i+1\}, \\ 2, & \text{if } j \notin \{i, i-1, i+1\}, \\ 1, & \text{if } j = i, \end{cases}$$

it follows from the classification of finite reflection groups that S_{n+1} has type A_n. The associated root system is $\Phi = \{\alpha_{i,j} : 1 \leq i \neq j+1 \leq n+1\}$, where $\alpha_{i,j} := \sum_{k \in I_{i,j}} \alpha_k = \varepsilon_i - \varepsilon_{j+1}$. Considering the simple system Δ, it has as positive system

$$\Phi^+ = \{\alpha_{i,j} : 1 \leq i \leq j \leq n\}.$$

In this case, the highest root is $\alpha_0 = \alpha_{1,n}$ and all roots have the same length. Moreover, this action fixes pointwise the line spanned by $\varepsilon_1 + \ldots + \varepsilon_{n+1}$, while leaves stable its orthogonal complement, the hyperplane generated by the simple roots. Thus, we

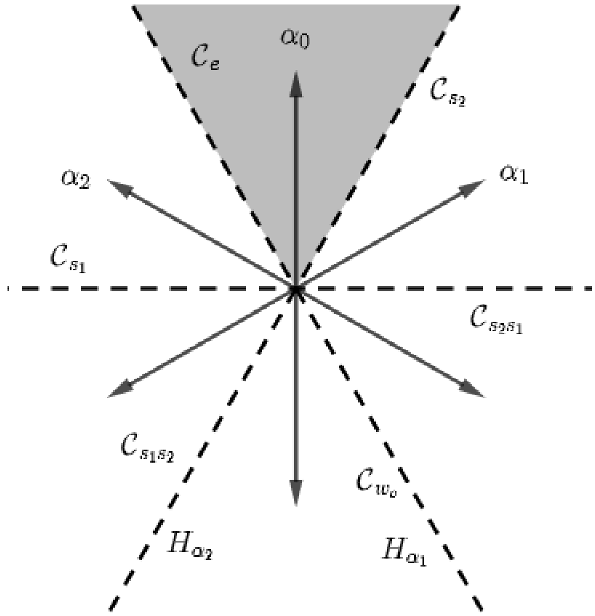


Figure 1.3.2 – Root system of type A_2.

may consider S_{n+1} acting on $\mathbb{R}^n$ when it is convenient. The fundamental weights $\{\omega_i : 1 \leq i \leq n\}$ satisfy

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \alpha_i = 2\omega_i - \omega_{i-1} - \omega_{i+1}, \quad 2 \leq i \leq n-1, \quad \text{and} \quad \alpha_n = 2\omega_n - \omega_{n-1}. \quad (1.3.2)$$

In particular, A_2 has simple system $\Delta = \{\alpha_1, \alpha_2\}$, with highest root $\alpha_0 = \alpha_1 + \alpha_2$ and positive system $\Phi^+ = \{\alpha_0, \alpha_1, \alpha_2\}$. Fix $\{i, j\} = I_{1,2}$ and denote $s_i := s_{\alpha_i}$. By Proposition 1.3.1, $s_i\alpha_j = \alpha_0$ and $s_i\alpha_0 = \alpha_j$. Since

$$s_i s_j s_i(\alpha_0) = s_i s_j(\alpha_j) = -s_i(\alpha_j) = -\alpha_0,$$

we have $s_{\alpha_0} = s_i s_j s_i = s_j s_i s_j$. In particular, $\ell(s_{\alpha_0}) = 3 = \#\Phi^+$, following that s_{α_0} is the longest element w_o of A_2 and, then, $w_o(\Phi^+) = \Phi^-$. Since the action is simply transitive, we must have $w_o(\alpha_i) = -\alpha_j$, $\{i, j\} = I$. Figure 1.3.2 exhibits the root system, some hyperplanes, and the Weyl chambers. Since $m(\alpha_1, \alpha_2) = 3$ and all roots are short, by (1.1.2),

$$(\alpha_1, \alpha_2) = -cos(\pi/3) = -1/2 \quad \text{and} \quad (\alpha_i, \alpha_0) = (\alpha_i, \alpha_i) + (\alpha_i, \alpha_j) = 1/2, \quad (1.3.3)$$

for $\{i, j\} = I_{1,2}$.

1.4 Connections with Lie Algebras

The exposition in this section follows (KAC, 1990).

A finite-dimensional complex algebra $\mathfrak{g}$ is said to be a *Lie algebra* if the multiplication $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies:

(i) $[x, x] = 0$,

(ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$,

for all $x, y, z \in \mathfrak{g}$. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a *subalgebra* if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, then $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$. If $\dim \mathfrak{g} > 1$ and the only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$, then $\mathfrak{g}$ is said to be *simple*. A Lie algebra is *semisimple* if it can be written as the direct sum of simple algebras

$$\mathfrak{g} = \bigoplus_{1 \le i \le k} \mathfrak{g}_i, \quad \text{with} \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}, \text{ if } i \neq j.$$

If $\mathfrak{g}$ is simple, then there exists a subalgebra $\mathfrak{h}$, with $[\mathfrak{h}, \mathfrak{h}] = \{0\}$, and $\Phi \subseteq \mathfrak{h}^*$ finite such that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}, \quad \text{with } \dim \mathfrak{g}_\alpha = 1. \tag{1.4.1}$$

Moreover, the set $\Phi \subseteq V$ is a crystallographic root system and the rank of Φ is the dimension of $\mathfrak{h}$, where $V \subseteq \mathfrak{h}^*$ is the real Euclidean space generated by Φ. The Weyl group $\mathcal{W}$ associated with Φ is said to be the Weyl group associated with $\mathfrak{g}$ and the type of $\mathfrak{g}$ is the type of $\mathcal{W}$. In the case that all roots are short, $\mathfrak{g}$ is said to be *simply laced*. Given a positive system Φ^+, denote

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha. \tag{1.4.2}$$

In this work, we are most interested in the Weyl group associated with the complex Lie algebra $\mathfrak{sl}_{n+1}$ of square matrices with trace zero, whose type is A_n. Fix $\Delta := \{\alpha_i : 1 \le i \le n\}$ a simple system, with corresponding fundamental weights $\{\omega_i : 1 \le i \le n\}$, then $\mathfrak{h}$ has a basis $\{h_i : 1 \le i \le n\}$ defined by

$$\omega_i(h_j) = \delta_{i,j}, \quad i, j \in I_{1,n}. \tag{1.4.3}$$

In this case, putting $s_i = s_{\alpha_i}$, $i \in I_{1,n}$, the action of $\mathcal{W}$ on V can also be given by

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \quad \text{for all } \lambda \in V.$$

For further use, it will be convenient to define $h_{\alpha_0} := \sum_{i \in I_{1,n}} h_i$, which satisfies

$$\omega_i(h_{\alpha_0}) = 1, \quad \text{for all } i \in I_{1,n}. \tag{1.4.4}$$

1.5 Affine Weyl Groups

We are now interested in the *affine reflection groups*, i.e., groups generated by *affine reflections*. An affine reflection is a reflection relative to a hyperplane that does not necessarily pass through the origin of V. Such groups are infinite and admit presentations as Coxeter groups. A particularly interesting class of such groups consists of the *affine Weyl groups*, which play an important role in the theory of Kac-Moody algebras.

Consider $Aff(V)$ the semidirect product of $GL(V)$ and the group of translations in V. It is easy to see that the group of translations is indeed normalized by $GL(V)$. Given a root system $\Phi \subseteq V$, for each root α and $k \in \mathbb{R}$, define the associated *affine hyperplane* as the set

$$H_{\alpha,k} := \{\lambda \in V : (\lambda, \alpha) = k\}.$$

The corresponding affine reflection is given by

$$s_{\alpha,k}\lambda := s_\alpha\lambda + k\alpha^\vee = \lambda - ((\lambda,\alpha) - k)\alpha^\vee \quad \text{for all} \quad \lambda \in V.$$

Note that $s_{\alpha,k}$ fixes the affine hyperplane $H_{\alpha,k}$ and sends the 0 vector to $k\alpha^\vee$. More generally, the restriction of $s_{\alpha,k}$ to $H_{\alpha,\ell}$ is the translation to $H_{\alpha,2k-\ell}$ along the direction determined by α. More precisely:

$$s_{\alpha,k}\lambda = \lambda + (k-\ell)\alpha^\vee \quad \text{for all} \quad \lambda \in H_{\alpha,\ell}.$$

It is not difficult to see that $s_{\alpha,0} = s_\alpha$.

If $\mathcal{W}$ is the Weyl group associated with the crystallographic root system Φ, the *affine Weyl group* $\widehat{\mathcal{W}}$ is defined to be the subgroup of $Aff(V)$ generated by the affine reflections $s_{\alpha,k}$, with $\alpha \in \Phi$, $k \in \mathbb{Z}$. That is, $\widehat{\mathcal{W}}$ is the semidirect product of $\mathcal{W}$ and the subgroup of translations given by elements of the coroot lattice $Q^\vee$. Let T_λ denote the translation by $\lambda \in Q^\vee$. Then, each $s_{\alpha,k}$ can be written uniquely as

$$s_{\alpha,k} = w_\lambda = T_\lambda w, \quad \text{where } w = s_\alpha \text{ and } \lambda = k\alpha^\vee. \tag{1.5.1}$$

Fix $\Delta := \{\alpha_1, \dots, \alpha_n\}$ a simple system for Φ and let α_0 be the highest root relative to Δ. Denote

$$\widehat{S} := \{s_i : 0 \le i \le n\}, \tag{1.5.2}$$

where $s_0 := s_{\alpha_0,1}$ and $s_i := s_{\alpha_i}$, for $1 \le i \le n$. It is possible to show that $\widehat{\mathcal{W}}$ is generated by the set $\widehat{S}$, subjected only to the relations

$$(s_i s_j)^{m(i,j)} = 1,$$

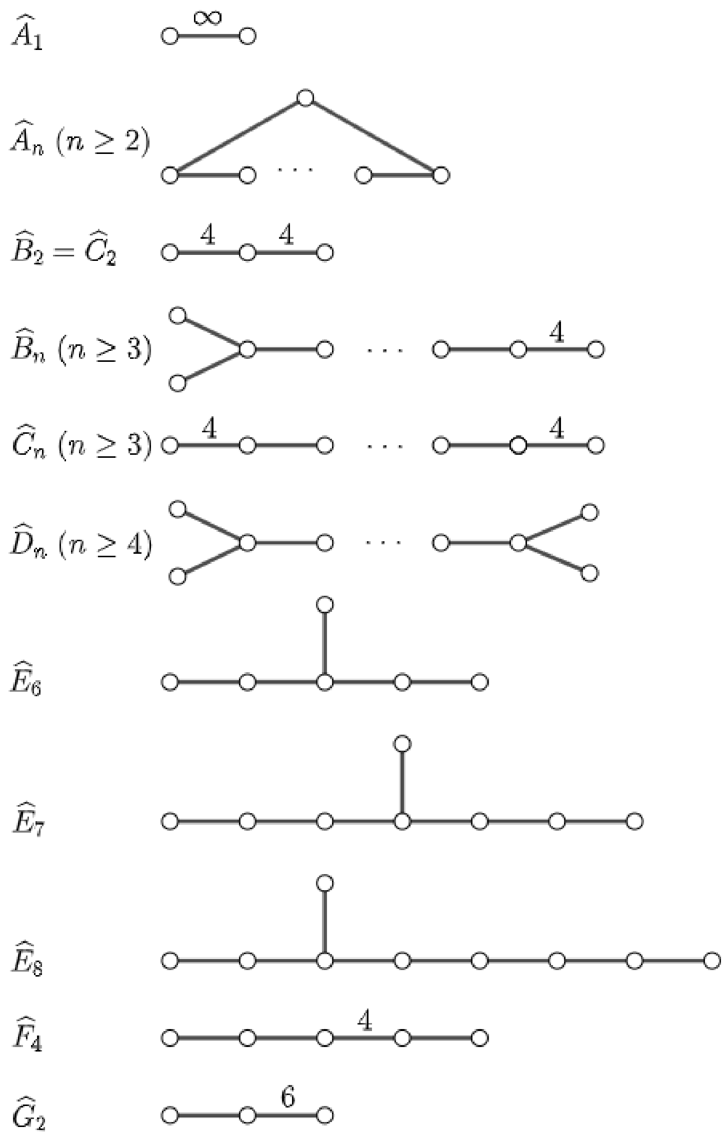


Figure 1.5.1 – Classification of affine Weyl groups.

where $m(i,j)$ is the order of the element $s_i s_j \in Aff(V)$, with $1 \leq i \leq n$ (HUMPHREYS, 1990, Theorem 4.6). That is, $(\widehat{\mathcal{W}}, \widehat{S})$ is a Coxeter system. All possible graphs arising from an affine Weyl group were classified and are exhibited in Figure 1.5.1, where the subscript n denotes the rank of the root system.

1.6 Alcoves

Let Φ be an irreducible root system. Although the underlying finite Weyl groups $\mathcal{W}$ and $\mathcal{W}^\vee$ coincide, the affine Weyl groups $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{W}}^\vee$ do not need to be isomorphic. These groups are isomorphic if and only if Φ has only short roots. Anyway, since $(\Phi^\vee)^\vee = \Phi$, we may work with dual root systems to characterize affine Weyl groups, preserving the indexing by roots instead of coroots. It turns out that this change makes easier to phrase some results that we are interested in, following (SHI, 1999).

Let $\widehat{\mathcal{W}}$ be an affine Weyl group, with positive system $(\Phi^\vee)^+ \subseteq V$ and simple system $\Delta = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$. Let also $\widehat{S} = \{s_i : i \in I_{0,n}\}$, where

$$s_i = s_{\alpha_i^\vee,0}, \; i \in I_{1,n}, \quad \text{and} \quad s_0 = s_{\alpha_0^\vee,1}.$$

Note $s_{\alpha_i} = s_{\alpha_i^\vee}$, for all $i \in I_{0,n}$. By abuse of notation, let $H_{\alpha,k}$ denote the affine hyperplane $H_{\alpha^\vee,k}$, for $\alpha \in \Phi, k \in \mathbb{Z}$.

The connected components of

$$V \setminus \bigcup_{\alpha\in\Phi, k\in\mathbb{Z}} H_{\alpha,k}$$

are called *alcoves*. Let us denote $\mathscr{A}$ the set of all alcoves. Define a *strip* as a set of the form $H^1_{\alpha,k} := \{\lambda \in V : k < (\lambda, \alpha^\vee) < k+1\}$, for $\alpha \in \Phi$. Note

$$H^1_{-\alpha,k} = H^1_{\alpha,-k-1}, \quad \text{for all } \alpha \in \Phi, k \in \mathbb{Z}. \tag{1.6.1}$$

Given a family $\boldsymbol{k} = (k_\alpha)_{\alpha\in\Phi^+}$ of integers, define

$$\mathcal{A}_{\boldsymbol{k}} := \bigcap_{\alpha\in\Phi^+} H^1_{\alpha,k_\alpha} = \{\lambda \in V : k_\alpha < (\lambda, \alpha^\vee) < k_\alpha + 1, \alpha \in \Phi^+\}.$$

Setting

$$k_{-\alpha} := -k_\alpha - 1, \quad \text{for } \alpha \in \Phi^- \tag{1.6.2}$$

and using 1.6.1, it follows that $\mathcal{A}_{\boldsymbol{k}} = \bigcap_{\alpha\in\Phi} H^1_{\alpha,k_\alpha}$. These sets are not alcoves in general, but every alcove coincide with one such set as proved in (SHI, 1987), where it was also proved a precise characterization of the families $\boldsymbol{k}$ such that $\mathcal{A}_{\boldsymbol{k}}$ is an alcove (SHI, 1987, Theorem 5.2). The next result is a simplification of this characterization given in (SHI, 1999, Theorem 1.1). The proof consists of handling a series of inequalities, including ones which depend on a case by case analysis according to type of the subsystem generated by a given pair of positive roots. As the steps of the proof are not relevant for our purposes, we have decided to omit the details here.

Lemma 1.6.1. Let $\boldsymbol{k} = (k_\alpha)_{\alpha\in\Phi^+}$ be a family of integers. The set $\mathcal{A}_{\boldsymbol{k}}$ is an alcove if and only if, for all $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = (\alpha^\vee + \beta^\vee)^\vee$, the following holds

$$k_\alpha + k_\beta \leq k_\gamma \leq k_\alpha + k_\beta + 1. \tag{1.6.3}$$

In particular, if Φ has only short roots, then $\gamma = \alpha + \beta$ in the lemma. Proceeding inductively, one can then easily check the following.

Corollary 1.6.2. Suppose Φ has only short roots and that $\alpha = \sum_{j\in I_{1,m}} \alpha_{i_j}$ for some $m \in \mathbb{Z}_{>1}$ and $i_j \in I_{1,n}$. Then, $\sum_{j\in I_{1,m}} k_{\alpha_{i_j}} \leq k_\alpha \leq \sum_{j\in I_{1,m}} k_{\alpha_{i_j}} + m - 1$.

If $\boldsymbol{k} = (k_\alpha)_{\alpha\in\Phi^+}$ satisfies (1.6.3), we shall say $\boldsymbol{k}$ is the *coordinate form* of the alcove $\mathcal{A}_{\boldsymbol{k}}$. We let $\boldsymbol{k} = \mathbf{0}$ denote the *fundamental alcove*, i.e., the one associated with the family $k_\alpha = 0$ for all $\alpha \in \Phi^+$. Straightforward computations show that

$$\mathcal{A}_{\mathbf{0}} = \{\lambda \in V : 0 < (\lambda, \alpha_i^\vee), \forall i \in I_{1,n}, (\lambda, \alpha_0^\vee) < 1\}. \tag{1.6.4}$$

Note $\mathcal{A}_{\boldsymbol{k}}$ is contained in the fundamental Weyl chamber if and only if

$$k_\alpha \geq 0, \quad \text{for all } \alpha \in \Phi^+. \tag{1.6.5}$$

From Corollary 1.6.2 immediately follows the next.

Corollary 1.6.3. Let $\boldsymbol{k} = (k_\alpha)_{\alpha\in\Phi^+}$. If Φ has just short roots, then $\mathcal{A}_{\boldsymbol{k}} \subseteq \mathcal{C}_e$ if and only if $k_{\alpha_i} \geq 0$, for all $i \in I_{1,n}$.

Example 1.6.4. Suppose $\widehat{\mathcal{W}}$ and, hence, $\widehat{\mathcal{W}}^\vee$ have type $\widehat{A}_2$. By Example 1.3.2, we have $\Phi^+ \smallsetminus \Delta = \{\alpha_0\}$. Then, the only inequality that must be satisfied is

$$k_{\alpha_1} + k_{\alpha_2} \leq k_{\alpha_0} \leq k_{\alpha_1} + k_{\alpha_2} + 1.$$

Thus, the triple $(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_0}) = (1,0,0)$ does not correspond to an alcove, for instance. Figure 1.6.1 shows alcoves labeled by their corresponding coordinate form.

The length function in an affine Weyl group has a geometric characterization in terms of alcoves. Let $\mathcal{H} = \{H_{\alpha,k} : \alpha \in \Phi^+, k \in \mathbb{Z}\}$. The hyperplane $H \in \mathcal{H}$ *separates* two alcoves if each one of them lies in different half-spaces relative to H. The number of such hyperplanes for two fixed alcoves is always finite, then it is well-defined the number $n(w)$ as the cardinality of the set

$$\mathcal{H}(w) := \{H \in \mathcal{H} : H \text{ separates } \mathcal{A}_e \text{ and } \mathcal{A}_w\}, \quad w \in \widehat{\mathcal{W}}^\vee.$$

For instance, $\mathcal{H}(s) = \{H_{s,0}\}$, for all $s \in \widehat{S}$.

The group $\widehat{\mathcal{W}}^\vee$ acts on the set of alcoves $\mathscr{A}$, via the correspondence given by $\mathcal{A} \mapsto w\mathcal{A}$, $\mathcal{A} \in \mathscr{A}$, $w \in \widehat{\mathcal{W}}^\vee$.

Theorem 1.6.5. (HUMPHREYS, 1990, Theorem 4.5) Let $w \in \widehat{\mathcal{W}}^\vee \smallsetminus \{e\}$, with a reduced expression $s_{i_1} \ldots s_{i_m}$, and denote $H_j := H_{s_{i_j}, \delta_{i_j,0}}$. Then $\mathcal{H}(w) = \{H_1, s_{i_1}H_2, \ldots, s_{j_1} \ldots s_{j_{m-1}} H_m\}$ and these hyperplanes are all distinct. In particular, $\ell(w) = n(w)$, for all $w \in \widehat{\mathcal{W}}^\vee$, and $\widehat{\mathcal{W}}^\vee$ acts simply transitively on $\mathscr{A}$.

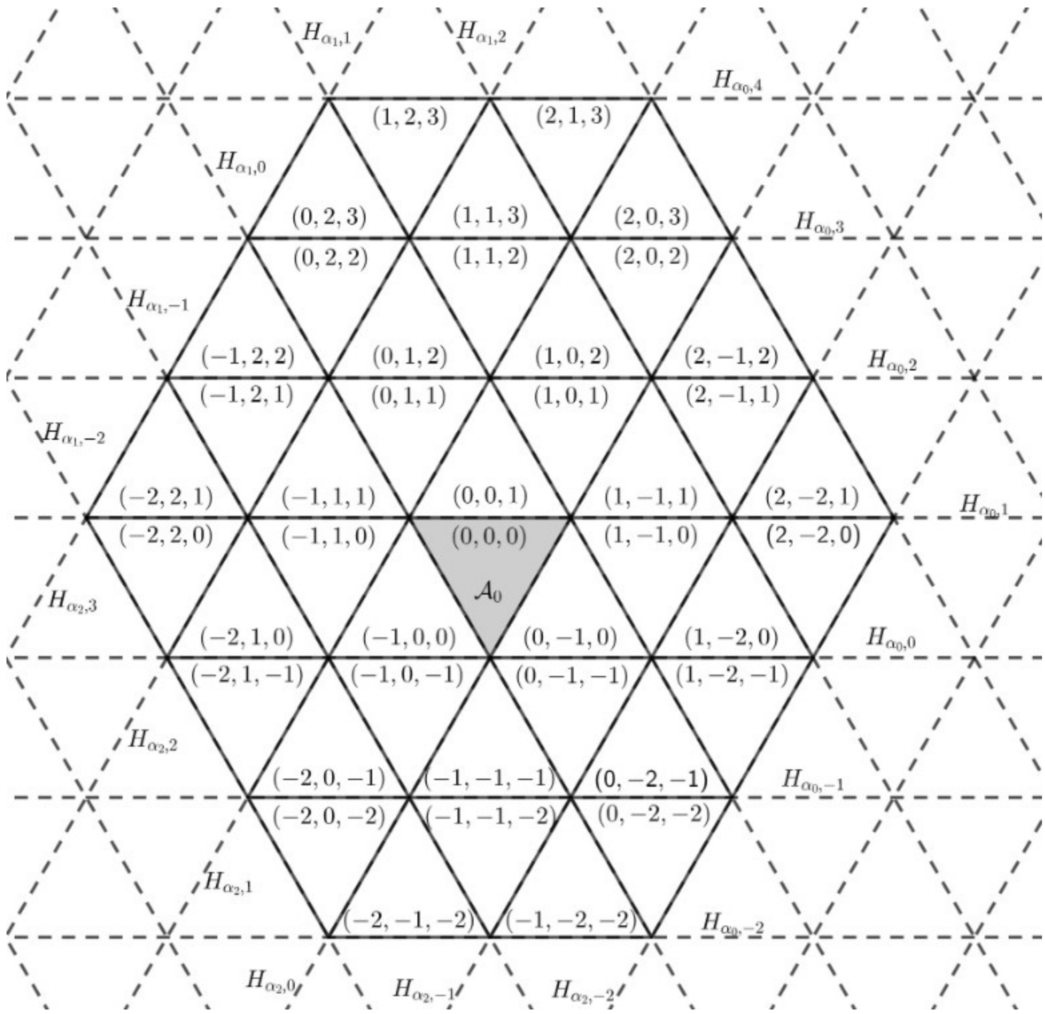


Figure 1.6.1 – Coordinate form for alcoves of type $\widehat{A}_2$.

Since the action is simply transitive, it allows us to define $\mathcal{A}_w := w(\mathcal{A}_\mathbf{0})$. In particular, $\mathcal{A}_\mathbf{0} = \mathcal{A}_e$. Moreover, there exists a well-defined map $k : \widehat{\mathcal{W}}^\vee \times \Phi^+ \to \mathbb{Z}$ determined by

$$\mathcal{A}_w = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha, k(w,\alpha)}.$$

The map $k_w : \Phi^+ \to \mathbb{Z}$, $\alpha \mapsto k(w,\alpha)$, or, equivalently, the family of integers $(k(w,\alpha))_{\alpha\in\Phi^+}$, is called the *alcove form* of w. Since the correspondence $w \mapsto \mathcal{A}_w$ is a bijection, w is determined by its alcove form.

Write $w_\lambda = T_\lambda w$, for $w \in \mathcal{W}^\vee$ and $\lambda \in Q$. By 1.5.1, $\widehat{\mathcal{W}}^\vee = \{w_\lambda : w \in \mathcal{W}^\vee, \lambda \in Q\}$. We want to characterize the alcove form of $w_\lambda \in \widehat{\mathcal{W}}^\vee$ in terms of w and λ. It will be useful to recall that $(w^{-1}\alpha)^\vee = w^{-1}\alpha^\vee$ and

$$(w\lambda, \alpha^\vee) = (\lambda, w^{-1}\alpha^\vee), \quad \text{for all } w \in \mathcal{W}^\vee, \lambda \in V, \alpha \in \Phi, \tag{1.6.6}$$

since $\mathcal{W}^\vee$ is a group of orthogonal transformations in V.

Lemma 1.6.6. (SHI, 1987, Lemma 3.1) If $w \in \mathcal{W}^\vee$, then $k(w,\alpha) = \begin{cases} 0, & \text{if } w^{-1}\alpha \in \Phi^+, \\ -1, & \text{if } w^{-1}\alpha \in \Phi^-. \end{cases}$

Proof. Let $\mu \in \mathcal{A}_e$ and $\alpha \in \Phi^+$. Then $w\mu \in \mathcal{A}_w$. By (1.6.6), $(w\lambda, \alpha^\vee) = (\lambda, w^{-1}\alpha^\vee)$, for all $\alpha \in \Phi^+$. If $w^{-1}\alpha \in \Phi^+$, then $0 < (\mu, w^{-1}\alpha^\vee) < 1$. Thus, $0 < (w\mu, \alpha^\vee) < 1$, following that $k(w,\alpha) = 0$. Otherwise, if $w^{-1}\alpha \in \Phi^-$, then $-w^{-1}\alpha \in \Phi^+$. Similarly, it follows that $-1 < (\mu, w^{-1}\alpha^\vee) < 0$ and, hence, $-1 < (w\mu, \alpha^\vee) < 0$, proving that $k(w\mu, \alpha) = -1$. □

Lemma 1.6.7. (SHI, 1999, (1.5.1)) Let $w \in \mathcal{W}^\vee$ and $\lambda \in Q$. Then $k(w_\lambda, \alpha) = k(w,\alpha) + (\lambda, \alpha^\vee)$.

Proof. Notice $\mathcal{A}_{w_\lambda} = \{\lambda\} + \mathcal{A}_w$. For all $\alpha \in \Phi^+$, $\mu \in \mathcal{A}_w$, $k(w,\alpha) < (\mu, \alpha^\vee) < k(w,\alpha) + 1$. Let $\eta = \lambda + \mu \in \mathcal{A}_{w_\lambda}$. Since $(\eta, \alpha^\vee) = (\lambda, \alpha^\vee) + (\mu, \alpha^\vee)$, we have

$$k(w,\alpha) + (\lambda, \alpha^\vee) < (\eta, \alpha^\vee) < k(w,\alpha) + (\lambda, \alpha^\vee) + 1,$$

from where the lemma follows. □

In particular, if $\widehat{S}$ is a generating set for $\widehat{\mathcal{W}}^\vee$ as in (1.5.2),

$$k(s_i, \alpha) = -\delta_{\alpha_i, \alpha} \quad \text{and} \quad k(s_0, \alpha) = \delta_{\alpha_0, \alpha}, \quad \text{for all } i \in I_{1,n}, \alpha \in \Phi^+. \tag{1.6.7}$$

This leads to the following formula for the alcove form of an element of $\widehat{\mathcal{W}}^\vee$ in terms of a reduced expression.

Proposition 1.6.8. Let $w = s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in \widehat{\mathcal{W}}^\vee$ and set $u_l = s_{\alpha_{i_1}} \dots s_{\alpha_{i_l}} \in \mathcal{W}^\vee$, for $1 \le l \le m$. Then

$$k(w,\alpha) = k(u_m, \alpha) + \sum_{j \in I_{1,m}} \delta_{i_j,0}(u_{j-1}\alpha_0, \alpha^\vee), \quad \text{for all } \alpha \in \Phi^+.$$

Proof. Write $w = u_\lambda$, $u \in \mathcal{W}^\vee$, $\lambda \in Q$. By Lemma 1.6.7, it suffices to check that $u = u_m$ and $\lambda = \sum_{j \in I_{1,m}} \delta_{i_j,0} u_{j-1}\alpha_0$. Indeed, let us proceed by induction on $\ell(w) = m \in \mathbb{Z}_{\ge 1}$. Recall $s_i = s_{\alpha_i} + \delta_{i,0}\alpha_0$, $i \in I_{0,n}$. Thus, for $m = 1$, the result is clear. If $m > 1$, then the induction hypothesis imply

$$\begin{aligned} w\mu = (s_{i_1} \dots s_{i_{m-1}})(s_{i_m}\mu) &= u_{m-1}s_{i_m}\mu + \sum_{j \in I_{1,m-1}} \delta_{i_j,0}u_{j-1}\alpha_0 \\ &= u_{m-1}(s_{\alpha_{i_m}}\mu + \delta_{i_m,0}\alpha_0) + \sum_{j \in I_{1,m-1}} \delta_{i_j,0}u_{j-1}\alpha_0 \\ &= u_m\mu + \delta_{i_m,0}u_{m-1}\alpha_0 + \sum_{j \in I_{1,m-1}} \delta_{i_j,0}u_{j-1}\alpha_0 = u_m\mu + \sum_{j \in I_{1,m}} \delta_{i_j,0}u_{j-1}\alpha_0 \end{aligned}$$

for all $\mu \in V$, as claimed. □

Corollary 1.6.9. (SHI, 1999, Section 1.6) For all $i \in I_{0,n}$, $w \in \widehat{\mathcal{W}}^\vee$, and $\alpha \in \Phi^+$, we have

$$k(s_i w, \alpha) = k(w, s_{\alpha_i}\alpha) + \delta_{i,0}(\alpha_0, \alpha^\vee). \tag{1.6.8}$$

Proof. Since $u^{-1}(v\alpha) = (v^{-1}u)^{-1}\alpha$, it follows from Lemma 1.6.6 that

$$k(u, v\alpha) = k(v^{-1}u, \alpha), \quad \text{for all } u, v \in \mathcal{W}^\vee. \tag{1.6.9}$$

Therefore, by Proposition 1.6.8,

$$\begin{aligned} k(s_i w, \alpha) &= k(s_{\alpha_i} u_m, \alpha) + \delta_{i,0}(\alpha_0, \alpha^\vee) + \sum_{j \in I_{1,m}} \delta_{i_j,0}(s_{\alpha_i} u_{j-1}\alpha_0, \alpha^\vee) \\ &\overset{(1.6.6)}{\underset{(1.6.9)}{=}} k(u_m, s_{\alpha_i}\alpha) + \delta_{i,0}(\alpha_0, \alpha^\vee) + \sum_{j \in I_{1,m}} \delta_{i_j,0}(u_{j-1}\alpha_0, (s_{\alpha_i}\alpha)^\vee) \\ &= k(w, s_{\alpha_i}\alpha) + \delta_{i,0}(\alpha_0, \alpha^\vee). \end{aligned}$$

□

The length and the left descent set of an element of $\widehat{\mathcal{W}}^\vee$ can be computed in terms of alcoves, as seen in the next theorem. A proof for type $\widehat{A}$ is given in the book (SHI, 1980, Proposition 6.4.1). It is claimed the proof works for all types in (SHI, 1987, Proposition 4.3).

Theorem 1.6.10. (SHI, 1999, Proposition 1.9) For all $w \in \widehat{\mathcal{W}}^\vee$, we have $\ell(w) = \sum_{\alpha \in \Phi^+} |k(w, \alpha)|$ and $D(w) = \{s_i : i = 0 \text{ and } k(w, \alpha_0) > 0 \text{ or } i \in I_{1,n} \text{ and } k(w, \alpha_i) < 0\}$. □

Corollary 1.6.11. An alcove $\mathcal{A}_w$ is contained in the fundamental Weyl chamber if and only if $D(w) = \{s_0\}$.

Proof. Theorem 1.6.10 implies that

$$D(w) = \{s_0\} \quad \Leftrightarrow \quad k(w, \alpha_j) > 0, \quad \text{for all } j \in I_{0,n}.$$

By Corollary 1.6.3, the result follows. □

Example 1.6.12. Suppose $\widehat{\mathcal{W}}^\vee$ has type $\widehat{A}_2$. Let us compute the alcove forms of $s_0 s_i$, $s_0 s_i s_j$, $s_0 w_o$, for $\{i, j\} = I_{1,2}$. Using (1.6.7) and (1.6.8), one easily checks that, for all $w \in \widehat{\mathcal{W}}^\vee$, we have

$$k(s_p u, \alpha_q) = \begin{cases} -k(u, \alpha_i) - 1, & \text{if } p = q = i, \\ -k(u, \alpha_0) + 1, & \text{if } p = q = 0, \\ k(u, \alpha_r), & \text{if } p = i, \{q, r\} = \{0, j\}, \\ -k(u, \alpha_r), & \text{if } p = 0, \{q, r\} = \{i, j\}. \end{cases} \tag{1.6.10}$$

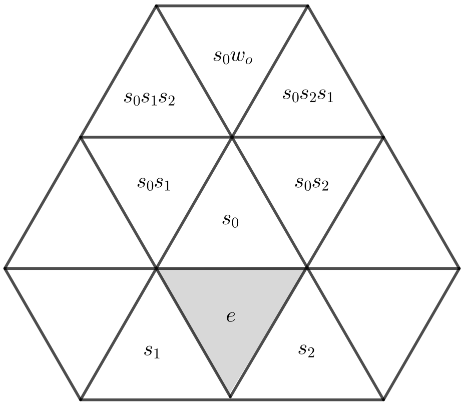


Figure 1.6.2 – Elements corresponding to alcoves of type $\widehat{A}_2$.

Now, iterating this, we get

$$k(s_0 s_i u, \alpha_q) = \begin{cases} -k(u, \alpha_j) + 1, & \text{if } q = 0, \\ -k(u, \alpha_0), & \text{if } q = i, \\ k(u, \alpha_i) + 1, & \text{if } q = j. \end{cases} \tag{1.6.11}$$

Plugging $w = e$, we see that the alcove form of $s_0 s_i$ is

$$k(s_0 s_i, \alpha_i) = 0 \quad \text{and} \quad k(s_0 s_i, \alpha_0) = k(s_0 s_i, \alpha_j) = 1. \tag{1.6.12}$$

Using (1.6.10) and (1.6.11), we get

$$k(s_0 s_i s_j u, \alpha) = \begin{cases} -k(u, \alpha_i), & \text{if } q = i, \\ k(u, \alpha_r) + 1 + \delta_{q,0}, & \text{if } \{q, r\} = \{0, j\}. \end{cases} \tag{1.6.13}$$

For $w = e$, we see the alcove form of $s_0 s_i s_j$ is

$$k(s_0 s_i s_j, \alpha_i) = 0, \quad k(s_0 s_i s_j, \alpha_j) = 1, \quad k(s_0 s_i s_j, \alpha_0) = 2. \tag{1.6.14}$$

Using (1.6.13) and (1.6.10), we have

$$k(s_0 w_o u, \alpha_q) = k(u, \alpha_q) + 1 + \delta_{q,0}, \quad \text{for all } u \in \widehat{\mathcal{W}}^\vee.$$

Hence, the alcove form of $s_0 w_o$ is

$$k(s_0 w_o, \alpha_i) = k(s_0 w_o, \alpha_j) = 1, \quad k(s_0 w_o, \alpha_0) = 2. \tag{1.6.15}$$

Comparison with Figure 1.6.1 gives us Figure 1.6.2. Note also Theorem 1.6.10 implies $\ell(s_0 s_i s_j) = 3$ and $\ell(s_0 w_o) = 4$, as expected.

1.7 Alcove Paths

By (1.6.4), the boundary of $\mathcal{A}_e$ is contained in $\bigcup_{i \in I_{0,n}} H_{\alpha_i,\delta_{i,0}}$. Moreover, since the action of $\widehat{\mathcal{W}}^\vee$ permutes the set of alcoves, the boundary of $\mathcal{A}_w$ is contained in $\bigcup_{i \in I_{0,n}} wH_{\alpha_i,\delta_{i,0}}$, for any $w \in \widehat{\mathcal{W}}^\vee$. These $n+1$ hyperplanes will be referred to as the *walls* of $\mathcal{A}_w$. For $i \in I_{0,n}$, the hyperplane $H_{\alpha_i,\delta_{i,0}}$ is the unique wall shared by $\mathcal{A}_e$ and $\mathcal{A}_{s_i}$.

The intersection of the closure of an alcove with one of its walls will be referred to as a *facet*. Let $\mathscr{F}$ be the set of facets, i.e., $\mathscr{F} = \{F \subseteq V : F \text{ is a facet of } \mathcal{A} \text{ for some } \mathcal{A} \in \mathscr{A}\}$. Since the action of $\widehat{\mathcal{W}}^\vee$ on $\mathscr{A}$ is simply transitive, it induces an action on $\mathscr{F}$. Moreover, given $F \in \mathscr{F}$, there exists a unique facet F' of $\mathcal{A}_e$ such that $F = wF'$ for some $w \in \widehat{\mathcal{W}}^\vee$. We say F is an *i-facet*, $i \in I_{0,n}$, if F' is contained in the wall shared by $\mathcal{A}_e$ and $\mathcal{A}_{s_i}$. We let $\mathscr{F}_i$ be the subset of $\mathscr{F}$ containing the i-facets.

The following lemma can be regarded as a motivation to the definition of the notion of alcove paths.

Lemma 1.7.1. (SHI, 1987, Lemma 6.1) Let $w, w' \in \widehat{\mathcal{W}}^\vee$. Then $w' = ws_i$, for some $i \in I_{0,n}$, if and only if $\mathcal{A}_w$ and $\mathcal{A}_{w'}$ share an i-facet.

Proof. Suppose $w' = ws_i$. Then $\mathcal{A}_w = w(\mathcal{A}_e)$ and $\mathcal{A}_{w'} = w(\mathcal{A}_{s_i})$. Since $\mathcal{A}_e$ and $\mathcal{A}_{s_i}$ share an i-facet, so do $\mathcal{A}_w$ and $\mathcal{A}_{w'}$. Conversely, if $\mathcal{A}_w$ and $\mathcal{A}_{w'}$ share an i-facet, there exists $v \in \widehat{\mathcal{W}}^\vee$ such that $\mathcal{A}_w = v(\mathcal{A}_e) = \mathcal{A}_v$ and $\mathcal{A}_{w'} = v(\mathcal{A}_{s_i}) = \mathcal{A}_{vs_i}$. Hence, $w = v$ and $w' = ws_i$. □

Two alcoves are said to be *adjacent* if they are distinct and share a facet. We write $\mathcal{A} \overset{i}{\sim} \mathcal{A}'$ if $\mathcal{A}$ and $\mathcal{A}'$ are adjacent and share an i-facet. In particular, $\mathcal{A}_e \overset{i}{\sim} \mathcal{A}_{s_i}$, $i \in I_{0,n}$.

An *alcove path* is a finite sequence of adjacent alcoves. We shall write

$$\mathcal{A}_1 \overset{i_1}{\to} \mathcal{A}_2 \overset{i_2}{\to} \dots \overset{i_l}{\to} \mathcal{A}_{l+1}$$

to denote such a given path if $\mathcal{A}_j \overset{i_j}{\sim} \mathcal{A}_{j+1}$, $j \in I_{1,l}$. The alcove path is said to be *reduced* if l is minimal among all alcove paths from $\mathcal{A}_1$ to $\mathcal{A}_{l+1}$. We let $\mathscr{P}(\mathcal{A}, \mathcal{A}')$ denote the set of alcove paths from $\mathcal{A}$ to $\mathcal{A}'$ and set $\mathscr{P}(\mathcal{A}) = \mathscr{P}(\mathcal{A}_e, \mathcal{A})$.

Given $w \in \widehat{\mathcal{W}}^\vee$, let

$$\mathscr{E}_w = \left\{ \iota : I_{1,l} \to I_{0,n} : l \in \mathbb{Z}_{\geq 0}, w = \overrightarrow{\prod_{1 \leq j \leq l}} s_{\iota(j)} \right\}.$$

We refer to $\mathscr{E}_w$ as the *set of expressions* for w. The subset of $\mathscr{E}_w$ containing the reduced expressions for w will be denoted $\mathscr{R}_w$. For $\iota \in \mathscr{E}_w$, the *length* of ι, denoted by $\ell(\iota)$, is the cardinality of the domain of ι. As usual, by abuse of notation, we let $i_j = \iota(j)$. Given

$\iota \in \mathscr{E}_w$, consider

$$v_l = \overrightarrow{\prod_{1\le j\le l}} s_{i_j} \quad \text{and} \quad u_l = \overrightarrow{\prod_{1\le j\le l}} s_{\alpha_{i_j}}, \quad 0 \le l \le \ell(\iota). \tag{1.7.1}$$

In particular, $v_0 = w_0 = e$ and $v_{\ell(\iota)} = w$. Lemma 1.7.1 implies

$$\mathcal{A}_{v_{l-1}} \overset{i_l}{\sim} \mathcal{A}_{v_l}, \quad 1 \le l \le \ell(\iota). \tag{1.7.2}$$

Hence, we can associate the following element of $\mathscr{P}(\mathcal{A}_w)$ with ι, which we denote by ϱ_ι:

$$\mathcal{A}_e \overset{i_1}{\to} \mathcal{A}_{v_1} \overset{i_2}{\to} \mathcal{A}_{v_2} \cdots \overset{i_{\ell(\iota)}}{\to} \mathcal{A}_w.$$

Consider also

$$\beta_l = u_{l-1}\alpha_{i_l}, \quad 1 \le l \le \ell(\iota).$$

Proposition 1.7.2. (LENART; POSTNIKOV, 2007, Lemma 5.3) The map $\mathscr{E}_w \to \mathscr{P}(\mathcal{A}_w)$, $\iota \mapsto \varrho_\iota$, is a bijection and $\iota \in \mathscr{R}_w$ if and only if ϱ_ι is reduced. Moreover, for $1 \le l \le \ell(\iota)$, the i_l-facet shared by $\mathcal{A}_{v_{l-1}}$ and $\mathcal{A}_{v_l}$ is contained in $H_{\beta_l,m}$ for some $m \in \mathbb{Z}$. □

As observed in (LENART; POSTNIKOV, 2007), the proof of Proposition 1.7.2 is essentially contained in (HUMPHREYS, 1990) and, therefore, we omit the details here. A streamlined proof is presented in (LENART; POSTNIKOV, 2007) nevertheless.

Example 1.7.3. Suppose $\widehat{\mathcal{W}}^\vee$ has type $\widehat{A}_2$. Figure 1.7.1 exhibits 1-facets in red, 2-facets in blue, and 0-facets in green, according to Lemma 1.7.1. Two different alcove paths from $\mathcal{A}_e$ to $\mathcal{A}_{w_o}$ are presented following Proposition 1.7.2. Moreover, since $\ell(w_o) = 3$, both alcove paths are reduced.

We end this section with an analysis of reduced paths contained in the fundamental Weyl chamber. We show that these paths can be described by certain families of finite non-decreasing sequences of non-negative integers indexed by Φ^+.

Lemma 1.7.4. Let $\mathcal{A}_w \subseteq \mathcal{C}_e$, $w \in \widehat{\mathcal{W}}^\vee$. Let also $s_{i_1} \dots s_{i_m}$ be a reduced expression for w and v_l, u_l, $0 \le l \le m$, as in (1.7.1). Then, for each $i \in I_{2,l}$, there exists $\tilde{\alpha}_i \in \Phi^+$ such that $k(v_i,\alpha) = k(v_{i-1},\alpha) + \delta_{\alpha,\tilde{\alpha}_i}$. In particular, any reduced alcove path in $\mathscr{P}(\mathcal{A}_w)$ is contained in the fundamental Weyl chamber.

Proof. From Proposition 1.7.2, it follows that the unique hyperplane crossed in the path $\mathcal{A}_{v_{l-1}} \overset{s_{i_l}}{\to} \mathcal{A}_{v_l}$ is orthogonal to β_l. Define $\tilde{\alpha}_l$ as the positive root in $\{\beta_l, -\beta_l\}$. Hence,

$$k(v_l,\alpha) = k(v_{l-1},\alpha) \pm \delta_{\alpha,\tilde{\alpha}_l}. \tag{1.7.3}$$

Moreover, notice that, by Theorem 1.6.10,

$$\ell(v_l) > \ell(v_{l-1}) \quad \Leftrightarrow \quad |k(v_l,\tilde{\alpha}_l)| > |k(v_{l-1},\tilde{\alpha}_l)|. \tag{1.7.4}$$

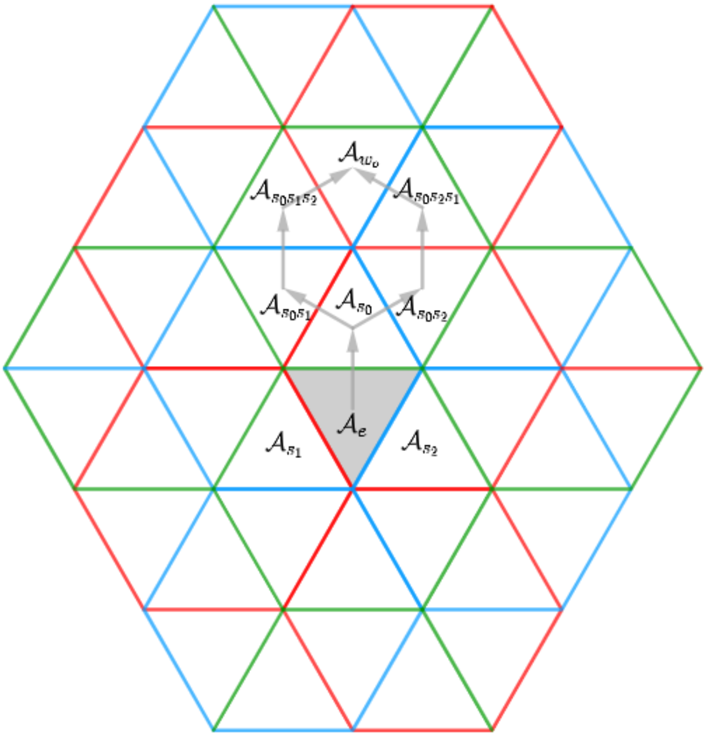


Figure 1.7.1 – Alcoves paths from $\mathcal{A}_e$ to $\mathcal{A}_{w_o}$.

Let us proceed by induction on $l \in I_{1,m}$ to prove that $0 \le k(v_{l-1}, \alpha) \le k(v_l, \alpha)$, for all $\alpha \in \Phi^+$. By Corollary 1.6.11, $v_1 = s_0$, then the claim holds for $l = 1$. Suppose, thus, $l > 1$. Then, the induction hypothesis and (1.7.4) ensures the result. In particular, by (1.7.3), the first part of the lemma follows. The second part follows immediately since $k(v_l, \alpha) \ge 0$, for all $\alpha \in \Phi^+, l \in I_{1,m}$. □

Let $w \in \widehat{\mathcal{W}}^\vee$, $\mathcal{A}_w \in \mathcal{C}_e$. Each pair $(\iota, \alpha) \in \mathscr{R}_w \times \Phi^+$ determines a sequence $(k_{\alpha,l})_{l \in I_{0,m}}$, where $m = \ell(w)$, $k_{\alpha,l} = k(v_l, \alpha)$, for v_l as in (1.7.1), $l \in I_{1,m}$, which will be referred to as the *α-sequence associated with ι*. These sequences satisfy

(i) $k_{\alpha,0} = 0$, $k_{\alpha,l} \in \mathbb{Z}_{\ge 0}$, for all $\alpha \in \Phi^+$,

(ii) for each $l \in I_{1,m}$, there exists $\tilde{\alpha}_l$ such that $k_{\alpha,l} = k_{\alpha,l-1} + \delta_{\alpha,\tilde{\alpha}_l}$, for all $\alpha \in \Phi^+$,

(iii) $k_{\alpha,l} + k_{\beta,l} \le k_{\gamma,l} \le k_{\alpha,l} + k_{\beta,l} + 1$, for all $l \in I_{1,m}$, $\alpha, \beta, \gamma \in \Phi^+$, with $\gamma = (\alpha^\vee + \beta^\vee)^\vee$.

Given $m \in \mathbb{Z}_{\ge 0}$, consider the set of Φ^+-families of $I_{1,m}$-families of integers. An element of this set will be denoted by $(k_\alpha)_{\alpha \in \Phi^+}$, while, for each α, $k_\alpha = (k_{\alpha,l})_{l \in I_{1,m}}$ with $k_{\alpha,l} \in \mathbb{Z}$. We shall say such a family is a *reduced Φ^+-family* if its members satisfy conditions (i)-(iii) above.

Proposition 1.7.5. There is a bijection between $\{\iota \in \mathscr{R}_w : w \in \widehat{\mathcal{W}}^\vee, \mathcal{A}_w \subseteq \mathcal{C}_e\}$ (or, equivalently, reduced alcove paths contained in the fundamental chamber) and reduced Φ^+-families.

Proof. We have already proved that α-sequences associated with $\iota \in \mathscr{R}_w$, $w \in \widehat{\mathcal{W}}^\vee$, with $\mathcal{A}_w \subseteq \mathcal{C}_e$, form a reduced Φ^+-family. Conversely, let $\mathcal{F} = \{(k_{\alpha,l})_{l \in I_{1,m}} : \alpha \in \Phi^+\}$ be a reduced Φ^+-family. Since it satisfies (i) and (iii), by (1.6.3) and (1.6.5), each $\mathcal{A}_{\boldsymbol{k}_l}$, $\boldsymbol{k}_l = (k_{\alpha,l})_{\alpha \in \Phi^+}$, $0 \leq l \leq m$, is an alcove in the fundamental chamber and $\mathcal{A}_{\boldsymbol{k}_0} = \mathcal{A}_e$. Let $w_l \in \widehat{\mathcal{W}}^\vee$ be the such that $\mathcal{A}_{\boldsymbol{k}_l} = \mathcal{A}_{w_l}$, $l \in I_{1,m}$. By (ii),

$$\mathcal{A}_e \overset{i_1}{\to} \mathcal{A}_{w_1} \overset{i_2}{\to} \dots \overset{i_m}{\to} \mathcal{A}_{w_m}$$

is an alcove path. From Proposition 1.7.2, it follows that $s_{i_1} \dots s_{i_l}$ is an expression for w_l, for each $l \in I_{1,m}$. Moreover, from (i), (ii), and Theorem 1.6.10, it follows that each expression is reduced. Proposition 1.7.2 also ensures that the corresponding path is reduced. Therefore, $\mathcal{F}$ is the family of α-sequences, $\alpha \in \Phi^+$, associated with $\iota \in \mathscr{R}_w$, for which $\iota(j) = i_j$. □

In particular, for all $u \in \widehat{\mathcal{W}}^\vee$, with $\mathcal{A}_u \subseteq \mathcal{C}_e$, the reduced alcove paths in $\mathscr{P}(\mathcal{A}_u)$ are contained in the fundamental Weyl chamber and can be completely characterized by reduced Φ^+-families. If $v \in \widehat{\mathcal{W}}^\vee$, with $\mathcal{A}_v \subseteq \mathcal{C}_w$, $w \in \mathcal{W}^\vee$, write $v = wu$, with $\mathcal{A}_u \subseteq \mathcal{C}_e$. Corollary 1.8.5 below ensures $\ell(v) = \ell(w) + \ell(u)$. Hence, any reduced path in $\mathscr{P}(\mathcal{A}_v)$ passing through the alcove $\mathcal{A}_w$ is composed by a reduced path in $\mathscr{P}(\mathcal{A}_w)$ followed by a reduced path in $\mathscr{P}(\mathcal{A}_w, \mathcal{A}_v)$, which is the image by w of a reduced path contained in the fundamental Weyl chamber.

Remark 1.7.6. These paths do not describe completely $\mathscr{R}_v$ when $\mathcal{A}_v \nsubseteq \mathcal{C}_e$. It may exist $s_{i_1} \dots s_{i_m} \in \mathscr{R}_v$ for which the first occurrence of $i_l = 0$ is such that $l < \ell(w)$. For example, in Figure 1.7.2, we see two distinct reduced alcove paths in $\mathscr{P}(\mathcal{A}_v)$. One corresponds to the reduced expression $s_1 s_2 s_0 s_2$ and it is obtainable by the method we have described, for $w = s_1 s_2$ and $u = s_0 s_2$, while the other one, corresponding to the expression $s_1 s_0 s_2 s_0$, only enters $\mathcal{C}_w$ in its last step.

1.8 Left-long Extracts of Affine Weyl Groups

The main motivation for our work comes from the study of certain structural aspects of representations of affine Kac-Moody algebras which involves the computation of specific elements in the orbits of the action of the corresponding affine Weyl group on affine dominant weights. In order to compute such orbit elements, it is useful to obtain a set of reduced expressions for the elements of the group as minimal as we can. In this section, inspired by Remark 1.7.6, we express each element v of an

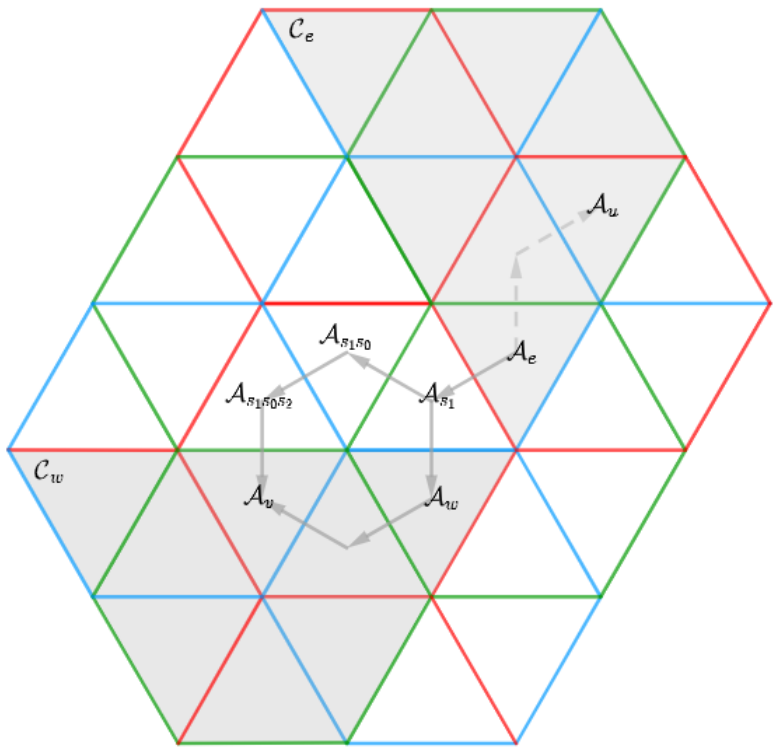


Figure 1.7.2 – Distinct types of alcoves paths from $\mathcal{A}_e$ to $\mathcal{A}_{s_1s_2s_0s_2}$.

affine Weyl group $\widehat{\mathcal{W}}$, as $v = wu$, for $w \in \mathcal{W}, \mathcal{A}_v \subseteq \mathcal{C}_w$, and $\mathcal{A}_u \subseteq \mathcal{C}_e$, giving rise to the concept of left-long extract. Arranging systematically the set of reduced expressions associated with left-long extracts, we describe a subset of $\mathscr{R}_v$, for which the positions of the occurrences of s_0 do not depend on the chosen reduced expression.

Let $\mathcal{W}$ be a Weyl group with rank n and $\widehat{\mathcal{W}}$ be the corresponding affine Weyl group, with generating sets S and $\widehat{S}$, respectively. It will be convenient to introduce the following notation. Given a sequence $w_1, \dots, w_m \in \widehat{\mathcal{W}}$, set

$$\overrightarrow{\prod_{1 \le j \le m}} w_j = w_1 w_2 \dots w_m \quad \text{and} \quad \overleftarrow{\prod_{1 \le j \le m}} w_j = w_m w_{m-1} \dots w_1.$$

Since $\widehat{\mathcal{W}}$ is generated by $\widehat{S}$, we have

$$\widehat{\mathcal{W}} = \{ s_{i_1} s_{i_2} \dots s_{i_k} \mid s_{i_j} \in \widehat{S}, 1 \le j \le k, k \in \mathbb{Z}_{\ge 0} \}.$$

Of course, the inconvenience of this presentation is that it does count all possible expressions for the elements of $\widehat{\mathcal{W}}$, eventually with many repetitions. Considering also its set of relations, the aim of this section is to describe a smaller set of expressions for the elements of $\widehat{\mathcal{W}}$, without so many repetitions, and more suitable to compute some linear representations in Chapter 3.

Given $w \in \widehat{\mathcal{W}}$ and $m \in \mathbb{Z}_{\geq 0}$, set

$$E_m(w) = \left\{ (\sigma_0, \ldots, \sigma_m) \in \mathcal{W}^{m+1} : w = \sigma_0 \overrightarrow{\prod_{1 \leq i \leq m}} (s_0 \sigma_i) \text{ and } \ell(w) = m + \sum_{i \in I_{0,m}} \ell(\sigma_i) \right\}.$$

One easily checks that

$$(\sigma_0, \ldots, \sigma_m) \in E_m(w) \text{ and } 0 < j < m \quad \Rightarrow \quad \sigma_j \neq e. \tag{1.8.1}$$

Moreover, any reduced expression for w gives rise to an element of $E_m(w)$ where m is the number of appearances of s_0 in the expression. In particular, $E_m(w) \neq \varnothing$ for some $m \in \mathbb{Z}$. Set also

$$E(w) = \bigcup_{m \geq 0} E_m(w).$$

We will refer to an element $\sigma \in E_m(w)$ as a $\widehat{\mathcal{W}}$*-extract* for w of *depth* $d(\sigma) := m$. We also define the (affine) *depth* of w to be

$$d(w) = \min\{m : E_m(w) \neq \varnothing\}.$$

If $\sigma = (\sigma_0, \ldots, \sigma_{d(\sigma)}) \in E(w)$ for some $w \in \widehat{\mathcal{W}}$, we will refer to the element σ_j as the j-th *component* of σ. If $j > 0$, we say σ_j is a *distinguished component*. Set also

$$\widehat{\sigma}_j = (\sigma_j, \sigma_{j+1}, \ldots, \sigma_{d(\sigma)}), \quad \tilde{\sigma}_j = \overrightarrow{\prod_{j < i \leq d(\sigma)}} (s_0 \sigma_i), \quad \text{and} \quad \bar{\sigma}_j = \sigma_j \tilde{\sigma}_j.$$

In particular, $\bar{\sigma}_0 = w$,

$$\widehat{\sigma}_j \in E_{d(\sigma)-j}(\bar{\sigma}_j) \quad \text{and} \quad d(\bar{\sigma}_j) = d(\sigma) - j \quad \text{for all} \quad 0 \leq j \leq d(\sigma). \tag{1.8.2}$$

Moreover,

$$\ell(\tilde{\sigma}_j) = d(\sigma) - j + \sum_{i \in I_{j+1,d(\sigma)}} \ell(\sigma_i) \quad \text{for all} \quad 0 \leq j \leq d(\sigma).$$

We shall say that the extract σ for w is *left-long* if

$$\tau \in E(w) \quad \Rightarrow \quad \ell(\tau_0) \leq \ell(\sigma_0).$$

Finally, we say σ is *fully left-long* (fll) if $\widehat{\sigma}_j$ is a left-long extract for $\bar{\sigma}_j$ for all $0 \leq j \leq d(\sigma)$.

Proposition 1.8.1. For every $w \in \widehat{\mathcal{W}}$, $E_{d(w)}(w)$ contains the unique fll element of $E(w)$.

Lemma 1.8.2. Suppose $\sigma \in E(w)$ is left-long and that $\sigma_0 = s_i w'$ for some $i \in I_{1,n}, w' \in \mathcal{W}$ such that $\ell(s_i \sigma_0) = \ell(w')$. Then, $\tau = (w', \sigma_1, \ldots, \sigma_{d(\sigma)})$ is a left-long element of $E(s_i w)$ which is fll if so is σ.

Proof. It is clear that $\tau \in E(s_i w)$. Suppose τ is not left-long and choose $\xi \in E(s_i w)$ such that $\ell(\xi_0) > \ell(w')$. In particular, we have $s_i w = w'\tilde{\sigma}_0 = \xi_0\tilde{\xi}_0$ and

$$\ell(\xi_0) + \ell(\tilde{\xi}_0) = \ell(s_i w) = \ell(w) - 1.$$

Since $w = s_i\xi_0\tilde{\xi}_0$, this implies $(s_i\xi_0, \xi_1, \ldots, \xi_{d(\xi)}) \in E(w)$ and $\ell(s_i\xi_0) = 1 + \ell(\xi_0) > 1 + \ell(w') = \ell(\sigma_0)$, contradicting the assumption that σ is left-long. The last claim is now obvious. □

Proof of Proposition 1.8.1. Let us begin by showing that $E_{d(w)}(w)$ contains a fll element of $E(w)$. We proceed by induction on $m := d(w)$, which clearly begins when $m = 0$. If $m > 0$, chose any left-long element $\varsigma \in E_m(w)$. It follows from (1.8.2) that $d(\bar{\varsigma}_1) = m - 1$ and then, by induction hypothesis, $E_{m-1}(\bar{\varsigma}_1)$ contains a fll element of $E(\bar{\varsigma}_1)$, say τ. It follows that $\sigma := (\varsigma_0, \tau_0, \ldots, \tau_{m-1}) \in E_m(w)$ is fll.

To prove uniqueness, we proceed by induction on $\ell(w)$ which clearly starts for $w = e$. Moreover, uniqueness is also clear if $w \in \mathcal{W}$ since $E_0(w) = \{w\}$ and $\ell(\sigma_0) < \ell(w)$ if $\sigma \in E(w) \setminus E_0(w)$. Thus, let $w \notin \mathcal{W}$ and suppose $\sigma, \tau \in E(w)$ are fll elements. In particular, $\ell(\sigma_0) = \ell(\tau_0)$ and $d(w) > 0$.

Assume first that $\ell(\sigma_0) \neq 0$. In this case, we have $\sigma_0 = s_i w'$ for some $i \in I_{1,n}$ and $w' \in \mathcal{W}$, such that $\ell(s_i\sigma_0) = \ell(w')$. Lemma 1.8.2 and the induction hypothesis imply $\sigma' := (s_i\sigma_0, \sigma_1, \ldots, \sigma_{d(\sigma)})$ is the fll in $E(s_i w)$. If $\ell(s_i\tau_0) < \ell(\tau_0)$, it follows that $\tau' := (s_i\tau_0, \tau_1, \ldots, \tau_{d(\tau)}) \in E(s_i w)$ is a left-long element (as $\ell(s_i\tau_0) = \ell(s_i\sigma_0)$) and, hence, also fll. Therefore, $\tau' = \sigma'$ and, hence,

$$d(\tau) = d(\sigma), \quad \tau_j = \sigma_j \quad \text{for} \quad j > 0, \quad \text{and} \quad s_i\tau_0 = s_i\sigma_0,$$

which clearly implies, $\tau = \sigma$. If $\ell(s_i\tau_0) > \ell(\tau_0)$, since $\ell(s_i w) < \ell(w)$, Proposition 1.1.1 implies there exists a reduced expression for $s_i w$ which starts with a reduced expression for τ_0. This reduced expression gives rise to $\xi \in E(s_i w)$ having $\ell(\xi_0) > \ell(\sigma'_0)$, yielding a contradiction since σ' is left-long.

Finally, assume $\sigma_0 = \tau_0 = e$. In particular, $\tilde{\sigma}_1 = \tilde{\tau}_1$ and the induction hypothesis implies this element admits a unique fll extract, which concludes the proof. □

If σ is the fll extract of w, set

$$\varepsilon(w) = \sigma, \quad \varepsilon_0(w) = \sigma_0, \quad \tilde{\varepsilon}_0(w) = \tilde{\sigma}_0.$$

In particular, $w = \varepsilon_0(w)\tilde{\varepsilon}_0(w)$. Set also

$$\mathcal{E} = \{\tilde{\varepsilon}_0(w) : w \in \widehat{\mathcal{W}}\}.$$

Therefore,

$$\widehat{\mathcal{W}} = \{wu : w \in \mathcal{W}, u \in \mathcal{E}\}. \tag{1.8.3}$$

Lemma 1.8.3. Let $w \in \widehat{\mathcal{W}}$, $\sigma \in E_m(w)$. Then σ is fully left-long if and only if $D(\tilde{\sigma}_j) = \{s_0\}$, for all $0 \le j < d(\sigma)$.

Proof. Assume σ is fll and suppose $\tilde{\sigma}_j = s_i w'$ with $i \neq 0$ and $\ell(w') = \ell(\tilde{\sigma}_j) - 1$. Then, $\bar{\sigma}_j = \sigma_j s_i w'$ and, since σ is fll, we must have $\ell(\sigma_j s_i) < \ell(\sigma_j)$. Letting

$$l = \ell(w) - \ell(\bar{\sigma}_j) = \ell(w) - \ell(\tilde{\sigma}_j) - \ell(\sigma_j),$$

we conclude that

$$\ell(w) \le l + \ell(\sigma_j s_i) + \ell(w') < \ell(w), \tag{1.8.4}$$

yielding a contradiction. Conversely, if σ is not fll, there exists $0 \le j \le d(\sigma)$ and $\tau \in E(\bar{\sigma}_j)$ such that $\ell(\tau_0) > \ell(\sigma_j)$. In particular,

$$\tilde{\sigma}_j = \sigma_j^{-1}\tau_0\tilde{\tau}_0.$$

We might as well assume $\tau = \varepsilon(\bar{\sigma}_j)$ and, hence, $D(\tilde{\tau}_0) = \{s_0\}$. Since $\sigma_j^{-1}\tau_0 \in \mathcal{W}$, it then follows from (1.2.1) that

$$\ell(\tilde{\sigma}_j) = \ell(\sigma_j^{-1}\tau_0) + \ell(\tilde{\tau}_0).$$

Also, $S \cap D(\sigma_j^{-1}\tau_0) \neq \varnothing$ since $\sigma_j^{-1}\tau_0 \neq e$ and Corollary 1.1.4 then implies $S \cap D(\sigma_j^{-1}\tau_0) \subseteq D(\tilde{\sigma}_j) \setminus \{s_0\}$.

□

In light of Corollary 1.1.3, it follows that $D(w) = \{s_0\}$, for all $w \in \mathcal{E}$. Conversely, let us show that

$$D(w) = \{s_0\} \quad \Rightarrow \quad w \in \mathcal{E}.$$

Indeed, if $w \notin \mathcal{E}$, it follows, in particular, that $w \neq \tilde{\varepsilon}(w)$ and, hence, $\varepsilon_0(w) \in \mathcal{W} \setminus \{e\}$. This implies there exists $i \in I_{1,n}$ such that $s_i \in D(w)$, reaching the desired contradiction. Thus, we have shown

$$w \in \mathcal{E} \quad \Leftrightarrow \quad D(w) = \{s_0\}. \tag{1.8.5}$$

Since $\mathcal{W} = \widehat{\mathcal{W}}_S$,

$$\mathcal{E} = \widehat{\mathcal{W}}^S.$$

In particular, (1.2.1) becomes

$$\ell(uv) = \ell(u) + \ell(v) \quad \text{for all} \quad u \in \mathcal{W}, v \in \mathcal{E} \tag{1.8.6}$$

and Proposition 1.2.1 says that, for all $w \in \widehat{\mathcal{W}}$, there exist unique $(u,v) \in \mathcal{W} \times \mathcal{E}$ such that $w = uv$. In fact, v is the shortest representative of $\mathcal{W}w$.

Note also that Lemma 1.8.3, together with (1.2.1), imply that, if $\sigma = \varepsilon(w)$, then

$$\ell(w) = d(w) + \sum_{j \in I_{0,d(w)}} \ell(\sigma_j).$$

Hence, in order to find a reduced expression for w, it suffices to describe reduced expressions for all the components of σ.

On the other hand, (1.8.5) together with Corollary 1.6.11 implies the following.

Proposition 1.8.4. Let $w \in \widehat{\mathcal{W}}$. The alcove $\mathcal{A}_w$ is contained in the fundamental chamber if and only if $w \in \mathcal{E}$.

This allows to recover, without using the theory of parabolic subgroups the following fact.

Corollary 1.8.5. For all $v \in \widehat{\mathcal{W}}$, there exist unique $(w, u) \in \mathcal{W} \times \mathcal{E}$ such that $v = wu$. In particular, $\ell(wu) = \ell(w) + \ell(u)$, for all $w \in \mathcal{W}$ and $u \in \mathcal{E}$.

Proof. After (1.8.3), it remains to prove uniqueness. Indeed, if $v = wu$ with $w \in \mathcal{W}$ and $u \in \mathcal{E}$, we have

$$\mathcal{A}_v = w\mathcal{A}_u \subseteq w\mathcal{C}_e = \mathcal{C}_w,$$

and we are done since $\mathcal{W}$ acts simply transitively on the set of chambers. For the last statement, since $wu = \varepsilon_0(wu)\tilde{\varepsilon}_0(wu)$, it follows from the first part that $w = \varepsilon_0(wu)$ and $u = \tilde{\varepsilon}_0(wu)$. The conclusion follows since $\ell(\varepsilon_0(v)) + \ell(\tilde{\varepsilon}_0(v)) = \ell(v)$ for all $v \in \widehat{\mathcal{W}}$, by definition. □

Chapter 2

Affine Weyl group of type $\widehat{A}_n$

The aim of this chapter is to describe from different points of view the affine Weyl group of type $\widehat{A}_n$, $n \geq 2$. We begin with a realization as a permutation group in Section 2.1 and then relate it with the respective alcove form of each element, following (BJÖRNER; BRENTI, 1996) and (SHI, 1999). By seeing them as Coxeter groups, following (STEMBRIDGE, 1997) and (AL HARBAT, 2021), we obtain special $\mathbb{W}$-factorizations of elements of maximal parabolic subgroups in Section 2.2. This allows us to present, in Section 2.3, a characterization of the fll extracts of elements of $\widehat{A}_n$, which can be codified in a special directed graph, which will be the subject of Section 2.4. We conjecture that the expressions arising from this characterization provide a minimal set of reduced expressions for the elements of $\widehat{A}_n$. In Section 2.5, we prove that indeed it does for depth 2. Finally, in Section 2.6, we briefly describe the connection of $\widehat{A}_n$ with the affine Kac-Moody algebra $\hat{\mathfrak{sl}}_{n+1}$, following (KAC, 1990).

Let $\widehat{\mathcal{W}}$ be an affine Weyl group of type $\widehat{A}_n$, $n \geq 2$. Recall that, as a Coxeter group, $\widehat{\mathcal{W}}$ has a set of generators $\widehat{S} = \{s_i : i \in I_{0,n}\}$, satisfying the relations

$$\begin{cases} s_i^2 = e, & \text{for all } i \in I_{0,n}, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & \text{for all } i \in I_{1,n}, \\ s_i s_j = s_j s_i, & \text{if } j \neq i-1, i+1, \end{cases} \tag{2.0.1}$$

while $\mathcal{W}$ is generated by $S := \{s_1, \ldots, s_n\}$.

2.1 Affine Symmetric Group

In this section we introduce the *affine symmetric group* $\widehat{S}_n$, which is an affine Weyl group of type $\widehat{A}_{n-1}$, $n \geq 2$. It has the same root system as the symmetric group S_n regarded as a Weyl group of type A_{n-1}, and it contains a subgroup isomorphic to S_n. Regarding $\widehat{S}_n$ as an affine Weyl group leads us to describe alcove forms for its elements and, hence, compare them with the expression of each element given by a permutation.

The set $\widehat{S}_n$ of permutations π of $\mathbb{Z}$ such that

$$\pi(x+n) = \pi(x) + n \quad \text{for all} \quad x \in \mathbb{Z}, \tag{2.1.1}$$

and

$$\sum_{x \in I_{1,n}} \pi(x) = \binom{n+1}{2}, \tag{2.1.2}$$

is a group under composition of functions. The elements of $\widehat{S}_n$ are called *affine permutations* and $\widehat{S}_n$ is called the affine symmetric group.

From (2.1.1), it follows that π is uniquely determined by its values on $I_{1,n}$, writing $\pi(i) = nr_i + k_i$, $r_i \in \mathbb{Z}$ and $k_i \in I_{1,n}$, we can denote

$$\pi = (r_1, \ldots, r_n \mid \bar{\pi}),$$

where $\bar{\pi} \in S_n$, $\bar{\pi}(i) = k_i$, and, by (2.1.2), $\sum_{i \in I_{1,n}} r_i = 0$. In this case, (BJÖRNER; BRENTI, 1996, Section 3)

$$\pi^{-1} = (-r_{(\bar{\pi})^{-1}(1)}, \ldots, -r_{(\bar{\pi})^{-1}(n)} \mid (\bar{\pi})^{-1}). \tag{2.1.3}$$

The group $\widehat{S}_n$ is generated by $S := \{s_0, s_1, \ldots, s_{n-1}\}$ where $s_i := (0, \ldots, 0 \mid (i \quad i+1))$ for $i \in I_{1,n-1}$ and $s_0 := (-1, 0, \ldots, 0, 1 \mid (1 \quad n))$. It is clear that the subgroup generated by $\{s_1, \ldots, s_n\}$ is isomorphic to S_n.

Let us describe an action of $\widehat{S}_n$ on $V = \mathbb{R}^n$. Recall the action of S_n on V which permutes the subscripts of the standard basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ presented in Example 1.3.2. That is, for each $1 \le i \le n$, s_i acts on V as the reflection in the direction determined by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, which fixes the origin. Make the affine permutation s_0 act on V as the affine reflection in the direction determined by $\alpha_0 = \varepsilon_1 - \varepsilon_n$, which sends 0 to α_0. Since $\widehat{S}_n$ is the group generated by affine reflections associated with the Weyl group S_n of type A_{n-1}, it follows that $\widehat{S}_n$ is an affine Weyl group of type $\widehat{A}_{n-1}$.

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. The next theorems give us formulae to the transition between the alcove form and the permutation form of an element of $\widehat{S}_n$.

Theorem 2.1.1. (SHI, 1999, Theorem 1.4) For $1 \le i < j \le n$ and $\pi \in \widehat{S}_n$, we have

$$k(\pi, \alpha_i + \ldots + \alpha_{j-1}) = \left\lfloor \frac{\pi^{-1}(j) - \pi^{-1}(i)}{n} \right\rfloor.$$

□

Theorem 2.1.2. (SHI, 1999, Theorem 5.2) Let $\pi \in \widehat{S}_n$ and $t \in I_{1,n}$. Then

$$\pi^{-1}(t) = t + \sum_{j \in I_{1,t-1}} k(\pi, \alpha_j + \ldots + \alpha_{t-1}) - \sum_{j \in I_{t+1,n}} k(\pi, \alpha_t + \ldots + \alpha_{j-1}).$$

□

2.2 $\mathbb{W}$-factorizations

In this section, we describe $\mathbb{W}$-factorizations, for special chains $\mathbb{W}$. This characterization will be helpful in the description of the fll extracts of elements of $\widehat{\mathcal{W}}$ in the next section.

Given $1 \le i \le j \le n$, set

$$s_{i,j} = \overrightarrow{\prod_{i \le k \le j}} s_k \quad \text{and} \quad s_{j,i} = \overleftarrow{\prod_{i \le k \le j}} s_k$$

Since each simple reflection appears exactly once in the definition of $s_{i,j}$, we have

$$\ell(s_{i,j}) = |i - j| + 1 \quad \text{for all} \quad 1 \le i, j \le n.$$

Consider the chain $\mathbb{W}$ of inclusions of maximal proper parabolic subgroups obtained by letting $W_i = \langle s_1, \ldots, s_i \rangle$. It is not difficult to check that $\mathbb{W}_i = \{e\} \cup \{s_{i,j} : 1 \le j \le i\}$. Therefore, the $\mathbb{W}$-factorization of a given element is characterized by a family $(i_k, j_k) \in I \times I, j_k \le i_k, 1 \le k \le m$ such that $i_{k+1} > i_k$ for $k < m$ where $0 \le m \le n$ is the number of nontrivial components of the factorization:

$$w = \overrightarrow{\prod_{1 \le k \le m}} s_{i_k, j_k}.$$

Moreover, since there are exactly $(n+1)!$ such sequences, all of them arise from $\mathbb{W}$-factorizations. We shall refer to s_{i_k,j_k} as the k-th nontrivial $\mathbb{W}$-component of w. Note the k-th nontrivial component of $s_{i,j}, i \le j$, is $s_{i+k-1} = s_{i+k-1,i+k-1}$, $1 \le k \le j - i + 1$. In other words, the number of nontrivial components is equal to the length. On the other hand, if $i \ge j$, $s_{i,j}$ has a unique nontrivial component.

The elements

$$a_j := s_{1,j}, \quad b_j := s_{n,j} \quad \text{and} \quad c_{j,k} := a_j b_k$$

will be of special relevance to us. For convenience, we set $a_0 = e = b_{n+1}$. Note the above discussion implies $c_{j,k}$ has $j + 1 - \delta_{k,n+1}$ nontrivial components and $\ell(c_{j,k}) = \ell(a_j) + \ell(b_k) = n + j - k + 1$. Consider also the reflections

$$\overrightarrow{t_i} = a_{i-1} s_i a_{i-1}^{-1},\ 1 \le i \le n, \quad \overleftarrow{t_i} = b_{i+1} s_i b_{i+1}^{-1},\ 1 \le i \le n, \qquad (2.2.1)$$
$$\text{and} \quad t_{i,j} = a_i \overleftarrow{t_j} a_i^{-1},\ 0 \le j < n,\ 1 \le i \le n.$$

Note $\overrightarrow{t_i} = a_{i-1} s_{i,1}$ and $\overleftarrow{t_i} = b_{i+1} s_{i,n}$, and, it will become clear in Lemma 2.3.2(v) that

$$\overrightarrow{t_i} = a_{i-1} s_{i,1} = s_{i,2} a_i \quad \overleftarrow{t_i} = b_{i+1} s_{i,n} = s_{i,n-1} b_i. \qquad (2.2.2)$$

The above discussion gives us the $\mathbb{W}$-factorization of $\overrightarrow{t_i}$. In particular,

$$\ell(\overrightarrow{t_i}) = 2\ell(a_{i-1}) + 1 = 2i - 1 \quad \text{and} \quad \mathrm{supp}(\overrightarrow{t_i}) = \{1, 2, \ldots, i\} \quad \text{for all} \quad 1 \le i < n.$$

Suppose $\mathbb{W}'$ is another chain satisfying $W'_{n-1} = W_{n-1}$ and $W'_{n-2} = \langle s_i : 1 < i < n \rangle$. Then, $\mathbb{W}'_{n-1} = \{e\} \cup \{s_{1,j} : 1 \le j < n\}$. Thus, the of $\mathcal{W}$ by W'_{n-2} is $\{c_{j,k} : 0 \le j < n, 1 \le k \le n+1\}$. Hence, every element of $\mathcal{W}$ has a unique factorization of the form

$$uc_{j,k} \quad \text{for some} \quad u \in \langle s_i : 1 < i < n \rangle,\ 0 \le j < n, 1 \le k \le n+1 \tag{2.2.3}$$

and, moreover $\ell(w) = \ell(u) + \ell(c_{j,k})$ for all such u, j, k (cf. (AL HARBAT, 2021, Lemma 2.3)).

Consider a sequence $(\tilde{s}_i)_{i \in I_{1,n}}$, where either $\tilde{s}_i = s_{i+j}$, $0 \le j \le n$ and $\bar{i} \cong_{n+1} \bar{j}$, $\tilde{s}_i = s_{n-j-i+1}$, $0 \le j \le n$ and $\bar{i} \cong_{n+1} \overline{n-j+1}$, for $i \in I_{1,n}$. The subgroup $\langle \tilde{s}_1, \dots, \tilde{s}_n \rangle$ of $\widehat{\mathcal{W}}$ is isomorphic to $\mathcal{W}$. Thus, if $\mathbb{W}$ is the chain of inclusions of maximal proper parabolic subgroups obtained by letting $W_i = \langle \tilde{s}_1, \dots, \tilde{s}_i \rangle$, the previous argument also ensures

$$\ell(\tilde{s}_1 \dots \tilde{s}_i \dots \tilde{s}_1) = 2i - 1 \quad \text{for all} \quad 1 \le i \le n.$$

In particular,

$$\ell(\overleftarrow{t_i}) = 2\ell(b_{i+1}) + 1 = 2(n-i) - 1 \quad \text{and} \quad \operatorname{supp}(\overleftarrow{t_i}) = \{i, i+1, \dots, n\} \quad \text{for all} \quad 1 \le i \le n.$$

The $\mathbb{W}$-factorization of the reflections $t_{j,i}$, $0 \le j < n$, $1 \le i \le n$ will be given in the Section 2.4.

2.3 Left and Right Sequences

As we have seen in Section 1.8, an element of an affine Weyl group $\widehat{\mathcal{W}}$ admits a unique fll extract. By (1.8.3), up to a reflection in $\mathcal{W}$, the action of $\widehat{\mathcal{W}}$ on V is described by the action of $\mathcal{E}$ on V. As we have observed in (1.8.1), not every element of $\mathcal{W}$ may appear as a distinguished component of an fll extract of an element of $\mathcal{E}$. Our goal in this section is to characterize such elements by presenting a unique special reduced expression for each one of them. Moreover, we show that there is a correspondence between those reduced expressions and certain pairs of monotonous finite sequences with non-negative integer entries.

Given $w \in \mathcal{W}$, with $\ell(w) = l > 0$ set

$$R(w) = \{(i_1, \dots, i_l) \in I^l \mid w = s_{i_1} s_{i_2} \dots s_{i_l}\}$$

and

$$p_n(w) = \max\{k \mid \exists\ (i_1, \dots, i_l) \in R(w), i_k = n, i_j \ne n \text{ for } j < k\}.$$

We set $p_n(w) = \infty$ if either $w = e$ or $\ell(w) > 0$ and w has a reduced expression with no occurrence of s_n. We shall say a reduced expression for w is *n-deferred* if

$$i_j \ne n \text{ for } j < p_n(w). \tag{2.3.1}$$

Proposition 2.3.1. If w is a distinguished component of an fll extract, then an n-deferred reduced expression for w is of the form $c_{j,k}$ for some choice of $0 \le j < n$ and $1 \le k \le n+1$.

Proof. If $\ell(w) = 0$, then $w = e = a_0$. For $\ell(w) > 0$, let $s_{i_1} \dots s_{i_l}$ be a reduced expression for w and note $i_1 \in \{1, n\}$. Indeed, let σ be an extract having w as a component, let w' be the component preceding w, and suppose $i_1 \notin \{1, n\}$. Then, $w' s_0 w = w' s_{i_1} s_0 s_{i_2} \dots s_{i_l}$ and, by definition of fll extract, we must have $\ell(w' s_{i_1}) < \ell(w')$. Note we also have $\ell(s_{i_2} \dots s_{i_l}) = \ell(w) - 1$. An argument similar to that leading to (1.8.4) yields a contradiction. By (2.2.3), $w = u c_{j,k}$ for a unique choice of j, k and $u \in \langle s_i : 1 < i < n \rangle$. Thus, if $u \ne e$, w would have a reduced decomposition with $i_1 \notin \{1, n\}$, yielding a contradiction. Thus, $u = e$, completing the proof of the proposition. □

Let σ be the fll extract of $w \in \widehat{\mathcal{W}}$. The above proposition says that, for each $1 \le i \le d := d(w)$, there exist $0 \le j_i < n$ and $1 \le k_i \le n+1$ such that $\sigma_i = a_{j_i} b_{k_i}$. We refer to the sequences $(j_1, \dots, j_d)$ and $(k_1, \dots, k_d)$ as the *left and right sequences* associated with w, respectively. We now investigate the behavior of these sequences. For this, denote $s_{n+1} := s_0$.

Lemma 2.3.2. The following hold

(i) $s_0 a_j s_0 = s_1 s_0 a_j$, for $1 \le j \le n-1$,

(ii) $s_0 b_k s_0 = s_n s_0 b_k$, for $2 \le k \le n$,

(iii) $a_j s_0 a_j = s_0 \overrightarrow{\prod_{1 \le l \le j}} s_l s_{l-1}$, for $1 \le j \le n-1$,

(iv) $b_k s_0 b_k = s_0 \overleftarrow{\prod_{k \le l \le n}} s_l s_{l+1}$, for $2 \le k \le n$,

(v) $s_{j,k} s_{k-1,j} = s_{k,j+1} s_{j,k}$, for $1 \le j < k \le n$,

(vi) $s_{j,k}^2 = s_{j+1,k} s_{j,k-1}$, for $1 \le j < k \le n$,

(vii) $c_{j,k} = \begin{cases} b_k a_j, & \text{if } (j,k) = (0,1) \text{ or } 0 \le j < k-1 \le n-1 \\ b_{k+1} a_{j+1}, & \text{if } 1 \le k \le j+1 \le n+1. \end{cases}$

Proof. Part (i) for $j = 1$ holds by (2.0.1). Otherwise, if $2 \le j \le n-1$, then

$$s_0 a_j s_0 = s_0 a_1 (s_{2,j} s_0) = (s_0 s_1 s_0) s_{2,j} = s_1 s_0 a_j.$$

Part (ii) follows from (2.0.1) if $k = n$. Supposing $2 \le k \le n-1$, we have

$$s_0 b_k s_0 = s_0 b_n (s_{n-1,k} s_0) = (s_0 s_n s_0) s_{n-1,k} = s_n s_0 b_k.$$

Let us proceed by induction on $1 \leq j \leq n-1$ to prove part (iii). Since (2.0.1), the result is clear for $j = 1$. Otherwise, assuming $1 < j < n-1$ and applying the induction hypothesis, we conclude that

$$a_{j+1}s_0a_{j+1} = a_j(s_{j+1})s_0a_{j-1}s_js_{j+1} = a_js_0a_{j-1}(s_{j+1}s_js_{j+1}) = (a_j)s_0(a_{j-1}s_j)s_{j+1}s_j = (a_js_0a_j)s_{j+1}s_j$$
$$= \left(s_0 \overrightarrow{\prod_{1\leq l\leq j}} s_ls_{l-1}\right)(s_{j+1}s_j) = s_0 \overrightarrow{\prod_{1\leq l\leq j+1}} s_ls_{l-1}.$$

To prove part (iv), let us proceed by induction on $p = n-k$, $0 \leq p \leq n-2$. If $p = 0$, then $k = n$ and the result follows from (2.0.1). If not, assuming that $0 < p < n-2$, then $k = n-p$ and it follows from the induction hypothesis that

$$b_{n-(p+1)}s_0b_{n-(p+1)} = b_{n-p}(s_{n-p-1})s_0b_{n-p+1}s_{n-p}s_{n-p-1} = b_{n-p}s_0b_{n-p+1}(s_{n-p-1}s_{n-p}s_{n-p-1})$$
$$= b_{n-p}s_0(b_{n-p+1}s_{n-p})s_{n-p-1}s_{n-p} = (b_{n-p}s_0b_{n-p})s_{n-p-1}s_{n-p}$$
$$= \left(s_0 \overleftarrow{\prod_{n-p\leq l\leq n}} s_ls_{l+1}\right)(s_{n-p-1}s_{n-p}) = s_0 \overleftarrow{\prod_{n-(p+1)\leq l\leq n}} s_ls_{l+1}.$$

Let $1 \leq j \leq n-1$. In order to prove part (v), we shall proceed by induction on $p = k-j$, $1 \leq p \leq n-1$, once more. If $p = 1$, then $k = j+1$, following that

$$s_{j,j+1}s_{j,j} = s_js_{j+1}s_j = s_{j+1}s_js_{j+1} = s_{j+1,j}s_{j+1,j+1}.$$

Otherwise, supposing $1 < p < n-1$, then $k = j+p$ and applying the induction hypothesis we obtain

$$s_{j,j+p+1}s_{j+p,j} = s_{j,j+p-1}(s_{j+p}s_{j+p+1}s_{j+p})s_{j+p-1,j} = (s_{j,j+p-1}s_{j+p+1})s_{j+p}(s_{j+p+1}s_{j+p-1,j})$$
$$= s_{j+p+1}s_{j,j+p-1}(s_{j+p}s_{j+p-1,j})s_{j+p+1} = s_{j+p+1}(s_{j,j+p-1}s_{j+p,j})s_{j+p+1}$$
$$= (s_{j+p+1}s_{j+p,j+1})(s_{j,j+p}s_{j+p+1}) = s_{j+p+1,j+1}s_{j,k+p+1}.$$

For part (vi), $1 \leq j \leq n-1$, we similarly proceed by induction on $p = k-j$, $1 \leq p \leq n-1$. If $p = 1$, then $k = j+1$ and

$$s_{j,j+1}s_{j,j+1} = s_js_{j+1}s_js_{j+1} = s_{j+1}s_js_{j+1}s_{j+1} = s_{j+1}s_j = s_{j+1,j+1}s_{j,j}.$$

In the case $1 < p < n-1$, then $k = j+p$ and the induction hypothesis implies

$$s_{j,j+p+1}s_{j,j+p+1} = s_{j,j+1}s_js_{j+2,j+p+1}s_{j+1,j+p+1} = s_{j+1}s_js_{j+1}s_{j+2,j+p+1}s_{j+1,j+p+1}$$
$$= s_{j+1,j}s_{j+1,j+p+1}s_{j+1,j+p+1} = s_{j+1,j}s_{j+1,j+p+1}s_{j+1,j+p+1} = s_{j+1,j}s_{j+2,j+p+1}s_{j+1,j+p}$$
$$= s_{j+1,j+p+1}s_{j,j+p}.$$

Finally, to prove part (vii), notice that it is immediate that if either $(j,k) = (0,1)$ or $0 \leq j < k-1 \leq n-1$, then $c_{j,k} = b_ka_j$. If $2 \leq k = j+1 \leq n$, then

$$c_{j,j+1} = (a_jb_{j+2})s_{j+1} = b_{j+2}(a_js_{j+1}) = b_{j+2}a_{j+1}.$$

Suppose then $1 \le k \le j \le n-1$. Thus, by part (v),

$$c_{j,k} = a_{k-1}(s_{k,j}b_{j+2})s_{j+1,k} = a_{k-1}b_{j+2}(s_{k,j}s_{j+1,k}) = a_{k-1}(b_{j+2}s_{j+1,k+1})s_{k,j+1} = (a_{k-1}b_{k+1})s_{k,j+1}$$
$$= b_{k+1}(a_{k-1}s_{k,j+1}) = b_{k+1}a_{j+1}.$$

□

The next proposition reveals a monotonous behavior of left and right sequences. Note the condition $j_i < k_i - 1$ is equivalent to saying that the set of simple reflections appearing in a decomposition of the distinguished component c_{j_i,k_i}, given by Corollary 1.1.6(ii), is not the whole set S.

Proposition 2.3.3. Let $(j_1,\dots,j_d)$ and $(k_1,\dots,k_d)$ be the left and the right sequences associated with w, respectively. Then $(j_1,\dots,j_d)$ is non-increasing and $(k_1,\dots,k_d)$ is non-decreasing. Moreover, if

$$i < d(w) \text{ and } j_i < k_i - 1 \quad \Rightarrow \quad j_i > j_{i+1} \tag{2.3.2}$$

and

$$1 < i \text{ and } j_i < k_i - 1 \quad \Rightarrow \quad k_{i-1} < k_i. \tag{2.3.3}$$

Proof. If $d \in \{0,1\}$, then the monotonicity of the left and the right sequences associated with v is trivial. Thus, we shall consider $d > 1$ and let $v_i = c_{j_i,k_i}$, for $1 \le i \le d$.

We begin by noting that

$$(j_i,k_i) \in U := \{(j,k) \mid 1 \le j \le n-1, 1 \le k \le n\} \cup \{(0,1)\} \quad \text{for all} \quad 1 \le i < d. \tag{2.3.4}$$

Indeed, since $v_i \ne e$, $(j_i,k_i) \ne (0,n+1)$. If it could be $k_i = n+1$ and $j_i > 0$, Lemma 2.3.2(i) would imply

$$s_0v_is_0v_{i+1} = (s_0a_{j_i}s_0)v_{i+1} = s_1s_0a_{j_i}v_{i+1},$$

yielding a contradiction with Lemma 1.8.3. Similarly, if it could be $j_i = 0$ and $k_i \notin \{1,n+1\}$, Lemma 2.3.2(ii) would imply

$$s_0v_is_0v_{i+1} = (s_0b_{k_i}s_0)v_{i+1} = s_ns_0b_{k_i}v_{i+1},$$

yielding a contradiction with Lemma 1.8.3 once more.

Let us show the left sequence is non-increasing. Thus, fix $1 \le i < d$ and assume we could have $j_{i+1} > j_i$. We will use Lemma 2.3.2 to reach a contradiction with Lemma 1.8.3 as before. Indeed, if $j_i = 0$, we have $k_i = 1$ by (2.3.4) and the assumption $j_{i+1} > j_i$ implies $a_{j_{i+1}} = s_1s_{2,j_{i+1}}$. It would then follow from Lemma 2.3.2(ii) that

$$s_0v_is_0v_{i+1} = s_0b_1s_0a_{j_{i+1}}b_{k_{i+1}} = s_0b_2s_1s_0s_1s_{2,j_{i+1}}b_{k_{i+1}} = s_0b_2s_0s_1s_0s_{2,j_{i+1}}b_{k_{i+1}}$$
$$= s_ns_0b_2s_1s_0s_{2,j_{i+1}}b_{k_{i+1}}.$$

If $j_i > 0$, by Lemma 2.3.2(vii), we must have

$$c_{j_i,k_i} = b_r a_q, \quad \text{for} \quad (q,r) = \begin{cases} (j_i, k_i), & \text{if } 1 \le j_i < k_i - 1 \le n-1 \\ (j_i+1, k_i+1), & \text{if } 1 \le k_i \le j_i + 1 \le n \end{cases}.$$

The assumption of $j_{i+1} > j_i$ implies $q \le j_i + 1 \le j_{i+1}$ and consequently $a_{j_{i+1}} = a_q s_{q+1,j_{i+1}}$. Hence, from (iii) and (ii) of Lemma 2.3.2, we would have

$$\begin{aligned} s_0 v_i s_0 v_{i+1} &= s_0(a_{j_i} b_{k_i}) s_0 a_{j_{i+1}} b_{k_{i+1}} = s_0 b_r a_q s_0 (a_{j_{i+1}}) b_{k_{i+1}} = s_0 b_r (a_q s_0 a_q) s_{q+1,j_{i+1}} b_{k_{i+1}} \\ &= (s_0 b_r s_0) \left(\overrightarrow{\prod_{1\le l \le q}} s_l s_{l-1} \right) s_{q+1,j_{i+1}} b_{k_{i+1}} = s_n s_0 b_r \left(\overrightarrow{\prod_{1\le l \le q}} s_l s_{l-1} \right) s_{q+1,j_{i+1}} b_{k_{i+1}}. \end{aligned}$$

Moreover, if $i < d_0(v)$ and $j_i < k_i - 1$, let us check that $j_i \ne j_{i+1}$. Indeed, from Lemma 2.3.2(vii), it follows that $b_{k_i} a_{j_i} = a_{j_i} b_{k_i}$. If we could have $j_i = j_{i+1}$, then (iii) and (ii) of Lemma 2.3.2 would imply that

$$\begin{aligned} s_0 v_i s_0 v_{i+1} &= s_0 (a_{j_i} b_{k_i}) s_0 a_{j_i} b_{k_{i+1}} = s_0 b_{k_i} (a_{j_i} s_0 a_{j_i}) b_{k_{i+1}} \\ &= (s_0 b_{k_i} s_0) s_1 s_0 s_2 s_1 \ldots s_{j_i} s_{j_i - 1} b_{k_{i+1}} = s_n s_0 b_{k_i} s_1 s_0 s_2 s_1 \ldots s_{j_i} s_{j_i-1} b_{k_{i+1}}, \end{aligned}$$

yielding another contradiction with Lemma 1.8.3.

Now we show the right sequence is non-decreasing. Evidently, there is nothing to do if $k_i = 1$. Thus, assume $k_i > 1$. By (2.3.4), we must have $j_i < n$. From Lemma 2.3.2(vii) it follows that

$$c_{j_{i+1},k_{i+1}} = b_r a_q, \quad \text{for} \quad (q,r) = \begin{cases} (j_{i+1}, k_{i+1}), & \text{if } 1 \le j_{i+1} < k_{i+1} - 2 \le n-1 \\ (j_{i+1}+1, k_{i+1}+1), & \text{if } 1 \le k_{i+1} \le j_{i+1} + 1 \le n \end{cases}.$$

If we could have $k_{i+1} < k_i$, it would imply that $r \le k_{i+1} + 1 \le k_i$ and hence $b_r = b_{k_i} s_{k_i - 1, r}$. Then, using (iv) and (i) of Lemma 2.3.2, we would have that

$$\begin{aligned} s_0 v_i s_0 v_{i+1} &= s_0 a_{j_i} b_{k_i} s_0 (a_{j_{i+1}} b_{k_{i+1}}) = s_0 a_{j_i} b_{k_i} s_0 (b_r) a_q = s_0 a_{j_i} (b_{k_i} s_0 b_{k_i}) s_{k_i - 1, r} a_q \\ &= (s_0 a_{j_i} s_0) \left(\overleftarrow{\prod_{k_i \le l \le n}} s_l s_{l+1} \right) s_{k_i-1,r} a_q = s_1 s_0 a_{j_i} \left(\overleftarrow{\prod_{k \le l \le n}} s_l s_{l+1} \right) s_{k_i - 1, r} a_q, \end{aligned}$$

yielding again contradiction with Lemma 1.8.3. Finally, if $i \le d_0(v)$ and $j_i < k_i - 1$, by Lemma 2.3.2(vii), $c_{j_i,k_i} = b_{k_i} a_{j_i}$. If we could have $k_{i-1} = k_i$, then, from (iv) and (i) of Lemma 2.3.2, it would follow that

$$\begin{aligned} s_0 v_{i-1} s_0 v_i &= s_0 a_{j_{i-1}} b_{k_i} s_0 (a_{j_i} b_{k_i}) = s_0 a_{j_{i-1}} (b_{k_i} s_0 b_{k_i}) a_{j_i} \\ &= (s_0 a_{j_{i-1}} s_0) s_n s_0 s_{n-1} s_n \ldots s_{k_i} s_{k_i+1} a_{j_i} = s_1 s_0 a_{j_{i-1}} s_n s_0 s_{n-1} s_n \ldots s_{k_i} s_{k_i+1} a_{j_i}, \end{aligned}$$

yielding a contradiction with Lemma 1.8.3 once more. □

2.4 Extract Graphs

Let $\mathcal{V} = \{(j,k) \in \mathbb{Z}^2 : 0 \le j < n, 1 \le k \le n+1\}$ equipped with the partial order defined by $(j',k') \le (j,k)$ if and only if the following conditions hold

(i) $j' \le j$ and $k' \ge k$;

(ii) $j' < j$ if $j < k-1$;

(iii) $k' > k$ if $j' < k'-1$.

Let also $\mathcal{G}$ be the directed graph having vertex set $\mathcal{V}$ and arrows $v \longrightarrow v'$ with $v > v'$, for which there is no $v'' \in \mathcal{V}$ satisfying $v < v'' < v'$. We refer to $\mathcal{G}$ as the *left-long $\widehat{A}_n$-extract graph* or, for short, the *extract graph of type $\widehat{A}_n$*. We say (j,k) is a *restricted* vertex if $j < k-1$. We refer to $v = (j,k)$ as a *terminal* vertex if v is a minimal element of $\mathcal{V}$. In other words, by (ii) and (iii), v is terminal if and only $k = n+1$ or $j = 0$ and $k \neq 1$ (cf. Proposition 2.4.6 below). Finally, let $\mathcal{M}$ be the set of functions $\mu : \mathcal{V} \to \mathbb{Z}_{\ge 0}$ satisfying the following properties

(i) The support of μ is a totally ordered subset of $\mathcal{V}$;

(ii) $\mu(v) \le 1$ if v is a restricted vertex.

Given $v = (j,k) \in \mathcal{V}$, set $w_v = s_0 c_{j,k}$ and, given $\mu \in \mathcal{M}$, set

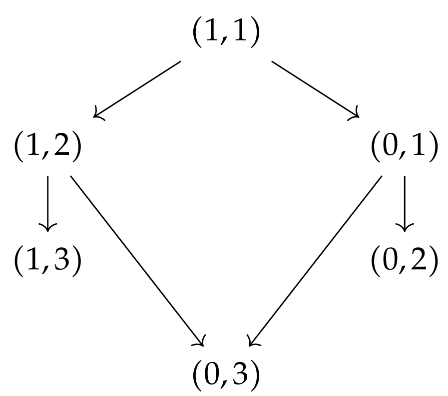
$$w^\mu = \overrightarrow{\prod_{v \in \mathrm{supp}(\mu)}} w_v^{\mu(v)}.$$

Proposition 2.3.3 implies

$$\mathcal{E} \subseteq \{w^\mu : \mu \in \mathcal{M}\}. \tag{2.4.1}$$

Conjecture 2.4.1. The equality holds in (2.4.1).

Example 2.4.2. The extract graph of type $\widehat{A}_2$ is



and the bottom three vertices are the restricted ones. Note

$$w_{(1,1)} = w_o, \quad w_{(1,2)} = s_1 s_2, \quad w_{(0,1)} = s_2 s_1, \quad w_{(1,3)} = s_1, \quad w_{(0,2)} = s_2, \quad w_{(0,3)} = e.$$

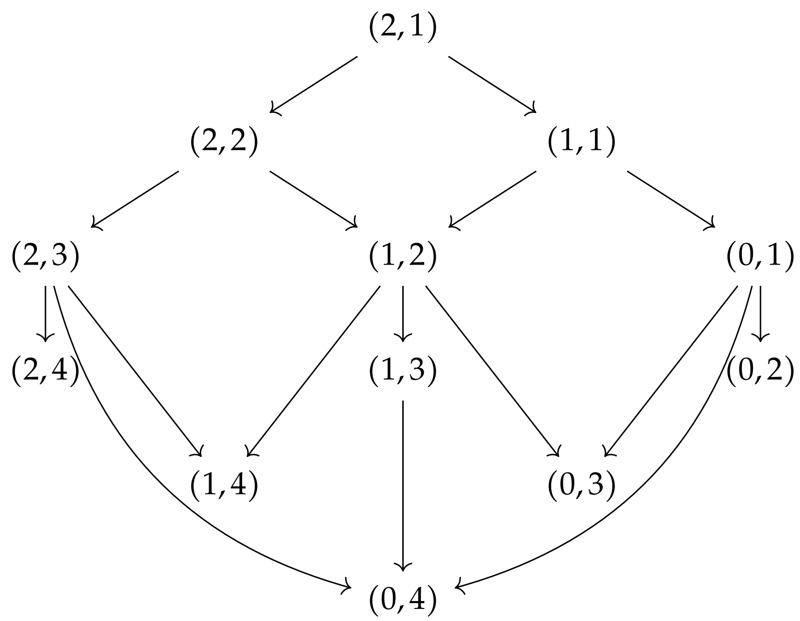
Thus, in this case, $\mathcal{V} \to \mathcal{W}, v \mapsto w_v$, is a bijection. Given $\{i,j\} = I_{1,2}, x, y \in \mathbb{Z}_{\geq 0}$, set

$$w_i^{x,y} = (s_0 w_o)^x (s_0 s_i s_j)^y. \tag{2.4.2}$$

The element $w_i^{x,y}$ is the one associated with the function μ satisfying $\mu(1,1) = x, \mu(\bar{i}, \bar{i}+1) = y$ where $\bar{i} \in \{0,1\}$, $\bar{i} \cong_2 i$, and $\mu(v) = 0$ for the other vertices. It follows that

$$\mathcal{E} \subseteq \{w_i^{x,y} u : i \in I_{1,2}, u \in \{e, s_0, s_0 s_i\}, x, y \in \mathbb{Z}_{\geq 0}\}. \tag{2.4.3}$$

Example 2.4.3. In general, the natural map $\mathcal{V} \to \mathcal{W}$ is only injective, since $\#\mathcal{V} = n(n+1) < \#\mathcal{W} = (n+1)!$. For instance, the extract graph of type $\widehat{A}_3$ is



and the restricted vertices are the ones in the bottom three rows.

Given $\mu \in \mathcal{M}$, let $d = \sum_{v \in \mathcal{V}} \mu(v)$ and let $\sigma^\mu \in \mathcal{W}^d$ be the element obtained by ordering the family

$$(s_0 w_v^{\mu(v)})_{v \in \mathcal{V}}$$

according to the order of $\operatorname{supp}(\mu)$. Here, the exponent $\mu(v)$ means the element appears that many times in the family. Let also $\varepsilon(\mu) =\in \mathcal{W} \times \mathcal{W}^d$.

Conjecture 2.4.4. For all $\varepsilon(w^\mu) = \varepsilon(\mu)$.

In particular, $d(w^\mu) = \sum_{v \in \mathcal{V}} \mu(v)$ and $\ell(w^\mu) = \sum_{v \in \mathcal{V}} \mu(v) \ell(w_v)$ and Conjecture 2.4.1 follows.

In light of (2.4.1), the description of the elements of $\mathcal{E}$ is deeply related to the study of oriented paths in extract graphs. If, moreover, Conjecture 2.4.1 is true, such study leads to a description of a minimal set of expressions for the elements of $\widehat{\mathcal{W}}$. Furthermore, if Conjecture 2.4.4 holds, then extract graphs also describe fll extracts. With this in mind, we end this section with a first step in the direction of studying the combinatorics of extract graphs by classifying the adjacent vertices.

Remark 2.4.5. Apparently, Conjecture 2.4.4 has been proved as the main result of the preprint (AL HARBAT, 2021). Our original strategy for proving this conjecture was based on the computation alcove forms and permutation forms. This strategy is carried out for $n = 2$ in Chapter 3. So far we have not found an efficient way of performing the same computations for rank higher than 2. The strategy of (AL HARBAT, 2021) is different, exploring strongly several consequences of the Exchange Property. After becoming aware of the existence of (AL HARBAT, 2021) at the end of January 2022, we have tried to understand this strategy. This gave rise to Section 2.5, where we present a proof of Conjecture 2.4.4 for all n but only for μ of depth at most 2, i.e., $\sum_{v \in \mathcal{V}} \mu(v) \leq 2$. As of this moment, we have not understood the extra details to complete the proof for higher depth. We are actively working on it.

We point out some further differences between the present work and (AL HARBAT, 2021). First, (AL HARBAT, 2021) deals only with type $\widehat{A}$ and does not consider the notion of fll extract for other types, although the author claims they will be addressed in forthcoming publications. The main result of (AL HARBAT, 2021) gives an explicit reduced expression for all elements of $\widehat{\mathcal{W}}$, termed canonical and, in the process of proving they are indeed reduced expressions, deduces several properties of such expressions which amount to the properties of what we would call fully right (instead of left) long extracts. By putting these properties in the conceptual forefront of the study, we are able to prove existence and uniqueness of fll extracts for all types, remaining to address the task of characterizing explicit expression for their components. In type A, this initiates with the concept of Left and Right sequences or, equivalently, Extract Graphs, and is completed once Conjecture 2.4.4 is proved.

If σ is the fll extract of w, the element corresponding to $\tilde{\sigma}_0$ is called the affine block of w in (AL HARBAT, 2021) while the set corresponding to $\mathcal{E}$ is denoted by $\mathscr{B}_n$ there. A few other interesting questions which we have not considered here are answered in (AL HARBAT, 2021). The connection with alcoves and, hence, Proposition 1.8.4, is not considered in (AL HARBAT, 2021). As far as we know, the notion of extract graph has not been considered in the literature, but here.

We now initiate the study of combinatorics of Extract Graphs. In order to characterize the incoming and outgoing arrows of a fixed vertex, it will be convenient to introduce some terminology. For $v = (j,k)$, let $l(v) := j - k$. If $l(v) = -1$, we will refer to v as a *critical vertex*. If v is neither a restricted nor a critical vertex, it will be referred to as a *free* vertex. Notice that v is a free if and only if $l(v) \geq 0$. Similarly, v is a restricted vertex if and only if $l(v) \leq -2$. Denote by

$$a(v) = \{v' \in \mathcal{V} : v \longrightarrow v' \text{ is an arrow in } \mathcal{G}\}.$$

Proposition 2.4.6. Let $v = (j,k) \in \mathcal{V}$.

(i) If v is free, then $a(v) = \{(j-1,k),(j,k+1)\} \cap \mathcal{V}$.

(ii) If v is critical, then

$$a(v) = (\{(j-1,k+p) : p \in I_{1,n+1-k}\} \cup \{(j-q,k) : q \in I_{1,j}\}) \cap \mathcal{V}.$$

(iii) If v is restricted, then

$$a(v) = (\{(j-1,k+p) : p \in I_{1,n+1-k}\} \cup \{(j-q,k+1) : q \in I_{2,j}\}) \cap \mathcal{V}.$$

In particular, v is terminal if and only $k = n+1$ or $j = 0$ and $k \neq 1$.

Proof. Let $v' = (j',k') \in \mathcal{V}$. Suppose v is free. In this case, $l(v) \geq 0$ and, if $(j-1,k),(j,k+1) \in \mathcal{V}$, they are minimal non-comparable free vertices and, hence, belong to $a(v)$. Notice that if v' is either free or critical, then $v' < v$ if and only if $v' \neq v$, $j' \leq j$, and $k \leq k'$. In the case $j' = j$, then $k' > k+1$ and $v' \leq (j,k+1) < v$. Otherwise, $j' \leq j-1$ and $k' \geq k$, following that $v' \leq (j-1,k) < v$. Similarly, if v' is restricted, then $v' < v$ if and only if $j' < j$ and $k' \geq k$, implying that $v' \leq (j-1,k) < v$. Therefore, part (i) holds.

For part (ii), assume v is critical. In particular, every element in $a(v)$ is restricted. In this case, $v' < v$ if and only if $j' < j$ and $k' \geq k$. Write $j' = j-1-q$ and $k' = k+p$, for some $p \in I_{0,n+1-k}$ and $q \in I_{0,j-1}$. If $k' = k+p$, for some $p \in I_{1,n+1-k}$, then $v' \leq (j-1,k+p) < v$. Otherwise, if $k' = k$, we have $j' = j-q$, for some $q \in I_{1,j}$. Since $(j-1,k+p), p \in I_{1,n+1-k}$, are all restrict, these elements are all non-comparable with each other. Similarly, the elements $(j-q,k), q \in I_{1,j}$, are all non-comparable with each other. Since $j-1 > j-q$ and $k+p > k$, $(j-1,k+p)$ and $(j-1,k)$ are also non-comparable for $p \in I_{1,n+1-k}, q \in I_{1,j}$. Thus, part (ii) holds.

Finally, if v is restricted, then $v' < v$ if and only if $j' < j$ and $k' > k$. One can do an an analysis similar to the one in the previous case and check part (iii). □

2.5 Proof of Conjecture 2.4.4 for Depth Two

Let $\mathcal{N}$ be the set of all functions from $\mathcal{V}$ to $\mathbb{Z}_{\geq 0}$ and define a partial order on $\mathcal{N}$ by

$$\nu \leq \mu \quad \Leftrightarrow \quad \nu(v) \leq \mu(v) \quad \text{for all} \quad v \in \mathcal{V}.$$

In particular,

$$\nu \leq \mu \quad \Rightarrow \quad \mathrm{supp}(\nu) \subseteq \mathrm{supp}(\mu). \tag{2.5.1}$$

Let us denote by 0 the function $\mu \in \mathcal{N}$ such that $\mu(v) = 0$ for all $v \in \mathcal{V}$. One easily checks that

$$\mu \in \mathcal{M} \text{ and } \nu \leq \mu \quad \Rightarrow \quad \nu \in \mathcal{M}.$$

We shall use the notation

$$(w_1^{m_1}, \ldots, w_k^{m_k}), \quad \text{where} \quad k \in \mathbb{Z}_{\geq 0}, m_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq k,$$

for an element of $\mathcal{W}^m, m = \sum_{j=1}^{k} m_j$, whose m_1 first components are equal to w_1, and so on. For instance, given $\mu \in \mathcal{N}$, suppose

$$\mathrm{supp}(\mu) = \{v_1, \ldots, v_k\} \quad \text{and} \quad v_j < v_{j+1} \quad \text{for all} \quad 1 \leq j < k,$$

and define

$$\varepsilon(\mu) = (e,\ (s_0 w_{v_1})^{\mu(v_1)}, \ldots,\ (s_0 w_{v_k})^{\mu(v_k)}).$$

Note this definition coincides with the one given just prior to 2.4.4. By Lemma 1.8.3 and (1.8.6), we have

$$D(w^\nu) = \{s_0\} \quad \text{for all} \quad 0 \neq \nu \leq \mu \quad \Rightarrow \quad \varepsilon(\mu) = \varepsilon(w^\mu).$$

In light of (2.5.1), in order to prove Conjecture 2.4.4, it then suffices to show

$$D(w^\mu) = \{s_0\} \quad \text{for all} \quad \mu \in \mathcal{M} \smallsetminus \{0\} \tag{2.5.2}$$

or, equivalently,

$$s \notin D(w^\mu) \quad \text{for all} \quad \mu \in \mathcal{M},\ s \in S. \tag{2.5.3}$$

Let $d = \sum_{v \in \mathcal{V}} \mu(v)$ and write $\varepsilon(\mu) = (e, \sigma_1, \ldots, \sigma_d)$. Given $0 \leq r \leq d$, let $\mu_r \in \mathcal{M}$ correspond to $(e, \sigma_1, \ldots, \sigma_r)$. In particular,

$$\mu_d = \mu \quad \text{and} \quad \mu_r < \mu \quad \text{for all} \quad r < d.$$

Let us simplify notation and set $w_r = w^{\mu_r}$.

Lemma 2.5.1 (cf. (AL HARBAT, 2021, Lemma 2.9)). Conjecture 2.4.4 holds if $d = 1$.

Proof. We have $w^\mu = s_0 c_{j,k}$ where (j,k) is the unique element in $\mathrm{supp}(\mu)$. There is nothing to do if $(j,k) = (0, n+1)$ since $w^\mu = s_0$ in that case. Otherwise, if (2.5.3) failed, Corollary 1.1.2 would imply that there exists $s \in S$ such that

$$\begin{array}{lll} \text{either} & s = s_0 a_{i-1} s_i a_{i-1}^{-1} s_0 & \text{with} \quad 1 \leq i \leq j \\ \text{or} & s = s_0 c_{j,i+1} s_i c_{j,i+1}^{-1} s_0 & \text{with} \quad k \leq i \leq n. \end{array}$$

Using (2.2.1), let us rewrite this as

$$s_0 s s_0 \in \{t_i\ ,\ t_{j,i'} : 1 \le i \le j\ ,\ k \le i' \le n\}.$$

It follows that $s_0 s s_0 \in \mathcal{W}$ and, hence, must be equal to s. Therefore, $s \notin \{s_1, s_n\}$. However, $s = t_i$ implies $s = s_i$ and $i = 1$, whereas $s = t_{j,i'}$ implies $s = s_i$ and $i = n$, yielding a contradiction in both cases. □

Corollary 2.5.2. Let $1 \le j \le n-1$, $2 \le k \le n$ and consider the elements

$$s_1 s_n s_0 s_n s_1, \quad s_0 \overrightarrow{t_j} s_0, \quad s_0 \overleftarrow{t_k} s_0, \quad b_1 s_0 b_1^{-1}, \quad \text{and} \quad \overleftarrow{t_n} s_0 \overleftarrow{t_n}.$$

The lengths of these elements are equal to the sum of the lengths of the factors in the corresponding definitions.

Proof. Note

$$s_1 s_n s_0 s_n s_1 = s_1 s_n s_0 a_1 b_n, \quad b_1 s_0 b_1^{-1} = b_1 s_0 a_{n-1} b_n, \quad \overleftarrow{t_n} s_0 \overleftarrow{t_n} = \overleftarrow{t_n} s_0 a_{n-1} b_1$$

and, by (2.2.2),

$$s_0 \overrightarrow{t_j} s_0 = a_j^{-1} s_0 a_j, \quad \text{and} \quad s_0 \overleftarrow{t_k} s_0 = b_k^{-1} s_0 b_k.$$

By Lemma 2.5.1, the right side of each expression is the fll extract of the corresponding element. Hence, the corollary follows. □

Now we turn to the proof of the case $d = 2$. Let $\sigma_1 = c_{j,k}$, $\sigma_2 = c_{j',k'}$. If (2.5.3) failed, Corollary 1.1.2 would imply that there exists $s \in S$, $1 \le i \le j'$, $k' \le i' \le n$, such that

$$s \in \{w_1 s_0 w_1^{-1}\ ,\ w_1 s_0 t_i s_0 w_1^{-1}\ ,\ w_1 s_0 t_{j,i'} s_0 w_1^{-1}\}. \tag{2.5.4}$$

Recall (2.2.1) and, given $v = (j,k) \in \mathcal{V}$, set $t_v = t_{j,k}$. In particular, $t_v = e$ if $v = (0, n+1)$ and, otherwise, $t_v = \overleftarrow{t_k}$ if $j = 0$ and $t_v = \overrightarrow{t_j}$ if $k = n+1$.

Lemma 2.5.3. Let $v = (j,k) \in \mathcal{V}$, $k \le n$. If v is not critical, there exists $\tilde{k} > k$ such that $t_v = \overleftarrow{t_{\tilde{k}}}$. If v is critical, then $t_v = \overleftarrow{t_1}$.

Proof. Suppose v is not critical. If v is restricted, then $c_{j,k} = b_k a_j$, following that $a_j \overleftarrow{t_k} = \overleftarrow{t_k} a_j$ and, hence,

$$t_{j,k} = a_j \overleftarrow{t_k} a_j^{-1} = \overleftarrow{t_k}.$$

Otherwise, if v is free, then $k - 1 < (j+1) - 1$ and, by (2.2.2) and Lemma 2.3.2(vi),

$$\begin{aligned} t_{j,k} = a_j \overleftarrow{t_k} a_j^{-1} = a_j s_{k,n-1} b_k a_j^{-1} = a_{k-1} s_{k,j}^2 \overleftarrow{t_{j+1}} s_{j,k}^2 a_{k-1}^{-1} = a_{k-1} s_{k+1,j} s_{k,j-1} \overleftarrow{t_{j+1}} s_{j-1,k} s_{j,k+1} a_{k-1}^{-1} \\ = s_{k+1,j} a_{k-1} \overleftarrow{t_{j+1}} a_{k-1}^{-1} s_{j,k+1} = s_{k+1,j} \overleftarrow{t_{j+1}} s_{j,k+1} = \overleftarrow{t_{k+1}}. \end{aligned}$$

If v is critical, then, by (2.2.2),

$$t_{j,k} = a_j \overleftarrow{t_{j+1}} a_j^{-1} = a_j s_{j+1,n-1} b_{j+1} a_j^{-1} = \overleftarrow{t_1}.$$

□

In light of (2.5.3), the next lemma ends the proof.

Lemma 2.5.4. Let $v, v' \in \mathcal{V}$, say $v = (j,k)$ and $v' = (j',k')$, be such that $v < v'$. Then, for $s \in S$, we have

$$s_0 c_{j,k} s_0 t s_0 c_{j,k}^{-1} s_0 \notin S \quad \text{for} \quad t \in \{s_0, \overrightarrow{t_{j'}}, t_{j',k'}\}.$$

Proof. Since v is not terminal, we have $k > n+1$ and $j > 0$ if $k > 1$. Denote

$$t_0 = s_0 c_{j,k} s_0 c_{j,k}^{-1} s_0, \quad t_a = s_0 c_{j,k} s_0 \overrightarrow{t_{j'}} s_0 c_{j,k}^{-1} s_0, \quad \text{and} \quad t_b = s_0 c_{j,k} s_0 t_{j',k'} s_0 c_{j,k}^{-1} s_0.$$

These are the elements we need to show are not in S according to the options for t.

Let us first prove that $t_0 \notin S$. Suppose $j < n-1$ and $k > 1$. In particular, since $k > 1$, $j > 0$. Note

$$\begin{aligned} t_0 &= s_0 s_1 (s_2 \cdots s_j) s_n (s_{n-1} \cdots s_k) s_0 (s_k \cdots s_{n-1}) s_n (s_j \cdots s_2) s_1 s_0 \\ &= s_0 s_1 s_n s_0 s_n s_1 s_0 = s_0 s_1 s_0 s_n s_0 s_1 s_0 = s_1 s_0 s_1 s_n s_1 s_0 s_1 = s_1 s_0 s_n s_0 s_1 = s_1 s_n s_0 s_n s_1. \end{aligned}$$

Corollary 2.5.2 then implies $t_0 \notin S$. In the case $k = 1$, if $j = 0$ then $t_0 = s_0 b_1 s_0 b_1^{-1} s_0$, and if $0 < j < n-1$ we also have

$$t_0 = s_0 a_j b_1 s_0 b_1^{-1} a_j^{-1} s_0 = s_0 b_2 a_{j+1} s_0 a_{j+1}^{-1} b_2^{-1} s_0 = s_0 b_2 s_1 s_{2,j+1} s_0 s_{j+1,2} s_1 b_2^{-1} s_0 = s_0 b_1 s_0 b_1^{-1} s_0,$$

whereas for $j = n-1$ and $k > 1$, we get

$$t_0 = s_0 a_{n-1} s_n s_{n-1,k} s_0 s_{k,n-1} s_n a_{n-1}^{-1} s_0 = s_0 b_1^{-1} s_0 b_1 s_0.$$

Since $s_0 b_1 s_0 b_1^{-1} s_0 = (s_0 b_1^{-1} s_0 b_1 s_0)^{-1}$, if one these elements is not in S, neither is the other. If we had $s_0 b_1 s_0 b_1^{-1} s_0 = s_i \in S$, it would follow that $b_1 s_0 b_1^{-1} = s_0 s_i s_0$, yielding a contradiction with Corollary 2.5.2. Finally, if $j = n-1$ and $k = 1$, then

$$t_0 = s_0 a_{n-1} b_1 s_0 b_1^{-1} a_{n-1}^{-1} = s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0,$$

contradicting Corollary 2.5.2.

Now we prove $t_a \notin S$. Since $j' < n$

$$t_a = s_0 c_{j,k} a_{j'}^{-1} s_0 a_{j'} c_{j,k}^{-1} s_0. \tag{2.5.5}$$

Also note

$$a_q a_{q'}^{-1} = a_{q'+1}a_{q'}^{-1}s_{q'+2,q} = \overrightarrow{t_{q'+1}}s_{q'+2,q} \overset{(2.2.2)}{=} a_{q'+1}^{-1}s_{2,q} \tag{2.5.6}$$

when $0 < q' < q \leq n-1$ $n-1 \geq q > q' > 0$, and

$$a_j b_k^{-1} = \begin{cases} b_k^{-1}a_j, & \text{if } j+1<k, \\ b_1^{-1}, & \text{if } j+1=k, \\ b_{k+1}^{-1}a_{j-1}, & \text{if } j+1>k. \end{cases} \tag{2.5.7}$$

Indeed, the first two equalities are immediate and the last follows from observing that $j-1 \geq k$ and then

$$a_j b_k^{-1} a_{j-1}^{-1} b_{k+1} = a_j(a_{j-1}b_k)^{-1}b_{k+1}a_j(b_{k+1}a_j)^{-1}b_{k+1} = e.$$

If $j'+1<k$, then $j<n-1$, $k>1$, $j'+1<k'$ and then we must have $j'<j$, following that

$$\begin{aligned} t_a &\overset{(2.5.7)}{=} s_0a_ja_{j'}^{-1}b_ks_0b_k^{-1}a_{j'}a_j^{-1}s_0 = s_0a_ja_{j'}^{-1}s_ns_0s_na_{j'}a_j^{-1}s_0 \overset{(2.5.6)}{=} s_0a_{j'+1}^{-1}s_{2,j}s_ns_0s_ns_{j,2}a_{j'+1}s_0 \\ &= s_0a_{j'+1}^{-1}s_ns_0s_na_{j'+1}s_0 = s_0s_na_{j'+1}^{-1}s_0a_{j'+1}s_ns_0 \overset{(2.5.5)}{=} s_0s_ns_0\overrightarrow{t_{j'+1}}s_0s_ns_0 \\ &= s_ns_0s_n\overrightarrow{t_{j'+1}}s_ns_0s_n = s_ns_0\overrightarrow{t_{j'+1}}s_0s_n. \end{aligned}$$

The assumption $s_ns_0\overrightarrow{t_{j'+1}}s_0s_n = s_i \in S$ would imply $s_0\overrightarrow{t_{j'+1}}s_0 = s_ns_is_n \in \mathcal{W}$, contradicting Corollary 2.5.2, so $t_a \notin S$. In the case $j'+1=k$, we get

$$\begin{aligned} t_a &= s_0a_jb_ka_{k-1}^{-1}s_0a_{k-1}b_k^{-1}a_j^{-1}s_0 \overset{(2.5.7)}{=} s_0a_jb_1s_0b_1^{-1}a_j^{-1}s_0 = s_0b_1s_{2,j+1}s_0s_{j+1,2}b_1^{-1}s_0 \\ &= \begin{cases} s_0b_1s_0b_1^{-1}s_0, & \text{if } j'<n-1, \\ s_0\vec{t}_ns_0\vec{t}_ns_0, & \text{if } j'=n-1, \end{cases} \end{aligned}$$

which we have already proved do not belong to S. Otherwise, if $j' \geq k$, we get

$$\begin{aligned} t_a &\overset{(2.5.7)}{=} s_0a_ja_{j'-1}^{-1}b_{k+1}s_0b_{k+1}^{-1}a_{j'-1}a_j^{-1}s_0 = s_0a_ja_{j'-1}^{-1}s_ns_0s_na_{j'-1}a_j^{-1}s_0 \overset{(2.5.6)}{=} s_0a_{j'}^{-1}s_{2,j}s_ns_0s_ns_{j,2}a_{j'}s_0 \\ &= \begin{cases} s_0a_{j'}^{-1}s_ns_0s_na_{j'}s_0, & \text{if } j<n-1, \\ s_0a_{j'}^{-1}b_2^{-1}s_0b_2a_{j'}s_0, & \text{if } j=n-1 \end{cases} = \begin{cases} s_ns_0\vec{t}_{j'}s_0s_n, & \text{if } j<n-1, \\ s_0b_1^{-1}s_0b_1s_0, & \text{if } j=n-1, j'=1, \\ s_0b_1^{-1}s_0\overrightarrow{t_{j'-1}}s_0b_1s_0, & \text{if } j=n-1, j'>1, \end{cases} \end{aligned}$$

where the last expression follows from (2.5.5). We have already checked that the first two expressions are not in S, so it remains to verify $s_0b_1^{-1}s_0\overrightarrow{t_{j'-1}}s_0b_1s_0 \notin S$. Indeed, if we had $s_0b_1^{-1}s_0\overrightarrow{t_{j'-1}}s_0b_1s_0 = s_i \in S$, it would imply

$$s_0\overrightarrow{t_{j'-1}}s_0 = b_1s_ib_1^{-1} = \begin{cases} b_1s_ib_1^{-1}, & \text{if } 1<i<n, \\ b_2s_0b_2^{-1}, & \text{if } i=1, \\ b_1s_0s_ns_0b_1^{-1}, & \text{if } i=n, \end{cases}$$

whereas Corollary 2.5.2 implies $s_0, s_1 \in \operatorname{supp}(s_0\overrightarrow{t_{j'-1}}s_0)$ and $s_n \notin \operatorname{supp}(s_0\overrightarrow{t_{j'-1}}s_0)$, yielding a contradiction.

Finally, let us prove $t_b \notin S$. Note

$$b_r b_{r'}^{-1} = b_{r'-1} b_{r'}^{-1} s_{r'-2,r} = \overleftarrow{t_{r'-1}} s_{r'-2,r} \overset{(2.2.2)}{=} b_{r'-1}^{-1} s_{n-1,r} \tag{2.5.8}$$

for $1 \le r < r' < n+1$. Suppose v' is not critical. Then, by Lemma 2.5.3, there exists $\tilde{k} > k$ for which $t_{j',k'} = \overleftarrow{t_{\tilde{k}}}$, such that

$$t_b = s_0 c_{j,k} b_{\tilde{k}}^{-1} s_0 b_{\tilde{k}} c_{j,k}^{-1} s_0. \tag{2.5.9}$$

If $j+1 < \tilde{k}$, since $1 \le k < \tilde{k} \le n$, then $j < n-1$, $\tilde{k} > 1$, and

$$\begin{aligned} t_b &\overset{(2.5.8)}{=} s_0 a_j b_{\tilde{k}-1}^{-1} s_{n-1,k} s_0 s_{k,n-1} b_{\tilde{k}-1} a_j^{-1} s_0 = s_0 b_{\tilde{k}-1}^{-1} a_j s_0 a_j^{-1} b_{\tilde{k}-1} s_0 \overset{(2.5.9)}{=} s_0 s_1 s_0 \overleftarrow{t_{\tilde{k}-1}} s_0 s_1 s_0 \\ &= s_1 s_0 \overleftarrow{t_{\tilde{k}-1}} s_0 s_1. \end{aligned}$$

Similarly as for the previous case, one can use Corollary 2.5.2 to prove $t_b \notin S$. For $\tilde{k} = j+2$, we have $k \le \tilde{k} - 1 = j+1$ and hence

$$\begin{aligned} t_b &= s_0 b_{k+1} a_{j+1} b_{j+2}^{-1} s_0 b_{j+2} a_{j+1}^{-1} b_{k+1}^{-1} s_0 \overset{(2.5.7)}{=} s_0 b_{k+1} b_1^{-1} s_0 b_1 b_{k+1}^{-1} s_0 \overset{(2.5.8)}{=} s_0 a_{n-1} b_k s_0 b_k^{-1} a_{n-1}^{-1} s_0 \\ &= \begin{cases} s_0 b_1^{-1} s_0 b_1 s_0, & \text{if } k > 1 \\ s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0, & \text{if } k = 1, \end{cases} \end{aligned}$$

which we have already seen do not belong to S. Otherwise, if $\tilde{k} \le j+1$, then $k \le j+1$ and

$$\begin{aligned} t_b &= s_0 b_{k+1} a_{j+1} b_{\tilde{k}}^{-1} s_0 b_{\tilde{k}} a_{j+1}^{-1} b_{k+1}^{-1} s_0 \overset{(2.5.7)}{=} s_0 b_{k+1} b_{\tilde{k}+1}^{-1} a_j s_0 a_j^{-1} b_{\tilde{k}+1} b_{k+1}^{-1} s_0 \\ &\overset{(2.5.8)}{=} s_0 b_{\tilde{k}}^{-1} s_{n-1,k+1} s_1 s_0 s_1 s_{k+1,n-1} b_{\tilde{k}} s_0 = \begin{cases} s_0 b_{\tilde{k}}^{-1} s_1 s_0 s_1 b_{\tilde{k}} s_0, & \text{if } k > 1 \\ s_0 b_{\tilde{k}}^{-1} a_{n-1}^{-1} s_0 a_{n-1} b_{\tilde{k}} s_0, & \text{if } k = 1 \end{cases} \\ &\overset{(2.5.9)}{=} \begin{cases} s_1 s_0 \overleftarrow{t_{\tilde{k}}} s_0 s_1, & \text{if } k > 1 \\ s_0 b_1 s_0 b_1^{-1} s_0, & \text{if } k = 1, \tilde{k} = n \\ s_0 b_1 s_0 \overleftarrow{t_{\tilde{k}+1}} s_0 b_1^{-1} s_0, & \text{if } k = 1, \tilde{k} < n. \end{cases} \end{aligned}$$

The first two expressions we have already proved are not in S and analogously to the respective case t_a, one can check $s_0 b_1 s_0 \overleftarrow{t_{\tilde{k}+1}} s_0 b_1^{-1} s_0 \notin S$ using Corollary 2.5.2. It remains to check $t_b \notin S$ when v' is critical. In this case, by Lemma 2.5.3, $t_{j',k'} = \overleftarrow{t_1}$, following that

$$t_b = s_0 c_{j,k} s_0 \overleftarrow{t_1} s_0 c_{j,k}^{-1} s_0. \tag{2.5.10}$$

If $k > 1$,

$$\begin{aligned} b_k s_0 \overleftarrow{t_1} s_0 b_k^{-1} &\overset{(2.2.2)}{=} s_n s_0 s_{n-1,k} a_{k-2} s_{k-1,n-1} s_n s_{n-1,k-1} a_{k-2}^{-1} s_{k,n-1} s_0 s_n \\ &= s_n s_0 a_{k-2} s_{n-1,k} s_{k-1,n-1} s_n s_{n-1,k-1} s_{k,n-1} a_{k-2}^{-1} s_0 s_n \\ &= s_n s_0 a_{k-2} s_{k-1,n-1} s_{n,k-1} s_n s_{k-1,n} s_{n-1,k-1} a_{k-2}^{-1} s_0 s_n \\ &= s_n s_0 a_{k-2} s_{k-1,n-2} s_{n-1} s_n s_{n-1} s_n s_{n-1} s_n s_{n-1} s_{n-2,k-1} a_{k-2}^{-1} s_0 s_n \\ &= s_n s_0 a_{k-2} s_{k-1,n-2} s_{n-1} s_{n-2,k-1} a_{k-2}^{-1} s_0 s_n = s_n s_0 \overrightarrow{t_{n-1}} s_0 s_n \\ &\overset{(2.5.5)}{=} s_n a_{n-1}^{-1} s_0 a_{n-1} s_n = b_1 s_0 b_1^{-1}, \end{aligned}$$

where the third equality follows from Lemma 2.3.2(v), and, hence, if $j < n-1$,

$$\begin{aligned} t_b &= s_0 a_j b_1 s_0 b_1^{-1} a_j^{-1} s_0 = s_0 a_j b_{j+2} a_{j+1}^{-1} s_0 a_{j+1} b_{j+2}^{-1} a_j^{-1} s_0 = s_0 b_{j+2} a_j a_{j+1}^{-1} s_0 a_{j+1} a_j^{-1} b_{j+2}^{-1} s_0 \\ &\overset{(2.2.2)}{=} s_0 b_{j+2} a_{j+1}^{-1} a_{2,j+1} s_0 s_{j+1,2} a_{j+1} b_{j+2}^{-1} s_0 = s_0 b_{j+2} a_{j+1}^{-1} s_0 a_{j+1} b_{j+2}^{-1} s_0 = s_0 b_1 s_0 b_1^{-1} s_0, \end{aligned}$$

whereas if $j = n-1$,

$$t_b = s_0 a_{n-1} b_1 s_0 b_1^{-1} a_{n-1}^{-1} s_0 \overset{(2.2.2)}{=} s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0,$$

which are both expressions we have known cannot belong to S. In the case $k = 1$ and $j < n-1$, as for the case $k > 1$, one can shows $a_{j+1} s_0 \overleftarrow{t_1} s_0 a_{j+1}^{-1} = b_1^{-1} s_0 b_1$ and, hence,

$$t_b = s_0 a_j b_1 s_0 \overleftarrow{t_1} s_0 b_1^{-1} a_j^{-1} s_0 = s_0 b_2 a_{j+1} s_0 \overleftarrow{t_1} s_0 a_{j+1}^{-1} b_2^{-1} = s_0 b_2 b_1^{-1} s_0 b_1 b_2^{-1} = s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0,$$

which is not in S. Finally, it $j = n-1$ and $k = 1$, we have

$$t_b = s_0 a_{n-1} b_1 s_0 \overleftarrow{t_1} s_0 b_1^{-1} a_{n-1}^{-1} s_0 = s_0 \overrightarrow{t_n} s_0 \overleftarrow{t_1} s_0 \overrightarrow{t_n} s_0 = s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0.$$

The assumption $s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0 \overrightarrow{t_n} s_0 = s_i \in S$ would imply $\overrightarrow{t_n} s_0 \overrightarrow{t_n} = s_0 s_i s_0 \overrightarrow{t_n} s_0$. If $1 < i < n$ it would follow that $\overrightarrow{t_n} s_0 \overrightarrow{t_n} = s_i \overrightarrow{t_n} s_0$, whereas if $i = 1$, it would imply

$$\overrightarrow{t_n} s_0 \overrightarrow{t_n} = s_1 s_0 s_1 \overrightarrow{t_n} s_0 = s_1 s_0 s_{2,n-1} b_1,$$

yielding a contradiction with Corollary 2.5.2 once more. Similarly, one can prove the case $i = n$ cannot occur, which ends the proof. □

2.6 Representation of Affine Kac-Moody Algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra of type A_n, i.e., $\mathfrak{g} \cong \mathfrak{sl}_{n+1}$. The affine Kac-Moody algebra $\hat{\mathfrak{g}} \cong \hat{\mathfrak{sl}}_{n+1}$ associated with $\mathfrak{g}$ is the Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ with

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\delta_{r,-s}(n+1)Tr(xy)c, \quad [c, \hat{\mathfrak{g}}] = \{0\}, \quad \text{and} \quad [d, x \otimes t^r] = rx \otimes t^r$$

for any $x, y \in \mathfrak{g}$ and $r, s \in \mathbb{Z}$, where $Tr(xy)$ denotes the trace of xy. The Lie algebra $\mathfrak{g}$ can be considered as a subalgebra of $\hat{\mathfrak{g}}$. Recall (1.4.1) and set $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Phi} \hat{\mathfrak{g}}_\alpha, \quad \text{where } \hat{\mathfrak{g}}_\alpha = \{x \in \hat{\mathfrak{g}} : [h, x] = \alpha(h)x, \text{ for all } h \in \hat{\mathfrak{h}}\}.$$

Denote $\hat{\mathfrak{n}}^+ = \mathfrak{n}^+ \oplus (\mathfrak{g} \otimes t\mathbb{C}[t])$ (recall (1.4.2)) and $\hat{\mathfrak{b}} = \hat{\mathfrak{n}}^+ \oplus \hat{\mathfrak{h}}$. Identify $\mathfrak{h}^*$ with the subspace $\{\lambda \in \hat{\mathfrak{h}}^* : \lambda(c) = \lambda(d) = 0\}$. Let $\Lambda_0, \delta \in \hat{\mathfrak{h}}^*$ be defined by

$$\Lambda_0(d) = 0 = \Lambda_0(\mathfrak{h}), \quad \Lambda_0(c) = 1, \quad \delta(c) = 0 = \delta(\mathfrak{h}), \quad \delta(d) = 1.$$

Also, set $h_0 = c - h_{\alpha_0}$ and

$$\Lambda_i = \omega_i + \Lambda_0, \quad \text{for } i \in I_{1,n}.$$

Then, $\Lambda_i(h_j) = \delta_{i,j}$ for all $i, j \in I_{0,n}$, and$\{h_i : i \in I_{0,n}\} \cup \{d\}$ is a basis of $\hat{\mathfrak{h}}$. Consider the linear endomorphisms of $\hat{\mathfrak{h}}^*$ defined by

$$s_0(\lambda) = \lambda - \lambda(h_0)(\delta - \alpha_0), \quad s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \; i \in I_{1,n} \quad \text{for all} \quad \lambda \in \hat{\mathfrak{h}}^*. \tag{2.6.1}$$

The group $\widehat{\mathcal{W}} :=< s_0, s_1, \ldots, s_n >$ is an affine Weyl group of type $\widehat{A}_n$. The subgroup $\mathcal{W} :=< s_1, \ldots, s_n >$ is the Weyl group of $\mathfrak{g}$, which has type A_n. For convenience of notation, set $\Lambda_{n+1} = \Lambda_0$. In this case, if $\Delta := \{\alpha_1, \ldots, \alpha_n\}$, then

$$\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}, \; i \in I_{1,n}, \quad \text{and} \quad \alpha_0 = -2\Lambda_0 + \Lambda_1 + \Lambda_n.$$

Moreover,

$$s_0(\Lambda_j) = \Lambda_j + \delta_{0,j}(\alpha_0 - \delta), \quad s_i(\Lambda_j) = \Lambda_j - \delta_{ij}\alpha_i, \; i \in I_{1,n}, \quad \text{and} \quad s_i(\delta) = \delta, i \in I_{0,n}. \tag{2.6.2}$$

The *orbit* of an element $\Lambda \in \hat{\mathfrak{h}}^*$ by $\widehat{\mathcal{W}}$ is the set $\widehat{\mathcal{W}}\Lambda = \{w\Lambda : w \in \widehat{\mathcal{W}}\}$. The *affine weight lattice* is the set

$$\widehat{P} = \{\lambda \in \hat{\mathfrak{h}}^* : \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I_{0,n}\} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \ldots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{C}\delta = \mathbb{Z}\Lambda_0 \oplus P \oplus \mathbb{C}\delta,$$

where P is the weight lattice of $\mathcal{W}$. Given $\Lambda \in \hat{\mathfrak{h}}^*$, the number $\ell := \Lambda(c)$ is called the level of Λ. Since $\alpha(c) = 0$ for every $\alpha \in \widehat{\Phi}^+$, it follows that $w\Lambda(c) = \Lambda(c)$ for all $\lambda \in \widehat{P}, w \in \widehat{\mathcal{W}}$. Hence, all elements of $\widehat{\mathcal{W}}\Lambda$ have the same level as Λ, for all $\Lambda \in \hat{\mathfrak{h}}^*$. It is well known that $\#(\widehat{\mathcal{W}}\Lambda \cap \widehat{P}^+) \leq 1$, for all $\Lambda \in \widehat{P}$. It will be useful do denote

$$\widehat{P}_+ = \{\Lambda \in \widehat{P} : \widehat{\mathcal{W}}\Lambda \cap \widehat{P}^+ \neq \varnothing\}. \tag{2.6.3}$$

If $\Lambda \in \widehat{P}$, there exist unique $\mu \in P$ and $d_\Lambda \in \mathbb{C}$ such that

$$\Lambda = \ell\Lambda_0 + \mu + d_\Lambda\delta.$$

Let $\pi : \widehat{P} \to P$ be given by $\pi(\Lambda_j) = \omega_j, \pi(\delta) = 0$. In other words,

$$\pi(\ell\Lambda_0 + \mu + d_\Lambda\delta) = \mu.$$

The *affine dominant weight lattice* is

$$\widehat{P}^+ = \{\lambda \in \widehat{P} : \lambda(h_i) \geq 0 \text{ for all } i \in I_{0,n}\} = \mathbb{Z}_{\geq 0}\Lambda_0 \oplus \mathbb{Z}_{\geq 0}\Lambda_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\Lambda_n \oplus \mathbb{C}\delta.$$

Hence,

$$\Lambda = \ell\Lambda_0 + \sum_{i \in I_{1,n}} m_i\omega_i + d_\Lambda\delta \in \widehat{P}^+ \quad \Leftrightarrow \quad m_i \geq 0, i \in I_{1,n}, \text{ and } \ell \geq \sum_{i \in I_{1,n}} m_i. \tag{2.6.4}$$

Given $\Lambda \in \widehat{P}^+$, let

$$\widehat{\mathcal{W}}^+_\Lambda = \{w \in \widehat{\mathcal{W}} \mid \pi(w(\Lambda)) \in P^+\} \quad \text{and} \quad \Gamma_\Lambda = \{w\Lambda \mid w \in \widehat{\mathcal{W}}^+_\Lambda\}. \tag{2.6.5}$$

Conjecture 2.6.1. If $\Lambda \in \widehat{P}^+$, then $\mathcal{E} \subseteq \widehat{\mathcal{W}}^+_\Lambda$.

Let $\Lambda \in \widehat{P}^+$. If the conjecture is true, then it implies

$$\Gamma_\Lambda = \mathcal{E}\Lambda. \tag{2.6.6}$$

Indeed, let $\Lambda = \ell\Lambda_0 + \lambda + d_\Lambda\delta$, with $\ell \in \mathbb{Z}_{\geq 0}, d_\Lambda \in \mathbb{C}, \lambda \in P^+$. Given $v \in \widehat{\mathcal{W}}^+_\Lambda$, write $v = wu$ for some $w \in \mathcal{W}$ and $u \in \mathcal{E}$ using (1.8.3). Since $w \in \mathcal{W}$, then $w\Lambda = \ell\Lambda_0 + w\lambda + d_\Lambda\delta$, following that $\pi(w\Lambda) \in P^+$ if and only if $w \in \mathcal{W}_\lambda := \{w' \in \mathcal{W} : w'\lambda = \lambda\}$. If the conjecture holds, $\mu = \pi(u\Lambda) \in P^+$. It follows that $\pi(v\Lambda) \in P^+$ if and only if $w \in \mathcal{W}_\mu$ and, hence, $v\Lambda = u\Lambda$. One of our goals for the next chapter is to prove Conjecture 2.6.1 and compute Γ_Λ when $n = 2$.

Chapter 3

Affine Weyl group of type $\widehat{A}_2$

In this last chapter, we restrict to the case $n = 2$. We first compute the alcove forms of the elements of $\mathcal{E}$ in Section 3.1 and, then, prove Conjecture 2.4.1. In Section 3.2, we use the alcove form to compute the permutation form of the elements of $\mathcal{E}$ and prove Conjecture 2.4.4. In Section 3.3, we prove Conjecture 2.6.1 and compute the sets Γ_Λ in (2.6.5). We end the work with some final remarks and an outline of how our conclusions may help in the solution of problems related to multiplicities in Damazure flags.

3.1 Alcoves of Type $\widehat{A}_2$

We now turn to the proof of Conjecture 2.4.1. Recall the elements defined in (2.4.2). We need to show

$$w_i^{x,y}u \in \mathcal{E} \quad \text{for all} \quad x,y \in \mathbb{Z}_{\geq 0},\ i \in I_{1,2},\ u \in \{e, s_0, s_0 s_i\}. \tag{3.1.1}$$

In light of Corollary 1.6.3 and Proposition 1.8.4, it suffices to show

$$k(w_i^{x,y}u, \alpha_q) \geq 0 \quad \text{for all} \quad q,i \in I_{1,2},\ x,y \in \mathbb{Z}_{\geq 0},\ u \in \{e, s_0, s_0 s_i\}, \tag{3.1.2}$$

but for further use, we will compute $k(w_i^{x,y}u, \alpha_q)$ also when $q = 0$.

The remainder of the section is dedicated to computing the alcove form of the elements $w_i^{x,y}u$ appearing in (3.1.2).

Lemma 3.1.1. Let $u \in \widehat{\mathcal{W}}$, $x, y \geq 0$, $\epsilon \in I_{0,1}$, $\{i, j\} = I_{1,2}$, $w = w_i^{x,2y+\epsilon}$. Then

$$k(wu, \alpha_q) = \begin{cases} (-1)^\epsilon k(u, \alpha_i) + x, & \text{if } q = i, \\ k(u, \alpha_{(1-\epsilon)j}) + x + \epsilon + 3y, & \text{if } q = j, \\ k(u, \alpha_{\epsilon j}) + 2(x+\epsilon) + 3y, & \text{if } q = 0. \end{cases}.$$

Proof. We begin by showing that

$$k(w_i^{x,0}u,\alpha_q) = k(u,\alpha) + (1+\delta_{q,0})x, \quad \text{for all } x \in \mathbb{Z}_{\geq 0}. \tag{3.1.3}$$

We proceed by induction on x, which clearly starts with $x = 0$. Moreover, (1.6.15) proves the case $x = 1$. Thus, assume $x > 1$. Using the induction hypothesis, we get

$$\begin{aligned} k((s_0w_o)^{x+1}u,\alpha_q) &= k(s_0w_o(s_0w_o)^x u,\alpha_q) \overset{(1.6.15)}{=} k((s_ow_o)^x u,\alpha_q) + 1 + \delta_{q,0} \\ &= k(w_o u,\alpha_q) + 1 + \delta_{q,0} + (1+\delta_{q,0})x \\ &= k(u,\alpha_q) + (1+\delta_{q,0})x + 1 + \delta_{q,0}, \end{aligned}$$

from where (3.1.3) follows.

Next, we prove

$$k(w_i^{0,2y}u,\alpha_q) = k(u,\alpha_q) + 3(1-\delta_{q,i})y \quad \text{for all} \quad y \in \mathbb{Z}_{\geq 0}, \tag{3.1.4}$$

which is clear for $y = 0$. Note

$$\begin{aligned} k(w_i^{0,2(y+1)}u,\alpha_q) &= k(s_0s_is_j(s_0s_is_j)^{2y+1}u,\alpha_q) \\ &\overset{(1.6.14)}{=} \begin{cases} -k((s_0s_is_j)^{2y+1}u,\alpha_i), & \text{if } q = i, \\ k((s_0s_is_j)^{2y+1}u,\alpha_r) + 1 + \delta_{q,0}, & \text{if } \{q,r\} = \{0,j\}. \end{cases} \end{aligned}$$

If $q = i$, another application of (1.6.14) and the induction hypothesis immediately imply (3.1.4). If $q \neq i$, since we also have $r \neq q$ above, (1.6.14) implies

$$k((s_0s_is_j)^{2y+1}u,\alpha_r) = k((s_0s_is_j)^{2y}u,\alpha_q) + 1 + \delta_{r,0}$$

and, therefore, since $\delta_{r,0} + \delta_{q,0} = 1$, we get

$$k(w_i^{0,2(y+1)}u,\alpha_q) = k(w_i^{0,2y}u,\alpha_q) + 3.$$

The induction hypothesis on r now implies (3.1.4). Applying (1.6.14) to (3.1.4), one easily checks that $k(w_i^{0,2y+1}u,\alpha_i) = -k(u,\alpha_i)$ and $k(w_i^{0,2y+1}u,\alpha_q) = k(u,\alpha_r) + 1 + \delta_{r,0} + 3y$, for $y \in \mathbb{Z}_{\geq 0}$, $\{q,r\} = I_{1,2}$.

Therefore, letting $w = w_i^{x,2y+\epsilon}$, we get

$$\begin{aligned} k(wu,\alpha_q) &\overset{(3.1.3)}{=} k(w_i^{0,2y+\epsilon}u,\alpha_q) + (1+\delta_{q,0})x \\ &= \begin{cases} (-1)^\epsilon k(u,\alpha_i), & \text{if } q = i, \\ k(u,\alpha_{(1-\epsilon)j}) + x + \epsilon + 3y, & \text{if } q = j, \\ k(u,\alpha_{\epsilon j}) + 2(x+\epsilon) + 3y, & \text{if } q = 0, \end{cases} \end{aligned}$$

from where the lemma follows. □

Claim (3.1.2) and, hence, (3.1.1), is immediate from the next proposition.

Proposition 3.1.2. Let $\{i,j\} = I_{1,2}$, $x,y \geq 0$, $\epsilon \in \{0,1\}$, and $u \in \{e, s_0, s_0 s_i\}$. Let also $\ell_r(u)$ be the number of appearances of s_r in the expression for u, $r \in \{0,i\}$. Then

$$k(w_i^{x,2y+\epsilon}u, \alpha_q) = \begin{cases} x, & \text{if } q = i, \\ x + 3y + \delta_{\epsilon,1}(\ell_0(u)+1) + \delta_{\epsilon,0}\ell_i(u), & \text{if } q = j, \\ 2x + 3y + \delta_{\epsilon,1}(\ell_i(u)+2) + \delta_{\epsilon,0}\ell_0(u), & \text{if } q = 0. \end{cases}$$

Proof. By (1.6.10) and (1.6.12), we have

$$k(s_0, \alpha_q) = \delta_{q,0}\ell_0(u) \quad \text{and} \quad k(s_0 s_i, \alpha_q) = \delta_{q,0}\ell_0(u) + \delta_{q,j}\ell_i(u).$$

This proves the proposition for $x = y = \epsilon = 0$. Plugging this back on Lemma 3.1.1, we are done. □

Let us use Proposition 3.1.2 to show

$$\ell(w_i^{x,y}u) = 4x + 3y + \ell(u) \quad \text{for all} \quad x,y \in \mathbb{Z}_{\geq 0},\ i \in I,\ u \in \{e, s_0, s_0 s_i\}. \tag{3.1.5}$$

One then easily sees that

$$\sum_{\alpha \in \Phi^+} |k(w_i^{x,2y+\epsilon}u, \alpha)| = \sum_{q \in I_{0,2}} k(w_i^{x,2y+\epsilon}u, \alpha_q) = 4x + 3(2y+\epsilon) + \ell_0(u) + \ell_i(u),$$

which is the right-hand side of (3.1.5). By Proposition 1.6.10, this is also the left-hand side.

In particular, $\{w_i^{x,y}u : i \in I_{1,n}, x,y \in \mathbb{Z}_{\geq 0}, u \in \{e, s_0 s_0 s_i\}\}$ is a minimal set of reduced expressions for the elements of $\widehat{\mathcal{W}}$. It induces then a minimal set of representatives of reduced paths for each $\mathcal{P}(\mathcal{A}_w)$, $w \in \widehat{\mathcal{W}}$.

3.2 Permutation Forms

Given $x,y \in \mathbb{Z}_{\geq 0}, \{i,j\} \in I_{1,2}, u \in \{e, s_0, s_0 s_i\}$, let

$$\sigma_{i,u}^{x,y} = \left(\underbrace{w_0, \ldots, w_0}_{x \text{ times}},\ \underbrace{s_i s_j, \ldots, s_i s_j}_{y \text{ times}},\ \underbrace{s_i^{\ell_i(u)}}_{\ell_0(u) \text{ times}} \right)$$

This is the element σ^μ defined in the paragraph preceding Conjecture 2.4.4, where $\mu(1,1) = x, \mu(\bar{i}, \bar{i}+1) = y$ with $\bar{i} \in \{0,1\}$, $\bar{i} \cong_2 i$, $\mu(v) = \delta_{w_v,u}$ if v is restricted vertex, and $\mu(v) = 0$ for the remaining vertices (cf. Example 2.4.2). Note $\sigma_{1,u}^{x,0} = \sigma_{2,u}^{x,0}$, for $x \in \mathbb{Z}_{\geq 0}$ $u \in \{e, s_0\}$, since they do not depend on i. It follows from (3.1.5) that

$$\sigma_{i,u}^{x,y} \in E_d(w_i^{x,y}u), \quad \text{where} \quad d = x + y + \ell_0(u). \tag{3.2.1}$$

We have just seen that the fll extract of $w_i^{x,y}u$ is of the form $\sigma_{i',u'}^{x',y'}$ for some $x',y' \in \mathbb{Z}_{\geq 0}, i' \in I, u' \in \{e, s_0, s_0s_{i'}\}$. Denote $w_{i,t}^{x,y} = w_i^{x,2y}u_{i,t}$, $v_{i,t}^{x,y} = (w_{i,t}^{x,y})^{-1}$, and

$$(u_{i,t})_{t=0}^5 = (e, s_0, s_0s_i, s_0s_is_j, s_0s_is_js_0, s_0s_is_js_0s_i). \tag{3.2.2}$$

Therefore, Conjecture 2.4.4 follows if we show that, for $(y,t) \notin \{(0,0),(0,1)\}$,

$$(x,y,i,t) \neq (x',y',i',t') \quad \Rightarrow \quad w_{i,t}^{x,y} \neq w_{i',t'}^{x',y'}. \tag{3.2.3}$$

This will be clear from the permutation forms of these elements which we compute in the remainder of this section.

From Theorem 2.1.2, it follows that

$$\begin{aligned} v_{i,t}^{x,y}(1) &= 1 - k(w_{i,t},\alpha_1) - k(w_{i,t},\alpha_0),\\ v_{i,t}^{x,y}(2) &= 2 + k(w_{i,t},\alpha_1) - k(w_{i,t},\alpha_2), \text{ and}\\ v_{i,t}^{x,y}(3) &= 3 + k(w_{i,t},\alpha_0) + k(w_{i,t},\alpha_2). \end{aligned} \tag{3.2.4}$$

Rephrasing Proposition 3.1.2 gives us

$$\begin{aligned} k(w_{i,t},\alpha_i) &= x,\\ k(w_{i,t},\alpha_j) &= x + 3y + \delta_{t,2} + \delta_{t,3} + 2(\delta_{t,4} + \delta_{t,5}), \text{ and}\\ k(w_{i,t},\alpha_0) &= 2x + 3y + \delta_{t,1} + \delta_{t,2} + 2(\delta_{t,3} + \delta_{t,4}) + 3\delta_{t,5}. \end{aligned} \tag{3.2.5}$$

Replacing (3.2.5) in (3.2.4) with $i = 1$, we get

$$\begin{aligned} v_{1,t}^{x,y}(1) &= -3(x+y) + \delta_{t,0} - \delta_{t,3} - \delta_{t,4} - 2\delta_{t,5},\\ v_{1,t}^{x,y}(2) &= -3y + 2(\delta_{t,0} + \delta_{t,1}) + \delta_{t,2} + \delta_{t,3}, \text{ and}\\ v_{1,t}^{x,y}(3) &= 3(x + 2y + 1) + t. \end{aligned}$$

Writing $v_{1,t}^{x,y}(s) = 3r_s + k_s$, $r_s \in \mathbb{Z}$, $\{k_1,k_2,k_3\} = I_{1,3}$, we obtain

$$(r_1,r_2,r_3) = \begin{cases} (-x-y, -y, x+2y), & \text{if } t = 0,\\ (-x-y-1, -y, x+2y+1), & \text{if } t \in I_{1,3},\\ (-x-y-1, -y-1, x+2y+2), & \text{if } t \in I_{4,5}, \end{cases}$$

and $(\bar{v}_{1,t})_{i=0}^5 = (e, (1\ \ 3), (1\ \ 3\ \ 2), (1\ \ 2), (1\ \ 2\ \ 3), (2\ \ 3))$, where $\bar{v}_{i,t}$ denotes the underlying finite permutation corresponding to $v_{i,t}^{x,y}$, which we see does not depend on x and y. Hence, by (2.1.3),

$$w_{1,0}^{x,y} = (x+y, y, -x-2y \mid e), \quad w_{1,1}^{x,y} = (-x-2y-1, y, x+y+1 \mid w_o),$$

$$w_{1,2}^{x,y} = (y, -x-2y-1, x+y+1 \mid s_1s_2), \quad w_{1,3}^{x,y} = (y, x+y+1, -x-2y-1 \mid s_1),$$

$$w_{1,4}^{x,y} = (-x-2y-2, x+y+1, y+1 \mid s_2 s_1), \quad w_{1,5}^{x,y} = (x+y+1, -x-2y-2, y+1 \mid s_2)$$

One can do similar computations with $i = 2$ to find

$$w_{2,0}^{x,y} = (x+2y, -y, -x-y \mid e), \quad w_{2,1}^{x,y} = (-x-y-1, -y, x+2y+1 \mid w_o),$$
$$w_{2,2}^{x,y} = (-x-y-1, x+2y+1, -y \mid s_2 s_1), \quad w_{2,3}^{x,y} = (x+2y+1, -x-y-1, -y \mid s_2),$$
$$w_{2,4}^{x,y} = (-y-1, -x-y-1, x+2y+2 \mid s_1 s_2), \quad w_{2,5}^{x,y} = (-y-1, x+2y+2, -x-y-1 \mid s_1).$$

Two distinct expressions $w_{i,t}^{x,y}$ and $w_{i',t'}^{x',y'}$ can only express the same element if the underlying finite permutations $\bar{v}_{i,t}$ and $\bar{v}_{i',t'}$ are the same. Equalities of the form $w_{i,t}^{x,y} = w_{i,t}^{x',y'}$, $i \in I_{1,2}$, $t \in I_{0,5}$, imply $x = x'$ and $y = y'$. Also, note $w_{1,t}^{x,y} = w_{2,t}^{x',y'}$ can only occur if $(y,t) \notin \{(0,0),(0,1)\}$, which is not the case to be considered. On the other hand, comparisons of the respective permutations of type $w_{1,t}^{x,y} = w_{2,t'}^{x',y'}$, with $\{t,t'\} = \{2,4\}$ or $\{t,t'\} = \{3,5\}$, always lead to the absurd $y = -1/2$. Therefore, (3.2.3) and, hence, Conjecture 2.4.4 follow.

3.3 Computation of Γ_Λ

The aim of this section is to compute the set Γ_Λ in (2.6.5). Using (2.6.2), one easily checks that

$$\begin{aligned} s_0(\Lambda_0) &= \Lambda_0 + \omega_1 + \omega_2 - \delta, \quad s_0(\omega_i) = -\omega_j + \delta, \quad s_0(\delta) = \delta \\ s_i(\Lambda_0) &= \Lambda_0 \quad s_i(\omega_j) = \omega_j, \quad s_i(\omega_i) = \omega_j - \omega_i, \quad s_i(\delta) = \delta, \end{aligned} \tag{3.3.1}$$

if $\{i,j\} = I_{1,2}$. Also, $\alpha_0 = \omega_1 + \omega_2$. Recall (3.2.2).

Lemma 3.3.1. Let $\Lambda = \ell\Lambda_0 + m_1\omega_1 + m_2\omega_2 + d\delta \in \widehat{P}$, $m_0 := m_1 + m_2$, and $\{i,j\} = I_{1,2}$. Then

(i) $w_i^{x,2y}\Lambda = \Lambda + x\ell\alpha_0 + 3y\ell\omega_j - ((x+y)(m_0+3y\ell) + x^2\ell + ym_j)\delta$, for any $x, y \geq 0$.

(ii) If $k \in I_{1,3}$, then $u_{i,k}\Lambda = \ell\Lambda_0 + (\ell - m_{p_k})\omega_i + (\ell + (-1)^{\delta_{p_k,j}} m_{q_k})\omega_j + (d - \ell + (-1)^{\delta_{p_k,i}} m_{r_k})\delta$, with $(p_k)_{k\in I_{1,3}} = (j,0,i)$, $(q_k)_{k\in I_{1,3}} = (i,i,0)$, $(r_k)_{k\in I_{1,3}} = (0,j,j)$.

(iii) If $k \in I_{4,5}$, then $u_{i,k}\Lambda = \ell\Lambda_0 + m_{p_k}\omega_i + (3\ell - m_{q_k})\omega_j + (d - 3\ell + m_{q_k} + (-1)^{\delta_{p_k,0}} m_{r_k})\delta$, with $(p_k)_{k\in I_{4,5}} = (j,0)$, $(q_k)_{k\in I_{4,5}} = (0,j)$, $(r_k)_{k\in I_{4,5}} = (i,i)$.

Proof. To prove part (i), we first proceed by induction on x to show

$$(s_0 w_o)^x \Lambda = \Lambda + x\ell\alpha_0 - x(m_0 + x\ell)\delta \tag{3.3.2}$$

which clearly starts with $x = 0$. Since $w_o\Lambda = \ell\Lambda_0 - m_2\omega_1 - m_1\omega_2 + d\delta$, it follows from (3.3.1) that

$$s_0 w_o \Lambda = \Lambda + \ell\alpha_0 - (m_0 + \ell)\delta. \tag{3.3.3}$$

Therefore, by the induction hypothesis, we have

$$\begin{aligned}(s_0w_o)^x\Lambda &= (s_0w_o)(s_0w_o)^{k-1}\Lambda = (s_0w_o)\left(\Lambda + (x-1)\ell\alpha_0 - (x-1)(m_0 + (x-1)\ell)\delta\right)\\ &\overset{(3.3.3)}{=} \Lambda + (x-1)\ell\alpha_0 - (x-1)(m_0+(x-1)\ell)\delta + (\ell\alpha_0 - (m_0 + 2(x-1)\ell + \ell)\delta)\\ &= \Lambda + x\ell\alpha_0 - (x(m_0) + \ell((x-1)^2 + 2x - 1))\delta = \Lambda + x\ell\alpha_0 - x(m_0 + x\ell)\delta,\end{aligned}$$

proving (3.3.2). Now we proceed by induction once again to prove

$$(s_0s_is_j)^{2y}\Lambda = \Lambda + 3y\ell\omega_j - y(m_i + 2m_j + 3y\ell)\delta, \tag{3.3.4}$$

for $y \geq 0$. The case $y = 0$ is immediate. Since (3.3.1) implies

$$s_i\Lambda = \ell\Lambda_0 - m_i\omega_i + m_0\omega_j + d\delta, \tag{3.3.5}$$

it follows from (3.3.3) that

$$s_0s_is_j\Lambda = s_0w_os_i\Lambda = \ell\Lambda_0 + (\ell - m_i)\omega_i + (\ell + m_0)\omega_j + (d - \ell - m_j)\delta. \tag{3.3.6}$$

Iterating once more we get

$$(s_0s_is_j)^2\Lambda = \ell\Lambda_0 + m_i\omega_i + (3\ell + m_j)\omega_j + (d - 3\ell - m_i - 2m_j)\delta, \tag{3.3.7}$$

which proves (3.3.4) with $y = 1$. A simple inductive argument proves (3.3.4). Therefore, applying (3.3.2) to (3.3.4), we have (i).

To prove part (ii), note (3.3.1) implies

$$u_{i,1}\Lambda = \ell\Lambda_0 + (\ell - m_2)\omega_1 + (\ell - m_1)\omega_2 - (\ell - m_0 - d)\delta, \tag{3.3.8}$$

while applying (3.3.5) to (3.3.8) gives us

$$u_{i,2}\Lambda = \ell\Lambda_0 + (\ell - m_0)\omega_i + (\ell + m_i)\omega_j - (\ell - m_j - d)\delta. \tag{3.3.9}$$

Thus, by (3.3.8), (3.3.9), and (3.3.6), it follows (ii).

Finally, to prove (iii), first apply (3.3.6) to (3.3.8) and obtain

$$u_{i,4} = \ell\Lambda_0 + m_j\omega_i + (3\ell - m_0) + (d - 3\ell + m_0 + m_i)\delta,$$

then apply (3.3.6) once again to (3.3.9) and get

$$u_{i,5} = \ell\Lambda_0 + m_0\omega_i + (3\ell - m_j)\omega_j + (d - 3\ell + m_j - m_i)\delta.$$

□

The following proposition answers Conjecture 2.6.1 in rank 2.

Proposition 3.3.2. If $\Lambda \in \widehat{P}^+$, then $\mathcal{E} \subseteq \widehat{\mathcal{W}}^+_\Lambda$. Moreover, $\Gamma_\Lambda = \mathcal{E}\Lambda$.

Proof. Using the notation of Lemma 3.3.1, $\Lambda \in \widehat{P}^+$ if and only if $\ell, m_1, m_2 \in \mathbb{Z}_{\geq 0}$ and $\ell \geq m_1 + m_2$. Fix $i \in I_{1,2}$ and let $u \in \{u_{i,k} : k \in I_{0,5}\}$. Parts (ii) and (iii) of Lemma 3.3.1 imply $\pi(u\Lambda) \in P^+$. On the other hand, if w is as in part (i) of the lemma and $\mu \in \widehat{P}$ satisfies $\pi(\mu) \in P^+$, then $\pi(w\mu) \in P^+$. In particular, this is true for $\mu = u\Lambda$. In light of (2.4.3), this completes the proof of the first claim. The second affirmation follows from (2.6.6). □

Finally, we present the computation of Γ_Λ.

Proposition 3.3.3. Let $\Lambda = \ell\Lambda_0 + m_1\omega_1 + m_2\omega_2 + d\delta \in \widehat{P}^+$. Denote $m_0 := m_1 + m_2$ and $\bar{\delta}_{k,l} := 1 - \delta_{k,l}$, $k, l \in \mathbb{Z}_{\geq 0}$. Then $\lambda \in \Gamma_\Lambda$ if and only if

$$\lambda = \ell f_i(x,y) + m_p\omega_i + (3\ell\bar{\delta}_{p,i} + (-1)^{\bar{\delta}_{p,i}} m_q)\omega_j + g_{p,q}(\Lambda, x, y)\delta, \tag{3.3.10}$$

or

$$\lambda = \ell f_i(x,y) + (\ell - m_p)\omega_i + (\ell + (-1)^{\delta_{p,j}} m_q)\omega_j + h_{p,q}(\Lambda, x, y)\delta, \tag{3.3.11}$$

for some $x, y \in \mathbb{Z}_{\geq 0}$, $\{i, j\} = I_{1,2}$, and $\{p, q, r\} = I_{0,2}$, where

$$f_i(x,y) = \Lambda_0 + x\omega_i + (x + 3y)\omega_j - (x^2 + 3xy + 3y^2)\delta,$$

$$g_{p,q}(\Lambda, x, y) = d - 3\ell\bar{\delta}_{p,i}(x + 2y + 1) - m_p(x + y) + (-1)^{\delta_{p,i}} m_q(x + 2y + \bar{\delta}_{p,i}) + (-1)^{\delta_{p,0}}\bar{\delta}_{p,i} m_r,$$

if $(p, q, r) \in \{(i, j, 0), (0, j, i), (j, 0, i)\}$, and

$$h_{p,q}(\Lambda, x, y) = d - \ell(2x + 3y + 1) + m_p(x + y) + (-1)^{\bar{\delta}_{p,j}} m_q(x + 2y) + (-1)^{\delta_{p,i}} m_r$$

if $(p, q, r) \in \{(j, i, 0), (0, i, j), (i, 0, j)\}$.

Proof. By Proposition 3.3.2, $\lambda \in \Gamma_\Lambda$ if and only if there exists $w = w_i^{w,2y} u_{i,k}$ such that $\lambda = w\Lambda$, for some $\{i, j\} = I_{1,2}$, $x, y \geq 0$, and $k \in I_{0,5}$. If $k = 0$, part (i) of Lemma 3.3.1 ensures λ is as in (3.3.10), whereas applying (i) to (iii) of the lemma gives (3.3.10) for $k \in I_{4,5}$. Otherwise, if $k \in I_{1,3}$, applying part (i) of the lemma to (ii) results (3.3.11). □

3.4 Final Remarks and Further Steps

In this last section, we first describe briefly how our results and approaches relate with multiplicities in Damazare flags. Then, we point out some further steps and discuss some challenges in increasing the rank and changing the type of the group.

Given $\Lambda \in \widehat{P}^+$, we fix and denote by $V(\Lambda)$ an integrable irreducible $\hat{\mathfrak{g}}$-module of highest weight Λ. For $\theta \in \widehat{P}_+$ (recall (2.6.3)), the *Demazure module* $D(\theta)$ is the $\hat{\mathfrak{b}}$-submodule of $V(\Lambda)$ generated by $V(\Lambda)_\theta$, where Λ is the unique element of $\widehat{\mathcal{W}}\theta \cap \widehat{P}^+$ and $V(\Lambda)_\mu$ denotes the weight space

$$V(\Lambda)_\mu = \{v \in V(\Lambda) : hv = \mu(h)v \text{ for all } h \in \hat{\}, \quad \text{for all } \mu \in \hat{\mathfrak{h}}^*.$$

It turns out that $D(\theta)$ is a $\mathfrak{g}$-submodule of $V(\Lambda)$ and, hence, also a $\mathfrak{g}[t]$-submodule, if, and only if, $\pi(\theta) \in -P^+$ or, equivalently,

$$\theta = w_o w\Lambda \quad \text{for some} \quad w \in \widehat{\mathcal{W}}_\Lambda^+.$$

As reviewed in (JAKELIĆ; MOURA, 2018, Section 2.3), we have

$$V(\Lambda) = \bigcup_{w \in \widehat{\mathcal{W}}_\Lambda^+} D(w_o w\Lambda). \tag{3.4.1}$$

Set

$$\Gamma_\Lambda^- = w_o\Gamma_\Lambda, \ \Lambda \in \widehat{P}^+ \quad \text{and} \quad \Gamma = \bigcup_{\Lambda \in \widehat{P}^+} \Gamma_\Lambda^- \subseteq \widehat{P}_+.$$

A $\mathfrak{g}[t]$-module V admits a Demazure flag if there exist $l > 0, \lambda_j \in S, j = 1, \ldots, l,$ and a sequence of inclusions

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{l-1} \subset V_l = V \quad \text{with} \quad V_j/V_{j-1} \cong D(\lambda_j) \ \forall \ 1 \le j \le l. \tag{3.4.2}$$

If $\lambda_j(c) = \ell$ for some ℓ and all j, such a sequence is said to be a *level-ℓ Demazure flag* for V. Let $\mathbb{V}$ be a Demazure flag of V as in (3.4.2) and, for a Demazure module D, define the multiplicity of D in $\mathbb{V}$ by

$$[\mathbb{V} : D] = \#\{1 \le j \le l : V_j/V_{j-1} \cong D\}.$$

As observed in (CHARI et al., 2014, Lemma 2.1), the multiplicity does not depend on the choice of the flag and, hence, by abuse of language, we shift the notation from $[\mathbb{V} : D]$ to $[V : D]$.

Given a $\hat{\mathfrak{g}}$-module V such that

$$V \cong \bigoplus_{\theta \in \widehat{P}^+} V(\theta)^{\oplus m_\theta} \quad \text{for some} \quad m_\theta \in \mathbb{Z}_{\ge 0},$$

we set

$$[V : V(\theta)] = m_\theta.$$

The following is the main result of (JAKELIĆ; MOURA, 2018).

Theorem 3.4.1. Suppose $\mathfrak{g}$ is simply laced and let $V = V(\Lambda_0) \otimes V(\Lambda)$ for some $\Lambda \in \widehat{P}^+$. Then,

$$[V : V(\theta)] = \sum_{\Psi \in \Gamma_\theta^-} \max_{\Upsilon \in \Gamma_\Lambda^-} [D(\Upsilon) : D(\Psi)] \quad \text{for all} \quad \theta \in \widehat{P}^+, \theta(c) = \Lambda(c) + 1.$$

In the particular case that $\mathfrak{g} = \mathfrak{sl}_2$ and $\Lambda(c) = 1$, the right-hand side of the above formula was expressed in terms of partitions with bounded parts in (JAKELIĆ; MOURA, 2018, Proposition 2.6.3). One important ingredient for doing this was the explicit description of the sets $\widehat{W}_\Lambda^+$ for $\Lambda \in \widehat{P}^+$ such that $\Lambda(c) \le 2$. Thus, the computations of Section 3.3 provide the necessary generalization for type $\widehat{A}_2$. The other relevant ingredient was the results of (CHARI et al., 2014) relating multiplicities in level-2 flags for level-1 Demazure modules to Gaussian binomials. For type $\widehat{A}_2$, a similar description is given in (WAND, 2015, Theorem 16), which may be helpful to extend the formulae involving partitions. The results of (WAND, 2015) were deeply expanded in (BISWAL et al., 2021).

The extracts graphs of type $\widehat{A}_n$ provide families of formulae for all elements of an affine Weyl group $\widehat{W}$ of type $\widehat{A}_n$. Thus, it is possible in theory to compute Γ_Λ for higher ranks, although quite impractical at the moment. Therefore, in our next steps, we intend to look for properties of extract graphs which might lead to some less explicit computations which still contains sufficient information in order to to reach nice formulae as in (JAKELIĆ; MOURA, 2018, Proposition 2.6.3).

Although the inclusion (2.4.1) gives a smaller set of expressions when compared with the whole set of expressions for elements of $\widehat{W}$, beyond type $\widehat{A}_2$, the methods we have used so far did not lead us to a proof that this is a minimal set. Apparently, (AL HARBAT, 2021) have proved this minimality for a slightly different set, but, unfortunately, we came across (AL HARBAT, 2021) too close to the deadline for submitting this dissertation and, hence, we did not have time to study it properly in order to include it here. This study is certainly one of the first things we will do in the next steps of our project. In particular, although (AL HARBAT, 2021) is focused on type $\widehat{A}$ only, we expect it should provide further intuition to generalize these constructions for other types.

Throughout this work we have intended to bring as many approaches for the problem as we could. The identifications with permutation forms, alcove forms, and alcove paths made possible to establish an equivalence of working with certain fll extracts, left and right sequences, elements corresponding to alcoves in the fundamental chamber, and reduced paths contained in the fundamental chamber. The connections as stated in this work have not lead to solutions for higher ranks and different types yet, but point towards certain approaches which might be worth exploring.

Bibliography

AL HARBAT, S. Canonical reduced expression for elements of affine Coxeter groups part i – type $\tilde{A}_n$. *arXiv:2105.07417v2*, 2021. Cited 6 times on pages 14, 42, 45, 52, 54, and 70.

BIDIGARE, P.; HANLON, P.; ROCKMORE, D. A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements. *Duke Mathematical Journal*, v. 99, n. 1, 1999. Cited on page 12.

BISWAL, R.; CHARI, V.; KUS, D. Demazure flags, q-Fibonacci polynomials and hypergeometric series. *Research in the Mathematical Sciences*, v. 5, n. 12, 2018. Cited on page 10.

BISWAL, R.; CHARI, V.; SCHNEIDER, L.; VISWANATHA, S. Demazure flags, Chebyshev polynomials, partial and mock theta functions. *Journal of Combinatorial Theory*, v. 140, p. 38–75, 2016. Cited on page 10.

BISWAL, R.; CHARI, V.; SHEREEN, P.; WAND, J. Macdonald polynomials and level two Demazure modules for affine $\mathfrak{sl}_{n+1}$. *Journal of Algebra*, v. 575, p. 159–191, 2021. Cited 2 times on pages 10 and 70.

BISWAL, R.; KUS, D. A combinatorial formula for graded multiplicities in excellent filtrations. *Transformation groups*, v. 26, p. 81–114, 2021. Cited on page 10.

BJÖRNER, A.; BRENTI, F. Affine permutations. *The Electronic Journal of Combinatorics*, v. 3, n. 2, p. R18, 1996. Cited 2 times on pages 42 and 43.

_____. *Combinatorics of Coxeter groups*. 1. ed. New York: Springer, 2005. (Graduate Texts in Mathematics 231). Cited 3 times on pages 15, 17, and 18.

BROWN, K. S.; DIACONIS, P. Random walks and hyperplane arrangements. *The Annals of Probability*, v. 26, n. 4, p. 1813–1854, 1998. Cited on page 12.

BURGER, T. *Characterization of certain elements in the orbits of the affine Weyl group action on the affine weight lattice for type $A_2^{(1)}$ via alcove walks*. M.Sc. Thesis — University of North Carolina Wilmington, 2017. Cited 2 times on pages 11 and 13.

CHARI, V.; SCHNEIDER, L.; SHEREEN, P.; WAND, J. Modules with Demazure flags and character formulae. *SIGMA*, v. 10, n. 32, p. 1–16, 2014. Cited 4 times on pages 10, 11, 69, and 70.

DEFOSSEUX, M. Fusion coefficients and random walks in alcoves. *Annales de l'Institut Henri Poincaré*, v. 52, n. 4, p. 1515–1534, 2016. Cited on page 12.

FEINGOLD, A. J. Tensor products of certain modules for the generalized Cartan matrix Lie algebra $A_1^{(1)}$. *Communications in Algebra*, v. 9, p. 1323–1341, 1981. Cited on page 10.

GAUSSENT, S.; LITTELMAN, P. LS-galleries, the path model and MV-cycles. *Duke Mathematical Journal*, v. 127, p. 35–88, 2005. Cited on page 12.

GESSEL, I. M.; ZEILBERGER, D. Random walk in a Weyl chamber. *Proceedings of the American Mathematical Society*, v. 115, n. 1, 1992. Cited on page 11.

GRABINER, D. J.; MAGYAR, P. Random walks in Weyl chambers and the decomposition of tensor powers. *Journal of Algebraic Combinatorics*, v. 2, p. 239–260, 1993. Cited on page 12.

HUMPHREYS, J. *Reflection groups and Coxeter groups*. 1. ed. New York: Cambridge University Press, 1990. (Cambridge Studies in Advanced Mathematics 29). Cited 9 times on pages 15, 16, 17, 18, 20, 21, 26, 28, and 34.

JAKELIĆ, D.; MOURA, A. Limits of multiplicities in excellent filtrations and tensor product decompositions for affine Kac-Moody algebras. *Algebras and Representation Theory*, v. 21, p. 239–258, 2018. Cited 6 times on pages 9, 10, 11, 14, 69, and 70.

JOSEPH, A. A decomposition theorem for Demazure crystals. *Journal of Algebra*, v. 265, n. 2, p. 562–578, 2003. Cited on page 10.

_____. Modules with a Demazure flag. In: _____. *Studies in Lie Theory*. Boston: Birkhäuser, 2006. (Progress in Mathematics), p. 131–169. Cited on page 10.

KAC, V. *Infinite dimensional Lie algebras*. 3. ed. Cambridge: Cambridge University Press, 1990. Cited 3 times on pages 15, 23, and 42.

KING, R.; WELSH, T. Tensor products for affine Kac-Moody algebras. *Group theoretical methods in physics*, v. 382, p. 508–511, 1991. Cited on page 10.

LAM, T. The shape of a random affine weyl group element and random core partitions. *The Annals of Probability*, v. 43, n. 4, p. 1643–1662, 2015. Cited on page 12.

LECOUVEY, C.; TARRAGO, P. Alcove random walks, k-Schur functions and the minimal boundary of the k-bounded partition poset. *hal-01691407*, v. 2, 2020. Cited on page 12.

LENART, C. Hall-Littlewood polynomials, alcove walks, and fillings of Young diagrams. *Discrete Mathematics*, v. 311, n. 4, p. 258–275, 2011. Cited 2 times on pages 12 and 13.

LENART, C.; POSTNIKOV, A. Affine Weyl groups in K-theory and representation theory. *Internacional Mathematics Research Notices*, v. 2007, n. 38, p. 1–65, 2007. Cited 3 times on pages 12, 15, and 34.

LEPOWSKY, J.; WILSON, R. A Lie theoretic interpretation and proof of the Rogers-Ramanujan identities. *Advances in Mathematics*, v. 45, p. 21–72, 1982. Cited on page 10.

LITTELMAN, P. Paths and root operators in representation theory. *Annals of Mathematics*, v. 142, p. 499–525, 1995. Cited on page 12.

LITTELMANN, P. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Inventiones mathematicae*, v. 116, p. 329–346, 1994. Cited 2 times on pages 10 and 12.

LUSZTIG, G. Some examples of square integrable representations of semisimple p-adic groups. *Transactions of the American Mathematical Society*, v. 277, n. 2, 1983. Cited on page 11.

MISRA, K.; WILSON, E. On tensor product decomposition of $\hat{\mathfrak{sl}}(n)$-modules. *Journal of Algebra and Its Applications*, v. 12, 2013. Cited on page 10.

_____. Tensor product decomposition of $\hat{\mathfrak{sl}}(n)$-modules and identities. *Contemporary Mathematics*, v. 627, p. 131–144, 2014. Cited on page 10.

NAOI, K. Weyl modules, Demazure modules and finite crystals for non-simply laced type. *Advances in Mathematics*, v. 229, n. 2, p. 875–934, 2012. Cited on page 10.

OKADO, M.; SCHILLING, A.; SHIMOZONO, M. A tensor product theorem related to perfect crystals. *Journal of Algebra*, v. 267, p. 212–245, 2003. Cited on page 10.

PARKINSON, J.; RAM, A. Alcove walks, buildings, symmetric functions and representations. *arXiv:0807.3602*, 2008. Cited on page 12.

RAM, A. Alcove walks, Hecke algebras, spherical functions and column strict tableaux. *Pure and Applied Mathematics Quarterly*, v. 2, n. 4, p. 135–183, 2006. Cited on page 12.

SCHWER, C. Galleries, Hall-Littlewood polynomials and structure constants of the spherical Hecke algebra. *International Mathematics Research Notices*, v. 2006, n. 75935, 2006. Cited on page 12.

SHI, J. *The Kazhdan-Lusztig cells in certain affine Weyl groups*. Berlin: Springer, 1980. (Lecture Notes in Mathematics). Cited 2 times on pages 11 and 31.

_____. Alcoves corresponding to an affine Weyl group. *Journal of the London Mathematical Society*, v. 2-35, n. 1, p. 42–55, 1987. Cited 6 times on pages 11, 15, 27, 30, 31, and 33.

_____. On two presentations of the affine Weyl groups of classical types. *Journal of Algebra*, v. 221, n. 1, p. 360–383, 1999. Cited 7 times on pages 11, 15, 27, 30, 31, 42, and 43.

STEMBRIDGE, J. Some combinatorial aspects of reduced words in finite Coxeter groups. *Transactions of the American Mathematical Society*, v. 349, n. 4, p. 1285–1332, 1997. Cited 3 times on pages 15, 19, and 42.

WAND, J. O. *Demazure Flags of Local Weyl Modules*. Ph.D. Dissertation — University of California Riverside, 2015. Cited on page 70.