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A Modal Approach to Logical Consistency

Uma abordagem modal para consistência lógica

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A MODAL APPROACH TO LOGICAL CONSISTENCY

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Resumo

Nesta Tese, investigamos o conceito de consistência por meio de suas formalizações semânticas e sintáticas. Em especial, mostramos que tais formalizações não capturam inteiramente o conceito pré-teórico, intuitivo, de consistência.

Nossa análise do conceito dá-se mediante o argumento de Kreisel, conhecido como *argumento de compressão*. Ao examinar o argumento original, bem como suas variantes, concluímos que as formalizações semânticas e sintáticas da consistência capturam apenas parcialmente o conceito informal de consistência, embora essas formalizações sejam explicações do conceito informal. Além disso, mostramos que o próprio conceito informal de consistência é teoricamente sofisticado para ser considerado como pré-teórico, ou intuitivo.

Uma vez que as formalizações semânticas e sintáticas sejam explicações de suas contrapartes informais, investigamos as propriedades gerais do conceito semântico bem como as do conceito sintático de consistência. Para tal, utilizamos extensivamente as lógicas modais, visto que elas são ferramentas tradicionalmente utilizadas na análise de conceitos formais, tais como o conceito de provabilidade aritmética. Nossa análise do conceito de consistência abrange teorias formais que não possuem necessariamente o mesmo poder expressivo de teorias aritméticas. Então, mostramos que as propriedades gerais dos conceitos formais de consistência são capturadas por lógicas modais não-normais consideravelmente fracas.

Palavras-chave: Kreisel, Georg, 1923-2015 - Crítica e interpretação, Consequência lógica, Modalidade (Lógica), Lógica simbólica e matemática.

Abstract

In this Thesis, we investigate the concept of consistency through its semantic and syntactic formalizations. Particularly, we show that such formalizations do not totally capture the pre-theoretical, intuitive concept of consistency.

Our analysis of this concept takes place through Kreisel's squeezing argument. In analyzing Kreisel's original argument, and its variants, we show that the semantic and syntactic formalizations of consistency partially capture the informal concept of consistency, even if they are explications of the informal concept. Moreover, we show that the informal concept of consistency is theorized enough to be called pre-theoretical or intuitive consistency.

Given that the semantic and syntactic formalizations of consistency are explications of the informal concept, we investigate the general properties of such formalizations. For such purposes, we extensively use modal logics since they are traditionally used tools in the analysis of formal concepts, such as the concept of provability in arithmetical theories. Our analysis of the concept of consistency comprehends formal theories which do not necessarily have the expressive power of arithmetical theories. Then, we show that considerably weak non-normal modal logics captures the general properties of the formal counterparts of consistency.

Keywords: Kreisel, Georg, 1923-2015 - Criticism and interpretation, Logical consequence, Modality (Logic), Symbolic and mathematical logic.

Contents

Chapter 1

Introduction: modal logic and formal concepts

This Thesis defends that the informal counterpart of the formal concept of consistency, present in mathematical and logical discussions, is not a primitive notion. The concept of consistency has different characterizations in literature. In some characterizations, consistency is understood as non-contradictoriness. The leibnizian account of modality, a proposition is possible if it is not self-contradictory (LOOK, 2013). In this sense, possible propositions are consistent. On the other hand, it is not necessary that consistency involves, at least directly, non-contradictoriness. For example, in Metaphysics, consistency means persistence over time and change. That is, a property P is consistent if it persists over time and change. In this sense, consistency has a narrow connection with the concept of essence (Fine (1994)) because P may be a property of an object O without which O would not exist. Instead, consistency is significantly theorized already based in its informal characterization. Here we will focus on the logical characterization of consistency, where it was largely investigated. The concept of consistency had a central role in development of modern logic due to the "crisis" of paradoxes in foundations of Mathematics in the beggining of 20th century. For example, the Russell's paradox posed a real challenge to Set Theory as a foundation of Mathematics until Zermelo's work in 1908.[1] According to Kneale & Kneale (1962) and Zach (2016), already in 1899, Hilbert had drawn attention to the need for the axiomatization of logic and mathematics as a whole in order to avoid paradoxes. According to Hilbert, from the axiomatization of a theory T it is possible to know if T is consistent or not by means of a finitary proof. Disregarding the question of whether or not Hilbert's program succeeded, his program had the importance of putting consistency as a central issue in Mathematics.

Given a theory T one can define consistency in two ways:

Definition 1.0.1 (Proof theory)**.** *T is* consistent *iff T does not prove a contradiction φ*

[1] See Kneale & Kneale (1962) and Enderton (1977) for more historical details.

and $\neg\varphi$.

Definition 1.0.2 (Model theory)**.** T *is* consistent *iff there is a model which makes true the formulas of* T.

Gödel's completeness theorem for First-Order Logic (FOL) establishes that the theorems of FOL are precisely the valid formulas. Thus, this theorem extensionally collapses the set of valid formulas (model-theoretical) with the set of theorems (proof-theoretical). Thus, ultimately, this theorem also collapses these two formal definitions of consistency.

Although consistency can be formally characterized by two different ways, such formal approaches are different in nature. The proof-theoretical counterpart of a theory T is finite by its nature. The syntactical proofs within T are finite, because proofs are defined as a finite sequences of formulas. But this is not necessarily the case with the model-theoretical counterpart of T, simply because models can be infinite. Moreover, there are theories such that their proof systems are not capable to capture the semantical validities of T. This conceptual difference between the semantical and the syntactical aspects of consistency inpired proposals of investigating consistency at the informal level, such as Kreisel (1967)'s squeezing argument, which aims to justify the use of informal concepts in Mathematics. One of the concepts which Kreisel concentrates is the concept of logical validity. Then, he considers the informal validity, which is stronger than the standard model-theoretical notion of validity because it comprehends structures whose domains are classes. In this sense, informal validity is reducible neither to syntactical validity nor to semantical validity. By establishing principles which connects informal validity to its formal counterparts, Kreisel shows that the completeness theorem collapses these three aspects of validity. In this sense, as Andrade-Lotero & Novaes (2012) observe, informal validity can be taken as a bridge that connects syntactical validity and semantical validity. This connection is attested by the completeness theorem, which proves the extensional equivalence of the set of valid formulas T and the set of theorems of T. But, as we will see, Kreisel's definition of informal validity comprises only validity in First-Order Logic in a way this concept of informal validity does not work for other logics, like Infinitary Logic and Second-Order Logic. This limitation is pointed out by Kennedy & Väänänen (2017), which propose to give broader definitions of informal validity capable to give squeezing arguments for other logics. They conclude that each logic has its own definition of informal validity, which corresponds to the formal validity definitions of the particular system.

As we can see, the concept of informal validity seems to have a local character, so being unable to cover the plethora of the existing formal logical systems. If there is such concept, may be it is not interestingly informative. On the other hand, there are broader notions of validity than logical validity, such as Myhill (1960)'s absolute provability of mathematical sentences. But, as Myhill himself argues, such notion covers only sentences

from set-theory and analysis, and it heavily relies on set-theoretical vocabulary. So, the existence of a notion of validity, and so consistency, which covers all domains of knowledge does not seem to make sense.

In this Thesis, we will treat consistency as a derived notion from the model-theoretical and proof-theoretical apparatus and we will investigate the minimal properties that such concept. This means that we will investigate the properties of consistency by means of a semantic predicate Con as the dual of validity predicate Val and a syntactic predicate Con' as the dual of a provability predicate $Prov$. So our discussion will be also a discussion about the validity and provability predicates. Second, we will show that these approaches can be formalized in modal logics. That is, the predicate of validity will be related to the modality $\Box$ and the predicate of consistency will be related to the modality $\Diamond$. So, we investigate what are the minimal properties that these modalities of consistency should satisfy in order to be called a modality of consistency. As we will see, in the systematization of consistency in complete logical theories, $\Box$ will stand for both Val and $Prov$.

Tarski (1956)'s undefinability result about truth in mathematical languages and Gödel's incompleteness theorems inaugurated a trend about the incorporation of semantic and syntactic concepts in the object language of formal theories. Gödel's (1931,1986b) formalization of the provability predicate in his proof of his incompleteness theorems instigated several investigations of provability in arithmetical theories. In the case of the validity predicate, one of the first investigations of this predicate can be traced to Myhill (1960)'s investigation of the predicate of absolute provability, *i.e*, provability which is not particular to a single formal theory. In general terms, the main investigations about the predicate of validity aim to give consistent formalizations of its general properties in a sufficient strong and theory of syntax, where it is possible to talk about expression, formulas and sentences. Generally such theory is also Peano Arithmetic (PA). By Montague (1963)'s theorem, such formalization is not possible if we maintain that validity is truth-preserving and that arithmetical theorems are valid in this wide sense. Then, the proposals that we find nowadays adopt theories weaker than PA in the formalization of truth or they adopt validity predicates. Both lines of research aim to establish what are the most general principles about validity as a semantic concept.

The sistematization of formal concepts by means of modal logic showed itself to be fruitful from Gödel (1986a)'s work where he showed that the modal logic S4 satisfies *Brouwer-Heyting-Kolmogorov* interpretation for Intuitionistic Logic. But the main application of modal logic to formal concepts was the *provability logics*. The main investigation on logics of provability was due to Solovay (1976), when he uses the logic KGL to interpret the provability predicate of Peano Arithmetic. Solovay's work started a trend about the use of modal logics to interpret provability in formal theories.[2] One example is the

[2]Urbaniak & Pawlowski (2018) present many applications of modal logics to formal theories.

logic of proofs presented by Artëmov & Straßen (1992), where they interpret the explicit provability predicates in a modal language. Thus, $\Box_x\varphi$ means that "x is a proof of φ". In addition to the application of modal logics to formal theories of arithmetic, there are also applications of modal logic to describe the Tarskian consequence operator, such as Naumov (2006)'s and Mortari e Feitosa (2011)'s approaches, which can be seen as logics of provability of classical propositional logic.

This Thesis is organized as follows. Chapter 2 presents the notation and some systems used in the course of the discussion. Thus, Classical Propositional Logic, First-Order Logic, First-Order Peano Arithmetic and Modal Logics will be presented. The presentation of First-Order Peano Arithmetic will be relevant to delineate some important results about Metamathematics. The presentation of Modal Logics is justified by the fact that we will use these logics to interpret the provability predicate as the modality of necessity.

Chapter 3 discusses the plausibility of taking the notion of consistency as a primitive notion, which does not depend on abstract entities. Thus, we present Kreisel (1967)'s squeezing argument as a proposal for considering the notion of informal validity. We will see that his characterization does not yield a primitive notion since it depends on mathematical abstract entities. Then we will present two proposals which defend that consistency is a primitive notion. First, we present Field (1991)'s version of the squeezing argument, which attempts to pose consistency as a primitive notion which is, in a certain sense, captured by a modality. As we will see, his argument faces some objections in such a way that fails to establish the primitiveness of consistency. Second, we present the paraconsistent approach to consistency. Our aim is to show that this approach does not succeed to justify that the connective of consistency $\circ$, present in Logics of Formal Inconsistency, has some counterintuitive aspects if it is interpreted as consistency in a more mathematical sense. We finish the chapter by discussing the difficulty of taking such notion as a primitive one. And, then, we suggest to treat consistency in two separated ways: consistency as a derived notion from the perspective of proof-theory as well as a derived notion from the perspective of model-theory.

Chapters 4 and 5 propose to consider consistency as a derived notion. Chapter 4 concentrates on the interpretation of the modality $\Box$ as standing for the syntactic provability predicate *Prov*. In order to do so, we first present the provability predicate $Prov_{\mathsf{PA}}$ formalizable in Peano Arithmetic as well as the fundamental results concerning this predicate such as *Gödel's Incompleteness Theorems*. Then we present the modal logic KGL, the modal logic that completely captures the predicate $Prov_{\mathsf{PA}}$. We will present the modal operator $\boxdot$ which captures the PA predicate of true provability. In the case of $\boxdot$ we will present the minimal logic of the modality $\mathsf{K}^{\boxdot}$, which captures the general properties of such modality. We will discuss an arithmetical completeness theorem for KD presented by Kurahashi (2018) with respect to Rosser provability predicate. Last we will discuss if it is possible to obtain a modality which captures the consistent provability predicate of

PA.

Chapter 5 concentrates on the interpretation of $\Box$ as the semantic validity predicate Val. Thus, we present Skyrms (1978)'s hierarchical interpretation of modal logics, where he defends that the logic S5 is the modal logic which formalizes the predicate Val in a hierarchical setting. But as we will see, S5 is not adequate to interpret validity in a non-hierarchical setting. Thus we present alternative proposals of Val, such as the Ketland (2012)'s proposal. As we will see, the non-normal modal logic S0.5 is sound but not complete with respect to Ketland's proposal. Last, we will present a validity predicate for FOL and we will prove that it is captured by the first-order modal logic QS0.5. In what concerns the latter result, we will discuss the status of Barcan Formula.

Chapter 6 discusses the systematization of consistency in non-classical logic. Given the plethora of non-classical logics, it is quite reasonable to wonder about the properties of this metatheoretical notion in this logics. Here we will concentrate our discussion to the case of Many-valued Logics, the logics which have more values than truth and falsity. Given these logics, call them L we will prove that the modal logics $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5} capture, respectively the predicate of logical validity in L's and hierarchical validity in L's. We also show that $\Box$ and $\Diamond$ in such logics may work as recovery operators.

Some parts of this thesis were already published in co-autorship:

- The discussion about informal pluralism in Chapter 3 is presented in "Squeezing arguments and the plurality of informal notions". (joint work with Giorgio Venturi). *Journal of Applied Logics - IfCoLog Journal*. Number 7. Volume 8. 2021.

- The basic logic $\mathsf{K}^{\boxplus}$ is originally presented in "A Non-Standard Kripke Semantics for the Minimal Deontic Logic" (joint work with Giorgio Venturi). *Logic and Logical Philosophy*. Number 1. Volume 30. 2021;

- Chapter 6 was based on "Many-valued logics and bivalent modalities" (joint work with Giorgio Venturi). Submitted.

Chapter 2

Logical preliminaries

In this section we specify the notation we will use througout this work.

2.1 Propositional Logics

2.1.1 Basic theory: Language and semantics

The language $\mathcal{L}_{\mathsf{L}}$ of a logic L is the set $\mathcal{L}_{\mathsf{L}} = \{\mathcal{V}, c_1^{k_1}, \ldots, c_m^{k_m}\}$, where $\mathcal{V} = \{p_i \mid i \in \mathbb{N}\}$ is the set of propositional variables,[1] $c_1^{k_1}, \ldots, c_m^{k_m}$ are connectives of such that the arity $c_i^{k_i}$ is k. The set of formulas of $\mathcal{L}_{\mathsf{L}}$, $For(\mathcal{L}_{\mathsf{L}})$ is defined inductively as follows

The set of formulas of $\mathcal{L}_{\mathsf{L}}$, $For(\mathcal{L}_{\mathsf{L}})$, is defined inductively as follows:

$$p_i \mid c_i^{k_i}(\varphi_1, \ldots, \varphi_k)$$

For $1 \leq i \leq n$ and $\varphi_1, \ldots, \varphi_k \in For(\mathcal{L}_{\mathsf{L}})$.

Definition 2.1.1. *A* matrix *for* L *is a structure* $M_{\mathsf{L}} = \langle V_n, o_1^{k_1}, \ldots, o_n^{k_n}, D_{\mathsf{L}} \rangle$ *where* $V_n = \{\frac{m}{n-1} | \ 0 \leq m \leq n-1, m, n \in \mathbb{N}\}$ *is the set of truth-values,* $o_1^{k_1}, \ldots, o_n^{k_n}$ *are operations on* V_n *such that the arity* $o_i^{k_i}$ *is* k, $D_{\mathsf{L}} \subset V_n$ *is the set of designated values* $\{\frac{r}{n-1}, \ldots, 1\}$, *where* $0 < r$. *For practical purposes we will assume that the values 1 and 0 denote the classical values of* truth *and* falsity. *A* valuation v *is a homomorphism* $v : \mathcal{V} \to V_n$ *which is extended to* $For(\mathcal{L}_{\mathsf{L}})$ *as follows:*

1 $v(c_m^{k_m}(\varphi_1, \ldots, \varphi_k)) = o_m^{k_m}(v(\varphi_1), \ldots, v(\varphi_k))$.

The definition of $o_m^{k_m}$ *is particular to the specific logic. The set of valuations* $v : For(\mathcal{L}_{\mathsf{L}}) \to V_n$ *is called the semantics of* $\mathcal{L}_{\mathsf{L}}$, sem_{L}.

Definition 2.1.2. *Let* $\varphi \in For(\mathcal{L}_{\mathsf{L}})$. *We say that* v *is a* model *for* φ *if* $v(\varphi) \in D_{\mathsf{L}}$. *Let* $\Gamma \subseteq For(\mathcal{L}_{\mathsf{L}})$. *We say that* v *is a model of* Γ *if* v *is a model of each* $\gamma \in \Gamma$. *If*

[1] For the sake of simplicity, we use the variables p,q,r,... instead of p_0,p_1,p_2,p_3, and so on.

$v(\varphi) \in D_{\mathsf{L}}$, for some (resp., for every) $v \in sem_{\mathsf{L}}$, we say that φ is satisfiable *(resp., a* tautology*) of* L. *If $v(\varphi) \notin D_{\mathsf{L}}$ for every $v \in sem_{\mathsf{L}}$, then φ is a* anti-tautology *of* L. *The* semantic consequence relation, *$\models_{sem_{\mathsf{L}}} \subseteq \wp(For(\mathcal{L}_{\mathsf{L}})) \times For(\mathcal{L}_{\mathsf{L}})$ is defined as follows: consider $\Gamma \cup \{\alpha\} \subseteq For(\mathcal{L}_{\mathsf{L}})$. We say that α is a* semantic consequence of *($\Gamma \models_{sem_{\mathsf{L}}} \alpha$) Γ iff: if $v(\gamma) \in D_{\mathsf{L}}$, for every $\gamma \in \Gamma$, then $v(\alpha) \in D_{\mathsf{L}}$.*

When the context is clear, we shall omit the subscript. It is clear that the definition of $\models_{\mathsf{L}}$ as preservation of designated values from premises to conclusion implies the following properties:[2]

(1) $\Gamma \cup \{\varphi\} \models_{\mathsf{L}} \varphi$ (Reflexivity);

(2) $\Gamma \models_{\mathsf{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \models_{\mathsf{L}} \varphi$ (Monotonicity);

(3) If $\Gamma \models_{\mathsf{L}} \varphi$ and $\Delta, \varphi \models_{\mathsf{L}} \psi$, then $\Delta, \Gamma \models_{\mathsf{L}} \psi$ (Transitivity);

(4) If $\Gamma \models_{\mathsf{L}} \varphi$, then, for every substitution σ, $\sigma(\Gamma) \models_{\mathsf{L}} \sigma(\varphi)$,
where $\sigma(\Gamma) = \{\sigma(\gamma) \mid \gamma \in \Gamma\}$ (Structurality).

Definition 2.1.3. *A logic* L *is* Tarskian structural *if its consequence relation satisfies (1)-(4).*

Classical Propositional Logic

The language $\mathcal{L}_{\mathsf{CPL}}$ of CPL is the set $\mathcal{L}_{\mathsf{CPL}} = \{\mathcal{V}, \neg, \rightarrow\}$ where $\neg$ is a unary connective of negation and $\rightarrow$ is a binary connective of implication. Then, its set of formulas is recursively defined as follows:

$$p_i \mid \neg\varphi \mid \varphi \rightarrow \psi$$

For $\varphi, \psi \in For(\mathcal{L}_{\mathsf{CPL}})$. A *matrix* for CPL is a structure $\mathcal{M} = \langle\{1,0\}, \{\neg, \rightarrow\}, \{1\}\rangle$, where 1 is the only distinguished value. The operators of the matrix have the following truth-tables:

	$\neg$
1	0
0	1

$\rightarrow$	1	0
1	1	0
0	1	1

Definition 2.1.4. A *valuation* v of CPL is a function $v : For(\mathcal{L}_{\mathsf{CPL}}) \rightarrow \{1,0\}$ governed by the following conditions:

[2] See Łoś & Suszko (1958) and Wójcicki (2013) for results concerning consequence relations.

(1) $v(\neg\varphi) = 1 - v(\varphi)$;

(2) $v(\varphi \to \psi) = min\{1, (1 - v(\varphi)) + v(\psi)\}$

The set of valuations $v : For(\mathcal{L}_{\mathsf{CPL}}) \to \{1, 0\}$ is called *semantics* of CPL, sem_{CPL}.

In Mendelson (2009) one finds the following axiomatization of CPL:

Definition 2.1.5. *(MENDELSON, 2009) The logic* CPL *is axiomatized as follows:*

(Ax1) $\varphi \to (\psi \to \varphi)$;

(Ax2) $(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$;

(Ax3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$;

(MP) *From* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *we infer* $\vdash \psi$;

The provability relation $\vdash_{\mathsf{CPL}} \subseteq \wp(For(\mathcal{L}_{\mathsf{CPL}})) \times For(\mathcal{L}_{\mathsf{CPL}})$ is defined as follows:

Definition 2.1.6. $\Sigma \vdash_{\mathsf{CPL}} \varphi$ *if there is a sequence* $\langle \varphi_1, \ldots, \varphi_{n-1}, \varphi_n \rangle$ *such that:*

(1) *For each* γ_j $(1 \leq j \leq n - 1)$*:*

(1.a) *is an axiom;*

(1.b) $\gamma_j \in \Gamma$*; or* (1.3) γ_j *is obtained by the application of inference rules to* γ_i*'s for* $i < j$.

(2) $\varphi_n = \varphi$.

If $\Gamma = \emptyset$*, we say that* φ *is a* theorem *of* CPL.

Mendelson (2009) proves that CPL is sound and complete with respect to $\mathcal{M}_{\mathsf{CPL}}$. Definition 2.1.6 can be easily extended to extensions of CPL such as first-order logic and modal logics.

2.2 First-Order Logic (FOL)

The language $\mathcal{L}_{\mathsf{FOL}}$ of FOL is the set $\mathcal{L}_{\mathsf{FOL}} = \{V, C, P, F, =, \neg, \to, \forall\}$ where $V = \{x_n | n \in \mathbb{N}\}$ is the set of individual variables, $C = \{c_n | n \in \mathbb{N}\}$ is the set of individual constants, $P = \{P^n_k | n, k \in \mathbb{N}\}$ is the set of relation symbols, $F = \{f^n_k | n, k \in \mathbb{N}\}$ function symbols, and $=$ is the symbol of identity. All the definitions in this section are taken from (MENDELSON, 2009).

Definition 2.2.1. *The terms of the language are recursively defined as follows:*

(1) variables and individual constants are terms;

(2) if f_k^n is a function symbol and $t_1, \ldots, t_n$ are terms, then $f_k^n(t_1, \ldots, t_n)$ is a term;

(3) an expression is a term if it can be shown to be a term on basis of 1 and 2.

The set of terms of $\mathcal{L}_{\mathsf{FOL}}$ is denoted by $Term$.

Definition 2.2.2. *The set $For(\mathcal{L}_{FOL})$ of well-formed formulas of $\mathcal{L}_{\mathsf{FOL}}$ are defined as follows:*

(1) if P_k^n is a relation symbol and $t_1, \ldots, t_n$ are terms, then $P_k^n(t_1, \ldots, t_n) \in For(\mathcal{L}_{\mathsf{FOL}})$;

(2) if $\varphi, \psi \in For(\mathcal{L}_1)$, then $\neg\varphi, \varphi \to \psi \in For(\mathcal{L}_{\mathsf{FOL}})$;

(3) if x_i is a individual variable and φ is a formula, then $\forall x_i\varphi \in For(\mathcal{L}_{\mathsf{FOL}})$ is a formula.

(4) an expression is a formula if it can be shown to be a term on basis of i-iii.

Let t be a term and φ a formula of $\mathcal{L}_{\mathsf{FOL}}$. We say that t is *free for* x_i in φ if no free occurrence of x_i in the formula φ lies in the scope of any quantifier $\forall x_j$, where x_j is a variable in t. Finally, a formula φ is a *sentence* or a *closed formula* if φ has no free variable occuring in φ.

Definition 2.2.3. *The set of sentences of $\mathcal{L}_{\mathsf{FOL}}$ is denoted by $Sent(\mathcal{L}_{\mathsf{FOL}})$ If x_i occurs free in φ, we write $\varphi(x_i)$.*

Mendelson provides the following axiom system for FOL:

Definition 2.2.4. FOL *is axiomatized as follows:*

(CPL) Ax1 - Ax3 *of Definition 2.1.5;*

(Ax4) $\forall x_i\varphi(x_i) \to \varphi(t)$ *where t is free for x_i;*

(Ax5) $\forall x_i(\varphi \to \psi) \to (\varphi \to \forall x_i\psi)$ *if φ does not contain free occurrences of x_i;*

(Ax6) $\forall x_i(x_i = x_i)$;

(Ax7) $x_i = x_j \to (\varphi(x_i, x_i) \to \varphi(x_i, x_j))$;

(MP) *From $\vdash \varphi$ and $\vdash \varphi \to \psi$ we infer $\vdash \psi$;*

(Gen) *from $\vdash \varphi$ we infer $\vdash \forall x_i\varphi$.*

It is clear that all tautologies of CPL are probable in FOL by using only the axioms Ax1 - Ax3 and the rule MP of Definition 2.1.5. The notion of proof of FOL is defined similarly as in Definition 2.1.6. Then the following proposition is immediate:

The existential quantifier is traditionally introduced as follows:

$$\exists x_i \varphi := \neg \forall x_i \neg \varphi$$

Theorem 2.2.5. *(MENDELSON, 2009) Every $\varphi' \in \mathcal{L}_{\mathsf{FOL}}$ which is an instance of a* CPL*-tautology φ is a theorem of* FOL*, and it may be proved using only axioms Ax1 - Ax3 of Definition 2.1.5 and modus ponens.*

Proof. Let ψ be a tautology CPL. Since CPL is complete, ψ is provable by the axioms Ax1 - Ax3 of Definition 2.1.5 and modus ponens. Call this proof $\mathcal{D}$. Given $\mathcal{D}$ uniformly substitute the formulas γ occur in the proof $\mathcal{D}$ by formulas γ' of FOL. The resulting proof, call it $\mathcal{D}'$, will be a proof of a substitution instance of the tautology ψ in FOL. Q.E.D.

By Proposition 2.2.5 it is possible to use tautology instances of CPL in FOL proofs. In each step of proofs where tautology instances are introduced we will write CPL.

We define now the semantics for the language $\mathcal{L}_{\mathsf{FOL}}$.

Definition 2.2.6. *An interpretation $\mathfrak{A} = \langle D, (\cdot)^{\mathfrak{A}} \rangle$ for the language $\mathcal{L}_{\mathsf{FOL}}$ is a pair where $D \neq \emptyset$ is a non-empty set called domain and $(\cdot)^{\mathfrak{A}}$ is a function such that:*

(1) For each relation symbol P_k^n of $\mathcal{L}_{\mathsf{FOL}}$ there is an assigment of a n-place relation $(P_k^n)^{\mathfrak{A}}$ in D;

(2) For each function symbol f_k^n of $\mathcal{L}_{\mathsf{FOL}}$ there is an assigment of a n-place operation $(f_k^n)^{\mathfrak{A}}$ in D;

(3) For each individual c_n of $\mathcal{L}_{\mathsf{FOL}}$ there is an assigment of some fixed element $(c_n)^{\mathfrak{A}}$ of D.

Let $s = (s_1, s_2, s_3 \ldots)$ be a denumerable sequence of objects of D and Seq be the set of all denumerable sequences s of D. Given a $s \in Seq$ we define a function $s^* : Term \to D$, which assigns to each term $t \in Term$ an element $s^*(t) \in D$ as follows:

Definition 2.2.7. *Let $s \in Seq$ be a sequence of objects of D. The function $s^* : Term \to D$ is defined as follows:*

(1) If $t \in Term$ is a variable x_i, let $s^*(x_i) = s_i$;

(2) If $t \in Term$ is a constant c_i, then $s^*(c_j) = (c_n)^{\mathfrak{A}}$;

(3) If $t \in Term$ is $f_k^n(t_1, \ldots t_n)$, where $f_k^n \in F$ and $t_1, \ldots t_n \in Term$, then $s^*(f_k^n(t_1, \ldots t_n)) = (f_k^n)^{\mathfrak{A}}(s^*(t_1), \ldots, s^*(t_n))$.

Now we define the notion of satisfiability

Definition 2.2.8. *Let $s \in Seq$ be a sequence and φ a formula $\mathcal{L}_{\mathsf{FOL}}$, the notion of satisfiability of a formula φ, $\mathfrak{A} \models_s \varphi$, is defined as follows:*

(1) If φ *is* $P^n_k(t_1, \ldots t_n)$*, then* $\mathfrak{A} \models_s P^n_k(t_1, \ldots t_n)$ *iff* $(s^*(t_1), \ldots, s^*(t_n)) \in (P^n_k)^{\mathfrak{A}}$*;*

(2) If φ *is* $t_1 = t_2$*, then* $\mathfrak{A} \models_s t_1 = t_2$ *iff* $s^*(t_1) = s^*(t_2)$

(3) If φ *is* $\neg\psi$*, then* $\mathfrak{A} \models_s \neg\psi$ *iff* $\mathfrak{A} \not\models_s \psi$*;*

(4) If φ *is* $\psi \to \gamma$*, then* $\mathfrak{A} \models_s \psi \to \gamma$ *iff* $\mathfrak{A} \not\models_s \psi$ *or* $\mathfrak{A} \models_s \gamma$*;*

(5) If φ *is* $\forall x_j \psi$*, then* $\mathfrak{A} \models_s \forall x_j \psi$ *iff every sequence* $s' \in Seq$ *that differs from* s *in at most the* i*-th component is such that* $\mathfrak{A} \models_{s'} \psi$*.*

We say that φ *is* true *in* $\mathfrak{A}$ *(*$\models_{\mathfrak{A}} \varphi$*) (or* $\mathfrak{A}$ *is a* model *of* φ*) iff every sequence* $s \in Seq$ *satisfies* φ*.* $\mathfrak{A}$ *is a model for* Γ *iff every* $\gamma \in \Gamma$ *is such that* $\mathfrak{A}$ *is a model of* γ*.* φ *is* logically valid *(respectively,* satisfiable*) iff* φ *is true for every (resp., some) model* $\mathfrak{A}$*. When* φ *is valid, we write* $\models \varphi$*. We say that* ψ *is a* logical consequence *of* Γ *iff for every* $\mathfrak{A}$*, if* $\mathfrak{A}$ *is a model of* Γ*,* $\mathfrak{A}$ *is a model of* ψ*.*

As a immediate consequence of Definition 2.2.8, we have the following consequence:

Proposition 2.2.9. $\mathfrak{A} \models \varphi$ *iff* $\mathfrak{A} \models \forall x_i \varphi$*.*

We now state without proof important lemmas about quantification which will be discussed in Chapter 5.

Lemma 2.2.10. *If the variables of* φ *occurr in the list* $x_{i_1}, \ldots, x_{i_n}$ *and if the sequences* s *and* s' *have the same components in* $ith_1, \ldots, ith_n$ *places of the sequence, then*

$$\mathfrak{A} \models_s \varphi \text{ iff } \mathfrak{A} \models_{s'} \varphi. \tag{2.1}$$

For all $\varphi \in \mathcal{L}_{\mathsf{FOL}}$*.*

Lemma 2.2.11. *Let* t *be free for* x_i *in* $\varphi(x_i)$*. Then:*

(A) *A sequence* $s = (s_1, s_2, \ldots)$ *satisfies* $\varphi(t)$ *iff the sequence* s'*, obtained from* s *by substituting* $s^*(t)$ *for* s_i *in the* i*th place, satisfies* $\varphi(x_i)$*.*

(B) *If* $\forall x_i \varphi(x_i)$ *is satisfied by the sequence* s*, then* $\varphi(t)$ *also is satisfied by* s*.*

In Mendelson (2009) one finds the characterization results with respect to FOL.

Theorem 2.2.12. *If* $\vdash \varphi$*, then* $\models \varphi$*.*

Theorem 2.2.13. *If* $\models \varphi$*, then* $\vdash \varphi$*.*

2.2.1 Peano Arithmetic (PA)

The language $\mathcal{L}_{\mathsf{PA}} = \{+, \cdot, \mathbf{s}, 0\}$ is built on $\mathcal{L}_{\mathsf{FOL}}$ where $+$, $\cdot$, $\mathbf{s}$ are takes as function symbols and 0 is taken as a constant.

Definition 2.2.14. *(MENDELSON, 2009)* PA *has the following axioms:*

(PA1) $x_i = x_j \rightarrow (x_i = x_l \rightarrow x_j = x_l)$;

(PA2) $x_i = x_j \rightarrow \mathbf{s}(x_i) = \mathbf{s}(x_j)$;

(PA3) $0 \neq \mathbf{s}(0)$;

(PA4) $\mathbf{s}(x_i) = \mathbf{s}(x_j) \rightarrow x_i = x_j$;

(PA5) $x_i + 0 = x_i$;

(PA6) $x_i + \mathbf{s}(x_j) = \mathbf{s}(x_i + x_j)$;

(PA7) $x_i \cdot 0 = 0$;

(PA8) $x_i \cdot \mathbf{s}(x_j) = \mathbf{s}(x_i \cdot x_j) + x_i$;

(PA9) $\varphi(0) \rightarrow (\forall x_i(\varphi(x_i) \rightarrow \varphi(\mathbf{s}(x_i)) \rightarrow \forall x_i \varphi(x_i))$ *for all* $\varphi(x_i)$.

Let $\exists x < y\varphi$ and $\forall x < y\varphi$ abbreviate $\exists x(x < y \wedge \varphi)$ and $\forall x(x < y \rightarrow \varphi)$. These formulas are called *bounded formulas*. Consider now the following definition:

Definition 2.2.15. *(BOOLOS, 1995) A* strict Σ_1-formula *is a member of the smallest class that contains all formulas* $x_i = x_j$, $0 = x_j$, $\mathbf{s}(x_i) = x_j$, $x_i + x_k = x_j$, $x_i.x_k = x_j$ *and contains* $\varphi \wedge \psi$, $\varphi \wedge \psi$, $\exists x\varphi$ *and* $\forall x < y\varphi$, *whenever it contains* φ *and* ψ. *A* Σ_1-formula *is one that is provably equivalent in* PA *to a strict* Σ_1-formula.

Boolos (1995) observes that the class of Σ_1-formulas are of specific interest since every recursive function can be represented as a Σ_1-formula. And it can be proved that:

Theorem 2.2.16. *If* φ *is a true* Σ_1*-sentence, then* $\vdash_{\mathsf{PA}} \varphi$.

The proof of Theorem 2.2.16 can be checked in (BOOLOS, 1995, p.25). Boolos says that this theorem establishes a kind of *partial* completeness of PA. The reason to say that Theorem 2.2.16 is a partial completeness due to the fact that PA fails to prove some arithmetical truths which are negations of Σ_1-formulas. One of these formulas is a formula which states the consistency of PA. As we will see further, PA is not able to prove its own consistency due to *Gödel's Incompleteness Theorems*.

2.3 Modal logic

Let $\mathcal{L}_{\mathsf{CPL}}$ be the language of CPL. The language $\mathcal{L}^{\Box}_{\mathsf{CPL}}$ is the set $\mathcal{L}^{\Box}_{\mathsf{CPL}} = \mathcal{L}_{\mathsf{CPL}} \cup \{\Box\}$, where $\Box$ is a unary connective of necessity. The set of formulas of $\mathcal{L}^{\Box}_{\mathsf{CPL}}$, $For(\mathcal{L}^{\Box}_{\mathsf{CPL}})$, is defined inductively as follows:

$$p_i \mid \neg\varphi \mid \varphi \rightarrow \psi \mid \Box\varphi$$

For $\varphi, \psi \in For(\mathcal{L}^{\Box}_{\mathsf{CPL}})$.

Modal semantics

The semantics for $\mathcal{L}^{\Box}_{\mathsf{CPL}}$ is defined as follows.

Definition 2.3.1. *A* frame *F is an ordered pair $F = \langle W, R\rangle$, where W is a nonempty set of possible worlds and $R \subseteq W \times W$ is a binary relation, a accessibility relation, defined on $W \times W$. A* model *$\mathcal{M}$ based on $F = \langle W, R\rangle$ is a frame $\mathcal{M} = \langle F, V\rangle$ where $V : \mathcal{V} \rightarrow \wp(W)$ is a valuation. The interpretation for the language $\mathcal{L}^{\Box}_{\mathsf{CPL}}$ is defined as follows:*

(1) $\mathcal{M}, w \models p$ *iff* $w \in V(p)$;

(2) $\mathcal{M}, w \models \neg\varphi$ *iff* $\mathcal{M}, w \not\models \varphi$;

(3) $\mathcal{M}, w \models \varphi \rightarrow \psi$ *iff* $\mathcal{M}, w \not\models \varphi$ *or* $\mathcal{M}, w \models \psi$;

(4) $\mathcal{M}, w \models \Box\varphi$ *iff for every* $y \in W$ *such that* wRy, $\mathcal{M}, y \models \varphi$.

φ is true *in a model $\mathcal{M}$ iff every $w \in W$ of $\mathcal{M}$, $\mathcal{M}, w \models \varphi$. φ is* valid *on a frame F iff it is true in every model $\mathcal{M}$ based on F. In order to put away confusions, when we refer to the satisfiability with respect to a specific modal logic, for example L, we write $\mathcal{M}, w \models_{\mathsf{L}}$. We shall not use the subscript when the context is clear.*

The modal operator $\Diamond$ is introduced as:

$$\Diamond\varphi \quad := \quad \neg\Box\neg\varphi$$

Then its semantic condition is stated as follows:[3]

(5) $\mathcal{M}, w \models \Diamond\varphi$ *iff there is* $y \in W$ *such that* wRy, $\mathcal{M}, y \models \varphi$.

Definition 2.3.2. *(HUGHES; CRESSWELL, 1996) The modal logic K is axiomatized as follows:*

(CPL) *All the propositional tautologies;*

[3] Of course, it is possible to define $\Box$ from $\Diamond$ as $\Box\varphi := \neg\Diamond\neg\varphi$.

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;

(MP) *From* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *we infer* $\vdash \psi$;

(Nec) *From* $\vdash \varphi$ *we infer* $\vdash \Box\varphi$;

The logic K is sound and complete with respect to the class $\mathbb{C}_{\mathbf{K}}$ of all frames. It is the weakest logic of the family of *normal modal logics*. Roughly speaking, normal modal logics are commonly understood as extensions of logic K. In (CHELLAS, 1980), Chellas provides the following general characterization for this family of logics as follows:

Definition 2.3.3. *A modal system* $\mathsf{L}^{\Box}$ *is* normal *if it contains:*

(Def) $\vdash_{\mathsf{L}\Box} \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$;

(RK) *From* $\vdash_{\mathsf{L}\Box} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \psi$ *we obtain* $\vdash_{\mathsf{L}\Box} (\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n) \to \Box\psi$.

It is easy to see that axiom K is provable from the rules presented in Definition 2.3.3.

Theorem 2.3.4. *(CHELLAS, 1980) Let* $\mathsf{L}^{\Box}$ *be a modal system which conforms according to Definition 2.3.3. Then* $\vdash_{\mathsf{L}\Box} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *and Nec is a rule of* $\mathsf{L}^{\Box}$.

Proof. Consider the following derivation:

1.	$((\varphi \to \psi) \wedge \varphi) \to \psi$	Taut
2.	$(\Box(\varphi \to \psi) \wedge \Box\varphi) \to \Box\psi$	RK,1
3.	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Taut,2

The case of necessitation is simple, just take the case $n = 0$ in the rule RK. This concludes the proof. Q.E.D.

Given Definition 2.3.3 we obtain non-normal modal logics by rejecting Def or RK. In what concerns RK, there is more than a way to reject it, because we can have a version of axiom K but not Nec. Or we can have rule Nec and not axiom K. In Chapter 4 we will give an example of the first case.

If we add constraints on the accessibility relation R we obtain stronger logics. Consider the following definitions:

Definition 2.3.5. *Let* W *be a set and* $R \subseteq W \times W$ *be a relation on* R.

- R *is* reflexive *iff for all* $w \in W$: wRw;
- R *is* serial *iff for every* $w \in W$ *there is* $y \in W$ *such that* wRy;
- R *is* symmetric *iff for all* $w, y \in W$: wRy *implies* yRw;
- R *is* transitive *iff for all* $w, y, z \in W$: wRy *and* yRz *imply* wRz;

- *R is* euclidean *iff for all $w, y, z \in W$: wRy and wRz imply yRz.*

It is easy to check that:

Proposition 2.3.6. *The following items are true.*

(A) $\Box\varphi \to \varphi$ (T) *is valid in the class* $\mathbb{C}_{\mathsf{KT}}$ *of all reflexive frames.*

(B) $\Box\varphi \to \Diamond\varphi$ (D) *is valid the in class* $\mathbb{C}_{\mathsf{KD}}$ *of all serial frames.*

(C) $\Diamond\varphi \to \Box\Diamond\varphi$ (B) *is valid the in class* $\mathbb{C}_{\mathsf{KB}}$ *of all symmetric frames.*

(D) $\Box\varphi \to \Box\Box\varphi$ (4) *is valid the in class* $\mathbb{C}_{\mathsf{K4}}$ *of all transitive frames.*

(E) $\varphi \to \Box\Diamond\varphi$ (5) *is valid the in class* $\mathbb{C}_{\mathsf{K5}}$ *of all euclidean frames.*

Now we present extensions of the basic logic K in the definition bellow:

Definition 2.3.7. *Let* K *be the logic defined as in Definition 2.3.2. Then:*

- *The logic* KT *is obtained by extending* K *with formula* T.
- *The logic* KD *is obtained by extending* K *with formula* D.
- *The logic* KB *is obtained by extending* K *with formula* B.
- *The logic* $\mathsf{K4}$ *is obtained by extending* K *with formula* 4.
- *The logic* $\mathsf{K5}$ *is obtained by extending* K *with formula* 5.
- *The logic* $\mathsf{S4}$ *is obtained by extending* K *with formulas* T *and* 4.
- *The logic* $\mathsf{S5}$ *is obtained by extending* K *with formulas* T *and* 5.

In standard introductory textbooks of modal logics such as Chellas (1980) and Hughes & Cresswell (1996) one finds soundness and adequacy results for the above systems. In normal modal logics, the operator $\Box$ is usually introduced as a primitive operator, and then we define $\Diamond$ as above. But, there are modal logics where the interdefinability between $\Box$ and $\Diamond$ does not hold. Most of them are modal logics whose basic propositional logic is non-classical. In Chapter 6, we will present such logics.

Chapter 3

Formalizations of Consistency

In this chapter we will defend that the notion of logical consistency is not a primitive notion, in the sense that it cannot be reduced in terms of simpler ones. In order to defend this, we will present Kreisel's squeezing argument to argue that the informal concept of validity, intended to bridge the model and proof-theoretical validity is not a primitive concept. Instead, it is indeed an informal concept which is a theoretical construct, defined in terms of more basic concepts. Further, we will see that, even if both formal approaches to consistency do not entirely capture informal consistency, they capture important aspects of this notion.

3.1 Is consistency a primitive concept?

Definitions 2.1.6 and 2.2.8 show two ways in which the concept of consistency can be reduced. The completeness theorem link these notions by showing that every valid formula is provable. Even so, it is widely accepted that there is an informal notion of validity which is irreducible to these formal approaches. Now we will turn to the discussion about informal validity.

3.1.1 Kreisel's squeezing argument

Informal rigour consists in the precise analysis of intuitive notions in order to "eliminate the doubtful properties of the intuitive notions when drawing conclusions about them" (KREISEL, 1967, p.138). In his famous work , Kreisel (1967) undertook this analysis to show that some intuitive concepts used in Mathematics are meaningful, and the informal validity appears among these concepts.[1] Although it is clear that validity can be defined by means of the metalogical frameworks, both proof-theoretically and model-theoretically, Kreisel defends that there is an informal concept of validity which is not fully reduced

[1] Another famous application of squeezing argument lies in the analysis of the informal concept of computability.

neither to syntax nor to semantics. Let φ be a first-order formula. So, he states these three aspects this concept may have:

(*Informal*) $Val(\varphi)$ means that φ is true in all structures;

(*Semantical*) $V(\varphi)$ means that φ is true in all structures in the cummulative hierarchy;

(*Syntactical*) $D(\varphi)$ means that φ is derivable by means of some fixed sets of formal rules.

The informality of Val lies in not specifying what structures we are dealing with. V and D are formal because they are theoretically precise. Then informality here is understood as not being defined within a well-structured conceptual framework. Given these three aspects which define validity, we present the reasons which justify the irreducibility of *Val*. Kreisel defends that *Val* is not reduced to D because nobody reasons according to formal rules. In his own words,

> (...) First (e.g. Bourbaki) 'ultimately' inference is nothing else but following formal rules, in other words D is primary (...) This is a specially peculiar idea, because 99 per cent of the readers, and 90 percent of the writers of Bourbaki, don't have the rules in their heads at all!! Nobody would expect a mathematician to work on groups if he did not know the definition of a group. (KREISEL, 1967, 153-154)

It is also generally accepted that the concept of validity is not properly captured by purely syntactic devices, such as proof systems. A well-known objection to the proof-theoretical approach to validity is due to Prior (1960). If one maintains that the meaning of a connective can be pure and solely described by means of its introduction and elimination rules, one should accept that this characterization is trivial and then everything is provable. The reason stems from the fact that such requeriments for meaning are very broad.[2] Prior (1960) presents the connective called "Tonk" characterized by the following rules:

(Tonk-intr) $\varphi \vdash \varphi \mathfrak{T} \psi$

(Tonk-elim) $\varphi \mathfrak{T} \psi \vdash \psi$

Now, the following demonstration shows that we can deduce anything with Tonk:

Argument 3.1.1. *Prior's argument can be summarized as follows:*

1.	φ	*Assumption*
2.	$\varphi \mathfrak{T} \bot$	*Tonk-intr, (1)*
3.	$\bot$	*Tonk-elim, (2)*

[2]This discussion presupposes natural deduction systems.

Then, for every assumption φ, we can deduce every ψ, $\varphi \vdash \psi$. So, we have a trivial theory. Therefore, the broad conception according to which the meaning of a connective is captured by its introduction and elimination rules lead us to triviality.

On the other hand, we think that this latter objection is very far from being definitive with respect to the proof-theoretical approach to validity. It is possible avoid Argument 3.1.1 by adding some constraints in the proper deductions. The requirement of *harmony* between introduction and elimination is able to block derivations like of Argument 3.1.1. Roughly speaking, such requirement means that whenever we apply an introduction rule of a connective c to $\varphi_1, \ldots, \varphi_n$, we gain nothing in applying the elimination rule of c right after we applied c to $\varphi_1, \ldots, \varphi_n$. It is clear that it blocks the step (3) in Argument 3.1.1.

The requirement of harmony is not the unique strategy to block Prior's argument. Belnap (1962) observes that there is an essential use of the transitivity of consequence relation in the derivation of Argument 3.1.1. So, if we dispense transitivity, Prior's argument does not work. There are other two requeriments: *conservativeness* and *uniqueness*. Conservativeness says that any extension of a deducibility statement of the form φ_1, ..., $\varphi_n \vdash \psi$ with a connective c produces deducibility statements where c must occur. That is, from φ_1, ..., $c(\varphi_i, \varphi_j)$, ..., $\varphi_n \vdash \psi$ we cannot obtain deducibility statements φ_1, ..., $\varphi_n \vdash \psi$ which does not have any occurrence of c. This requirement also blocks Argument 3.1.1. Last, there is the requirement of *uniqueness*, which says that any connective must be uniquely determined by its inference rules. As Naibo and Petrolo (2015) observe, the connective *tonk* alone satisfies uniqueness, but it fails to be conservative.[3]

In last instance, Prior's argument shows that the wittgensteinian claim that *meaning is use* should be taken carefully. That is, it is necessary to adopt some constraints in order to defend that the meaning of the logical constants can be given by their inference rules. According to Peregrin (2012), this inferential approach to logical validity seems to give more a know how of the inference rules rather than a knowledge that such rules represent. One advantages of the proof-theoretical approach, according to Peregrin, is that its makes possible to explain the emergence of the logical constants and logic itself. So, this syntactical approach figures as a promising way to capture our informal notion of validity.

On the other hand, one may resist to proof-theoretical approach by claiming that it rules out systems which cannot be characterized by a proof system. Second-Order Logic (SOL), for example, does not have a complete proof-system due to its high expressive power. So, if we defend that the meaning of the logical constants is given by its inference

[3]There are many ways of solving this problem raised by Prior. We refer the reader to Stevenson (1961) for another proposal for solving the challenge posed by Prior. The very idea of *Proof-Theoretical Semantics* shows that it is possible to maintain that the meaning of the logical operations may be given by its inference rules. Of course, it is necessary to adopt constraints in order to block Argument 3.1.1, such as the harmony requirement. Here we will not discuss such an approach, but we refer the reader to Schroeder-Heister (2018) for a more detailed discussion.

rules, the case of SOL is put aside. The proof-theoretical approach may not be enough if one defends the logical status of SOL.

It is important to stress that Kreisel did not *prove* the relation between Val, V and D. He recognizes that talking about structures is too vague, whereas talking about structures in the cummulative hierarchy of sets (the sets of ZFC, for example) and provability in deductive systems are precise concepts. Then it makes no sense in trying to *prove* the relation of a formal concept with a informal one. What Kreisel did was to argue for the plausibility of the relations of these three concepts. For him, even if informal validity is not essentially proof-theoretical, it follows immediately that $D(\varphi) \Rightarrow Val(\varphi)$ (*Soundness Assumption*, SA henceforth). Then, derivability is a *sufficient condition* for informal validity. Kreisel's argument for SA is itself interesting.

> Let us go back to the fact (which is not in doubt) that one reasons in mathematical practice, using the notion of consequence or of logical consequence, freely and surely, ((...) the 'crises' in the past in classical mathematics were not due to lack of precision in the notion of consequence.) Also it is generally agreed that at the time of Frege who formulated rules for first-order logic, Bolzano's set-theoretic definition of consequece had been forgotten (and had to be rediscovered by Tarski) yet one recognizes the validity of Frege's rules. (KREISEL, 1967, pp. 153)

The interesting point is that in 1967 model-theoretical soundness results were already available. Even so, Kreisel takes Frege's deductive system as an example. Frege (1972) himself did not provide a model-theoretical semantics for his system in 1879. What he did was to use some principles, such as the Principle of Non-Contradiction and Excluded Middle, to prove the validity of the axioms of his system. These last two principles are taken for granted. They are constituents of the pure thinking.

At this point it must be clear that the SA stands for an *informal soundness* instead the metatheoretical soundness. Informal validity and model-theoretical validity are different: the former is a informal concept and the latter is a formal one. Of course, someone can informally show that the axioms of FOL are sound with respect to this informal notion of validity. In such informal demonstration, one uses some classical principles and inference rules, such as (informal) modus ponens, to show that FOL axioms and rules are informally valid.

But, one can question the informal principles we estipulate to argue in favor of SA. In other words, how can we know that the principles of Val are valid? This may seen a silly question, but it involves the issue of justification of basic principles of logic. Feigl (1981) calls attention for this problem. First, he holds that we cannot defend that the basic principles of logic are the most basic laws of nature. On contrary, it is attributed to them factual content. The problem is that factual statements are inductively justified. Moreover, inductive justifications presuppose deductive rules. Then, we would be

committed to circularity. Therefore, those basic principles cannot be empirical. Second, we cannot consider the logical axioms and rules as norms for the correct reasoning. This move does not provide us what guarantees the validity of these basic principles.[4] Feigl points that a possible way to solve this problem is to appeal to the analytic character of the deducibility relation. The problem, according to him, is that analytic inferences pressupose logical rules. And this is also circular.

According to Feigl, if we consider justification as standing for *pragmatical vindication*, it is possible to shed light in the problem of justification of the basic principles of logic. Feigl understands a justification as a *pragmatical vindication* if it provides means towards to an end. If we follow the axioms and rules of FOL, it is because we want to avoid at all costs inconsistencies and to yield true conclusions from true premises.[5] Note that this kind of justification differs from justification as a validation of a principle. In the case of (classical) Mathematics, we follow, even if silent, the rules of FOL.

Of course, this is not to say that Kreisel adopts the concept of justification as pragmatical vindication to argue for SA. What we did here was to show that it is plausible to accept SA.

It is also the case that informal validity is not essentially model-theoretical because Val leaves open the possibility for the structure to be class sized. Then a formula φ can be informally valid without being model-theoretically valid, because the domains of the models of first-order language are set sized. Thus, Val cannot be reduced to the V. According to Etchemendy (1999) models can be understood in two ways: in a *representational* point of view and in an *interpretational* one. From a representational point of view, the truth of a sentence is related to the possible configurations of the world. Taking truth-tables as a guiding example, each line of the truth-table represents the possible changes in the world. In this case, logical validity is seen as truth in all possible configurations of the world. On the other hand, the interpretational point of view treats logical validity in the traditional way, by varying the interpretation of the non-logical vocabulary. So, if a sentence remains true in all reinterpretations of the non-logical vocabulary, then it is logically valid. However, Etchemendy (1999) argues that the model-theoretical approach fails to satisfy to both perspectives. The representational point of view fails to capture the formality of the consequence relation. And the interpretational point of view faces the difficulty of justifying what is taken as a logical constant and what is not. To sum up, the model-theoretical approach fails to capture the intuitive notion of logical validity.

[4] In our view, this objection does not work since it is plausible to maintain that the logical principles are codifications of a established and well-grounded mathematical practice, the truth-preserving reasoning. See, Kennedy & Väänänen (2017) for arguments in this direction.

[5] It is important to keep in mind that not every logical system is adequate to formalize truth preserving reasoning, when truth is understood in general terms. Intuionistic logic, for example, is better understood as describe a very narrow notion of truth, which is equated to the presence of a constructive proof.

A very common objection to the proof-theoretical approach to informal validity concerns the arbitrariness of the choice of proof systems which characterize this informal notion. In the case of FOL, we have sequent calculus, Hilbert's axiomatic systems, natural deduction systems, and tableaux. As we know, all of them are sound and complete with respect to the models of FOL. That being said, what is the proof system which captures the informal notion of validity? As Field (1991) and Etchemendy (1999) argue, it is quite arbitrary to say that informal validity is tied to a particular set of axioms and/or inference rules. We think that the same question can be raised to the model-theoretical approach. In fact, there are many model-theories with respect to a particular logic L, and some of them do not give us intuitions about the connectives of the language of L. According to Copeland (1986), there are two stages in the development of a model-theoretical semantics for L. The first is the construction of a mathematical formalism, and the second concerns to the interpretation of this formalism as capturing the meaning of the logical constants. Copeland says that many model-theories stop in the first stage, like the case of algebraic semantics for intuitionistic logic. One of the first model-theories capable to shed light upon the logical constants of intuitionistic logic was Kripke semantics. In this case, we can say that Kripke semantics yields the second stage. We can also cite the case of the paraconsistent logics. Many Costa (1974)'s paraconsistent systems $\mathsf{C_n}$ are characterized in terms of bivaluation semantics, which are simple devices to prove soundness and completeness of the formal systems $\mathsf{C_n}$. But they lack the intuitive account of the logical constants of these systems. Finally, we give the example of fuzzy logics, which are logics intending to deal with degrees of truth. One common objection to fuzzy logics is that their metatheories are classical. This objection threats who proposes these logics as alternatives to classical logic. As Bacon (2013) defends, if it is possible to give a non-classical metatheory for a non-classical logic, then the objection to non-classical logics that they are metatheoretically bivalent disappears.[6] To sum up, the possibility of giving more than one semantical framework for logical systems shows that model-theory is also subjected to the objection faced by proof-theoretical approaches to logic.

Even so, if one takes for granted that Logic applies to Mathematics, we have $Val(\varphi) \Rightarrow V(\varphi)$ (*Logic and Mathematics Assumption*). Moreover, the definition of Val encompasses set-theoretical structures. So, if a formula is valid in Val sense, then so is in V sense. Then, set-theoretical validity is a *necessary condition* for informal validity. The argument below shows that the completeness theorem extensionally collapses these three notions of logical validity:

[6] One consequence of his position is that model-theory and semantics are not the same thing. Semantics has a more general character and it has to do with the *intended meaning* of the logical constants of the language. It is possible that the case of classical logic is one of the only logics which have a model-theory, Tarski's semantics, giving a good account of the meaning of its logical constants. Of course, it does not prevent other model-theories for classical logic. So, in general, when we use "semantics" to refer to "model theory" we are doing so in a very loose way.

Argument 3.1.2. *Kreisel's argument can be summarized as follows:*

1.	$D(\varphi) \Rightarrow Val(\varphi)$	*Soundness assumption*
2.	$Val(\varphi) \Rightarrow V(\varphi)$	*LM assumption*
3.	$V(\varphi) \Rightarrow D(\varphi)$	*Completeness Theorem*
4.	$V(\varphi) \Rightarrow Val(\varphi)$	*Logic (1),(3)*
5.	$D(\varphi) \Leftrightarrow Val(\varphi) \Leftrightarrow V(\varphi)$	*Logic (1)-(4)*

From Kreisel's definitions of validity we can define the notion of consistency as the dual of the notion of validity as follows:

(*Informal*) $Con_{inf}(\varphi)$ means that there is a structure where φ is true ($Con_{inf}(\varphi) := \neg Val(\neg\varphi)$);

(*Semantical*) $Con_{sem}(\varphi)$ means that there is a structure in the cummulative hierarchy φ where is true ($Con_{sem}(\varphi) := \neg V(\neg\varphi)$);

(*Syntactical*) $Con_{syn}(\varphi)$ means that $\neg\varphi$ is not derivable by means of some fixed sets of formal rules ($Con_{syn}(\varphi) := \neg D(\neg\varphi)$).

An argument similar to Argument 3.1.2 can be done in order to show that $Con_{inf}(\varphi)$, $Con_{sem}(\varphi)$ and $Con_{syn}(\varphi)$ are extentionally equivalent.

Squeezing arguments, such as given in Argument 3.1.2, give a philosophical understanding of the completeness theorem since, as Andrade-Lotero & Novaes (2012) observe, the informal notion of validity is a bridge between the formal notions of validity. The importance of this theorem is evident when looking for versions of this argument for other logics. In the case of FOL, Argument 3.1.2 is immediate since there is a completeness theorem for FOL. If we consider logics which do not have a corresponding complete proof-system, Kreisel's argument cannot be applied directly. SOL is an example of logic which does not have a complete proof system, and this absence may pose some difficulties in order to establish a version of the squeezing argument for this logic. Despite this absence, Kennedy & Väänänen (2017) show that it is still possible to provide a version of this argument for extensions of FOL, such as SOL and Infinitary Logics.

In search of versions of the squeezing argument for strong logics, Kennedy & Väänänen (2017) observe that Kreisel seems to implicitly assume in his paper that the completeness theorem for a logic L is enough to get a squeezing argument for L. In the same paper, Kennedy & Väänänen exemplify some completeness results for some logics stronger than FOL. The case of SOL is particularly interesting, because it is widely known that this logic is a extension of FOL but it is not complete with respect to its *standard semantics* (sometimes called *full semantics*).[7] That is, let us denote, as Kennedy & Väänänen do, φ^2 a second-order formula and:

[7]We will not enter in formal details about this logic. For those details, one can check Mendelson (2009).

(*Semantical*) $V(\varphi^2)$ means that φ^2 is true in all set-theoretical structures;

(*Informal*) $Val(\varphi^2)$ means that φ^2 is informally true in all structures, including class-sized structures and including structures which do not have set-theoretical definition;

(*Syntactical*) $D(\varphi^2)$ is provable in the axiom system for SOL.

Since it is not the case that $V(\varphi^2) \Rightarrow D(\varphi^2)$, we have neither $Val(\varphi^2) \Leftrightarrow V(\varphi^2)$ nor $V(\varphi^2) \Leftrightarrow D(\varphi^2)$. So, we do not have a squeezing argument for SOL if we consider its full semantics. On the other hand, SOL is complete with respect to Henkin's general models and this may completely change the scenario with respect to the possibility of a squeezing argument for SOL. Consider the following definition of semantical validity given by Kennedy & Väänänen which corresponds to validity in general models:

(*Semantical*) $V'(\varphi^2)$ means that φ^2 is valid with respect to set-theoretically defined general models.

$V'(\varphi^2)$ is also a necessary condition for $Val(\varphi^2)$. Thus, we have the following schema:

$$D(\varphi^2) \Rightarrow Val(\varphi^2) \Rightarrow V'(\varphi^2) \Rightarrow V(\varphi^2)$$

The completeness theorem of SOL establishes that $V'(\varphi^2) \Rightarrow D(\varphi^2)$, obtaining

$$Val(\varphi^2) \Leftrightarrow D(\varphi^2) \Leftrightarrow V'(\varphi^2)$$

But Kennedy & Väänänen recognize that the implication $Val(\varphi^2) \Rightarrow V'(\varphi^2)$ is a problematic one, since from the informal level the general models and the full models are not discernible. On the other hand, in the mathematical practice, we refer only to definable relations and sets, which are, as they say, "known" by the general models. In sum, the squeezing argument for SOL is possible if the difference between general and full models is not attached to the informal validity. Such a possibility shows how broad Kreisel's argument is.

Smith (2011) points out that even if Kreisel considers $Val(\varphi)$ as an informal conception of validity, $Val(\varphi)$ is not an intuitive characterization of our pre-theoretical notion of validity. It is instead is a result of a necessary refinement of an intuitive concept to the success of the squeezing argument. That is, the informal notion Val must be theoretically robust enough to make the Argument 3.1.2 work. This observation also applies to Kennedy & Väänänen's arguments for strong logics.

That Val may fail to capture our intuitive/pre-theoretical notion of logical validity, which we barely know what it is, does not constitute a crucial problem for Kreisel's original argument. As argued in Kennedy & Väänänen, the informal notion adopted in natural mathematical language is semantic, close to a model-theoretical approach. Then,

from this perspective, Kreisel's informal notion seems to capture a notion of validity from mathematical practice. And indeed this seemed to be Kreisel's goal in his 1967's paper. If Kreisel's objective were to capture the informal notion of consequence of natural language, then Val may fail to do so because it is not straightforward that mathematical structures are able to completely capture the material inferences. In this respect, Andrade-Lotero & Novaes say:

> It seems to us that the idea of developing logic in connection with mathematical structures is essentially related to the logicist project of providing logical foundations to mathematics, and it is not obvious that this assumption should hold irrestrictly also beyond the scope of the logicist program. After all, deductive validity is a notion that goes well beyond its possible interpretations onto mathematical structures.(ANDRADE-LOTERO; NOVAES, 2012, p.396)

The example of informal inferences is illuminating because one may think that the intuitive definition of valid informal inference requires a connection between sentences we take as premises and the sentence we take as the conclusion. But the required connection is difficult to characterize even in informal terms, as the following argument shows:

Argument 3.1.3. *Consider the example given by Haack (1978, pp. 25):*

The President signed the treaty with a red pen.
∴ The President signed the treaty.

This argument can be considered as intuitively valid, but is clearly invalid from a formal standpoint, even according to Kreisel's definition of informal validity. Although there is a connection between the premise and the conclusion, it is quite difficult to grasp it in a formal system like FOL. In light of this, some people may consider it as an objection to Kreisel's argument. But this is far from being correct because, as we said before, Kreisel's approach aims to justify the use of informal notions in Mathematics, not in the every day discourse. The only thing we can say about his approach is that Val does not capture the intuitive meaning of validity present in natural language and that informal concepts may not be intuitive because the informal concepts can be a result of a formal refinement.

Besides noting that Kreisel's definition of Val is not an intuitive notion, it is also important to note that Val is not a primitive notion in the sense that it is not defined in terms of more basic entities. From his definition, it is very clear that Val is defined in terms of more basic entities like structures which are, ultimately, sets and classes. Then, Val cannot be assumed, for example, by a nominalist who wants to avoid abstract entities at all costs.

Last, but not least, it is important to stress that Kreisel (1967)'s objective was not to establish that Val is *the* informal notion to be captured by V and D. That is, Argument

3.1.2 does not imply that Val is the unique informal notion captured by its formal counterparts. His argument establishes that the informal notion, once properly articulated, extensionally collapses with its formal counterparts. Even if it was not Kreisel's objective in determining whether Val is unique or not, we think it to be relevant to ask whether it is the case. If such notion is not the only notion captured by V and D, there may be other informal notions, still theorized, which may provide a more intuitive understanding of the formal notions of validity.

Squeezing arguments gained attention in the literature, due its simplicity and schematic form, in order to capture intuitive notions of validity from natural language. In the next section, we will direct our attention to this latter approach, and we will see that the plurality of informal notions to be captured by V and D suggests that formal consequence relations are underdetermined by their informal counterparts.

3.1.2 Variants of squeezing argument

Shapiro (2005) formulates a version of squeezing argument to defend that the model and the proof-theoretical accounts of logical consequence capture relevant informal notions of logical consequence in natural language. For such, he delineates some definitions of informal logical consequence, which we will present now. We will use the notation F to mean that F is a sentence of natural language which is a counterpart of a formal sentence φ and $Prem$ is a counterpart in natural language of a set Γ of formal sentences.

Definition 3.1.4. *The relation $Val_B(Prem, F)$ holds whenever F is logical consequence of $Prem$ in the* blended *sense; that is, it is not possible to every member of $Prem$ to be true and F be false, and this impossibility holds in virtue of the meaning of the logical terms. $Val_B(F)$ means that F is valid in the blended sense.*

This blended notion of informal consequence captures elements of formality and necessity of consequence relation and it is intended to be the informal counterpart of the model-theoretical consequence relation. Now, it is clear that every φ provable in FOL is valid in the sense of Val_B. So, we say that the deductive system of FOL is *faithful* with respect to Val_B. It is also clear that every valid F, which is a natural language correspondent of φ, has a valid formalization φ in FOL. In this sense, we say that V is *adequate* to Val_B. Then assuming that F and $Prem$ are counterparts of φ and Γ in natural language, we can present the following squeezing argument for the informal notion Val_B:

Argument 3.1.5. *Shapiro's argument can be summarized as follows:*

1.	$D(\varphi) \Rightarrow Val_B(F)$	*Faithfulness*
2.	$Val_B(F) \Rightarrow V(\varphi)$	*Adequacy*
3.	$V(\varphi) \Rightarrow D(\varphi)$	*Completeness Theorem*
4.	$V(\varphi) \Rightarrow Val_B(\varphi)$	*Logic (1),(3)*
5.	$D(\varphi) \Leftrightarrow Val_B(F) \Leftrightarrow V(\varphi)$	*Logic (1)-(4)*

Shapiro (2005) presents another informal notion of validity, which tends more to a deductive validity.

Definition 3.1.6. *The relation $Val_{Ded}(Prem, F)$ holds whenever F is logical consequence of $Prem$ in the* deductive sense*; that is, there is a deduction of F from $Prem$ by a chain of legitimate gap-free (self-evident) rules of inference. $Val_{Ded}(F)$ means that F is valid in the deductive sense.*

Argument 3.1.7. *Shapiro's second argument can be summarized as follows:*

1.	$D(\varphi) \Rightarrow Val_{Ded}(F)$	*Faithfulness*
2.	$Val_{Ded}(F) \Rightarrow V(\varphi)$	*Adequacy*
3.	$V(\varphi) \Rightarrow D(\varphi)$	*Completeness Theorem*
4.	$V(\varphi) \Rightarrow Val_{Ded}(\varphi)$	*Logic (1),(3)*
5.	$D(\varphi) \Leftrightarrow Val_{Ded}(F) \Leftrightarrow V(\varphi)$	*Logic (1)-(4)*

Griffiths (2014) raises some objections to Shapiro's version of squeezing arguments. First, neither Val_B nor Val_B give account for the totality of natural language, but only for a well-behaved fragment of natural language formalizable in FOL. Second, the Argument 3.1.5 assumes the φ is the formalization of F in FOL. That is, as Griffiths points, F can be takes as the *reading* of φ, where reading is understood as the reverse process of formalization. That being said, he argues that Argument 3.1.5 only works in virtue of the correspondence between F and φ. Then, there is nothing special about both Val_B and Val_B.

As a consequence of Arguments 3.1.5 and 3.1.7, we obtain:

$$Val_{Ded}(F) \Leftrightarrow Val_B(F) \tag{3.1}$$

Considering the correspondence between F and φ, we obtain:

$$Val_{Ded}(F) \Leftrightarrow Val(\varphi) \Leftrightarrow Val_B(F) \tag{3.2}$$

But this extensional equivalence holds because F is taken to be the reading of first-order formulas. If we F were a reading of a second-order formula, the biconditional 3.1 may not hold, because (full) SOL is not complete.

We can now advance a further objection to Shapiro's argument(s). We agree with Griffiths that the three notions Val, Val_B, and Val_{Ded}, are coextensive (3.2). But then, if the formal notions of classical FOL capture the three of them, which one can be seen as the informal or the intuitive content of the formal notions? This is a relevant question due to the fact that the Val, Val_B, and Val_{Ded} are meant to be intensional objects: properties of formulas.[8] If these squeezing arguments are able to show that we can capture these

[8]One could say that the objections also works against Kreisel. It would do so if Kreisel's interests were natural language. But, as we highlighted before, his interests were only mathematical.

notions by means of extensional concepts (V and D), however, we are left in the dark with respect to which one of these represents the intentional concept we associate to logical (classical FOL) validity. In this sense, logical validity is therefore underdetermined by its formal counterparts, even if these manifest a perfect correspondence between syntax and (formal) semantics.

One could resist to these objections by simply arguing that FOL captures informal notions of validity which have semantic or syntactic aspects. Then, the completeness theorem shows that these notions are extensionally equivalent, despite their intensional difference. This response, however, misses our main point: Argument 3.1.5 and Argument 3.1.7 only hold in virtue of the correspondence between F and φ, not in virtue of the intrinsic characteristics of Val_B and Val_{Ded}. To make our point clear, consider the following notion presented in Griffiths (2014):

Definition 3.1.8. *The relation $Val_{Nec}(Prem, F)$ holds whenever F is logical consequence of $Prem$ in the* modal *sense if and only if necessarily, every member of $Prem$ is true, F is true. $Val_{Nec}(F)$ means that F is valid in the modal sense.*

In his aforementioned paper, Griffiths presents a squeezing argument for Val_{Nec} in order to show that there is nothing distinctive about Val_B (and Val_{Ded}) since Val_{Nec} holds by the same reason as the validity in the blended sense. There may be other informal notions logical validity which are captured by V and D, but are extensionally different from the ones presented here.[9] Given this abundance of options, and since formal logic is mute on this topic, any choice is therefore moved by pre-theoretical choices, which therefore suggests a form of informal pluralism with respect to our pre-formal notion of validity.

A consequence of this phenomenon is that the formal consequence relation of FOL is not able to capture the intuitive notion of logical consequence. Kreisel's argument and its variants neither capture such intuitive notion nor they capture an unique one, even if they capture relevant informal notions, which regulate our inferential practice. Moreover, following Griffiths's analysis of Shapiro's argument, we cannot hold that these informal notions capture the whole of our inferential practices, but only a small, formalizable fragment of natural language inferences. Probably, to capture all inferences of natural language in a system like FOL, we should be able to extend it to the point of doubting that it remains formal.[10]

[9] It is not our purpose to give an exhaustive list of these notions.

[10] Glanzberg (2015) argues that natural language is not determined by a consequence relation in the same sense of formal logic.

3.1.3 Field's view on consistency

It is common to see in the literature that model-theoretical approach to validity/consistency gives an account of the modal component of informal validity, such as given in Definition 3.1.8. However, such formal approach faces some objections when it is taken to formalize our pre-theoretical notion of validity. As we saw earlier, models for FOL are structures whose domains are sets. This implies, according to Field (2008), that logically valid sentences may not be actually true. That is, models do not reflect reality. Adopting Field's vocabulary, we can say that technical soundness is not genuine soundness. But, then, in what sense can we say that model-theoretical validity captures genuine validity? According to Field, Kreisel's argument offers a good way to look to this question.

Field deals with squeezing argument in some of his works. Field (1991) uses Kreisel's argument to argue that consistency can be treated as a primitive concept, irreducible model-theoretical notions nor to proof-theoretical ones. Field says that both model theory and proof theory are platonistic since they use abstract concepts such as models and proofs. And the idea of proposing consistency as a primitive notion is to make metalogic possible for nominalists. On the other hand, he recognizes that consistency can be partially treated in terms of models, but he defends that the model-theoretical approach to consistency does not capture what consistency is in its full totality. That is, even if consistency is tractable by means of model-theoretical tools, it cannot be totally reducible to a model-theoretical concept. In order to defend such view, Field raises an objection to the model-theoretical approach to consistency. For example, let Γ be a set containing all the truths about set theory which are statable inside set theory. Since Γ is the set of all truths statable in set theory, Γ should be consistent. Thus, a natural model for Γ should be set of all sets. However, due to the set-theoretical result which says that there is no set of all sets, it seems that there is no such model for Γ. At the same time, if Γ set is stated in the language of FOL, there is a model for Γ due to a variation of *Löwenheim-Skolem-Theorem*. In his own words:

> (...) Second, we must go from this conclusion to the existence of a model that makes all members of Γ true, and here is where the fancy model theory comes in: we use one of the fancy model-theoretic arguments that underlie the classical completeness theorem for first order logic. These arguments are pretty fancy (they are variations on the Skolem-Löwenheim theorem), and the models of set theory they produce are quite unnatural (for instance, in being countable, and in there being no guarantee that what gets assigned to '$\in$' looks very much like membership). The fact is that *it is only by virtue of an "accident of first order logic" that the Tarski account of consequence gives the intuitively desirable results.* (FIELD, 1991, pps. 3-4)

According to the above passage, the fact that Γ has a model stems from an accident of FOL. From Field's words, it seems that this accident basically consists in the possibility

of using a variation of *Löwenheim-Skolem-Theorem* to define such model. Since this possibility is an "accidental feature" of FOL, it is better to avoid it if one wants to contemplate other logics different than FOL. So, according to him, consistency cannot be purely model-theoretical. In a more recent work, Field (2008) also objects the model-theoretical approach to validity/consistency. His argument goes as follows: the concepts of validity and consistency are defined in terms of the model-theoretical concept of truth. Since models have sets as their domains and the actual world is not a set, then model-theoretical truth does not yield actual truth. As a consequence, model-theoretical validity does not capture informal validity.

On the other hand, Field argues that consistency cannot be reduced to proof-theoretical notions due to the fact that it would be particular to a given formal system. Thus, consistency also cannot be purely proof-theoretical. It is very important to Field that consistency cannot be reduced to these concepts, because, according to him, if we can treat consistency as a primitive notion, then we do not need to deal with abstract mathematical entities in metalogic, such as models and proofs. In this case, a nominalist can freely talk about this notion. Moreover, Field defends that the primitiveness of consistency explains better the significance of the completeness theorem for FOL.

Then, as an alternative, he proposes to treat consistency (logical validity) as a primitive notion, in the sense that "we cannot clarify its meaning by giving a definition of it in more basic terms" (FIELD, 1991). Field proposes to understand the meaning of consistency by means of some principles which govern it together, in the same sense that we understand the meaning of the logical connectives according to the rules governing them. According to Field, the *Model Theoreical Possibility Principle* (MTP) and the *Modal Soundness* (MS) are two principles that govern the meaning of consistency. MTP and MS are stated as follows:

(MTP) If there is a model in which Γ is true, then Γ is consistent;

(MS) If Γ is consistent, it is formally irrefutable in a formal system S.

MTP is a sufficient condition for consistency, while MS is a necessary condition. But neither MTP nor MS alone fully characterize the concept of consistency, which is taken as primitive. Since MTP and MS are a sufficient and necessary conditions for consistency, we have:

$$MTP \Rightarrow \text{Primitive consistency} \Rightarrow MS.$$

In this point, Field proposes a version of the squeezing argument, which he attributes to Kreisel, in order to characterize primitive consistency. Analogously to Kreisel's argument, the completeness theorem plays the fundamental role, since it proves that if a set Γ is formally irrefutable in a formal system, then Γ has a model. Then we obtain

that $MS \Rightarrow MTP$. So, the completeness theorem proves that MTP, consistency and MS extentionally coincide. This version of squeezing argument establishes that MTP and MS govern the primitive consistency, but neither MTP nor MS alone capture primitive consistency. Besides the principles MTP and MS, he argues that there are two rules governing consistency: the *C-rules* and *I-rules*. The *C-rules* are used to show that something is consistent, while *I-rules* are used to show that one thing implies another thing.

We will summarize now the structure of Field's argument. Let $Con_{prim}(\Gamma)$ mean that Γ is primitively consistent, $I^S(\Gamma)$ mean that Γ is formally irrefutable in a formal system S, and $M(\Gamma)$ mean that there is a model in which Γ is true. Then:

Argument 3.1.9. *Field's argument can be summarized as follows:*[11]

1. $Con_{prim}(\Gamma) \Rightarrow I^S(\Gamma)$ *MS*
2. $M(\Gamma) \Rightarrow Con_{prim}(\Gamma)$ *MTP*
3. $I^S(\Gamma) \Rightarrow M(\Gamma)$ *Completeness Theorem*
4. $I^S(\Gamma) \Rightarrow Con_{prim}(\Gamma)$ *Logic (2),(3)*

Since it is assumed that consistency is neither a model-theoretical nor a proof-theoretical notion, Field proposes to treat consistency as a modal operator. In this sense, consistency is taken as the $\Diamond$ operator of modal logics and it is understood as a logical notion. So, $\Diamond\varphi$ means that "φ is consistent". From consistency, we can define a dual concept of logical truth, whose primitiveness is also defended by Field. Thus, it makes no difference whether we take $\Box$ or $\Diamond$ as primitive since they are interdefinable. Specifically, he leaves open if these modalities belong to $\mathsf{S4}$ or to $\mathsf{S5}$. He says that if we accept the axiom $\Diamond\varphi \rightarrow \Box\Diamond\varphi$, we are assuming that the concept of logical truth and logical consistency have "a certain vagueness or indeterminacy", since second-order quantifiers, in general, have no recursive proof procedure. According to Field, the reason for treating consistency and validity as modalities is that the laws governing such concepts include modal axioms, not that they are essentially modal. Lastly, Field's squeezing argument applies only to theories which have an available proof system. If we consider a a theory built in second-order logic, the squeezing argument cannot be applied since second-order logic, in general, does not have a complete proof system.

As mentioned before, Field attributes his view to Kreisel. But, the way Kreisel deals with the squeezing argument is different from Field's account, even if the structure of the Argument 3.1.2 and Argument 3.1.9 are similar. The difference between those arguments becomes clear when we analyse their details, because Field does not seem to understand consistency in the same way as Kreisel does. It is clear that Field's primitive consistency is not *ipsis litteris* Kreisel's definition, since Kreisel deals with mathematical structures (set or class sized). That is, Field requires consistency to be taken as a primitive notion, not

[11] Field (2008) gives a similar version of Argument 3.1.9, but for primitive validity. Since he understands consistency as the dual of validity, the difference lies just in the presentation.

being defined in terms of abstract entities, whereas Val is defined in terms of structures, which are defined in terms of sets, as we showed in the Section 3.1.1. The problem is that Field ignores that Kreisel's argument works because the informal validity has a precise definition. That is, $Val(\varphi) \Rightarrow V(\varphi)$ and $D(\varphi) \Rightarrow Val(\varphi)$ hold because Val is properly defined.

The obscurity involving primitive consistency may raise objections against Field's view. Akiba (1996) raises some objections against Field's proposal on primitive consistency. One of his main objections is outlined as follows. By comparing Kreisel's and Field's squeezing arguments, Akiba argues that Kreisel's approach to informal validity/consistency is *interpretational* in the sense that validity and consistency are defined in terms of structures which interpret the non-logical vocabulary of the language of FOL, and he argues that Kreisel's consistency clearly satisfies the principles MTP and MS. Moreover, Kreisel's approach allows a procedure to determine what is consistent/valid and what is not. On the other hand, Field does not explain the procedure of how the I-rules as well as the C-rules are applied in order to determine whether or not a statement is consistent/valid or not.[12] Since Field attributes his argument to Kreisel and he did not provide a clear explanation about what primitive consistency is, we have a good reason to consider Kreisel's definition as taken by Field. But, if this is the case, Field's approach depends on abstract entities.[13]

Second, it seems that Field confuses informality with primitiveness. Primitive concepts, by definition, are not defined in terms of more basic terms. The most we can do with respect to them is to stipulate some basic principles which govern them. The principles MS and MTP hold if we consider Kreisel's informal notion. We cannot be sure that MS and MTP still hold in the case that consistency is taken as primitive. Field does not provide a convincing argument to defend that MS and MTP hold when consistency is taken as primitive. On the other hand, informal concepts are not necessarily primitive. The proper definition of informal validity Val shows this. The reason for which Val is considered as informal is due to the fact that Kreisel did not specify the size of the structures which define Val. Shapiro's definition is informal in a different way, because he does not clarify the meaning of necessity and possible worlds occurring in the definition of validity in the blended sense. And his definition is intended to capture validity in a fragment of natural language. But both notions are articulated enough so that it is possible to provide a squeezing argument for Val and Val_B, as well as their variants. As Smith (2011) argues, such refinement is necessary for the success of these arguments. Then, while informal concepts can be defined in terms of simpler ones, primitive concepts cannot. This is a confusion committed by Field.

[12] In a more recent work, Field (2008) does not make use of these rules in his formulation of the argument. Then, I-rules and C-rules do not play a relevant role in Field's argument.

[13] Akiba (1996) raises other objections against Field's view on primitive consistency comparing Field's approach to consistency to Etchemendy's approach.

The strategy of taking consistency as a primitive notion is not restricted to nominalists like Field. Balaguer (1995) argues that it is possible to consider Field's and Kreisel's definition of informal consistency as a primitive notion in order to explain how we do acquire mathematical knowledge. But we claim that Balaguer commits a confusion similar to Field with respect to the distinction between informal concepts and primitive ones, since Kreisel's definition is not primitive, but it is defined in terms of structures.

To sum up, even if Field is right in defending that consistency is not totally reducible to formal notions, he fails in showing that it is primitive.

3.1.4 Consistency from a paraconsistent point of view

The idea of incorporating metatheoretical concepts in the object language has become a fruitful field of investigation with the development of provability logics.[14] Field's proposal can be seen as an attempt to incorporate (primitive) consistency in the object language of logic because he defends that consistency and validity obey modal principles. In recent years, many works in this direction were done, and the case of *Logics of Formal Inconsistency* (LFIs) became widely known in the literature (CARNIELLI; CONIGLIO; MARCOS, 2007). In this section, we will argue that the consistency operator present in LFIs fails to be interpreted as primitive metatheoretical consistency in light of squeezing arguments.[15]

The presence of a contradiction φ and $\neg\varphi$ in a classical theory T leads T to triviality due to the *principle of explosion* ($\varphi, \neg\varphi \vdash_T \psi$). In order to deal with contradictions in a non-trivial way, *paraconsistent logics* are often employed. Paraconsistent logics are logics which reject this principle.[16] LFIs are paraconsistent logics which, at the same time that they transgress explosion, they respect the *gentle principle of explosion* ($\circ\varphi, \varphi, \neg\varphi \vdash_T \psi$), where $\circ\varphi$ means that 'φ is consistent'. It is distinctive in this class of paraconsistent logics the incorporation of the notion of consistency into their object language. The gentle principle of explosion follows the following intuition: if a theory T admits φ, $\neg\varphi$ and $\circ\varphi$, then T is trivial. That is, explosion holds for consistent formulas. In general, LFIs have the interesting property of recapturing classical inferences due to *Derivability Adjustment Theorem*.[17]

[14] We will focus in provability logics in the next chapter.

[15] In Chapter 6, we argue that these logics capture the interpretation of classicality, as developed by Omori and Sano (2014).

[16] Here we understand contradictions as the presence of a pair of contradictory formulas of the form $\{\varphi, \neg\varphi\}$ and as formulas like $\varphi \wedge \neg\varphi$. In this sense, contradictions make essential use of the connective of negation. On the other hand, as Novaes (2007) shows, contradictions do not need to be reduced to situations involving negation. For example, 'Aristotle is dead' and 'Aristotle is alive' are contradictory statements because they cannot be true together and they cannot be false together, but none of the statements make use of negation. But, by terminological issues, we will understand contradictory statements as involving the use of the negation operator.

[17] This theorem can be checked in (CARNIELLI; CONIGLIO; MARCOS, 2007).

In what follows we present a LFI system which was proposed to capture the concept of inconsistency, LFI1 presented in Carnielli et al (2004). The language of LFI1 is $\mathcal{L}^{\bullet} = \mathcal{L} \cup \{\bullet\}$, where $\mathcal{L}$ is the same language of $\mathcal{L}_{\mathsf{CPL}}$ and $\circ$ is a unary connective of consistency. The set of formulas of LFI1, $For(\mathcal{L}^{\bullet})$ is inductively defined as:

$$p_{i \in \mathbb{N}} \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \bullet\varphi \mid \varphi \leftrightarrow \psi$$

For $\varphi, \psi \in For(\mathcal{L}^{\bullet})$. $\bullet\varphi$ means that "φ is inconsistent". From these connectives, we define the consistency connective $\circ\varphi$ as:

$$\circ\varphi \equiv \neg \bullet \varphi$$

Definition 3.1.10. *(CARNIELLI; MARCOS; AMO, 2004) The logic* LFI1 *is axiomatized as follows:*

(Ax1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(Ax2) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\varphi \rightarrow \gamma))$

(Ax3) $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$

(Ax4) $(\varphi \wedge \psi) \rightarrow \varphi$

(Ax5) $(\varphi \wedge \psi) \rightarrow \psi$

(Ax6) $\varphi \rightarrow (\varphi \vee \psi)$

(Ax7) $\psi \rightarrow (\varphi \vee \psi)$

(Ax8) $(\varphi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\varphi \vee \psi) \rightarrow \gamma))$

(Ax9) $\varphi \vee \neg\varphi$

(Ax10) $\neg\neg\varphi \leftrightarrow \varphi$

(Ax11) $\circ\varphi \rightarrow (\varphi \rightarrow (\neg\varphi \rightarrow \psi))$

(Ax12) $\bullet\varphi \rightarrow (\varphi \wedge \neg\varphi)$

(Ax13) $\bullet(\varphi \wedge \psi) \leftrightarrow ((\bullet\varphi \wedge \psi) \vee (\bullet\psi \wedge \varphi))$

(Ax14) $\bullet(\varphi \vee \psi) \leftrightarrow ((\bullet\varphi \wedge \neg\psi) \vee (\bullet\psi \wedge \neg\varphi))$

(Ax15) $\bullet(\varphi \rightarrow \psi) \leftrightarrow (\varphi \wedge \bullet\psi)$

(MP) *From* φ *and* $\varphi \rightarrow \psi$ *we infer* ψ

The logic LFI1 is characterized by the matrix $M_{\mathsf{LFI1}} = \langle\{1, \frac{1}{2}, 0\}, \bullet, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \{1, \frac{1}{2}\}\rangle$, whose operations have the following tables:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\leftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	1

	$\neg$	$\bullet$
1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	1
0	1	0

As a consequence, $\circ\varphi$ has the following truth-table:

	$\circ$
1	1
$\frac{1}{2}$	0
0	1

Carnielli et al (2004) proved that LFI1 is sound and complete with respect to these truth-tables. Although LFI is a label covering a wide range of logics, we choose to analyse the case of LFI1, because of its truth-functionality.

In general, LFIs intend to be a logical basis for non-trivial inconsistent theories, thus being able to deal with contradictory reasoning in a "remarkably natural and elegant way", as Carnielli (2011) argues. Diverging from traditional approaches to contradictions, which defends their existence in the real world, such as Priest's *Dialetheism*, LFIs face contradictions in an epistemological perspective as the following passage shows:[18]

> At this point we would like to call attention to the fact that logics of formal inconsistency, although neutral with respect to real contradictions, are perfectly well suited to the idea that we do not know whether or not there are real contradictions, despite the fact that we have to deal with contradictions. When a physicist considers two theories to be inconsistent when put together, (s)he is doing exactly the kind of thing that logics of formal inconsistency are designed for – using classical logic in the theories taken separately, but restricting the principle of explosion with respect to the contradiction yielded by combining them together. Thus we affirm that *logics of formal inconsistency act primarily within the epistemic domain of logic, without any commitment to the existence of real contradictions.* (CARNIELLI; RODRIGUES, 2012, pp. 13)

This epistemological approach to contradictions shows itself to be interesting by two reasons. First, it is certainly more scientifically guided and philosophically plausible to accept that contradictions occur only in the level of information than accepting that there is a real contradiction. The second reason is that, in general, paraconsistent logics are too weak to represent many forms of reasoning present which are difficult to neglect their validity. For example, some paraconsistent logics do not validate modus ponens as well

[18]For an ontological defense of paraconsistency, check Priest (2006) for more details.

as contraposition, which are forms of reasoning present in mathematical reasoning. But, differently from these paraconsistent logics, LFIs are able to recover these inference rules by estipulating that the sentences in question are consistent. Then, these logics do not put ourselves very distant from the classical reasoning, present in Mathematics. In sum, this approach can shed new light on the nature of contradictions.

This broad character of $\circ$ in LFIs makes it difficult to say what aspect of consistency it captures, even in the epistemological realm. For us, it is clear that the proposed meaning for $\circ$ does not capture the metatheoretical concept of consistency.[19] In order to show that $\circ$ does not capture metatheoretical consistency, we will give an indirect argument. We will give a squeezing argument for LFI1, and then we will compare the informal notion of the argument with the proposed informal reading of $\circ$. The argument runs as follows. Let D_{LFI1} and V_{LFI1} respectively denote validity in a formal deductive system for a LFI1 and validity in an adequate model-theory. Lastly, let Val_{Des} be defined as follows:

$Val_{Des}(\varphi)$: φ is designated in all models.[20]

The informality of Val_{Des} lies in the non specification of, for example, the cardinality of the set of truth-values V of the matrix. Indeed, Val_{Des} may stand as an informal notion of validity for many-valued logics, in general.[21] It is clear that every LFI1-theorem is informally valid. Then (1) $D_{\mathsf{LFI1}}(\varphi) \Rightarrow Val_{Des}(\varphi)$. Moreover, if φ is designated in all models, φ is designated in the three-valued matrices of LFI1. Then, (2) $Val_{Des}(\varphi) \Rightarrow V_{\mathsf{LFI1}}(\varphi)$. The squeezing argument runs as follows:

Argument 3.1.11. *The squeezing argument for* LFI1 *can be summarized as follows:*

1.	$D_{\mathsf{LFI1}}(\varphi) \Rightarrow Val_{Des}(\varphi)$	*(1)*
2.	$Val_{Des}(\varphi) \Rightarrow V_{\mathsf{LFI1}}(\varphi)$	*(2)*
3.	$V_{\mathsf{LFI1}}(\varphi) \Rightarrow D_{\mathsf{LFI1}}(\varphi)$	*Completeness Theorem*
4.	$V_{\mathsf{LFI1}}(\varphi) \Rightarrow Val_{Des}(\varphi)$	*Logic 2,3*
5.	$D_{\mathsf{LFI1}}(\varphi) \Leftrightarrow Val_{Des}(\varphi) \Leftrightarrow V_{\mathsf{LFI1}}(\varphi)$	*Logic 1-4*

Argument 3.1.11 establishes that the informal notion Val_{Des} extensionally collapses with the formal notions D_{LFI1} and V_{LFI1} when LFI1-formulas are at issue. Consider the dual of $Val_{Des}(\varphi)$, defined as

$Con_{Des}(\varphi)$: φ is designated in some model.

[19] Mendonça & Carnielli (2020) recognize that the operator $\circ$ does not faithfully capture model-theoretical consistency. In their paper, they argue that the operator $\circ$ captures a particular form of consistency, present in information theories.

[20] Of course, in each case it is necessary to take for granted that Val_{Des} adapts to the recursive definitions of the valuations $v \in sem_{\mathsf{L}}$ for each many-valued logic L.

[21] In the last chapter of this thesis, we will concentrate on this family of logics.

Now we present some remarkable differences between Con_{Des} and $\circ$. Consider the following (informal) assertion:

Assertion 3.1.12. *Let $\wedge$ and $\rightarrow$ be interpreted as the in truth-tables of* LFI1*. Then, the meta-scheme*

$$(Con_{Des}(\varphi) \wedge Con_{Des}(\neg\varphi)) \rightarrow Con_{Des}(\varphi \wedge \neg\varphi) \tag{3.3}$$

is not valid.

Proof. Consider an instance of the meta-scheme 3.3 where $\varphi = \circ p$. Now, take two models M and M' which respectively attribute 1 and $\frac{1}{2}$ to p. Then, $Con_{Des}(\circ p)$ and $Con_{Des}(\neg \circ p)$ are true, and hence $(Con_{Des}(\circ p) \wedge Con_{Des}(\neg \circ p))$. Since there is no model for $\circ p \wedge \neg \circ p$ in LFI1, $Con_{Des}(\circ p \wedge \neg \circ p)$. Therefore, the meta-schema 3.3 is not valid.

Q.E.D.

It is easy to check that the following formulas are LFI1 theorems:

1 $\vdash_{\mathsf{LFI1}} (\circ\varphi \wedge \circ\psi) \rightarrow \circ(\varphi \wedge \psi)$;

2 $\vdash_{\mathsf{LFI1}} \circ\bot$.

Then, as Assertion 3.1.12 shows, the informal notion of consistency expressed by Con_{Des} does not coincide with the notion of consistency expressed by $\circ$. Since Val_{Des} is the informal bridge between D_{LFI1} and V_{LFI1}, we can conclude that $\circ$ is independent from model-theoretical and proof-theoretical definitions of consistency.

The Assertion 3.1.12 can be extended to other LFIs which extends the logic mbC with axiom:

(cc) $\circ \circ \varphi$.

So, this shows that our argument affects a wide class of LFIs.[22] Our conclusion is compatible with some proponents of the LFIs about $\circ$, who defend that:

> Taking into account that a primitive concept is one that is not defined in terms of other concepts, the idea of consistency viewed as a primitive concept is rendered formal by means of a propositional operator (or a primitive connective) governed by certain logic axioms.
>
> Consistency, in this sense, would certainly be a notion totally independent of model theoretical and proof-theoretical means. (BUENO-SOLER; CARNIELLI, 2017, p. 12)

[22] There are LFIs which validates both the meta-scheme 3.3 and the formula $(\circ\varphi \wedge \circ\neg\varphi) \rightarrow \circ(\varphi \wedge \neg\varphi)$, such as mbCciw (Carnielli and Coniglio (2016)). So, Assertion 3.1.12 cannot be used as a definitive argument to all LFIs. Of course, one has to verify other principles envolving $\circ$ in these LFIs in order to say that $\circ$ may stand for Con_{Des}. I thank Professor Coniglio for these observations.

In this section we analysed so far the interpretation of $\circ$ as standing for a metatheoretical concept of consistency and we argued that $\circ$ fails in doing so. This conclusion was reached by formulating a variation of the so called squeezing argument in an informal notion of validity for LFI1. Then, we saw that such informal notion considerably differs from the proposed meaning for $\circ$. We think that the same procedures can be applied to other LFIs to evaluate whether their operator $\circ$ succeeds in interpreting metatheoretical consistency, but we will leave this question open. Further, in Chapter 6, we will show that the connective $\circ$ of LFI1 captures an interesting formal notion of classicality.

3.2 Consistency as a derivative notion

The discussion raised in Section 3.1 shows the difficulty of stipulating a concept as primitive. We showed that squeezing arguments do not reach pre-theoretical concepts. Instead, they capture notions which previously went through an idealization process. Indeed, they only work for notions which are theorized enough to be captured. All informal notions analysed in this chapter exemplify this. Of course, some are sharper than others, but all of them are theorized enough to be captured by the formal notions of validity previously presented. The fact that it is possible to have squeezing arguments for the same logic with different informal notions is due to the fact that formalization itself is not a discrete process, but continuous. That is, we do not obtain a totally formalized notion $F(I)$ from a totally informal notion I at once, like a magic. It is more plausible to accept these informal notions are themselves products of such continuous process.

On the other hand, the choice for a single notion is guided by theoretical preferences. Something similar happens in the relation between informal proofs and their formalizations. For a single informal proof there may be several formalizations of it. According to Marfori (2010), it suggests that informal proofs are underdetermined with respect to its formalizations. The choice for a single formalization is also guided by theoretical preferences.

We hold that squeezing arguments are the best way to establish if an informal notion of validity is captured by formal notions of validity. If, for example, an informal notion of validity Val'' fails in $D_{\mathsf{L}}(\varphi) \Rightarrow Val''(\varphi)$, we know that this informal notion is not adequate for interpreting logic L. On the other hand, if informal notions must be theorized enough for the success of their corresponding squeezing arguments, then we have good reasons to say that validity and consistency are not primitive notions. Indeed, the many intentionally different notions of validity captured by the formal notions of validity are result of some idealization process. As a consequence, such sharpening makes those notions conceptually robust. Any attempt to define intuitive/primitive consistency will result in a robust concept which depends on the existence of other concepts. For this reason, we do not believe in the existence of such primitive notions of validity (and consistency).

One could say that all the above informal notions of validity (resp, consistency) are pre-conceptions resulting from the contemporary development of logic and that, therefore, we are unable to obtain a primitive notion of logical validity. So, he/she might suggest that looking for notions of logical consequence prior to the contemporary development of logic may shed light towards to a primitive notion of validity. However, as Smiley (1998) points, even the ancient logic tradition approaches to logical validity fall in the same detail: each approach to logical validity is couched in a web of diverse theories. He gives the example of Aristotle's syllogistic. Even if Aristotle himself did not provide a precise definition of logical consequence, we can draw its distinctive properties. Aristotle's theory of consequence presupposes modal elements as well as the formality of logical consequence. The latter feature requires, as we know nowadays, the distinction of logical and non-logical constants. In sum, even in the ancient approach to logic the question of the non-primitiveness of logical validity is salient.

As Sorensen (2003) remarks, the deductive practice present in Greek culture came from the Greeks's proof centred mathematical practice. For example, the pythagoreans, who saw the activity of proving as the emulation of the gods's perfection, defended that the proofs of mathematical statements should become public in order to make it possible for the mathematical community to check the reasoning step by step. Making a proof accessible to the public highlights, according to Novaes (2016), the public character of the proofs and the dialogical dimension of deductive practice, where the persuasion plays an important role. In the act of proving a mathematical statement it is necessary for the prover to provide convincing deductive arguments where each step is convincingly justified. In this sense, Novaes argues that informal logical deductions have this dialogical component. In a dialogical situation where the prover wants to show for the skeptic that a certain conclusion follows from a certain set of premises, the prover asks for the skeptic to take these premises for granted. Then, the prover must show that each step of his deduction is truth preserving. The skeptic, by its turn, tries to give a counterexample in each step of prover's reasoning. So, if each step deductively follows from the premises, the skeptic is not able to provide such counterexamples; on contrary, this counterexample is available and then the argument is invalid. In this sense, logic is inherently dialogical. Of course, as she remarks, our logical practice internalizes the skeptic: when we are to show that a certain conclusion indeed follows from the premises, we must assure that each step of the argument is truth preserving. That is, the steps of the proof are not susceptible to counterexamples.

There are, in fact, several notions of informal validity. Many of them are captured by the same formal notions of validity, while others are not. That is, the same formal system captures different informal notions of validity. For Bezerra & Venturi (2021), this constitutes what we call *informal pluralism*. Informal pluralism asserts that we do not have purely logical reasons to choose what is *the* informal notion captured by the formal

notions.

As an example of informal notion which is not adequately captured by classical FOL, consider the following informal notion:

$Val_I(\varphi)$: φ is constructively provable.

The informality of Val_I stems from the absence of a specification of the methods of construction. And, as Iemhoff (2016) observes, different interpretations of constructibility may lead us to different conceptions of constructivism. For example, under Markov's (Dalen & Toelstra (1988)) interpretation of constructivism, every algorithm must terminate, whereas Brouwer's intuitionism allows the construction of infinite sequences of objects.

> (...) the ideal mathematician may construct longer and longer initial segments $\alpha(0), \ldots, \alpha(n)$ of an infinite sequence of natural numbers a where a is not a priori determined by some fixed process of producing the values, so the construction of a is never finished: a is an example of a choice sequence. (DALEN; TROELSTRA, 1988, pp.5)

Let D_I and V_I now stand for deductibility in an intuitionistic proof system and V_I for structures whose internal logic is Intuitionistic Logic (IL). We now argue for the coextensivity of D_I, Val_I and V_I.

Although Val_I is theoretically irreducible to both D_I and V_I, nonetheless (A') Val_I is sound with respect to IL, as Dalen (1986) argues, and also (B') the constructions allowed by Val_I can clearly be carried out in structures whose internal logic is IL. Indeed, the methods of constructions codified by IL represent a qualification of the (in principle) more general notion of constructibility. Therefore, we can argue that D_I captures a more restricted version of constructibility than Val_I. We can, therefore, run an analogous of Kreisel's squeezing argument.

Argument 3.2.1. *A squeezing argument for* IL *can be summarized as follows:.*

1.	$D_I(\varphi) \Rightarrow Val_I(\varphi)$	*(A')*
2.	$Val_I(\varphi) \Rightarrow V_I(\varphi)$	*(B')*
3.	$V_I(\varphi) \Rightarrow D_I(\varphi)$	*Completeness*
4.	$D_I(\varphi) \Leftrightarrow Val_I(\varphi) \Leftrightarrow V_I(\varphi)$	*from (1)-(3)*

Accepting instances of informal validity such as Val_I and Val amount us to accept a version of logical pluralism which says that the consequence relations of two different logical systems preserve different things. That is, incompatibility of two different systems is apparent because their consequence relation preserve different things. The Argument 3.2.1 shows that classical and intuitionistic logic do not need to be seen as rivals. Indeed, they can be seen as talking about different things: while classical logic may be

seen as giving a good account of preservation of truth, intuitionistic logic can be see as giving a good account of preservation of proof-constructibility. As a consequence, different logics preserve different informal notions of logical validity. For example, if D_{FOL} is the derivability relation of classical FOL, we know that $D_{\mathsf{FOL}}(\varphi \vee \neg\varphi) \Rightarrow Val_I(\varphi \vee \neg\varphi)$ do not hold because intuitionistic logic does not validate excluded middle. So, given two different logics L and L^*, we have that they do not share informal notions of logical validity since squeezing arguments establishes that $V_{\mathsf{L}}(\varphi) \Leftrightarrow Val_{\mathsf{L}}(\varphi) \Leftrightarrow D_{\mathsf{L}}(\varphi)$ and $V_{\mathsf{L}^*}(\varphi) \Leftrightarrow Val_{\mathsf{L}^*}(\varphi) \Leftrightarrow D_{\mathsf{L}^*}(\varphi)$.[23]

Logical pluralism received several objections in the literature. One of the main objections against such philosophical view about logics was formulated by Quine (1986). Quine defends that there is no dispute between logics because they simply talk about different things. The following fragment illustrates Quine's view:

> My view of this dialogue is that neither party knows what he is talking about. They think they are talking about negation, '$\sim$', 'not'; but surely the notation ceased to be recognizable as negation when they took to regarding some conjunctions of the form '$p. \sim p$' as true, as stopped regarding such sentences as implying all others. Here, evidently, is the deviant logician's predicament: when he tries to deny doctrine he only changes the subject. (QUINE, 1986, pg. 81)

That is, let L_1 and L_2 be two logical systems which are substantially different each other, in the sense that they disagree on at least one logical principle. According to Quine, L_1 and L_2 do not need to be as rivalling each other since they talk about different things. As a consequence, even if L_1 and L_2 share a connective c, the meaning of c is different in the systems L_1 and L_2. Although the informal pluralism defended here has some similarities with Quine's view, they are different views. As the passage above suggests, Quine is considering the case where both systems talk about truth. So, it seems that the possibility of L_1 and L_2 are preserving different notions than truth is not under consideration. This difference is important for the meaning of the connectives. For example, from the perspective of informal pluralism, $\vee$ can be considered as disjunction in both logics CPL and IL. Since both systems preserve different informal notions of validity, it is to be expected that CPL validates $\varphi \vee \neg\varphi$ whereas does not.[24]

[23] This shows that squeezing arguments have an interesting application, as a criteria to know whether an informal notion can be taken to interpret the formal notions of validity of a formal system. That is, in order to know that a certain informal validity notion Val^* is adequate to interpret V_{L} and D_{L}, we try to formulate a squeezing argument for this informal notion. If we can conclude $V_{\mathsf{L}}(\varphi) \Leftrightarrow Val^*(\varphi) \Leftrightarrow D_{\mathsf{L}}(\varphi)$, then Val^* is an adequate notion to interpret the formal notions of validity of L. If we cannot, then Val^* is not adequate for L.

[24] In Bezerra and Venturi (2021) we discussed the informal pluralism present in translation of logics. It is a well-known result that there is a translation between IL and $\mathsf{S4}$. Moreover, it is well-known that the latter system has different interpretations in the literature: epistemologic (STALNAKER, 2006), informal provability (BURGESS, 1999). So, these translation results between these logics can be used to argue that IL can also be epistemologically interpreted as well as in terms of informal provability. Such discussion can be extended to the relation between classical and intuitionistic logic due to the

Logical pluralism is attested by our logical practice. Many systems are often proposed to deal with several problems. In this sense, in principle is possible to provide at least one informal notion of validity for each logic we find in the literature. If we have a plurality of informal notions of validity (and consistency) and we do not have purely logical reasons to say that only one is the correct, the we have that logical validity, even in its informal characterization, is a *local* notion. In fact, if there are many logical systems capturing at least one informal notion of validity and we do not have an ultimate reason to choose one of them as the true, then we have good reasons to defend that there are many notions of validity. The plurality of logical systems, then, are able to offer an elucidative characterization of these informal notions.

It is clear, however, that both formal accounts of validity/consistency fail to capture primitive consistency since we do not know what primitive consistency is . They partially capture the informal notions of validity when we consider a fixed class of formulas. We say partially because the informal notions themselves, such as Kreisel's informal notion Val, cannot be totally captured by the model-theoretical and proof-theoretical approaches to validity. In the case of Kreisel's Val, we saw that it also comprehends class-sized structures while V only considers set-sized structures. But Val, V and D coincide when we fix the vocabulary of FOL. In this sense, we say that the informal notions of validity are partially captured by the formal definitions of validity. In what follows, we will defend that the formal approaches can be seen as explications of informal notion of logical validity/consistency.

In this Chapter, we focused in the relation between informal notions of logical validity and their corresponding formalizations. But as we argued before, the informal notions presented here do not account of all valid inferences since they are extensionally equivalent to the formal ones. In Chapter 5, we will present a more general notion of validity and how this notion can be formalized.

3.2.1 Model-theory and proof-theory as explanations of validity

Despite the difficulties that both approaches to validity face, we can see them, as Griffiths (2014) proposes, as *explications* of validity. The explanation here is understood in the sense of Carnap (1950). The informal notions of validity presented in this chapter has at least one occurrence of an imprecise concept, in the sense that its reference it is not clear at first. As an example, the definition of Val_{Nec} has the occurrence of the concept of necessity, which is not clear what is its reference. In this sense, such informal notions play the role of *explicandum*.

By their turn, the model and proof-theoretical definitions of validity occur in precise and well-defined frameworks. In this sense, they are the *explicata* of informal validity. In

double negation translation between such systems (GLIVENKO, 1929). Here we will not develop such discussion and it will be left for further investigation.

the case of FOL, of complete systems in general, we have that two *explicata*, E_1 and E_2, each one corresponding to a formal approach. E_1 and E_2 explain two sides of the same coin.

Carnap (1950, pp. 7) delineates some criteria to classify something as an explanation. In his own words:

1. *Similarity*: The explicatum is to be *similar* to the explicandum in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used;

2. *Exactness*: The characterization of the explicatum, that is, the rules of its use (for instance, in the form of a definition), is to be given in an *exact* form, so as to introduce the explicatum into a well-connected system of scientific concepts.

3. *Fruitfulness*. The explicatum is to be a fruitful concept, that is, useful for the formulation of many universal statements (empirical laws in the case of a nonlogical concept, logical theorems in the case of a logical concept).

4. *Simplicity*. The explicatum should be as *simple* as possible; this means as simple as the more important requirements as the requirements 1, 2 and 3 permit.

It is clear that the formal approaches to informal validity fulfil the above requirements. The only one which we have to devote more attention is the first. As we can see, the squeezing argument fail to capture an unique notion of validity. Some of them may have considerable differences with respect to its formalizations. Consider, for example, the case of Val_{Nec}. This informal notion is considerably different from V. So, it may suggests that the requirement 1 is violated. However, Carnap himself recognizes that it is not always that this similarity will occur. He allows "considerable differences". This is usual in most cases of formal theories. That is, the formalizations are not similar in the way that clause 1 suggests. In the cases of formal theories, we can appeal, as Novaes & Reck (2017) do, to a weaker requirement, which is the *material adequacy*. The squeezing arguments show that if we consider a restrict class of sentences, we can show that these informal notions can be captured by their model and proof-theoretical counterparts. In this sense, we can say that V and D are similar to their informal counterparts.

In the next chapters we will adopt both perspectives in order to obtain the general properties which govern them. By model-theoretical approach we mean the analysis of consistency as the dual of the semantic validity predicate Val.[25] By proof-theoretical approach we mean the analysis of consistency as the dual of the syntactic provability

[25] As we will see in the last chapter, the view that consistency is the dual of validity works in the classical case. If we look for proper fragments of classical logic, we will see that consistency should be defined independently of validity. But, of course, this does not mean that consistency will become a primitive notion in Field's sense.

predicate Pr. So it is to be expected that the discussion about consistency will become a discussion about validity/provability since we are taking consistency as the dual of validity. Then we will wonder whether there are modal logics which captures the general properties of Val and $Prov$. So, the predicate Val (respec., $Prov$) will stand for the modal operator $\Box$ and the predicate Con will stand for $\Diamond$. We want to keep this characterization as general as possible, for we want to include theories which do not necessarily contain arithmetic. As we will see, the logical validity predicate are captured by considerably weak modal logics, which sugests that such predicate keeps very general properties.

Chapter 4

Necessity as provability

In this Chapter, we will present some formalizations of the notion of provability by means of modal logics. First, we will give a general overview of Gödel's incompleteness results and of the early development of logics of provability. Then we will discuss the modal approach of formal and informal provability and we will point the principles that each notion satisfy. Second, we will present the modal logic KGL, the modal logic which captures the provability predicate of PA, and the results that show that this formal interpretation is well justified. Last, we will present to modalities which capture the predicates of *true provability* and *consistent provability*. In both cases, we will provide the axiomatizations of both modalities in the class of all frames as well as their characterization results.

4.1 On Hilbert's Program

Hilbert's Program is a program about provability. It aims to prove the consistency of classical mathematics via finitary proofs. The crisis in the foundations of mathematics generated by the paradoxes in the early 20th century showed the need for research on the foundations of mathematics. In the case of Set Theory, for example, Russell's paradox showed that the concept of set could not be as wide as it was in Cantor's naïve Set Theory. Zermelo's axiomatization of Set Theory in 1908, ZFC, became one of the most (if not the most) prominent foundational theories. Even if one cannot show the consistency of ZFC axioms inside ZFC, one can easily show that Russell's set cannot be formulated in ZFC without violating the axioms of Zermelo's theory.[1] Already in 1905, Hilbert (1905) saw that the axiomatic method was capable of overcoming the difficulties posed by the paradoxes. Later, in the decade of 1920, Hilbert and his students developed such axiomatic approach to mathematical theories, which gave rise to *Proof Theory*. Such development, along with the formalization of logic, should be able in principle to show

[1] 1This can be checked in any introductory book of Set Theory, such as Enderton (1977).

that contradictions do not raise in this approach to mathematics.[2]

As Zach (2007) points out, the development of an axiomatic theory allows to make explicit the logical relationships between the basic concepts of the theory and dispensing with any appeal to intuition. In addition to dispensing with any appeal to intuition concerning mathematical theories, Hilbert's *Formalism*, according to Franks (2009), aimed to rid mathematics of any philosophical inclinations, as the following passage shows:

> One must see him [Hilbert] deliberately offering mathematical explanations where philosophical ones were wanted. He did this, not to provide philosophical foundations, but to liberate mathematics from any apparent need for them (...) The legitimacy of Hilbert's philosophical stance lies precisely in its ability to generate an arena for the scientific study of mathematics. (FRANKS, 2009, pp. 7)

What matters, therefore, are the logical relationships between the concepts expressed by the axioms. As we will see below, the axiomatic development of mathematical theories must be simultaneous to the development of formal logic. Different from the logicist school, which saw that logic has a priority over mathematics, Hilbert saw that logic and mathematics are closely connected. But this question about such priority does not raise for Hilbert.[3]

As is known, Hilbert's basic requirement for an axiomatic theory is its consistency. Such requirement plays a central role because it is a existential condition for mathematical objects. That is, the objects of consistent mathematical theories exist. In general, the axiomatic method must be able to show that any axiomatic mathematical theory is free of contradictions. In addition to the requirement for consistency, Hilbert demanded that demonstration methods in axiomatic theories be finitary.

Obviously, the requirement for consistency is not only present in the works of Hilbert, but also in the works of Frege and Zermelo. There is a subtle difference between Frege's and Hilbert's conceptions of consistency. As Schmidt & Venturi (2021) observe, such difference is expressed by their views about existence of mathematical objects. For Frege, mathematical objects are consistent because they exist. Then, the axioms of Freges's theory are intended to grasp such pre-existing reality.[4] Hilbert, by its turn, defended that mathematical objects exist because they are consistent. The postulates of the axiomatic theory express the meaning of the objects of the theory. In this case, the consistency of the theory is a precondition for the existence of the objects characterized by the theory.

[2]This last aspect is important, as this concern in the development of logic has driven studies in metatheory. Further, we will come back to this issue.

[3]There are views according to which mathematics has priority over logic. For example, according to Franks (2009), Peirce defended such view.

[4]Frege's conception of axiomatics comes, in a certain sense, from Euclids's axiomatic method. As Mancosu et al (2009) point, consistency proofs were not necessary in the ancient approach to axiomatics because there was the assumption that the axioms are true of some reality. In this case, such proofs are not necessary.

In other words, consistency proofs of mathematical theories guarantee that the objects of such theories exist.

According to Hilbert (2002), the inconsistency of Cantor's theory of transfinite numbers was because the deductive logic methods do not work for infinite collections if such methods are adopted without any restrictions being made. Specific axiomatic treatment for sets is necessary so that the logic can be correctly applied. In Hilbert's words:

> Kant already taught -and indeed it is part and parcel of his doctrine- that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone; that is why the efforts of Frege and Dedekind were bound to fail. (HILBERT, 2002, pp. 376)

As the passage above attests, it is necessary to provide proper axioms for mathematics as well as a solid logical background. Logic itself is not capable of handling all mathematical reasoning.

The elementary number theory plays an important role in this discussion. By applying simple finite operations (successor, addition and multiplication) on sequence of strokes, which do not have themselves a meaning, one can generate infinitely many strokes. Although these strokes have no meaning, they play the role of numerals. That is, by means of finite operations on strokes, one can generate a structure like ω-sequence. According to Hilbert (2002), no contradiction can arise by this methods of generating strokes. Such theory, according to Hilbert's finitist point of view, is the contentful part of mathematics. Then, the formal theory which describes such structure must be a finitary arithmetic for which we can prove consistency results.

As Silva (2003) observes, although Hilbert is not clear about what is the formal axiomatic theory will formalize the elementary number theory, Skolem's *Primitive Recursive Arithmetic* (PRA) seems to fulfill the requirements. Informally speaking, PRA has as axioms the primitive recursive functions (*e.g.* successor, addition and multiplication), it does not make use of unbounded quantification and the variables range over only in finite domains.[5]

The consistency proof aimed by Hilbert must proceed by direct proof. That is, there is no appeal to a model which validates the axioms of PRA. The attitude of providing a model in order to prove the consistency might go in the opposite direction of that of grounding the axiomatic method as a foundation of mathematics.

Now, in what concerns Set Theory, the procedure must be the same. That is, one should provide a finitary direct proof of consistency of Set Theory. As Silva (2003) notes, proving the consistency of Set Theory, which is a theory of the infinite, by finitary methods does not mean giving up the infinite. It only means that the bases of the Set Theory are themselves finitary. To sum up, Hilbert's program aims to provide consistent finitary bases

[5]We invite the reader to Skolem (2002) to a presentation of PRA.

to foundations of mathematics by means of axiomatic theories. As a result, Mathematics would be seen as a stock of finitary provable formulas. In what follows, we briefly present Gödel's incompleteness theorems in order to show that Hilbert's program cannot totally succeed.

4.1.1 Provability in mathematical theories

In this subsection, we will give an informal presentation of Gödel incompleteness results and we will discuss some of its consequences. If T is a theory which contains PA, T is strong enough to represent the (finitary) recursive primitive functions in the following sense:

Theorem 4.1.1. *Let T be a theory extending PA and n-ary f a recursive function. We say that T* represents *f if there is a n-ary predicate F of T such that:*

$$\vdash_T F(x_1, \ldots, x_n, y) \text{ iff } f(x_1, \ldots, x_n) = y \tag{4.1}$$

By being capable of representing primitive recursive functions, T is capable to talk about its own syntax, T is capable to express names $\ulcorner\varphi\urcorner$ of its sentences φ, as well as the recursive primitive provability predicate $Prov_T$, where $\exists y Prov_T(y, \ulcorner\varphi\urcorner)$ means that y is the code of the proof of φ in T. From $\exists y Pr_T(y, \ulcorner\varphi\urcorner)$, one defines $Prov_T(\ulcorner\varphi\urcorner)$ as $Prov_T(\ulcorner\varphi\urcorner) := \exists y Pr_T(y, \ulcorner\varphi\urcorner)$. The Diagonalisation Lemma makes it possible for T to provide certain statements about equivalences.[6]

Lemma 4.1.2 (Diagonalisation Lemma)**.** *Let T be a theory extending PA. For any formula $A(x)$ of T with a free variable x there is a sentence C such that $\vdash_T C \leftrightarrow A(\ulcorner C\urcorner)$.*

Given Lemma 4.1.2 it is possible to construct for a sentences C a materially equivalent to the sentence $A(\ulcorner C\urcorner)$.[7] Particularly, one can construct a sentence A which is materially equivalent to $\neg Prov_T(\ulcorner A\urcorner)$. That is, T is capable to construct the following sentence:

$$A \leftrightarrow \neg Prov_T(\ulcorner A\urcorner) \tag{4.2}$$

The next two theorems are of fundamental importance in foundations of mathematics. If they are not the two most important results in the foundations of mathematics, they are certainly among the most important ones. Formal theories like our T are capable to *represent* all the (*finitary*) recursive operations of elementary number theory. That

[6] In a loose way of speaking, Diagonalisation Lemma provides a way of making self-referential statements.

[7] It is very common so see in the philosophical literature scholars saying that Lemma 4.1.2 allows self-reference in arithmetical sentences. However, as Raatikainen (2020) argues, it is an informal way of speech. It is not immediate at all that such Lemma provides a way to obtain self-reference in the formal system. Diagonalisation Lemma only says that they are provably equivalent, not that they have the same meaning. So, self-reference is an informal way of speak about this Lemma.

is, all the finitary operations of the elementary number theory can be codified within T. *Gödel's Incompleteness Theorems* give a robust answer about the feasibility of Hilbert's program. The *First incompleteness theorem* shows that if T is consistent, then there is a sentence that T cannot prove nor disprove. The *Second incompleteness theorem* shows that T cannot prove its consistency. Of course, if T is inconsistent, then all sentences are provable, including the sentence about its consistency. So, the important case here is when T is consistent. Then the next two theorems assume the consistency of T.

Theorem 4.1.3 (First Incompleteness Theorem)**.** *Let $Prov_T$ be a provability predicate such that for all sentences A:*

(Completeness) If $\vdash_T A$ then $\vdash_T Prov_T(\ulcorner A \urcorner)$;

(Soundness) If $\vdash_T Prov_T(\ulcorner A \urcorner)$ then $\vdash_T A$.

Moreover, if we let $\varphi \leftrightarrow \neg Prov_T(\ulcorner \varphi \urcorner)$, then

(i) $\nvdash_T \varphi$

(ii) $\nvdash_T \neg\varphi$

Proof. Suppose that $\vdash_T \varphi$. Then:

1.	$\vdash_T \varphi$	Hyp.
2.	$\vdash_T \varphi \leftrightarrow \neg Prov_T(\ulcorner \varphi \urcorner)$	Hyp.
3.	$\vdash_T \varphi \rightarrow \neg Prov_T(\ulcorner \varphi \urcorner)$	CPL 2
4.	$\vdash_T \neg Prov_T(\ulcorner \varphi \urcorner)$	MP 1,3
5.	$\vdash_T Prov_T(\ulcorner \varphi \urcorner)$	Completeness 1

Contradiction. Then $\nvdash_T \varphi$. Now suppose that $\vdash_T \neg\varphi$. Then:

1.	$\vdash_T \neg\varphi$	Hyp.
2.	$\vdash_T \varphi \leftrightarrow \neg Prov_T(\ulcorner \varphi \urcorner)$	Hyp.
3.	$\vdash_T \neg Prov_T(\ulcorner \varphi \urcorner) \rightarrow \varphi$	CPL 2
4.	$\vdash_T \neg\varphi \rightarrow (\neg Prov_T(\ulcorner \varphi \urcorner) \rightarrow \neg\varphi)$	CPL
5.	$\vdash_T \neg Prov_T(\ulcorner \varphi \urcorner) \rightarrow \neg\varphi$	MP 1,4
6.	$\vdash_T \neg Prov_T(\ulcorner \varphi \urcorner)$	CPL 3,5
7.	$\vdash_T \neg\varphi$	Soundness 6

Contradiction. Then $\nvdash_T \neg\varphi$. This concludes the proof. Q.E.D.

The full proof of the Theorem 4.1.3 and of the Lemma 4.1.2 can be found in Smoryński (1985). The Theorem 4.1.3 establishes that the unprovability of the sentence C, which is equivalent to its unprovability, depends on the consistency of T. That is, if T does not prove a contradiction, then C is not provable. The sentence C is true in the standard model of T, but not provable in T. According to Smoryński, the second incompleteness

theorem shows that the sentence which express the consistency of the system T is one of the sentences that T cannot prove.

Theorem 4.1.4 (Second Incompleteness Theorem)**.** *Let* $0 = 1$ *be a sentence denoting an absurd and* $Con = \neg Prov_T(\ulcorner 0 = 1 \urcorner)$*. Then* $\nvdash_T Con$*.*

According to Smoryński, Theorem 4.1.4 could be proved in a similar way than the Theorem 4.1.3. But to avoid excessive labour, it is convenient to introduce the so called *Gödel-Löb Derivability Conditions*, which are stated as follows:

Fact 4.1.5. *Let* $Prov_T$ *be a provability predicate of the theory* T*.* $Prov_T$ *satisfies the following conditions:*

(DC1) *If* $\vdash_T \varphi$*, then* $\vdash_T Prov_T(\ulcorner \varphi \urcorner)$

(DC2) $\vdash_T Prov_T(\ulcorner \varphi \to \psi \urcorner) \to (Prov_T(\ulcorner \varphi \urcorner) \to Prov_T(\ulcorner \psi \urcorner))$

(DC3) $\vdash_T Prov_T(\ulcorner \varphi \urcorner) \to Prov_T(\ulcorner Prov_T(\ulcorner \varphi \urcorner) \urcorner)$

(DC4) *If* $\vdash_T Prov_T(\ulcorner Prov_T(\ulcorner \varphi \urcorner) \to \varphi \urcorner)$*, then* $\vdash_T Prov_T(\ulcorner \varphi \urcorner)$

Theories T, obviously including PA, which contain arithmetic are capable to code their own sentences, as well as the proofs within itself. So, if T proves a sentence φ, then there is a code n which codifies the proof of φ in T. What condition (DC1) asserts is that T is able to prove that n is a proof of φ. Then $\vdash_T Prov_T(\ulcorner \varphi \urcorner)$. The condition (DC2) asserts that the provability is preserved under modus ponens. The condition (DC3) asserts that if φ is provable, then it is provable that φ is provable. Finally, the condition (DC4) is the *Löb's Theorem* (1955), which asserts that $Prov_T(\ulcorner \varphi \urcorner) \to \varphi$ is provable only in the case that φ is already provable. So, given that $Prov_T$ satisfies Löb conditions, the second incompleteness theorem is easily provable by means of a simple formal derivation. (DC4) can be seen as a generalization of Theorem 4.1.4, because it says that the reflection schema only holds already proved formulas.[8] Before presenting the proof for Theorem 4.1.4, we prove the following auxiliary result:

Proposition 4.1.6. *The following schemas are theorems of* T*:*

(A) $\vdash_T Prov_T(\ulcorner \varphi \wedge \psi \urcorner) \leftrightarrow (Prov_T(\ulcorner \varphi \urcorner) \wedge Prov_T(\ulcorner \psi \urcorner))$*;*

(B) $\vdash_T Prov_T(\ulcorner \varphi \leftrightarrow \psi \urcorner) \to (Prov_T(\ulcorner \varphi \urcorner) \leftrightarrow Prov_T(\ulcorner \psi \urcorner))$*.*

Proof. Consider the following formal derivations:

[8] Smoryński (1985, pg.10) says: "Consistency is an expression of faith in the system which the Second Incompleteness Theorem asserts the system cannot prove; Löb's Theorem generalises this by characterizing provable instances of a more general expression of faith."

1.	$\vdash_T \varphi \to (\psi \to (\varphi \wedge \psi))$	CPL
2.	$\vdash_T Prov_T(\ulcorner \varphi \to (\psi \to (\varphi \wedge \psi)) \urcorner)$	DC1,1
3.	$\vdash_T Prov_T(\ulcorner \varphi \to (\psi \to (\varphi \wedge \psi)) \urcorner) \to$	DC2
	$(Prov_T(\ulcorner \varphi \urcorner) \to Prov_T(\ulcorner \psi \to (\varphi \wedge \psi) \urcorner)$	DC2(3).
4.	$\vdash_T (Prov_T(\ulcorner \varphi \urcorner) \to Prov_T(\ulcorner \psi \to (\varphi \wedge \psi) \urcorner)$	MP 2,3
5.	$\vdash_T Prov_T(\ulcorner \psi \to (\varphi \wedge \psi) \urcorner) \to (Prov_T(\ulcorner \psi \urcorner) \to Prov_T(\ulcorner \varphi \wedge \psi \urcorner))$	DC2
6.	$\vdash_T Prov_T(\ulcorner \varphi \urcorner) \to (Prov_T(\ulcorner \psi \urcorner) \to Prov_T(\ulcorner \varphi \wedge \psi \urcorner))$	CPL 4,5
7.	$\vdash_T (Prov_T(\ulcorner \varphi \urcorner) \wedge Prov_T(\ulcorner \psi \urcorner)) \to (Prov_T(\ulcorner \varphi \wedge \psi \urcorner))$	CPL,6

For the converse:

1.	$\vdash_T (\varphi \wedge \psi) \to \varphi$	CPL
2.	$\vdash_T (\varphi \wedge \psi) \to \psi$	CPL
3.	$\vdash_T Prov_T(\ulcorner (\varphi \wedge \psi) \to \varphi \urcorner)$	DC1,1
4.	$\vdash_T Prov_T(\ulcorner (\varphi \wedge \psi) \to \psi \urcorner)$	DC1,2
5.	$\vdash_T Prov_T(\ulcorner (\varphi \wedge \psi) \to \varphi \urcorner) \to (Prov_T(\ulcorner \varphi \wedge \psi \urcorner) \to Prov_T(\ulcorner \varphi \urcorner))$	DC2
6.	$\vdash_T Prov_T(\ulcorner (\varphi \wedge \psi) \to \psi \urcorner) \to (Prov_T(\ulcorner \varphi \wedge \psi \urcorner) \to Prov_T(\ulcorner \psi \urcorner))$	DC2
7.	$\vdash_T Prov_T(\ulcorner \varphi \wedge \psi \urcorner) \to Prov_T(\ulcorner \varphi \urcorner)$	MP3,5
8.	$\vdash_T Prov_T(\ulcorner \varphi \wedge \psi \urcorner) \to Prov_T(\ulcorner \varphi \urcorner)$	MP4,6
9.	$\vdash_T Prov_T(\ulcorner \varphi \wedge \psi \urcorner) \to (Prov_T(\ulcorner \varphi \urcorner) \wedge Prov_T(\ulcorner \psi \urcorner))$	CPL 7,8

The proof of (B) is left for the reader. This concludes the proof.

Q.E.D.

Now we present the proof of Theorem 4.1.4:

Proof. By Diagonalisation Lemma, we obtain a sentence φ which is provably equivalent to its non-provability, $\varphi \leftrightarrow \neg Prov(\ulcorner \varphi \urcorner)$. We have to show that $Con \to \varphi$.

1.	$\vdash_T \varphi \leftrightarrow \neg Prov(\ulcorner\varphi\urcorner)$	Lemma 4.1.2
2.	$\vdash_T \neg\neg Prov(\ulcorner\varphi\urcorner) \leftrightarrow \neg\varphi$	CPL 1
3.	$\vdash_T \neg\neg Prov(\ulcorner\varphi\urcorner) \leftrightarrow Prov(\ulcorner\varphi\urcorner)$	CPL
4.	$\vdash_T Prov(\ulcorner\varphi\urcorner) \leftrightarrow \neg\varphi$	CPL 3,2
5.	$\vdash_T Prov(\ulcorner Prov(\ulcorner\varphi\urcorner) \leftrightarrow \neg\varphi\urcorner)$	DC1,4
6.	$\vdash_T Prov_T(\ulcorner Prov(\ulcorner\varphi\urcorner) \leftrightarrow \neg\varphi\urcorner) \rightarrow$ $(Prov_T(\ulcorner Prov(\ulcorner\varphi\urcorner)\urcorner) \leftrightarrow Prov_T(\ulcorner\neg\varphi\urcorner))$	Th. 4.1.6(cont. 6)
7.	$\vdash_T Prov_T(\ulcorner Prov(\ulcorner\varphi\urcorner)\urcorner) \leftrightarrow Prov_T(\ulcorner\neg\varphi\urcorner)$	MP 5,6
8.	$\vdash_T Prov_T(\ulcorner\varphi\urcorner) \rightarrow Prov_T(\ulcorner Prov_T(\ulcorner\varphi\urcorner)\urcorner)$	DC3
9.	$\vdash_T Prov_T(\ulcorner\varphi\urcorner) \rightarrow Prov_T(\ulcorner\neg\varphi\urcorner)$	CPL 7,8
10.	$\vdash_T Prov_T(\ulcorner\varphi\urcorner) \rightarrow Prov_T(\ulcorner\varphi\urcorner)$	CPL
11.	$\vdash_T (\varphi \wedge \neg\varphi) \rightarrow 0 = 1$	CPL
12.	$\vdash_T Prov_T(\ulcorner(\varphi \wedge \neg\varphi) \rightarrow 0 = 1\urcorner) \rightarrow$ $(Prov_T(\ulcorner\varphi \wedge \neg\varphi\urcorner) \rightarrow Prov_T(\ulcorner 0 = 1\urcorner))$	DC2 (cont.)
13.	$\vdash_T Prov_T(\ulcorner(\varphi \wedge \neg\varphi) \rightarrow 0 = 1\urcorner)$	DC1, 11
14.	$\vdash_T (Prov_T(\ulcorner\varphi \wedge \neg\varphi\urcorner) \rightarrow Prov_T(\ulcorner 0 = 1\urcorner))$	MP 12,13
15.	$\vdash_T (Prov_T(\ulcorner\varphi\urcorner) \wedge Prov_T(\ulcorner\neg\varphi\urcorner)) \rightarrow Prov_T(\ulcorner\varphi \wedge \neg\varphi\urcorner)$	Th. 4.1.6
16.	$\vdash_T (Prov_T(\ulcorner\varphi\urcorner) \wedge Prov_T(\ulcorner\neg\varphi\urcorner)) \rightarrow Prov_T(\ulcorner 0 = 1\urcorner)$	CPL 14,15
17.	$\vdash_T Prov_T(\ulcorner\varphi\urcorner) \rightarrow (Prov_T(\ulcorner\varphi\urcorner) \wedge Prov_T(\ulcorner\neg\varphi\urcorner))$	CPL 9,10
18.	$\vdash_T Prov_T(\ulcorner\varphi\urcorner) \rightarrow Prov_T(\ulcorner 0 = 1\urcorner)$	CPL 16,17
19.	$\vdash_T \neg Prov_T(\ulcorner 0 = 1\urcorner) \rightarrow \neg Prov_T(\ulcorner\varphi\urcorner)$	CPL 18

This means that $Con \rightarrow \varphi$. So, if $\vdash_T Con$, the sentence $\varphi \leftrightarrow \neg Prov(\ulcorner\varphi\urcorner)$ would also be provable in T, which we know that it is not the case, by Theorem 4.1.3. This concludes the proof.

Q.E.D.

Gödel's theorems had a considerable impact on foundations of mathematics. As we said before, they show that Hilbert's program cannot succeed. Both incompleteness theorems can also be proved for axiomatic Set Theory such as ZFC. Then, foundational theories like ZFC cannot prove their own consistency. According to Gödel (1995), it is the second incompleteness theorem which asserts the incompleteness of mathematical axiomatic theories. In view of this result, it is impossible to say that such axiomatic theories contain all mathematics. Even if one believes, or is certain, that such axiomatic theories are consistent, he/she cannot prove the consistency of these theorems. It means that we cannot totally capture the infinitary mathematics by finitary means.

There are many interpretations of Gödel results in the literature. A very standard interpretation of Theorem 4.1.3 states that there are true sentences which are not provable in T. Let $S_G := A \leftrightarrow \neg Prov_T(\ulcorner A\urcorner)$. What does it mean to say that S_G is true? According to Verbrugge (2017), S_G is true in the standard model, i.e., in the model $\mathbb{N}$ of natural

numbers. On the other hand, if $T = \mathsf{PA}$, T has *non-standard models*, i.e., a model which has at least one number which is different from all natural numbers. And S_G may not be true in non-standard models. Then 'S_G is true' actually means S_G is true in the standard model.[9]

According to Raatikainen (2005), the best way to understand the truth of S_G is to consider the following biconditional:

$$S_G \text{ is true iff } T \text{ is consistent} \tag{4.3}$$

It is clear that biconditional 4.3 is not provable in T. Now, if T is inconsistent, S_G is true because everything would be true. On the other hand, if T is consistent, then S_G is true. Under this semantic interpretation, first incompleteness theorem is also a problem for intuitionists because the identification between truth and provability is not valid.

In what concerns the philosophical interpretations of Gödel's theorems, there are some things which are important to discuss. According to Raatikainen (2018), Gödel (1995)'s own interpretation is a very cautious one. His interpretation is known as *Gödel's Disjunction* (GD), which can be stated as (RAATIKAINEN, 2018):

(GD) Either the human mind (even within the realm of pure mathematics) can surpass the power of any finite computing machine, or there are absolutely undecidable mathematical problems.

The second disjunct means that there are mathematical problems which cannot be solved in finitary formal systems. There are some interpretations that seem to be compatible with GD which will be useful for our discussion. Some interpretation maintain that Gödel's theorems show that there are mathematical inferences which are not formalizable in formal systems which extend PA.[10] For example, Marfori (2010) argues that:

> Incompleteness results in mathematical logic seriously undermined the formalist-reductionist project and showed that the project could not provide the desired secure foundations for mathematics, at least not in its original formulation(s). More specifically, the incompleteness theorems undermined the claim that mathematical provability was indeed reducible to provability within a formal system, and accordingly a fundamental part of the project of giving axiomatic foundations to mathematics. (MARFORI, 2010, pp. 263)

Marfori's passage follows Myhill (1960)'s interpretation according to which mathematical provability has an absolute/informal sense that is not captured by formal provability.

[9] In the literature there are many discussions about how to know if sentences like S_G are true. We will not enter in such discussion.

[10] Here we will focus on mathematical provability. Of course, if we go for natural language, there will be of course inferences which are not adequately formalizable in a mathematical language. For example, if someone says "I smell smoke", she/he can safely infer that there is a fire principle nearby.

In these wide understanding of absolute provability, he defends that sentences like S_G are provable. In his work, however, he does not provide a definition of absolute provability. He only says that absolute provability is not reducible to proof-theoretical concepts nor to model-theoretical concepts.

Of course, there is a clear distinction between informal provability and formal provability: while the latter is particular to a specific formal system, the former does not depend on a particular formal system. Informal provability also seems to encompass epistemological elements which are somewhat rejected by the formal approach. So it is to be expected that formal and informal provability do not coincide. But, in what concerns to mathematical provability, whether formal or informal, it is somewhat difficult to believe that such informal proofs cannot be formalized. Here we are not saying that such informal proofs must be necessarily formalizable in a first-order theory. We are just saying that it is difficult to accept that such inferences cannot be formalizable at all in a formal system, whether it is a first-order, higher-order systems or even infinitary logics.

In order to defend the status of informal provability, Pawlowski (2018b) points the difficulty of converting informal proofs into formal ones, in the sense that something can be lost in this process of formalization. Even if it may be true, this is the cost of the precision present in formal theories. A formal proof of the statement F_T in a formal theory T may be much more difficult to grasp than the proof of F in an informal mathematical language. But such translation has advantages, such as the clear logical relations in the steps of the formal proofs. Moreover, this problem of losing "something" is characteristic of the process of translations, even between natural languages.

It is a matter of fact the existence of informal notions of validity. On the other hand, we defend that such notions must be sharp enough to be understood as a provability relation. As we defended in Chapter 3, it is very difficult to grasp what is characteristic of an informal inference. There we also defended that the notion of validity is a notion which is theorized enough to be called primitive. And, because it is theorized, it is very difficult to accept the existence of an absolute notion of provability not amenable to formal analysis. It can encompass many formal systems, but not all.

Now we turn to the analysis of the provability predicate by means of modal logics.

4.2 The logics of provability

As the name suggests, the logics of provability are intended to formalize, in terms of modalities, the concept of provability of mathematical theories such as, for example, PA. We can say that the interest of studying provability of these theories is influenced by Gödel's two incompleteness theorems. Roughly speaking, the first theorem says that if T is a theory which contains arithmetic, then T has non-provable true sentences. And the second establishes that T cannot prove its own consistency. On the other hand,

differently from the alethic concepts of necessity and possibility, the logics of provability give a precise meaning to modal operators since the concept of provability in a formal theory is rigorously defined, leaving no room for ambiguities.

4.3 The birth of logics of provability

One of the most successful achievements of modal logics was the so called *Provability Logics*. One of the first investigations in this direction was due to Gödel (1986a) in interpreting the Intuitionistic Propositional Calculus (IPC) into a modal system.[11] The formula '$\Box\varphi$' (originally written "$B\varphi$") was intended to mean that "φ is provable" (B stands for "beweisbar"). The resulting modal logic of this translation was the logic $\mathcal{G}$, nowadays known as S4 (Definition 2.3.7).

Under the provability interpretation of $\Box$, the axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ says that provability is preserved under modus ponens. The axiom $\Box\varphi \to \varphi$ says that whatever it is provable it is true. The axiom $\Box\varphi \to \Box\Box\varphi$ says that if φ is provable, then it is provable that φ is provable. The latter axiom is a kind of introspection principle. The necessitation rule has a similar meaning as in the latter case.

Gödel (1986a) points out that S4 does not coincide with provability in formal systems. That is, $\Box\varphi$ cannot stand for "φ is provable in a formal system " which contains arithmetic. According to Gödel (1986a), the reason for this is the following:

> It is to be noted that for the notion "provable in a certain formal system S" not all the formulas provable in $\mathcal{G}$ hold. For example, $B(Bp \to p)$ never holds for that notion, that is, holds for no system S that contains arithmetic. For otherwise, for example, $B(0 \neq 0) \to 0 \neq 0$ and therefore also $\neg B(0 \neq 0)$ would be provable in S, that is, the consistency of S would be provable. (GÖDEL, 1986a, p. 301-302)

In other words, this interpretation would contradict the *second incompleteness theorem*, which says that the consistency of first-order arithmetic cannot be proved. The problem is to take $B\varphi \to \varphi$ as an axiom of the system $\mathcal{G}$. The incompatibility of $\mathcal{G}$ with the interpretation of provability of arithmetical theories justifies the search for modal logics which are compatible with respect to this interpretation.

As Dean (2014) observes, it is interesting that Gödel uses the logic S4 in order to give a provability interpretation for IPC. As we know, *Brouwer Heyting Kolmogorov* (BHK) interpretation for IPC interprets the logical constants of intuitionistic logic in terms of constructive provability.[12] So, according to this interpretation, $\vdash_{\mathsf{IPC}} \varphi$ means that φ is constructively provable. On the other hand, Gödel at the same time defends that $\Box$ operator of S4 cannot be interpreted as provability in theories which contains arithmetic.

[11] For the axiomatization of IPC check (MOSCHOVAKIS, 2018).

[12] For a presentation and discussion of BHK interpretation, consult Dalen e Troelstra (1988).

Both interpretations contradict each other when taken together. As Bezerra & Venturi (2021) show, we can run a squeezing argument for each informal interpretation of IPC. The possibility of presenting these two squeezing arguments for IPC corroborates which the view, originally defended by Barrio (2018), that formal systems do not have canonical interpretations.

Even if S4 cannot be taken to interpret provability in theories which contain arithmetic, there is a sense that S4 can be said to capture something akin to provability. We can say that S4 captures a broader concept of provability, which we can call *absolute*.[13] According to Crocco (2019), Gödel understands absolute provability in two ways: in a weak sense and in a strict sense. In the weak sense, absolute provability means independence from formal languages and systems. It encompass a transfinite hierarchy which it is highly set theoretical, in the sense that it uses set-theoretical language. In the strict sense, absolute provability means independence from any formal languages and objects.[14] According to Crocco, we have "no clear analysis of what absolute proofs can be" in the strict sense. It is more plausible to accept that S4 captures absolute provability in this weak sense.

According to Myhill (1960), absolute provability is reducible neither proof-theoretical constructs nor to model-theoretical ones. In the course of his paper, Myhill does not give a definition of what he understands by absolute provability, but it seems that he understands such concept in a similar way as Halldén (1963) does. So, absolute provability of φ is understood as *giving compelling logical grounds to belive* φ. Myhill argues that the undecidable statements of PA are provable in the absolute sense, as the following passage attests:

> I am asserting that there is an absolute sense of 'provable', neither syntactical nor semantical nor psychological, and that in this sense of 'provable', Gödel undecidable statements are provable. The proof which I assert not to be formalizable in elementary arithmetic is as follows: The axioms of elementary arithmetic are true, and the rules of inference are truth-preserving. Therefore, every theorem of elementary arithmetic is true. Therefore $0 = 1$ is not a theorem of elementary arithmetic. Therefore a certain statement p (the arithmetization of the statement that $0 = 1$ is not a theorem) is true. (MYHILL, 1960, pp. 463)

Definition 4.3.1. *Let* φ *be formula in the language* $\mathcal{L}_{\mathsf{PA}}$. *The formula* $\mathrm{AbPr}(\ulcorner\varphi\urcorner)$ *means that* φ *is* absolutely provable. AbPr *has the following properties:*

1. $\mathrm{AbPr}(\ulcorner\varphi\urcorner) \to \varphi$;
2. $\mathrm{AbPr}(\ulcorner\varphi \to \psi\urcorner) \to (\mathrm{AbPr}(\ulcorner\varphi\urcorner) \to \mathrm{AbPr}(\ulcorner\psi\urcorner))$;
3. *if* φ *is a theorem, then* $\mathrm{AbPr}(\ulcorner\varphi\urcorner)$.

[13] Based on Crocco (2019) we prefer not to refer to Gödel's broader concept of provability as informal provability because of translation issues from German to English.

[14] This distinction can be found in Gödel (1990).

The introspection principle $\mathrm{AbPr}(\ulcorner\varphi\urcorner) \rightarrow \mathrm{AbPr}(\ulcorner\mathrm{AbPr}(\ulcorner\varphi\urcorner)\urcorner)$ is not valid, according to Myhill, because the interest at issue is the absolute provability of arithmetical sentences. And formulas of the form $\mathrm{AbPr}(\ulcorner\varphi\urcorner)$ are not expected to be arithmetical because the predicate AbPr is intended to be in a higher hierarchy than arithmetical statements.

By Gödel's incompleteness theorems, such predicate cannot consistently extend PA. Myhill's observations is an informal version of Montague's theorem, Montague (1963), about modal predicates in arithmetical theories. Roughly speaking, if T extends PA with a predicate P satisfying the properties 1-3 of Definition 4.3.1, then T is trivial. In the next Chapter, we will present Montague's theorem and discuss its philosophical significance.

Although the concept expressed by AbPr is reducible neither to model-theoretical nor to proof-theoretical concepts, Myhill's sense of absolute provability is highly formal. As Crocco (2019) points, his notion of absolute provability is particular to mathematics. That is, it does not extrapolates to other areas of reasoning. And the Myhill's reason to defend that absolute provability is irreducible is that we cannot assure that our present mathematical theories capture an idealized notion of proof. The following passage resumes well Myhill's point.

> (...) (M)athematics is in a state of slow, jerky motion (which has incidentally become *more* rapid rather than less since the advent of formalization). I hope that this will make it appear reasonable that there is an ideal of proof to which historically given mathematical formalisms are better and better approximations. If one thinks seriously and realistically about contemporary mathematics compared with the mathematics of the eighteenth century, and if one prognosticates even conservatively about what the mathematics of the twenty-second century is likely to be like, it becomes just a little silly to think of equating 'correct provability' with, for example, provability in Zermelo-Frankel set-theory. (MYHILL, 1960, pp. 464)

So, as the above passage shows, the provability in our present mathematical theories are approximations to this ideal of proof. For this reason, we cannot say that, for example, provability in PA captures the whole concept of provability.

At this point, it should be clear that Kreisel's informal notion of validity does not collapse with Myhill's notion of informal provability. Even if both are highly mathematical, they have considerable differences. As Crocco (2019) observes, while Kreisel's informal notion of validity is a result of a conceptual analysis that aims "to eliminate the doubtful properties of the intuitive notions", and that coincides with the formal notions of validity when first-order formulas are taken into consideration, Myhill's notion of absolute provability refers to an ideal of proof which is partially captured by the mathematical formalism.

The concept of informal provability inspired some attempts in the literature in order to clarify such concept. Halldén (1963) defines informal provability as provability *by any correct logical means*. In his work, informal provability also comprehends scientific

theories other than mathematics, and the logic which regulates informal provability is also S4. Inspired in Halldén's arguments, Burgess (1999) argues that S4 is the logic of demonstrability, in the sense that the latter notion may not take into consideration the theoretical limitations of a particular theory, whereas (formal) provability is particular to the theory in question.

Because it is not a precise concept, some authors defend that question about the logic which captures the most general principles of informal provability is still an open problem. There is a consensus, however, that the logic S4 is at least (informally) sound with respect to informal provability, when we are not exclusively dealing with absolute provability of arithmetical sentences. On the other hand, it is an open problem whether S4 is (informally) complete with respect to informal provability. Even if there are arguments in favour of S4's informal completeness, Leitgeb (2009) points out that this is an open problem yet because it is debatable if we have a complete understanding about all our intuitions about proofs. He defends that an investigation on formal proofs may shed lights upon our understanding of informal proofs.

> Speaking of intuitions of proofs is certainly not void of content. Proof theorists seem to rely on quite clear non semantic representations of proofs in terms of trees or other graphs; while the proofs in question are of course formal ones, something like this might also be true of informal proofs. If so, then some formal results on the geometry or topology of formal proofs might become applicable in the realm of informal provability, too. (LEITGEB, 2009, pp. 30)

As above quote shows, the informal rigour *à la* Kreisel is needed in order to understand our informal notion of proofs. But the present case is not analogous to Kreisel's informal notion of validity. Kreisel's informal notion is sufficiently sharp, capturing the same validities as its formal counterparts in first-order logic. In the present case, we cannot say the same unless we provide a definition of what informal provability is. Even so, the suggestion to look for formal results in order to understand informal proofs shows that formal tools are not only simple systematizations of our informal concepts. The results about those formal tools improves our understanding of informal notions.[15]

4.4 The modal logic of arithmetical provability: KGL

The most known provability logic is called KGL and it is defined as follows:

[15]The present discussion supposes that our activity of proving has classical logic as the basic logic. There are works suggesting that informal provability is better understood if we change the basic logic. For example, Pawlowski (2018a) proposes that the logic of informal provability is non-determinitic in the sense that the truth-value of a complex formula is not totally determined by the truth-values of its constituents. In the last Chapter, we develop a similar, but different, proposal. There we will use non-classical logics to discover the most general properties of model-theoretical validity, but without suggesting what is *the* logic of validity.

Definition 4.4.1. *(BOOLOS, 1995) The logic* KGL *is obtained by adding to* K *the axiom:*

$$\text{(GL) } \Box(\Box\varphi \to \varphi) \to \Box\varphi$$

The axiom is also called Löb's Axiom. Boolos (1995) proves that the formula (4) $\Box\varphi \to \Box\Box\varphi$ is a theorem of KGL.

Theorem 4.4.2. $\vdash_{\mathsf{KGL}} \Box\varphi \to \Box\Box\varphi$

As a consequence of the Theorem 4.4.2, the logic K4 is a subsystem of KGL.

Definition 4.4.3. *Let W be a set and $R \subseteq W \times W$ be a relation on W.*

i *A relation is* conversely well-founded *iff for every non-empty set X there is $x \in X$ such that for no $y \in X$ where wRy. That is, there is no infinite ascending chains.*

Theorem 4.4.4. *(BOOLOS, 1995) For all frames $F = \langle W, R\rangle$, $F \models \Box(\Box\varphi \to \varphi) \to \Box\varphi$ iff R is transitive conversely well-founded.*

Theorem 4.4.5 (Soundness)**.** KGL *is sound with respect to transitive conversely well-founded frames.*

Differently from several normal modal logics, the completeness of KGL cannot be proved using the method of canonical models. Roughly speaking, the frame of the canonical model of KGL does not validate all of its theorem. But it can be proved that KGL is complete with respect to the class of finite transitive and conversely well-founded frames.[16]

Theorem 4.4.6 (Completeness)**.** KGL *is sound with respect to transitive conversely well-founded frames.*

4.4.1 KGL and arithmetical provability

KGL is a very interesting modal logic because it is one of the few modal systems whose interpretation is indisputable. For example, take the example of deontic logics and epistemic logics. It is difficult to assert what is the right deontic (respectively, epistemic) logic which capture our deontic (respectively, epistemic) informal intuitions, which are vague in a certain instance. Then, there may be the case that a theorem of a deontic logic may not reflect our deontic intuitions. On the other hand, provability in a formal theory is not a vague concept, since it is defined within a well-structured conceptual framework.

According to Verbrugge (2017), the origins of the axiom KGL lies in Henkin's question about sentences which express its own provability. For example, $A \leftrightarrow Pr(\ulcorner A \urcorner)$, where A is a formula of PA. As we said before, Löb answered this question by proving that sentences of the form $Pr(\ulcorner A \urcorner) \to A$ can be proved in PA just in case A is already provable in PA.

[16] For an exposition of this proof check Hughes & Cresswell (1996).

Theorem 4.4.7 (Löb's theorem)**.** *If* $\vdash_{\mathsf{KGL}} \Box\varphi \to \varphi$*, then* $\vdash_{\mathsf{KGL}} \varphi$*.*[17]

Theorems like Theorem 4.4.7 show that the logic KGL can represent the most important results concerning the provability predicate of PA. But the fact that KGL is able to capture these results in the modal language is not an accident. Consider the following definition given by Boolos et al (2002):

Definition 4.4.8. *(BOOLOS; BURGESS; JEFFREY, 2002) Let* $\mathcal{L}_{\mathsf{PA}}$ *be the language of arithmetic and* ϕ *a realization which assigns to sentence letters sentences of* $\mathcal{L}_{\mathsf{PA}}$*. We associate each modal sentence* α *a sentence* α^{ϕ} *as follows:*

$$\begin{array}{lcl} p^{\phi} & = & \phi(p), \text{ where } p \text{ is a sentential letter} \\ \bot^{\phi} & = & \mathbf{0} = \mathbf{1} \\ (\alpha \to \beta)^{\phi} & = & \alpha^{\phi} \to \beta^{\phi} \\ (\Box\alpha)^{\phi} & = & Prov(\ulcorner \alpha \urcorner) \end{array}$$

From the Definition 4.4.8, Solovay (1976) proved two results which show the relations between KGL and PA:

Theorem 4.4.9. *(SOLOVAY, 1976, Arithmetical soundness) If* $\vdash_{\mathsf{KGL}} \alpha$*, then ,for every* ϕ*,* $\vdash_{\mathsf{PA}} \alpha^{\phi}$*.*

Proof. The proof of Theorem 4.4.9 is simple. Consider any ϕ according to Definition 4.4.8. The proof proceeds by induction on the length of PA proofs. First, we have to show that KGL axioms are provable in PA modulo realization ϕ and that the rules of KGL preserve theoremhood. The cases of propositional axioms are immediate. The inference rules are also straightforward. Let us focus on the modal axioms and rules. Since we are dealing with PA we will omit bellow the subscript PA in $Prov_{\mathsf{PA}}$.

$\vdash_{\mathsf{KGL}} \alpha$ such that $\alpha = (\Box\varphi \to \varphi) \to (\Box\varphi \to \Box\psi)$. By applying the function ϕ to α, we obtain $Prov(\ulcorner \varphi^{\phi} \to \psi^{\phi} \urcorner) \to (Prov(\ulcorner \varphi^{\phi} \urcorner) \to Prov(\ulcorner \psi^{\phi} \urcorner))$. As we can see, α^{ϕ} is the derivability condition (DC2) of Fact 4.1.5.

$\vdash_{\mathsf{KGL}} \alpha$ such that $\alpha = \Box\varphi \to \Box\Box\varphi$. By applying the function ϕ to α, we obtain $Prov(\ulcorner \varphi^{\phi} \urcorner) \to Prov(\ulcorner Prov(\ulcorner \varphi^{\phi} \urcorner) \urcorner)$, which is derivability condition (DC3) of Fact 4.1.5.

$\vdash_{\mathsf{KGL}} \alpha$ such that $\alpha = \Box(\Box\varphi \to \varphi) \to \Box\varphi$. By Löb's Theorem, it suffices to prove that $Prov(\ulcorner Prov(\ulcorner Prov(\ulcorner \psi^{\phi} \urcorner) \to \psi^{\phi} \urcorner) \to Prov(\ulcorner \psi^{\phi} \urcorner) \urcorner) \to (Prov(\ulcorner Prov(\ulcorner \psi^{\phi} \urcorner) \to \psi^{\phi} \urcorner) \to \psi^{\phi} \urcorner))$. In the following derivation, we write $Pr(\psi^{\phi})$ instead of $Prov(\ulcorner \psi^{\phi} \urcorner)$ in order to simplify the notation. Let $\gamma = \psi^{\phi}$. Then:

[17] It is worth saying that Löb (1955) proved this result in PA, not using modal logic, but his proof has a strong modal character.

1.	$Pr(Pr(Pr(\gamma) \to \gamma) \to Pr(\gamma)) \to (Pr(Pr(Pr(\gamma) \to \gamma)) \to Pr(Pr(\gamma)))$	(DC2)
2.	$Pr(Pr(\gamma) \to \gamma) \to \gamma) \to (Pr(Pr(\gamma)) \to Pr(\gamma))$	(DC2)
3.	$Pr(Pr(\gamma) \to \gamma) \to Pr(Pr(Pr(\gamma) \to \gamma)))$	(DC3)
4.	$Pr(Pr(Pr(\gamma) \to \gamma) \to Pr(\gamma)) \to (Pr(Pr(\gamma) \to \gamma) \to Pr(Pr(\gamma)))$	CPL 1,3
5.	$Pr(Pr(Pr(\gamma) \to \gamma) \to Pr(\gamma)) \to (Pr(Pr(\gamma) \to \gamma) \to Pr(\gamma))$	CPL 2,4

This concludes the proof. Q.E.D.

It is sometimes convenient to adopt instead KGL the logic K4LR, which is K4 extended with *Löb rule* (LR):[18]

(LR) From $\vdash \Box\varphi \to \varphi$, we obtain $\vdash \varphi$

Is is possible to prove that KGL and K4LR are in fact the same logic.

Theorem 4.4.10. KGL *and* K4LR *are equivalent.*

Proof. First, we prove that the rule LR is provable in KGL.

1.	$\Box\varphi \to \varphi$	Hyp.
2.	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	GL
3.	$\Box(\Box\varphi \to \varphi)$	Nec, 1
4.	$\Box\varphi$	MP 2,3
5.	φ	MP 1,4

Second, we prove that the axiom GL is provable in K4LR. Actually, if we consider the modal version of the proof given in Theorem 4.4.9 in the step $\alpha = \Box(\Box\varphi \to \varphi) \to \Box\varphi$, we have our desired proof. Then, KGL and K4LR are equivalent systems.

Q.E.D.

Even if it is not our objective here to investigate in detail the applications of KGL, we will spend a few words about interesting applications of KGL in the analysis of formal notion of provability.

The Theorem 4.4.9 is itself has interesting consequences, since it allows establishing facts about PA by means of KGL. That is, the theorems of KGL are theorems about provability in PA. The possibility of establishing facts about provability of PA in KGL allows the investigation of *fixed points* from a modal point of view. Roughly speaking, given a sentence φ where an atomic sentence p occurs, it is possible to find a sentence ψ containing atomic sentences occurring in φ excepting p and

$$\vdash_{\mathsf{KGL}} \boxdot(p \leftrightarrow \varphi) \leftrightarrow (p \leftrightarrow \psi)$$

[18]The rule LR is presented in Chellas (1980) under the name Gr.

where $\boxdot\alpha := \Box\alpha \wedge \alpha$. The fixed point theorem has many proofs in the literature. We refer the reader to Reidhaar-Olson (1989)'s proof due to its simplicity and to the fact that it gives a simple procedure to calculate fixed points. As an example, given the algorithm given by Reidhaar-Olson in her paper, one can prove that the fixed point for the formula $\neg\Box\Box p$ is $\neg\Box\Box\bot$.

Results like Theorem 4.4.7 witness that KGL can provide a modal version of important metamathematical results. The following theorems show that KGL can give a modal proof of the two most fundamental metamathematical results, the incompleteness theorems. Let the formula '$\neg\Box\bot$' mean that PA is consistent, that is, that does not prove a contradiction. Let φ be the sentence 'I am not provable'. It is statable in modal logic as:

$$\varphi \leftrightarrow \neg\Box\varphi$$

Then we have the following theorem:

Theorem 4.4.11. *If* $\vdash_{\mathsf{KGL}} \varphi \leftrightarrow \neg\Box\varphi$, *then* $\vdash_{\mathsf{KGL}} \neg\Box\bot \rightarrow \neg\Box\varphi$.

The proof of Theorem 4.4.11 can be found in Benthem (2010), and it is quite similar to the proof of Theorem 4.1.3.

Theorem 4.4.12. *(SOLOVAY, 1976, Arithmetical completeness) For every* ϕ, *if* $\vdash_{\mathsf{PA}} \varphi^{\phi}$, *then* $\vdash_{\mathsf{KGL}} \varphi^{\phi}$.

Let $\varphi \in \mathcal{L}^{\Box}$ such that $\nvdash_{\mathsf{KGL}} \varphi$. Solovay's theorem proceeds by defining a finite, transitive and conversely well-founded model $\mathcal{M} = \langle W, R, V\rangle$ which is embeddable in PA. Given such embedding, Solovay shows that $\nvdash_{\mathsf{PA}} \varphi^{\phi}$. We refer the reader to Solovay's own paper and (BOOLOS, 1995, Chapter 9) for the proof of Theorem 4.4.12. Then, Theorems 4.4.9 and 4.4.12 establishes that KGL is the logic of provability of PA

These results are important for two reasons. First, it shows that KGL is the logic which captures the *formal* concept of provability in PA. Thus it constitutes a response to Quine's criticism against modalities. Second, it shows that the concept of provability in PA can be described in a decidable system.

4.4.2 KGL and consistency operator

In the Section 4.2 we pointed out that a provability interpretation of $\Box$, or better, B, cannot contain the schema $\Box\varphi \rightarrow \varphi$ (T) if $\Box$ stands for provability in mathematical theories which contain arithmetic. Interestingly, if we add the axiom T to KGL, we obtain a trivial system, as the following theorem shows:

Theorem 4.4.13. *Let* KGLT *be the modal system resultant of extending* KGL *with formula* T. KGLT *is trivial.*

Proof. Consider the following proof:

1.	$\Box\bot \to \bot$	T
2.	$\Box(\Box\bot \to \bot)$	Nec 1
3.	$\Box(\Box\bot \to \bot) \to \Box\bot$	GL
4.	$\Box\bot$	MP 2, 3
5.	$\bot$	MP 1, 4

This concludes the proof. Q.E.D.

Besides being incompatible with the axiom T, GL is also incompatible with a weaker axiom $\Box\varphi \to \Diamond\varphi$ (D) which, according to the provability interpretation of modalities, expresses a form of consistency. In this interpretation, the axiom D states that if φ is provable, then $\neg\varphi$ is not provable.

Theorem 4.4.14. *Let* KGLD *be the modal system resultant of extending* KGL *with formula* D. KGLD *is trivial.*

Proof. Consider the following proof:

1.	$\Box\top \to \Diamond\top$	**D**
2.	$\Box\top$	N
3.	$\Diamond\top$	MP 1, 2
4.	$\top \to \Diamond\top$	CPL 3
5.	$\neg\Diamond\top \to \bot$	CPL 4
6.	$\Box\bot \to \neg\Diamond\top$	K-theorem
7.	$\Box\bot \to \bot$	CPL 5, 6
8.	$\Box(\Box\bot \to \bot)$	Nec 7
9.	$\Box(\Box\bot \to \bot) \to \Box\bot$	GL
10.	$\Box\bot$	MP 8, 9
11	$\bot$	MP 8, 10

This concludes the proof.

Q.E.D.

Theorem 4.4.14 establishes, in last instance, that KGL does not have theorems of the form $\Diamond\top$. Despite this incompatibility, the consistency can also be formalized in the language of KGL as the non provability of a contradiction, $\neg\Box\bot$, as showed by the Theorem 4.4.11. This formalization of consistency is close to Hilbert's definition of consistency, which refers exclusively to proof in formal systems. So, if consistency is understood as $\neg\Box\bot$ in KGL, what does $\Diamond$ stand for? Boolos (1980) shows that the connective $\Diamond$ can interpreted as *omega consistency* (or simply ω-*consistency*) which is defined as:

Definition 4.4.15 ((CHANG; KEISLER, 1990)). *Let T be a first-order arithmetical theory and $\mathbb{N}$ be a ω-model (i.e., the model of natural numbers). We say that $\boldsymbol{T}$ is ω-consistent if there is no formula $\varphi(x)$ of T such that:*

$\mathbb{N} \models \varphi(1)$, $\mathbb{N} \models \varphi(2)$,..., $\mathbb{N} \models \varphi(n)$, ..., *for all n, but* $\mathbb{N} \not\models \forall x\varphi(x)$.

Using the Solovay's methods, Boolos (1980) shows that KGL is also the modal logic of ω-consistency. As we said before, all these results are interesting because they show how a decidable theory can talk about the provability of an arithmetical theory, which we know to be undecidable.

4.5 The modal logic of true provability

The incompatibility between KGL and $\Box\varphi \to \varphi$ may suggest a mismatch between arithmetical provability and arithmetical necessity, where necessity is understood as truth in all possible worlds. As Gilbert & Venturi (2020) note, such mismatch is not surprising because KGL-operator $\Box$ captures $Prov_{\mathsf{PA}}(y) := \exists x Pr(x, y)$, which is a local notion, whereas alethic necessity requires that everything necessary be *actually* true. Because of the failure of axiom T, Goldblatt (1978) proposes a modal system where $\Box\varphi$ interprets φ is PA-*provable and true*. Such logic is S4Grz, which is defined as follows:

Definition 4.5.1. *(GRZEGORCZYK, 1967) The logic* S4Grz *is the logic which extends* S4 *(Definition 2.3.7) with the following axiom:*

$$(\mathrm{Grz})\ \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$$

Definition 4.5.2. *Let $R \subseteq W \times W$ be a relation on W. We say that R is* antisymmetric *iff wRy and yRw implies $w = y$.*

As in the case of KGL, the completeness of S4Grz cannot be proved by the method of canonical models because it its canonical frame does not validate all of its theorems. But it can be proved that S4Grz is characterized by the class of finite, reflexive, transitive and antisymmetric frames. The proof of non-canonicity of S4Grz as well as the proof of its characterization results for finite frames can be found in Hughes & Cresswell's paper (1982).[19]

Now, since KGL interprets provability in PA and it is incompatible with **T**, how the question is about the relation between KGL and S4Grz. Consider the following translation:

Definition 4.5.3. *Define $\circledast : \mathcal{L}^{\Box} \to \mathcal{L}^{\Box}$ as follows:*

[19] In Hughes & Cresswell (1982), S4Grz appears under the name K1.1.

$$\begin{array}{lcl} p^{\circledast} & = & p \\ (\neg\varphi)^{\circledast} & = & \neg\varphi^{\circledast} \\ (\varphi \to \psi)^{\circledast} & = & \varphi^{\circledast} \to \psi^{\circledast} \\ (\Box\varphi)^{\circledast} & = & \Box\varphi^{\circledast} \wedge \varphi^{\circledast} \end{array}$$

Given translation $\circledast$, one can prove by induction of the complexity of φ the following lemma:

Lemma 4.5.4. *Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{N} = \langle W, S, V\rangle$ be two models for the language such that wSy iff wRy and $w \neq y$. Then, for every $\varphi \in \mathcal{L}^{\Box}$, and every $w \in W$:*

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{N}, w \models \varphi^{\circledast}.$$

Given Lemma 4.5.4, it is provable that:

Theorem 4.5.5. *For every $\varphi \in \mathcal{L}^{\Box}$:*

$$\vdash_{\mathsf{S4Grz}} \varphi \text{ iff } \vdash_{\mathsf{KGL}} \varphi^{\circledast}.$$

Therefore, as a consequence of Lemma 4.5.4 and Theorem 4.5.5, we have a version of arithmetical completeness theorem for S4Grz:

Theorem 4.5.6. *(GOLDBLATT, 1978) For every realization ϕ, if $\vdash_{\mathsf{S4Grz}} \varphi$ then $\vdash_{\mathsf{PA}} (\varphi^{\circledast})^{\phi}$.*

So, in light of Theorem 4.5.6, $\Box\varphi$ stands for $\varphi^{\phi} \wedge Prov(\ulcorner\varphi^{\phi}\urcorner)$.

4.5.1 The modality $\boxdot$

In the Section 4.5 we presented the logic of true provability, S4Grz. We saw that this logic has the same theorems as KGL under translation $\circledast$, and its modality $\Box$ means $\Box\varphi \wedge \varphi$. Now, we will present a systematic investigation of the modality $\boxdot$, which says that φ *is necessary and true*.

$$\boxdot\varphi := \Box\varphi \wedge \varphi$$

So, we will take $\boxdot$ as a primitive operator in the language. We consider the language $\mathcal{L}^{\boxdot} = \mathcal{L} \cup \{\boxdot\}$ whose set of formulas $For(\mathcal{L}^{\boxdot})$ is inductively defined as follows:[20]

$$p_i \mid \neg\varphi \mid \varphi \to \psi \mid \boxdot\varphi$$

[20]Since the modal logics of this chapter extend CPL, we will omit the subscript CPL in $\mathcal{L}^{\Box}$.

For $\varphi, \psi \in For(\mathcal{L}^{\boxdot})$. The modality $\boxdot$ is not new in the literature. For example, as we said before, it was presented in the context of provability logics, such as in the works of Goldblatt (1978) and Boolos (1995). This modality also appears in Pelletier (1984) and French & Humberstone (2009) in the context of translations between modal systems.[21]

The semantic clause for $\boxdot$ is given by the following definition:

Definition 4.5.7. *The semantics for the language $\mathcal{L}^{\boxdot}$ is a Kripke semantics, as presented in Definition 2.3.1, differing in the clause* **4***:*

4" $\mathcal{M}, w \models \boxdot\varphi$ *iff* $\mathcal{M}, w \models \varphi$ *and for all* $y \in W$ *such that* wRy, $\mathcal{M}, y \models \varphi$.

Consider the following definition:

Definition 4.5.8. *(FAN; WANG; DITMARSCH, 2015) Given two logical languages $\mathcal{L}_1$ and $\mathcal{L}_2$ that are interpreted in the same class $\mathbb{M}$ of models.*

- $\mathcal{L}_2$ *is* at least as expressive as $\mathcal{L}_1$, $\mathcal{L}_1 \preceq \mathcal{L}_2$, *iff for every* $\varphi_1 \in \mathcal{L}_1$ *there is* $\varphi_2 \in \mathcal{L}_2$ *such that for all pointed models* $(\mathcal{M}, w) \in \mathbb{M}$, $\mathcal{M}, w \models \varphi_1$ *iff* $\mathcal{M}, w \models \varphi_1$*;*
- $\mathcal{L}_1$ *and* $\mathcal{L}_2$ are equally expressive, $\mathcal{L}_1 \equiv \mathcal{L}_2$, *iff* $\mathcal{L}_1 \preceq \mathcal{L}_2$ *and* $\mathcal{L}_2 \preceq \mathcal{L}_1$*;*
- $\mathcal{L}_2$ *is* less expressive than $\mathcal{L}_1$ *iff* $\mathcal{L}_1 \preceq \mathcal{L}_2$ *and not-*$\mathcal{L}_2 \preceq \mathcal{L}_1$

Given Definition 4.5.7, we will show that $\Box$ and $\boxdot$ are not equally expressive.

Theorem 4.5.9. *$\mathcal{L}^{\boxdot}$ is less expressive than $\mathcal{L}^{\Box}$ in the class of all models.*

Proof. Consider the translation $h : \mathcal{L}^{\boxdot} \to \mathcal{L}^{\Box}$ defined as follows:

$$\begin{array}{lcl} h(p) & = & p \\ h(\neg\varphi) & = & \neg h(\varphi) \\ h(\varphi \to \psi) & = & h(\varphi) \to h(\psi) \\ h(\boxdot\varphi) & = & \Box h(\varphi) \wedge h(\varphi) \end{array}$$

We can prove that for all $\varphi \in \mathcal{L}^{\Box}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models h(\varphi)$. Then $\mathcal{L}^{\Box} \preceq \mathcal{L}^{\boxdot}$. On the other hand, consider two one world models $\mathcal{N}_1 = \langle W_1, R_1, V_1\rangle$, where $W_1 = \{w\}$, $R_1 = \{< w, w >\}$ and V_1 be arbitrary over propositional variables, and $\mathcal{N}_2 = \langle W_2, R_2, V_2\rangle$, where $W_2 = \{w\}$, $R_2 = \emptyset$ and $V_2(p_i) = V_1(p_i)$, for all p_i. It is easy to see that for all $\varphi \in \mathcal{L}^{\boxdot}$, $\mathcal{N}_1, w \models \varphi$ iff $\mathcal{N}_2, w \models \varphi$. Then it is easy to see that while $\Box$ differentiates both models with the formula $\Box\bot$, $\boxdot$ does not because formulas of the form $\boxdot\bot$ are always false. By the semantic definition of $\boxdot$, we can see that $\boxdot$ cannot distinguish reflexive worlds from non-reflexive ones. Then, $\mathcal{L}^{\boxdot} \prec \mathcal{L}^{\Box}$. Q.E.D.

[21] To be honest, Pelletier does not explicitly use the modality $\boxdot$. Instead, he directly uses $\Box\varphi \wedge \varphi$.

From the semantical definition of the operator $\boxdot$ it is clear that it carries an intrinsic property of reflexivity, and this justifies the validity of the axiom $\mathrm{T}^{\boxdot}$ in the class $\mathbb{C}_{\mathsf{K}}$. As a consequence, if one defines the translation $f : \mathcal{L}^{\Box} \to \mathcal{L}^{\boxdot}$

$$\begin{array}{lcl} f(p) & = & p \\ f(\neg\varphi) & = & \neg f(\varphi) \\ f(\varphi \to \psi) & = & f(\varphi) \to f(\psi) \\ f(\Box\varphi) & = & \boxdot f(\varphi) \end{array}$$

which replaces all the occurrences if $\Box$ by $\boxdot$, then it is provable by induction on the complexity of φ following proposition:

Proposition 4.5.10. *Given $\varphi \in For(\mathcal{L}^{\Box})$ and $F = \langle W, R\rangle$ is a reflexive frame. Then, for all $w \in W$,*

$$F, w \models \varphi \text{ iff } F, w \models f(\varphi).$$

Even if $\boxdot$ is not a new modality it is not of our knowledge its minimal axiomatization. Here, we provide the axiomatization of the minimal axiomatization of $\boxdot$ in the language $\mathcal{L}^{\boxdot}$ is the following:

Definition 4.5.11. *The axiom system for logic $\mathsf{K}^{\boxdot}$ is defined as follows:*

(CPL) *All propositional tautologies;*

($\mathrm{K}^{\boxdot}$) $\boxdot(\varphi \to \psi) \to (\boxdot\varphi \to \boxdot\psi)$;

($\mathrm{T}^{\boxdot}$) $\boxdot\varphi \to \varphi$;

(MP) *from $\vdash \varphi$ and $\vdash \varphi \to \psi$ we infer $\vdash \psi$;*

($\mathrm{Nec}^{\boxdot}$) *from $\vdash \varphi$ we infer $\vdash \boxdot\varphi$*

Theorem 4.5.12. $\vdash \boxdot(\varphi \wedge \psi) \leftrightarrow (\boxdot\varphi \wedge \boxdot\psi)$

The proof of Theorem 4.5.12 runs as in Theorem 4.6.14 and Theorem 4.6.14. It is also immediate to prove that:

Theorem 4.5.13. $\vdash \boxdot\varphi \leftrightarrow (\boxdot\varphi \wedge \varphi)$.

Proof. Consider the following formal deduction:

1. $(\boxdot\varphi \wedge \varphi) \to \boxdot\varphi$ CPL
2. $\boxdot\varphi \to \boxdot\varphi$ CPL
3. $\boxdot\varphi \to \varphi$ $\mathrm{T}^{\boxdot}$
4. $\boxdot\varphi \to (\boxdot\varphi \wedge \varphi)$ CPL 2,3
5. $\boxdot\varphi \leftrightarrow (\boxdot\varphi \wedge \varphi)$ CPL 1,4

This concludes the proof. Q.E.D.

Given the semantic characterization of the language $\mathcal{L}^{\boxdot}$ given in Definition 4.5.7, we can prove by induction of the length of proofs that $\mathsf{K}^{\boxdot}$ is sound in the class $\mathbb{C}_{\mathsf{K}}$.

Theorem 4.5.14. $\mathsf{K}^{\boxdot}$ *is sound with respect to the class* $\mathbb{C}_{\mathsf{K}}$.

Proof. We will show that the axioms of $\mathrm{K}^{\boxdot}$ are valid in the class $\mathbb{C}_{\mathsf{K}}$. The non-modal axioms and rules are given by the soundness of CPL. Then we will focus only in the modal axioms and rules.

I) Axiom $\boxdot(\varphi \to \psi) \to (\boxdot\varphi \to \boxdot\psi)$.

Suppose that for every $\mathcal{M} = \langle W, R, v\rangle$, for every $w \in W, \mathcal{M}, w \models \boxdot(\varphi \to \psi)$ and $\mathcal{M}, w \models \boxdot\varphi$. Then, ($\mathcal{M}, w \models \varphi \to \psi$ and for every $y \in W$ such that $wRy, \mathcal{M}, y \models \varphi \to \psi$) and ($\mathcal{M}, w \models \varphi$ and for every $y \in W$ such that $wRy, \mathcal{M}, y \models \varphi$). By semantic modus ponens, $\mathcal{M}, w \models \psi$. Since $\mathcal{M}, y \models \varphi \to \psi$ and $\mathcal{M}, y \models \varphi$, for every $y \in W$ such that wRy, we obtain $\mathcal{M}, y \models \psi$. Then, $\mathcal{M}, w \models \boxdot\psi$. Therefore, $\mathcal{M}, w \models \boxdot(\varphi \to \psi) \to (\boxdot\varphi \to \boxdot\psi)$.

II) Axiom $\boxdot\varphi \to \varphi$.. The validity of this axiom immediately follows from the semantic definition of $\boxdot$.

III) Rule $\mathrm{Nec}^{\boxdot}$.

Suppose that $\mathcal{M}, w \models \varphi$ for every model $\mathcal{M} = \langle W, R, v\rangle$, for every $w \in W$. Then so ($\mathcal{M}, w \models \varphi$ and $\mathcal{M}, y \models \varphi$, for every $y \in W$ such that wRy). Therefore, $\mathcal{M}, w \models \boxdot\varphi$. This concludes the proof. Q.E.D.

Now, consider the following definition:

Definition 4.5.15. *The canonical model* $\mathcal{M} = \langle W^{\boxdot}, R^{\boxdot}, V^{\boxdot}\rangle$ *is defined as follows:*

(1) $W^{\boxdot}$ *is the set of maximal consistent set of formulas in* $\mathcal{L}^{\boxdot}$*;*

(2) For all $w, y \in W^{\boxdot}$ *the relation* $R^{\boxdot} \subseteq W^{\boxdot} \times W^{\boxdot}$ *is defined as follows:* $wR^{\boxdot}y$ *iff* $\lambda(w) \subseteq y$*, where* $\lambda(w) = \{\varphi \mid \boxdot\varphi \in w\}$.

(3) the function $V^{\boxdot} : \mathcal{V} \to \wp(W)$ *is defined as:* $w \in V^{\boxdot}(p)$ *iff* $p \in w$.

Theorem 4.5.16. *Let* w *be a maximal consistent set of formulas of* $For(\mathcal{L}^{\boxdot})$*. Then:*

(A) $\varphi \in w$ *iff* $\vdash \varphi$*;*

(B) $\neg\varphi \in w$ *iff* $\varphi \notin w$*;*

(C) $\varphi \to \psi \in w$ *iff* $\varphi \in w$ *implies* $\psi \in w$.

The proof of Theorem 4.5.16 can be found in (CHELLAS, 1980, pp.53, Theorem 2.18).

Definition 4.5.17. *(HUGHES; CRESSWELL, 1996) Let* **S** *be a modal system and* Δ *be a set of formulas in the language of* **S**. *We say that* Δ *is* **S***-consistent if there is no* $\gamma_1, ..., \gamma_n \in \Delta$ *such that*

$$\vdash \neg(\gamma_1 \wedge ... \wedge \gamma_n).$$

Lemma 4.5.18 (Lindenbaum Lemma)**.** *Let Δ be* **S***-consistent set of formulas. Then there is a maximal* **S***-consistent set of formulas Γ such that $\Delta \subseteq \Gamma$.*

Lemma 4.5.19. *Let w be a maximal consistent of formulas in $\mathcal{L}^{\boxdot}$. If $\boxdot\varphi \notin w$, then $\varphi \notin w$ or $\lambda(w) \cup \{\neg\varphi\}$ is consistent.*

Proof. Suppose that $\boxdot\varphi \notin w$, by Theorem 4.5.13, we have (i) $\varphi \notin w$ or (ii) the consistency of $\lambda(w) \cup \{\neg\varphi\}$. We will show (ii). Suppose that $\lambda(w) \cup \{\neg\varphi\}$ is inconsistent. Then by Definition 4.5.17 there is $\{\gamma_1, ..., \gamma_n\} \subseteq \lambda(w)$ such that:

(1)	$\vdash \neg(\gamma_1 \wedge ... \wedge \gamma_n \wedge \neg\varphi)$	Def. 4.5.17
(2)	$\vdash (\gamma_1 \wedge ... \wedge \gamma_n) \rightarrow \varphi$	CPL 1
(3)	$\vdash \boxdot(\gamma_1 \wedge ... \wedge \gamma_n) \rightarrow \boxdot\varphi$	$\mathrm{RK}^{\boxdot}$, 2
(4)	$\vdash (\boxdot\gamma_1 \wedge ... \wedge \boxdot\gamma_n) \rightarrow \boxdot(\gamma_1 \wedge ... \wedge \gamma_n)$	Theorem
(5)	$\vdash (\boxdot\gamma_1 \wedge ... \wedge \boxdot\gamma_n) \rightarrow \boxdot\varphi$	CPL 3,4
(6)	$\vdash \neg(\boxdot\gamma_1 \wedge ... \wedge \boxdot\gamma_n \wedge \neg \boxdot \varphi)$	CPL 5

Thus, $\{\boxdot\gamma_1, ..., \boxdot\gamma_n, \neg\boxdot\varphi\}$ is inconsistent, contradicting the consistency of w. Q.E.D.

Lemma 4.5.20. *Let $\mathcal{M}$ be a canonical model for $\mathsf{K}^{\boxdot}$. The for every $w \in W^{\boxdot}$ and every formula $\varphi \in \mathsf{K}^{\boxdot}$:*

$$\mathcal{M}, w \models \varphi \textit{ iff } \varphi \in w.$$

Proof. The proof runs by induction on φ. The atomic case is given by definition and the boolean cases follow from Theorem 4.5.16. Then let $\varphi = \boxdot\psi$. Suppose that $\boxdot\psi \in w$ and $\lambda(w) \subseteq y$ for all $y \in W^{\boxdot}$. Because $\boxdot\psi \rightarrow \psi \in w$ we conclude $\psi \in w$. Moreover, $\psi \in y$. By definition of $R^{\boxdot}$, we obtain $wR^{\boxdot}y$. By applying the induction hypothesis twice, we conclude $\mathcal{M}, w \models \psi$ and $\mathcal{M}, y \models \psi$. Then, $\mathcal{M}, w \models \boxdot\psi$.

Conversely, suppose that $\boxdot\psi \notin w$. By Lemma 4.5.19, $\psi \notin w$ or $\lambda(w) \cup \{\neg\psi\}$ is consistent. Consider the second disjunct. By Lindenbaum lemma, we extend $\lambda(w) \cup \{\neg\psi\}$ to a maximal consistent set of formulas y in the language $\mathcal{L}^{\boxdot}$. Then, $\lambda(w) \cup \{\neg\psi\} \subseteq y$. So, $\neg\psi \in y$. Because y is maximally consistent, we obtain $\psi \notin y$. By applying induction hypothesis twice, (i) $\mathcal{M}, w \nvDash \psi$ or (ii) $\mathcal{M}, y \nvDash \psi$. In both cases, we obtain $\mathcal{M}, w \nvDash \boxdot\psi$. Q.E.D.

Theorem 4.5.21. *If φ is valid, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Then there is a maximal consistent set of formulas such that $\varphi \notin w$. Then, by Theorem 4.5.16, $\neg\varphi \in w$. By Lemma 4.5.20, $\mathcal{M}, w \models \neg\varphi$. Therefore, $\mathcal{M}, w \nvDash \varphi$. Q.E.D.

Given a sound and complete axiomatization for $\boxdot$ in the class of all models, we also conjecture to be possible to prove that $\boxdot$ collapses, in the class of all models, all logics between the logics K and KT. This collapse would happen due to the fact that $\boxdot$ does not distinguish worlds which are related to themselves and worlds which are not related to themselves. From the semantical point of view, this phenomenon can be captured by the following definition:

Definition 4.5.22. *(MARCOS, 2005a) Let $F = \langle W, R\rangle$ and $F^m = \langle W, R\rangle$ be frames such that $R^m \subseteq R$ and $R - R^m \subseteq \{(x, x)|w \in W\}$. Then F^m is said to be a mirror reduction of F. Two frames F_1 and F_2 are mirror-related, $F_1 \sim_m F_2$, if they are mirror reductions of a common frame.*

With the minimal $\boxdot$-logic $\mathsf{K}^{\boxdot}$, and Definition 4.5.22 at hands the following proposition is provable:

Proposition 4.5.23. *Let $F = \langle W, R\rangle$ and $F^m = \langle W, R^m\rangle$ be frames such that F^m is a mirror reduction of F. Then, for every model $\mathcal{M}$ based on F, every model $\mathcal{M}^m$ based on F^m, and every $w \in W$:*

$$\mathcal{M}, w \models \varphi \textit{ iff } \mathcal{M}^m, w \models \varphi$$

for every $\varphi \in For(\mathcal{L}^{\boxdot})$.

It is important to stress here the importance of a minimal axiomatization for $\boxdot$-logic in the class $\mathbb{C}_{\mathsf{K}}$. In the presence of a such sound and complete axiomatization, the Proposition 4.5.23 allows to prove that all logics KX such that $\mathsf{K} \subseteq \mathsf{KX} \subseteq \mathsf{KT}$ collapse in the language $\mathcal{L}^{\boxdot}$. Now we present the proof of Proposition 4.5.23:

Proof. The boolean cases are straightfoward. We will consider the case where $\varphi = \boxdot\psi$. Suppose that $\mathcal{M}, w \models \boxdot\psi$. Then $\mathcal{M}, w \models \psi$ and, for every $y \in W$, if wRy then $\mathcal{M}, y \models \psi$. Since $R^m \subseteq R$, then if wR^my then $\mathcal{M}, y \models \psi$. By I.H., $\mathcal{M}^m, w \models \psi$ and $\mathcal{M}^m, y \models \psi$, for every $y \in W$ such that wR^my. Therefore, $\mathcal{M}^m, w \models \boxdot\psi$.

For the converse, suppose that $\mathcal{M}, w \not\models \boxdot\psi$. Then, we have the following cases:

1. $\mathcal{M}, w \not\models \psi$ and, for all $y \in W$, if wRy then $\mathcal{M}, y \models \psi$;
2. $\mathcal{M}, w \not\models \psi$ and, for some $y \in W$ such that wRy, $\mathcal{M}, y \not\models \psi$;
3. $\mathcal{M}, w \models \psi$ and for some $y \in W$ such that wRy, $\mathcal{M}, y \not\models \psi$.

The cases 1 and 2 are straightfoward because $\mathcal{M}, w \not\models \psi$. Thus, applying I.H., we obtain $\mathcal{M}^m, w \not\models \psi$. Then $\mathcal{M}^m, w \not\models \boxdot\psi$. So we will concentrate on the case 3. So, there is some $y \in W$ such that wRy and $\mathcal{M}, y \not\models \psi$. Remove now some reflexive arrows from R, possibly including wRw. Let R^m be the result of this removal. Clearly, $R^m \subseteq R$. Since

$\mathcal{M}, w \models \psi$, such removal makes no difference since ψ is true at w. So, by I.H., we obtain $\mathcal{M}^m, w \models \psi$. Also, for some $y \in W$ such that $wR^m y$, $\mathcal{M}, y \not\models \psi$. By I.H., we obtain $\mathcal{M}^m, y \not\models \psi$ for some $y \in W$ such that $wR^m y$. Therefore, we conclude $\mathcal{M}^m, w \not\models \boxdot\psi$. This concludes the proof.

Q.E.D.

As a consequence of the Proposition 4.5.23 we have the following corollary:

Corollary 4.5.24. *Let $F_1 = \langle W, R\rangle$ and $F_2 = \langle W, R\rangle$ be two frames such that $F_1 \sim_m F_2$. Then, for every $\varphi \in For(\mathcal{L}^{\boxdot})$,*

$$F_1, w \models \varphi \text{ iff } F_2, w \models \varphi.$$

Therefore, given a sound and complete axiomatization for the minimal $\boxdot$-logic, the Proposition 4.5.23 and Corollary 4.5.24 will establish the collapse of all logics between K and KT in the language $\mathcal{L}^{\boxdot}$.

The modality $\divideontimes$ can be defined as:

$$\divideontimes\varphi \quad := \quad \neg \boxdot \neg\varphi$$

In the provability interpretation of $\boxdot$, $\divideontimes\varphi$ means that "φ is either provable or true". By the results of this section, we obtain that $\divideontimes$ validates the same principles as $\Diamond$ in the class of all frames.

4.6 The modal logic of consistent provability

In the provability interpretation of modal logics, the $\Box\varphi \rightarrow \Diamond\varphi$ expresses a form of consistency because it says that the provability of φ implies the non provability of $\neg\varphi$. From Theorem 4.4.14, it is clear that there is no consistent extension of KGL with the axiom D. So, the operator $\Box$ of KD cannot interpret the predicate $Prov_{\mathsf{PA}}$ as KGL does. Consider the following definition:

Definition 4.6.1. *Let T be a theory in the language of $\mathcal{L}_{\mathsf{PA}}$ and $Prov_T$ be a provability predicate of T. we say that $Prov_T$ is a* standard *provability predicate if it satisfies the conditions* DC1-DC4 *of the Claim 4.1.5. $Prov_T$* non-standard *if it is not standard.*

It is obvious that $Prov_{\mathsf{PA}}$ is standard. In the literature about provability in formal theories, the search for *non-standard* provability predicates became and their associated provability logics became a relevant problem. Consider the following definition

Definition 4.6.2. *(KURAHASHI, 2018) Let $Pr_T(x, y)$ be a predicate of theory T in the language of $\mathcal{L}_{\mathsf{PA}}$ and $\neg(x)$ be the operation which gives the Gödel number of a negated formula whose Gödel number is x. For each Pr_T the formula*

$$\exists y(Pr_T(y,x) \wedge \neg Pr_T(y,\neg(x))) \tag{4.4}$$

is a Rosser provability predicate *of* T.

As a consequence of Definition 4.6.2, the following proposition hilds

Proposition 4.6.3. *(KURAHASHI, 2018) Let* $Prov_T^R$ *be a Rosser provability predicate. Then, if* $\vdash_T \neg\varphi$, *then* $\vdash \neg Prov_T^R(\ulcorner\varphi\urcorner)$.

By a variation of Solovay's theorem, Kurahashi proves that the logic KD is the logic which completely captures $Prov_T^R$. Consider the following definition

Definition 4.6.4. *(KURAHASHI, 2018) Let* $\mathcal{L}_{\mathsf{PA}}$ *be the language of arithmetic and* $\clubsuit$ *a realization which assigns to sentence letters sentences of* $\mathcal{L}_{\mathsf{PA}}$. *We associate each modal sentence* φ *a sentence* φ^{ϕ} *as follows:*

$$\begin{array}{lcl} p^{\clubsuit} & = & \phi(p), \text{ where } p \text{ is a sentential letter} \\ \bot^{\clubsuit} & = & \mathbf{0}=\mathbf{1} \\ (\varphi\to\psi)^{\clubsuit} & = & \varphi^{\clubsuit}\to\psi^{\clubsuit} \\ (\Box\varphi)^{\clubsuit} & = & Pr_T^R(\ulcorner\varphi^{\clubsuit}\urcorner) \end{array}$$

Theorem 4.6.5. *(KURAHASHI, 2018) There exists a Rosser provability predicate* $Pr_T^R(x)$ *of* T *such that the following conditions hold:*

1. *For any* $\varphi\in\mathcal{L}^{\Box}$: *if* $\vdash_{\mathsf{KD}}\varphi$, *then* $\vdash_T\varphi^{\clubsuit}$, *for any realization* $\clubsuit$ *based on* $Pr_T^R(x)$.
2. *There exists a realization* $\clubsuit$ *based on* $Pr_T^R(x)$ *such that for any* $\varphi\in\mathcal{L}^{\Box}$: $\vdash_{\mathsf{KD}}\varphi$ *iff* $\vdash_T\varphi^{\clubsuit}$.

The logic KD is characterized by the class $\mathbb{C}_{\mathsf{KD}}$ of serial frames. Then, it is easily probable that:

Theorem 4.6.6. $\vdash_{\mathsf{KD}}\Box\varphi\leftrightarrow(\Box\varphi\wedge\Diamond\varphi)$

Proof. Consider the following formal derivation:

1. $(\Box\varphi\wedge\Diamond\varphi)\to\Box\varphi$ CPL
2. $\Box\varphi\to\Diamond\varphi$ D
3. $\Box\varphi\to\Box\varphi$ CPL
4. $\Box\varphi\to(\Box\varphi\wedge\Diamond\varphi)$ CPL 2,3
5. $\Box\varphi\leftrightarrow(\Box\varphi\wedge\Diamond\varphi)$ CPL 1,4

This concludes the proof. Q.E.D.

Theorem 4.6.4 establishes that the logic KD captures a Rosser provability predicate, in the lines of Definition 4.6.2. Now, consider the following translation

Definition 4.6.7. *Define* $\oplus : \mathcal{L}^{\Box} \to \mathcal{L}^{\Box}$ *as follows:*

$$\begin{array}{lcl} p^{\oplus} & = & p \\ (\neg\varphi)^{\oplus} & = & \neg\varphi^{\oplus} \\ (\varphi \to \psi)^{\oplus} & = & \varphi^{\oplus} \to \psi^{\oplus} \\ (\Box\varphi)^{\oplus} & = & \Box\varphi^{\oplus} \wedge \Diamond\varphi^{\oplus} \end{array}$$

Consider the following definition:

Definition 4.6.8. *The modal system* KD4 *is obtained by extending the logic* KD *with the axiom* $\Box\varphi \to \Box\Box\varphi$.

Even if KGL is not compatible with axiom D, we wonder, inspired in Goldblatt (1978), if there is a logic L extending KD4 such that the following is provable:

Question 4.6.9. *Let* L *be a modal logic extending* KD4. *For every* $\varphi \in \mathcal{L}^{\Box}$*:*

$$\vdash_{\mathsf{L}} \varphi \text{ iff } \vdash_{\mathsf{KGL}} \varphi^{\oplus}.$$

If there is such logic according to Question 4.6.9, then L is the logic of consistent provability. This means that $\Box\varphi$ in such L means $Prov_{\mathsf{PA}}(\ulcorner\varphi^{\phi}\urcorner) \wedge \neg Prov_{\mathsf{PA}}(\ulcorner\neg\varphi^{\phi}\urcorner)$.

4.6.1 The modality $\boxplus$

In the Section 4.6, we introduced the translation $\oplus$ where $\Box\varphi$ is translated as $\Box\varphi \wedge \Diamond\varphi$. Now, we will present a systematic investigation of the modality $\boxplus$, which says that φ is *necessary and possible*:

$$\boxplus\varphi := \Box\varphi \wedge \Diamond\varphi$$

In a provability interpretation of modal logics $\boxplus\varphi$ expresses the notion of consistent provability. The modality $\boxplus$ is non-normal, because, as we will see below, it does not validate necessitation in the class of all models. Now, we will consider a modal logic where $\boxplus$ is taken as the only primitive modal operator of the modal language. Thus, we consider the language $\mathcal{L}^{\boxplus}$ as the set $\mathcal{L}^{\boxplus} = \mathcal{L} \cup \{\boxplus\}$. The set of formulas of $\mathcal{L}^{\boxplus}$, $For(\mathcal{L}^{\boxplus})$, is defined inductively as follows:

$$p_i \mid \neg\varphi \mid \varphi \to \psi \mid \boxplus\varphi$$

For $\varphi, \psi \in For(\mathcal{L}^{\boxplus})$.

Now we present the deductive system for the minimal logic which contains the operator $\boxplus$.

Definition 4.6.10. *The logic* $\mathsf{K}^{\boxplus}$ *is defined as follows:*

(CPL) *All propositional tautologies;*

(K$^{\boxplus}$) $\boxplus(\varphi \to \psi) \to (\boxplus\varphi \to \boxplus\psi)$;

(D$^{\boxplus}$) $\boxplus\varphi \to \neg \boxplus \neg\varphi$;

(MP) *from* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *we infer* $\vdash \psi$;

(RK$^{\boxplus}$) *from* $\vdash \varphi \to \psi$ *we infer* $\vdash \boxplus\varphi \to \boxplus\psi$.

Definition 4.6.11. *The semantics for the language* $\mathcal{L}^{\boxplus}$ *is also a Kripke semantics of Definition 2.3.1, differing in the clause* **4***:*

4' $\mathcal{M}, w \models \boxplus\varphi$ *iff (for every* $z \in W$ *such that* wRz, $\mathcal{M}, z \models \varphi$*) and (there is* $x \in W$ *such that* wRx, $\mathcal{M}, x \models \varphi$*).*

From the semantic definition of $\boxplus$, it is immediate to see that in the class $\mathbb{C}_{\mathsf{K}}$ of all frames, the necessitation rule is not a valid schema due to the presence of dead ends. Thus, in order to validate the inference rule of necessitation, the class of frames must be totally serial.

Despite the non-normality of the operator $\boxplus$, one can define the normal operator in $\mathsf{K}^{\boxplus}$ as follows:[22]

$$\Box\varphi := \boxplus\top \to \boxplus\varphi$$

In fact, one can prove that in the class of serial frames, $\boxplus\varphi$ and $\Box\varphi$ are interchangeable. First, consider the following definition:

Definition 4.6.12. *Define the following translation, that we call the* $\boxplus$*-translation, from* $For(\mathcal{L}^{\boxplus})$ *to* $For(\mathcal{L}^{\Box})$*:*

$$\begin{array}{lcl} (p)^{\boxplus} & = & p \\ (\neg\varphi)^{\boxplus} & = & \neg\varphi^{\boxplus} \\ (\varphi \to \psi)^{\boxplus} & = & \varphi^{\boxplus} \to \psi^{\boxplus} \\ (\Box\varphi)^{\boxplus} & = & \boxplus\varphi^{\boxplus} \end{array}$$

Proposition 4.6.13. *Given* $\varphi \in For(\mathcal{L}^{\Box})$ *and* $F = \langle W, R\rangle$ *is a serial frame, then, for all* $w \in W$*,*

$$F, w \models \varphi \text{ iff } F, w \models \varphi^{\boxplus}$$

The proof easily follows by induction on the complexity of formulas.

Consider now the following theorems which will be useful for the characterization results for K$^{\boxplus}$.

[22] We thank Lloyd Humberstone for suggesting this definition in a private communication.

Theorem 4.6.14. $\vdash \boxplus(\varphi \wedge \psi) \to (\boxplus\varphi \wedge \boxplus\psi)$

Proof. We give a formal deduction:

1.	$(\varphi \wedge \psi) \to \varphi$	CPL
2.	$\boxplus(\varphi \wedge \psi) \to \boxplus\varphi$	$\mathrm{RK}^{\boxplus}$ 1
3.	$(\varphi \wedge \psi) \to \psi$	CPL
4.	$\boxplus(\varphi \wedge \psi) \to \boxplus\psi$	$\mathrm{RK}^{\boxplus}$ 3
5.	$\boxplus(\varphi \wedge \psi) \to (\boxplus\varphi \wedge \boxplus\psi)$	CPL 2,4

This concludes the proof. Q.E.D.

Theorem 4.6.15. $\vdash (\boxplus\varphi \wedge \boxplus\psi) \to \boxplus(\varphi \wedge \psi)$

Proof. We give a formal deduction:

1.	$\varphi \to (\psi \to (\varphi \wedge \psi))$	CPL
2.	$\boxplus\varphi \to \boxplus(\psi \to (\varphi \wedge \psi))$	$\mathrm{RK}^{\boxplus}$ 1
3.	$\boxplus(\varphi \to (\varphi \wedge \psi)) \to (\boxplus\varphi \to \boxplus(\varphi \wedge \psi))$	$\mathrm{K}^{\boxplus}$
4.	$\boxplus\varphi \to (\boxplus\psi \to \boxplus(\varphi \wedge \psi))$	CPL 2,3
5.	$(\boxplus\varphi \wedge \boxplus\psi) \to \boxplus(\varphi \wedge \psi)$	CPL 4

This concludes the proof. Q.E.D.

These two results also establish that $\mathsf{K}^{\boxplus}$ is the minimal logic of the operator $\boxplus$. It is easy to prove that this logic is sound on the class of all frames.

Theorem 4.6.16. *The logic $\mathsf{K}^{\boxplus}$ is sound with respect to $\mathbb{C}_{\mathsf{K}}$.*

The system $\mathsf{K}^{\boxplus}$ can be seen as a *regular modal system* in the sense of Segerberg (1971): a modal system $\mathsf{L}^{\Box}$ is called *regular* if $\mathsf{L}^{\Box}$ contains the modal axiom K, the rule MP and the rule RK; indeed it is the case that $\mathsf{E2}^0 \subseteq \mathsf{K}^{\boxplus} \subseteq \mathsf{E2} = \mathsf{E2}^0 + T$, where $\mathsf{E2}^0$ is the minimal regular system.

The proof of completeness is inspired by Steinsvold (2011). We define the canonical model of $\mathsf{K}^{\boxplus}$ as follows:

Definition 4.6.17. *The canonical model $\mathcal{M} = \langle W^{\boxplus}, R^{\boxplus}, V^{\boxplus}\rangle$ is defined as follows:*

(1) $W^{\boxplus}$ is the set of maximal consistent sets of formulas in $\mathcal{L}^{\boxplus}$ such that $W^{\boxplus} = W^{s} \cup W^{\neg s}$ where:

(1.1) $w \in W^{s}$ if and only if $\exists\varphi(\boxplus\varphi \in w)$;

(1.2) $w \in W^{\neg s}$ if and only if $\neg\exists\varphi(\boxplus\varphi \in w)$.

(2) Let $w, y \in W^{\boxplus}$. The relation $R^{\boxplus} \subseteq W^{\boxplus} \times W^{\boxplus}$ is defined as follows:

(2.1) if $w \in W^{s}$, then let $wR^{\boxplus}y$ iff $\lambda(w) \subseteq y$ (where $\lambda(w) = \{\varphi \mid \boxplus\varphi \in w\}$);

(2.2) if $w \in W^{\neg s}$, then there is no $y \in W^{\boxplus}$ such that $wR^{\boxplus}y$.

(3) The function $V^{\boxplus} : Var \to \wp(W^{\boxplus})$ is defined as follows: $w \in V^{\boxplus}(p)$ iff $p \in w$.

Proposition 4.6.18. *Let w be a maximally $\mathsf{K}^{\boxplus}$-consistent set of formulas such that $\lambda(w) \neq \emptyset$. If $\neg \boxplus \varphi \in w$, then $\lambda(w) \cup \{\neg\varphi\}$ is $\mathsf{K}^{\boxplus}$-consistent.*

Proof. Suppose that $\lambda(w) \cup \{\neg\varphi\}$ is not $\mathsf{K}^{\boxplus}$-consistent. Since $\lambda(w) \neq \emptyset$, then, by Definition 4.5.17 there is $\{\gamma_1, ..., \gamma_n\} \subseteq \lambda(w)$ such that:

(1)	$\vdash \neg(\gamma_1 \wedge ... \wedge \gamma_n \wedge \neg\varphi)$	Def. 4.5.17
(2)	$\vdash (\gamma_1 \wedge ... \wedge \gamma_n) \to \varphi$	Taut,1
(3)	$\vdash \boxplus(\gamma_1 \wedge ... \wedge \gamma_n) \to \boxplus\varphi$	$RK^{\boxplus}$, 2
(4)	$\vdash (\boxplus\gamma_1 \wedge ... \wedge \boxplus\gamma_n) \to \boxplus(\gamma_1 \wedge ... \wedge \gamma_n)$	Theorem
(5)	$\vdash (\boxplus\gamma_1 \wedge ... \wedge \boxplus\gamma_n) \to \boxplus\varphi$	Taut 3,4
(6)	$\vdash \neg(\boxplus\gamma_1 \wedge ... \wedge \boxplus\gamma_n \wedge \neg \boxplus \varphi)$	Taut,5

Thus, $\{\boxplus\gamma_1, ..., \boxplus\gamma_n, \neg\boxplus\varphi\}$ is inconsistent, contradicting the consistency of w. Q.E.D.

Theorem 4.6.19. *Let $\mathcal{M}$ be a canonical model for $\mathsf{K}^{\boxplus}$. Then, for every $w \in W^{\boxplus}$ and every formula φ of $\mathsf{K}^{\boxplus}$:*

$$\mathcal{M}, w \models \varphi \text{ iff } \varphi \in w.$$

Proof. The atomic case and the Boolean cases are straightforward. Then let $\boxplus\psi \in w$. Therefore $w \in W^s$ and so not only $wR^{\boxplus}y$ iff $\lambda(w) \subseteq y$. Then, $\psi \in y$. By induction hypothesis, $\mathcal{M}, y \models \psi$ for all y, $wR^{\boxplus}y$. Moreover, since $\boxplus\psi \to \psi$ is an axiom, we conclude that $\neg \boxplus \neg\psi \in w$. By Proposition 4.6.18, we conclude that $\lambda(w) \cup \{\psi\}$ is consistent. Then, by the Lindenbaum lemma, we can extend $\lambda(w) \cup \{\psi\}$ to a maximal consistent set z of $\mathsf{K}^{\boxplus}$ formulas. Then, $\psi \in z$. y induction hypothesis, $\mathcal{M}, z \models \psi$ for some z such that $wR^{\boxplus}z$. Therefore, $\mathcal{M}, w \models \boxplus\psi$.

Conversely, suppose that $\boxplus\psi \notin w$. Then $\neg \boxplus \psi \in w$. There are two possibilities: $w \in W^s$ or $w \in W^{\neg s}$. If $w \in W^s$, then $\lambda(w) \neq \emptyset$ and, by the Proposition 4.6.18, $\lambda(w) \cup \{\neg\psi\}$ is $\mathbf{B}_K$-consistent. So, applying Lindenbaum lemma, we can extend $\lambda(w) \cup \{\neg\psi\}$ to a maximal, $\mathbf{B}_K$-consistent set y. Then $\lambda(w) \cup \{\neg\psi\} \subseteq y$. So $\lambda(w) \subseteq y$ and $\neg\psi \in y$. Moreover, by $\mathsf{K}^{\boxplus}$-consistency of y, we have $\psi \notin y$. By induction hypothesis, $\mathcal{M}, y \nvDash \psi$ and so $\mathcal{M}, w \nvDash \boxplus\psi$.

On the other hand, if $w \in W^{\neg s}$, then there is no $y \in W^{\boxplus}$ such that $wR^{\boxplus}y$. Therefore, by the semantic definition of the $\boxplus$-operator, $\mathcal{M}, w \nvDash \boxplus\psi$.

Q.E.D.

Generalized completeness for $\boxplus$-logics

As noted earlier, the non-normality of the operator $\boxplus$ lies in the failure of necessitation rule in the class of all models. But, if we concentrate on serial frames and their restrictions, the operator $\boxplus$ becomes normal and we regain the necessitation rule. The possibility of regaining normality above serial frames allows us to characterize the $\boxplus$-counterparts of normal modal logics which extend the logic KD. In this section we will apply a method that we will call the *generated subframe method*, first introduced in Goldblatt & Mares (2006) for the study of quantified modal logic and then adapted in Gilbert & Venturi (2016) to the study of non-normal modal logics. We therefore provide characterization results for $\boxplus$-logics, whose theorems are valid in serial frames.

For this method to work, we need first to isolate the basic normal modal logic to which, later, apply a theorem-preserving translation between $\mathcal{L}^{\Box}$ and $\mathcal{L}^{\boxplus}$. Then, the same translation will be used to characterize all extensions of this logic. Because of the theorem-preserving character of the translation, we need to preserve the application of the necessitation rule and therefore we are forced to consider KD as our basic logic.

We then define the $\boxplus$-logic that will allow us to define the appropriate translation.

Definition 4.6.20. *The logic* $\mathsf{B_D}$ *is axiomatized as follows:*

(CPL) *All instances of propositional tautologies.*

($\mathrm{K}^{\boxplus}$) $\boxplus(\varphi \to \psi) \to (\boxplus\varphi \to \boxplus\psi)$

($\mathrm{D}^{\boxplus}$) $\boxplus\varphi \to \neg \boxplus \neg\varphi$

(MP) *From* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *we infer* $\vdash \psi$

($\mathrm{Nec}^{\boxplus}$) *From* $\vdash \varphi$ *we infer* $\vdash \boxplus\varphi$

By minimality of $\mathsf{K}^{\boxplus}$ we have that $\mathsf{K}^{\boxplus} \subseteq \mathsf{B_D}$. Moreover, it is immediate to see that $\mathsf{B_D}$ is sound with respect to serial frames.

Theorem 4.6.21. $\mathsf{B_D}$ *is sound with respect to* $\mathbb{C}_{\mathrm{D}}$. *Q.E.D.*

For completeness we again use a canonical model construction. The proof of the completeness theorem for $\mathsf{B_D}$ follows closely that of $\mathsf{K}^{\boxplus}$. The only difference, that is actually a simplification, consists in noting that there is no world of the canonical model for $\mathsf{B_D}$ which does not contain formulas of form $\boxplus\varphi$.

Theorem 4.6.22. *The logic* $\mathsf{B_D}$ *is complete with respect to the class of serial frames. Q.E.D.*

Now we define the $\boxplus$ translation:

Definition 4.6.23. *Define the following translation, that we call the $\boxplus$-translation, from $Form(\mathcal{L}^{\Box})$ to $Form(\mathcal{L}^{\boxplus})$:*

$$\begin{array}{rcl} (p)^{\boxplus} & = & p \\ (\neg\varphi)^{\boxplus} & = & \neg\varphi^{\boxplus} \\ (\varphi \wedge \psi)^{\boxplus} & = & \varphi^{\boxplus} \wedge \psi^{\boxplus} \\ (\varphi \vee \psi)^{\boxplus} & = & \varphi^{\boxplus} \vee \psi^{\boxplus} \\ (\varphi \to \psi)^{\boxplus} & = & \varphi^{\boxplus} \to \psi^{\boxplus} \\ (\Box\varphi)^{\boxplus} & = & \boxplus\varphi^{\boxplus} \end{array}$$

Moreover, for a normal modal logic L, define $\mathsf{L}^{\boxplus}$ be the smallest logic in the language $\mathcal{L}^{\boxplus}$ extending $\mathsf{B_D}$ and containing $\varphi^{\boxplus}$ for every $\varphi \in \mathsf{L}$. We will call $\mathsf{L}^{\boxplus}$ the $\boxplus$-counterpart of L.

It is easy to see that $\mathsf{B_D}$ is indeed the minimal logic of $\boxplus$ in serial frames.

Theorem 4.6.24. $\mathsf{KD}^{\boxplus} = \mathsf{B_D}$.

Proof. That $\mathsf{B_D} \subseteq \mathsf{KD}^{\boxplus}$ is obvious. On the other hand, we prove by induction on the length of proofs that $\mathsf{KD}^{\boxplus} \subseteq \mathsf{B_D}$. If α is an instance of a K-axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, then $\alpha^{\boxplus}$ is:

$$\boxplus(\varphi^{\boxplus} \to \psi^{\boxplus}) \to (\boxplus\varphi^{\boxplus} \to \boxplus\psi^{\boxplus})$$

which is an axiom of Bd. If α is an instance of a D-axiom $\Box\varphi \to \neg\Box\neg\varphi$, then $\alpha^{\boxplus}$ is:

$$\boxplus\varphi^{\boxplus} \to \neg \boxplus \neg\varphi^{\boxplus}$$

which is a theorem of the minimal $\boxplus$-logic and therefore also of $\mathsf{B_D}$. For Modus Ponens, there is nothing to prove, while in the case of necessitation, notice that by definition the logic $\mathsf{B_D}$ is closed under the $\boxplus$-translation of necessitation. Therefore $\mathsf{B_D} = \mathsf{KD}^{\boxplus}$. Q.E.D.

From now on, we will refer to $\mathsf{B_D}$ as the *minimal $\boxplus$-logic with respect to the $\boxplus$-translation* and it will be denoted by $\mathsf{KD}^{\boxplus}$.

We are now in the position to apply the generated subframe method. In order to do so, recall the following, standard, definitions and results Blackburn et al (2001).

Definition 4.6.25 (Bounded Morphism)**.** *Let $F_1 = \langle W_1, R_1\rangle$ and $F_2 = \langle W_2, R_2\rangle$ be frames. Then $f : W_1 \to W_2$ is a bounded morphism from F_1 to F_2 when the following two conditions are met:*

(forth) xR_1y implies $f(x)R_2f(y)$;

(back) if $f(x)R_2z$, then there is a w s.t. xR_1w and $f(w) = z$.

When there is a surjective bounded morphism from F_1 onto F_2, written $F_1 \twoheadrightarrow F_2$, F_2 is said to be a bounded morphic image of F_1.

Definition 4.6.26 (Generated Subframe)**.** *Let $F_1 = \langle W_1, R_1\rangle$ and $F_2 = \langle W_2, R_2\rangle$ be frames. F_2 is a generated subframe of F_1, written $F_2 \rightarrowtail F_1$, when F_2 is a subframe of F_1 and the following condition holds:*

if $x \in W_2$ and xR_1y, then $y \in W_2$.

Theorem 4.6.27. *Let F_1 and F_2 be frames and α a modal formula.*

If $F_1 \rightarrowtail F_2$, then $F_2 \models \alpha$ implies $F_1 \models \alpha$;

If $F_1 \twoheadrightarrow F_2$, then $F_1 \models \alpha$ implies $F_2 \models \alpha$.

Definition 4.6.28 (Canonical Logic)**.** *A normal modal logic L is said to be canonical when the frame of its canonical model is an L-frame; that is, when all L-theorems are valid on the canonical frame.*

Our goal is to prove the following theorem.

Theorem 4.6.29. *Let L be a normal modal logic that is canonical and that is compatible with the axiom* D*. Furthermore, let its canonical frame be contained in the class $\mathbb{C}_{\mathsf{LD}}$. Then $\mathsf{L}^{\boxplus}$ is also complete with respect to $\mathbb{C}_{\mathsf{LD}}$.*

Remark 4.6.30. Notice that requiring the compatibility with D is used to avoid the trivial cases when the $\boxplus$-translation produces inconsistent set of sentences. Indeed, the $\boxplus$-counterpart of a logic is a set of formula of $\mathcal{L}^{\boxplus}$ that extends the logic $\mathsf{KD}^{\boxplus}$, which, in turn, characterizes the class of serial frames. However, notice that the above theorem is giving some more. Indeed, completeness results are preserved upwards with respect to the $\subseteq$-relation between classes of frames. For this reason, Theorem 4.6.29 shows that $\mathcal{L}^{\boxplus}$ is also complete with respect to $\mathbb{C}_{\mathsf{L}}$.

In order to prove Theorem 4.6.29 we construct an isomorphism between the canonical model for $\mathsf{L}^{\boxplus}$ and a generated subframe of the canonical model for LD. Specifically, we will construct an injective bounded morphism from the canonical model of $\mathsf{L}^{\boxplus}$ to that of LD.

Consider a mapping from Var onto $For(\mathcal{L}^{\boxplus})$:

$$\begin{array}{ccc} Var & \rightarrow & For(\mathcal{L}^{\boxplus}) \\ p & \mapsto & p^* \end{array}$$

This map exists since our sets of formulas are countable. Now extend it recursively to a map:

$$\begin{array}{ccc} For(\mathcal{L}^{\Box}) & \rightarrow & For(\mathcal{L}^{\boxplus}) \\ \alpha & \mapsto & \alpha^* \end{array}$$

where α^* is defined similarly to Definition 4.6.23:

$$\begin{array}{rcl} (\neg\varphi)^* & = & \neg\varphi^* \\ (\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\ (\Box\varphi)^* & = & \boxplus\varphi^* \end{array}$$

We call the above function the $*$-map.

Because of how $\mathsf{L}^{\boxplus}$ is defined, the $*$-map preserves theoremhood. Indeed, notice that the characteristic axiom D is mapped, by the $*$-translation to the axiom $\boxplus\varphi \to \neg \boxplus \neg\varphi$ of $\mathsf{KD}^{\boxplus}$, which is included in $\mathsf{L}^{\boxplus}$. Moreover, by construction of $\mathsf{L}^{\boxplus}$, if φ is a theorem of L, then φ^* is a theorem of $\mathsf{L}^{\boxplus}$. Finally, notice that $\mathsf{L}^{\boxplus}$ is closed under the rules of MP and $\mathrm{Nec}^{\boxplus}$, therefore the closure of the axioms for LD with respect to the deduction rules, is preserved under the $*$-translation. All this shows that if φ is a theorem of LD, then φ^* is a theorem of $\mathsf{L}^{\boxplus}$.

Now, let $F_{\mathsf{L}^{\boxplus}} = \langle W_{\mathsf{L}^{\boxplus}}, R_{\mathsf{L}^{\boxplus}} \rangle$ be the canonical frame for $\mathsf{L}^{\boxplus}$, as defined before, and let $F_{\mathsf{LD}} = \langle W_{\mathsf{LD}}, R_{\mathsf{LD}} \rangle$ be the canonical frame for LD as it is usually defined.

We can then define the following function:

$$\begin{array}{rcl} f : W_{\mathsf{L}^{\boxplus}} & \to & W_{\mathsf{LD}} \\ w & \mapsto & \{\varphi : \varphi^* \in w\} = f(w) \end{array}$$

for any maximal $\mathsf{L}^{\boxplus}$-consistent $w \in W_{\mathsf{L}^{\boxplus}}$.

The proof of the next proposition is *verbatim* the same as in Gilbert & Venturi (2016).

Proposition 4.6.31. *The set $f(w)$ is maximal and LD-consistent.* *Q.E.D.*

Proposition 4.6.32. *The function f is injective.* *Q.E.D.*

Proposition 4.6.33. *If $wR_{\mathsf{L}^{\boxplus}}z$ then $f(w)R_{\mathsf{LD}}f(z)$.* *Q.E.D.*

We report the proof next proposition, since this is where $\mathsf{KD}^{\boxplus}$ is used.

Proposition 4.6.34. *If $f(w)R_{\mathsf{LD}}x$, there is a z such that $wR_{\mathsf{L}^{\boxplus}}z$ and $f(z) = x$.*

Proof. Define the following set:

$$z_0 = \{\varphi^* : \boxplus\varphi^* \in w\} \cup \{\psi^* : \psi \in x\}$$

We claim that z_0 is $\mathsf{L}^{\boxplus}$-consistent. If it is not, then there is $\varphi^*, \psi^* \in For(\mathcal{L}^{\boxplus}) \cap z_0$ such that

$$\mathsf{L}^{\boxplus} \vdash \varphi^* \wedge \psi^* \to \bot$$

Then, we have the following deductions:

1. $\mathsf{L}^{\boxplus} \vdash \varphi^* \to \neg\psi^*$

2. $\mathsf{L}^{\boxplus} \vdash \varphi^* \rightarrow (\boxplus(\varphi^* \rightarrow \neg\psi^*) \rightarrow (\boxplus\varphi^* \rightarrow \boxplus\neg\psi^*))$

3. $\mathsf{L}^{\boxplus} \vdash \boxplus(\varphi^* \rightarrow \neg\psi^*)$

where (2) is an instance of a Theorem of $\mathsf{KD}^{\boxplus}$, being of $\mathsf{K}^{\boxplus}$.

Now, because $\varphi^* \in w$, we obtain $\boxplus\neg\psi^* \in w$. Then, $(\Box\neg\psi)^* \in w$ and, therefore, $\Box\neg\psi \in f(w)$. Then, $\neg\psi \in x$, contradicting the consistency of x.

We extend now the set z_0 to some maximal consistent set z. We show that $\lambda(w) \subseteq x$ and that $f(z) = x$.

Assume that $\varphi \in \lambda(w)$. Since φ is a formula of $\mathcal{L}^{\boxplus}$, we know that there is a $p \in Var$ such that $p^* = \varphi$. Thus, $\boxplus p^* \in w$. Then, $p \in z_0 \subseteq x$. So, $\varphi \in z$.

To see that $f(z) = x$ is the case, it is enough to note that $x \subseteq f(z)$. Moreover, $f(z) \subseteq x$ holds because x is maximal. Q.E.D.

Propositions 4.6.32, 4.6.33 and 4.6.34 show that the function f is an injective bounded morphism from the canonical frame of $\mathsf{L}^{\boxplus}$ to that of LD. Symbolically, we have

$$F_{\mathsf{L}^{\boxplus}} \cong F_{sub} \rightarrowtail F_{\mathsf{LD}}$$

(where F_{sub} is the subframe of F_{LD}). Therefore, $F_{\mathsf{L}^{\boxplus}}$ is actually an LD-frame, from Theorem 4.6.27.

Finally, assume that some formula φ is not a theorem of $\mathsf{L}^{\boxplus}$. Then, clearly, it is not valid on the canonical frame $F_{\mathsf{L}^{\boxplus}}$. In turn, we then know that there is a generated subframe of F_{LD}, call it F_{sub}, on which φ is not valid (since $F_{\mathsf{L}^{\boxplus}} \cong F_{sub}$). This then implies that φ is not valid on F_{LD} (because $F_{sub} \rightarrowtail F_{\mathsf{LD}}$ and so $F_{\mathsf{LD}} \models \varphi$ implies $F_{sub} \models \varphi$). Therefore, on the assumption that LD is canonical, we have that $\mathsf{L}^{\boxplus}$ is complete with respect to classes of frames $\mathbb{C}_{\mathsf{LD}}$ containing the canonical frame of L, as desired. This completes the proof of Theorem 4.6.29.

Generalized Soundness

In this section we will provide a general soundness result for $\boxplus$-counterparts, thus providing the last ingredient for a general characterization theorem. We can reformulate Proposition 4.6.13 in terms of the $\boxplus$-translation.

Lemma 4.6.35. *Let $\mathcal{M} = \langle W, R, V\rangle$ a model based on the serial frame $F = \langle W, R\rangle$. Then, for every $w \in W$,*

$$F \models \varphi \textit{ iff } F \models \varphi^{\boxplus}.$$

Q.E.D.

Consider now the following definition, that is the adaptation of robustness with respect to reflexivity, as defined in Gilbert & Venturi (2016).

Definition 4.6.36. *Let $\mathbb{C}_{\mathsf{L}}$ be a class of L-frames. We say that $\mathbb{C}_{\mathsf{L}}$ is* robust *with respect to seriality when the following condition holds: if $F = \langle W, R\rangle \in \mathbb{C}_{\mathsf{L}}$ and $F^d = \langle W, R^d\rangle$ is the result of adding (w, w) to R where w is a dead end in F, then $F^d = \langle W, R^d\rangle \in \mathbb{C}_{\mathsf{L}}$.*

It is immediate from this definition that from a frame F which is not serial, we obtain a frame F^d which is serial.

Theorem 4.6.37. *Let L be a normal modal logic that is sound with respect to a class $\mathbb{C}_{\mathsf{L}}$ that is robust with respect to seriality. Then, $\mathsf{L}^{\boxplus}$ is sound with respect to $\mathbb{C}_{\mathsf{LD}}$.*

Proof. Suppose that L is sound with respect to a class $\mathbb{C}_{\mathsf{L}}$ that is robust with respect to seriality. Thus, we can assume $\mathbb{C}_{\mathsf{LD}} \neq \emptyset$. Assume that $\mathsf{L}^{\boxplus}$ is not sound with respect to $\mathbb{C}_{\mathsf{LD}}$. Then there is a frame $F \in \mathbb{C}_{\mathsf{LD}}$ and a formula $\varphi \in \mathsf{L}$, such that $F \not\models \varphi^{\boxplus}$. Therefore, by Lemma 4.6.35, $F \not\models \varphi$, which contradicts the fact that $\varphi \in \mathsf{L}$. Q.E.D.

From Theorem 4.6.29 and Theorem 4.6.37, we obtain a general characterization result.

Corollary 4.6.38. *Let L be a normal modal logic that is canonical and that is sound and complete with respect to a class $\mathbb{C}_{\mathsf{L}}$, which is robust with respect to seriality. Then $\mathsf{L}^{\boxplus}$ is sound and complete with respect to $\mathbb{C}_{\mathsf{LD}}$.*

Notice that the condition on the compatibility with the axiom D has now been subsumed under the robustness with respect to seriality.

Main results and axiomatizations

We have some immediate corollaries of Theorem 4.6.38.

Corollary 4.6.39. *The logic $\mathsf{K4}^{\boxplus}$ is sound and complete with respect to the class of all serial and transitive frames.* *Q.E.D.*

Corollary 4.6.40. *The logic $\mathsf{KB}^{\boxplus}$ is sound and complete with respect to the class of all serial and symmetric frames.* *Q.E.D.*

Indeed, it is obvious to see that the class of frames which are transitive and symmetric are robust with respect to seriality. For what concerns euclidean frames we need to work a bit more.

Proposition 4.6.41. *Euclidean frames are robust with respect to seriality.*

Proof. First, recall the definition of euclidean relation:

$$\text{for every } w, y \text{ and } z, \text{ if } wRy \text{ and } wRz, \text{ then } yRz$$

We need to show that if there is an euclidean frame $F = \langle W, R\rangle$ that is not serial, then $F^d = \langle W, R^d\rangle$ is also euclidean. Notice that if y is a dead end of F, then y is an isolated point. Otherwise, if there were an $x \in W$ accessing y, we would have xRy and xRy, which implies yRy, since R is euclidean. But then, in F^d, the world y is such that yRy, which is perfectly compatible with being an euclidean frame. Therefore, euclidean frames are robust under seriality. Q.E.D.

We therefore get that the following corollary.

Corollary 4.6.42. *The logic* $\mathsf{K5}^{\boxplus}$ *is sound and complete with respect to the class of all serial and eculidian frames. Q.E.D.*

Moreover, since reflexivity implies seriality, we get the following results.

Corollary 4.6.43. *The logic* $\mathsf{KT}^{\boxplus}$ *is sound and complete with respect to the class of all reflexive frames. Q.E.D.*

Corollary 4.6.44. *The logic* $\mathsf{S4}^{\boxplus}$ *is sound and complete with respect to the class of all reflexive and transitive frames. Q.E.D.*

Corollary 4.6.45. *The logic* $\mathsf{S5}^{\boxplus}$ *is sound and complete with respect to the class of all reflexive symmetric, and transitive frames. Q.E.D.*

Notice that the notion of robustness with respect to reflexivity, defined in Gilbert & Venturi (2016), is a stronger notion since it requires that the addition of *all* reflexive arrows to a frame results in a frame of the similar kind. On the contrary, the robustness under seriality requires only the addition of *some* reflexive arrows: namely those added to dead-ends. This simple observation shows that if a class of frames is robust with respect to reflexivity, then it is also robust with respect to seriality. Therefore, the generated subframe method is here applied to an actually larger—because of Corollary 4.6.42—class of logics than those to which it is applied in Gilbert & Venturi's aforementioned paper.

Of course there are logics to which this method cannot be applied. For example the logic whose axiom is $\neg\Diamond\top$, since, being incompatible with D, it characterizes a class of frames that is not robust with respect to seriality.

Now, in what concerns the axiomatization of $\boxplus$-logics, we can prove the following theorem:

Theorem 4.6.46. *Let* L *be a normal modal logic such that* $\mathsf{L} = \mathsf{K} + \varphi$. *Let* $\mathsf{KD}^{\boxplus} + \varphi^{\boxplus}$ *be the smallest* $\boxplus$*-counterpart containing all the instances of the axiom* $\varphi^{\boxplus}$. *Then,* $\mathsf{L}^{\boxplus} = \mathsf{KD}^{\boxplus} + \varphi^{\boxplus}$.

Proof. The direction $\mathsf{KD}^{\boxplus} + \varphi^{\boxplus} \subseteq \mathsf{L}^{\boxplus}$ is obvious by definition of $\mathsf{L}^{\boxplus}$. While for the other direction we only need to check that $\boxplus$-translation preserves theoremhood, since $\mathsf{KD}^{\boxplus}$ is the minimal $\boxplus$ logic in serial frames.

which is the case, as already argued in Theorem 4.6.24. Q.E.D.

Among others, we get the following corollaries.

Corollary 4.6.47. *The logic* $\mathsf{KD}^{\boxplus} + \boxplus\varphi \to \varphi$ *is sound and complete with respect to the class* $\mathbb{C}_{\mathsf{KT}}$. *Q.E.D.*

Corollary 4.6.48. *The logic* $\mathsf{KD}^{\boxplus} + \boxplus\varphi \to \varphi + \boxplus\varphi \to \boxplus\boxplus\varphi$ *is sound and complete with respect to the class* $\mathbb{C}_{\mathsf{S4}}$. *Q.E.D.*

Corollary 4.6.49. *The logic* $\mathsf{KD}^{\boxplus} + \boxplus\varphi \to \varphi + \neg\boxplus\neg\varphi \to \boxplus\neg\boxplus\neg\varphi$ *is sound and complete with respect to the class* $\mathbb{C}_{\mathsf{S5}}$. *Q.E.D.*

4.6.2 $\boxplus$ and consistent provability

In this subsection we will argue that it is not obvious that the logic which captures $Prov_{\mathsf{PA}}(\ulcorner\varphi^{\phi}\urcorner) \wedge \neg Prov_{pa}(\ulcorner\neg\varphi^{\phi}\urcorner)$ exists. That is, we will argue that the response to the Question 4.6.9 seems to be negative.

Given the translations of Definitions 4.6.7 and 4.6.12 we prove the following results:

Lemma 4.6.50. *Let* $\varphi \in For(\mathcal{L}^{\Box})$ *be any modal formula. For any model* $\mathcal{M} = \langle W, R, v\rangle$ *we have that:*

$$\mathcal{M}, w \models \varphi^{\boxplus} \textit{ iff } \mathcal{M}, w \models \varphi^{\oplus}$$

For all $w \in W$.

Proof. The proof runs by induction on φ. The non modal cases are immediate. We will focus in the case where $\varphi = \Box\psi$. Then:
$\mathcal{M}, w \models (\Box\psi)^{\boxplus}$ iff $\mathcal{M}, w \models \boxplus\psi^{\boxplus}$ iff (for all $y \in W$ such that wRy, $\mathcal{M}, w \models \psi^{\boxplus}$) and (exists $y \in W$ such that wRy, $\mathcal{M}, w \models \psi^{\boxplus}$). By induction hypothesis, (for all $y \in W$ such that wRy, $\mathcal{M}, w \models \psi^{\oplus}$) and (exists $y \in W$ such that wRy, $\mathcal{M}, w \models \psi^{\oplus}$) iff $\mathcal{M}, w \models \Box\psi^{\oplus}$ and $\mathcal{M}, w \models \Diamond\psi^{\oplus}$ iff $\mathcal{M}, w \models \Box\psi^{\oplus} \wedge \Diamond\psi^{\oplus}$ iff $\mathcal{M}, w \models (\Box\psi)^{\oplus}$. Q.E.D.

As a consequence of Lemma 4.6.50 we have the following corollary:

Corollary 4.6.51. *Let* $\varphi \in For(\mathcal{L}^{\Box})$ *be any modal formula. Then:*

1. *For all frames* $F = \langle W, R\rangle$*:* $F, w \models \varphi^{\boxplus}$ *iff* $F, w \models \varphi^{\oplus}$*, for all* $w \in W$*;*
2. $\mathbb{C}_{\mathsf{L}} \models \varphi^{\boxplus}$ *iff* $\mathbb{C}_{\mathsf{L}} \models \varphi^{\oplus}$.

The next theorem shows that the possibility of self-embedding a logic L in itself forces a form of seriality, via $\boxplus$-translation.

Theorem 4.6.52. *Let* L *be a normal modal logic characterized by a class of frames* $\mathbb{C}_{\mathsf{L}}$. *Then, the following are equivalent:*

1. $\mathsf{KD} \subseteq \mathsf{L}$
2. *For every* $\varphi \in For(\mathcal{L}^{\Box})$, *if* $\vdash_{\mathsf{L}} \varphi^{\oplus}$ *then* $\vdash_{\mathsf{L}} \varphi$;
3. *For every* $\varphi \in For(\mathcal{L}^{\Box})$, $\vdash_{\mathsf{L}} \varphi^{\oplus}$ *iff* $\vdash_{\mathsf{L}} \varphi$;
4. *For every* $\varphi \in For(\mathcal{L}^{\Box})$, $\vdash_{\mathsf{L}} \varphi^{\oplus} \leftrightarrow \varphi$

Proof. (1) implies (2). Suppose that $\vdash_{\mathsf{L}} \varphi^{\oplus}$. By soundness of $L, \mathbb{C}_{\mathsf{L}} \models \varphi^{\oplus}$. By Lemma 4.6.50, $\mathbb{C}_{\mathsf{L}} \models \varphi^{\boxplus}$. By hypothesis, $\mathsf{KD} \subseteq \mathsf{L}$, which means that all frames in $\mathbb{C}_{\mathsf{L}}$ are at least serial. Then, by Proposition 4.6.13 $\mathbb{C}_{\mathsf{L}} \models \varphi$. Therefore, by completeness, we obtain $\vdash_{\mathsf{L}} \varphi$.
(2) implies (1). By definition translation $\oplus$, $\vdash_{\mathsf{L}} (\Box\varphi \rightarrow \neg\Box\neg\varphi)^{\oplus}$ implies that $\vdash_{\mathsf{L}} (\Box\varphi^{\oplus} \wedge \Diamond\varphi^{\oplus}) \rightarrow (\Box\varphi^{\oplus} \vee \Diamond\varphi^{\oplus})$, which is a truth-functional tautology, clearly provable in L.
(2) implies (3). Suppose that $\vdash_{\mathsf{L}} \varphi^{\oplus}$. By hypothesis, (2), $\vdash_{\mathsf{L}} \varphi$. Now suppose that $\vdash_{\mathsf{L}} \varphi$. Since (1) and (2) are equivalent, we know that L extends KD. Then the class of frames $\mathbb{C}_{\mathsf{L}}$ which characterizes L is obviously robust with respect to seriality, then $\vdash_{\mathsf{L}} \varphi^{\boxplus}$, because $\vdash_{\mathsf{L}} \Box\varphi \leftrightarrow (\Box\varphi \wedge \Diamond\varphi)$. and $\vdash_{\mathsf{L}} \boxplus\varphi \leftrightarrow (\Box\varphi \wedge \diamond\varphi)$. By soundness and Lemma 4.6.49, $\vdash_{L} \varphi^{\oplus}$.
(3) trivially implies (2).
(4) implies (3) by applications of modus ponens.
(1) implies (4). In other words, if $\mathsf{KD} \subseteq \mathsf{L}$, then

$$\vdash_{\mathsf{L}} \varphi^{\oplus} \leftrightarrow \varphi \tag{4.5}$$

For all $\varphi \in For(\mathcal{L}^{\Box})$. The proof of 4.5 runs by induction. When $\varphi = \Box\psi$, the result follows from Theorem 4.6.6.

Q.E.D.

Now turning to the Question 4.6.9 seems to be negative, we ask whether there is a logic L such that for all $\varphi \in For(\mathcal{L}^{\Box})$.

$$\vdash_{\mathsf{KGL}} \varphi^{\oplus} \text{ iff } \vdash_{\mathsf{L}} \varphi \tag{4.6}$$

If this logic L proves $\vdash_{\mathsf{L}} \varphi \leftrightarrow \varphi^{\oplus}$, then L extends KD, by Theorem 4.6.52, Ł extends KD, which means that $\vdash_{\mathsf{L}} \Box\top \leftrightarrow (\Box\top \wedge \Diamond\top)$. The problem is, as Theorem 4.4.14 shows, that KGL does not prove formulas of the form $\Diamond\top$. So, since KGL does not prove formulas $\Diamond\top$, then KGL does not prove formulas $(\Box\varphi)^{\oplus}$. It means that the logic of consistent provability which captures $Prov_{\mathsf{PA}}(\ulcorner\varphi^{\phi}\urcorner) \wedge \neg Prov_{\mathsf{PA}}(\ulcorner\neg\varphi^{\phi}\urcorner)$ cannot be normal. Therefore, the answer

to the Question 4.6.9 is negative. On the other hand, if L is non-normal, the question is open.

The above discussion does not show that the logics which extend KD as well as the logics in the language $\mathcal{L}^{\boxplus}$ do not have provability interpretation. In fact, as the Theorem 4.6.5 shows, the logic KD, and so the logics $\mathsf{K}^{\boxplus}$ and $\mathsf{KD}^{\boxplus}$, capture Rosser provability predicate, which is a consistent provability predicate.

Chapter 5

Necessity as validity

In this Chapter, we will focus on the validity predicate. First, we will give a general overview of Tarski's theorem about undefinability of truth, and then we will present Montague's theorem and Curry's theorem, which show that the predicate of validity is as problematic as naïve truth is in arithmetical theories. Second, we will present alternative formalisations of the predicate of validity and their relations with modal logics. Last, we will present a validity theory for FOL and its relation with quantified modal logic.

5.1 Tarski's theorem and semantic concepts

Tarski's theorem establishes that the arithmetical truth predicate cannot be expressible (i.e., representable in the sense of Definition 4.1.1) in strong mathematical theories. In particular, it cannot be expressible in arithmetical theories which extends Q.[1] That is, given a predicate $Tr(\cdot)$ satisfying transparency:

$$Tr(\ulcorner\varphi\urcorner) \leftrightarrow \varphi \tag{5.1}$$

which says that 'φ' *is true* iff φ (*e.g.* 'Snow is white' is true if and only if snow is white), one can obtain the *liar sentence* via Diagonalisation Lemma:

$$\neg Tr(\ulcorner\varphi\urcorner) \leftrightarrow \varphi \tag{5.2}$$

Sentence 5.2 states that φ is equivalent to its non-truth. Since we are dealing with classical logic, sentence 5.2 states that φ is equivalent to its falsity. Then we can prove Tarski's theorem as follows:[2]

[1] The arithmetic Q is a subsystem of PA, obtained by dropping the axiom of induction (PA**9**) from PA.

[2] The full proof of Tarski's theorem can be consulted in Smullyan (1992). Tarski's theorem also provides a proof of Gödel's first incompleteness theorem. Roughly speaking, since the provability predicate is representable in PA and the truth predicate is not, then there are true sentences which are not provable.

Theorem 5.1.1. *Let T be a theorem extending* Q *with a predicate $Tr(\cdot)$ satisfying (5.1). Then, T is inconsistent.*

Proof. Consider the following derivation:

1.	$\vdash_T \neg Tr(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$	(5.2)
2.	$\vdash_T Tr(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$	(5.1)
3.	$\vdash_T Tr(\ulcorner\varphi\urcorner) \leftrightarrow \neg Tr(\ulcorner\varphi\urcorner)$	CPL 1,2
4.	$\vdash_T Tr(\ulcorner\varphi\urcorner) \wedge \neg Tr(\ulcorner\varphi\urcorner)$	CPL 3
5.	$\vdash_T (Tr(\ulcorner\varphi\urcorner) \wedge \neg Tr(\ulcorner\varphi\urcorner)) \rightarrow \bot$	CPL
6.	$\vdash_T \bot$	MP 4,5

This concludes the proof.

Q.E.D.

So Theorem 5.1.1 establishes that the arithmetical truth predicate is inconsistent with arithmetical theories. Tarski's original solution was the adoption of a hierarchy of languages, where the predicate of the true sentences of a theory T is only expressible in an expressively stronger theory T^1. In T^1 we have the resources to talk about the syntax as well as the semantical concepts of T. The same reasoning applies to T^1, thus generating an transfinite hierarchy of languages T^ω. This means that no language $\mathcal{L}_{T^n}$ is capable to talk about itself using its own expressive resources. As a immediate consequence of Tarski's result, the concept of arithmetical truth is *local*, in the sense that "truth" means "truth-in-$\mathcal{L}_{T^n}$." That is, there is no truth predicate which applies to every language. In this sense, as Peregrin (1999) says, truth is ineffable.

Tarski's undefinability result gave rise to a research program about the incorporation of semantic concepts into the object language of formal theories. For example, several theories of truth were proposed in order to accommodate the truth predicate in a way to avoid the troublesome inference of Theorem 5.1.1.[3] Such result also motivated the search for other semantic concepts, such as satisfiability and validity. In the next section, we will present some results which show that the concept of validity is not less problematic than the notion of truth.

5.2 Naïve validity and Montague's theorem

As defined in Definition 2.2.8, model-theoretic validity is commonly defined as truth in all (set-theoretic) models. In Section 3.1.1 we discussed that Kreisel's definition of informal

[3]The main strategies to deal with liar sentence are the paracomplete and paraconsistent theories of truth. The paracomplete account (e.g., Kripke's fixed point theory) proceeds by showing that sentence 5.2 is neither true nor false. The paraconsistent account proceeds by showing that the inconsistency caused by such sentence does not imply that the theory itself is trivial. We invite the reader to consult Barrio (2014) for a nice presentation of theories of truth.

validity pressuposes the language of FOL in his version of the squeezing argument. So, $Val(\varphi)$ is the informal notion corresponding to FOL. As a consequence, it does not capture some validities which are valid in an informal standpoint, such as the Argument 3.1.3 presented in Section 3.1.1:

The President signed the treaty with a red pen.
$\therefore$ The President signed the treaty.

This example shows that first-order validity does not capture all valid inferences, which shows that validity does not coincide with logical validity. The former is broader than the latter. Such broader notion of validity will be called *naïve validity* and will be denoted by Val. Naïve validity is expected to incorporate semantic concepts as well as principles of formal logic. We can understand naïve validity in the lines of Halldén (1963) as *giving compelling logical grounds to believe* φ. The naive validity predicate Val should satisfy the following schemas:

(Val-K) $\text{Val}(\ulcorner\varphi \to \psi\urcorner) \to (\text{Val}(\ulcorner\varphi\urcorner) \to \text{Val}(\ulcorner\psi\urcorner))$;

(Val-T) $\text{Val}(\ulcorner\varphi\urcorner) \to \varphi$;

(Val-4) $\text{Val}(\ulcorner\varphi\urcorner) \to \text{Val}(\ulcorner\text{Val}(\ulcorner\varphi\urcorner)\urcorner)$;

(Val-Nec) If φ is valid, then $\text{Val}(\ulcorner\varphi\urcorner)$.

According to such understanding of naïve validity, it is clear that it satisfies the above principles. It is clear that validity is preserved under modus ponens. Second, if we provide compelling logical grounds to believe φ, then φ is the case. Third, if we give logical compelling grounds to believe φ, then we can assume that the principles we assumed in the justification of the validity of φ as themselves valid. Last, logical provable formulas are valid in the naïve sense.

It is worth to note that the concept expressed by the predicate Val is wider than Kreisel's informal notion Val. Val is a set-theoretical informal notion for FOL, which collapses with formal notions of validity when first-order formulas are taken into consideration. Val does not validate inferences such as that of the Argument 3.1.3. By its turn, Val is wide enough to encompass analytical inferences as well as valid natural language inferences. One could respond by saying that Val is still able to capture such inferences given some adequate assumptions. For example, in the case of Argument 3.1.3, one may correctly point that there is an implicit premise. So the argument has the following form:

The President signed the treaty with a red pen.
If the President signed the treaty with a red pen, then she/he signed the treaty.
$\therefore$ The President signed the treaty.

Even if we recognize that this is correct, Val needs all the enthymemes in order to make the Argument 3.1.3 to work, whereas Val does not.[4] The fact that Val does not need such enthymemes present in the inferences is enough for our argument.

Before we continue, it is important to make it explicit what we mean by "logical" in "compeling logical grounds". By logical we are assuming the axioms and inference rules of FOL. So, the compelling logical grounds are purely deductive. So, this definition of naïve validity does not comprehend all kinds of inferences, given that there are non deductive inferences which are not valid according to Val. As an example, consider the case of informal logic. As Pugliese (2020) observes, informal logic does not have at its disposal a mechanical procedure capable to distinguish valid arguments from invalid ones because informal logic includes inferences which are not necessarily deductive. In this Thesis we deal only with deductive inferences. We speculate that a predicate of validity which includes non deductive inferences may not satisfy Val-T, because inductive inferences does not guarantee truth.

In the analysis of predicates such as validity, PA is generally considered as the basic theory, because it has sufficient expressive power to talk about its own syntax. Let $\mathsf{PA}^{\mathrm{Val}}$ be the theory obtained by adding to the language $\mathcal{L}_{\mathsf{PA}}$ the one-place predicate $\mathrm{Val}(x)$, where x is the Gödel number of an expression of $\mathcal{L}_{\mathsf{PA}}$. However, the addition of the predicate Val with those three properties to PA is troublesome due to the following result proved by Montague (1963) . Let P be any predicate. Then:

Theorem 5.2.1. *(MONTAGUE, 1963) Suppose that T is any theory such that*

(i) T is an extension of Q*;*

(ii) $\vdash_T P(\ulcorner\varphi\urcorner) \rightarrow \varphi$;

(iii) $\vdash_T P(\ulcorner\varphi\urcorner)$ whenever φ is a sentence such that $\vdash_T \varphi$;

Then, T is inconsistent.

Proof. Consider the following proof:

[4] As Glanzberg (2015) points, natural language has many verbal inferences. For example, if we say that "Roberto Dinamite kicked the ball", we infer that "German Cano used his foot." In order to capture this kind of inferences in a first-order setting, we must add some enthymemes. But, in order to make FOL to encompass all these inferences, it may not be formal any more.

1.	$\vdash_T \varphi \leftrightarrow \neg P(\ulcorner\varphi\urcorner)$	Diagonalization
2.	$\vdash_T \varphi \rightarrow \neg P(\ulcorner\varphi\urcorner)$	CPL, 1
3.	$\vdash_T P(\ulcorner\varphi\urcorner) \rightarrow \varphi$	(ii)
4.	$\vdash_T \neg\neg P(\ulcorner\varphi\urcorner) \rightarrow \neg\varphi$	CPL, 2
5.	$\vdash_T P(\ulcorner\varphi\urcorner) \rightarrow \neg\neg P(\ulcorner\varphi\urcorner)$	CPL
6.	$\vdash_T P(\ulcorner\varphi\urcorner) \rightarrow \neg\varphi$	CPL 5, 4
7.	$\vdash_T \neg P(\ulcorner\varphi\urcorner)$	CPL 3, 6
8.	$\vdash_T \neg P(\ulcorner\varphi\urcorner) \rightarrow \varphi$	CPL, 1
9.	$\vdash_T \neg\varphi \rightarrow \neg\neg P(\ulcorner\varphi\urcorner)$	CPL, 8
10.	$\vdash_T \neg\varphi \rightarrow \neg P(\ulcorner\varphi\urcorner)$	CPL, 3
11.	$\vdash_T \neg\neg\varphi$	CPL 9, 10
12.	$\vdash_T \neg\neg\varphi \rightarrow \varphi$	CPL
13.	$\vdash_T \varphi$	MP 11, 12
14.	$\vdash_T P(\ulcorner\varphi\urcorner)$	(iii), 1

This concludes the proof. Q.E.D.

It is clear that if we replace P by Val in the Theorem 5.2.1 we obtain that the extention of PA with the predicate Val is inconsistent. This result is generalized by Murzi (2014), where he proves that the validity predicate is neither definable nor expressible in PA.

We can also prove a similar result to Theorem 5.2.1 by taking the dual of Val, the predicate Con of informal consistency. Con satisfies the following properties:

(Con-T) $\varphi \rightarrow \mathrm{Con}(\ulcorner\varphi\urcorner)$;

(Con-4) $\mathrm{Con}(\ulcorner\varphi\urcorner) \rightarrow \mathrm{Con}(\ulcorner\mathrm{Con}(\ulcorner\varphi\urcorner)\urcorner)$;

(Con-Intr) If $\neg\varphi$ is valid, then $\neg\mathrm{Con}(\ulcorner\varphi\urcorner)$.

Such principles are sound in an intuitive reading. (Con-T) says that if φ is the case, then it is consistent. In other words, (Con-T) says that truth implies consistency. The converse implication is clearly invalid. As Woleński (2010) argues, the consistency of a story does not imply its truth. The second principle, (Con-Intr), says that the negation of valid formulas is inconsistent. The following result is an easy modification of Theorem 5.2.1.

Theorem 5.2.2. *Let T be a theory which extends* PA *with predicate R such that:*

(i) $\vdash_T \varphi \rightarrow R(\ulcorner\varphi\urcorner)$*;*

(ii) $\vdash_T \neg R(\ulcorner\varphi\urcorner)$*, whenever* $\vdash_T \neg\varphi$*.*

Then, T is inconsistent.

Proof. By Diagonalization Lemma, we obtain the sentence $\varphi \leftrightarrow \neg R(\ulcorner \varphi \urcorner)$. Then we obtain the following deduction:

1.	$\vdash_T \varphi \leftrightarrow \neg R(\ulcorner \varphi \urcorner)$	Diag.
2.	$\vdash_T \varphi \rightarrow R(\ulcorner \varphi \urcorner)$	(i)
3.	$\vdash_T \varphi \rightarrow \neg R(\ulcorner \varphi \urcorner)$	CPL 1
4.	$\vdash_T \neg \varphi$	CPL 2,3
5.	$\vdash_T \neg \varphi \rightarrow (\neg R(\ulcorner \varphi \urcorner) \rightarrow \neg \varphi)$	CPL
6.	$\vdash_T \neg R(\ulcorner \varphi \urcorner) \rightarrow \varphi$	MP 4,5
7.	$\vdash_T \neg R(\ulcorner \varphi \urcorner) \rightarrow \varphi$	CPL 1
8.	$\vdash_T R(\ulcorner \varphi \urcorner)$	CPL 6,7
9.	$\vdash_T \neg \varphi \rightarrow \neg \neg R(\ulcorner \varphi \urcorner)$	CPL 7
10.	$\vdash_T \neg R(\ulcorner \varphi \urcorner)$	(ii), 4

This concludes the proof.

Q.E.D.

Theorem 5.2.1 has a philosophical importance in the early discussion about the legitimacy of modal logics. As it is known, Quine was one of the main critics of the use of modal logics. Quine (1966) outlined three ways in which we can be involved. The first concerns the use of modalities as predicates (syntactical approach); the second concerns the use of modalities as sentence operators; and the third concerns the use of modalities as operators in first-order modal logic. Among these three ways, Quine argues that the first is the least pernicious, since there are some predicates of sentences which are, according to him, legitimate, such as the *theoremhood* predicate

> Something very much to the purpose of the semantical predicate 'Nec' is regularly needed in the theory of proof. When, e.g., we speak of the completeness of a deductive system of quantification theory, we have in mind some concept of *validity* as norm with which to compare the class of obtainable theorems. The notion of validity in such contexts is not identifiable with truth. A true statement is not a valid statement of quantification theory unless not only it but all other statements similar to it in quantificational structure are true. Definition of such a notion of validity presents no problem, and the importance of the notion for proof theory is incontestable. (QUINE, 1966, pp.153)

On the other hand, if the unique reasonable way to use modalities can lead to inconsistencies, then, as Montague (1963) argues, modal logics "must be sacrified".[5] Then, as Slater (1995) suggests, Montague's theorem establishes that predicate approaches to modalities are doomed to failure.

The second problem with considering validity as being captured by the above principles is that one can prove another inconsistency with PA. First, observe that CPL validates the principle of contraction:

[5](MONTAGUE, 1963, pg. 161).

(Contr) $\vdash_{\mathsf{CPL}} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$

The following theorem is a version of *Curry Paradox*, the *validity-Curry paradox*, which also shows that the addition of the predicate Val satisfying the properties (Val-K), (Val-T) and (Val-Nec) is inconsistent with Q (hence, with PA) as the following theorem shows.[6]

Let $\mathsf{PA}^{\mathrm{Val}}$ be the theory obtained by adding the predicate Val to PA, satisfying the properties (Val-K), (Val-T) and (Val-Nec). Then we have the following result.

Theorem 5.2.3. $\mathsf{PA}^{\mathrm{Val}}$ *is inconsistent.*

Proof. By Diagonalization Lemma (Lemma 4.1.2), we obtain the sentence $\varphi \leftrightarrow (\mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot)$, which says that its own validity implies $\bot$. Then we have the following proof:

1.	$\vdash_T \varphi \leftrightarrow (\mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot)$	Hyp.
2.	$\vdash_T \varphi \to \varphi$	CPL
3.	$\vdash_T \varphi \to (\mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot)$	CPL 1,2
4.	$\vdash_T \mathrm{Val}(\ulcorner\varphi\urcorner) \to \varphi$	Val - T
5.	$\vdash_T (\mathrm{Val}(\ulcorner\varphi\urcorner) \to (\mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot)) \to (\mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot)$	Contr.
6.	$\vdash_T \mathrm{Val}(\ulcorner\varphi\urcorner) \to (\mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot)$	CPL 3,4
7.	$\vdash_T \mathrm{Val}(\ulcorner\varphi\urcorner) \to \bot$	MP 5,6
8.	$\vdash_T \varphi$	CPL, 1,7
9.	$\vdash_T \mathrm{Val}(\ulcorner\varphi\urcorner)$	Val-Nec 8
10.	$\vdash_T \bot$	MP 7,9

This concludes the proof. Q.E.D.

Theorem 5.2.3 has an interesting philosophical significance, because it poses a problem to the non-classical solutions to logical paradoxes, specially to the paracomplete and paraconsistent approaches. Both strategies are adopted in order to restrict the behaviour of the negation operator, and so blocking the troublesome steps of Theorem 5.1.1 and Theorem 5.2.1. On the other hand validity-Curry does not make use of negation operator. Since negation does not play a significant role in Theorem 5.2.3, the legitimacy of such non-classical solutions becomes questionable.[7]

At this point, one could wonder if the concept of informal validity is itself inconsistent with PA. We think, however, that this move is not straightforward. Indeed, the naïve

[6] The proof of the Theorem 5.2.3 is an adaptation of the proof given in Shapiro & Beall (2018). There, they used sequent calculus to prove this theorem. Curry (1942) proved the original result for certain classes of systems called combinatorial systems. His theorem was known by the fact that this does not make use of the connective of negation.

[7] Curry paradox motivates the *substructural* solutions of logical paradoxes. Such substructural theories are obtained by subtracting some structural rules of sequent calculus such as (left and right) *weakening*, (left and right) *contraction*, and (left and right) interchange. For good presentation of sequent calculus, we refer the reader to Smullyan (1995, Chapter 11) and for a substructural solution to validity Curry we refer the reader to Barrio et al (2016).

validity predicate is inconsistent with arithmetical theories where it is possible to prove Diagonalization lemma. In this case, one is authorized to say that this naïve validity is too general to allow diagonalization. On the other hand, there are proposals in the literature which challenges the idea that the names of sentences must necessarily be given by Gödel numbers. As Skyrms (1978), Niemi (1972) and Schweizer (1992) observe, Theorem 5.2.1 can be blocked if we accept that names of sentences can be given by other devices than Gödel numbers.[8] Schweizer (1992) proves that PA is consistent with a predicate N which satisfies the same principles as the propositional S5. The point here is that the predicate N is defined to have sentence names as arguments instead of Gödel numbers of sentences.[9]

Now, if one wants to maintain that the names of the sentences must be given by Gödel numbers, then it is necessary to restrict the notion of validity by adopting weaker principles which govern the behaviour of the predicate. For example, Ketland (2012) proves that the predicate of logical validity is consistent with PA. In this case, we do not appeal to intuitive principles of general validity, but the principles which govern the predicate of logical validity exclusively depend on the deductive power of the logic of the theory. That is, such predicate of logical validity captures the general principles of validity of the logical basis of PA, which is FOL. This means that we do not use our intuitions about validity to talk about logical validity. Here we must be attained to the deductive capabilities of the logic at issue.

In what follows, we will present Skyrms's metalinguistic models as Schweizer approaches and how they block the paradoxes of proved in Theorem 5.2.1 and Theorem 5.2.3. So, we will present Skyrms's result that his metalinguistic models are captured by the modal logic S5. Second, we will present Ketland's proposal for logical validity and its relations with the modal logic S0.5. Last, by adopting Skyrms and Schweizer formalisms, we will present a validity theory for pure FOL. We will not use Gödel numbers because we will use a rather weak validity theory.

5.3 Skyrms's approach

Skyrms (1978) proposes to establish a connection between the concepts of necessity and validity. His work is twofold. First, he proposes a construction of a hierarchy of languages $\mathcal{L}^n$ where $\mathcal{L}^0$ is the base language, which is supposed to contain at least CPL. Moreover, each $\mathcal{L}^k$ $(0 < k \leq n)$ is expressively stronger than its predecessors in the hierarchy, and $\mathcal{L}^k$ contains validity predicates and sentence names $\overline{\varphi}$ which describe the valid formulas of

[8] Schweizer approach goes in the direction of Gupta (1982)'s solution to liar paradox in the context of theories of truth.

[9] As we will see, even if Skyrms and Schweizer adopt alternative devices to deal with validity, it is to be noted that their notion of validity is not informal. They adopt a very restrict notion of formal validity, defined as truth in all models. But, of course, their move does not mean that informal validity is itself inconsistent. With a similar move, it should be possible to show that informal validity is consistent.

the weaker languages. Thus, in this case, necessity is interpreted as predicates of formulas in terms of a predicate *Val*. That is, if a formula φ is valid in the logic corresponding to the language $\mathcal{L}^n$, then $Val(\overline{\varphi})$ is a sentence of the language $\mathcal{L}^{n+1}$. On the other hand, he shows that this predicate has a modal counterpart. That is, the validity predicate Val can be treated as a modality operator in the sense that this predicate satisfies some familiar modal axioms. Skyrms proves that under certain interpretations of validity predicate, the modal axioms in question are the axioms of the modal logic $\mathsf{S5}$.

Let T be a theory which extends at least CPL. The hierarchical language relative to T takes $\mathcal{L}_T$ as the base language, which we denote by $\mathcal{L}_T^0$ for convenience. Informally, given $\mathcal{L}_T^0$, we construct a hierarchy of increasingly stronger languages $\mathcal{L}_T^1, \mathcal{L}_T^2, \ldots, \mathcal{L}_T^i, \ldots$, where each $\mathcal{L}_T^i$, for $i > 0$, contains the formulas φ of the languages $\mathcal{L}_T^k$, $0 \leq k < i$, as well as sentence names $\overline{\varphi}$ for each φ and the predicate Val. Informally, given the base language $\mathcal{L}_T^0$, we construct a hierarchy of increasingly stronger languages $\mathcal{L}_T^1, \mathcal{L}_T^2, \ldots, \mathcal{L}_T^k, \ldots$, where each $\mathcal{L}_T^k$ $(0 < k \leq n)$ is expressively stronger than its predecessors in the hierarchy, and $\mathcal{L}_T^k$ contains validity predicates and sentence names $\overline{\varphi}$ which describe the valid formulas of the weaker languages in the hierarchy. Taking the union of all the $\mathcal{L}_T^k$'s, we obtain $\mathcal{L}_T^\omega$. Formally:

Definition 5.3.1. *Let $\mathcal{L}_{\mathsf{L}}^0$ be the language of L which extends at least the language of CPL. From $\mathcal{L}_{\mathsf{L}}^0$ we define inductively the languages $\mathcal{L}_{\mathsf{L}}^n$, for $n \in \mathbb{N}$.*

(1) $For(\mathcal{L}_{\mathsf{L}}^n) \subseteq For(\mathcal{L}_{\mathsf{L}}^{n+1})$*;*

(2) If $\varphi \in For(\mathcal{L}_{\mathsf{L}}^n)$*, then* $\mathcal{L}_{\mathsf{L}}^{n+1}$ *contains* $\overline{\varphi}$ *and* $Val(\overline{\varphi}) \in For(\mathcal{L}_{\mathsf{L}}^{n+1})$*;*

$\mathcal{L}_{\mathsf{L}}^\omega = \bigcup_{n \in \omega} \mathcal{L}_{\mathsf{L}}^n$.

Since Skyrms has a general assumption about L, he makes use of a broad notion of *extensional model* which is defined as follows:

Definition 5.3.2. *An* extensional model *is a function* $v : For(\mathcal{L}_{\mathsf{L}}) \to \{1, 0\}$ *which is defined as follows:*

(1) $v(\varphi) = 1$ *or* $v(\varphi) = 0$*, for all* $\varphi \in \mathcal{L}_{\mathsf{L}}$*;*

(2) $v(\neg\varphi) = 1$ *iff* $v(\varphi) = 0$*;*

(3) $v(\varphi \to \psi) = 1$ *iff* $v(\varphi) = 0$ *or* $v(\psi) = 1$.

Of course, if the language of L includes quantifiers, we should extend Definition 5.3.2 by adequating it to the definitions of first order models.[10] Given the extensional models for L, the models for $\mathcal{L}_{\mathsf{L}}^\omega$ are defined as follows.

[10]This is done in Schweizer (1992).

Definition 5.3.3. *(SKYRMS, 1978) The models v^n of the language $\mathcal{L}^n_{\mathsf{L}}$ are induced by an extensional model v^0 of $\mathcal{L}^0_{\mathsf{L}}$ as follows:*

(1) The model v^0 of $\mathcal{L}^0_{\mathsf{L}}$ is v of Definition 5.3.2.

(2) The model v^{n+1} of $\mathcal{L}^{n+1}_{\mathsf{L}}$ is induced by a model v^0 of $\mathcal{L}^0_{\mathsf{L}}$ is the smallest extension of v^n of $\mathcal{L}^n_{\mathsf{L}}$ such that:

(2.1) $v^{n+1}(Val(\overline{\varphi})) = 1$ if $v'^n(\varphi) = 1$ for all models v'^n of $\mathcal{L}^n_{\mathsf{L}}$;
otherwise $v^{n+1}(Val(\overline{\varphi})) = 0$;

(2.2) The interpretation of $\neg$ and $\rightarrow$ are given by the truth-tables of L.

The model v^ω of the language $\mathcal{L}^\omega_{\mathsf{L}}$ induced by the model v^n of $\mathcal{L}^\omega_{\mathsf{L}}$ is the union of the models v^n of $\mathcal{L}^n_{\mathsf{L}}$.

An interesting way to know what are the valid principles of the models of Definition 5.3.3 is by stablishing a translation r from the sentences of $\mathcal{L}^{\Box}_{\mathsf{L}}$ to the sentences of $\mathcal{L}^\omega_{\mathsf{L}}$ defined as follows.[11]

Definition 5.3.4. *Let $r : \mathcal{L}^{\Box\Diamond}_{\mathsf{L}} \rightarrow \mathcal{L}^\omega_{\mathsf{L}}$ be a function defined as follows:*

If φ is a modal free sentence, then $r(\varphi) = \varphi$;

If $\varphi = \neg\psi$, then $r(\neg\psi) = \neg r(\psi)$;

If $\varphi = \gamma \rightarrow \psi$, then $r(\gamma \rightarrow \psi) = r(\varphi) \rightarrow r(\psi)$;

If $\varphi = \Box\psi$, then $r(\Box\psi) = Val(\overline{r(\psi)})$.

Given the translation r of Definition 5.3.4, one can prove that the models v^ω validates the principles of $\mathsf{S5}$ extending L:

Theorem 5.3.5. *(SKYRMS, 1978) If $\vdash_{\mathsf{S5}} \varphi$, then $r(\varphi)$ is true in all models v^ω of $\mathcal{L}^\omega_{\mathsf{L}}$.*

Proof. We will show that the translation of $\mathsf{S5}$-axioms and rules are valid in models v^ω of $\mathcal{L}^\omega_{\mathsf{L}}$. The propositional axiom and rules of L are immediate, since $\mathcal{L}^\omega_{\mathsf{L}}$ and $\mathcal{L}^{\Box\Diamond}_{\mathsf{L}}$ share the same base language. Then we only look for $\mathsf{S5}$ axioms and necessitation rule. Applying the translation to $\mathsf{S5}$ principles, we obtain:

(K^r) $Val(\overline{r(\varphi) \rightarrow r(\psi)}) \rightarrow (Val(\overline{r(\varphi)}) \rightarrow Val(\overline{r(\psi)}))$;

(T^r) $Val(\overline{r(\varphi)}) \rightarrow r(\varphi)$;

(5^r) $r(\varphi) \rightarrow Val(\overline{Con(\overline{r(\varphi)})})$;

[11] Note that $\mathcal{L}^{\Box}_{\mathsf{L}}$ is not necessarily the propositional modal language. The only assumption about $\mathcal{L}^{\Box}_{\mathsf{L}}$ is that it contains at least the modal propositional language.

(Nec^r) If $r(\varphi)$ is valid in models v^ω, then is so $Val(\overline{r(\varphi)})$.

Suppose that $v^\omega(Val(\overline{r(\varphi) \to r(\psi)})) = v^\omega(Val(\overline{r(\varphi)})) = 1$ in every model v^ω of $\mathcal{L}^\omega_{\mathsf{L}}$. Then, $Val(\overline{r(\varphi) \to r(\psi)})$ and $Val(\overline{r(\varphi)})$ are true in v^{k+1} of $\mathcal{L}^{k+1}_{\mathsf{L}}$, where $k+1 \in \omega$ and $\mathcal{L}^k_{\mathsf{L}}$ is the first metalanguage which contains $r(\varphi) \to r(\psi)$ and $r(\varphi)$. By definition, v^{k+1} of $\mathcal{L}^{k+1}_{\mathsf{L}}$ is induced by v^0 of $\mathcal{L}^0_{\mathsf{L}}$. So, if $v^k(r(\varphi) \to r(\psi)) = v^k(r(\varphi)) = 1$, by "metalinguistic" modus ponens, we infer $v^k(r(\psi)) = 1$, for all v^k of $\mathcal{L}^k_{\mathsf{L}}$. Then $v^{k+1}(Val(\overline{r(\psi)})) = 1$. So, we obtain $v^{k+1}(Val(\overline{r(\varphi) \to r(\psi)}) \to (Val(\overline{r(\varphi)}) \to Val(\overline{r(\psi)}))) = 1$. Thus, $v^\omega(Val(\overline{r(\varphi) \to r(\psi)}) \to (Val(\overline{r(\varphi)}) \to Val(\overline{r(\psi)}))) = 1$.

If $r(\varphi)$ is valid, then $r(\varphi)$ is true in every v^ω of $\mathcal{L}^\omega_{\mathsf{L}}$, where v^ω is induced by v^0 of $\mathcal{L}^0_{\mathsf{L}}$. By Definition 5.3.3, $v^k(\overline{r(\varphi)}) = 1$, for every model v^k of $\mathcal{L}^n_{\mathsf{L}}$, and so $v^{k+1}(Val(\overline{r(\varphi)})) = 1$. Therefore, $v^\omega(Val(\overline{r(\varphi)})) = 1$ for every v^ω of $\mathcal{L}^\omega_{\mathsf{L}}$.

The verification of the other principles is similar.

Q.E.D.

We know that $\mathsf{S5}$ is characterized by models $\mathcal{M} = \langle W, R, V\rangle$ where R is an equivalence relation. Since every world $w \in W$ is related to every other world, we can define $\mathsf{S5}$ as pairs $\mathcal{M} = \langle W, V\rangle$ by dropping R. Then, truth for modal formulas is defined as follows:

($\Box$-$\mathsf{S5}$) $\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, y \models \varphi$ for every $y \in W$;

($\Diamond$-$\mathsf{S5}$) $\mathcal{M}, w \models \Diamond\varphi$ iff $\mathcal{M}, y \models \varphi$ for some $y \in W$.

Consequently, as Hughes & Cresswell (1996) observe, every formula $\Box\varphi$ is true throughout $\mathcal{M}$ or it is false throughout $\mathcal{M}$ due to the extensionality of the model.

Now, it is possible to prove that $\mathsf{S5}$, based on L, captures all valid principles in models of Definition 5.3.3. Consider the following definitions:

Definition 5.3.6. *A* metalinguistic model *for $\mathcal{L}^{\Box\Diamond}_{\mathsf{L}}$ is a pair $\langle v^0, \mathsf{L}\rangle$, where v^0 is a model for the logic L with language $\mathcal{L}^0_{\mathsf{L}}$, such that v^0 induces models v^ω as in Definition 5.3.3. $\langle v^0, \mathsf{L}\rangle$ is a model for $\varphi \in For(\mathcal{L}^{\Box\Diamond}_{\mathsf{L}})$ iff $r(\varphi)$ is true in $\langle v^0, \mathsf{L}\rangle$.*

Definition 5.3.7. *The* metalinguistic counterpart *of a model $\mathcal{M} = \langle W, V\rangle$ is a model $\langle v^0, \mathsf{L}\rangle$ whose language $\mathcal{L}^0_{\mathsf{L}}$ is the non-modal fragment of $\mathcal{L}^{\Box\Diamond}_{\mathsf{L}}$. v^0 is taken to be the restriction of the valuation V in each world $w \in W$ to $\mathcal{L}^0_{\mathsf{L}}$ (i.e., L). The set of models v^0 will be taken to include each restriction of V in each $w \in W$ to $\mathcal{L}^0_{\mathsf{L}}$.*

Theorem 5.3.8. *(SKYRMS, 1978) If $\varphi \in For(\mathcal{L}^{\Box\Diamond})$ has a model $\mathcal{M} = \langle W, V\rangle$, then the metalinguistic counterpart of $\mathcal{M}$ is a metalinguistic model for φ.*

Proof. The proof runs by induction on φ. When φ is a non-modal sentence, the result follows from the common base language $\mathcal{L}^0_{\mathsf{L}}$. The boolean case is also straightforward.

Let $\varphi = \Box\psi$. Suppose that $\mathcal{M}, w \models \Box\psi$. Then, for all $y \in W$, $\mathcal{M}, w \models \psi$. By Definition 5.3.7, for each world $x_i \in W$, v_i^ω is the metalinguistic counterpart of v_{x_i}, where v_i^ω is induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$. By induction hypothesis, $v_i^\omega(r(\psi)) = 1$ for all models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$. So, we have that $v_i^n(r(\psi)) = 1$ for all models v_i^ω of $\mathcal{L}_{\mathsf{L}}^n$. Then, $v_i^{n+1}(Val(\overline{r(\psi)})) = 1$ for all models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$. Therefore, $v_i^\omega(Val(\overline{r(\psi)})) = 1$.

Q.E.D.

Already in the propositional level, we can see how the paradoxes showed in Theorem 5.2.1 and Theorem 5.2.3 are avoided using Skyrms's metalinguistic models. Both theorems made essential use of Diagonalization Lemma to obtain the sentences:

(A) $\varphi \leftrightarrow \neg Val(\ulcorner\varphi\urcorner)$

(B) $\varphi \leftrightarrow (Val(\ulcorner\varphi\urcorner) \rightarrow \bot)$

In an arithmetical theory like PA^{Val} the biconditionals A and B express, in last instance, a numerical equality between the codes of the formulas connected by the connective $\leftrightarrow$. For example, formula B means that there is a provable numerical equality between the number which corresponds to φ and the number which corresponds to $(Val(\ulcorner\varphi\urcorner) \rightarrow \bot)$. And, again, this is only possible because PA proves Diagonalization Lemma.

On the other hand, in a metalinguistic model the lines of Definition 5.3.3 where $\mathcal{L}^0 = \mathcal{L}_{\mathsf{PA}}^0$, the model v_{PA}^0 will induce a hierarchy of models v_{PA}^{n+1} of $\mathcal{L}_{\mathsf{PA}}^{n+1}$, where the valid sentences φ of $\mathcal{L}_{\mathsf{PA}}^n$ will have their corresponding sentence names $\overline{\varphi}$ and $v_{\mathsf{PA}}^{n+1}(Val(\overline{\varphi})) = 1$.[12] Since the sentence names are introduced as primitive objects, having no internal structure, it is not immediate that the following biconditionals are provable:

(A') $\varphi \leftrightarrow \neg Val(\overline{\varphi})$

(B') $\varphi \leftrightarrow (Val(\overline{\varphi}) \rightarrow \bot)$

As Schweizer (1992) observes, the predicate Val was defined to have only sentence names as its arguments, but not arithmetical codes of sentences. This syntactical restriction precludes sentences like $Val(\ulcorner\varphi\urcorner)$. One could argue, however, that PA^{Val} is still powerful enough to formalize a sentence like A. Fortunately, this sentence will be harmless since, by hypothesis, sentence names where distinguished from arithmetical names. Since T^r is a valid schema of PA^{Val} and (A) is also provable in PA^{Val}, one obtains:

$$\vdash_{\mathsf{PA}^{Val}} Val(\overline{\varphi}) \rightarrow \neg Val(\ulcorner\varphi\urcorner)$$

But this is not a contradiction. Last, but not least, the following equality is not a theorem of PA^{Val}. Then:

[12] Roughly speaking, this is basically Schweizer (1992)'s construction.

$$\nvdash_{\mathsf{PA}^{Val}} \overline{\varphi} = \ulcorner \varphi \urcorner$$

This equality is not provable because sentence names do not have internal structure. In Schweizer's words, they are not descriptive. For these reasons, one could not expect the possibility of diagonalizing on the predicate Val. Because it is not immediate that Diagonalization Lemma can be applied to predicate Val due to its formation clauses, we should not expect a proof *à la* Solovay that $\mathsf{S5}$ is the logic of validity predicate of PA^{Val} because Solovay's proof makes essential use of this lemma.

In what concerns Skyrms's results, Otte (1982) observes that if it were possible to represent a quotation function, call it $\mathfrak{q}$, which gives a sentence name $\overline{\varphi}$ for every φ, then it would be possible to prove a variation of Diagonalization Lemma such as the following:

Lemma 5.3.9. *Let T be a theory extending* PA *with the predicate $Val(\cdot)$ in the lines of Definition 5.3.1. For any formula $A(x)$ of T with a free variable x there is a sentence C such that $\vdash_T C \leftrightarrow A(\overline{C})$.*

Given Lemma 5.3.9, Montague's theorem would be provable again. But Skyrms's results (and Schweizer's) only works because his metalanguage is severely restricted in such a way that $\overline{\varphi}$ only occurs in the predicate Val. That is, it is not obvious at all that Lemma 5.3.9 is provable in T because of the severe restrictions that Skyrms's metalanguage suffers.[13]

Now it is important to observe that Skyrms's proposal of taking $\mathsf{S5}$ as the resulting modal logic makes sense only in the hierarchical approach, because in a non-hierarchical approach, $\mathsf{S5}$ cannot be taken as the logic of validity. If we consider, for example, the fist-order logic taken solely and directly as the object of investigation, axiom 5 surely does not make sense since first-order logic does not have a decision procedure to determine formulas of the form $\Diamond\varphi$, since this logic is undecidable. The properties K and T clearly hold with respect to the interpretation of Val. The case of axiom 4 is debatable, as we will see below. Thus, in a more general perspective, the modal logic which captures the validity predicate is weaker than $\mathsf{S5}$.

Following Skyrms and Schweizer proposals, we obtain a modal theory which is consistent with PA. Then, this implies that the validity predicate is not inconsistent with arithmetic, but with arithmetization of syntax because of Diagonalization Lemma (Lemma 4.1.2). The same reasoning applies to other predicates. Since it is not obvious to apply Diagonalization Lemma to modal predicates due to the primitiveness of sentence names, we have a consistent way to talk about validity in an arithmetical theory.

[13] As Hazen (1984) observes, even if $\mathfrak{q}$ were representable and Lemma 5.3.9 were provable, it would still be possible to maintain the core of Skyrms's construction by assuming that the validity predicate is contextually definable. That is, given a language $\mathcal{L}_{\mathsf{L}}$ of type n, the validity predicate can only be defined in a language $\mathcal{L}_{\mathsf{L}}$ of type $n+1$.

5.4 Ketland's approach and Lemmon's modal system

The strategy for blocking Theorem 5.2.1 presented in Section 5.3 basically consists in extending the language of $\mathcal{L}$ with sentence names for the validity predicate. In this section, we present a different move proposed by Ketland (2012), which explores a more restrict definition of validity, while maintaining arithmetical names as names for sentences. In his characterization of such predicate, Ketland considers the schemas (Val-K) and (Val-T) as axioms for the predicate Val with the following introduction rule of this predicate:

(Val-Nec") Given a *logical* derivation of φ, infer $Val(\ulcorner\varphi\urcorner)$.

Ketland considers as logical derivation a derivation which uses only the *logical* axioms of the formal system. For example, the logical axioms of first-order logic are exactly the axioms of Definition 2.2.4. Since FOL is complete, every (logically) provable formula is logically valid. In this sense, the predicate *Val* can be introduced only in the conclusions φ of a derivation. So, given a logical derivation $\langle\varphi_1, ..., \varphi_n, \psi\rangle$, we can conclude that $\langle\varphi_1, ..., \varphi_n, \psi, Val(\ulcorner\psi\urcorner)\rangle$ since ψ is derivable (i.e. valid). In his aforementioned paper, the formal system which formalizes Val is called *V-logic* (hereafter VL) defined as follows:

Definition 5.4.1. VL *is obtained by extending* PA *with predicate* Val *satisfying (*Val-K*), (*Val-T*) and (*Val-Nec"*).*

As an example of how predicate Val works, consider the following theorem:

Theorem 5.4.2. $\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi)\urcorner) \rightarrow (Val(\ulcorner\varphi\urcorner) \wedge Val(\ulcorner\neg\neg\varphi\urcorner))$

Proof. Consider the following derivation:

1.	$\vdash_{\mathsf{FOL}} (\varphi \wedge \neg\neg\varphi) \rightarrow \varphi$	FOL
2.	$\vdash_{\mathsf{FOL}} (\varphi \wedge \neg\neg\varphi) \rightarrow \neg\neg\varphi$	FOL
3.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi) \rightarrow \varphi\urcorner)$	Nec" 1
4.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi) \rightarrow \neg\neg\varphi\urcorner)$	Nec" 2
5.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi) \rightarrow \varphi\urcorner) \rightarrow (Val(\ulcorner(\varphi \wedge \neg\neg\varphi)\urcorner) \rightarrow Val(\ulcorner\varphi\urcorner))$	K
6.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi) \rightarrow \neg\neg\varphi\urcorner) \rightarrow (Val(\ulcorner(\varphi \wedge \neg\neg\varphi)\urcorner) \rightarrow Val(\ulcorner\neg\neg\varphi\urcorner))$	K
7.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi)\urcorner) \rightarrow Val(\ulcorner\varphi\urcorner)$	MP 3,5
8.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi)\urcorner) \rightarrow Val(\ulcorner\neg\neg\varphi\urcorner)$	MP 4,6
9.	$\vdash_{\mathsf{VL}} Val(\ulcorner(\varphi \wedge \neg\neg\varphi)\urcorner) \rightarrow (Val(\ulcorner\varphi\urcorner) \wedge Val(\ulcorner\neg\neg\varphi\urcorner))$	FOL 8,9

This concludes the proof. Q.E.D.

As one can check we used two different subscripts in Theorem 5.4.2 in the provability relation $\vdash$ to stress that we can only apply the rule Val-Nec" to formulas provable in FOL. The logical proof ends already in the step 2. From step 3 to set 9, we have a proof

in *V-logic*, which is a first-order theory, where its characteristic axioms are non-logical axioms. That is, they are not logically valid.

Given the intuitions concerning this proposed predicate, Ketland argues that VL is consistent.[14] According to him, logical validity is not susceptible to the inconsistency proved by the Theorem 5.2.1 because its proof is not a logical proof, but a proof which uses arithmetical resources of PA, such as the Diagonalization Lemma. In this case, we cannot apply the introduction rule of Val in the step 13 of the proof of Theorem 5.2.1. The same argument applies to the Theorem 5.2.3 since it makes essential use of Diagonalization Lemma. Then, we cannot apply the rule of introduction of logical validity in the step 9 of Theorem 5.2.3. So, if Val is only applicable to logical proofs, it does not give rise to the inconsistency proved by Beall & Murzi and to the paradox proved by validity Curry. As a consequence, assertions concerning logical validity are not logically valid themselves in the sense of being provable in the basic logical system. According to Cook (2014), the reason why the axioms of VL are not logically valid lies in the fact that the theory VL itself dos not validate the substitutivity of equivalents, as the following example shows:

$$\nvdash_{\mathsf{VL}} Val(\ulcorner Val(\ulcorner \varphi \urcorner) \leftrightarrow Val(\ulcorner \neg\neg\varphi \urcorner)\urcorner) \tag{5.3}$$

To see that this is not valid, consider the following demonstration:

Theorem 5.4.3. $\vdash_{\mathsf{VL}} Val(\ulcorner \varphi \urcorner) \leftrightarrow Val(\ulcorner \neg\neg\varphi \urcorner)$

Proof. Consider the following formal derivation:

1.	$\vdash_{\mathsf{FOL}} \varphi \rightarrow \neg\neg\varphi$	FOL
2.	$\vdash_{\mathsf{FOL}} \neg\neg\varphi \rightarrow \varphi$	FOL
3.	$\vdash_{\mathsf{VL}} Val(\ulcorner \varphi \rightarrow \neg\neg\varphi \urcorner)$	Nec",1
4.	$\vdash_{\mathsf{VL}} Val(\ulcorner \neg\neg\varphi \rightarrow \varphi \urcorner)$	Nec", 2
5.	$\vdash_{\mathsf{VL}} Val(\ulcorner \varphi \rightarrow \neg\neg\varphi \urcorner) \rightarrow (Val(\ulcorner \varphi \urcorner) \rightarrow Val(\ulcorner \neg\neg\varphi \urcorner))$	K
6.	$\vdash_{\mathsf{VL}} Val(\ulcorner \neg\neg\varphi \rightarrow \varphi \urcorner) \rightarrow (Val(\ulcorner \neg\neg\varphi \urcorner) \rightarrow Val(\ulcorner \varphi \urcorner))$	K
7.	$\vdash_{\mathsf{VL}} Val(\ulcorner \varphi \urcorner) \rightarrow Val(\ulcorner \neg\neg\varphi \urcorner)$	MP 3,5
8.	$\vdash_{\mathsf{VL}} Val(\ulcorner \neg\neg\varphi \urcorner) \rightarrow Val(\ulcorner \varphi \urcorner)$	MP 4,6
9.	$\vdash_{\mathsf{VL}} Val(\ulcorner \varphi \urcorner) \leftrightarrow Val(\ulcorner \neg\neg\varphi \urcorner)$	FOL 7,8

This concludes the proof. Q.E.D.

Since the rule Val-Nec" can be applied only formulas proved by FOL, we cannot apply it to the step 9. Then, 5.3 is not a valid formula in VL. Since φ and $\neg\neg\varphi$ are equivalent formulas, we would expect the same with respect to $Val(\ulcorner \varphi \urcorner)$ and $Val(\ulcorner \neg\neg\varphi \urcorner)$. But, as we see, the substitutivity fails for Val. In last instance, for failing the substitutivity of equivalents, Cook argues that validity is not a logical notion.

From the axioms of *Val* we have the following consequences:

[14] If PA is consistent, of course.

Proposition 5.4.4. *The following statements are true with respect to the predicate Val:*

(A) $\vdash_{\mathsf{VL}} \varphi \rightarrow \neg Val(\ulcorner\neg\varphi\urcorner)$

(B) $\vdash_{\mathsf{VL}} Val(\ulcorner\varphi\urcorner) \rightarrow \neg Val(\ulcorner\neg\varphi\urcorner)$

(C)) $\vdash_{\mathsf{VL}} \neg Val(\ulcorner\neg\top\urcorner)$

(D) $\vdash_{\mathsf{VL}} Val(\ulcorner\top\urcorner)$

(E) $\vdash_{\mathsf{VL}} \neg Val(\ulcorner\neg(\varphi \wedge \psi)\urcorner) \rightarrow (\neg Val(\ulcorner\neg\varphi\urcorner) \wedge \neg Val(\ulcorner\neg\psi\urcorner))$

(F) $\vdash_{\mathsf{VL}} \neg Val(\ulcorner\neg(\varphi \vee \psi)\urcorner) \leftrightarrow (\neg Val(\ulcorner\neg\varphi\urcorner) \vee \neg Val(\ulcorner\neg\psi\urcorner))$

(G) $\vdash_{\mathsf{VL}} Val(\ulcorner\neg\varphi\urcorner) \rightarrow \neg Val(\ulcorner\varphi\urcorner)$

(H) $\vdash_{\mathsf{VL}} (Val(\ulcorner\neg\varphi\urcorner) \wedge \neg Val(\ulcorner\neg\psi\urcorner)) \rightarrow (\neg Val(\ulcorner\varphi\urcorner) \wedge \neg Val(\ulcorner\neg\psi\urcorner))$

Where $\neg Val(\ulcorner\neg\varphi\urcorner)$ is a statement of consistency. It is important here to point that the item (C) of Theorem 5.4.4 does not establish that PA is consistent. What (C) establishes is that *the logic of* PA does not prove contradictions. That is, FOL is consistent. Consider the following theorem:

Theorem 5.4.5. $\vdash_{\mathsf{VL}} \neg Val(\ulcorner 0 = \mathbf{s}(0)\urcorner)$

Proof. Let take $\mathbf{s}(0) = 1$. Then, consider the following demonstration:

1.	$\vdash_{\mathsf{VL}} Val(\ulcorner 0 = 1\urcorner) \rightarrow 0 = 1$	T
2.	$\vdash_{\mathsf{VL}} 0 \neq 1 \rightarrow \neg Val(\ulcorner 0 = 1\urcorner)$	FOL 1
3.	$\vdash_{\mathsf{VL}} 0 \neq 1$	PA3
4.	$\vdash_{\mathsf{VL}} \neg Val(\ulcorner 0 = 1\urcorner)$	MP 2,3

This concludes the proof. Q.E.D.

What Theorem 5.4.5 shows is that the logic of PA does not prove contradictions. But this does not mean that PA proves its own consistency. We can say that this predicate of validity expresses a weaker notion of provability, which is restricted to the first order calculus, whereas $Prov_{\mathsf{PA}}$ comprehends a stronger notion of provability which, by its turns, comprehends the whole theory of PA. This observation reinforces our claim that validity and provability, and so consistency, are local notions. The fact that PA is able to express more than one notion of provability is due to its strong expressiveness.

Ketland's results shows that we do not need to any appeal to our intuitions about naïve validity to evaluate whether a predicate of logical validity is good or not. In the case of the predicate of logical validity, the response is immediate: if Val correctly captures the deductive principles of FOL, then the predicate is a good one. If not, it is not an adequate

predicate. Indeed, this can be generalized to all logical systems: let T_{L}^{Val} be a validity theory whose logical base system is L. The predicate Val of validity is adequate for a logic L if the following biconditional is true:

$$\vdash_{T_{\mathsf{L}}^{Val}} Val(\ulcorner\varphi\urcorner) \text{ iff } \vdash_{\mathsf{L}} \varphi. \tag{5.4}$$

Since the rule Val-Nec" is applicable only in case of logical derivations, then it cannot to be applied to $Val(\ulcorner\psi\urcorner)$. Such restriction, according to Ketland (2012), breaks the analogy of the predicate Val with the modal operator $\Box$. But it is important to note that Ketland is taking into consideration the operator $\Box$ of normal modal logics. So, if we consider non-normal modal logics, it may be possible to establish a connection with his approach to logical validity. There is a non-normal modal logic, which is at least sound, but not complete, with respect to Ketland's approach of the predicate Val. This modal logic is one of the Lemmon (1957)'s systems, and it is called $\mathsf{S0.5}$. The soundness and the non-completeness of $\mathsf{S0.5}$ with respect to Ketland's validity predicate will be discussed in the next section.

In what follows, we will present a validity theory for FOL inspired in the methods of Schweizer (1993) and Stern (2014). We will directly deal with FOL instead of dealing with first-order PA because we want to provide a validity theory for pure FOL. Their method enjoy sufficient generality to provide such theory for FOL. Further, we discuss how the results presented below could be adapted to provide a characterization results for VL. First we will present Lemmon (1957)'s non-normal modal logic $\mathsf{S0.5}$, whose modality $\Box$ interprets "it is tautologous by truth tables that". After presenting $\mathsf{S0.5}$, we presenting its first-order extension $\mathsf{QS0.5}$ and so we prove that $\mathsf{QS0.5}$ captures first-order validity predicate.

5.4.1 The modal logic S0.5

Now we will present the modal system $\mathsf{S0.5}$, whose modality $\Box$ means that *it is tautologous (by truth-tables) that*. Its axiom system is presented as follows:

Definition 5.4.6. *(LEMMON, 1959) The system* $\mathsf{S0.5}$ *has the following axiom system:*

(CPL) *All propositional tautologies;*

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$*;*

(T) $\Box\varphi \to \varphi$*;*

(MP) *From* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *we can infer* $\vdash \psi$*;*

(N') *If* $\vdash \varphi$ *is and* φ *is a tautology, then we infer* $\vdash \Box\varphi$*.*

As we can see, this system is a non-normal counterpart of the normal modal system KT. The rule N' can only be applied in the case that φ is a classical tautology. The modality $\Diamond$ can be defined as usual:

$$\Diamond\varphi \quad := \quad \neg\Box\neg\varphi$$

According to Hughes & Cresswell (1968), S0.5 is the weakest modal logic to prove the equivalences

$$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \tag{5.5}$$

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi \tag{5.6}$$

But, since rule N' is restricted to (CPL) tautologies, we cannot apply rule N' to the formulas 5.5 and 5.6. This makes sense in the informal reading of $\Box$ and $\Diamond$, because it is possible to evaluate whether a formula φ is a tautology/consistent by using this method, while it is not possible to evaluate by truth tables the statement "φ is a tautology/consistent." As we will see below, this restriction of N' to propositional tautologies shows some oddities of S0.5 with respect to the other modal logics.

The following Theorem easily follows from the axioms of S0.5. It will be useful in the next section.

Theorem 5.4.7. *The following items are provable in* S0.5*:*

(A) *If* $\vdash \varphi \rightarrow \psi$ *and* $\varphi \rightarrow \psi$ *is a tautology, we obtain* $\vdash \Box\varphi \rightarrow \Box\psi$*;*

(B) $\Box(\varphi \rightarrow (\psi \wedge \neg\psi)) \vdash \Box\neg\varphi$*;*

(C) $\Box(\varphi \rightarrow \psi), \Box(\psi \rightarrow \gamma) \vdash \Box(\varphi \rightarrow \gamma)$*;*

(D) $\vdash \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$

(E) $\vdash \Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow (\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n)$ *(*$n \geq 2$*)*

Proof. For (A), consider the following derivation:

1.	$\vdash \varphi \rightarrow \psi$	Hyp.
2.	$\vdash \Box(\varphi \rightarrow \psi)$	N' 1
3.	$\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	N' 1
4.	$\vdash (\Box\varphi \rightarrow \Box\psi)$	MP 2,3

For (B):

1.	$\vdash (\varphi \to (\psi \wedge \neg\psi)) \to \neg\varphi$	CPL
2.	$\vdash \Box((\varphi \to (\psi \wedge \neg\psi)) \to \neg\varphi)$	N' 1
3.	$\vdash \Box((\varphi \to (\psi \wedge \neg\psi)) \to \neg\varphi) \to (\Box(\varphi \to (\psi \wedge \neg\psi)) \to \Box\neg\varphi)$	K
4.	$\vdash \Box(\varphi \to (\psi \wedge \neg\psi)) \to \Box\neg\varphi$	MP 2,3
5.	$\vdash \Box(\varphi \to (\psi \wedge \neg\psi))$	Hyp.
6.	$\vdash \Box\neg\varphi$	MP 4,5

For (C):

1.	$\vdash (\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma))$	CPL
2.	$\vdash \Box((\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma)))$	N' 1
3.	$\vdash \Box((\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma))) \to$	...
...	$(\Box(\varphi \to \psi) \to \Box((\psi \to \gamma) \to (\varphi \to \gamma)))$	cont K
4.	$\vdash \Box(\varphi \to \psi) \to \Box((\psi \to \gamma) \to (\varphi \to \gamma))$	MP 2,3
5.	$\vdash \Box(\varphi \to \psi)$	Hyp.
6.	$\vdash \Box((\psi \to \gamma) \to (\varphi \to \gamma))$	MP 4,5
7.	$\vdash \Box((\psi \to \gamma) \to (\varphi \to \gamma)) \to (\Box(\psi \to \gamma) \to \Box(\varphi \to \gamma))$	K
8.	$\vdash \Box(\psi \to \gamma) \to \Box(\varphi \to \gamma)$	MP 6,7
9.	$\vdash \Box(\psi \to \gamma)$	Hyp.
10.	$\vdash \Box(\varphi \to \gamma)$	MP 8,9

The deduction for(D) is exactly the same as in Theorems 4.6.14 and 4.6.15

The proof of (E) runs by induction on n. When $n = 2$, the result follows by Theorem 5.4.7 (C). by induction hypothesis, we have:

1.	$\vdash \Box(\varphi_1 \wedge \ldots \wedge \varphi_k) \to (\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_k)$	I.H.
2.	$\vdash \Box\varphi_{k+1} \to \Box\varphi_{k+1}$	CPL
3.	$\vdash (\Box(\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \Box\varphi_{k+1}) \to ((\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_k) \wedge \Box\varphi_{k+1})$	CPL 1,2
4.	$\vdash \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1}) \to (\Box(\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \Box\varphi_{k+1})$	Th. 5.4.7 (D)
5.	$\vdash \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1}) \to \Box(\varphi_1 \wedge \ldots \varphi_k)$	CPL 4
6.	$\vdash \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1}) \to \Box\varphi_{k+1}$	CPL 4
7.	$\vdash \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1}) \to (\Box\varphi_1 \wedge \ldots \Box\varphi_k)$	CPL 1,5
8.	$\vdash \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1}) \to ((\Box\varphi_1 \wedge \ldots \Box\varphi_k) \wedge \Box\varphi_{k+1})$	CPL 6,7
9.	$\vdash ((\Box\varphi_1 \wedge \ldots \Box\varphi_k) \wedge \Box\varphi_{k+1}) \to (\Box\varphi_1 \wedge \ldots \Box\varphi_k \wedge \Box\varphi_{k+1})$	CPL
10.	$\vdash \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1}) \to (\Box\varphi_1 \wedge \ldots \Box\varphi_k \wedge \Box\varphi_{k+1})$	CPL 8,9
11.	$\vdash (\varphi_1 \wedge \ldots \wedge \varphi_{k+1}) \to ((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1})$	CPL
12.	$\vdash \Box(\varphi_1 \wedge \ldots \wedge \varphi_{k+1}) \to \Box((\varphi_1 \wedge \ldots \wedge \varphi_k) \wedge \varphi_{k+1})$	Th. 5.4.7 (A), 11
13.	$\vdash \Box(\varphi_1 \wedge \ldots \wedge \varphi_k \wedge \varphi_{k+1}) \to (\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_k \wedge \Box\varphi_{k+1})$	CPL 10 ,12

The converse derivation is similar. Q.E.D.

The item (A) of Theorem 5.4.7 is called (RK'). The semantic interpretation for S0.5 is given by the following definition:

Definition 5.4.8 ((LEMMON, 1959), (HUGHES; CRESSWELL, 1996)). *An* S0.5*-frame is a triple* $F = \langle W, R, N\rangle$*, where* W *is a set of worlds,* N *is the set of normal worlds such that* $N \subseteq W$ *and* $N \neq W$*, and* R *is a reflexive relation over* N *such that for every* $y \in W$ *there is* $x \in N$ *such that* xRy*. A model* $\mathcal{M} = \langle F, V\rangle$ *based on* F *is a structure where* $V : Var \to \wp(W)$ *is a valuation. The interpretation of the basic Boolean connectives are the same as in Definition 2.3.1. The difference lies in the definition of the modal operator* $\Box$*:*

4" *For any* $w \in W$*,* $\mathcal{M}, w \models \Box\varphi$ *if* $w \in N$ *and for every* $y \in W$ *such that* wRy*,* $\mathcal{M}, y \models \varphi$*. If* $w \notin N$*,* $\mathcal{M}, w \models \Box\varphi$ *or* $\mathcal{M}, w \nvDash \Box\varphi$*.*

The definitions of truth in a model and validity are defined in terms of worlds $w \in N$*.*

As Definition 5.4.8 shows, modal formulas are arbitrarily evaluated in non-normal worlds. For this reason, $\Box\varphi$ and $\neg\Diamond\neg\varphi$ are not equivalent in all worlds.

Theorem 5.4.9. *In* S0.5 *the following schemas are not valid:*

(A) $\Box(\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi)$*;*

(B) $\Box(\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi)$*;*

(C) $\Diamond(\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi)$*;*

(D) $\Diamond(\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi)$.

Proof. For (A) consider a model $\mathcal{M} = \langle W, N, R, v\rangle$ such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$, $v_y(\Box\varphi) = 1$ and $v_y(\Diamond\neg\varphi) = 1$. By definition of negation, $v_y(\neg\Diamond\neg\varphi) = 0$. So, we obtain $v_y(\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi) = 0$. Therefore, $v_y(\Box(\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi)) = 0$. The reasoning is the same for the remaining cases. This concludes the proof. Q.E.D.

Now we will prove that S0.5 is sound with respect to its Kripke semantics of Definition 5.4.8.

Theorem 5.4.10. *If* $\vdash \varphi$*, then* $\models \varphi$.

Proof. We will show that the axioms of S0.5 are valid and that its inference rules preserve validity. Since it is a well known fact that CPL is sound with respect to the boolean clauses of Definition 5.4.8, we will focus in the modal axioms and inference rules.

I) Axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

Suppose that $\mathcal{M}, w \models \Box(\varphi \to \psi)$ and $\mathcal{M}, w \models \Box\varphi$ for every model $\mathcal{M} = \langle W, N, R, v\rangle$, for every $w \in N$. Then, for all $y \in W$ such that $wRy, \mathcal{M}, y \models \varphi \to \psi$ and $\mathcal{M}, y \models \varphi$. By

a semantic version of modus ponens, we obtain $\mathcal{M}, y \models \psi$, for all $y \in W$ such that wRy. Then, $\mathcal{M}, w \models \Box\psi$. Therefore, $\mathcal{M}, w \models \Box(\varphi \to \psi) \to (\Box\varphi \to \psi)$.

II) Axiom $\Box\varphi \to \varphi$.

Suppose that $\mathcal{M}, w \models \Box\varphi$ for every model $\mathcal{M} = \langle W, N, R, v\rangle$, for every $w \in N$. Since R is reflexive, we obtain $\mathcal{M}, w \models \varphi$. Therefore, $\mathcal{M}, w \models \Box\varphi \to \varphi$.

III) Rule N'.

Suppose that φ is a tautology. Then for every model $\mathcal{M} = \langle W, N, R, v\rangle$, for every $y \in W$, $\mathcal{M}, y \models \varphi$. Then so is for every $y \in W$ such that wRy, for $w \in W$. Therefore, $\mathcal{M}, w \models \Box\varphi$.

This concludes the proof. Q.E.D.

The canonical model for S0.5 is defined in the same way as in Definition 5.4.12.[15] We prove now the completeness of S0.5.

First, consider the following definition:

Definition 5.4.11. *(HUGHES; CRESSWELL, 1996) Let $\Gamma \subseteq For(\mathcal{L}^{\Box\Diamond}_{\mathsf{S0.5}})$ be a set of formulas of* S0.5*. We say that Γ is* CPL-consistent *if there is no $\{\varphi_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that $\vdash \neg(\varphi_1 \wedge \ldots \wedge \gamma_n)$, and each $\varphi_i \in \Gamma$, $1 \leq i \leq n$ is a substitution instance of a* CPL*-tautology.*

Definition 5.4.12. *The canonical model $\mathcal{M} = \langle W, N, R, V\rangle$ of* S0.5 *is defined as follows:*

A.i *$w \in N$ is a maximal consistent set of* S0.5*-formulas;*

A.ii *every $w \in W$ ($w \notin N$) is a maximal* CPL-consistent *set of* S0.5*-formulas.*

B *The accessibility relation $R \subseteq N \times W$ is defined as follows:*

B.i *Let $w, y \in W$. For $w \in N$: if $\Box\varphi \in w$, then we let wRy iff $\lambda(w) \subseteq y$ (where $\lambda(w) = \{\varphi | \Box\varphi \in w\}$;*

B.ii *For $w \notin N$: $\Box\varphi \in w$ or $\Box\varphi \notin w$.*

C *$\mathcal{M}, w \models p$ iff $p \in w$.*

Proposition 5.4.13. *Let $w \in N$ be a maximal consistent set of* S0.5 *such that $\neg\Box\psi \in w$. Then, $\lambda(w) \cup \{\neg\psi\}$ is (*CPL*)-consistent.*

Proof. If $\lambda(w) \cup \{\neg\psi\}$ is CPL-inconsistent, then there are $\gamma_1, ..., \gamma_n \in \lambda(w)$ such that $\neg(\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \neg\psi)$ is an instance of a CPL tautology:

[15] Cresswell (1966) proved that S0.5 is sound and complete with respect to S0.5-models. The proof given by Cresswell is slightly different than the proof given here. In this paper, he proposes a semantics where there is only one non-normal world and there is no accessibility relation.

(1)	$\vdash \neg(\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \neg\psi)$	Def. 5.4.11.
(2)	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \psi$	CPL (1)
(3)	$\vdash \Box((\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \psi)$	N' (2)
(4)	$\vdash \Box((\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \psi) \rightarrow (\Box(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \Box\psi)$	K
(5)	$\vdash \Box(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \Box\psi$	MP (3),(4)
(6)	$\vdash (\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \rightarrow \Box(\gamma_1 \wedge \ldots \wedge \gamma_n)$	Theorem 5.4.7 (E)
(7)	$\vdash (\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \rightarrow \Box\psi$	CPL (5),(6)
(8)	$\vdash \neg(\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n \wedge \neg\Box\psi)$	CPL (7)

Then, the set $\{\Box\gamma_1, \ldots, \Box\gamma_n, \neg\Box\psi\}$ is S0.5-inconsistent, which contradicts the consistency of w. Therefore, $\lambda(w) \cup \{\neg\psi\}$ is CPL-consistent. Q.E.D.

Lemma 5.4.14. *Let $\mathcal{M}$ be the canonical model for* S0.5*. Then, for every $w \in W$ and every formula φ of* S0.5*:*

$$\mathcal{M}, w \models \varphi \textit{ iff } \varphi \in w.$$

Proof. The boolean cases are straightforward. So, we will consider only the case $\varphi = \Box\psi$. If $w \in N$ and $\Box\psi \in w$, then $\psi \in y$, for every $y \in W$ such that $\lambda(w) \subseteq y$. By definition, wRy, for all y. Then, by induction hypothesis, we obtain $\mathcal{M}, y \models \psi$. Moreover, $\mathcal{M}, w \models \psi$ since $\Box\psi \rightarrow \psi \in w$. Then, $\mathcal{M}, w \models \Box\psi$.

Conversely, suppose that $\neg\Box\psi \in w$ for $w \in N$. Then, by Proposition 5.4.13, $\lambda(w) \cup \{\gamma_1, \ldots, \gamma_n, \neg\psi\}$ is (CPL-) consistent. So, by the Lindenbaum Lemma, $\lambda(w) \cup \{\neg\psi\} \subseteq y$ where y is a maximal (CPL-) consistent set. Thus, we obtain, $\neg\psi \in y$. So, $\psi \notin y$. By definition, wRy for some $y \in W$. Then, by induction hypothesis, $\mathcal{M}, y \nvDash \psi$. Then, $\mathcal{M}, w \nvDash \Box\psi$. Therefore, $\mathcal{M}, w \models \neg\Box\psi$. This concludes the proof.

For $w \notin N$, it is arbitrary, once formulas of the form $\Box\psi$ behaves like propositional variables in worlds $w \notin N$. Q.E.D.

Cresswell (1966) presents the following semantic interpretation for S0.5:

Definition 5.4.15. *An* S0.5*-model is a structure $\langle w^*, W, V\rangle$, where $W \neq \emptyset$ is a set of worlds, $w^* \in W$ is a distinguished world. $V : Var \rightarrow \wp(W)$ is a valuation. The interpretation of the basic Boolean connectives are the same as in Definition 2.3.1. The difference lies in the definition of the modal operator $\Box$:*

4* *For w^* $\mathcal{M}, w^* \models \Box\varphi$ iff for every $y \in W$, $\mathcal{M}, y \models \varphi$. For $w \neq w^*$, $\mathcal{M}, w \models \Box\varphi$ or $\mathcal{M}, w \nvDash \Box\varphi$, arbitrarily.*

The definitions of truth in a model and validity are defined in terms of the world w^*.

The characterization results for S0.5 with respect to the semantic definition of Definition 5.4.15 can be found in Cresswell's paper.[16] Such definition will be helpful.

According to the intended meaning of $\Box$ and $\Diamond$, $\Box\varphi/\Diamond\varphi$ is true iff φ is a tautology/logically consistent (by truth-tables). But once we apply $\Box$ or $\Diamond$ to φ, $\Box\varphi/\Diamond\varphi$ is no more tautological/logically consistent by the truth-table method. That is $\Box\varphi$ and $\Diamond\varphi$ are not tautological and logically consistent according to the truth-tables. That is why we cannot allow validities of the form $\Box\ldots\Box\varphi$. Then, in order to block such valid iterations of modalities, we adopted the division of the set W into normal worlds and non-normal worlds. Before proving the adequacy of the informal interpretation of $\Box$, we will argue that S0.5 modal axioms are at least (informally) sound with respect to their informal interpretation.

Argument 5.4.16. S0.5 *modal axioms are informally sound with respect to their informal interpretation.*

$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$*: The axiom* K *says that tautologousness (by truth-tables) is preserved by modus ponens. In fact, if* $\varphi \to \psi$ *and* φ *are tautologies (by truth-tables), then* ψ *is a tautology because classical implication is truth-reserving under modus ponens.*

$\Box\varphi \to \varphi$*: The axiom* T *says that tautological formulas are actually true. In fact, if* φ *is true in all assignments, then* φ *is true. Note that this axiom holds for theories which are sound/non-trivial. If* Th *is a theory which contains arithmetic, then* T *cannot be valid in the interpretation on the pain of contradicting Theorem 4.1.4. The same argument holds for the axiom* $\Box\varphi \to \Diamond\varphi$*. In the case of* CPL*, which is sound and complete, if* φ *is a tautology, then* φ *is satisfiable/consistent. That is, there is at least one line of the truth-table where* φ *receives 1.*

If φ *is a* CPL*-tautology, then* $\vdash \Box\varphi$*. This case is straightforward. If* φ *is a tautology, then* φ *is a tautology by truth-table method.*

This is reinforced by the following proposition:

Proposition 5.4.17. $\Box\varphi$ *is* S0.5*-valid iff* φ *is a* CPL*-tautology.*

Proof. Suppose that $\Box\varphi$ is S0.5-valid. Then, for every model $\mathcal{M} = \langle W, N, R, v\rangle$ and every $w \in N$, it is the case that $\mathcal{M}, w \models \Box\varphi$. So, for every $y \in W$ such that wRy, $\mathcal{M}, y \models \varphi$. We know that every $y \in W - N$ are maximal consistent set of CPL-formulas. Clearly, only tautologies are invariant over maximal consistent sets of CPL-formulas. Hence, it is also the case that $\mathcal{M}, w \models \varphi$. Therefore, φ is a tautology. The converse is immediate. Q.E.D.

In light of Proposition 5.4.17 and Argument 5.4.16, one could say that S0.5 is the logic of CPL-tautologies. However, there are some objections against such interpretation. Since

[16] In Pietruszczak (2009) one finds several completeness results for non-normal modal logics characterized by slight modifications of semantics. There, Pietruszczak also deals with S0.5.

CPL is a decidable system, S0.5 should also work as the logic of non-tautologies of CPL, in the sense that for every $p_i \in \mathcal{V}$, $\neg\Box p_i$ should be a theorem. Moreover, he defends that formulas like $\neg\Box\Box p_i$ should be valid under such informal interpretation. None of them are theorems of S0.5. Then, according to Routley (1968), this constitutes a semantical incompleteness of S0.5 with respect to this informal interpretation of $\Box$.

Following Routley's objection, Urquhart (2010) presents the following definition:

Definition 5.4.18. *(URQUHART, 2010) Let $\mathcal{L}_{\mathsf{PA}}$ be the language of arithmetic and ρ a realization which assigns to sentence letters sentences of $\mathcal{L}_{\mathsf{PA}}$. We associate each modal sentence α a sentence α^ρ as follows:*

$$\begin{array}{lcl} p^\rho & = & \phi(p), \text{ where } p \text{ is a sentential letter} \\ \bot^\rho & = & \mathbf{0} = \mathbf{1} \\ (\alpha \to \beta)^\rho & = & \alpha^\rho \to \beta^\rho \\ (\Box\alpha)^\rho & = & Taut(\ulcorner \alpha^\rho \urcorner) \end{array}$$

He argues that S0.5 axioms are sound with respect to translation ρ. However, S0.5 is not complete because $\neg\Box\Box p$ is provable in any realization ρ. But it is not a S0.5-theorem. This establishes, as Routley (1968) argues, a semantical incompleteness of S0.5 with respect to interpretation ρ. As a consequence, S0.5 may not completely capture Ketland's validity predicate by similar reasons. Then, S0.5 is at most sound with respect to $Taut(\ulcorner \alpha^\rho \urcorner)$.

We note that if we extend S0.5 with formulas like $\neg\Box p_i$ and $\neg\Box\Box p_i$, we should give up the rule of uniform substitution. If the resulting system contains uniform substitution, then it is trivial. Just substitute p_i by any tautology. Then one obtains $\neg\Box\top$. Then by the necessitation rule of S0.5, one obtains $\Box\top$. Then, we obtain a contradiction.[17]

So in what sense does S0.5 capture the tautology predicate? In Chapter 6, we will prove that even if p is not a tautology, it is true in some subsets V of valuations of CPL. Of course, it is not true in all subsets V. Then, in some V, $\neg\Box p$ is false, whereas $\neg\Box p$ is true in some V'. Then, $\neg\Box p$ will not be valid under such scenario. Based on such intuition, we can say that S0.5 captures tautologicity when we consider subsets of classical valuations. In such chapter we will prove this result not only for CPL, but for all of its many-valued fragments.

In the next section, we will show that the first-order extension of S0.5, called QS0.5, captures the predicate of validity in a first-order setting. In a certain sense, we can argue that S0.5 captures the minimal properties of validity predicate because we are assuming few things about the basic theory.[18] To be specific, we want to discuss the status of *Barcan Formula* with respect to the validity interpretation of $\Box$.

[17] In Urquhart (2010), one can find a logical system, called TS which contains validities of the form $\neg\Box\Box p_i$ as well as $\neg\Box p_i$. But, different from S0.5, TS is not finitely axiomatizable.

[18] S0.5 is at least sound with respect to the validity theory for FOL. Under this interpretation, $\neg\Box p$ could not be valid since in FOL with identity there are atomic formulas which are valid, such as $x = x$.

5.5 Validity interpretation of Quantified S0.5: QS0.5

In this section, we introduce the first-order extension of S0.5, which we call QS0.5 in the following definition.[19]

Definition 5.5.1. *The system* QS0.5 *has the following axiom system:*

(FOL) CPL, *Ax4, Ax5 and inference rules of* FOL*;*

(S0.5) *All theorems of* S0.5*;*

(BF) $\forall x_i \Box \varphi \rightarrow \Box \forall x_i \varphi$

($\mathbf{N}^{\mathsf{FOL}}$) *If* $\vdash \varphi$ *and* φ *is* FOL*-provable, then we infer* $\vdash \Box \varphi$.

After presenting the models for QS0.5 we justify the absence of the axioms of identity in the logic QS0.5. In what follows we present the semantics for QS0.5, where we deal only with *constant domains*.[20] That is, the domain remain fixed in all worlds of the structure.

Definition 5.5.2. *An interpretation for* QS0.5 *is a structure* $\mathfrak{M} = \langle W, N, R, D, (\cdot)^{\mathfrak{M}} \rangle$ *where* $W \neq \emptyset$ *is a set of worlds,* $N \subseteq W$*, is a set of normal worlds* R *is a reflexive relation such that for every* $w \in N$ *there is a* $y \in W$ *such that* wRy*,* $D \neq \emptyset$ *is the domain of the structure.* $(\cdot)^{\mathfrak{M}}$ *is an assignment defined as in Definition 2.2.6, with the following modification: if* P^n_k *is a n-ary predicate, then* $(P^n_k)^{\mathfrak{M}} = \{(s_1, \ldots, s_n, w) \mid s_1, \ldots, s_n \in D \text{ and } w \in W\}$*. Let* $s \in Seq$ *be a sequence of objects and* s^* *be a function defined in Definition 2.2.7. The notion of satisfiability of* φ *in* $\mathfrak{M}$ *in a world* $w \in W$ *is defined as follows:*

1. *If* φ *is* $P^n_k(t_1, \ldots t_n)$*, then* $\mathfrak{M}, w \models_s P^n_k(t_1, \ldots t_n)$ *iff* $(s^*(t_1), \ldots, s^*(t_n)) \in (P^n_k)^{\mathfrak{M}}$*;*
2. *If* φ *is* $\neg\psi$*, then* $\mathfrak{M}, w \models_s \neg\psi$ *iff* $\mathfrak{M}, w \not\models_s \psi$*;*
3. *If* φ *is* $\psi \rightarrow \gamma$*, then* $\mathfrak{M}, w \models_s \psi \rightarrow \gamma$ *iff* $\mathfrak{M}, w \not\models_s \psi$ *or* $\mathfrak{M}, w \models_s \gamma$*;*
4. *If* φ *is* $\forall x_j \psi$*, then* $\mathfrak{M}, w \models_s \forall x_j \psi$ *iff every sequence* $s' \in Seq$ *that differs from* s *in at most the* i*-th component is such that* $\mathfrak{M} \models_{s'} \psi$*.*
5. *If* $w \in N$*, then:*
   (a) *For any* $w \in W$*,* $\mathfrak{M}, w \models_s \Box\varphi$ *iff for every* $y \in W$ *such that* wRy*,* $\mathfrak{M}, y \models_s \varphi$*.*
6. *If* $w \notin N$*, then:*
   (a) $\mathfrak{M}, w \models_s \Box\varphi$ *iff* $(s^*(t_1, \ldots, t_n), w) \in (\varphi)^{\mathfrak{M}}$*.*

[19] In Priest (2008a), one finds a presentation of QS0.5 by means of a tableaux system. Here we present this logic with a hilbertian proof system.

[20] The choice of constant domains is a matter of simplicity.

We say that φ is true *in $\mathfrak{M}$ ($\models_{\mathfrak{M}} \varphi$) (or $\mathfrak{M}$ is a* model *of φ) iff every sequence $s \in Seq$ satisfies φ in every $w \in N$. The notion of logical validity and logical consequence are defined as in Definition 2.2.6.*

There are important remarks things to say about Definitions 5.5.1 and 5.5.2. In Definition 5.4.8 the semantic clause for $\Box\varphi$ in non-normal worlds says that the value of the modal formulas in such worlds is arbitrary. Such strategy could be directly adapted to the first-order case. But, as Priest (2008a) notes, this would significantly affect the behaviour of the quantifiers in non-normal worlds. Consider, for example, a non-normal world where formulas $\Box P_n^m(t_1, \ldots, c_k, \ldots t_m)$ and $\Box P_n^m(t'_1, \ldots, c_j, \ldots t'_m)$ occur. Moreover suppose that $(t_i)^{\mathfrak{M}} = (t'_i)^{\mathfrak{M}}$, $1 \leq i \leq n$. If the values of modal formulas in non-normal worlds were arbitrary, then $\Box P_n^m(t_1, \ldots, c_k, \ldots t_m)$ and $\Box P_n^m(t'_1, \ldots, c_j, \ldots t'_m)$ would have different values. Then it was necessary to treat modal formulas like atomic formulas in such worlds.

As a title of example of how modalities work in non-normal worlds, consider the following formula:

$$\Box(\Box F(x_1) \leftrightarrow \Box\neg\neg F(x_1)) \tag{5.7}$$

To see that the formula 5.7 is not valid, consider the model $\mathcal{M} = \langle W, N, R, D(\cdot)^{\mathcal{M}}\rangle$ such that $W = \{w_0, w_1\}$, $N = \{w_0\}$, $R = \{(w_0, w_0), (w_0, w_1)\}$, $D = \{d_1, d_2\}$, $(\Box F(x_1), w_1)^{\mathcal{M}} = \{d_1\}$ and $(\Box\neg\neg F(x_1), w_1)^{\mathcal{M}} = \{d_2\}$. Now, consider $s^* : Term \to D$ such that $s^*(x_1) = d_1$ and $s^*(x_k) = d_2$, for $k \geq 2$. Then, $\mathcal{M}, w_1 \models_s \Box F(x_1)$ and $\mathcal{M}, w_1 \not\models_s \Box\neg\neg F(x_1)$. Therefore, $\mathcal{M}, w_0 \not\models_s \Box(\Box F(x_1) \leftrightarrow \Box\neg\neg F(x_1))$.

Second, the logic QS0.5 was not presented with identity predicate, because such predicate brings some undesired consequences in the logical theory. The usual way to define the satisfiability for formulas of the form $t_1 = t_2$ is the following way:

(=) If φ is $t_1 = t_2$, then $\mathfrak{M}, w \models_s t_1 = t_2$ iff $s^*(t_1) = s^*(t_2)$

Thus, if $s^*(t_1) = s^*(t_2)$, then $\mathfrak{M}, w \models_s t_1 = t_2$ for all worlds $w \in W$. As a consequence, the following formulas would be valid:

$$x = y \to \Box x = y \tag{5.8}$$

$$x \neq y \to \Box x \neq y \tag{5.9}$$

Then, 5.8 says that all identities are valid and 5.9 all non-identities are valid. Those formulas are clearly non-valid according to the interpretation we are proposing for QS0.5. That being said, we will drop the identity relation to avoid such complications.

Last, but not least, we discuss our choice for constant domains (CD) and we justify the validity of *Barcan Formula* (BF) in the present interpretation. The axiom (BF) is

one of the most controversial among the modal axioms in quantified modal logic. Its validity is immediate once one accepts the CD approach to quantified modal logic. The main issue with (BF) is its alleged counter-intuitiveness. For example, under the temporal interpretation of the modalities $\Box$ and $\Diamond$. Consider the dual of (BF), (BF$\Diamond$):

$$\Diamond\exists x_i\varphi(x_i) \rightarrow \exists x_i\Diamond\varphi(x_i)$$

In the temporal reading of modalities, $\Diamond$ is interpreted as "it was the case that". Let $\varphi(x_i)$ mean "x_i writes Dom Casmurro". Under such interpretation the antecedent is true, because it was the case that Machado de Assis wrote Dom Casmurro. But the consequence is false, because Machado de Assis is not among us any more. In other readings of modalities in quantified modal logic, such as the ontological, there is still dispute about its validity. The defenders of CD approach are called *possibilists* and the defenders of the variable domains approaches are called *actualists*. Possibilism is the claim that the domain contains all the possible objects. In this case, the same domain is shared by all possible worlds. Actualism is the claim that each world has its own domain.[21]

In general, actualists deny the general validity of (BF), because they reject the idea that the same object exists across all worlds.[22] So, the most direct strategy taken by actualist is the adoption of variable domains (VD). It is a well known fact that (BF) is not valid in models of VD. On the other hand, such a device brings its own problems, such as possible counterexamples to $\forall x_i\varphi \rightarrow \varphi(t)$ $(Ax4)$. There are ways to overcome such obstacles, such as the *existence predicates* $E(x)$, which postulate the existence of objects in the domain, as well as the adoption of free logics as the basic logic. So it is possible to regain a version of $(Ax4)$ with the formula $(\forall x_i\varphi \wedge E(t)) \rightarrow \varphi(t)$.

According to Cresswell (1991), the CD approach has advantages over the VD, because it keeps intact the functioning of the quantifiers, whereas the VD approach must adopt strategies to recover $(Ax4)$. Moreover, it is possible to translate modal system which adopt VD in systems which adopts CD. But the converse it is not always possible. Thus, because of the simplicity and the technical advantages, we adopted CD approach to QS0.5.

The Theorems 4.4.9 and 4.4.12 show that KGL completely capture PA predicate $Prov$. So it is natural to ask whether such results can be extended to the first-order case. As Montagna (1984) and Boolos (1995, Chapter 17) show, this is not the case. Montagna proves that first-order KGL, call it QKGL, is not arithmetically complete, because there are arithmetical sentences which are not QKGL theorems. Moreover, as Boolos argues, the formula (BF) is not true in theories which contain PA. Let τ be a translation from QKGL into PA, and φ^τ be the formula $\neg Prf(x, \bot)$, then $\forall x\Box\varphi$ is true, whereas $\Box\forall x\varphi$ is false. Then, (BF) is not arithmetically sound. On the other hand, the converse of (BF):

$$\text{(CBF) } \Box\forall x\varphi \rightarrow \forall x\Box\varphi$$

[21] We invite the reader to consult Garson (2013) for a detailed discussion of both approaches.
[22] Lewis (1968)'s *Counterpart Theory* is a strong defence of the actualist approach to modalities.

is valid under translation τ. So, it is clear that the inadequacy of (BF) with respect to the arithmetical provability is given by the example presented in the last paragraph. On the other hand, we will argue that (BF) captures some aspects of logical validity if we look for weak validity predicates. In other words, even if (BF) is incompatible with arithmetical provability, (BF) may capture a weaker notion of first-order validity.

5.5.1 Validity interpretation for QS0.5

In the Section 5.3 we presented Skyrms's result that S5 is the logic of validity in a hierarchical setting. Now, inspired in Schweizer (1992) and Stern (2014) methods we will provide a validity interpretation for the logic QS0.5. Now we will show in what sense(BF) is compatible with such interpretation of modalities.

So, the resulting systematization will be the metalinguistic models for FOL, where we have a validity predicate for names for formulas. We will show that the non-normal modal logic QS0.5 captures the general principles of the predicate of logical validity.

The language $\mathcal{L}_{\mathsf{FOL}}^{Val}$ of the validity theory for FOL, which we will call FOL^{Val}, is defined as follows:

Definition 5.5.3. *The language $\mathcal{L}_{\mathsf{FOL}}^{Val}$ is defined as the smallest set such that:*

(1) Terms of $\mathcal{L}_{\mathsf{FOL}}^{Val}$:

(1.a) if t is a term of $\mathcal{L}_{\mathsf{FOL}}$, then t is a term of $\mathcal{L}_{\mathsf{FOL}}^{Val}$;

(1.b) if φ is a formula of $\mathcal{L}_{\mathsf{FOL}}$, then $\overline{\varphi}$ is an individual constant of $\mathcal{L}_{\mathsf{FOL}}^{Val}$.

(2) Formulas of $\mathcal{L}_{\mathsf{FOL}}^{Val}$:

(2.a) if φ is a formula of $\mathcal{L}_{\mathsf{FOL}}$, then φ is a formula of $\mathcal{L}_{\mathsf{FOL}}^{Val}$;

(2.b) if $\overline{\varphi}$ is a formula name of φ, then $Val(\overline{\varphi})$ is a formula of $\mathcal{L}_{\mathsf{FOL}}^{Val}$.

Since validity is a predicate of formulas, clause (*2.b*) blocks cases $Val(t)$, where $t \neq \overline{\varphi}$, for all $\varphi \in \mathcal{L}_{\mathsf{FOL}}$.

Definition 5.5.4. *$\mathcal{W}$ is a non-empty set of first-order models $\mathfrak{A}_i = \langle D_0, (\cdot)^{\mathfrak{A}_i} \rangle$ such that:*

1. *Every $\mathfrak{A}_i \in \mathcal{W}$ shares the same domain D_0;*
2. *If c is a constant of $\mathcal{L}_{\mathsf{FOL}}$, then $(c)^{\mathfrak{A}_i} = (c)^{\mathfrak{A}_j}$, for every $\mathfrak{A}_i, \mathfrak{A}_i \in \mathcal{W}$;*
3. *If f_n^k is a function term of $\mathcal{L}_{\mathsf{FOL}}$, then $(f_n^k)^{\mathfrak{A}_i} = (f_n^k)^{\mathfrak{A}_j}$, for every $\mathfrak{A}_i, \mathfrak{A}_i \in \mathcal{W}$.*

Definition 5.5.5. *(SCHWEIZER, 1993) Let $\overline{\varphi(x_i)}$ be the formula name of $\varphi(x_i)$. We say that x_i is* metalinguistically free *in $\overline{\varphi(x_i)}$ if x_i occurs free in $\varphi(x_i)$.*

It is important to observe that $\overline{\varphi(x_i)}$ is always a closed term, even if $\varphi(x_i)$ is a open formula.

Definition 5.5.6. *A model for $\mathcal{L}^{Val}_{\mathsf{FOL}}$ is a structure $\langle \mathfrak{A}^+_0, \mathcal{W}\rangle$ where $\mathcal{W}$ is a set of first-order models $\mathfrak{A}$ defined as in Definition 5.5.4. The model $\mathfrak{A}^+_0 = \langle D^+_0, (\cdot)^{\mathfrak{A}^+_0}\rangle$ is defined as follows:*

(1) $D^+_0 = \{\overline{\varphi}|\ \varphi \in For(\mathcal{L}_{\mathsf{FOL}})\} \cup D_0$, where D_0 is a non empty set of objects defined as in Definition 2.2.6 shared by all $\mathfrak{A}_i \in \mathcal{W}$;

(2) $(c_j)^{\mathfrak{A}^+_i}$ is a fixed element of $D_0 - \{\overline{\varphi}|\ \varphi \in For(\mathcal{L}_{\mathsf{FOL}})\}$ such that $(c_j)^{\mathfrak{A}^+_i} = (c_j)^{\mathfrak{A}_i}$, for all $\mathfrak{A}_i \in \mathcal{W}$;

(3) $(P^k_n)^{\mathfrak{A}^+_0}$ is a set of k-tuples in $D^n_0 - \{\overline{\varphi}|\ \varphi \in For(\mathcal{L}_{\mathsf{FOL}})\}$;

(4) $(f^k_n)^{\mathfrak{A}^+_0}$ is a k-ary operation in $D_0 - \{\overline{\varphi}|\ \varphi \in For(\mathcal{L}_{\mathsf{FOL}})\}$ such that $(f^k_n)^{\mathfrak{A}^+_i} = (f^k_n)^{\mathfrak{A}_i}$, for all $\mathfrak{A}_i \in \mathcal{W}$;

(5) $(\overline{\varphi})^{\mathfrak{A}^+_0} = \varphi$

(6) $(Val)^{\mathfrak{A}^+_0} = \{\varphi|\ \varphi \in For(\mathcal{L}_{\mathsf{FOL}})\ \&\ \forall \mathfrak{A}^*_j \in \mathcal{W}, \mathfrak{A}^*_j \models \varphi\}$.

Definition 5.5.7. *Let $s \in Seq^+$ be a sequence of objects in D^+_0. The function $(s^+)^* : Term \to D^+_0$ is defined as follows:*

1. *If $t \in Term$ is a variable x_i, let $(s^+)^*(x_i) = s_i$, for $s_i \in D_0 - \{\overline{\varphi}|\ \varphi \in For(\mathcal{L}_{\mathsf{FOL}})\}$;*
2. *If $t \in Term$ is a constant c_i, then $(s^+)^*(c_i) = (c_i)^{\mathfrak{A}^+_0}$;*
3. *$(s^+)^*(\overline{\varphi}) = (\overline{\varphi})^{\mathfrak{A}^+_i}$.*

Definition 5.5.8. *Let $s \in Seq$ be a sequence of objects in D^+ and φ a formula of $\mathcal{L}^{Val}_{\mathsf{FOL}}$ the notion of satisfiability of φ in $\mathfrak{A}^+$ is defined as follows:*

1. *If φ is atomic, then $\mathfrak{A}^+_0 \models_{s^+} P^k_n(t_1, \ldots t_n)$ iff $((s^+)^*(t_1), \ldots, (s^+)^*(t_n)) \in (P^k_n)^{\mathfrak{A}^+_0}$;*
2. *If φ is $\neg\psi$, then $\mathfrak{A}^+_0 \models_{s^+} \neg\psi$ iff $\mathfrak{A}^+_0 \not\models_{s^+} \psi$;*
3. *If φ is $\psi \to \gamma$, then $\mathfrak{A}^+_0 \models_{s^+} \varphi \to \psi$ iff $\mathfrak{A}^+_0 \not\models_{s^+} \psi$ or $\mathfrak{A}^+_0 \models_{s^+} \varphi$;*
4. *If φ is $\forall x_i\psi$, then $\mathfrak{A}^+_0 \models_{s^+} \forall x_i\psi$ iff $\mathfrak{A}^+_0 \models_{s^+} \psi$, for every sequence s'^+ which differs from s^+ in the ith element;*
5. *If φ is $Val(\overline{\psi})$, then $\mathfrak{A}^+_0 \models_{s^+} Val(\overline{\psi})$ iff $\mathfrak{A}^+_0 \models_{s^+} \psi$ and $(s^+)^*(\overline{\varphi}) \in (Val)^{\mathfrak{A}^+_i}$.*

A formula φ is true in $\langle \mathfrak{A}^+_0, \mathcal{W}\rangle$ if $\mathfrak{A}^+_0 \models_{s^+} \varphi$, for every s^+. φ is valid if it is true in every $\langle \mathfrak{A}^+_0, \mathcal{W}\rangle$.

As we can see, Definitions 5.5.6 and 5.5.8 leave the satisfiability of first-order formulas untouched. The functions $(\cdot)^{\mathfrak{A}_0^+}$ and $(s^+)^*$ were defined in order to preserve the interpretation of the pure first-order language in the models $\mathfrak{A}_0^+$. So, the pure first-order axioms remain valid according to the models $\mathfrak{A}_0^+$. Second, the model $\mathfrak{A}_0^+$ was defined in such a way that it has the same domain of the set of models $\mathcal{W}$. In this sense, $\mathfrak{A}_0^+$ is the validity theory of the models $\mathfrak{A}_i \in \mathcal{W}$. That is, $\mathfrak{A}_0^+$ captures the general principles about predicate Val in every set $\mathcal{W}$ of first-order models.

From the Definitions 5.5.6 and 5.5.8, we obtain the following validities:

Theorem 5.5.9. *The following schemata are true for any models $\langle \mathfrak{A}_0^+, \mathcal{W} \rangle$ of $\mathcal{L}_{\mathsf{FOL}}^{Val}$:*

1. $Val(\overline{\varphi \to \psi}) \to (Val(\overline{\varphi}) \to Val(\overline{\psi}))$;
2. $Val(\overline{\varphi}) \to \varphi$;
3. $\forall x_i Val(\overline{\varphi}) \to Val(\overline{\forall x_i \varphi})$;
4. *If φ is a* FOL *validity, then $Val(\overline{\varphi})$ is valid.*

Proof. 2. Suppose that $\mathfrak{A}_0^+ \models Val(\overline{\varphi})$. Then, for every $s^+ \in Seq^+$, $\mathfrak{A}_0^+ \models_{s^+} Val(\overline{\varphi})$. By definition, we obtain $\mathfrak{A}_0^+ \models_{s^+} \psi$ and $(s^+)^* \in (Val)^{\mathfrak{A}_i^+}$. Therefore, $\mathfrak{A}_0^+ \models \psi$.

3. Suppose that $\mathfrak{A}_0^+ \models \forall x_i Val(\overline{\varphi})$. Then, for every $s^+ \in Seq^+$, $\mathfrak{A}_0^+ \models_{s^+} \forall x_i Val(\overline{\varphi})$. So, $\mathfrak{A}_0^+ \models_{s^+} Val(\overline{\varphi})$ for every sequence $s'^+ \in Seq^+$ which differs from s^+ in the ith element. By definition, for every $s'^+ \in Seq^+$, $\mathfrak{A}_0^+ \models_{s'^+} \varphi$ and $(s'^+)^*(\overline{\varphi}) \in (Val)^{\mathfrak{A}_0^+}$. Since, $(s'^+)^*(\overline{\varphi}) = (\overline{\varphi})^{\mathfrak{A}_0^+}$, then $(\overline{\varphi})^{\mathfrak{A}_0^+} \in (Val)^{\mathfrak{A}_0^+}$. Then, for every $\mathfrak{A} \in \mathcal{W}$, $\mathfrak{A} \models \varphi$. So, we have that $\mathfrak{A} \models \forall x_i \varphi$. By definition, $(\overline{\forall x_i \varphi})^{\mathfrak{A}_0^+} \in (Val)^{\mathfrak{A}_0^+}$ and $\mathfrak{A}_0^+ \models_{s^+} Val(\overline{\forall x_i \varphi})$, for every $s^+ \in Seq^+$. Therefore, $\mathfrak{A}_0^+ \models Val(\overline{\forall x_i \varphi})$.

4. Suppose that φ is a FOL validity. Then, for every first-order $\mathfrak{A}$, $\mathfrak{A} \models \varphi$, including the models $\mathfrak{A} \in \mathcal{W}$ which share the same domain. So, $(\overline{\varphi})^{\mathfrak{A}_0^+} \in (Val)^{\mathfrak{A}_0^+}$. Moreover, the truth of φ is regulated by the recursive clauses of Definition 2.2.8. Then, φ is also true in $\mathfrak{A}_0^+$, whose recursive clauses for the operators are the same as in first-order models. Therefore, $\mathfrak{A}_0^+ \models Val(\overline{\varphi})$. Q.E.D.

Consider now the translation $t : \mathcal{L}_{\mathsf{FOL}}^{\Box} \to \mathcal{L}_{\mathsf{FOL}}^{Val}$ defined as follows:

$$\begin{array}{lcl} r(P_n^k(t_1, \ldots, t_n)) & = & P_n^k(t_1, \ldots, t_n) \\ r(\neg\varphi) & = & \neg r(\varphi) \\ r(\varphi \to \psi) & = & r(\varphi) \to r(\psi) \\ r(\Box\varphi) & = & Val(\overline{r(\varphi)}) \end{array}$$

The translation r considered here has one restriction: r is defined for formulas without iterations of modalities. Such restriction is harmless because QS0.5 has no valid formulas with iterated modalities, by the same reason that S0.5 does not have such theorems. This

restriction is necessary to the present case because the predicate can only be applied to $\overline{\varphi}$, where φ is a first-order formula.

For the two following lemmas, remember that the assignment function s^+ was defined in such a way that it behaves as s with respect to the objects in domain D_0. That is, if t is a term of $\mathcal{L}_{\mathsf{FOL}}$, then $(s^+)^*(t)$ is not a formula name. It is an object in the first order domain.

Lemma 5.5.10. *For every model $\langle \mathfrak{A}_0^+, \mathcal{W}\rangle$ for $\mathcal{L}_{\mathsf{FOL}}^{Val}$, there is a model $\mathcal{M} = \langle W, N, R, D, (\cdot)^{\mathcal{M}}\rangle$ such that for every $\mathfrak{A} \in \mathcal{W} \cup \{\mathfrak{A}_0^+\}$ there is a world $w \in W$ such that:*

$$\mathcal{M}, w \models_s \varphi \text{ iff } \mathfrak{A} \models_{s'} r(\varphi) \tag{5.10}$$

For all $s \in Seq$ and all $s \in \{Seq, Seq^+\}$.

Proof. Given $\langle \mathfrak{A}_0^+, \mathcal{W}\rangle$ we define $\mathfrak{M} = \langle W, N, R, D, (\cdot)^{\mathfrak{M}}\rangle$ as follows:

- $W - N$ is the collection of worlds w_i such that for every $\mathfrak{A}_i \in \mathcal{W}$, $\mathcal{M}, w \models_s P_n^k(t_1, \ldots, t_n)$ iff $\mathfrak{A}_i \models_s P_n^k(t_1, \ldots, t_n)$;
- $N = \{w_0\}$ is the normal worlds such that $\mathcal{M}, w_0 \models_s P_n^k(t_1, \ldots, t_n)$ iff $\mathfrak{A}_0^+ \models_s P_n^k(t_1, \ldots, t_n)$;
- $R = \{(w_0, w_i) \mid w_i \in W\} \cup \{(w_0, w_0)\}$;
- D is the domain of objects shared by all $\mathfrak{A}_i \in \mathcal{W}$;
- $(\cdot)^{\mathfrak{A}_0^+}$ is defined over D as usual.

The proof of 5.10 runs by induction on φ. When φ is atomic, the result follows by definition. We will focus when $\varphi = \Box\psi$. Since the satisfiability of modal formulas in non-normal worlds is arbitrary, we will focus on the case where w is a normal world.

$\mathfrak{M}, w_0 \models_s \Box\psi$ iff for all $y \in W$ such that $w_0 R y$ $\mathfrak{M}, w_0 \models_s \psi$ and $\mathfrak{M}, y \models_s \psi$ iff by I.H. $\mathfrak{A}_0^+ \models_{s^+} r(\psi)$ and $\mathfrak{A}_i \models_s r(\psi)$ for all $\mathfrak{A}_i \in \mathcal{W}$ iff $\mathfrak{A}_0^+ \models_{s+} r(\psi)$ and $\mathfrak{A}_i \models r(\psi)$ for all $\mathfrak{A}_i \in \mathcal{W}$ iff $\mathfrak{A}_0^+ \models_{s^+} r(\psi)$ and $(\overline{r(\psi)})^{\mathfrak{A}_0^+} \in (Val)^{\mathfrak{A}_0^+}$ iff $\mathfrak{A}_0^+ \models_{s^+} Val(\overline{r(\psi)})$.

Q.E.D.

Lemma 5.5.11. *For every model $\mathfrak{M} = \langle W, N, R, D, (\cdot)^{\mathcal{M}}\rangle$ there is a model $\langle \mathfrak{A}_0^+, \mathcal{W}\rangle$ such that for every $w_i \in W$ there is $\mathfrak{A} \in \mathcal{W} \cup \{\mathfrak{A}_0^+\}$ such that for all $\varphi \in For(\mathcal{L}_{\mathsf{FOL}}^{Val})$*

$$\mathfrak{A}_i \models \varphi \text{ iff } \mathfrak{M}, w_i \models r^{-1}(\varphi). \tag{5.11}$$

For all $s \in Seq$ and all $s \in \{Seq, Seq^+\}$.

Proof. Given $\mathfrak{M} = \langle W, N, R, D, (\cdot)^{\mathfrak{M}}\rangle$ we define the model $\langle \mathfrak{A}_0^+, \mathcal{W}\rangle$ as follows:

- $\mathcal{W}$ is the set of all models $\mathfrak{A}_i$ in the lines of Definition 5.5.4 such that, for $w_i \in W - N$, $\mathfrak{A}_i \models_s P^k_n(t_1, \ldots, t_k)$ iff $\mathfrak{M}, w_i \models_s P^k_n(t_1, \ldots, t_k)$;
- $\mathfrak{A}^+_0$ is a models of $\mathcal{L}^{Val}_{\mathsf{FOL}}$ such that $\mathfrak{A}^+_0 \models_{s^+} P^k_n(t_1, \ldots, t_k)$ iff $\mathfrak{M}, w_0 \models_s P^k_n(t_1, \ldots, t_k)$.

The proof of 5.11 runs by induction on φ. When φ is atomic, the result follows by definition. We will focus in the model $\mathfrak{A}^+_0$ because the satisfiability of formulas $Val(\overline{\psi})$ is not defined in models $\mathfrak{A}_i$.
$\varphi = Val(\overline{\psi})$

$\mathfrak{A}^+_0 \models_{s^+} Val(\overline{\psi})$ iff $\mathfrak{A}^+_0 \models_{s^+} \psi$ and $(\overline{\psi})^{\mathfrak{A}^+_0} \in (Val)^{\mathfrak{A}^+_0}$ iff for all $\mathfrak{A}_i \in \mathcal{W}, \mathfrak{A}_i \models \psi$ iff for every sequence s $\mathfrak{A}_i \models_s \psi$ iff by I.H. $\mathfrak{M}, w_0 \models_s \psi$ and $\mathfrak{M}, w_i \models_s \psi$ for all $w_i \in W$ such that $w_0 R w_i$ iff $\mathfrak{M}, w_0 \models_s \Box\psi$ iff $\mathfrak{M}, w_0 \models_s r^{-1}(\Box\psi)$. This concludes the proof.

Q.E.D.

5.5.2 Characterization results

The results proved in this section follows the general strategy of Mendelson (2009) and Hughes & Cresswell (1996). As we discussed in the beginning of the Section 5.5, it is necessary to treat modal formulas as predicates in non-normal worlds in order to preserve the behaviour of the quantifiers. In order to show that the semantic definitions given in Definition 5.5.2 does not affect the quantifiers, we prove the following results:

Proposition 5.5.12. *If the variables of a term t occur in the list $x_{i_1}, \ldots, x_{i_n}$, and if s and s' have the same components in $ith_1, \ldots, ith_n$ places of the sequence, then $s^*(t) = s'^*(t)$.*

Proof. The proof runs by complexity of the terms.

t is a variable x_{i_j}, where $1 \leq j \leq n$. By assumption s and s' have the same components in $ith_1, \ldots, ith_n$ places. Then $s^*(x_{i_j}) = s_i = s'_i = s'^*(x_{i_j})$.

If t is a constant x_{i_j}, the result is immediate.

If t is a function term of the form $f^k_m(x_{i_1}, \ldots, x_{i_n})$, then:

$$\begin{aligned} s^*(f^k_m(x_{i_1}, \ldots, x_{i_n})) &= (f^k_m)^{\mathfrak{M}}(s^*(x_{i_1}), \ldots, s^*(x_{i_n})) \\ &= (f^k_m)^{\mathfrak{M}}(s'^*(x_{i_1}), \ldots, s'^*(x_{i_n})) \\ &= s'^*(f^k_m(x_{i_1}, \ldots, x_{i_n})) \end{aligned}$$

This concludes the proof. Q.E.D.

Lemma 5.5.13. *If the variables of φ occurr in the list $x_{i_1}, \ldots, x_{i_n}$ and if the sequences s and s' have the same components in $ith_1, \ldots, ith_n$ places of the sequence, then*

$$\mathfrak{M}, w \models_s \varphi \text{ iff } \mathfrak{M}, w \models_{s'} \varphi. \tag{5.12}$$

For all $\varphi \in \mathcal{L}^{\Box}_{\mathsf{FOL}}$, for all $w \in W$.

Proof. If $\varphi = P^k_m(x_{i_1}, \ldots, x_{i_n})$, then:

$\mathfrak{M}, w \models_s P^k_m(x_{i_1}, \ldots, x_{i_n})$ iff $(s^*(x_{i_1}), \ldots, s^*(x_{i_n}), w) \in (P^k_m)^{\mathfrak{M}}$ iff (Proposition 5.5.12) $(s'^*(x_{i_1}), \ldots, s'^*(x_{i_n}), w) \in (P^k_m)^{\mathfrak{M}}$ iff $\mathfrak{M}, w \models_{s'} P^k_m(x_{i_1}, \ldots, x_{i_n})$.
If $\varphi = \forall x_j \psi$, then:

$\mathfrak{M}, w \models_s \forall x_j \psi$ iff $\mathfrak{M}, w \models_{s''} \psi$ for every $s'' \in Seq$ which differs from s in at most in the ith component iff (by IH) $\mathfrak{M}, w \models_{s''} \psi$ for every $s'' \in Seq$ which differs from s' in at most in the ith component iff $\mathfrak{M}, w \models_{s'} \forall x_j \psi$.

If $\varphi = \Box\psi$ and $w \in N$, then:

$\mathfrak{M}, w \models_s \Box\psi$ iff $\mathfrak{M}, y \models_s \psi$ for every $y \in W$ such that wRy iff $\mathfrak{M}, y \models_{s'} \psi$ for every $y \in W$ such that wRy iff $\mathfrak{M}, w \models_{s'} \Box\psi$.

If $w \notin N$, then:

$\mathfrak{M}, w \models_s \Box\psi$ iff $(s^*(x_{i_1}), \ldots, s^*(x_{i_n}), w) \in (\psi)^{\mathfrak{M}}$ iff (by I.H.) $(s'^*(x_{i_1}), \ldots, s'^*(x_{i_n}), w) \in (\psi)^{\mathfrak{M}}$ iff $\mathfrak{M}, w \models_{s'} \Box\psi$. Q.E.D.

Definition 5.5.14. *Let t be a term and φ be a formula. We say that t* is free for x_i *in φ if no free occurrence of x_i in φ lies within the scope of any quantifier $\forall x_j$, where x_j is a variable in t.*

Lemma 5.5.15. *If t and u are terms, $s \in Seq$, t' results from t by replacing all occurrences of x_i by u, and s' results from s by replacing the ith component of s by s^*, then $s^*(t') = s'^*(t)$.*

Proof. The proof runs by induction in the complexity of terms. If t is a constant, then it is not free and the result is trivial.

If t is a variable x_i, then $s^*(u/x_i) = s'_i$ and $s'^*(x_i) = s'_i$. Therefore, $s^*(u) = s'^*(x_i)$.

If t is a variable x_j, for $j \neq i$. Therefore, $s^*(x_j) = s'^*(x_j)$.

If t is a $f^n_k(t_1, \ldots, t_n)$, then:

$$\begin{array}{rcll} s^*(f^n_k(t'_1, \ldots, t'_i, \ldots, t'_n) & = & (f^n_k)^{\mathfrak{M}}(s^*(t'_1), \ldots, s^*(t'_i), \ldots, s^*(t'_n)) & \text{Def.} \\ & = & (f^n_k)^{\mathfrak{M}}(s'^*(t_1), \ldots, s^*(u), s'^*(t_n)) & \text{I.H.} \\ & = & s'^*(f^k_n(t_1, \ldots, t_i, \ldots, t_n)) & \text{Def.} \end{array}$$

This concludes the proof. Q.E.D.

Lemma 5.5.16. *Let t be free for x_i in $\varphi(x_i)$. Then $\mathfrak{M}, w \models_s \varphi(t)$, where $s = (s_1, \ldots, s_i, \ldots)$ iff $\mathfrak{M}, w \models_{s'} \varphi(t)$, where s' is obtained from s by substituting $s^*(t)$ for s_i in the ith-place, for all $w \in W$, for all formulas.*

Proof. The proof goes by induction on the complexity of formulas.

$\varphi(t)$ is $P^n_k(t_1, \ldots, t, \ldots, t_n)$. Then:
$\mathfrak{M}, w \models_s P^n_k(t_1, \ldots, t, \ldots, t_n)$ iff $(s^*(t_1), \ldots, s^*(t), \ldots, s^*(t_n)) \in (P^n_k)^{\mathfrak{M}}$ iff (by Lemma 5.5.15) $s^*(t_1), \ldots, s^*(t), s_i/s^*(t_n) \in (P^n_k)^{\mathfrak{M}}$ iff $\mathfrak{M}, w \models_s P^n_k(t_1, \ldots, x_i, \ldots, t_n)$.

The case of boolean connectives is straightforward. We focus on the modal and on the quantifiers.

$\varphi(t)$ is $\Box\psi(t)$. Then we have two cases to analyse: when $w \in N$ and $w \notin W$. When $w \notin W$, the verification is similar to the atomic case since in modal formulas are evaluated as atomic formulas. Then we will focus in the case where $w \in N$.
$\mathfrak{M}, w \models_s \Box\psi(t)$ iff for every $y \in W$, wRy, $\mathfrak{M}, y \models_s \psi(t)$ iff (by I.H.) $\mathfrak{M}, y \models_{s'} \psi(t)$ iff $\mathfrak{M}, w \models_{s'} \Box\psi(t)$.

$\varphi(t)$ is $\forall x_i\psi(t)$. Then, we have two cases: when $x_i = t$ and $x_i \neq t$. In the first case, the result is trivial because t would not be free.

Consider the case where $x_i \neq t$. Suppose that $\mathcal{M}, w \models_s \forall x_i\psi(t)$, then $\mathcal{M}, w \models_{s''} \psi(t)$ for every sequence s'' which differs from s in the ith place. For each s'' we obtain a sequence s''' by substituting $s''^*(t)$ by s_i. Since t does not contain x_i, by Lemma 5.5.13, we obtain $s''^*(t) = s'''^*(t)$. By induction, $\mathfrak{M}, w \models_s \psi(t)$. Then we obtain that $\mathcal{M}, w \models_s \forall x_i\psi(s_i)$, because the sequences also satisfy $\psi(x_i)$. The converse is similar.

Q.E.D.

Again, we see that this modified version of Lemma 5.5.16 guarantees the validity of (Ax4). So, the models for QS0.5 keep intact the behaviour of the quantifiers. We now prove that QS0.5 is sound with respect to its semantic definition given in Definition 5.5.2.

Theorem 5.5.17. *If* $\vdash_{\mathsf{QS0.5}} \varphi$ *then* $\models_{\mathsf{QS0.5}} \varphi$.

Proof. Here we will deal only with Barcan Formula, since the validity of the axioms K, T and the rule N' was proved in Theorem 5.4.10 and the validity of non-modal FOL axioms are sound with-respect to the non-modal clauses of Definition 5.4.8.

Suppose that $\mathfrak{M}, w \models_s \forall x_i\Box\varphi$, for every model $\mathfrak{M}$, for every $w \in N$ and every sequence s. Then, $\mathfrak{M}, w \models_{s'} \Box\varphi$ for every sequence s' which differs from s in the ith element of the sequence. By the semantic definition of $\Box$, $\mathfrak{M}, y \models_{s'} \varphi$ for all $y \in W$ such that wRy, for all s'. So we obtain $\mathfrak{M}, y \models_s \forall x_i\varphi$, for all $y \in W$ such that wRy. Therefore, $\mathfrak{M}, w \models_s \Box\forall x_i\varphi$. Q.E.D.

Definition 5.5.18. *If* x_i *and* x_j *are distinct, then* $\varphi(x_i)$ *and* $\varphi(x_j)$ *are said to be* similar *iff* x_j *is free for* x_i *in* $\varphi(x_i)$ *and* $\varphi(x_i)$ *does not have free occurrences of* x_j.

Theorem 5.5.19. *If $\varphi(x_i)$ and $\varphi(x_j)$ are similar, then $\vdash \forall x_i\varphi(x_i) \leftrightarrow \forall x_j\varphi(x_j)$.*

Proof. Consider the following derivation.

1.	$\vdash \forall x_j\varphi(x_j) \rightarrow \varphi(x_i)$	Ax4
2.	$\vdash \forall x_i(\forall x_j\varphi(x_j) \rightarrow \varphi(x_i))$	Gen 1
3.	$\vdash \forall x_i(\forall x_j\varphi(x_j) \rightarrow \varphi(x_i)) \rightarrow (\forall x_j\varphi(x_j) \rightarrow \forall x_i\varphi(x_i))$	Ax5
4.	$\vdash \forall x_j\varphi(x_j) \rightarrow \forall x_i\varphi(x_i)$	MP 2,3
5.	$\vdash \forall x_i\varphi(x_j) \rightarrow \varphi(x_j)$	Ax4
6.	$\vdash \forall x_j(\forall x_i\varphi(x_j) \rightarrow \varphi(x_j))$	Gen 5
7.	$\vdash \forall x_j(\forall x_j\varphi(x_i) \rightarrow \varphi(x_j)) \rightarrow (\forall x_i\varphi(x_i) \rightarrow \forall x_j\varphi(x_j))$	Ax5
8.	$\vdash \forall x_i\varphi(x_i) \rightarrow \forall x_j\varphi(x_j)$	MP 6,7
9.	$\vdash \forall x_j\varphi(x_j) \leftrightarrow \forall x_i\varphi(x_i)$	CPL 5,8

This concludes the proof. Q.E.D.

We now state a theorem which will be useful further.

Theorem 5.5.20. *The following formulas are* FOL*-theorems:*

(A) $\vdash \forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$;

(B) $\vdash \forall x(\varphi \wedge \psi) \rightarrow (\forall x\psi \wedge \varphi)$;

(C) $\vdash \forall x(\varphi \wedge \psi) \rightarrow (\forall x\varphi \wedge \forall x\psi)$;

(D) $\vdash \forall x\varphi \leftrightarrow \neg\exists x\varphi$

Proof. (A) Suppose that the variable x_i occurs free in φ. Then, we have the following derivation:

1.	$\vdash \forall x_i(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x_i\psi)$	Ax5
2.	$\vdash \forall x_i\varphi \rightarrow \varphi$	Ax4
3.	$\vdash (\forall x_i\varphi \rightarrow \varphi) \rightarrow ((\varphi \rightarrow \forall x_i\psi) \rightarrow (\forall x_i\varphi \rightarrow \forall x_i\psi))$	CPL
4.	$\vdash (\varphi \rightarrow \forall x_i\psi) \rightarrow (\forall x_i\varphi \rightarrow \forall x_i\psi)$	MP 2,3
5.	$\vdash \forall x_i(\varphi \rightarrow \psi) \rightarrow (\forall x_i\varphi \rightarrow \forall x_i\psi)$	CPL 1,4

For (B), consider the following derivation:

1.	$\vdash (\varphi \wedge \psi) \rightarrow \varphi$	CPL
2.	$\vdash (\varphi \wedge \psi) \rightarrow \psi$	CPL
3.	$\vdash \forall x_i((\varphi \wedge \psi) \rightarrow \varphi)$	Gen 1
4.	$\vdash \forall x_i((\varphi \wedge \psi) \rightarrow \psi)$	Gen 2
5.	$\vdash \forall x_i((\varphi \wedge \psi) \rightarrow \varphi) \rightarrow (\forall x_i(\varphi \wedge \psi) \rightarrow \forall x_i\varphi)$	Th. 5.5.20 (A)
6.	$\vdash \forall x_i((\varphi \wedge \psi) \rightarrow \psi) \rightarrow (\forall x_i(\varphi \wedge \psi) \rightarrow \forall x_i\psi)$	Th. 5.5.20 (A)
7.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow \forall x_i\varphi$	MP 3,5
8.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow \forall x_i\psi$	MP 4,6
9.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow (\forall x_i\varphi \wedge \forall x_i\psi)$	CPL 7,8

For (C), consider the following derivation:

1.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow (\forall x_i\varphi \wedge \forall x_i\psi)$	Thm 5.5.20
2.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow \forall x_i\varphi$	CPL 1
3.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow \forall x_i\psi$	CPL 1
4.	$\vdash \forall x_i\varphi \rightarrow \varphi$	Ax4
5.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow \varphi$	CPL 2,4
6.	$\vdash \forall x_i(\varphi \wedge \psi) \rightarrow (\forall x_i\psi \wedge \varphi)$	CPL 3,5

(D) is left for the reader. This concludes the proof. Q.E.D.

Definition 5.5.21. *A* closed term *t is a term without variables A theory T is a* scapegoat theory *if, for any $\varphi(x)$ that has x as its only free variable, there is a closed term t such that*

$$\exists x\neg\varphi(x) \rightarrow \neg\varphi(t).$$

So its proof in QS0.5 is exactly the same as in the (non-modal) quantificational case. Our proof is based on Mendelson (2009)'s proof, but the only difference is that we do not rely on Deduction Theorem, because its original formulation does not hold in general for modal logics. Instead, we use Hughes & Cresswell's (1996) definition of consistency 4.5.17.

Lemma 5.5.22. *Every consistent theory T has a consistent extension T' such that T' is a scapegoat theory and T' contains denumerably many closed items.*

Proof. First, we add the denumerable set $\{b_1, b_2 \ldots\}$ of individual constants to the language of QS0.5. The resulting theory is called $\mathsf{QS0.5}_0$ with the extended language $\mathcal{L}^{+}_{\mathsf{QS0.5}}$, which has all the axioms of QS0.5, which now also involve the new constants.

Fact 5.5.23. $\mathsf{QS0.5}_0$ *is consistent.*

Proof of the Fact 5.5.23. Suppose that $\mathsf{QS0.5}_0$ is inconsistent. Then there are $\gamma_1, \ldots, \gamma_n$ of $\mathsf{QS0.5}_0$ such that

$$\vdash_{\mathsf{QS0.5}_0} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n) \tag{5.13}$$

In the proof 5.13 we replace the occurrence of individual constants $b_i \in \{b_1, \ldots\}$ by a variable which does not occur in the proof. The resulting proof will be a proof in the original QS0.5, which we know to be consistent. Contradiction. Then, $\mathsf{QS0.5}_0$ is consistent. *End of the proof of Fact 5.5.23.*

Let $\psi_1(x_{i_1}), \psi_2(x_{i_2}), \ldots, \psi_k(x_{i_k}), \ldots$ be an enumeration of all formulas of $\mathsf{QS0.5}^{+}$ which have one free variable. Such enumeration is possible because the language $\mathsf{QS0.5}_0$ is enumerable. Let now $b_{i_1}, b_{i_2}, \ldots, b_{i_k}, \ldots$ be a sequence of such new individual constants such that each b_{i_k} does not occur in the formulas $\psi_k(x_{i_k})$ and each b_{i_k} differs from each other. Consider the formula

$$(S_k)\ \exists x_{i_k}\neg\psi_k(x_{i_k}) \rightarrow \neg\psi(b_{i_k})$$

Now we define the theories $\mathsf{QS0.5}_n$ as follows:

$$\mathsf{QS0.5}_n = \mathsf{QS0.5}_{n-1} \cup \{\exists x_{i_n}\neg\psi_n(x_{i_n}) \rightarrow \neg\psi(b_{i_n})\}$$

$\mathsf{QS0.5}_\omega = \bigcup_{n\in\omega} \mathsf{QS0.5}_n$. To prove that $\mathsf{QS0.5}_\omega$ is consistent, we prove by induction on n that each $\mathsf{QS0.5}_n$ is consistent. When $n = 0$, we have that $\mathsf{QS0.5}_0$ is consistent by Fact 5.5.23.

Suppose that $\mathsf{QS0.5}_{n-1}$ is consistent and that $\mathsf{QS0.5}_n$. Then, by Definition 4.5.17 there are formulas $\gamma_1, \ldots, \gamma_m$ of $\mathsf{QS0.5}_{n-1}$ such that:

1.	$\vdash \neg(\gamma_1 \wedge \ldots \wedge \gamma_m \wedge (\exists x_{i_n}\neg\psi_k(x_{i_n}) \rightarrow \neg\psi(b_{i_n})))$	Def. 4.5.17
2.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \neg(\exists x_{i_n}\neg\psi_k(x_{i_n}) \rightarrow \neg\psi(b_{i_n}))$	CPL 1
3.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow (\exists x_{i_n}\neg\psi_k(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$	CPL 2
4.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \exists x_{i_n}\neg\psi_k(x_{i_n})$	CPL 3
5.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \neg\neg\psi(b_{i_n})$	CPL 3
6.	$\vdash \neg\neg\psi(b_{i_n}) \rightarrow \psi(b_{i_n})$	CPL
7.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \psi(b_{i_n})$	CPL 5,6
8.	$\vdash \forall x_{i_n}(\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \psi(x_{i_n})$	Gen 7

Since $\gamma_1 \wedge \ldots \wedge \gamma_m$ do not have free occurrences of x_{i_n}, then

9.	$\vdash \forall x_{i_n}(\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \psi(x_{i_n}) \rightarrow ((\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \forall x_{i_j}\psi(x_{i_j}))$	*Ax 5*
10.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \rightarrow \forall x_{i_j}\psi(x_{i_j})$	MP 8,9
11.	$\vdash \forall x_{i_j}(x_{i_j}) \leftrightarrow \forall x_{i_n}\psi(x_{i_n})$	Th. 5.5.19
12.	$\vdash \neg(\gamma_1 \wedge \ldots \wedge \gamma_m)$	CPL 4,10

Which contradicts the consistency pf $\mathsf{QS0.5}_{n-1}$. Then, $\mathsf{QS0.5}_n$ is consistent. Since we are considering an arbitrary n, then every $\mathsf{QS0.5}_n$ is consistent. Therefore, $\mathsf{QS0.5}_\omega$ is consistent. Q.E.D.

Definition 5.5.24. *Let $w \subseteq For(\mathcal{L}_{\mathsf{QS0.5}})$ be a set of* QS0.5 *formulas. We say that w is* FOL-consistent *if there is no $\{\varphi_1, \ldots, \varphi_n\} \subseteq w$ such that $\vdash \neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$ and each $\varphi_i \in w$ is a substitution instance of a* FOL*-validity.*

Definition 5.5.25. *The canonical model for* QS0.5 *in the extended language $\mathcal{L}^+_{\mathsf{QS0.5}}$ is a structure of the form $\langle W, N, R, D, (\cdot)^{\mathfrak{M}}\rangle$ where:*

1. *N is the set of normal worlds where each $w \in N$ is a maximal consistent set of formulas of $\mathcal{L}^+_{\mathsf{QS0.5}}$ and w is a scapegoat theory;*

2. *Each $y \in W - N$ is a maximal* FOL*-consistent set of formulas and y is a scapegoat theory;*

3. *R is an accessibility relation defined in normal worlds defined as for propositional* $\mathsf{S0.5}$*;*

4. *D is the set of closed terms of* $\mathcal{L}^{+}_{\mathsf{QS0.5}}$*;*

5. *The interpretation function $(\cdot)^{\mathfrak{M}}$ is defined as follows:*

   (a) *if t is a constant c_i, then $(c_i)^{\mathfrak{M}} = c_i$;*

   (b) *if t is $f^n_k(t_1, \ldots, t_n)$ and $t_1, \ldots, t_n$ are closed terms, then $(f^n_k(t_1, \ldots, t_n))^{\mathfrak{M}} = f^n_k(t_1, \ldots, t_n)$.*

6. *Let P^n_k be a n-ary predicate and $t_1, \ldots, t_n$ closed terms, then $((t_1)^{\mathfrak{M}}, \ldots, (t_n)^{\mathfrak{M}}) \in (P^n_k)^{\mathfrak{M}}$ iff $P^n_k(t_1, \ldots, t_n) \in w$.*

7. *if $w \in W - N$ then:*

   (a) *$\mathfrak{M}, w \models \Box\psi$ iff $((t_1)^{\mathfrak{M}}, \ldots, (t_n)^{\mathfrak{M}}) \in (\psi)^{\mathfrak{M}}$;*

   (b) *$\mathfrak{M}, w \models \Diamond\psi$ iff $((t_1)^{\mathfrak{M}}, \ldots, (t_n)^{\mathfrak{M}}) \in (\psi)^{\mathfrak{M}}$.*

Theorem 5.5.26. *Let $w \in N$. If w is a maximal consistent set of formulas of modal predicate logic and w is a scapegoat theory, and φ is a formula such that $\Box\varphi \notin w$, then there is a* FOL*-consistent set y which is a scapegoat theory such that $\lambda(w) \cup \{\neg\varphi\} \subseteq y$.*

Proof. First we define the sequence of formulas $\varphi_0, \varphi_1, \ldots$, where $\varphi_0 = \neg\psi$. Given φ_n we define φ_{n+1} as follows:

$$\varphi_{n+1} := \varphi_n \wedge (\exists x_{i_n} \neg\psi_n(x_{i_n}) \rightarrow \neg\psi(b_{i_n}))$$

Where b_{i_n} is the first individual constant such that

$$\lambda(w) \cup \{\varphi_{n+1}\} \tag{5.14}$$

is consistent. Since w is a scapegoat theory, b_{i_n} is not new in $\lambda(w)$. Even so, we can show that for every n there is such b_{i_n}. When $n = 0$, we have $\lambda(w) \cup \{\neg\psi\}$ which follows as in the propositional case. We now prove that $\lambda(w) \cup \{\varphi_n\}$ is consistent, for every n. For the inductive case, suppose that there is no such b_{i_n}. Then for every b_{i_n} there is a $\{\beta_1, \ldots, \beta_m\} \subseteq \lambda(w)$ such that:

1.	$\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_m \wedge \varphi_n \wedge (\exists x_{i_n} \neg\psi(x_{i_n}) \rightarrow \neg\psi(b_{i_n})))$	Def. 4.5.17
2.	$\vdash (\beta_1 \wedge \ldots \wedge \beta_m \wedge \varphi_n) \rightarrow \neg(\exists x_{i_n} \neg\psi(x_{i_n}) \rightarrow \neg\psi(b_{i_n}))$	CPL 1
3.	$\vdash \neg(\exists x_{i_n} \neg\psi(x_{i_n}) \rightarrow \neg\psi(b_{i_n})) \rightarrow (\exists x_{i_n} \neg\psi_n(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$	CPL
4.	$\vdash (\beta_1 \wedge \ldots \wedge \beta_m \wedge \varphi_n) \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$	CPL 2,3
5.	$\vdash (\beta_1 \wedge \ldots \wedge \beta_m) \rightarrow (\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$	CPL 4
6.	$\vdash \Box(\beta_1 \wedge \ldots \wedge \beta_m) \rightarrow \Box(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$	RK' 5
7.	$\vdash \Box(\beta_1 \wedge \ldots \wedge \beta_m) \rightarrow (\Box\beta_1 \wedge \ldots \wedge \Box\beta_m)$	Th 5.4.7 (E)
8.	$\vdash (\Box\beta_1 \wedge \ldots \wedge \Box\beta_m) \rightarrow \Box(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$	CPL 6,7

Since $\beta_1, \ldots, \beta_m \in \lambda(w)$, then $\Box\beta_1 \wedge \ldots \Box\beta_m \in w$. Then, we have: Q.E.D.

9. $\vdash \Box(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(b_{i_n}))$ MP

Let x_j be a variable which does not occur in φ_n, $\exists x_{i_n} \neg\psi(x_{i_n})$ and $\psi(b_{i_n})$. By Gen:

10. $\vdash \forall x_j \Box(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)))$
11. $\vdash \forall x_j \Box(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j))) \rightarrow \Box\forall x_j(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)))$
12. $\vdash \Box\forall x_j(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)))$

Where step 11 is an instance of (BF) and 12 is obtained by MP from 10 and 11. Since x_j does not occur in φ_n, then we have the following instance of $(Ax5)$

13. $\vdash \forall x_j(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j))) \rightarrow (\varphi_n \rightarrow \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)))$

Since the formula of step 13 is a substitution instance of a QS0.5-axiom, then it is FOL-valid (in the language $\mathcal{L}_{\mathsf{QS0.5}}$). Then we can apply the rule RK' to this formula. So:

14. $\vdash \Box\forall x_j(\varphi_n \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j))) \rightarrow \Box(\varphi_n \rightarrow \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)))$

15. $\vdash \Box(\varphi_n \rightarrow \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)))$ MP 12,14
16. $\vdash \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)) \rightarrow (\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \forall x_j \neg\neg\psi(x_j))$ Th 5.5.20 (C)

Because $\neg\neg\psi(x_j)$ and $\neg\neg\psi(x_{i_n})$ are similar, we obtain:

17. $\vdash \forall x_j \neg\neg\psi(x_j) \leftrightarrow \forall x_{i_n} \neg\neg\psi(x_{i_n})$ Thm 5.5.19
18. $\vdash \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)) \rightarrow \exists x_{i_n} \neg\psi(x_{i_n})$ CPL 16
19. $\vdash \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)) \rightarrow \forall x_j \neg\neg\psi(x_j)$ CPL 16
20. $\vdash \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)) \rightarrow \forall x_{i_n} \neg\neg\psi(x_{i_n})$ CPL 17, 19
21. $\vdash \forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)) \rightarrow (\forall x_{i_n} \neg\neg\psi(x_{i_n}) \wedge \exists x_{i_n} \neg\psi(x_{i_n}))$ CPL 19, 20
22. $\vdash \Box(\forall x_j(\exists x_{i_n} \neg\psi(x_{i_n}) \wedge \neg\neg\psi(x_j)) \rightarrow (\forall x_{i_n} \neg\neg\psi(x_{i_n}) \wedge \exists x_{i_n} \neg\psi(x_{i_n})))$ N^{FOL} 21
23. $\vdash \Box(\varphi_n \rightarrow (\forall x_{i_n} \neg\neg\psi(x_{i_n}) \wedge \exists x_{i_n} \neg\psi(x_{i_n})))$ Th 5.4.7 (C)
24. $\vdash \Box\neg\varphi_n$ Th 5.4.7 23

Which contradicts the consistency of w. Then $\lambda(w) \cup \{\varphi_{n+1}\}$ is consistent. Now, let y be the union of all $\lambda(w)$ and φ_n, for every n. Since, for every n, $\lambda(w) \cup \{\varphi_n\}$ is consistent. For every i and j, if $i \geq j$, then $\vdash \varphi_i \rightarrow \varphi_j$. Then their union is consistent. Then the result follows.

Lemma 5.5.27. *For any scapegoat theory $w \in W$, any sentence φ of $\mathcal{L}^{+}_{\mathsf{QS0.5}}$:*

$$\mathfrak{M}, w \models \varphi \textit{ iff } \varphi \in w$$

Proof. The proof runs by induction on φ. When φ is atomic the result follows by Definition 5.5.25. The boolean cases are simple. Thus, we will concentrate in the cases where $\varphi = \forall x_j \psi$ and $\varphi = \Box\psi$

$\varphi = \forall x_j \psi$. We have to cases to analuse: when (A) ψ is closed and (B) the case where ψ is open.

A. Suppose that $\forall x_j \psi \in w$. By *Ax4*, $\forall x_j \psi \rightarrow \psi \in w$. Then, by MP, $\psi \in w$. By I.H., $\mathfrak{M}, w \models \psi$. Since ψ is closed, then every sequence s satifies ψ. Then, $\mathfrak{M}, w \models \forall x_j \psi$.

Suppose now that $\forall x_j \psi \notin w$. Since w is maximal consistent, we obtain $\neg\forall x_j \psi \in w$. By Theorem 5.5.20 (E), we have $\neg\forall x_i \psi \leftrightarrow \exists x_j \neg\psi \in w$. Then we have that $\exists x_j \neg\psi \in w$. Because w is a scapegoat theory, $\neg\psi \in w$. Because w is consistent, $\psi \notin w$. By I.H., $\mathfrak{M}, w \nvDash \psi$. Therefore, $\mathfrak{M}, w \nvDash \forall x_j \psi$.

B. If ψ is open, then the variable x_j is free in ψ, since φ is a sentence. Suppose that $\mathcal{M}, w \models \forall x_j \psi$ and consider that $\forall x_j \psi \notin w$. Because w is complete, we have $\neg\forall x_j \psi \in w$. By Theorem 5.5.20 (E), we have $\neg\forall x_i \psi \leftrightarrow \exists x_j \neg\psi \in w$. Then, $\exists x_j \neg\psi \in w$. Because w is a scapegoat theory, $\neg\psi \in w$. Since $\mathcal{M}, w \models \forall x_j \psi$, we obtain $\mathcal{M}, w \models \psi$ because $\mathcal{M}, w \models \forall x_j \psi \rightarrow \psi$. By I.H., $\psi \in w$. Contradiction. Then if $\mathcal{M}, w \models \forall x_j \psi$, then $\forall x_j \psi \in w$.

Suppose that $\forall x_j \psi \in w$ and $\mathcal{M}, w \nvDash \forall x_j \psi$. By definition, there is a $s \in Seq$ such that the ith element does not satisfy ψ. $\mathfrak{M}, w \nvDash_s \psi$. By substituting $s^*(t)$ by s_i in the sequence s, where t is a closed term, we obtain a sequence s' such that by Lemma 5.5.16, $\mathfrak{M}, w \nvDash_{s'} \psi$. Since $\forall x_j \psi \in w$ and $\forall x_j \psi \rightarrow \psi \in w$, then $\psi \in w$. By I.H, $\psi \notin w$, contradicting the consistency w. Therefore, if $\forall x_j \psi \in w$, then $\mathcal{M}, w \models \forall x_j \psi$.

$\varphi = \Box\psi$. If $w \in N$, then suppose that $\Box\psi \in w$ and $\lambda(w) \subseteq y$. By $\Box\psi \rightarrow \psi \in w$, we obtain $\psi \in w$. Since $\lambda(w) \subseteq y$, for all $y \in W$, we have that $\psi \in y$. By I.H., $\mathfrak{M}, w \models \psi$ and $\mathfrak{M}, y \models \psi$ such that wRy. Then, $\mathfrak{M}, w \models \Box\psi$.

Suppose that $\Box\psi \notin w$. By maximality of w, $\neg\Box\psi \in w$. Then, by Lemma 5.5.26 there is a FOL-consistent set y which is a scapegoat theory such that $\lambda(w) \cup \{\neg\psi\} \subseteq y$. Then $\neg\psi \in y$ and $\psi \notin y$. By I.H. $\mathfrak{M}, w \nvDash \psi$ for wRy. Therefore, $\mathfrak{M}, w \nvDash \Box\psi$.

If $w \notin W$, the result is given by Definition 5.5.25. This concludes the proof. Q.E.D.

It is clear that the canonical model $\mathfrak{M}$ of $\mathsf{QS0.5}$ is denumerable, because the language $\mathcal{L}_{\mathsf{QS0.5}}$ is itself denumerable and $\mathcal{L}^{+}_{\mathsf{QS0.5}}$ is obtained by adding to $\mathcal{L}_{\mathsf{QS0.5}}$ a denumerable set $\{b_1, b_2, \ldots\}$ of constants. The next proposition is a direct consequence of Lindenbaum Lemma and Lemmas 5.5.22, 5.5.26 and 5.5.27.

Proposition 5.5.28. *Every consistent theory T has a model.*

The proof of Proposition 5.5.28 can be consulted in (MENDELSON, 2009, pp. 88, Proposition 2.17).

Theorem 5.5.29. *If φ is logically valid, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Then by Proposition 4.5.16 (1), $\varphi \notin w$, where w is a maximal consistent set of sentences. Since w is a scapegoat theory in the extended language $\mathcal{L}^{+}_{\mathsf{QS0.5}}$, then by Proposition 5.5.28, $\mathfrak{M}, w \nvDash \varphi$, where $\mathfrak{M}$ is a denumerable model. Q.E.D.

Chapter 6

Logical validity for non-classical logics

This work concentrates in describing the concept of consistency by means of modal logics. But it is clear that we are dealing with this concept being defined in the classical setting. Thus, one might wonder whether these properties that logical consistency exhibits would remain we changed logic. More precisely: what are the properties that logical validity and logical consistency have in a non-classical logics? We think that looking some fragments of classical logic may shed light into this question. Since they are fragments of classical logic, some principles of logical validity and logical consistency may fail in these logics. But, if some principles remain, then we have reason to think that this remaining one constitutes the core of logical validity. As a methodological choice, we now analyze the case of a very popular family of non-classical logics: the *many-valued logics* (MVLs).

The last few years witnessed a growing interest in MVL in the philosophical literature. Their applications in the analysis of semantical paradoxes (*e.g*, Priest (1979), da Ré et al (2018)) and in the study of rationality (*e.g.*, Belnap (1977), Kubyshkina & Zaitsev (2016), Bezerra (2020)) show how interesting those logics can be for clarifying some philosophical problems. Such applications, even if indirectly, respond to a very common objection against these logics, which is about the interpretation of the intermediate values.

Despite its philosophical interest, those logics were challenged due to their *metatheoretical bivalence*. As Suszko (1977) observes, the concepts of tautology and logical consequence only take into account whether a value t is *designated* (truth-like) or *non-designated* (false-like). In other words, it is the bipartition of the set of truth values that is preponderant in the analysis of the concepts of tautology and logical consequence, not the many truth values that a MVL can have.

In this Chapter we extend the analysis done in Chapter 5 to MVLs. Those modalities $\Box$ and $\Diamond$ are intended to respectively interpret the concepts of "it is logically valid that" and "it is logically consistent that" can be called *suszkian modalities*, because they only take into account designatedness of the truth-values at issue. By analysing those modalities in

a wide family of logics, we can know what are the most general properties those concepts may have in a modal framework.

First, we introduce the family of many-valued modal logics $\mathfrak{L}^{\mathsf{S0.5}}$, which results from the addition of $\mathsf{S0.5}$ modalities to the MVL L's, and then its characterization results. These logics follow the same intuition as $\mathsf{S0.5}$. Then, we introduce a stronger modality to logics L, still bivalent, which intends to capture a more hierarchical notion of validity, such as investigated by Skyrms (1978). This family of logics will be denoted by $\mathfrak{L}^{\mathsf{S5}}$, because they are extended with the modality of classical $\mathsf{S5}$. Second, we discuss whether validity theories based on MVLs which use Gödel naming device are consistent. Third, we show how the modalities $\Box$ and $\Diamond$ allow us to talk about anti-validities and logical consistency. Fourth, we show that these modalities allow us to define recovery operators from a modal point of view. Fifth, we explore a validity theory based on the *Strict Tolerant Logic*. Last, we show characterization results for the logics of the families $\mathfrak{L}^{\mathsf{S0.5}}$ and $\mathfrak{L}^{\mathsf{S5}}$.

Before moving on, we state our minimal requirement about the logics we will deal with in this Chapter:

Assumption 6.0.1. *(RESCHER, 1969) A n-valued connective $c_m^{k_m}$ is called* normal *if its operation $o_m^{k_m}$ agrees with the two values ones when only the truth-values 1 and 0 are involved. A logic L is normal if its connectives are normal.*[1]

Then, the logics to be investigated here a normal in the sense of Assumption 6.0.1.[2] This assumption guarantees that when we have only classical values, the logic is classical propositional logic (CPL).

6.1 The modal logics $\mathfrak{L}^{\mathsf{S0.5}}$

Here, since we are dealing with a large family of systems, the language $\mathcal{L}$ will have the subscript of the corresponding logic L. Consider the following definition:

Definition 6.1.1. *Given a language $\mathcal{L}_{\mathsf{L}}$, we define its modal extension by $\mathcal{L}_{\mathsf{L}}^{\Box\Diamond} = \mathcal{L}_{\mathsf{L}} \cup \{\Box, \Diamond\}$.*

Definition 6.1.2. *Fix an n-valued normal logic L, with corresponding language $\mathcal{L}_{\mathsf{L}}$ and matrix $M_{\mathsf{L}} = \langle V_n, o_1^{k_1}, \ldots, o_m^{k_m}, D_{\mathsf{L}} \rangle$. An* M_{L}-modal model *is a structure of the form $\mathcal{M}_{\mathsf{L}} = \langle W, N, R, v \rangle$ where W is a set of worlds, $N \subseteq W$ is a set of normal worlds, R is a reflexive relation on N such that for every $y \in W$ there is $x \in N$ such that xRy, and v is an assignment such that for every $w \in W$, $v_w(p) \in V_n$. The function v is recursively extended in the standard way:*

[1] This concept does not coincide with normality of modal logics.

[2] We warn the reader that this sense of normality differs from normality in modal logics.

1 $v_w(c_m^{k_m}(\varphi_1, \ldots, \varphi_k)) = c_m^{k_m}(v_w(\varphi_1), \ldots, v_w(\varphi_k))$.

Now the interpretation of the modal operators runs as follows:

2.1 *For any* $w \in W$, $v_w(\Box\varphi) = 1$ *if* $w \in N$ *and for all* $y \in W$ *such that* wRy, $v_y(\varphi) \in D_{\mathsf{L}}$; *otherwise* $v_w(\Box\varphi) = 0$;

2.2 *If* $w \notin N$, *the value of* $v_w(\Box\varphi)$ *is arbitrary in* V_n;

3.1 *For any* $w \in W$, $v_w(\Diamond\varphi) = 1$ *if* $w \in N$ *and for some* $y \in W$ *such that* wRy, $v_y(\varphi) \in D_{\mathsf{L}}$; *otherwise* $v_w(\Diamond\varphi) = 0$;

3.2 *If* $w \notin N$, *the value* $v_w(\Diamond\varphi)$ *is arbitrary in* V_n.

A formula $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\Box\Diamond})$ *is true in a* M_{L}*-modal model* $\mathcal{M}$ *iff for every* $w \in N$ $v_w(\varphi) \in D_{\mathsf{L}}$. *A formula* $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\Box\Diamond})$ *is* M_{L}*-valid iff it is true in every* M_{L}*-modal model.*

Definition 6.1.3. *The* Suszkian modal counterpart of L, *that we indicate by* $\mathsf{L}^{\mathsf{S0.5}}$, *is the modal logic in the language* $\mathcal{L}_{\mathsf{L}}^{\Box\Diamond}$ *that consists of all* M_{L}*-valid formulas.*

Definition 6.1.4. *We denote the family of n-valued logics* $\mathsf{L}^{\mathsf{S0.5}}$ *as* $\mathfrak{L}^{\mathsf{S0.5}}$.

As we argued in the Subsection 5.4.1, once we apply $\Box$ or $\Diamond$ to φ, $\Box\varphi/\Diamond\varphi$ is no more tautological/logically consistent by the truth-table method. Then we cannot allow validities of the form $\Box \ldots \Box\varphi$. So, to block such valid iterations of modalities, it was necessary to adopt the division of the set W into normal worlds and non-normal worlds.

Note that our construction differs from Priest (2008b). There he takes the normal operators $\Box$ and $\Diamond$ and change the basic logic. Moreover, he allows formulas $\Box\varphi$ and $\Diamond\varphi$ to receive intermediate values.[3] Our definition, on the other hand, Definition 6.1.2 imposes that these formulas can only be true or false in worlds $w \in N$, i.e., worlds which determine the validity in models.[4] In a metatheoretical point of view, our proposal makes sense because a formula is designated or not.[5] They are, in some sense, two-valued (SUSZKO, 1977). In this sense, one may call $\Box$ and $\Diamond$ of *suszkian modalities*.

Allowing $\Box\varphi$ and $\Diamond\varphi$ to receive intermediate values in normal worlds would give us principles which are not valid in our proposed interpretation for $\Box$ and $\Diamond$. Let us consider

[3]The logics investigated here can be seen as non-normal counterparts of the modal many-valued logics investigated by Priest (2008b).

[4]One can note that the modal operator $\Box$ in logics $\mathfrak{L}^{\mathsf{S0.5}}$ similarly works as a recovery operator, in the sense of Carnielli et al (2019) and Coniglio & Peron (2013). We will concentrante on this issue in Section 6.5.

[5]Schotch et al (1978) introduce a study of non-classically based modal logic based, where they consider the three-valued logic Ł$_3$. In this work, they provide an axiomatization in the class of all models for Ł$_3M_2$, which is obtained by extending Ł$_3$ to the modal language where formulas $\Box\varphi$ only receives classical values. They suggest that this logic captures the idea that the modal discourse is essentially two-valued.

such possibility of allowing intermediate values in a concrete example to see why it deviates from our proposal. Suppose for a moment that the clauses 2.1 and 3.1 of Definition 6.1.2 were defined for $w \in N$ as follows. Let *Glb* be the *greatest lower bound* and *Lub* be the *lowest upper bound*. :

2.1' $v_w(\Box\varphi) = Glb\{v_y(\varphi) \mid wRy\}$

3.1' $v_w(\Box\varphi) = Lub\{v_y(\varphi) \mid wRy\}$

Consider now the logic LP ((ASENJO, 1966), (PRIEST, 1979)), defined as follows:

Definition 6.1.5. LP *is characterized by the matrix* $\mathcal{M}_{\mathsf{LP}} = (\{1, \frac{1}{2}, 0\}, \neg, \vee, \{1, \frac{1}{2}\})$ *whose operations have the following truth-tables:*

$\vee$	*1*	$\frac{1}{2}$	*0*
1	*1*	*1*	*1*
$\frac{1}{2}$	*1*	$\frac{1}{2}$	$\frac{1}{2}$
0	*1*	$\frac{1}{2}$	*0*

	$\neg$
1	*0*
$\frac{1}{2}$	$\frac{1}{2}$
0	*1*

The connective $\rightarrow$ can be defined as $\varphi \rightarrow \psi := \neg\varphi \vee \psi$. Then, $\rightarrow$ has the following truth-table:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

Consider now that the M_{LP}-modal model for $\mathsf{LP}^{\mathsf{S0.5}}$ is a structure $\mathcal{M} = \langle W, N, R, v\rangle$ where the modal operators are defined according to the clauses 2.1' and 3.1'. Then the reader can easily check that the principle K is valid in the models $\mathcal{M}$ according to these models. The problem is that, as we will see in the Proposition 6.1.24, the definitions of $\Box$ and $\Diamond$ given by the clauses 2.1' and 3.1' say that validity is preserved under modus ponens. But the implication connective of LP does not validate such rule, as the following proposition shows:

Proposition 6.1.6. $\varphi, \varphi \rightarrow \psi \nvDash_{\mathsf{LP}} \psi$.

Proof. Consider a LP valuation $v \in sem_{\mathsf{LP}}$ such that $v(p) = \frac{1}{2}$ and $v(q) = 0$. By the truth-table of $\rightarrow$, we obtain $v(p \rightarrow q) = \frac{1}{2}$. Then, such valuation shows that modus ponens is not truth-preserving in LP. Q.E.D.

If $\Box$ is taken to interpret logical validity, then the clauses 2.1' and 3.1' cannot be taken to define the meaning of $\Box$ and $\Diamond$. These two latter clauses go in the contrary direction to Proposition 6.1.6. Then, we claim that a faithful modal interpretation of logical validity in many-valued logics must be defined like in Definition 6.1.2.

It is easy to see that when $n = 2$, $\mathsf{L}^{\mathsf{S0.5}}$ is just $\mathsf{S0.5}$, whose soundness and completeness were proved in Chapter 5. Here we will provide a tableaux proof system for the logics $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$, by adapting Carnielli (1987)'s labelled tableaux for many-valued logics to modal many-valued logics.

Notation 6.1.7. *We denote a truth-value* $\frac{m}{n-1}$*, for* $0 \leq m \leq n-1$*, by* t^m*.*

Definition 6.1.8. *Let* φ *be a formula and* $[t^m]$ *be a label, for* $t^m \in \mathcal{V}_n$ *and* $i \in \mathbb{N}$*. A* signed formula *has the form* $[t^m]\varphi, i$*.*

Definition 6.1.9. *Let* $[t^m]\varphi$ *be a signed formula and* $i \in \mathbb{N}$*. Given* $[t^m]\varphi, i$*, we construct a* tree *(a* tableau*) for* $[t^m]\varphi, i$ *as follows:*

(i) $[t^m]\varphi, i$ *is the* root*/initial* node *of the tree;*

(ii) *We expand the root of the tree into* branches *b by applying the rule for* $[t^m]\varphi, i$*. Every such b contains signed formulas resulting from the application of the rule for* $[t^m]\varphi, i$*, and possibly* $k\mathfrak{r}l$*, where* $\mathfrak{r}$ *is a rule and* $k, l \in \mathbb{N}$*, in the case that* φ *is a modal formula.*

(iii) *The* endpoints *of the tree are nodes which contains formulas for which there is no rule to be applied.*

The Definition 6.1.9 can be extended for sets of formulas Γ in the obvious way.

Definition 6.1.10. *Let* $\mathcal{T}$ *be a tableau and b be a branch of* $\mathcal{T}$*. We say that b is* complete *if every rule which can be applied is applied.* $\mathcal{T}$ *is complete if its branches are complete.*

Definition 6.1.11. *Let* $\mathcal{T}$ *be a tableau and b be a branch of* $\mathcal{T}$*. We say that b* closes *(i) if there is a formula* φ *such that* $[t^m]\varphi, j$ *and* $[t^l]\varphi, j$ *with* $m \neq l$ *occurring in b; (ii) if* $[t^m]\Box\varphi, 0$ *or* $[t^m]\Diamond\varphi, 0$ *for* $0 < m < n-1$ *occurs in b.*

Let $j \in \mathbb{N}$ and $0 \leq r, s \leq n-1$. The general rule of boolean connectives, the (c_n^k-rule) is defined as follows:

$$\frac{[t^m]c_m^{k_m}(\varphi_1, \ldots, \varphi_k), j}{\bigvee\{\bigwedge_r^s [t^r]\varphi, j \mid t^r \in V_n, j \in \mathbb{N}, v_j(c_m^{k_m}(t_1^r, \ldots, t_k^s)) = t^m\}}$$

The notations $\bigvee$ and $\bigwedge$ mean, respectively, the expansion into different branches and the expansion into a single branch; v_j is a homomorphism from the language $\mathcal{L}_{\mathsf{L}}$ of L to

the matrix M_{L}. This homomorphism must be required in c_n^k-rule in order to maintain the correspondence between the semantics of L and its proof system.

$$\frac{[t^{n-1}]\Box\varphi, 0}{\bigvee\{[t^m]\varphi, j \mid t^m \in D \subset V_n, 0 \leq b < m \leq n-1\}}(0\mathfrak{r}j)$$

where b is the greatest element of the set of non-designated values.

$$\frac{[t^0]\Box\varphi, 0}{\bigvee\{[t^m]\varphi, j \mid t^m \in V_n - D, 0 \leq m < r \leq n-1\}}(0\mathfrak{r}j, j \text{ new})$$

where r is the least designated value.

$$\frac{[t^{n-1}]\Diamond\varphi, 0}{\bigvee\{[t^m]\varphi, j \mid t^m \in D \subset V_n, 0 \leq b < m \leq n-1\}}(0\mathfrak{r}j, j \text{ new})$$

$$\frac{[t^0]\Diamond\varphi, 0}{\bigvee\{[t^m]\varphi, j \mid t^m \in V_n - D, 0 \leq m < r \leq n-1\}}(0\mathfrak{r}j)$$

The rule $\mathfrak{r}$ obbeys the following constraint:

$$\frac{\cdot}{0\mathfrak{r}0}(\text{rule } \rho)$$

In the usual labelled tableaux for non-modal propositional logics, the endpoints are only inhabited by atomic formulas. But in our tableaux, there are modal formulas which inhabit the endpoints and there is no rule to be applied to these formulas, specially formulas of the form $[t^m]M\varphi, i$ ($M \in \{\Box, \Diamond\}$), for $i > 0$. We will give an example of this when we present the logic $Ł_3^{\mathsf{S0.5}}$ (Example 6.1.13). Now we define the notion of proof.

Definition 6.1.12. $\Sigma \vdash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$ *if every tableaux* $\mathcal{T}$*'s satisfying the following four conditions close:*

1. *For each* $\sigma_i \in \Sigma$, $[t^m]\sigma_i, 0$, *where* $t^m \in D \subset V_n$;
2. *If* $\sigma_i \in \Sigma$ *and* $\sigma_i = \Box\psi$, *then* $[t^{n-1}]\Box\psi, 0$;
3. *If* $\sigma_i \in \Sigma$ *and* $\sigma_i = \Diamond\psi$, *then* $[t^{n-1}]\Diamond\psi, 0$;

4. $[t^j]\varphi, 0$*, where* $t^j \in V_n \setminus D$*.*

To illustrate the above definitions and rules consider the following example:

Example 6.1.13. *The modal three-valued Łukasiewicz logic* $\text{Ł}_3^{\mathsf{S0.5}}$ *is characterized by the structure* $\mathcal{M} = \langle W, R, v\rangle$ *whose connectives* $\neg$ *and* $\rightarrow$ *of* $M_{\text{Ł}_3} = \langle\{1, \frac{1}{2}, 0\}, \neg, \rightarrow, \{1\}\rangle$ *are interpreted by the following truth-tables:*

	$\neg$
1	*0*
$\frac{1}{2}$	$\frac{1}{2}$
0	*1*

$\rightarrow$	*1*	$\frac{1}{2}$	*0*
1	*1*	$\frac{1}{2}$	*0*
$\frac{1}{2}$	*1*	*1*	$\frac{1}{2}$
0	*1*	*1*	*1*

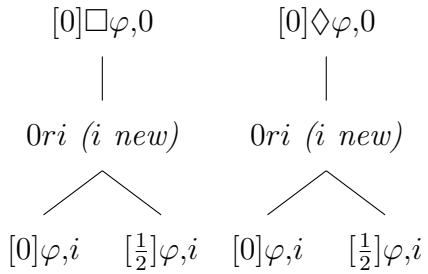
Based on the general tableau rules we presented above, the following rules define the deductive system for Ł_3*:*

$$\begin{array}{ccc} [1]\neg\varphi,i & [\tfrac{1}{2}]\neg\varphi,i & [0]\neg\varphi,i \\ | & | & | \\ [0]\varphi,i & [\tfrac{1}{2}]\varphi,i & [1]\varphi,i \end{array}$$

$$\begin{array}{c} [1]\varphi \rightarrow \psi,i \\ \diagup \quad \diagup \quad | \quad \diagdown \quad \diagdown \quad \diagdown \\ \begin{array}{cccccc} [1]\varphi,i & [\frac{1}{2}]\varphi,i & [\frac{1}{2}]\varphi,i & [0]\varphi,i & [0]\varphi,i & [0]\varphi,i \\ [1]\psi,i & [1]\psi,i & [\frac{1}{2}]\psi,i & [1]\psi,i & [\frac{1}{2}]\psi,i & [0]\psi,i \end{array} \end{array}$$

$$\begin{array}{cc} [\tfrac{1}{2}]\varphi \rightarrow \psi,i & [0]\varphi \rightarrow \psi,i \\ \diagup \quad \diagup & | \\ \begin{array}{cc} [1]\varphi,i & [\frac{1}{2}]\varphi,i \\ [\frac{1}{2}]\psi,i & [0]\psi,i \end{array} & \begin{array}{c} [1]\varphi,i \\ [0]\psi,i \end{array} \end{array}$$

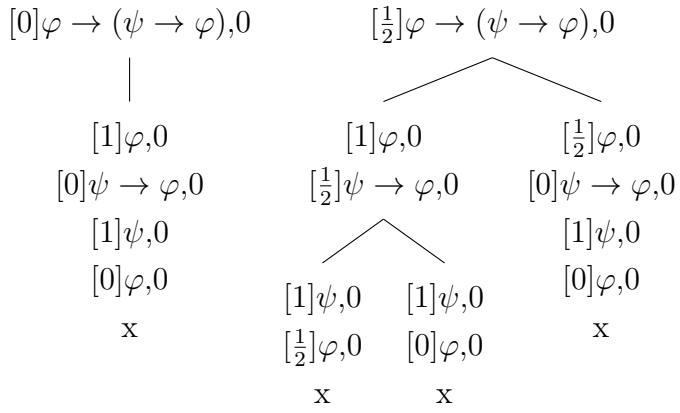
$$\begin{array}{cc} [1]\Box\varphi,0 & [1]\Diamond\varphi,0 \\ | & | \\ 0ri & 0ri \text{ (i new)} \\ [1]\varphi,i & [1]\varphi,i \end{array}$$



To see how the tableaux for $Ł_3^{S0.5}$ work, consider the following examples:

Theorem 6.1.14. $\vdash_{Ł_3^{S0.5}} \varphi \to (\psi \to \varphi)$.

Proof. Consider the following tableaux:



Since all tableaux close, then $\vdash_{Ł_3^{S0.5}} \varphi \to (\psi \to \varphi)$. This concludes the proof. Q.E.D.

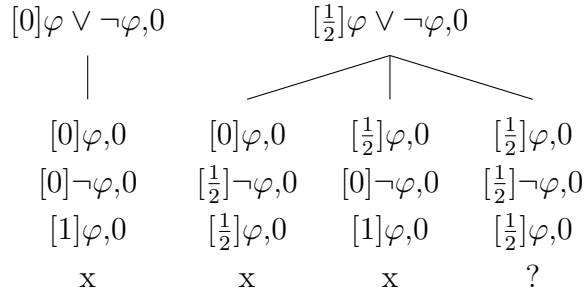
Now, consider following definition:

$$\varphi \vee \psi \quad := \quad (\varphi \to \psi) \to \psi$$

Then:

Theorem 6.1.15. $\nvdash_{Ł_3^{S0.5}} \varphi \vee \neg\varphi$.

Proof. Consider the following tableaux:



Since there is at least one open tableau for $\varphi \vee \neg\varphi$, then $\nvdash_{Ł_3^{S0.5}} \varphi \vee \neg\varphi$. This concludes the proof. Q.E.D.

To give a concrete example of endpoints inhabited by modal formulas, consider the following application of the rule for $[1]\Box\varphi, 0$:

$$\begin{array}{c} [1]\Box\neg(\Box p \to \Box\neg p),0 \\ | \\ 0r1 \\ [1]\neg(\Box p \to \Box\neg p),1 \\ [0]\Box p \to \Box\neg p,1 \\ [1]\Box p,1 \\ [0]\Box\neg p,1 \end{array}$$

Since there is no rule available for $[1]\Box p,1$ or $[1]\Box\neg p,1$, the verification stops and the branch is open. So the endpoints of the tableaux for $Ł_3^{\mathsf{S0.5}}$ in general will contain other formulas than atomic ones for which no rule can be applied.

The characterization results for the logics in $\mathfrak{L}^{\mathsf{S0.5}}$ will be proved in the subsection 6.7.1.

6.1.1 $\mathfrak{L}^{\mathsf{S0.5}}$ and (tauto)logical validity

In this subsection, we will define a *theory of validity and consistency* for L. The validity theory for a logic L is obtained by adding to the language of L *sentence names* $\overline{\varphi}$, in the sense of Skyrms (1978), for each $\varphi \in For(\mathcal{L}_\mathsf{L})$ and sentence predicates of validity Val and consistency Con.

Definition 6.1.16. *The set $For(\mathcal{L}_\mathsf{L}^{VC})$ is defined as follows:*

1. $For(\mathcal{L}_\mathsf{L}^{VC}) \subseteq For(\mathcal{L}_\mathsf{L}^{VC})$
2. *if $\varphi \in For(\mathcal{L}_\mathsf{L})$ and $\overline{\varphi}$ is a sentence name of φ, then $Val(\overline{\varphi}) \in For(\mathcal{L}_\mathsf{L}^{VC})$ and $Con(\overline{\varphi}) \in For(\mathcal{L}_\mathsf{L}^{VC})$.*

Consider the following definition:

Definition 6.1.17. *Let $\varphi \in For(\mathcal{L}_\mathsf{L}^{VC})$. We say that φ is a* quasi-atomic formula *if φ has one of the following forms:*

1. $\varphi = p \in \mathcal{V}$;
2. $\varphi = Val(\overline{\psi})$;
3. $\varphi = Con(\overline{\psi})$;

The set of quasi-atomic formulas is denoted by $\mathcal{V}^q$.

Such definition will be important when we present the translation from $\mathcal{L}_{\mathsf{L}}^{VC}$ to the modal language $\mathcal{L}_{\mathsf{L}}^{\Box\Diamond}$.

Definition 6.1.18. *Let* $v \in sem_{\mathsf{L}}$ *be a valuation for a logic* L *as defined in Definition 2.1.1. Given* v *we extend it to the homomorphism* $v^* : \mathcal{V}^q \to V_n$ *as follows:*

1' $v^*(c_m^{k_m}(\varphi_1, \ldots, \varphi_k)) = o_m^{k_m}(v^*(\varphi_1), \ldots, v^*(\varphi_k))$.

The set of all valuations $v^* : \mathcal{V}^q \to V_n$ *is called* $sem^*_{\mathcal{L}_{\mathsf{L}}^{VC}}$.

The valuation v^* behaves like $v \in sem_{\mathsf{L}}$ excepting that it takes formulas $Val(\overline{\varphi})$ and $Con(\overline{\varphi})$ as atomic propositions. Now we define the models for the language $\mathcal{L}_{\mathsf{L}}^{VC}$ as follows:

Definition 6.1.19. *A model for* $\mathcal{L}_{\mathsf{L}}^{VC}$ *is a structure* $M_{\mathsf{L}}^{VC} = \langle v_0^+, \mathcal{A}^* \rangle$, *where* M_{L} *is a matrix for* L, $\mathcal{A}^*$ *is a set of valuations* $v_i^* \in sem_{\mathsf{L}}^{VC}$ *and* $v_0^+ : \mathcal{V} \to V_n$ *is an assignment recursively extended as follows:*

1. $v_0^+(c_m^{k_m}(\varphi_1, \ldots, \varphi_k)) = o_m^{k_m}(v_0^+(\varphi_1), \ldots, v_0^+(\varphi_k))$

2. $v_0^+(Val(\overline{\varphi})) = 1$ *if* $v_0^+(\varphi) \in D_{\mathsf{L}}$ *and for all* $v_i^* \in \mathcal{A}^*$, $v_i(\varphi) \in D_{\mathsf{L}}$;
   otherwise, $v_0^+(Val(\overline{\varphi})) = 0$;

3. $v_0^+(Con(\overline{\varphi})) = 1$ *if* $v_0^+(\varphi) \in D_{\mathsf{L}}$ *or for some* $v_i^* \in \mathcal{A}^*$, $v_i^*(\varphi) \in D_{\mathsf{L}}$;
   otherwise, $v_0^+(Con(\overline{\varphi})) = 0$;

A formula $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{VC})$ *is true in* M_{L}^{VC} *iff* $v_0^+(\varphi) \in D$. *A formula* $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{VC})$ *is valid if it is true in every model* M_{L}^{VC}.

Remember that the sentence names $\overline{\varphi}$ introduced above are not Gödel codes $\ulcorner\varphi\urcorner$. While the latter are defined within an arithmetical theory, the former are introduced as primitive objects in $\mathcal{L}^{VC}$. Of course there will be cases where $\ulcorner\varphi\urcorner$ and $\overline{\varphi}$ will be present in the same theory. Those cases are theories **Th** whose their basic logic is strong enough to validate, for example, arithmetical axioms. Many logics L do not enjoy this inferential power to be a basis for a first-order arithmetical theory. For this reason, we prefer to uniformly introduce $\overline{\varphi}$ instead of considering the possibility to define arithmetical codes for the syntax.[6][7] Consider the following translation:

[6] In paraconsistent set theory there is an interesting parallel, naïve set theory based on LP is not able to prove Cantor's Theorem due to its weak implication connective (PRIEST, 2006). Then, in order to build a naïve set theory based on LP strong enough to prove Cantor's Theorem, it is necessary to extend LP with a strong conditional. In Weber (2012), one can check on proposal in this direction.

[7] There is also the possibility of obtaining such naming device by restricting the valuations in order to validate the self-referential biconditionals. Pailos (2020) adopts such procedure to propose a validity theory based on LP. Our approach here is simpler, since we are adopting such naming devices as primitive objects in the language.

Definition 6.1.20. *Let $t : \mathcal{L}_{\mathsf{L}}^{\Box\Diamond} \to \mathcal{L}_{\mathsf{L}}^{VC}$ be a function defined as follows:*

$$\begin{array}{lcl} t(p) & = & p \\ t(c_m^k(\varphi_1, \ldots, \varphi_k)) & = & c_m^k(t(\varphi_1), \ldots, t(\varphi_k)) \\ t(\Box\varphi) & = & Val(\overline{t(\varphi)}) \\ t(\Diamond\varphi) & = & Con(\overline{t(\varphi)}) \end{array}$$

Fact 6.1.21. *The t-translation is an injective function, whose inverse t^{-1} is also injective.*

Given translation t, we now discuss the reason to use valuations $v^* \in sem^*_{\mathcal{L}_{\mathsf{L}}^{VC}}$ instead of valuations $v \in sem_{\mathsf{L}}$ in Definition 6.1.19. Let us consider again the logic $\mathsf{S0.5}$ (Definitions 5.4.6 and 5.4.8). It is easy to check that the following formula is a $\mathsf{S0.5}$-theorem:

$$\Box(\Box\varphi \vee \neg\Box\varphi) \tag{6.1}$$

Even if formulas of the form $\Box\varphi$ receive arbitrary values in non-normal worlds, the formula 6.1 is valid due to its form. The formula $\Box\varphi \vee \neg\Box\varphi$ is a substitution instance of a CPL-tautology. On the other hand, if we use valuations v instead of v^*, the following formula would not be valid:

$$Val(\overline{Val(\overline{\varphi}) \vee \neg Val(\overline{\varphi})}) \tag{6.2}$$

Since $\mathcal{L}_{\mathsf{CPL}}$ does not contain any predicate, the valuations v do not assign truth-values to formulas $Val(\overline{\varphi})$. Then, the formula 6.2 would not be valid in models of Definition 6.1.19. As a consequence, the Definition 6.1.19 would not be captured by $\mathsf{S0.5}$. The use of valuations v^* give an account of cases like formula 6.2. The valuations v^* preserve the tautologies validated by L-valuations $v \in sem_{\mathsf{L}}$. Now we will prove that the logics $\mathsf{L}^{\mathsf{S0.5}}$ capture the predicates Val and Con of $\mathcal{L}_{\mathsf{L}}^{VC}$ given translation t.

Lemma 6.1.22. *For every model $M_{\mathsf{L}}^{VC} = \langle v_0^+, \mathcal{A}_{\mathsf{L}}^* \rangle$ for $\mathcal{L}_{\mathsf{L}}^{VC}$ there is $\mathcal{M} = \langle W, N, R, v \rangle$ for $\mathsf{L}^{\mathsf{S0.5}}$ such that, for every $v \in \mathcal{A}_{\mathsf{L}}^* \cup \{v_0^+\}$ there is a $x \in W$, so that the following holds:*

$$v_x(\varphi) = v(t(\varphi))$$

For all $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\Box\Diamond})$.

Proof. Given a model $M_{\mathsf{L}}^{VC} = \langle v_0^+, \mathcal{A}_{\mathsf{L}}^* \rangle$ we define $\mathcal{M} = \langle W, N, R, v \rangle$ as follows:

- W is the collection of words w_i such that $v_{w_i}(p) = v_i^*(p)$, for $v_i^* \in \mathcal{A}_{\mathsf{L}}^*$
- $N = \{w_0^+\}$ is the set of normal worlds such that $v_0^+(p) = v_0^+(p)$,
- $R = \{(w_0^+, w_i) |\ w_i \in W\} \cup \{(w_0^+, w_0^+)\}$.

The proof of $v_{w_i}(\varphi) = v_i(t(\varphi))$ follows from the fact that both valuations are homomorphisms.

For $v_{w_0^+}(\varphi) = v_0^+(t(\varphi))$ we proceed by induction. We only deal with the modal cases. So, $v_{w_0^+}(\Box\varphi) = 1$ iff for every $v_{w_i} \in W$, $v_{w_i}(\varphi) \in D_{\mathsf{L}}$ and $v_{w_0^+}(\varphi) \in D_{\mathsf{L}}$ iff (by inductive hypothesis) for every $v_i^* \in \mathcal{A}_{\mathsf{L}}^*$, $v_i^*(t(\varphi)) \in D_{\mathsf{L}}$ and $v_0^+(t(\varphi)) \in D_{\mathsf{L}}$ iff $v_0^+(Val(\overline{t(\varphi)}) = 1$; otherwise we get 0. The case of $\Diamond\psi$ equally depends on the definitions and the inductive hypothesis.

Q.E.D.

Lemma 6.1.23. *For every $\mathcal{M} = \langle W, N, R, v\rangle$ for $\mathsf{L}^{\mathsf{S0.5}}$ there is $M_{\mathsf{L}}^{VC} = \langle v_0^+, \mathcal{A}_{\mathsf{L}}^*\rangle$ for $\mathcal{L}_{\mathsf{L}}^{VC}$ such that, for every $w \in W$ there is a $v \in \mathcal{A}_{\mathsf{L}}^* \cup \{v_0^+\}$ the following holds:*

$$v_x(\varphi) = v_w(t^{-1}(\varphi))$$

For all $\varphi \in For(\mathcal{L}^{VC})$.

Proof. Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{L}-modal model for $\mathsf{L}^{\mathsf{S0.5}}$. Without loss of generality we can assume that $N \neq \varnothing$; otherwise there are no modal formulas that are valid in $\mathcal{M}$ and, thus, the proof is trivial. Given a $w_0^* \in N$, we know that $w_0^* R w_i$, for all $w_i \in W$, by Definition 6.1.2. Notice that the normal words display the same set of modal validities, since they are all connected with all non-normal worlds. Then fix, w_0^* a world in N.

We now define a model $M_{\mathsf{L}}^{VC} = \langle v_0^+, V\rangle$ as follows:

- $\mathcal{A}_{\mathsf{L}}^*$ is a collection of valuations $v_x^* \in sem_{\mathsf{L}^{VC}}^*$, for $x \in W - N$,
- v_0^+ is a valuation of the language $\mathcal{L}_{\mathsf{L}}^{VC}$ such that $v_0^+(p) = v_{w_0^*}(p)$ in the lines of Definition 6.1.19.

Thus, consider the case $v = w_0^*$ and when $\varphi = Val(\overline{\psi})$. We proceed by induction. Then, $v_0^*(Val(\overline{\psi})) = 1$ iff $v_0^+(\psi)) \in D_{\mathsf{L}}$ and $v_x(\psi)) \in D_{\mathsf{L}}$ for every $v_x^* \in \mathcal{A}_{\mathsf{L}}^*$ iff, by Inductive Hypothesis, $v_x(\psi)) \in D_{\mathsf{L}}$ for every $x \in W$ iff $v_{w_0^*}(\Box\psi) = 1$, which, by definition of the t-translation, is equivalent to say $v_{w_0^*}(t^{-1}(Val(\overline{\psi})) = 1$. The case of $Con(\overline{\psi})$ is similar. Q.E.D.

Because $\mathfrak{L}^{\mathsf{S0.5}}$ comprehends a wide class of many-valued logics, the majority of the characteristic modal principles are not valid in this family of logics, since we have to take into consideration the idiosyncrasies of each system $\mathsf{L}^{\mathsf{S0.5}}$. The following propositions illustrate this point:

Proposition 6.1.24. *The axiom* K *is not valid in* $\mathsf{LP}^{\mathsf{S0.5}}$.

Proof. Let $M_{\mathsf{LP}} = (\{1, \frac{1}{2}, 0\}, \neg, \vee, \{1, \frac{1}{2}\})$ be the matrix presented in Definition 6.1.5. Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{LP}-modal model for $\mathsf{LP}^{\mathsf{S0.5}}$ such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$ and $v_w(q) = v_y(q) = 0$. Then, $v_w(p \to q) = v_y(p \to q) = \frac{1}{2}$. Since p and $p \to q$ takes a designated value in every world of W, then $v_w(\Box p) = v_w(\Box(p \to q)) = 1$. On the other hand, $v_w(\Box q) = 0$. Therefore, $v_w(\Box(p \to q) \to (\Box p \to \Box q)) = 0$. Q.E.D.

Proposition 6.1.25. Nec *is not valid in* $\mathsf{K_3}^{\mathsf{S0.5}}$.

Proof. The matrix of the logic $\mathsf{K_3}$, Kleene (1938), is obtained from $M_{\mathsf{LP}} = (\{1, \frac{1}{2}, 0\}, \neg, \vee, \{1, \frac{1}{2}\})$ by taking only 1 as designated value. The matrix of $M_{\mathsf{K_3}}$ of $\mathsf{K_3}^{\mathsf{S0.5}}$ has no boolean operation o^2_k such that $o^2_m(\frac{1}{2}, \frac{1}{2}) \in \{1, 0\}$. Since $\frac{1}{2} \notin D$, then there is no tautology in $\mathsf{K_3}^{\mathsf{S0.5}}$. Then $\mathsf{K_3}^{\mathsf{S0.5}}$ has no theorem of the form $\Box\varphi$. Q.E.D.

Proposition 6.1.26. *The axiom* T *is not valid in* $\mathsf{RM_3}^{\mathsf{S0.5}}$.

Proof. Let $M_{\mathsf{RM_3}} = (\{1, \frac{1}{2}, 0\}, \neg, \vee, \{1, \frac{1}{2}\})$ be the matrix of the logic $\mathsf{RM_3}$ (Anderson & Belnap (1975)) whose operations have the following truth-tables:

$\to$	1	$\frac{1}{2}$	0
1	1	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

Let $\mathcal{M} = \langle W, N, R, v\rangle$ be a$M_{\mathsf{RM_3}}$-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$. By definition of $\Box$, we obtain $v_w(\Box p) = 1$. By the definition of $\to$, we obtain $v_w(\Box p \to p) = 0$. Q.E.D.

Consider now the following definition:

Definition 6.1.27. *(RESCHER, 1969) A truth-value r is called* infectious *if, whenever it is an input of a truth-function, r is an output, for every truth-function of a given matrix. A logic* L *is called infectious if its characteristic matrix has at least one infectious value.*[8]

Proposition 6.1.28. $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$ *and* $\Diamond(\varphi \wedge \psi) \to (\Diamond\varphi \wedge \Diamond\psi)$ *are not valid in logics* $\mathsf{L}^{\mathsf{S0.5}}$ *which have infectious designated values.*

Proof. Let $M_{\mathsf{H_3}} = (\{1, \frac{1}{2}, 0\}, \neg, \vdash, \vee, \{1, \frac{1}{2}\})$ be the matrix of $\mathsf{H_3}$ (Halldén (1949)) whose operations have the following truth-tables:

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

	$\vdash$
1	1
$\frac{1}{2}$	0
0	1

[8] We refer the reader to Szmuc (2016) for a systematic treatment of these logics.

The connective $\rightarrow$ can be defined as $\varphi \rightarrow \psi := \neg\varphi \vee \psi$. Then, $\rightarrow$ has the following truth-table:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	1

Consider now the $\{\neg, \vee, \rightarrow\}$-fragment of $\mathsf{H_3}$, usually called *Paraconsistent Weak Kleene Logic* (PWK).[9]

Let M_{PWK} be a matrix for an infectious logic L and let $\mathcal{M} = \langle W, N, R, v\rangle$ be a $M_{\mathsf{H_3}}$-modal model for $\mathsf{H_3}^{\mathsf{S0.5}}$ such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$, and $v_w(q) = v_y(q) = 0$. So $v_w(p \wedge q) = v_y(p \wedge q) = \frac{1}{2}$. By truth-definition of $\Box$, we obtain $v_w(\Box(p \wedge q)) = v_w(\Box p) = 1$ and $v_w(\Box q) = 0$. Then, $w_w(\Box p \wedge \Box q) = 0$. Therefore, $v_w(\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)) = 0$. The case of $\Diamond(\varphi \wedge \psi) \rightarrow (\Diamond\varphi \wedge \Diamond\psi)$ is similar.

Q.E.D.

Definition 6.1.29. *(RÉ; SZMUC, 2021) A truth-value r is called* immune *if, whenever it is an input of a truth-function along with a truth-value r', r' is the output.*

Proposition 6.1.30. $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ *is not valid in immune logics which the intermediate values taken as designated values.*

Proof. Let L be a logic characterized by the matrix $M_{\mathsf{L}} = (\{1, \frac{1}{2}, 0\}, \neg, \wedge, \rightarrow, \{1, \frac{1}{2}\})$ whose operations have the following truth-tables:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0
0	0	0	0

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\mathcal{M} = \langle W, N, R, v\rangle$ be a M_{L}-modal model for $\mathsf{L}^{\mathsf{S0.5}}$ such that $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = 1$, and $v_w(q) = v_y(q) = \frac{1}{2}$. By definition of $\Box$, $v_w(\Box p) = 1$ and $v_w(\Box p) = 1$. So, $v_w(\Box p \wedge \Box q) = 1$ On the other hand, $v_w(p \wedge q) = 0$. Then $v_w(\Box(p \wedge q)) = 0$. Therefore, $v_w((\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)) = 0$. Q.E.D.

Proposition 6.1.31. *The substitutivity of equivalents is not valid in logics $\mathsf{L}^{\mathsf{S0.5}}$.*

Proof. In the Chapter 5 we proved that $\mathsf{S0.5}$ does not validate substitutivity of equivalents. Since $\mathsf{S0.5}$ is stronger than each $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$, then no logic $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$ satisfy such rule. Q.E.D.

Proposition 6.1.32. $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ *is not generally valid in logics $\mathsf{L}^{\mathsf{S0.5}}$.*

[9] The reader can find a systematic study of PWK in Bonzio et al. (2017).

Proof. Let $\mathsf{LP}^{\mathsf{S0.5}}$ be the logic presented in Proposition 6.1.24. $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$. Then, $v_w(\Box\neg p) = v_w(\Diamond p) = 1$. By applying the negation, we obtain $v_w(\neg\Box\neg p) = 0$. Therefore , $v_w(\Diamond p \rightarrow \neg\Box\neg p) = 0$. Q.E.D.

The Proposition 6.1.31 and Proposition 6.1.32 justify the introduction of both modal operators as primitive. From a philosophical point of view it could be defended that logical validity and logical consistency are somewhat independent. That is, weakening the logic those concepts may show to be independent each other.[10]

Definition 6.1.2 covers a myriad of many-valued modal systems $\mathsf{L}^{\mathsf{S0.5}}$s. So an axiomatization *à la* Hilbert of the most general modal principles which all systems $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$ satisfy would constitute an import result about validity and consistency of logics L. But, as Propositions 6.1.24 - 6.1.32 show, many modal principles interact with the truth-functional connectives and these latter significantly vary according to the logic L. So, it is not immediate for us how to obtain such a general axiomatization.

Proposition 6.1.33. *The following principles hold for any* $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$.

1. $\models_{\mathsf{L}^{\mathsf{S0.5}}} \Box\varphi \rightarrow \Diamond\varphi$;
2. $\Box\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$;
3. $\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \Diamond\varphi$;
4. $\Box\varphi \rightarrow \Box\psi, \Box\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \Box\psi$
5. *If* ψ *is a* L*-tautological consequence of* φ*, then* $\Box\varphi \models_{\mathsf{L}^{\mathsf{S0.5}}} \Box\psi$.

Proof. 1. Suppose that every $\mathcal{M}$ for $\mathsf{L}^{\mathsf{S0.5}}$ is such that $v_w(\Box\varphi) = 1$, for every $w \in N$. Then, for every $y \in W$, such that wRy, $v_y(\varphi) \in D_{\mathsf{L}}$. Then, since R is reflexive over N, $v_w(\varphi) \in D_{\mathsf{L}}$. So there is $y \in W$ such that $v_y(\varphi) \in D_{\mathsf{L}}$. Therefore, $v_w(\Diamond\varphi) = 1$. By Assumption 6.0.1, we obtain $v_w(\Box\varphi \rightarrow \Diamond\varphi) = 1$.

2. Suppose $\mathcal{M}$ for $\mathsf{L}^{\mathsf{S0.5}}$ is such that $v_w(\Box\varphi) = 1$, for every $w \in N$. Then, for every $y \in W$, such that wRy, $v_y(\varphi) \in D_{\mathsf{L}}$. Then, since R is reflexive over N, we obtain $v_w(\varphi) \in D$.

3. Same reasoning as *2.*

4. This follows from normality assumption, since the reasoning only involves the classical values 1 and 0.

[10] Some concepts which are identical in classical logic become different each other in non-classical logics. Paraconsistent logics are good examples for this. By invalidating the inference $\varphi, \neg\varphi \vdash \psi$, they show that inconsistency and triviality are not identical concepts.

5. Suppose that ψ is a tautological consequence of φ in $\mathsf{L}^{\mathsf{S0.5}}$. Then, for all $w \in N$, $v_w(\varphi) \in D_{\mathsf{L}}$ implies $v_w(\varphi) \in D_{\mathsf{L}}$. Since ψ is a tautological consequence of φ, it is also the case in worlds $y \in W - N$. By definition of models for $\mathsf{L}^{\mathsf{S0.5}}$, every $y \in W$ is accessed by a normal world $w \in N$. Then, $v_w(\Box\varphi) = 1$ implies $v_w(\Box\psi) = 1$. Q.E.D.

Proposition 6.1.33 shows that the operators $\Box$ and $\Diamond$ fulfill the minimal requirements of their "intended interpretations". Items (2) and (3) are quite general since they do not involve operators other than $\Box$ and $\Diamond$. That is, their validity do not depend on other operators, such as $\neg$, $\rightarrow$, $\wedge$, and $\vee$. The item (1), on the other hand, shows that logical validity and logical consistency are connected by means of implication. Since $\Box\varphi$ and $\Diamond\varphi$ can only either true (1) or false (0), then $\rightarrow$ has as arguments only classical values. By our Assumption 6.0.1, $\Box\varphi \rightarrow \Diamond\varphi$ is valid in all logics in the family $\mathfrak{L}^{\mathsf{S0.5}}$. Then axiom D is one of the most general principle which connects $\Box$ and $\Diamond$ by means of truth-functional connectives.

6.2 The modal logics $\mathfrak{L}^{\mathsf{S5}}$

The modalities investigated in §6.1 are considerably weak. As one can see, imposing other properties in accessibility relation will not make any difference since R is defined only for normal worlds and the non-normal worlds assign arbitrary values to modal formulas. In this present section we will deal with stronger modalities which are intended to capture a stronger notion of modalities. Then we will extend Skyrm's result (SKYRMS, 1978) in proving that these modalities capture a strong notion of logical validity.

Definition 6.2.1. *Fix an n-valued normal logic* L*, with corresponding language* $\mathcal{L}_{\mathsf{L}}$ *and matrix* $M_{\mathsf{L}} = \langle V_n, o_1^1, \ldots, o_m^k, D_{\mathsf{L}} \rangle$*. An* M_{L}*-standard modal model is a structure of the form* $\mathcal{M} = \langle W, R, v \rangle$*, where* $W \neq \emptyset$ *is a set of worlds,* $R \subseteq W \times W$ *is an equivalence relation,* v *is an assignment, and* v_w *is recursively defined as follows:*

1 *The boolean cases are defined as in Definition 6.1.2;*

2 $v_w(\Box\varphi) = 1$ *iff for all* $y \in W$*,* wRy *implies* $v_y(\varphi) \in D_{\mathsf{L}}$*; otherwise* $v_w(\Box\varphi) = 0$

3 $v_w(\Diamond\varphi) = 1$ *iff there is* $y \in W$*,* wRy *and* $v_y(\varphi) \in D_{\mathsf{L}}$*; otherwise* $v_w(\Diamond\varphi) = 0$

A formula $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\Box\Diamond})$ *is true in a* M_{L}*-standard modal model iff for every* $w \in W$ $v_w(\varphi) \in D_{\mathsf{L}}$*.* φ *is valid iff it is true in every* M_{L}*-standard modal model.*

Definition 6.2.2. *The* standard Suszkian modal counterpart *of* L*, that we indicate by* L^{S5}*, is the modal logic in the language* $\mathcal{L}_{\mathsf{L}}^{\Box\Diamond}$ *that consists of all valid formulas in the* M_{L}*-standard modal model.*

Definition 6.2.3. *We denote the family of many-valued logics* L^{S5} *as* $\mathfrak{L}^{\mathsf{S5}}$*.*

Now, since modal formulas are now evaluated in all worlds, the possibility of iterations of modalities is regained. Note that Definition 6.2.1 still gives classical values to modal formulas. Again, always being designated or not is the only thing that matters to evaluate the validity (or non-validity) of a formula.

The proof system for the logics $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$ is a modification of the proof system presented in last section. Now we present the definitions and rules where there are modifications.

Definition 6.2.4. *Let $\mathcal{T}$ be a tableau and b be a branch of $\mathcal{T}$. We say that b closes (i) if there is a formula φ such that $[t^m]\varphi, j$ and $[t^l]\varphi, j$ with $m \neq l$ occurring in b; (ii) if $[t^m]\Box\varphi, i$ or $[t^m]\Diamond\varphi, i$ for $0 < m < n-1$ occurs in b.*

The modal rules suffer the following modification: $[t^{n-1}]\Box\varphi$, $[t^{n-1}]\Diamond\varphi$, $[t^0]\Box\varphi$ and $[t^0]\Diamond\varphi$ are now defined for every node i. Moreover, the following rules are added:

$$\frac{irj \qquad jrk}{irk}(\text{rule } \tau) \qquad \frac{irj}{jri}(\text{rule } \sigma)$$

The characterization results for the logics in $\mathfrak{L}^{\mathsf{S0.5}}$ will be proved in the subsection 6.7.2.

6.2.1 $\mathfrak{L}^{\mathsf{S5}}$ and hierarchical validity

Now we will prove that the logics $\mathfrak{L}^{\mathsf{S5}}$ capture the concept of logical validity in a hierarchical framework. The hierarchical language relative to L takes $\mathcal{L}_{\mathsf{L}}$ as the base language, which we denote by $\mathcal{L}^0_{\mathsf{L}}$ for convenience. Informally, given $\mathcal{L}^0_{\mathsf{L}}$, we construct a hierarchy of increasingly stronger languages $\mathcal{L}^1_{\mathsf{L}}, \mathcal{L}^2_{\mathsf{L}}, \ldots, \mathcal{L}^i_{\mathsf{L}}, \ldots$, where each $\mathcal{L}^i_{\mathsf{L}}$, for $i > 0$, contains the formulas φ of the languages $\mathcal{L}^k_{\mathsf{L}}$, $0 \leq k < i$, as well as sentence names $\overline{\varphi}$ for each φ and the predicate Val. Informally, given the base language $\mathcal{L}^0_{\mathsf{L}}$, we construct a hierarchy of increasingly stronger languages $\mathcal{L}^1_{\mathsf{L}}, \mathcal{L}^2_{\mathsf{L}}, \ldots, \mathcal{L}^k_{\mathsf{L}}, \ldots$, where each $\mathcal{L}^k_{\mathsf{L}}$ $(0 < k \leq n)$ is expressively stronger than its predecessors in the hierarchy, and $\mathcal{L}_k$ contains validity predicates and sentence names $\overline{\varphi}$ which describe the valid formulas of the weaker languages in the hierarchy. Taking the union of all the $\mathcal{L}^k_{\mathsf{L}}$'s, we obtain $\mathcal{L}^\omega_{\mathsf{L}}$. More formally

Definition 6.2.5. *Let $\mathcal{L}^0_{\mathsf{L}}$ be a propositional language defined as in Chapter 2. From $\mathcal{L}^0_{\mathsf{L}}$ we define inductively the languages $\mathcal{L}^n_{\mathsf{L}}$, for $n \in \mathbb{N}$.*

1. *$For(\mathcal{L}^n_{\mathsf{L}}) \subseteq For(\mathcal{L}^{n+1}_{\mathsf{L}})$;*

2. If $\varphi \in For(\mathcal{L}^n_{\mathsf{L}})$, then $\mathcal{L}^{n+1}_{\mathsf{L}}$ contains $\overline{\varphi}$ and $Val(\overline{\varphi})$, $Con(\overline{\varphi}) \in For(\mathcal{L}^{n+1}_{\mathsf{L}})$;

$\mathcal{L}^{\omega}_{\mathsf{L}} = \bigcup_{n\in\omega} \mathcal{L}^n_{\mathsf{L}}$.

The models for $\mathcal{L}^{\omega}_{\mathsf{L}}$ are defined as follows. Let $v \in sem_{\mathsf{L}}$, call it v^0 be a valuation of a logic L with language $\mathcal{L}^0_{\mathsf{L}}$. Each v^0 induces, inductively, a model v^n for the hierarchy $\mathcal{L}^n_{\mathsf{L}}$. Then, a model v^0 for $\mathcal{L}^0_{\mathsf{L}}$ induces a model v^{ω} for $\mathcal{L}^{\omega}_{\mathsf{L}}$. Formally:

Definition 6.2.6. *Fix an n-valued normal logic L, with corresponding language $\mathcal{L}_{\mathsf{L}}$ and matrix $M_{\mathsf{L}} = \langle V_n, o^1_1, \ldots, o^k_m, D_{\mathsf{L}}\rangle$. The models v^n for the language $\mathcal{L}^n_{\mathsf{L}}$ are induced by a model v^0 of $\mathcal{L}^0_{\mathsf{L}}$ as follows:*

(1) The models v^0_i of $\mathcal{L}^0_{\mathsf{L}}$ are valuations $v_i \in sem_{\mathsf{L}}$ defined in Definition 2.1.1.

(2) The models v^{n+1}_i of $\mathcal{L}^{n+1}_{\mathsf{L}}$ induced by a model v^0_i of $\mathcal{L}^0_{\mathsf{L}}$ is the smallest extension of v^n_i of $\mathcal{L}^n_{\mathsf{L}}$ such that:

(2.1) If $\varphi \in \mathcal{L}^n_{\mathsf{L}}$ and $\varphi = c^k_m(\varphi_1, \ldots, \varphi_k)$, then:

(2.1.1) $v^{n+1}_i(c^{k_m}_m(\varphi_1, \ldots, \varphi_k)) = (o^{k_m}_m(v^{n+1}_i(\varphi_1), \ldots, v^{n+1}_i(\varphi_k)))$.

(2.2) If $\varphi \in \mathcal{L}^n_{\mathsf{L}}$, then:

(2.2.1) $v^{n+1}_i(Val(\overline{\varphi})) = 1$ *if* $v^n_j(\varphi) \in D_{\mathsf{L}}$ *for all models* v^n_j *of* $\mathcal{L}^n_{\mathsf{L}}$*;*
otherwise $v^{n+1}(Val(\overline{\varphi})) = 0$*;*

(2.2.2) $v^{n+1}_i(Con(\overline{\varphi})) = 1$ *if* $v^n_j(\varphi) \in D_{\mathsf{L}}$ *for some model* v^n_j *of* $\mathcal{L}^n_{\mathsf{L}}$*;*
otherwise $v^{n+1}_i(Con(\overline{\varphi})) = 0$*;*

Now, the model v^{ω}_i of $\mathcal{L}^{\omega}_{\mathsf{L}}$ induced by a v^0_i is the union of all v^n_i induced by v^0_i of $\mathcal{L}^0_{\mathsf{L}}$. $\varphi \in For(\mathcal{L}^{\omega}_{\mathsf{L}})$ is true *in a model v^{ω}_i of $\mathcal{L}^{\omega}_{\mathsf{L}}$ if $v^{\omega}_i(\varphi) \in D_{\mathsf{L}}$. $\varphi \in For(\mathcal{L}^{\omega}_{\mathsf{L}})$ is* valid *if φ is true in every model v^{ω}_i of $\mathcal{L}^{\omega}_{\mathsf{L}}$.*

S5 is characterized by models $\mathcal{M} = \langle W, R, v\rangle$ where R is an equivalence relation. Since every world $w \in W$ is related to every other world in its equivalence relation, we can define the models for S5 as pairs $\mathcal{M} = \langle W, v\rangle$ by dropping R. Then, truth for modal formulas is defined as follows:

(□-S5) $v_w(\Box\varphi) = 1$ iff $v_y(\varphi) = 1$ for every $y \in W$;

(◊-S5) $v_w(\Box\varphi) = 1$ iff $v_y(\varphi) = 1$ for some $y \in W$.

Consequently, as Hughes & Cresswell (1996) observe, every formula $\Box\varphi$ is true throughout $\mathcal{M}$ or it is false throughout $\mathcal{M}$ due to the extensionality of the model. Since the metatheory of the logics $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$ is classical set-theory, such logics can also be characterized by models $\mathcal{M} = \langle W, v\rangle$. Then the two clauses above suffer the obvious modifications:

(□-S5) $v_w(\Box\varphi) = 1$ iff $v_y(\varphi) \in D$ for every $y \in W$;

(◊-S5) $v_w(\Box\varphi) = 1$ iff $v_y(\varphi) \in D$ for some $y \in W$.

The following results to be proved will make use of these latter models.

Notation 6.2.7. *$\mathcal{A}_{\mathsf{L}}$ is a set of models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ such that each v_i^ω is induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$.*

Now, consider the translation $t : \mathcal{L}_{\mathsf{L}}^{\Box\Diamond} \to \mathcal{L}_{\mathsf{L}}^\omega$ as defined as in Definition 5.3.4 with the obvious modifications to language $\mathcal{L}_{\mathsf{L}}^\omega$. Then:

Lemma 6.2.8. *Let $\mathcal{A}_{\mathsf{L}}$ be a set of models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ which are induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$. For every set $\mathcal{A}_{\mathsf{L}}$ of models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ we define a model $\mathcal{M} = \langle W, v\rangle$ for L^{S5} such that for all $v_i^\omega \in \mathcal{A}_{\mathsf{L}}$ there is $x_i \in W$*

$$v_{x_i}(\varphi) = v_i^\omega(t(\varphi)).$$

For all $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\Box\Diamond})$.

Proof. Given the set $\mathcal{A}_{\mathsf{L}}$ of models v^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ induced by v^0 of $\mathcal{L}_{\mathsf{L}}^0$ we define a model $\mathcal{M} = \langle W, v\rangle$ for L^{S5} as follows:

(•) W is a collection of worlds x_i ($i \in \mathbb{N}$) such that $v_{x_i}(p) = v_i^\omega(p)$, where each $v_i^\omega \in \mathcal{A}_{\mathsf{L}}$ is induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$ according to Definition 6.2.6.

We proceed by induction to prove that $v_{x_i}(\varphi) = v_i^\omega(t(\varphi))$. The cases of boolean connectives are straightforward. We will focus on the case where $\varphi = \Box\psi$. So, $v_{x_i}(\Box\psi) = 1$ iff for all $x_j \in W$ such that $v_{x_j}(\psi) \in D_{\mathsf{L}}$. By construction, each model v_j^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ correspond to the valuation v of $\mathcal{M}$ on $x_j \in W$ in the lines of (•). By I.H., we obtain that $v_j^\omega(t(\psi)) \in D_{\mathsf{L}}$. Particularly, $v_i^\omega(t(\psi)) \in D_{\mathsf{L}}$. Then $v_i^n(t(\psi)) \in D_{\mathsf{L}}$, where $n \in \omega$ and v_i^n of $\mathcal{L}_{\mathsf{L}}^n$ is also induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$ iff $v_i^{n+1}(Val(\overline{t(\psi)})) = 1$ iff $v_i^\omega(Val(\overline{t(\psi)})) = 1$, by the definition of v_i^ω. Q.E.D.

Lemma 6.2.9. *For every model $\mathcal{M} = \langle W, v\rangle$ for L^{S5} we define a set $\mathcal{A}_{\mathsf{L}}$ of models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$ such that for all $x_j \in W$ there is $v_i^\omega \in \mathcal{A}_{\mathsf{L}}$:*

$$v_{x_i}(\varphi) = v_i^\omega(t^{-1}(\varphi)).$$

For all $\varphi \in For(\mathcal{L}_{\mathsf{L}}^\omega)$.

Proof. Let $\mathcal{M} = \langle W, v\rangle$ be a model for L^{S5}. Take a world $x_i \in W$. Now we define the models $v_i^\omega \in \mathcal{A}_{\mathsf{L}}$ of $\mathcal{L}_{\mathsf{L}}^\omega$ as follows:

(•′) $v_i^0 \in sem_{\mathsf{L}}$ is defined as the restriction of v_{x_i} of $\mathcal{M}$ to atomic propositions. Then, $v_{x_i}(p) = v_i^0(p)$. Since v_i^0 induces v_i^ω according to Definition 6.2.6, we obtain $v_{x_i}(p) = v_i^\omega(p)$.

Now we proceed by induction to prove $v_{x_i}(\varphi) = v_i^{\omega}(t^{-1}(\varphi))$. The boolean connectives are straightforward. We focus on the case where $\varphi = Val(\overline{\psi})$. So, consider $v_i^{\omega}(Val(\overline{\psi}) = 1$. By definition, $v_i^{n+1}(Val(\overline{\psi}) = 1$ for a model v_i^{n+1} of the language $\mathcal{L}_{\mathsf{L}}^{k+1}$, the first metalanguage where $\overline{\psi}$ appears. Then $v_i^{n+1}(Val(\overline{\psi}) = 1$ iff $v_j^n(\psi) \in D_{\mathsf{L}}$ for all models v_j^n of the language $\mathcal{L}_{\mathsf{L}}^n$ iff $v_j^{\omega}(\psi) \in D_{\mathsf{L}}$ for each v_j^{ω} of $\mathcal{L}_{\mathsf{L}}^{\omega}$ extending v_j^n iff by induction hypothesis $v_{w_j}(\psi) \in D_{\mathsf{L}}$ for all $w_j \in W$ iff $v_{w_j}(\Box\psi) = 1$ iff $v_i^{\omega}(t^{-1}(\Box\psi))$.

Q.E.D.

It is to be noted that many non-valid principles in logics $\mathsf{L}^{\mathsf{S0.5}}$ are still non valid in L^{S5} and the reason is the same: many of these principles depends on the particularities of the system. That is, if a logic $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$ has an implication like $\rightarrow$ of $\mathsf{RM_3}$, then the principle **T** will not be valid in ${\mathsf{RM_3}}^{Val}$. It is clear that when $n = 2$, $\mathsf{L}^{\mathsf{S5}} = \mathsf{S5}$.

Proposition 6.2.10. *The following principles hold for every* $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$*:*

(1) - (4) of Proposition 6.1.33;

(5) $\Box\varphi \models_{\mathsf{L}^{\mathsf{S5}}} \Box\Box\varphi$*;*

(6) $\varphi \models_{\mathsf{L}^{\mathsf{S5}}} \Box\Diamond\varphi$*;*

(7) If ψ *is a logical consequence of* φ*, Then* $\Box\varphi \models_{\mathsf{L}^{\mathsf{S5}}} \Box\psi$*.*

Proof. 5. Suppose that there is a M_{L}-standard modal model $\mathcal{M} = \langle W, R, v\rangle$ such that $v_w(\Box\varphi) \in D$ and $v_w(\Box\Box\varphi) \notin D$ for some $w \in W$. By truth-definitions, we know that $v_w(\Box\varphi) = 1$ and $v_w(\Box\Box\varphi) = 0$. So, there is a $y \in W$ such that wRy and $v_y(\Box\varphi) \notin D$. Again, $v_y(\Box\varphi) = 0$. Then there is a $z \in W$ such that yRz, $v_z(\varphi) = 0$. Since R is transitive, we obtain wRz, which implies that $v_w(\Box\varphi) = 0$, a contradiction.

6. Suppose that there is a M_{L}-standard modal model $\mathcal{M} = \langle W, R, v\rangle$ such that $v_w(\varphi) \in D$ and $v_w(\Box\Diamond\varphi) \notin D$, for some $w \in W$. The truth definitions for modal operators imply that $v_w(\varphi) = 1$ and $v_w(\Box\Diamond\varphi) = 0$. Then, there is $y \in W$ such that wRy, $v_y(\Diamond\varphi) \notin D$. Again, $v_y(\Diamond\varphi) = 0$. For all $z \in W$ such that yRz, $v_z(\varphi) \notin D$. Since R is symmetric, we obtain a contradiction.

7. The argument is similar to (3), but it holds for all $w \in W$.

Q.E.D.

6.3 Some remarks on validity paradoxes: a thought experiment

As we saw in the Chapter 5, Montague's theorem (5.2.1) poses a problem for the predicate approach to modalities. It means that we cannot have a predicate of validity satisfying its intuitive properties without risk of triviality. To avoid Montague's result as well as validity Curry (Theorem 5.2.3), the following strategies are commonly adopted:

1. Restriction of some principles of validity: Ketland (2012);
2. Gödel numbers are not privileged names of validity predicate: Skyrms (1978) and Schweizer (1992);
3. Weakening the basic logic: Weber (2014), Shapiro (2015) and Barrio et al (2016).

In this Section, we will explore the third route. As we argued before, the pursuit of non-classical logic was highly motivated by problems which do not seem to be adequately solved by employing classical logic. In the case of paradoxes/inconsistencies, such logics were proposed in order to block the steps in the proof which give rise to the paradoxes. For example, if one adopts a theory Th_{L} which has a non-contractive conditional, the proof of Theorem 5.2.3 will not work in Th_{L}, at least not directly. Of course, there may be alternative proofs of the problematic result. Then non-classical logics show themselves as good tools for philosophical analysis.

On the other hand, all the results of the Subsections 6.1.1 and 6.2.1 were proved by means of the device of sentence names, which blocks self-reference in all validity theories based on many-valued logics. If the introduction of names is a solution already in the classical case, there seems to be no good reason to consider non-classical logics. We said in the beginning of this chapter that our aim is to establish what are the most general principles of validity in this class of logics. Since arithmetization of language may not be available to every arithmetical theory based on many-valued logics, the use of sentence names is totally instrumental.

If one wants to take a non-classical as an alternative to classical logic, such devices may not be enough. By adopting a logic as an alternative to the classical logic, it is reasonable to require that his/her basic logic has sufficient inferential power to do arithmetization of syntax when the arithmetical language is taken into consideration. Of course, such attitude itself excludes the majority of MVLs, because many of them are remarkably weak. Such is the case, for example, with LP. As we saw in the Proposition 6.1.24, modus ponens fails in this logic. Without such rule, it is quite difficult, or even impossible, to do (first-order) mathematics by taking LP as a basic logic. That is why LP-supporters extend this logic with a strong conditional.[11] In the context of truth-theories, Picollo

[11] See footnotes of 6.1.1.

(2020) shows that some "multilations" of classical logic can significantly affect the way we do mathematics. In her work, she shows that truth theories based on LP°, which is LP enriched with the consistency operator, does not validate certain instances of the axiom of induction.[12] Her result shows that it is far from being obvious that the move to non-classical logics can be done without serious costs.

In what follows, we explore the validity predicate, which we will call Val^*, which has Gödel numbers instead of sentence names as arguments. In doing so, we restrict the range of many-valued logics we will look at, since only the logics which are strong enough for arithmetization will be taken into consideration. We will take for granted the arithmetization of the language, because we are interested only in the possibility of obtaining names $\ulcorner\varphi\urcorner$ for each φ.

6.3.1 The thought experiment: changing $\overline{\varphi}$ to $\ulcorner\varphi\urcorner$

Let Th_{L}^{Val} be a theory in the arithmetical language which extends L^{Val}. Moreover, suppose that Th_{L}^{Val} is capable to talk about its own syntax. The question is: when L is weaker than classical FOL, is Th_{L}^{Val} trivial?

First, by a reasoning close to what we did in Section 5.4, we can define a realization from the language $\mathcal{L}^{\Box\Diamond}$ of L^{S5} to Th_{L}^{Val} in such a way that:

$$\begin{array}{lcl} r(p)^{*'} & = & \varphi \in Sent(\mathcal{L}_1^{Val}) \\ r(c_m^k(\varphi_1,\ldots,\varphi_k)) & = & c_m^k(r(\varphi_1),\ldots,r(\varphi_k)) \\ r(\Box\varphi) & = & Val(\overline{r(\varphi)}) \\ r(\Diamond\varphi) & = & Con(\overline{r(\varphi)}) \end{array}$$

In this present section we only deal with logics $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$. The reason behind this choice is simple: the validity and consistency predicates interpreted by the modalities of $\mathsf{L}^{\mathsf{S0.5}}$ are to weak to give rise to paradoxes even if we consider Gödel codes instead of sentence names, as discussed in Section 5.4.

If, on the other hand, if Th_{L}^{Val} is an arithmetical theory, then Th_{L}^{Val} can formalize a predicate Val^* such that its arguments are Gödel numbers. The question is: is $Th_{\mathsf{L}}^{Val^*}$ consistent?

Suppose that a realization r' is defined like r except in the last two clauses, where we have:

$$\begin{array}{lcl} r'(\Box\varphi) & = & Val^*(\ulcorner r'(\varphi)\urcorner) \\ r'(\Diamond\varphi) & = & Con^*(\ulcorner r'(\varphi)\urcorner) \end{array}$$

To ease our analysis, we present the following definitions:

Definition 6.3.1. *$Th_{\mathsf{L}}^{Val^*}$ is Montague-trivial if Th_{L}^{Val} proves Theorem 5.2.1.*

[12] It can be showed that LP° is a LFI.

Definition 6.3.2. *$Th_{\text{Ł}}^{Val^*}$ is Curry-trivial if $Th_{\text{Ł}}^{Val^*}$ proves Theorem 5.2.3.*

Now we will analyse some examples to show that some important theories $Th_{\text{Ł}}^{Val}$ are trivial.

Łukasiewicz logic and the revenge problem

Definition 6.3.3. *The modal three-valued Łukasiewicz logic Ł_3^{S5} is characterized by the structure $\mathcal{M} = (W, R, v)$ where $M_{\text{Ł}_3}$ was given in Example 6.1.13.*

The connectives $\wedge$, $\vee$ and $\leftrightarrow$ are defined as follows:

$$\begin{array}{lcl} \varphi \vee \psi & := & (\varphi \rightarrow \psi) \rightarrow \psi \\ \varphi \wedge \psi & := & \neg(\neg\varphi \vee \neg\psi) \\ \varphi \leftrightarrow \psi & := & (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \end{array}$$

They are interpreted by the following truth-tables:

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\leftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

By analysing Theorem 5.2.1, one sees that the following propositional inference rules are needed to carry out the proof of this theorem.

1. $\varphi \rightarrow \psi, \psi \rightarrow \gamma \vdash_{\mathbf{T}} \varphi \rightarrow \gamma$;
2. $\varphi \rightarrow \psi, \varphi \rightarrow \neg\psi \vdash_{\mathbf{T}} \neg\varphi$

By the semantics for Ł_3^{S5}, we can show that *reductio rule* is not valid. To invalidate reductio, just consider a two world structure $\mathcal{M} = \langle W, R, M_{\text{Ł}_3}, v\rangle$ such that $v_w(\varphi) = v_y(\varphi) = \frac{1}{2}$ and $\{(w,y),(y,w),(w,y),(w,w),(y,y)\} \subseteq R$. So, since this rule does not hold, we can only say that the proof given in Theorem 5.2.1 does not work for Ł_3^{S5}.

However, this is not enough to say that the theory $Th_{\text{Ł}_3}^{Val^*}$ is immune to Montague's theorem. As Haack (1978) observes, many non-classical solutions to inconsistencies and paradoxes faces the *revenge paradoxes*. Revenge paradoxes can be understood as a strengthening of an already existing paradox in order to cover a larger class of logics. In the case of $Th_{\text{Ł}_3}^{Val^*}$, we can prove a strengthened paradox in two ways: the first goes by defining a classicality connective in the basic logic and the second goes by defining the connective of classical negation in the basic logic Ł_3.[13]

[13] In the study of theories of truth, MVLs were proposed to overcome the logical paradoxes, such as the liar paradox. Such logics are not, in general, immune to revenge paradoxes, as Haack argues. There are, of course, promising proposals of truth-theories based on such logics which are able to deal with paradoxes, such as Cobreros et al (2012). On the other hand, if we consider theories **T** based on MVL which are expressive enough to talk about its own syntax, it is possible to prove a more general version of Tarski's indefinability theorem (1956). Ketland (2003) proves that theories like **T** cannot express its *truth degrees* predicate. It means that such theories cannot express their own semantic concepts.

In the first case, we define the following connective of classicality:

$$\star\varphi := (\neg(\neg\varphi \to \varphi) \to \neg(\varphi \to \neg\varphi)) \to \neg(\varphi \to \neg\varphi)$$

Whose corresponding operation has the following truth-table:

	$\star$
1	1
$\frac{1}{2}$	0
0	1

Given $\star$ we can recover the classicality of the formulas. For every formula of $\varphi \in For(\mathcal{L}_{Ł_3})$ of Ł$_3$, we can recover classical inferences by setting $\star\varphi \wedge \varphi$. Inspired in the recovery project of the proponents of paraconsistent logics, we now state a version of Costa (1974) *Derivability Adjustment Theorem.*

Theorem 6.3.4. *For every* $\Gamma \subseteq For(\mathcal{L})$*, for every* $\varphi \in For(\mathcal{L})$*,*

$$\Gamma \models_{\mathsf{CPL}} \varphi \textit{ iff } \Gamma, \{\star p_1, \ldots \star p_n\} \models_{Ł_3} \varphi \tag{6.3}$$

where $\{p_1, \ldots, p_n\}$ *is the set of propositional variables which occur in* $\Gamma \cup \{\varphi\}$*.*

Since Theorem 6.3.4 establishes that φ inferentially behaves like classical logic, we apply the Diagonalization Lemma (Lemma 4.1.2) to obtain the sentence:

$$(\star\varphi \wedge \varphi) \leftrightarrow \neg Val^*(\ulcorner \star\varphi \wedge \varphi \urcorner) \tag{6.4}$$

Its modal translation is:

$$(\star\varphi \wedge \varphi) \leftrightarrow \neg\Box(\star\varphi \wedge \varphi) \tag{6.5}$$

Then we prove that $Th_{Ł_3}^{Val^*}$ is Montague-trivial.

Theorem 6.3.5. $Th_{Ł_3}^{Val^*}$ *is Montague-trivial.*

Proof. By a self-reference procedure we obtain the sentence $(\star\varphi \wedge \varphi) \leftrightarrow \neg\Box(\star\varphi \wedge \varphi)$. Since Ł$_3$ recovers classical logic, we can prove a (modal) version of Theorem 5.2.1 by substituting the occurrences of φ by $\star\varphi \wedge \varphi$. To ease the reading, we abbreviate by fixing $\varphi^c = \star\varphi \wedge \varphi$. We now present a *metaproof* in Ł$_3^{\mathsf{S5}}$.[14]

[14]We say metaproof because the proof system we presented was a tableau system. In this proof we deal with an axiomatic-like reasoning.

1.	$\varphi^c \leftrightarrow \neg\Box(\varphi^c)$	Diagonalization
2.	$\varphi^c \rightarrow \neg\Box(\varphi^c)$	Ł$_3$, 1
3.	$\Box(\varphi^c) \rightarrow \varphi^c$	**T**
4.	$\neg\neg\Box(\varphi^c) \rightarrow \neg\varphi^c$	Ł$_3$, 2
5.	$\Box(\varphi^c) \rightarrow \neg\neg\Box(\varphi^c)$	Ł$_3$
6.	$\Box(\varphi^c) \rightarrow \neg\varphi^c$	Ł$_3$ 5, 4
7.	$\neg\Box(\varphi^c)$	Ł$_3$ 3, 6
8.	$\neg\Box(\varphi^c) \rightarrow \varphi^c$	Ł$_3$, 1
9.	$\neg\varphi^c \rightarrow \neg\neg\Box(\varphi^c)$	Ł$_3$, 8
10.	$\neg\varphi^c \rightarrow \neg\Box(\varphi^c)$	Ł$_3$, 3
11.	$\neg\neg\varphi^c$	Ł$_3$ 9, 10
12.	$\neg\neg\varphi^c \rightarrow \varphi^c$	Ł$_3$
13.	φ^c	MP 11, 12
14.	$\Box(\varphi^c)$	Nec, 11

This concludes the proof. Q.E.D.

The second strategy is to define a classical negation in the logic Ł$_3$. The definition goes as follows:

$$\sim\varphi \quad := \quad \varphi \rightarrow \neg\varphi$$

Whose corresponding operation has the following truth-table:

	$\sim$
1	0
$\frac{1}{2}$	1
0	1

With the negation $\sim$ we can prove another version of Theorem 6.3.5 using $\sim$ instead $\neg$ and φ instead of $\star\varphi \wedge \varphi$.

From the truth conditions of the implication connective of Ł$_3$ it is easy to see that contraction is not a valid rule. Consider a two world structure $\mathcal{M} = \langle W, R, v\rangle$ such that $W = \{x, y\}$ and $R = \{(w, y), (y, w), (w, y), (w, w), (y, y)\}$ such that $v_w(p) = v_y(p) = \frac{1}{2}$ and $v_w(q) = v_y(q) = 0$. Then, from the truth-tables of the operation $\rightarrow$, we obtain that:

$$\varphi \rightarrow (\varphi \rightarrow \psi) \nvDash_{\text{Ł}_3^{\text{S5}}} \varphi \rightarrow \psi \tag{6.6}$$

Again, because of the revenge problem, we cannot say that the failure of 6.6 blocks validity-Curry in $Th^*_{\text{Ł}_3}$. The logic Ł$_3$ is expressively strong enough to define an implication connective which validates contraction. Consider the following definition:

$$\varphi \sqsupset \psi \quad := \quad \varphi \to (\varphi \to \psi)$$

which is characterized by the following truth-table:

$\sqsupset$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	1	1	1
0	1	1	1

The connective $\sqsupset$ is contractive. Moreover, we can show the following facts about $\sqsupset$:

Proposition 6.3.6. *The following items hold for $\sqsupset$:*

1. $\models_{\text{Ł}_3^{\text{S5}}} \varphi \sqsupset (\psi \sqsupset \varphi)$;
2. $\models_{\text{Ł}_3^{\text{S5}}} (\varphi \sqsupset \psi) \sqsupset ((\psi \sqsupset \gamma) \sqsupset (\varphi \sqsupset \psi))$;
3. $\varphi \sqsupset (\varphi \sqsupset \psi) \models_{\text{Ł}_3^{\text{S5}}} \varphi \sqsupset \psi$;
4. *(EPSTEIN, 1990)* $\Gamma \models_{\text{Ł}_3} \varphi \sqsupset \psi$ *iff* $\Gamma \cup \{\varphi\} \models_{\text{Ł}_3} \psi$.

Proposition 6.3.6 shows that $\sqsupset$ satisfies some minimal constraints that an operator should satisfy to be called a conditional connective. That being said, it is possible to prove an analogous result to Theorem 5.2.3. only by replacing $\to$ by $\sqsupset$. Then,

Theorem 6.3.7. *$Th_{\text{Ł}_3}^{Val^*}$ is Curry-trivial.*

Proof. Replace $\to$ by $\sqsupset$ in the proof of Theorem 5.2.3. Q.E.D.

The logic of paradox: LP

The versions of the liar paradox shows that predicates, such as truth predicate Tr and the validity predicate Val cannot live in harmony with laws like the principle of non-contradiction and principle of explosion. Then, the classical strategy was to adopt a stratified hierarchy of languages in order to talk about such semantic concepts without the risk of inconsistencies. However, Priest (1979) argues that such strategies of blocking the occurrence of paradoxical sentences do not constitute a solution. For him, a legitimate solution is to show that the conclusion of the paradoxical result does not follow from the premisses.

The logic he proposes to deal with semantical paradoxes is the logic LP, which was defined in Definition 6.1.5. This logic is sometimes taken to be a basis for inconsistent theories due to its weak deductive power. However, such weakness is object of critics.[15] As showed in Proposition 6.1.24, the implication connective of $\to$ does not validate modus

[15] Picollo (2020) is a clear example of critic.

ponens. This failure motivates logicians as well as philosophers, including Priest (2006) himself, to search for stronger logics with a stronger implication.

At the same time, however, the search for strong conditionals must be aware of Curry paradox, because it does depend only on the properties of implication (plus arithmetic, of course). Then this required conditional must be contractive. In the case of validity, for example, a possible route is to accept that the conditional which governs the predicate Val^* is the original LP's conditional because it does not have modus ponens. Then, it is not obvious how to prove that the validity theory extending the logic LP is trivial. Such strategy was suggested by Goodship (1996).

Recall the definition of LP given by matrices presented in Definition 6.1.5. Then:

Definition 6.3.8. *The modal logic* $\mathsf{LP}^{\mathsf{S5}}$ *is characterized by the structure* $\mathcal{M} = \langle W, R, v\rangle$ *where* M_{LP} *was given in Definition 6.1.5.*

The logic $\mathsf{LP}^{\mathsf{S5}}$ are remarkably weak, as the proposition bellow shows:

Proposition 6.3.9. *(PRIEST, 1979) In* LP*, the following inferences are not valid:*

1. $\varphi \to \psi, \varphi \to \neg\psi \nvDash_{\mathsf{LP}} \neg\varphi$;
2. $\varphi \to \psi, \neg\psi \nvDash_{\mathsf{LP}} \neg\varphi$;
3. $(\varphi \to \psi) \wedge \varphi \nvDash_{\mathsf{LP}} \psi$
4. $(\varphi \to \psi), \varphi \nvDash_{\mathsf{LP}} \psi$
5. $\varphi \to \psi, \psi \to \gamma \nvDash_{\mathsf{LP}} \varphi \to \gamma$

In the proof of Theorem 5.2.1, it was used several propositional inferences which are not valid in $\mathsf{LP}^{\mathsf{S5}}$, such as reductio rule and modus ponens. But, as Proposition 6.3.9 shows, the inferences of the items 1, 2 and 4 are not truth-preserving in LP. This means that the modal version of the proof of Theorem 5.2.1 does not work for $\mathsf{LP}^{\mathsf{S5}}$, and so it does not work for $Th^{Val^*}_{\mathsf{LP}}$. Of course, this does show that $Th^{Val^*}_{\mathsf{LP}}$ is not trivial. It has been argued that theories such as $Th^{Val^*}_{\mathsf{LP}}$ do not suffer the revenge problem due to the weak deductive power of LP. This by itself brings a dilemma: theories based on LP are not trivial, but LP is not able to be a basis for a strong and expressive mathematical theory.

As a title of example, in the case of naïve set theory based of LP, call it ST_{LP}, Restall (1992) argues that the absence of modus ponens and transitivity of implication are really obstacles if one wants to adopts ST_{LP}. Such theory validates all the axioms of ZF, excepting foundation, but they inferentially differ due to the weak conditional connective. Another problem with comes with the deductive weakness of $\to$ of LP lies in the definition of $=$. The symbol is usually defined in set theory as follows:

$$x = y := \forall z(z \in x \leftrightarrow z \in y) \tag{6.7}$$

From the axioms of classical FOL it is possible to prove that from $x = y$ one obtains $\varphi(x) \leftrightarrow \varphi(y)$. But this does not work in LP because the transitivity of implication fails. Let a be a set such that $x \in a \leftrightarrow \varphi(x)$ and $y \in a \leftrightarrow \varphi(y)$. If we have $x \in a \leftrightarrow y \in a$, we should derive $\varphi(x) \leftrightarrow \varphi(y)$. But, since the implication connective of LP is not transitive, then we cannot perform such inference.[16]

This last problem is essentially attached to the choice of the basic logic. If the possibility of constructing a expressive mathematical theory is a *desideratum*, then one has to search for another logic stronger than LP.[17]

6.4 Tautologies, anti-tautologies and logical contingencies

In the debate about the metatheoretical concepts of truth, validity, and consistency in classical logic, many concepts extensionally collapse, such as contradiction and triviality, and contradiction and anti-tautology. In non-classical logics, such concepts may not be equivalent in such a way that it is necessary to give a more detailed analysis of these concepts, even within classical metatheory.

In the philosophical debate about non-classical logic's legitimacy, it is often claimed the need for a non-classical metatheory for such logics. Such logics are often objected due to their classical metatheory. That is, if one wants to adopt a non-classical logics as a basis for foundational theories (such as Set Theory), then he/she cannot rely on a classical metatheory for his/her preferred logic. Rosenblatt (2021) argues that if one wants to provide a validity theory based on a non-classical logic, he/she must introduce the predicate which captures the invalidities of the theory. That is, such theory must have a predicate which shows the invalidity of an inference of φ from Γ if this inference is in fact invalid. In his paper, Rosenblatt also defends the construction of non-classical metatheories for non-classical logics. But he adopts a proof-theoretical perspective, in the sense that the meaning of the logical constants are given by their sequent rules.

In this Section, we will introduce modalities that capture the concepts of anti-tautologies and contradictions. Such modalities will be bivalent because we are still in a classical

[16] Check Restall (1992) for details.

[17] This issue of the choice of basic logic for strong mathematical theories is discussed in Badia et al (2020). There, they discuss the deductive strength a propositional logic should have in order to prove independence results, like Rosser's Undecidability Theorem. Of course, if a logic L is a fragment of classical logic, then the arithmetical theory based on L is trivially undecidable since classical PA is undecidable. But one should want to prove such a result in the arithmetic based on L. Then, in order to do so, such logic must satisfy some inference rules to perform the proof, such as transitivity, weakening, modus ponens, De Morgan. We invite the reader to check Badia et al. (2020).

metatheory. Although we recognize the importance of the debate about non-classical metatheories, we also think that we can provide an analysis of the metatheory of non-classical logics in a classical framework by introducing concepts such as invalidities, taken as primitives. Such analysis shed light on how such predicates should behave in a non-classical framework. We will focus on the logic $Ł_3^{S0.5}$, and then we will present some properties that such modalities have. Last, we will discuss how these modalities behave in other non-classical logics.[18]

Definition 6.4.1. *Let φ be a formula of L. Then:*

1. *φ is a* logical contingency *if there are valuations $v_i, v_j \in sem_{\mathsf{L}}$ such that $v_i(\varphi) \in D_{\mathsf{L}}$ and $v_j(\varphi) \notin D_{\mathsf{L}}$;*

2. *φ is a* logical falsity *if for every $v_i \in sem_{\mathsf{L}}$, $v_i(\varphi) = 0$;*

3. *φ is a* non-theorem *if φ is either a logical contingency or a logical falsity.*

The fact that non-theorems are, in a way, neglected in formal logic studies is not exclusive to this work. There is a long tradition in logic that has given priority to the investigation of validity, as they understand that the objective of logic is primarily to study valid inferences. On the other hand, as Goranko (1994) notes, the study of non-valid inferences was of fundamental importance already in the works of Aristotle. The study of non-valid inferences gave rise to the *Refutation systems*, which intends to capture non-validities in a given formal systems.[19] Here, we will use the formal interpretation of modalities given in this Chapter to study anti-tautologies and the non-theorems. We will focus on the logic $Ł_3^{S0.5}$ (Example 6.1.13.)

In $Ł_3^{S0.5}$, $\Box\varphi$ receives 1 in normal worlds w whenever φ receives 1 in the worlds accessible to w. So consider the following definition:

Definition 6.4.2. *Let $\mathcal{M} = (W, N, R, v)$ be $M_{Ł_3}$-modal model and $w \in N$:*

1. *$v_w(\blacksquare\varphi) = 1$ iff for all $y \in W$ such that wRy, $v_y(\varphi) = 0$;*

2. *$v_w(\blacklozenge\varphi) = 1$ iff for some $y \in W$ such that wRy, $v_y(\varphi) = 0$;*

3. *$v_w(\boxminus\varphi) = 1$ iff for all $y \in W$ such that wRy, $v_y(\varphi) \in \{0, \frac{1}{2}\}$;*

4. *$v_w(\ominus\varphi) = 1$ iff for some $y \in W$ such that wRy, $v_y(\varphi) \in \{0, \frac{1}{2}\}$.*

Such operators are defined in $Ł_3^{S0.5}$ as follows:

[18] The arguments for $Ł_3^{S0.5}$ are similar.

[19] We invite the reader to consult Wybraniec-Skardowska (2018) for a historical overview about refutation systems. In the aforementioned paper, Goranko provides refutation systems for modal logics like KGL, S4 and S4Grz. Here we will not investigate such approach to formal systems.

$$\begin{array}{lcl} \blacksquare\varphi & := & \Box\neg\varphi \\ \bowtie\varphi & := & \neg\Diamond\neg\varphi \\ \blacklozenge\varphi & := & \neg\bowtie\varphi \\ \boxminus\varphi & := & \neg\Diamond\varphi \\ \ominus\varphi & := & \neg\Box\varphi \end{array}$$

In CPL all anti-tautologies are equivalent, because the only non-designated value is 0.[20]

This is not the case with non-modal $\text{Ł}_3^{\mathsf{S0.5}}$, as the formulas below show:

$$p \wedge \neg p \tag{6.8}$$

$$\neg(p \to p) \tag{6.9}$$

By analysing the truth conditions of these formulas, we obtain that formula (6.8) always receive 0 or $\frac{1}{2}$, whereas formula (6.9) always receive 0. Therefore:

$$\nvdash_{\text{Ł}_3} (p \wedge \neg p) \to \neg(p \to p) \tag{6.10}$$

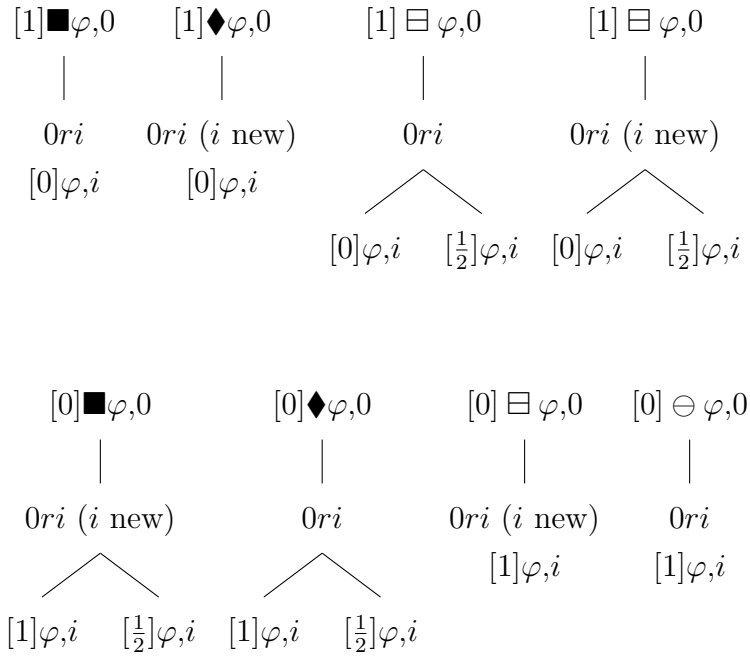
Such non equivalences justify the study of modalities of Definition 6.4.2. Since the substituvity of equivalents is not valid in $\text{Ł}_3^{\mathsf{S0.5}}$, we will consider for a moment the modal logic $\text{Ł}_3^{\mathsf{S0.5}}$ with these modal operators taken as primitive, even if $\text{Ł}_3^{\mathsf{S0.5}}$ is able to define each one of these operators. By the definition of $\blacksquare$, $\blacksquare\varphi$ means that "φ is an anti-tautology".[21]. So, as a consequence we have that:

$$\text{If } \vdash_{\text{Ł}_3} \varphi, \text{ then } \vdash_{\text{Ł}_3^{\mathsf{S0.5}}} \blacksquare\varphi. \tag{6.11}$$

Note that schema 6.11 does not hold for the operator $\boxminus$ because of the non-designated value $\frac{1}{2}$. Thus, while $\boxminus$ captures the concept of anti-tautology, $\blacksquare$ captures the concept of logical falsity. The tableau rules are straightforward:

[20]Such collapse happens if we consider the standard truth-table semantics for CPL. But if we consider alternative semantics for this logic it may be the case that not all contradictions (or not all tautologies) are equivalent to each other. Piazza & Pulcini (2020) provide a many-valued semantics for CPL based on a (trivial) sequent calculus for Kleene's four valued logic. In such (proof-theoretic) semantics for classical logic, there are non-equivalent contradictions since they can get different truth-values. We invite the reader to read their paper for technical details.

[21]Such distinction resembles Malinowski (1990)'s distinction between designated and anti-designated values. In his work, he defines a matrix of the form $(\{1, \frac{1}{2}, 0\}, \neg, \to, \{1\}, \{0\})$, where $\{1\}$ is the set of designated values and $\{0\}$ is the set of anti-designated values. In his work, he proves that Ł_3 characterized by such structure cannot be reduced by a two-valued model



Theorem 6.4.3. *The following formulas are provable in* $Ł_3^{S0.5}$*:*

1. $\nvdash \blacksquare\varphi \leftrightarrow \boxminus\varphi$;
2. $\vdash \blacksquare\varphi \rightarrow \boxminus\varphi$;
3. $\vdash \blacksquare\varphi \rightarrow \neg\varphi$;
4. $\vdash \blacksquare\varphi \rightarrow \Box\varphi$;
5. $\vdash \boxminus\varphi \rightarrow \Box\varphi$;
6. $\vdash \blacksquare\varphi \rightarrow \blacksquare\varphi$;
7. $\vdash \boxminus\varphi \rightarrow \ominus\varphi$;
8. $\vdash \blacksquare(\varphi \rightarrow \psi) \rightarrow (\blacksquare\psi \rightarrow \Box\varphi)$;
9. $\vdash \blacksquare(\varphi \vee \psi) \rightarrow (\blacksquare\psi \wedge \blacksquare\varphi)$;
10. $\vdash \boxminus(\varphi \vee \psi) \rightarrow (\boxminus\psi \wedge \boxminus\varphi)$.

All items of Theorem 6.4.3 are easily provable by means of the above tableaux rules. The list is not exhaustive, but it shows some interesting properties of the operators of Definition 6.4.2. They state obvious facts about the semantic conditions of $Ł_3^{S0.5}$ operators. For example, 8 says that if $\varphi \rightarrow \psi$ and ψ are logical falsities, i.e. always false, then φ is a tautology. The difference between $\blacksquare$ and $\boxminus$ may not seem to be meaningful, but it shows that classical logic collapses many metalogical concepts. Thus, when we look to non-classical logic, we should keep in mind that such collapses may disappear. In the case

of Ł$_3$, we saw that logical falsities and anti-tautologies do not collapse. Of course, there are some paracomplete logics, like $\mathsf{K_3}$ (Example 6.1.25) which the classical collapse still holds because every formula φ receive the value $\frac{1}{2}$ in at least one valuation $v \in sem_{\mathsf{K_3}}$. In the dual case, in some paraconsistent logics some tautologies are not equivalent. Consider the following definition:

Definition 6.4.4. *The logic* $\mathsf{RM_3}^\circ$ *is obtained by extending the language of* $\mathsf{RM_3}$ *(Definition 6.1.26) with the connective* $\circ$*, defined as in Definition 3.1.10.*[22]

It is clear that any iteration of $\circ$ results in a tautology. Then it is easy to check that the following holds.

Theorem 6.4.5. $\nvDash_{\mathsf{RM_3}^\circ} \circ \circ p \rightarrow \neg(p \wedge \neg p)$.

Proof. Consider a valuation $v \in sem_{\mathsf{RM_3}^\circ}$ such that $v(p) = \frac{1}{2}$. Then, $v(\circ \circ p) = 1$ and $v(\neg(p \wedge \neg p)) = \frac{1}{2}$. Therefore, $v(\circ \circ p \rightarrow \neg(p \wedge \neg p)) = 0$. Q.E.D.

Thus, by an analogous reasoning that we used in the case of Ł$_3^{\mathsf{S0.5}}$, one should introduce different modalities which capture two different concepts of tautologies: one for formulas which only receives the value 1 and for formulas which receives both 1 and $\frac{1}{2}$. We said that the search for the general properties of predicates of logical validity and logical consistency were our guiding intuition in the study of MVLs. From a logical pluralist perspective, such logics deserve to be investigated as well as any other logics we take as our preferred one. Of course, MVLs do not need to be faced as rivals of classical logic. As we argued in Bezerra & Venturi (2021), different formal systems may be seen as preserving different informal notions of validity.

6.4.1 The logical contingencies

In the Chapter 5, we discussed an aspect of incompleteness with respect to the intended interpretation of S0.5. Even if $\Box\varphi$ is true in S0.5 if and only if φ is a CPL-tautology, there are some formulas that should be valid in S0.5 that are not, as $\neg\Box p_i$, for p_i atomic. Then we argued that S0.5 captures a specific sense of tautology, when we only consider subsets of classical valuations. It is clear that it also happens with all logics $\mathsf{L}^{\mathsf{S0.5}}$.

Since the logics $\mathsf{L}^{\mathsf{S0.5}}$ capture tautologicity when we consider subsets of valuations of L, it turns out that such modal logics do not give a complete account of logical contingencies, because they do not validate formulas like $\Diamond p_i \wedge \Diamond\neg p_i$, which says that "it is contingent that p_i". We cannot require that S0.5 as well as its many-valued fragments mirror all the metatheoretical concepts at once, given that such concepts are too broad to be captured by

[22]Note that $\mathsf{RM_3}^\circ$ and LFI1 are equivalent sytems. From the connectives of LFI1 it is possible to define the connective $\rightarrow_{\mathsf{RM_3}^\circ}$ as $(\varphi \rightarrow_{\mathsf{LFI1}} \psi) \wedge (\neg\psi \rightarrow \neg\varphi)$. Reciprocally, from the connectives of $\mathsf{RM_3}^\circ$, we define the connective $\rightarrow_{\mathsf{LFI1}}$ as $(\varphi \rightarrow_{\mathsf{RM_3}^\circ} \psi) \vee_{\mathsf{RM_3}^\circ} \psi$.

a modal theory. Indeed this is a general flaw that most modal approaches to contingency suffer. For example, in the pioneering work of Montgomery & Routley (1966) about contingency and non-contingency modalities for normal modal logics, such formulas are not theorems.

This shows that such metalogical predicates cannot be totally reduced to modalities, even if modal operators capture relevant aspects of such predicates within a decidable formal system. That is, modal logics provide nice tools in the analysis of such predicates mainly due to their simple semantic structure and their associate proof theories. As Field (1991) argues, the analysis of metalogical notions through modal logics does not imply that such notions are essentially modal, but it shows that their general properties can be studied in modal systems, because they validate familiar modal principles.

6.5 Recovery project

The idea of recovering classical inferences in non-classical logics matches well in the contemporary debate of Logical Pluralism. In such a point of view, a logician is not forced to give up all classical reasoning when she/he adopts a logic different from classical logic. The possibility of representing classical inferences in a non-classical framework suggests that classical logic is adequate to represent certain forms of reasoning.

Even in a non-pluralist point of view, there is sometimes the need for recapturing classical reasoning in non-classical logics. As Priest (2006) recognizes, there are situations where Disjunctive Syllogism is valid. However, as Antunes (2020) argues, Priest's proposal is not adequate, at least directly, to recover classical inferences because of the weak expressiveness of his proposed logic.

In the logical literature, we can find different ways of representing classical inferences in non-classical logics. One of the most known recovery strategies comes from Intuitionistic Logic. Kolmogorov (1992), Gödel (1986c) and Glivenko (1929) translations of Intuitionistic Logic into Classical Logic provide a way of representing Classical Logic by means of double negation.

The development of paraconsistent logics, mainly leaded by the Brazilian (Costa (1974)) and Belgian (Batens (2000)) schools of paraconsistency, inaugurated a trend in the logical literature of introducing an operator in the object language, which is able to recover classical inferences once some 'consistent assumptions' are made.

The introduction of such connectives was vastly investigated in the field of non-classical logics. In paraconsistent logics (Carnielli et al (2007)), such connectives *recover* the explosive character of propositions. In the case of paracomplete logics (Marcos (2005b)), they *recover* the determinedness of propositions. And in the paranormal case (Omori (2020)), they *recover* both explosiveness and determinedness. Because these connective are able to recover commonly lost properties in non-classical logics, they are called *recovery*

operators. In sum, recovery operators are devices to recover inferences which we lose when depart from classical logic. It is not uncommon to see they being understood as incorporating metatheoretical concepts in the object language of the logic.

Such operators were vastly investigated from a technical point of view.[23]. Although the point of introducing such connectives is somewhat clear, their informal interpretation is still an open problem. For example, Ferguson (2018) argues that, in LFIs, "the notion of consistency too broad to draw decisive conclusions with respect to the validity of many theses involving the consistency connective." We will briefly discuss their relation with the modalities $\Box$ and $\Diamond$ of the logics $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5} discussed in Sections 6.1.1 and 6.2.1. So, the following discussion concentrates only in many-valued logics L which have recovery operators.[24]

The Argument 3.1.11 establishes that $\circ$ fails in capturing its informal interpretation of consistency.[25] There is a mismatch between the connective $\circ$ of $\mathsf{LFI1}$ and its intended interpretation. In the light of the results of Sections 6.1.1 and 6.2.1, we know for example, that the modalities and $\Diamond$ of logics $\mathsf{LFI1}^{\mathsf{S0.5}}$ and $\mathsf{LFI1}^{\mathsf{S5}}$ capture well-justified notions of metatheoretical consistency. The principles validated by $\Diamond$ in both logics have significant differences from the principles validated by $\circ$.

On the other hand, it is clear that $\circ$ echoes some intuitions about consistency. Even if such connective validates some counter-intuitive principles about consistency, $\circ$ is a nice device to label formulas which behaves like in classical logic. We think that the best interpretation for $\mathsf{LFI1}$'s $\circ$ goes in direction of Omori's classicality interpretation. That is, $\circ\varphi$ is interpreted as "φ has a classical value." This interpretation is more general than consistency, because a formula can be false under all interpretations. In this sense φ has a classical value but it is not consistent. On the other hand, φ can be semantically consistent without receiving a classical value.

It is immediate that this interpretation makes sense of $\mathsf{LFI1}$-axioms. Particularly, it gives a nice interpretation for the theorems:

1. $\vdash_{\mathsf{LFI1}} \circ\bot$
2. $\vdash_{\mathsf{LFI1}} (\circ\varphi \wedge \circ\psi) \rightarrow \circ(\varphi \wedge \psi)$

We will leave the question of whether this interpretation applies to other LFIs open. The reason to leave this question open is simple: the meaning of the connectives is local. That is, we have to analyse at least the inferential behavior of $\circ$ in the particular logic L before fixing an informal interpretation. That is, we have to analyse how $\circ$ behaves before $\vdash_{\mathsf{L}}$ (and $\models_{\mathsf{L}}$). For example, it is difficult to defend that the connective $\circ$ in the logic mbC

[23]We refer the reader to check Corbalán (2012) for a wide investigation of such connectives.

[24]Many MVLs have neither available recovery nor expressive connectives to define such operators. But, as Caleiro *et al* (2015) show, it is always possible to extend them with recovery operators.

[25]Subsection 3.1.4 of the Chapter 3.

since the only axiom and inference schema which involve $\circ$ are $\circ\varphi \rightarrow (\varphi \rightarrow (\neg\varphi \rightarrow \psi))$ and $\circ\varphi, \varphi, \neg\varphi \vdash_{\mathsf{mbC}} \psi$. They alone are not sufficient to characterize the meaning of consistency and that of classicality.

The reasoning raised in the last paragraph applies to all logical systems. In order to assert that a connective $\clubsuit$ has a particular informal interpretation, one should first look at the axioms or the inferential principles which govern $\clubsuit$. Without such interpretation exercise, any discussion on the meaning of $\clubsuit$ will be a pointless verbal dispute. Modal logics are paradigmatic cases. There are modal logics which are difficult to fix informal interpretation. The axioms and rules which govern the basic logic K are too wide in such a way that they are compatible with several interpretations. The axiom K and the rule Nec are compatible with provability, alethic, epistemic, deontic and temporal interpretations of the modalities $\Box$ and $\Diamond$. But, the logic K itself is too general to be interpreted. On the other hand, there are modal systems which are interpreted in more than one way. Such is the case with the modal logic $\mathsf{S4.2}$, which has at least two informal interpretations: epistemic (Stalnaker (2006)) and set-theoretical (Hamkins & Löwe (2008)).

Returning to our main point, we can say that $\mathsf{LFI1}$'s $\circ$ can be interpreted as "having a classical value"' because the axioms and rules which govern the behaviour of $\circ$ "allow" such interpretation. Now we will discuss that the modalities $\Box$ and $\Diamond$ can also serve to study such recovering operators. In what follows, we will analyse these operators in two logics: $Ł_3^{\mathsf{S5}}$ and $\mathsf{LP}^{\mathsf{S5}}$.

6.5.1 $\Box$ and $\Diamond$ and recovery operators

Because of the bivalence of the modalities investigated, we argue that $\Box$ and $\Diamond$ can starting points to the investigation of recovery operators. By looking at the truth-table of the connectives $\star$ and $\circ$ we know that $v(\star p) = 1$ whenever $v(p) \in \{1, 0\}$. We will apply the same idea to the case of modalities to be investigated in this Subsection.

The logic $Ł_3^{\mathsf{S5}}$

The logic $Ł_3^{\mathsf{S5}}$ is defined in Definition 6.3.3. Based on the intuition given by the connective $\star$, we define the connective of non-contingency as follows:

$$\Delta\varphi \quad := \quad \Box\varphi \vee \Box\neg\varphi$$

Therefore, the truth condition of $\Delta\varphi$ is given by the following clause:

$v_w(\Delta\varphi) = 1$ iff (for every $y \in W$ such that wRy, $v_y(\varphi) = 1$) or (for every $z \in W$ such that wRz, $v_z(\varphi) = 0$); otherwise, $v_w(\Delta\varphi) = 0$.

The connective Δ was introduced by Montgomery & Routley (1966) and investigated by Cresswell (1988) and Humberstone (1995). By the semantic condition of Δ, $\Delta\varphi$ receives

the truth-value 1 if and only if φ receives 1 or 0 in all worlds y and z accessible to w. Thus, $\Delta\varphi$ reflects the intuitions of the truth-functional connective $\star$. With the connective Δ in hands, we now prove that it is possible to recover inferences of $\mathsf{S5}$ under certain assumptions.

Notation 6.5.1. *Let t be a truth-value and v be a valuation. $v[\Gamma] = t$ means $v(\gamma) = t$, for every $\gamma \in \Gamma$.*

Theorem 6.5.2. *For every $\Gamma \subseteq For(\mathcal{L}^{\Box\Diamond}_{\text{Ł}_3})$, for every $\varphi \in For(\mathcal{L}^{\Box\Diamond}_{\text{Ł}_3})$,*

$$\Gamma \models_{\mathsf{S5}} \varphi \textit{ iff } \Gamma, \{\Delta p_1, \ldots, \Delta p_n\} \models_{\text{Ł}_3^{\mathsf{S5}}} \varphi \tag{6.12}$$

where $\{p_1, \ldots p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.

Proof. Suppose that $\Gamma \models_{\mathsf{S5}} \varphi$ and that $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$. By definition of $\models_{\mathsf{S5}}$, for every M_{CPL}-standard model $\mathcal{M} = \langle W, R, v\rangle$, for every $w \in W$, $v_w[\Gamma] = 1$ implies $v_w(\varphi) = 1$. Also, we clearly have that $v_w(p_i) \in \{1, 0\}$ for $p_i \in \{p_1, \ldots p_n\}$. Now, turning to $M_{\text{Ł}_3}$-standard models $\mathcal{N} = \langle W', R', v'\rangle$, we have the following cases:

(1) for all $y \in W'$, $v'_y(p_i) = 1$, for $p_i \in \{p_1, \ldots, p_n\}$;

(2) for all $y \in W'$, $v'_y(p_i) = 0$, for $p_i \in \{p_1, \ldots, p_n\}$;

(3) for some $z, x, u \in W'$, $v'_x(p_i) = 1$, $v'_u(p_i) = 0$ and $v'_z(p_i) = \frac{1}{2}$;

(4) for all $y \in W'$, $v'_y(p_i) = \frac{1}{2}$ for all $p_i \in \{p_1, \ldots, p_n\}$.

By the cases (1) and (2), $v_w(\Delta p_i) = 1$ and then the result follows because all propositional variables have only classical values in all worlds $y \in W$. The cases (3) and (4) are also immediate because $v_w(\Delta p_i) = 0$ for some (respec., for all) $p_i \in \{p_1, \ldots, p_n\}$. Therefore, $\Gamma, \{\Delta p_1, \ldots, \Delta p_n\} \models_{\text{Ł}_3^{\mathsf{S5}}} \varphi$.

Conversely, suppose that $\Gamma, \{\Delta p_1, \ldots, \Delta p_n\} \models_{\text{Ł}_3^{\mathsf{S5}}} \varphi$. Then, for every $M_{\text{Ł}_3}$-standard model $\mathcal{N} = \langle W', R', v'\rangle$, for every $w \in W$, $v_w[\Gamma] = 1$ and $v_w(\Delta p_i) = 1$, for $p_i \in \{p_1, \ldots, p_n\}$, imply $v_w(\varphi) = 1$. By the semantic definition of Δ, we have the following possibilities:

1. all variables $p_i \in \{p_1, \ldots, p_n\}$ receive 1 in every $y \in W'$ such that $wR'y$;

2. all variables $p_i \in \{p_1, \ldots, p_n\}$ receive 0 in every $y \in W'$ such that $wR'y$.

Since these propositional variables only receive classical values, we obtain $\Gamma \models_{\mathsf{S5}} \varphi$.

Q.E.D.

A similar result with respect to the logic $\text{Ł}_3^{\mathsf{S0.5}}$ can be similarly stated. First, consider the following definition.

Definition 6.5.3. *Let $\varphi \in For(\mathcal{L}_{\text{Ł}_3}^{\Box\Diamond})$ be a $\text{Ł}_3^{\mathsf{S0.5}}$ formula. The modal degree of φ, $md(\varphi)$, is defined as follows:*

1. *if $\varphi = p$, then $md(p) = 0$;*
2. *if $\varphi = \neg\psi$, then $md(\neg\psi) = md(\psi)$;*
3. *if $\varphi = \psi \to \gamma$, then $md(\psi \to \gamma) = max(md(\psi), md(\gamma))$;*
4. *if $\varphi = \Box\psi$, then $md(\Box\psi) = md(\psi) + 1$;*
5. *if $\varphi = \Diamond\psi$, then $md(\Diamond\psi) = md(\psi) + 1$.*

Theorem 6.5.4. *For every $\Gamma \subseteq For(\mathcal{L}_{\text{Ł}_3}^{\Box\Diamond})$ such that $md(\gamma) \leq 1$, for every $\gamma \in \Gamma$, and for every $\varphi \in For(\mathcal{L}_{\text{Ł}_3}^{\Box\Diamond})$ such that $md(\varphi) \leq 1$,*

$$\Gamma \models_{\mathsf{S0.5}} \varphi \text{ iff } \Gamma, \{\Box p_1 \vee \Box\neg p_1, \ldots, \Box p_n \vee \Box\neg p_n\} \models_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi \tag{6.13}$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.

Proof. Suppose that $\Gamma \models_{\mathsf{S0.5}} \varphi$ and that $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$. By definition of $\models_{\mathsf{S0.5}}$, for every M_{CPL}-modal model $\mathcal{M} = \langle W, N, R, v\rangle$, for every $w \in N$, $v_w[\Gamma] = 1$ implies $v_w(\varphi) = 1$. Also, we clearly have that $v_w(p_i) \in \{1, 0\}$ for $p_i \in \{p_1, \ldots p_n\}$. Now, turning to $M_{\text{Ł}_3}$-modal models $\mathcal{N} = \langle W', N', R', v'\rangle$, we have the following cases:

(1) for all $y \in W'$, $v'_y(p_i) = 1$, for $p_i \in \{p_1, \ldots, p_n\}$;

(2) for all $y \in W'$, $v'_y(p_i) = 0$, for $p_i \in \{p_1, \ldots, p_n\}$;

(3) for some $z, x, u \in W'$, $v'_x(p_i) = 1$, $v'_u(p_i) = 0$ and $v'_z(p_i) = \frac{1}{2}$.

(4) for all $y \in W'$, $v'_y(p_i) = \frac{1}{2}$ for all $p_i \in \{p_1, \ldots, p_n\}$.

By hypothesis, $md(\gamma) \leq 1$, for all $\gamma \in \Gamma$, and $md(\varphi) \leq 1$. We will analyse the cases where $md(\gamma) = md(\varphi) = 0$ and $md(\gamma) = md(\varphi) = 1$. The other cases follow a similar reasoning.

If, for every $\gamma \in \Gamma$, $md(\gamma) = md(\varphi) = 0$, then only truth-functional operators occur on these formulas and propositional variables. So, by the cases (1) and (2), $v'_w(\Box p \vee \Box\neg p) = 1$ and then the result follows because all propositional variables have only one classical value in all $y \in W$. The cases (3) and (4) are also immediate, because $v'_w(\Box p_k \vee \Box\neg p_k) = 0$, for some (respec., for all) $p_k \in \{p_1, \ldots, p_n\}$. Then, $\Gamma, \{\Box p_1 \vee \Box\neg p_1, \ldots, \Box p_n \vee \Box\neg p_n\} \models_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi$.

If for every $\gamma \in \Gamma$, $md(\gamma) = md(\varphi) = 1$, then for every $\gamma \in \Gamma$ there is at least one $\psi = M\sigma$ such that $\psi \in Sb(\gamma)$, where $M \in \{\Box, \Diamond\}$ and σ is non-modal, and φ has at least one $\theta = M\tau$ such that $\theta \in Sb(\varphi)$, and τ is non-modal. Given that $md(\sigma) = md(\tau) = 0$, the values of σ and θ in all worlds $y \in W'$ accessible to w are uniquely determined by truth-functional operators and propositional variables of the set $\{p_1, \ldots, p_n\}$ which occur in σ and τ. In the case that the propositional variables $p_i \in \{p_1, \ldots, p_n\}$ only receive classical values, then every $\gamma \in \Gamma$ and φ only receive classical values. By the cases (1) and (2), we have that $v_w(\Box p_i \vee \Box\neg p_i) = 1$ and then the result follows, since we supposed that $\Gamma \models_{\mathsf{S0.5}} \varphi$. In the case (3), $v_w(\Box p_i \vee \Box\neg p_i) = 0$, for some $p_i \in \{p_1, \ldots, p_n\}$ and the result trivially follows. In the case that all propositional variables p_i receive the value $\frac{1}{2}$ at worlds $y \in W'$, the result also follows because $v_w(\Box p_i \vee \Box\neg p_i) = 0$. Therefore, $\Gamma, \{\Box p_1 \vee \Box\neg p_1, \ldots, \Box p_n \vee \Box\neg p_n\} \models_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi$.

Conversely, suppose that $\Gamma, \{\Box p_1 \vee \Box\neg p_1, \ldots, \Box p_n \vee \Box\neg p_n\} \models_{\text{Ł}_3^{\mathsf{S0.5}}} \varphi$. Then, for every $M_{\text{Ł}_3}$-modal model $\mathcal{N} = \langle W', N', R', v'\rangle$, for every $w \in N'$, $v'_w[\Gamma] = 1$, and $v'_w(\Box p_i \vee \Box\neg p_i) = 1$ for $p_i \in \{p_1, \ldots, p_n\}$, imply $v'_w(\varphi) = 1$. By the semantic definition of $\Box$, we have the following possibilities:

1. all variables $p_i \in \{p_1, \ldots, p_n\}$ receive 1 in every $y \in W'$ such that $wR'y$;

2. all variables $p_i \in \{p_1, \ldots, p_n\}$ receive 0 in every $y \in W'$ such that $wR'y$;

Given that $md(\gamma) \leq 1$, for all $\gamma \in \Gamma$, and $md(\varphi) \leq 1$, all subformulas $M\sigma$ and $M\tau$ of the formulas γ and φ, respectively, are such that σ and τ have modal degree 0. This means that the values of σ and τ in all worlds $y \in W'$ are determined by the occurrences of truth-functional operators and by the values of the propositional variables of the set $p_i \in \{p_1, \ldots, p_n\}$. Then so is γ, for every γ, and φ. Since these propositional variables only receive classical values, we obtain that $\Gamma \models_{\mathsf{S0.5}} \varphi$.

Q.E.D.

First, Theorem 6.5.4 does not generalise to any modal degree. To see why Theorem 6.5.4 does not work for formulas with modal degree greater than 1, consider the following formula

$$\Box(\Box p \vee \neg\Box p) \tag{6.14}$$

This formula is valid in classical $\mathsf{S0.5}$, but it is not valid in $\text{Ł}_3^{\mathsf{S0.5}}$. According to Theorem 6.5.4, the formula 6.14 would be recoverable as follows:

$$\Box p \vee \Box\neg p \models_{\text{Ł}_3^{\mathsf{S0.5}}} \Box(\Box p \vee \neg\Box p) \tag{6.15}$$

However, the inference 6.15 is not valid. Consider the $M_{\text{Ł}_3}$-modal model $\mathcal{M} = \langle W, N, R, v\rangle$, where $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$, $v_w(p) = v_y(p) = 1$

and $v_y(\Box p) = \frac{1}{2}$. Then, $v_w(\Box p) = 1$, and so $v_w(\Box p \vee \Box\neg p) = 1$. On the other hand, $v_y(\Box p \vee \neg\Box p) = \frac{1}{2}$. So, we obtain $v_w(\Box(\Box p \vee \neg\Box p)) = 0$. Therefore, $\Box p \vee \Box\neg p \not\models_{\text{Ł}_3^{\mathsf{S0.5}}} \Box(\Box p \vee \neg\Box p)$.

The failure of substitutivity of equivalents is the reason to not introduce Δ in Theorem 6.5.4. Now, if one asks for a general result, encompassing all $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$ and $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$, he or she should provide a Δ-like modality which gives only classical values to the formulas in the scope of Δ-like in all the accessible worlds. This problem becomes apparent when we deal with logics which have more than one designated value. And now we turn to this problem.

The logic $\mathsf{LP}^{\mathsf{S5}}$

The logic $\mathsf{LP}^{\mathsf{S5}}$ was defined in Definition 6.3.8. Different from Ł_3^{S5}, we cannot use $\Box$ in order to define Δ in $\mathsf{LP}^{\mathsf{S5}}$ because the truth condition of $\Box$ includes cases where a formula can receive the value $\frac{1}{2}$ since it is a designated value. In Ł_3^{S5} it is simpler because 1 is the only designated value. Then it is necessary to find a way to define a modality like Δ which only takes 1 or 0 in each world $w \in W$. Consider the following definition:

$$\circledcirc\varphi \quad := \quad \neg\Diamond\neg\varphi$$

As a consequence, the truth condition of $\circledcirc\varphi$ is given by the following clause:

$v_w(\circledcirc\varphi) = 1$ iff for every $y \in W$ such that wRy, $v_y(\varphi) = 1$; otherwise, $v_w(\circledcirc\varphi) = 0$

Then we define:

$$\blacktriangle\varphi \quad := \quad \circledcirc\varphi \vee \circledcirc\neg\varphi$$

$v_w(\blacktriangle\varphi) = 1$ iff (for every $y \in W$ such that wRy, $v_y(\varphi) = 1$) or (for every $z \in W$ such that wRz, $v_z(\varphi) = 0$); otherwise, $v_w(\blacktriangle\varphi) = 0$.

With the connective $\blacktriangle$ in hands, we now prove a similar result we proved for Ł_3^{S5}.

Theorem 6.5.5. *For every $\Gamma \subseteq For(\mathcal{L}_{\text{Ł}_3}^{\Box\Diamond})$, for every $\varphi \in For(\mathcal{L}_{\text{Ł}_3}^{\Box\Diamond})$,*

$$\Gamma \models_{\mathsf{S5}} \varphi \textit{ iff } \Gamma, \{\Delta p_1, \ldots, \Delta p_n\} \models_{\mathsf{LP}^{\mathsf{S5}}} \varphi \tag{6.16}$$

where $\{p_1, \ldots p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.

The proof of Theorem 6.5.5 is similar to the proof of Theorem 6.5.2.

Now, if one asks for a general result, encompassing all $\mathfrak{L}^{\mathsf{S0.5}}$ and $\mathfrak{L}^{\mathsf{S5}}$, he or she should provide a $\Box$-like modality which gives the value 1 at w if φ receives the value 1 in all accessible worlds to w. That is, such operator, call it $\Box^*$ should behave as follows:

A) For logics $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$:

- For any $w \in W$, $v_w(\Box^*\varphi) = 1$ if $w \in N$ and for all $y \in W$ such that wRy, $v_y(\varphi) = 1$; otherwise $v_w(\Box\varphi) = 0$;

B) For logics $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S5}}$:

- $v_w(\Box^*\varphi) = 1$ if for all $y \in W$ such that wRy, $v_y(\varphi) = 1$; otherwise $v_w(\Box\varphi) = 0$;

Such modalities behave like the Rosser & Turquette (1952) J-connectives , which are truth-functional connectives able to identify the values on the truth-table in the family of finite-valued Łukasiewicz logics Ł$_n$. For example: $v(J(\varphi)) = 1$ iff $v(\varphi) = 1$; othewise $v(J(\varphi)) = 0$. The same reasoning applies to the other truth-values. With modalities $\Box^*$ in hand it is possible to define modalities like Δ. Then it is possible to prove such recovery results. But this goes beyond the interest of this paper.

By Suszko's reduction result, we could have worked with the bivalent counterpart of each many-valued logic L. For example, we could have used the bivaluation semantics which characterizes the logic Ł$_3$ and define modal structures for Ł$_3^{\mathsf{S0.5}}$ and Ł$_3^{\mathsf{S5}}$ upon such bivalent semantics.[26][27] The reason for presenting here a matricial semantics for Ł$_3$ instead of a bivalent semantics stems from the fact that matrix semantics are more user-friendly than bivalent semantics. Moreover, not every many-valued semantics can be directly characterized by a bivalent semantics because many of them do not have expressive power enough to define a recovery operator in order to distinguish the intermediate values from a bivalent perspective. Such thing happen with many logics which have more than three values. In these cases, we have to extend the basic language of these logics with a recovery operator.[28] This shows how recovery operators and reductive results *à la* Suszko interact each other.

In the beginning of this Chapter, we said that the modalities $\Box$ and $\Diamond$ of the logics $\mathsf{L}^{\mathsf{S0.5}}$'s and L^{S5}'s can be called suszkian modalities because they only receive truth and falsity. This reflects the classicality of the metatheory of MVLs. The validity of this formula in $\mathsf{LP}^{\mathsf{S5}}$ is not adequate for our present interpretation for $\Box$. From the metatheoretical point of view, a formula φ is valid/consistent or non-valid/consistent. There are no intermediate values in those metatheoretical statements if we maintain a classical metatheory.

Because of the bivalent character of the modalities it was possible to show that they are able to define modalities such as Δ which work as recovery operators, allowing to recover inferences of classical $\mathsf{S5}$. In what concerns reductive results by means of modalities instead of truth-functional connectives like $\star$ is beyond our interest in the present work.

[26] We refer the reader to check Malinowski (1993) for a bivalent semantics for Ł$_3$.

[27] Rosenblatt (2015) undertook such enterprise of in truth-theories based on non-classical logics.

[28] The problem of bivalent reduction of MVLs with weak expressive power is discussed in Caleiro et al (2005).

Our main point in this section was to show that metatheoretical concepts, such as validity and consistency, once introduced in the object language, are capable to recover inferences in modal logic based on classical logic.

Last, but not least, we turn to the relation between the modalities investigated here and the meaning of recovery operators. It is not difficult to find in the literature arguments defending that recovery operators, such as $\star$, internalize metalogical concepts in the object language of the logic. In the case of $Ł_3$ it is said that $\star$ internalize a form of decidability. But it is clear that $\star$ and Δ differ in meaning, as the following validities show:

1. $\models_{Ł_3^{S5}} (\star(\varphi \vee \psi) \wedge (\varphi \vee \psi)) \rightarrow ((\star\varphi \wedge \varphi) \vee (\star\psi \wedge \psi))$
2. $\not\models_{Ł_3^{S5}} (\Delta(\varphi \vee \psi) \wedge (\varphi \vee \psi)) \rightarrow ((\Delta\varphi \wedge \varphi) \vee (\Delta\psi \wedge \psi))$

As a consequence, $\star$ does not have the same interpretation as Δ. Since the results of this chapter show that $\Box$ and $\Diamond$, and then Δ, have a well justified interpretation of validity and consistency (and then decidability), we cannot say the same with respect to $\star$. That is, it is not obvious that $\star$ incorporates a metalogical notion in the object language of logic. This same argument can be applied to the LFIs since $\circ$ is interpreted as consistency. As we argued in Chapter 3, the informal interpretation of $\circ$ does not do justice to the behaviour of $\Diamond$.

6.6 A brief look to Strictly Tolerant Logic

All the logics we investigated in this chapter are tarskian structural logics in the sense of Definition 2.1.3. In this Section, we will investigate a many-valued logic widely discussed in the last few years, the *Strictly Tolerant Logic* (ST), formulated by Cobreros et al (2012). Here we will concentrate on the propositional fragment of ST and on the semantical presentation of it given by Barrio et al (2020a, 2020b).

Definition 6.6.1. *The logic* ST *has the language* $\mathcal{L}_{\mathsf{ST}}$ *is characterized by the matrix* $M_{\mathsf{ST}} = \langle\{1, \frac{1}{2}, 0\}, \neg, \vee, \{1\}, \{1, \frac{1}{2}\}\rangle$ *whose operations have the same truth tables of* LP *(Definition 6.1.5).*

Let sem_{ST} *be the set of all valuations* v *of* ST*. We say that* v *is a model for* φ *if* v *assigns* 1 *or* $\frac{1}{2}$ *to* φ*.* φ *is a tautology of* ST *if every valuation* $v \in sem_{\mathsf{ST}}$ *is a model for* φ*. The relation* $\models_{\mathsf{ST}} \subseteq \wp(For(\mathcal{L}_{\mathsf{ST}})) \times For(\mathcal{L}_{\mathsf{ST}})$ *is defined as follows:*

$$\Gamma \models_{\mathsf{ST}} \varphi \text{ iff: if } v(\gamma) = 1, \text{ for each } \gamma \in \Gamma, \text{ then } v(\varphi) \in \{1, \frac{1}{2}\}. \tag{6.17}$$

The logic ST has some peculiarities which are important to remark. First, as we can see in Definition 6.6.1, the matrix M_{ST} has two sets of designated values. We can understand

$\{1\}$ as the set of designated valued of premises and $\{1, \frac{1}{2}\}$ the set of designated values of conclusions. Second, the relation $\models_{\mathsf{ST}}$ is not transitive, as the following proposition shows:

Proposition 6.6.2. $\models_{\mathsf{ST}}$ *is not transitive. That is, it does not satisfy*

$$\textit{If } \varphi \models_{\mathsf{ST}} \psi \textit{ and } \psi \models_{\mathsf{ST}} \gamma, \textit{ then } \varphi \models_{\mathsf{ST}} \gamma. \tag{6.18}$$

Proof. To invalidate the above schema, let $\varphi = p$, $\psi = q$ and $\gamma = r$ such that $v(p) = 1$, $v(q) = \frac{1}{2}$ and $v(r) = 0$. By 6.17 of Definition 6.6.1, we obtain $p \models_{\mathsf{ST}} q$ and $q \models_{\mathsf{ST}} r$, but $p \not\models_{\mathsf{ST}} r$. Then, 6.18 is not valid. Q.E.D.

By invalidating transitivity of consequence relation ST is not a tarskian logic, in the sense of Definition 2.1.3.

Surprisingly, ST is too close to CPL. In fact, by having the same truth-tables as LP, ST has the same tautologies as CPL. Also, because of the definition of $\models_{\mathsf{ST}}$, ST behaves inferentially like CPL. On the other hand, we cannot say that ST and CPL are the same system because, ST is non-transitive and it is able to be a basis for a naïve theory of truth, whereas classical logic is not.

The difference between ST and CPL lies in the validation of rules which validates other rules. As Barrio et al (2016) define, a *metarule* is a rule which establishes that if a certain rule is valid, then so is another rule. The transitivity is a clear example: given $\varphi \models_{\mathsf{CPL}} \psi$ and $\psi \models_{\mathsf{CPL}} \gamma$, we obtain $\varphi \models_{\mathsf{CPL}} \gamma$. But it does not happen with ST because the consequence relation of ST is not transitive. Then, although ST and CPL have the same tautologies and the same inference relations, they differ on the metarules. As Barrio et al (2015) prove, a metarule is valid ST if and only if the corresponding inference rule is valid in LP. For example, consider the rule of explosion and its corresponding metarule:

1. $\varphi, \neg\varphi \models \psi$;
2. $\models \varphi$ and $\models \neg\varphi$, then $\models \psi$.

As we know, 1 is invalid in LP. Just consider a valuation $v \in sem_{\mathsf{LP}}$ such that $v(\varphi) = \frac{1}{2}$ and $v(\psi) = 0$. By definition of consequence in ST, 1 is a valid rule. But, in using the same counterexample of LP we invalidate 2 in ST. The metarule of modus ponens is also invalid in ST:

3 $\models \varphi$ and $\models \varphi \to \psi$, then $\not\models \psi$.

Further we will provide a proof system for ST. Now we will analyse this logic from the point of view of the modalities we introduced in this chapter. We will see that, although CPL and ST coincide in inferences and tautologies, their modal expansions are very different.

6.6.1 The modal logic $\mathsf{ST}^{\mathsf{S5}}$

The logic $\mathsf{ST}^{\mathsf{S5}}$ has the language $\mathcal{L}^{\Box\Diamond}_{\mathsf{ST}}$ and it is semantic characterized as follows:

Definition 6.6.3. *The logic* ST *is characterized by the structure* $\mathcal{M} = \langle W, R, M_{\mathsf{ST}}, v\rangle$*, where* $W \neq \emptyset$ *is a set of worlds,* R *is an equivalence relation,* M_{ST} *is a matrix defined as in Definition 6.6.1 and* v_w *is recursively defined as follows:*

1 *The boolean cases are defined as in Definition 6.6.1;*

2 $v_w(\Box\varphi) = 1$ *iff for all* $y \in W$*,* wRy *implies* $v_y(\varphi) \in \{1, \frac{1}{2}\}$*; otherwise* $v_w(\Box\varphi) = 0$

3 $v_w(\Diamond\varphi) = 1$ *iff there is* $y \in W$*,* wRy *and* $v_y(\varphi) \in \{1, \frac{1}{2}\}$*; otherwise* $v_w(\Diamond\varphi) = 0$

A formula $\varphi \in For(\mathcal{L}^{\Box\Diamond})$ *is true in a* $\mathsf{ST}^{\mathsf{S5}}$*-model* $\mathcal{M}$ *iff for every* $w \in W$ $v_w(\varphi) \in \{1, \frac{1}{2}\}$*.* φ *is valid iff it is true in every* $\mathsf{ST}^{\mathsf{S5}}$*-model. The relation* $\Gamma \models_{\mathsf{ST}^{\mathsf{S5}}} \varphi$ *is defined as in Definition 6.6.1.*

As we said before, although ST and CPL are close, their modal extensions, $\mathsf{S5}$ and ST are remarkably different. By taking the counterexample of Proposition 6.1.24, we can show that:

Proposition 6.6.4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *is not valid in* $\mathsf{ST}^{\mathsf{S5}}$*.*

Although $\mathsf{ST}^{\mathsf{S5}}$ validates explosion rule, it does not validate the modal validity explosion rule

Proposition 6.6.5. $\Box\varphi, \Box\neg\varphi \models \psi$ *is not valid in* $\mathsf{ST}^{\mathsf{S5}}$*.*

The proposition below show $\mathsf{ST}^{\mathsf{S5}}$ and $\mathsf{LP}^{\mathsf{S5}}$ are really different each other. The principle in question can be called *validity detachment*.[29]

Proposition 6.6.6. $\varphi, \Box(\varphi \to \psi) \models \psi$ *is valid in* $\mathsf{ST}^{\mathsf{S5}}$ *but invalid in* $\mathsf{LP}^{\mathsf{S5}}$*.*

The reason why $\mathsf{ST}^{\mathsf{S5}}$ validates validity detachment is simple: we only consider the case where φ takes 1 because of the definition of consequence relation of $\mathsf{ST}^{\mathsf{S5}}$.

Definition 6.6.7. *Let* $\mathcal{L}_{\mathsf{ST}}$ *be the language of* ST *and* $M_{\mathsf{ST}} = \langle\{1, \frac{1}{2}, 0\}, \neg, \to, \{1, \frac{1}{2}\}\rangle$ *be a matrix for* $\mathcal{L}_{\mathsf{ST}}$*. The models* v^n_i *for the language* $\mathcal{L}^n_{\mathsf{ST}}$ *are induced by a model* v^0_i *of* $\mathcal{L}^0_{\mathsf{ST}}$ *as follows:*

(1) The models v^0_i *of* $\mathcal{L}^0_{\mathsf{ST}}$ *are valuations* $v_i \in sem_{\mathsf{ST}}$ *defined in Definition 6.6.1.*

[29] In the literature about the predicate of validity, it is usual to use a binary predicate $Val(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$ to mean that the inference from φ to ψ is valid. The way that *validity detachment* is usually stated is $\varphi, Val(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \psi$, where $\Rightarrow$ is the deducibility relation in sequent calculus (Murzi (2014)). Here, we formalize $Val(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$ as $\Box(\varphi \to \psi)$ because we are working on unary modalities.

(2) The models v_i^{n+1} of $\mathcal{L}_{\mathsf{ST}}^{n+1}$ induced by a model v_i^0 of $\mathcal{L}_{\mathsf{ST}}^0$ is the smallest extension of v_i^n of $\mathcal{L}_{\mathsf{ST}}^n$ such that:

(2.1) If $\varphi \in \mathcal{L}_{\mathsf{ST}}^n$ and $\varphi = c_m^{k_m}(\psi_1, \ldots, \psi_k)$ for $c_m^{k_m} \in \{\neg, \rightarrow\}$, then:

(2.1.1) $v_i^{n+1}(\neg\varphi) = 1 - v_i^{n+1}(\varphi)$;

(2.1.2) $v_i^{n+1}(\varphi \rightarrow \psi) = max(v_i^{n+1}(\neg\varphi), v_i^{n+1}(\psi))$.

(2.2) If $\varphi \in \mathcal{L}_{\mathsf{ST}}^n$, then:

(2.2.1) $v_i^{n+1}(Val(\overline{\varphi})) = 1$ *if* $v_j^n(\varphi) \in \{1, \frac{1}{2}\}$ *for all models* v_j^n *of* $\mathcal{L}_{\mathsf{ST}}^n$;
otherwise $v^{n+1}(Val(\overline{\varphi})) = 0$;

(2.2.2) $v_i^{n+1}(Con(\overline{\varphi})) = 1$ *if* $v_j^n(\varphi) \in \{1, \frac{1}{2}\}$ *for some model* v_j^n *of* $\mathcal{L}_{\mathsf{ST}}^n$;
otherwise $v_i^{n+1}(Con(\overline{\varphi})) = 0$;

Now, the model v_i^ω of $\mathcal{L}_{\mathsf{ST}}^\omega$ induced by a v_i^0 is the union of all v_i^n induced by v_i^0 of $\mathcal{L}_{\mathsf{ST}}^0$. $\varphi \in For(\mathcal{L}_{\mathsf{ST}}^\omega)$ is true *in a model v_i^ω of $\mathcal{L}_{\mathsf{ST}}^\omega$ if $v_i^\omega(\varphi) \in \{1, \frac{1}{2}\}$. $\varphi \in For(\mathcal{L}_{\mathsf{ST}}^\omega)$ is* valid *if φ is true in every model v_i^ω of $\mathcal{L}_{\mathsf{ST}}^\omega$. The notion of logical consequence is defined as in Definition 6.6.1.*

Lemma 6.6.8. *Let $\mathcal{A}_{\mathsf{ST}}$ be a set of models v_i^ω of $\mathcal{L}_{\mathsf{ST}}^\omega$ which are induced by v_i^0 of $\mathcal{L}_{\mathsf{ST}}^0$. For every set $\mathcal{A}_{\mathsf{ST}}$ of models v_i^ω of $\mathcal{L}_{\mathsf{ST}}^\omega$ we define a model $\mathcal{M} = \langle W, v\rangle$ for $\mathsf{ST}^{\mathsf{S5}}$ such that for all $v_i^\omega \in \mathcal{A}_{\mathsf{ST}}$ there is $x_i \in W$*

$$v_{x_i}(\varphi) = v_i^\omega(t(\varphi)).$$

For all $\varphi \in For(\mathcal{L}_{\mathsf{ST}}^{\Box\Diamond})$.

The proof of Lemma 6.6.8 is similar to the proof of Lemma 6.2.8.

Lemma 6.6.9. *For every model $\mathcal{M} = \langle W, v\rangle$ for L^{S5} we define a set $\mathcal{A}_{\mathsf{ST}}$ of models v_i^ω of $\mathcal{L}_{\mathsf{L}}^\omega$ induced by v_i^0 of $\mathcal{L}_{\mathsf{L}}^0$ such that for all $x_j \in W$ there is $v_i^\omega \in \mathcal{A}_{\mathsf{ST}}$:*

$$v_{x_i}(\varphi) = v_i^\omega(t^{-1}(\varphi)).$$

For all $\varphi \in For(\mathcal{L}_{\mathsf{ST}}^\omega)$.

The proof of Lemma 6.6.9 is similar to the proof of Lemma 6.2.9.
Lemmas 6.6.8 and 6.6.9 imply:

Theorem 6.6.10. *For every $\varphi \in For(\mathcal{L}_{\mathsf{ST}}^{\Box\Diamond})$:*

1. $\models_{\mathsf{ST}^{\mathsf{S5}}} \varphi$ *iff* $t(\varphi)$ *is valid in models* M_{ST}^*
2. *Let* $t(\Gamma) = \{t(\gamma \mid \gamma \in \Gamma\}$. *Then,* $\Gamma \models_{\mathsf{ST}^{\mathsf{S5}}} \varphi$ *iff* $t(\Gamma) \models_{\mathcal{L}_{\mathsf{ST}}^\omega} t(\varphi)$.

In the debate about the internalization of the predicate of validity, there is the question of whether the validity predicate captures all metainferences of the theory. If a validity theory of a logic L is not capable to capture all of its metainferences, then we may ask whether the validity predicate of such theory is adequate for L. In our approach, such adequacy is guaranteed by the Theorem 6.6.10.

A tableau system for $\mathsf{ST}^{\mathsf{S5}}$

In the same line of the logics $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$, we provide a proof system for $\mathsf{ST}^{\mathsf{S5}}$. The tableau rules are stated as follows:

$$\begin{array}{ccc} [1]\neg\varphi,i & [\tfrac{1}{2}]\neg\varphi,i & [0]\neg\varphi,i \\ | & | & | \\ [0]\varphi,i & [\tfrac{1}{2}]\varphi,i & [1]\varphi,i \end{array}$$

$$\begin{array}{ccccc} & & [1]\varphi\to\psi,i & & \\ \swarrow & \swarrow & \downarrow & \searrow & \searrow \\ [1]\varphi,i & [\tfrac{1}{2}]\varphi,i & [0]\varphi,i & [0]\varphi,i & [0]\varphi,i \\ [1]\psi,i & [1]\psi,i & [1]\psi,i & [\tfrac{1}{2}]\psi,i & [0]\psi,i \end{array}$$

$$\begin{array}{cccc} [\tfrac{1}{2}]\varphi\to\psi,i & & & [0]\varphi\to\psi,i \\ \swarrow & \downarrow & \searrow & | \\ [1]\varphi,i & [\tfrac{1}{2}]\varphi,i & [\tfrac{1}{2}]\varphi,i & [1]\varphi,i \\ [\tfrac{1}{2}]\psi,i & [\tfrac{1}{2}]\psi,i & [0]\psi,i & [0]\psi,i \end{array}$$

$$\begin{array}{cccc} [1]\Box\varphi,i & & & [1]\Diamond\varphi,i \\ | & & & | \\ irj & & & irj\ (i \text{ new}) \\ \swarrow & \searrow & & \swarrow & \searrow \\ [1]\varphi,i & [\tfrac{1}{2}]\varphi,j & & [1]\varphi,i & [\tfrac{1}{2}]\varphi,j \end{array}$$

$$\begin{array}{cc} [0]\Box\varphi,i & [0]\Diamond\varphi,i \\ | & | \\ irj\ (i \text{ new}) & irj \\ [0]\varphi,i & [0]\varphi,i \end{array}$$

The definitions of closed branch and closed tableau runs as in Definition 6.2.4. Now we define the notion of proof for the tableaux of $\mathsf{ST}^{\mathsf{S5}}$:

Definition 6.6.11. $\Sigma \vdash_{\mathsf{ST}^{\mathsf{S5}}} \varphi$ *if there is a closed tableau* $\mathcal{T}$*'s such that:*

1. *For each* $\sigma_i \in \Sigma$, $[1]\sigma_i, 0$;

2. $[0]\varphi, 0$.

Theorem 6.6.12. *If* $\Sigma \vdash_{\mathsf{ST}^{\mathsf{S5}}} \varphi$, *then* $\Sigma \models_{\mathsf{ST}^{\mathsf{S5}}} \varphi$.

Proof. Suppose that $\Sigma \nvDash \mathsf{ST}^{\mathsf{S5}}\varphi$. Then there is a model $\mathcal{M} = \langle W, R, v\rangle$ for $\mathsf{ST}^{\mathsf{S5}}$ such that for some $w \in W$, $v_w(\sigma_i) = 1$, for all $\sigma_i \in \Sigma$, and $v_w(\varphi) = 0$. Let $\mathcal{T}$ be a tableau with the premisses $\sigma \in \Sigma$ and the conclusion φ. Suppose that the function $f : \mathbb{N} \to W$ shows $\mathcal{M}$ to be faithful to a branch b of $\mathcal{T}$. The premisses of $\sigma \in \Sigma$ begin with (i) $[1]\sigma_i, 0$ and (ii) $[0]\varphi, 0$. By Lemma 6.7.8 applying the rules to (i) and to (ii), we extend b with at least one extension b' such that $\mathcal{M}$ is faithful to b'. Since $\mathcal{M}$ is faithful to b', then the whole branch b remains open. Therefore $\Sigma \nvdash_{\mathsf{ST}^{\mathsf{S5}}} \varphi$. Q.E.D.

Theorem 6.6.13. *If* $\Sigma \models_{\mathsf{ST}^{\mathsf{S0.5}}} \varphi$, *then* $\Sigma \vdash_{\mathsf{ST}^{\mathsf{S0.5}}} \varphi$.

Proof. Suppose that $\Sigma \nvdash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$. Then there is an open branch b which contains in its initial list premisses $[1]\sigma, 0$, for $\sigma_i \in \Sigma$ and $[0]\varphi, 0$. Let $\mathcal{M} = \langle W, R, v\rangle$ be the induced interpretation by b. So, if $[1]\sigma_i, 0$ and $[0]\varphi, 0$ occur on b, then, by Lemma 6.7.10, $v_{w_i}(\sigma_k) = 1$ and $v_{w_i}(\varphi) = 0$. Therefore $\Sigma \nvDash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$. Q.E.D.

6.7 Soundness and completeness for $\mathfrak{L}^{\mathsf{S0.5}}$ and $\mathfrak{L}^{\mathsf{S5}}$

In this section we provide the characterization results for the family of modal logics $\mathsf{L}^{\mathsf{S0.5}}$ and for L^{S5}. The following proofs and definitions follows Priest (2008b).

6.7.1 The logics $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$

Definition 6.7.1. *Let* $\mathcal{M} = \langle W, N, R, v\rangle$ *be a* M_{L}*-modal model for* $\mathsf{L}^{\mathsf{S0.5}}$ *and* b *any branch of a tableau* $\mathcal{T}$. *Then* $\mathcal{M}$ *is* faithful *to* b *iff there is a map* $h : \mathbb{N} \to W$ *such that:* $N = \{0\}$ *and for all* $i > 0$*: (i) if* $[t^m]\Box\varphi, i$ *is on* b, *then* $v_{h(i)}(\Box\varphi) = t^m$. *For every node of* b, *if* $[t^m]\varphi, i$ *is on* b, $v_{h(i)}(\varphi) = t^m$.

Lemma 6.7.2. *Let* b *be any branch and* $\mathcal{M} = \langle W, N, R, v\rangle$ *be a* M_{L}*-modal model for* $\mathsf{L}^{\mathsf{S0.5}}$. *If* $\mathcal{M}$ *is faithful to* b, *and a rule is applied to it, then it produces at least one extension,* b', *such that* $\mathcal{M}$ *is faithful to* b'.

Proof. The proof runs by case-by-case checking.

1. Boolean connectives.

Suppose that $[t^m]c_n^k(\varphi_1,\ldots,\varphi_k), i$ occurs in b, where $i \in \mathbb{N}$ and $0 \leq m \leq n-1$. Applying $c_n^k - rule$ to it, the branch b is extended with at least one b' such that $\bigwedge_r^s[t^r]\varphi_j, i$ occurs in b', for $1 \leq j \leq k$. That is, $[t^r]\varphi_1, i \,\&\ldots\&[t^s]\varphi_k, i$ occur in b'. By hypothesis, $\mathcal{M}$ is faithful to b, then $v_{h(i)}(\varphi_1) = t^r \,\&\ldots\&\, v_{h(i)}(\varphi_k) = t^s$. Since c_n^k is interpreted as the operation o_n^k of the matrix M_{L}, then $v_{h(i)}(\varphi_1,\ldots,\varphi_k) = t^m$.

2. Modal connectives. Suppose that $[1]\Box\varphi, 0$ occurs in b. Applying the corresponding rule to it, we extend b with at least one a branch b' such that $0\mathfrak{r}j$ and $[t^m]\varphi, j$ occur in b', and $0 \leq k < m \leq n-1$ where k is the greatest non-designated value. Since $\mathcal{M}$ is faithful to b, $v_{h(0)}(\Box\varphi) = 1$. $h(0) \in N$ and, then, it is a normal world. If $0\mathfrak{r}j$, for all j, then $h(0)Rh(j)$. Then, $v_{h(j)}(\varphi) = t^m$ such that $t^m \in D$. Therefore, $\mathcal{M}$ is faithful to b'.

Suppose that $[0]\Box\varphi, 0$ occurs in b. Applying the corresponding rule to it, we extend b with at least one a branch b' such that $0\mathfrak{r}j$ and $[t^m]\varphi, j$, j being new, occur in b', and $0 \leq m < k \leq n-1$, where k is the least designated value. Since $\mathcal{M}$ is faithful to b, $v_{f(h)}(\Box\varphi) = 0$. $h(0) \in N$ and, then, it is a normal world. Then w_0Ry, for some $y \in W$. Consider a slight variation of h, call it h', which only differs to h in $h'(j) = y$. Since h and h' only differ in j, $\mathcal{M}$ is also faithful to b with respect to h'. Then $h'(i)Rh'(j)$ and $v_{h'(j)}(\varphi) = t^m$, $t^m \notin D$.

The argument for $[1]\Diamond\varphi, 0$ and $[0]\Diamond\varphi, 0$ is respectively similar to $[0]\Box\varphi, 0$ and $[1]\Box\varphi, 0$.

The cases $i > 0$ are immediate from Definition 6.7.1. Now we analyse the rule ρ. Suppose that $0\mathfrak{r}j$ occurs on b. By applying rule ρ, we extend b with b' where $0r0$ occurs on b'. Since $\mathcal{M}$ is faithful to b, $h(0)Rh(j)$. Since R is reflexive over $N \subseteq W$, we obtain $h(0)Rh(0)$. Therefore, $\mathcal{M}$ is faithful to b'. Q.E.D.

Theorem 6.7.3. *If* $\Sigma \vdash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$, *then* $\Sigma \models_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$.

Proof. Suppose that $\Sigma \nvDash \varphi$. Then there is a model $\mathcal{M} = \langle W, N, R, v\rangle$ such that for some $w \in N$, $\mathcal{M}, w \models \sigma$, for all $\sigma \in \Sigma$, and $\mathcal{M}, w \nvDash \varphi$. Let $\mathcal{T}$ be a tableau with the premisses $\sigma \in \Sigma$ and the conclusion φ. Suppose that the function $f : \mathbb{N} \to W$ shows $\mathcal{M}$ to be faithful to a branch b of $\mathcal{T}$. The premisses of $\sigma \in \Sigma$ begin with (i) $[t^m]\sigma, 0$, for $0 \leq k < m \leq n-1$ and $[t^j]\varphi, 0$, where $0 \leq j < r \leq n-1$. By Lemma 6.7.2 applying the rules to (i) and to (ii), we extend b with at least one extension *b'* such that $\mathcal{M}$ is faithful to *b'*. Since $\mathcal{M}$ is faithful to *b'*, then the whole branch b remains open. Therefore $\Sigma \nvDash \varphi$. Q.E.D.

Definition 6.7.4. *Let b an open branch of a tableau. The interpretation* $\mathcal{M} = \langle W, N, R, v\rangle$ induced by b *is defined as:* $W = \{w_i|$ *i occurs on b*$\}$; w_iRw_j *iff* $i\mathfrak{r}j$ *occurs on b; if* $[t^m]p$ *occurs on b,* $v_{w_i}(p) = t^m$ *(*$0 \leq m \leq n-1$*).* $w_0 = 0$ $w_0 \in N$ *and for all* $i > 0$*: if* $[t^m]\Box\varphi, i$ *occurs on b,* $v_{w_i}(\Box\varphi) = t^m$ *(*$0 \leq m \leq n-1$*).*

Lemma 6.7.5. *Let b be any open complete branch of a tableau. Let $\mathcal{M} = \langle W, w_0, R, M_{\mathsf{L}}, v\rangle$ be the interpretation induced by b. Then,*

For every node, $[t^m]\varphi, i$, $v_{w_i}(\varphi) = t^m$ $(0 \leq m \leq n-1)$

Proof. The proof is by induction on φ. The atomic case is immediate from Definition 6.7.4.

$\varphi = c_n^k(\psi_1, \ldots, \psi_k)$

If $[t^m]c_n^k(\psi_1, \ldots, \psi_k), i$ is on b, then $\bigwedge_r^s[t^r]\varphi_{1\leq r\leq k}, i$ occurs on at least an extension of b, b'. By I.H., $v_{w_i}(\varphi_1) = t^s$ $(0 \leq s \leq n-1)$ & ... & $v_{w_i}(\varphi_k) = t^r$. The rule $c_n^k - rule$ requires the correspondence between the boolean connectives and the operators of the matrix M_{L}. Then, $v_{w_i}(c^k(\psi_1, \ldots, \psi_k)) = t^m$.

$\varphi = \Box\psi$. When $i = 0$, we have:

If $[1]\Box\psi, 0$ occurs on b, then $w_0 = 0$. By applying the correspondent rule, we extend b with b' where $0\mathfrak{r}j$, for every $j \in \mathbb{N}$, and $[t^m]\psi$ occur, for $0 \leq b < m \leq n-1$. Moreover $0\mathfrak{r}j$ iff w_0Rw_j, for all $w_j \in W$. By I.H., $v_{w_j}(\psi) \in D \subset V_n$. Therefore, $v_{w_0}(\Box\psi) = 1$.

If $[0]\Box\psi, 0$ occurs on b, then $w_0 = 0$. By applying the correspondent rule, we extend b with b' where $0\mathfrak{r}j$, for a new $j \in \mathbb{N}$, and $[t^m]\psi$ occur, for $0 \leq m < k \leq n-1$. Moreover $0\mathfrak{r}j$ iff w_0Rw_j, for some $w_j \in W$. Suppose there is a $w_{j'} \neq w_j$. If, by I.H., $v_{w_{j'}}(\psi) \in V_n - D$, then, $v_{w_0}(\Box\psi) = 0$.

If $[1]\Diamond\psi, 0$ occurs on b, then $0 \in N$. By applying the correspondent rule, we extend b with b' where $0\mathfrak{r}j$, for a new $j \in \mathbb{N}$, and $[\frac{m}{n-1}]\psi$ occur, for $0 \leq m < k \leq n-1$. Moreover $0\mathfrak{r}j$ iff w_0Rw_j, for some $w_j \in W$. Suppose there is a $w_{j'} \neq w_j$. If, by I.H., $v_{w_{j'}}(\psi) \in D$, then, $v_{w_0}(\Diamond\psi) = 1$.

The case $[0]\Diamond\psi, 0$ is similar to $[1]\Box\psi, 0$. When $i > 0$, $[t^m]\Box\psi, i$ and $[t^m]\Diamond\psi, i$ are arbitrary.

For the rule ρ: suppose that $0Rj$ occurs on b. Then, by rule ρ, $0\mathfrak{r}0$. By definition of R, w_0Rw_0. This concludes the proof.

Q.E.D.

Theorem 6.7.6. *If $\Sigma \models_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$, then $\Sigma \vdash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$.*

Proof. Suppose that $\Sigma \nvdash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$. Then there is an open branch b which contains in its initial list premisses $[t^m]\sigma, 0$, where $0 \leq k < m \leq n-1$, such that $\sigma_k \in \Sigma$ and $[t^j]\varphi, 0$, where $0 \leq j < k \leq n-1$. Let $\mathcal{M} = \langle W, w_0, R, M_{\mathsf{L}}, v\rangle$ be the induced interpretation by

b. So, if $[t^m]\sigma, 0$ and $t^j\varphi, 0$ occur on b, then $v_{w_i}(\sigma_k) = t^m$ and $v_{w_i}(\varphi) = t^j$. Therefore $\Sigma \nvDash_{\mathsf{L}^{\mathsf{S0.5}}} \varphi$. Q.E.D.

6.7.2 The logics $\mathsf{L}^{\mathsf{S5}} \in \mathfrak{L}^{\mathsf{S5}}$

Definition 6.7.7. *Let $\mathcal{M} = \langle W, R, v\rangle$ be a M_{L}-standard modal model for L^{S5} and b be any branch of a tableau $\mathcal{T}$. Then $\mathcal{M}$ is faithful to b iff there is a map $h : \mathbb{N} \to W$ such that $h(i) = w_i$. For every node of b, if $t^m\varphi, i$ is on b, then $v_{h(i)}(\varphi) = t^m$.*

Lemma 6.7.8. *Let b be any branch of a tableau, $\mathcal{M} = \langle W, R, , v\rangle$ be a M_{L}-standard modal model for L^{S5}. If $\mathcal{M}$ is faithful to b and a rule is applied to it, then it produces at least one extension b'.*

Proof. The proof of Lemma 6.7.8 is similar to the proof of Lemma 6.7.2. We will only analyse the case of the rules τ and σ. For τ, suppose that irj and jrk. Applying the rule τ, we extend b with b' where irk occur in b'. Since $\mathcal{M}$ is faithful to b, we obtain $h(i)Rh(j)$ and $h(j)Rh(k)$. Since R is transitive, we obtain $h(i)Rh(k)$. Therefore, $\mathcal{M}$ is faithful to b'. Q.E.D.

Definition 6.7.9. *Let b an open branch of a tableau. The interpretation induced by b is defined as $W = \{w_i |\ i$ occurs on $b\}$; w_iRw_j iff irj is on b; if $[t^m]p, i$ occurs on b, then $v_{w_i}(p) = t^m$.*

Lemma 6.7.10. *Let b be any open complete branch of a tableau. Let $\mathcal{M} = \langle W, R, v\rangle$ be the interpretation induced by b. Then, for every node and $0 \leq m \leq n-1$, if $[t^m]\varphi, i$ occurs on b, then $v_{w_i}(\varphi) = t^m$.*

Proof. The proof is similar to Lemma 6.7.5. We will only analyse the rules τ and σ. The rule ρ was previously analysed. For the rule τ, suppose that $i\mathfrak{r}j$ and $i\mathfrak{r}k$ are on b. Then, we obtain w_iRw_j and w_jRw_k. By the rule τ, we obtain $i\mathfrak{r}k$. By definition of R, w_iRw_k.

For the rule σ, suppose that $i\mathfrak{r}j$ is on b. Then w_iRw_j. By the rule ρ, we obtain $j\mathfrak{r}i$. By definition of R, w_jRw_i. This concludes the proof.

Q.E.D.

By similar arguments given in Theorem 6.7.3 and Theorem 6.7.6, we can prove soundness and completeness for logics L^{S5}.

Chapter 7

Final considerations

In this Thesis, we investigated some formalizations of the formal concepts of validity and consistency in order to know their general properties from a modal point of view. As the squeezing arguments show, even if model-theoretical and proof-theoretical validities capture important aspects of their informal counterpart, they fail to capture intuitive, or even pre-theoretic validity. That is, these formal notions have an important regulative role in our inferential practice, but they do not exhaust the totality of intuitive validity. The informal notions themselves present in the squeezing arguments are significantly theorized to be called intuitive. For example, Kreisel's informal notion Val, is a purely formal concept, in such a way that outside mathematical reasoning, one could legitimately say that Val is not our intuitive/pre-theoretic notion of validity. By its turn, intuitive validity is very difficult to grasp due to its high generality, to the point that it is legitimate to raise suspicions about its existence.[1]

The informal notions of validity present in squeezing arguments (Chapter 3) capture only a small fragment of natural language which is formalizable in logical systems. So, different inferential aspects of natural language are captured by different informal notions of validity. This means that these informal notions also share the local character that their formal counterparts have. In this sense, the informal notions investigated in Chapter 3 do not capture intuitive validity. Even so, they improve our understanding of deductive inferences, in such a way that they are useful in the analysis of ordinary reasoning. That is, because they are conceptually sharpened, they provide a better understanding of our inferential practice. In the case of FOL, its corresponding informal notions give the general principles of truth preserving reasoning. In the case of intuitionistic logic, its corresponding informal notions give the general principles of constructive reasoning. In the same way that the informal notions of logical validity provide a better understanding of our inferential practice, the formal notions of logical validity provide a better understanding of informal validity because they explain their corresponding informal notions within a

[1] As Halbach (2020) does in his paper.

well structured linguistic framework.

As we highlighted before, we are dealing with a strictly deductive notion of validity. In this Thesis, we did not deal with a notion of validity which involves inductive inferences. As we argued, a notion of validity which includes inductive inferences does not validate at least the principle Val-T because inductive inferences are not truth-preserving in general. Of course, the fact that they are neither deductive not truth-preserving is not a problem for such kind of reasoning. It would be a huge mistake to misjudge inductive inferences, which are widely present in scientific reasoning. Our choice for the deductive notion of validity was purely guided by theoretical preferences. The axiomatization of a wider notion of validity constitutes an interesting research program and it will be pursued in a further work.

"Which modal logic is the right one?" This is the title of Burgess (1999)'s paper, where he discuss the validity/provability interpretation of modal logics. When validity is understood as provability, Solovay's original result already responds to this question, because KGL captures the provability predicate $Prov_{\mathsf{PA}}$. But, with the development of the investigations about provability interpretation of modal logics, logicians established several *à la* Solovay results. For example, Goldblatt (1978) proves, $\mathsf{S4Grz}$ is the modal logic of true provability of PA. Artëmov & Straßen (1992) prove that their logic of proofs captures the explicit provability predicate $\exists y Pr_{\mathsf{PA}}(y, x)$ of PA. Although inconsistent with KGL, Kurahashi (2018) proves KD captures a non-standard Rosser provability predicate. More alternatively, if we consider Skyrm's naming devices instead of Gödel numbers, we have that $\mathsf{S4}$ captures the provability predicate Pr whose arguments are the sentence names of Section 5.3. So, such results show that Burgess's question needs to be more precise, as the following questions illustrate:

- What is the modal logic which captures the provability predicate $Prov_{\mathsf{PA}}$?
- What is the modal logic which captures the predicate of true provability of PA?
- What is the modal logic which captures Rosser provability predicate?
- What is the modal logic which captures the provability predicate whose arguments are sentence names?

If we consider weaker predicates as well as weaker base theories, such as logical validity as defined in in Section 5.5 and in Chapter 6, we obtain different modal systems. Thus, we have a plethora of modal logics capturing notions of logical validity. That is, each modal logic $\mathsf{L}^{\mathsf{S0.5}}$ captures their local notion of logical validity and logical consistency. On the other hand, if we consider a hierarchical notion of logical validity, by using Skyrms (1978)'s formalism, we can prove that each L^{S5} capture their corresponding local notion of hierarchical validity.

Now if one asks about a more general notion of logical validity, not essentially tied to a particular formal system, we have good reasons to believe that $\mathsf{S4}$ is at least sound with respect to this broader notion of validity, which is also taken as informal provability, as we argued in Chapter 4. But, as we know, the corresponding predicate cannot be representable in the arithmetical theories extending Q if we use the standard naming devices. In face of his incompleteness results, Gödel (1986a) himself defends that $\mathsf{S4}$ is adequate to stand for informal provability.

It is important to observe the role of intuitions in the axiomatization of the local notion of logical validity and in the axiomatization of the notion of informal provability. In the case of informal provability, the role of intuitions is remarkably strong. The fact that informal provability encompasses epistemological elements and that it is intuitively plausible to include analytical inferences in informal provability witness the role of intuitions in this case. This is totally different with logical validity and hierarchical validity, which heavily depends on the deductive capabilities of the logical system in question. The logics $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5} testify our claim.

The arithmetical completeness theorems for provability logics and the results proved for the logics $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5} assure that each logic captures well grounded notions of validity. This does not happen with the informal provability interpretation of $\mathsf{S4}$, since the acceptance of a principle about informal provability will depend on what one counts as an intuitive principle about informal provability. For example, as Leitgeb (2009), if statements about unprovability should be taken as axioms of informal provability, then the logic of informal provability is stronger than $\mathsf{S4}$.[2] This means that the absence of a completeness theorem with respect to the validity/provability interpretation makes it harder to assert that a modal logic really captures informal provability, because it will ultimately depend on the theoretical inclinations one has. For example, if one does not accept that statements about unprovability should be taken as axioms, then $\mathsf{S4}$ is the correct logic of informal provability. In general, if one wants to know whether a modal logic L captures our intuitions about informal provability, he/she should argue that the axioms at issue are plausible under such reading, or she/he should present a proof that L captures a formal validity/provability predicate.[3] That is, there is no general receipt. One has to analyse logic by logic.

Our general characterization results for $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5} are restricted to many-valued

[2]The discussion about the inclusion of statements about unprovability in informal provability is philosophically interesting. In the aforementioned discussion about a system which captures the non-valid inferences of a logical systems, there are some reasons to not include such rules about rejected propositions. For example, as Goranko (1994) observes, accepted arguments always yield true conclusions from true premises, whereas we cannot say anything in general about the rejected ones. As a consequence, any schematic approach to rejected arguments may be faced as useless. On the other hand, the philosophical reasons to include such rules may show the relations of some metatheoretical concepts of the formal system.

[3]Of course, there will be modal systems which do not have adequate formal/informal provability interpretation,, such as Lewis's strict implication system $\mathsf{S1}$.

logics Ls whose consequence relations are Tarskian structural in the sense of Definition 2.1.3 and whose connectives are normal in the sense of Assumption 6.0.1. So, the most direct extension of the results presented in Chapter 6 would be (1) logics which are not Tarskian; and (2) logics which do not have normal connectives. As an example of the first, we take Malinowski (1990)'s *q-consequence relations*, which are defined according to the matrices of the form $M_q = (V_n, o_1, \ldots o_n, D^+, D^-)$, where D^+ is the usual set of designated values and D^- is the set of *anti-designated values*. In these matrices, it may be the case that $D^+ \cup D^- \subset V_n$. Given the matrices M_q, one defines the relation $\models^q_{\mathsf{L}} \subseteq \wp(For(\mathcal{L}_{\mathsf{L}})) \times For(\mathcal{L}_{\mathsf{L}})$ as follows:

$$\Gamma \models^q_{\mathsf{L}} \varphi \text{ iff: if } v(\gamma) \notin D^-, \text{ for each } \gamma \in \Gamma, \text{ then } v(\gamma) \in D^+ \tag{7.1}$$

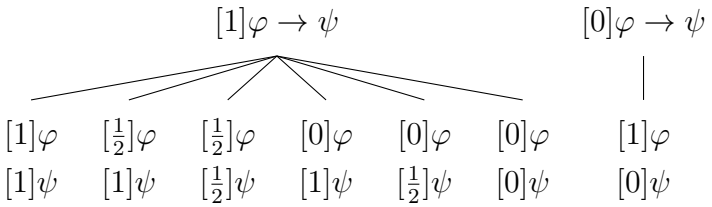
Malinowski (1990) applies such matrix to the logic Ł$_3$, obtaining Ł$^q_3$. In his paper, Malinowski shows that $\models^q_{Ł_3}$ is not reflexive. Because it is not reflexive, it is not immediate that the metatheory of the logics L^q are bivalent. As a consequence, the validity theory for the logics L^q must be adapted, as we did in the case of STT. The extension of our results for non-Tarskian consequence relations allows a wider understanding of the validity predicates, and it will be pursued in a future work.

The results proved in this thesis were also restricted for logics which have a complete deductive system. So, for example, it was enough to introduce only one (logical) validity predicate in the validity theory of $\mathsf{S0.5}$, instead of also introducing a logical provability predicate Pr, because we would have that $Val(\overline{\varphi}) \leftrightarrow Prov(\overline{\varphi})$. In the case that we have a logic L whose deductive system does not capture all the model-theoretical validities, we would have the following schemas:

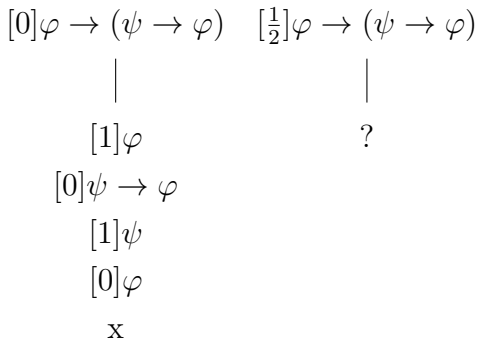
(Sound) $Prov(\overline{\varphi}) \rightarrow Val(\overline{\varphi})$

But the converse would not be valid since L is not complete. Here we will discuss two examples, one simple and one more complex. Given the tableaux of Chapter 6, it is quite simple to produce an (non-modal) incomplete logical system. Let L be the logic whose matrix $M_{\mathsf{L}} = \langle \{1, \frac{1}{2}, 0\}, \neg, \rightarrow, \{1\} \rangle$ where $\neg$ and $\rightarrow$ are interpreted in the same way as in Ł$_3$ (Example 6.1.13). The following rules define the deductive system for L:

$$\begin{array}{ccc} [1]\neg\varphi & [\frac{1}{2}]\neg\varphi & [0]\neg\varphi \\ \Big| & \Big| & \Big| \\ [0]\varphi & [\frac{1}{2}]\varphi & [1]\varphi \end{array}$$



The definitions of closed branch and proof are similar to Definitions 6.1.11 and 6.1.12, excepting that we are not considering the indexes which stand for the worlds. Given semantic definitions for the truth-functional operators of L, it is immediate that $\varphi \to (\psi \to \varphi)$ is a tautology of L. Hence, $Val(\overline{\varphi \to (\psi \to \varphi)})$ is true in the language $\mathcal{L}_{\mathsf{L}}^{Val}$ according to the formalism of Definition 6.1.19. On the other hand, $\varphi \to (\psi \to \varphi)$ is not provable, as the following tableaux show:



Since we do not have the rule for $[\frac{1}{2}]\varphi \to \psi$, the tableau does not close. Let $Prov$ be a provability predicate which extends $\mathcal{L}_{\mathsf{L}}^{Val}$ such that $Prov(\overline{\varphi})$ is true whenever φ is provable in L. Since $\varphi \to (\psi \to \varphi)$ is a theorem of L, then $Prov(\overline{\varphi \to (\psi \to \varphi)})$ is false. Therefore, $Val(\overline{\varphi}) \to Prov(\overline{\varphi})$ is not valid.

In this case, the solution is immediate because we can extend L with the rule for $[\frac{1}{2}]\varphi \to \psi$, and thus we obtain a complete system, which collapses with Ł$_3$. There are more complicated cases where the logic is inherently incomplete, such as the case of SOL. As we discussed in Chapter 3, full SOL does not have a complete deductive system and the reason for this is that its set of theorems is not recursively enumerable. In this case, no matter how we extend the axiomatic system of full SOL we will not obtain a completeness theorem for this logic, due to the high expressivity of SOL's language. It is necessary to adopt a weaker semantics, Henkin's general semantics, in order to obtain a complete deductive system for SOL. Given the lack of completeness theorem for full SOL, if $\nvdash_{\mathsf{SOL}} \varphi$ we do not know whether φ is really non-valid or that the deductive system is not capable to prove φ. So, the axiomatization of Val in the language $\mathcal{L}_{\mathsf{SOL}}^{Val}$ is left as an open problem.[4]

[4]As a side note, observe that the lack of a complete deductive system for SOL does not prevent some limitative results for the predicate of validity Val in the second-order language of arithmetic. For example, let $\mathsf{PA}^{\Box}$ be Peano Arithmetic extended with a predicate $\Box$ whose arguments are Gödel numbers

Last, we turn to the proof system adopted in Chapter 6. Labelled tableau systems is the most straightforward strategies in order to define a proof systems for the logics $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5}. Their simplicity, both operational and metalogical, were the main reason to adopt them for the logics at issue. But, one could argue that such proof systems are not interesting in a proof-theoretical perspective, because it does not allow a direct comparison with alternative approaches, such as Hilbert systems allow. A general axiomatization for the logics $\mathsf{L}^{\mathsf{S0.5}}$ and L^{S5} would constitute an interesting result about the predicates Val and Con of logics L. On the other hand, given the multiplicity of logical systems covered by $\mathfrak{L}^{\mathsf{S0.5}}$ and $\mathfrak{L}^{\mathsf{S5}}$ and the fact that the interpretation of the truth-functional connectives significantly varies according to the logic, it is not immediate for us how to obtain such a general result. Of course, the axiomatization of some important modal many-valued systems, such as $\text{Ł}_3^{\mathsf{S0.5}}$ (Ł_3^{S5}), $\mathsf{K_3}^{\mathsf{S0.5}}$ (resp., $\mathsf{K_3}^{\mathsf{S5}}$) and $\mathsf{LP}^{\mathsf{S0.5}}$ (resp., $\mathsf{LP}^{\mathsf{S5}}$) constitute interesting results about these families of modal systems.[5] But this will be investigated in a future work.

of formulas. Halbach et al (2003) provide limitative results for $\mathsf{PA}^{\Box}$ by means of purely model-theoretical arguments. We speculate that the same procedure can be applied to the case of $\mathsf{PA}^{\Box}$ in the language of SOL.

[5]The logic $\mathsf{K_3}$ is not usually presented by means of axiomatic systems because this logic has no tautologies. But, by adapting Shramko (2021)'s results to $\mathsf{K_3}$, it is still possible to provide a Hilbert-style axiomatization of this logic where the set of premises Γ is non-empty. That is, it is possible to reproduce a natural deduction proof system with such Hilbert axiomatization by providing only deduction rules for each connective of the language. So the relation $\Gamma \vdash_{\mathsf{K_3}} \varphi$ will be such that Γ is never empty, given that $\mathsf{K_3}$ has no theorems.

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