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A weighted composition semigroup related to the Riemann hypothesis

Um semigrupo de composição ponderado relacionado à hipótese de Riemann

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Resumo

A hipótese de Riemann (HR) é considerada o problema aberto mais importante na matemática, que afirma que os zeros não triviais da função zeta de Riemann se encontram na "linha crítica". Este problema foi estudado durante cerca de um século e meio, mas ainda não se tem uma demonstração para ele. O principal objectivo desta tese é estudar o semigrupo de operadores de composição ponderada $(W_n)_{n\geqslant 1}$, onde

$$W_n f(z) = (1 + z + \cdots + z^{n-1}) f(z^n)$$

sobre o espaço de Hardy H^2 do disco aberto unitário. Mostramos uma nova reformulação da HR envolvendo este semigrupo. Encontramos uma nova família de vetores cíclicos a fim de generalizar o critério Báez-Duarte em H^2. Também fazemos uma abordagem da HR através do critério Báez-Duarte no espaço Hardy do semi-plano $H^2(\mathbb{C}_{1/2})$.

Palavras-chave: Espaços de Hardy; Hipótese de Riemann; Operador de composição ponderado; Subespaços invariantes; Vetores cíclicos; Problema de Completude de Dilatação Periódica (PCDP).

Abstract

The Riemann hypothesis (RH) is considered to be the most important open problem in mathematics, which states that the non-trivial zeros of the Riemann zeta function lie on the "critical line". This problem has been studied for about a century and a half, but there is still no proof for it. The main purpose of this thesis is to study the semigroup of weighted composition operators $(W_n)_{n\geqslant 1}$, where

$$W_n f(z) = (1 + z + \cdots + z^{n-1}) f(z^n)$$

on the Hardy space H^2 of the open unit disk. We show a new reformulation of the RH involving this semigroup. We find a new family of cyclic vectors in order to generalize the Báez-Duarte criterion in H^2. We also make an approach of the RH through the Báez-Duarte criterion in the Hardy space of the half-plane $H^2(\mathbb{C}_{1/2})$.

Keywords: Hardy spaces; Riemann hypothesis; Weighted composition operator; Invariant subspaces; Cyclic vectors; Periodic Dilation Completeness Problem (PDCP).

List of symbols

$\mathbb{N}$	the set of natural numbers.
$\mathbb{N}_0$	the set of non-negative integers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{D}$	the set $\{z : \|z\| < 1\}$.
$\mathbb{T}$	the set $\{z : \|z\| = 1\}$.
$\mathbb{C}_\alpha$	the set $\{z \in \mathbb{C} : \text{Re } z > \alpha\}$.
$\text{Hol}(\mathbb{D})$	the space of all holomorphic functions on $\mathbb{D}$.
$H^2(\mathbb{D})$	the Hardy-space of the unit disk.
H_0^2	the set $\{f \in H^2(\mathbb{D}) : f(0) = 0\}$.
$H^2(\mathbb{C}_\alpha)$	the Hardy-space of the half-plane $\mathbb{C}_\alpha$.
$H^\infty(\mathbb{D})$	the set of analytic functions on $\mathbb{D}$ which are bounded.
$\mathcal{H}^2$	the Hilbert space of Dirichlet series.
$\mathcal{H}$	Hilbert space.
$\mathcal{L}(\mathcal{H})$	the bounded operators on $\mathcal{H}$.
$\mathcal{A}$	the weighted Bergman space of the unit disk.
D_{δ_ζ}	the local Dirichlet spaces at $\zeta \in \mathbb{T}$.
$L^2(0,1)$	the space of square-integrable functions on $(0,1)$.
$\mathcal{V}$	the functions which are a.e constant on each sub-interval $\left(\frac{1}{n+1}, \frac{1}{n}\right]$, $n = 0, 1, 2....$
$\\| \cdot \\|_2$	the norm of $H^2(\mathbb{D})$.
$\\| \cdot \\|_\infty$	the norm of $H^\infty(\mathbb{D})$.
$\\| \cdot \\|_{H^2(\mathbb{C}_\alpha)}$	the norm of $H^2(\mathbb{C}_\alpha)$.
$\\| \cdot \\|_{\mathcal{H}^2}$	the norm of $\mathcal{H}^2$.
C_ϕ	composition operators.

$T_{\varphi,\phi}$	weighted composition operators.
μ	the Möbius function.
ζ	the Riemann zeta function.
$[\ x\ ]$	the integer part of x.
$\{x\}$	fractional part of x.
$\chi_{(a,b)}$	characteristic function of the interval (a, b).
dA	the normalized area measure of $\mathbb{D}$.
$\log(t)$	logarithm on the basis e of $t > 0$.

Contents

Introduction

The Riemann hypothesis (RH) is a mathematical conjecture which was first published in 1859 by Bernhard Riemann. This conjecture is considered to be the most important open problem in mathematics: It states that the zeros of the Riemann zeta function in the complex plane that have real part between 0 and 1 lie on the "critical line" Re(s)=1/2. This problem is one of the millennium problems.

In 1950, Bertil Nyman introduced in his doctoral thesis, [15], a reformulation for the RH, equivalent to the density of the space spanned by a family of functions $\{f_\lambda : 0 < \lambda \leqslant 1\}$ on $L^2(0,1)$, where $f_\lambda(x) = \{\lambda/x\} - \lambda\{1/x\}$. Five years later, [5], Arne Beurling makes a generalization of Nyman Theorem, where, besides generalizing the space $L^2(0,1)$ to the spaces $L^p(0,1)$, $1 < p < 1$, he adds another equivalence stating that the RH is equivalent to $\chi_{(0,1)}$ being in the closure of the space spanned by the family mentioned above. These three equivalences are known today as the Nyman-Beurling criterion for the RH.

In 2003, [2], Báez-Duarte showed a stronger version of the Nyman-Beurling criterion in $L^2(0,1)$. He proved that the uncountable family $\{f_\lambda : 0 < \lambda \leqslant 1\}$ may be replaced by the sequence $\{f_{1/k} : k \geqslant 1\}$.

Applying an unitary operator between Hilbert spaces, Bagchi, [3], gave an equivalent version of the Báez-Duarte criterion in a weighted sequence space l^2_w, where the functions $f_{1/k}$ were replaced by sequences $r_k = (k\{1/k\}, k\{2/k\}, k\{3/k\}, \dots)$ and the characteristic function $\chi_{(0,1)}$ was replaced by the constant sequence $\mathbf{1} = (1,1,1,\dots)$.

Recently, Noor, [22], gave the Hardy space H^2 version of the Báez-Duarte Theorem. Let $\mathcal{N}$ denote the linear span of functions

$$h_k(z) = \frac{1}{1-z}\log\left(\frac{1+z+\cdots+z^{k-1}}{k}\right)$$

for $k \geqslant 2$. Then the Báez-Duarte criterion may be stated as follows:

"*The RH holds if and only if the constant* 1 *belongs to the closure of* $\mathcal{N}$ *in* H^2".

In [22], a semigroup of weighted composition operators $\mathcal{W} = (W_n)_{n\geqslant 1}$ on H^2 was introduced, where

$$W_n f(z) = (1+z+\cdots+z^{n-1})f(z^n) = \frac{1-z^n}{1-z}f(z^n).$$

Each W_n is bounded on H^2, $W_1 = I$ and $W_nW_m = W_{nm}$ for each $m,n \geqslant 1$. The connection of $\mathcal{W}$ to the RH stems from the fact that subspace $\mathcal{N}$ is invariant under $\mathcal{W}$, that is,

$W_n(\mathcal{N}) \subset \mathcal{N}$ for all $n \geqslant 2$. A vector $f \in H^2$ is called a cyclic vector for an operator semigroup $\{S_n : n \geqslant 1\}$ if the span of $\{S_n f : n \geqslant 1\}$ is dense in H^2. Noor showed that the constant 1 appearing in the Báez-Duarte criterion in H^2 may be replaced by any cyclic vector for the semigroup $\mathcal{W}$.

The semigroup $\mathcal{W}$ is also related to an open problem in harmonic analysis known as the Periodic Dilation Completeness problem (PDCP). The PDCP asks which 2-periodic functions ϕ on $(0,\infty)$ have the property that the span of its dilates $\{\phi(nx) : n \geqslant 1\}$ is dense in $L^2(0,1)$. Such ϕ are called PDCP functions. This difficult open problem was first considered independently by Wintner, [23], and Beurling, [6].

The PDCP has an equivalent reformulation in H^2. N. Nikolski, [14], proved that solving this problem is equivalent to characterizing the cyclic vectors of a semigroup $\mathcal{T} = (T_n)_{n\geqslant 1}$ on $H_0^2 := H^2 \ominus \mathbb{C}$, defined by $T_n f(z) = f(z^n)$. Although the semigroups $\mathcal{T}$ and $\mathcal{W}$ are not unitarily equivalent (see details in section 2.3), they are semiconjugate; this is, $T_n(I-S) = (I-S)W_n$, where S is the shift of multiplication by z in H^2. This relation allowed Noor to guarantee that cyclic vectors of $\mathcal{W}$ are properly embedded into the PDCP functions. The multiplicative semigroup $\mathcal{T}$ is completely characterized, up to unitary equivalence, by four properties (see [16, Theorem 2.2]):

1) For each prime p, T_p defines a shift operator.

2) For different primes p and q, T_p and T_q double commute; this is, T_p and T_q commute, and so do T_p and T_q^*.

3) The semigroup $\mathcal{T}$ is such that

$$\dim \bigcap_{k=1}^{\infty} \operatorname{Ker} T_{p_k}^* = 1.$$

4) The semigroup $\mathcal{T}$ is such that

$$\bigcap_{n=1}^{\infty} \left(\bigvee_{k=n}^{\infty} T_{p_k} \right) = \{0\},$$

where $\{p_k\}_{k=1}^{\infty}$ is the sequence of prime numbers.

In order to study the semigroup $(W_n/\sqrt{n})_{n\geqslant 1}$, we shall see that this semigroup satisfies condition 1), 3) and 4), but not condition 2).

This thesis will be divided into four chapters. In the first chapter we shall provide basic results that will be required throughout this work. We shall discuss the Riemann hypothesis, some Hilbert spaces of holomorphic functions and properties related to the shift and weighted composition operators.

Chapter two will be concerned with the study of the semigroup $\mathcal{W}$. In the first section we shall provide the explicit form of each operator W_n^* for $n \geqslant 1$. In the second section we shall see that the subspace spanned by $\{h_n - h_m : n, m \geqslant 2\}$ is not dense in $H^2(\mathbb{D})$, and, in addition, the dimension of this subspace is closely related to the RH. The last section is focused on the invariant subspaces; in particular, a new reformulation of the RH involving the invariance of $\mathcal{N}^\perp$ under W_k, for any $k \geqslant 2$, is presented.

In chapter three, generalizations of the Báez-Duarte criterion in H^2 will be presented. For a first reformulation, a result of Yang, [24], will be used. For a second generalization, a question about cyclic vectors for the semigroup $\mathcal{W}$ will be solved.

In chapter four we shall discuss about the Báez-Duarte criterion in the Hardy space of the half-plane $H^2(\mathbb{C}_{1/2})$. We shall prove that in a Hardy space of a smaller half-plane, the function E can be approximated by a linear combination of functions G_k (see [3, Theorem 2]).

1 Preliminaries

In this chapter some necessary concepts for the development of this thesis will be given. We shall start by introducing the Riemann zeta function and a famous conjecture related to it: the Riemann hypothesis. We shall provide some reformulations related to this conjecture in the Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{C}_\alpha)$. Then we shall present a shift semigroup of weighted composition operators in $H^2(\mathbb{D})$ having a relation with the RH and another important open problem: The Periodic Dilation Completeness Problem from Harmonic analysis.

1.1 The Riemann zeta function

In this section we shall introduce the Riemann zeta function and its extension as a meromorphic[1] function in the complex plane. Also, some properties that satisfy this function will be presented.

The results of this section can be found at [20].

Definition 1.1.1. *The Riemann zeta function is initially defined for real $s > 1$ by the convergent series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function can be extended analytically to a meromorphic function in the complex plane. First, we extend the series defining ζ to a half-plane of $\mathbb{C}$ and, after that, it is extended by analytic continuation to the entire complex plane. These classical facts are summarized in the following results.

Proposition 1.1.2. *The series defining $\zeta(s)$ converges for $Re(s) > 1$, and the function ζ is holomorphic in this half-plane.*

Theorem 1.1.3. *The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$.*

Now we present some properties related to the zeros of the Riemann zeta function.

Theorem 1.1.4. *The only zeros of ζ outside the strip $0 \leqslant Re(s) \leqslant 1$ are at the negative even integers, $-2, -4, -6, \ldots$.*

[1] A meromorphic function on an open subset Ω of the complex plane is a function that is holomorphic on all of Ω, except for a set of isolated points, which are poles of the function.

Theorem 1.1.5. *The zeta function has no zeros on the line $Re(s) = 1$.*

The zeros of ζ located outside the strip $0 \leqslant \text{Re}(s) \leqslant 1$ are called the trivial zeros of the zeta function.

1.1.1 The Riemann Hypothesis

In this subsection we shall present a conjecture related to the zeros of the Riemann zeta function, known as the Riemann hypothesis and considered the most important open problem in mathematics. In addition, some results on number theory will be introduced.

We begin by denoting the space $\mathbb{C}_\alpha$, $\alpha \in \mathbb{R}$, as the half-plane

$$\mathbb{C}_\alpha = \{s = \sigma + it : \sigma > \alpha, -\infty < t < \infty\}.$$

The set $\{s = \sigma + it : \sigma = 1/2, -\infty < t < \infty\}$ is called the **critical line**.

By the analytic continuation of the zeta function, it is shown that the zeros of ζ occur in symmetric pairs about the critical line; that is, if ρ is a zero, so is $1 - \rho$, $\overline{\rho}$ and $1 - \overline{\rho}$. In particular, since ζ has no zero on the line $\text{Re}(s) = 1$, it follows that it has no zero on the line $\text{Re}(s) = 0$.

At this point, we know that all of the non-trivial zeros of the Riemann zeta function must lie in the strip $0 < \text{Re}(s) < 1$. Riemann, in his famous paper [18], introduced a conjecture about this problem, which is stated as follows.

Conjecture 1. ***(Riemann hypothesis[2]).*** *The non-trivial zeros of the Riemann zeta function lie on the critical line.*

In view of the symmetry mentioned above, the RH is equivalent to saying that ζ has no zeros on $\mathbb{C}_{1/2}$. In chapter 4, there is a particular interest on this half-plane.

1.2 Spaces of holomorphic functions and the PDCP

1.2.1 The Hardy-Hilbert space of the unit disk

Let $\mathbb{D}$ be the open unit disk on the complex plane. The Hardy-Hilbert space of the disk, to be denoted $H^2(\mathbb{D})$, is the Banach space of all analytic functions f on $\mathbb{D}$ having power series representation with square-summable complex coefficients. More precisely,

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2(\mathbb{D}) \quad \text{if and only if} \quad \|f\|_2^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

[2] Sometimes we will denote the Riemann hypothesis by RH.

The inner product inducing the $H^2(\mathbb{D})$ norm is given by

$$\langle f, g\rangle_2 = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)},$$

where $(\hat{f}(n))_{n=0}^{\infty}$ and $(\hat{g}(n))_{n=0}^{\infty}$ are the sequences of Maclaurin coefficients for f and g, respectively.

There is an alternative definition of the Hardy–Hilbert space.

Theorem 1.2.1. *(See [12, Theorem 1.1.12]) Let f be an analytic on $\mathbb{D}$. Then $f \in H^2(\mathbb{D})$ if and only if*

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

Moreover, for $f \in H^2(\mathbb{D})$,

$$\|f\|_2^2 = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

For any $f \in H^2$ and $\zeta \in \mathbb{T}$, the radial limit $f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta)$ exists $m-$a.e on $\mathbb{T}$, where m denotes the normalized Lebesgue measure on $\mathbb{T}$.

A space of analytic functions arises in order to study operators on $H^2(\mathbb{D})$.

Definition 1.2.2. *The space $H^\infty(\mathbb{D})$ consists of all functions that are analytics and bounded on $\mathbb{D}$. The norm of a function f on $H^\infty(\mathbb{D})$ is defined by $\|f\|_\infty = \sup\{f(z) : z \in \mathbb{D}\}$.*

The space $(H^\infty(\mathbb{D}), \|\cdot\|_\infty)$ defines a Banach space. Furthermore, every element of $H^\infty(\mathbb{D})$ belongs to $H^2(\mathbb{D})$. In fact,

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leqslant \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \|f\|_\infty^2 d\theta = \|f\|_\infty^2.$$

In particular, $\|f\|_2 \leqslant \|f\|_\infty$.

1.2.2 A weighted Bergman space

The weighted Bergman space of the unit disk, to be denoted by $\mathcal{A}$, is the Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ defined on $\mathbb{D}$ for which the inner product is given by

$$\langle f, g\rangle := \sum_{n=0}^{\infty} \frac{a_n\overline{b_n}}{(n+1)(n+2)}.$$

There also exists an area integral form of the corresponding $\mathcal{A}$-norm given by

$$\|f\|_{\mathcal{A}}^2 = \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)dA(z),$$

where dA is the normalized area measure on $\mathbb{D}$. The text [10] is a modern reference for such weighted Bergman spaces.

1.2.3 Local Dirichlet spaces

Another important family of proper subspaces of $H^2(\mathbb{D})$ are known as the local Dirichlet spaces. Let δ_ζ be the Dirac measure at $\zeta \in \mathbb{T}$. The local Dirichlet space at $\zeta \in \mathbb{T}$, denoted by $\mathcal{D}_{\delta_\zeta}$, consists of all $f \in H^2$ satisfying

$$\mathcal{D}_\zeta(f) = \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|z-\zeta|^2} dA(z) < \infty, \tag{1.2.1}$$

where dA is the normalized area measure of $\mathbb{D}$. Then $\mathcal{D}_{\delta_\zeta}$ is a Hilbert space with the norm $\|f\|^2_{\mathcal{D}_{\delta_\zeta}} = \|f\|_2^2 + \mathcal{D}_\zeta(f)$. The book [13] is a reference for such local Dirichlet spaces.

These subspaces can be redefined as follows.

Theorem 1.2.3. *(See [13, Theorem 7.2.1]) Let $\zeta \in \mathbb{T}$ and $f \in \mathrm{Hol}(\mathbb{D})$. Then $\mathcal{D}_\zeta(f) < \infty$ if and only if*

$$f(z) = a + (z-\zeta)g(z)$$

for some $g \in H^2$ and $a \in \mathbb{C}$. In this case $\mathcal{D}_\zeta(f) = \|g\|_2^2$ and

$$a = f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta).$$

Each local Dirichlet space $\mathcal{D}_{\delta_\zeta}$ is a proper dense subspace of H^2 and it has the property that evaluation at the boundary $f \mapsto f^*(\zeta)$ is a bounded linear functional (see [13, Theorem 8.1.2 (ii)]).

An alternative way to verify the integral condition (1.2.1) is as follows (see [13, Page 115, exercise 5]). For completeness we give a proof.

Theorem 1.2.4. *Let f be holomorphic on $\mathbb{D}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ and $\zeta \in \mathbb{T}$. If $\sum_{n=0}^\infty a_n \zeta^n$ converges and*

$$\sum_{k=0}^\infty \left| \sum_{n=k+1}^\infty a_n \zeta^n \right|^2 < \infty, \tag{1.2.2}$$

then $\mathcal{D}_\zeta(f) < \infty$.

Proof. We illustrate here the case $\zeta = 1$ (the generalized case is made by a change of variable $z \mapsto \zeta z$). Define

$$g(z) := \sum_{k=0}^\infty \left(\sum_{n=k+1}^\infty a_n \right) z^k.$$

By (1.2.2), it is easy to see that g is in H^2. Now, notice that

$$\begin{aligned}
(z-1)g(z) &= (z-1)\sum_{k=0}^{\infty}\left(\sum_{n=k+1}^{\infty} a_n\right) z^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=k+1}^{\infty} a_n\right) z^{k+1} - \sum_{k=0}^{\infty}\left(\sum_{n=k+1}^{\infty} a_n\right) z^k \\
&= \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} a_n\right) z^k - \sum_{k=0}^{\infty}\left(\sum_{n=k+1}^{\infty} a_n\right) z^k \\
&= \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} a_n - \sum_{n=k+1}^{\infty} a_n\right) z^k - \sum_{n=1}^{\infty} a_n \\
&= \sum_{k=1}^{\infty} a_k z^k - \sum_{n=1}^{\infty} a_n \\
&= \sum_{k=0}^{\infty} a_k z^k - \sum_{n=0}^{\infty} a_n = f(z) - \sum_{n=0}^{\infty} a_n.
\end{aligned}$$

Then

$$f(z) = (z-1)g(z) + \sum_{n=0}^{\infty} a_n.$$

By Theorem 1.2.3, the assertion follows. □

1.2.4 The Hilbert space of Dirichlet series

In this section we are concerned with Dirichlet series, which are series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s \in \mathbb{C},$$

where $(a_n)_{n=1}^{\infty}$ is a complex sequence.

The Hilbert space of Dirichlet functions, to be denoted by $\mathcal{H}^2$, is the Hilbert space of all analytic functions on $\mathbb{C}_{1/2}$ having Dirichlet series representation with square-summable complex coefficients. More precisely,

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^2 \text{ if and only if } \|f\|_{\mathcal{H}^2}^2 := \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

It is a complex Hilbert space when endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}^2} = \sum_{n=1}^{\infty} a_n \overline{b_n},$$

where $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$.

There is a natural semigroup $\mathcal{S} = (S_n)_{n\geqslant 1}$ in $\mathcal{H}^2$ defined by

$$S_n f(s) = n^{-s} f(s).$$

The semigroups $\mathcal{S} = (S_n)_{n\geqslant 1}$ and $\mathcal{T} = (T_n)_{n\geqslant 1}$ are unitarily equivalent. Defining the unitary operator $R : H_0^2 \to \mathcal{H}^2$ by:

$$R : z^k \mapsto k^{-s},$$

it is not difficult to verify that $S_n R = RT_n$, for each $n \geqslant 1$.

This semigroup was extensively studied in [16]. In particular as an immediate consequence of Theorem 2.2 of this paper, we can state:

Theorem 1.2.5. *The semigroup $\mathcal{S}$ is the only multiplicative semigroup, up to unitary equivalence, satisfying the following properties:*

1) For each prime p, S_p defines a shift operator.

2) For different primes p and q, S_p and S_q double commute; this is, S_p and S_q commute, and so do S_p and S_q^*.

3) The semigroup $\mathcal{S}$ is such that

$$\dim \bigcap_{k=1}^{\infty} \mathrm{Ker} S_{p_k}^* = 1.$$

4) The semigroup $\mathcal{S}$ is such that

$$\bigcap_{n=1}^{\infty} \left(\bigvee_{k=n}^{\infty} S_{p_k} \right) = \{0\},$$

where $\{p_k\}_{k=1}^{\infty}$ is the sequence of prime numbers.

Throughout the next chapters, we shall see that the multiplicative semigroup $(W_n/\sqrt{n})_{n\geqslant 1}$ satisfies condition 1), 3) and 4), but not condition 2).

1.2.5 Hardy space of the half-plane

The Hardy-Hilbert space of the half-plane $\mathbb{C}_\alpha$, to be denoted $H^2(\mathbb{C}_\alpha)$, is the Hilbert space of all analytic functions F in $\mathbb{C}_\alpha$ such that

$$\|F\|^2_{H^2(\mathbb{C}_\alpha)} := \sup_{x>\alpha} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x+it)|^2 dt < \infty.$$

It is known that any F in $H^2(\mathbb{C}_\alpha)$ has, almost everywhere on the line $\{\text{Re}(s) = \alpha\}$, a non-tangential boundary value F^* given by (see [11, Chapter 8]):

$$F^*(\alpha + it) = \lim_{x \to \alpha} F(x + it).$$

This F^* is such that

$$\|F\|^2_{H^2(\mathbb{C}_\alpha)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F^*(\alpha + it)|^2 dt.$$

Thus $H^2(\mathbb{C}_\alpha)$ may be identified (via the isometric embedding $F \mapsto F^*$) with a closed subspace of the L^2-space of the line $\{\text{Re}(s) = \alpha\}$ with respect to the Lebesgue measure scaled by the factor $\frac{1}{2\pi}$.

1.2.6 A reformulation of the Riemann hypothesis on $H^2(\mathbb{C}_{1/2})$

In this subsection we introduce the unitary equivalent version of the Baez-Duarte criterion for $H^2(\mathbb{C}_{1/2})$ given in [3].

For each $k \geqslant 2$, define

$$G_k(s) = (k^{-s} - k^{-1}) \frac{\zeta(s)}{s} \quad \text{and} \quad E(s) = \frac{1}{s}, \quad s \in \mathbb{C}_{1/2},$$

which belong to $H^2(\mathbb{C}_{1/2})$ (see [3, Page 139]). In terms of these notations, the formulation of the Baez-Duarte criterion in $H^2(\mathbb{C}_{1/2})$ is the following.

Theorem 1.2.6. *Riemann hypothesis is true if and only if E belongs to the closed linear span of $\{G_k : k \geqslant 2\}$ in $H^2(\mathbb{C}_{1/2})$.*

In Chapter 4 we shall give an approach to the RH through this reformulation.

1.2.7 Weighted composition operators on $H^2(\mathbb{D})$

We begin by introducing the definition of composition operators, which are operators induced by analytic self-maps of $\mathbb{D}$. More precisely, if ϕ is an analytic function mapping $\mathbb{D}$ into itself, we define the composition operator C_ϕ by

$$(C_\phi f)(z) = f(\phi(z)),$$

for all $f \in H^2(\mathbb{D})$ and $z \in \mathbb{D}$.

Composition operators are always bounded linears operators on $H^2(\mathbb{D})$ (see [12, Theorem 5.1.5]).

A weighted composition operator is nothing more than a composition operator followed by a multiplication. More precisely, if ψ is analytic on $\mathbb{D}$ and ϕ analytic self-map of $\mathbb{D}$, we define the weighted composition operator $T_{\psi,\phi}$ by

$$(T_{\psi,\phi} f)(z) = \psi(z) f(\phi(z)),$$

for all $f \in H^2(\mathbb{D})$ and $z \in \mathbb{D}$.

In particular, if $\psi \in H^\infty$, $T_{\psi,\phi}$ defines a bounded linear operator on $H^2(\mathbb{D})$. This comes from the fact that the composition operator is bounded and the multiplication by a function in $H^\infty(\mathbb{D})$ induces a bounded linear operator on $H^2(\mathbb{D})$ (see [12, Chapter 3]).

Example 1.2.7. *i) For each $n \geqslant 1$, let T_n be the linear operator on $H^2(\mathbb{D})$ defined by*

$$T_n f(z) = f(z^n).$$

Making $\phi_n(z) = z^n$, it is easy to see that T_n is a composition operator, since $T_n = C_{\phi_n}$. Note also that $T_1 = I$ and $T_m T_n = T_{mn}$; this means that $\mathcal{T} = (T_n)_{n\geqslant 1}$ defines a multiplicative semigroup on $H^2(\mathbb{D})$.

ii) For each $n \geqslant 1$, let W_n be the linear operator on $H^2(\mathbb{D})$ defined by

$$W_n f(z) = (1 + z + \cdots + z^{n-1}) f(z^n).$$

Making $\psi_n(z) = 1 + z + \cdots + z^{n-1}$, it is easy to see that W_n is a weighted composition operator, since $W_n = \psi_n C_{\phi_n}$. Note also that $W_1 = I$ and $W_m W_n = W_{mn}$; this means that $\mathcal{W} = (W_n)_{n\geqslant 1}$ defines a multiplicative semigroup on $H^2(\mathbb{D})$.

The semigroup $\mathcal{T}$ when restricted to $H_0^2 := \{f \in H^2(\mathbb{D}) : f(0) = 0\}$ was extensively studied in [14]. The semigroup $\mathcal{W}$ was first introduced in [22] and its study is related to the RH. For this reason, a major part of this thesis is devoted to the study of this semigroup.

1.2.8 A reformulation of the Riemann hypothesis on $H^2(\mathbb{D})$

In [4], Balazard and Saias introduced an equivalent version of the Báez-Duarte criterion in the weighted space l^2_ω with inner product given by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \frac{x(n)\overline{y(n)}}{n(n+1)}$$

for sequences $x, y \in l^2_\omega$. For each $k \geqslant 2$, let r_k denote the sequence defined by $r_k(n) = k\{n/k\}$. Then the Báez-Duarte criterion in l^2_ω may be stated as follows.

Theorem 1.2.8. *The RH is true if and only if $\mathbf{1} := (1, 1, 1, \dots)$ belongs to the closed linear span of $\{r_k : k \geqslant 2\}$ in l^2_ω.*

In [22], Noor gave a unitary equivalent version of Theorema 1.2.8 for the Hardy space $H^2(\mathbb{D})$ by using the unitary operator $\Phi := T^{-1} \circ \Psi : l^2_\omega \to H^2(\mathbb{D})$, where $\Psi : l^2_\omega \to \mathcal{A}$ and $T : H^2(\mathbb{D}) \to \mathcal{A}$ are unitary operators defined by

$$\Psi(x(1), x(2), \dots) = \sum_{k=0}^{\infty} x(k+1) z^k \tag{1.2.3}$$

and

$$Tf(z) = \frac{((1-z)f(z))'}{1-z}. \tag{1.2.4}$$

The operator Ψ was first introduced, in a equivalent form, in [3, page 145], whereas T can be found in [13, Lemma 7.2.3]. Such operator T is initially defined on the whole space of holomorphic functions on $\mathbb{D}$, $\mathrm{Hol}(\mathbb{D})$, but when restricted to H^2 defines an isometric isomorphism onto the weighted Bergman space $\mathcal{A}$.

Noor showed that

$$R(z) := \Psi \mathbf{1} = \frac{1}{1-z} \quad \text{and} \quad R_k(z) := \Psi r_k = \frac{1}{1-z}[\log(1+z+\cdots+z^{k-1})]'.$$

Then

$$\Phi \mathbf{1} = T^{-1}R = -1 \quad \text{and} \quad \Phi r_k = T^{-1}R_k = h_k,$$

where

$$h_k(z) = \frac{1}{1-z}\log\left(\frac{1+z+\cdots+z^{k-1}}{k}\right),$$

which belongs to $H^2(\mathbb{D})$. Then the Báez-Duarte criterion in $H^2(\mathbb{D})$ can be stated as follows.

Theorem 1.2.9. *(See [22, Theorem 6]) The Riemann hypothesis holds if and only if the constant* 1 *belongs to the closed linear span of* $\{h_k : k \geqslant 2\}$*.*

Noor also related the Riemann hypothesis with the multiplicative semigroup $\mathcal{W}$. It is easy to see that

$$W_n h_k = h_{nk} - h_n, \quad \text{for all } k, n \geqslant 1 \quad (\text{where } h_1 \equiv 0).$$

Hence the linear span of $\{h_k : k \geqslant 2\}$ is invariant under $\mathcal{W}$. A vector $f \in H^2(\mathbb{D})$ is called a cyclic vector for an operator semigroup $\{S_n : n \geqslant 1\}$ if the span of $\{S_n f : n \geqslant 1\}$ is dense in $H^2(\mathbb{D})$. Hence the following result generalizes Theorem 1.2.9.

Theorem 1.2.10. *(See [22, Theorem 8]) The following statements are equivalent:*

1) *the Riemann hypothesis,*
2) *the closed linear span of* $\{h_k : k \geqslant 2\}$ *contains a cyclic vector for* $\mathcal{W}$*,*
3) *span*$\{h_k : k \geqslant 2\}$ *is dense in* $H^2(\mathbb{D})$*.*

Items 1) and 2) say that the constant 1 appearing in 1.2.9 may be replaced by any cyclic vector of $\mathcal{W}$. At this point, there is only one known cyclic vector: the constant 1. In chapter 3 we shall discover a family of cyclic vectors for $\mathcal{W}$.

We finish this subsection with two important results of [22] that will be useful in Chapter 2 and 4. Let S be the shift operator on $H^2(\mathbb{D})$ defined by $Sf(z) = zf(z)$.

The Möbius function is defined on $\mathbb{N}$ by $\mu(n) = (-1)^k$ if n is the product of k distinct primes and $\mu(n) = 0$ otherwise. For the proof of the following Lemma, Noor required of the Prime Number Theorem in equivalent forms.

Lemma 1.2.11. *(See [22, Lemma 11]) The series* $\sum_{k=2}^{\infty} \frac{\mu(k)}{k}(I-S)h_k$ *converges to* $1-z$ *in* $H^2(\mathbb{D})$*, where* μ *is the Möbius function*

Theorem 1.2.12. *(See [22, Theorem 12]) Let* $\mathcal{N}$ *be the linear span of* $\{h_k : k \geqslant 2\}$*. Then*

$$\mathcal{N}^{\perp} \cap \mathcal{D}_{\delta_1} = \{0\}.$$

1.2.9 The Periodic Dilation Completeness Problem (PDCP)

Aurel Wintner in 1944, [23], and, independently, Arne Beurling in 1945, [6], formulated the following completeness problem.

Periodic Dilation Completeness Problem (PDCP). To characterize functions $\psi \in L^2(0,1)$ for which the dilation system $\{\psi(kx) : k = 1, 2, \dots\}$ is complete in $L^2(0,1)$, where ψ is identified with its extension to an odd 2-periodic function on $\mathbb{R}$. Such a function ψ is said to be a PDCP function.

Partial results have been achieved, but the problem is still open. See [14], [9], [22] and [8] for references. In particular, Noor showed that the cyclic vectors of the semigroup $\mathcal{W}$ are properly embedded into PDCP functions.

Theorem 1.2.13. *There exists an injective linear map* $V : H^2(\mathbb{D}) \to L^2(0,1)$ *such that if* f *is a cyclic vector for* $\mathcal{W}$ *in* $H^2(\mathbb{D})$*, then* Vf *is a PDCP function.*

The embedding of Theorem 1.2.13 is given by $V = UP(I-S)$, where P is the orthogonal projection of $H^2(\mathbb{D})$ onto H_0^2 and $U : H_0^2 \to L^2(0,1)$ is the unitary operator defined by

$$U : z^n \longmapsto \sqrt{2}\sin(n\pi x),$$

with $e_n := \sqrt{2}\sin(n\pi x)$, $n \geqslant 1$, an orthonormal basis of $L^2(0,1)$.

1.2.10 Shift operators

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. An operator S in $\mathcal{L}(\mathcal{H})$ is a shift operator if S is an isometry and $S^{*n} \to 0$ strongly, that is, $\|S^{*n}f\| \to 0$ for all f in $\mathcal{H}$.

There is an alternative definition for the shift operators (see [19]).

Theorem 1.2.14. *An isometry $S \in \mathcal{L}(\mathcal{H})$ is a shift operator if and only if* $\bigcap_{j=0}^{\infty} S^j\mathcal{H} = \{0\}$.

We next introduce a decomposition of a Hilbert space in terms of a shift operators.

Theorem 1.2.15. *(See [19, Chapter 1]) Let $S \in \mathcal{L}(\mathcal{H})$ be a shift operator. A subspace M of $\mathcal{H}$ reduces S if and only if*

$$M = \sum_{j=0}^{\infty} \bigoplus S^j M_0,$$

where M_0 is a subspace of $\mathrm{Ker}S^*$. *Each $f \in M$ has a unique representation $f = \sum_{j=0}^{\infty} S^j k_j$, where $k_j \in M_0$, $j \geqslant 0$.*

Definition 1.2.16. *Two shift semigroups $\mathcal{S} = (S_k)_{k\geqslant 1}$ and $\mathcal{V} = (V_k)_{k\geqslant 1}$ defined on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, are said to be unitarily equivalent if there exists an unitary operator $R : \mathcal{H} \to \mathcal{K}$ such that $R^{-1}V_kR = S_k$, for each $k \geqslant 1$.*

We define the multiplicity of a shift operator $S \in \mathcal{L}(\mathcal{H})$ to be the dimension of the Ker S^*.

Theorem 1.2.17. *(See [19, Chapter 1]) Two shift operators are unitarily equivalent if and only if they have the same multiplicity.*

In chapter 2 we shall see that the semigroup $(W_k/\sqrt{k})_{k\geqslant 1}$ defines a semigroup of shift operators.

2 A semigroup related to the Riemann hypothesis

This chapter is concerned with the study of the semigroup $\mathcal{W}$. In the first part we shall see that the subspace spanned by $\{h_n - h_m : n, m \geqslant 2\}$ is not dense in $H^2(\mathbb{D})$, and, in addition, the dimension of this subspace is closely related to the RH. The second part is focused on the invariant subspaces; in particular, a new reformulation of the RH involving the invariance of $\mathcal{N}^\perp$ under W_k, for any $k \geqslant 2$, is presented.

From now on we shall denote $H^2 := H^2(\mathbb{D})$.

2.1 Explicit form of W_n^*

The main objective of this section is to provide the explicit form of each element of the semigroup $\mathcal{W}^* = (W_n^*)_{n\geqslant 1}$. We also prove that each operator W_n, $n \geqslant 1$, defines an isometry.

By the definition of W_n, it is not difficult to see that

$$W_n = (I + S + \cdots + S^{n-1})T_n, \quad \forall n \geqslant 1,$$

where $T_n f(z) = f(z^n)$ and $Sf(z) = zf(z)$. Taking adjoint, we obtain

$$W_n^* = T_n^* + T_n^* S^* + \cdots + T_n^* S^{*(n-1)}, \tag{2.1.1}$$

where S^* is given by $(S^*f)(z) = \dfrac{f(z) - f(0)}{z}$ (see [19, Page 1]). So if we find the explicit form of T_n^*, we are done. To do that, we use the following result.

Theorem 2.1.1. *(See [7, Section 2]) Let φ be an analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the adjoint of a composition operator C_φ on H^2 is given by*

$$(C_\varphi^* f)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})} z} d\theta.$$

Since T_n is a composition operator with symbol $\varphi(z) = z^n$, Theorem 2.1.1 can be applied to T_n^*. Indeed,

$$\begin{aligned}
(T_n^* f)(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{1-e^{-ni\theta}z}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ni\theta}f(e^{ni\theta})}{e^{ni\theta}-z}d\theta \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta^{n-1}f(\zeta)}{\zeta^n - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta^{n-1}f(\zeta)}{(\zeta-\sqrt[n]{z})(\zeta-\sqrt[n]{z}\omega_n)\dots(\zeta-\sqrt[n]{z}\omega_n^{n-1})}d\zeta,
\end{aligned}$$

where $\sqrt[n]{z}$ is the principal n-th root of z and $\omega_n = e^{2\pi i/n}$. By the Residue Theorem, we obtain

$$\begin{aligned}
(T_n^* f)(z) &= \frac{(\sqrt[n]{z})^{n-1}f(\sqrt[n]{z})}{(\sqrt[n]{z}-\sqrt[n]{z}\omega_n)\dots(\sqrt[n]{z}-\sqrt[n]{z}\omega_n^{n-1})} \\
&+ \frac{(\sqrt[n]{z})^{n-1}\omega_n^{n-1}f(\sqrt[n]{z}\omega_n)}{(\sqrt[n]{z}\omega_n-\sqrt[n]{z})(\sqrt[n]{z}\omega_n-\sqrt[n]{z}\omega_n^2)\dots(\sqrt[n]{z}\omega_n-\sqrt[n]{z}\omega_n^{n-1})} \\
&+\dots+ \frac{(\sqrt[n]{z})^{n-1}\omega_n^{(n-1)^2}f(\sqrt[n]{z}\omega_n^{n-1})}{(\sqrt[n]{z}\omega_n^{n-1}-\sqrt[n]{z})\dots(\sqrt[n]{z}\omega_n^{n-1}-\sqrt[n]{z}\omega_n^{n-2})} \\
&= \frac{f(\sqrt[n]{z})}{(1-\omega_n)\dots(1-\omega_n^{n-1})} \\
&+ \frac{\omega_n^{n-1}f(\sqrt[n]{z}\omega_n)}{(\omega_n-1)(\omega_n-\omega_n^2)\dots(\omega_n-\omega_n^{n-1})} \\
&+\dots+ \frac{\omega_n^{(n-1)^2}f(\sqrt[n]{z}\omega_n^{n-1})}{(\omega_n^{n-1}-1)\dots(\omega_n^{n-1}-\omega_n^{n-2})} \\
&= \frac{f(\sqrt[n]{z})}{(1-\omega_n)\dots(1-\omega_n^{n-1})} \\
&+ \frac{f(\sqrt[n]{z}\omega_n)}{(1-\omega_n^{-1})(1-\omega_n)\dots(1-\omega_n^{n-2})} \\
&+\dots+ \frac{f(\sqrt[n]{z}\omega_n^{n-1})}{(1-\omega_n^{-(n-1)})\dots(1-\omega_n^{-1})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(T_n^* f)(z) &= \frac{f(\sqrt[n]{z})+f(\sqrt[n]{z}\omega_n)+\dots+f(\sqrt[n]{z}\omega_n^{n-1})}{(1-\omega_n)\dots(1-\omega_n^{n-1})} \\
&= \frac{f(\sqrt[n]{z})+f(\sqrt[n]{z}\omega_n)+\dots+f(\sqrt[n]{z}\omega_n^{n-1})}{n},
\end{aligned}$$

where the last equality comes from

$$\begin{aligned}
z^n - 1 &= (z-1)(z-\omega_n)\dots(z-\omega_n^{n-1}) \\
\Longleftrightarrow 1+z+\dots+z^{n-1} &= (z-\omega_n)\dots(z-\omega_n^{n-1}).
\end{aligned}$$

In order to use relation (2.1.1), note that, by using the definition of S^*, we can verify

$$
\begin{aligned}
(S^{*n} f)(z) &= \frac{f(z) - \hat{f}(0) - \hat{f}(1)z - \cdots - \hat{f}(n-1)z^{n-1}}{z^n} \\
&= \frac{f(z) - \sum_{j=0}^{n-1} \hat{f}(j) z^j}{z^n}.
\end{aligned}
$$

Then, for $n \geqslant 2$,

$$
\begin{aligned}
(W_n^* f)(z) &= \frac{1}{n}\left(\sum_{j=0}^{n-1} f(\omega_n^j \sqrt[n]{z}) + \sum_{j=0}^{n-1} (S^* f)(\omega_n^j \sqrt[n]{z}) + \cdots + \sum_{j=0}^{n-1} (S^{*(n-1)} f)(\omega_n^j \sqrt[n]{z}) \right) \\
&= \frac{1}{n}\left(\sum_{j=0}^{n-1} f(\omega_n^j \sqrt[n]{z}) + \sum_{j=0}^{n-1} \frac{f(\omega_n^j \sqrt[n]{z}) - \hat{f}(0)}{\omega_n^j \sqrt[n]{z}} \right. \\
&\quad \left. + \cdots + \sum_{j=0}^{n-1} \frac{f(\omega_n^j \sqrt[n]{z}) - \sum_{l=0}^{n-2} \hat{f}(l)(\omega_n^j \sqrt[n]{z})^l}{(\omega_n^j \sqrt[n]{z})^{n-1}} \right).
\end{aligned}
\tag{2.1.2}
$$

To simplify this expression, note that

$$
\sum_{j=0}^{n-1} \frac{\hat{f}(0)}{\omega_n^j \sqrt[n]{z}} = \frac{\hat{f}(0)}{\sqrt[n]{z}} \sum_{j=0}^{n-1} \omega_n^{-j} = \frac{\hat{f}(0)}{\sqrt[n]{z}} (1 + \omega_n + \cdots + \omega_n^{n-1}) = 0
$$

$$\vdots$$

$$
\begin{aligned}
\sum_{j=0}^{n-1} \frac{\sum_{l=0}^{n-2} \hat{f}(l)(\omega_n^j \sqrt[n]{z})^l}{(\omega_n^j \sqrt[n]{z})^{n-1}} &= \frac{1}{(\sqrt[n]{z})^{n-1}} \sum_{j=0}^{n-1} \sum_{l=0}^{n-2} \frac{\hat{f}(l)(\omega_n^j \sqrt[n]{z})^l}{(\omega_n^j)^{n-1}} \\
&= \frac{1}{(\sqrt[n]{z})^{n-1}} \sum_{l=0}^{n-2} \hat{f}(l)(\sqrt[n]{z})^l \sum_{j=0}^{n-1} \omega_n^{j(l-n+1)} \\
&= \frac{1}{(\sqrt[n]{z})^{n-1}} \sum_{l=0}^{n-2} \hat{f}(l)(\sqrt[n]{z})^l \sum_{j=0}^{n-1} \omega_n^{j(l+1)} = 0,
\end{aligned}
$$

where the last equality comes from the identity (see [1, Theorem 8.1])

$$
\sum_{k=0}^{n-1} \omega_n^{km} = \begin{cases} n & if\ n|m \\ 0 & otherwise. \end{cases}
\tag{2.1.3}
$$

Then, going back to (2.1.2):

$$
(W_n^* f)(z) = \frac{1}{n}\left(\sum_{j=0}^{n-1} f(\omega_n^j \sqrt[n]{z}) + \frac{1}{\sqrt[n]{z}} \sum_{j=0}^{n-1} \frac{f(\omega_n^j \sqrt[n]{z})}{\omega_n^j} + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}} \sum_{j=0}^{n-1} \frac{f(\omega_n^j \sqrt[n]{z})}{\omega_n^{j(n-1)}} \right).
$$

We state now some properties involving these semigroups.

Theorem 2.1.2. *The following statements hold:*

1. $W_n^* T_n = I$,
2. $W_n^* S^k T_n = I$, *for* $k = 0, 1, \ldots, n-1$,
3. $W_n^* S^n T_n = S$,
4. $W_n^* W_n = nI$,

Proof. 1. Notice that

$$\begin{aligned}
(W_n^* T_n f)(z) &= \frac{1}{n}\left(\sum_{j=0}^{n-1}(T_n f)(\omega_n^j \sqrt[n]{z}) + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{(T_n f)(\omega_n^j \sqrt[n]{z})}{\omega_n^{j(n-1)}}\right) \\
&= \frac{1}{n}\left(\sum_{j=0}^{n-1} f(z) + \frac{1}{\sqrt[n]{z}}\sum_{j=0}^{n-1}\frac{f(z)}{\omega_n^j} + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{f(z)}{\omega_n^{j(n-1)}}\right) \\
&= f(z),
\end{aligned}$$

Where the last equality comes by using the identity (2.1.3). Hence $W_n^* T_n = I$.

2. Let us take $0 \leqslant k \leqslant n-1$. Then

$$\begin{aligned}
(W_n^* S^k T_n f)(z) &= \frac{1}{n}\left(\sum_{j=0}^{n-1}(S^k T_n f)(\omega_n^j \sqrt[n]{z}) + \cdots + \frac{1}{(\sqrt[n]{z})^k}\sum_{j=0}^{n-1}\frac{(S^k T_n f)(\omega_n^j \sqrt[n]{z})}{\omega_n^{jk}}\right. \\
&\quad \left. + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{(S^k T_n f)(\omega_n^j \sqrt[n]{z})}{\omega_n^{j(n-1)}}\right) \\
&= \frac{1}{n}\left(\sum_{j=0}^{n-1}(\omega_n^j \sqrt[n]{z})^k f(z) + \cdots + \frac{1}{(\sqrt[n]{z})^k}\sum_{j=0}^{n-1}(\omega_n^j \sqrt[n]{z})^k \frac{f(z)}{\omega_n^{jk}}\right. \\
&\quad \left. + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}(\omega_n^j \sqrt[n]{z})^k \frac{f(z)}{\omega_n^{j(n-1)}}\right) \\
&= \frac{1}{n}\left((\sqrt[n]{z})^k f(z)\sum_{j=0}^{n-1}\omega_n^{jk} + \cdots + \sum_{j=0}^{n-1} f(z)\right. \\
&\quad \left. + \cdots + \frac{(\sqrt[n]{z})^k f(z)}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{1}{\omega_n^{j(n-k-1)}}\right) \\
&= f(z),
\end{aligned}$$

where the last equality comes by using identity (2.1.3). Hence $W_n^* S^k T_n = I$.

3. Notice that

$$\begin{aligned}(W_n^* S^n T_n f)(z) &= \frac{1}{n}\left(\sum_{j=0}^{n-1}(S^n T_n f)(\omega_n^j \sqrt[n]{z}) + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{(S^n T_n f)(\omega_n^j \sqrt[n]{z})}{\omega_n^{j(n-1)}}\right)\\ &= \frac{1}{n}\left(\sum_{j=0}^{n-1}(\omega_n^j \sqrt[n]{z})^n f(z) + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{(\omega_n^j \sqrt[n]{z})^n f(z)}{\omega_n^{j(n-1)}}\right)\\ &= \frac{1}{n}\left(\sum_{j=0}^{n-1} z f(z) + \frac{1}{\sqrt[n]{z}}\sum_{j=0}^{n-1}\frac{z f(z)}{\omega_n^j} + \cdots + \frac{1}{(\sqrt[n]{z})^{n-1}}\sum_{j=0}^{n-1}\frac{z f(z)}{\omega_n^{j(n-1)}}\right)\\ &= z f(z),\end{aligned}$$

where the last equality comes by using identity (2.1.3). Hence $W_n^* S^n T_n = S$.

4. Notice that

$$\begin{aligned}W_n^* W_n &= W_n^*(I + S + \cdots + S^{n-1})T_n\\ &= W_n^* T_n + W_n^* S T_n + \cdots + W_n^* S^{n-1} T_n\\ &= I + I + \cdots + I = nI.\end{aligned}$$

Hence $W_n^* W_n = nI$.

□

Property 4) of Theorem 2.1.2 says that the operator $V_n := W_n/\sqrt{n}$, for each $n \geqslant 1$, defines an isometry.

2.2 Properties of the semigroup $\mathcal{W}$

In section 2.1 we described the explicit form of each W_n^*; due to the complexity of that expression, it sometimes will be useful to work with the coefficient form. We start this section by finding this expression.

As introduced in Section 1.2.7, each W_n is defined by

$$W_n f(z) = (1 + z + \cdots + z^{n-1}) f(z^n).$$

Let $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ be the expansion of an arbitrary function in H^2. Then

$$\begin{aligned}W_n f(z) &= (1 + z + \cdots + z^{n-1}) \sum_{k=0}^{\infty} \hat{f}(k) z^{nk}\\ &= \sum_{k=0}^{\infty} \hat{f}(k)(z^{nk} + z^{nk+1} + \cdots + z^{nk+n-1}).\end{aligned}$$

Now, for any other $g \in H^2$ of the form $g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k$, we have

$$\langle W_n f, g \rangle_2 = \sum_{k=0}^{\infty} \hat{f}(k)(\overline{\hat{g}(nk)} + \overline{\hat{g}(nk+1)} + \cdots + \overline{\hat{g}(nk+n-1)}) = \langle f, W_n^* g \rangle_2.$$

Hence we can identify the adjoint operators as follows:

$$W_n^* g(z) = \sum_{k=0}^{\infty} B_n(k) z^k, \tag{2.2.1}$$

where $B_n(k) = \hat{g}(nk) + \hat{g}(nk+1) + \cdots + \hat{g}(nk+n-1)$.

Theorem 2.2.1. *The semigroup $\mathcal{W}$ is such that*

$$\bigcap_{n=2}^{\infty} \mathrm{Ker} W_n^* = \bigcap_{n=1}^{\infty} \mathrm{Ker} W_{p_n}^* = \langle 1 - z \rangle,$$

where $\mathcal{P} := \{p_n\}_{n \geqslant 1}$ is the set of all primes.

Proof. Suppose $f \in \mathrm{Ker} W_n^*$ for all $n \geqslant 2$. Computing the first term of the power series (2.2.1), we have

$$B_n(0) = \hat{f}(0) + \hat{f}(1) + \cdots + \hat{f}(n-1) = 0, \quad \forall n \geqslant 2.$$

This implies that $\hat{f}(n) = B_{n+1}(0) - B_n(0) = 0$, for all $n \geqslant 2$, and hence

$$f(z) = \hat{f}(0) + \hat{f}(1) z = \hat{f}(0)(1 - z).$$

This implies that

$$\bigcap_{n=2}^{\infty} \mathrm{Ker} W_n^* \subset \langle 1 - z \rangle.$$

The other containment is straightforward. Now, to verify

$$\bigcap_{k=1}^{\infty} \mathrm{Ker} W_{p_n}^* = \bigcap_{k=2}^{\infty} \mathrm{Ker} W_n^*,$$

we use the multiplicative property of $\{W_n^* : n \geqslant 2\}$. Indeed, by the fundamental theorem of arithmetic, each natural number can be represented as a product of prime power: $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$. Then

$$W_n^* = W_{p_1}^{*r_1} W_{p_2}^{*r_2} \ldots W_{p_k}^{*r_k}.$$

□

We generalize this result as follows:

Theorem 2.2.2. *The semigroup $\mathcal{W}$ is such that*

$$\bigcap_{n=k+1}^{\infty} \mathrm{Ker} W_n^* = \mathrm{span}\left\{1-z, 1-z^2, \ldots, 1-z^k\right\}.$$

Proof. Follows as Theorem 2.2.1. □

Let $\{\mathcal{M}_i\}_{i\in I}$ be a family of subsets of a Hilbert space $\mathcal{H}$; we denote $\bigvee_{i\in I} \mathcal{M}_i$ as the closed linear span of $\bigcup_{i\in I} \mathcal{M}_i$.

Corollary 2.2.3. *The semigroup $\mathcal{W}$ is such that*

$$\bigvee_{n=k+1}^{\infty} \mathrm{Im} W_n = \left\{1-z, 1-z^2, \ldots, 1-z^k\right\}^{\perp}.$$

Proof. Since

$$\begin{aligned}\bigcap_{n=k+1}^{\infty} \mathrm{Ker} W_n^* &= \bigcap_{n=k+1}^{\infty} \mathrm{Im} W_n^{\perp} \\ &= \left(\bigcup_{n=k+1}^{\infty} \mathrm{Im} W_n\right)^{\perp} = \left(\bigvee_{n=k+1}^{\infty} \mathrm{Im} W_n\right)^{\perp},\end{aligned}$$

the corollary follows as an immediate consequence of Theorem 2.2.2. □

The following result provides us a tool to know when an element in H^2 is not a cyclic vector for $\mathcal{W}$.

Theorem 2.2.4. *If $f \perp 1-z$, then f is not cyclic vector for $\mathcal{W}$.*

Proof. By Corollary 2.2.3, $\bigvee_{n=2}^{\infty} \mathrm{Im} W_n = \{1-z\}^{\perp}$. Then, for every $f \in H^2$:

$$\overline{\mathrm{span}\{W_n f : n \geqslant 2\}} \subset \bigvee_{n=2}^{\infty} \mathrm{Im} W_n = \{1-z\}^{\perp}.$$

In particular, if $f \perp 1-z$, we have

$$\overline{\mathrm{span}\{W_n f : n \geqslant 1\}} \subset \{1-z\}^{\perp} \subsetneq H^2.$$

□

Another consequence is the following. For every $k \geqslant 2$, set

$$\mathcal{N}_k := W_k \mathcal{N} = \mathrm{span}\{h_{nk} - h_k : n \geqslant 2\}.$$

Since

$$h_n - h_m = (h_n - h_{nm}) + (h_{nm} - h_m),$$

it is easy to see that

$$\bigvee_{k=2}^{\infty} \mathcal{N}_k = \overline{\operatorname{span}\{h_n - h_m : n, m \geqslant 2\}}.$$

Next theorem shows that this is a proper subspace of H^2. This is an unexpected result since RH holds if and only if the subspace $\mathcal{N} = \operatorname{span}\{h_n : n \geqslant 2\}$ is dense in H^2.

Theorem 2.2.5. $\bigvee_{k=2}^{\infty} \mathcal{N}_k$ *is a closed proper subspace of* H^2.

Proof. This is an immediate consequence of $\bigvee_{k=2}^{\infty} \mathcal{N}_k$ being a subspace of $\bigvee_{k=2}^{\infty} \operatorname{Im} W_k$. □

These subspaces are also related to RH as follows.

Theorem 2.2.6. *Let $n \geqslant 1$. Then the RH is true if and only if* $\dim\left(\bigvee_{k=n+1}^{\infty} \mathcal{N}_k\right)^{\perp} = n$.

Proof. Suppose RH is true. Since $\overline{\mathcal{N}} = H^2$ and $W_k/\sqrt{k}$ is an isometry, we obtain

$$\mathcal{N}_k^{\perp} = \overline{W_k \mathcal{N}}^{\perp} = (W_k \overline{\mathcal{N}})^{\perp} = \operatorname{Im} W_k^{\perp} = \operatorname{Ker} W_k^*.$$

Then

$$\left(\bigvee_{k=n+1}^{\infty} \mathcal{N}_k\right)^{\perp} = \bigcap_{k=n+1}^{\infty} \mathcal{N}_k^{\perp} = \bigcap_{k=n+1}^{\infty} \operatorname{Ker} W_k^*.$$

By Theorem 2.2.2,

$$\left(\bigvee_{k=n+1}^{\infty} \mathcal{N}_k\right)^{\perp} = \operatorname{span}\left\{1 - z, 1 - z^2, \ldots, 1 - z^n\right\}.$$

For the converse, suppose $\dim\left(\bigvee_{k=n+1}^{\infty} \mathcal{N}_k\right)^{\perp} = n$. By Theorem 2.2.2,

$$\operatorname{span}\{1 - z, \ldots, 1 - z^n\} = \left(\bigvee_{k=n+1}^{\infty} \operatorname{Im} W_k\right)^{\perp} \subset \left(\bigvee_{k=n+1}^{\infty} \mathcal{N}_k\right)^{\perp}.$$

By our assumption

$$\mathcal{N}^{\perp} \subset \left(\bigvee_{k=n+1}^{\infty} \mathcal{N}_k\right)^{\perp} = \operatorname{span}\{1 - z, \ldots, 1 - z^n\}.$$

Since $\mathcal{N}^{\perp} \cap \mathcal{D}_{\delta_1} = \{0\}$, we conclude that $\mathcal{N}^{\perp} = \{0\}$. Hence RH is true. □

Corollary 2.2.7. *RH is true if and only if the orthogonal complement of* $\{h_n - h_m : n, m \geqslant 2\}$ *is one dimensional.*

Proof. Just take $n = 1$ in Theorem 2.2.6. □

2.3 Invariant subspaces

The main objective of this section is to obtain a new reformulation for the RH in terms of the invariance of the closure of $\mathcal{N}$ under W_n^*, for any $n \geqslant 2$. Other properties of this semigroup are also presented.

Lemma 2.3.1. *The semigroup $\mathcal{W}$ is such that*

$$\bigcap_{n=1}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{p_k}\right)=\bigcap_{n=2}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{k}\right)=\{0\}.$$

Proof. By Corollary 2.2.3,

$$\bigvee_{k=n}^{\infty}\mathrm{Im}W_k=\left\{1-z,1-z^2,\dots,1-z^{n-1}\right\}^{\perp}.$$

If $f\in\bigcap_{n=2}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{k}\right)$, then $f\perp 1-z^n$, for every $n\geqslant 1$. This implies that

$$\hat{f}(0)=\hat{f}(1)=\cdots=\hat{f}(n)=\dots.$$

Thus $f\equiv 0$. Therefore

$$\bigcap_{n=2}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{k}\right)=\{0\}.$$

Using the multiplicative property of $\{W_k : k\geqslant 1\}$, it follows that

$$\bigcap_{n=1}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{p_k}\right)=\bigcap_{n=2}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{k}\right).$$

This completes the proof. □

Corollary 2.3.2. *For any $k\geqslant 2$, $\bigcap_{n=0}^{\infty}W_k^n(H^2)=\{0\}$.*

Proof. Just notice that $W_k^n=W_{k^n}$. So

$$\bigcap_{n=0}^{\infty}W_k^n(H^2)\subset\bigcap_{n=2}^{\infty}\left(\bigvee_{k=n}^{\infty}\mathrm{Im}W_{k}\right)=\{0\}.$$

□

A consequence of Corollary 2.3.2 is that $V_k := W_k/\sqrt{k}$ is not only an isometry, but a shift operator too, for every $k\geqslant 2$ (see Theorem 1.2.14). Even though $(V_k)_{k\geqslant 1}$ is a shift semigroup satisfying Theorem 2.2.1 and Lemma 2.3.1, the operators V_p and V_q^* do not always commute, for different primes p and q. In particular,

$$W_2W_3^*1=1+z\neq 2=W_3^*W_21.$$

So $\mathcal{V}=(V_k)_{k=1}$ and $\mathcal{T}=(T_k)_{k=1}$ are not unitarily equivalent semigroups (see Theorem 1.2.5).

Theorem 2.3.3. KerW_n^* *is infinite-dimensional for all* $n \geqslant 2$.

Proof. It is not difficult to verify that KerW_n^* contains the functions

$$z \mapsto z^{jn} + z^{jn+1} + \cdots + z^{jn+n-2} - (n-1)z^{jn+n-1}.$$

Since all of the above functions are mutually orthogonal, KerW_n^* is infinite-dimensional. □

By Theorem 1.2.17, we derive that the shift operators $V_n = W_n/\sqrt{n}$, for $n \geqslant 2$, are unitarily equivalent to each other.

Lemma 2.3.4. Ker W_n^* *is a subset of the local Dirichlet space* $\mathcal{D}_{\delta_1}$ *for all* $n \geqslant 2$.

Proof. Let $f \in$ KerW_n^*. Our goal is to prove that $\mathcal{D}_1(f) < \infty$ using (1.2.2). By (2.2.1) we have

$$B_n(k) = \hat{f}(nk) + \hat{f}(nk+1) + \cdots + \hat{f}(nk+n-1) = 0$$

for all $k \geqslant 0$. Using this relation and the fact that $\lim_{j \to 0} |\hat{f}(j)| = 0$ (since $f \in H^2$), we obtain that

$$\sum_{j=0}^{\infty} \hat{f}(j) = 0,$$

which is the first condition in (1.2.2). For the second condition, let k_i be the unique positive integer such that $nk_i \leqslant i+1 \leqslant nk_i + n - 1$ for each $i \geqslant 0$. Then by (1.2.2) we have

$$\begin{aligned}
\mathcal{D}_1(f) &= \sum_{i=0}^{\infty} \left| \sum_{j=i+1}^{\infty} \hat{f}(j) \right|^2 = \sum_{i=0}^{\infty} \left| \sum_{j=i+1}^{nk_i+n-1} \hat{f}(j) \right|^2 \leqslant \sum_{i=0}^{\infty} \left(\sum_{j=nk_i}^{nk_i+n-1} |\hat{f}(j)| \right)^2 \\
&\leqslant 2^n \sum_{i=0}^{\infty} \sum_{j=nk_i}^{nk_i+n-1} |\hat{f}(j)|^2 = 2^n n \sum_{i=0}^{\infty} \sum_{j=ni}^{n(i+1)-1} |\hat{f}(j)|^2 \\
&= 2^n n \sum_{j=0}^{\infty} |\hat{f}(j)|^2 < \infty.
\end{aligned}$$

Therefore $f \in \mathcal{D}_{\delta_1}$. □

The new reformulation of the RH can be stated as follows.

Theorem 2.3.5. *Let* $n \geqslant 2$. *Then the Riemann hypothesis is true if and only if the closure of* $\mathcal{N}$ *is* W_n^**-invariant.*

Proof. Suppose the RH is true. Then the closure of $\mathcal{N}$ is H^2, so the assertion follows. Conversely, suppose the closure of $\mathcal{N}$ is invariant under W_n^* for a given $n \geqslant 2$. Note that the closure of $\mathcal{N}$ is also invariant under W_n; therefore, the closure of $\mathcal{N}$ reduces W_n^*, what is equivalent, $\mathcal{N}^{\perp}$ reduces W_n. Since $W_n/\sqrt{n}$ is a shift operator, it follows by Theorem 1.2.15 that there exists a subspace M_0 of KerW_n^* such that

$$\mathcal{N}^{\perp} = \sum_{j=0}^{\infty} \bigoplus \frac{W_n^j}{\sqrt{n}^j} M_0 = \sum_{j=0}^{\infty} \bigoplus W_n^j M_0.$$

In particular, $M_0 \subset \mathcal{N}^{\perp}$, and by Lemma 2.3.4, $M_0 \subset \mathcal{D}_{\delta_1}$. Since $\mathcal{N}^{\perp} \cap \mathcal{D}_{\delta_1} = \{0\}$, $M_0 = \{0\}$, which means that $\mathcal{N}^{\perp} = \{0\}$. Hence RH is true. □

Theorem 2.3.6. *W_n has no eigenvectors for any $n \geqslant 2$.*

Proof. For $n \geqslant 2$, suppose there exists $f \neq 0$ in H^2 and $\lambda \in \mathbb{C}$ such that $W_n f = \lambda f$. By the injectivity of W_n, $\lambda \neq 0$. Notice that

$$W_{n^2} f = \lambda^2 f, \ldots, W_{n^k} f = \lambda^k f, \ldots$$

Since $n^k \to \infty$ as $k \to \infty$, follows that

$$f \in \bigcap_{n=2}^{\infty} \left(\bigvee_{k=n}^{\infty} \mathrm{Im} W_k \right) = \{0\},$$

which is a contradiction. □

As an important consequence of this Theorem, we derive the following Corollary.

Corollary 2.3.7. *W_n does not have non-trivial finite-dimensional invariant subspace, for any $n \geqslant 2$.*

Proof. It is an immediate consequence of Theorem 2.3.6. □

The natural question now is to know if any W_n^* has a non-trivial infinite-dimensional invariant subspace, for $n \geqslant 2$. Notice that

$$\mathcal{N}_k \subset \mathrm{Im} W_k = (\mathrm{Ker} W_k^*)^{\perp}.$$

Then $\mathrm{Ker} W_k^* \subset \mathcal{N}_k^{\perp}$. Thus $\mathcal{N}_k^{\perp}$ is a non-trivial infinite-dimensional invariant subspace for each W_n^*, $n \geqslant 2$ (see Theorem 2.3.3).

2.3.1 Eigenvectors and the spectrum of W_n^*

In this subsection we provide a characterization for all common eigenvectors of $\mathcal{W}^*$ and we found out who the spectrum of W_k and W_k^* are, for every $k \geqslant 2$.

Theorem 2.3.8. *A non-zero function f is a common eigenvector of $\mathcal{W}^*$ if and only if satisfy the following properties:*

i) $\hat{f}(n) = (\lambda_{n+1} - \lambda_n) \hat{f}(0), \quad \forall n \geqslant 1,$

ii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|^2 < \infty$,

iii) $\lambda_{mn} = \lambda_m \lambda_n$ $(\lambda_1 = 1)$, $m, n \in \mathbb{N}$,

where λ_k *is the corresponding eigenvalues for* W_k^*.

Proof. Suppose f is a common eigenvector of $\mathcal{W}^*$ with eigenvalues $\{\lambda_k\}_{k\geqslant 1}$. This means that

$$W_k^* f = \lambda_k f, \ \forall k \geqslant 1.$$

Computing the first term of the power series (2.2.1), we have

$$B_k(0) = \hat{f}(0) + \hat{f}(1) + \cdots + \hat{f}(k-1) = \lambda_k \hat{f}(0).$$

This implies that

$$\hat{f}(k) = B_{k+1}(0) - B_k(0) = (\lambda_{k+1} - \lambda_k)\hat{f}(0), \quad \forall k \geqslant 1.$$

Hence item i) follows. Item ii) follows by the fact that f is in H^2 and $f \neq 0$. Finally, item iii) follows by the fact that $W_{mn}^* = W_m^* W_n^*$. Conversely, suppose f is a non-zero function satisfying property i), ii) and iii). By item i) and ii), $f \in H^2$. Without loss of generality, let us take $\hat{f}(0) = 1$; then

$$\begin{aligned}
(W_k^* f)(z) &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{k-1} \hat{f}(kn+j) \right) z^n \\
&= \sum_{j=0}^{k-1} \hat{f}(j) + \sum_{n=1}^{\infty} \left(\sum_{j=0}^{k-1} \hat{f}(kn+j) \right) z^n \\
&= \lambda_k + \sum_{n=1}^{\infty} (\lambda_{kn+k} - \lambda_{kn})\, z^n \\
&= \lambda_k \left\{ 1 + \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n)\, z^n \right\} \\
&= \lambda_k f(z), \quad \forall k \geqslant 2.
\end{aligned}$$

□

In particular,

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right) z^n + 1 = (1-z) \sum_{n=0}^{\infty} \frac{1}{n+1} z^n \\
&= \frac{(1-z)}{z} \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} = \frac{(z-1)}{z} \log(1-z)
\end{aligned}$$

is a common eigenvector of $\mathcal{W}^*$.

Let $\sigma(T)$ be the spectrum of a given operator T and $B(a,r)$, $\overline{B}(a,r)$ the open and closed ball of radius r center at a in the complex plane, respectively.

Theorem 2.3.9. $\sigma(W_k^*) = \sigma(W_k) = \overline{B}(0, \sqrt{k})$.

Proof. Note first, by using properties of the spectrum, that

$$\sigma(W_k^*) \subset \overline{B}(0, \|W_k^*\|) = \overline{B}(0, \sqrt{k}).$$

To prove the other containment, let $\lambda \in B(0, \sqrt{k})$. We must show that λ is an eigenvalue for W_k^*. Indeed, note that $W_k^* f = \lambda f$ is equivalent to

$$\begin{aligned} \hat{f}(0) + \hat{f}(1) + \cdots + \hat{f}(k-1) &= \lambda \hat{f}(0) \\ \hat{f}(k) + \hat{f}(k+1) + \cdots + \hat{f}(2k-1) &= \lambda \hat{f}(1) \\ \hat{f}(2k) + \hat{f}(2k+1) + \cdots + \hat{f}(3k-1) &= \lambda \hat{f}(2) \\ &\vdots \end{aligned}$$

So a f satisfying this can be defined like this:

$$\begin{aligned} \hat{f}(0) = 1, \quad \hat{f}(1) = \hat{f}(2) = \cdots = \hat{f}(k-1) &= \frac{\lambda - 1}{k-1} \\ \hat{f}(k) = \hat{f}(k+1) = \cdots = \hat{f}(k^2-1) &= \frac{\lambda}{k}\left(\frac{\lambda-1}{k-1}\right) \\ \hat{f}(k^2) = \hat{f}(k^2+1) = \cdots = \hat{f}(k^3-1) &= \frac{\lambda^2}{k^2}\left(\frac{\lambda-1}{k-1}\right). \\ \vdots \end{aligned}$$

We only need to show is that $f \in H^2$:

$$\begin{aligned} \sum_{k=0}^{\infty} \left|\hat{f}(k)\right|^2 &= 1 + (k-1)\frac{|\lambda-1|^2}{(k-1)^2} + k(k-1)\frac{|\lambda|^2}{k^2}\frac{|\lambda-1|^2}{(k-1)^2} + k^2(k-1)\frac{|\lambda|^4}{k^4}\frac{|\lambda-1|^2}{(k-1)^2} + \ldots \\ &= 1 + \frac{|\lambda-1|^2}{(k-1)} + \frac{|\lambda|^2}{k}\frac{|\lambda-1|^2}{(k-1)} + \frac{|\lambda|^4}{k^2}\frac{|\lambda-1|^2}{(k-1)} + \ldots \\ &= 1 + \frac{|\lambda-1|^2}{k-1}\sum_{k=0}^{\infty}\left(\frac{|\lambda|^2}{k}\right)^n, \end{aligned}$$

where the last summation converges because $|\lambda| < \sqrt{k}$. Thus $B(0, \sqrt{k}) \subset \sigma(W_k^*)$. Since the spectrum of an operator is closed, we conclude that $\sigma(W_k^*) = \overline{B}(0, \sqrt{k})$. To verify that $\sigma(W_k) = \overline{B}(0, \sqrt{k})$, we simply use the fact that

$$\sigma(W_k) = \left\{\overline{\lambda} : \lambda \in \sigma(W_k^*)\right\}.$$

□

3 Generalization of the Báez-Duarte criterion in H^2

Recently, the mathematician Jongho Yang, [24], generalized the classical Nyman-Beurling criterion of the RH in $L^2(0,1)$ by replacing $\chi_{(0,1)}$ with $\chi_{(a,b)}$ for any $0 \leqslant a < b \leqslant 1$. With this in mind and a contribution of Bagchi, [3], to the Báez-Duarte criterion in $L^2(0,1)$, we shall generalize the Noor criterion of the RH in H^2. The second part is also focused on generalizing the Báez-Duarte criterion in H^2, but this time with an unexplored tool provided by Noor: cyclic vectors for the semigroup $\{W_n : n \geqslant 1\}$.

We start this chapter by introducing the Jongho Yang's generalization of the Nyman-Beurling criterion. This criterion can be stated, in a slighted modified form, as follows:

Theorem 3.0.1. *(See [24, Theorem 1.2]) Let $f_\lambda(x) = \{\lambda/x\} - \lambda\{1/x\}$. Then the Riemann hypothesis is true if and only if $\chi_{(a,b]}$ belongs to the closed linear span of $\{f_\lambda : 0 < \lambda \leqslant 1\}$ for any $0 \leqslant a < b \leqslant 1$ in $L^2(0,1]$.*

In [3], Bagchi showed that in addition to the Baéz-Duarte criterion, RH is equivalent to the density of the linear span of $\left\{f_{1/k} : k \geqslant 1\right\}$ in the closed subspace of $L^2(0,1]$, let us call $\mathcal{V}$, consisting of the functions which are a.e. constant on each sub-interval $\left(\dfrac{1}{n+1}, \dfrac{1}{n}\right]$, $n = 0, 1, 2....$ It is shown that in fact these functions $f_{1/k}(x) = \{1/kx\} - 1/k\{1/x\}$ are in $\mathcal{V}$ since

$$f_{1/k}(x) = f_{1/k}\left(\frac{1}{n}\right) = \left\{\frac{n}{k}\right\}, \ \forall x \in \left(\frac{1}{n+1}, \frac{1}{n}\right].$$

Theorem 3.0.2. *(See [3, Theorem 7]) The following statements are equivalents:*

i) The Riemann hypothesis,

ii) $\chi_{(0,1]}$ belongs to the closed linear span of $\left\{f_{1/k} : k \geqslant 1\right\}$, and

iii) the linear span of $\left\{f_{1/k} : k \geqslant 1\right\}$ is dense in $\mathcal{V}$.

Combining these two results, we can generalize the original Báez-Duarte criterion as follows.

Theorem 3.0.3. *The Riemann hypothesis is true if and only if $\chi_{(0,\frac{1}{n}]}$ belongs to the closed linear span of $\left\{f_{1/k} : k \geqslant 1\right\}$ for any $n \in \mathbb{N}$.*

Proof. It is clear that $\chi_{(0,\frac{1}{n}]} \in \mathcal{V}$. Suppose the RH is true; by Theorem 3.0.2 follows that $\chi_{(0,\frac{1}{n}]}$ belongs to the closed linear span of $\{f_{1/k} : k \geqslant 1\}$. The converse is straightforward by Theorem 3.0.1. □

Theorem 3.0.4. *The Riemann hypothesis is true if and only if $\chi_{(\frac{1}{n+1},\frac{1}{n}]}$ belongs to the closed linear span of $\{f_{1/k} : k \geqslant 1\}$ for any $n \in \mathbb{N}$.*

Proof. It follows as Theorem 3.0.3. □

Our purpose is to transfer Theorem 3.0.3 and Theorem 3.0.4 to H^2, so we need to choose a suitable unitary operator from $\mathcal{V}$ to H^2. In subsection 1.2.8, Noor transferred the Báez-Duarte criterion in l^2_ω to H^2 through the unitary operator $\Phi := T^{-1} \circ \Psi : l^2_\omega \to H^2$ (see 1.2.3 and 1.2.4). If we consider the canonical isometric isomorphism $\Upsilon : \mathcal{V} \to l^2_\omega$ given by

$$\Upsilon f = (f(1), f(1/2), f(1/3), \dots),$$

then the operator $\Phi \circ \Upsilon = T^{-1} \circ \Psi \circ \Upsilon : \mathcal{V} \to H^2$ defines a unitary operator. The operator $\Psi \circ \Upsilon : \mathcal{V} \to \mathcal{A}$ is such that

$$(\Psi \circ \Upsilon)f(z) = \sum_{k=0}^{\infty} f\left(\frac{1}{k+1}\right) z^k.$$

We shall use these operators to generalize the Báez-Duarte criterion in H^2.

Theorem 3.0.5. *The Riemann hypothesis is true if and only if the polinomial $1+z+\cdots+z^n$ belongs to the closed linear span of $\{h_k : k \geqslant 2\}$ in H^2 for any $n \in \mathbb{N}_0$.*

Proof. In order to prove this, we must show that there exists a constant c_n such that

$$(\Phi \circ \Upsilon)\chi_{(0,\frac{1}{n+1}]} = c_n(1 + z + \cdots + z^n), \quad \text{for each} \quad n \in \mathbb{N}_0. \tag{3.0.1}$$

In fact, note that

$$(\Psi \circ \Upsilon)\chi_{(0,\frac{1}{n+1}]} = \sum_{k=n}^{\infty} z^k = \frac{z^n}{1-z}$$

and

$$T(1 + z + \cdots + z^n) = \frac{((1-z)(1 + z + \cdots + z^n))'}{1-z}$$
$$= \frac{(1-z^{n+1})'}{1-z} = -(n+1)\frac{z^n}{1-z}.$$

Therefore

$$(\Phi \circ \Upsilon)\chi_{(0,\frac{1}{n+1}]} = (T^{-1} \circ \Psi \circ \Upsilon)\chi_{(0,\frac{1}{n+1}]} = -\frac{1 + z + \cdots + z^n}{n+1}.$$

This proves (3.0.1) and the theorem (as a consequence of Theorem 3.0.3). □

For $n \geqslant 1$, define

$$p_n(z) = nz^n - \sum_{k=0}^{n-1} z^k.$$

In [13, Lemma 7.2.2] it was shown that $\{p_n : n \geqslant 1\}$ is an orthogonal basis for H^2. In terms of these polinomials, we have the following generalization for the RH.

Theorem 3.0.6. *The Riemann hypothesis is true if and only if the polinomial p_n belongs to the closed linear span of $\{h_k : k \geqslant 2\}$ in H^2 for any $n \in \mathbb{N}$.*

Proof. Notice that

$$(\Psi \circ \Upsilon)\chi_{(\frac{1}{n+1},\frac{1}{n}]} = z^{n-1}$$

and

$$\begin{aligned}(Tp_n)(z) &= \frac{((1-z)(nz^n - \sum_{k=0}^{n-1} z^k))'}{1-z} \\ &= \frac{((n+1)z^n - nz^{n+1} - 1)'}{1-z} \\ &= n(n+1)\frac{z^{n-1} - z^n}{1-z} \\ &= n(n+1)z^{n-1}.\end{aligned}$$

Therefore

$$(\Phi \circ \Upsilon)\chi_{(\frac{1}{n+1},\frac{1}{n}]} = \frac{p_n}{n(n+1)}.$$

As a consequence of Theorem 3.0.4, the proof is complete. □

Due to Theorem 1.2.10, it is natural to ask whether $1 + z + \cdots + z^n$ and $p_n(z)$, for $n \geqslant 1$, are cyclic vectors for $\mathcal{W}$. By Corollary 2.2.4, it is not difficult to see that each $1 + z + \cdots + z^n$, for $n \geqslant 1$, is not a cyclic vector. The same holds for each p_n, $n \geqslant 2$. It remains to be known whether $p_1(z) = 1 - z$ is a cyclic vector or not. We shall answer this question in the next subsection.

3.1 Cyclic vectors for the semigroup $\mathcal{W}$

In this subsection we are concerned on generalizing the Báez-Duarte criterion in H^2 by finding a new family of cyclic vectors for the semigroup $\mathcal{W}$. As a consequence we provide a generalization of the original Báez-Duarte criterion in $L^2(0,1)$.

Let $p_{m,\lambda}(z) := z^m + \cdots + z - \lambda$, $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

Theorem 3.1.1. *Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $|\lambda + 1| > \sqrt{m+1}$. Then $p_{m,\lambda}$ is a cyclic vector for $\mathcal{W}$.*

Proof. Note first that

$$\begin{aligned}(W_n p_{m,\lambda})(z) &= W_n(z^m + \cdots + z - \lambda) \\ &= \sum_{j=nm}^{n(m+1)-1} z^j + \cdots + \sum_{j=n}^{2n-1} z^j - \sum_{j=0}^{n-1} \lambda z^j \\ &= \sum_{j=n}^{n(m+1)-1} z^j - \sum_{j=0}^{n-1} \lambda z^j.\end{aligned}$$

So, let us take $f \in H^2$ such that $f \perp W_n p_{m,\lambda}$, for each $n \geqslant 1$; this is

$$\sum_{j=n}^{n(m+1)-1} \hat{f}(j) = \sum_{j=0}^{n-1} \lambda \hat{f}(j), \quad \forall n \geqslant 1. \tag{3.1.1}$$

Looking at the particular cases $n = (m+1)^k$ in (3.1.1), $k \geqslant 0$, we get:

$$\begin{aligned}
k = 0 \qquad & \sum_{j=1}^{m} \hat{f}(j) = \lambda \hat{f}(0) \\
k = 1 \qquad & \sum_{j=m+1}^{(m+1)^2-1} \hat{f}(j) = \lambda \sum_{j=0}^{m} \hat{f}(j) \\
& = \lambda(\lambda+1)\hat{f}(0) \\
k = 2 \qquad & \sum_{j=(m+1)^2}^{(m+1)^3-1} \hat{f}(j) = \lambda \sum_{j=0}^{(m+1)^2-1} \hat{f}(j) \\
& = \lambda^2(\lambda+1)\hat{f}(0) + \lambda(\lambda+1)\hat{f}(0) \\
& = \lambda(\lambda+1)^2 \hat{f}(0)
\end{aligned}$$

and for the general case we use induction to obtain

$$\sum_{j=(m+1)^k}^{(m+1)^{k+1}-1} \hat{f}(j) = \lambda(\lambda+1)^k \hat{f}(0).$$

By the geometric sum formula with common ratio $\lambda + 1$ we have

$$\begin{aligned}\sum_{j=0}^{(m+1)^{k+1}-1} \hat{f}(j) &= \sum_{j=0}^{k} \lambda(\lambda+1)^k \hat{f}(0) + \hat{f}(0) \\ &= \left(\frac{1-(\lambda+1)^{k+1}}{1-(\lambda+1)}\right) \lambda \hat{f}(0) + \hat{f}(0) \\ &= \left((\lambda+1)^{k+1} - 1\right) \hat{f}(0) + \hat{f}(0) \\ &= (\lambda+1)^{k+1} \hat{f}(0).\end{aligned}$$

Thus

$$\sum_{j=0}^{(m+1)^{k+1}-1} \hat{f}(j) = (\lambda+1)^{k+1} \hat{f}(0).$$

Using the Cauchy-Schwarz inequality, we have

$$|\hat{f}(0)| \leqslant \left(\frac{\sqrt{m+1}}{|\lambda+1|}\right)^{k+1} \|f\|,$$

and

$$\left(\frac{\sqrt{m+1}}{|\lambda+1|}\right)^{k+1} \longrightarrow 0 \text{ as } k \longrightarrow \infty \Longleftrightarrow \sqrt{m+1} < |\lambda+1|.$$

So, in these conditions, $\hat{f}(0) = 0$. By using the same arguments, for each $r \geqslant 1$, taking $n = (r+1)(m+1)^k$ in (3.1.1), it can be shown that $\hat{f}(r) = 0$; thus, $f \equiv 0$. Therefore, $p_{m,\lambda}(z) = z^m + \cdots + z - \lambda$ is a cyclic vector for $|\lambda+1| > \sqrt{m+1}$. □

The new generalization of the Báez-Duarte criterion in H^2 can be stated as follows.

Theorem 3.1.2. *Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $|\lambda+1| > \sqrt{m+1}$. The Riemann hypothesis is true if and only if $p_{m,\lambda}$ belongs to the closed linear span of $\{h_k : k \geqslant 2\}$ in H^2.*

Proof. Immediate consequence of Theorem 3.1.1 and Theorem 1.2.10. □

Transfering Theorem 3.1.2 to $L^2(0,1]$, we obtain a new generalization of the original Báez-Duarte criterion. Let $f_{m,\lambda} := (\lambda+1)\chi_{\left(\frac{1}{m+1},1\right]} + (\lambda-m)\chi_{\left(0,\frac{1}{m+1}\right]}$.

Theorem 3.1.3. *Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $|\lambda+1| > \sqrt{m+1}$. The Riemann hypothesis is true if and only if $f_{m,\lambda}$ belongs to the closed linear span of $\left\{f_{1/k} : k \geqslant 1\right\}$ in $L^2(0,1]$.*

Proof. Notice that

$$\begin{aligned}
(Tp_{m,\lambda})(z) &= \frac{((1-z)(z^m + \cdots + z - \lambda))'}{1-z} \\
&= \frac{(z - z^{m+1} - \lambda + \lambda z)'}{1-z} \\
&= \frac{(1+\lambda) - (m+1)z^m}{1-z} \\
&= ((1+\lambda) - (m+1)z^m) \sum_{n=0}^{\infty} z^n \\
&= (1+\lambda) + \cdots + (1+\lambda)z^{m-1} + \sum_{n=m}^{\infty} (\lambda - m) z^n.
\end{aligned}$$

Then

$$(\Upsilon^{-1} \circ \Phi^{-1}) p_{m,\lambda} = (\Upsilon^{-1} \circ \Psi^{-1} \circ T) p_{m,\lambda} = f_{m,\lambda}.$$

As a consequence of Theorem 3.1.2, the proof is complete. □

To finish this section, we present a family of PDCP functions. For every $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, let $g_{m,\lambda} := (1+\lambda)e_1 - e_{m+1}$, where $e_k(x) = \sqrt{2}\sin(k\pi x)$.

Theorem 3.1.4. *Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $|\lambda + 1| > \sqrt{m+1}$. Then $g_{m,\lambda}$ is a PDCP function.*

Proof. In order to use the embedding of Theorem 1.2.13 and Theorem 3.1.1, we see that

$$P(I-S)p_{m,\lambda} = P(z - z^{m+1} - \lambda + \lambda z) = (1+\lambda)z - z^{m+1}.$$

Applying the unitary operator $U : z^n \longmapsto e_n$, the desired result follows. $\square$

4 An approach to the Riemann hypothesis through the Báez-Duarte criterion in $H^2(\mathbb{C}_\alpha)$

In section 1.2.5 it was introduced the Báez-Duarte criterion in $H^2(\mathbb{C}_{1/2})$. This criterion says that the Riemann hypothesis is true if and only if the function E belongs to the closed linear span of $\{G_k : k \geqslant 2\}$ (see Theorem 1.2.6). In this chapter we prove that this convergence is in fact true but in a Hardy space with a smaller half-plane. We also prove this convergence explicitly.

In order to reach the goal, we shall make use of Lemma 1.2.11. First note that

$$\begin{aligned} h_k(z) &= \frac{1}{1-z} \log\left(\frac{1+z+\cdots+z^{k-1}}{k}\right) \\ &= \frac{1}{1-z}\left(\log(1-z^k) - \log(1-z) - \log k\right), \end{aligned}$$

so

$$(I-S)h_k(z) = \log(1-z^k) - \log(1-z) - \log k.$$

Applying the orthogonal projection $P : H^2 \to H_0^2$, we have

$$\begin{aligned} g_k(z) := P(I-S)h_k(z) &= \log(1-z^k) - \log(1-z) \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^{nk}}{n}. \end{aligned} \tag{4.0.1}$$

Applying the unitary operator $R : H_0^2 \to \mathcal{H}^2$ (see Section 1.2.4), we have

$$\begin{aligned} (Rg_k)(s) &= \sum_{n=1}^{\infty} \frac{n^{-s}}{n} - \sum_{n=1}^{\infty} \frac{(nk)^{-s}}{n} \\ &= \sum_{n=1}^{\infty} \frac{n^{-s}}{n} - k^{-s} \sum_{n=1}^{\infty} \frac{n^{-s}}{n} \\ &= (1-k^{-s}) \sum_{n=1}^{\infty} n^{-(s+1)} \\ &= (1-k^{-s})\zeta(s+1), \quad s \in \mathbb{C}_{1/2}. \end{aligned}$$

To take these functions to the Hardy space of a half-plane, we consider the following bounded linear operator (see [17, Page 1622])

$$\mathcal{M} : \mathcal{H}^2 \longrightarrow H^2(\mathbb{C}_{1/2})$$

defined by

$$(\mathcal{M}f)(s) = f(s)/s, \quad s \in \mathbb{C}_{1/2}.$$

By making a translation, we can define the linear operator

$$\mathcal{M}_1 : \mathcal{H}^2 \longrightarrow H^2(\mathbb{C}_{3/2})$$

given by

$$(\mathcal{M}_1 f)(s) = f(s-1)/s, \quad s \in \mathbb{C}_{3/2}.$$

Let us see that this operator is well defined and bounded. In fact, for $f \in \mathcal{H}^2$ and $\sigma > 3/2$,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{M}_1 f\,(\sigma+it)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|f(\sigma-1+it)|^2}{|\sigma+it|^2}dt \\
&\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|f(\sigma-1+it)|^2}{|\sigma-1+it|^2}dt \\
&\leqslant \sup_{x>1/2} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|f(x+it)|^2}{|x+it|^2}dt \\
&= \|\mathcal{M}f\|^2_{H^2(\mathbb{C}_{1/2})}.
\end{aligned}$$

Taking supremum over all $\sigma > 3/2$, we derive that $\mathcal{M}_1 f$ is in $H^2(\mathbb{C}_{3/2})$. And, by the boundedness of $\mathcal{M}$, we have

$$\|\mathcal{M}_1 f\|_{H^2(\mathbb{C}_{3/2})} \leqslant \|\mathcal{M}f\|_{H^2(\mathbb{C}_{1/2})} \leqslant \|\mathcal{M}\|\,\|f\|_{\mathcal{H}^2}, \quad \forall f \in \mathcal{H}^2.$$

Therefore, $\mathcal{M}_1$ is also a bounded linear operator. The operator $\mathcal{M}$ satisfies the following properties:

$$(\mathcal{M}_1 1)(s) = \frac{1}{s}$$

and

$$\begin{aligned}
(\mathcal{M}_1 R g_k)(s) &= \frac{(Rg_k)(s-1)}{s} \\
&= (1-k^{1-s})\frac{\zeta(s)}{s} \\
&= -k(k^{-s}-k^{-1})\frac{\zeta(s)}{s} \\
&= -kG_k(s).
\end{aligned}$$

With all this in mind, we state the following theorem.

Theorem 4.0.1. *The series* $\sum_{k=2}^{\infty} \mu(k) G_k$ *converges to* E *in* $H^2(\mathbb{C}_{3/2})$.

Proof. By Lemma 1.2.11 we have that

$$\sum_{k=2}^{n} \frac{\mu(k)}{k}(I-S)h_k \longrightarrow 1-z \quad \text{in} \quad H^2.$$

By the continuity of P and R, we have

$$\sum_{k=2}^{n} \frac{\mu(k)}{k} g_k \longrightarrow -z \quad \text{in} \quad H_0^2$$

and

$$\sum_{k=2}^{n} \frac{\mu(k)}{k} Rg_k \longrightarrow -1 \quad \text{in} \quad \mathcal{H}^2.$$

Applying the linear operator $\mathcal{M}_1$, we have

$$\sum_{k=2}^{\infty} \mu(k) G_k \longrightarrow E \quad \text{in} \quad H^2(\mathbb{C}_{3/2}).$$

□

Corollary 4.0.2. *E belongs to the closed linear span of $\{G_k : k \geqslant 2\}$ in $H^2(\mathbb{C}_{3/2})$.*

Proof. Follows as a consequence of Theorem 4.0.1. □

We turn next to improve Theorem 4.0.1. The objective will be to show that this convergence still holds in a Hardy space with a larger half-plane. Our motivation to do this will be Lemma 1.2.11. So we start by generalizing functions (4.0.1) in the following way. For $\tau > 1/2$ and $k \geqslant 1$, define

$$g_k^\tau(z) = k^{\tau-1} \sum_{n=1}^{\infty} \frac{z^n}{n^\tau} - \sum_{n=1}^{\infty} \frac{z^{nk}}{n^\tau}.$$

It is clear that $g_k^\tau \in H_0^2$. Notice that

$$\begin{aligned}
(Rg_k^\tau)(s) &= k^{\tau-1} \sum_{n=1}^{\infty} \frac{n^{-s}}{n^\tau} - k^{-s} \sum_{n=1}^{\infty} \frac{n^{-s}}{n^\tau} \\
&= k^{\tau-1} \sum_{n=1}^{\infty} n^{-(s+\tau)} - k^{-s} \sum_{n=1}^{\infty} n^{-(s+\tau)} \\
&= (k^{\tau-1} - k^{-s})\zeta(s+\tau), \quad s \in \mathbb{C}_{1/2}.
\end{aligned}$$

Let us define now the linear operator

$$\mathcal{M}_\tau : \mathcal{H}^2 \longrightarrow H^2(\mathbb{C}_{1/2+\tau})$$

given by

$$\mathcal{M}_\tau : f \longrightarrow f(s-\tau)/s.$$

Following the same ideas as in the case of $\mathcal{M}_1$, it is possible to prove that the operator $\mathcal{M}_\tau$ is well defined and bounded. Such operator $\mathcal{M}_\tau$ satisfies the following properties:

$$(\mathcal{M}_\tau 1)(s) = \frac{1}{s}$$

and

$$\begin{aligned}(\mathcal{M}_\tau R g_k^\tau)(s) &= \frac{(Rg_k^\tau)(s-\tau)}{s} \\ &= (k^{\tau-1} - k^{\tau-s})\frac{\zeta(s)}{s} \\ &= -k^\tau(k^{-s} - k^{-1})\frac{\zeta(s)}{s} \\ &= -k^\tau G_k(s).\end{aligned}$$

The following lemma is inspired from Lemma 1.2.11. The idea of the proof is still maintained.

Lemma 4.0.3. *The series* $\sum_{k=2}^{\infty} \frac{\mu(k)}{k^\tau} g_k^\tau$ *coverges to* $-z$ *in* H_0^2 *for each* $\tau > 1/2$.

Proof. Since $g_1^\tau = 0$, we have

$$\begin{aligned}\sum_{k=2}^{n} \frac{\mu(k)}{k^\tau} g_k^\tau &= \sum_{k=1}^{n} \frac{\mu(k)}{k^\tau} g_k^\tau(z) \\ &= \sum_{k=1}^{n} \frac{\mu(k)}{k^\tau}\left\{k^{\tau-1}\sum_{j=1}^{\infty}\frac{z^j}{j^\tau} - \sum_{j=1}^{\infty}\frac{z^{jk}}{j^\tau}\right\} \\ &= \sum_{k=1}^{n} \frac{\mu(k)}{k}\sum_{j=1}^{\infty}\frac{z^j}{j^\tau} - \sum_{k=1}^{n} \frac{\mu(k)}{k^\tau}\sum_{j=1}^{\infty}\frac{z^{jk}}{j^\tau}\end{aligned}$$

and

$$\left\|\sum_{k=1}^{n} \frac{\mu(k)}{k}\sum_{j=1}^{\infty}\frac{z^j}{j^\tau}\right\|_{H_0^2} = \left|\sum_{k=1}^{n} \frac{\mu(k)}{k}\right| \left\|\sum_{j=1}^{\infty}\frac{z^j}{j^\tau}\right\|_{H_0^2} \longrightarrow 0$$

as $n \to \infty$, because (see [1, Theorem 4.16])

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0.$$

So, we only need to prove that

$$\left\|-\sum_{k=1}^{n} \frac{\mu(k)}{k^\tau}\sum_{j=1}^{\infty}\frac{z^{jk}}{j^\tau} + z\right\|_{H_0^2} \longrightarrow 0 \tag{4.0.2}$$

as $n \to \infty$. Indeed,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\mu(k)}{k^\tau} \sum_{j=1}^{\infty} \frac{z^{jk}}{j^\tau} &= \sum_{k=1}^{n} \mu(k) \sum_{j=1}^{\infty} \frac{z^{jk}}{j^\tau k^\tau} \\
&= \mu(1) \sum_{j=1}^{\infty} \frac{z^j}{j^\tau} + \mu(2) \sum_{j=1}^{\infty} \frac{z^{2j}}{2^\tau j^\tau} + \cdots + \mu(n) \sum_{j=1}^{\infty} \frac{z^{nj}}{n^\tau j^\tau} \\
&= z\mu(1) + \frac{z^2}{2^\tau}(\mu(1) + \mu(2)) + \frac{z^3}{3^\tau}(\mu(1) + \mu(3)) + \ldots \\
&= \sum_{j=1}^{\infty} \frac{z^j}{j^\tau} \left(\sum_{\substack{d|j, \\ 1 \leqslant d \leqslant n}} \mu(d) \right) \\
&= \sum_{j=1}^{n} \frac{z^j}{j^\tau} \left(\sum_{d|j} \mu(d) \right) + \sum_{j=n+1}^{\infty} \frac{z^j}{j^\tau} \left(\sum_{\substack{d|j, \\ 1 \leqslant d \leqslant n}} \mu(d) \right) \\
&= \sum_{j=1}^{n} \frac{z^j}{j^\tau} \left[\frac{1}{j} \right] + \sum_{j=n+1}^{\infty} \frac{z^j}{j^\tau} \left(\sum_{\substack{d|j, \\ 1 \leqslant d \leqslant n}} \mu(d) \right). \qquad (4.0.3)
\end{aligned}$$

The last equality comes from the identity (see [1, Theorem 2.1])

$$\sum_{d|j} \mu(d) = \left[\frac{1}{j} \right].$$

Going back to (4.0.3), we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\mu(k)}{k^\tau} \sum_{j=1}^{\infty} \frac{z^{jk}}{j^\tau} &= z + \sum_{j=n+1}^{\infty} \left(\frac{1}{j^\tau} \sum_{\substack{d|j, \\ 1 \leqslant d \leqslant n}} \mu(d) \right) z^j \\
&= z + \phi_n(z),
\end{aligned}$$

where

$$\phi_n(z) = \sum_{j=n+1}^{\infty} \left(\frac{1}{j^\tau} \sum_{\substack{d|j, \\ 1 \leqslant d \leqslant n}} \mu(d) \right) z^j.$$

If we prove that $\|\phi_n\|_{H_0^2} \longrightarrow 0$, as $n \to \infty$, we shall be done.

If $\sigma(n)$ denotes the number of divisor of n, it follows that

$$\begin{aligned}\|\phi_n\|^2_{H^2_0} &= \sum_{j=n+1}^{\infty} \frac{1}{j^{2\tau}} \left| \sum_{\substack{d|j,\\ 1\leqslant d\leqslant n}} \mu(d) \right|^2 \\ &\leqslant \sum_{j=n+1}^{\infty} \frac{1}{j^{2\tau}} \left(\sum_{d|j} 1 \right)^2 \\ &= \sum_{j=n+1}^{\infty} \frac{\sigma(j)^2}{j^{2\tau}}.\end{aligned}$$

The function σ satisfies the relation $\sigma(n) = o(n^\epsilon)$[1], for every $\epsilon > 0$ (see [1, page 296]). In particular, $\sigma(n) \lesssim n^\epsilon$ for some $\epsilon > 0$ such that $\tau > 1/2 + \epsilon$. Therefore

$$\|\phi_n\|^2_{H^2_0} \lesssim \sum_{j=n+1}^{\infty} \frac{j^{2\epsilon}}{j^{2\tau}} \leqslant \sum_{j=n+1}^{\infty} \frac{1}{j^{2\tau-2\epsilon}}.$$

Since $2\tau - 2\epsilon > 1$, we have

$$\|\phi_n\|_{H^2_0} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This proves (4.0.2) and the lemma. □

The refinement of Theorem 4.0.1 can be stated as follows.

Theorem 4.0.4. *The series* $\sum_{k=2}^{\infty} \mu(k) G_k$ *converges to* E *in* $H^2(\mathbb{C}_{1/2+\tau})$ *for every* $\tau > 1/2$.

Proof. By Lemma 4.0.3, we have

$$\sum_{k=2}^{n} \frac{\mu(k)}{k^\tau} g_k^\tau \longrightarrow -z \ \text{ in } \ H^2_0.$$

Applying the linear operator $\mathcal{M}_\tau \circ R$, we obtain that

$$-\sum_{k=2}^{n} \frac{\mu(k)}{k^\tau} k^\tau G_k \longrightarrow -E \ \text{ in } \ H^2(\mathbb{C}_{1/2+\tau}).$$

This means that

$$\sum_{k=2}^{n} \mu(k) G_k \longrightarrow E \ \text{ in } \ H^2(\mathbb{C}_{1/2+\tau}).$$

□

Corollary 4.0.5. *E belongs to the closed linear span of $\{G_k : k \geqslant 2\}$ in $H^2(\mathbb{C}_{1+\epsilon})$ for every $\epsilon > 0$.*

[1] The notation $f(n) = o(g(n))$ as $n \to \infty$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Proof. Follows as a consequence of Theorem 4.0.4. □

The most interesting fact of Theorem 4.0.4 is that we can know explicitly the coefficients; so the question is: this convergence is still maintained in the Hardy space $H^2(\mathbb{C}_{1/2})$? If so, the RH would be true.

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