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A class of singular elliptic equations

Uma classe de equações elípticas singulares

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Supervisor: Marcelo da Silva Montenegro

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This thesis is a consequence of a choice I made ten years ago. It is the solid product of my time spent working, making wrong decisions and being without awareness of what I was actually doing. This work took its toll. I hope to someday realize it was worth it, and I also hope to make correct choices again, but this time as a fully aware adult.

Resumo

Nessa tese estudamos problemas elípticos com diferentes tipos de singularidades e termos não lineares. Consideramos problemas definidos em regiões limitadas contidas em espaços de dimensão finita. Quando essa região estiver contida no plano, admitimos termos com crescimento exponencial. Ao longo do trabalho assumimos que as singularidades possuem crescimento polinomial ou logarítmico perto da origem. Em regiões de dimensão maior, estudamos dois problemas singulares. O primeiro é um problema que envolve um termo não linear de crescimento polinomial subcrítico. Resolvemos esse problema aproximando o termo singular de maneira adequada. O segundo admite pesos que são singulares perto da fronteira do domínio no qual o problema está definido. Para resolvê-lo, obtemos uma subsolução que é estritamente positiva no interior dessa região. Os problemas discutidos nesse trabalho têm uma ampla gama de aplicações. Eles modelam fenômenos de catálise heterogênea e processos enzimáticos. Também existem aplicações em mecânica dos fluidos e fluxos pseudoplásticos. Problemas desse tipo também estão relacionados com equações de Schrödinger e de Klein-Gordon.

Palavras-chave: equações elípticas singulares; cálculo das variações; existência de solução; operador laplaciano; crescimento crítico.

Abstract

In this thesis we study elliptic problems with different types of singularities and nonlinearities. We consider problems defined in bounded regions contained in finite dimensional spaces. When this region is contained in the plane, we admit terms with exponential growth. Throughout this work we assume that the singularities have either polynomial or logarithmic growth near the origin. In regions contained in higher dimensions, we study two singular problems. The first one involves a nonlinearity with subcritical polynomial growth. We solve it by considering a suitable approximation of the singular term. The second one admits weights that are allowed to be singular near the boundary of the domain in which the problem is defined. To solve it, we obtain a subsolution that is strictly positive in the interior of the region. The problems discussed in this work have a wide range of applications. They model problems in heterogeneous catalysis and in enzymatic processes. There are also applications in fluid mechanics and pseudoplastic flows. These problems are also related to Schrödinger and Klein-Gordon equations.

Keywords: singular elliptic equations; calculus of variations; existence of solution; laplacian operator; critical growth.

Contents

Introduction

In this work we study different classes of elliptic equations. We are concerned with the existence of solutions of certain problems that admit a term that is unbounded near the origin. This term generates difficulties which are surpassed by using a perturbation argument. We study two different types of problems. The first one is defined in a bounded bidimensional region, and the second one in a bounded region of higher dimension. On one hand, problems of the first type possess a further term that is allowed to have exponential growth. On the other hand, the solutions of these problems are not strictly positive.

Problems of the second type do not possess elements with exponential growth. However, in some situations they have terms that are singular near the boundary of the region in which these problems are set. Furthermore, in some cases, we obtain positive solutions.

When we tackle problems of the first and second type, we use an approximation scheme. For problems of the first type and some problems of the second type, we approach the singular term by a sequence of smooth functions, thus creating a family of perturbed problems. By doing this, we cancel the effects of the singularity near the origin. For some problems of the second type, we use a different form of approximation. First, we obtain a subsolution that is positive inside the region in which the problem is defined. Next, instead of perturbing the singular term (as done in problems of first type), we consider a sequence of subdomains that are contained in the interior of the original region in which the problem is defined. By doing this, we cancel out the effects of the terms that are singular near the boundary of this region. Also, the fact that the subsolution is strictly positive in these subdomains is useful to avoid the singularities near the origin.

For both types of problems, we split our ideas in two steps.

Step 1: We obtain solutions for a smooth perturbation of the original singular problem (either by perturbing the singular term or by perturbing the domain)

Step 2: We show that the solutions obtained in Step 1 converge to a solution of the original singular problem.

The main ingredient in Step 1 are results of calculus of variations. Indeed, we obtain solutions that are mountain passes and local minima of certain perturbed functionals. Furthermore, these solutions are nonnegative and enjoy suitable regularity properties, which are very helpful when studying convergence in Step 2.

There is a vast literature on nonsingular problems. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain. The problem

$$\begin{cases} -\Delta u = f(u) & \text{in}\, \Omega \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{1}$$

has been extensively studied for continuous functions f, see [4], [11] [21], [30], [36], [43] and [45]. When $N > 2$, the Sobolev Embedding implies that $u^p \in L^1(\Omega)$ for all $1 \leq p \leq \dfrac{2N}{N-2}$. Consequently, initially it makes sense to look for solutions of problem (1) when f is continuous and satisfies

$$|f(s)| \leq a_1 + a_2|s|^p, \text{ for all } s \in \mathbb{R},$$

where $a_1, a_2 > 0$ are positive constants and $0 < p \leq \dfrac{2N}{N-2} - 1$, the case $p = \dfrac{2N}{N-2} - 1$ being *critical* and the case $0 < p < \dfrac{2N}{N-2} - 1$ being *subcritical*. In the subcritical context, problem (1) is solvable, provided f is continuous, see [21]. The critical case is more delicate. The Pohozaev inequality, see [36], implies that the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in}\, \Omega \\ u > 0 & \text{in}\ \Omega \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{2}$$

is not solvable in Ω. Nonetheless, in [4], the authors showed that there exists $\Lambda_1 > 0$ such that the problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in}\, \Omega \\ u > 0 & \text{in}\ \Omega \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{3}$$

with $0 < q < 1 < p \leq \dfrac{N+2}{N-2}$ has at least two solutions provided $0 < \lambda < \Lambda_1$.

When $N = 2$, the Sobolev imbedding implies that for each $u \in H_0^1(\Omega)$, the function u^p belongs to $L^1(\Omega)$ for all $0 < p < \infty$, so that problem (1) is solvable when f has polynomial growth. However, a stronger result holds. The Trudinger-Moser inequality, see [56], asserts that if $\Omega \subset \mathbb{R}^2$ is a bounded domain, then $e^{\alpha u^2}$ belongs to $L^1(\Omega)$ for all $\alpha > 0$ and $u \in H_0^1(\Omega)$, so that we may study problem (1) with f satisfying the following condition: There exists $\alpha > 0$ such that

$$|f(s)| \leq C \exp(\alpha s^2) \text{ for all } s \in \mathbb{R}. \tag{4}$$

For example, in [29], the authors studied the problem

$$\begin{cases} -\Delta u = h(u)e^{\alpha u^2} & \text{in}\, \Omega \\ u > 0 & \text{in}\ \Omega \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{5}$$

where h is a continuous function that satisfies certain conditions and $h(0) = 0$. In [27], the authors the authors obtained two solutions for the problem

$$\begin{cases} -\Delta u = \lambda u^q + e^{\alpha u^2} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

provided λ and α are sufficiently small and $0 < q < 1$. Observe that the right hand side of (6) does not vanish for $u = 0$, but the right hand side of (5) does. See [1], [2], [3], [7], [15] [30] and [31] for related results.

In this work, we study problems of the form

$$\begin{cases} -\Delta u = g(x, u)\chi_{\{u>0\}} + f(x, u) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded smooth domain and g is singular at the origin with

$$\lim_{s \to 0^+} g(x, s) = -\infty \text{ for all } x \in \Omega.$$

Problem (7) is a singular version of (2)–(6). The singularity g at the right hand side of (7) prevents us from solving this problem directly. To overcome this difficulty, we consider smooth perturbations which are treatable in a similar way as problem (1). Singular problems arise in several physical models such as fluid mechanics and pseudoplastic flows, see [12], [18], [19], [41] and [57]. Problem (7) with $g(x, s) = \log s$ is associated to some phase field models, see [20], [23], [35] and [44]. See also [61] where a reaction diffusion equation with logarithmic singularity is studied. These problems are also related to Schrödinger and Klein-Gordon equations, see [16], [24], [48], [64] and [67].

Solutions of problem (7) are not strictly positive in general, so that a free boundary might arise. For example, in [53] the authors showed that problem (7) with $g(x, s) = \log s$ and $f(x, s) = \lambda s^p$ for $x \in \Omega$ and $s \geq 0$ possesses a nontrivial solution $u_\lambda \geq 0$ that vanishes in a set of positive measure, provided λ is sufficiently small. Equation (7) is related to problems in heterogenous catalysis, see [6] and [33]. In this context, the regions of Ω in which $u = 0$ are the regions of the catalyst pellet in which no reaction takes place. See [32] for results on the free boundary of a related problem with logarithmic singularity. The free boundary of singular elliptic equations was also studied in [40] and [60].

Elliptic problems with singularity of the form $u^{-\beta}$ have been studied in the last decades. In [34] the authors considered the problem $-\Delta u = -u^{-\beta} + \lambda f(x)$ in Ω, $u = 0$ in $\partial\Omega$, with $f \geq 0$, $f \in L^1(\Omega)$. The sub-supersolution method was used and positive solutions were obtained when $0 < \beta < 1$ for large values of λ. Multiplicity of solutions was discussed in [17] with the assumption that $f \equiv 1$. There, the authors showed that the unidimensional problem

$$\begin{cases} -u'' = -u^{-\beta} + \lambda \text{ in } (-L, L) \\ u > 0 \text{ in } (-L, L) \\ u(-L) = u(L) = 0, \end{cases} \tag{8}$$

has at most one classical solution if $\beta = 1/2$. However, when $0 < \beta < 1/3$, two distinct solutions were obtained.

The problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{9}$$

was studied in [25], [26], [54], [62] and [70]. In [70], the authors showed that (9) does not admit a classical solution for $\beta \geq 1$. In [25] the authors obtained one positive solution of problem (9) with $0 < p < 1$ for large values of λ. This result was extended in [54], where the authors obtained two nonnegative solutions of problem (9) for large values of λ. In [26] the authors studied (9) with $p \geq 1$. When $p > 1$, they obtained one solution for each $\lambda > 0$. When $p = 1$, they showed that problem (9) is solvable for $\lambda > \lambda_1$. The more general equation $-\Delta u + K(x)u^{-\beta} = \lambda u^p$ with $0 < p < 1$ and zero boundary condition was studied in [62], where K is assumed to be of class $C^{2,\alpha}(\overline{\Omega})$.

The problem

$$\begin{cases} -\Delta u = K(x)u^{-\beta} \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{10}$$

was studied in [50], [69] and [71]. In [50], the authors showed that this problem has a positive solution provided K is a continuous function such that $K > 0$ in $\overline{\Omega}$. There, the authors showed that this equation is solvable in $H_0^1(\Omega)$ if and only if $\beta < 3$. This result was refined in [71] for $K(x) = \delta(x)^\gamma$ with $\delta(x) = dist(x, \partial\Omega)$. Finally, in [69], the authors showed that problem (10) is solvable in $H_0^1(\Omega)$ if and only if there exists $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega K(x)|u_0|^{1-\beta}\, dx < \infty.$$

Versions of the problem

$$\begin{cases} -\Delta u = u^{-\beta} + \lambda u^p \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{11}$$

were considered in [13], [14], [47] and [66]. In [13] the authors considered a version of problem (11) for $p = 1$. In [14] the authors established results for (a problem more general than) (11) with $\beta \geq 3$. In [47] the authors considered $0 < \beta \leq 1$, $1 < p \leq \dfrac{N+2}{N-2}$ and they used variational methods to study the equation $-\Delta u = \lambda u^{-\beta} + u^p$ with zero boundary condition. They showed that this problem possesses at least two distinct positive solutions provided λ is sufficiently small. This result was generalized in [66] for the equation $-\Delta u = K(x)u^{-\beta} + \lambda u^p$ where $K \geq 0$ is nontrivial and $K \in L^2(\Omega)$. See also [8], [22], [46], [58] and [59] where more general elliptic and quasilinear singular problems were discussed.

In Chapter 2, we study the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu f(u) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $0 < p < \infty$, $\lambda \geq 0$, $\mu > 0$, $f : [0, \infty) \to \mathbb{R}$ is a continuous and differentiable function with $f(0) = 0$ and $0 < \beta < 1$. By a solution of problem (12) we mean a function $u \in H_0^1(\Omega)$ such that

$$u^{-\beta}\chi_{\{u>0\}} \in L^1_{loc}(\Omega)$$

and

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} \Big(-u^{-\beta} + \lambda u^p + \mu f(u) \Big)\varphi,$$

for every $\varphi \in C_c^1(\Omega)$. Our main contribution is that we allow the nonlinearity f to have exponential growth at infinity. For example, for each $\lambda \geq 0$ we prove that there exists $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu(e^u - 1) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

has a solution for all $0 < \mu < \mu_0$. We should compare equation (13) with (5) and (6). Equations (5) and (6) have positive solutions, whereas the solutions of problem (13) are not shown to be strictly positive, and might give rise to a free boundary $\partial\{u > 0\}$. The method we use relies on the Ambrosetti-Rabinowitz condition, so that when $p > 1$, we may consider $f \equiv 0$ or $f(s) = s^q$ for $q > 0$ in (12). However, when $0 < p \leq 1$ in (12), the function $s \to s^p$ ceases to satisfy this condition. As a consequence, we have to impose stricter assumptions on f, so that we can consider $f(s) = s^r$ only for $r > 3$. Summarizing, when $p > 1$ we show that for each $\lambda > 0$ fixed, there exists $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^q & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution for all $0 < \mu < \mu_0$. An analogous statement holds for the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^r & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < p \leq 1$. Note also that problem (9) is a particular case of (12) with $f = 0$.

In Chapter 3, we assume that Ω is a bounded smooth domain in $\mathbb{R}^N$ for $N \geq 3$ and we consider the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u + u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{14}$$

with $1 < p < \dfrac{N+2}{N-2}$. We obtain solutions of (14) for each $\lambda \geq 0$, and consequently the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

is solvable for $1 < p < \dfrac{N+2}{N-2}$. This result is already known, see [26]. Thus, our contribution is when $\lambda > 0$.

In Chapter 4, we consider problem (12) with the logarithmic function $\log(s)$ replacing the term $-u^{-\beta}$ and we assume again that $f(0) = 0$ and Ω is a smooth bounded domain in $\mathbb{R}^2$. We allow f to have exponential growth, and this is our main contribution. For example, for each $\lambda \geq 0$ we prove that there exists $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p + \mu(e^u - 1) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

with $1 < p < \infty$ has a solution for all $0 < \mu < \mu_0$. We may consider $\lambda = 0$, but we do not have results when $0 < p < 1$ in (15). We are also able to consider $f = 0$ in (15), so that we generalize and improve the result of [52], where the authors showed that the problem

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

with $p > 1$ has a solution for sufficiently large λ. In this work, we obtain solutions of (16) for all $\lambda > 0$, provided $p > 1$.

Some results of this chapter were published in [38].

In Chapter 5, we extend the results of Chapter 4. Indeed, we consider problems of the form

$$\begin{cases} -\Delta u = (\log u + f(u))\chi_{\{u>0\}} & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and f is a function that must be superlinear. By a solution of problem (17), we mean a function $u \in H_0^1(\Omega)$ such that

$$(\log u)\chi_{\{u>0\}} \in L^1_{loc}(\Omega)$$

and

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega\cap\{u>0\}} (\log u + f(u))\varphi,$$

for every $\varphi \in C_c^1(\Omega)$. We remark that f satisfies:

- For each $\alpha > 0$ there exists $C = C_\alpha > 0$ such that (4) holds;
- We no longer assume that $f(0) = 0$;
- We allow f to change sign.

For example, we obtain a solution for the problem

$$\begin{cases} -\Delta u = (\log u + e^u + \lambda)\chi_{\{u>0\}} & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for all $\lambda \in \mathbb{R}$. We may also consider $f(u) = u^p + \lambda$ for $p > 1$, so that the problem

$$\begin{cases} -\Delta u = (\log u + u^p + \lambda)\chi_{\{u>0\}} & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is solvable for all $\lambda \in \mathbb{R}$. We also show that the problem

$$\begin{cases} -\Delta u = (\log u + u + e^u)\chi_{\{u>0\}} & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is solvable.

Chapters 2-5 have more or less the same structure, but there are nuances. In these chapters, we consider an associated perturbed functional of the form

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_\epsilon(u)\,dx - \int_\Omega F(u)\,dx,$$

and we show that I_ϵ satisfy the Palais-Smale condition and the hypothesis of the Mountain-Pass Theorem, so that it admits a critical point u_ϵ. This is done in the same spirit of [30] and [63]. Next, we show that u_ϵ is bounded in $H_0^1(\Omega)$ by a constant that does not depend on ϵ, as in [37], [52] and [54]. This bound is essential, and only holds for the specific solutions we obtain. The final, crucial step is to study the convergence of these solutions as $\epsilon \to 0$. To do this, we rely on the Moser iteration scheme, see [45] and [56], and in gradient estimates for the critical points of I_ϵ, similar in essence to [5], [37] and [52].

In Chapter 6, we consider problem (17) with f having critical growth in the sense of Trudinger-Moser, which states that there exists $\alpha > 0$ such that

$$\lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = \infty \text{ for all } 0 < \kappa < \alpha, \text{ and } \lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = 0 \text{ for all } \kappa > \alpha. \tag{18}$$

Examples of functions with critical growth are $f(s) = e^{s^2}$ and $f(s) = s^\tau e^{s^2}$ with $\tau > 0$. See also Remark 6.1 at page 102.

In Section 6.1, we obtain $\alpha_0 > 0$ such that problem (17) is solvable for $0 < \alpha < \alpha_0$. For example, we conclude that for each $p > 1$, there exists $\alpha_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = (\log u + u^p \exp(\alpha u^2))\chi_{\{u>0\}} & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is solvable for $0 < \alpha < \alpha_0$. Next, in Section 6.2, we study the parametrized problem

$$\begin{cases} -\Delta u = (\log u + \lambda f(u))\chi_{\{u>0\}} & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where f satisfies (18) for some $\alpha > 0$. The goal here is to obtain results without controlling the value of α. Such an approach has issues. Indeed, we can only show that problem (19) is solvable provided λ is sufficiently large and

$$|\Omega| < c, \tag{20}$$

for some constant $c > 0$ that (sadly) depends on Ω. We are not able to give examples of admissible sets Ω, but in Section 6.3 we obtain rather explicit values for the admissibility constant c when $f(s) = s\exp(\alpha s^2)$ and $f(s) = \exp(\alpha s^2)$.

In Chapter 7, we study the problem

$$\begin{cases} -\Delta u = a(x)g(u) + \lambda b(x)\, u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{21}$$

where $0 < p < 1$. We will assume that the function g is allowed to be singular near the origin and that there exists $0 < q, \beta < 1$ such that

$$\lim_{s\to 0} g(s) = -\infty, \quad \lim_{s\to 0} |s^\beta g(s)| < \infty, \text{ and } \lim_{s\to\infty} \frac{|g(s)|}{s^q} < \infty,$$

so that g is sublinear at ∞. Our contribution is that the weights $a(x)$ and $b(x)$ are allowed to be singular near the boundary of Ω, and g is allowed to change sign. We show that under certain conditions on g, a and b, problem (21) has a solution for large values of λ. We also show that this solution is unique in a certain class provided that p is small enough.

Our approach is as follows: first we show that problem (21) possesses a subsolution $\underline{u}$ satisfying $\underline{u} > 0$ in Ω and $\underline{u} = 0$ on $\partial\Omega$ provided λ is large enough. Next, we consider a sequence $\emptyset \neq \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \ldots \subset\subset \Omega$ of smooth domains Ω_k such that $\cup_{k=1}^{\infty}\Omega_k = \Omega$. We then use variational methods to obtain positive solutions u_k of the problem

$$\begin{cases} -\Delta u = a(x)g(u) + \lambda b(x)\, u^p & \text{in } \Omega_k \\ u > 0 & \text{in } \Omega_k \\ u = \underline{u} & \text{on } \partial\Omega_k, \end{cases} \tag{22}$$

Such solutions are obtainable because problem (22) is set in the interior of Ω, so that the eventual singularities of a and b on the boundary $\partial\Omega$ and of g have no effect. We show that the limit $u = \lim\limits_{k\to\infty} u_k$ is a solution of problem (21). Next, we show that this solution is unique in a certain class.

Our uniqueness result can be extended for a wide range of singular and nonsingular equations, for example we may consider $g \equiv 0$ or $g \equiv 1$ in (21). Observe also that problem (21) with $g(u) = u^{-\beta}$ for $0 < \beta < 1$ is very similar to (8), the key difference being the presence of the term u^p. We thus get a existence and uniqueness result for a modified version of the problems discussed in [17].

We should also compare equation (21) with the ones studied in Chapters 2-6. The term u^p in (21) is sublinear, and we are able to get a positive solution, thanks to the existence of a positive subsolution. On the other hand, in Chapters 2-6 we consider superlinear nonlinearities, and we obtain nonnegative (but not strictly positive) solutions thanks to the fact that these nonlinearities satisfy a version of the Ambrosetti-Rabinowitz condition (this condition is not satisfied for sublinear terms).

Some of the results of this chapter were published in [55].

Some open problems

In problem (12) studied in Chapter 2, we may consider $f(s) = \lambda s^p + \mu s^q$ for $0 < q < 1 < p$, but in Chapters 4-6 we cannot consider such f, because, as far as we know, the mountain pass ceases to exist. Thus, problems of the form

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^q + \mu f(u) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\Omega \subset \mathbb{R}^2$, $0 < q < 1$ and $\lambda, \mu > 0$ are left open.

In Chapters 2-6, we do not study the regularity of the solutions $u \in H_0^1(\Omega)$ that we obtain. We know, however, that u must be locally Lipschitz continuous, because it is obtained as a uniform limit of smooth functions. Results of [53] and [62] suggest that u might be of class $C^{1,\nu}$ for some $0 < \nu < 1$. The positivity of u is another open problem.

Throughout these chapters, we merely show that u is nontrivial and nonnegative. We do not study under which circumstances u is positive, and there are examples of solutions which are shown not to be positive in Ω. Here is an example: for all $\lambda > 0$, let u_λ be a solution of problem (12) with $f = 0$ and $0 < p < 1$, so that

$$\int_\Omega \nabla u_\lambda \nabla \varphi = \int_{\Omega \cap \{u_\lambda > 0\}} (-u_\lambda^{-\beta} + \lambda u_\lambda^p)\varphi \, dx,$$

for all $\varphi \in C_c^1(\Omega)$. We will show that u_λ cannot be strictly positive in Ω if the parameter λ is sufficiently small. Indeed, assume by contradiction that $u_\lambda > 0$ in Ω. By an approximation argument, we get

$$\int_\Omega \nabla u_\lambda \nabla \varphi = \int_\Omega (-u_\lambda^{-\beta} + \lambda u_\lambda^p)\varphi \, dx,$$

for all $\varphi \in C^1(\Omega)$. Choosing $\varphi = \phi_1$, where ϕ_1 is the first eigenfunction of $-\Delta$ with $\|\phi_1\|_{H_0^1(\Omega)} = 1$, we obtain

$$\lambda_1 \int_\Omega u_\lambda \phi_1 = \int_\Omega (-u_\lambda^{-\beta} + \lambda u_\lambda^p)\phi_1 \, dx.$$

Consequently,

$$\lambda \int_\Omega u_\lambda^p \phi_1 \, dx = \int_\Omega (\lambda_1 u_\lambda + u_\lambda^{-\beta})\phi_1.$$

Since there exists $c_1 > 0$ independent of λ such that $\lambda_1 s + s^{-\beta} \geq c_1 > 0$ for all $s > 0$, we get

$$\lambda \int_\Omega u_\lambda^p \phi_1 \, dx \geq c_1 \int_\Omega \phi_1.$$

But the function u_λ remains bounded as $\lambda \to 0$, see [25]. We have thus proven that there exists $\lambda^* > 0$ such that the set $\{u_\lambda > 0\}$ has positive measure if $0 < \lambda < \lambda^*$. Consequently, such problems admits a free boundary $\partial\{u > 0\}$, see [25] and [32].

A further question related to the discussion developed in Chapters 2-6 is the following. Can we obtain similar results for related problems with more general singularities? For example, let $0 < \beta_1, \beta_2 < 1$ and suppose that Ω is a smooth bounded domain in $\mathbb{R}^2$ and that f is a continuous function with exponential or polynomial growth. For what values of $\mu > 0$ is the problem

$$\begin{cases} -\Delta u = (u^{-\beta_1} - u^{-\beta_2})\chi_{\{u>0\}} + \mu f(u) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

solvable?

1 Main contributions of this work

In this chapter, we compare results found in the literature with ours.

1.1 Contribution of Chapter 2

In Chapter 2 we study a class of singular elliptic equations in a bounded smooth domain $\Omega \subset \mathbb{R}^2$. We extend results proven in [4], [26] and [63]. In [4], the nonsingular elliptic problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

was studied for $0 < q < 1 < p$. The authors showed that problem (1.1) has two solutions for sufficiently small λ. In [63], the authors considered the problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

They obtained two solutions for problem (1.2) provided λ is sufficiently small. In [26] the authors studied the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

with $0 < \beta < 1 < p$. They showed that problem (1.3) has a solution for each $\lambda > 0$. We should compare equations (1.1)–(1.3). Equation (1.3) has the singular term $u^{-\beta}$, but the solution obtained is not strictly positive in Ω. Problems (1.1) and (1.2) do not have singular terms, but the solutions of these problems are strictly positive in Ω. In this chapter, we study problems of the form

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu f(u) & \text{in } \Omega \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $0 < \beta < 1$, $0 < p < \infty$, $\mu > 0$ and f is a differentiable function with $f(0) = 0$. Our main contribution is that we allow f to have exponential growth. We may consider nonlinearities f of the form

$$f(s) = e^s - 1 \quad \text{ and } f(s) = s^k e^s \text{ for } k \geq 1, s \geq 0,$$

and when $p > 1$, we may consider

$$f(s) = 0, \quad \text{and} \quad f(s) = s^q \text{ for } s \geq 0 \text{ and } 0 < q < 1.$$

With these choices of f, problem (1.4) becomes

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu(e^u - 1) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^k e^u \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

and

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^q \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.7}$$

We prove that problems (1.5) and (1.6) have a nontrivial solution for each $\lambda \geq 0$ and $0 < p < \infty$, provided μ is small enough. When $\lambda = 0$, problems (1.5) and (1.6) are singular versions of problem (1.2). Observe that the right hand side of (1.2) does not vanish at the origin but the nonlinear terms of (1.5) and (1.6) do. We also show that problem (1.3) has a nonnegative and nontrivial solution for each $\lambda > 0$ and $p > 1$, thus obtaining the result of [26]. Furthermore, we obtain a solution for problem (1.7) with $0 < q < 1 < p$, where both concave and convex nonlinearities are present. We show that for each $\lambda > 0$ there exists $\mu_0 > 0$ such that problem (1.7) has a nonnegative solution provided $0 < \mu < \mu_0$. Finally, when $0 < p \leq 1$, we show that for each $\lambda > 0$ there exists $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^r \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.8}$$

with $r > 3$, has a nonnegative solution provided $0 < \mu < \mu_0$. Observe that problems (1.7) and (1.8) are singular versions of problem (1.1), studied in [4].

1.2 Contribution of Chapter 3

In this chapter we study a singular elliptic problem in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$. We extend results of [11] and [37]. In [11] the authors considered the nonsingular problem

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

They proved that problem (1.9) is solvable when $N \geq 4$ and $0 < \lambda < \lambda_1$, where λ_1 denotes the first eigenvalue of $-\Delta$. Furthermore, the authors showed that problem (1.9) is not solvable for $\lambda > \lambda_1$. When $N = 3$, this problem is much more delicate. The authors showed that if Ω is a ball, then there exists a solution of problem (1.9) if and only if $\lambda_1/4 < \lambda < \lambda_1$. In [37] the authors studied the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + u^{\frac{N+2}{N-2}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.10}$$

for $0 < p < 2^* - 1$ and $p \neq 1$. If $0 < p < 1$, they obtained two distinct solutions of (1.10) for small values of λ. When $1 < p < 2^* - 1$, they obtained one solution of (1.10) for large values of λ. Problems (1.9) and (1.10) should be compared. Equation (1.10) has a singular term, but the subcritical term in the right hand side is not allowed to be linear. On the other hand, problem (1.9) is not singular, but admits a linear term. In this work, we study the equation

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u + u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.11}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geq 3$, $0 < \beta < 1$, $\lambda \geq 0$ and $1 < p < 2^* - 1$, with $2^* - 1 = \dfrac{N+2}{N-2}$.

We show that problem (1.11) possesses at least one solution for each $\lambda \geq 0$ provided $1 < p < 2^* - 1$. Therefore, we conclude that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.12}$$

is solvable for $1 < p < 2^* - 1$. This result is already known and was proved in [26].

1.3 Contribution of Chapter 4

In this chapter we study an elliptic problem in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ with a singularity of logarithmic type. We extend the results of [52] and [53]. In [53], the problem

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda f(x,u) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.13}$$

was studied. The authors assumed that $f \geq 0, f \not\equiv 0$ nondecreasing and

$$\lim_{s\to\infty} \frac{f(x,s)}{s} = 0 \text{ uniformly for } x \in \Omega.$$

With these hypothesis, the authors obtained a solution u_λ of problem (1.13) for each $\lambda > 0$. Moreover, the authors proved that $u_\lambda > 0$ in Ω provided λ is sufficiently large. The superlinear case was treated in [52], where the problem

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.14}$$

was studied with $p < 2^* - 1$. If $0 < p < 1$ in (1.14), the authors obtained two distinct nontrivial solutions when the parameter λ is large. If $1 < p < 2^* - 1$ in (1.14), they obtained one solution for sufficiently large λ.

In this chapter, we study problems of the form

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p + \mu f(u) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.15}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\lambda \geq 0$ and $\mu > 0$ are positive parameters, $p > 1$, $f(0) = 0$ and f is allowed to have exponential growth. We may consider nonlinearities f of the form

$$f(s) = e^s - 1 \quad \text{ and } f(s) = s^k e^s \text{ for } k \geq 1, s \geq 0,$$

and when $p > 1$, we may consider

$$f(s) = 0, \quad \text{ for } s \geq 0,$$

and obtain problem (1.14). With these choices of f, problem (1.15) becomes

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p + \mu(e^u - 1) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.16}$$

and

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p + \mu u^k e^u \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.17}$$

We prove that problems (1.16) and (1.17) have a nontrivial solution for each $\lambda \geq 0$ and $1 < p < \infty$, provided μ is small enough. When $\lambda = 0$, problems (1.16) and (1.17) are singular versions of problem (1.2). We also prove that problem (1.14) has a nontrivial nonnegative solution for each $\lambda > 0$ provided $p > 1$ (this is the case $f = 0$ in (1.15)), thus improving the result of [52]. We should also compare problem (1.15) with (1.4). The term $\log u$ is less singular then $-u^{-\beta}$ near the origin, but the function $\log u$ changes sign and is unbounded at infinity. We obtain results for problem (1.4) with $0 < p < 1$, but we can not consider this case in (1.15). Also, when $p > 1$, we may consider $f(s) = s^q$ with $0 < q < 1$

in (1.4) but we may not take such f in (1.15). We remark that, unlike the sublinear case (1.13) studied in [53], we do not obtain results on the positivity of the solution u, so that the set $\{u = 0\}$ may have positive measure.

1.4 Contribution of Chapter 5

The aim of this chapter is to generalize and extend the results obtained in Chapter 4. Indeed, we consider problems of the form

$$\begin{cases} -\Delta u = (\log u + f(u))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.18}$$

where $f : [0, \infty) \to \mathbb{R}$ is a function that is allowed to have exponential growth. Here we make two major improvements over the problems discussed in Chapter 4.

- We no longer make use of parameters, as in (1.15);
- We do not assume that $f(0) = 0$.

We obtain solutions for a large class of problems. For example, we show that the problem

$$\begin{cases} -\Delta u = (\log u + \lambda e^u + \mu)\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

possesses a nontrivial solution for all $\lambda > 0$ and $\mu \in \mathbb{R}$, compare with problem (1.2) studied in [63].

Furthermore, we allow f to change sign. For example, we may consider

$$f(s) = \lambda s^p - \mu s^q, \text{ with } 0 < q < 1 < p.$$

With this choice of f, we obtain a solution for the problem

$$\begin{cases} -\Delta u = (\log u + \lambda u^p - \mu u^q)\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

for all $\lambda > 0$ and $\mu \geq 0$. This equation is a singular version of the problem (1.1) treated in [4].

1.5 Contribution of Chapter 6

In this chapter, we study problem (1.18) with f having critical growth. Indeed, in Chapters 2, 4 and 5, we assume that for each $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$|f(s)| \leq C_\alpha \exp(\alpha s^2) \text{ for all } s \geq 0.$$

In Chapter 6, we assume the following stricter condition: that there exists $\alpha > 0$ such that

$$\lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = \infty \text{ for all } 0 < \kappa < \alpha, \text{ and } \lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = 0 \text{ for all } \kappa > \alpha. \tag{1.19}$$

Elliptic problems involving functions satisfying (1.19) are of interest even in the nonsingular case, see for example [27], where the problem

$$\begin{cases} -\Delta u = \lambda u^q + e^{\alpha u^2} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.20}$$

with $\lambda > 0$, $0 < q < 1$ was studied, and the authors obtained two solutions provided α and λ are sufficiently small.

In Section 6.1, we show that, under certain conditions on f, there exists $\alpha_0 > 0$ such that problem (1.18) is solvable provided $0 < \alpha < \alpha_0$. The main problem we have in mind here is

$$\begin{cases} -\Delta u = (\log u + u^\tau \exp(\alpha u^2))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.21}$$

with $\tau > 1$. We remark, however, that we do not require $f(0) = 0$. As a consequence, we also obtain the following result: for each $\mu \in \mathbb{R}$, there exists $\alpha_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = (\log u + u^\tau \exp(\alpha u^2) + \mu)\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

is solvable for $0 < \alpha < \alpha_0$.

Next, in Section 6.2, we consider the parametrized problem

$$\begin{cases} -\Delta u = (\log u + \lambda f(u))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.22}$$

where f satisfies (1.19) for some $\alpha > 0$. Here, we cease to control the value of α, and as a consequence, two drawbacks appear. The first one, is that we have use the parameter λ in order to obtain suitable energy estimates, which only hold for large values of λ. The second, main drawback, is that we can only show that problem (1.22) is solvable provided that Ω satisfies a certain admissibility condition. Indeed, we show that, under certain hypothesis on f, there exists $\lambda_0 > 0$ such that problem (1.22) is solvable for $\lambda > \lambda_0$ provided $|\Omega| < c$, where c is a constant that depends on λ and α. The issue here is that λ_0 depends on Ω, and consequently, so does c. We are unable to obtain examples of sets Ω which are admissible. However, in Section 6.3, we give estimates for the value of c for the problems

$$\begin{cases} -\Delta u = (\log u + \lambda u \exp(\alpha u^2))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = (\log u + \lambda \exp(\alpha u^2))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

1.6 Contribution of Chapter 7

In [54] the authors studied the problem

$$\begin{cases} -\Delta u = -u^{-\beta} + \lambda u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded smooth domain, with $0 < \beta < 1$ and $0 < p < 1$. The authors proved two results:

• Problem (1.23) admits a positive solution $u_\lambda \in H_0^1(\Omega)$ provided λ is sufficiently large. Furthermore, there exists $\underline{u} \in H_0^1(\Omega)$ with $\underline{u} > 0$ in Ω such that $u_\lambda \geq \underline{u}$.

• There exists $0 < p_0 < 1$ such that the solution u_λ is unique in the set $\{u \in H_0^1(\Omega) : u \geq \underline{u}\}$, provided $0 < p < p_0$.

We also refer to [42], where the authors studied the problem

$$\begin{cases} -\Delta u = a(x)g(u) + \lambda f(x,u) + \mu b(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.24}$$

with $\lambda, \mu \geq 0$. There, the authors assumed that there exists $0 < \nu < 1$ such that

$$a, b \in C^\nu(\overline{\Omega}) \text{ with } a, b > 0 \text{ in } \Omega. \tag{1.25}$$

As for the functions f and g, it was assumed that

$$g \text{ is nonpositive, nondecreasing and } g \in C^\nu(0,\infty), \tag{1.26}$$

$$f : \Omega \times [0,\infty) \to [0,\infty) \text{ is nonnegative and } f \in C^\nu(\overline{\Omega} \times (0,\infty)), \tag{1.27}$$

$$\lim_{s\to\infty} g(s) = -\infty. \tag{1.28}$$

and there exist constants $C > 0$, $\delta_0 > 0$ and $0 < \beta < 1$ such that

$$|g(s)| = -g(s) \leq Cs^{-\beta} \text{ for } 0 < s < \delta_0. \tag{1.29}$$

The authors also assumed that the mapping

$$s \to \frac{f(x,s)}{s} \text{ is nonincreasing for all } x \in \overline{\Omega}, \tag{1.30}$$

and that
$$\lim_{s\to 0}\frac{f(x,s)}{s}=\infty,\quad \lim_{s\to\infty}\frac{f(x,s)}{s}=0,\ \text{uniformly for } x\in\overline{\Omega}. \tag{1.31}$$
With this hypothesis, they showed that there exist $\lambda^*>0$ and $\mu^*>0$ such that problem (1.24) is solvable provided $\lambda>\lambda^*$ or $\mu>\mu^*$. Conditions (1.30) and (1.31) mean that f is a generalization of the function $f(x,s)=s^p$ for $0<p<1$. Condition (1.29) imply that $\int_0^1 g(t)\,dt<\infty$, and the authors proved that if

$$\int_0^1 g(t)\,dt=\infty,$$

then problem (1.24) is not solvable.

In this chapter we consider the problem

$$\begin{cases} -\Delta u = a(x)g(u)+\lambda b(x)\,u^p & \text{in}\,\Omega\\ u>0 & \text{in}\,\Omega\\ u=0 & \text{on}\,\partial\Omega, \end{cases} \tag{1.32}$$

where $\Omega\subset\mathbb{R}^N, N\geq 1$ is a bounded smooth domain, $\lambda>0$ is a positive parameter, a and b are positive weight functions, $g:(0,\infty)\to\mathbb{R}$ is continuous, satisfies (1.28)-(1.29) and $0<p<1$. Observe that problem (1.32) may be obtained from problem (1.24) by considering $\mu=0$ and $f(x,s)=s^p$. On the other hand,

- we allow g to change sign

and

- we allow the weights a and b to be singular near the boundary $\partial\Omega$ of Ω.

Indeed, instead of (1.25), we assume that

$$a,b\in C(\Omega),\min_{\Omega}\{a,b\}>0,$$

and that there exist $C>0$ and $0<\sigma<1$ with $\sigma+\beta<1$ such that

$$\max\{a(x),b(x)\}\delta(x)^{\sigma}<C \text{ for all } x\in\Omega,$$

where
$$\delta(x)=dist(x,\partial\Omega)=\inf_{y\in\partial\Omega}|x-y|.$$
However, we add the assumption that the nonlinearity g is sublinear at infinity, that is, there are constants $0<q<1$ and $C_2\geq 0$ such that

$$\lim_{s\to\infty}\frac{|g(s)|}{s^q}=C_2. \tag{1.33}$$

Under these hypothesis, we show that there exists $\lambda_0>0$ such that problem (1.32) has a subsolution $\underline{u}\in H_0^1(\Omega)$ for $\lambda>\lambda_0$. Next, we obtain a solution $u_\lambda>\underline{u}$ provided $\lambda>\lambda_0$.

If we further assume that g is of class C^1 and that there exists $0 < \gamma < 2 - \sigma$ such that

$$|g'(s)| \leq C_4 |s|^{-\gamma} \quad \text{for every } s > 0, \tag{1.34}$$

then a uniqueness result hods: the solution u_λ is unique in the class of the functions $\{u \in H_0^1(\Omega) : u \geq \underline{u}\}$. We thus generalize the results of [54]. Our uniqueness result is very general, and also holds for nonsingular problems. For example, we may consider $a \equiv b \equiv 1$ in Ω and $g(s) = s^q$ with $0 < q < 1$ in (1.32), so that the problem

$$\begin{cases} -\Delta u = u^q + \lambda\, u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in the class $\{u \geq \underline{u}\}$ for small values of p. We may also take $g \equiv 0$ in (1.32), so that the problem

$$\begin{cases} -\Delta u = \lambda\, u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in the class $\{u \geq \underline{u}\}$ for small values of p.

2 A problem in the plane with terms of exponential growth

In this chapter we study the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu f(u) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $0 < \beta < 1$, $0 < p < \infty$ and f is allowed to have exponential growth. The aim of this chapter is to show that problem (2.1) has a nonnegative solution for every $\lambda \geq 0$ when the parameter $\mu > 0$ is small. We will suppose that f satisfies

$$f(s) = 0 \text{ for } s \leq 0, \quad f \in C^{1,\nu}(0,\infty) \cap C[0,\infty) \text{ for some } 0 < \nu < 1, \tag{2.2}$$

and that for each $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that

$$|f(s)| \leq C_\alpha \exp\left(\alpha s^2\right), \text{ for every } s \geq 0. \tag{2.3}$$

Examples of function f satisfying (2.3) for all $\alpha > 0$ are $f(s) = e^s$ and $f(s) = s^\tau e^s$ with $\tau > 1$.

We will use a perturbation technique. For each $0 < \epsilon < 1$, we consider the problem

$$\begin{cases} -\Delta u + g_\epsilon(u) = \lambda u^p + \mu f(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

with the perturbation g_ϵ given by

$$g_\epsilon(s) = \begin{cases} \dfrac{s^q}{(s+\epsilon)^{q+\beta}} \text{ for } s \geq 0 \\ 0 \text{ for } s < 0, \end{cases} \tag{2.5}$$

where $0 < q < \min\{1, p\}$. Our aim is to obtain solutions of (2.4) that converge to solutions of (2.1). Observe that $g_\epsilon(0) = 0$ and $g_\epsilon \in C^\infty(0,\infty) \cap C(\mathbb{R})$ converges pointwisely to $s^{-\beta}$ for $s > 0$. We define the functional $I_{\epsilon,\lambda,\mu} : H_0^1(\Omega) \to \mathbb{R}$ by

$$I_{\epsilon,\lambda,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega G_\epsilon(u) - \frac{\lambda}{1+p}\int_\Omega (u^+)^{1+p} - \mu \int_\Omega F(u)\, dx, \tag{2.6}$$

where $F(u) = \displaystyle\int_0^u f(s)\, ds$ and $G_\epsilon(u) = \displaystyle\int_0^u g_\epsilon(s)\, ds$. From the fact that f and g_ϵ are continuous functions that satisfy (2.2), (2.3) and (2.5), we conclude $I_{\epsilon,\lambda,\mu}$ is of class C^1

and

$$I'_{\epsilon,\lambda,\mu}(u)(v) = \int_\Omega \nabla u \nabla v + \int_\Omega g_\epsilon(u)v - \lambda \int_\Omega (u^+)^p v - \mu \int_\Omega f(u)v, \text{ for all } u, v \in H_0^1(\Omega), \quad (2.7)$$

see Theorem B.16. Consequently, if $u_\epsilon \in H_0^1(\Omega)$ is a critical point of $I_{\epsilon,\lambda,\mu}$ then

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda \int_\Omega (u_\epsilon^+)^p v + \mu \int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega). \quad (2.8)$$

Choosing $v = u_\epsilon^-$ in (2.8) and using (2.2), we obtain

$$-\int_\Omega |\nabla(u_\epsilon)^-|^2 = 0.$$

Hence, $u_\epsilon \geq 0$ in Ω. We conclude that critical points $u_\epsilon \in H_0^1(\Omega)$ of $I_{\epsilon,\lambda,\mu}$ are nonnegative and

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda \int_\Omega u_\epsilon^p v + \mu \int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega). \quad (2.9)$$

Therefore, critical points of $I_{\epsilon,\lambda,\mu}$ are weak solutions of problem (2.4). Furthermore, if $u_\epsilon \in L^\infty(\Omega)$, then for each $0 < \epsilon < 1$ fixed

$$\sup_\Omega (|g_\epsilon(u_\epsilon)| + \lambda u_\epsilon^p + \mu |f(u_\epsilon)|) < \infty,$$

and consequently

$$\Delta u_\epsilon \in L^\infty(\Omega).$$

We conclude from Elliptic Regularity Theory (Theorem B.14) that $u_\epsilon \in W^{2,r}(\Omega)$ for all $r > 1$. Thus, the Sobolev Embedding (Theorem B.13) implies that $u_\epsilon \in C^{1,\nu}(\overline{\Omega})$, where $0 < \nu < 1$ is given by (2.2). Summarizing, we have

Lemma 2.1. *Suppose that f satisfies* (2.2) *and* (2.3)*. The following assertions hold:*

(i) Critical points of $I_{\epsilon,\lambda,\mu}$ are nonnegative weak solutions of problem (2.4).

(ii) If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a nonnegative weak solution of problem (2.4)*, then u is smooth and $u \in C^{1,\nu}(\overline{\Omega})$, with ν given by* (2.2).

Remark 2.1. *The fact that critical points of $I_{\epsilon,\lambda,\mu}$ are nonnegative is key for our purposes. If instead of* (2.2)*, we assume that*

$$f(0) \neq 0, \quad f \in C^{1,\nu}(0,\infty) \cap C[0,\infty),$$

then we can no longer assume that $f(s) = 0$ for $s < 0$, because then the functional $I_{\epsilon,\lambda,\mu}$ would cease to be of class C^1. However, we will solve this issue in Chapter 5.

Throughout this chapter, we will use the Trudinger-Moser inequality, see Theorem B.6. We observe that

Lemma 2.2. *Assume that f satisfies (2.3) and that $f(s) = 0$ for $s < 0$. The following assertions hold*

(i) For each $\alpha > 0$ there exists a constant $C > 0$ that depends on α such that

$$\max\{|f(s)|, |F(s)|\} \leq C \exp(\alpha s^2) \text{ for } s \in \mathbb{R}. \tag{2.10}$$

(ii) If there exist a sequence (u_n) in $H_0^1(\Omega)$ and a constant $D > 0$ such that

$$\|u_n\|_{H_0^1(\Omega)} < D \text{ for all } n \in \mathbb{N},$$

then there exists $u \in H_0^1(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$,

$$\int_\Omega f(u_n)dx \to \int_\Omega f(u)dx \text{ as } n \to \infty, \tag{2.11}$$

and

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u)dx \text{ as } n \to \infty. \tag{2.12}$$

Proof of Lemma 2.2. First we prove item (i). Suppose that f satisfies (2.3) and fix $\alpha > 0$. From the fact that $f(s) = 0$ for $s < 0$, we can find a constant $C_1 > 0$ depending on α such that

$$|f(s)| \leq C_1 \exp(\alpha s^2) \text{ for } s \in \mathbb{R}.$$

From hypothesis (2.3) there is a constant $C_2 > 0$ such that

$$|F(s)| \leq \int_0^s |f(t)|\, dt \leq C_2 \int_0^s \exp\left(\frac{\alpha}{2} t^2\right) dt \leq C_2 |s| \exp\left(\frac{\alpha}{2} s^2\right) \text{ for } s \in \mathbb{R}.$$

Since there exists a constant $C_3 > 0$ depending only on α such that

$$|s| \leq C_3 \exp\left(\frac{\alpha}{2} s^2\right) \text{ for } s \in \mathbb{R},$$

we obtain

$$|F(s)| \leq C_2 C_3 \exp\left(\alpha s^2\right) \text{ for } s \in \mathbb{R}.$$

This proves (2.10).

Now we prove (ii). Since the sequence (u_n) is uniformly bounded in $H_0^1(\Omega)$, we know that there exists $u \in H_0^1(\Omega)$ such that up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and

$$u_n \to u \text{ in } L^r(\Omega) \text{ for all } r > 1. \tag{2.13}$$

Assertions (2.11) and (2.12) follow from (2.3) and from Theorems B.6 and B.8. Indeed, choose $\alpha > 0$ such that $2\alpha D^2 < 4\pi$. From Hölder's inequality we have

$$\int_\Omega |u_n f(u_n)|\, dx \leq C \int_\Omega |u_n| \exp(\alpha u_n^2)\, dx \leq C \left(\int_\Omega \exp(2\alpha u_n^2)\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_n|^2\, dx\right)^{\frac{1}{2}}.$$

From (B.8) and (2.13) we get a constant $\widetilde{C} > 0$ such that

$$\int_\Omega |u_n f(u_n)|\, dx \leq \widetilde{C}.$$

Then, (2.11) follows by Theorem B.8. Assertion (2.12) follows analogously: we only need to use item (i) and replace f by F in the previous estimates. We have proved item (ii). □

Throughout this chapter, we will need estimates on the perturbation g_ϵ which are uniform in ϵ. We have

Lemma 2.3. *The following assertions hold*

(i)

$$0 < g_\epsilon(s) < s^{-\beta} \text{ and } 0 < G_\epsilon(s) \leq \frac{1}{1-\beta} s^{1-\beta} \text{ for } s \geq 0. \tag{2.14}$$

(ii)

$$s g_\epsilon'(s) = \frac{q s^q}{(s+\epsilon)^{q+\beta}} - \frac{(q+\beta)s^{q+1}}{(s+\epsilon)^{q+\beta+1}}. \tag{2.15}$$

(iii) Let $0 < \overline{q} < 1$ be a constant such that $0 < q < \overline{q}$, where q is given by (2.5). Then, for each $M > 0$, there exists $0 < \overline{\delta} = \overline{\delta}(M) < 1$ such that

$$G_\epsilon(s) \geq \frac{M}{1+\overline{q}} s^{\overline{q}+1} \text{ for } 0 \leq s < \overline{\delta}. \tag{2.16}$$

Proof of Lemma 2.3. Items (i) and (ii) are clear from the definition of g_ϵ, see (2.5). We now prove item (iii). Note that

$$g_\epsilon(s) = \frac{s^q}{(s+\epsilon)^{q+\beta}} \geq \frac{s^q}{(s+1)^{q+\beta}} = \frac{s^{q-\overline{q}}}{(s+1)^{q+\beta}} s^{\overline{q}} \text{ for } s \geq 0.$$

Hence,

$$g_\epsilon(s) \geq \frac{1}{2^{q+\beta}} s^{q-\overline{q}} s^{\overline{q}} \text{ for } 0 \leq s < 1.$$

Since $0 < q < \overline{q}$, we know that for each $M > 0$ there exists $\overline{\delta} = \overline{\delta}(M) < 1$ such that

$$g_\epsilon(s) \geq M s^{\overline{q}} \text{ for } 0 \leq s < \overline{\delta} < 1.$$

We thus obtain

$$G_\epsilon(s) = \int_0^s g_\epsilon(t)\, dt \geq \int_0^s M t^{\overline{q}}\, dt = \frac{M}{1+\overline{q}} s^{\overline{q}+1} \text{ for } 0 \leq s < \overline{\delta} < 1.$$

This proves item (iii). □

Remark 2.2. *In other works, for example [25] and [26], the authors consider the simpler perturbation*

$$\overline{g}_\epsilon(s) = \begin{cases} \dfrac{s}{(s+\epsilon)^{1+\beta}} & \text{for } s \geq 0 \\ 0 \text{ for } s < 0. \end{cases}$$

Observe, however, that the function $\overline{g}_\epsilon$ does not satisfy item (iii) of Lemma 2.3.

The functions $j_\epsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$j_\epsilon(s) = \lambda(s^+)^p + \mu f(s) - g_\epsilon(s),$$

and $J_\epsilon(s) = \int_0^s j_\epsilon(t)\,dt$ will play an important role in this chapter, because

$$I_{\epsilon,\lambda,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u) \text{ for all } u \in H_0^1(\Omega). \tag{2.17}$$

We will denote the functions j_ϵ and J_ϵ merely by j and J respectively. In order to get important properties of j and J we assume that f satisfies the following condition.

There exists a constant $s_0 > 0$ such that

$$\min\{f(s), F(s)\} \geq 0 \text{ for every } s \geq s_0. \tag{2.18}$$

Under certain circumstances we also assume that there exist constants $A > 0$, and $\gamma > 2$ such that

$$F(s) \geq A|s|^\gamma \text{ for every } s \geq s_0. \tag{2.19}$$

Condition (2.19) will also be important later, when we proceed to obtain an element $\phi_0 \in H_0^1(\Omega)$ with negative energy. For now we establish elementary properties of j and J.

Lemma 2.4. *The following assertions hold.*

(i) Suppose that f satisfies (2.2). For each $R > 0$, there exists a constant $C > 0$ that does not depend on ϵ such that

$$\max\{|J(s)|, |sj(s)|\} \leq C \textit{ for all } s \leq R.$$

(ii) Suppose that $\lambda > 0$, $\mu > 0$ and that f satisfies conditions (2.2) and (2.18). Then, there exists $R_\lambda > 0$ such that $J(s) \geq 0$ for all $s \geq R_\lambda$.

(iii) Suppose that $\lambda = 0$, $\mu > 0$ and that f satisfies (2.2), (2.18) and (2.19). Then, there exists $R_\mu > 0$ such that $J(s) \geq 0$ for all $s \geq R_\mu$.

Proof of Lemma 2.4. First we prove item (i). Note that

$$J(s) = \frac{\lambda}{p+1}s^{p+1} + \mu F(s) - G_\epsilon(s) \text{ for } s \geq 0, \tag{2.20}$$

and

$$sj(s) = \lambda s^{p+1} + \mu s f(s) - s g_\epsilon(s) \text{ for } s \geq 0.$$

Then, it follows from Lemma 2.3 that

$$|J(s)| \leq \frac{\lambda}{p+1}R^{p+1} + \mu \sup_{0\leq s\leq R}|F(s)| + \frac{1}{1-\beta}R^{1-\beta} \text{ for } 0 \leq s \leq R,$$

and

$$|sj(s)| \leq \lambda R^{p+1} + \mu \sup_{0\leq s\leq R} |sf(s)| + R^{1-\beta} \text{ for } 0 \leq s \leq R.$$

This proves item (i). Item (ii) follows from (2.18), (2.20) and Lemma 2.3. Indeed,

$$J(s) \geq \frac{\lambda}{p+1} s^{p+1} - \frac{1}{1-\beta} s^{1-\beta} \text{ for } s \geq s_0.$$

Hence, $J(s) \geq 0$ provided

$$s^{p+\beta} \geq \frac{p+1}{\lambda(1-\beta)}.$$

We have proved item (ii). Now we prove item (iii). Applying (2.19), Lemma 2.3 and the condition $\lambda = 0$, we get

$$J(s) \geq \mu A s^{\gamma} - \frac{1}{1-\beta} s^{1-\beta} \text{ for } s \geq s_0.$$

Hence, $J(s) > 0$ provided

$$s^{\gamma+\beta-1} \geq \frac{1}{\mu A(1-\beta)}.$$

We have proved Lemma 2.4. □

2.1 Properties and solutions of the perturbed problem

In this section, we study the perturbed problem (2.4). The first aim of this section is to show that under certain conditions on f, there exist constants $0 < \theta < 1/2$ and $R_{\theta,\lambda,\mu} > 0$ that do not depend on ϵ such that

$$J(s) \leq \theta s j(s) \text{ for } s \geq R_{\theta,\lambda,\mu}.$$

Using this result and Lemma 2.4, we will be able to show that the functional $I_{\epsilon,\lambda,\mu}$ given by (2.17) satisfies a compactness condition. We make the following assumptions on f.

- When $p > 1$ in (2.1) we assume that

$$pf(s) \leq sf'(s) \text{ for all } s \geq s_0, \tag{2.21}$$

or that there exists constants $C > 0$ and $\widetilde{p} < p$ such that

$$|pf(s) - sf'(s)| \leq Cs^{\widetilde{p}} \text{ for all } s \geq s_0. \tag{2.22}$$

Observe that $f = 0$ satisfies (2.21) and $f(s) = s^{\tau}$ satisfies (2.22) when $0 < \tau < 1$.

- When $0 < p \leq 1$ in (2.1) we suppose that

$$\lim_{s\to\infty} s^{1-p} f'(s) = \infty, \tag{2.23}$$

and that there exists a constant $0 < \nu_1 < 1$ such that

$$\frac{sf'(s)}{f(s)} \geq 3 + \nu_1 \quad \text{for all } s \geq s_0. \tag{2.24}$$

• When $\lambda = 0$ in (2.1) we assume that there exists a constant $0 < \nu_2 < 1$ such that

$$\frac{sf'(s)}{f(s)} \geq 2 + \nu_2 \quad \text{for all } s \geq s_0, \tag{2.25}$$

and

$$\lim_{s\to\infty} sf'(s) = \infty. \tag{2.26}$$

Under these assumptions, we get

Lemma 2.5. *Suppose that f satisfies (2.2), (2.18) and that one of the following assertions hold:*

(i) $\lambda = 0$, $\mu > 0$ and f satisfies (2.19), (2.25) and (2.26).

(ii) $\lambda > 0$, $\mu > 0$, $0 < p \leq 1$, and f satisfies (2.23) and (2.24).

(iii) $\lambda > 0$, $p > 1$, and f satisfies one of the conditions (2.21) or (2.22).

Then there exist constants $0 < \theta < \dfrac{1}{2}$ and $R_{\theta,\lambda,\mu} > 0$ such that

$$0 \leq J(s) \leq \theta sj(s) \text{ for } s \geq R_{\theta,\lambda,\mu}.$$

Consequently, item (i) of Lemma 2.4 implies that there exists $D_{\theta,\lambda,\mu} > 0$ such that

$$|J(s)| \leq D_{\theta,\lambda,\mu} + \theta sj(s) \text{ for all } s \in \mathbb{R}.$$

Proof of Lemma 2.5. From Lemma 2.4, we know that $J(s) \geq 0$ for large value of s. For each $0 < \theta < \dfrac{1}{2}$, let $B_\epsilon(s) = J(s) - \theta sj(s)$. We only need to show that $B_\epsilon(s) \leq 0$ when s is large. We have

$$B'_\epsilon(s) = (1-\theta)j(s) - \theta sj'(s).$$

Hence,

$$B'_\epsilon(s) = -(1-\theta)g_\epsilon(s) + \theta sg'_\epsilon(s) + \lambda((1-\theta)s^p - \theta ps^p) + \mu((1-\theta)f(s) - \theta sf'(s)).$$

From (2.15) we obtain

$$|sg'_\epsilon(s)| \leq q|s|^{-\beta} + (q+\beta)|s|^{-\beta} \to 0 \text{ as } s \to \infty.$$

It is also clear that

$$(1-\theta)g_\epsilon(s) \to 0 \text{ as } s \to \infty.$$

Hence, for each $0 < \tau < 1$ there exists $T_\tau > 0$ such that

$$B'_\epsilon(s) < \tau + \lambda((1-\theta-\theta p))s^p + \mu((1-\theta)f(s) - \theta sf'(s)) \text{ for } s \geq T_\tau. \tag{2.27}$$

Case 1: Suppose that $p > 1$ and (iii) holds.

In this case, we choose θ such that $\frac{1}{p+1} < \theta < \frac{1}{2}$. Hence, $1 - \theta - \theta p < 0$. Using (2.18), we obtain

$$B'_\epsilon(s) < \tau + \lambda((1 - \theta - \theta p))s^p + \mu\theta(pf(s) - sf'(s)) \text{ for } s \geq \max\{s_0, T_\tau\}.$$

Choosing $\tau = 1/2$ and using the fact that f satisfies one of the conditions (2.21) or (2.22), we find a constant $T_{\theta,\lambda,\mu} > s_0$ such that

$$B'_\epsilon(s) < -1 \text{ for } s \geq T_{\theta,\lambda,\mu}. \tag{2.28}$$

Note that

$$B_\epsilon(s) = \frac{\lambda}{p+1}(s^+)^{p+1} + \mu F(s) - G_\epsilon(s) - \theta s(\lambda(s^+)^p + \mu f(s) - g_\epsilon(s)).$$

Hence,

$$B_\epsilon(T_{\theta,\lambda,\mu}) \leq R_1,$$

where

$$R_1 = \frac{\lambda(T_{\theta,\lambda,\mu})^{p+1}}{p+1} + \mu F(T_{\theta,\lambda,\mu}) + \theta(T_{\theta,\lambda,\mu})^{1-\beta}.$$

From (2.28), we conclude that there exists a constant $R_2 > 0$ such that

$$B_\epsilon(s) \leq -s + R_2 \text{ for } s \geq T_{\theta,\lambda,\mu}.$$

Hence, $B_\epsilon(s) \leq 0$ for $s \geq \max\{R_2, T_{\theta,\lambda,\mu}\}$. This proves (iii).

Case 2: Suppose that $0 < p \leq 1$ and that (ii) holds. We claim that there exists constants $0 < \theta < \frac{1}{2}$ and $T_2 > 0$ such that

$$\lambda((1 - \theta - \theta p))s^p + \mu((1 - \theta)f(s) - \theta sf'(s)) < -1 \text{ for } s \geq T_2. \tag{2.29}$$

Indeed, (2.29) holds if and only if

$$\lambda((1 - \theta - \theta p))s^p + \mu(1 - \theta)f(s) + 1 < \mu\theta sf'(s) \text{ for } s \geq T_2.$$

Consequently, it is enough to show that

$$sf'(s) > \frac{3}{\mu\theta}, \quad sf'(s) > \frac{3(1 - \theta)f(s)}{\theta}, \text{ and } s^{1-p}f'(s) > \frac{3\lambda(1 - \theta - \theta p)}{\mu\theta},$$

for sufficiently large s. Claim (2.29) then follows by choosing θ such that

$$1 < \frac{(1 - \theta)}{\theta} < 1 + \frac{\nu_1}{3}.$$

and by (2.23) and (2.24). Then, from (2.27) we obtain

$$B'_\epsilon(s) < \tau - 1 \text{ for } s \geq \max\{T_\tau, T_2\}.$$

Choosing $\tau = 1/2$, we find a constant $\widehat{T_{\theta,\lambda,\mu}} > 0$ such that

$$B'_\epsilon(s) < -\frac{1}{2} \text{ for } s \geq \widehat{T_{\theta,\lambda,\mu}}.$$

(ii) then follows by a similar argument given in (iii).

Case 3: Suppose that $\lambda = 0$ and that (i) holds. We will show that there exists $T_3 > 0$ such that

$$\mu((1-\theta)f(s) - \theta s f'(s)) < -1 \text{ for } s \geq T_3. \tag{2.30}$$

Indeed, (2.30) holds if and only if

$$\mu(1-\theta)f(s) + 1 < \mu\theta s f'(s) \text{ for } s \geq R_{\tau_1}.$$

Consequently, it is enough to show that

$$sf'(s) > \frac{2}{\mu\theta} \text{ and } sf'(s) > \frac{2(1-\theta)f(s)}{\theta}, \tag{2.31}$$

for sufficiently large s. (2.31) then follows by (2.25), (2.26) and by choosing θ such that

$$1 < \frac{(1-\theta)}{\theta} < 1 + \frac{\nu_2}{2}.$$

Hence, (2.30) follows. Then, from (2.27) and from the fact that $\lambda = 0$, we obtain

$$B'_\epsilon(s) < \tau - 1 \text{ for } s \geq \max\{T_\tau, T_3\}.$$

Choosing $\tau = 1/2$, we find a constant $T_{\theta,\mu} > 0$ such that

$$B'_\epsilon(s) < -\frac{1}{2} \text{ for } s \geq T_{\theta,\mu}.$$

(i) then follows by a similar argument given in (iii).

□

We now obtain a compactness result. We follow ideas of [30] and [63]. The main ingredient is a lemma due to Lions, see Theorem B.7.

Lemma 2.6. *Fix $0 < \epsilon < 1$ and assume that f satisfies (2.2), (2.3) and (2.18). Suppose that one of the following conditions hold:*

(i) $\lambda = 0$, f satisfies (2.19), (2.25) and (2.26).

(ii) $\lambda > 0$, $0 < p \leq 1$, f satisfies (2.23) and (2.24).

(iii) $\lambda > 0$, $p > 1$, f satisfies one of the conditions (2.21) or (2.22).

Then the functional $I_{\epsilon,\lambda,\mu}$ defined in (2.17) satisfies the Palais-Smale condition at every level $c \neq 0$.

Proof of Lemma 2.6. If f satisfies (2.3), then j also satisfies (2.3) and for each $\alpha > 0$ there exists a constant $C_{\epsilon,\alpha} > 0$ depending only on ϵ and α such that

$$\max\{|j(s)|, |J(s)|\} \leq C_{\epsilon,\alpha} \exp(\alpha s^2) \text{ for } s \in \mathbb{R}. \tag{2.32}$$

Throughout this proof we denote $\|\cdot\|_{H_0^1(\Omega)}$ by $\|\cdot\|$. Let $(v_n^\epsilon)_{n\in\mathbb{N}}$ be a Palais-Smale sequence for $I_{\epsilon,\lambda,\mu}$ in $H_0^1(\Omega)$ at the level c. That is

$$\frac{1}{2}\|v_n^\epsilon\|^2 - \int_\Omega J(v_n^\epsilon)\,dx \to c \text{ as } n \to \infty, \tag{2.33}$$

and there is a sequence $\tau_n \to 0$ such that

$$\left|\int_\Omega \nabla v_n^\epsilon \nabla w\,dx - \int_\Omega j(v_n^\epsilon) w\,dx\right| \leq \tau_n \|w\| \text{ for each } w \in H_0^1(\Omega). \tag{2.34}$$

Let $0 < \theta < 1/2$ and $D_{\theta,\lambda,\mu} > 0$ be given by Lemma 2.5. We have

$$|J(v_n^\epsilon)| < D_{\theta,\lambda,\mu} + \theta v_n^\epsilon j(v_n^\epsilon).$$

Using (2.33) we obtain a constant $D_1 > 0$ that does not depend on $\epsilon > 0$ such that

$$\frac{1}{2}\|v_n^\epsilon\|^2 < D_1 + \theta \int_\Omega v_n^\epsilon j(v_n^\epsilon)\,dx.$$

Taking $w = v_n^\epsilon$ in (2.34) we also conclude that

$$\int_\Omega j(v_n^\epsilon) v_n^\epsilon\,dx < \|v_n^\epsilon\|^2 + \tau_n \|v_n^\epsilon\|.$$

Hence,

$$\frac{1}{2}\|v_n^\epsilon\|^2 < D_1 + \theta \|v_n^\epsilon\|^2 + \tau_n \theta \|v_n^\epsilon\|.$$

Since $\theta < 1/2$, there is a constant $D > 0$ such that

$$\|v_n^\epsilon\| < D. \tag{2.35}$$

It follows from (2.32) and Lemma 2.2 that there exist a subsequence $(v_{n_k}^\epsilon)$ in $H_0^1(\Omega)$ that we continue to denote by (v_n^ϵ) and an element $v^\epsilon \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n^\epsilon \rightharpoonup v^\epsilon \text{ weakly in } H_0^1(\Omega); \\ v_n^\epsilon \to v^\epsilon \text{ in } L^r(\Omega) \text{ for every } r > 1; \\ v_n^\epsilon \to v^\epsilon \text{ a.e in } \Omega; \\ |v_n^\epsilon| \leq h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega); \\ \int_\Omega j(v_n^\epsilon)\,dx \to \int_\Omega j(v^\epsilon)\,dx; \\ \int_\Omega J(v_n^\epsilon)\,dx \to \int_\Omega J(v^\epsilon)\,dx. \end{cases} \tag{2.36}$$

For simplicity of notation we will denote the sequence (v_n^ϵ) and the function v^ϵ merely by (v_n) and v respectively. From (2.33), (2.34) and (2.36), we get

$$\lim_{n\to\infty} \|v_n\|^2 = 2(c + \int_\Omega J(v)\,dx), \tag{2.37}$$

and

$$\lim_{n\to\infty}\int_\Omega v_n j(v_n) = 2(c+\int_\Omega J(v)\,dx). \tag{2.38}$$

We will now split the proof in two cases.

Case I. Suppose that $c \neq 0$ and $v \neq 0$. We will show that $I_{\epsilon,\lambda,\mu}(v) = c$. Assume by contradiction that $I_{\epsilon,\lambda,\mu}(v) < c$. Then

$$\|v\|^2 < 2\left(c+\int_\Omega J(v)\,dx\right). \tag{2.39}$$

Let $W_n = v_n/\|v_n\|$ and $W = v/\sqrt{2(c+\int_\Omega J(v)\,dx)}$. Hence, $W_n \rightharpoonup W$ weakly in $H_0^1(\Omega)$, $\|W_n\| = 1$ and $\|W\| < 1$. From Theorem B.7, we get a constant $k_2 > 0$ such that

$$\sup_n \int_\Omega \exp(tW_n^2) < k_2 \text{ for every } 0 < t < \frac{4\pi}{(1-\|W\|^2)}.$$

We know from (2.32) that for each $\widehat{r} > 1$ there is a constant $C_{\widehat{r},\epsilon,\alpha} > 0$ such that

$$|j(v_n)|^{\widehat{r}} < C_{\widehat{r},\epsilon,\alpha}\exp\left(\widehat{r}\alpha v_n^2\right) = C_{\widehat{r},\epsilon,\alpha}\exp\left(\widehat{r}\alpha\|v_n\|^2 W_n\right). \tag{2.40}$$

We want to choose α and t such that

$$\widehat{r}\alpha\|v_n\|^2 < t < \frac{4\pi}{1-\frac{\|v\|^2}{2(c+\int_\Omega J(v))}} = \frac{4\pi(c+\int_\Omega J(v))}{c-I_{\epsilon,\lambda,\mu}(v)}.$$

To do that, we fix $t > 0$ such that

$$t < \frac{4\pi(c+\int_\Omega J(v))}{c-I_{\epsilon,\lambda,\mu}(v)},$$

and take $\alpha > 0$ so small that

$$\alpha < \frac{t}{\widehat{r}D^2},$$

where D is given by (2.35). With these choices of α and t we obtain from (2.40) and Theorem B.7

$$\int_\Omega |j(v_n)|^{\widehat{r}}\,dx < \int_\Omega C_{\widehat{r},\epsilon,\alpha}\exp\left(\widehat{r}\alpha\|v_n\|^2 W_n\right) < C_{\widehat{r},\epsilon,\alpha}k_2 \text{ for each } \widehat{r} \geq 1, \quad n\in\mathbb{N}.$$

Hence, from (2.38),from Hölder's inequality, and from (2.36), we get

$$2\left(c+\int_\Omega J(v)\,dx\right) = \lim_{n\to\infty}\int_\Omega v_n j(v_n)\,dx = \lim_{n\to\infty}\int_\Omega v j(v_n)\,dx.$$

On the other hand, from (2.34),

$$\left|\int_\Omega \nabla v_n \nabla v\,dx - \int_\Omega j(v_n)v\,dx\right| \leq \tau_n\|v\| \text{ for each } n\in\mathbb{N}.$$

Hence,

$$-\|v\|\tau_n + \int_\Omega j(v_n)v\,dx \leq \int_\Omega \nabla v_n\nabla v\,dx \leq \|v\|\tau_n + \int_\Omega j(v_n)v\,dx \text{ for each } n\in\mathbb{N}.$$

Letting $n \to \infty$, we get

$$\|v\|^2 = \lim_{n\to\infty} \int_\Omega \nabla v_n \nabla v \, dx = \lim_{n\to\infty} \int_\Omega v j(v_n)\, dx = 2\left(c + \int_\Omega J(v)\, dx\right).$$

This contradicts (2.39). We have thus proven that $I_{\epsilon,\lambda,\mu}(v) \geq c$. On the other hand, (2.36) and (2.37) imply that

$$\|v\|^2 \leq \lim_{n\to\infty} \|v_n\|^2 = 2\left(c + \int_\Omega J(v)\, dx\right).$$

Hence, $I_{\epsilon,\lambda,\mu}(v) \leq c$. Therefore, we must have $I_{\epsilon,\lambda,\mu}(v) = c$. As a consequence, using (2.37), we obtain

$$\lim_{n\to\infty} \|v_n\|^2 = 2\left(c + \int_\Omega J(v)\, dx\right) = \|v\|^2.$$

Then, it follows that $v_n \to v$ strongly in $H_0^1(\Omega)$.

Case 2. Assume that $c \neq 0$ and $v = 0$. We will prove that this cannot happen. We first show that

$$\int_\Omega |j(v_n)|^{\widetilde{r}}\, dx < \infty \text{ for each } \widetilde{r} > 1. \tag{2.41}$$

Fix a constant $0 < d < 1$. From $v = 0$ and (2.37), we know that for large n,

$$\|v_n\|^2 < 2c + d.$$

From (2.32), we know that for each $\alpha > 0$ there is a constant $C_{\widetilde{r},\alpha,\epsilon} > 0$ such that

$$\int_\Omega |j(v_n)|^{\widetilde{r}}\, dx < C_{\widetilde{r},\alpha,\epsilon} \int_\Omega \exp\{(\widetilde{r}\alpha v_n^2)\}\, dx.$$

Choosing

$$\alpha = \frac{2\pi}{\widetilde{r}(2c+d)},$$

it follows from (B.8) that there is a constant $C_{\widetilde{r},\epsilon} > 0$ such that, for a sufficiently large n,

$$\int_\Omega |j(v_n)|^{\widetilde{r}}\, dx < C_{\widetilde{r},\epsilon} k_1.$$

This proves the claim (2.41). We now apply Hölder's inequality and use the fact that $v_n \to 0$ strongly in $L^2(\Omega)$ to get

$$\left|\int_\Omega v_n j(v_n)\, dx\right| \leq \left(\int_\Omega |j(v_n)|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_n|^2\, dx\right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty.$$

On the other hand,

$$\left|\int_\Omega |\nabla v_n|^2\, dx - \int_\Omega j(v_n) v_n\, dx\right| \leq \tau_n \|v_n\| \text{ for all } n \in \mathbb{N}.$$

This means that

$$-\tau_n D + \int_\Omega j(v_n) v_n\, dx \leq \int_\Omega |\nabla v_n|^2\, dx \leq \tau_n D + \int_\Omega j(v_n) v_n\, dx \text{ for all } n \in \mathbb{N}.$$

Hence,

$$\|v_n\| \to 0 = \|v\|.$$

This contradicts the fact that $\|v_n\| \to 2c \neq 0$. This proves the result. $\square$

We have proved that the functional $I_{\epsilon,\lambda,\mu}$ satisfies the Palais-Smale condition. Also, we know that $I_{\epsilon,\lambda,\mu}$ is of class C^1. We turn our attention into showing that problem (2.4) possesses a nontrivial solution $u_\epsilon \geq 0$. We will obtain a solution that is a Mountain-Pass. To do that we need to prove that there exist constants $a_1 > 0$ and $0 < \rho < 1$ such that

$$I_{\epsilon,\lambda,\mu}(u) \geq a_1 \text{ for } \|u\|_{H_0^1(\Omega)} = \rho,$$

and that there exists an element $\phi_0 \in H_0^1(\Omega)$ such that

$$\|\phi_0\|_{H_0^1(\Omega)} \geq 1 \text{ and } I_{\epsilon,\lambda,\mu}(\phi_0) < 0. \tag{2.42}$$

The lemma below guarantees that (2.42) holds for $\phi_0 = N_0\phi_1$ when $N_0 > 0$ is large. It is here that condition (2.19) comes into play.

Lemma 2.7. *Suppose that f satisfies (2.2), (2.18) and that one of the following assertions hold:*

(i) $\lambda = 0$, $\mu > 0$ and f satisfies (2.19).

(ii) $\lambda > 0$, $\mu > 0$, $0 < p \leq 1$ and f satisfies (2.19).

(iii) $\lambda > 0$, $\mu > 0$ and $p > 1$.

Then there exist constants $N_0 > 0$ and $a_2 > 0$ such that

$$I_{\epsilon,\lambda,\mu}(N_0\phi_1) < -1 \textit{ for every } 0 < \epsilon < 1, \tag{2.43}$$

and

$$\sup_{0\leq s\leq 1} I_{\epsilon,\lambda,\mu}(sN_0\phi_1) < a_2 \textit{ for every } 0 < \epsilon < 1. \tag{2.44}$$

Proof of Lemma 2.7. First we prove (i). Note that

$$I_{\epsilon,0,\mu}(s\phi_1) = \frac{s^2}{2} + \int_\Omega G_\epsilon(s\phi_1) - \mu \int_\Omega F(s\phi_1) \text{ for all } s \geq 0.$$

Since $0 \leq g_\epsilon(s) \leq s^{-\beta}$, it follows that $G_\epsilon(s) \leq \dfrac{s^{1-\beta}}{1-\beta}$ for all $s \geq 0$. Let $s_0 > 0$ be given by (2.18). We have

$$I_{\epsilon,0,\mu}(s\phi_1) \leq \frac{s^2}{2} + \frac{s^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta} - \mu \int_{\Omega\cap\{s\phi_1\leq s_0\}} F(s\phi_1) - \mu \int_{\Omega\cap\{s\phi_1 > s_0\}} F(s\phi_1) \text{ for all } s \geq 0.$$

Using (2.19) and the continuity of F we obtain a constant $c_1 > 0$ such that

$$I_{\epsilon,0,\mu}(s\phi_1) \leq \frac{s^2}{2} + \frac{s^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta} + c_1 - \mu A s^\gamma \int_{\Omega\cap\{s\phi_1 > s_0\}} \phi_1^\gamma \text{ for all } s \geq 0.$$

Hence, there are constants $c_2, c_3 > 0$ such that

$$I_{\epsilon,0,\mu}(s\phi_1) \leq c_1 + \frac{s^2}{2} + c_2 s^{1-\beta} - c_3 s^\gamma \text{ for all } s \geq \frac{2s_0}{\sup \phi_1}.$$

Thus, $I_{\epsilon,0,\mu}(s\phi_1) < -1$ provided

$$c_1 + \frac{s^2}{2} + c_2 s^{1-\beta} - c_3 s^\gamma < -1.$$

Since $\gamma > 2 > 1-\beta$, we know that

$$\lim_{s\to\infty} \left(c_1 + \frac{s^2}{2} + c_2 s^{1-\beta} - c_3 s^\gamma \right) = -\infty,$$

and therefore $\lim_{s\to\infty} I_{\epsilon,0,\mu}(s\phi_1) = -\infty$. This proves (2.43). Let $N_0 > 2$ be such that $I_{\epsilon,0,\mu}(N_0\phi_1) < -1$. It follows that

$$I_{\epsilon,0,\mu}(sN_0\phi_1) \leq \frac{s^2 N_0^2}{2} + \int_\Omega G_\epsilon(sN_0\phi_1)\, dx - \mu \int_\Omega F(sN_0\phi_1) \text{ for all } s \geq 0.$$

From (2.18) the exists $c_4 > 0$ such that

$$I_{\epsilon,0,\mu}(sN_0\phi_1) < c_4 + \frac{s^2 N_0^2}{2} + \frac{N_0^{1-\beta} s^{1-\beta}}{1-\beta} \int_\Omega \phi_1^{1-\beta}\, dx \text{ for all } s \geq 0.$$

Hence,

$$\sup_{s\in[0,1]} I_{\epsilon,0,\mu}(sN_0\phi_1) < a_2,$$

where

$$a_2 = c_4 + \frac{N_0^2}{2} + \frac{N_0^{1-\beta}}{1-\beta} \int_\Omega \phi_1^{1-\beta}\, dx.$$

This proves (2.44). The proof of (i) is complete.

The proof of (ii) is very similar to the one given in (i) because

$$I_{\epsilon,\lambda,\mu}(s\phi_1) \leq \frac{s^2}{2} + \int_\Omega G_\epsilon(s\phi_1) - \mu \int_\Omega F(s\phi_1) \text{ for all } s \geq 0.$$

The result then follows by using condition (2.19) as in the proof of (i).

Now we prove (iii). We have

$$I_{\epsilon,\lambda,\mu}(s\phi_1) = \frac{s^2}{2} + \int_\Omega G_\epsilon(s\phi_1) - \frac{\lambda}{p+1} s^{p+1} \int_\Omega \phi_1^{p+1} - \mu \int_\Omega F(s\phi_1) \text{ for all } s \geq 0.$$

From (2.18) we obtain a constant c_5 such that

$$I_{\epsilon,\lambda,\mu}(s\phi_1) \leq c_5 + \frac{s^2}{2} + \int_\Omega G_\epsilon(s\phi_1) - \frac{\lambda}{p+1} s^{p+1} \int_\Omega \phi_1^{p+1} \text{ for all } s \geq 0.$$

Hence, there are constants $c_6, c_7 > 0$ such that

$$I_{\epsilon,\lambda,\mu}(s\phi_1) \leq c_5 + \frac{s^2}{2} + c_6 s^{1-\beta} - c_7 s^{p+1} \text{ for all } s \geq 0.$$

Thus, $I_{\epsilon,\lambda,\mu}(s\phi_1) < -1$ provided

$$c_5 + \frac{s^2}{2} + c_6 s^{1-\beta} - c_7 s^{p+1} < -1.$$

Since $p+1 > 2 > 1-\beta$, we know that

$$\lim_{s\to\infty}\left(c_5 + \frac{s^2}{2} + c_6 s^{1-\beta} - c_7 s^{p+1}\right) = -\infty,$$

and therefore $\lim_{s\to\infty} I_{\epsilon,\lambda,\mu}(s\phi_1) = -\infty$. This proves (2.43). Let $N_0 > 2$ be such that $I_{\epsilon,\lambda,\mu}(N_0\phi_1) < -1$. It follows that

$$I_{\epsilon,\lambda,\mu}(sN_0\phi_1) = \frac{s^2N_0^2}{2} + \int_\Omega G_\epsilon(sN_0\phi_1)\,dx - \frac{\lambda}{p+1}\int_\Omega (sN_0\phi_1)^{p+1} - \mu\int_\Omega F(sN_0\phi_1) \text{ for all } s \geq 0.$$

From (2.18) the exists $c_8 > 0$ such that

$$I_{\epsilon,\lambda,\mu}(sN_0\phi_1) < c_8 + \frac{s^2N_0^2}{2} + \frac{N_0^{1-\beta}s^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta}\,dx \text{ for all } s \geq 0.$$

Hence,

$$\sup_{s\in[0,1]} I_{\epsilon,\lambda,\mu}(sN_0\phi_1) < a_2,$$

where

$$a_2 = c_8 + \frac{N_0^2}{2} + \frac{N_0^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta}\,dx.$$

This proves Lemma 2.7. □

Next we obtain solutions for the perturbed problem (2.4).

Proposition 2.1. *Fix $\lambda \geq 0$ and let a_2 be given by Lemma 2.7. Suppose that f satisfies (2.2), (2.3) and (2.18) and that one of the following conditions hold:*

(i) $\lambda = 0$ and f satisfies (2.19),(2.25) and (2.26).

(ii) $\lambda > 0$, $0 < p \leq 1$ and f satisfies (2.19), (2.23) and (2.24).

(iii) $\lambda > 0$, $p > 1$ and f satisfies one of the conditions (2.21) or (2.22).

Then, there exists a constant $\mu_0 > 0$ such that problem (2.4) possesses a nonnegative nontrivial solution u_ϵ for each $0 < \mu < \mu_0$. Moreover, there exist constants $a_1 > 0$ and $D > 0$ that do not depend on ϵ such that

$$0 < a_1 \leq I_{\epsilon,\lambda}(u_\epsilon) \leq a_2,$$

and

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D.$$

Proof of Proposition 2.1. Let $\overline{\delta}$ be given by (2.16) and suppose that $\lambda > 0$. Note that

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{2}\int_{\Omega} |\nabla u|^2 + \int_{\{u<\overline{\delta}\}} G_{\epsilon}(u) - \frac{\lambda}{p+1}\int_{\Omega}(u^+)^{p+1} - \mu\int_{\Omega} F(u) \text{ for every } u \in H_0^1(\Omega).$$

Choosing $M = \dfrac{\lambda(1+\overline{q})}{1+p}$ in (2.16), we obtain

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\int_{\Omega} |\nabla u|^2 - \frac{\lambda}{p+1}\int_{\{u>\overline{\delta}\}} u^{p+1} - \mu\int_{\Omega} F(u)\,dx \text{ for every } u \in H_0^1(\Omega).$$

Observe that there exists a constant $C_1 > 0$ such that

$$s^{p+1} \leq C_1 s^{p+2} \text{ for } s \geq \overline{\delta}.$$

Hence, there exists a constant $C_2 > 0$ such that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_2\int_{\Omega} |u|^{2+p} - \mu\int_{\Omega} F(u)\,dx \text{ for every } u \in H_0^1(\Omega).$$

Hence, from the Sobolev embedding there is a constant $C_3 > 0$ such that

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_3\|u\|^{p+2}_{H_0^1(\Omega)} - \mu\int_{\Omega} F(u)\,dx \text{ for every } u \in H_0^1(\Omega).$$

Therefore,

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} - \mu\int_{\Omega} F(u)\,dx \text{ for } \|u\|_{H_0^1(\Omega)} \leq \rho, \tag{2.45}$$

where

$$\rho = \left(\frac{1}{4C_3}\right)^{\frac{1}{p}}.$$

Note that (2.45) also holds when $\lambda = 0$ and $\rho = 1$. Hence, from now on we will assume that $\lambda \geq 0$. Let $0 < \alpha < \dfrac{4\pi}{\rho^2}$. From (2.3) and Lemma 2.2 we get a constant $C_4 > 0$ such that

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} - \mu C_4\int_{\Omega} \exp\left(\alpha\|u\|^2_{H_0^1(\Omega)}\left(\frac{u}{\|u\|_{H_0^1(\Omega)}}\right)^2\right)dx \text{ for } \|u\|_{H_0^1(\Omega)} \leq \rho.$$

Hence, from the Trudinger-Moser inequality, (B.8), we get

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} - \mu C_4 k_1 \text{ for } \|u\|_{H_0^1(\Omega)} \leq \rho.$$

Choosing

$$\mu_0 = \frac{\rho^2}{8C_4k_1},$$

we obtain

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{4}\left(\|u\|^2_{H_0^1(\Omega)} - \frac{\rho^2}{2}\right), \text{ for every } 0 < \mu < \mu_0, \quad \|u\|_{H_0^1(\Omega)} \leq \rho.$$

Hence,

$$I_{\epsilon,\lambda,\mu}(u) \geq a_1 \text{ for } \|u\|_{H_0^1(\Omega)} = \rho,$$

where

$$a_1 = \frac{\rho^2}{4}.$$

Let $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = N_0\phi_1\}$. By the Mountain Pass Theorem (Theorem B.18), we conclude that there is a sequence (u_n^ϵ) in $H_0^1(\Omega)$ and a number

$$c_\epsilon = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} I_{\epsilon,\lambda,\mu}(\gamma(s)),$$

such that

$$\lim_{n\to\infty} I_{\epsilon,\lambda,\mu}(u_n^\epsilon) = c_\epsilon \text{ and } \lim_{n\to\infty} I'_{\epsilon,\lambda,\mu}(u_n^\epsilon) = 0.$$

That is,

$$\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} - \int_\Omega J(u_n^\epsilon)\, dx \to c_\epsilon. \tag{2.46}$$

And there is a sequence $\tau_n \to 0$ such that

$$\left| \int_\Omega \nabla u_n^\epsilon \nabla v\, dx - \int_\Omega j(u_n^\epsilon) v\, dx \right| \leq \tau_n \|v\|_{H_0^1(\Omega)} \text{ for each } v \in H_0^1(\Omega). \tag{2.47}$$

We will now show that there is a constant $D > 0$ that does not depend on ϵ such that

$$\|u_n^\epsilon\|_{H_0^1(\Omega)} < D. \tag{2.48}$$

Fix $0 < \theta < \dfrac{1}{2}$ and assume that one of the conditions (i), (ii) or (iii) hold. Then, we may apply Lemma 2.5 to obtain a constant $D_{\theta,\lambda,\mu} > 0$ depending only on θ, λ and μ such that

$$|J(u_n^\epsilon)| < D_{\theta,\lambda,\mu} + \theta u_n^\epsilon j(u_n^\epsilon).$$

Since $a_1 \leq c_\epsilon \leq a_2$, we know from (2.46) that there is a constant $D_1 > 0$ such that

$$\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq D_1 + \theta \int_\Omega u_n^\epsilon j(u_n^\epsilon)\, dx.$$

Taking $v = u_n^\epsilon$ in (2.47) we also conclude that

$$\int_\Omega j(u_n^\epsilon) u_n^\epsilon\, dx < \|u_n^\epsilon\|^2_{H_0^1(\Omega)} + \tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)}.$$

Hence,

$$\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} < D_1 + \theta\|u_n^\epsilon\|^2_{H_0^1(\Omega)} + \tau_n\theta\|u_n^\epsilon\|_{H_0^1(\Omega)}.$$

Since $\theta < \dfrac{1}{2}$, (2.48) follows. We conclude that there is $u_\epsilon \in H_0^1(\Omega)$, $\|u_\epsilon\|_{H_0^1(\Omega)} < D$ such that

$$u_n^\epsilon \rightharpoonup u_\epsilon \text{ weakly in } H_0^1(\Omega).$$

We know that (u_n^ϵ) is a Palais-Smale sequence at a positive level. By Lemma 2.6, it follows that u_ϵ is a critical point of $I_{\epsilon,\lambda,\mu}$. The result then follows from Lemma 2.1 □

2.2 Convergence of the perturbed solutions

In this section, we study the convergence of the solutions u_ϵ of problem (2.4) obtained in Proposition 2.1. This proposition guarantees that there exists a constant $D > 0$ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D, \text{ for each } 0 < \epsilon < 1.$$

Hence, there exist $u \in H_0^1(\Omega)$ and a sequence (ϵ_n) in $(0,1)$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} u_{\epsilon_n} \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_{\epsilon_n} \to u \text{ in } L^r(\Omega) \text{ for every } r > 1, \\ u_{\epsilon_n} \to u \text{ a.e in } \Omega, \\ |u_{\epsilon_n}| \leq h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega). \end{cases} \tag{2.49}$$

Under additional conditions on f, we can apply regularity results discussed in Appendix A to conclude that u_{ϵ_n} are smooth and that u is continuous. Assume that there exist constants $0 < q_0 < 1$ and $0 < q_1 < 1$ such that

$$\lim_{s \to 0} \frac{|f(s)|}{s^{q_0}} < \infty, \tag{2.50}$$

and

$$\lim_{s \to 0^+} s^{1-q_1}|f'(s)| < \infty. \tag{2.51}$$

From Lemma A.2 we obtain a constant $K_1 > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(\Omega)} < K_1 \text{ for all } 0 < \epsilon_n < 1.$$

Then, it follows from Lemma 2.1 that $u_{\epsilon_n} \in C^{1,\nu}(\overline{\Omega})$, where ν is given by (2.2). We proceed by obtaining gradient estimates for the functions u_{ϵ_n}. Let Ω' be a smooth subdomain of Ω such that $\Omega' \subset \overline{\Omega'} \subset \Omega$. We would like to get a bound of the form

$$\sup_{\Omega'} |\nabla u_{\epsilon_n}(x)| < C < \infty,$$

for some constant $C > 0$ that does not depend on ϵ. Actually, we obtain a sharp result. There exists a function $Z \in C[0,\infty)$ with $Z(0) = 0$ such that

$$|\nabla u_{\epsilon_n}(x)| \leq MZ(u_{\epsilon_n}(x)) \text{ for all } x \in \Omega'.$$

It follows from Lemma A.3 that there exist a constant $M > 0$ that depends on Ω' but not on ϵ, and a universal constant $\epsilon_0 > 0$ such that

$$|\nabla u_{\epsilon_n}(x)|^2 \leq M(u_{\epsilon_n}(x)^{1-\beta} + u_{\epsilon_n}(x)) \leq 2MK_1 \text{ for every } x \in \Omega', \quad 0 < \epsilon < \epsilon_0. \tag{2.52}$$

Hence, it follows from the Arzela-Ascoli Theorem (Theorem B.5) that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, so that u is continuous and $0 \leq u \leq K_1$. In this section, we will show that u is a solution of problem (2.1) in the sense that

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u > 0\}} \Big(- u^{-\beta} + \lambda u^p + \mu f(u) \Big) \varphi, \tag{2.53}$$

for every $\varphi \in C_c^1(\Omega)$ and

$$u^{-\beta}\chi_{\{u>0\}} \in L^1_{loc}(\Omega).$$

We have

Lemma 2.8. *The function u is nontrivial and $u^{-\beta}\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.*

Proof of Lemma 2.8. First we show that u is nontrivial. Since u_{ϵ_n} is a critical point of $I_{\epsilon_n,\lambda,\mu}$, we have

$$\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n} = \lambda \int_\Omega u_{\epsilon_n}^{p+1} + \mu \int_\Omega f(u_{\epsilon_n})u_{\epsilon_n},$$

and

$$I_{\epsilon_n,\lambda,\mu}(u_{\epsilon_n}) = \frac{1}{2}\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega G_{\epsilon_n}(u_{\epsilon_n}) - \frac{\lambda}{p+1}\int_\Omega u_{\epsilon_n}^{p+1} - \mu \int_\Omega F(u_{\epsilon_n}) > a_1,$$

where a_1 is given by Proposition 2.1. Hence,

$$\begin{aligned} I_{\epsilon_n,\lambda,\mu}(u_{\epsilon_n}) = &\int_\Omega \left(G_{\epsilon_n}(u_{\epsilon_n}) - \frac{1}{2}g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\right) dx + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u_{\epsilon_n}^{p+1}\,dx \\ &+ \mu \int_\Omega \left(\frac{1}{2}f(u_{\epsilon_n})u_{\epsilon_n} - F(u_{\epsilon_n})\right) dx > a_1. \end{aligned} \tag{2.54}$$

Recall that $0 \le u_{\epsilon_n} \le K_1$ in Ω and consequently $0 \le u \le K_1$ in Ω. Hence, from the Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f(u_{\epsilon_n})u_{\epsilon_n}\,dx = \int_\Omega f(u)u\,dx.$$

From Lemma 2.2, we get

$$\lim_{n\to\infty} \int_\Omega F(u_{\epsilon_n})\,dx = \int_\Omega F(u)\,dx.$$

It is also clear that

$$\lim_{n\to\infty} \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\,dx = \int_\Omega u^{1-\beta}\,dx$$

and

$$\lim_{n\to\infty} \int_\Omega G_{\epsilon_n}(u_{\epsilon_n})\,dx = \frac{1}{1-\beta}\int_\Omega u^{1-\beta}\,dx.$$

Taking the above limits into account and letting $n \to \infty$ in (2.54) we obtain

$$\int_\Omega \left(\frac{u^{1-\beta}}{1-\beta} - \frac{u^{1-\beta}}{2}\right) dx + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u^{p+1}\,dx + \mu\int_\Omega \left(\frac{1}{2}f(u)u - F(u)\right) dx \ge a_1.$$

We proved that u is nontrivial. Now let $V \subset \Omega$ be a open set such that $\overline{V} \subset \Omega$. We will show that

$$\int_V u^{-\beta}\chi_{\Omega^+}\,dx < \infty.$$

Take $\zeta \in C_c^1(\Omega)$ such that $0 \le \zeta \le 1$ and $\zeta \equiv 1$ in V. Since u_{ϵ_n} is a nonnegative critical point of $I_{\epsilon_n,\lambda,\mu}$, we obtain

$$\int_\Omega g_{\epsilon_n}(u_{\epsilon_n})\zeta = \lambda \int_\Omega u_{\epsilon_n}^p \zeta + \mu \int_\Omega f(u_{\epsilon_n})\zeta - \int_\Omega \nabla u_{\epsilon_n} \nabla \zeta.$$

Since $u_{\epsilon_n} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, we get

$$\int_\Omega g_{\epsilon_n}(u_{\epsilon_n})\zeta \to \int_\Omega (\lambda u^p + \mu f(u))\zeta - \int_\Omega \nabla u \nabla \zeta \text{ as } n \to \infty. \tag{2.55}$$

Define the set $\Omega_\rho = \{x \in \Omega : u(x) \ge \rho\}$ for $\rho > 0$. It follows from (2.55) and by the definition of ζ that there exists a constant $C > 0$ that does not depend on ϵ nor on ρ such that

$$\int_{V \cap \Omega_\rho} \frac{u_{\epsilon_n}^q}{(u_{\epsilon_n} + \epsilon_n)^{q+\beta}} \le \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})\zeta < C \text{ for all } 0 < \epsilon_n < \epsilon_0, \quad \rho > 0,$$

where ϵ_0 is given by (2.52). Letting $n \to \infty$ and using Fatou's Lemma, we get

$$\int_V u^{-\beta} \chi_{\Omega_\rho} < C.$$

Letting $\rho \to 0$ and applying Fatou's Lemma again, we conclude that

$$\int_V u^{-\beta} \chi_{\{u>0\}} < \infty.$$

Since V was arbitrarily chosen, Lemma 2.8 is proved. □

We state the main result of the chapter.

Theorem 2.1. *Suppose that f satisfies (2.2), (2.3), (2.18), (2.50) and (2.51). The following assertions hold:*

(i) Fix $\lambda > 0$, suppose that $p > 1$ and that f satisfies one of the conditions (2.21) or (2.22). Then there exists $\mu_0 > 0$ such that problem (2.1) has a nontrivial nonnegative solution for every $0 < \mu < \mu_0$.

(ii) Fix $\lambda > 0$, suppose that $0 < p \le 1$ and that f satisfies (2.19), (2.23) and (2.24). Then there exists $\mu_0 > 0$ such that problem (2.1) has a nontrivial nonnegative solution for every $0 < \mu < \mu_0$.

(iii) Suppose that f satisfies (2.19), (2.25) and (2.26). Then, there exists $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \mu f(u) \text{ in } \Omega \\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

has a nontrivial nonnegative solution for every $0 < \mu < \mu_0$.

Before proving Theorem 2.1, we remark that we allow the nonlinearity f to change sign. Furthermore, we obtain

Corollary 2.1. (*i*) *Let $p > 1$. The problem*

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p \ \text{in } \Omega \\ u \not\equiv 0 \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \end{cases} \tag{2.56}$$

has a nonnegative and nontrivial solution for each $\lambda > 0$.

(*ii*) *Suppose that $0 < q < 1 < p$. For each $\lambda > 0$, there exists $\mu_0 > 0$ such that the problem*

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^q \ \text{in } \Omega \\ u \not\equiv 0 \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \end{cases}$$

has a nontrivial nonnegative solution for each $0 < \mu < \mu_0$.

(*iii*) *Suppose that $0 < p < 1$ and $r > 3$. For each $\lambda > 0$, there exists $\mu_0 > 0$ such that the problem*

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu u^r \ \text{in } \Omega \\ u \not\equiv 0 \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \end{cases}$$

has a nontrivial nonnegative solution for each $0 < \mu < \mu_0$.

(*iv*) *Suppose that $p > 0$ and let $i \in \{1, 2\}$. For each $\lambda \geq 0$, there exists $\mu_0 > 0$ such that the problem*

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + \mu f_i(u) \ \text{in } \Omega \\ u \not\equiv 0 \ \text{in } \Omega \\ u = 0 \ \text{on } \partial\Omega \end{cases}$$

has a nontrivial nonnegative solution for each $0 < \mu < \mu_0$, where $f_1(s) = s^k e^s$, $k > 0$ and $f_2(s) = e^s - 1$.

Item (*i*) of Corollary 2.1 was proved in [26]. Items (*ii*) and (*iii*) should be compared with [4], where nonsingular elliptic equations with concave and convex nonlinearities were studied. Item (*iv*) gives examples of f with exponential growth for which problem (2.1) is solvable.

Proof of Theorem 2.1. We follow ideas given in in [37] and [54]. Let (ϵ_n) and (u_{ϵ_n}) be the sequences defined in (2.49), and let u be given by Lemma 2.8. We will prove that u is a solution of (2.1). The nontriviality and continuity of u is guaranteed by Lemma 2.8. Also recall that $u_{\epsilon_n} \to u$ in $C^0_{loc}(\Omega)$. Let $\varphi \in C^1_c(\Omega)$. Since $u_{\epsilon_n} \in C^1(\overline{\Omega})$ is a solution of problem (2.4), we know that

$$\int_\Omega \nabla u_{\epsilon_n} \nabla \varphi = \int_\Omega (-g_{\epsilon_n}(u_{\epsilon_n}) + \lambda u_{\epsilon_n}^p + \mu f(u_{\epsilon_n}))\varphi. \tag{2.57}$$

We would like to let $n \to \infty$ in (2.57). Since the term $-g_{\epsilon_n}(u_{\epsilon_n})$ does not converge pointwisely to $u^{-\beta}\chi_{\{u>0\}}$, we need to consider an auxiliar function η that vanishes near the origin . Throughout this proof, we will denote the functions u_{ϵ_n} merely by u_ϵ and we will let $\epsilon \to 0$. Let $\eta \in C^\infty(\mathbb{R})$, $0 \le \eta \le 1$, $\eta(s) = 0$ for $s \le 1/2$, $\eta(s) = 1$ for $s \ge 1$. For $m > 0$ we define the function $\varrho := \varphi\eta(u_\epsilon/m)$. Note that ϱ belongs to $C_c^1(\Omega)$, because $u_\epsilon \in C^1(\overline{\Omega})$.

From continuity, the set $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ is open. Let $\tilde{\Omega}$ be an open set such that $\overline{support(\varphi)} \subset \tilde{\Omega}$ and $\overline{\tilde{\Omega}} \subset \Omega$. Let $\Omega_0 = \Omega_+ \cap \tilde{\Omega}$. Since $u_\epsilon \to u$ uniformly in $\tilde{\Omega}$, we know that for every $m > 0$ there is an $\epsilon_1 > 0$ such that

$$u_\epsilon(x) \le m/2 \text{ for every } x \in \tilde{\Omega} \setminus \Omega_0 \text{ and } 0 < \epsilon \le \epsilon_1. \tag{2.58}$$

Replacing φ by ϱ in (2.57) we obtain

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) = \int_{\tilde{\Omega}} (-g_\epsilon(u_\epsilon) + \lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m). \tag{2.59}$$

We break the previous integral as

$$A_\epsilon := \int_{\Omega_0} (-g_\epsilon(u_\epsilon) + \lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m)$$

and

$$B_\epsilon := \int_{\tilde{\Omega}\setminus\Omega_0} (-g_\epsilon(u_\epsilon) + \lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m).$$

Clearly, $B_\epsilon = 0$, whenever $0 < \epsilon \le \epsilon_1$ by (2.58) and the definition of η. We claim that

$$A_\epsilon \to \int_{\Omega_0} (-u^{-\beta} + \lambda u^p + \mu f(u))\varphi\eta(u/m) \text{ as } \epsilon \to 0. \tag{2.60}$$

Indeed, $u_\epsilon \to u$ uniformly in Ω_0. Then,

$$\int_{\Omega_0} (\lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m)\, dx \to \int_{\Omega_0} (\lambda u^p + \mu f(u))\varphi\eta(u/m)\, dx \text{ as } \epsilon \to 0.$$

Hence, we only need to show that

$$\int_{\Omega_0} -g_\epsilon(u_\epsilon)\varphi\eta(u_\epsilon/m) \to \int_{\Omega_0} -u^{-\beta}\varphi\eta(u/m) \text{ as } \epsilon \to 0.$$

If $u \le m/4$ then, for $\epsilon > 0$ sufficiently small, we have $u_\epsilon \le m/2$. Consequently, from the definition of η,

$$0 = \int_{\Omega_0\cap\{u\le m/4\}} -u^{-\beta}\varphi\eta(u/m) = \lim_{\epsilon\to 0}\int_{\Omega_0\cap\{u\le m/4\}} -g_\epsilon(u_\epsilon)\varphi\eta(u_\epsilon/m).$$

If $u > m/4$, then $u_\epsilon \ge m/8$ for $\epsilon > 0$ small enough. We then apply the Dominated Convergence Theorem as $\epsilon \to 0$ to get

$$\int_{\Omega_0\cap\{u>m/4\}} -u^{-\beta}\varphi\eta(u/m) = \lim_{\epsilon\to 0}\int_{\Omega_0\cap\{u>m/4\}} -g_\epsilon(u_\epsilon)\varphi\eta(u_\epsilon/m).$$

We have proved claim (2.60). Hence,

$$\lim_{\epsilon\to 0}\int_{\tilde{\Omega}} (-g_\epsilon(u_\epsilon) + \lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m) = \int_{\Omega_0} (-u^{-\beta} + \lambda u^p + \mu f(u))\varphi\eta(u/m).$$

We take the limit in m to conclude that

$$\int_{\Omega_0}(-u^{-\beta}+\lambda u^p+\mu f(u))\varphi\eta(u/m)\to\int_{\Omega_0}(-u^{-\beta}+\lambda u^p+\mu f(u))\varphi \text{ as } m\to 0, \tag{2.61}$$

since $\eta(u/m)\leq 1$ and $u^{-\beta}\chi_{\Omega^+}+\lambda u^p+\mu f(u)\in L^1(\tilde{\Omega})$, according to Lemma 2.8.

We proceed with the integral on the left side of (2.59),

$$\int_{\Omega}\nabla u_\epsilon\nabla(\varphi\eta(u_\epsilon/m)):=\int_{\tilde{\Omega}}(\nabla u_\epsilon\nabla\varphi)\eta(u_\epsilon/m)+C_\epsilon. \tag{2.62}$$

Consequently,

$$\int_{\tilde{\Omega}}(\nabla u_\epsilon\nabla\varphi)\eta(u_\epsilon/m)\to\int_{\tilde{\Omega}}(\nabla u\nabla\varphi)\eta(u/m) \text{ as } \epsilon\to 0,$$

since $u_\epsilon\rightharpoonup u$ in $H_0^1(\Omega)$ and $u_\epsilon\to u$ uniformly in $\tilde{\Omega}$. Consequently, by the Dominated Convergence Theorem,

$$\int_{\tilde{\Omega}}(\nabla u\nabla\varphi)\eta(u/m)\to\int_{\tilde{\Omega}}\nabla u\nabla\varphi \text{ as } m\to 0. \tag{2.63}$$

We claim that

$$C_\epsilon:=\int_{\tilde{\Omega}}\frac{|\nabla u_\epsilon|^2}{m}\eta'(u_\epsilon/m)\varphi\to 0 \text{ as } \epsilon\to 0 \quad (\text{ and then as } m\to 0). \tag{2.64}$$

Let $Z_0(u)=u^{1-\beta}+u$. The estimate $|\nabla u_\epsilon|^2\leq MZ_0(u_\epsilon)$ in $\tilde{\Omega}$ provided by (2.52) yields

$$\begin{aligned}\limsup_{\epsilon\to 0}|C_\epsilon| &\leq\frac{M}{m}\lim_{\epsilon\to 0}\int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u_\epsilon\leq m\}}Z_0(u_\epsilon)|\eta'(u_\epsilon/m)\varphi|\\ &\leq M\lim_{\epsilon\to 0}\int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u_\epsilon\leq m\}}\frac{Z_0(u_\epsilon)|\eta'(u_\epsilon/m)\varphi|}{u_\epsilon}.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{\epsilon\to 0}|C_\epsilon| &\leq M\sup|\eta'|\sup|\varphi|\lim_{\epsilon\to 0}\int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u_\epsilon\leq m\}}\frac{Z_0(u_\epsilon)}{u_\epsilon}\\ &\leq M\sup|\eta'|\sup|\varphi|\int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u\leq m\}}(1+u^{-\beta}),\end{aligned}$$

for every $m>0$.

Thus invoking Lemma 2.8 and letting $m\to 0$, (2.64) is proved. As an immediate consequence of (2.59)–(2.64), we have

$$\int_{\Omega}\nabla u\nabla\varphi=\int_{\Omega\cap\{u>0\}}\Big(-u^{-\beta}+\lambda u^p+\mu f(u)\Big)\varphi,$$

for every $\varphi\in C_c^1(\Omega)$. This concludes the proof of Theorem 2.1. □

3 A problem in higher dimension

In this chapter we study the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u + u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 < \beta < 1$, $\lambda \geq 0$ and $1 < p < 2^* - 1$, with $2^* = \dfrac{2N}{N-2}$. We will again use a perturbation argument. Consider the problem

$$\begin{cases} -\Delta u + g_\epsilon(u) = \lambda u + u^p \text{ in } \Omega \\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.2}$$

where $g_\epsilon(u) \to u^{-\beta}$ for $u > 0$ pointwisely as $\epsilon \to 0$ and is again given by

$$g_\epsilon(s) = \begin{cases} \dfrac{s^q}{(s+\epsilon)^{q+\beta}} \text{ for } s \geq 0 \\ 0 \text{ for } s < 0, \end{cases} \tag{3.3}$$

with $0 < q < 1/2$. We define the functional $I_{\epsilon,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated to problem (3.2) by

$$I_{\epsilon,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_\epsilon(u) - \frac{\lambda}{2}\int_\Omega (u^+)^2 - \frac{1}{1+p}\int_\Omega (u^+)^{1+p}, \tag{3.4}$$

where $G_\epsilon(s) = \displaystyle\int_0^s g_\epsilon(t)\,dt$. Consequently, if $u_\epsilon \in H_0^1(\Omega)$ is a critical point of $I_{\epsilon,\lambda}$ then

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda\int_\Omega (u_\epsilon^+)v + \int_\Omega (u_\epsilon^+)^p v, \text{ for all } v \in H_0^1(\Omega). \tag{3.5}$$

Choosing $v = u_\epsilon^-$ in (3.5), we obtain

$$-\int_\Omega |\nabla(u_\epsilon)^-|^2 = 0.$$

Hence, $u_\epsilon \geq 0$ in Ω. We conclude that critical points $u_\epsilon \in H_0^1(\Omega)$ of $I_{\epsilon,\lambda}$ are nonnegative and

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda\int_\Omega u_\epsilon v + \int_\Omega u_\epsilon^p v, \text{ for all } v \in H_0^1(\Omega). \tag{3.6}$$

Therefore, critical points of $I_{\epsilon,\lambda}$ are weak solutions of problem (3.2). We also have the following estimates (uniform for ϵ) on the functions g_ϵ and G_ϵ.

Lemma 3.1. *The following assertions hold*

(i)

$$0 < g_\epsilon(s) < s^{-\beta} \textit{ and } 0 < G_\epsilon(s) \leq \frac{1}{1-\beta}s^{1-\beta} \textit{ for } s \geq 0. \tag{3.7}$$

(ii)

$$sg'_\epsilon(s) = \frac{qs^q}{(s+\epsilon)^{q+\beta}} - \frac{(q+\beta)s^{q+1}}{(s+\epsilon)^{q+\beta+1}}. \tag{3.8}$$

(iii)

$$G_\epsilon(s) \geq \frac{1}{2}g_\epsilon(s)s, \ \textit{for every } s \geq 0.$$

(iv) For each $M > 0$ there exists $\overline{\delta} = \overline{\delta}(M) > 0$ such that

$$G_\epsilon(s) \geq \frac{M}{2}s^2 \ \textit{for } 0 \leq s < \overline{\delta} < 1.$$

Proof of Lemma 3.1. Items (i) and (ii) are clear from the definition of g_ϵ, see (3.3). Now we prove item (iii). Let $\widetilde{B}_\epsilon(s) = G_\epsilon(s) - \frac{1}{2}g_\epsilon(s)s$. We have that $\widetilde{B}_\epsilon(0) = 0$ and

$$\widetilde{B}'_\epsilon(s) = g_\epsilon(s) - \frac{1}{2}g_\epsilon(s) - \frac{s}{2}g'_\epsilon(s) = \frac{1}{2}\left(g_\epsilon(s) - sg'_\epsilon(s)\right).$$

Therefore, $\widetilde{B}'_\epsilon(s) \geq 0$ if and only if

$$g_\epsilon(s) \geq sg'_\epsilon(s).$$

From (3.8), this inequality will be true if

$$\frac{s^q}{(s+\epsilon)^{q+\beta}} \geq \frac{qs^q}{(s+\epsilon)^{q+\beta}}. \tag{3.9}$$

Since $q < 1/2$, (3.9) holds for each $s \geq 0$. We conclude that $\widetilde{B}_\epsilon$ is nondecreasing. This proves item (iii).

We now prove item (iv). Note that

$$g_\epsilon(s) = \frac{s^q}{(s+\epsilon)^{q+\beta}} \geq \frac{s^q}{(s+1)^{q+\beta}} = \frac{s^{q-\frac{1}{2}}}{(s+1)^{q+\beta}}s^{\frac{1}{2}} \text{ for } s \geq 0.$$

Hence,

$$g_\epsilon(s) \geq \frac{1}{2^{q+\beta}}s^{q-\frac{1}{2}}s^{\frac{1}{2}} \text{ for } 0 \leq s < 1.$$

Therefore, from the fact that $0 < q < \frac{1}{2}$, it follows that for each $M > 0$ there exists $\overline{\delta} = \overline{\delta}(M) < 1$ such that

$$g_\epsilon(s) \geq M\sqrt{s} > Ms \text{ for } 0 \leq s < \overline{\delta} < 1.$$

The result then follows from item (iii). □

With these estimates we are able to obtain bounds for weak solutions of problem (3.2). Indeed, let u_ϵ be a nontrivial critical point of $I_{\epsilon,\lambda}$. Then, (3.4) and (3.6) yield

$$I_{\epsilon,\lambda}(u_\epsilon) = \int_\Omega \left(G_\epsilon(u_\epsilon) - \frac{1}{2}g_\epsilon(u_\epsilon)u_\epsilon\right)dx + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u_\epsilon^{p+1}\,dx.$$

Hence, from item (*iii*) of Lemma 3.1, and the fact that $p > 1$, we get

$$I_{\epsilon,\lambda}(u_\epsilon) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega u_\epsilon^{p+1}\, dx > 0.$$

We conclude that

Lemma 3.2. *The following assertions hold:*

(i) Critical points of $I_{\epsilon,\lambda}$ are nonnegative weak solutions of problem (3.2).

(ii) If $u_\epsilon \in H_0^1(\Omega)$ is a nonnegative nontrivial weak solution of problem (3.2), then $I_{\epsilon,\lambda}(u_\epsilon) > 0$. Furthermore, if there exists a constant $C > 0$ such that $0 < I_{\epsilon,\lambda}(u_\epsilon) < C$, then there exists a constant $D > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D.$$

Proof of Lemma 3.2.: We only need to prove item (*ii*). If $I_{\epsilon,\lambda}(u_\epsilon) < C$, we get

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega u_\epsilon^{p+1}\, dx < C.$$

Hence, $\|u_\epsilon\|_{L^{p+1}(\Omega)} < C_2$ for some constant $C_2 > 0$. The result then follows from the fact that

$$\frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2 \leq \frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2 + \int_\Omega G_\epsilon(u_\epsilon) = I_{\epsilon,\lambda}(u_\epsilon) + \frac{\lambda}{2}\int_\Omega u_\epsilon^2 - \frac{1}{1+p}\int_\Omega u_\epsilon^{1+p}.$$

□

As in Chapter 2, we define

$$j_\epsilon(s) = \lambda(s^+) + (s^+)^p - g_\epsilon(s) \text{ for } s \in \mathbb{R},$$

so that

$$I_{\epsilon,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u) \text{ for all } u \in H_0^1(\Omega), \tag{3.10}$$

where $J_\epsilon(s) = \int_0^s j_\epsilon(t)\, dt$. Throughout this chapter, we will denote j_ϵ and J_ϵ merely by j and J respectively.

We finish this section by giving a result similar to Lemma 2.3.

Lemma 3.3. *The following assertions hold*

(i) For each $R > 0$, there exists a constant $C > 0$ that does not depend on ϵ such that

$$\max\{|J(s)|, |sj(s)|\} \leq C \textit{ for all } s \leq R.$$

(ii) There exists a constant $R_2 > 0$ such that $J(s) \geq 0$ for all $s \geq R_2$.

Proof of Lemma 3.3. Note that

$$J(s) = \frac{\lambda}{2}(s^+)^2 + \frac{(s^+)^{p+1}}{p+1} - G_\epsilon(s) \text{ for } s \in \mathbb{R}, \tag{3.11}$$

and

$$sj(s) = \lambda(s^+)^2 + (s^+)^{p+1} - sg_\epsilon(s) \text{ for } s \in \mathbb{R}.$$

Then, it follows from item (i) of Lemma 3.1 that

$$|J(s)| \leq \frac{\lambda}{2}R^2 + \frac{R^{p+1}}{p+1} + \frac{1}{1-\beta}R^{1-\beta} \text{ for } 0 \leq s \leq R,$$

and

$$|sj(s)| \leq \lambda R^2 + R^{p+1} + R^{1-\beta} \text{ for } 0 \leq s \leq R.$$

This proves (v). Item (vi) is a consequence of the fact that

$$J(s) \geq \frac{s^{p+1}}{p+1} - \frac{1}{1-\beta}s^{1-\beta} \text{ for } s \in \mathbb{R}.$$

This proves Lemma 3.3. □

In Section 3.1 we study the perturbed problem (3.2) when $1 < p < 2^* - 1$. Next, we study the convergence of these solutions.

3.1 Solutions to the perturbed problem

The goal of this section is to show that problem (3.2) possesses a nonnegative nontrivial solution u_ϵ in the subcritical case $1 < p < 2^* - 1$. The structure of this section is very similar to Section 2.1. First we study compactness of the functional $I_{\epsilon,\lambda}$ and then we obtain solutions u_ϵ of problem (3.2). The following result is analogous to Lemma 2.5.

Lemma 3.4. *Suppose that $1 < p < 2^* - 1$. For each $\frac{1}{1+p} < \theta < 1/2$ there exists $R_{\theta,\lambda} > 0$ such that*

$$0 \leq J(s) \leq \theta sj(s) \text{ for } s \geq R_{\theta,\lambda}. \tag{3.12}$$

Proof of Lemma 3.4. Let $B_\epsilon(s) = J(s) - \theta sj(s)$. We have

$$B_\epsilon'(s) = (1-\theta)j(s) - \theta sj'(s).$$

Hence,

$$B_\epsilon'(s) = -(1-\theta)g_\epsilon(s) + \theta sg_\epsilon'(s) + ((1-\theta)s^p - \theta ps^p) + \lambda((1-\theta)s - \theta s).$$

From Lemma 3.3 we obtain

$$|sg_\epsilon'(s)| \leq q|s|^{-\beta} + (q+\beta)|s|^{-\beta} \to 0 \text{ as } s \to \infty.$$

and

$$(1-\theta)g_\epsilon(s) \to 0 \text{ as } s \to \infty.$$

Hence, there exists $R > 0$ such that

$$|(1-\theta)g_\epsilon(s)| + |sg'_\epsilon(s)| < \frac{1}{2} \text{ for } s \geq R.$$

Therefore,

$$B'_\epsilon(s) < \frac{1}{2} + ((1-\theta-\theta p))s^p + \lambda(1-2\theta)s \text{ for } s \geq R. \tag{3.13}$$

Since $\theta > \dfrac{1}{p+1}$, it follows that $1-\theta-\theta p < 0$. From the fact that $p > 1$, we conclude that the right hand side of (3.13) converges to $-\infty$ as $s \to \infty$. Hence, there exists a constant $T_{\theta,\lambda} > 0$ such that

$$B'_\epsilon(s) < -1 \text{ for } s \geq T_{\theta,\lambda}.$$

Note that

$$B_\epsilon(2T_{\theta,\lambda}) \leq R_1,$$

where

$$R_1 = \frac{(2T_{\theta,\lambda})^{p+1}}{p+1} + \frac{\lambda(2T_{\theta,\lambda})^2}{2} + \theta(2T_{\theta,\lambda})^{1-\beta}.$$

Therefore, there exists a constant $R_{\theta,\lambda} > 0$ such that

$$B_\epsilon(s) \leq -s + R_{\theta,\lambda} \text{ for } s \geq 2T_{\theta,\lambda}.$$

Hence, $B_\epsilon(s) \leq 0$ for $s \geq \max\{R_{\theta,\lambda}, 2T_{\theta,\lambda}\}$. This proves Lemma 3.4. □

As in Section 2.1, we turn our attention into showing that problem (3.2) possesses a nontrivial solution $u_\epsilon \geq 0$ that is a Mountain Pass. To do that we need to prove that there exist constants $a_1 > 0$ and $0 < \rho < 1$ such that

$$I_{\epsilon,\lambda,\mu}(u) \geq a_1 \text{ for } \|u\|_{H_0^1(\Omega)} = \rho,$$

and that there exists an element $\phi_0 \in H_0^1(\Omega)$ such that

$$\|\phi_0\|_{H_0^1(\Omega)} \geq 1 \text{ and } I_{\epsilon,\lambda,\mu}(\phi_0) < 0. \tag{3.14}$$

The lemma below guarantees that (3.14) holds for $\phi_0 = N_0\phi_1$, where $\phi_1 \in H_0^1(\Omega)$ is the first eigenfunction of the operator $-\Delta$ with $\|\phi_1\|_{H_0^1(\Omega)} = 1$. We have

Lemma 3.5. *Suppose that $1 < p < 2^* - 1$ and that $\lambda \geq 0$. There exist constants $N_0 > 0$, $a_2 > 0$ and $b_1 > 0$ such that*

$$I_{\epsilon,\lambda}(N_0\phi_1) < -b_1 < 0, \text{ for every } 0 < \epsilon < 1, \tag{3.15}$$

and

$$\sup_{0\leq s\leq 1} I_{\epsilon,\lambda}(sN_0\phi_1) < a_2 \text{ for every } \lambda \geq 0,\ 0 < \epsilon < 1. \tag{3.16}$$

Moreover, these constants do not depend on λ.

Proof of Lemma 3.5. For each $t > 0$, we have

$$I_{\epsilon,\lambda}(t\phi_1) = \frac{t^2}{2} + \int_\Omega G_\epsilon(t\phi_1) - \frac{\lambda t^2}{2}\int_\Omega \phi_1^2\,dx - \frac{t^{p+1}}{p+1}\int \phi_1^{1+p}\,dx.$$

Hence,

$$I_{\epsilon,\lambda}(t\phi_1) \le \frac{t^2}{2} + t^{1-\beta}\int_\Omega |\phi_1|^{1-\beta} - \frac{t^{p+1}}{p+1}\int \phi_1^{1+p}\,dx. \tag{3.17}$$

Since $p+1 > 2 > 1-\beta$, inequality (3.15) then follows by taking t large enough in (3.17). We also have

$$I_{\epsilon,\lambda}(sN_0\phi_1) \le \frac{s^2N_0^2}{2} + \int_\Omega G_\epsilon(sN_0\phi_1) - \frac{s^{p+1}N_0^{p+1}}{p+1}\int \phi_1^{1+p}\,dx, \text{ for every } s \ge 0.$$

Consequently, we get

$$I_{\epsilon,\lambda}(s\phi_1) \le \frac{s^2N_0^2}{2} + \frac{s^{1-\beta}N_0^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta}, \text{ for every } s \ge 0.$$

We conclude that

$$\sup_{0\le s\le 1} I_{\epsilon,\lambda}(sN_0\phi_1) < a_2,$$

where

$$a_2 = \frac{N_0^2}{2} + \frac{N_0^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta}.$$

This proves (3.16). We have proved Lemma 3.5. □

We now get a compactness result.

Lemma 3.6. *Fix $0 < \epsilon < 1$ and suppose that $1 < p < 2^* - 1$. The functional $I_{\epsilon,\lambda}$ defined in (3.10) satisfies the Palais-Smale condition.*

Proof of Lemma 3.6. Throughout this proof we denote $\|\cdot\|_{H_0^1(\Omega)}$ by $\|\cdot\|$. Let $(v_n^\epsilon)_{n\in\mathbb{N}}$ be a Palais-Smale sequence for $I_{\epsilon,\lambda}$ in $H_0^1(\Omega)$. That is, there exists $c \in \mathbb{R}$ such that

$$\frac{1}{2}\|v_n^\epsilon\|^2 - \int_\Omega J(v_n^\epsilon)\,dx \to c \text{ as } n \to \infty, \tag{3.18}$$

and there is a sequence $\tau_n \to 0$ such that

$$\left|\int_\Omega \nabla v_n^\epsilon \nabla w\,dx - \int_\Omega j(v_n^\epsilon)w\,dx\right| \le \tau_n\|w\| \text{ for each } w \in H_0^1(\Omega). \tag{3.19}$$

We will show that there is a constant $D > 0$ that does not depend on ϵ such that

$$\|v_n^\epsilon\| < D. \tag{3.20}$$

Fix $\dfrac{1}{p+1} < \theta < \dfrac{1}{2}$. From Lemma 3.4 there is a constant $R_{\theta,\lambda} > 0$ depending only on θ and λ such that

$$0 \le J(t) \le \theta t j(t) \text{ for } t \ge R_{\theta,\lambda}.$$

From itens (v) and (vi) of Lemma 3.3, we may find a constant $D_{\theta,\lambda} > 0$ such that

$$J(v_n^\epsilon) < D_{\theta,\lambda} + \theta v_n^\epsilon j(v_n^\epsilon).$$

We know from (3.18) that there is a constant $D_1 > 0$ such that

$$\frac{1}{2}\|v_n^\epsilon\|^2 \leq D_1 + \theta \int_\Omega v_n^\epsilon j(v_n^\epsilon)\, dx.$$

Taking $w = v_n^\epsilon$ in (3.19) we also conclude that

$$\int_\Omega j(v_n^\epsilon) v_n^\epsilon \, dx < \|v_n^\epsilon\|^2 + \tau_n \|v_n^\epsilon\|.$$

Hence,

$$\frac{1}{2}\|v_n^\epsilon\|^2 < D_1 + \theta\|v_n^\epsilon\|^2 + \tau_n\theta\|v_n^\epsilon\|.$$

Since $\theta < \dfrac{1}{2}$, (3.20) follows. Since J_ϵ has subcritical growth at infinity (see Theorem B.16), Lemma 3.6 follows. □

Next, we obtain solutions for the perturbed problem (3.2).

Proposition 3.1. *Suppose that $\lambda \geq 0$ and let $a_2 > 0$ be given by Lemma 3.5. Then, there is a nonnegative solution u_ϵ of problem (3.2) and there exist constants $a_1 > 0$ and $D > 0$ that do not depend on ϵ such that*

$$0 < a_1 \leq I_{\epsilon,\lambda}(u_\epsilon) \leq a_2,$$

and

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D.$$

Proof of Proposition 3.1. Let $0 < \overline{\delta} < 1$ be given by item (iv) of Lemma 3.1. Note that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\{u<\overline{\delta}\}} G_\epsilon(u) - \frac{\lambda}{2}\int_\Omega (u^+)^2 - \frac{1}{p+1}\int_\Omega (u^+)^{p+1} \text{ for every } u \in H_0^1(\Omega).$$

Choosing $M = \lambda$ in item (iv) of Lemma 3.3, we obtain

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\{u>\overline{\delta}\}} u^2 - \frac{1}{p+1}\int_\Omega (u^+)^{p+1} \text{for every } u \in H_0^1(\Omega).$$

Observe that there exists a constant $C_1 > 0$ such that

$$s^2 \leq C_1 s^{p+1} \text{ for } s \geq \overline{\delta}.$$

Hence, there exists a constant $C_2 > 0$ such that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_2\int_\Omega |u|^{1+p} \text{ for every } u \in H_0^1(\Omega).$$

Hence, from the Sobolev embedding there is a constant $C_3 > 0$ such that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_3\|u\|^{p+1}_{H_0^1(\Omega)}.$$

Therefore,

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{4}\|u\|^2_{H^1_0(\Omega)} \text{ for } \|u\|_{H^1_0(\Omega)} \leq \rho,$$

where

$$\rho = \left(\frac{1}{4C_3}\right)^{\frac{1}{p-1}}.$$

Also,

$$I_{\epsilon,\lambda}(u) \geq a_1 \text{ for } \|u\|_{H^1_0(\Omega)} = \rho,$$

where

$$a_1 = \frac{\rho^2}{4}.$$

Let $\Gamma = \{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = N_0\phi_1\}$. By the Mountain Pass Theorem (Theorem B.18) we conclude that there is a sequence (u_n^ϵ) in $H^1_0(\Omega)$ and a number

$$c_\epsilon = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} I_{\epsilon,\lambda}(\gamma(s)),$$

such that

$$\lim_{n\to\infty} I_{\epsilon,\lambda}(u_n^\epsilon) = c_\epsilon \text{ and } \lim_{n\to\infty} I'_{\epsilon,\lambda}(u_n^\epsilon) = 0.$$

Since (u_n^ϵ) is a Palais-Smale sequence, we conclude from Lemma 3.6 that up to a subsequence, there exists $u_\epsilon \in H^1_0(\Omega)$ such that $u_n^\epsilon \to u_\epsilon$ strongly in $H^1_0(\Omega)$. From the fact that $I_{\epsilon,\lambda}$ is of class C^1, we conclude that $I'_{\epsilon,\lambda}(u_\epsilon) = 0$. Therefore, u_ϵ is a critical point of $I_{\epsilon,\lambda}$. From Lemma 3.5 we know that $a_1 \leq c_\epsilon \leq a_2$. Consequently,

$$a_1 \leq I_{\epsilon,\lambda}(u_\epsilon) \leq a_2.$$

From Lemma, 3.2, we conclude that $u_\epsilon \geq 0$ and that there exists $D > 0$ such that

$$\|u_\epsilon\|_{H^1_0(\Omega)} < D.$$

This proves the result. □

3.2 Convergence of the perturbed solutions

In this section. we study the convergence of the solutions u_ϵ of problem (3.2) obtained in Proposition 3.1, which implies that there exists a constant $D > 0$ such that

$$\|u_\epsilon\|_{H^1_0(\Omega)} < D, \text{ for each } 0 < \epsilon < 1.$$

Hence, there exist $u \in H^1_0(\Omega)$ and a sequence (ϵ_n) in $(0,1)$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} u_{\epsilon_n} \rightharpoonup u \text{ weakly in } H^1_0(\Omega), \\ u_{\epsilon_n} \to u \text{ in } L^r(\Omega) \text{ for every } r > 1, \\ u_{\epsilon_n} \to u \text{ a.e in } \Omega, \\ |u_{\epsilon_n}| \leq h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega). \end{cases} \tag{3.21}$$

We may apply Lemma A.1 to obtain a constant $K_1 > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(\Omega)} < K_1 \text{ for all } 0 < \epsilon_n < 1.$$

Since u_{ϵ_n} is a solution of problem (3.2), we get

$$\Delta u_{\epsilon_n} \in L^\infty(\Omega) \text{ for all } n \in \mathbb{N}.$$

Then, it follows from elliptic regularity theory and from rhe Sobolev Embedding (Theorems B.13 and B.14) that $u_{\epsilon_n} \in C^1(\overline{\Omega})$. As in Chapter 2, we proceed to obtain gradient estimates for the solutions u_{ϵ_n}. Lemma A.3 implies that there exists a constant $\epsilon_0 > 0$ such that for each smooth subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there exists a constant $M > 0$ that depends on Ω' but not on ϵ such that

$$|\nabla u_\epsilon(x)|^2 \leq M(u_\epsilon(x)^{1-\beta} + u_\epsilon(x)) \leq 2MK_1 \text{ for every } x \in \Omega', \quad 0 < \epsilon < \epsilon_0. \tag{3.22}$$

Hence, it follows from the Arzela-Ascoli Theorem (Theorem B.5) that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, so that u is continuous and $0 \leq u \leq K_1$.

As in Chapter 2, will show that u is a nontrivial solution of (3.1) in the sense that

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} \big(- u^{-\beta} + \lambda u + u^p \big)\varphi, \tag{3.23}$$

for every $\varphi \in C^1_c(\Omega)$ and

$$u^{-\beta}\chi_{\{u>0\}} \in L^1_{loc}(\Omega).$$

First, we prove the following result.

Lemma 3.7. *The sequence (u_{ϵ_n}) of solutions obtained in Proposition 3.1 and defined in (3.21) has a subsequence which converges weakly in $H^1_0(\Omega)$ to a nontrivial function $u \in H^1_0(\Omega)$ and $u^{-\beta}\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.*

Proof of Lemma 3.7. First we show that u is nontrivial. Since u_{ϵ_n} is a nonnegative critical point of $I_{\epsilon_n,\lambda}$, we have

$$\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n} = \lambda \int_\Omega u^2_{\epsilon_n} + \int_\Omega u^{p+1}_{\epsilon_n}\, dx,$$

and

$$I_{\epsilon_n,\lambda}(u_{\epsilon_n}) = \frac{1}{2}\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega G_{\epsilon_n}(u_{\epsilon_n}) - \frac{\lambda}{2}\int_\Omega u^2_{\epsilon_n} - \frac{1}{p+1}\int_\Omega u^{p+1}_{\epsilon_n} > a_1,$$

where a_1 is given by Proposition 3.1. Hence,

$$I_{\epsilon_n,\lambda}(u_{\epsilon_n}) = \int_\Omega \left(G_{\epsilon_n}(u_{\epsilon_n}) - \frac{1}{2}g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\right) dx + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u^{p+1}_{\epsilon_n}\, dx > a_1. \tag{3.24}$$

It is also clear that

$$\lim_{n\to\infty} \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\, dx = \int_\Omega u^{1-\beta}\, dx,$$

and

$$\lim_{n\to\infty}\int_\Omega G_{\epsilon_n}(u_{\epsilon_n})\,dx = \frac{1}{1-\beta}\int_\Omega u^{1-\beta}\,dx.$$

Taking the above claims into account and letting $n \to \infty$ in (3.24) we obtain

$$\int_\Omega \left(\frac{u^{1-\beta}}{1-\beta} - \frac{u^{1-\beta}}{2}\right) dx + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u^{p+1}\,dx \geq a_1.$$

We proved that u is nontrivial.

Let $V \subset \Omega$ be a open set such that $\overline{V} \subset \Omega$. Take $\zeta \in C_c^1(\Omega)$ such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ in V. Since u_ϵ is a critical point of $I_{\epsilon,\lambda}$, we obtain

$$\int_{\{u_\epsilon<1-\epsilon\}} g_\epsilon(u_\epsilon)\zeta = \int_\Omega \lambda u_\epsilon\zeta + \int_\Omega u_\epsilon^p\zeta - \int_\Omega \nabla u_\epsilon\nabla\zeta - \int_{\{u_\epsilon\geq 1-\epsilon\}} g_\epsilon(u_\epsilon)\zeta.$$

Since $u_\epsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_\epsilon \to u$ uniformly in compact subsets of Ω, we get

$$\int_{\{u_\epsilon<1-\epsilon\}} g_\epsilon(u_\epsilon)\zeta \to \int_\Omega (\lambda u + u^p)\zeta - \int_\Omega \nabla u\nabla\zeta + \int_{\{u\geq 1\}} u^{-\beta}\zeta \text{ as } \epsilon \to 0. \tag{3.25}$$

Define the set $\Omega_\rho = \{x \in \Omega : u(x) \geq \rho\}$ for $\rho > 0$. It follows from (3.25) that there exists a constant $C > 0$ that does not depend on ϵ nor on ρ such that

$$\int_{V\cap\Omega_\rho} \frac{u_\epsilon^q}{(u_\epsilon+\epsilon)^{q+\beta}}\chi_{\{u_\epsilon<1-\epsilon\}}\zeta \leq \int_{\{u_\epsilon<1-\epsilon\}} g_\epsilon(u_\epsilon)\zeta < C \text{ for all } 0 < \epsilon < \epsilon_0, \quad \rho > 0,$$

where ϵ_0 is given by (3.22). Letting $\epsilon \to 0$ and using Fatou's Lemma, we then get

$$\int_V u^{-\beta}\chi_{\Omega_\rho} < C.$$

Letting $\rho \to 0$ and applying Fatou's Lemma again, we conclude that

$$\int_V u^{-\beta}\chi_{\{u>0\}} < \infty.$$

Since V was arbitrarily chosen, Lemma 3.7 is proved. □

We now prove the main result of this chapter.

Theorem 3.1. *If $1 < p < 2^* - 1$ in (3.1), then problem (3.1) has a nontrivial nonnegative solution for each $\lambda \geq 0$.*

By taking $\lambda = 0$ in item (i), we conclude that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

is solvable. This is a version of the result in [26].

Proof of Theorem 3.1. We proceed in a similar way as in Chapter 2. Let (ϵ_n) and (u_{ϵ_n}) be the sequences defined in (3.21), and let u be given by Lemma 3.7. We will prove that u

is a solution of (3.1). The nontriviality and continuity of u is guaranteed by Lemma 3.7. This Lemma also implies that $u^{-\beta}\chi_{\{u>0\}} \in L^1_{loc}(\Omega)$, so that we only need to prove (3.23). Also recall that $u_{\epsilon_n} \to u$ in $C^0_{loc}(\Omega)$. Let $\varphi \in C^1_c(\Omega)$. Since $u_{\epsilon_n} \in C^1(\overline{\Omega})$ is a solution of problem (3.2), we know that

$$\int_\Omega \nabla u_{\epsilon_n} \nabla \varphi = \int_\Omega (-g_{\epsilon_n}(u_{\epsilon_n}) + \lambda u_{\epsilon_n} + u^p_{\epsilon_n})\varphi. \tag{3.26}$$

We would like to let $n \to \infty$ in (3.26). Since the term $-g_{\epsilon_n}(u_{\epsilon_n})$ does not converge pointwisely to $u^{-\beta}\chi_{\{u>0\}}$, we need to consider an auxiliar function η that vanishes near the origin . Throughout this proof, we will denote the functions u_{ϵ_n} merely by u_ϵ and we will let $\epsilon \to 0$. Let $\eta \in C^\infty(\mathbb{R})$, $0 \le \eta \le 1$, $\eta(s) = 0$ for $s \le 1/2$, $\eta(s) = 1$ for $s \ge 1$. For $m > 0$ we define the function $\varrho := \varphi\eta(u_\epsilon/m)$. Note that ϱ belongs to $C^1_c(\Omega)$, because $u_\epsilon \in C^1(\overline{\Omega})$.

Replacing φ by ϱ in (3.26) we obtain

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) = \int_{\tilde{\Omega}} (-g_\epsilon(u_\epsilon) + \lambda u_\epsilon + u^p_\epsilon)\varphi\eta(u_\epsilon/m). \tag{3.27}$$

Arguing as in the proof of Theorem 1.1, we get

$$\lim_{m\to 0}\lim_{\epsilon\to 0}\int_\Omega (-g_\epsilon(u_\epsilon) + \lambda u_\epsilon + u_\epsilon)^p)\varphi\eta(u_\epsilon/m) = \int_\Omega (-u^{-\beta} + \lambda u + u^p)\varphi,$$

and

$$\lim_{m\to 0}\lim_{\epsilon\to 0}\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) := \int_\Omega \nabla u_\epsilon \nabla\varphi.$$

This proves Theorem 3.1. □

4 A problem with logarithmic singularity

In this chapter we study the problem

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p + \mu f(u) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\lambda \geq 0$ and $\mu > 0$ are positive parameters, $p > 1$ and f is allowed to have exponential growth. The structure of this chapter is very similar to Chapter 2. We will again study a perturbed problem of the form

$$\begin{cases} -\Delta u + g_\epsilon(u) = \lambda u^p + \mu f(u) \,\text{in}\, \Omega \\ u \geq 0 \text{ in } \Omega \\ u = 0 \,\text{on}\, \partial\Omega. \end{cases} \tag{4.2}$$

Here, $g_\epsilon \in C^\infty(0, \infty)$ is defined by

$$g_\epsilon(s) = \begin{cases} -\log\left(s + \dfrac{\epsilon}{s+\epsilon}\right) \text{ for } s \geq 0 \\ 0 \text{ for } s < 0, \end{cases} \tag{4.3}$$

so that $g_\epsilon(0) = 0$ for all $\epsilon > 0$ and $g_\epsilon(s) \to -\log(s)$ pointwisely for $s > 0$ as $\epsilon \to 0$. We shall see that the behaviour of this perturbation near the origin prevent us from considering $0 < p < 1$ in (4.1). We will again assume that f satisfies the following conditions.

$$f(s) = 0 \text{ for } s \leq 0, \quad f \text{ is of class } C^{1,\nu}(0,\infty) \cap C[0,\infty) \text{ for some } 0 < \nu < 1, \tag{4.4}$$

and that for each $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that

$$|f(s)| \leq C_\alpha \exp\left(\alpha s^2\right), \text{ for every } s \geq 0. \tag{4.5}$$

As in Chapter 2, we define the functional $I_{\epsilon,\lambda,\mu} : H_0^1(\Omega) \to \mathbb{R}$ by

$$I_{\epsilon,\lambda,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,dx + \int_\Omega G_\epsilon(u) - \frac{\lambda}{p+1}\int_\Omega (u^+)^{p+1} \,dx - \mu \int_\Omega F(u) \,dx, \tag{4.6}$$

where $F(u) = \int_0^u f(s)\,ds$ and $G_\epsilon(u) = \int_0^s g_\epsilon(s)\,ds$. From the fact that f and g_ϵ are continuous functions that satisfy (4.3)–(4.5), we conclude $I_{\epsilon,\lambda,\mu}$ is of class C^1 and

$$I'_{\epsilon,\lambda,\mu}(u)(v) = \int_\Omega \nabla u \nabla v + \int_\Omega g_\epsilon(u)v - \lambda\int_\Omega (u^+)^p v - \mu\int_\Omega f(u)v, \text{ for all } u, v \in H_0^1(\Omega). \tag{4.7}$$

Consequently, if $u_\epsilon \in H_0^1(\Omega)$ is a critical point of $I_{\epsilon,\lambda,\mu}$ then

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda\int_\Omega (u_\epsilon^+)^p v + \mu\int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega). \tag{4.8}$$

Choosing $v = u_\epsilon^-$ in (4.8) and using (4.4), we obtain

$$-\int_\Omega |\nabla(u_\epsilon)^-|^2 = 0.$$

Hence, $u_\epsilon \geq 0$ in Ω. We conclude that critical points $u_\epsilon \in H_0^1(\Omega)$ of $I_{\epsilon,\lambda,\mu}$ are nonnegative and

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda \int_\Omega u_\epsilon^p v + \mu \int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega). \tag{4.9}$$

Therefore, critical points of $I_{\epsilon,\lambda,\mu}$ are weak solutions of problem (4.2). Furthermore, if $u_\epsilon \in L^\infty(\Omega)$, then for each $0 < \epsilon < 1$ fixed

$$\sup_\Omega (|g_\epsilon(u_\epsilon)| + \lambda u_\epsilon^p + \mu |f(u_\epsilon)|) < \infty,$$

and consequently

$$\Delta u_\epsilon \in L^\infty(\Omega).$$

We conclude from Elliptic Regularity Theory (Theorem B.14) that $u_\epsilon \in W^{2,r}(\Omega)$ for all $r > 1$. Thus, the Sobolev Embedding (Theorem B.13) implies that $u_\epsilon \in C^{1,\nu}(\overline{\Omega})$, where $0 < \nu < 1$ is given by (4.4). Summarizing, we have

Lemma 4.1. *Suppose that f satisfies (4.4) and (4.5). The following assertions hold:*

(i) Critical points of $I_{\epsilon,\lambda,\mu}$ are nonnegative weak solutions of problem (4.2).

(ii) If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a nonnegative weak solution of problem (4.2), then u is smooth and $u \in C^{1,\nu}(\overline{\Omega})$, with ν given by (4.4).

Now we summarize the properties of the perturbation g_ϵ defined in (4.3). We remark that the estimates below are uniform in ϵ.

Lemma 4.2. *The following assertions hold.*

(i) We have

$$0 \leq -g_\epsilon(s) \leq s \text{ for } s \geq 1-\epsilon, \tag{4.10}$$

$$0 \leq G_\epsilon(s) \leq 2, \text{ for every } 0 \leq s \leq 1-\epsilon \text{ and } 0 < \epsilon < 1, \tag{4.11}$$

and

$$|G_\epsilon(s)| \leq \frac{s^2}{2} + s + 2 \text{ for all } s \geq 1-\epsilon. \tag{4.12}$$

(ii) For each $p_0 > 2$ there exists a constant $k_0 > 0$ such that

$$G_\epsilon(s) \geq -k_0 s^{p_0} \text{ for all } s \geq 1-\epsilon,\ 0 < \epsilon < 1/2. \tag{4.13}$$

(iii) There exists a constant $C > 0$ that does not depend on $0 < \epsilon < 1$ such that

$$|s g_\epsilon(s)| \leq C(1+s^2) \text{ for all } s \geq 0. \tag{4.14}$$

(iv) We have

$$\lim_{s\to 0^+} g_\epsilon'(s) = \frac{1}{\epsilon} - 1. \tag{4.15}$$

Remark: *Observe that the perturbation G_ϵ does not possess a lower bound near the origin as in item (iii) of Lemma 2.3. Consequently, we do not obtain results for the case $0 < p < 1$ in* (4.1)

Proof of Lemma 4.2. By definition of g_ϵ, we have $g_\epsilon(s) \geq 0$ for $0 \leq s \leq 1-\epsilon$ and $g_\epsilon(s) \leq 0$ for $s \geq 1-\epsilon$. Assertion (4.10) follows from the fact that

$$-g_\epsilon(s) = \log\left(s + \frac{\epsilon}{s+\epsilon}\right) \leq \log(s+\epsilon) \leq s \text{ for } s \geq 1-\epsilon.$$

Now we observe that $-\sqrt{t}\log t < 1$ for $0 \leq t \leq 1$. Hence,

$$0 \leq g_\epsilon(s) \leq -\log(\epsilon + s) \leq (s+\epsilon)^{-\frac{1}{2}} \leq s^{-\frac{1}{2}} \text{ for } 0 \leq s \leq 1-\epsilon.$$

Consequently (4.11) follows, because

$$0 \leq G_\epsilon(s) = \int_0^s g_\epsilon(t)\,dt = 2s^{1/2} \leq 2 \text{ for every } 0 \leq s \leq 1-\epsilon \text{ and } 0 < \epsilon < 1.$$

Inequality (4.12) holds. Indeed, using (4.11) and the fact that $\log t \leq t$ for all $t \geq 0$, we get

$$\begin{aligned} |G_\epsilon(s)| &\leq 2 + \int_{1-\epsilon}^s |g_\epsilon(t)|dt = 2 + \int_{1-\epsilon}^s \log\left(t + \frac{\epsilon}{t+\epsilon}\right) dt \\ &\leq 2 + \int_{1-\epsilon}^s \left(t + \frac{\epsilon}{t+\epsilon}\right) dt \leq 2 + \int_0^s (t+1)dt = 2 + \frac{s^2}{2} + s, \end{aligned}$$

where $s \geq 1-\epsilon$ and $0 < \epsilon < 1$. Note that for each $p_0 > 2$ there exists $k_0 > 0$ such that

$$\frac{s^2}{2} + s + 2 \leq k_0 s^{p_0} \text{ for all } s \geq \frac{1}{2}.$$

Thus, from (4.12) we obtain

$$G_\epsilon(s) \geq -\left(\frac{s^2}{2} + s + 2\right) \geq -k_0 s^{p_0} \text{ for } s \geq 1-\epsilon \text{ and for every } 0 < \epsilon < 1/2,$$

proving (4.13). Now we prove (4.14). For each $0 < \epsilon < 1$ and $0 \leq s \leq 1-\epsilon$ there exists a constant $C > 0$ independent of ϵ such that

$$|g_\epsilon(s)s| \leq -s\log\left(s + \frac{\epsilon}{s+\epsilon}\right) \leq (-\log s)s \leq C.$$

On the other hand, for $s \geq 1-\epsilon$ we have

$$|g_\epsilon(s)s| \leq s\log\left(s + \frac{\epsilon}{s+\epsilon}\right) \leq s\log(s+\epsilon) \leq s^2.$$

We conclude that there exists a constant $C > 0$ such that

$$|g_\epsilon(s)s| \leq C(1+s^2) \text{ for each } s \geq 0,\ 0 < \epsilon < 1.$$

Inequality (4.14) then follows from the fact that $g_\epsilon(s) = 0$ for $s \leq 0$. Finally, (4.15) is a consequence of

$$\frac{d}{ds}g_\epsilon(s) = -\frac{d}{ds}\log\left(s + \frac{\epsilon}{s+\epsilon}\right) = -\left(\frac{s+\epsilon}{s^2+s\epsilon+\epsilon}\right)\left(1 - \frac{\epsilon}{(s+\epsilon)^2}\right) \text{ for all } s > 0.$$

□

As in Chapter 2, we have the following convergence result.

Lemma 4.3. *Assume that f satisfies (4.5) and that $f(s) = 0$ for $s < 0$. The following assertions hold*

(i) For each $\alpha > 0$ there exists a constant $C > 0$ that depends on α such that

$$\max\{|f(s)|, |F(s)|\} \leq C \exp\left(\alpha s^2\right) \text{ for } s \in \mathbb{R}. \tag{4.16}$$

(ii) If there exist a sequence (u_n) in $H_0^1(\Omega)$ and a constant $D > 0$ such that

$$\|u_n\|_{H_0^1(\Omega)} < D \text{ for all } n \in \mathbb{N},$$

then there exists $u \in H_0^1(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$,

$$\int_\Omega f(u_n)dx \to \int_\Omega f(u)dx \text{ as } n \to \infty, \tag{4.17}$$

and

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u)dx \text{ as } n \to \infty. \tag{4.18}$$

Proof of Lemma 4.3. See the proof of Lemma 2.2. □

We again consider the functions $j_\epsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$j_\epsilon(s) = \lambda(s^+)^p + \mu f(s) - g_\epsilon(s),$$

and $J_\epsilon(s) = \int_0^s j_\epsilon(t)\, dt$. Observe that

$$I_{\epsilon,\lambda,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u). \tag{4.19}$$

We will again denote j_ϵ and J_ϵ by j and J respectively and we will assume that there exists a constant $s_0 > 0$ such that

$$\min\{f(s), F(s)\} \geq 0 \text{ for every } s \geq s_0. \tag{4.20}$$

Using this assumption, we obtain

Lemma 4.4. *The following assertions hold.*

(i) Suppose that f satisfies (4.4). For each $R > 0$, there exists a constant $C > 0$ that does not depend on ϵ such that

$$\max\{|J(s)|, |sj(s)|\} \leq C \text{ for all } s \leq R.$$

(ii) Suppose that $\lambda \geq 0$, $\mu > 0$ and that f satisfies conditions (4.4) and (4.20). Then, there exists $S > 1$ such that $J(s) \geq 0$ for all $s \geq S$.

Proof of Lemma 4.4. First we prove item (i). Note that

$$J(s) = \frac{\lambda}{p+1} s^{p+1} + \mu F(s) - G_\epsilon(s) \text{ for } s \geq 0, \tag{4.21}$$

and

$$sj(s) = \lambda s^{p+1} + \mu s f(s) - s g_\epsilon(s) \text{ for } s \geq 0.$$

Then, it follows from Lemma 4.2 that

$$|J(s)| \leq \frac{\lambda}{p+1} R^{p+1} + \mu \sup_{0 \leq s \leq R} |F(s)| + \frac{R^2}{2} + R + 2 \text{ for } 0 \leq s \leq R,$$

and

$$|sj(s)| \leq \lambda R^{p+1} + \mu \sup_{0 \leq s \leq R} |sf(s)| + C(1 + R^2) \text{ for } 0 \leq s \leq R.$$

This proves item (i). Item (ii) follows from (4.21) and from the fact that there must exist $s_1 > 1$ independent of ϵ such that $G_\epsilon(s) < 0$ for $s > s_1$. Choosing $S = \max\{s_0, s_1\}$, and using (4.20), we obtain from (4.21) that

$$J(s) \geq 0 \text{ for } s \geq S.$$

We have proved Lemma 4.4. □

4.1 Existence of solutions of the perturbed problem

The first aim of this section is to show that there exist constants $0 < \theta < 1/2$ and $R_{\theta,\lambda,\mu} > 0$ that do not depend on ϵ such that

$$J(s) \leq \theta s j(s) \text{ for } s \geq R_{\theta,\lambda,\mu}.$$

The proof of this result is not entirely analogous to the one given in Chapter 2, because now the perturbation g_ϵ is unbounded at ∞.

- When $\lambda > 0$ and $p > 1$ in (4.1) we assume that there exists $s_0 > 0$ such that

$$pf(s) \leq sf'(s) \text{ for all } s \geq s_0, \tag{4.22}$$

or that there exists constants $C > 0$ and $\widetilde{p} < p$ such that

$$|pf(s) - sf'(s)| \leq C s^{\widetilde{p}} \text{ for all } s \geq s_0. \tag{4.23}$$

Observe that $f = 0$ satisfies (4.22) and $f(s) = s^\tau$ satisfies (4.23) when $0 < \tau < 1$.

- When $\lambda = 0$ in (4.1), we will assume that there exists $0 < \nu_1 < 1$ such that

$$\lim_{s \to \infty} f'(s) = \infty \text{ and } \lim_{s \to \infty} \frac{sf'(s)}{f(s)} > 2 + \nu_1. \tag{4.24}$$

Observe that (4.24) implies (4.20).

Lemma 4.5. *Suppose that f satisfies (4.4), (4.20) and that one of the following assertions hold:*

(i) $\lambda > 0$, $\mu > 0$ and $p > 1$ in (4.1) and f satisfies (4.22) or (4.23).

(ii) $\lambda = 0$ and $\mu > 0$ in (4.1), and f satisfies (4.24).

Then there exist constants $0 < \theta < \frac{1}{2}$ and $R_{\theta,\lambda,\mu} > 0$ such that

$$0 \leq J(s) \leq \theta s j(s) \text{ for } s \geq R_{\theta,\lambda,\mu}. \tag{4.25}$$

Consequently, item (i) of Lemma 4.4 implies that there exists $D_{\theta,\lambda,\mu} > 0$ such that

$$|J(s)| \leq D_{\theta,\lambda,\mu} + \theta s j(s) \text{ for all } s \in \mathbb{R}.$$

Proof of Lemma 4.5. Let $0 < \theta < \frac{1}{2}$ be a positive constant to be chosen later. Define

$$B_\epsilon(s) = J(s) - \theta s j(s).$$

We first claim that there exists $T_{\theta,\lambda,\mu} > 0$ that does not depend on ϵ such that

$$B_\epsilon'(s) < -\frac{\theta}{6} \text{ for } s \geq T_{\theta,\lambda,\mu}. \tag{4.26}$$

First we recall that

$$g_\epsilon'(s) = -\left(\frac{(s+\epsilon)^2 - \epsilon}{(s^2 + s\epsilon + \epsilon)(s+\epsilon)}\right) < 0 \text{ for } s \geq 1,$$

and

$$B_\epsilon'(s) = (1-\theta) j(s) - \theta s j'(s).$$

Consequently,

$$B_\epsilon'(s) = (1-\theta)(\mu f(s) + \lambda s^p - g_\epsilon(s)) - \theta \mu s f'(s) - \theta \lambda p s^p - \theta s \left(\frac{(s+\epsilon)^2 - \epsilon}{(s^2 + s\epsilon + \epsilon)(s+\epsilon)}\right) \text{ for } s \geq 1.$$

Hence, $B_\epsilon'(s) < -\frac{\theta}{6}$ if and only if

$$(1-\theta)(\lambda s^p + \mu f(s) - g_\epsilon(s)) + \frac{\theta}{6} < \theta \left(\lambda p s^p + \mu s f'(s) + \frac{s^3 + 2s^2\epsilon + s\epsilon^2 - \epsilon s}{s^3 + 2s^2\epsilon + s\epsilon^2 + s\epsilon + \epsilon^2}\right).$$

Note that $2s^2\epsilon > \epsilon s$ if $s > 1/2$. Hence,

$$\frac{s^3 + 2s^2\epsilon + s\epsilon^2 - \epsilon s}{s^3 + 2s^2\epsilon + s\epsilon^2 + s\epsilon + \epsilon^2} > \frac{s^3}{s^3 + 2s^3 + s^3 + s^3 + s^3} = \frac{1}{6} \text{ for all } s \geq 1, \text{ with } 0 < \epsilon < 1.$$

Hence, it suffices to know for which values of $s \geq 1$ the following inequality holds

$$(1-\theta)(\lambda s^p + \mu f(s) - g_\epsilon(s)) + \frac{\theta}{6} < \theta \left(\lambda p s^p + \mu s f'(s) + \frac{1}{6}\right).$$

We need to solve the inequality

$$(1-\theta)(\lambda s^p + \mu f(s) - g_\epsilon(s)) < \theta\left(\lambda p s^p + \mu s f'(s)\right).$$

From (4.10), it is enough to show that

$$\lambda s^p + \mu f(s) + s < \frac{\theta}{1-\theta}\left(\lambda p s^p + \mu s f'(s)\right),$$

which is equivalent to

$$s + \lambda s^p\left(1 - \frac{\theta p}{1-\theta}\right) + \frac{\mu}{1-\theta}((1-\theta)f(s) - \theta s f'(s)) < 0. \tag{4.27}$$

We now split the proof in two cases.

Case 1: When $\lambda > 0$, we choose θ such that

$$1 < \frac{1-\theta}{\theta} < p.$$

Consequently, $\theta < 1/2$ and $1-\theta < \theta p$, so that from (4.27) we only need to prove that

$$s + \lambda s^p\left(1 - \frac{\theta p}{1-\theta}\right) + \frac{\mu\theta}{1-\theta}(pf(s) - sf'(s)) < 0.$$

Claim (4.26) then follows from (4.22), (4.23) and from the facts that $p > 1$ and

$$1 - \frac{\theta p}{1-\theta} < 0.$$

Case 2: When $\lambda = 0$, inequality (4.27) becomes

$$s + \mu f(s) < \frac{\mu\theta s f'(s)}{1-\theta}.$$

This inequality will be true if

$$f'(s) > \frac{2(1-\theta)}{\mu\theta} \text{ and } sf'(s) > \frac{(1-\theta)f(s)}{\theta}.$$

We choose θ such that

$$1 < \frac{1-\theta}{\theta} < 1 + \frac{\nu_1}{2},$$

where ν_1 is given by (4.24). This choice of θ and hypothesis (4.24) guarantee that (4.26) holds when s is large enough.

Now we prove (4.25). First note that if s is large enough, we may use (4.10) and (4.12) to obtain

$$\begin{aligned} B_\epsilon(s) &= \frac{\lambda}{p+1}s^{p+1} + \mu F(s) - G_\epsilon(s) - \theta s(\lambda s^p + \mu f(s) - g_\epsilon(s)) \\ &< \frac{\lambda}{p+1}s^{p+1} + \mu F(s) + \frac{3s^2}{2} + s + 2 - \theta\mu s f(s). \end{aligned}$$

Hence,

$$B_\epsilon(2T_{\theta,\lambda,\mu}) < \widetilde{\beta_{\theta,\lambda,\mu}},$$

where

$$\widetilde{\beta_{\theta,\lambda,\mu}} = \frac{\lambda}{p+1}(2T_{\theta,\lambda,\mu})^{p+1} + \mu F(2T_{\theta,\lambda,\mu}) + \frac{3(2T_{\theta,\lambda,\mu})^2}{2} + 2T_{\theta,\lambda,\mu} + 2 - 2\theta\mu T_{\theta,\lambda,\mu} f(2T_{\theta,\lambda,\mu}).$$

Using (4.26), there is a constant $\beta = \beta_{\theta,\lambda,\mu}$ such that

$$B_\epsilon(s) < -\frac{\theta}{6}s + \beta \text{ for } s \geq 2T_{\theta,\lambda,\mu}.$$

We conclude that $B_\epsilon(s) < 0$ if $s > \max\{\frac{6\beta}{\theta}, 2T_{\theta,\lambda,\mu}\}$. Choosing $R_{\theta,\lambda,\mu} = \frac{6\beta}{\theta}$, (4.25) follows. □

The following compactness result is analogous to Lemma 2.6, see page 37.

Lemma 4.6. *Fix $0 < \epsilon < 1$ and suppose that f satisfies (4.4), (4.5) and (4.20). Assume that one of the following assertions hold:*

(i) $\lambda > 0$, $\mu > 0$ and $p > 1$ in (4.1) and f satisfies (4.22) or (4.23).

(ii) $\lambda = 0$ and $\mu > 0$ in (4.1), and f satisfies (4.24).

Then the functional $I_{\epsilon,\lambda,\mu}$ defined in (4.6) satisfies the Palais-Smale condition at every level $c \neq 0$.

Proof of Lemma 4.6. The proof is similar to the one given in Lemma 2.6. The only difference is that we use Lemmas 4.4 and 4.5 instead of Lemmas 2.4 and 2.5. Recall that $j(s) = 0$ for $s \leq 0$, since $f(s) = 0$ for $s \leq 0$. Hence, if f satisfies (4.5), then j also satisfies (4.5) and for each $\alpha > 0$ there exists a constant $C_{\epsilon,\alpha} > 0$ depending only on ϵ and α such that

$$\max\{|j(s)|, |J(s)|\} \leq C_{\epsilon,\alpha} \exp(\alpha s^2) \text{ for } s \in \mathbb{R}. \tag{4.28}$$

Let (v_n^ϵ) be a Palais–Smale sequence for $I_{\epsilon,\lambda,\mu}$ in $H_0^1(\Omega)$ at the level c. Throughout this proof we denote v_n^ϵ by v_n and the norm $\|\cdot\|_{H_0^1(\Omega)}$ by $\|\cdot\|$. Thus (v_n) satisfies

$$\frac{1}{2}\|v_n\|^2 - \int_\Omega J(v_n)\,dx \to c \text{ as } n \to \infty, \tag{4.29}$$

and there is a sequence $\tau_n \to 0$ such that

$$\left|\int_\Omega \nabla v_n \nabla w\,dx - \int_\Omega j(v_n) w\,dx\right| \leq \tau_n \|w\| \text{ for each } w \in H_0^1(\Omega). \tag{4.30}$$

From Lemma 4.5 there exist constants $0 < \theta < 1/2$ and $D_{\theta,\lambda,\mu} > 0$ depending only on θ, λ and μ such that

$$|J(v_n)| < D_{\theta,\lambda,\mu} + \theta v_n j(v_n).$$

Therefore, there is a constant $D_1 = D_1(\theta) > 0$ that does not depend on $\epsilon > 0$ such that

$$\frac{1}{2}\|v_n\|^2 < D_1 + \theta \int_\Omega v_n j(v_n)\,dx.$$

Taking $w = v_n$ in (4.30) we also conclude that

$$\int_\Omega j(v_n)v_n\,dx < \|v_n\|^2 + \tau_n\|v_n\|.$$

Hence,

$$\frac{1}{2}\|v_n\|^2 < D_1 + \theta\|v_n\|^2 + \tau_n\theta\|v_n\|.$$

Since $\theta < 1/2$, there is a constant $D > 0$ such that

$$\|v_n\| < D. \tag{4.31}$$

It follows from (4.28) and Lemma 4.3 that there exist a subsequence (v_{n_k}) in $H_0^1(\Omega)$ that we continue to denote by (v_n) and an element $v \in H_0^1(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \\ v_n \to v \text{ in } L^r(\Omega) \text{ for every } r > 1, \\ v_n \to v \text{ a.e in } \Omega, \\ |v_n| \le h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega), \\ \displaystyle\int_\Omega j(v_n)\,dx \to \int_\Omega j(v)\,dx, \\ \displaystyle\int_\Omega J(v_n)\,dx \to \int_\Omega J(v)\,dx. \end{cases} \tag{4.32}$$

From (4.29), (4.30) and (4.32), we get

$$\lim_{n\to\infty} \|v_n\|^2 = 2(c + \int_\Omega J(v)\,dx), \tag{4.33}$$

and

$$\lim_{n\to\infty} \int_\Omega v_n j(v_n) = 2(c + \int_\Omega J(v)\,dx). \tag{4.34}$$

The result then follows by the same argument given in the proof of Lemma 2.6. □

As in Chapter 2, we will show that problem (4.2) possesses a nontrivial solution $u_\epsilon \ge 0$ that is a Mountain Pass. Recall that a function $u_\epsilon \in H_0^1(\Omega)$ is a weak solution of problem (4.2) if

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \lambda \int_\Omega u_\epsilon^p v + \mu \int_\Omega f(u_\epsilon)v \text{ for all } v \in H_0^1(\Omega). \tag{4.35}$$

We proceed to obtain constants $a_1 > 0$ and $0 < \rho < 1$ and an element $\phi_0 \in H_0^1(\Omega)$ such that

$$I_{\epsilon,\lambda,\mu}(u) \ge a_1 \text{ for } \|u\|_{H_0^1(\Omega)} = \rho,$$

and

$$\|\phi_0\|_{H_0^1(\Omega)} \ge 1 \text{ and } I_{\epsilon,\lambda,\mu}(\phi_0) < 0. \tag{4.36}$$

To obtain this element, we will assume that

$$\text{there exist constants } A, s_0 > 0 \text{ and } \gamma > 2 \text{ such that } F(s) \ge A|s|^\gamma \text{ for all } s \ge s_0. \tag{4.37}$$

Lemma 4.7. *Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\phi > 0$ in Ω and $\|\phi\|_{H_0^1(\Omega)} = 1$ and suppose that f satisfies (4.4) and (4.20). Assume that one of the following conditions hold*

(i) $\lambda > 0$ and $\mu > 0$, or

(ii) $\lambda = 0$ and $\mu > 0$ in (4.1), and f satisfies (4.37).

Then, there exists a constant $N_0 = N_0(\lambda, \mu)$ such that

$$I_{\epsilon,\lambda,\mu}(N_0\phi) < 0. \tag{4.38}$$

Also, there exists a constant $a_2 > 0$ such that

$$\sup_{s\in[0,1]} I_{\epsilon,\lambda,\mu}(sN_0\phi) < a_2. \tag{4.39}$$

Proof of Lemma 4.7. Using (4.11) and the fact that $g_\epsilon(s) \leq 0$ for all $s \geq 1-\epsilon$, we have $G_\epsilon(s) \leq 2$ for all $s \geq 0$. Hence,

$$I_{\epsilon,\lambda,\mu}(t\phi) \leq \frac{t^2}{2} + 2|\Omega| - \frac{\lambda t^{p+1}}{p+1}\int_\Omega \phi^{p+1} - \mu\int_\Omega F(t\phi). \tag{4.40}$$

We split the proof in two cases.

Case 1: Suppose that $\lambda > 0$. Then, (4.20) implies that

$$I_{\epsilon,\lambda,\mu}(t\phi) \leq \frac{t^2}{2} + 2|\Omega| - \frac{\lambda t^{p+1}}{p+1}\int_\Omega \phi^{p+1} - \mu\int_{\Omega\cap\{t\phi_1<s_0\}} F(t\phi).$$

Hence, there are constants $c_1, c_2 > 0$ depending on λ and μ such that

$$I_{\epsilon,\lambda,\mu}(t\phi) \leq \frac{t^2}{2} + c_1 - c_2 t^{p+1} \text{ for all } t \geq 0.$$

Inequality (4.38) then follows from the facts that $p+1 > 2$ and by letting $t \to \infty$.

Case 2: Suppose that $\lambda = 0$ and (4.37) holds.

From (4.37) and (4.40) we obtain constants $c_3, c_4 > 0$ depending on μ such that

$$I_{\epsilon,\lambda,\mu}(t\phi) \leq \frac{t^2}{2} + c_3 - c_4 t^\gamma \text{ for all } t \geq \frac{2s_0}{\sup_\Omega \phi},$$

where $s_0 > 0$ is given by (4.37). Thus, $I_{\epsilon,\lambda,\mu}(t\phi) < 0$ provided

$$\frac{t^2}{2} + c_3 - c_4 t^\gamma < 0,$$

which is true for sufficiently large t, because $\gamma > 2$. Hence, there exists $N_0 > 2$ such that $I_{\epsilon,\lambda,\mu}(N_0\phi) < 0$. This proves (4.38).

Now we see that

$$I_{\epsilon,\lambda,\mu}(sN_0\phi) = \frac{s^2N_0^2}{2} + \int_\Omega G_\epsilon(sN_0\phi)\,dx - \frac{\lambda s^{p+1}N_0^{p+1}}{p+1}\int_\Omega \phi^{p+1} - \mu\int_\Omega F(sN_0\phi)\,dx.$$

Since $0 \le sN_0\phi \le N_0 \sup_\Omega \phi$ for all $0 \le s \le 1$, there exists $c_5 > 0$ depending on λ and μ such that

$$I_{\epsilon,\lambda,\mu}(sN_0\phi) < \frac{s^2N_0^2}{2} + c_3 \text{ for all } 0 \le s \le 1.$$

Hence,

$$\sup_{s\in[0,1]} I_{\epsilon,\lambda,\mu}(sN_0\phi) < a_2,$$

where

$$a_2 = \frac{N_0^2}{2} + c_3.$$

We have proved Lemma 4.7. □

We conclude with

Proposition 4.1. *Suppose that f satisfies (4.4), (4.5) and (4.20). Assume also that one of the following conditions hold*

(i) $\lambda > 0$, $\mu > 0$ and $p > 1$ in (4.1) and f satisfies (4.22) or (4.23).

(ii) $\lambda = 0$ and $\mu > 0$ in (4.1), and f satisfies (4.24) and (4.37).

Let a_2 be given by Lemma 4.7. Then, there exist constants $\mu_0 > 0$ and $a_1 > 0$ such that for each $0 < \mu < \mu_0$, problem (4.2) has a weak solution $u_\epsilon \ge 0$, with $0 < a_1 < I_{\epsilon,\lambda,\mu}(u_\epsilon) < a_2$. Also, there is a constant $D > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D.$$

Proof of Proposition 4.1. Since $g_\epsilon(s) \ge 0$ for $0 \le s \le 1-\epsilon$, we have

$$I_{\epsilon,\lambda,\mu}(u) \ge \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_{\Omega\cap\{u\ge 1-\epsilon\}} G_\epsilon(u) - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\,dx - \mu\int_\Omega F(u)\,dx,$$

for all $u \in H_0^1(\Omega)$. Hence, item (ii) of Lemma 4.2, implies

$$I_{\epsilon,\lambda,\mu}(u) \ge \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - k_0\int_\Omega |u|^{p_0} - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1} - \mu\int_\Omega F(u)\,dx.$$

Using the Sobolev embedding, we obtain

$$I_{\epsilon,\lambda,\mu}(u) \ge \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - k_0C_{p_0}^{p_0}\|u\|^{p_0}_{H_0^1(\Omega)} - \frac{\lambda C_{p+1}^{p+1}}{p+1}\|u\|^{p+1}_{H_0^1(\Omega)} - \mu\int_\Omega F(u)\,dx,$$

where C_i is the best constant of the immersion $H_0^1(\Omega) \hookrightarrow L^i(\Omega)$. Assume that

$$\|u\|^2_{H_0^1(\Omega)} < \rho^2,$$

where

$$\rho = \begin{cases} \min\left\{\left(\dfrac{1}{8k_0C_{p_0}^{p_0}}\right)^{\frac{1}{p_0-2}}, \left(\dfrac{p+1}{8\lambda C_{p+1}^{p+1}}\right)^{\frac{1}{p-1}}\right\} & \text{if } \lambda > 0 \\ \left(\dfrac{1}{8k_0C_{p_0}^{p_0}}\right)^{\frac{1}{p_0-2}} & \text{if } \lambda = 0. \end{cases}$$

We obtain

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{8}\|u\|^2_{H^1_0(\Omega)} - \mu \int_\Omega F(u)\,dx \text{ for } \|u\|_{H^1_0(\Omega)} < \rho. \tag{4.41}$$

We may assume that $\rho < 1$. Let $\alpha = \dfrac{4\pi}{\rho^2}$. By Lemma 4.3 we conclude that there is a constant $C > 0$ such that

$$I_{\epsilon,\lambda,\mu}(u) \;\; \geq \;\; \frac{1}{8}\|u\|^2_{H^1_0(\Omega)} - \mu C \int_\Omega \exp(\alpha u^2)\,dx \text{ for } \|u\|_{H^1_0(\Omega)} < \rho.$$

Using (B.8), we obtain a constant $k_1 > 0$ such that

$$I_{\epsilon,\lambda,\mu}(u) \;\; \geq \;\; \frac{1}{8}\|u\|^2_{H^1_0(\Omega)} - \mu C k_1 \text{ for } \|u\|_{H^1_0(\Omega)} < \rho.$$

Choosing

$$\mu_0 = \frac{\rho^2}{16Ck_1},$$

we have

$$I_{\epsilon,\lambda,\mu}(u) \geq \frac{1}{8}\left(\|u\|^2_{H^1_0(\Omega)} - \frac{\rho^2}{2}\right) \text{ for } 0 < \mu < \mu_0,\; \|u\|_{H^1_0(\Omega)} < \rho.$$

From Lemma 4.7, we obtain

$$I_{\epsilon,\lambda,\mu}(0) = 0, \quad I_{\epsilon,\lambda,\mu}(N_0\phi) < 0,$$

and

$$I_{\epsilon,\lambda,\mu}(u) \geq a_1 \text{ for } \|u\|_{H^1_0(\Omega)} = \rho,$$

where

$$a_1 = \frac{\rho^2}{16}.$$

Let $\Gamma = \{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = N_0\phi\}$. By the Mountain Pass Theorem, we conclude that there is a sequence (u^ϵ_n) in $H^1_0(\Omega)$ and a number

$$c_\epsilon := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} I_{\epsilon,\lambda,\mu}(\gamma(s)),$$

such that

$$\lim_{n\to\infty} I_{\epsilon,\lambda,\mu}(u^\epsilon_n) = c_\epsilon \text{ and } \lim_{n\to\infty} I'_{\epsilon,\lambda,\mu}(u^\epsilon_n) = 0.$$

That is,

$$\frac{1}{2}\|u^\epsilon_n\|^2_{H^1_0(\Omega)} - \int_\Omega J(u^\epsilon_n)\,dx \to c_\epsilon.$$

And there is a sequence $\tau_n \to 0$ such that

$$\left|\int_\Omega \nabla u^\epsilon_n \nabla v\,dx - \int_\Omega j(u^\epsilon_n) v\,dx\right| \leq \tau_n \|v\|_{H^1_0(\Omega)} \text{ for each } v \in H^1_0(\Omega). \tag{4.42}$$

It is clear that $c_\epsilon \geq a_1 > 0$. Using Lemma 4.7 we also obtain

$$c_\epsilon \leq \sup_{s\in[0,1]} I_{\epsilon,\lambda,\mu}(sN_0\phi) < a_2.$$

Hence, for a sufficiently large n,

$$0 < a_1 \leq I_{\epsilon,\lambda,\mu}(u_n^\epsilon) < a_2. \tag{4.43}$$

Arguing as in the proof of Lemma 4.6, we may use (4.42), (4.43) and Lemma 4.5 to obtain a constant $D > 0$ that does not depend on ϵ such that

$$\|u_n^\epsilon\|_{H_0^1(\Omega)} < D.$$

We conclude that there is $u_\epsilon \in H_0^1(\Omega)$ with $\|u_\epsilon\|_{H_0^1(\Omega)} < D$ such that

$$u_n^\epsilon \rightharpoonup u_\epsilon \text{ weakly in } H_0^1(\Omega).$$

We know that (u_n^ϵ) is a Palais–Smale sequence at a positive level. It follows from (4.43) and Lemma 4.6 that up to a subsequence, $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$. Hence, $I'_{\epsilon,\lambda,\mu}(u_\epsilon) = 0$, and the result follows from Lemma 4.1 □

4.2 Convergence of the perturbed solutions

In this section. we study the convergence of the solutions u_ϵ of problem (4.2) obtained in Proposition 4.1. This proposition guarantees that there exists a constant $D > 0$ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D, \text{ for each } 0 < \epsilon < 1.$$

Hence, there exist $u \in H_0^1(\Omega)$ and a sequence (ϵ_n) in $(0,1)$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} u_{\epsilon_n} \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_{\epsilon_n} \to u \text{ in } L^r(\Omega) \text{ for every } r > 1, \\ u_{\epsilon_n} \to u \text{ a.e in } \Omega, \\ |u_{\epsilon_n}| \leq h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega). \end{cases} \tag{4.44}$$

As in Chapter 2, under additional conditions on f, we can apply regularity results discussed in Appendix A to conclude that u_{ϵ_n} is smooth for all $n \in \mathbb{N}$ and that u is continuous. Indeed, assume that

$$\lim_{s\to 0} |f'(s)| < \infty. \tag{4.45}$$

From Corollary A.1, we know that there exists a constant $K_1 > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(\Omega)} < K_1 \text{ for all } 0 < \epsilon_n < 1.$$

Then, it follows from elliptic regularity theory and from the Sobolev Embedding that $u_{\epsilon_n} \in C^1(\overline{\Omega})$, see Lemma 4.1. Lemma A.5 implies that there exists a constant $\epsilon_0 > 0$ such that for each smooth subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there exists a constant $M > 0$ that depends on Ω' but not on ϵ such that

$$|\nabla u_{\epsilon_n}(x)|^2 \leq MZ(u_{\epsilon_n}(x)) \text{ for every } x \in \Omega', \quad 0 < \epsilon < \epsilon_0, \tag{4.46}$$

where

$$Z(t) = \begin{cases} t^2 + t - t\log t & \text{for } 0 \le t \le \dfrac{1}{2} \\ \dfrac{1}{4} + \dfrac{1}{2}(1+\log 2) + \left(t - \dfrac{1}{2}\right)(1+\log 2) & \text{for } t \ge \dfrac{1}{2}. \end{cases}$$

Hence, it follows from the Arzela-Ascoli Theorem that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, so that u is continuous and $0 \le u \le K_1$. In this section, we show that u is a solution of (4.1) in the sense that

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} \Big(\log u + \lambda u^p + \mu f(u)\Big)\varphi, \tag{4.47}$$

for every $\varphi \in C_c^1(\Omega)$ and

$$(\log u)\chi_{\{u>0\}} \in L^1_{loc}(\Omega).$$

First, we show that

Lemma 4.8. *The function u is nontrivial and the function $(\log u)\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.*

Proof of Lemma 4.8. First we show that u is nontrivial. Since u_{ϵ_n} is a critical point of $I_{\epsilon_n,\lambda,\mu}$, we have

$$\|u_{\epsilon_n}\|^2_{H_0^1(\Omega)} + \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n} = \lambda \int_\Omega u_{\epsilon_n}^{p+1} + \mu \int_\Omega f(u_{\epsilon_n})u_{\epsilon_n},$$

and

$$I_{\epsilon_n,\lambda,\mu}(u_{\epsilon_n}) = \frac{1}{2}\|u_{\epsilon_n}\|^2_{H_0^1(\Omega)} + \int_\Omega G_{\epsilon_n}(u_{\epsilon_n}) - \frac{\lambda}{p+1}\int_\Omega u_{\epsilon_n}^{p+1} - \mu \int_\Omega F(u_{\epsilon_n}) > a_1.$$

Hence,

$$\begin{aligned} I_{\epsilon_n,\lambda,\mu}(u_{\epsilon_n}) = &\int_\Omega \left(G_{\epsilon_n}(u_{\epsilon_n}) - \frac{1}{2}g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\right) dx + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u_{\epsilon_n}^{p+1}\,dx \\ &+ \mu \int_\Omega \left(\frac{1}{2}f(u_{\epsilon_n})u_{\epsilon_n} - F(u_{\epsilon_n})\right) dx > a_1, \end{aligned} \tag{4.48}$$

where a_1 is given by Proposition 4.1. Recall that $0 \le u_{\epsilon_n} \le K_1$ in Ω and consequently $0 \le u \le K_1$ in Ω. Hence, from the Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f(u_{\epsilon_n})u_{\epsilon_n}\,dx = \int_\Omega f(u)u\,dx.$$

From Lemma 4.3, we get

$$\lim_{n\to\infty} \int_\Omega F(u_{\epsilon_n})\,dx = \int_\Omega F(u)\,dx.$$

From the Dominated Convergence Theorem and Lemma 4.2, we have

$$\lim_{n\to\infty} \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\,dx = \int_\Omega g(u)u\chi_{\{u>0\}}\,dx, \text{ where } g(s) = -\log s$$

and

$$\lim_{n\to\infty}\int_\Omega G_{\epsilon_n}(u_{\epsilon_n})\,dx = \int_\Omega G(u)\,dx,\ \text{where } G(s) = -\int_0^s \log t\,dt.$$

Taking the above claims into account and letting $n\to\infty$ in (4.48) we obtain

$$\int_\Omega \left(G(u) - \frac{1}{2}g(u)u\chi_{\{u>0\}}\right)\,dx + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega u^{p+1}\,dx + \mu\int_\Omega\left(\frac{1}{2}f(u)u - F(u)\right)\,dx \geq a_1.$$

We proved that u is nontrivial. Now let Let $V\subset\Omega$ be a open set such that $\overline{V}\subset\Omega$. We will show that

$$\int_V |\log u|\chi_{\{u>0\}}\,dx < \infty.$$

Indeed, take $\zeta\in C_c^1(\Omega)$ such that $0\leq\zeta\leq 1$ and $\zeta\equiv 1$ in V. Since u_{ϵ_n} is a critical point of $I_{\epsilon_n,\lambda,\mu}$, we obtain

$$\int_{\{u_{\epsilon_n}<1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta = \lambda\int_\Omega u_{\epsilon_n}^p\zeta + \int_\Omega \mu f(u_{\epsilon_n})\zeta - \int_\Omega \nabla u_{\epsilon_n}\nabla\zeta - \int_{\{u_{\epsilon_n}\geq 1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta.$$

The Dominated Convergence Theorem, Lemma 4.2 and (4.44) imply that

$$\int_\Omega u_{\epsilon_n}^p\zeta \to \int_\Omega u^p\zeta,$$

$$\int_\Omega f(u_{\epsilon_n})\zeta \to \int_\Omega f(u)\zeta,$$

$$\int_\Omega \nabla u_{\epsilon_n}\nabla\zeta \to \int_\Omega \nabla u\nabla\zeta \text{ as } \epsilon\to 0,$$

and

$$\int_{\{u_{\epsilon_n}\geq 1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta \to \int_{\{u\geq 1\}} \log(u)\zeta \text{ as } \epsilon\to 0.$$

Hence,

$$\int_{\{u_{\epsilon_n}<1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta \to \lambda\int_\Omega u^p\zeta + \mu\int_\Omega f(u)\zeta - \int_\Omega \nabla u\nabla\zeta + \int_{\{u\geq 1\}} \log(u)\zeta \text{ as } \epsilon\to 0. \quad (4.49)$$

Define the set $\Omega_\rho = \{x\in\Omega : u(x)\geq\rho\}$ for $\rho>0$. Then

$$\int_{V\cap\Omega_\rho} -\log\left(\frac{u_\epsilon^2+\epsilon u_\epsilon+\epsilon}{u_\epsilon+\epsilon}\right)\chi_{\{u_\epsilon<1-\epsilon\}}\zeta \leq \int_{\{u_\epsilon<1-\epsilon\}} -\log\left(\frac{u_\epsilon^2+\epsilon u_\epsilon+\epsilon}{u_\epsilon+\epsilon}\right)\zeta < \infty.$$

Hence,

$$\int_{V\cap\Omega_\rho}\left|\log\left(\frac{u_{\epsilon_n}^2+\epsilon_n u_{\epsilon_n}+\epsilon_n}{u_{\epsilon_n}+\epsilon_n}\right)\right| < C < \infty.$$

The constant C does not depend on ϵ and does not depend on ρ. It follows from Fatou's Lemma that

$$\int_V |\log u|\chi_{\Omega_\rho} < \infty,$$

independently of ρ. Letting $\rho\to 0$ and applying Fatou's Lemma again, we conclude that

$$\int_V |\log u|\chi_{\Omega^+} < \infty,$$

for every open set $V\subset\Omega$ such that $\overline{V}\subset\Omega$. □

We state the main result of this chapter.

Theorem 4.1. *Suppose that f satisfies (4.4), (4.5), (4.20) and (4.45). Assume also that one of the following conditions hold*

(i) $\lambda > 0$, $\mu > 0$ and $p > 1$ in (4.1) and f satisfies (4.22) or (4.23).

(ii) $\lambda = 0$ and $\mu > 0$ in (4.1), and f satisfies (4.24) and (4.37).

Then there exists $\mu_0 > 0$ such that for each $0 < \mu < \mu_0$, problem (4.1) has a nonnegative nontrivial solution u.

We remark that when $p > 1$, we may consider $f \equiv 0$, but we may not consider $f(s) = s^q$ for $0 < q < 1$ in view of (4.45). This condition is relevant, because one of the hypothesis of Lemma A.4 is that there exists $0 < \epsilon_0, \delta < 1$ such that

$$g_\epsilon(s) \geq f(s) \text{ for all } s \leq \delta,\ 0 < \epsilon < \epsilon_0,$$

which is a consequence of (4.45). However, it might be possible to replace this condition for a weaker one, so that $f(s) = s^q$ becomes admissible. As an immediate consequence of Theorem 4.1, we get

Corollary 4.1. *Suppose that $p > 1$ in (4.1). Then, the problem*

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \lambda u^p \ \textit{in } \Omega \\ u \geq 0, u \not\equiv 0 \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega \end{cases}$$

is solvable for each $\lambda > 0$.

This is an improvement of the result in [52]. Furthermore, Theorem 4.1 provides solutions for a large class of singular problems. Indeed,

Corollary 4.2. *There exists $\mu_0 > 0$ such that the problems*

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \mu(e^u - 1) \ \textit{in } \Omega \\ u \geq 0, u \not\equiv 0 \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + u^2 + \mu(e^u - 1) \ \textit{in } \Omega \\ u \geq 0, u \not\equiv 0 \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + \mu u^k e^u \ \textit{in } \Omega \\ u \geq 0, u \not\equiv 0 \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + u^2 + \mu u^k e^u \ \textit{in } \Omega \\ u \geq 0, u \not\equiv 0 \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = (\log u)\chi_{\{u>0\}} + u^{k+1}(1+\mu e^u) \ \ in \ \Omega \\ u \geq 0, u \not\equiv 0 \ \ in \ \Omega \\ u = 0 \ \ on \ \partial\Omega, \end{cases}$$

are solvable for $0 < \mu < \mu_0$ *and* $k \geq 1$.

We finish the section by proving Theorem 4.1.

Proof of Theorem 4.1.

We follow ideas given in [52]. Let (ϵ_n) and (u_{ϵ_n}) be the sequences defined in (4.44), and let u be given by Lemma 4.8. We will prove that u is a solution of (4.1). The nontriviality and continuity of u is guaranteed by Lemma 4.8. Also recall that $u_{\epsilon_n} \to u$ in $C^0_{loc}(\Omega)$. Let $\varphi \in C^1_c(\Omega)$. Since $u_{\epsilon_n} \in C^1(\overline{\Omega})$ is a solution of problem (4.2), we know that

$$\int_\Omega \nabla u_{\epsilon_n} \nabla \varphi = \int_\Omega (-g_{\epsilon_n}(u_{\epsilon_n}) + \lambda u^p_{\epsilon_n} + \mu f(u_{\epsilon_n}))\varphi. \tag{4.50}$$

We would like to let $n \to \infty$ in (4.50). Since the term $-g_{\epsilon_n}(u_{\epsilon_n})$ does not converge pointwisely to $\log(u)\chi_{\{u>0\}}$, we need to consider an auxiliar function η that vanishes near the origin . Throughout this proof, we will denote the functions u_{ϵ_n} merely by u_ϵ and we will let $\epsilon \to 0$. Let $\eta \in C^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, $\eta(s) = 0$ for $s \leq 1/2$, $\eta(s) = 1$ for $s \geq 1$. For $m > 0$ we define the function $\varrho := \varphi\eta(u_\epsilon/m)$. Note that ϱ belongs to $C^1_c(\Omega)$, because $u_\epsilon \in C^1(\overline{\Omega})$.

From continuity, the set $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ is open. Let $\tilde{\Omega}$ be an open set such that $\overline{supp(\varphi)} \subset \tilde{\Omega}$ and $\overline{\tilde{\Omega}} \subset \Omega$. Let $\Omega_0 = \Omega_+ \cap \tilde{\Omega}$. Since $u_\epsilon \to u$ uniformly in $\tilde{\Omega}$, we know that for every $m > 0$ there is an $\epsilon_1 > 0$ such that

$$u_\epsilon(x) \leq m/2 \text{ for every } x \in \tilde{\Omega} \setminus \Omega_0 \text{ and } 0 < \epsilon \leq \epsilon_1. \tag{4.51}$$

Replacing φ by ϱ in (4.50) we obtain

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) = \int_{\tilde{\Omega}} (-g_\epsilon(u_\epsilon) + \lambda u^p_\epsilon + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m). \tag{4.52}$$

We break the previous integral as

$$A_\epsilon := \int_{\Omega_0} (-g_\epsilon(u_\epsilon) + \lambda u^p_\epsilon + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m),$$

and

$$B_\epsilon := \int_{\tilde{\Omega}\setminus\Omega_0} (-g_\epsilon(u_\epsilon) + \lambda u^p_\epsilon + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m).$$

Clearly, $B_\epsilon = 0$, whenever $0 < \epsilon \leq \epsilon_1$ by (4.51) and the definition of η. We claim that

$$A_\epsilon \to \int_{\Omega_0} (\log u + \lambda u^p + \mu f(u))\varphi\eta(u/m) \text{ as } \epsilon \to 0. \tag{4.53}$$

Indeed, $u_\epsilon \to u$ uniformly in Ω_0. Then,

$$\int_{\Omega_0} (\lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m)\,dx \to \int_{\Omega_0} (\lambda u^p + \mu f(u))\varphi\eta(u/m)\,dx \text{ as } \epsilon \to 0.$$

Hence, we only need to show that

$$\int_{\Omega_0} -g_\epsilon(u_\epsilon)\varphi\eta(u_\epsilon/m) \to \int_{\Omega_0} \log(u)\varphi\eta(u/m) \text{ as } \epsilon \to 0.$$

If $u \le m/4$ then, for $\epsilon > 0$ sufficiently small, we have $u_\epsilon \le m/2$. Consequently, from the definition of η,

$$0 = \int_{\Omega_0\cap\{u\le m/4\}} \log(u)\varphi\eta(u/m) = \lim_{\epsilon\to 0}\int_{\Omega_0\cap\{u\le m/4\}} -g_\epsilon(u_\epsilon)\varphi\eta(u_\epsilon/m).$$

If $u > m/4$, then $u_\epsilon \ge m/8$ for $\epsilon > 0$ small enough. We then apply the Dominated Convergence Theorem as $\epsilon \to 0$ to get

$$\int_{\Omega_0\cap\{u>m/4\}} \log(u)\varphi\eta(u/m) = \lim_{\epsilon\to 0}\int_{\Omega_0\cap\{u>m/4\}} -g_\epsilon(u_\epsilon)\varphi\eta(u_\epsilon/m).$$

We have proved claim (4.53). Hence,

$$\lim_{\epsilon\to 0}\int_{\tilde\Omega}(-g_\epsilon(u_\epsilon) + \lambda u_\epsilon^p + \mu f(u_\epsilon))\varphi\eta(u_\epsilon/m) = \int_{\Omega_0}(\log u + \lambda u^p + \mu f(u))\varphi\eta(u/m).$$

We take the limit in m to conclude that

$$\int_{\Omega_0}(\log u + \lambda u^p + \mu f(u))\varphi\eta(u/m) \to \int_{\Omega_0}(\log u + \lambda u^p + \mu f(u))\varphi \text{ as } m \to 0, \tag{4.54}$$

since $\eta(u/m) \le 1$ and $(\log u)\chi_{\Omega^+} + \lambda u^p + \mu f(u) \in L^1(\tilde\Omega)$, according to Lemma 4.8.

We proceed with the integral on the left hand side of (4.52),

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) := \int_{\tilde\Omega}(\nabla u_\epsilon\nabla\varphi)\eta(u_\epsilon/m) + C_\epsilon. \tag{4.55}$$

Consequently,

$$\int_{\tilde\Omega}(\nabla u_\epsilon\nabla\varphi)\eta(u_\epsilon/m) \to \int_{\tilde\Omega}(\nabla u\nabla\varphi)\eta(u/m) \text{ as } \epsilon \to 0,$$

since $u_\epsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_\epsilon \to u$ uniformly in $\tilde\Omega$. Consequently, by the Dominated Convergence Theorem,

$$\int_{\tilde\Omega}(\nabla u\nabla\varphi)\eta(u/m) \to \int_{\tilde\Omega}\nabla u\nabla\varphi \text{ as } m \to 0. \tag{4.56}$$

We claim that

$$C_\epsilon := \int_{\tilde\Omega}\frac{|\nabla u_\epsilon|^2}{m}\eta'(u_\epsilon/m)\varphi \to 0 \text{ as } \epsilon \to 0 \quad \text{(and then as } m \to 0). \tag{4.57}$$

The estimate $|\nabla u_\epsilon|^2 \le MZ(u_\epsilon)$ in $\tilde\Omega$ given in (4.46) and recalling that $\eta'(u_\epsilon/m) = 0$ if $u_\epsilon \ge m$ yield

$$\begin{aligned}\limsup_{\epsilon\to 0}|C_\epsilon| &\le \frac{M}{m}\lim_{\epsilon\to 0}\int_{\tilde\Omega\cap\{\frac{m}{2}\le u_\epsilon\le m\}} Z(u_\epsilon)|\eta'(u_\epsilon/m)\varphi| \\ &\le M\lim_{\epsilon\to 0}\int_{\tilde\Omega\cap\{\frac{m}{2}\le u_\epsilon\le m\}} \frac{Z(u_\epsilon)|\eta'(u_\epsilon/m)\varphi|}{u_\epsilon}.\end{aligned}$$

Since $Z(t) = t(1+\log 2) + \dfrac{1}{4}$ if $\dfrac{1}{2} \leq t < 1$, we may find a constant $\ell > 0$ such that $\eta'(t) \leq \dfrac{\ell}{Z(t)}$ for $1/2 < t < 1$. Hence

$$\limsup_{\epsilon\to 0} |C_\epsilon| \leq M\ell \sup|\varphi| \lim_{\epsilon\to 0} \int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u_\epsilon\leq m\}} \frac{Z(u_\epsilon)}{u_\epsilon Z(u_\epsilon/m)}.$$

But if $1/2 \leq u_\epsilon/m \leq 1$ then $Z(u_\epsilon/m) \geq 1/2$. Hence,

$$\begin{aligned}\limsup_{\epsilon\to 0} |C_\epsilon| &\leq 2M\ell \sup|\varphi| \lim_{\epsilon\to 0} \int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u_\epsilon\leq m\}} \frac{Z(u_\epsilon)}{u_\epsilon} \\ &\leq 2M\ell \sup|\varphi| \int_{\tilde{\Omega}\cap\{\frac{m}{2}\leq u\leq m\}} (u+1-\log u))\chi_{\{u>0\}},\end{aligned}$$

for every $m > 0$. Thus invoking Lemma 4.8 and letting $m \to 0$, (4.57) is proved. As an immediate consequence of (4.52)–(4.57), we have

$$\int_\Omega \nabla u \nabla\varphi = \int_{\Omega\cap\{u>0\}} \Big(\log u + \lambda u^p + \mu f(u)\Big)\varphi,$$

for every $\varphi \in C_c^1(\Omega)$. This concludes the proof of Theorem 4.1. □

5 Log–exp problems without parameters

In this chapter we study problems of the form

$$\begin{cases} -\Delta u = (\log u + f(u))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and f is allowed to have exponential growth. We will again study a perturbed problem of the form

$$\begin{cases} -\Delta u + g_{\epsilon,f(0)}(u) = f(u) \,\text{in}\, \Omega \\ u \geq 0 \text{ in } \Omega \\ u = 0 \,\text{on}\, \partial\Omega. \end{cases} \tag{5.2}$$

Here, $g_{\epsilon,b} \in C^\infty(0,\infty)$ is defined by

$$g_{\epsilon,b}(s) = \begin{cases} -\log\left(s + \dfrac{\epsilon e^{-b}}{s+\epsilon}\right) \text{ for } s \geq 0 \\ b \text{ for } s < 0, \end{cases} \tag{5.3}$$

where $b = f(0)$, so that $g_{\epsilon,b}(0) = b = f(0)$ for all $\epsilon > 0$ and $g_\epsilon(s) \to -\log(s)$ pointwisely for $s > 0$ as $\epsilon \to 0$. The perturbation $g_{\epsilon,b}$ is a generalization of the perturbation considered in Chapter 4, and we no longer need to assume that $f(0) = 0$. We will assume that f satisfies the following assumptions:

- $f(s) = f(0)$ for all $s \leq 0, f \in C^{1,\nu}(0,\infty) \cap C(\mathbb{R})$ and $\sup\limits_{0\leq s\leq 1} |sf'(s)| < \infty,$ (5.4)

for some $0 < \nu < 1$, and that for each $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that

$$|f(s)| \leq C_\alpha \exp\left(\alpha s^2\right), \text{ for every } s \geq 0. \tag{5.5}$$

As in the previous chapters, we work with the functional $I_\epsilon : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega G_{\epsilon,f(0)}(u) - \int_\Omega F(u)\,dx, \tag{5.6}$$

where $F(u) = \displaystyle\int_0^u f(s)\,ds$ and $G_{\epsilon,b}(u) = \displaystyle\int_0^s g_{\epsilon,b}(s)\,ds$. From the fact that f and $g_{\epsilon,b}$ are continuous functions that satisfy (5.3), (5.4) and (5.5), we conclude I_ϵ is of class C^1 and

$$I'_\epsilon(u)(v) = \int_\Omega \nabla u \nabla v + \int_\Omega g_{\epsilon,f(0)}(u)v - \int_\Omega f(u)v, \text{ for all } u,v \in H_0^1(\Omega). \tag{5.7}$$

Consequently, if $u_\epsilon \in H_0^1(\Omega)$ is a critical point of I_ϵ then

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_{\epsilon,f(0)}(u_\epsilon)v = \int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega). \tag{5.8}$$

Consequently, critical points u_ϵ of I_ϵ are weak solutions of problem (5.2).

Remark 5.1. *It is important to choose $b = f(0)$ in (5.3) to guarantee that critical points of the functional I_ϵ are nonnegative.*

Indeed, choosing $v = -u_\epsilon^-$ in (5.8) and assuming that $I'_\epsilon(u_\epsilon) = 0$, we get

$$\begin{aligned}\int_\Omega |\nabla u_\epsilon^-|^2 &= \int_{\Omega\cap\{u_\epsilon<0\}} g_{\epsilon,f(0)}(u_\epsilon)u_\epsilon^- - \int_{\Omega\cap\{u_\epsilon<0\}} f(u_\epsilon)u_\epsilon^- \\ &= f(0)\int_{\Omega\cap\{u_\epsilon<0\}} u_\epsilon^- - f(0)\int_{\Omega\cap\{u_\epsilon<0\}} u_\epsilon^- \\ &= 0\end{aligned}$$

Consequently $u_\epsilon \geq 0$ in Ω.

Furthermore, if $u_\epsilon \in L^\infty(\Omega)$, then for each $0 < \epsilon < 1$

$$\sup_\Omega(|g_\epsilon(u_\epsilon)| + |f(u_\epsilon)|) < \infty,$$

and consequently

$$\Delta u_\epsilon \in L^\infty(\Omega).$$

We conclude from Elliptic Regularity Theory (Theorem B.14) that $u_\epsilon \in W^{2,r}(\Omega)$ for all $r > 1$. Thus, the Sobolev Embedding (Theorem B.13) implies that $u_\epsilon \in C^{1,\nu}(\overline{\Omega})$, where $0 < \nu < 1$ is given by (5.4). Summarizing, we have

Lemma 5.1. *Suppose that f satisfies (5.4) and (5.5). The following assertions hold:*

(i) Critical points of I_ϵ are nonnegative weak solutions of problem (5.2).

(ii) If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a nonnegative weak solution of problem (5.2), then u is smooth and $u \in C^{1,\nu}(\overline{\Omega})$, with ν given by (5.5).

Now we summarize the properties of the perturbation $g_{\epsilon,b}$ defined in (5.3).

Lemma 5.2. *Let $b \in \mathbb{R}$ and $\delta > 0$. There exists a constant $0 < \epsilon_0 < 1/2$ that depends on b and δ such that the following assertions hold for each $0 < \epsilon < \epsilon_0$.*

(i) If $b < 0$, then $g_{\epsilon,b}(s) \leq 0$ for $s \leq s_{b,\epsilon}$, $g_{\epsilon,b}(s) \geq 0$ for $s_{b,\epsilon} \leq s \leq S_{b,\epsilon}$, and $g_{\epsilon,b}(s) \leq 0$ for $s \geq S_{b,\epsilon}$, where

$$S_{b,\epsilon} = \frac{(1-\epsilon) + \sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b}-1)}}{2}$$

and

$$s_{b,\epsilon} = \frac{(1-\epsilon) - \sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b}-1)}}{2}.$$

Also, $0 < s_{b,\epsilon} < 1/2$, and

$$s_{b,\epsilon} < \delta \text{ for all } 0 < \epsilon < \epsilon_0.$$

Moreover,

$$g_{\epsilon,b}(s) \geq b - \log 2 \text{ for } s \leq e^{-b}.$$

(ii) If $b \geq 0$, then $g_{\epsilon,b}(s) \geq 0$ for $s \leq S_{b,\epsilon}$ and $g_{\epsilon,b}(s) \leq 0$ for $s \geq S_{b,\epsilon}$.

(iii) The following inequality holds

$$\frac{1}{2} < S_{b,\epsilon} < \frac{3}{2} \text{ for all } 0 < \epsilon < \epsilon_0.$$

(iv) For each constant $R > 0$, there exists $\widetilde{C}_b > 0$ such that

$$|sg_{\epsilon,b}(s)| \leq \widetilde{C}_b \text{ and } |G_{\epsilon,b}(s)| \leq \widetilde{C}_b \text{ for } 0 \leq s \leq R.$$

(v) For each $p_0 > 2$, there exists a constant $k_0 > 0$ such that

$$G_{\epsilon,b}(s) \geq -k_0 s^{p_0} \text{ for all } s \geq \delta \text{ and } 0 < \epsilon < \epsilon_0.$$

(vi)

$$g'_{\epsilon,b}(s) = -\frac{(s+\epsilon)^2 - \epsilon e^{-b}}{(s+\epsilon)(s^2 + s\epsilon + \epsilon e^{-b})} \text{ for all } s > 0,$$

so that

$$g'_{\epsilon,b}(0) = \frac{1}{\epsilon} - e^b \to \infty \text{ as } \epsilon \to 0.$$

Proof of Lemma 5.2. First we prove item (i). Assume that $b < 0$ and observe that $g_{\epsilon,b} \geq 0$ if and only if

$$s + \frac{\epsilon e^{-b}}{s+\epsilon} \leq 1,$$

or equivalently,

$$s^2 + (\epsilon - 1)s + \epsilon(e^{-b} - 1) \leq 0.$$

This inequality holds if and only if

$$s_{b,\epsilon} = \frac{1 - \epsilon - \sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1)}}{2} \leq s \leq \frac{1 - \epsilon + \sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1)}}{2} = S_{b,\epsilon}.$$

Observe that these quantities are well defined if $(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1) \geq 0$, which holds provided

$$0 < \epsilon < \frac{(1-\epsilon)^2}{4(e^{-b} - 1)}.$$

The assumption $0 < \epsilon < 1/2$ assure us that $s_{b,\epsilon}$ and $S_{b,\epsilon}$ are well defined when

$$0 < \epsilon < \epsilon_0 = \frac{1}{16(e^{-b} - 1)}.$$

We also have

$$s_{b,\epsilon} \leq \frac{1-\epsilon}{2} < \frac{1}{2} \text{ for all } 0 < \epsilon < \epsilon_0.$$

We now show that there exists $\epsilon_0 > 0$ such that $s_{b,\epsilon} < \delta$ for all $0 < \epsilon < \epsilon_0$. Indeed, $s_{a,b,\epsilon} < \delta$ if and only if

$$1 - \epsilon - \sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1)} < 2\delta.$$

Equivalently,

$$1 - \epsilon - 2\delta < \sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1)}.$$

This result clearly holds if $1 - \epsilon - 2\delta < 0$. Otherwise, we must consider $\epsilon > 0$ such that

$$\begin{aligned}
(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1) &> (1 - \epsilon - 2\delta)^2 \\
(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1) &> (1-\epsilon)^2 - 4(1-\epsilon)\delta + 4\delta^2 \\
-4\epsilon(e^{-b} - 1) &> 4\epsilon\delta - 4\delta + 4\delta^2 \\
4\epsilon((e^{-b} - 1) + \delta) &< 4\delta(1-\delta) \\
\epsilon &< \frac{\delta(1-\delta)}{e^{-b} - 1 + \delta}.
\end{aligned}$$

We further have

$$g_{\epsilon,b}(s) \geq b - \log 2 \text{ if and only if } \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \leq -b + \log 2,$$

which is equivalent to

$$s + \frac{\epsilon e^{-b}}{s+\epsilon} \leq 2e^{-b},$$

which is true provided

$$s \leq e^{-b}.$$

We have proved item (i). The proof of (ii) is very similar. We now prove (iii), thus proceeding with the estimates for $S_{b,\epsilon}$. We know that $S_{b,\epsilon} < 3/2$ provided

$$\begin{aligned}
\sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1)} &< 2 + \epsilon \\
(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1) &< (2+\epsilon)^2 \\
1 - 2\epsilon + \epsilon^2 - 4\epsilon(e^{-b} - 1) &< 4 + 4\epsilon + \epsilon^2.
\end{aligned}$$

Hence, we need to solve

$$\begin{aligned}
3 + 2\epsilon(3 + 2(e^{-b} - 1)) &> 0 \\
3 + 2\epsilon(1 + 2e^{-b}) &> 0,
\end{aligned}$$

which is clearly true. Furthermore, $S_{a,b,\epsilon} > 1/2$ provided

$$\begin{aligned}
\sqrt{(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1)} &> \epsilon \\
(1-\epsilon)^2 - 4\epsilon(e^{-b} - 1) &> \epsilon^2 \\
1 - 2\epsilon + \epsilon^2 - 4\epsilon(e^{-b} - 1) &> \epsilon^2.
\end{aligned}$$

Consequently,

$$\begin{aligned} 1 - 2\epsilon - 4\epsilon(e^{-b} - 1) > 0 \\ 1 > \epsilon(2 + 4(e^{-b} - 1)) = \epsilon(4e^{-b} - 2). \end{aligned}$$

This result is obviously true for small values of ϵ. We have proved (iii). Now we prove (iv). Let $R > S_{b,\epsilon}$ be a constant. In what follow, $C_i > 0$, $i = 1, 2, 3, \dots$ denote positive constants that do not depend on ϵ. Observe that

$$\begin{aligned} |sg_{\epsilon,b}(s)| = \left| s \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \right| &= s \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \\ &\leq s \log(s + e^{-b}) \text{ for } 0 \leq s \leq s_{b,\epsilon}. \end{aligned}$$

Consequently, $|sg_{\epsilon,b}(s)| \leq s \log(2e^{-b})$ if $s \leq e^{-b}$ and $|sg_{\epsilon,b}(s)| \leq s \log(2s) \leq C$ if $e^{-b} \leq s \leq s_{b,\epsilon} < 1/2$. We thus obtain

$$|sg_{\epsilon,b}(s)| \leq C_1 \text{ for } 0 \leq s \leq s_{b,\epsilon}.$$

Also,

$$\begin{aligned} |sg_{\epsilon,b}(s)| = \left| s \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \right| &= -s \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \\ &\leq -s \log s < C_2 \text{ for } s_{b,\epsilon} \leq s \leq S_{b,\epsilon} < 3/2. \end{aligned}$$

Finally,

$$\begin{aligned} |sg_{\epsilon,b}(s)| = \left| s \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \right| &= s \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \\ &\leq s \log(s + e^{-b}) \text{ for } S_{b,\epsilon} \leq s \leq R. \end{aligned}$$

Consequently, $|sg_{\epsilon,b}(s)| \leq s \log(2e^{-b})$ if $S_{b,\epsilon} \leq s \leq e^{-b}$ and $|sg_{\epsilon,b}(s)| \leq s \log(2s) \leq C$ if $e^{-b} \leq s \leq R$. We thus obtain

$$|sg_{\epsilon,b}(s)| \leq C_3 \text{ for } 0 \leq s \leq R.$$

We proceed to prove estimates for $G_{\epsilon,b}$. Note that

$$\begin{aligned} |G_{\epsilon,b}(s)| &\leq \int_0^s |g_{\epsilon,b}(t)|\, dt \\ &\leq \int_0^s \log\left(t + \frac{\epsilon e^{-b}}{t+\epsilon}\right) dt \\ &\leq \int_0^s \log\left(t + e^{-b}\right) dt \\ &= (s + e^{-b}) \log(s + e^{-b}) - (s + e^{-b}) - (e^{-b} \log(e^{-b}) - e^{-b}) \leq C_4 \text{ for } 0 \leq s \leq s_{b,\epsilon} < 1/2, \end{aligned}$$

so that $|G_{\epsilon,b}(s)| \leq C_4$ for $0 \leq s \leq s_{b,\epsilon}$. Furthermore,

$$\begin{aligned}
|G_{\epsilon,b}(s)| &\leq \left| G_{\epsilon,b}(s_{b,\epsilon}) + \int_{s_{b,\epsilon}}^{s} g_{\epsilon,b}(t)\,dt \right| \\
&\leq C_4 + \int_{s_{b,\epsilon}}^{s} |g_{\epsilon,b}(t)|\,dt \\
&\leq C_4 + \int_{s_{b,\epsilon}}^{s} -\log\left(t + \frac{\epsilon e^{-b}}{t+\epsilon}\right)\,dt \\
&\leq C_4 + \int_{s_{b,\epsilon}}^{s} -\log t\,dt \\
&= C_4 - (s\log s - s) + (s_{b,\epsilon}\log s_{b,\epsilon} - s_{b,\epsilon}) \leq C_5 \text{ for } s_{b,\epsilon} \leq s \leq S_{b,\epsilon} < 3/2.
\end{aligned}$$

Finally

$$\begin{aligned}
|G_{\epsilon,b}(s)| &\leq \left| G_{\epsilon,b}(S_{b,\epsilon}) + \int_{S_{b,\epsilon}}^{R} g_{\epsilon,b}(t)\,dt \right| \\
&\leq |G_{\epsilon,b}(S_{b,\epsilon})| + \int_{S_{b,\epsilon}}^{R} |g_{\epsilon,b}(t)|\,dt \\
&\leq C_5 + \int_{S_{b,\epsilon}}^{R} \log\left(t + \frac{\epsilon e^{-b}}{t+\epsilon}\right)\,dt \\
&\leq C_5 + \int_{S_{b,\epsilon}}^{R} \log\left(t + e^{-b}\right)\,dt \\
&= C_5 + (R+e^{-b})\log(R+e^{-b}) - (R+e^{-b}) \\
&- ((S_{b,\epsilon}+e^{-b})\log(S_{b,\epsilon}+e^{-b}) - (S_{b,\epsilon}+e^{-b})) \leq C_6 \text{ for } S_{b,\epsilon} \leq s \leq R.
\end{aligned}$$

Now we prove (v). From (i), we may choose $\epsilon_0 > 0$ such that $s_{b,\epsilon} < \delta$. Consequently, from (iv),

$$\begin{aligned}
G_{\epsilon,b}(s) &\geq G_{\epsilon}(s_{b,\epsilon}) \\
&\geq -C \\
&\geq -\frac{C}{\delta^{p_0}} s^{p_0} \text{ for } \delta \leq s \leq S_{b,\epsilon}.
\end{aligned}$$

On the other hand, from (iii) and (iv), we have

$$\begin{aligned}
G_{\epsilon,b}(s) &= G_\epsilon(S_{b,\epsilon}) + \int_{S_{b,\epsilon}}^s g_{\epsilon,b}(t)\,dt \\
&\geq -C - \int_{S_{b,\epsilon}}^s |g_{\epsilon,b}(t)|\,dt \\
&= -C - \int_{S_{b,\epsilon}}^s \log\left(t + \frac{\epsilon e^{-b}}{t+\epsilon}\right) dt \\
&\geq -C - \int_{S_{b,\epsilon}}^s \log\left(t + e^{-b}\right) dt \\
&\geq -C - \int_{S_{b,\epsilon}}^s t + e^{-b}\,dt \\
&\geq -C - \int_0^s t + e^{-b}\,dt \\
&= -C - \left(\frac{s^2}{2} + e^{-b}s\right) \quad \text{for } s \geq S_{b,\epsilon} \geq \delta.
\end{aligned}$$

Consequently,

$$\begin{aligned}
G_{\epsilon,b}(s) &\geq -\left(\frac{Cs^{p_0}}{\delta^{p_0}} + \frac{s^{p_0}}{2\delta^{p_0-2}} + \frac{e^{-b}s^{p_0}}{\delta^{p_0-1}}\right) \\
&= -k_0 s^{p_0} \text{ for } s \geq S_{b,\epsilon} \geq \delta.
\end{aligned}$$

This proves item (v). Item (vi) is a direct consequence of (5.3). We have proved Lemma 5.2. □

We further have

Lemma 5.3. *There exist constants $C_b > 0$ and $R_b > 0$ that do not depend on ϵ satisfying the following assertions.*

(i) $|sg'_{\epsilon,b}(s)| \leq 1$ *for all* $s \geq 0$,

(ii) $|sg'_{\epsilon,b}(s)| = -sg'_{\epsilon,b}(s) \geq \frac{1}{6}$ *for all* $s \geq R_b$ *and* $0 < \epsilon < 1$. *If* $b \geq 0$, *then* $R_b = 2$.

(iii) $0 \leq -g_{\epsilon,b}(s) \leq s$ *for all* $s \geq R_b$ *and* $0 < \epsilon < 1$.

(iv) $G_{\epsilon,b}(s) \leq 0$ *for all* $s \geq R_b$ *and* $0 < \epsilon < 1$.

Proof of Lemma 5.3. First, we prove (i). Observe that

$$-g'_{\epsilon,b}(s) = \frac{s^2 + 2\epsilon s + \epsilon^2 - \epsilon e^{-b}}{s^3 + 2\epsilon s^2 + s(\epsilon^2 + \epsilon e^{-b}) + \epsilon^2 e^{-b}}.$$

Consequently,

$$|sg'_{\epsilon,b}(s)| \leq \frac{s^3 + 2\epsilon s^2 + (\epsilon^2 + \epsilon e^{-b})s}{s^3 + 2\epsilon s^2 + (\epsilon e^{-b} + \epsilon^2)s + \epsilon^2 e^{-b}} \leq 1.$$

This proves (i). Furthermore, $-g'_{\epsilon,b}(s) \geq 0$ if $2\epsilon s \geq \epsilon e^{-b}$, which holds for $s \geq \dfrac{e^{-b}}{2}$. Also,

$$\epsilon^2 s \leq s^3 \text{ for } 0 < \epsilon < 1, s > 1,$$

$$\epsilon e^{-b} s \leq s^3 \text{ for } 0 < \epsilon < 1 \text{ and } s \geq e^{-b/2}.$$

and

$$\epsilon^2 e^{-b} \leq s^3 \text{ for } 0 < \epsilon < 1 \text{ and } s \geq e^{-b/3}.$$

We conclude that

$$-sg'_{\epsilon,b}(s) \geq \frac{s^3}{s^3 + 2s^3 + 2s^3 + s^3} = \frac{1}{6} \text{ for } s \geq \max\left\{\frac{e^{-b}}{2}, e^{-b/2}, e^{-b/3}, 1\right\}.$$

This proves (ii). We now prove (iii). Indeed, from items (i), (ii) and (iii) of Lemma 5.2, there exists $R_b > 0$ such that

$$0 \leq -g_{\epsilon,b}(s) = \log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \leq \log(s + e^{-b}) \leq s \text{ for all } s \geq R_b \geq S_{b,\epsilon}.$$

Now we prove (iv). Indeed, from Lemma item (iv) of 5.2,

$$\begin{aligned}
G_{\epsilon,b}(s) &= G_{\epsilon,b}(S_{b,\epsilon}) + \int_{S_{b,\epsilon}}^{s} g_{\epsilon,b}(t)\,dt \\
&\leq C_4 - \int_{S_{b,\epsilon}}^{s} \log\left(t + \frac{\epsilon e^{-b}}{t+\epsilon}\right) dt \\
&\leq C_4 - \int_{S_{b,\epsilon}}^{s} \log t\,dt \\
&= C_4 - (s\log s - s) + (S_{b,\epsilon}\log(S_{b,\epsilon}) - S_{b,\epsilon}) \\
&\leq C_5 - s(\log s - 1).
\end{aligned}$$

Consequently, $G_\epsilon(s) \leq 0$ if $s(\log s - 1) \geq C_5$. This proves the result. □

We remark that the estimates given in Lemmas 5.2 and 5.3 are uniform in ϵ.

5.1 Existence of solutions for the perturbed problem

In this chapter, we obtain critical points of the functional I_ϵ given by (5.6). We again consider the functions $j_\epsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$j_\epsilon(s) = f(s) - g_{\epsilon,f(0)}(s),$$

and $J_\epsilon(s) = \int_0^s j_\epsilon(t)\,dt$. Observe that

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u). \tag{5.9}$$

Also, (5.3) and (5.4) imply that $j_\epsilon(s) = J_\epsilon(s) = 0$ for $s \leq 0$. We assume further that f satisfies the following conditions:

- There exist constants $0 < \epsilon_0, \delta < 1$ such that

$$g_{\epsilon_0, f(0)}(s) \geq f(s) \text{ for all } s \leq \delta, \tag{5.10}$$

This condition is satisfied when, for example

$$\lim_{s \to 0^+} |f'(s)| < \infty.$$

The existence of ϵ_0 satisfying condition (5.10) implies that

$$g_{\epsilon, f(0)}(s) \geq f(s) \text{ for all } s \leq \delta, 0 < \epsilon \leq \epsilon_0.$$

- There exist constants $0 < \theta < 1/2$ and $s_0 > 0$ such that

$$\min\{f(s), F(s)\} \geq 0 \text{ for } s \geq s_0, \tag{5.11}$$

and

$$(1-\theta)(f(s)+s) \leq \theta s f'(s) \text{ for } s \geq s_0. \tag{5.12}$$

- There exist constants $A > 0$ and $\gamma > 2$ such that

$$F(s) \geq A|s|^\gamma \text{ for } s \geq s_0. \tag{5.13}$$

Condition (5.10) will be used to show that the origin is a local minimum for the functional I_ϵ for all $0 < \epsilon < 1$. Condition (5.13) will imply that there exists an element $\phi_0 \in H_0^1(\Omega)$ with $I_\epsilon(\phi_0) < 0$ for all $0 < \epsilon < 1$. From the Mountain Pass Theorem, we will get a Palais-Smale sequence (u_n^ϵ) for I_ϵ. Conditions (5.11) and (5.12) will be important to show that the sequence (u_n^ϵ) is bounded in $H_0^1(\Omega)$ by a constant that does not depend on ϵ. We will then conclude that it converges to a critical point u_ϵ of I_ϵ. First, we get

Lemma 5.4. *Assume that (5.4) and (5.5) hold. The following assertions are true.*

(i) If (5.10) holds, then $j_\epsilon(s) \leq 0$ for $s \leq \delta$ and $0 < \epsilon < \epsilon_0$.

(ii) Assume further that (5.11) and (5.12) hold. Then, there exists $R^*_{f(0)} > 0$ and $\epsilon_0 > 0$ such that

$$0 \leq J_\epsilon(s) \leq \theta s j_\epsilon(s) \text{ for } s \geq R^*_{f(0)} \text{ and } 0 < \epsilon < \epsilon_0.$$

We thus obtain a constant $C = C_{f(0)} > 0$ such that

$$|J_\epsilon(s)| \leq C + \theta s j_\epsilon(s) \text{ for all } s \geq 0 \text{ and } 0 < \epsilon < \epsilon_0.$$

Proof of Lemma 5.4. It is clear that condition (5.10) implies (i). We prove (ii). Observe that $J_\epsilon(s) \geq 0$ for large s provided $F(s) - G_{\epsilon, f(0)}(s) \geq 0$. This follows from (5.11) and Lemma 5.3. Now let $B_\epsilon(s) = F(s) - G_{\epsilon, f(0)}(s) - \theta s(f(s) - g_{\epsilon, f(0)}(s))$, where $0 < \theta < \dfrac{1}{2}$ is given by (5.12). We will show that $B_\epsilon(s) \leq 0$ for large values of s. Indeed,

$$B'_\epsilon(s) = (1-\theta)(f(s) - g_{\epsilon, f(0)}(s)) - \theta s f'(s) + \theta s g'_{\epsilon, f(0)}(s).$$

Therefore, from item (ii) of Lemma 5.3, we conclude that

$$B'_\epsilon(s) \le (1-\theta)(f(s)+s) - \theta s f'(s) - \frac{\theta}{6} \text{ for } s \ge R,$$

where $R = R_{f(0)}$ is a constant that does not depend on ϵ. Therefore, $B'_\epsilon(s) \le -\theta/6$ for $s \ge R$ provided

$$(1-\theta)(f(s)+s) - \theta s f'(s) \le 0 \text{ for } s \ge R,$$

which follows from (5.12). Furthermore,

$$B_\epsilon(R) = F(R) - G_{\epsilon,f(0)}(R) - \theta R f(R) + \theta R g_{\epsilon,f(0)}(R).$$

Consequently, from item (iv) of Lemma 5.2 and (5.4) , we obtain a constant $C > 0$ that does not depend on ϵ such that

$$|B_\epsilon(R)| \le C_{f(0)} + |F(R)| + |Rf(R)| = C, \text{ for all } 0 < \epsilon < \epsilon_0.$$

We conclude that

$$B_\epsilon(s) \le -\frac{\theta s}{6} + \widetilde{C} \text{ for all } 0 < \epsilon < \epsilon_0,$$

where the constant $\widetilde{C}$ does not depend on ϵ. Hence, $B_\epsilon(s) \le 0$ for $s \ge \dfrac{6\widetilde{C}}{\theta}$. Consequently, there must exist $R^* > 0$ such that

$$0 \le J_\epsilon(s) \le \theta s j_\epsilon(s) \text{ for all } s \ge R^* \text{ and } 0 < \epsilon < \epsilon_0.$$

From Lemma 5.2, we know that there exists a constant $C_2 > 0$ such that $|sj_\epsilon(s)| \le C_2$ and $|J_\epsilon(s)| \le C_2$ for all $0 \le s \le R^*$ and $0 < \epsilon < \epsilon_0$. Consequently,

$$|J_\epsilon(s)| \le C_3 + \theta s j_\epsilon(s) \text{ for } s \in \mathbb{R} \text{ and } 0 < \epsilon < \epsilon_0,$$

where $C_3 > 0$ is a constant that does not depend on ϵ. This proves the estimate. □

We also need the following energy estimates.

Lemma 5.5. *Assume that f is a function that satisfies* (5.4), (5.5) *and* (5.13). *Then, there exist $\phi_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that*

$$I_\epsilon(\phi_0) < 0 \textit{ for all } 0 < \epsilon < 1 \tag{5.14}$$

Furthermore, there exists $C > 0$ that does not depend on ϵ such that

$$\sup_{t\in[0,1]} I_\epsilon(t\phi_0) < C \textit{ for all } \textit{ for all } 0 < \epsilon < 1.$$

Proof of Lemma 5.5. Indeed, let $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be a function such that $\phi > 0$ in Ω and $\sup_\Omega \phi > 1/2$. Then

$$I_\epsilon(t\phi_0) = \frac{t^2}{2}\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega G_{\epsilon,f(0)}(t\phi) - \int_\Omega F(t\phi)\,dx \text{ for all } t > 0.$$

Consequently, from (5.13),

$$I_\epsilon(t\phi) = \frac{t^2}{2}\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega G_{\epsilon,f(0)}(t\phi) - \int_{\{t\phi\le s_0\}} F(t\phi)\,dx - At^\gamma \int_{\{t\phi>s_0\}} \phi^\gamma\,dx \text{ for all } t>0.$$

We choose $t_0 > 0$ so large that

$$\frac{s_0}{t_0} = \frac{1}{4}, \text{ so that } \{\phi > 1/4\} \subset \{t\phi > s_0\} \text{ for all } t > t_0.$$

From (5.4), item (iv) of Lemma 5.2 and item (iv) of Lemma 5.3, we conclude that

$$I_\epsilon(t\phi) \le \frac{t^2}{2}\int_\Omega |\nabla\phi|^2\,dx + C_1 - At^\gamma \int_{\{\phi>1/4\}} \phi^\gamma\,dx \text{ for all } t \ge t_0,$$

where $C_1 > 0$ is a positive constant that does not depend on t. Inequality (5.14) then follows by choosing $\phi_0 = T\phi$, where $T > 0$ is so large that $I_\epsilon(T\phi) < 0$. Furthermore, (5.4) implies that there exists $C_2 > 0$ such that

$$|F(s)| \le C_2 \text{ for } 0 \le s \le \sup_\Omega \phi_0.$$

Consequently, from item (iv) of Lemma 5.2,

$$\begin{aligned} I_\epsilon(t\phi_0) &= \frac{t^2}{2}\|\phi_0\|^2_{H_0^1(\Omega)} + \int_\Omega G_{\epsilon,f(0)}(t\phi_0) - \int_\Omega F(t\phi_0) \\ &\le \frac{t^2}{2}\|\phi_0\|^2_{H_0^1(\Omega)} + \int_\Omega G_{\epsilon,f(0)}(t\phi_0) + C_2|\Omega| \\ &\le \frac{1}{2}\|\phi_0\|^2_{H_0^1(\Omega)} + C_3 \text{ for all } 0 \le t \le 1, 0 < \epsilon < 1. \end{aligned}$$

We have proved the result. □

Now we obtain the main result of this section. As in the preceding chapters, the idea is to apply the Mountain-Pass Theorem to obtain a critical point of I_ϵ. We remark that (5.5) implies that

$$\max\{|f(s)|, |F(s)|\} \le C_\alpha \exp(\alpha s^2) \text{ for all } s \ge 0 \text{ and } \alpha > 0, \tag{5.15}$$

see Lemma 2.2.

Proposition 5.1. *Suppose that f satisfies (5.4), (5.5) and (5.10) – (5.13). Then, there exist $D > 0$ and $\epsilon_0 > 0$ such that I_ϵ has a critical point $u_\epsilon \in H_0^1(\Omega)$ satisfying*

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} \le D \text{ for all } 0 < \epsilon < \epsilon_0.$$

Moreover, there exist constants $a_1, a_2 > 0$ such that

$$a_1 < I_\epsilon(u_\epsilon) < a_2. \tag{5.16}$$

Proof of Proposition 5.1. Let $0 < \epsilon_0, \delta < 1$ be given by (5.10). As a consequence of item (i) of Lemma 5.4 and item (v) of Lemma 5.2, we get

$$\begin{aligned} I_\epsilon(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\Omega\cap\{u\geq\delta\}} J_\epsilon(u) \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\Omega\cap\{u\geq\delta\}} G_\epsilon(u) - \int_{\Omega\cap\{u\geq\delta\}} F(u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_{\Omega\cap\{u\geq\delta\}} u^{p_0} - \int_{\Omega\cap\{u\geq\delta\}} F(u), \text{ for all } u \in H_0^1(\Omega), 0 < \epsilon < \epsilon_0. \end{aligned}$$

Choosing $p_0 = 3$, using the Trudinger-Moser inequality, (5.15), the Sobolev embedding and Hölder's inequality, we obtain

$$\begin{aligned} I_\epsilon(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_{\Omega\cap\{u\geq\delta\}} u^3 - C_2\int_{\Omega\cap\{u\geq\delta\}} \exp(u^2). \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_\Omega |u|^3 - \frac{C_3}{\delta^3}\int_\Omega |u|^3\exp(u^2) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_\Omega |u|^3 - \frac{C_3}{\delta^3}\left(\int_\Omega |u|^6\right)^{\frac{1}{2}}\left(\int_\Omega \exp(2u^2)\right)^{\frac{1}{2}} \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_4\|u\|^3_{H_0^1(\Omega)} \text{ for } \|u\|^2_{H_0^1(\Omega)} < 2\pi, \end{aligned}$$

where C_i are positive constants that do not depend on ϵ. Consequently, there exists $\rho > 0$ that does not depend on ϵ such that

$$I_\epsilon(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} \text{ for all } u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1(\Omega)} \leq \rho. \tag{5.17}$$

We know that I_ϵ is a functional of class C^1. Consequently, Lemma 5.5, (5.17) and the Mountain-Pass Theorem imply that there exists a Palais-Smale sequence (u_n^ϵ) for I_ϵ at level

$$c_\epsilon = \inf_{\Psi\in\Gamma}\max_{0\leq t\leq 1} I_\epsilon(\Psi(t)),$$

where

$$\Gamma = \{\Psi \in C([0,1], H_0^1(\Omega)) : \Psi(0) = 0 \text{ and } \Psi(1) = \phi_0\}.$$

From (5.17) we obtain

$$c_\epsilon \geq \frac{\rho^2}{4}, \text{ for all } 0 < \epsilon < \epsilon_0.$$

Furthermore, Lemma 5.5 imply that $c_\epsilon \leq C$ for some constant $C > 0$ that does not depend on ϵ. Consequently,

$$\frac{\rho^2}{4} < |I_\epsilon(u_n^\epsilon)| < C,$$

for sufficiently large n. We thus get, from Lemma 5.4,

$$\begin{aligned} \frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} &\leq C + \int_\Omega J_\epsilon(u_n^\epsilon(x))\,dx \\ &\leq C + C_1|\Omega| + \theta\int_\Omega u_n^\epsilon j_\epsilon(u_n^\epsilon)\,dx, \end{aligned}$$

where $0 < \theta < 1/2$ is given by (5.12). Furthermore, $I'_\epsilon(u_n^\epsilon) \to 0$ as $n \to \infty$. Consequently, there exists a sequence (τ_n^ϵ) in $(0,1)$ such that $\tau_n^\epsilon \to 0$ and

$$\left| \int_\Omega \nabla u_n^\epsilon \nabla v \, dx - \int_\Omega j_\epsilon(u_n^\epsilon) v \, dx \right| \leq \tau_n^\epsilon \|v\|_{H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega) \text{ and } n \in \mathbb{N}. \tag{5.18}$$

Taking $v = u_n^\epsilon$ in (5.18), we get

$$-\tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)} \leq \|u_n^\epsilon\|^2_{H_0^1(\Omega)} - \int_\Omega j_\epsilon(u_n^\epsilon) u_n^\epsilon \, dx \leq \tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)} \text{ for all } n \in \mathbb{N}.$$

Consequently,

$$\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq C + C_1|\Omega| + \theta\tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)} + \theta\|u_n^\epsilon\|^2_{H_0^1(\Omega)}.$$

We thus get

$$\left(\frac{1}{2} - \theta\right) \|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq C + C_1|\Omega| + \theta\tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)}.$$

Hence, (u_n^ϵ) must be uniformly bounded in $H_0^1(\Omega)$. Letting $n \to \infty$ we get

$$\left(\frac{1}{2} - \theta\right) \lim_{n\to\infty} \|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq C + C_1|\Omega|.$$

Consequently,

$$\lim_{n\to\infty} \|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \left(\frac{2}{1-2\theta}\right)(C + C_1|\Omega|).$$

We thus obtain a constant $D > 0$ independent of ϵ such that

$$\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq D \text{ for all } 0 < \epsilon < \epsilon_0.$$

Hence, there must exist $u_\epsilon \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{cases} u_n^\epsilon \rightharpoonup u_\epsilon \text{ weakly in } H_0^1(\Omega), \\ u_n^\epsilon \to u_\epsilon \text{ in } L^p(\Omega) \text{ for every } p > 1, \\ u_n^\epsilon \to u_\epsilon \text{ a.e in } \Omega. \end{cases} \tag{5.19}$$

We recall that since f satisfies (5.5), there exists a constant $C_\epsilon > 0$ such that

$$\max\{|j_\epsilon(s)|, |J_\epsilon(s)|\} \leq C_\epsilon \exp\left(\frac{2\pi}{D}s^2\right) \text{ for } s \in \mathbb{R}. \tag{5.20}$$

Let $1 < r < 2$. From (B.8), we get

$$\int_\Omega |j_\epsilon(u_n^\epsilon)|^r \, dx \leq C_\epsilon \int_\Omega \exp\left(2\pi r \left(\frac{u_n^\epsilon}{\|u_n^\epsilon\|_{H_0^1(\Omega)}}\right)^2\right) \leq C_\epsilon k_1 \text{ for all } n \in \mathbb{N}.$$

From (5.18), (5.19) and Hölder's inequality, we get

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 \, dx &= \lim_{n\to\infty} \int_\Omega j_\epsilon(u_n^\epsilon) u_n^\epsilon \, dx \\ &= \lim_{n\to\infty} \int_\Omega j_\epsilon(u_n^\epsilon) u_\epsilon \, dx \\ &= \lim_{n\to\infty} \int_\Omega \nabla u_n^\epsilon \nabla u_\epsilon \, dx \\ &= \int_\Omega |\nabla u_\epsilon|^2 \, dx. \end{aligned}$$

We conclude that $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$. Hence, $I'_\epsilon(u_\epsilon) = 0$ and

$$\frac{\rho^2}{4} \leq I_\epsilon(u_\epsilon) \leq C.$$

The fact that $u_\epsilon \geq 0$ in Ω is a consequence of Lemma 5.1. The result then follows by taking $a_1 = \dfrac{\rho^2}{4}$ and $a_2 = C$. □

5.2 Convergence of the perturbed solutions

In this section. we study the convergence of the solutions u_ϵ of problem (5.2) obtained in Proposition 5.1. This proposition guarantees that there exists a constant $D > 0$ such that

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} < D, \text{ for each } 0 < \epsilon < \epsilon_0. \tag{5.21}$$

Hence, there exist $u \in H_0^1(\Omega)$ and a sequence (ϵ_n) in $(0, \epsilon_0)$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} u_{\epsilon_n} \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_{\epsilon_n} \to u \text{ in } L^r(\Omega) \text{ for every } r > 1, \\ u_{\epsilon_n} \to u \text{ a.e in } \Omega, \\ |u_{\epsilon_n}| \leq h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega). \end{cases} \tag{5.22}$$

As in Chapters 2 and 4, we will apply regularity results discussed in Appendix A to conclude that u_{ϵ_n} is smooth for all $n \in \mathbb{N}$ and that u is continuous. Indeed, if (5.21) holds, then Corollary A.1, implies that there exists a constant $K_1 > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(\Omega)} < K_1 \text{ for all } 0 < \epsilon_n < \epsilon_0. \tag{5.23}$$

Then, it follows from elliptic regularity theory and from the Sobolev Embedding that $u_{\epsilon_n} \in C^1(\overline{\Omega})$, see Lemma 5.1. Lemma A.5 implies that there exists a constant $\epsilon_0 > 0$ such that for each smooth subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there exists a constant $M > 0$ that depends on Ω' but not on ϵ such that

$$|\nabla u_{\epsilon_n}(x)|^2 \leq MZ(u_{\epsilon_n}(x)) \text{ for every } x \in \Omega', \quad 0 < \epsilon < \epsilon_0, \tag{5.24}$$

where

$$Z(t) = \begin{cases} t^2 + t - t\log t & \text{for } 0 \leq t \leq \dfrac{1}{2} \\ \dfrac{1}{4} + \dfrac{1}{2}(1 + \log 2) + \left(t - \dfrac{1}{2}\right)(1 + \log 2) & \text{for } t \geq \dfrac{1}{2}. \end{cases}$$

Hence, it follows from the Arzela-Ascoli Theorem that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, so that u is continuous and $0 \leq u \leq K_1$. In this section, we show that u is a solution of (5.1) in the sense that

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u > 0\}} \Big(\log u + f(u)\Big)\varphi, \tag{5.25}$$

for every $\varphi \in C_c^1(\Omega)$ and

$$(\log u)\chi_{\{u>0\}} \in L^1_{loc}(\Omega).$$

First, we show that

Lemma 5.6. *The function u is nontrivial and the function $(\log u)\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.*

Proof of Lemma 5.6. First we show that u is nontrivial. Indeed, since $u_{\epsilon_n} \geq 0$ is a critical point of I_{ϵ_n} and $0 < a_1 < I_{\epsilon_n}(u_\epsilon)$, we have

$$\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega g_{\epsilon_n,f(0)}(u_{\epsilon_n})u_{\epsilon_n} = \int_\Omega f(u_{\epsilon_n})u_{\epsilon_n},$$

and

$$I_{\epsilon_n}(u_{\epsilon_n}) = \frac{1}{2}\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega G_{\epsilon_n,f(0)}(u_{\epsilon_n}) - \int_\Omega F(u_{\epsilon_n}) > a_1.$$

Hence,

$$I_{\epsilon_n}(u_{\epsilon_n}) = \int_\Omega \left(G_{\epsilon_n,f(0)}(u_{\epsilon_n}) - \frac{1}{2}g_{\epsilon_n,f(0)}(u_{\epsilon_n})u_{\epsilon_n}\right) dx + \int_\Omega \left(\frac{1}{2}f(u_{\epsilon_n})u_{\epsilon_n} - F(u_{\epsilon_n})\right) dx > a_1. \tag{5.26}$$

We will show that

$$\int_\Omega g_{\epsilon_n,f(0)}(u_{\epsilon_n})u_{\epsilon_n}\,dx \to \int_\Omega (-\log u)\chi_{\{u>0\}}u\,dx. \tag{5.27}$$

First, observe that if $g_{\epsilon,f(0)}(s) < 0$ then

$$|sg_\epsilon(s)| = s\log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \leq s\log(s + e^{-b}),$$

and if $g_{\epsilon,f(0)}(s) \geq 0$ then

$$|sg_{\epsilon,f(0)}(s)| = -s\log\left(s + \frac{\epsilon e^{-b}}{s+\epsilon}\right) \leq -s\log s.$$

Consequently,

$$|sg_{\epsilon,f(0)}(s)| \leq g_1(s) \text{ for all } s \geq 0 \text{ and } 0 < \epsilon < 1,$$

where

$$g_1(s) = \max\{|s\log(s+e^{-b})|, |s\log s|\} \in L^1(0, K_1),$$

with K_1 given by (5.23). Now we prove (5.27). Indeed, fix $x \in \Omega$ such that $u(x) > 0$. We know that $u_{\epsilon_n}(x) \to u(x)$. Hence,

$$g_{\epsilon_n,f(0)}(u_{\epsilon_n}(x))u_{\epsilon_n}(x) = -u_{\epsilon_n}(x)\log\left(u_{\epsilon_n}(x) + \frac{\epsilon_n e^{-b}}{u_{\epsilon_n}(x)+\epsilon_n}\right) \to -u(x)\log u(x),$$

pointwisely as $n \to \infty$. On the other hand, if $u(x) = 0$, then $u_{\epsilon_n}(x) \to 0$ as $n \to \infty$, so that

$$|g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}| \leq g_1(u_{\epsilon_n}) \to 0 = -u(\log u)\chi_{\{u>0\}} \text{ pointwisely as } n \to \infty.$$

Inequality (5.27) then follows from the dominated convergence Theorem. We proceed to show that

$$\int_\Omega G_{\epsilon_n,f(0)}(u_{\epsilon_n}) \to \int_\Omega G(u)\,dx,$$

where

$$G(s) = -\int_0^s \log t\,dt.$$

Indeed,

$$G_{\epsilon_n,f(0)}(u_{\epsilon_n}(x)) = -\int_0^{u_{\epsilon_n}(x)} \log\left(s+\frac{\epsilon_n e^{-b}}{s+\epsilon_n}\right) ds = -\int_0^\infty \log\left(s+\frac{\epsilon_n e^{-b}}{s+\epsilon_n}\right)\chi_{\{s\le u_{\epsilon_n}(x)\}}\,ds.$$

Observe that

$$\left(\log\left(s+\frac{\epsilon_n e^{-b}}{s+\epsilon_n}\right)\chi_{\{s\le u_{\epsilon_n}(x)\}}\right) \to (\log s)\chi_{\{s\le u(x)\}} \text{ as } n\to\infty \text{ for } s\notin\{0,u(x)\}$$

and

$$\left|\log\left(s+\frac{\epsilon_n e^{-b}}{s+\epsilon_n}\right)\chi_{\{s\le u_{\epsilon_n}(x)\}}\right| \le |g_2(s)|\chi_{\{0\le s\le K_1\}} \text{ for all } n\in\mathbb{N},$$

where

$$g_2(s) = \max\{|\log(s+e^{-b})|, |\log s|\} \in L^1(0,K_1).$$

The Dominated Convergence Theorem then implies that

$$\lim_{n\to\infty} G_{\epsilon_n,f(0)}(u_{\epsilon_n}(x)) = -\int_0^{u(x)} \log s\,ds \text{ for all } x\in\Omega.$$

From item (iv) of Lemma 5.2, we know that there exists $C_1>0$ such that

$$|G_{\epsilon,f(0)}(s)| \le C_1 \text{ for all } 0\le s\le K_1, 0<\epsilon<\epsilon_0.$$

Applying the Dominated Convergence Theorem again, we get

$$\int_\Omega G_{\epsilon_n,f(0)}(u_{\epsilon_n})\,dx = \int_\Omega G(u)\,dx.$$

Moreover, it follows directly from the Dominated Convergence Theorem that

$$\int_\Omega \left(\frac{1}{2}f(u_{\epsilon_n})u_{\epsilon_n} - F(u_{\epsilon_n})\right) dx \to \int_\Omega \left(\frac{1}{2}f(u)u - F(u)\right) dx.$$

Letting $n\to\infty$ in (5.26), we get

$$\int_\Omega \left(G(u) - \frac{1}{2}(-\log u)u\chi_{\{u>0\}}\right) dx + \int_\Omega \left(\frac{1}{2}f(u)u - F(u)\right) dx > a_1, \tag{5.28}$$

so that u is nontrivial. We proceed to show that $(\log u)\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$. Let $V\subset\Omega$ be a open set such that $V\subset\overline{V}\subset\Omega$. Take $\zeta\in C^1_c(\Omega)$ such that $0\le\zeta\le 1$ and $\zeta\equiv 1$ in V. Since u_{ϵ_n} is a critical point of I_{ϵ_n}, we obtain

$$\int_{\{u_{\epsilon_n}<S_{f(0),\epsilon_n}\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta = \int_\Omega f(u_{\epsilon_n})\zeta - \int_\Omega \nabla u_{\epsilon_n}\nabla\zeta - \int_{\{u_{\epsilon_n}\ge S_{f(0),\epsilon_n}\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta.$$

(5.22), (5.23) and the fact that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω imply that

$$\int_\Omega f(u_{\epsilon_n})\zeta \to \int_\Omega f(u)\zeta,$$

$$\int_\Omega \nabla u_{\epsilon_n} \nabla\zeta \to \int_\Omega \nabla u \nabla \zeta \text{ as } n \to \infty,$$

and

$$\left| \int_{\{u_{\epsilon_n} \geq S_{f(0),\epsilon_n}\}} g_{\epsilon_n, f(0)}(u_{\epsilon_n})\zeta \right| \leq \int_{\Omega\cap\{1/2 \leq u_{\epsilon_n} \leq K_1\}} |g_{\epsilon_n,f(0)}(u_{\epsilon_n})| < C_1, \text{ for all } n \in \mathbb{N},$$

where $C_1 > 0$ is a positive constant that does not depend on ϵ. Hence, there exists $C_2 > 0$ such that

$$\left| \int_{\{u_{\epsilon_n} < S_{f(0),\epsilon_n}\}} g_{\epsilon_n,f(0)}(u_{\epsilon_n})\zeta \right| \leq C_2 \text{ for all } n \in \mathbb{N}. \tag{5.29}$$

We split the proof in two cases

Case 1: First, assume that $f(0) \geq 0$. Item (ii) of Lemma 5.2 implies that $g_{\epsilon_n,f(0)}(s) \geq 0$ for $0 \leq s \leq S_{f(0),\epsilon_n}$. From item (iii) of Lemma 5.2 we may also assume that $S_{f(0),\epsilon_n} > 1/2$, so that, from (5.29),

$$\int_{\{u_{\epsilon_n} < 1/4\}} |g_{\epsilon_n,f(0)}(u_{\epsilon_n})|\zeta \, dx \leq C_3 \text{ for all } n \in \mathbb{N}.$$

Consequently,

$$\int_\Omega |g_{\epsilon_n,f(0)}(u_{\epsilon_n})|\zeta \, dx \leq C_4 \text{ for all } n \in \mathbb{N},$$

where $C_4 > 0$ is a positive constant independent on n. Define the set $\Omega_\rho = \{x \in \Omega : u(x) \geq \rho\}$ for $\rho > 0$. Then

$$\int_{V\cap\Omega_\rho} \left| \log\left(u_{\epsilon_n} + \frac{\epsilon_n e^{-b}}{u_{\epsilon_n} + \epsilon_n} \right) \right| \zeta < C_4.$$

It follows from Fatou's Lemma and (5.29) that

$$\int_V |\log u| \chi_{\Omega_\rho} < C_4,$$

independently of ρ. Letting $\rho \to 0$ and applying Fatou's Lemma again, we conclude that

$$\int_V |\log u| \chi_{\Omega^+} < \infty,$$

for every open subset $V \subset \overline{V} \subset \Omega$.

Case 2: When $f(0) < 0$, we should recall (from item (i) of Lemma 5.2) that $f(0) - \log 2 \leq g_\epsilon(s) \leq 0$ for $0 \leq s \leq \min\{s_{f(0),\epsilon_n}, e^{-f(0)}\}$, so that

$$\int_{\Omega\cap\{0\leq u_{\epsilon_n} < s_{f(0),\epsilon_n}\}} |g_{\epsilon_n}(u_{\epsilon_n})|\zeta \, dx \leq (|f(0)| + \log 2)|\Omega| = C_5.$$

From (5.29), we obtain a constant C_6 such that

$$\int_{\Omega\cap\{s_{f(0),\epsilon_n} \leq u_{\epsilon_n} < S_{f(0),\epsilon_n}\}} |g_{\epsilon_n}(u_{\epsilon_n})|\zeta \, dx \leq C_6.$$

Consequently, we obtain $C_7 > 0$ such that

$$\begin{aligned}\int_\Omega |g_{\epsilon_n,f(0)}(u_{\epsilon_n})|\zeta\,dx \le& \int_{\Omega\cap\{0\le u_{\epsilon_n}<s_{f(0),\epsilon_n}\}} |g_{\epsilon_n,f(0)}(u_{\epsilon_n})|\zeta\,dx \\ &+ \int_{\Omega\cap\{s_{f(0),\epsilon_n}\le u_{\epsilon_n}<S_{f(0),\epsilon_n}\}} |g_{\epsilon_n,f(0)}(u_{\epsilon_n})|\zeta\,dx \\ &+ \int_{\Omega\cap\{S_{f(0),\epsilon_n}u_{\epsilon_n}\le K_1\}} |g_{\epsilon_n,f(0)}(u_{\epsilon_n})|\zeta\,dx < C_7.\end{aligned}$$

The result then follows as in Case 1. □

We now state the main result of this chapter.

Theorem 5.1. *Assume that f is a function that satisfies (5.4), (5.5) and (5.10) – (5.13). Then problem (5.1) has a nontrivial nonnegative solution.*

Observe that this theorem extends the result of the previous chapter. Indeed, we no longer assume that $f(0) = 0$ and we make no use of parameters. For example, we get

Corollary 5.1. *(i) The problem*

$$\begin{cases} -\Delta u = (\log u + \mu e^u)\chi_{\{u>0\}} & in\ \Omega \\ u \ge 0,\ \ u \not\equiv 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

is solvable for all $\mu > 0$

(ii) The problem

$$\begin{cases} -\Delta u = (\log u + \mu u^p)\chi_{\{u>0\}} & in\ \Omega \\ u \ge 0,\ \ u \not\equiv 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

is solvable for all $p > 1$ and $\mu > 0$.

(iii) The problem

$$\begin{cases} -\Delta u = (\log u + \mu u^p e^u)\chi_{\{u>0\}} & in\ \Omega \\ u \ge 0,\ \ u \not\equiv 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

is solvable for all $p > 1$ and $\mu > 0$.

Observe also that we allow f to change sign. Indeed,

Corollary 5.2. *(i) The problem*

$$\begin{cases} -\Delta u = (\log u + e^u - \mu)\chi_{\{u>0\}} & in\ \Omega \\ u \ge 0,\ \ u \not\equiv 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

is solvable for all $\mu > 0$.

(*ii*) *The problem*

$$\begin{cases} -\Delta u = (\log u + u^p - \mu)\chi_{\{u>0\}} \ \textit{in}\ \Omega \\ u \geq 0,\ \ u \not\equiv 0 \ \textit{in}\ \Omega \\ u = 0 \ \textit{on}\ \partial\Omega, \end{cases}$$

is solvable for all $\mu > 0$, $p > 1$

(*iii*) *The problem*

$$\begin{cases} -\Delta u = (\log u + u^p e^u - \mu)\chi_{\{u>0\}} \ \textit{in}\ \Omega \\ u \geq 0,\ \ u \not\equiv 0 \ \textit{in}\ \Omega \\ u = 0 \ \textit{on}\ \partial\Omega, \end{cases}$$

is solvable for all $\mu > 0$ *and* $p \geq 1$.

(*iv*) *The problem*

$$\begin{cases} -\Delta u = (\log u + \lambda u^p - \mu u^q)\chi_{\{u>0\}} \ \textit{in}\ \Omega \\ u \geq 0, u \not\equiv 0 \ \textit{in}\ \Omega \\ u = 0 \ \textit{on}\ \partial\Omega, \end{cases}$$

is solvable for all $\lambda > 0$, $0 < q < 1 < p$ *and* $\mu \geq 0$.

Proof of Theorem 5.1. The proof is essentially the same of Theorem 4.1, see page 79. The nontriviality of u is guaranteed by Lemma 5.6. Let u_ϵ be an arbitrary solution of problem (5.2) and let $\varphi \in C_c^1(\Omega)$. We have

$$\int_\Omega \nabla u_\epsilon \nabla \varphi = \int_\Omega (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi. \tag{5.30}$$

We again introduce the auxiliary function $\eta \in C^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, $\eta(s) = 0$ for $s \leq 1/2$, $\eta(s) = 1$ for $s \geq 1$. For $m > 0$ the function $\varrho := \varphi\eta(u_\epsilon/m)$ belongs to $C_c^1(\Omega)$, because u_ϵ is smooth, according to Lemma 5.1.

From continuity, the set $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ is open. Let $\tilde{\Omega}$ be an open set such that $\overline{support(\varphi)} \subset \tilde{\Omega}$ and $\overline{\tilde{\Omega}} \subset \Omega$. Let $\Omega_0 = \Omega_+ \cap \tilde{\Omega}$. Since $u_\epsilon \to u$ uniformly in $\tilde{\Omega}$, we know that for every $m > 0$ there is an $\epsilon_1 > 0$ such that

$$u_\epsilon(x) \leq m/2 \text{ for every } x \in \tilde{\Omega} \setminus \Omega_0 \text{ and } 0 < \epsilon \leq \epsilon_1. \tag{5.31}$$

Replacing φ by ϱ in (5.30) we obtain

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) = \int_{\tilde{\Omega}} (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi\eta(u_\epsilon/m).$$

We break the previous integral as

$$A_\epsilon := \int_{\Omega_0} (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi\eta(u_\epsilon/m)$$

and

$$B_\epsilon := \int_{\tilde{\Omega}\setminus\Omega_0} (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi\eta(u_\epsilon/m).$$

Clearly, $B_\epsilon = 0$, whenever $0 < \epsilon \leq \epsilon_1$ by (5.31) and the definition of η. Furthermore,

$$A_\epsilon \to \int_{\Omega_0} (\log u + f(u))\varphi\eta(u/m) \text{ as } \epsilon \to 0,$$

because $u_\epsilon \to u$ uniformly in Ω_0. Next, we take the limit in m to conclude that

$$\int_{\Omega_0} (\log u + f(u))\varphi\eta(u/m) \to \int_{\Omega_0} (\log u + f(u))\varphi \text{ as } m \to 0,$$

since $\eta(u/m) \leq 1$ and $(\log u)\chi_{\Omega^+} + f(u) \in L^1(\tilde{\Omega})$, according to Lemma 5.6. The result is then obtained by following the proof of Theorem 4.1. □

6 Critical Log–exp problems

In this chapter we study problems of the form

$$\begin{cases} -\Delta u = (\log u + f(u))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and f has critical growth in the following sense.

Definition 1. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. We say that f has **critical** growth if there exists $\alpha > 0$ such that*

$$\lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = \infty \text{ for all } 0 < \kappa < \alpha, \text{ and } \lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = 0 \text{ for all } \kappa > \alpha. \tag{6.2}$$

Examples of functions with critical growth are $f(s) = e^{s^2}$ and $f(s) = s^\tau e^{s^2}$ with $\tau \geq 1$.

Remark 6.1. *In Chapter 5 we studied problem (6.1) with functions satisfying: for all $\alpha > 0$ there exists $C_\alpha > 0$ such that*

$$|f(s)| \leq C_\alpha \exp(\alpha s^2) \text{ for all } s \geq 0, \tag{6.3}$$

see conditions (2.3), (4.5) and (5.5). Observe that f satisfies (6.3) for all $\alpha > 0$ if and only if

$$\lim_{s\to\infty} \frac{|f(s)|}{\exp(\alpha s^2)} = 0 \text{ for all } \alpha > 0. \tag{6.4}$$

*Functions satisfying (6.4) are called **subcritical**. Examples of functions of subcritical growth are $f(s) = e^s$, $f(s) = s^p$ and $f(s) = s^p e^s$ with $p > 0$.*

Remark 6.2. *If f is a function of critical growth, then problem (6.1) was studied in Chapter 5. The novelty in this chapter is that we consider problem (6.1) with f having critical growth. The main difficulty here is that the associated functional I_ϵ lacks compactness. To overcome this difficulty, we need to obtain sharper estimates in $H_0^1(\Omega)$ for the solutions of the approximated problem.*

We will again study a perturbed problem of the form

$$\begin{cases} -\Delta u + g_{\epsilon,f(0)}(u) = f(u) \,\text{in}\, \Omega \\ u \geq 0 \text{ in } \Omega \\ u = 0 \,\text{on}\, \partial\Omega. \end{cases} \tag{6.5}$$

Here, $g_{\epsilon,b} \in C^\infty(0,\infty)$ is defined by

$$g_{\epsilon,b}(s) = \begin{cases} -\log\left(s + \dfrac{\epsilon e^{-b}}{s+\epsilon}\right) \text{ for } s \geq 0 \\ b \text{ for } s < 0, \end{cases} \tag{6.6}$$

where $b = f(0)$, so that $g_{\epsilon,b}(0) = b = f(0)$ for all $\epsilon > 0$ and $g_{\epsilon,b}(s) \to -\log(s)$ pointwisely for $s > 0$ as $\epsilon \to 0$. We will further assume that

$$f(s) = f(0) \text{ for all } s \leq 0, f \in C^{1,\nu}(0,\infty) \cap C(\mathbb{R}) \text{ and } \sup_{0\leq s\leq 1} |sf'(s)| < \infty, \tag{6.7}$$

for some $0 < \nu < 1$. As in Chapter 5, we work with the functional $I_\epsilon : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega G_{\epsilon,f(0)}(u) - \int_\Omega F(u)\,dx, \tag{6.8}$$

where $F(u) = \int_0^u f(s)\,ds$ and $G_{\epsilon,b}(u) = \int_0^s g_{\epsilon,b}(s)\,ds$. From the fact that f is a function with critical growth satisfying (6.7) and $g_{\epsilon,b}$ is continuous satisfying (6.6), we conclude I_ϵ is of class C^1 and

$$I'_\epsilon(u)(v) = \int_\Omega \nabla u \nabla v + \int_\Omega g_{\epsilon,f(0)}(u)v - \int_\Omega f(u)v, \text{ for all } u,v \in H_0^1(\Omega). \tag{6.9}$$

Consequently, if $u_\epsilon \in H_0^1(\Omega)$ is a critical point of I_ϵ then

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_{\epsilon,f(0)}(u_\epsilon)v = \int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega). \tag{6.10}$$

Consequently, critical points u_ϵ of I_ϵ are weak solutions of problem (6.5). Arguing as in the proof of Lemma 5.1, we obtain

Lemma 6.1. *Suppose that f is a function of critical growth that satisfies* (6.7). *The following assertions hold:*

(i) Critical points of I_ϵ are nonnegative weak solutions of problem (6.5).

(ii) If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a nonnegative weak solution of problem (6.5), *then u is smooth and $u \in C^{1,\nu}(\overline{\Omega})$, with ν given by* (6.7).

In Section 6.1, we assume that f satisfies certain assumptions which are uniform for α. The goal is to obtain a nontrivial solution of problem (6.1) when f has critical growth for small values of α. Next, in Section 6.2, we study a parametrized version of problem (6.1) and remove the hypothesis that α is small. The drawback of not controlling the value of α is that a certain abstract admissibility condition for Ω appears. We are unable to give examples of sets Ω which are admissible. However, in Section 6.3, we give more concrete examples of the result given in Section 6.2.

6.1 Problems without parameters

In this section we study problem (6.1) when f satisfies the following assumptions, analogous to the ones stated in Chapter 5. We assume that:

- There exist $0 < \alpha < 1$ and constants $C_1, C_2 > 0$, $\zeta > 0$ that do not depend on α such that

$$|f(s)| \leq C_1 s^\zeta \exp\left(\alpha s^2\right) + C_2 \text{ and } |F(s)| \leq C_1 s^\zeta \exp\left(\alpha s^2\right) \text{ for all } s \geq 0; \tag{6.11}$$

$$\bullet\ f(0) \text{ is a constant that does not depend on } \alpha; \tag{6.12}$$

- There exist constants $0 < \epsilon_0, \delta < 1$ which do not depend on α such that

$$g_{\epsilon_0, f(0)}(s) \geq f(s) \text{ for all } s \leq \delta. \tag{6.13}$$

Again observe that the existence of ϵ_0 satisfying condition (6.13) implies that

$$g_{\epsilon, f(0)}(s) \geq f(s) \text{ for all } s \leq \delta, 0 < \epsilon \leq \epsilon_0.$$

- There exist constants $0 < \theta < 1/2$ and $s_0 > 0$ which do not depend on α such that

$$\min\{f(s), F(s)\} \geq 0 \text{ for } s \geq s_0, \tag{6.14}$$

and

$$(1-\theta)(f(s) + s) \leq \theta s f'(s) \text{ for } s \geq s_0. \tag{6.15}$$

- There exist constants $A > 0$ and $\gamma > 2$ which do not depend on α such that

$$F(s) \geq A|s|^\gamma \text{ for } s \geq s_0. \tag{6.16}$$

Condition (6.12) imply that the estimates for $g_{\epsilon, f(0)}$ given by Lemmas 5.2 and 5.3 do not depend on α. Conditions (6.11) and (6.13) will be used when showing that the origin is a local minimum of I_ϵ. Condition (6.16) will again imply that there exists an element $\phi_0 \in H_0^1(\Omega)$ such that $I_\epsilon(\phi_0) < 0$ for all $\epsilon > 0$. We will thus be able to apply the Mountain Pass Theorem, which will give a Palais-Smale sequence (u_n^ϵ) for I_ϵ. Conditions (6.14) and (6.15) will imply that (u_n^ϵ) converges in $H_0^1(\Omega)$ to a critical point u_ϵ, if the constant α given by (6.11) is sufficiently small, say $0 < \alpha < \alpha_0$, where $\alpha_0 > 0$ is an adequate constant to be chosen later. The fact that the constants defined in (6.11)-(6.16) do not depend on α yields estimates uniform in α, which will imply that α_0 is well defined.

The prototype of function f with critical growth and satisfying conditions (6.11) – (6.16) is $f(s) = s^\tau \exp(\alpha s^2)$ with $\tau > 1$. We again are interested in the quantity $j_\epsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$j_\epsilon(s) = f(s) - g_{\epsilon, f(0)}(s),$$

and $J_\epsilon(s) = \int_0^s j_\epsilon(t)\,dt$, so that

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u). \tag{6.17}$$

We may use Lemmas 5.2 and 5.3 to obtain the following version of Lemma 5.4.

Lemma 6.2. *Assume that (6.7), (6.11) and (6.12) hold. The following assertions are true.*

(i) If (6.13) holds, then $j_\epsilon(s) \leq 0$ for $s \leq \delta$ and $0 < \epsilon < \epsilon_0$.

(ii) Assume further that (6.14) and (6.15) hold. Then, there exists $R^* > 0$ and $\epsilon_0 > 0$ such that

$$0 \leq J_\epsilon(s) \leq \theta s j_\epsilon(s) \text{ for } s \geq R^* \text{ and } 0 < \epsilon < \epsilon_0.$$

We thus obtain a constant $C > 0$ such that

$$|J_\epsilon(s)| \leq C + \theta s j_\epsilon(s) \text{ for all } s \geq 0 \text{ and } 0 < \epsilon < \epsilon_0.$$

Furthermore, C does not depend on α (given by (6.11)).

Proof of Lemma 6.2 . The proof is entirely analogous to the one given in Lemma 5.4, see page 90. We point out that condition (6.12) implies that the constants defined in Lemmas 5.2 and 5.3 do not depend on α. □

As in Chapter 5, we obtain the following energy estimate.

Lemma 6.3. *Assume that f is a function that satisfies (6.7), (6.11), (6.12) and (6.16). Then, there exist $\phi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that*

$$I_\epsilon(\phi_0) < 0 \text{ for all } 0 < \epsilon < 1 \tag{6.18}$$

Furthermore, there exists $C > 0$ that does not depend on ϵ nor on α such that

$$\sup_{t\in[0,1]} I_\epsilon(t\phi_0) < C \text{ for all } 0 < \epsilon < 1.$$

Proof of Lemma 6.3. The proof of this result is entirely analogous to the proof of Lemma 5.5, see page 91. We point out that due to (6.12) and (6.16), the constant $C > 0$ does not depend on α (given by (6.11)). □

We now obtain critical points u_ϵ of the functional I_ϵ

Proposition 6.1. *Suppose that f is a function of critical growth that satisfies (6.7), (6.11) and (6.12) – (6.16). Then, there exists $\alpha_0 > 0$ such that I_ϵ has a critical point $u_\epsilon \geq 0$ satisfying*

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} \leq \frac{7\pi}{2\alpha} \text{ for all } 0 < \epsilon < \epsilon_0 \text{ and } 0 < \alpha < \alpha_0.$$

Furthermore, there exists constants $a_1, a_2 > 0$ such that

$$0 < a_1 < I_\epsilon(u_\epsilon) < a_2 \text{ for all } 0 < \epsilon < \epsilon_0 \text{ and } 0 < \alpha < \alpha_0. \tag{6.19}$$

Proof of Proposition 6.1. Let $0 < \epsilon_0, \delta < 1$ be given by (6.13). As a consequence of item (i) of Lemma 6.2 and item (v) of Lemma 5.2, we get

$$\begin{aligned} I_\epsilon(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\Omega\cap\{u\geq\delta\}} J_\epsilon(u) \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\Omega\cap\{u\geq\delta\}} G_{\epsilon,f(0)}(u) - \int_{\Omega\cap\{u\geq\delta\}} F(u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_{\Omega\cap\{u\geq\delta\}} u^{p_0} - \int_{\Omega\cap\{u\geq\delta\}} F(u), \text{ for all } u \in H_0^1(\Omega), 0<\epsilon<\epsilon_0. \end{aligned}$$

Choosing $p_0 = 3$, using (B.8), (6.11) and Hölder's inequality, we obtain from the Sobolev Embedding (and from the fact that $0 < \alpha < 1$) that

$$\begin{aligned} I_\epsilon(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_{\Omega\cap\{u\geq\delta\}} u^3 - C_2\int_{\Omega\cap\{u\geq\delta\}} u^\zeta \exp(\alpha u^2). \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_\Omega |u|^3 - \frac{C}{\delta^3}\int_\Omega |u|^{3+\zeta}\exp(u^2) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_\Omega |u|^3 - \frac{C}{\delta^3}\left(\int_\Omega |u|^{6+2\zeta}\right)^{\frac{1}{2}}\left(\int_\Omega \exp(2u^2)\right)^{\frac{1}{2}} \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_4\|u\|^3_{H_0^1(\Omega)} - C_5\|u\|^{3+\zeta}_{H_0^1(\Omega)} \text{ for } \|u\|^2_{H_0^1(\Omega)} < 2\pi. \end{aligned}$$

where C_i are positive constants that do not depend on α. Consequently, there exists $\rho > 0$ that does not depend on ϵ nor on α such that

$$I_\epsilon(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} \text{ for all } u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1(\Omega)} \leq \rho. \tag{6.20}$$

We know that I_ϵ is a functional of class C^1. Consequently, Lemma 6.3, (6.20) and the Mountain-Pass Theorem imply that there exists a Palais-Smale sequence (u_n^ϵ) for I_ϵ at level

$$c_\epsilon = \inf_{\Psi\in\Gamma}\max_{0\leq t\leq 1} I_\epsilon(\Psi(t)),$$

where

$$\Gamma = \{\Psi \in C([0,1], H_0^1(\Omega)) : \Psi(0) = 0 \text{ and } \Psi(1) = \phi_0\}.$$

From (6.20) we obtain

$$c_\epsilon \geq \frac{\rho^2}{4}, \text{ for all } 0 < \epsilon < \epsilon_0.$$

Furthermore, Lemma 6.3 imply that $c_\epsilon \leq C$ for some constant $C > 0$ that does not depend on α. Consequently,

$$|I_\epsilon(u_n^\epsilon)| < C,$$

for sufficiently large n. We thus get, from Lemma 6.2,

$$\begin{aligned} \frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} &\leq C + \int_\Omega J_\epsilon(u_n^\epsilon(x))\,dx \\ &\leq C + C_1|\Omega| + \theta\int_\Omega u_n^\epsilon j_\epsilon(u_n^\epsilon)\,dx, \end{aligned}$$

where $0 < \theta < 1/2$ is given by (6.15) and $C_1 > 0$ does not depend on α. Furthermore, $I'_\epsilon(u_n^\epsilon) \to 0$ as $n \to \infty$. Consequently, there exists a sequence (τ_n^ϵ) in $(0,1)$ such that $\tau_n^\epsilon \to 0$ and

$$\left| \int_\Omega \nabla u_n^\epsilon \nabla v \, dx - \int_\Omega j_\epsilon(u_n^\epsilon) v \, dx \right| \leq \tau_n^\epsilon \|v\|_{H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega) \text{ and } n \in \mathbb{N}. \tag{6.21}$$

Taking $v = u_n^\epsilon$ in (6.21), we get

$$-\tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)} \leq \|u_n^\epsilon\|^2_{H_0^1(\Omega)} - \int_\Omega j_\epsilon(u_n^\epsilon) u_n^\epsilon \, dx \leq \tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)} \text{ for all } n \in \mathbb{N}.$$

Consequently,

$$\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq C + C_1|\Omega| + \theta\tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)} + \theta \|u_n^\epsilon\|^2_{H_0^1(\Omega)}.$$

We thus get

$$\left(\frac{1}{2} - \theta\right) \|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq C + C_1|\Omega| + \theta\tau_n \|u_n^\epsilon\|_{H_0^1(\Omega)}.$$

Hence, (u_n^ϵ) must be uniformly bounded in $H_0^1(\Omega)$. Letting $n \to \infty$ we get

$$\left(\frac{1}{2} - \theta\right) \lim_{n\to\infty} \|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq C + C_1|\Omega|.$$

Consequently,

$$\lim_{n\to\infty} \|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \left(\frac{2}{1-2\theta}\right)(C + C_1|\Omega|).$$

If we choose $\alpha_0 > 0$ so small that

$$\left(\frac{2}{1-2\theta}\right)(C + C_1|\Omega|) < \frac{3\pi}{\alpha} \text{ for all } 0 < \alpha < \alpha_0,$$

we get

$$\lim_{n\to\infty} \|u_n^\epsilon\|^2_{H_0^1(\Omega)} < \frac{7\pi}{2\alpha} \text{ for all } 0 < \alpha < \alpha_0$$

Consequently, there exist constants $r_1, r_2 > 1$ independent of n such that

$$r_1 r_2 \alpha \|u_n^\epsilon\|^2_{H_0^1(\Omega)} < 4\pi \text{ for all } n \in \mathbb{N} \text{ and } 0 < \alpha < \alpha_0.$$

Furthermore, there must exist $u_\epsilon \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n^\epsilon \rightharpoonup u_\epsilon \text{ weakly in } H_0^1(\Omega), \\ u_n^\epsilon \to u_\epsilon \text{ in } L^p(\Omega) \text{ for every } p > 1, \\ u_n^\epsilon \to u_\epsilon \text{ a.e in } \Omega. \end{cases} \tag{6.22}$$

Observe that since f satisfies (6.11) for some $\alpha > 0$, then there exists a constant $C_{\epsilon,\alpha} > 0$ such that

$$\max\{|j_\epsilon(s)|, |J_\epsilon(s)|\} \leq C_{\epsilon,\alpha} \exp\left(r_1 \alpha s^2\right) \text{ for } s \in \mathbb{R}. \tag{6.23}$$

Consequently, from (B.8),

$$\int_\Omega |j_\epsilon(u_n^\epsilon)|^{r_2} \, dx \leq C_{\epsilon,\alpha} \int_\Omega \exp\left(\alpha r_1 r_2 (u_n^\epsilon)^2\right) \leq C_{\epsilon,\alpha} k_1 \text{ for all } n \in \mathbb{N}.$$

From (6.21) and Hölder's inequality, we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx &= \lim_{n\to\infty}\int_\Omega j_\epsilon(u_n^\epsilon)u_n^\epsilon\,dx\\ &= \lim_{n\to\infty}\int_\Omega j_\epsilon(u_n^\epsilon)u_\epsilon\,dx\\ &= \lim_{n\to\infty}\int_\Omega \nabla u_n^\epsilon\nabla u_\epsilon\,dx\\ &= \int_\Omega |\nabla u_\epsilon|^2\,dx.\end{aligned}$$

We conclude that $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$. Hence, $I_\epsilon'(u_\epsilon) = 0$ and

$$0 < \frac{\rho^2}{4} \le I_\epsilon(u_\epsilon) \le C.$$

The fact that $u_\epsilon \ge 0$ in Ω is a consequence of Lemma 6.1. The result then follows by taking $a_1 = \dfrac{\rho^2}{4}$ and $a_2 = C$. □

We now replicate ideas given in Chapter 5 to study the convergence of the solutions u_ϵ of problem (6.5) obtained in Proposition 6.1. This proposition guarantees that there exists a constant $\alpha_0 > 0$ and $\epsilon_0 > 0$ such that

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} < \frac{7\pi}{2\alpha}, \text{ for each } 0<\epsilon<\epsilon_0, 0<\alpha<\alpha_0$$

Hence, there exist $u \in H_0^1(\Omega)$ and a sequence (ϵ_n) in $(0,\epsilon_0)$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} u_{\epsilon_n} \rightharpoonup u \text{ weakly in } H_0^1(\Omega),\\ u_{\epsilon_n} \to u \text{ in } L^r(\Omega) \text{ for every } r>1,\\ u_{\epsilon_n} \to u \text{ a.e in } \Omega,\\ |u_{\epsilon_n}| \le h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega).\end{cases} \tag{6.24}$$

As in Chapter 5, we will apply regularity results discussed in Appendix A to conclude that u_{ϵ_n} is smooth for all $n \in \mathbb{N}$ and that u is continuous. Indeed, if (6.7) holds, then Corollary A.1, implies that there exists a constant $K_1 > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(\Omega)} < K_1 \text{ for all } 0<\epsilon_n<\epsilon_0. \tag{6.25}$$

Then, it follows from elliptic regularity theory and from the Sobolev Embedding that $u_{\epsilon_n} \in C^1(\overline{\Omega})$, see Lemma 6.1. Lemma A.5 implies that there exists a constant $\epsilon_0 > 0$ such that for each smooth subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there exists a constant $M > 0$ that depends on Ω' but not on ϵ such that

$$|\nabla u_{\epsilon_n}(x)|^2 \le MZ(u_{\epsilon_n}(x)) \text{ for every } x \in \Omega', \quad 0<\epsilon<\epsilon_0,$$

where

$$Z(t) = \begin{cases} t^2 + t - t\log t \quad \text{for } 0 \le t \le \dfrac{1}{2}\\ \dfrac14 + \dfrac12(1+\log 2) + \left(t - \dfrac12\right)(1+\log 2) \quad \text{for } t \ge \dfrac12.\end{cases}$$

Hence, it follows from the Arzela-Ascoli Theorem that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, so that u is continuous and $0 \le u \le K_1$. We may now mimic the approach given in Chapter 5 to prove that u is a solution of (6.1) in the sense that

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega\cap\{u>0\}} \Big(\log u + f(u)\Big)\varphi,$$

for every $\varphi \in C_c^1(\Omega)$ and

$$(\log u)\chi_{\{u>0\}} \in L^1_{loc}(\Omega).$$

Indeed, we have

Lemma 6.4. *The function u is nontrivial and the function $(\log u)\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.*

Proof of Lemma 6.4. The proof of this result is entirely analogous to the one given in Chapter 5, Lemma 5.6, see page 96. □

We conclude that

Theorem 6.1. *Assume that f is a function that satisfies (6.7), (6.11) and (6.12) – (6.16). Then, there exists $\alpha_0 > 0$ such that problem (6.1) has a nontrivial nonnegative solution provided $0 < \alpha < \alpha_0$.*

Proof of Theorem 6.1. The proof of this result is entirely analogous to the ones given in Chapters 4 and 5, Theorems 4.1 and 5.1, see pages 79 and 100 respectively. □

As an immediate consequence, we get

Corollary 6.1. *Let $f(s) = s^\tau \exp(\alpha s^2)$ with $\tau > 1$. There exists $\alpha_0 > 0$ such that problem (6.1) has a nontrivial nonnegative solution provided $0 < \alpha < \alpha_0$.*

Corollary 6.2. *For each $\mu \in \mathbb{R}$ and $\tau > 1$, there exists $\alpha_0 > 0$ such that the problem*

$$\begin{cases} -\Delta u = (\log u + u^\tau \exp(\alpha u^2) + \mu)\chi_{\{u>0\}} \ \textit{in } \Omega \\ u \ge 0, u \not\equiv 0 \ \textit{in } \Omega \\ u = 0 \ \textit{on } \partial\Omega, \end{cases}$$

has a nontrivial nonnegative solution provided $0 < \alpha < \alpha_0$.

6.2 Problems with parameters

In this section, we study the problem

$$\begin{cases} -\Delta u = (\log u + \lambda f(u))\chi_{\{u>0\}} \text{ in } \Omega \\ u \ge 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.26}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and f has critical growth. Our goal is to study whether problem (6.26) can be solvable independently of the value of α. In Section 6.1, we showed that problem (6.26) has a nontrivial solution provided that the constant $0 < \alpha < 1$ given by (6.11) is small. Here, we want to drop this condition. In doing so, an admissibility condition appears. We consider the perturbed problem

$$\begin{cases} -\Delta u + g_{\epsilon,\lambda f(0)}(u) = \lambda f(u) \text{ in } \Omega \\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.27}$$

where $g_{\epsilon,b}$ is given by (6.6). In this context, we consider the functional $I_{\epsilon,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_{\epsilon,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_{\epsilon,\lambda f(0)}(u) - \lambda \int_\Omega F(u) \text{ for } u \in H_0^1(\Omega). \tag{6.28}$$

We assume that f has critical growth, so that there exists $\alpha > 0$ and $C > 0$ such that

$$\max\{|f(s)|, |F(s)|\} \leq C \exp(\alpha s^2) \text{ for all } s \geq 0. \tag{6.29}$$

Moreover, we assume that there exists $\delta > 0$ such that

$$f(s) > 0 \text{ for all } 0 < s < \delta. \tag{6.30}$$

We will use condition (6.30) to obtain an element $\phi_0 \in H_0^1(\Omega)$ with $I_\epsilon(\phi_0) < 0$ for all $\epsilon > 0$. Assumptions (6.7) and (6.29) imply that the functional $I_{\epsilon,\lambda}$ is of class C^1 and

$$I'_{\epsilon,\lambda}(u)(v) = \int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_{\epsilon,\lambda f(0)}(u)v - \lambda \int_\Omega f(u)v \text{ for all } u, v \in H_0^1(\Omega), 0 < \epsilon < 1. \tag{6.31}$$

Furthermore, critical points of $I_{\epsilon,\lambda}$ are nonnegative weak solutions of problem (6.27), according to Lemma 6.1.

Let $\phi_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi_1 > 0$ in Ω be the first eigenfunction of $-\Delta$ satisfying $\|\phi_1\|_{H_0^1(\Omega)} = 1$. We get

Lemma 6.5. *Let $\alpha, \lambda > 0$ and $0 < \theta < 1/2$. Assume that (6.7), (6.29) and (6.30) hold. There exists $t_0 > 0$ that does not depend on λ and $\epsilon_0 > 0$ such that*

$$\frac{t^2}{2} + \int_\Omega G_{\epsilon,\lambda f(0)}(t\phi_1)\, dx < \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha}, \text{ for all } 0 \leq t \leq t_0,\ 0 < \epsilon < \epsilon_0. \tag{6.32}$$

Proof of Lemma 6.5. Continuity and (6.30) implies that $f(0) \geq 0$. Consequently, from items (*ii*) and (*iii*) of Lemma 5.2, we get $g_{\epsilon,\lambda f(0)}(s) \geq 0$ for $0 \leq s \leq \frac{1}{2}$, provided ϵ is sufficiently small. Hence, $s \to G_{\epsilon,\lambda f(0)}(s)$ is nondecreasing for $0 \leq s \leq 1/2$. Furthermore,

$$g_{\epsilon,\lambda f(0)}(s) = -\log\left(s + \frac{\epsilon e^{-\lambda f(0)}}{s+\epsilon}\right) \leq -\log s \text{ for } 0 \leq s \leq 1/2.$$

Consequently,

$$0 \le G_{\epsilon,\lambda f(0)}(s) \le \int_0^s -\log t\, dt = -(t\log t - t)|_{t=0}^s = s - s\log s \text{ for } 0 \le s \le 1/2.$$

Since $\lim_{s\to 0}(s - s\log s) = 0$, we may choose $t_1 > 0$ such that

$$s - s\log s < \left(\frac{1-2\theta}{2}\right)\frac{\pi}{2\alpha|\Omega|} \text{ for all } 0 \le s \le t_1.$$

We choose $t_0 > 0$ such that

$$\frac{t_0^2}{2} < \left(\frac{1-2\theta}{2}\right)\frac{\pi}{2\alpha} \text{ and } t_0\phi_1(x) < t_1 \text{ for all } x \in \Omega.$$

We conclude that

$$\begin{aligned} G_{\epsilon,\lambda f(0)}(t\phi_1) &\le t\phi_1 - t\phi_1\log(t\phi_1) \le t_0\phi_1 - t_0\phi_1\log(t_0\phi_1) \\ &< \left(\frac{1-2\theta}{2}\right)\frac{\pi}{2\alpha|\Omega|} \text{ for all } 0 \le t \le t_0. \end{aligned}$$

Consequently,

$$\frac{t^2}{2} + \int_\Omega G_{\epsilon,\lambda f(0)}(t\phi_1)\,dx < \frac{t_0^2}{2} + \left(\frac{1-2\theta}{2}\right)\int_\Omega \frac{\pi}{2\alpha|\Omega|}\,dx = \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha} \text{ for all } 0 \le t \le t_0.$$

This proves (6.32). □

Let $I_{\epsilon,\lambda}$ be given by (6.28). From Lemma 6.5, we get

$$\begin{aligned} I_{\epsilon,\lambda}(t\phi_1) &= \frac{t^2}{2} + \int_\Omega G_{\epsilon,\lambda f(0)}(t\phi_1)\,dx - \lambda\int_\Omega F(t\phi_1)\,dx \\ &\le \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha} - \lambda\int_\Omega F(t\phi_1)\,dx \text{ for all } 0 \le t \le t_0. \end{aligned} \tag{6.33}$$

Hence,

$$I_{\epsilon,\lambda}(t_0\phi_1) < 0,$$

provided $\lambda \ge \lambda_0$, where

$$\lambda_0 = \left(\frac{1-2\theta}{2}\right)\left(\frac{\pi}{\alpha\int_\Omega F(t_0\phi_1)}\right). \tag{6.34}$$

Observe that λ_0 depends on Ω. We now obtain

Lemma 6.6. *Let λ_0 be given by (6.34) and fix $\lambda > \lambda_0$. There exists $0 < t_\lambda < t_0$ such that*

$$I_{\epsilon,\lambda}(t_\lambda\phi_1) < 0 \textit{ and } |I_{\epsilon,\lambda}(t\phi_1)| \le \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} \textit{ for all } 0 \le t \le t_\lambda. \tag{6.35}$$

Proof of Lemma 6.6. Indeed, we choose $0 < t_\lambda < t_0$ such that

$$\left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha} < \lambda \int_\Omega F(t_\lambda \phi_1)\,dx < \left(\frac{1-2\theta}{2}\right)\frac{2\pi}{\alpha}.$$

This choice of t_λ is possible because F is continuous with $F(0) = 0$ and

$$\lambda_0 \int_\Omega F(t_0\phi_1) = \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha}.$$

The choice of t_λ and (6.33) imply that

$$I_{\epsilon,\lambda}(t_\lambda\phi_1) \leq \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha} - \lambda\int_\Omega F(t_\lambda\phi_1)\,dx < 0$$

and

$$\begin{aligned}
|I_{\epsilon,\lambda}(t\phi_1)| &\leq \frac{t^2}{2} + \int_\Omega G_{\epsilon,\lambda f(0)}(t\phi_1)\,dx + \lambda\int_\Omega F(t\phi_1)\,dx \\
&\leq \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha} + \lambda\int_\Omega F(t\phi_1)\,dx \\
&\leq \left(\frac{1-2\theta}{2}\right)\frac{\pi}{\alpha} + \lambda\int_\Omega F(t_\lambda\phi_1)\,dx \\
&\leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} \text{ for all } 0 \leq t \leq t_\lambda < t_0.
\end{aligned}$$

This proves (6.35). □

We further assume that

- There exist constants $0 < \epsilon_0, \delta_\lambda < 1$ (which may depend on α and λ) such that

$$g_{\epsilon_0,\lambda f(0)}(s) \geq \lambda f(s) \text{ for all } s \leq \delta_\lambda. \tag{6.36}$$

- There exist constants $0 < \theta < 1/2$ and $s_\lambda > 0$ such that

$$\min\{f(s), F(s)\} \geq 0 \text{ for } s \geq s_\lambda. \tag{6.37}$$

and

$$(1-\theta)(\lambda f(s) + s) \leq \theta\lambda s f'(s) \text{ for } s \geq s_\lambda, \tag{6.38}$$

with θ independent of λ. If f has critical growth and satisfies (6.29) and (6.36) – (6.38), then the argument used in the proof of Lemma 5.4 yields a constant $C = C_{\lambda,\alpha} > 0$ such that

$$|J_{\epsilon,\lambda}(s)| \leq C_{\lambda,\alpha} + \theta s j_{\epsilon,\lambda}(s) \text{ for all } s \in \mathbb{R}, \tag{6.39}$$

where $j_{\epsilon,\lambda}(s) = \lambda f(s) - g_{\epsilon,\lambda f(0)}(s)$, $J_{\epsilon,\lambda}(s) = \int_0^s j_{\epsilon,\lambda}(t)\,dt$ and $0 < \theta < 1/2$ is given by (6.38). We now prove

Proposition 6.2. *Let $\lambda_0 > 0$ be given by (6.34). Assume that f is a function with critical growth that satisfies (6.7), (6.29), (6.30) and (6.36) − (6.38). Then, there exist $u_\epsilon \in H_0^1(\Omega)$ such that $I'_{\epsilon,\lambda}(u_\epsilon) = 0$ and constants $a_1, a_2, c > 0$ such that*

$$0 < a_1 \leq I_{\epsilon,\lambda}(u_\epsilon) < a_2,$$

provided $\lambda > \lambda_0$ and $|\Omega| < c$. The constants a_1, a_2 and c do not depend on ϵ but depend on λ and α (given by (6.29)). Furthermore,

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} \leq \frac{7\pi}{2\alpha} \text{ for all } 0 < \epsilon < \epsilon_0. \tag{6.40}$$

Proof of Proposition 6.2 . Recall that $\delta_\lambda > 0$ is given by (6.36). As a consequence of Lemma 5.2, we get

$$\begin{aligned} I_{\epsilon,\lambda}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_{\epsilon,\lambda}(u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\Omega\cap\{u\geq\delta_\lambda\}} J_{\epsilon,\lambda}(u) \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\Omega\cap\{u\geq\delta_\lambda\}} G_{\epsilon,\lambda f(0)}(u) - \lambda\int_{\Omega\cap\{u\geq\delta_\lambda\}} F(u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_{\Omega\cap\{u\geq\delta_\lambda\}} u^{p_0} - \lambda\int_{\Omega\cap\{u\geq\delta_\lambda\}} F(u). \end{aligned}$$

Choosing $p_0 = 3$ in Lemma 5.2 and using (6.29), we obtain from the Sobolev Embedding

$$\begin{aligned} I_{\epsilon,\lambda}(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_{\Omega\cap\{u\geq\delta_\lambda\}} u^3 - \lambda C\int_{\Omega\cap\{u\geq\delta_\lambda\}} \exp(\alpha u^2) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_\Omega |u|^3 - \frac{\lambda C}{\delta_\lambda^3}\int_\Omega |u|^3\exp(\alpha u^2) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - k_0\int_\Omega |u|^3 - \frac{\lambda C}{\delta_\lambda^3}\left(\int_\Omega u^6\right)^{\frac{1}{2}}\left(\int_\Omega \exp(2\alpha u^2)\right)^{\frac{1}{2}} \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_1\|u\|^3_{H_0^1(\Omega)} - C_2\|u\|^3_{H_0^1(\Omega)} \text{ for } \|u\|^2_{H_0^1(\Omega)} < \frac{2\pi}{\alpha}, \end{aligned}$$

where the constants C_i do not depend on ϵ. Consequently, there exists $\rho > 0$ that does not depend on ϵ such that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} \text{ for all } u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1(\Omega)} \leq \rho. \tag{6.41}$$

We know that I_ϵ is a functional of class C^1. Consequently, (6.35), (6.41) and the Mountain-Pass Theorem imply that there exists a Palais-Smale sequence (u_n^ϵ) for $I_{\epsilon,\lambda}$ at level

$$c_{\epsilon,\lambda} = \inf_{\Psi\in\Gamma}\max_{0\leq t\leq 1} I_{\epsilon,\lambda}(\Psi(t)),$$

where

$$\Gamma = \{\Psi \in C([0,1], H_0^1(\Omega)) : \Psi(0) = 0 \text{ and } \Psi(1) = t_\lambda\phi_1\}.$$

From (6.41) we obtain

$$c_{\epsilon,\lambda} \geq \frac{\rho^2}{4}, \text{ for all } 0 < \epsilon < \epsilon_0.$$

Furthermore, (6.35) implies that $c_{\epsilon,\lambda} \leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha}$. Consequently,

$$|I_{\epsilon,\lambda}(u_n^\epsilon)| < \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} \text{ for all } n \in \mathbb{N}.$$

Inequality (6.39) implies that

$$\begin{aligned}\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} &\leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} + \int_\Omega J_{\epsilon,\lambda}(u_n^\epsilon(x))\,dx \\ &\leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} + C_{\lambda,\alpha}|\Omega| + \theta\int_\Omega u_n^\epsilon j_{\epsilon,\lambda}(u_n^\epsilon)\,dx\end{aligned}$$

Furthermore, $I'_{\epsilon,\lambda}(u_n^\epsilon) \to 0$ as $n \to \infty$. Consequently, there exists a sequence (τ_n^ϵ) in $(0,1)$ such that $\tau_n^\epsilon \to 0$ and

$$\left|\int_\Omega \nabla u_n^\epsilon \nabla v\,dx - \int_\Omega j_{\epsilon,\lambda}(u_n^\epsilon)v\,dx\right| \leq \tau_n^\epsilon\|v\|_{H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega) \text{ and } n \in \mathbb{N}. \tag{6.42}$$

Taking $v = u_n^\epsilon$ in (6.42), we get

$$-\tau_n\|u_n^\epsilon\|_{H_0^1(\Omega)} \leq \|u_n^\epsilon\|^2_{H_0^1(\Omega)} - \int_\Omega j_{\epsilon,\lambda}(u_n^\epsilon)u_n^\epsilon\,dx \leq \tau_n\|u_n^\epsilon\|_{H_0^1(\Omega)} \text{ for all } n \in \mathbb{N}.$$

Consequently,

$$\frac{1}{2}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} + C_{\lambda,\alpha}|\Omega| + \theta\tau_n\|u_n^\epsilon\|_{H_0^1(\Omega)} + \theta\|u_n^\epsilon\|^2_{H_0^1(\Omega)}.$$

We thus get

$$\left(\frac{1}{2}-\theta\right)\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} + C_{\lambda,\alpha}|\Omega| + \theta\tau_n\|u_n^\epsilon\|_{H_0^1(\Omega)}.$$

Consequently, (u_n^ϵ) must be uniformly bounded in $H_0^1(\Omega)$. Letting $n \to \infty$ we get

$$\left(\frac{1}{2}-\theta\right)\lim_{n\to\infty}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \left(\frac{1-2\theta}{2}\right)\frac{3\pi}{\alpha} + C_{\lambda,\alpha}|\Omega|.$$

Consequently,

$$\lim_{n\to\infty}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \frac{3\pi}{\alpha} + \left(\frac{2}{1-2\theta}\right)C_{\lambda,\alpha}|\Omega|.$$

If we choose $|\Omega|$ so small that

$$\left(\frac{2}{1-2\theta}\right)C_{\lambda,\alpha}|\Omega| < \frac{\pi}{2\alpha}, \tag{6.43}$$

we get

$$\lim_{n\to\infty}\|u_n^\epsilon\|^2_{H_0^1(\Omega)} \leq \frac{7\pi}{2\alpha}.$$

Consequently, there exists $r > 1$ independent of n such that

$$r\alpha \|u_n^\epsilon\|^2_{H_0^1(\Omega)} < 4\pi \text{ for all } n \in \mathbb{N}.$$

Furthermore, there must exist $u_\epsilon \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n^\epsilon \rightharpoonup u_\epsilon \text{ weakly in } H_0^1(\Omega), \\ u_n^\epsilon \to u_\epsilon \text{ in } L^p(\Omega) \text{ for every } p \geq 1, \\ u_n^\epsilon \to u_\epsilon \text{ a.e in } \Omega. \end{cases} \tag{6.44}$$

We recall that if f satisfies (6.29) for some $\alpha > 0$, then there exists $C_{\epsilon,\alpha,\lambda} > 0$ such that

$$\max\{|j_{\epsilon,\lambda}(s)|, |J_{\epsilon,\lambda}(s)|\} \leq C_{\epsilon,\alpha,\lambda} \exp\left(\alpha s^2\right) \text{ for } s \in \mathbb{R}. \tag{6.45}$$

Consequently, from (B.8),

$$\int_\Omega |j_{\epsilon,\lambda}(u_n^\epsilon)|^r \, dx \leq C_{\epsilon,\alpha,\lambda} \int_\Omega \exp\left(\alpha r (u_n^\epsilon)^2\right) \leq C_{\epsilon,\alpha,\lambda} k_1 \text{ for all } n \in \mathbb{N}.$$

From (6.42), (6.44) and Hölder's inequality, we get

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 \, dx &= \lim_{n\to\infty} \int_\Omega j_{\epsilon,\lambda}(u_n^\epsilon) u_n^\epsilon \, dx \\ &= \lim_{n\to\infty} \int_\Omega j_{\epsilon,\lambda}(u_n^\epsilon) u_\epsilon \, dx \\ &= \lim_{n\to\infty} \int_\Omega \nabla u_n^\epsilon \nabla u_\epsilon \, dx \\ &= \int_\Omega |\nabla u_\epsilon|^2 \, dx. \end{aligned}$$

We conclude that $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$. Consequently, $I'_{\epsilon,\lambda}(u_\epsilon) = 0$ and

$$0 < \frac{\rho^2}{4} \leq I_{\epsilon,\lambda}(u_\epsilon) \leq \left(\frac{1-2\theta}{2}\right) \frac{3\pi}{\alpha}.$$

This proves the result. □

We again replicate ideas given in Section 6.1 and in Chapter 5 to study the convergence of the solutions u_ϵ of problem (6.27) obtained in Proposition 6.2. This proposition guarantees that there exists a constant $\alpha_0 > 0$ and $\epsilon_0 > 0$ such that

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} < \frac{7\pi}{2\alpha}, \text{ for each } 0 < \epsilon < \epsilon_0, 0 < \alpha < \alpha_0,$$

provided Ω satisfies the admissibility condition

$$|\Omega| < c.$$

Hence, there exist $u \in H_0^1(\Omega)$ and a sequence (ϵ_n) in $(0, \epsilon_0)$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} u_{\epsilon_n} \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_{\epsilon_n} \to u \text{ in } L^r(\Omega) \text{ for every } r > 1, \\ u_{\epsilon_n} \to u \text{ a.e in } \Omega, \\ |u_{\epsilon_n}| \leq h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega). \end{cases} \tag{6.46}$$

As in Chapter 5, we will apply regularity results discussed in Appendix A to conclude that u_{ϵ_n} is smooth for all $n \in \mathbb{N}$ and that u is continuous. Indeed, if (6.40) holds, then Lemma A.4, implies that there exists a constant $K_1 > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(\Omega)} < K_1 \text{ for all } 0 < \epsilon_n < \epsilon_0. \tag{6.47}$$

Then, it follows from elliptic regularity theory and from the Sobolev Embedding that $u_{\epsilon_n} \in C^1(\overline{\Omega})$, see Lemma 6.1. Lemma A.5 implies that there exists a constant $\epsilon_0 > 0$ such that for each smooth subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there exists a constant $M > 0$ that depends on Ω' but not on ϵ such that

$$|\nabla u_{\epsilon_n}(x)|^2 \leq MZ(u_{\epsilon_n}(x)) \text{ for every } x \in \Omega', \quad 0 < \epsilon < \epsilon_0, \tag{6.48}$$

where

$$Z(t) = \begin{cases} t^2 + t - t\log t & \text{for } 0 \leq t \leq \dfrac{1}{2} \\ \dfrac{1}{4} + \dfrac{1}{2}(1 + \log 2) + \left(t - \dfrac{1}{2}\right)(1 + \log 2) & \text{for } t \geq \dfrac{1}{2}. \end{cases}$$

Hence, it follows from the Arzela-Ascoli Theorem that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω, so that u is continuous and $0 \leq u \leq K_1$. We may now mimic the approach given in Chapter 5 to prove that u is a solution of (6.26) in the sense that

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} \Big(\log u + f(u)\Big)\varphi, \tag{6.49}$$

for every $\varphi \in C_c^1(\Omega)$ and

$$(\log u)\chi_{\{u>0\}} \in L^1_{loc}(\Omega).$$

Indeed, we have

Lemma 6.7. *The function u is nontrivial and the function $(\log u)\chi_{\Omega_+}$ belongs to $L^1_{loc}(\Omega)$, where $\Omega_+ = \{x \in \Omega : u(x) > 0\}$.*

Proof of Lemma 6.7. The proof of this result is entirely analogous to the one given in Chapter 5, Lemma 5.6, see page 96. □

We conclude that

Theorem 6.2. *Assume that f is a function with critical growth that satisfies (6.7), (6.29),(6.30) and (6.36) − (6.38). There exists $\lambda_0 > 0$ such that problem (6.26) has a nontrivial nonnegative solution provided $\lambda > \lambda_0$ and $|\Omega| < c$, where $c = c_{\alpha,\theta,\lambda} > 0$ is a constant depending on α, λ and θ.*

Proof of Theorem 6.2. The proof of this result is entirely analogous to the ones given in Chapters 4 and 5, Theorems 4.1 and 5.1, see pages 79 and 100 respectively. □

6.3 The admissibility condition for specific problems

In this section we look more closely at the admissibility condition $|\Omega| < c$ given by Theorem 6.2. We do not give explicit examples of sets Ω satisfying this condition, but in some cases we can obtain nice values for c. First we consider $f(s) = s\exp(\alpha s^2)$ and next we take $f(s) = \exp(\alpha s^2)$.

Theorem 6.3. *Let $\alpha \geq 3/4$ and assume that $f(s) = s\exp(\alpha s^2)$. There exists $\lambda_0 > 1$ such that problem (6.26) has a nontrivial nonnegative solution for $\lambda > \lambda_0$, and $|\Omega| < c_{\lambda,\alpha}$, where*

$$c_{\lambda,\alpha} = \frac{\pi}{8\alpha}\left(\frac{1}{\left(\frac{\lambda}{2\alpha}e^{4/\alpha} + \frac{1}{4e}\right)}\right) = \frac{\pi}{2}\left(\frac{e}{2\lambda e^{4/\alpha} + \alpha}\right).$$

Proof of Theorem 6.3 . Let $f(s) = s\exp(\alpha s^2)$, so that

$$F(s) = \frac{1}{2\alpha}\left(\exp(\alpha s^2) - 1\right) \text{ and } f'(s) = \exp(\alpha s^2) + 2\alpha s^2\exp(\alpha s^2).$$

Since $f(0) = 0$, we consider the perturbation

$$g_\epsilon(s) = \begin{cases} -\log\left(s + \dfrac{\epsilon}{s+\epsilon}\right) \text{ for } s \geq 0 \\ 0 \text{ for } s < 0, \end{cases}$$

and $G_\epsilon(s) = \int_0^s g_\epsilon(t)\,dt$. We will mimic the proof of Lemma 5.4 and see where this leads us. Indeed, let $B_\epsilon(s) = \lambda F(s) - G_\epsilon(s) - \frac{1}{4}\lambda s f(s) + \frac{1}{4}s g_\epsilon(s)$. We have

$$B_\epsilon'(s) = \frac{3}{4}(\lambda f(s) - g_\epsilon(s)) - \frac{1}{4}\lambda s f'(s) + \frac{1}{4}s g_\epsilon'(s).$$

Therefore, from Lemma 5.3, we conclude that

$$B_\epsilon'(s) \leq \frac{3}{4}(\lambda f(s) + s) - \frac{1}{4}\lambda s f'(s) - \frac{1}{24} \text{ for } s \geq 2.$$

Therefore, $B_\epsilon'(s) \leq -1/24$ for $s \geq 2$ provided

$$3(\lambda f(s) + s) - \lambda s f'(s) \leq 0 \text{ for } s \geq 2. \tag{6.50}$$

Observe that

$$3(\lambda f(s) + s) \leq \lambda s f'(s) \text{ if and only if } 3\lambda s\exp(\alpha s^2) + 3s \leq \lambda(s\exp(\alpha s^2) + 2\alpha s^3\exp(\alpha s^2))$$

Equivalently,

$$2\lambda\exp(\alpha s^2) + 3 \leq 2\alpha\lambda s^2\exp(\alpha s^2).$$

This inequality holds provided

$$2\lambda\exp(\alpha s^2) \leq \alpha\lambda s^2\exp(\alpha s^2) \text{ and } 3 \leq \alpha\lambda s^2\exp(\alpha s^2).$$

Assuming that $\lambda > 1$ it holds for $s \geq s_0$ with

$$\alpha s_0^2 = 3, \tag{6.51}$$

so that $s_0 = \sqrt{3/\alpha}$. Assuming that $\alpha > 3/4$, we conclude that (6.50) holds, and consequently,

$$B_\epsilon'(s) \leq -1/24 \text{ for } s \geq 2. \tag{6.52}$$

Furthermore,

$$B_\epsilon(2) = \lambda F(2) - G_\epsilon(2) - \frac{1}{2}\lambda f(2) + \frac{1}{2}g_\epsilon(2).$$

Observe that

$$\begin{aligned} -G_\epsilon(2) &= \int_0^2 \log\left(t + \frac{\epsilon}{at+\epsilon}\right) dt \\ &\leq \int_0^2 \log(t+1)\, dt \\ &= \int_1^3 \log t\, dt \\ &= 3\log 3 - 3 \leq 0, \end{aligned} \tag{6.53}$$

$$g_\epsilon(2) = -\log\left(2 + \frac{\epsilon}{2a+\epsilon}\right) \leq -\log 2 \leq 0, \tag{6.54}$$

$$F(2) = \frac{1}{2\alpha}\left(e^{4\alpha} - 1\right) \tag{6.55}$$

and

$$f(2) = 2e^{4\alpha}. \tag{6.56}$$

From (6.53), (6.54), (6.55) and (6.56), we conclude that

$$B_\epsilon(2) \leq \lambda\left(\frac{1}{2\alpha}\left(e^{4\alpha}-1\right) - e^{4\alpha}\right) \leq 0,$$

since $\alpha \geq 3/4$. From (6.52), we conclude that

$$B_\epsilon(s) \leq 0 \text{ for } s \geq 2.$$

Consequently,

$$J_{\epsilon,\lambda}(s) \leq \frac{1}{4}s j_{\epsilon,\lambda}(s) \text{ for } s \geq 2, \tag{6.57}$$

where $j_{\epsilon,\lambda}(s) = \lambda f(s) - g_\epsilon(s)$. Furthermore,

$$\begin{aligned} J_{\epsilon,\lambda}(s) &= \lambda F(s) - G_\epsilon(s) - \frac{1}{4}s j_{\epsilon,\lambda}(s) + \frac{1}{4}s j_{\epsilon,\lambda}(s) \\ &= \lambda\left(F(s) - \frac{1}{4}s f(s)\right) - G_\epsilon(s) + \frac{1}{4}s g_\epsilon(s) + \frac{1}{4}s j_{\epsilon,\lambda}(s) \text{ for } 0 \leq s \leq 2. \end{aligned}$$

Since

$$F(s) - \frac{1}{4}s f(s) = \frac{1}{2\alpha}\left(e^{\alpha s^2} - 1\right) - \frac{s}{4}e^{\alpha s^2} = e^{\alpha s^2}\left(\frac{1}{2\alpha} - \frac{s}{4}\right) - \frac{1}{2\alpha} \text{ for } 0 \leq s \leq 2,$$

$$-G_\epsilon(s) \le 0 \text{ for } 0 \le s \le 2,$$

and

$$\frac{1}{4} s g_\epsilon(s) = -\frac{1}{4} s \log\left(s + \frac{\epsilon}{s+\epsilon}\right) \le -\frac{1}{4} s \log s \text{ for } 0 \le s \le 2,$$

we conclude that

$$J_{\epsilon,\lambda}(s) \le \frac{\lambda}{2\alpha} e^{4/\alpha} + \frac{1}{4} \sup_{0\le s\le 2} (-s\log s) + \frac{1}{4} s j_{\epsilon,\lambda}(s). \text{ for } 0 \le s \le 2.$$

Consequently,

$$J_{\epsilon,\lambda}(s) \le \frac{\lambda}{2\alpha} e^{4/\alpha} + \frac{1}{4e} + \frac{1}{4} s j_{\epsilon,\lambda}(s). \text{ for } 0 \le s \le 2.$$

We get from (6.57) that

$$J_{\epsilon,\lambda}(s) \le \frac{\lambda}{2\alpha} e^{4/\alpha} + \frac{1}{4e} + \frac{1}{4} s j_{\epsilon,\lambda}(s). \text{ for } s \ge 0.$$

Consequently, (6.39) holds for

$$C_{\lambda,\alpha} = \frac{\lambda}{2\alpha} e^{4/\alpha} + \frac{1}{4e}.$$

Consequently, (6.43) implies that $|\Omega|$ must satisfy

$$4\left(\frac{\lambda}{2\alpha} e^{4/\alpha} + \frac{1}{4e}\right) |\Omega| < \frac{\pi}{2\alpha}.$$

This proves Theorem 6.3. □

Theorem 6.4. *Let $\alpha \ge 3$ and assume that $f(s) = \exp(\alpha s^2)$. There exists $\lambda_0 > 0$ such that problem (6.26) has a nontrivial nonnegative solution for $\lambda > \lambda_0$, and $|\Omega| < c_{\lambda,\alpha}$, where*

$$c_{\lambda,\alpha} = \frac{\pi}{8\alpha}\left(\frac{1}{18\lambda \exp(\alpha(24B)^2) + (24B+1)\log(24B+1) + \frac{1}{4e}}\right),$$

with

$$B = \frac{1}{12} + \frac{3\lambda}{2} e^{4\alpha}.$$

Proof of Theorem 6.4 . Now we consider $f(s) = \exp(\alpha s^2)$ and study versions of Lemma 5.4 in this context. Observe that

$$0 \le F(s) \le s\exp(\alpha s^2) \text{ and } f'(s) = 2\alpha s \exp(\alpha s^2).$$

Since $f(0) = 0$, we consider the perturbation

$$g_\epsilon(s) = \begin{cases} -\log\left(s + \dfrac{\epsilon e^{-\lambda}}{s+\epsilon}\right) \text{ for } s \ge 0 \\ \lambda \text{ for } s < 0, \end{cases}$$

and $G_\epsilon(s) = \int_0^s g_\epsilon(t)\,dt$. Note that

$$3(\lambda f(s) + s) \le s\lambda f'(s) \text{ if and only if } 3\lambda\exp(\alpha s^2) + 3s \le 2\alpha\lambda s^2 \exp(\alpha s^2). \tag{6.58}$$

This inequality holds provided

$$3\lambda \exp(\alpha s^2) \leq \alpha\lambda s^2 \exp(\alpha s^2) \text{ and } 3s \leq \alpha\lambda s^2 \exp(\alpha s^2).$$

Hence, it holds provided

$$\alpha s^2 \geq 3, \text{ and } 3 \leq \alpha\lambda s. \tag{6.59}$$

Assume that $\alpha \geq 3$ and $\lambda \geq 1$. In this case, inequality (6.58) holds for $s \geq 2$. We will mimic the proof of Lemma 5.4. Let $B_\epsilon(s) = \lambda F(s) - G_\epsilon(s) - \theta\lambda s f(s) + \theta s g_\epsilon(s)$. We have

$$B'_\epsilon(s) = (1-\theta)(\lambda f(s) - g_\epsilon(s)) - \theta\lambda s f'(s) + \theta s g'_\epsilon(s).$$

Therefore, from Lemma 5.3, we conclude that

$$B'_\epsilon(s) \leq (1-\theta)(\lambda f(s) + s) - \theta s\lambda f'(s) - \frac{\theta}{6} \text{ for } s \geq 2.$$

Therefore, $B'_\epsilon(s) \leq -\theta/6$ for $s \geq 2$ provided

$$(1-\theta)(\lambda f(s) + s) - \theta s\lambda f'(s) \leq 0 \text{ for } s \geq 2.$$

Choosing $\theta = 1/4$, this is equivalent to

$$3(\lambda f(s) + s) - s\lambda f'(s) \leq 0 \text{ for } s \geq 2,$$

which holds from (6.58). Consequently,

$$B'_\epsilon(s) \leq -\frac{1}{24} \text{ for } s \geq 2. \tag{6.60}$$

Furthermore,

$$B_\epsilon(2) = \lambda F(2) - G_\epsilon(2) - \frac{\lambda}{2} f(2) + \frac{1}{2} g_\epsilon(2).$$

We have

$$\begin{aligned} -G_\epsilon(2) &= \int_0^2 \log\left(t + \frac{\epsilon e^{-\lambda}}{t+\epsilon}\right) dt \\ &\leq \int_0^2 \log(t + e^{-\lambda})\, dt \\ &\leq \int_0^2 \log(t+1)\, dt \\ &= \int_1^3 \log t\, dt \\ &= 3\log 3 - 3 \leq 0, \end{aligned} \tag{6.61}$$

$$g_\epsilon(2) = -\log\left(2 + \frac{\epsilon e^{-\lambda}}{2+\epsilon}\right) \leq -\log 2 \leq 0, \tag{6.62}$$

$$F(2) = \int_0^2 e^{\alpha t^2}\, dt \leq 2e^{4\alpha}. \tag{6.63}$$

and

$$f(2) = e^{4\alpha}. \tag{6.64}$$

From (6.61), (6.62), (6.63) and (6.64), we conclude that

$$B_\epsilon(2) \leq \left(2\lambda - \frac{\lambda}{2}\right) e^{4\alpha}.$$

From (6.60), we conclude that

$$B_\epsilon(s) \leq -\frac{s}{24} + B \text{ for all } s \geq 2,$$

where

$$B = \frac{1}{12} + \left(2\lambda - \frac{\lambda}{2}\right) e^{4\alpha}.$$

We conclude that

$$B_\epsilon(s) \leq 0 \text{ for all } s \geq 24B.$$

Consequently,

$$J(s) \leq \frac{1}{4} sj(s) \text{ for } s \geq 24B, \tag{6.65}$$

where $j(s) = \lambda f(s) - g_\epsilon(s)$. We now study the behaviour of j and J in the interval $[0, 24B]$. Indeed,

$$J(s) = \lambda\left(F(s) - \frac{1}{4} sf(s)\right) - \left(G_\epsilon(s) - \frac{1}{4} sg_\epsilon(s)\right) + \frac{1}{4} sj(s) \text{ for } 0 \leq s \leq 24B.$$

Observe that

$$-G_\epsilon(s) \leq (s+1)\log(s+1) - (s+1) \leq (24B+1)\log(24B+1) \text{ for all } 0 \leq s \leq 24B.$$

Next, we have

$$-sg_\epsilon(s) = s \log\left(s + \frac{\epsilon e^{-\lambda}}{s+\epsilon}\right) \geq s\log s \geq -\frac{1}{e},$$

for all $0 \leq s \leq 24B$. Consequently,

$$-\frac{1}{4} sg_\epsilon(s) \geq -\frac{1}{4e} \text{ for all } 0 \leq s \leq 24B.$$

Furthermore,

$$F(s) = \int_0^s \exp(\alpha t^2)\, dt \leq s \exp(\alpha s^2) \text{ for all } 0 \leq s \leq 24B,$$

and

$$sf(s) = s\exp(\alpha s^2) \text{ for all } 0 \leq s \leq 24B.$$

Hence,

$$J(s) \leq \left(\frac{3\lambda}{4}\right) s \exp(\alpha s^2) + (24B+1)\log(24B+1) + \frac{1}{4e} + \frac{1}{4} sj(s) \text{ for } 0 \leq s \leq 24B.$$

From (6.65), we conclude that

$$J(s) \leq C_{\lambda,\alpha} + \frac{1}{4} s j(s) \text{ for } s \geq 0,$$

where

$$C_{\lambda,\alpha} = 18\lambda \exp(\alpha(24B)^2) + (24B+1)\log(24B+1) + \frac{1}{4e}$$

Consequently, (6.43) implies that $|\Omega|$ must satisfy

$$4\left(18\lambda \exp(\alpha(24B)^2) + (24B+1)\log(24B+1) + \frac{1}{4e}\right)|\Omega| < \frac{\pi}{2\alpha}.$$

This proves Theorem 6.4. □

7 A problem with singular weights

In this chapter we consider the problem

$$\begin{cases} -\Delta u = a(x)g(u) + \lambda b(x)\, u^p & \text{in}\, \Omega \\ u > 0 & \text{in}\, \Omega \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{7.1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded smooth domain, $\lambda > 0$ is a positive parameter, a and b are positive weight functions that may be singular near the boundary $\partial\Omega$ of Ω, $g : (0, \infty) \to \mathbb{R}$ is singular at the origin and $0 < p < 1$.

We assume that the singular function $g : (0, \infty) \to \mathbb{R}$ is C^1 and behaves in a certain way near the origin. We assume that

$$\lim_{s \to 0^+} g(s) = -\infty, \tag{7.2}$$

and there exist constants $0 < \beta < 1$ and $C_1 \geq 0$ such that

$$\lim_{s \to 0^+} |g(s)s^\beta| = C_1. \tag{7.3}$$

We will also assume that g has sublinear growth, that is, there are constants $0 < q < 1$ and $C_2 \geq 0$ such that

$$\lim_{s \to \infty} \frac{|g(s)|}{s^q} = C_2. \tag{7.4}$$

Observe that the function g is a generalization of $-u^{-\beta}$. Other examples of g that we can have in mind are

- $g(s) = \log s$;
- $g(s) = s^{-\beta} \log s + \rho s^q$, where $\rho \geq 0$;
- $g(s) = -s^{-\beta} + \log s$;
- $g(s) = -\dfrac{1}{s^{-\alpha} + s^{-\beta}} + \rho s^q$, where $0 < \alpha, \beta, q < 1$ and $\rho \geq 0$.

We assume that the weights a and b satisfy

$$a, b \in C(\Omega) \text{ and } \min_{x \in \Omega}\{a(x), b(x)\} > 0. \tag{7.5}$$

These weights are allowed to be singular near the boundary of Ω provided that there are constants $\sigma > 0$ and $C_3 > 0$ such that

$$\sigma + \beta < 1, \tag{7.6}$$

$$a(x)\delta(x)^\sigma \leq C_3 \text{ for every } x \in \Omega, \tag{7.7}$$

and

$$b(x)\delta(x)^{\sigma} \leq C_3 \text{ for every } x \in \Omega, \tag{7.8}$$

where

$$\delta(x) = dist(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|.$$

The function δ plays an important role in this chapter. We recall the classical Hardy inequality, see [10], that states that there exists a constant $\Lambda > 0$ such that

$$\Lambda \int_{\Omega} \frac{\varphi^2}{\delta^2} \leq \int_{\Omega} |\nabla\varphi|^2 \text{ for every } \varphi \in C_c^{\infty}(\Omega). \tag{7.9}$$

Our first result asserts that problem (7.1) possesses a positive subsolution. To construct this subsolution, we introduce the auxiliary functions Y and ϕ_1, solutions of the problems

$$\begin{cases} -\Delta Y = 1 & \text{in } \Omega \\ Y > 0 & \text{in } \Omega \\ Y = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta \phi_1 = \lambda_1 \phi_1 & \text{in } \Omega \\ \phi_1 > 0 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively. Here $\lambda_1 > 0$ denotes the first eigenvalue of the Laplacian operator. We know that these functions have certain regularity properties. From Lemma B.2, there are constants $A_1, A_2 > 0$ and $B_1, B_2 > 0$ such that

$$A_1\delta(x) \leq Y(x) \leq B_1\delta(x) \text{ for all } x \in \Omega$$

and

$$A_2\delta(x) \leq \phi_1(x) \leq B_2\delta(x) \text{ for all } x \in \Omega.$$

Hence, there must exist constants $A, B, c > 0$ with c small such that $\phi_1 - 2cY > 0$ in Ω and

$$A\delta(x) \leq \psi(x) \leq B\delta(x) \text{ for all } x \in \Omega, \tag{7.10}$$

where $\psi(x) = \phi_1 - cY$. Hence, from (7.9) we get

$$\Lambda \int_{\Omega} \frac{\varphi^2}{\psi^2} \leq \int_{\Omega} |\nabla\varphi|^2 \text{ for every } \varphi \in C_c^{\infty}(\Omega). \tag{7.11}$$

Inequalities (7.9) and (7.11) hold (by extension) for all $\varphi \in H_0^1(\Omega)$. We now show that problem (7.1) possesses a subsolution of the form $\underline{u} = K\psi^{\nu}$ provided that $\lambda > 0$ is sufficiently large.

Lemma 7.1. *Suppose that* (7.2)*-*(7.8) *hold. There exists* $\lambda_0 > 0$ *(that does not depend on* p*) such that problem* (7.1) *has a positive subsolution* $\underline{u}$ *for all* $\lambda > \lambda_0$ *and* $0 < p < 1$.

Proof Let $K > 0$ and $\nu > 1$ be constants to be fixed later and take $\underline{u} = K\psi^\nu$. Then,

$$-\Delta\underline{u} = -K\nu\psi^{\nu-1}\Delta\psi - K\nu(\nu-1)\psi^{\nu-2}|\nabla\psi|^2.$$

Then, $\underline{u}$ is a subsolution of (7.1) if and only if

$$-\Delta\underline{u} = -K\nu\psi^{\nu-1}\Delta\psi - K\nu(\nu-1)\psi^{\nu-2}|\nabla\psi|^2 \leq a(x)g(\underline{u}) + \lambda b(x)\underline{u}^p \text{ in } \Omega. \tag{7.12}$$

We have

$$-\Delta\psi(x) = \lambda_1\phi_1(x) - c < 0 \text{ near the boundary } \partial\Omega.$$

Then we choose a smooth subdomain $\Omega' \subset\subset \Omega$ such that

$$-\Delta\psi(x) < 0 \text{ for all } x \in \Omega \setminus \Omega'.$$

Then, to prove (7.12) in $\Omega \setminus \Omega'$ we just need to show that

$$-K\nu(\nu-1)\psi^{\nu-2}|\nabla\psi|^2 \leq a(x)g(\underline{u}) \text{ in } \Omega \setminus \Omega'.$$

But by Hopf's Lemma, we can assume that there exists a positive number $\eta_1 > 0$ such that $|\nabla\psi|^2 > \eta_1$ in $\Omega \setminus \Omega'$. Therefore, it is enough to prove that

$$-K\nu(\nu-1)\psi^{\nu-2}\eta_1 \leq a(x)g(\underline{u}) \text{ in } \Omega \setminus \Omega'.$$

But by condition (7.4) and by choosing Ω' sufficiently close to $\partial\Omega$, we can also assume that there exists a constant $D_1 > 0$ such that

$$|g(\underline{u})| \leq D_1\underline{u}^{-\beta} = D_1K^{-\beta}\psi^{-\nu\beta} \text{ in } \Omega \setminus \Omega'.$$

Hence, it is enough to prove that

$$-K\nu(\nu-1)\psi^{\nu-2}\eta_1 \leq -D_1a(x)K^{-\beta}\psi^{-\nu\beta} \text{ in } \Omega \setminus \Omega'.$$

Therefore, we need to find $K > 0$ and $\nu > 1$ such that

$$a(x)\psi(x)^{2-\nu(1+\beta)} \leq \frac{\eta_1\nu(\nu-1)K^{1+\beta}}{D_1} \text{ in } \Omega \setminus \Omega'.$$

But by (7.5) and (7.10) we need only to show that

$$\delta(x)^{-\sigma+2-\nu(1+\beta)} \leq \frac{\eta_1}{D_1}\nu(\nu-1)K^{1+\beta} \text{ in } \Omega \setminus \Omega'. \tag{7.13}$$

Then, we choose $\nu > 1$ such that

$$1 < \nu < \frac{2-\sigma}{1+\beta},$$

so that the left side of (7.13) is bounded. Once such ν is fixed, we take $K > 0$ large enough so that (7.13) holds. Observe that the choices of ν and K do not depend on p nor on λ.

Thus we have proven that there exist constants $K > 0$, $\nu > 1$ and a smooth subdomain $\Omega' \subset\subset \Omega$ such that (7.12) holds in $\Omega \setminus \Omega'$. We proceed to prove that (7.12) holds in Ω' for $\lambda > 0$ sufficiently large. Let $m_b > 0$ be defined by $m_b = \min_{x \in \Omega} b(x)$. We will find $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$

$$-K\nu\psi^{\nu-1}\Delta\psi \leq a(x)g(\underline{u}) + \lambda m_b \underline{u}^p \text{ in } \Omega'.$$

Hereafter $D_i > 0, i = 2, 3, \ldots$ denote various constants. By the boundedness of $-\Delta\psi$ and by the fact that $\sup_\Omega \psi < \infty$, it is enough to prove that

$$\begin{aligned} KD_2 &\leq a(x)g(\underline{u}) + \lambda m_b \underline{u}^p \\ &= a(x)g(\underline{u}) + \lambda K^p m_b \psi^{\nu p} \text{ in } \Omega'. \end{aligned}$$

Note that there exists $0 < \eta_2 < 1$ such that $\psi > \eta_2$ in Ω'. Since $\underline{u} \geq K\eta_2^\nu \geq \min\{\eta_2, \eta_2^2\}$ in Ω', we have $|g(\underline{u})| \leq D_K$ in Ω'. Also, the function a is bounded in Ω'. Hence, one is led to verify

$$KD_2 \leq -D_K + \lambda K^p m_b \eta_2^{\nu p} \text{ in } \Omega'.$$

Observe that $\eta_2^{\nu p} > \eta_2^2$, because $0 < \nu p < 2$. Hence, we only need to verify

$$KD_2 \leq -D_K + \lambda K^p D_5, \tag{7.14}$$

where $D_5 > 0$ does not depend on p nor on λ. Indeed (7.14) holds if we choose $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$

$$\frac{D_2 + D_K}{D_5} \leq \frac{\lambda}{K} \text{ in } \Omega'. \tag{7.15}$$

Observe that we may choose λ_0 independently of p. The proof is complete. □

In the next two sections we establish the main results of this chapter.

7.1 Existence of solution

Recall that $u \in H_0^1(\Omega)$ is a solution of problem (7.1) if

$$\int_\Omega \nabla u \nabla\varphi \, dx = \int_\Omega f(x,u)\varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega),$$

where $f(x,s) = a(x)g(s) + \lambda b(x)s^p$ for $x \in \Omega$ and $s \geq 0$. We have shown that there exists a subsolution $\underline{u}$ for problem (7.1) provided $\lambda > \lambda_0$. Consequently, $\underline{u}$ must satisfy the inequality

$$\int_\Omega \nabla\underline{u}\nabla\varphi \, dx \leq \int_\Omega f(x,\underline{u})\varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega), \varphi \geq 0. \tag{7.16}$$

In this section we will consider the truncation

$$\widehat{f}(x,s) = \begin{cases} f(x,\underline{u}(x)) & \text{if } s \leq \underline{u}(x) \\ f(x,s) & \text{if } s \geq \underline{u}(x), \end{cases}$$

and we will look for solutions of the auxiliary perturbed problems

$$\begin{cases} -\Delta u = \widehat{f}(x,u), & \text{in } \Omega_k \\ u = \underline{u} & \text{on } \partial\Omega_k, \end{cases} \tag{7.17}$$

where $\emptyset \neq \Omega_1 \subset\subset \Omega_2 \subset\subset \ldots \subset\subset \Omega$ is a sequence of smooth domains such that $\bigcup_{k=1}^{\infty} \Omega_i = \Omega$. Observe that u_k is a solution of problem (7.17) if and only if $v_k = u_k - \underline{u}$ is a solution of

$$\begin{cases} -\Delta v = \widehat{f}(x, v + \underline{u}) + \Delta \underline{u}, & \text{in } \Omega_k \\ v = 0 & \text{on } \partial\Omega_k. \end{cases} \tag{7.18}$$

Using variational arguments, we will prove that problem (7.18) has a nonnegative solution $v_k \in H_0^1(\Omega_k)$. Define the functional $\widetilde{I}_k : H_0^1(\Omega_k) \to \mathbb{R}$ by

$$\widetilde{I}_k(v) = \frac{1}{2}\int_{\Omega_k} |\nabla v|^2\,dx - \int_{\Omega_k} \widetilde{F}(x,v) + \int_{\Omega_k} \nabla \underline{u} \nabla v\,dx, \tag{7.19}$$

where $\widetilde{F}(x,v) = \int_0^v \widehat{f}(x, s + \underline{u}(x))\,ds$. From Theorem B.16, we know that the functional $\widetilde{I}_k$ is of class C^1 and

$$\widetilde{I}_k'(v)(w) = \int_{\Omega_k} \nabla v \nabla w\,dx - \int_{\Omega_k} \widehat{f}(x, v + \underline{u})w + \int_{\Omega_k} \nabla \underline{u} \nabla w\,dx \text{ for all } v, w \in H_0^1(\Omega_k).$$

Hence, if v is a critical point of $\widetilde{I}_k$, then

$$\int_{\Omega_k} \nabla v \nabla w\,dx = \int_{\Omega_k} \widehat{f}(x, v + \underline{u})w - \int_{\Omega_k} \nabla \underline{u} \nabla w\,dx \text{ for all } w \in H_0^1(\Omega_k). \tag{7.20}$$

Therefore, critical points of $\widetilde{I}_k$ are weak solutions of problem (7.18). We have

Lemma 7.2. *The functional $\widetilde{I}_k$ is coercive, that is, $\widetilde{I}_k(v) \to \infty$ as $\|v\|_{H_0^1(\Omega)} \to \infty$.*

Consequently, $\widetilde{I}_k$ possesses a nonnegative critical point $v_k \in H_0^1(\Omega_k)$.

Proof of Lemma 7.2. Observe that

$$\widetilde{F}(x,v) = \begin{cases} v f(x, \underline{u}(x)) & \text{if } v < 0 \\ \widehat{F}(x, v + \underline{u}(x)) - \widehat{F}(x, \underline{u}(x)) & \text{if } v \geq 0, \end{cases} \tag{7.21}$$

where $\widehat{F}(x,v) = \int_0^v \widehat{f}(x,s)\,ds$. We will estimate the term $\widehat{F}$. We have

$$\begin{aligned} \widehat{F}(x,s) &= \int_0^s \widehat{f}(x,t)\,dt = \int_0^{\underline{u}} \widehat{f}(x,t)\,dt + \int_{\underline{u}}^s \widehat{f}(x,t)\,dt = \underline{u} f(x, \underline{u}(x)) + \int_{\underline{u}}^s f(x,t)\,dt \\ &= \underline{u}(x)\left(a(x) g(\underline{u}(x)) + \lambda b(x) \underline{u}^p(x)\right) + a(x)\int_{\underline{u}(x)}^s g(t)\,dt + \lambda b(x) \int_{\underline{u}(x)}^s t^p\,dt. \\ &= a(x)\int_{\underline{u}(x)}^s g(t)\,dt + \frac{\lambda b(x) s^{p+1}}{p+1} + a(x)\underline{u}(x) g(\underline{u}(x)) + \lambda b(x) \underline{u}^{p+1}(x)\left(1 - \frac{1}{p+1}\right). \end{aligned}$$

Next, we split $\widehat{F}$ into three integrals $(I), (II)$ and (III), and estimate each of them. Let

$$(I) = a(x) \int_{\underline{u}(x)}^{s} g(t)\, dt,$$

$$(II) = \frac{\lambda b(x) s^{p+1}}{p+1},$$

and

$$(III) = a(x)\underline{u}(x) g(\underline{u}(x)) + \lambda b(x) \underline{u}^{p+1}(x) \left(1 - \frac{1}{p+1}\right).$$

Then,

$$\widehat{F}(x, v) = (I) + (II) + (III) \tag{7.22}$$

Estimate of $|(I)|$: There exists a constant $a_k > 0$ such that $a(x) \leq a_k$ for all $x \in \Omega_k$. By conditions (7.3) and (7.4), we can find positive constants $A, B, C > 0$ indepedent of k such that

$$\begin{aligned} |(I)| \leq a_k \int_{\underline{u}(x)}^{s} |g(t)|\, dt &\leq a_k \int_0^s \left(A|t|^{-\beta} + B|t|^q + C\right)\, dt \\ &\leq \frac{A a_k |s|^{1-\beta}}{1-\beta} + \frac{B a_k |s|^{1+q}}{1+q} + C a_k |s|. \end{aligned}$$

Given $\epsilon > 0$ we are able to find positive constants $A_k, B_k, C_k > 0$ such that for every $s \in \mathbb{R}$

$$|s|^{1-\beta} \leq \frac{\epsilon(1-\beta)}{6 A a_k} |s|^2 + A_k$$

$$|s|^{1+q} \leq \frac{\epsilon(1+q)}{6 B a_k} |s|^2 + B_k$$

and

$$|s| \leq \frac{\epsilon}{6 C a_k} |s|^2 + C_k$$

Hence, we can find a constant $D_k > 0$ such that

$$|(I)| \leq \frac{\epsilon}{2} |s|^2 + D_k. \tag{7.23}$$

Estimate of $|(II)|$ and $|(III)|$: Similarly, we can find constants $E_k, F_k > 0$ such that

$$|(II)| \leq \frac{\epsilon}{2} |s|^2 + E_k, \tag{7.24}$$

and

$$|(III)| \leq F_k. \tag{7.25}$$

Then substituting (7.23), (7.24) and (7.25) in (7.22) we find a constant $c_k > 0$ that depends on k such that

$$|\widehat{F}(x, s)| \leq \epsilon |s|^2 + c_k. \tag{7.26}$$

Let $v \in H_0^1(\Omega_k)$. From (7.21) and (7.26), we get

$$\begin{aligned}\widetilde{I}_k(v) &= \frac{1}{2}\int_{\Omega_k} |\nabla v|^2\,dx - \int_{\Omega_k} \widetilde{F}(x,v) + \int_{\Omega_k} \nabla \underline{u}\nabla v\,dx \\ &\geq \frac{1}{2}\int_{\Omega_k} |\nabla v|^2\,dx - \int_{\Omega_k} |v||f(x,\underline{u}(x))| - \int_{\Omega_k} |\widehat{F}(s, v+\underline{u}(x))| - \int_{\Omega_k} |\widehat{F}(x,\underline{u}(x))| \\ &\geq \frac{1}{2}\|v\|^2_{H_0^1(\Omega_k)} - c_k\|v\|_{L^1(\Omega_k)} - \int_{\Omega_k} \epsilon|v+\underline{u}(x)|^2 - d_k.\end{aligned}$$

Using that $|a+b|^2 \leq \dfrac{|a|^2+|b|^2}{2}$, and the Sobolev Embedding we obtain constants $C, D > 0$ such that

$$\begin{aligned}\widetilde{I}_k(v) &\geq \frac{1}{2}\|v\|^2_{H_0^1(\Omega_k)} - c_k\|v\|_{L^1(\Omega_k)} - \frac{\epsilon}{2}\int_{\Omega_k}|v|^2 - d_k \\ &\geq \left(\frac{1}{2} - C\epsilon\right)\|v\|^2_{H_0^1(\Omega_k)} - Dc_k\|v\|_{H_0^1(\Omega_k)} - d_{k,\epsilon}.\end{aligned}$$

The coercivity of $\widetilde{I}_k$ then follows by taking $\epsilon > 0$ sufficiently small. We also obtain that $\widetilde{I}_k$ is bounded from below.

We now claim that each Palais-Smale sequence of $\widetilde{I}_k$ must be uniformly bounded in $H_0^1(\Omega)$. Indeed, if (v_n) is a sequence in $H_0^1(\Omega_k)$ such that $|I_k(v_n)| < C$ for some positive constant $C > 0$, then (7.4), (7.19) and (7.21) imply that there exist constants $a_k, b_k, c_k > 0$ that do not depend on n such that

$$\frac{1}{2}\|v_n\|^2_{H_0^1(\Omega_k)} \leq a_k\|v_n\|_{H_0^1(\Omega_k)} + b_k\|v_n\|^{1+q}_{H_0^1(\Omega_k)} + c_k\|v_n\|^{1+p}_{H_0^1(\Omega_k)}$$

This proves the claim. From Theorem B.16, we conclude that $\widetilde{I}_k$ satisfies the Palais-Smale condition. Next, we apply Theorem B.17 to obtain a function $v_k \in H_0^1(\Omega_k)$ such that $v_k = \inf_{v\in H_0^1(\Omega_k)} \widetilde{I}_k(v)$. Hence, v_k is a solution to problem (7.18) and $u_k := v_k + \underline{u}$ is a solution to problem (7.17). We proceed to show that $v_k \geq 0$ in Ω_k. Choosing $w = -v_k^-$ in (7.20) and using (7.16) we obtain

$$\begin{aligned}-\int_{\Omega_k} |\nabla v_k^-|^2\,dx &= -\int_{\{v_k<0\}} v_k^-\widehat{f}(x, v_k+\underline{u})\,dx + \int_{\Omega_k}\nabla\underline{u}\nabla v_k^-\,dx \\ &= -\int_{\{v_k<0\}} v_k^- f(x,\underline{u})\,dx + \int_{\Omega_k}\nabla\underline{u}\nabla v_k^-\,dx \leq 0.\end{aligned}$$

This proves that $v_k \geq 0$ almost everywhere in Ω_k and therefore $u_k \geq \underline{u}$ almost everywhere in Ω_k. □

We now prove the main result of this section.

Theorem 7.1. *Suppose that (7.2)-(7.8) hold. Let* $\mu = \max\{p,q\}$. *Suppose that*

$$\sigma < \frac{2N - (1+\mu)(N-2)}{2N}. \tag{7.27}$$

Then, there exists $\lambda_0 > 0$ (that does not depend on p) such that problem (7.1) has a solution for each $\lambda > \lambda_0$ and for all $0 < p < 1$.

Proof of Theorem 7.1. First we will prove that $||v_k||_{H_0^1(\Omega_k)}$ is bounded by a constant that does not depend on k. Choosing $w = v_k$ in (7.20) and the fact that $v_k \geq 0$ in Ω_k we get

$$\begin{aligned} ||v_k||^2_{H_0^1(\Omega_k)} &= \int_{\Omega_k} f(x, v_k + \underline{u}) v_k \, dx - \int_{\Omega_k} \nabla \underline{u} \nabla v_k \, dx \\ &= \int_{\Omega_k} a(x) g(v_k + \underline{u}) v_k \, dx + \lambda \int_{\Omega_k} b(x)(v_k + \underline{u})^p v_k \, dx - \int_{\Omega_k} \nabla \underline{u} \nabla v_k \, dx \\ &= (J) + (JJ) + (JJJ). \end{aligned}$$

Let's estimate (J). By conditions (7.2) and (7.4) we can find constants $A, B, s_1 > 0$ such that $g(s) < 0$ for every $0 < s < s_1$ and $|g(s)| \leq A + B|s|^q$ for every $s \geq s_1$. Therefore,

$$\begin{aligned} (J) &= \int_{\Omega_k} a(x) g(v_k + \underline{u}) v_k \, dx \leq \int_{\Omega_k \cap \{v_k + \underline{u} > s_1\}} a(x) g(v_k + \underline{u}) v_k \, dx \\ &\leq A \int_{\Omega_k} a(x) v_k \, dx + B \int_{\Omega_k} a(x) |v_k + \underline{u}|^q v_k \, dx \\ &\leq A \int_{\Omega_k} \delta^{-\sigma} v_k \, dx + B \int_{\Omega_k} \delta^{-\sigma} |v_k + \underline{u}|^q v_k \, dx. \end{aligned}$$

We choose a number $r > 1$ such that $\dfrac{2N}{2N - (1+\mu)(N-2)} \leq r < \dfrac{1}{\sigma}$. Then

$$\int_{\Omega_k} \delta^{-\sigma} |v_k + \underline{u}|^q v_k \, dx \leq \left(\int_{\Omega_k} \delta^{-\sigma r} \, dx \right)^{\frac{1}{r}} \left(\int_{\Omega_k} (v_k + \underline{u})^{\frac{r(q+1)}{r-1}} \, dx \right)^{\frac{q(r-1)}{(q+1)r}} \left(\int_{\Omega_k} v_k^{\frac{r(q+1)}{r-1}} \, dx \right)^{\frac{r-1}{(q+1)r}}. \tag{7.28}$$

Since $0 < \sigma r < 1$ we have from Lemma B.1 that

$$\left(\int_{\Omega_k} \delta^{-\sigma r} \, dx \right)^{\frac{1}{r}} < C.$$

Then, inserting in (7.28) we obtain

$$\int_{\Omega_k} \delta^{-\sigma} |v_k + \underline{u}|^q v_k \, dx \leq C ||v_k + \underline{u}||^q_{L^{\frac{r(q+1)}{r-1}}} ||v_k||_{L^{\frac{r(q+1)}{r-1}}}.$$

Therefore, by Minkowski inequality,

$$\int_{\Omega_k} \delta^{-\sigma} |v_k + \underline{u}|^q v_k \, dx \leq C \left(||v_k||^{q+1}_{L^{\frac{r(q+1)}{r-1}}(\Omega_k)} + ||v_k||_{L^{\frac{r(q+1)}{r-1}}(\Omega_k)} \right).$$

Also, note that by the Hölder inequality

$$\int_{\Omega_k} \delta^{-\sigma} v_k \, dx \leq C ||v_k||_{L^{\frac{r}{r-1}}(\Omega_k)},$$

so that

$$(J) \leq A ||v_k||_{L^{\frac{r}{r-1}}(\Omega_k)} + B \left(||v_k||^{q+1}_{L^{\frac{r(q+1)}{r-1}}(\Omega_k)} + ||v_k||_{L^{\frac{r(q+1)}{r-1}}(\Omega_k)} \right).$$

Similarly

$$(JJ) \leq C \left(||v_k||^{p+1}_{L^{\frac{r(p+1)}{r-1}}(\Omega_k)} + ||v_k||_{L^{\frac{r(p+1)}{r-1}}(\Omega_k)} \right).$$

It is also clear that

$$(JJJ) \leq D||v_k||_{H_0^1(\Omega_k)}.$$

Note that the constants A, B, C, D do not depend on k. Using that $\mu \geq p$ and $\mu \geq q$ we conclude that

$$||v_k||^2_{H_0^1(\Omega_k)} \leq A\left(||v_k||_{L^{\frac{r(1+\mu)}{r-1}}(\Omega_k)} + ||v_k||^{q+1}_{L^{\frac{r(\mu+1)}{r-1}}(\Omega_k)} + ||v_k||^{p+1}_{L^{\frac{r(\mu+1)}{r-1}}(\Omega_k)} + \|v_k\|_{H_0^1(\Omega_k)}\right).$$

By the choice of r, we have $\dfrac{r(\mu+1)}{r-1} \leq \dfrac{2N}{N-2}$. Then, the Sobolev Embedding Theorem implies that

$$||v_k||^2_{H_0^1(\Omega_k)} \leq A\left(||v_k||_{H_0^1(\Omega_k)} + ||v_k||^{q+1}_{H_0^1(\Omega_k)} + ||v_k||^{p+1}_{H_0^1(\Omega_k)}\right).$$

Hence, there must exist a constant $C > 0$ that does not depend on k such that

$$\|v_k\|_{H_0^1(\Omega_k)} < C \text{ for all } k \in \mathbb{N}.$$

Since $u_k = v_k + \underline{u}$, and by construction $\underline{u} \in H_0^1(\Omega)$, we conclude that there exists $D > 0$ such that

$$\|u_k\|_{H_0^1(\Omega)} < D \text{ for all } k \in \mathbb{N}.$$

Therefore, there must exist $u \in H_0^1(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and

$$\begin{cases} u_k \rightharpoonup u, & \text{in } H_0^1(\Omega) \\ u_k \to u & \text{in } L^\tau(\Omega), \text{for all } 1 \leq \tau < \dfrac{2N}{N-2} \\ u_k \to u & \text{almost everywhere in } \Omega. \end{cases}$$

Since $u_k \geq \underline{u}$ for all $k \in \mathbb{N}$, we conclude that $u \geq \underline{u}$ in Ω. We prove in the sequel that u is a weak solution of (7.1). Indeed, let $\varphi \in C_c^\infty(\Omega)$ be a test function and choose $k' \in \mathbb{N}$ such that *support*$(\varphi) \subset\subset \Omega_{k'}$. Then,

$$\int_{\Omega_{k'}} \nabla u_k \nabla \varphi = \int_{\Omega_{k'}} (a(x)g(u_k) + \lambda b(x)u_k^p)\varphi \quad \text{for every } k \geq k'. \tag{7.29}$$

Since $u_k \geq \underline{u} \geq c_{k'}$, we get $|g(u_k)| \leq A + B|u_k|^q$ and $g(u_k) \to g(u)$ almost everywhere in $\Omega_{k'}$. Also, $|a(x)| \leq a_{k'}, |b(x)| \leq b_{k'}$ in $\Omega_{k'}$. Hence, $a(x)g(u_k)\varphi \to a(x)g(u)\varphi$ almost everywhere in $\Omega_{k'}$ as $k \to \infty$. Observe that

$$|a(x)g(u_k)\varphi| \leq C_{k'}|g(u_k)\varphi| \leq C_{k'}(|\varphi| + |u_k|^q\varphi) \leq D_{k'}(|\varphi| + |u_k|^{q+1} + |\varphi|^{q+1}) \in L^1(\Omega).$$

Hence, by the Dominated Convergence Theorem,

$$\int_{\Omega_{k'}} a(x)g(u_k)\varphi \to \int_{\Omega_{k'}} a(x)g(u)\varphi.$$

Similarly,

$$\int_{\Omega_{k'}} b(x)u_k^p\varphi \to \int_{\Omega_{k'}} b(x)u^p\varphi.$$

Hence, letting $k \to \infty$ in (7.29) we obtain

$$\int_{\Omega_{k'}} \nabla u \nabla \varphi = \int_{\Omega_{k'}} (a(x)g(u) + \lambda b(x)u^p)\varphi.$$

The proof of Theorem 7.1 is complete. □

7.2 Uniqueness of solution

In this section we get a uniqueness result for the solutions of problem (7.1). Moreover, we may weaken our hypothesis, so that we may consider a singularity g such that

$$\lim_{s\to 0^+} g(s) = +\infty.$$

For example, our uniqueness result is applicable for the problem

$$\begin{cases} -\Delta u = a(x)u^{-\beta} + \lambda b(x)\, u^p & \text{in}\, \Omega \\ u > 0 & \text{in}\, \Omega \\ u = 0 & \text{on}\, \partial\Omega. \end{cases}$$

Here, the Hardy inequality (7.11) will play an important role.

Theorem 7.2. *Let $\lambda, K > 0$ and $\nu > 1$ be constants. Suppose that g is of class C^1 and that (7.5), (7.7) and (7.8) hold for $0 < \sigma < 2$. Assume also that there are constants $0 < \gamma < 2 - \sigma$ and $C_4 > 0$ such that*

$$|g'(s)| \leq C_4 |s|^{-\gamma} \quad \textit{for every}\, s > 0. \tag{7.30}$$

Then, there exists $C > 0$ that does not depend on p nor on λ such that if

$$1 < \nu < \min\left\{2 - \sigma, \frac{2-\sigma}{\gamma}\right\} \qquad \textit{and} \qquad K > C,$$

then there exists $0 < p_0 < 1$ small depending on λ and K such that if $0 < p < p_0$ there is at most one solution of problem (7.1) in the class of functions $u \geq \underline{u} = K\psi^\nu$.

This theorem is applicable for a large class of problems. For example, we may consider $g \equiv 0$ or $g \equiv 1$ in problem (7.1). Moreover, we may choose $a \equiv 1$ and $b \equiv 1$. Consequently, we get

Corollary 7.1. *Fix $\lambda > 0$ and $0 < \beta, q < 1$. There exists $p_0 > 0$ such that the problems*

$$\begin{cases} -\Delta u = \lambda\, u^p & \textit{in}\, \Omega \\ u > 0 & \textit{in}\, \Omega \\ u = 0 & \textit{on}\, \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = u^q + \lambda\, u^p & \textit{in}\, \Omega \\ u > 0 & \textit{in}\, \Omega \\ u = 0 & \textit{on}\, \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = \lambda \delta(x)^{-1/3}\, u^p & \textit{in}\, \Omega \\ u > 0 & \textit{in}\, \Omega \\ u = 0 & \textit{on}\, \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = \delta(x)^{-1/3}(-u^{-1/2} + \lambda\, u^p) & in\, \Omega \\ u > 0 & in\, \Omega \\ u = 0 & on\, \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = \delta(x)^{-1/3}(u^{-1/2} + \lambda\, u^p) & in\, \Omega \\ u > 0 & in\, \Omega \\ u = 0 & on\, \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = (u^{-\beta} + \lambda\, u^p) & in\, \Omega \\ u > 0 & in\, \Omega \\ u = 0 & on\, \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u = (-u^{-\beta} + \lambda\, u^p) & in\, \Omega \\ u > 0 & in\, \Omega \\ u = 0 & on\, \partial\Omega, \end{cases}$$

have at most one solution for $0 < p < p_0$ in the class of functions $u \geq \underline{u}$, provided K is large enough.

Theorem 7.2 also implies that the solution obtained in Theorem 7.1 is unique in the class $u \geq \underline{u}$.

Corollary 7.2. *Suppose that (7.2)-(7.8), (7.27) and (7.30) hold. Let λ_0 be given by Theorem 7.1. Then, for each $\lambda > \lambda_0$ there exists $p_0 > 0$ such that the solution u given by Theorem 7.1 is unique in the class $u \geq \underline{u}$ provided $0 < p < p_0$.*

In the proof of Theorem 7.2, we do not use the fact that $\underline{u}$ is a subsolution of problem (7.1) for large values of λ.

Proof of Theorem 7.2. Suppose that $u \geq \underline{u}$ and $v \geq \underline{u}$ are solutions of (7.1), so that $u, v \in H_0^1(\Omega)$ and

$$-\Delta u = f(x,u) \text{ and } -\Delta v = f(x,v) \text{ in } \Omega.$$

Observe that

$$f(x,u) - f(x,v) = a(x)(g(u) - g(v)) + \lambda b(x)(u^p - v^p). \tag{7.31}$$

We will estimate the right hand side of (7.31). Note that

$$g(u) - g(v) = \int_0^1 \frac{d}{dt} g(tu + (1-t)v)\, dt = (u - v) \int_0^1 g'(tu + (1-t)v)\, dt.$$

Hence, using (7.7), on the set $\{u - v \geq 0\}$,

$$|g(u) - g(v)| \leq C_4(u - v) \int_0^1 |t(u - v) + v|^{-\gamma}\, dt \leq C_4(u - v) \int_0^1 |v|^{-\gamma}\, dt,$$

and

$$|g(u) - g(v)| \leq C_4(u - v)|v|^{-\gamma}.$$

By concavity of the function $s \to s^p$ we have

$$u^p - v^p \leq pv^{p-1}(u-v),$$

so that

$$|f(x,u) - f(x,v)| \leq C_4 a(x)(u-v)v^{-\gamma} + \lambda b(x) p v^{p-1}(u-v) \text{ on the set } \{u - v \geq 0\}.$$

Since $v \geq \underline{u}$, by (7.5), (7.7) and (7.8) we get

$$\begin{aligned}
|f(x,u) - f(x,v)| &\leq C_4 a(x)(u-v)\underline{u}^{-\gamma} + \lambda b(x) p \underline{u}^{p-1}(u-v) \\
&= C_4 a(x)(u-v)(K\psi^\nu)^{-\gamma} + \lambda b(x) p (K\psi^\nu)^{p-1}(u-v) \\
&= \psi^{-2}(u-v)\left(C_4 a(x) K^{-\gamma}\psi^{2-\gamma\nu} + \lambda p b(x) K^{p-1}\psi^{2-\nu(1-p)}\right) \\
&\leq C\psi^{-2}(u-v)\left(\delta^{-\sigma} K^{-\gamma}\psi^{2-\gamma\nu} + \delta^{-\sigma}\lambda p K^{p-1}\psi^{2-\nu(1-p)}\right) \\
&\leq C\psi^{-2}(u-v)\left(K^{-\gamma}\psi^{2-\gamma\nu-\sigma} + \lambda p K^{p-1}\psi^{2-\nu(1-p)-\sigma}\right) \\
&\leq C\psi^{-2}(u-v)\left(K^{-\gamma}\psi^{2-\gamma\nu-\sigma} + \lambda p K^{p-1}\psi^{2-\nu(1-p)-\sigma}\right).
\end{aligned} \tag{7.32}$$

Notice that $2 - \gamma\nu - \sigma > 0$ and $2 - \nu(1-p) - \sigma > 0$, and let $\Lambda > 0$ be given by (7.11). If K is so large that

$$CK^{-\gamma}\psi^{2-\gamma\nu-\sigma} < \Lambda \text{ in } \Omega,$$

then there exists $p_0 > 0$ such that if $0 < p < p_0$ then

$$C(K^{-\gamma}\psi^{2-\gamma\nu-\sigma} + \lambda p K^{p-1}\psi^{2-\nu(1-p)-\sigma}) \leq \Lambda \text{ in } \Omega.$$

Hence,

$$|f(x,u) - f(x,v)| \leq \Lambda\psi^{-2}(u-v) \text{ on } \{u - v \geq 0\}.$$

An analogous argument yields

$$|f(x,v) - f(x,u)| \leq \Lambda\psi^{-2}(v-u) \text{ on } \{v - u \geq 0\},$$

so that

$$|f(x,u) - f(x,v)| \leq \Lambda\psi^{-2}|u-v| \text{ in } \Omega.$$

Let $w = u - v$. From the facts that $w \in H_0^1(\Omega)$ and $-\Delta w = f(x,u) - f(x,v)$, we know that

$$\int_\Omega \nabla w \nabla\varphi \, dx = \int_\Omega (f(x,u) - f(x,v))\varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega).$$

We claim that this equality can be extended for all $\varphi \in H_0^1(\Omega)$. Indeed, let $\varphi_0 \in H_0^1(\Omega)$ and let (φ_n) be a sequence in $C_c^\infty(\Omega)$ such that $\varphi_n \to \varphi_0$ in $H_0^1(\Omega)$ as $n \to \infty$. We know that

$$\int_\Omega \nabla w \nabla\varphi_n \, dx = \int_\Omega (f(x,u) - f(x,v))\varphi_n \, dx \text{ for all } n \in \mathbb{N}. \tag{7.33}$$

Observe that

$$|(f(x,u) - f(x,v))(\varphi_n - \varphi_0)| \leq \Lambda(\psi^{-1}|u-v|)(\psi^{-1}|\varphi_n - \varphi_0|),$$

so that from Hölder's inequality, we get

$$\int_\Omega |(f(x,u)-f(x,v))(\varphi_n-\varphi_0)| \le \Lambda \left(\int_\Omega \frac{|u-v|^2}{\psi^2}\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\varphi_n-\varphi_0|^2}{\psi^2}\right)^{\frac{1}{2}}.$$

From Hardy's inequality (7.11), we conclude that there exists a constant $C>0$ such that

$$\begin{aligned}&\int_\Omega |(f(x,u)-f(x,v))(\varphi_n-\varphi_0)|\,dx\\ &\le C\left(\int_\Omega |\nabla(|u-v|)|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla(|\varphi_n-\varphi_0|)|^2\,dx\right)^{\frac{1}{2}} \to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence, letting $n\to\infty$ in (7.33) we get

$$\int_\Omega \nabla w\nabla\varphi_0\,dx = \int_\Omega (f(x,u)-f(x,v))\varphi_0\,dx \text{ for all } \varphi_0\in H_0^1(\Omega).$$

We proved the claim. Taking $\varphi_0 = w^+$, we get

$$\int_\Omega |\nabla w^+|^2\,dx = \int_\Omega (f(x,u)-f(x,v))(w^+)\,dx.$$

From (7.32), we get

$$\int_\Omega |\nabla w^+|^2\,dx \le \int_\Omega C\psi^{-2}\left(K^{-\gamma}\psi^{2-\gamma\nu-\sigma}+\lambda pK^{p-1}\psi^{2-\nu(1-p)-\sigma}\right)(w^+)^2\,dx.$$

Using the Hardy inequality (7.11), we conclude that

$$\begin{aligned}0 &\le \int_\Omega \frac{1}{\psi^2}\left(\Lambda - C(K^{-\gamma}\psi^{2-\gamma\nu-\sigma}+\lambda pK^{p-1}\psi^{2-\nu(1-p)-\sigma})\right)(w^+)^2\,dx\\ &\le \int_\Omega |\nabla w^+|^2\,dx - \int_\Omega \frac{(w^+)^2}{\psi^2}\left(C(K^{-\gamma}\psi^{2-\gamma\nu-\sigma}+\lambda pK^{p-1}\psi^{2-\nu(1-p)-\sigma})\right)dx \le 0.\end{aligned}$$

Hence, $w^+\equiv 0$ and then $(u-v)^+\equiv 0$. Similarly, $(v-u)^+\equiv 0$. Therefore, $u\equiv v$. □

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APPENDIX A – Regularity results

Here we give regularity results used in Chapters 2-6. In these chapters, we studied problems of the form

$$\begin{cases} -\Delta u + g_\epsilon(u) = \widetilde{f}(u) \text{ in } \Omega \\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{A.1}$$

where g_ϵ is a smooth perturbation of a singular term and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain for $N \geq 2$. We will assume that $\widetilde{f}$ is a nonlinear function such that

$$\widetilde{f} \in C[0,\infty) \cap C^{1,\nu}(0,\infty) \text{ for some } 0 < \nu < 1. \tag{A.2}$$

In Chapters 2 and 3 we considered

$$g_\epsilon(s) = \begin{cases} \dfrac{s^q}{(s+\epsilon)^{q+\beta}} \text{ for } s \geq 0 \\ 0 \text{ for } s < 0, \end{cases} \tag{A.3}$$

where $0 < q < 1$.

We say that a function $u_\epsilon \in H_0^1(\Omega)$ is a solution of problem (A.1) provided

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \int_\Omega \widetilde{f}(u_\epsilon)v \text{ for all } v \in H_0^1(\Omega). \tag{A.4}$$

Lemma A.1. *Let g_ϵ and $\widetilde{f}$ be given by (A.3) and*

$$\widetilde{f}(s) = \lambda s + s^p \textit{ for } \lambda > 0, 1 < p < 2^* - 1 \textit{ and } s \geq 0,$$

respectively. Suppose also that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain for $N \geq 3$. Let $u_\epsilon \in H_0^1(\Omega)$ be a nonnegative solution of problem (A.1) and assume that there exists a constant $D > 0$ independent of ϵ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} \leq D \textit{ for each } 0 < \epsilon < 1. \tag{A.5}$$

Then $u_\epsilon \in L^\infty(\Omega)$ and there exists a constant $K_1 > 0$ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq K_1 \textit{ for each } 0 < \epsilon < 1. \tag{A.6}$$

Proof of Lemma A.1. This result follows from a version of the Moser iteration technique, see [56]. Indeed, note that

$$\frac{\lambda s}{g_\epsilon(s)} = \frac{(s+\epsilon)^{q+\beta}}{s^q}(\lambda s) \leq (s+1)^{q+\beta}(\lambda s^{1-q}) \to 0 \text{ as } s \to 0.$$

Hence, the exists $0 < \delta_\lambda < 1$ that does not depend on ϵ such that

$$\frac{\lambda s}{g_\epsilon(s)} < 1 \text{ for } s \leq \delta_\lambda. \tag{A.7}$$

Consequently, from (A.4), we get

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_{\{u_\epsilon > \delta_\lambda\}} g_\epsilon(u_\epsilon) v \leq \lambda \int_{\{u_\epsilon > \delta_\lambda\}} u_\epsilon v + \int_\Omega u_\epsilon^p v \text{ for all } v \in H_0^1(\Omega).v \geq 0.$$

Since $g_\epsilon \geq 0$ and $p > 1$, we conclude that there exists a constant $C_{\delta,\lambda} > 0$ such that

$$\int_\Omega \nabla u_\epsilon \nabla v \leq C_{\delta,\lambda} \int_\Omega u_\epsilon^p v \text{ for all } v \in H_0^1(\Omega), v \geq 0. \tag{A.8}$$

For $L > 1$ we define,

$$u_{L,\epsilon}(x) = \begin{cases} u_\epsilon(x), & \text{if} \quad u_\epsilon(x) \leq L \\ \\ L, & \text{if} \quad u_\epsilon(x) \geq L, \end{cases}$$

$$z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)} u_\epsilon \quad \text{and} \quad w_{L,\epsilon} = u_\epsilon u_{L,\epsilon}^{\sigma-1},$$

with $\sigma > 1$ to be determined later. Note that $z_{L,\epsilon} \in H_0^1(\Omega)$, $z_{L,\epsilon} \geq 0$ and

$$\nabla z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)} \nabla u_\epsilon + 2(\sigma-1) u_\epsilon u_{L,\epsilon}^{2\sigma-3} \nabla u_{L,\epsilon}.$$

Taking $v = z_{L,\epsilon}$ in (A.8) we obtain

$$\int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 + 2(\sigma-1) \int_\Omega u_\epsilon u_{L,\epsilon}^{2\sigma-3} \nabla u_\epsilon \nabla u_{L,\epsilon} < C_{\lambda,\delta} \int_\Omega u_\epsilon^{p+1} u_{L,\epsilon}^{2(\sigma-1)}.$$

Since $\sigma > 1$ and

$$\int_\Omega u_\epsilon u_{L,\epsilon}^{2\sigma-3} \nabla u_\epsilon \nabla u_{L,\epsilon} = \int_{\{u_\epsilon < L\}} u_\epsilon^{2(\sigma-1)} |\nabla u_\epsilon|^2 \geq 0,$$

we conclude that

$$\int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 < C_{\lambda,\delta} \int_\Omega u_\epsilon^{p+1} u_{L,\epsilon}^{2(\sigma-1)} < C_{\lambda,\delta} \int_\Omega u_\epsilon^{p-1} u_\epsilon^{2\sigma}. \tag{A.9}$$

On the other hand, from the Sobolev embedding, we know that there is a constant $C_1 > 0$ such that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1} \, dx \right)^{\frac{2}{p+1}} \leq C_1 \int_\Omega |\nabla w_{L,\epsilon}|^2 \, dx.$$

Since

$$\nabla w_{L,\epsilon} = u_{L,\epsilon}^{\sigma-1} \nabla u_\epsilon + (\sigma-1) u_\epsilon u_{L,\epsilon}^{\sigma-2} \nabla u_{L,\epsilon},$$

it follows that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1} \, dx \right)^{\frac{2}{p+1}} \leq C_1 \int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 \, dx + C_1 (\sigma-1)^2 \int_\Omega u_\epsilon^2 u_{L,\epsilon}^{2(\sigma-2)} |\nabla u_{L,\epsilon}|^2 + 2C_1(\sigma-1) \int_\Omega u_\epsilon u_{L,\epsilon}^{2\sigma-3} \nabla u_\epsilon \nabla u_{L,\epsilon}.$$

From the definition of $u_{L,\epsilon}$, we conclude that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq C_1\sigma^2 \int_\Omega u_{L,\epsilon}^{2(\sigma-1)}|\nabla u_\epsilon|^2\,dx.$$

Using (A.9), we obtain

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq C_1\sigma^2 C_{\delta,\lambda}\int_\Omega u_\epsilon^{p-1}u_\epsilon^{2\sigma}. \tag{A.10}$$

Now observe that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,dx\right)^{\frac{2}{p+1}} = \left(\int_\Omega u_\epsilon^{p+1}u_{L,\epsilon}^{(p+1)(\sigma-1)}\,dx\right)^{\frac{2}{p+1}} \geq \left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,dx\right)^{\frac{2}{p+1}}.$$

Hence, there is a constant $\widetilde{C_{\delta,\lambda}} > 0$ such that

$$\left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,dx\right)^{\frac{2}{p+1}} \leq \sigma^2\widetilde{C_{\delta,\lambda}}\int_\Omega u_\epsilon^{p-1}u_\epsilon^{2\sigma}. \tag{A.11}$$

Let $\alpha_1, \alpha_2 > 1$ be constants such that $\dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} = 1$ and $p+1 < \alpha_1(p-1) < 2^*$. From (A.11) and Hölder's inequality it follows that

$$\left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,dx\right)^{\frac{2}{p+1}} \leq \sigma^2\widetilde{C_{\delta,\lambda}}\left(\int_\Omega u_\epsilon^{\alpha_1(p-1)}\,dx\right)^{\frac{1}{\alpha_1}}\left(\int_\Omega u_\epsilon^{2\sigma\alpha_2}\,dx\right)^{\frac{1}{\alpha_2}}.$$

Using (A.5) and the Sobolev Embedding, we obtain a constant $\widetilde{C} > 0$ such that

$$\int_\Omega u_\epsilon^{\alpha_1(p-1)}\,dx \leq \widetilde{C}.$$

Hence, there exists a constant $\widehat{C} > 0$ that does not depend on σ nor on ϵ such that

$$\left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,dx\right)^{\frac{2}{p+1}} \leq \widehat{C}\sigma^2\left(\int_\Omega u_\epsilon^{2\sigma\alpha_2}\,dx\right)^{\frac{1}{\alpha_2}}. \tag{A.12}$$

Letting $L \to \infty$ in (A.12) and using Fatou's Lemma, we conclude that

$$\left(\int_\Omega u_\epsilon^{\sigma(p+1)}\,dx\right)^{\frac{2}{p+1}} \leq \widehat{C}\sigma^2\left(\int_\Omega u_\epsilon^{2\sigma\alpha_2}\,dx\right)^{\frac{1}{\alpha_2}} \text{ for each } \sigma > 1,$$

provided $u_\epsilon \in L^{2\sigma\alpha_2}(\Omega)$. Equivalently,

$$\|u_\epsilon\|_{L^{\sigma(p+1)}} \leq C^{\frac{1}{\sigma}}\sigma^{\frac{1}{\sigma}}\|u_\epsilon\|_{L^{2\sigma\alpha_2}} \text{ for each } \sigma > 1, \tag{A.13}$$

where $C = \sqrt{\widehat{C}}$. Observe that the choices of α_1 and α_2 imply that $\sigma(p+1) > 2\sigma\alpha_2$. The result now follows from an iterative argument. Indeed, take

$$\sigma_1 = \frac{p+1}{2\alpha_2}.$$

Using the Sobolev embedding and (A.5) we obtain a constant $\widetilde{D} > 0$ such that

$$\|u_\epsilon\|_{L^{\sigma_1(p+1)}(\Omega)} \leq C^{\frac{1}{\sigma_1}}\sigma_1^{\frac{1}{\sigma_1}}\|u_\epsilon\|_{L^{p+1}(\Omega)} \leq \widetilde{D}C^{\frac{1}{\sigma_1}}\sigma_1^{\frac{1}{\sigma_1}}.$$

Now take $\sigma_2 = \sigma_1^2$ in (A.13). We get

$$\|u_\epsilon\|_{L^{\sigma_1^2(p+1)}(\Omega)} \leq C^{\frac{1}{\sigma_1^2}} \sigma_2^{\frac{1}{\sigma_2}} \|u_\epsilon\|_{L^{\sigma_1(p+1)}} \leq \widetilde{D}C^{\frac{1}{\sigma_1}+\frac{1}{\sigma_1^2}} \left(\sigma_1^{\frac{1}{\sigma_1}} \sigma_2^{\frac{1}{\sigma_2}}\right).$$

Taking $\sigma_k = \sigma_1^k$ in (A.13), we get

$$\|u_\epsilon\|_{L^{\sigma_1^k(p+1)}(\Omega)} \leq \widetilde{D}C^{\sum_{i=1}^k \frac{1}{\sigma_1^i}} \left(\Pi_{i=1}^k \sigma_i^{\frac{1}{\sigma_i}}\right). \tag{A.14}$$

It is clear that

$$\lim_{k\to\infty} \left(\Pi_{i=1}^k \sigma_i^{\frac{1}{\sigma_i}}\right) = \lim_{k\to\infty} \left(\Pi_{i=1}^k \sigma_1^{\frac{i}{\sigma_1^i}}\right) < \infty \text{ and } \lim_{k\to\infty} C^{\sum_{i=1}^k \frac{1}{\sigma_1^i}} < \infty.$$

Letting $k \to \infty$ in (A.14), it follows from Theorem B.10 that $u_\epsilon \in L^\infty(\Omega)$ and we obtain a constant $K_1 > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq K_1.$$

This proves (A.6). We have proved the Lemma. □

Now we state a similar result for the problem studied in Chapter 2. We have

Lemma A.2. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and let $u_\epsilon \in H_0^1(\Omega)$ be a nonnegative weak solution of problem (A.1) with $\widetilde{f}$ satisfying (A.2) and g_ϵ given by (A.3). Also, assume that for each $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that*

$$|\widetilde{f}(s)| \leq C_\alpha \exp\left(\alpha s^2\right) \text{ for every } s \geq 0, \tag{A.15}$$

and that there exists $0 < q_0 < 1$ such that

$$\lim_{s\to 0} \frac{|\widetilde{f}(s)|}{s^{q_0}} < \infty, \tag{A.16}$$

so that $\widetilde{f}(0) = 0$. Then, the following assertion holds: If there exists a constant $D > 0$ independent on ϵ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} \leq D \text{ for each } 0 < \epsilon < 1, \tag{A.17}$$

then $u_\epsilon \in L^\infty(\Omega)$ and there exists a constant $K_1 > 0$ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq K_1 \text{ for each } 0 < \epsilon < 1. \tag{A.18}$$

Proof of Lemma A.2. From (A.4), we know that

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \int_\Omega \widetilde{f}(u_\epsilon)v \text{ for all } v \in H_0^1(\Omega).$$

From (A.16) and from the fact that we may assume without loss of generality that $0 < q < q_0$, we get

$$\frac{|\widetilde{f}(s)|}{g_\epsilon(s)} \leq \left(\frac{(s+\epsilon)^{q+\beta}}{s^q}\right)(|\widetilde{f}(s)|) \leq (s+1)^{q+\beta}\left(\frac{|\widetilde{f}(s)|}{s^{q_0}} s^{q_0-q}\right) \to 0 \text{ as } s \to 0.$$

Hence, the exists $\delta > 0$ that does not depend on ϵ such that

$$\frac{|\widetilde{f}(s)|}{g_\epsilon(s)} < \frac{1}{2} \text{ for } s \le \delta.$$

Consequently,

$$\int_\Omega \nabla u_\epsilon \nabla v + \frac{1}{2}\int_\Omega g_\epsilon(u_\epsilon)v < \int_{\Omega\cap\{u\ge\delta\}} \widetilde{f}(u_\epsilon)v \text{ for all } v \in H_0^1(\Omega), v \ge 0.$$

Let $\alpha > 0$ be such that $2\alpha D^2 < 4\pi$. Using (A.15) we obtain a constant $C_1 = C_\delta > 0$ such that

$$\int_\Omega \nabla u_\epsilon \nabla v < C_1 \int_\Omega u_\epsilon \exp{(\alpha u_\epsilon^2)} v \text{ for all } v \in H_0^1(\Omega), v \ge 0. \tag{A.19}$$

For $L > 1$ we define, as before

$$u_{L,\epsilon}(x) = \begin{cases} u_\epsilon(x), & \text{if} \quad u_\epsilon(x) \le L \\ \\ L, & \text{if} \quad u_\epsilon(x) \ge L, \end{cases}$$

$$z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)} u_\epsilon \quad \text{and} \quad w_{L,\epsilon} = u_\epsilon u_{L,\epsilon}^{\sigma-1}$$

with $\sigma > 1$ to be determined later. In the course of the present proof, $C_1, C_2, C_3, \ldots$ denote various positive constants independent on ϵ.

Choosing $\varphi = z_{L,\epsilon}$ in (A.19) we have

$$\int_\Omega \nabla u_\epsilon \nabla z_{L,\epsilon} \le C_1 \int_\Omega u_\epsilon \exp{(\alpha u_\epsilon^2)} z_{L,\epsilon}. \tag{A.20}$$

We now estimate the left-hand side of equation (A.20). Note that

$$\nabla z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)} \nabla u_\epsilon + 2(\sigma-1) u_{L,\epsilon}^{2\sigma-3} u_\epsilon \nabla u_{L,\epsilon}.$$

Hence

$$\int_\Omega \nabla u_\epsilon \nabla z_{L,\epsilon} = \int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 \, dx + 2(\sigma-1)\int_\Omega u_{L,\epsilon}^{2\sigma-3} u_\epsilon \nabla u_{L,\epsilon} \nabla u_\epsilon.$$

Since $\nabla u_{L,\epsilon} = 0$ on $\{u_\epsilon > L\}$ we obtain

$$\int_\Omega u_{L,\epsilon}^{2\sigma-3} u_\epsilon \nabla u_{L,\epsilon} \nabla u_\epsilon = \int_{\{u_\epsilon \le L\}} u_\epsilon^{2\sigma-2} |\nabla u_\epsilon|^2 \, dx \ge 0.$$

We conclude that

$$\int_\Omega \nabla u_\epsilon \nabla z_{L,\epsilon} \ge \int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2.$$

Substituting in (A.20) we obtain

$$\int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 \le C_1 \int_\Omega u_\epsilon \exp{(\alpha u_\epsilon^2)} z_{L,\epsilon} = C_1 \int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)}. \tag{A.21}$$

Now we note that

$$\nabla w_{L,\epsilon} = u_{L,\epsilon}^{\sigma-1} \left((\sigma-1)\nabla u_{L,\epsilon} + \nabla u_\epsilon\right).$$

Hence,

$$|\nabla w_{L,\epsilon}| \leq \sigma u_{L,\epsilon}^{\sigma-1}|\nabla u_\epsilon|.$$

Substituting in (A.21) we conclude that

$$\frac{1}{\sigma^2}\int_\Omega |\nabla w_{L,\epsilon}|^2 \leq C_1 \int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)}.$$

Let $r > 1$ to be fixed later. Using the Sobolev embedding $L^r(\Omega) \hookrightarrow H_0^1(\Omega)$, there is a constant $C_2 > 0$ such that

$$\left(\int_\Omega |w_{L,\epsilon}|^r\right)^{\frac{2}{r}} \leq C_2\sigma^2 \int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)}. \tag{A.22}$$

Note that

$$\int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)} = \int_{\{u_\epsilon \leq L\}} u_\epsilon^{2\sigma} \exp{(\alpha u_\epsilon^2)} + \int_{\{u_\epsilon \geq L\}} (L^{\sigma-1}u_\epsilon)^2 \exp{(\alpha u_\epsilon^2)}.$$

Hence,

$$\int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)} \leq \int_\Omega u_\epsilon^{2\sigma} \exp{(\alpha u_\epsilon^2)}.$$

Replacing in (A.22) we get

$$\left(\int_\Omega |w_{L,\epsilon}|^r\right)^{\frac{2}{r}} \leq C_2\sigma^2 \int_\Omega u_\epsilon^{2\sigma} \exp{(\alpha u_\epsilon^2)}.$$

From Hölder's inequality we obtain

$$\left(\int_\Omega |w_{L,\epsilon}|^r\right)^{\frac{2}{r}} \leq C_2\sigma^2 \left(\int_\Omega \exp{(2\alpha u_\epsilon^2)}\right)^{\frac{1}{2}} \left(\int_\Omega u_\epsilon^{4\sigma}\right)^{\frac{1}{2}}.$$

From the choice of α, we conclude from (B.8) that

$$\left(\int_\Omega |w_{L,\epsilon}|^r\right)^{\frac{2}{r}} \leq C_3\sigma^2 \left(\int_\Omega u_\epsilon^{4\sigma}\right)^{\frac{1}{2}}.$$

Since $u_{L,\epsilon}^\sigma \leq w_{L,\epsilon}$, we have

$$\left(\int_\Omega u_{L,\epsilon}^{r\sigma}\right)^{\frac{2}{r}} \leq \left(\int_\Omega |w_{L,\epsilon}|^r\right)^{\frac{2}{r}} \leq C_3\sigma^2 \left(\int_\Omega u_\epsilon^{4\sigma}\right)^{\frac{1}{2}}.$$

Letting $L \to \infty$ it follows from Fatou's Lemma that

$$\left(\int_\Omega u_\epsilon^{r\sigma}\right)^{\frac{2}{r}} \leq C_3\sigma^2 \left(\int_\Omega u_\epsilon^{4\sigma}\right)^{\frac{1}{2}}.$$

This means that

$$\|u_\epsilon\|_{L^{r\sigma}(\Omega)}^{2\sigma} \leq C_3\sigma^2 \|u_\epsilon\|_{L^{4\sigma}(\Omega)}^{2\sigma}.$$

Taking $r = 8$ in the equation above, we get

$$\|u_\epsilon\|_{L^{8\sigma}(\Omega)} \leq C_3^{\frac{1}{2\sigma}} \sigma^{\frac{1}{\sigma}} \|u_\epsilon\|_{L^{4\sigma}(\Omega)} \text{ for all } \sigma > 1.$$

The result then follows by the iteration argument given in the proof of Lemma A.1. □

We now prove the gradient estimate used in Chapters 2 and 3.

Lemma A.3. *Suppose that g_ϵ is given by (A.3) and assume that $\widetilde{f}$ satisfies (A.2). Let $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geq 2$. For each $0 < \epsilon < 1$, let $u_\epsilon \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a nonnegative solution of problem (A.1) and assume that there exist constants $K_1 > 0$ and $0 < q_1 < 1$ such that*

$$\|u_\epsilon\|_{L^\infty(\Omega)} < K_1 \text{ for every } 0 < \epsilon < 1, \tag{A.23}$$

and

$$\lim_{s \to 0^+} s^{1-q_1} |\widetilde{f}'(s)| < \infty. \tag{A.24}$$

Let ψ be such that (an example of one such ψ is $\psi = \phi_1^2$ where ϕ_1 is the first normalized eigenfunction of $-\Delta$)

$$\psi \in C^2(\overline{\Omega}),\ \ \psi > 0 \text{ in } \Omega,\ \ \psi = 0 \text{ on } \partial\Omega \text{ and } \frac{|\nabla \psi|^2}{\psi} \text{ is bounded in } \Omega.$$

Then there exist constants $M > 0$ and $\epsilon_0 > 0$ such that

$$\psi(x)|\nabla u_\epsilon(x)|^2 \leq M(u_\epsilon(x)^{1-\beta} + u_\epsilon(x)) \text{ for every } x \in \Omega, \quad 0 < \epsilon < \epsilon_0.$$

Proof of Lemma A.3. From (A.24) we obtain constants $C_1 > 0$ and $0 < t_0 < 1$ such that

$$|\widetilde{f}'(s)| \leq C_1 s^{q_1 - 1} \text{ for } 0 \leq s \leq t_0. \tag{A.25}$$

From (A.23) we obtain that Δu_ϵ is bounded in $L^\infty(\Omega)$. Thus, by standard elliptic regularity, u_ϵ belongs to $C^{1,\nu}(\overline{\Omega})$. We define

$$\overline{h}_\epsilon(u) = g_\epsilon(u) - \widetilde{f}(u).$$

We shall denote u_ϵ simply by u. Define the functions

$$Z_a(u) = u^{1-\beta} + u + a, \qquad w = \frac{|\nabla u|^2}{Z_a(u)}, \qquad v = w\psi,$$

where $a > 0$ is small. We will argue by contradiction, thus we assume that

$$\sup_{\Omega} v > \widetilde{M}, \tag{A.26}$$

where $\widetilde{M} > 0$ will be chosen later independent of $0 < \epsilon < 1$ and $a > 0$.

The function v is continuous in $\overline{\Omega}$, hence it attains its maximum at some point $x_0 \in \overline{\Omega}$. Thus, by (A.26) we obtain

$$v(x_0) > \widetilde{M}.$$

Then $x_0 \in \Omega$, because $v = 0$ on $\partial\Omega$. Hence

$$\nabla v(x_0) = 0$$

and

$$\Delta v(x_0) \leq 0. \tag{A.27}$$

We will compute Δv and evaluate it at the point x_0. As we shall see this leads to the absurd $\Delta v(x_0) > 0$ if one fixes $\widetilde{M}$ large enough.

Note that $u(x_0) > 0$, since otherwise x_0 would be a critical point of u and $w(x_0) = 0$. By continuity, there must exist an open ball $B \subset \Omega$ centered at x_0 such that $u > 0$ in B. Since u is positive in B, we know that $\overline{h}_\epsilon(u) \in C^{1,\nu}(B)$. Since u satisfies the equation $-\Delta u + \overline{h}_\epsilon(u) = 0$ in B, we conclude that $u \in C^3(B)$.

The computations already carried out in [52] and [54] lead to the following expression evaluated at x_0

$$\begin{aligned} \Delta v \geq \frac{1}{Z_a(u)} \Big[& \psi w^2 \Big(\frac{1}{2} Z_a'(u)^2 - Z_a(u) Z_a''(u) \Big) \\ & + w \Big(2\psi Z_a(u) \overline{h}_\epsilon'(u) - \psi \overline{h}_\epsilon(u) Z_a'(u) - K_0 Z_a(u) \Big) \\ & - K_0 Z_a'(u) Z_a(u)^{1/2} \psi^{1/2} w^{3/2} \Big], \end{aligned} \tag{A.28}$$

where

$$K_0 = \max \left(\sup_\Omega \left(\frac{|\nabla \psi|}{\psi^{1/2}} \right), \sup_\Omega \left(\Delta \psi - 2 \frac{|\nabla \psi|^2}{\psi} \right) \right) > 0.$$

We will show that if $v(x_0)$ is large enough then the right hand side of (A.28) must be positive, which would contradict (A.27).

For this purpose we need to establish the following estimates uniformly for every ϵ sufficiently small.

$$Z_a'(u) Z_a(u)^{1/2} \leq C(\frac{1}{2} Z_a'(u)^2 - Z_a''(u) Z_a(u)), \tag{A.29}$$

$$Z_a(u) |\overline{h}_\epsilon'(u)| \leq C(\frac{1}{2} Z_a'(u)^2 - Z_a''(u) Z_a(u)), \tag{A.30}$$

$$Z_a'(u) |\overline{h}_\epsilon(u)| \leq C(\frac{1}{2} Z_a'(u)^2 - Z_a''(u) Z_a(u)), \tag{A.31}$$

$$Z_a(u) \leq C(\frac{1}{2} Z_a'(u)^2 - Z_a''(u) Z_a(u)), \tag{A.32}$$

for every $0 \leq u \leq K_1$. The constant C depends only on K_1, but not on ϵ nor on a.

Assuming for a moment that (A.29)–(A.32) are true. Inequality (A.28) implies that

$$\begin{aligned} \Delta v &\geq \frac{\frac{1}{2} Z_a'(u)^2 - Z_a''(u) Z_a(u)}{Z_a(u)} \Big(\psi w^2 - C(w + \psi^{1/2} w^{3/2}) \Big) \\ &= \frac{\frac{1}{2} Z_a'(u)^2 - Z_a''(u) Z_a(u)}{Z_a(u) \psi} \Big(v^2 - C(v + v^{3/2}) \Big). \end{aligned}$$

Thus if $v(x_0) = \sup v > \widetilde{M}$ for some large enough $\widetilde{M}$ independent on $0 < \epsilon < 1$ we obtain a contradiction with (A.27).

We prove now the relations (A.29)–(A.32). Note that

$$Z_a(u) = u^{1-\beta} + u + a,$$

$$Z_a'(u) = (1-\beta)u^{-\beta} + 1, \qquad Z_a''(u) = -\beta(1-\beta)u^{-\beta-1}.$$

Hence,

$$\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u) \geq \frac{(1-\beta)^2}{2}\left(u^{-2\beta}+1\right) + a\beta(1-\beta)u^{-1-\beta}. \text{ for } u>0. \tag{A.33}$$

We first prove (A.32). Indeed, there is a constant $C>0$ such that

$$Z_a(u) = u^{1-\beta} + u + a \leq C \text{ for } 0 \leq u \leq K_1.$$

Hence, (A.32) follows from (A.33).

We now prove (A.31). Note that there exists a constant $\widetilde{C}>0$ such that

$$\begin{aligned} Z_a'(u)|\overline{h}_\epsilon(u)| &\leq ((1-\beta)u^{-\beta}+1)(g_\epsilon(u) + |\widetilde{f}(u)|) \\ &\leq (1-\beta)u^{-2\beta} + (1-\beta)u^{-\beta}\sup_{0\leq s\leq K_1}|\widetilde{f}(s)| + u^{-\beta} + \sup_{0\leq s\leq K_1}|\widetilde{f}(s)| \\ &\leq \widetilde{C}(1+u^{-2\beta}). \end{aligned}$$

Inequality (A.31) then follows from (A.33).

Now we prove (A.30). Note that

$$\overline{h}_\epsilon'(u) = \frac{u^{q-1}}{(u+\epsilon)^{q+\beta+1}}(q\epsilon - \beta u) - \widetilde{f}'(u).$$

We split the proof of (A.30) in three cases.

Case I. Suppose that $0<u<\min\{\frac{q\epsilon}{2\beta}, t_0\}$, where $0<t_0<1$ is given by (A.25). We define

$$\omega_\epsilon(u) = \frac{u^{q-1}}{(u+\epsilon)^{q+\beta+1}}(q\epsilon-\beta u) - C_1u^{q_1-1},$$

where $C_1>0$ is given by (A.25). We claim that there exists $\epsilon_0>0$ such that $\omega_\epsilon(u)>0$ for each $0<\epsilon<\epsilon_0$. Indeed, assume by contradiction that $\omega_\epsilon(u)<0$ for some $0<u<\frac{q\epsilon}{2\beta}$. We then have

$$q\epsilon u^{q-1} < \beta u^q + C_1u^{q_1-1}(u+\epsilon)^{q+\beta+1} < \beta u^q + C_1u^{q_1-1}\epsilon^{q+\beta+1}\left(1+\frac{q}{2\beta}\right)^{q+\beta+1}.$$

Now take $\epsilon_0>0$ such that

$$C_1\epsilon^{q+\beta+1}\left(1+\frac{q}{2\beta}\right)^{q+\beta+1} < \frac{\epsilon q}{2} \text{ for } 0<\epsilon<\epsilon_0.$$

We then get

$$q\epsilon u^{q-1} < \beta u^q + \frac{\epsilon q u^{q_1-1}}{2} < \beta u^q + \frac{\epsilon q u^{q-1}}{2}.$$

Hence,

$$\frac{q\epsilon u^{q-1}}{2} < \beta u^q,$$

which implies that

$$u > \frac{q\epsilon}{2\beta}.$$

This contradicts our initial assumption. The claim is proven. Since

$$\frac{q\epsilon u^{q-1}}{(u+\epsilon)^{q+\beta+1}} \leq q\frac{u^q}{u(u+\epsilon)^q}\frac{\epsilon}{(u+\epsilon)^{\beta+1}} \leq \frac{q}{u^{\beta+1}},$$

we obtain

$$|\overline{h}'_\epsilon(u)| = \overline{h}'_\epsilon(u) \leq \frac{q + C_1 u^{q_1+\beta}}{u^{\beta+1}} \text{ for } 0 < u < \min\{\frac{q\epsilon}{2\beta}, t_0\}.$$

Hence,

$$|\overline{h}'_\epsilon(u)| \leq \frac{2q}{u^{\beta+1}} \text{ for } 0 < u < \min\{\frac{q\epsilon}{2\beta}, t_0, t_1\},$$

where $t_1 > 0$ is chosen such that

$$C_1 u^{q_1+\beta} < q \text{ for } 0 \leq u \leq t_1.$$

Therefore,

$$Z_a(u)|\overline{h}'_\epsilon(u)| \leq (u^{1-\beta} + u + a)\left(\frac{2q}{u^{\beta+1}}\right) \leq \frac{2q}{u^{2\beta}} + \frac{2qa}{u^{\beta+1}} \text{ for } 0 \leq u \leq \min\{\frac{q\epsilon}{2\beta}, t_0, t_1\}.$$

Comparing with (A.33), it follows that there exists a constant $C > 0$ that does not depend on a such that

$$Z_a(u)|\overline{h}'_\epsilon(u)| \leq C\left(\frac{1}{2}Z'_a(u)^2 - Z''_a(u)Z_a(u)\right) \text{ for } 0 < u < \min\{\frac{q\epsilon}{2\beta}, t_0, t_1\}, \quad 0 < \epsilon < \epsilon_0. \tag{A.34}$$

Case II. Suppose that $\frac{\epsilon q}{2\beta} \leq u \leq \min\{t_0, t_1\}$. We have

$$|\overline{h}'_\epsilon(u)| \leq \frac{u^{q-1}|q\epsilon - \beta u|}{(u+\epsilon)^{q+\beta+1}} + C_1 u^{q_1-1} \text{ for } \frac{q\epsilon}{2\beta} \leq u \leq t_0.$$

Note that $|q\epsilon - \beta u| \leq \beta u$ if $2\beta u \geq q\epsilon$. We then obtain

$$|\overline{h}'_\epsilon(u)| \leq \frac{\beta u^q + C_1 u^{q+q_1+\beta}\left(1 + \frac{2\beta}{q}\right)^{q+\beta+1}}{(u+\epsilon)^{q+\beta+1}} \text{ for } \frac{q\epsilon}{2\beta} \leq u \leq t_0.$$

Now, observe that there exists $0 < t_2 < \min\{t_0, t_1\}$ that does not depend on ϵ such that

$$C_1 u^{q_1+\beta}\left(1 + \frac{2\beta}{q}\right)^{q+\beta+1} < \beta \text{ for } 0 \leq u \leq t_2.$$

Therefore,

$$|\overline{h}'_\epsilon(u)| \leq \frac{2\beta}{u^{\beta+1}} \text{ for } \frac{q\epsilon}{2\beta} \leq u < t_2.$$

Comparing with (A.33) we obtain

$$Z_a(u)|\overline{h}_\epsilon'(u)| \leq C\left(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)\right) \text{ for } \frac{q\epsilon}{2\beta} \leq u < t_2. \tag{A.35}$$

Case III. Assume that $t_2 \leq u \leq K_1$. Since there exists a constant $C > 0$ such that $|\overline{h}_\epsilon'(u)| \leq C$ for $t_2 \leq u \leq K_1$, it follows from (A.32) that

$$Z_a(u)|\overline{h}_\epsilon'(u)| \leq C\left(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)\right) \text{ for } t_2 \leq u \leq K_1. \tag{A.36}$$

Hence, (A.30) follows from (A.34), (A.35) and (A.36).

We now prove (A.29). Observe that

$$Z_a'(u)Z_a(u)^{1/2} = ((1-\beta)u^{-\beta} + 1)\sqrt{u^{1-\beta} + u + a}.$$

Hence

$$Z_a'(u)Z_a(u)^{1/2} \leq \sqrt{3K_1}((1-\beta)u^{-\beta} + 1).$$

When $0 \leq u \leq 1$ we know that $u^2 \leq u$. Hence $u^{-\beta} \leq u^{-2\beta}$. Therefore, from (A.33), there exist constants $C_1 > 0$ and $C_2 > C_1$ such that

$$Z_a'(u)Z_a(u)^{1/2} \leq C_1(u^{-2\beta} + 1) \leq C_2\left(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)\right) \text{ for } 0 \leq u \leq 1. \tag{A.37}$$

If $1 \leq u \leq K_1$, we know that there exists a constant $C_3 > 0$ such that $Z'(u)Z(u)^{1/2} \leq C_3$. Hence, from (A.33), there exists a constant $C_4 > 0$ such that

$$Z_a'(u)Z_a(u)^{1/2} \leq C_4\left(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)\right) \text{ for } 1 \leq u \leq K_1. \tag{A.38}$$

Inequality (A.29) then follows from (A.37) and (A.38). We proved that (A.26) is false. Hence, there exists $M > 0$ that does not depend on ϵ nor on a such that

$$v(x) = \frac{|\nabla u(x)|^2\psi(x)}{u(x)^{1-\beta} + u(x) + a} \leq M \text{ for all } x \in \Omega.$$

The result follows by letting $a \to 0$. We have proved Lemma A.3. □

We now obtain regularity results for the perturbed problem studied in Chapters 4, 5 and 6. There, we studied problem (A.1) with

$$g_\epsilon(s) = \begin{cases} -\log\left(s + \dfrac{\epsilon e^{-\widetilde{f}(0)}}{s+\epsilon}\right) \text{ for } s \geq 0 \\ \widetilde{f}(0) \text{ for } s < 0. \end{cases} \tag{A.39}$$

We now assume that the function $\widetilde{f}$ satisfies (A.2) and (A.15) for some $\alpha > 0$. Furthermore, we will assume that there exist constants $0 < \epsilon_0, \delta < 1$ such that

$$g_\epsilon(s) \geq \widetilde{f}(s) \text{ for } 0 < \epsilon < \epsilon_0 \text{ and } s < \delta. \tag{A.40}$$

We recall that item (vi) of Lemma 5.2 states

$$\lim_{s\to 0^+} g'_\epsilon(s) = \frac{1}{\epsilon} - 1.$$

Consequently, condition (A.40) is satisfied if

$$\lim_{s\to 0} |\widetilde{f}'(s)| < \infty.$$

We have

Lemma A.4. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and let $u_\epsilon \in H_0^1(\Omega)$ be a nonnegative weak solution of problem (A.1) with g_ϵ given by (A.39). Assume that $\widetilde{f}$ satisfies (A.2), (A.40), and that (A.15) holds for some $\alpha > 0$.*

Then, the following assertion holds: If

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} \le \frac{7\pi}{2\alpha} \text{ for each } 0 < \epsilon < \epsilon_0, \tag{A.41}$$

then $u_\epsilon \in L^\infty(\Omega)$ and there exists a constant $K_1 > 0$ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \le K_1 \text{ for each } 0 < \epsilon < \epsilon_0. \tag{A.42}$$

Proof of Lemma A.4. We know that u_ϵ is a weak solution of problem (A.1). Hence, from (A.4) we get

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v = \int_\Omega \widetilde{f}(u_\epsilon)v \text{ for all } v \in H_0^1(\Omega), v \ge 0.$$

Let $\delta > 0$ be given by (A.40). We have

$$\int_\Omega \nabla u_\epsilon \nabla v = \int_\Omega \left(\widetilde{f}(u_\epsilon) - g_\epsilon(u_\epsilon)\chi_{\{u_\epsilon<\delta\}}\right) v - \int_{\{u_\epsilon\ge\delta\}} g_\epsilon(u_\epsilon)v.$$

Using (A.40) and that there exists $C_\delta > 0$ such that $|g_\epsilon(s)| \le C_\delta s$ for $s \ge \delta$, we get

$$\int_\Omega \nabla u_\epsilon \nabla v \le \int_{\{u_\epsilon\ge\delta\}} \widetilde{f}(u_\epsilon)v + C_\delta \int_{\{u_\epsilon\ge\delta\}} u_\epsilon v.$$

Hence,

$$\int_\Omega \nabla u_\epsilon \nabla \varphi \le \frac{1}{\delta}\int_{\{u_\epsilon\ge\delta\}} u_\epsilon|f(u_\epsilon)|v + C_\delta \int_\Omega u_\epsilon v.$$

From (A.15), it follows that there exists $\alpha > 0$ and $C_\alpha > 0$ such that

$$\int_\Omega \nabla u_\epsilon \nabla v \le \frac{C}{\delta}\int_{\{u_\epsilon\ge\delta\}} u_\epsilon \exp(\alpha u_\epsilon^2)v + C_\delta \int_\Omega u_\epsilon v.$$

Since $\exp s \ge 1$ for $s \ge 0$, we have

$$\int_\Omega \nabla u_\epsilon \nabla v \le \frac{C}{\delta}\int_{\{u_\epsilon\ge\delta\}} u_\epsilon \exp(\alpha u_\epsilon^2)v + C_\delta \int_\Omega u_\epsilon \exp(\alpha u_\epsilon^2)v.$$

Hence, there is a constant $C > 0$ that does not depend on ϵ such that

$$\int_\Omega \nabla u_\epsilon \nabla v \leq C \int_\Omega u_\epsilon \exp{(\alpha u_\epsilon^2)} v \text{ for every } v \in H_0^1(\Omega), v \geq 0. \tag{A.43}$$

For $L > 1$ we define,

$$u_{L,\epsilon}(x) := \begin{cases} u_\epsilon(x) & \text{if} \quad u_\epsilon(x) \leq L \\ \\ L & \text{if} \quad u_\epsilon(x) \geq L, \end{cases}$$

$$z_{L,\epsilon} := u_{L,\epsilon}^{2(\sigma-1)} u_\epsilon \quad \text{and} \quad w_{L,\epsilon} := u_\epsilon u_{L,\epsilon}^{\sigma-1}.$$

with $\sigma > 1$ to be determined later. In the course of the present proof, C, C_q, $\widetilde{C}$ and $\widetilde{C_q}$ denote various positive constants which are independent of ϵ.

Choosing $v = z_{L,\epsilon}$ in (A.43) we have

$$\int_\Omega \nabla u_\epsilon \nabla z_{L,\epsilon} \leq C \int_\Omega u_\epsilon \exp{(\alpha u_\epsilon^2)} z_{L,\epsilon}. \tag{A.44}$$

We now estimate the left-hand side of equation (A.44)). Note that

$$\nabla z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)} \nabla u_\epsilon + 2(\sigma-1) u_{L,\epsilon}^{2\sigma-3} u_\epsilon \nabla u_{L,\epsilon}.$$

Hence

$$\int_\Omega \nabla u_\epsilon \nabla z_{L,\epsilon} = \int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 \, dx + 2(\sigma-1) \int_\Omega u_{L,\epsilon}^{2\sigma-3} u_\epsilon \nabla u_{L,\epsilon} \nabla u_\epsilon.$$

Since $\nabla u_{L,\epsilon} = 0$ on $\{u_\epsilon > L\}$ we obtain

$$\int_\Omega u_{L,\epsilon}^{2\sigma-3} u_\epsilon \nabla u_{L,\epsilon} \nabla u_\epsilon = \int_{\{u_\epsilon \leq L\}} u_\epsilon^{2\sigma-2} |\nabla u_\epsilon|^2 \, dx \geq 0.$$

We conclude that

$$\int_\Omega \nabla u_\epsilon \nabla z_{L,\epsilon} \geq \int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2.$$

Substituting in (A.44) we obtain

$$\int_\Omega u_{L,\epsilon}^{2(\sigma-1)} |\nabla u_\epsilon|^2 \leq C \int_\Omega u_\epsilon \exp{(\alpha u_\epsilon^2)} z_{L,\epsilon} = C \int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)}. \tag{A.45}$$

Now we note that

$$\nabla w_{L,\epsilon} = u_{L,\epsilon}^{\sigma-1} \left((\sigma-1) \nabla u_{L,\epsilon} + \nabla u_\epsilon\right).$$

Hence,

$$|\nabla w_{L,\epsilon}| \leq \sigma u_{L,\epsilon}^{\sigma-1} |\nabla u_\epsilon|.$$

Substituting in (A.45) we conclude that

$$\frac{1}{\sigma^2} \int_\Omega |\nabla w_{L,\epsilon}|^2 \leq C \int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)}.$$

Let $q > 1$ to be fixed later. Using the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, there is a constant $C_q > 0$ such that

$$\left(\int_\Omega |w_{L,\epsilon}|^q\right)^{\frac{2}{q}} \leq C_q\sigma^2 \int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)}. \tag{A.46}$$

By construction $u_{L,\epsilon} \leq u_\epsilon$, then

$$\int_\Omega u_\epsilon^2 \exp{(\alpha u_\epsilon^2)} u_{L,\epsilon}^{2(\sigma-1)} \leq \int_\Omega u_\epsilon^{2\sigma} \exp{(\alpha u_\epsilon^2)}.$$

Replacing in (A.46) we get

$$\left(\int_\Omega |w_{L,\epsilon}|^q\right)^{\frac{2}{q}} \leq C_q\sigma^2 \int_\Omega u_\epsilon^{2\sigma} \exp{(\alpha u_\epsilon^2)}. \tag{A.47}$$

From Hölder's inequality we obtain

$$\left(\int_\Omega |w_{L,\epsilon}|^q\right)^{\frac{2}{q}} \leq C_q\sigma^2 \left(\int_\Omega \exp{(r\alpha u_\epsilon^2)}\right)^{\frac{1}{r}} \left(\int_\Omega u_\epsilon^{2r'\sigma}\right)^{\frac{1}{r'}},$$

where $r > 1$ is chosen such that $r\alpha\|u_\epsilon\|^2 < 4\pi$ and $1/r + 1/r' = 1$. We conclude from the Trudinger-Moser inequality that

$$\left(\int_\Omega |w_{L,\epsilon}|^q\right)^{\frac{2}{q}} \leq \widetilde{C}_q\sigma^2 \left(\int_\Omega u_\epsilon^{2r'\sigma}\right)^{\frac{1}{r'}}.$$

Since $u_{L,\epsilon}^\sigma \leq w_{L,\epsilon}$, we have

$$\left(\int_\Omega u_{L,\epsilon}^{q\sigma}\right)^{\frac{2}{q}} \leq \left(\int_\Omega |w_{L,\epsilon}|^q\right)^{\frac{2}{q}} \leq \widetilde{C}_q\sigma^2 \left(\int_\Omega u_\epsilon^{2r'\sigma}\right)^{\frac{1}{r'}}.$$

Letting $L \to \infty$ it follows from Fatou's Lemma that

$$\left(\int_\Omega u_\epsilon^{q\sigma}\right)^{\frac{2}{q}} \leq \widetilde{C}_q\sigma^2 \left(\int_\Omega u_\epsilon^{2r'\sigma}\right)^{\frac{1}{r'}}.$$

This means that

$$\|u_\epsilon\|_{L^{q\sigma}(\Omega)}^{2\sigma} \leq \widetilde{C}_q\sigma^2 \|u_\epsilon\|_{L^{2r'\sigma}(\Omega)}^{2\sigma}.$$

Taking $q = 4r'$ in the equation above, we get a constant $\widetilde{C} > 0$ such that

$$\|u_\epsilon\|_{L^{4r'\sigma}(\Omega)} \leq \widetilde{C}^{\frac{1}{2\sigma}} \sigma^{\frac{1}{\sigma}} \|u_\epsilon\|_{L^{2r'\sigma}(\Omega)} \text{ for all } \sigma > 1. \tag{A.48}$$

The result follows by considering a suitable sequence of values of σ in the above inequality and iterating. Indeed, we first choose $\sigma_1 = 2$. Using the Sobolev embedding we obtain

$$\|u_\epsilon\|_{L^{8r'}(\Omega)} \leq \widetilde{C}^{\frac{1}{4}} 2^{\frac{1}{2}} \|u_\epsilon\|_{L^{4r'}(\Omega)} \leq \sqrt{\frac{7\pi}{2\alpha}} C \widetilde{C}^{\frac{1}{4}} 2^{\frac{1}{2}},$$

for some constant $C > 0$. Now take $\sigma_2 = 4$ in (A.48). We get

$$\|u_\epsilon\|_{L^{16r'}(\Omega)} \leq \widetilde{C}^{\frac{1}{8}} 4^{\frac{1}{4}} \|u_\epsilon\|_{L^{8r'}(\Omega)} \leq \sqrt{\frac{7\pi}{2\alpha}} C \widetilde{C}^{\frac{1}{8}+\frac{1}{4}} (4^{\frac{1}{4}} 2^{\frac{1}{2}}).$$

Taking $\sigma_k = 2^k$ in (A.48) we obtain,

$$\|u_\epsilon\|_{L^{2^{k+2}r'}(\Omega)} \leq \sqrt{\frac{7\pi}{2\alpha}} C\widetilde{C}^{\sum_{i=1}^{k}\frac{1}{2^{i+1}}} \left(\Pi_{i=1}^{k}(2^i)^{\frac{1}{2^i}}\right). \tag{A.49}$$

It is clear that $\Pi_{i=1}^{\infty}(2^i)^{\frac{1}{2^i}} = 4$, which is a consequence of $\displaystyle\sum_{i=1}^{\infty}\frac{i}{2^i} = 2$. Moreover, $\displaystyle\sum_{i=1}^{\infty}\frac{1}{2^{i+1}} = \frac{1}{2}$. Thus letting $k \to \infty$ in (A.49) and using Theorem B.10, it follows that $u_\epsilon \in L^\infty(\Omega)$ and we obtain a constant $K_1 > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq K_1.$$

This proves the result. □

Lemma A.4 possesses the following variants.

Corollary A.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and let $u_\epsilon \in H_0^1(\Omega)$ be a nonnegative weak solution of problem (A.1) with g_ϵ given by (A.39). Assume that $\widetilde{f}$ satisfies (A.2) and (A.40). We have:*

(i) Instead of (A.15), assume that f satisfies the following condition: There exists $C_1, C_2 > 0$ and $\zeta > 0$ such that

$$|f(s)| \leq C_1 s^\zeta \exp(\alpha s^2) + C_2 \textit{ for all } s \geq 0. \tag{A.50}$$

Further assume that

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} < \frac{7\pi}{2\alpha} \textit{ for all } 0 < \epsilon < \epsilon_0$$

Then, $u_\epsilon \in L^\infty(\Omega)$ and inequality (A.42) holds.

(ii) Assume that f satisfies condition (A.15) for all $\alpha > 0$. If there exists $D > 0$ such that

$$\|u_\epsilon\|^2_{H_0^1(\Omega)} < D \textit{ for all } 0 < \epsilon < \epsilon_0,$$

then (A.42) holds.

Proof of Corollary A.1 . We prove (i). Indeed, condition (A.50) implies that for each $r > 1$ there exists $C_r > 0$ such that

$$|f(s)| \leq C_r \exp(\alpha r s^2) \text{ for all } s \geq 0$$

We then choose $r_1, r_2 > 1$ such that

$$r_1 r_2 \alpha \|u_\epsilon\|^2_{H_0^1(\Omega)} < 4\pi \text{ for all } 0 < \epsilon < \epsilon_0,$$

and mimic the proof of Lemma A.4.

Item (ii) follows by taking $\alpha > 0$ such that

$$D < \frac{7\pi}{2\alpha},$$

and applying Lemma A.4. □

We now prove the gradient estimate used in Chapters 4, 5 and 6.

Lemma A.5. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and let $u_\epsilon \in H_0^1(\Omega)$ be a nonnegative weak solution of problem (A.1) with g_ϵ given by (A.39). Assume that $\widetilde{f}$ satisfies (A.2) and that*

$$\sup_{s\in[0,1]} |s\widetilde{f}'(s)| < \infty.$$

Assume further that there exists a constant $K_1 > 0$ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} < K_1 \text{ for every } 0 < \epsilon < 1. \tag{A.51}$$

Again, let ψ be such that

$$\psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega \text{ and } \frac{|\nabla\psi|^2}{\psi} \text{ is bounded in } \Omega.$$

Then there exists a constant $M > 0$ such that

$$\psi(x)|\nabla u_\epsilon(x)|^2 \le MZ(u_\epsilon(x)) \text{ for every } x \in \Omega, 0 < \epsilon < 1/2, \tag{A.52}$$

where

$$Z(t) = \begin{cases} t^2 + t - t\log t & \text{for } 0 \le t \le \dfrac{1}{2} \\ \dfrac{1}{4} + \dfrac{1}{2}(1+\log 2) + \left(t - \dfrac{1}{2}\right)(1+\log 2) & \text{for } t \ge \dfrac{1}{2}. \end{cases}$$

Proof of Lemma A.5. The proof is similar to Lemma A.3. From (A.51), we conclude $\Delta u_\epsilon \in L^\infty(\Omega)$. Thus, by standard elliptic regularity, u_ϵ belongs to $C^{1,\nu}(\overline{\Omega})$. We define

$$\overline{h}_\epsilon(s) = -\log\left(s + \frac{\epsilon e^{-\widetilde{f}(0)}}{s+\epsilon}\right) - \widetilde{f}(s) \text{ for } s \ge 0.$$

We shall denote u_ϵ simply by u. As in the proof of Lemma A.3, let $0 < a < 1$ be small and define the functions

$$Z_a(u) = Z(u) + a, \qquad w = \frac{|\nabla u|^2}{Z_a(u)}, \qquad v = w\psi.$$

Note that v is C^2 at all points $x \in \Omega$ such that $u(x) > 0$. Indeed, let $x \in \Omega$ be one such point. By continuity, there must exist an open ball $B \subset \Omega$ centered at x such that $u > 0$ in $\overline{B}$. Consequently, we know that $g_\epsilon(u) \in C^{1,\nu}(B)$ and $f(u) \in C^{1,\nu}(B)$. Hence, $\overline{h}_\epsilon(u) \in C^{1,\nu}(B)$. Since u satisfies the equation $-\Delta u + \overline{h}_\epsilon(u) = 0$ in B, we conclude that $u \in C^3(B)$, implying that $Z_a(u)$ and w are C^2 in B.

Our aim is to prove the estimate by contradiction, thus we assume that

$$\sup_\Omega v > \widetilde{M}, \tag{A.53}$$

where $\widetilde{M} > 0$ will be chosen later independently of a and $0 < \epsilon < \epsilon^*$.

The function v is continuous in $\overline{\Omega}$, hence it attains its maximum at some point $x_0 \in \overline{\Omega}$. Thus, by (A.53) we obtain

$$v(x_0) > \widetilde{M}.$$

Then $x_0 \in \Omega$, because $v = 0$ on $\partial\Omega$. Note that $u(x_0) > 0$, because otherwise $\nabla u(x_0) = 0$ and then $v(x_0) = 0$. Hence, v is C^2 at x_0,

$$\nabla v(x_0) = 0$$

and

$$\Delta v(x_0) \leq 0. \tag{A.54}$$

We will compute Δv and evaluate it at the point x_0. As we shall see this leads to the absurd $\Delta v(x_0) > 0$ if one fixes $\widetilde{M}$ large enough. To do that, we observe that Z_a satisfies the following crucial properties

$$Z_a \in C^2(0,\infty), \quad Z_a(t) > 0, \quad Z_a'(t) > 0 \text{ and } Z_a''(t) \leq 0 \text{ for all } t > 0.$$

Consequently, the computations already carried out in [52, Section 3] are applicable, and they lead to the following expression evaluated at x_0

$$\begin{aligned}\Delta v \geq \frac{1}{Z_a(u)}\Big[&\psi w^2\Big(\frac{1}{2}Z_a'(u)^2 - Z_a(u)Z_a''(u)\Big)\\ &+ w\Big(2\psi Z_a(u)\overline{h}_\epsilon'(u) - \psi\overline{h}_\epsilon(u)Z_a'(u) - KZ_a(u)\Big)\\ &- KZ_a'(u)Z_a(u)^{1/2}\psi^{1/2}w^{3/2}\Big],\end{aligned} \tag{A.55}$$

where

$$K = 4\max\left(\sup_\Omega\left(\frac{|\nabla\psi|}{\psi^{1/2}}\right), \sup_\Omega\left|\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right|\right) > 0.$$

We will show that if $v(x_0)$ is large enough then the right hand side of (A.55) must be positive, which would contradict (A.54).

For this purpose we need to establish the following estimates uniformly for every $0 < \epsilon < \epsilon^*$.

$$Z_a'(u)Z_a(u)^{1/2} \leq C(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)), \tag{A.56}$$

$$Z_a(u)|\overline{h}_\epsilon'(u)| \leq C(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)), \tag{A.57}$$

$$Z_a'(u)|\overline{h}_\epsilon(u)| \leq C(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)), \tag{A.58}$$

$$Z_a(u) \leq C(\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)), \tag{A.59}$$

for every $0 < u \le K_1$. The constant C depends only on K_1, but not on ϵ nor on a either.

Assuming for a moment that (A.56)–(A.59) are true. Inequality (A.55) implies that

$$\begin{aligned}\Delta v &\ge \frac{\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)}{Z_a(u)}\Big(\psi w^2 - C(w + \psi^{1/2}w^{3/2})\Big)\\ &= \frac{\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u)}{Z_a(u)\psi}\Big(v^2 - C(v + v^{3/2})\Big).\end{aligned}$$

Thus if $v(x_0) = \sup v > \widetilde{M}$ for some large enough $\widetilde{M}$ independent of a and $0 < \epsilon < \epsilon^*$ we obtain a contradiction with (A.54). Hence, there must exist $M > 0$ independent of a such that $\psi|\nabla u|^2 \le MZ_a(u)$ in Ω. The result then follows by letting $a \to 0$.

We prove now the relations (A.56)–(A.59). The only difference from [52] is the proof of (A.57) and (A.58). The key ingredient is that the function $s \to |sf'(s)|$ is bounded for $0 \le s \le 1/2$.

Case 1. If $u(x_0) \ge \dfrac{1}{2}$, then the left hand sides of (A.56)–(A.59) are uniformly bounded in the interval $[1/2, K_1]$. Since

$$\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u) = \frac{1}{2}\left(1 + \log 2\right)^2 \quad \text{for } u \ge \frac{1}{2}, \tag{A.60}$$

the right hand sides of (A.56)–(A.59) are also uniformly bounded. This proves (A.56)–(A.59).

Case 2. If $0 < u(x_0) \le 1/2$, then

$$Z_a(u) = u^2 + u - u\log u + a, \quad Z_a'(u) = 2u - \log u, \quad Z_a''(u) = 2 - \frac{1}{u} \text{ for } 0 < u \le \frac{1}{2}.$$

We conclude that

$$\frac{1}{2}Z_a'(u)^2 - Z_a''(u)Z_a(u) \ge \frac{1}{2}(\log u)^2 - \log u + \frac{1}{4} + \frac{a}{u} \text{ for } 0 < u \le \frac{1}{2}, 0 < a < \frac{1}{8}. \tag{A.61}$$

Note that

$$Z_a'(u)Z_a(u)^{1/2} = (2u - \log u)(u^2 + u - u\log u + a)^{1/2} \text{ for } 0 < u \le \frac{1}{2}.$$

Since $(u^2 + u - u\log u + a)$ is bounded for $0 < u \le \dfrac{1}{2}$, and $(2u - \log u) \le 1 - \log u$, we conclude that (A.56) holds. Inequality (A.59) is also clear. To prove (A.57) and (A.58), note that

$$\begin{aligned}
Z_a(u)|\overline{h}_\epsilon'(u)| &\leq u\left(u+1-\log u+\frac{a}{u}\right)\left(\left|\frac{(u+\epsilon)^2-\epsilon e^{-\widetilde{f}(0)}}{(u+\epsilon)(u^2+u\epsilon+\epsilon e^{-\widetilde{f}(0)})}\right|+|\widetilde{f}'(u)|\right)\\
&\leq u\left(u+1-\log u+\frac{a}{u}\right)\left(\frac{(u+\epsilon)^2+\epsilon e^{-\widetilde{f}(0)}}{(u+\epsilon)(u^2+u\epsilon+\epsilon e^{-\widetilde{f}(0)})}+|\widetilde{f}'(u)|\right)\\
&\leq \left(3/2-\log u+\frac{a}{u}\right)\left(\frac{u^3+2\epsilon u^2+(\epsilon^2+\epsilon e^{-\widetilde{f}(0)})u}{u^3+2\epsilon u^2+(\epsilon e^{-\widetilde{f}(0)}+\epsilon^2)u+\epsilon^2 e^{-\widetilde{f}(0)}}+u|\widetilde{f}'(u)|\right)\\
&\leq \left(3/2-\log u+\frac{a}{u}\right)(1+u|\widetilde{f}'(u)|)\\
&\leq C\left(1-\log u+\frac{a}{u}\right).
\end{aligned}$$

This estimate and (A.61) prove (A.57). We now prove (A.58). We have

$$Z_a'(u)|\overline{h}_\epsilon(u)| \leq (2u-\log u)\left(\left|\log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right)\right|+|\widetilde{f}(u)|\right) \text{ for } 0<u\leq\frac{1}{2}.$$

We split the proof in two cases. If

$$\left|\log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right)\right| = \log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right),$$

then

$$\begin{aligned}
Z_a'(u)|\overline{h_\epsilon}(u)| &\leq (2u-\log u)\left(\log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right)+|\widetilde{f}(u)|\right)\\
&\leq (2u-\log u)\left(\log\left(u+e^{-\widetilde{f}(0)}\right)+|\widetilde{f}(u)|\right)\\
&\leq (2u-\log u)\left(\log\left(1/2+e^{-\widetilde{f}(0)}\right)+|\widetilde{f}(u)|\right)\\
&\leq C(1-\log u) \text{ for } 0<u\leq\frac{1}{2}.
\end{aligned}$$

On the other hand, if

$$\left|\log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right)\right| = -\log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right),$$

then

$$\begin{aligned}
Z_a'(u)|\overline{h}_\epsilon(u)| &\leq (2u-\log u)\left(-\log\left(u+\frac{\epsilon e^{-\widetilde{f}(0)}}{u+\epsilon}\right)+|\widetilde{f}(u)|\right)\\
&\leq 2(1-\log u)\left(-\log u+|\widetilde{f}(u)|\right) \text{ for } 0<u\leq\frac{1}{2}.
\end{aligned}$$

We conclude that

$$Z_a'(u)|\overline{h}_\epsilon(u)| \leq \sup_{0\leq s\leq 1/2}|\widetilde{f}(s)|+\log^2 u-(1+\sup_{0\leq s\leq 1/2}|\widetilde{f}(s)|)\log u \text{ for } 0<u\leq\frac{1}{2}.$$

This estimate and (A.61) prove (A.58). □

APPENDIX B – Some basic notions

B.1 Basic notions and notation

We introduce some basic notation that will be used throughout the text. We define

(a) $\Omega \subset \mathbb{R}^N$, $N \geq 1$ denotes a bounded and open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$;

(b) We say that Ω is smooth if $\partial\Omega$ is of class C^k for $k \geq 2$, see Definition 2;

(c) $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$;

(d) The set $\overline{\{f(x) \neq 0\}} \subset \overline{\Omega}$ is called the support of the function $f : \Omega \to \mathbb{R}$;

(e) Let $k \in \mathbb{N}, k \geq 1$. The space $C_c^k(\Omega)$ ($C_c^\infty(\Omega)$) denotes the space of functions of class C^k (C^∞) with compact support in Ω. The space of functions f of class C^k such that $\sup_{x\in\Omega} |\nabla^k f(x)| < \infty$ is denoted by $C^k(\overline{\Omega})$;

(f) The space $C(\Omega)$ denote the space of continuous functions defined in Ω;

(g) Let $0 < \tau < 1$. We say that f is uniformly Hölder continuous with exponent τ if f is continuous and

$$\sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^\tau} < \infty.$$

This space is denoted by $C^\tau(\overline{\Omega})$;

(h) Let $k \in \mathbb{N}$ and $0 < \tau < 1$. The space $C^{k,\tau}(\overline{\Omega})$ is the space of functions of class C^k such that each derivative of order k is belongs to $C^\tau(\overline{\Omega})$;

(i) We say that $u_{\epsilon_n} \to u$ in $C^0_{loc}(\Omega)$ if $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω;

(j) Let $A \subset \mathbb{R}^N$ be a set. The function $\chi_A : \mathbb{R}^N \to \mathbb{R}$ is defined by $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in \mathbb{R}^N \setminus A$;

(k) For $N > 2$, we define $2^* = \dfrac{2N}{N-2}$.

(l) The function $\phi_1 \in H_0^1(\Omega)$ is the first eigenfunction of the operator $-\Delta$ with $\|\phi_1\|_{H_0^1(\Omega)} = 1$, so that

$$\int_\Omega \nabla\phi_1 \nabla v \, dx = \lambda_1 \int_\Omega \phi_1 v \, dx \text{ for all } v \in H_0^1(\Omega),$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$.

Let $N \in \mathbb{N}$, $N \geq 1$. We will equip the space $\mathbb{R}^N$, with the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^N}$ and with the Lebesgue measure $|\cdot|$, see [9] and [39].

Let $\Omega \subset \mathbb{R}^N$ be a bounded and open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We will equip it with the Lebesgue σ-algebra $\mathcal{L}_\Omega = \mathcal{L}_{\mathbb{R}^N} \cap \Omega = \{B \cap \Omega : B \in \mathcal{L}_{\mathbb{R}^N}\}$. We say that a property holds almost everywhere in Ω if it holds in a set $V \subset \Omega$ such that $|\Omega \setminus V| = 0$.

A map $f : \Omega \to \mathbb{R}$ is called *Lebesgue measurable* if $f^{-1}(B) \in \mathcal{L}_\Omega$ for all $B \in \mathcal{L}_\Omega$. Let

$$\mathcal{A} = \{f : \Omega \to \mathbb{R} : f \text{ is Lebesgue measurable }\}$$

be the set of measurable functions defined in Ω. Suppose that $f, g \in \mathcal{A}$. We define the equivalence relation $\sim$ in $\mathcal{A}$ as

$$f \sim g \text{ if and only if } f - g = 0 \text{ almost everywhere in } \Omega.$$

Let $p \geq 1$. We define

$$L^1_{loc}(\Omega) = \{f \in \mathcal{A}/\sim : \int_V |f| < \infty \text{ for all } V \subset \overline{V} \subset \Omega\},$$

(here $\overline{V}$ stands for the closure of V in Ω) and

$$L^p(\Omega) = \{f \in L^1_{loc}(\Omega) : \int_\Omega |f|^p < \infty\}.$$

We endow the space $L^p(\Omega)$ with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}.$$

We say that $f \in L^p(\Omega)$ is *weakly differentiable* if for each $i \in \{1, 2, \ldots, N\}$ there exists a measurable function $f_{x_i} : \Omega \to \mathbb{R}$ such that

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} = -\int_\Omega f_{x_i} \varphi. \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We define $\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_N})$. The Sobolev space $W^{1,p}(\Omega)$ is the set defined by

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) : f_{x_i} \in L^p(\Omega) \text{ for all } i \in \{1, 2, \ldots, N\}\}.$$

We equip the space $W^{1,p}(\Omega)$ with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |f|^p + \int_\Omega |\nabla f|^p\right)^{\frac{1}{p}}.$$

We denote the space $W^{1,2}(\Omega)$ merely by $H^1(\Omega)$. We then define the space $H^1_0(\Omega)$ as

$$H^1_0(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}},$$

where $\overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$ denotes the closure of $C_c^\infty(\Omega)$ in $L^1_{loc}(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(\Omega)}$. The Sobolev Embedding Theorem imply that the quantity

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}} \tag{B.1}$$

defines a norm in $H_0^1(\Omega)$, which is equivalent to $\|\cdot\|_{H^1(\Omega)}$. We always equip $H_0^1(\Omega)$ with the norm given by (B.1).

Finally, we say that $f \in L^\infty(\Omega)$ if $f : \Omega \to \mathbb{R}$ is a Lebesgue measurable function and if there exists a constant $C > 0$ such that $|f(x)| \leq C$ for almost all $x \in \Omega$. We also define the norm

$$\|f\|_{L^\infty(\Omega)} = \inf\{C : |f(x)| \leq C \text{ almost everywhere in } \Omega\}.$$

The spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $L^\infty(\Omega)$ equipped with their respective norms are Banach spaces. The spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ equipped with their respective norms are Hilbert spaces, see [9] and [36]. We also recall the concept of *weak convergence* and *dual space*. Let E be a Banach space with norm $\|\cdot\|_E$. We say that $f : E \to \mathbb{R}$ is a linear functional if for each $x, y \in E$ and $c \in \mathbb{R}$ we have

$$f(x+y) = f(x) + f(y), \qquad \text{and} \qquad f(cx) = cf(x).$$

Also, f is *continuous* if there exists a constant $C > 0$ such that

$$|f(x)| \leq C\|x\|_E \text{ for all } x \in E.$$

We denote by E^* the *dual space* of E, that is, the space of all continuous linear functionals defined on E. We may equip E^* with the norm

$$\|f\|_{E^*} = \sup_{\|x\|_E \leq 1} |f(x)|.$$

It is known that E^* equipped with the norm $\|\cdot\|_{E^*}$ is a Banach space. We denote by $\mathcal{L}(E, E^*)$ the space formed by the bounded linear mappings between E and E^*. That is, $L : E \to E^* \in \mathcal{L}(E, E^*)$ if and only if there exists a constant $C > 0$ such that

$$\|L(x)\|_{E^*} \leq C\|x\|_E \text{ for all } x \in E$$

and

$$L(x+y) = L(x) + L(y), \quad L(cx) = cL(x), \text{ for all } x, y \in E, c \in \mathbb{R}.$$

We say that a sequence (x_n) in E converges weakly to an element $x \in E$, and we write $x_n \rightharpoonup x$ if

$$f(x_n) \to f(x) \text{ for all } f \in E^*.$$

We will denote the space $H_0^1(\Omega)^*$ merely by $H^{-1}(\Omega)$. Let $f \in H^{-1}(\Omega)$. The *Riesz Representation Theorem*, see [9, Section 5.2], asserts that there exists $w \in H_0^1(\Omega)$ such that

$$f(u) = \int_\Omega \nabla w \nabla u \text{ for all } u \in H_0^1(\Omega).$$

Hence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ if and only if

$$\int_\Omega \nabla w \nabla u_n \to \int_\Omega \nabla w \nabla u \text{ as } n \to \infty \text{ for all } w \in H_0^1(\Omega).$$

We remark that if $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\|u_n\|_{H_0^1(\Omega)} \to \|u\|_{H_0^1(\Omega)}$, then $u_n \to u$ strongly in $H_0^1(\Omega)$, see [9, Exercise 5.19]. Finally, we give the concept of *Fréchet differentiability*, see [68]. Let $I : H_0^1(\Omega) \to \mathbb{R}$ be a functional. We say that I has a Fréchet derivative $f \in H^{-1}(\Omega)^*$ at $u \in H_0^1(\Omega)$ if

$$\lim_{h \to 0} \frac{1}{\|h\|_{H_0^1(\Omega)}} (I(u+h) - I(u) - (f, h)) = 0.$$

We write $f = I'(u)$ and we say that the functional I is of class C^1 if $I'(u)$ is well defined for all $u \in H_0^1(\Omega)$ and if the mapping $u \to I'(u)$ is continuous in $H_0^1(\Omega)$. Moreover, we say that I satisfies the Palais-Smale condition if any sequence (u_n) in $H_0^1(\Omega)$ such that

$$I(u_n) \text{ is bounded and } I'(u_n) \to 0 \text{ as } n \to \infty \tag{B.2}$$

possesses a convergent subsequence. A sequence (u_n) satisfying (B.2) is called a Palais-Smale sequence for I. Moreover, if there exists $c \in \mathbb{R}$ such that $I(u_n) \to c$, then (u_n) is called a Palais-Smale sequence for I at level c.

We now study smoothness of bounded sets in $\mathbb{R}^N$.

Definition 2. *A bounded domain Ω in $\mathbb{R}^N$ and its boundary are of class C^k if at each point $x_0 \in \partial\Omega$ there exists a ball B centered at x_0 and a one to one mapping Ψ of B onto $D \subset \mathbb{R}^N$ such that*

$$\Psi \in C^k(B), \quad \Psi^{-1} \in C^k(D), \quad \Psi(B \cap \Omega) \subset \mathbb{R}_+^N, \text{ and } \psi(B \cap \partial\Omega) \subset \Gamma,$$

where $\Gamma = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N = 0\}$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and let $\delta : \Omega \to \mathbb{R}_+$ be given by $\delta(x) = \inf_{y \in \partial\Omega} |x - y|$. We have

Lemma B.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is a domain of class C^k for some $k \geq 1$. Then*

$$\int_\Omega \delta(x)^{-\sigma} < \infty \text{ for all } 0 < \sigma < 1.$$

Proof of Lemma B.1. Fix $0 < \sigma < 1$ and let $x_0 \in \partial\Omega$. From the smoothness of Ω, we know that there exist a ball B centered at x_0 and applications $\Psi : B \to D$ and $\Psi^{-1} : D \to B$ such that

$$\Psi \in C^k(B), \quad \Psi^{-1} \in C^k(D), \quad \Psi(B \cap \Omega) \subset \mathbb{R}^N_+, \text{ and } \Psi(B \cap \partial\Omega) \subset \Gamma,$$

where $\Gamma = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N = 0\}$ and $\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$. We claim that

$$\int_{\widetilde{B}\cap\Omega} \delta(x)^{-\sigma} < \infty, \tag{B.3}$$

where $\widetilde{B}$ is a ball centered at x_0 such that $\widetilde{B} \subset \overline{\widetilde{B}} \subset B$. Let $\widetilde{B}$ be one such ball. Claim (B.3) is a consequence of the Change of Variables Theorem and from the fact that Ψ is Lipschitz continuous in the interior of B. Indeed, we know that there must exist a ball $\widehat{B}$ centered at x_0 and such that $\widetilde{B} \subset \overline{\widetilde{B}} \subset \widehat{B} \subset \overline{\widehat{B}} \subset B$. Let $d = dist(\partial\widehat{B}, \partial\widetilde{B})$. From the fact that $\Psi \in C^k(B)$, we know that $\Psi \in C^k(\overline{\widehat{B}})$ and therefore,

$$|\Psi(x) - \Psi(y)| \le C|x - y| \text{ for all } x, y \in \widehat{B}, \tag{B.4}$$

where $C = \sup\limits_{x\in\widehat{B}} |\nabla\Psi(x)|$. Observe that

$$\int_{\widetilde{B}\cap\Omega} \frac{1}{\delta(x)^\sigma}\, dx = \int_{\widetilde{B}\cap\Omega} \frac{1}{(\inf_{y\in\partial\Omega} |x-y|)^\sigma}\, dx.$$

Let $y \in \partial\Omega \setminus \widehat{B}$. We know that in this case

$$|x - y| \ge d \text{ for all } x \in \widetilde{B} \cap \Omega.$$

Therefore

$$\int_{\widetilde{B}\cap\Omega} \frac{1}{\delta(x)^\sigma}\, dx = \int_{\widetilde{B}\cap\Omega} \frac{1}{\left(\min\left\{d, \inf_{y\in\partial\Omega\cap\widehat{B}} |x-y|\right\}\right)^\sigma}\, dx.$$

Hence, we need to show that

$$\int_{\widetilde{B}\cap\Omega} \left(\frac{1}{\inf_{y\in\partial\Omega\cap\widehat{B}} |x-y|}\right)^\sigma dx < \infty.$$

From (B.4) and from the fact that $\Psi(y) \in \Gamma$ for all $y \in \partial\Omega \cap B$, we get

$$\begin{aligned}\int_{\widetilde{B}\cap\Omega} \left(\frac{1}{\inf_{y\in\partial\Omega\cap\widehat{B}} |x-y|}\right)^\sigma dx &\le C^\sigma \int_{\widetilde{B}\cap\Omega} \left(\frac{1}{\inf_{y\in\partial\Omega\cap\widehat{B}} |\Psi(x)-\Psi(y)|}\right)^\sigma dx \\ &\le C^\sigma \int_{\widetilde{B}\cap\Omega} \frac{1}{\Psi_N(x)^\sigma}\, dx.\end{aligned}$$

Using the change of variables $x \to \Psi(x) = (\Psi_1(x), \dots, \Psi_N(x)) = (z_1, \dots z_N)$, and the fact that $\Psi(\widetilde{B} \cap \Omega) \subset D \cap \mathbb{R}^N_+$ is a bounded set, we get

$$\begin{aligned}\int_{\widetilde{B}\cap\Omega} \left(\frac{1}{\inf_{y\in\partial\Omega\cap\widehat{B}} |x-y|} \right)^{\sigma} dx &\le C_2 \int_{D\cap\mathbb{R}^+_N} \frac{1}{z_N^{\sigma}}\, dz \\ &\le C_3 \int_0^M \frac{1}{z_N^{\sigma}}\, dz_N \\ &= \frac{C_3}{1-\sigma} [z_N^{1-\sigma}]_0^M < \infty,\end{aligned}$$

where M is a positive constant chosen such that $0 < z_N < M$ for all $z = (z_1, \dots, z_N) \in D \cap \mathbb{R}^N_+$. This proves the claim (B.3). Lemma B.1 follows from the compactness of $\overline{\Omega}$. $\square$

The following result is a consequence of Höpf's Lemma, see Theorem B.9.

Lemma B.2. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$ be a bounded smooth domain and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a function such that $-\Delta u > 0$ in Ω and $u = 0$ in $\partial\Omega$. Let $\delta(x) = dist(x, \partial\Omega)$. Then, there exists constants $A > a > 0$ such that*

$$a\delta(x) \le u(x) \le A\delta(x) \text{ for every } x \in \Omega. \tag{B.5}$$

Proof of Lemma B.2. From smoothness, we may assume that $\Omega = B_1(0) \cap \mathbb{R}^N_+$ is the upper half of the ball centered at the origin with radius 1, and that $u = 0$ in Γ, where $\Gamma = \partial B_1(0) \cap \{x_n = 0\}$. We need to show that there exist constants $A > a > 0$ such that

$$a\delta_\Gamma(x) \le u(x) \le A\delta_\Gamma(x) \text{ for every } x \in B_{1/2}(0) \cap \mathbb{R}^N_+, \tag{B.6}$$

where $\delta_\Gamma(x) = dist(x, \Gamma) = x_n$. From Höpf's Lemma, we know that there exists an open strip T containing $\Gamma \cap B_{1/2}(0)$ and a constant $c > 0$ such that

$$\frac{\partial u}{\partial x_n}(x) > c \text{ for every } x \in T.$$

It is clear that (B.6) holds in $(B_{1/2}(0) \cap \mathbb{R}^N_+) \setminus T$. We know that there are constants $c_1, c_2 > 0$ such that

$$c_1 e^{x_n} < \frac{\partial u}{\partial x_n}(x) < c_2 e^{x_n} \text{ for all } x \in T. \tag{B.7}$$

Define, for each $x = (x_1, \dots x_n) \in T \cap B_{1/2}(0)$

$$\psi(x) = u(x) - c_1(e^{x_n} - 1),$$

and

$$g(t) = \psi(x_1, x_2, \dots, x_{n-1}, tx_n).$$

It is clear that $g(0) = \psi(0) = 0$ and $g(1) = \psi(x)$. From (B.7), we have

$$g(1) - g(0) = \int_0^1 g'(t)\, dt = \int_0^1 x_n \frac{\partial \psi}{\partial x_n}(x_1, \dots, tx_n)\, dt \ge 0.$$

Hence, $\psi(x) \geq 0$ for all $x \in T$, which implies that

$$u(x) \geq c_1(e^{x_n} - 1) = c_1 x_n \left(\frac{e^{x_n} - 1}{x_n}\right) \text{ for all } x \in T.$$

Using the fact that $(e^s - 1)/s \geq 1$ for $s \geq 0$, we obtain

$$u(x) \geq c_1 x_n = c_1 \delta_\Gamma(x) \text{ for all } x \in T.$$

Analogously,

$$u(x) \leq c_2 \delta_\Gamma(x) \text{ for all } x \in T.$$

This proves (B.6). Inequality (B.5) then follows from a change of variables and from the compactness of $\overline{\Omega}$. □

B.2 Results used throughout the text

First, we give some basic results

Theorem B.1 (Fatou's Lemma). *Let (f_n) be a sequence in $L^1(\Omega)$ such that $f_n \geq 0$ almost everywhere in Ω and $\sup_n \int_\Omega f_n < \infty$. Then,*

$$\int_\Omega f(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x),$$

where $f(x) = \liminf_{n\to\infty} f_n(x)$.

Theorem B.2 (Dominated Convergence Theorem). *Let (f_n) be a sequence in $L^1(\Omega)$ such that $f_n(x) \to f(x)$ almost everywhere in Ω and assume that there exists $g \in L^1(\Omega)$ such that $|f_n(x)| \leq g(x)$ almost everywhere in Ω. Then, $f \in L^1(\Omega)$ and $f_n \to f$ in $L^1(\Omega)$.*

Theorem B.3 (Generalized Dominated Convergence Theorem). *Let (f_n) and (g_n) be sequences in $L^1(\Omega)$ such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ almost everywhere in Ω. Assume that $|f_n(x)| \leq g_n(x)$ almost everywhere in Ω. If $g \in L^1(\Omega)$ and*

$$\int_\Omega g_n(x)\,dx \to \int_\Omega g(x)\,dx,$$

then $f \in L^1(\Omega)$ and

$$\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx.$$

Theorem B.4 (Hölder's inequality). *Let $1 < p < \infty$ and $1 < q < \infty$ be positive constants such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg|\,dx \leq \left(\int_\Omega |f|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q\,dx\right)^{\frac{1}{q}}.$$

Next, we give a version of the Arzelà-Ascoli Theorem.

Theorem B.5. *Let Ω be a domain in $\mathbb{R}^N$ for $N \geq 1$ and let (u_n) be a sequence in $C^1(\Omega)$. Assume that for each smooth bounded subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there are constants $K_1, K_2 > 0$ such that*

$$\sup_{x\in\Omega'} |u_n(x)| \leq K_1 \text{ and } \sup_{x\in\Omega'} |\nabla u_n(x)| \leq K_2 \text{ for all } n \in \mathbb{N}.$$

Then, there exists a continuous function $u \in C(\Omega)$ and a subsequence (u_{n_k}) such that $u_{n_k} \to u$ uniformly in compact subsets of Ω.

We now state the Trudinger-Moser inequality, see [56].

Theorem B.6 (Trudinger-Moser inequality).

$$\exp(\alpha w^2) \in L^1(\Omega) \text{ for every } w \in H_0^1(\Omega) \text{ and } \alpha > 0,$$

and there is a constant $k_1 > 0$ such that

$$\sup_{\|w\|_{H_0^1(\Omega)}\leq 1} \int_\Omega \exp(\alpha w^2) \leq k_1 \text{ for every } \alpha \leq 4\pi \text{ and } w \in H_0^1(\Omega). \tag{B.8}$$

The next result was proven in [51, Section I.7, Remark I.18].

Theorem B.7. *Let (u_n) be a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_n\|_{L^2(\Omega)} = 1$ such that $u_n \rightharpoonup u \not\equiv 0$ weakly in $H_0^1(\Omega)$. Then for every*

$$0 < t < \frac{4\pi}{(1 - \|\nabla u\|^2_{L^2(\Omega)})}$$

we have

$$\sup_n \int_\Omega \exp(tu_n^2) < k_2 \text{ for some constant } k_2 > 0 \text{ independent on } n.$$

The next theorem is proven in [30, Lemma 2.1].

Theorem B.8. *Let (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(u_n(x))$ and $f(u(x))$ are also L^1 functions. If there exists a constant $C > 0$ such that*

$$\int_\Omega |u_n(x) f(u_n(x))| \leq C \text{ for all } n \in \mathbb{N},$$

then $f(u_n)$ converges in L^1 to $f(u)$.

We now state a version of the Höpf's Lemma, see [45, Lemma 3.4]

Theorem B.9 (Höpf's Lemma). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a function such that $-\Delta u > 0$ in Ω and $u = 0$ in $\partial\Omega$. Let $x \in \partial\Omega$. Then,*

$$\frac{\partial u(x)}{\partial \nu} < 0,$$

where ν is the unit vector that is orthogonal to $\partial\Omega$ at x.

For the next result, see [45, Exercise 7.1].

Theorem B.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and suppose that $u \in L^p(\Omega)$ for all $p > 1$. Then,*

$$\lim_{p\to\infty} \Phi_p(u) = \sup_{\Omega} |u|,$$

where

$$\Phi_p(u) = \left(\frac{1}{|\Omega|} \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}.$$

We now state some Sobolev embeddings. We refer the reader to [9], [36] and [45].

Theorem B.11. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 3$ and suppose that $u \in H_0^1(\Omega)$. Then $u \in L^p(\Omega)$ for all $1 \leq p \leq \dfrac{2N}{N-2}$. Furthermore, there exists a constant $C > 0$ that depends on p, N and Ω such that*

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}.$$

Also, if $1 \leq p < \dfrac{2N}{N-2}$, then $H_0^1(\Omega)$ is compactly embedded in $L^p(\Omega)$. Consequently if (u_n) is a bounded sequence in $H_0^1(\Omega)$, then there exists a subsequence (u_{n_k}) in $H_0^1(\Omega)$ and an element $u \in H_0^1(\Omega)$ such that $u_{n_k} \to u$ in $L^p(\Omega)$.

Theorem B.12. *Let Ω be a bounded open subset of $\mathbb{R}^2$, and suppose that $u \in H_0^1(\Omega)$. Then $u \in L^p(\Omega)$ for all $1 \leq p < \infty$. Furthermore, there exists a constant $C > 0$ that depends on p and Ω such that*

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}.$$

Moreover, $H_0^1(\Omega)$ is compactly embedded in $L^p(\Omega)$. Consequently if (u_n) is a bounded sequence in $H_0^1(\Omega)$, then there exists a subsequence (u_{n_k}) in $H_0^1(\Omega)$ and an element $u \in H_0^1(\Omega)$ such that $u_{n_k} \to u$ in $L^p(\Omega)$.

Theorem B.13. *Let Ω be a smooth bounded open subset of $\mathbb{R}^N$, and suppose that $u \in W^{2,p}(\Omega)$ for all $p \geq 1$. Then, $u \in C^{1,\nu}(\overline{\Omega})$ for all $0 < \nu < 1$. Moreover, the embedding*

$$C^1(\overline{\Omega}) \hookrightarrow W^{2,p}(\Omega)$$

is continuous, so that there exists a constant $C > 0$ that depends on p and Ω such that

$$\|u\|_{C^1(\overline{\Omega})} \leq C\|u\|_{W^{2,p}(\Omega)}.$$

The next two results are found in [45] and [49].

Theorem B.14. *Let Ω be a smooth bounded open subset of $\mathbb{R}^N$ with $N \geq 2$, and suppose that $f \in L^p(\Omega)$ for some $1 < p < \infty$. Then, there exists a unique function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

and there exists a constant $C > 0$ that does not depend on u nor on f such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

In particular, if $p > \dfrac{N}{2}$, then $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$.

Theorem B.15. *Suppose that $f \in C^\alpha(\overline{\Omega})$ then, there exists a unique function $u \in C^{2,\alpha}(\Omega)$ satisfying*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, if $f \in C^{k,\alpha}(\overline{\Omega})$, then $u \in C^{k+2,\alpha}(\overline{\Omega})$.

The next results are found in [21, Remark 2.2.1, Proposition 2.2.1], [65, Appendix C] and [68].

Theorem B.16. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 3$ and let $f : \Omega \to \mathbb{R} \to \mathbb{R}$ be a function satisfying*

(i) $f(\cdot, s)$ is measurable in Ω for every fixed $s \in \mathbb{R}$.

(ii) $f(x, \cdot)$ is continuous in $\mathbb{R}$ for almost all $x \in \Omega$.

(iii) There exist constants $c, d > 0$ and $0 \leq \sigma \leq \dfrac{N+2}{N-2}$ such that

$$|f(x,s)| \leq c|s|^\sigma + d \text{ for each } x \in \Omega, s \in \mathbb{R}.$$

Let $I : H_0^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(x,u)\,dx,$$

where $F(x,s) = \displaystyle\int_0^s f(x,t)\,dt$. Then, I is of class C^1 and

$$I'(u)(v) = \int_\Omega \nabla u \nabla v\,dx - \int_\Omega f(x,u)v\,dx.$$

The same conclusion holds when $N = 2$ and f satisfies (i), (ii) and the following condition

(iv) There exist constants $\alpha > 0$ and $C > 0$ such that

$$|f(x,s)| \leq C\exp\{\alpha s^2\} \text{ for each } x \in \Omega, s \in \mathbb{R}.$$

We also remark that if f satisfies $(i) - (iii)$ with $0 \leq \sigma < \dfrac{N+2}{N-2}$, then the functional I satisfies the Palais-Smale condition, see (B.2), provided each Palais-Smale sequence for I is uniformly bounded in $H_0^1(\Omega)$.

Theorem B.17. *Suppose that* $I : H_0^1(\Omega) \to \mathbb{R}$ *is a functional of class* C^1 *that is bounded from below and that satisfies the Palais-Smale condition. Then, there exists* $u_0 \in H_0^1(\Omega)$ *such that* $I(u_0) = \inf_{u \in H_0^1(\Omega)} I(u)$ *and* u_0 *is a critical point of* I.

For the next result, see [28, Theorem 5.1]

Theorem B.18 (The Mountain-Pass Theorem). *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^N$ *and* $I \in C^1(H_0^1(\Omega), \mathbb{R})$. *Suppose that there exists* $e \in H_0^1(\Omega)$ *and* $r > 0$ *such that* $r < \|e\|$ *and*

$$\inf_{\|u\|=r} I(u) > I(0) \geq I(e).$$

Then, for each $\epsilon > 0$ *there exists* $u \in H_0^1(\Omega)$ *such that*

$$c - 2\epsilon \leq I(u) \leq c + 2\epsilon,$$

and

$$\|I'(u)\| < 2\epsilon,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \, \gamma(1) = e\}.$$