UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

FELIPE CÉSAR FREITAS MONTEIRO

Moduli Problems in Algebraic Geometry

Problemas de Moduli em Geometria Algébrica

Campinas

2022

Felipe César Freitas Monteiro

Moduli Problems in Algebraic Geometry

Problemas de Moduli em Geometria Algébrica

Dissertação apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Matemática.

Dissertation presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Mathematics.

Supervisor: Marcos Benevenuto Jardim

Este trabalho corresponde à versão final da Dissertação defendida pelo aluno Felipe César Freitas Monteiro e orientada pelo Prof. Dr. Marcos Benevenuto Jardim.

Campinas

2022

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

M764m Monteiro, Felipe César Freitas, 1996-
Moduli problems in algebraic geometry / Felipe César Freitas Monteiro. – Campinas, SP : [s.n.], 2022.

Orientador: Marcos Benevenuto Jardim.
Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Geometria algébrica. 2. Álgebra homológica. I. Jardim, Marcos Benevenuto, 1973-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Problemas de moduli em geometria algébrica
Palavras-chave em inglês:
Algebraic geometry
Homological algebra
Área de concentração: Matemática
Titulação: Mestre em Matemática
Banca examinadora:
Marcos Benevenuto Jardim [Orientador]
Gaia Comaschi
Eduardo de Sequeira Esteves
Data de defesa: 15-02-2022
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0003-4182-6071
- Currículo Lattes do autor: http://lattes.cnpq.br/9326360061745042

Dissertação de Mestrado defendida em 15 de fevereiro de 2022 e aprovada pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). MARCOS BENEVENUTO JARDIM

Prof(a). Dr(a). GAIA COMASCHI

Prof(a). Dr(a). EDUARDO DE SEQUEIRA ESTEVES

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Acknowledgements

I would like to thank my advisor, for all helpful discussions that lead to this text, the financial support through the grant #2019/20843-9, São Paulo Research Foundation (FAPESP), and my mom.

Resumo

O objetivo desta dissertação é estudar a teoria abstrata de problemas de moduli em Geometria Algébrica, descrevendo a Teoria Geométrica dos Invariantes (ou GIT, sigla em inglês) de David Mumford como uma abordagem geral para construção de espaços de moduli nesse contexto. Começamos com as definições de problemas de moduli e espaços de moduli usando linguagem categórica, com exemplos, e depois desenvolvemos a Teoria Geométrica dos Invariantes nos capítulos 2 e 3, sobre um corpo de característica zero. Finalmente, no último capítulo, aplicamos as ferramentas desenvolvidas para revisar a construção do espaço de moduli de fibrados vetoriais (semi)estáveis sobre curvas algébricas projetivas suaves.

Assumimos conhecimentos básicos da Teoria de Esquemas para os primeiros 3 capítulos, e no último também precisamos usar as ferramentas da álgebra homológica e cohomologia de feixes. A exposição segue as referências clássicas para o assunto, em especial as notas de aula da Prof. Victoria Hoskins (veja (HOSKINS, 2015)).

Palavras-chave: Espaços de moduli; Geometria Algébrica; Teoria Geométrica dos Invariantes; Álgebra Homológica;

Abstract

The purpose of this dissertation is to study the abstract theory of moduli problems in Algebraic Geometry, describing David Mumford's Geometric Invariant Theory (or GIT) as a general framework for building moduli spaces in this context. We start by defining moduli problems and spaces using categorical language, with various examples, and then study GIT in chapters 2 and 3, over a field of characteristic zero. Afterwards, in the last chapter, we apply the developed tools to review the construction of the moduli space of (semi)stable vector bundles over smooth projective algebraic curves.

We assume basic knowledge of scheme theory for most of the first three chapters, and in the fourth we also need to use tools from homological algebra and sheaf cohomology. The exposition follows the classical references for the subject, specially Prof. Victoria Hoskins' lecture notes (see (HOSKINS, 2015)).

Keywords: moduli spaces; algebraic geometry; geometric invariant theory; homological algebra;

Contents

	Introduction	**10**
1	**MODULI PROBLEMS**	**12**
1.1	**Yoneda's lemma**	**12**
1.2	**Moduli Problems in Algebraic Geometry**	**15**
2	**ALGEBRAIC INVARIANT THEORY**	**24**
2.1	**Algebraic Groups**	**25**
2.2	**Algebraic Group Actions**	**34**
2.3	**Representations of Algebraic Groups**	**43**
2.4	**Reductive Groups**	**46**
2.5	**Hilbert fourteenth problem**	**52**
3	**GEOMETRIC INVARIANT THEORY**	**56**
3.1	**Categorical Quotients**	**56**
3.2	**Affine Geometric Invariant Theory**	**62**
3.3	**Projective Geometric Invariant Theory**	**72**
3.3.1	Linear actions	72
3.3.2	Linearizations and the general case	79
3.3.3	Functoriality	88
3.4	**Criteria for (semi)stability**	**90**
3.5	**Applications**	**100**
4	**VECTOR BUNDLES OVER A CURVE**	**103**
4.1	**Coherent sheaves over curves**	**103**
4.2	**Vector bundles and torsion sheaves**	**109**
4.3	**Stability and slope**	**114**
4.4	**Study of the moduli functor**	**124**
4.5	**Quot scheme**	**126**
4.6	**The GIT setup**	**133**
4.7	**Analysis of stability**	**137**
4.8	**Le Potier criterion**	**142**
4.9	**Construction of the moduli space**	**145**
	BIBLIOGRAPHY	**151**

APPENDIX A – CATEGORY THEORY 155
A.1 Category Theory 155
A.2 Abelian Categories 159

APPENDIX B – SHEAVES AND SCHEMES 160

APPENDIX C – HOMOLOGICAL ALGEBRA 165
C.1 Exact and Derived Functors 165
C.2 Sheaf Cohomology 168

Introduction

Classification problems have always been a major part of mathematics, and moduli problems and spaces arise as a geometric realization of classification. Informally, given any problem of classification, a **moduli space** for this problem is a geometric space of the equivalence classes of the object. This usually gives a nice way of understanding properties of the object which is being classified, as it is displayed in a lot of modern mathematical achievements.

The usage of the word **moduli** in the context of classification, a Latin word which is the plural of **modulus** (or measure, parameter), starts with Riemann's work in the XIX century on Riemann surfaces. Riemann stated and calculated, in several different ways, that the number of moduli (or number of parameters) for a Riemann surface of genus $p \geq 2$ is given by $3p - 3$. In Riemann's work, the word has a vague idea of minimum number of coordinates, although there were no proper definitions for dimension or manifolds at the time. Throughout the XX century, the notion of moduli spaces evolved, and the formalization of these notions today uses categorical language. For more about the early history of moduli problems, we refer the interested reader to (A'CAMPO; JI; PAPADOPOULOS, 2016).

The objective of this dissertation is to study Geometric Invariant Theory, or GIT, a framework for constructing moduli spaces using algebraic group actions on schemes developed by David Mumford in (MUMFORD; FOGARTY; KIRWAN, 1994), and then study some applications, with a particular focus on the construction of the moduli space of vector bundles over smooth projective curves. We are heavily influenced by Victoria Hoskins's lecture notes (see (HOSKINS, 2015)) and by Peter Newstead textbook (see (NEWSTEAD, 2012)), among other classical references such as David Mumford's book (MUMFORD; FOGARTY; KIRWAN, 1994).

In the first chapter, we use category theory to properly define the concept of moduli problems, focusing our study on the category $\mathcal{C} = \mathrm{Sch}_k$ of finite type k–schemes, over an algebraically closed field k.

In the second chapter, we start the study of Algebraic Invariant Theory. Assuming basic knowledge about Algebraic Geometry (for example (HARTSHORNE, 1977), Chapter II) and introducing the language of algebraic group actions, we study the following problem, known as the **Hilbert Fourteenth Problem**: starting with a k–scheme X and an algebraic group action of an algebraic group G, when does the k–algebra of G–invariant regular functions over X is finitely generated?

In the third chapter, we develop Mumford's Geometric Invariant Theory, using the finitely generated k–algebra of invariants to construct quotients for the algebraic G action on X, when G is reductive and X is either affine or projective over k. To build a well-defined quotient space in the affine case, we need to restrict the study of the G–action to the open set of points with closed orbits and finite dimensional stabilizers (called **stable** points). In the projective case, which locally resembles the affine one, we first study the case when X is embedded in an ambient space, and then use invertible sheaves to generalize this condition. At the end of this chapter, we study some useful criteria to recognize stable (and semistable) points in applications.

In the fourth chapter, we focus our study into the moduli problem of vector bundles over smooth curves. To get a bounded moduli problem, we study the slope stability of vector bundles over curves. This is an important construction for modern research topics in algebraic geometry, and it is the basis for both the generalization for vector bundles over varieties with arbitrary dimensions and also for more abstract theories such as Bridgeland stability conditions on arbitrary triangulated categories. This is the heaviest chapter of this dissertation in terms of prerequisites, as we use tools from homological algebra and sheaf cohomology.

In the appendix, we recollect some basic definitions and theorems of algebraic geometry and category theory needed for this dissertation.

1 Moduli Problems

In this chapter, we review some category theory to define and study basic properties of abstract moduli problems.

1.1 Yoneda's lemma

We assume the reader is familiar with category theory. For some definitions, see the appendix A. If $\mathcal{C}, \mathcal{D}$ are categories, we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the category of (covariant) functors between $\mathcal{C}$ and $\mathcal{D}$, and by $\mathcal{C}^{\mathrm{op}}$ the opposite category to $\mathcal{C}$.

Definition 1.1.1. Let us fix a locally small category $\mathcal{C}$. The **functor of points** of an object $X \in \mathcal{C}$ is a contravariant functor:

$$\begin{aligned} h_X \doteq \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} &\to \mathrm{Sets} \\ Y &\mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X) \\ Y \xrightarrow{f} Z &\mapsto f^* : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X), \end{aligned}$$

where, whenever $g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$, we have the commutative triangle:

$$\begin{tikzcd} Z \arrow[r, "g"] & X \\ Y \arrow[u, "f"] \arrow[ur, swap, "f^* g \doteq g \circ f"] & \end{tikzcd}$$

Moreover, any morphism $f : X \to Y$ induces a natural transformation between functors $h_f : h_X \to h_Y$, which is given, over each object $Z \in \mathcal{C}$, by:

$$\begin{aligned} h_{f,Z} : h_X(Z) &\to h_Y(Z) \\ g &\mapsto f \circ g. \end{aligned}$$

Thus, we can view the correspondence $X \mapsto h_X$ as a covariant functor:

$$\begin{aligned} h : \mathrm{Sch} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}) \\ X &\mapsto h_X \\ X \xrightarrow{f} Y &\mapsto h_X \xrightarrow{h_f} h_Y \end{aligned}$$

To prove that h_f is a natural transformation, we only need to check functoriality: if $B \xrightarrow{g} A$ is a morphism in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} h_X(A) & \xrightarrow{h_{f,A}} & h_Y(A) \\ \downarrow{\scriptstyle g^*} & & {\scriptstyle g^*}\downarrow \\ h_X(B) & \xrightarrow[h_{f,B}]{} & h_Y(B) \end{array}$$

commutes, since whenever $\alpha \in h_X(A)$, we can write:

$$g^* \circ h_{f,A}(\alpha) = g^*(f \circ \alpha) = (f \circ \alpha) \circ g,$$

and, on the other hand

$$h_{f,B} \circ g^*(\alpha) = h_{f,B}(\alpha \circ g) = f \circ (\alpha \circ g).$$

Definition 1.1.2. A functor $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ is **representable** if there is an object X in $\mathcal{C}$ such that $F \simeq h_X$.

Example 1.1. In the category of finite type schemes over k, $\mathcal{C} = \mathrm{Sch}_k$, we could consider the functor of regular functions:

$$\begin{aligned} \mathcal{O} : \mathrm{Sch}_k &\to \mathrm{Sets} \\ X &\mapsto \mathcal{O}(X) = \Gamma(X, \mathcal{O}_X) \\ X \xrightarrow{f} Y &\mapsto \mathcal{O}(Y) \xrightarrow{f^*} \mathcal{O}(X) \end{aligned}$$

which is representable by the object $\mathbb{A}^1_k$.

Remark 1.1.3. We can also consider the dual notion, of whether or not a covariant functor $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Sets})$ is representable by an object X in $\mathcal{C}$, i.e., when we have an isomorphism $F \simeq \mathrm{Hom}(X, -)$. For $\mathcal{C} = \mathrm{Vect}_k$ the category of vector spaces over k, we could fix a pair of vector spaces V, W and consider:

$$\begin{aligned} \mathrm{Bil}(V \times W, -) : \mathrm{Vect} &\to \mathrm{Sets} \\ A &\mapsto \mathrm{Bil}(V \times W, A) = \{\text{ bilinear maps } V \times W \to A\} \\ A \xrightarrow{T} B &\mapsto \mathrm{Bil}(V \times W, A) \xrightarrow{T_*} \mathrm{Bil}(V \times W, B). \end{aligned}$$

By the universal property of the tensor product, we have an isomorphism:

$$\mathrm{Bil}(V \times W, Z) \simeq \mathrm{Hom}_{\mathrm{Vect}}(V \otimes W, Z).$$

Thus, in this case, the object $V \otimes W$ represents the covariant functor $\mathrm{Bil}(V \times W, -)$.

As most of the categorical phenomena we are interested in this dissertation are of contravariant nature, we show Yoneda's lemma for this case.

Theorem 1.1.4 (Yoneda's lemma)**.** *For any object $C \in \mathcal{C}$ and any contravariant functor $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$, there is a bijection:*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}}(h_C, F) &\simeq F(C) \\ \eta &\mapsto \eta_C(\mathrm{Id}_C). \end{aligned}$$

Proof. First, to see this is surjective, let us fix $s \in F(C)$ any element. We define a natural transformation $\eta = \eta(s) : h_C \to F$ by setting, whenever C' is another object of $\mathcal{C}$, the morphism

$$\begin{aligned} \eta_{C'} : h_C(C') &\to F(C') \\ C' \xrightarrow{f} C &\mapsto F(f)(s). \end{aligned}$$

This procedure defines a natural transformation since it is compatible with morphisms in $\mathcal{C}$, i.e., whenever X, Y are objects in $\mathcal{C}$ and $\phi : X \to Y$ is a morphism in $\mathcal{C}$, we can consider the following diagram:

$$\begin{tikzcd} h_C(Y) \arrow[r, "\eta_Y"] \arrow[d, "\phi^*"'] & F(Y) \arrow[d, "F(\phi)"] \\ h_C(X) \arrow[r, "\eta_X"] & F(X) \end{tikzcd}$$

which commutes because, whenever $f : Y \to C \in h_C(Y)$, we can write:

$$(\eta_X \circ \phi^*)(f) = \eta_X(f \circ \phi) = F(f \circ \phi)(s),$$

and on the other hand

$$(F(\phi) \circ \eta_Y)(f) = F(\phi)(F(f(s))) = F(f \circ \phi)(s),$$

as F is contravariant.

To prove injectivity, we show that if $\eta, \eta' \in \mathrm{Hom}_{Fun}(h_C, F)$ are two natural transformations such that $\eta_C(\mathrm{Id}_C) = \eta'_C(\mathrm{Id}_C)$, then $\eta = \eta'$. As η and η' are natural transformations, we have two commutative diagrams:

$$\begin{tikzcd} h_C(C) \arrow[r, "\eta_C"] \arrow[d, "h_C(g)"'] & F(C) \arrow[d, "F(g)"] \\ h_C(C') \arrow[r, "\eta_{C'}"] & F(C') \end{tikzcd} \qquad \begin{tikzcd} h_C(C) \arrow[r, "\eta'_C"] \arrow[d, "h_C(g)"'] & F(C) \arrow[d, "F(g)"] \\ h_C(C') \arrow[r, "\eta'_{C'}"'] & F(C') \end{tikzcd}$$

whenever $g : C' \to C$ is a morphism in $\mathcal{C}$. Thus, we can write:

$$\begin{aligned} (F(g) \circ \eta_C)(\mathrm{Id}_C) &= (\eta_{C'} \circ h_C(g))(\mathrm{Id}_C) = \eta_{C'}(g) \\ (F(g) \circ \eta'_C)(\mathrm{Id}_C) &= (\eta'_{C'} \circ h_C(g))(\mathrm{Id}_C) = \eta'_{C'}(g), \end{aligned}$$

and

$$\eta_{C'}(g) = F(g) \circ \eta_C(\mathrm{Id}_C) = F(g) \circ \eta'_C(\mathrm{Id}_C) = \eta'_{C'}(g),$$

concluding $\eta_{C'} = \eta'_{C'}$ for each object C' in $\mathcal{C}$. □

Remark 1.1.5. As one would expect, there is also a dual analogue for the Yoneda's lemma, where we consider the covariant functors $h^C \doteq \mathrm{Hom}(C, -)$ and there is a bijection

$$\mathrm{Hom}_{\mathrm{Fun}}(h^C, F) \simeq F(C)$$

whenever $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Sets})$ is a functor. For more, see for example (RIEHL, 2016).

Corollary 1.1.5.1. The functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ is fully faithfull.

Proof. By definition, h is fully faithfull if, for every pair C, C' of objects in $\mathcal{C}$, the induced morphism:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(C, C') &\to \mathrm{Hom}_{\mathrm{Fun}}(h_C, h_{C'}) \\ f &\mapsto h_f \end{aligned}$$

is bijective. This follows from Yoneda's lemma taking $F = h_{C'}$. □

Since h is fully faithfull, we usually refer to it as **Yoneda's embedding**.

1.2 Moduli Problems in Algebraic Geometry

In this section, we follow definitions as given in (BALAJI; NATIONALBIBLIOTHEK, 2010) and (HOSKINS, 2015).

A **moduli problem** consists informally of a classification problem. We define a **naive moduli problem** as a pair $(\mathcal{A}, \simeq)$ consisting of a collection of objects $\mathcal{A}$ of a category and an equivalence relation $\simeq$ on $\mathcal{A}$. Throughout this text, we are mostly interested in moduli problems in Algebraic Geometry using basic scheme theory, so we will work over the category of finite type schemes over k.

Our aim is to find an algebraic space (e.g., a scheme) M whose k−points are in bijection with the set of equivalence classes $\mathcal{A}/\simeq$. In this case, we say M is a **naive moduli space** for this naive moduli problem.

Definition 1.2.1. Let $(\mathcal{A}, \simeq)$ be a naive moduli problem. An **extended moduli problem** for $\mathcal{A}$ is a contravariant functor $F \in \mathrm{Fun}(\mathrm{Sch}_k^{op}, \mathrm{Sets})$ satisfying:

- $F(\operatorname{Spec} k) = \mathcal{A}$;
- For each object $T \in \mathrm{Sch}_k$, the set $F(T)$ is given an equivalence relation $\simeq_T$ such that $\simeq_{\operatorname{Spec} k}$ coincides with $\simeq$ on $F(\operatorname{Spec} k) = \mathcal{A}$;

- For each morphism $\phi : T_1 \to T_2$, the corresponding map of sets

$$F(\phi) : F(T_2) \to F(T_1)$$

takes $\simeq_2$ −equivalent objects in $F(T_2)$ to $\simeq_1$ −equivalent objects in $F(T_1)$.

For each object T, the elements of $F(T)$ are called the **families of objects of $\mathcal{A}$ parametrized by** the space T. For each morphism $\phi : T_1 \to T_2$, the corresponding morphism $F(\phi)$ is also called **pullback** of ϕ, and denoted $F(\phi) = \phi^*$. The functor F is also called the **functor of families of objects** associated to the naive moduli problem $(\mathcal{A}, \simeq)$. For a family $\mathcal{F}$ of objects of $\mathcal{A}$ over S and a point $s : \operatorname{Spec} k \to S$, we write $\mathcal{F}_s \doteq s^*\mathcal{F}$ to denote the corresponding family over $\operatorname{Spec} k$.

Any extended moduli problem F defines a **moduli functor** $\mathcal{M}$ by passing to equivalence classes:

$$\mathcal{M}(S) \doteq F(S)/ \simeq_S$$
$$\mathcal{M}(f : T \to S) \doteq F(f) = f^* : \mathcal{M}(S) \to \mathcal{M}(T).$$

Although we use the name (and notation) of pullbacks to denote the image of the functors F and $\mathcal{M}$ on arrows, the appropriate notion of pullback of families for a given moduli problem $\mathcal{M}$ should be clear from context.

A scheme $M \in \mathrm{Sch}_k$ is called a **fine moduli space** for a moduli functor $\mathcal{M}$ if the scheme M represents the functor $\mathcal{M}$, i.e., $\mathrm{Hom}_{\mathrm{Sch}_k}(-, M) \simeq \mathcal{M}$. In this case, it follows that $(\mathcal{A}/\simeq) \simeq \mathcal{M}(\operatorname{Spec} k) \simeq \mathrm{Hom}(\operatorname{Spec} k, M)$, so M is a naive moduli space.

If we denote the natural isomorphism by $\eta : \mathrm{Hom}_{\mathrm{Sch}_k}(-, M) \xrightarrow{\simeq} \mathcal{M}$, there is a distinguished element $\mathcal{U} \doteq (\eta_M)(\mathrm{Id}_M) \in \mathcal{M}(M)$, which is called the **universal family** over M. For any scheme $S \in \mathrm{Sch}_k$, we have

$$\mathrm{Hom}_{\mathrm{Sch}_k}(S, M) \simeq \mathcal{M}(S),$$

and thus for any family $[\mathcal{F}] \in \mathcal{M}(S)$, there is a corresponding morphism $f : S \to M$, which satisfies $f^*\mathcal{U} \simeq_S \mathcal{F}$, as $\mathrm{Id}_M \circ f = f$.

Example 1.2. Let V be a k−vector space of dimension $n + 1$. We will define a **family of lines through the origin in** V over a scheme S to be a line bundle L over S which is a subbundle of the trivial bundle $V \times S$ over S. We say that two such families are equivalent if they are equal, and denote the corresponding moduli functor by $\mathcal{M}$.

The candidate of the universal family in this case is the **tautological bundle** over the projective space $\mathbb{P}^n$, the line subbundle of $V \times \mathbb{P}^n$ which over each point assigns the corresponding line in the affine space $\mathbb{A}^{n+1} = V$. This can be identified with the sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$.

If $f : S \to \mathbb{P}^n$ is a morphism of schemes, the line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by the pullback global sections $f^*(x_0), \ldots, f^*(x_n)$, and this determines a surjection

$$\mathcal{O}_S^{n+1} \twoheadrightarrow f^*\mathcal{O}_{\mathbb{P}^n}(1).$$

When dealing with locally free sheaves, the pullback commutes with dualizing, so

$$f^*\mathcal{O}_{\mathbb{P}^n}(-1) \simeq (f^*\mathcal{O}_{\mathbb{P}^n}(1))^\vee,$$

and the surjection induces a dual inclusion

$$\mathcal{L} \doteq f^*\mathcal{O}_{\mathbb{P}^n}(-1) \hookrightarrow (\mathcal{O}_S^{n+1})^\vee \simeq \mathcal{O}_S^{n+1}$$

which determines a family of lines through the origin in V over S, by definition.

Conversely, let $L \subset V \times S$ be a family of lines through V over S. Then, dual to the inclusion, there is a surjection $q : V^\vee \times S \twoheadrightarrow L^\vee$, and as the vector bundle $V^\vee \times S$ is generated by global sections $\sigma_0, \ldots, \sigma_n$ corresponding to the dual basis for the standard basis on V, we conclude the dual line bundle $L^\vee$ is generated by global sections $q \circ \sigma_0, \ldots, q \circ \sigma_n$, which induce a unique morphism

$$\begin{aligned} f : S &\to \mathbb{P}^n \\ s &\mapsto [q \circ \sigma_0(s) : \ldots : q \circ \sigma_n(s)] \end{aligned}$$

such that $f^*\mathcal{O}_{\mathbb{P}^n}(-1)$ corresponds to $L \subset V \times S$. Hence, we have constructed a bijection $\mathrm{Hom}(-, \mathbb{P}^n) \simeq \mathcal{M}$ and the projective space $\mathbb{P}^n$ is the fine moduli space for this functor, with universal family $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Theorem 1.2.2. *Consider the moduli problem of d–dimensional linear subspaces in a fixed vector space $V = \mathbb{A}^n$, where a family over S is a rank d vector subbundle $\mathcal{E}$ of $V \times S$, and the equivalence relation is given by equality. We denote the associated functor as $\mathcal{G}(d, n)$. This functor is representable by the Grassmanian variety $Gr(d, n)$.*

Proof. Let $T \subset V \times Gr(d, n)$ be the tautological vector bundle over $Gr(d, n)$, which assigns to each point the corresponding linear subspace of V,$\mathcal{E} \subset V \times S$ be a family over S and $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of S which trivializes $\mathcal{E}$, i.e., such that

$$\mathcal{E}\big|_{U_\alpha} \simeq \mathbb{A}^d \times U_\alpha.$$

This determines morphisms

$$\mathcal{E}\big|_{U_\alpha} \simeq \mathbb{A}^d \times U_\alpha \to \mathbb{A}^n \times U_\alpha \simeq V \times S\big|_{U_\alpha}$$

whenever $\alpha \in \Lambda$, which are in turn determined by $n \times d$ matrices with coefficients in $\mathcal{O}(U_\alpha)$, of rank d. That is, a morphism $U_\alpha \to M_{n\times d}^d(k)$, where $M_{n\times d}^d(k)$ is the variety of

$n \times d$ matrices of rank d over k. Taking wedge product of the d–rows in this matrix defines a mapping

$$f_\alpha : U_\alpha \to \mathbb{P}(\bigwedge^d(V)).$$

To show that these morphisms obtained locally can be glued to a global morphism $f : S \to \mathbb{P}(\bigwedge^d(V))$, we only need to show that they agree on the intersections, i.e., whenever $\alpha, \beta \in \Lambda$,

$$f_\alpha\big|_{U_\alpha \cap U_\beta} = f_\beta\big|_{U_\alpha \cap U_\beta},$$

but this follows from the compatibility condition of local trivializations for vector bundles.

Via the classical construction of the Grassmanian as a projective variety (see, for example, (REID; SHAFAREVICH, 2013)), this map has image $Gr(d,n)$. Moreover, we also have $f^*\mathcal{T} = \mathcal{E}$, since locally f_α corresponds to the inclusion of $\mathcal{E}$ as $\mathbb{A}^d \subset \mathbb{A}^n$ over each open set U_α. Moreover, we get an isomorphism

$$\begin{aligned} \eta_S : \mathrm{Gr}(d,n) &\to \mathrm{Hom}(S, \mathrm{Gr}(d,n)) \\ \mathcal{E} &\mapsto f_{\mathcal{E}}, \end{aligned}$$

which is functorial, so $\mathrm{Gr}(d,n)$ is the fine moduli space for the moduli problem $\mathcal{G}(d,n)$. □

Example 1.3. Let us consider the naive moduli problem given by classifying locally free sheaves over a fixed scheme X up to isomorphism. This can be extended in two different ways. The natural notion for a family over S is a locally free sheaf $\mathcal{F}$ over $X \times S$, which is flat over S. We could consider the equivalence relation $\sim_S$ of families over S as just isomorphism of locally free sheaves, but we could also consider a more flexible one, where we only ask

$$\mathcal{F} \sim_S \mathcal{G} \iff \mathcal{F} \simeq \mathcal{G} \otimes \pi_S^* \mathcal{L}$$

for a line bundle $\mathcal{L}$ over S. Since line bundles are locally trivial, this means $\mathcal{F} \sim_S \mathcal{G}$ if and only if there is a cover $\{S_i\}_{i \in I}$ of S such that

$$\mathcal{F}\big|_{X \times S_i} \simeq \mathcal{G}\big|_{X \times S_i}.$$

We revisit this example in Chapter 4, after developing the GIT theory, to study this moduli problem when X is a smooth projective curve over k.

Example 1.4. Let S be a Noetherian scheme and X be a finite type scheme over S. If we denote by Sch_S the category of locally Noetherian schemes over S, we define the contravariant functor:

$$\begin{aligned}\mathrm{Hilb}_{X/S} : \mathrm{Sch}_S &\to \mathrm{Sets}\\ T &\mapsto \mathrm{Sub}_{X/S}(T)\{Y \subset X \times_T T : Y \text{ is flat over } T\}\\ T \xrightarrow{f} P &\mapsto (\mathrm{Id} \times_S f)^* : \mathrm{Sub}_{X/S}(P) \to \mathrm{Sub}_{X/S}(T),\end{aligned}$$

called the **Hilbert Functor**. We could also stratify this functor into the disjoint union

$$\mathrm{Hilb}_{X/S}(T) \doteq \bigsqcup_{i=0}^{\infty} \mathrm{Sub}^i_{X/S}(T)$$

where $\mathrm{Sub}^i_{X/S}(T) \subset \mathrm{Sub}_{X/S}(T)$ denotes the subset of fixed dimension i flat families. This functor is representable, and the corresponding scheme is called **Hilbert scheme** (for a construction, see (FANTECHI; GOTTSCHE; ILLUSIE, 2005)).

Unfortunately, there are many natural moduli problems which do not admit a fine moduli space. We list some pathologies which usually prevent a moduli problem from admitting a fine moduli space:

1. Moduli problems which jump in families, in the sense that we can construct a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_{s'}$ for all $s, s' \in \mathbb{A}^1 \setminus \{0\}$, but $\mathcal{F}_0 \nsim \mathcal{F}_s$ for $s \in \mathbb{A}^1 \setminus \{0\}$.
2. Moduli problems which are unbounded, in the sense that there is no family over a scheme S which parametrizes all objects in the moduli problem.
3. The existence of automorphisms, and although this is an important one, we do not treat in here. For an example, see the case of elliptic curves, in (HARRIS; MORRISON, 1998), Chapter I. This is the phenomena that eventually lead Mumford to define Algebraic Stacks (see more on (OLSSON, 2016)), as a more general algebro-geometric space and an alternative for the existence of a fine moduli space.

For an example of the first behaviour, we let $n \geq 1$ be an integer and consider the set

$$\mathcal{A}(n) = \{(V, \phi) : V \in \mathrm{Vect}_k,\ \dim V = n,\ \phi \in \mathrm{End}(V)\},$$

with the following equivalence relation:

$$(V, \phi) \sim (V', \phi') \iff \exists\, h : V \to V' \text{ isomorphism such that } h \circ \phi = \phi' \circ h.$$

To extend this naive moduli problem, let us fix a k–scheme $S \in \mathrm{Sch}_k$. A family over S for the extended moduli problem will be a rank n vector bundle F over S, with an endomorphism $\phi : F \to F$. Then the equivalence relation will be

$$(F, \phi) \sim_S (G, \phi') \iff \exists\, h : F \to G \text{ isomorphism such that } h \circ \phi = \phi' \circ h.$$

We denote this extended moduli problem by End_n.

Let us consider the case $n = 2$ and the family corresponding to the locally free sheaf $\mathcal{O}_{\mathbb{A}^1}^{\oplus 2}$ over $\mathbb{A}^1$ and we fix the endomorphism ϕ defined on fibers as

$$\phi_s \doteq \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

whenever $s \in \mathbb{A}^1$. For $s, t \neq 0$, these matrices are similar and so $\phi_t \sim \phi_s$. However, $\phi_0 \nsim \phi_1$, as this matrices have different Jordan normal forms.

For an example of the second kind of pathological behaviour, let us consider the moduli problem of classifying vector bundles over $\mathbb{P}^1$ of fixed rank 2 and degree 0. We claim that there is no family F over a Noetherian scheme S with the property that for any rank 2 and degree zero vector bundle $\mathcal{E}$ on $\mathbb{P}^1$, there is a k–point $s \in S$ such that $\mathcal{F}_s \simeq \mathcal{E}$.

Let us suppose such family $\mathcal{F}$ over S exists. Whenever $n \in \mathbb{N}$, we can consider the rank 2 degree zero vector bundle corresponding to the locally free sheaf

$$\mathcal{E}(n) = \mathcal{O}_{\mathbb{P}^1}(n) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-n).$$

As will be shown in the Grothendieck's theorem (see 4.3.13), every rank 2 degree zero vector bundle over $\mathbb{P}^1$ is of this form. Furthermore, we can compute the dimension

$$\dim \Gamma(\mathbb{P}^1, \mathcal{E}(n)) = \dim_k(k[x_0, x_1]_n \bigoplus k[x_0, x_1]_{-n}) = \begin{cases} 2 & \text{if } n = 0, \\ n+1 & \text{if } n \geq 1. \end{cases}$$

Consider the subschemes

$$S_n \doteq \{s \in S : \dim \Gamma(\mathbb{P}^1, \mathcal{F}_s) \geq n\} \subset S,$$

which are closed by the semi-continuity theorem (see (HARTSHORNE, 1977), III Theorem 12.8). So there is a descending chain of closed subschemes

$$\ldots \subsetneq S_4 \subsetneq S_3 \subsetneq S_2 = S,$$

which does not stabilize, since $\mathcal{E}(n) \in S_{n+1} \setminus S_{n+2}$ whenever $n \geq 1$. This contradicts the fact that S is a Noetherian scheme.

As demonstrated by these examples, not always we have a fine moduli space for a moduli problem. We can relax this definition, to get a weaker version of a moduli space.

Definition 1.2.3. A **coarse moduli space** for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation $\eta : \mathcal{M} \to h_M$ such that

- The map $\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective.
- For any scheme N and natural transformation $\nu : \mathcal{M} \to h_N$, there is a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$, so the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\eta"] \arrow[dr, "\nu"'] & h_M \arrow[d, dashed, "h_f"] \\ & h_N \end{tikzcd}$$

  commutes.

Proposition 1.2.4. Let (M, η) be a coarse moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if

1. There is a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{Id}_M$ and,
2. If $\mathcal{F}, \mathcal{G}$ are families over S, we have

$$\mathcal{F} \sim_S \mathcal{G} \iff \eta_S(\mathcal{F}) = \eta_S(\mathcal{G}).$$

Proof. If (M, η) is a fine moduli space, properties 1 and 2 hold. Now, suppose (M, η) is a coarse moduli space for $\mathcal{M}$ and it satisfies properties 1 and 2. We need to prove that the natural transformation η is an isomorphism of functors, and to do this it suffices to verify that, whenever N is an object in Sch_k, the map

$$\eta_N : \mathcal{M}(N) \to h_M(N) = \operatorname{Hom}(N, M)$$

is bijective. To see it is surjective, given any $f \in \operatorname{Hom}(N, M)$, we can consider the morphism

$$\mathcal{M}(M) \xrightarrow{\mathcal{M}(f)} \mathcal{M}(N),$$

and using 1, there is a natural choice of family $\mathcal{M}(f)(\mathcal{U}) \in \mathcal{M}(N)$. We claim that $\eta_N(\mathcal{M}(f)(\mathcal{U})) = f$. Since η is a natural transformation, the diagram

$$\begin{tikzcd} \mathcal{M}(M) \arrow[r, "\eta_M"] \arrow[d, "\mathcal{M}(f)"'] & h_M(M) \arrow[d, "f^*"] \\ \mathcal{M}(N) \arrow[r, "\eta_N"] & h_M(N) \end{tikzcd}$$

commutes, so $f^* \circ \eta_M(\mathcal{U}) = \eta_N \circ \mathcal{M}(f)(\mathcal{U}) = \eta_N(\mathcal{F}(f))$, and thus

$$f^* \circ \eta_M(\mathcal{U}) = f^*(\mathrm{Id}_M) = f,$$

as we wanted to show. Injectivity follows from property 2, as $\mathcal{M}(N)$ is given by the set of equivalence classes with respect to $\sim_N$. $\square$

We can also formalize the jump phenomena observed before using coarse moduli spaces, in the next proposition.

Proposition 1.2.5. Let $\mathcal{M}$ be a moduli problem and suppose there exists a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_1$ for all $s \neq 0$ but $\mathcal{F}_0 \nsim \mathcal{F}_1$. Then, for any scheme M and natural transformation $\eta : \mathcal{M} \to h_M$, the map

$$\eta_{\mathbb{A}^1}(\mathcal{F}) : \mathbb{A}^1 \to M$$

is constant. In particular, there is no coarse moduli space for this moduli problem.

Proof. Let $f \doteq \eta_{\mathbb{A}^1}(\mathcal{F}) : \mathbb{A}^1 \to M$. For any k–point

$$\operatorname{Spec} k \xrightarrow{s} \mathbb{A}^1 \xrightarrow{f} M$$

we have $f \circ s = \eta_{\operatorname{Spec} k}(\mathcal{F}_s)$. Since $\mathcal{F}_s \sim_S \mathcal{F}_1$ whenever $s \neq 0$, the function $f\big|_{\mathbb{A}^1 \setminus \{0\}}$ is constant, and we denote by

$$\operatorname{Spec} k \xrightarrow{m} M$$

the closed point corresponding to this image. By continuity, it follows that:

$$\mathbb{A}^1 \setminus \{0\} \subset f^{-1}(m) \Rightarrow \mathbb{A}^1 = \overline{\mathbb{A}^1 \setminus \{0\}} \subset f^{-1}(m),$$

since $f^{-1}(m)$ is closed, and thus f is constant.

In particular, the morphism

$$\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k) = M(k)$$

cannot be a bijection, since $\mathcal{F}_0 \nsim \mathcal{F}_1$ but they correspond to the same closed point $m \in M(k)$. □

Even when a moduli problem $\mathcal{M}$ admits a fine moduli space, which is the best case scenario, its representation as a k–scheme is not always easy to understand. When dealing with various moduli problems in the next chapters, we will follow the following general strategy:

1. Given a particular naive moduli problem $(\mathcal{A}, \sim)$, we search for a **space of parameters** P, which for us will be a k–scheme with a surjective map of sets

$$P(k) \twoheadrightarrow \mathcal{A}/\sim.$$

2. We search for an algebraic group G which acts algebraically on P and corresponds to the equivalence relation $\sim$ in k–points, i.e., two k–points $p, q \in P(k)$ lie in the same G–orbit if and only if they correspond to isomorphic families in $\mathcal{A}$. This induces a bijection

$$P(k)/G \xrightarrow{\simeq} \mathcal{A}/\sim.$$

3. We take the quotient in the category of schemes, when G and P satisfy some hypothesis, using **Geometric Invariant Theory** (GIT), a method developed by David Mumford which will be explained in the next chapters.

Having a easy description for the moduli space as an scheme helps to find, not only the answers to a lot of questions concerning the classification of said objects, but also the right answers to ask.

The objective of the next two chapters is to study Mumford's GIT theory as a framework of systematic construction for fine moduli spaces as schemes.

2 Algebraic Invariant Theory

Introduction

Let X be an algebraic variety inside some affine space $\mathbb{A}^n$ over k and we let $G = \mathrm{GL}(n)$ be the general linear group of dimension n over k, which acts by linear automorphisms on X, induced by the embedding of X into $\mathbb{A}^n$.

This action also induces an action of G on the ring of regular functions $\mathcal{O}(X)$ given by change of coordinates. More explicitely, given any $T \in \mathrm{GL}(n)$ and regular function $f \in \mathcal{O}(X)$, we define $T \cdot f \doteq f \circ T$ and

$$\mathcal{O}(X)^G \doteq \{f \in \mathcal{O}(X) : T \cdot f = f \ \forall T \in \mathrm{GL}(n)\}$$

is called the **fixed subalgebra** for this G-action over X. This is also called the algebra of invariants of $\mathcal{O}(X)$ for this action, and a possible question is the following:

Conjecture 2.1. Is $\mathcal{O}(X)^G$ finitely generated over k?

Hilbert showed that this is the case when $X = \mathbb{A}^n$. In 1900, in the famous paper "Mathematical Problems" (HILBERT, 1902), Hilbert cites a generalization of this for any algebraic subgroup of the general linear group and for any affine algebraic variety:

Conjecture 2.2 (Hilbert's Fourteenth Problem)**.** If $X \subset \mathbb{A}^n$ is an affine variety and $G \subset \mathrm{GL}(n)$ is an affine algebraic subgroup acting by linear change of coordinates, is the algebra $\mathcal{O}(X)^G$ finitely generated over k?

Since $\mathcal{O}(X)$ is a finitely generated k–algebra, by hypothesis, there is a surjection on top of the diagram

$$\begin{tikzcd} k[x_1,\dots,x_n] \arrow[r, two heads] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ k[x_1,\dots,x_n]^G \arrow[r, dashed] & \mathcal{O}(X)^G \end{tikzcd},$$

which induces a surjection at the bottom whenever G satisfies some additional hypothesis (namely, the group G has to be reductive, see 2.5.4). The vertical arrows making this commute, which exist given these hypothesis, are called **Reynolds operators**.

In general, however, the above conjecture is false, and Masayoshi Nagata gave a counterexample in (NAGATA, 1965). For a survey of modern counterexamples for this problem, see (FREUDENBURG, 2001).

In this chapter, we review all the background in algebraic groups, their algebraic actions and representations, eventually finding sufficient conditions for G (namely, we ask G to be a linearly reductive group over k) to get a positive answer for 2.2, in 2.5.4. We follow the exposition of (HOSKINS, 2015), and usually refer to (MILNE, 2012) and (MILNE, 2017) for technical facts about algebraic group theory.

In this Sch_k, we denote by $S \times S = S \times_k S$ the fiber product, since this is the product in the slice category over the object $\operatorname{Spec} k$. We also use Grp to denote the category of groups, and Sets the category of sets. Given a k–scheme X, we denote by $X(k) = h_X(\operatorname{Spec} k) = \operatorname{Hom}(\operatorname{Spec} k, X)$ its set of k–points. Whenever $f : A \to B$ is a map of sets, then we denote by $\operatorname{Im} f$ the set $f(A) \subset B$.

2.1 Algebraic Groups

Definition 2.1.1. An **algebraic group** (over k) is a k–scheme G together with morphisms $m : G \times G \to G$, $i : G \to G$ and $e : \operatorname{Spec} k \to G$ such that the following diagrams in Sch_k

$$\begin{tikzcd} G \times G \times G \arrow[r, "(Id_G{,}m)"] \arrow[d, "(m{,}Id_G)"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} G \arrow[r, "(Id_G{,}i)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(i{,}Id_G)"'] \arrow[d] \\ \operatorname{Spec} k \arrow[r, "e"'] & G & \operatorname{Spec} k \arrow[l, "e"] \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} \operatorname{Spec} k \times G \arrow[r, "(e{,}Id_G)"] \arrow[dr, "\simeq"'] & G \times G \arrow[d, "m"] & G \times \operatorname{Spec} k \arrow[l, "(Id_G{,}e)"'] \arrow[dl, "\simeq"] \\ & G & \end{tikzcd} \tag{2.3}$$

commute, where the isomorphisms in (2.3) are given by the projections.

Remark 2.1.2. This construction can done in any category $\mathcal{C}$ with binary products and terminal object, and in this more general context it is called a **group object** (see, for example, (FANTECHI; GOTTSCHE; ILLUSIE, 2005), 2.2). In the category of sets, for example, the group objects are the usual groups, and if we consider the category of manifolds with smooth mappings, the group objects are Lie groups.

To see that these axioms in fact correspond with a group structure, we have the following universal property of group objects, using Yoneda's Lemma and the language of representable functors.

Lemma 2.1.3. Let G be an algebraic group and $h_G = \mathrm{Hom}_{\mathrm{Sch}_k}(-, G) : \mathrm{Sch}_k \to \mathrm{Sets}$ the functor of points of G. There is a unique functor $\mathscr{H}_G : \mathrm{Sch} \to \mathrm{Grp}$ such that the diagram

$$\begin{tikzcd} \mathrm{Sch}_k \arrow[r, "h_G"] \arrow[d, "\mathscr{F}_G"'] & \mathrm{Sets} \\ Grp \arrow[ur, "forget"'] & \end{tikzcd} \tag{2.4}$$

commutes.

Proof. Let $G = (G, m, i, e)$ denote the structure morphisms of the algebraic group and S be any k−scheme. We define a group structure on the set $h_G(S) = \mathrm{Hom}_{\mathrm{Sch}_k}(S, G)$ as follows: if $f, g \in h_G(S)$,

$$\begin{aligned} f + g &\doteq m \circ (f, g) \\ -f &\doteq i \circ f \\ 0 &\doteq e \circ \mathrm{str} \end{aligned}$$

where $\mathrm{str} : S \to \mathrm{Spec}\, k$ is the structural morphism and (f, g) denotes the product arrow from $G \times G$ to $S \times S$. To see that $(h_G, +)$ is a group with the structure mentioned above, we note that:

1. The operation $+$ is associative, since given $f, g, h \in h_G(S)$ we can write

$$\begin{aligned} (f+g)+h &= m \circ (f+g, h) \\ &= m \circ (m \circ (f, g), h) \\ &= (m \circ (m, Id_G))(f, g, h) \qquad \text{(using 2.1)} \\ &= (m \circ (Id_G, m))(f, g, h) \\ &= m \circ (f, g+h) \\ &= f + (g+h). \end{aligned}$$

2. 0 is the neutral element, since if $f \in h_G(S)$ we have

$$\begin{aligned} f + 0 &= m \circ (f, e \circ \mathrm{str}) \\ &= m \circ (Id_G, e)(f, \mathrm{str}) \qquad \text{(using 2.3)} \\ &= f. \end{aligned}$$

3. The element $-f$ is the inverse of $f \in h_G(S)$ with respect with $+$, since

$$\begin{aligned} f + (-f) &= m \circ (f, -f) \\ &= m \circ (f, i \circ f) \\ &= m \circ (Id_G, i)(f) \qquad \text{(using 2.2)} \\ &= e \circ \mathrm{str}_G \circ f \end{aligned}$$

and because $\mathrm{Spec}\, k$ is the terminal object, the two arrows

$$\mathrm{str}_G \circ f : S \to \mathrm{Spec}\, k \text{ and } \mathrm{str} : S \to \mathrm{Spec}\, k$$

must coincide, implying $f + (-f) = e \circ \mathrm{str}_G \circ f = e \circ \mathrm{str}_S = 0$.

Let $\mathscr{H}_G(S) \doteq (h_G(S), +)$. Given any morphism $\phi \in \mathrm{Hom}_k(S, T)$, we have the pullback

$$\begin{aligned} \phi^* : h_G(T) &\to h_G(S) \\ f &\mapsto f \circ \phi, \end{aligned}$$

which induces a group morphism between $(h_G(T), +)$ and $(h_G(S), +)$, since whenever $f, g \in h_G(T)$,

$$\begin{aligned} \phi^*(f + g) &= (f + g) \circ \phi \\ &= m \circ (f, g) \circ \phi \\ &= m \circ (f \circ \phi, g \circ \phi) \\ &= f \circ \phi + g \circ \phi = \phi^*(f) + \phi^*(g), \end{aligned}$$

and we can again use the property of the terminal object to conclude that the arrows $\mathrm{str}_T \circ \phi$ and $\mathrm{str}_S : S \to \mathrm{Spec}\, k$ coincide, and this means that $\phi^*(0_T) = e \circ \mathrm{str}_T \circ \phi = e \circ \mathrm{str}_S = 0_S$. Thus, $\mathscr{H}_G$ defines a functor between Sch_k and Grp satisfying the diagram (2.4). □

Definition 2.1.4. Let $G = (G, m_G, i_G, e_G)$ and $H = (H, m_H, i_H, e_H)$ be algebraic groups. A morphism of k–schemes $\phi : G \to H$ is a **morphism of algebraic groups** if the following diagrams commute:

$$\begin{tikzcd} G \times G \arrow[r, "(\phi,\phi)"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_H"] \\ G \arrow[r, "\phi"] & H \end{tikzcd} \quad \begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathrm{Spec}\, k \arrow[u, "e_G"] \arrow[ur, "e_H"'] & \end{tikzcd} \quad \begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "i_G"'] & H \arrow[d, "i_H"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd}$$

We will denote by AlgGrp_k the category of algebraic groups and morphisms of algebraic groups.

Proposition 2.1.5. The functor

$$\begin{aligned} \mathrm{AlgGrp} &\to \mathrm{RepFun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Grp}) \\ G &\mapsto \mathscr{H}_G \end{aligned}$$

is an equivalence of categories, where $\mathrm{RepFun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Grp})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Grp})$ of contravariant representable functors.

Proof. Let $\mathscr{F} : \mathrm{Sch}_k \to \mathrm{Grp}$ be a contravariant representable functor. By definition, there must be a k–scheme X such that $\mathscr{F} \simeq h_X$. Let's prove that X admits an algebraic group structure such that $\mathscr{F} \simeq \mathscr{H}_X$, which shows the functor $G \mapsto \mathscr{H}_G$ is essentially surjective. This is enough, since by Yoneda's lemma this correspondence is also fully faithfull, and these properties imply the functor establishes an equivalence between categories (see (RIEHL, 2016), Theorem 1.5.9).

Again by Yoneda's lemma, $h_X \times h_X \simeq h_{X \times X}$ and if we can find natural transformations $m : h_X \times h_X \to h_X$, $i : h_X \to h_X$ and $e : h_{(\mathrm{Spec}\, k)} \to h_X$ commuting the diagrams in 2.1.1 at level of objects, we are done. Given any k–scheme S, there are operations m_S, i_S, e_S in $h_X(S)$, defined in the proof of the universal property of algebraic groups (2.1.3), so that m_S and i_S define morphisms satisfying 2.1 and 2.2. Moreover, since $h_{\mathrm{Spec}\, k}(S) \simeq (0, +)$, using the universal property of $\mathrm{Spec}\, k$, and the map

$$(0, +) \xrightarrow{e_S} h_X(S)$$

induces a unique map

$$h_{\mathrm{Spec}\, k}(S) \to h_X(S)$$

which must satisfy diagram 2.3, proving the statement. □

Definition 2.1.6. An algebraic group G is called an **affine algebraic group** if G is an affine scheme.

Since there is an equivalence of categories

$$\begin{aligned} \mathrm{AffSch}_k &\simeq \mathrm{Alg}_k \\ X &\mapsto \mathcal{O}(X) \\ \mathrm{Spec}\, A &\leftarrow\!\shortmid A, \end{aligned}$$

where AffSch_k and Alg_k denote the category of affine k–schemes and the category of k–algebras of finite type, respectively, if we restrict this to the subcategory of affine algebraic groups $\mathrm{AffAlgGrp}_k$ inside AlgGrp_k, we should expect to get a corresponding algebraic object on Alg_k via this equivalence.

Definition 2.1.7. A k–algebra A together with k–algebra morphisms $m^* : A \otimes A \to A$, $i^* : A \to A$ and $e^* : A \to k$ such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "Id_A \otimes m^*"'] & A \arrow[r, "Id_A \otimes i^*"] & A \otimes A & A \arrow[l, "i^* \otimes Id_A"'] \\
A \otimes A \arrow[u, "m^* \otimes Id_A"] & A \arrow[l, "m^*"] \arrow[u, "m^*"'] & k \arrow[u] & A \arrow[l, "e^*"] \arrow[r, "e^*"'] \arrow[u, "m^*"] & k \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "e^* \otimes Id_A"'] \arrow[r, "Id_A \otimes e^*"] & A \otimes k \\
& A \arrow[ul, "\simeq"] \arrow[u, "m^*"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

is called a **Hopf algebra**. Given two Hopf algebras (A, m_A^*, i_A^*, e_A^*) and (B, m_B^*, i_B^*, e_B^*), a morphism of k–algebras $\phi : A \to B$ is a **morphism of Hopf algebras** if the following diagrams commute:

$$\begin{tikzcd}
A \times A \arrow[r, "(\phi,\phi)"] & B \times B & A \arrow[r, "\phi"] \arrow[d, "e_A^*"'] & B \arrow[dl, "e_B^*"] & A \arrow[r, "\phi"] \arrow[d, "i_A^*"] & B \arrow[d, "i_B^*"] \\
A \arrow[r, "\phi"] \arrow[u, "m_A^*"] & B \arrow[u, "m_B^*"'] & k & & A \arrow[r, "\phi"'] & B
\end{tikzcd}$$

The category of Hopf Algebras over k will be denoted by $\mathrm{HopfAlg}_k$.

This definition is dual to the definition of algebraic group, and the correct one so that we have the result:

Theorem 2.1.8. *The following*

$$\mathrm{HopfAlg}_k \simeq \mathrm{AffAlgGrp}_k$$
$$A \mapsto \operatorname{Spec} A$$
$$(\mathcal{O}_G(G), m^*, i^*, e^*) \mapsfrom (G, m, i, e)$$

is an equivalence of categories.

Proof. By definition, $\mathrm{HopfAlg}_k$ and $\mathrm{AffAlgGrp}_k$ are subcategories of Alg_k and AlgGrp_k, respectively, which satisfy exactly dual diagrams, so we can just use the equivalence given by Spec and the definition of each object to conclude. □

We can use this equivalence of categories to produce interesting examples of group schemes:

1. **The additive group** $G_a \doteq \operatorname{Spec} k[t]$. To see that this is an affine algebraic group, we can give $k[t]$ a Hopf algebra structure by defining:

$$\begin{aligned} m^* : k[t] &\to k[t] \otimes k[t] \\ t &\mapsto t \otimes 1 + 1 \otimes t \\ i^* : k[t] &\to k[t] \\ t &\mapsto -t \\ e^* : k[t] &\to k \\ t &\mapsto 0 \end{aligned}$$

These morphisms satisfy the diagrams on (2.1.7), and the name comes from the following universal property: if $(A, +, \cdot)$ is a k–algebra, we have

$$h_{\mathbb{G}}(\operatorname{Spec} A) = \operatorname{Hom}_k(\operatorname{Spec} A, \operatorname{Spec} k[t]) = \operatorname{Hom}_k(k[t], A) = (A, +),$$

since the map

$$\begin{aligned} \phi : A &\to \operatorname{Hom}_k(k[t], A) \\ a &\mapsto \phi_a : t \mapsto a \end{aligned}$$

is a bijection such that $\phi_{a+b} = \phi_a + \phi_b$ and $\phi_0 = 0$. This means that the representable functor $h_{\mathbb{G}_a} : \mathrm{AffSch}_k \to \mathrm{Grp}$ can be interpreted (via equivalence of categories) as the functor $\mathrm{Alg}_k \to \mathrm{Grp}$ that associates the additive group of a k–algebra.

Geometrically, since $\operatorname{Spec} k[t] \simeq \mathbb{A}^1_k$ and $\mathbb{A}^1_k(k) \simeq k$, we could also define the usual operation

$$\begin{aligned} m : k \times k &\to k \\ (a, b) &\mapsto a + b \end{aligned}$$

and this indeed induces an algebraic group structure in $\mathbb{A}^1_k$ since it defines a morphism of algebraic varieties satisfying the diagrams in 2.1.1. We note that

$$\begin{aligned} m^*(t)(a,b) &= t \circ m(a,b) = a + b = (t \otimes 1 + 1 \otimes t)(a,b) \\ i^*(t)(a) &= t \circ i(a) = -a = (-t)(a) \\ e^*(t)(a) &= t \circ e(a) = t \circ 0 = 0 = 0 \circ (a) \end{aligned}$$

for each pair $(a,b) \in \mathbb{A}^1_k(k) \times \mathbb{A}^1_k(k)$, and thus (m^*, i^*, e^*) induce the same Hopf algebra structure associated to the algebraic group $(\mathbb{A}^1_k, +)$.

2. **The multiplicative group** $\mathbb{G}_m \doteq \operatorname{Spec} k[t, t^{-1}]$. We can give $k[t, t^{-1}]$ a Hopf algebra structure by defining

$$\begin{aligned} m^* : k[t, t^{-1}] &\to k[t] \otimes k[t, t^{-1}] \\ t &\mapsto t \otimes t \\ i^* : k[t, t^{-1}] &\to k[t, t^{-1}] \\ t &\mapsto t^{-1} \\ e^* : k[t, t^{-1}] &\to k \\ t &\mapsto 1 \end{aligned}$$

This defines a Hopf algebra, and again the name comes from a universal property: if $(A, +, \cdot)$ is a $k-$algebra, we have

$$h_{\mathbb{G}_m}(\operatorname{Spec} A) = \operatorname{Hom}_k(\operatorname{Spec} A, \operatorname{Spec} k[t, t^{-1}]) = \operatorname{Hom}_k(k[t, t^{-1}], A) = (A^\times, \cdot).$$

Moreover, since $\operatorname{Spec} k[t, t^{-1}] \simeq \mathbb{A}^1 \setminus \{0\}$ and $(\mathbb{A}^1_k \setminus \{0\})(k) = k \setminus \{0\}$, we could also define the usual operation

$$\begin{aligned} m : (k \setminus \{0\}) \times (k \setminus \{0\}) &\to (k \setminus \{0\}) \\ (a, b) &\mapsto a \cdot b \end{aligned}$$

and, this induces the same Hopf algebra structure as defined before.

3. **General Linear Group**. Let

$$A \doteq \frac{k[T_{11}, \ldots, T_{nn}]}{(\det(T_{ij}) - 1)}$$

Note that $\operatorname{Spec} A$ can be viewed as the affine scheme corresponding to $\mathrm{GL}_n(k) \subset \mathbb{A}^{n^2}_k$, and we can define the operations:

$$\begin{aligned} m^* : A \otimes A &\to A \\ T_{ij} &\mapsto \sum_{l=1}^{n} T_{il} \otimes T_{lj} \\ i^* : A &\to A \\ T_{ij} &\mapsto a_{ij}, \text{where } A = (T_{ij})^{-1} \\ e^* : A &\to k \\ T_{ij} &\mapsto \det T_{ij} \end{aligned}$$

This defines a Hopf algebra, and if R is a $k-$algebra the associated representable functor will satisfy

$$h_{\mathrm{GL}_n(k)}(\operatorname{Spec} R) \simeq \operatorname{Hom}_k(A, R) = \mathrm{GL}_n(R).$$

Using this last example, we can produce all the classical algebraic matrix subgroups of $\mathrm{GL}_n(R)$, such as the special linear group $\mathrm{SL}_n(R)$, the orthogonal group $SO_n(R)$ and the symplectic group $Sp_n(R)$. For more examples, see (MILNE, 2012).

Definition 2.1.9. Let G be an affine algebraic group over k. We define the **character group** of G as the group $X^*(G) \doteq \mathrm{Hom}(G, \mathbb{G}_m)$, with pointwise induced group structure. The **cocharacter group** of G is defined by the group $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$.

In the following, we compute some examples of character groups.

Lemma 2.1.10. $X^*(\mathbb{G}_m) = X_*(\mathbb{G}_m) \simeq \mathbb{Z}$.

Proof. Let

$$\begin{aligned} \theta : \mathbb{Z} &\to X^*(\mathbb{G}_m) \\ n &\mapsto \theta(n) : t \mapsto t^n. \end{aligned}$$

First, we need to show that this is well defined, i.e., that the function $\theta(n)$ is a morphism of algebraic groups. To do this, we consider its dual in the category of Hopf algebras, given explicitly by the map

$$\begin{aligned} \theta(n)^* : k[t, t^{-1}] &\to k[t, t^{-1}] \\ t &\mapsto t^n. \end{aligned}$$

To see this is a morphism of Hopf algebras, we need to check the compatibility between $\theta(n)^*$ and the operations m^*, i^* and e^* defined on $\mathcal{O}(\mathbb{G}_m)$. But this follows from associativity, by

$$\begin{aligned} m^*(t)^n &= (t \otimes t)^n = t^n \otimes t^n = m^*(t^n), \\ (e^*(t))^n &= (1)^n = 1^n = e^*(t^n), \\ i^*(t)^n &= (t^{-1})^n = (t^n)^{-1} = i^*(t^n). \end{aligned}$$

The map θ itself defines a morphism of groups. Furthermore, if the function $\theta(n)$ is the neutral element in the group $X^*(\mathbb{G}_m)$, then we must have $n = 0$, so θ is injective.

To show surjectivity, we let $\phi \in \mathrm{End}(\mathbb{G}_m)$ be any element. Since $\phi^* : \mathcal{O}(G) \to \mathcal{O}(G)$ is a morphism of Hopf algebras, ϕ is completely determined by its value on t. We write

$$\phi^*(t) = \sum_{|i|<m} a_i t^i$$

for a general element of $k[t, t^{-1}]$ and show ϕ must be in the image of θ using the condition that ϕ commutes with m^* and e^*. First, the equation $m^*(\phi^*(t)) = \phi^*(m^*(t))$

can be rewritten as:

$$\phi^*(t) \otimes \phi^*(t) = \sum_{|i|,|j|<m} a_i a_j t^i \otimes t^j = \sum_{|i|<m} a_i t^i \otimes t^i$$

and this already implies

$$\sum_{|i|<m} (a_i^2 - a_i) t^i \otimes t^i + \sum_{i \neq j, |i|,|j|<m} a_i a_j t^i \otimes t^j = 0$$

which means that $a_i a_j = 0$ whenever $i \neq j$, so that at most one of the coefficients satisfies $a_n \neq 0$.

On the other hand, if we use $e^*(t) = e^* \circ \phi^*(t)$, we get

$$1 = \sum_{|i|<m} a_i e^*(t^i) = \sum_{|i|<m} a_i$$

and conclude that at least one coefficient must be non-zero, and the sum of all coefficients must be one. With the previous observation, we get $a_n = 1$ and in this case $\phi = \theta(n)$. □

Definition 2.1.11. An affine algebraic group T over k is a **torus** if $T \simeq \mathbb{G}_m^n$ for some $n > 0$.

Note that, if $T \simeq \mathbb{G}_m^n$ is a torus, we can use the universal property of products to deduce:

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \simeq \prod_{i=1}^{n} \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \prod_{i=1}^{n} \mathbb{Z}.$$

Proposition 2.1.12. $X^*(\mathrm{GL}_n(\mathbb{C})) = \mathrm{Hom}(\mathrm{GL}_n(\mathbb{C}), \mathbb{G}_m(\mathbb{C})) \simeq \mathbb{Z}$.

Proof. If $\mathcal{X} \in X^*(\mathrm{GL}_n(\mathbb{C})$, because $\mathbb{G}_m$ is an abelian group, then

$$\mathcal{X}([g,h]) = \mathcal{X}(gh - hg) = 1$$

whenever $g, h \in \mathrm{GL}_n(\mathbb{C})$. By basic group theory (see 6.7, Lemma 1 in (JACOBSON, 2009)), we can compute the commutators

$$[\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})] = [\mathrm{SL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})] = \mathrm{SL}_n(\mathbb{C}),$$

so $\mathcal{X}$ must factor by the quotient in the commutative diagram

$$\begin{tikzcd} \mathrm{GL}_n(\mathbb{C}) \arrow[r, "\mathcal{X}"] \arrow[d, two heads, "\pi"'] & \mathbb{G}_m(\mathbb{C}) \\ \mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C}) \arrow[ur, dashed] & \end{tikzcd}$$

Moreover, we have $\mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C}) \simeq \mathbb{G}_m(\mathbb{C})$, since $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C})$ is a surjective map which factors through the quotient, by the isomorphism theorem.

By the universal property of quotients, we have an isomorphism

$$\mathrm{Hom}(\mathrm{GL}_n(\mathbb{C}), \mathbb{G}_m(\mathbb{C})) \simeq \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z},$$

and using the determinant map we get an explicit isomorphism:

$$\begin{aligned} \mathbb{Z} &\to X^*(\mathrm{GL}_n(\mathbb{C})) \\ n &\mapsto \mathcal{X}_n : g \mapsto \frac{1}{(\det g)^n}. \end{aligned}$$

□

2.2 Algebraic Group Actions

Definition 2.2.1. An **algebraic action** of an affine algebraic group G over k on a k–scheme X is a morphism of k–schemes $\sigma : G \times X \to X$ such that the following diagrams commute

$$\begin{tikzcd} \operatorname{Spec} k \times X \arrow[rr, "(e, Id_X)"] \arrow[dr, "\simeq"'] & & G \times X \arrow[dl, "\sigma"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} G \times G \times X \arrow[r, "(Id_G, \sigma)"] \arrow[d, "(m_G, Id_X)"'] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

Let $\sigma_X : G \times X \to X$ and $\sigma_Y : G \times Y \to Y$ be algebraic actions of G on the k–schemes X and Y. We say that a morphism $f : X \to Y$ is G–**equivariant** if the diagram

$$\begin{tikzcd} G \times X \arrow[r, "(Id_G, f)"'] \arrow[d, "\sigma_X"'] & G \times Y \arrow[d, "\sigma_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. If Y is given the trivial action $\sigma_Y = \pi_Y$, then we say f is a G–**invariant morphism**.

When X is affine, an algebraic action $\sigma : G \times X \to X$ induces a coaction in the k–algebras:

$$\begin{aligned} \sigma^* : \mathcal{O}(X) &\to \mathcal{O}(G \times X) \simeq \mathcal{O}(G) \otimes \mathcal{O}(X) \\ f &\mapsto \sum_i h_i \otimes f_i. \end{aligned}$$

More concretely, using the isomorphism $\mathcal{O}(G \times X) \simeq \mathcal{O}(G) \otimes \mathcal{O}(X)$ we can write

$$\sigma^*(f)(g,x) = f \circ \sigma(g,x) = \sum_i h_i(g) \cdot f_i(x)$$

whenever $g \in G$ and $x \in X$, so fixing $g \in G$ defines a mapping

$$\begin{aligned} G &\to \operatorname{Aut}_k(\mathcal{O}(X)) \\ g &\mapsto \phi_g \end{aligned}$$

given by

$$\phi_g(f) \doteq \sum_i h_i(g) f_i,$$

whenever $\sigma^*(f) = \sum h_i \otimes f_i$. In other words, this is the action

$$(g \cdot f)(x) = f(g \cdot x),$$

which uses the action σ as a change of coordinates for morphisms $f \in \mathcal{O}(X)$.

Lemma 2.2.2. Let G be an affine algebraic group over k acting algebraically on an affine k–scheme X, and let G act on $\mathcal{O}(X)$ as change of coordinates, as above. For any finite dimensional vector subspace W of $\mathcal{O}(X)$, there is a finite dimensional G–invariant vector subspace V of $\mathcal{O}(X)$ which contains W.

Proof. Since W is finite dimensional, we let $\{f_1, \dots, f_r\}$ be a basis of W and define

$$V \doteq \operatorname{span}\{g \cdot f_i : g \in G, i = 1, \dots, r\} \subset \mathcal{O}(X).$$

By definition, V is G–invariant and $W \subset V$. All we need to prove is that V is finite dimensional. To do this, whenever $i = 1, \dots, r$, let

$$\sigma^*(f_i) = \sum_{j=1}^{n_i} a_{ij} \otimes b_{ij}$$

with $a_{ij} \in \mathcal{O}(G)$ and $b_{ij} \in \mathcal{O}(X)$. Using this notation, the action has the form

$$g \cdot f_i = \sum_{j=1}^{n_i} a_{ij}(g) b_{ij}.$$

Let W' be the vector space generated by the set $\{b_{ij} : i = 1, \dots, r, j = 1, \dots, n_i\}$. Since W' is finite dimensional over k and

$$g \cdot f_i = \sum_{j=1}^{n_i} a_{ij}(g) b_{ij} \in W'$$

whenever $g \in G$, then $V \subset W'$ and this means that V is also finite dimensional. $\square$

Definition 2.2.3. We say that an algebraic action of an affine algebraic group G over k on a k–algebra is **rational** if every element of A is contained in a finite dimensional G–invariant linear subspace of A.

By the previous lemma, the induced action of G on the k–algebra $\mathcal{O}(X)$ is rational, whenever G acts algebraically on X. This is a key observation, and using this fact we can prove the following characterization of affine algebraic groups of finite type over k:

Theorem 2.2.4. *Any affine algebraic group of finite type over k is a* ***linear algebraic group****, i.e., a closed algebraic subgroup of* $\mathrm{GL}_N(k)$ *for some* $N > 0$.

Proof. Since G is of finite type over k, $\mathcal{O}(G)$ is finitely generated over k. Let W denote the vector space generated by a finite choice of generators of $\mathcal{O}(G)$ as a k–algebra. The vector space W will be finite dimensional and since the action of G on $\mathcal{O}(G)$ is rational, there is a finite dimensional vector space V which is G–invariant, $W \subset V$ and it is the vector space with smallest dimension which satisfies these two properties.

Let $\{f_1, \dots, f_n\}$ be a basis of V, and $m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ the dual of the multiplication. Since V is G–invariant, $m^*(f_i) \in \mathcal{O}(G) \otimes V$ and hence we can write:

$$m^*(f_i) = \sum_{j=1}^{n} a_{ij} \otimes f_j$$

for $a_{ij} \in \mathcal{O}(G)$. The coefficients define the following k–algebra morphism:

$$\begin{aligned} \rho^* : \mathcal{O}(M_n(k)) &\to \mathcal{O}(G) \\ x_{ij} &\mapsto a_{ij}, \end{aligned}$$

which corresponds to a map of k–schemes $\rho : G \to M_n(k)$. To prove that ρ is a closed embedding, we only need to show that the associated map ρ^* of k–algebras is surjective.

Using the structure of algebraic group of $G = (G, m, i, e)$, we can write:

$$\begin{aligned} f_i &= (Id_{\mathcal{O}(G)} \otimes e^*) \circ m^*(f_i) \\ &= (Id_{\mathcal{O}(G)} \otimes e^*) \sum_{j=1}^{n} a_{ij} \otimes f_j = \sum_{j=1}^{n} e^*(f_j) a_{ij}, \end{aligned}$$

which means that $\{f_1, \dots, f_n\} \subset \operatorname{Im} \rho^*$, and thus $V \subset \operatorname{Im} \rho^*$. Since ρ^* is also a k–algebra morphism, so its image must be a k–algebra containing every generator of $\mathcal{O}(G)$, which means ρ^* is surjective.

Note that the k–scheme $M_n(k)$ is just the affine space $\mathbb{A}_k^{n^2}$, and the multiplication of matrices is a morphism $m_n : M_n(k) \times M_n(k) \to M_n(k)$, since each coordinate is polynomial on the coefficients. This means that we can consider $M_n(k)$ as an **algebraic**

semigroup over k with this operation, and only this, since not every matrix has an inverse under multiplication.

We show ρ is a morphism of algebraic semigroups using the corresponding diagram in the category of k–algebras:

$$\begin{tikzcd}
\mathcal{O}(M_n(k)) \arrow[r, "m_n^*"] \arrow[d, "\rho^*"'] & \mathcal{O}(M_n(k)) \otimes \mathcal{O}(M_n(k)) \arrow[d, "\rho^* \otimes \rho^*"] \\
\mathcal{O}(G) \arrow[r, "m^*"] & \mathcal{O}(G) \otimes \mathcal{O}(G)
\end{tikzcd}$$

To see this commutes, we must show that for each $x_{ij} \in \mathcal{O}(M_n(k))$, the expressions

$$\rho^* \otimes \rho^* \circ m_n^*(x_{ij}) = \rho^* \otimes \rho^* \left(\sum_k x_{ik} \otimes x_{kj} \right) = \sum_k a_{ik} \otimes a_{kj}$$

and $m^* \circ \rho^*(x_{ij}) = m^*(a_{ij})$ coincide. Consider the dual of the associativity diagram (2.1) of the group G, given by $m \circ (Id_G, m) = m \circ (m, Id_G)$, applied on the element $f_i \in \mathcal{O}(G)$. On the left side, we get:

$$\begin{aligned}
(Id_{\mathcal{O}(G)} \otimes m^*) \circ m^*(f_i) &= (Id_{\mathcal{O}(G)} \otimes m^*) \left(\sum_{j=1}^n a_{ij} \otimes f_j \right) \\
&= \sum_{j=1}^n a_{ij} \otimes \left(\sum_{k=1}^n a_{jk} \otimes f_k \right) \\
&= \sum_{j,k=1}^n a_{ij} \otimes a_{jk} \otimes f_k,
\end{aligned}$$

and on the right side:

$$m^* \otimes Id_{\mathcal{O}(G)} \left(\sum_{j=1}^n a_{ij} \otimes f_j \right) = \sum_{j=1}^n m^*(a_{ij}) \otimes f_j.$$

Since the set $\{f_1, \dots f_n\}$ is linear independent in $\mathcal{O}(G)$, comparing the coefficients on both sides yields the equality

$$m^*(a_{ij}) = \sum_{j,k} a_{ij} \otimes a_{jk}$$

whenever $i, j = 1, \dots, n$, what we wanted to prove.

Since G is a group, every element has an inverse and we conclude that the image of ρ must be contained in the group $\mathrm{GL}_n(k)$ inside $M_n(k)$. □

The naive set-theoretical formulations of orbit and stabilizer have natural adaptations to algebraic geometry. To define them, we recall some useful notions.

Definition 2.2.5 ((EISENBUD; HARRIS; HARRIS, 2000), p. 26)**.** A subset of X is **locally closed** if it is an open subscheme of a closed subscheme of X.

This is the natural way of algebraic geometry to construct schemes or, more generally, locally ringed spaces, as subspaces of already defined ones. If $(X, \mathcal{O}_X)$ is any locally ringed space and $F \subset X$ is an open subset of a closed subset of X, we can consider the induced sheaf $\mathcal{O}_X|_F$, since restriction to both open and closed subsets are well defined.

Another useful way of constructing morphisms of schemes is to consider base changes: if

$$\begin{tikzcd} X' \arrow[r, "B(\pi)"] \arrow[d, "B(f)"'] & X \arrow[d, "f"] \\ S' \arrow[r, "\pi"] & S \end{tikzcd}$$

is a pull-back diagram in Sch_k, then we call $B(f)$ the **base change** of the arrow f by π and $B(\pi)$ the **base change** of the arrow $B(\pi)$ by f. For more on how base changes behave in algebraic geometry, see (LIU; ERNE, 2006), Chapter 3.

Definition 2.2.6 ((LIU; ERNE, 2006), 3.13)**.** We say a map $f : X \to Y$ of schemes is **closed** if f is closed as a map of topological spaces. Moreover, we say that f is **universally closed** if every base change of f is closed.

Theorem 2.2.7 (see (LIU; ERNE, 2006), Prop 3.16)**.** *If*

$$\begin{tikzcd} X' \arrow[r, "B(\pi)"] \arrow[d, "B(f)"'] & X \arrow[d, "f"] \\ S' \arrow[r, "\pi"] & S \end{tikzcd}$$

is a pull-back in Sch_k *and* f *is a closed immersion, then* $B(f)$ *is a closed immersion. In particular, closed immersions are universally closed.*

Definition 2.2.8. Let $\sigma : G \times X \to X$ be an algebraic action of an affine algebraic group G on a scheme X. Let $x : \operatorname{Spec} k \to X$ be a k-point. We define:

1. The **orbit** $G \cdot x$ of x is the (set-theoretic) image of the map $\sigma_x = \sigma(-, x) : G \to X$. This is usually just a set, but in the next lemma we prove $G \cdot x$ admits a locally closed structure inside X.

2. The **stabilizer** G_x of x is the fiber product in Sch_k of σ_x and x, as in the diagram below:

$$\begin{tikzcd} G_x \arrow[r, "i"] \arrow[d] & G \arrow[d, "\sigma_x"] \\ \operatorname{Spec} k \arrow[r, "x"] & X \end{tikzcd}$$

where, since x is a closed immersion and i is a base change of x, i is also a closed immersion of G_x into G, so that G_x is a closed subscheme of X.

Theorem 2.2.9. *The orbits of closed points are locally closed subsets of X, hence can be identified with the corresponding reduced locally closed subschemes.*

Moreover, the boundary of an orbit $\overline{G\cdot x} \setminus G\cdot x$ is a union of orbits of stricly smaller dimension. In particular, each orbit closure contains a closed orbit (of minimal dimension).

Proof. Let $x \in X(k)$. The orbit $G\cdot x$ is the image of the morphism σ_x, and by a theorem of Chevalley ((HARTSHORNE, 1977), II Exercise 3.19) it is **constructible**, i.e., there exists a dense open set U of the closure $\overline{G\cdot x}$ such that $U \subset G\cdot x$. Since G acts transitively on $G\cdot x$ through σ_x, every point of $G\cdot x$ is contained by a translation of U, and all the translations are again inside $G\cdot x$, so this implies that $G\cdot x$ is an open set inside the closed set $\overline{G\cdot x}$.

This implies that $G\cdot x$ is a locally closed subset, and hence we can have the corresponding reduced scheme associated with G, G_{red}, acting on the reduced scheme associated with $G\cdot x$, which is transitive on k–points. In particular, the dimension of $G\cdot x$ is the same at every point, by the transitivity of the action.

To show that the boundary of the orbit is G–invariant, let

$$y \in \overline{G\cdot x} \setminus G\cdot x$$

be a point at the boundary, $g \in G$ and V any open set around the point $g\cdot y$. Since G acts on X by algebraic automorphisms, in particular G acts continuously such that we can take the open neighbourhood $g^{-1}V$ around the point $g^{-1}(gy) = y$. Because $y \in \overline{G\cdot x}$, there exists a point

$$z = h\cdot x \in g^{-1}V \cap G\cdot x.$$

This means that

$$gz = gh\cdot x \in V \cap G\cdot x,$$

in particular $V \cap G\cdot x \neq \varnothing$, so that $g\cdot y \in \overline{G\cdot x}$. Because $y \notin G\cdot x$, we could not have $g\cdot y \in G\cdot x$, so that $g\cdot y \in \overline{G\cdot x} \setminus G\cdot x$.

In particular, we can write the boundary $\overline{G\cdot x} \setminus G\cdot x$ as a union of disjoint orbits, which are also finite since X is affine, hence quasi-compact.

Since $G \cdot x$ is locally closed and the boundary $\overline{G \cdot x} \setminus G \cdot x$ is the complement of a dense open subset, the boundary is closed and of strictly lower dimension than $G \cdot x$. Indeed, if the complement $\overline{G \cdot x} \setminus G \cdot x$ is irreducible, we are done because non-trivial irreducible subsets have strictly smaller dimension. If not, let $F \subset \overline{G \cdot x}$ be a closed irreducible set such that $G \cdot x \subset F$. Then $F \cap \overline{G \cdot x}$ is an open dense set subset of F, and this means $F = \overline{G \cdot x}$, so that $\dim G \cdot x < \dim F = \dim \overline{G \cdot x}$.

This means that orbits of minimal dimension are closed (if not, there would be orbits on the boundary with even smaller dimension) and we conclude each orbit closure contains a closed orbit. □

An algebraic action of an affine algebraic group G on X is **closed** if all G–orbits are closed on X. To illustrate the previous theorem, we propose the following examples:

Example 2.1. Let $G_m = \operatorname{Spec} k[t, t^{-1}]$ act on $\mathbb{A}^2_k$ by the rule

$$t \cdot (x, y) \doteq (tx, t^{-1}y)$$

whenever $(x, y) \in \mathbb{A}^2_k(k)$ and $t \in \mathbb{G}_m(k)$. The orbits of this action are:

- Conics $\{(x, y) : x \cdot y = \alpha\}$, whenever $\alpha \in \mathbb{A}^1 \setminus \{0\}$ and the origin, which are closed orbits.
- The punctured x–axis and y–axis, which are not closed.

Note that the puncture axes contain the origin in their closure, with strictly smaller dimension, and the corresponding coaction morphism can be written as

$$\begin{aligned}\sigma^* : k[x, y] &\to k[t, t^{-1}] \otimes k[x, y]\\ x &\mapsto t \otimes x\\ y &\mapsto t^{-1} \otimes y.\end{aligned}$$

Example 2.2. Let $G = \mathbb{G}_m$ act on $X = \mathbb{A}^{n+1}_k$ via

$$t \cdot (x_0, \ldots, x_n) \doteq (tx_0, \ldots, tx_n).$$

Note that there are no closed orbits for this action, but if we consider the restriction to the open set $\mathbb{A}^{n+1}_k \setminus \{0\}$, we get an action which is algebraic, closed and the orbits are the punctured lines at the origin

$$l_{(x_0, \ldots, x_n)}(k) = \{t \cdot (x_0, \ldots, x_n) \in \mathbb{A}^{n+1}(k) : t \in k \setminus \{0\}\}.$$

The open subscheme $X = \mathbb{A}^{n+1} \setminus \{0\}$ is famously not affine and moreover

$$\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^{n+1} \setminus \{0\}) \simeq k[x_0, \ldots, x_n].$$

Computing the G–invariant functions, we get

$$\mathcal{O}(X)^G \simeq \{f \in k[x_0, \dots, x_n] : f(x_0, \dots, x_n) = f(t \cdot x_0, \dots, t \cdot x_n) \ \forall t \in k \setminus \{0\}\} \simeq k$$

In this case, the topological quotient X/G should coincide with the projective n–space over k, and as we know the only global regular functions over the projective space are the constant ones. We discuss this example in more detail in the next chapter.

There are also other schemes which can be defined in this context, and the following lemma again uses the technique of base change to produce them.

Lemma 2.2.10. Let G be an affine algebraic group over k acting algebraically on a k–scheme X.

(i) If Y and Z are subschemes of X, with Z closed, then the set

$$\{g \in G : gY \subset Z\}$$

is closed.

(ii) If X is a variety (i.e., separable over k), for any subgroup $H \leq G$ its **fixed point locus**

$$X^H \doteq \{x \in X : Hx = x\}$$

is closed.

Proof. (i) We fix $y \in Y$ and consider the fiber product

$$\begin{tikzcd} G \times_X Z \arrow[r, "\pi_1"] \arrow[d] & G \arrow[d, "\sigma_y"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where $i : Z \to X$ denotes the closed immersion. Since π_1 is a base change of i, this means that π_1 is also a closed immersion, hence the image

$$\pi_1(G \times_X Z) = \{g \in G : \sigma_y(g) \in Z\}$$

is a closed set in G. Moreover,

$$\{g \in G : gY \subset Z\} = \bigcap_{y \in Y} G_y(Z),$$

and thus this set is also closed.

(ii) Given $h \in H$, we have an automorphism $\sigma_h = \sigma(h, -) \in \mathrm{Aut}(X)$, and as before, we can consider the fiber product

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_X"] \\ X \times X \arrow[r, "(\sigma_h, \mathrm{Id}_X)"] & X \times X \end{tikzcd}$$

and by construction the set $P(k)$ coincides with the graph of σ_h, so the inclusion $\mathrm{Gr}(\sigma_h) \subset X \times X$ is a base change of the diagonal morphism Δ_X. Since X is separable over k, the diagonal $\Delta_X \subset X \times X$ is a closed immersion, so σ_h is also a closed immersion and because

$$\begin{aligned} X^H &= \{x \in X : \sigma(H, x) = \{x\}\} \\ &= \bigcap_{h \in H} \{x \in X : \sigma(h, x) = x\} \\ &= \bigcap_{h \in H} \pi_1^{-1}(\mathrm{Gr}(\sigma_h) \cap \Delta_X), \end{aligned}$$

the claim follows.

□

The following can be thought as an orbit-stabilizer type theorem in our algebro-geometric context.

Proposition 2.2.11. Let G be an affine algebraic group acting algebraically on an affine k–scheme X. For $x \in X(k)$, we have

$$\dim G = \dim(G_x) + \dim(G \cdot x)$$

Proof. Since the dimension is a topological invariant of a scheme, we can assume G and X are reduced. The orbit $G \cdot x$ can be seen as a locally closed subscheme of X according to the previous theorem, which is reduced by definition. We can use a theorem (B.0.12, in appendix B) to conclude that there is an open dense subset U of $G \cdot x$ such that the restriction $\sigma_x : \sigma_x^{-1}(U) \to U$ is flat.

If $g \in G$, we can consider the base change

$$\begin{tikzcd} G \arrow[r, "m(g,-)"] \arrow[d, "\sigma_x"] & G \arrow[d, "\sigma_x"] \\ G \cdot x \arrow[r, "\sigma_g"] & G \cdot x \end{tikzcd}$$

and writing it over the open set U, we get that the morphism $\sigma_x : \sigma_x^{-1}(\sigma_g(U)) \to \sigma_g(U)$ is also flat, where σ_g is the automorphism induced by the action of G onto $G \cdot x$, since flatness is also stable by base change (see B.0.10 in appendix B). This action is also transitive, what means that the open sets $\{\sigma_g(U) : g \in G\}$ cover the set $G \cdot x$, and hence the map σ_x is itself flat.

By definition, $G_x = \sigma_x^{-1}(x)$ and we can use the formula in appendix B.0.11 to compute dimensions of fibers of flat morphisms and conclude

$$\dim G_x = \dim G - \dim G \cdot x.$$

□

Proposition 2.2.12. Let G be an affine algebraic group over k acting algebraically on a k–scheme X by a morphism $\sigma : G \times X \to X$. Then the dimension of the stabilizer subgroup (resp. orbit) viewed as a function $X \to \mathbb{N}$ is **upper semi-continuous** (resp. **lower semi-continuous**), which means that whenever $n \in \mathbb{N}$, the sets

$$\{x \in X : \dim G_x \geq n\} \text{ and } \{x \in X : \dim(G \cdot x) \leq n\}$$

are closed in X.

Proof. Consider the morphism $\Gamma \doteq (\pi_X, \sigma) : G \times X \to X \times X$ and the fiber product P in the corresponding diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \Delta_X \\ G \times X & \xrightarrow[\Gamma]{} & X \times X. \end{array}$$

Then the k–points of the fiber product P consist precisely of pairs (g, x) such that $g \in G_x$, and if $p = (g, x) \in P$ we denote the fiber over p by $P_{\varphi(p)} = \varphi^{-1}(\varphi(p))$.

By a theorem of Chevalley (see (GROTHENDIECK, 1964), 13.1.3), the function which assigns the dimension of the fiber at each point is upper semi-continuous. This implies the first claim, and the second follows from the previous result. □

2.3 Representations of Algebraic Groups

We give an overview of Representation Theory of Algebraic Groups. For most of this section, the reference is (MILNE, 2017).

Definition 2.3.1. A **linear representation** of an algebraic group G over k is a morphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$ over k, where V is a k–vector space. Since this is equivalent to consider an action of G on the vector space V by k–linear automorphisms, linear representations can be seen as G–modules. If $W \subset V$ is a subspace, then it induces a **subrepresentation** $\nu : G \to \mathrm{GL}(W)$ of ρ by restriction, whenever W is a $\rho(G)$–invariant subspace.

Dually, if (A, m^*, i^*, e^*) is any k–Hopf algebra and there is a morphism

$$\rho^* : V \to V \otimes A,$$

then the pair (V, ρ^*) is called a (right) A- **comodule** whenever it satisfies:

$$(\mathrm{Id}_V \otimes m^*) \circ \rho^* = (\rho^* \otimes \mathrm{Id}_A) \circ \rho^*,$$
$$(\mathrm{Id}_V \otimes e^*) \circ \rho^* = \mathrm{Id}_V .$$

These two conditions are satisfied whenever ρ^* comes from a linear representation, and this is precisely the dual version of a G-module.

Let $\rho : G \to \mathrm{GL}(V)$ be a G–module, with the associated $\mathcal{O}(G)$–co-module $\rho^* : V \to \mathcal{O}(G) \otimes_k V$. Then a subspace $V' \subset V$ is G–invariant if and only if $\rho^*(V') \subset \mathcal{O}(G) \otimes_k V'$. In this case, the pair $(V', \rho^*|_{V'})$ is also a $\mathcal{O}(G)$–comodule, which is called a $\mathcal{O}(G)$–**subcomodule** of (V, ρ^*). A comodule (V, ρ^*) is called **irreducible** if it's only subcomodules are the trivial ones ($V' = (0)$ or $V' = V$). Dually, the representation $\rho : G \to \mathrm{GL}(V)$ is **irreducible** if, whenever $V' \subset V$ induces a subrepresentation, we conclude $V' = (0)$ of $V' = V$.

A linear representation of G is **faithfull** if the corresponding morphism $\rho : G \to \mathrm{GL}(V)$ is injective.

If $\rho : G \to \mathrm{GL}(V)$ and $\eta : G \to \mathrm{GL}(W)$ are linear representations of G, a linear map $\phi : V \to W$ is a **morphism of representations** whenever the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\eta(g)"] \\ V \arrow[r, "\phi"] & W \end{tikzcd}$$

commutes for every $g \in G$. This notion defines a category of linear representations of G, which we will denote by $\mathrm{Rep}(G)$. We also denote $\mathrm{Hom}_G \doteq \mathrm{Hom}_{\mathrm{Rep}(G)}$.

Proposition 2.3.2. The category $\mathrm{Rep}(G)$ is additive and it has kernels.

Proof. Let us first define the additive structure on $\mathrm{Rep}(G)$. If $\rho : G \to \mathrm{GL}(V)$ and $\eta : G \to \mathrm{GL}(W)$ are two representation, the hom-set is defined by

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(\rho, \eta) = \{\phi \in \mathrm{Hom}_{\mathrm{Vect}_k}(V, W) : \eta(g) \circ \phi = \phi \circ \rho(g) \ \forall\ g \in G\},$$

and thus it forms a subgroup of the additive group $\mathrm{Hom}_{\mathrm{Vect}_k}(V,W)$, as the category Vect_k is additive. The compatibility with the composition in $\mathrm{Rep}(G)$ follows from the compatibility in Vect_k. If $T \in \mathrm{Hom}_G(V,W)$, note that $\ker T \subset V$ is a G–invariant subspace, since $T(g \cdot x) = g \cdot T(x)$ whenever $x \in V$, and $g \cdot 0 = 0$ for every $g \in G$. □

Proposition 2.3.3. For a finite dimensional linear representation of a torus $\rho : T \to \mathrm{GL}(V)$, there is a **weight space decomposition**, a direct sum

$$V \simeq \bigoplus_{\alpha \in X^*(T)} V_\alpha$$

where each subspace $V_\alpha \doteq \{v \in V : t \cdot v = \alpha(t)v \ \forall t \in T\}$ is called a **weight space** of weight α. Moreover, whenever $V_\alpha \neq 0$, we call α a **weight** for this action of T.

Proof. First, let $T \simeq \mathbb{G}_m$. We can consider the induced map of algebras

$$\rho^* : V \to V \otimes \mathcal{O}(\mathbb{G}_m) \simeq V \otimes k[t,t^{-1}]$$

which gives V a co-module structure such that the diagram

$$\begin{tikzcd} V \arrow[r, "\rho*"] \arrow[d, "\rho^*"] & V \otimes k[t,t^{-1}] \arrow[d, "\mathrm{Id}_V \otimes m^*"] \\ V \otimes k[t,t^{-1}] \arrow[r, "\rho^* \otimes \mathrm{Id}_V"] & V \otimes k[t,t^{-1}] \otimes k[t,t^{-1}] \end{tikzcd}$$

commutes. For any $m \in \mathbb{Z}$, let $V_m \doteq \{v \in V : \rho^*(v) = v \otimes t^m\}$. This is trivially a subrepresentation of (V,ρ^*). Given $v \in V$, we can write $\rho^*(v)$ in coordinates:

$$\rho^*(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m$$

where $f_m : V \to V$ is k–linear, and since ρ^* also compatible with e^* via $(\mathrm{Id}_V \otimes e^*) \circ \rho^* = \mathrm{Id}_V$, we can rewrite this as

$$\sum_{m \in \mathbb{Z}} f_m(v) \otimes 1^m = \sum_{m \in \mathbb{Z}} f_m(v) = v.$$

Moreover, using compatibility with m^*, we have

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \rho^*(f_m(v) \otimes t^m) = (\rho^* \otimes \mathrm{Id}_{k[t,t^{-1}]})(\rho^*(v)) &= (\mathrm{Id}_V \otimes m^*)(\rho^*(v)) \\ &= \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m \otimes t^m, \end{aligned}$$

and because $\{t^m : m \in \mathbb{Z}\}$ is a linear independent set in $k[t,t^{-1}]$, this means that $\rho^*(f_m(v)) = f_m(v) \otimes t^m$, which implies each coordinate $f_m(v)$ of v lies inside V_m. Thus, we get that

$$v = \sum_{m \in \mathbb{Z}} f_m(v)$$

and this already gives V as the sum of subspaces V_α, since $X^*(T) \simeq \mathbb{Z}$. Finally, to see that this is a direct sum, note that whenever $m, n \in \mathbb{Z}$:

$$\sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m \otimes t^m = \sum_{m \in \mathbb{Z}} \rho^*(f_m(v)) \otimes t^m = \sum_{m,n \in \mathbb{Z}} f_n(f_m(v)) \otimes t^n \otimes t^m,$$

and we can compare coefficients again, concluding that $f_n \circ f_m = 0$ whenever $m \neq n$ and $f_n \circ f_n = f_n$.

For a general torus $T \simeq \mathbb{G}_m^n$, if $\rho : T \to \mathrm{GL}(V)$ is a representation, by the universal property of products there are induced representations $\rho_i : \mathbb{G}_m \to \mathrm{GL}(V)$ commuting

$$\begin{tikzcd} T \arrow[r, "\rho"] \arrow[d, "\pi_i"'] & \mathrm{GL}(V) \\ T_i \simeq \mathbb{G}_m \arrow[ur, "\rho_i"'] & \end{tikzcd},$$

where $\pi_i : T \to \mathbb{G}_m$ is the i–th projection. By the claim for $T_i \simeq \mathbb{G}_m$, we can conclude that

$$V \simeq \bigoplus_{i=1}^{n} \bigoplus_{m_i \in \mathbb{Z}} V_{m_i} \simeq \bigoplus_{\alpha \in X_*(T)} V_\alpha,$$

since $X^*(T) \simeq \mathbb{Z}^n$. □

This result will be important, as we will see, to diagonalize (and characterize) torus actions on our applications.

Another important result on representation theory is the following, which describes irreducible representations over algebraically closed fields.

Theorem 2.3.4 (Schur's Lemma). *Let $\rho : G \to \mathrm{GL}(V)$ a representation of an algebraic group G. If V is irreducible and k is algebraically closed, then $\mathrm{End}_G(V) \simeq k$.*

Proof. Let $T \in \mathrm{End}_G(V)$. Since k is algebraically closed, T has an eigenvector, say $T(v) = \lambda v$, $\lambda \in k$, so that $L \doteq T - \lambda I : V \to V$ is another G-morphism, with non-zero kernel. Because every kernel of G–morphisms is also a representation of G and V is irreducible, $\ker L$ must be equal to V, implying that $T = \lambda I$ and $\mathrm{End}_G(V) \simeq k$. □

2.4 Reductive Groups

In this section, we review some of the theory of unipotent and reductive groups, following (MILNE, 2017) and (MILNE, 2012). Consider the following interpretation of the Jordan normal form for the group $G = \mathrm{GL}_n(k)$.

Theorem 2.4.1 (Jordan-Chevalley Decomposition)**.** *Every matrix $g \in \mathrm{GL}_n(k)$ has a unique decomposition of the form:*

$$g = g_{ss} \cdot g_u = g_u \cdot g_{ss}$$

where g_{ss} is semisimple (diagonalizable, if k is algebraically closed) and g_u is unipotent (that is, $g_u - I_n$ is nilpotent).

Proof. The existence follows using the Jordan normal form, choosing a basis where g is written in Jordan' s canonical form, just let g_u be the matrix obtained by dividing all entries of each Jordan block by its diagonal element (non-zero since $g \in \mathrm{GL}_n(k)$) and let g_{ss} the matrix containing only the diagonal elements of g. Then

$$g = g_{ss} \cdot g_u = g_u \cdot g_{ss}.$$

For the proof of uniqueness and even more modern interpretations (such as the Tannaka Duality) of Jordan's canonical form, we refer the reader to (MILNE, 2012), Chapter 9. □

We describe an analogous decomposition in the general case when G is an affine algebraic group.

Definition 2.4.2. Let G be an affine algebraic group over k. An element $g \in G$ is **semisimple** if there exists a faithfull linear representation $\rho : G \to \mathrm{GL}_n(k)$ such that $\rho(g)$ is semisimple. Analogously, we say that an element $g \in G$ is **unipotent** if there exists a faithfull linear representation $\rho : G \to \mathrm{GL}_n(k)$ such that $\rho(g)$ is unipotent.

Theorem 2.4.3 ((MILNE, 2017), Prop. 9.14)**.** *Let G be an affine algebraic group over k. For every $g \in G(k)$, there exists a unique semisimple element g_{ss} and a unique unipotent element g_u such that*

$$g = g_{ss} \cdot g_u = g_u \cdot g_{ss}.$$

Furthermore, this decomposition is functorial with respect with group morphisms. In particular, if $g \in G(k)$ is semisimple (resp. unipotent), then for all representations $\rho : G \to \mathrm{GL}_n$ the element $\rho(g)$ is semisimple (resp. unipotent).

The proof of this works whenever k is a perfect field, and can be found in (MILNE, 2017).

Definition 2.4.4. An affine algebraic group over k is **unipotent** if every non-trivial linear representation has a non-zero $G-$invariant vector.

Proposition 2.4.5. Let G be an affine algebraic group. The following are equivalent:

(i) G is unipotent.

(ii) Every linear representation $\rho : G \to \mathrm{GL}(V)$ admits a basis such that $\rho(G)$ is contained in the group of upper triangular matrices with diagonal entries equal to one, or the **unitary group** $\mathbb{U}_n$.

(iii) G is isomorphic to a subgroup of the unitary group $\mathbb{U}_n$.

Proof. $(i) \Rightarrow (ii)$ We proceed by induction on the dimension of V. If $\dim V = 1$, the claim is trivial. If $\dim V = n$, as G is unipotent, the fixed vector subspace $V^G \subset V$ is non-zero, so we can choose a basis $\{e_1, \ldots, e_m\}$ of V^G and the quotient V/V^G is of strictly smaller dimension.

By the induction hypothesis, this induced representation has image in the unitary group. More explicitly, there is a basis $\{\overline{e_{m+1}}, \cdots, \overline{e_n}\}$ such that

$$\overline{\rho}(G) \subset \mathbb{U}_{n-m} \subset \mathrm{GL}(V/V^G).$$

Thus, we can choose representatives forming a basis $\{e_1, \cdots, e_n\}$, such that

$$\rho(G) \subset I_m \oplus U_{n-m} \subset \mathbb{U}_n,$$

since the space V^G is fixed.

$(ii) \Rightarrow (iii)$ By 2.2.4, every affine algebraic group G admits a faithfull representation $G \to \mathrm{GL}(V)$, and (ii) gives an isomorphism between G and a subgroup $\rho(G) \subset \mathbb{U}_n$.

$(iii) \Rightarrow (i)$ This follows from the verification that unitary groups (and their subgroups) are unipotent, and for this we refer to (MILNE, 2012), Chapter XV, Theorem 2.4. $\square$

Remark 2.4.6. If G is a unipotent affine algebraic group, then every element $g \in G(k)$ is unipotent. The converse is true if G is smooth (see (MILNE, 2017), Chapter XV, Corollary 2.6)

Definition 2.4.7. An affine algebraic group G is **reductive** if G is smooth and every smooth unipotent normal algebraic subgroup $H \subset G$ is trivial.

Example 2.3. (i) Every smooth simple affine algebraic group over k is reductive, as simple groups don't have non-trivial normal subgroups.

(ii) Every compact connected Lie group has a complexification, which is a complex reductive algebraic group. Furthermore, the converse is also true: every complex reductive algebraic group arises in this way, moreover there is an equivalence between the categories of complex compact connected Lie groups and of algebraic reductive groups over $\mathbb{C}$ (see, for example, section 4.5 in (LEE, 2001)).

Definition 2.4.8. Let G be an affine algebraic group.

(a) G is **linearly reductive** if every finite dimensional linear representation $G \to \mathrm{GL}(V)$ decomposes as a direct sum of irreducible ones.

(b) G is **geometrically reductive** if, for every finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every non-zero G–invariant vector $v \in V$, there is a G–invariant non-constant homogeneous polynomial $f \in \mathcal{O}(V)$ such that $f(v) \neq 0$.

As we saw in 2.3.3, any algebraic torus T is linearly reductive. The groups $\mathrm{SL}(V)$ and $\mathrm{GL}(V)$ are all linearly reductive over $\mathrm{char}\, k = 0$. For a proof of this using Lie algebras, see for example (MUKAI et al., 2003). In the proof of 2.3.3, we also proved implicitly that the (finite) product of linearly reductive groups is also linearly reductive.

Proposition 2.4.9. For an affine algebraic group G, the following statements are equivalent:

(i) G is linearly reductive.

(ii) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ any G–invariant subspace $V' \subset V$ admits a G–stable complement, i.e., there exists a subrepresentation $V'' \subset V$ such that $V = V' \oplus V''$.

(iii) For any surjection of finite dimensional G–representations $\phi : V \to W$, the induced map $\phi^G : V^G \to W^G$ is surjective.

(iv) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every non-zero G–invariant point $v \in V$, there exists a G–invariant linear form $f : V \to k$ such that $f(v) \neq 0$.

(v) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and any surjective G–invariant linear form $f : V \to k$, there exists $v \in V^G$ such that $f(v) \neq 0$.

Proof. $(i) \Rightarrow (ii)$ Given $\rho : G \to \mathrm{GL}(V)$ and $V' \subset V$ a G–invariant space, since G is linearly reductive, we can consider a decomposition:

$$V = \bigoplus_{\alpha \in \Lambda} V_\alpha$$

of V into irreducible representations V_α. Since $V' \subset V$ is G–invariant, we can consider the induced representation of ρ in V', which induces a decomposition

$$V' = \bigoplus_{\alpha \in \Lambda} V'_\alpha$$

such that $V'_\alpha \subset V_\alpha$ whenever $\alpha \in \Lambda$. Thus, to get a G–stable complement of V', we just complete basis of V'_α to V_α whenever $\alpha \in \Lambda$, such that

$$V = \bigoplus_{\alpha \in \Lambda} V'_\alpha \oplus V''_\alpha ,$$

and set $V'' \doteq \bigoplus_\alpha V''_\alpha$.

To see $(ii) \Rightarrow (iii)$, let $\phi : (V, \rho) \to (W, \eta)$ be a surjection of G–representations and $V' \doteq \ker(\phi) \subset V$. Now, by (ii), V' has a G–stable complement V'', which must be isomorphic to W by the first isomorphism theorem.

Since V' and V'' are G–invariant, we have $V^G = (V')^G \oplus (V'')^G$, so the induced map $\phi^G : V^G \to (V'')^G \simeq W^G$ is surjective.

$(iii) \Rightarrow (iv)$ Any non-zero G–invariant vector determines a G–invariant form $\phi : V^\vee \to k$, since

$$V^G \simeq \mathrm{Hom}_G(k, V) \simeq \mathrm{Hom}_G(V^\vee, k),$$

if we let G act trivially on k. Also, we can use the fact V is finite dimensional, to have $V \simeq V^\vee$ and get an induced morphism $f \doteq \phi^G : V^G \to k$ which is also surjective, implying the existence of a non-zero vector $v \in V^G$ such that $f(v) \neq 0$.

The equivalence $(iv) \iff (v)$ follows from the previous duality observation.

$(v) \Rightarrow (i)$ Since $\dim V = 1$ implies that V is already irreducible, we let $\dim V = n > 1$ and proceed by induction.

Suppose that $\{0\} \neq V' \subsetneq V$ is a G–invariant proper subset which is also minimal in dimension. Unless V is already irreducible, in which case we are done, there is such V'. If we take a basis $\{e_1, \cdots, e_k\}$ to be a basis of V', and complete it to a basis $\{e_1, \cdots, e_n\}$ for V, we can define a linear functional

$$f \doteq e_1^* + \cdots + e_k^* \in V^\vee$$

Since this is surjective and G–invariant, there must be a G–invariant vector v_0 where $f(v_0) \neq 0$, by hypothesis. In such case, we can consider the complement V'' such that

$$V = \langle v_0 \rangle \oplus V''.$$

Since $\dim V'' < \dim V$, we can apply the induction hypothesis to conclude. □

Example 2.4. As an example of a group which is not linearly reductive, let $G = \mathbb{G}_a$ and consider the representation

$$\begin{aligned} \mathbb{G}_a &\to \mathrm{GL}_2(k) \\ t &\mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

In this case, the projection

$$\begin{aligned}\pi : V = k^2 &\to k = W\\ (x, y) &\mapsto x\end{aligned}$$

is a surjective morphism of G–representations, taking G to act trivially on k, but the restriction to invariant subspaces

$$\{(0, b) : b \in k\} = V^G \xrightarrow{\pi^G} W^G = k$$

is not surjective.

Theorem 2.4.10. *Every finite group is linearly reductive over k whenever* $\operatorname{char} k$ *does not divide $|G|$.*

Proof. Using 2.4.9 we prove that, whenever $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional linear representation, every G–invariant subspace $V' \subset V$ admits a G–stable complement. Let W be any complement for V', such that $V = V' \oplus W$, and denote the corresponding projection by $\pi : V \to V'$. We define

$$R \doteq \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1},$$

which is a well-defined k–linear endomorphism of V by our hypothesis on $\operatorname{char} k$. We claim that $R\big|_{V'} = \mathrm{Id}_{V'}$. Indeed, if $v \in V'$,

$$R(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi(\rho(g)^{-1}(v)) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \rho(g)^{-1}(v) = v,$$

since V' is G–invariant. Moreover, if $h \in G$,

$$\begin{aligned}\rho(h) \circ R \circ \rho(h)^{-1} &= \rho(h) \circ \left(\frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho^{-1}(h) \right) \circ \rho(h)^{-1}\\ &= \frac{1}{|G|} \sum_{g \in G} \rho(h)\rho(g)\pi\rho(g)^{-1}\rho(h)^{-1}\\ &= \frac{1}{|G|} \sum_{g \in G} \rho(hg)\pi\rho(hg)^{-1}\\ &= R,\end{aligned}$$

which means that the diagram

$$\begin{tikzcd} V \arrow[r, "R"] \arrow[d, "\rho(h)"'] & V^G \arrow[d, "\rho(h)"] \\ V \arrow[r, "R"] & V^G \end{tikzcd}$$

commutes. These two properties will be generalized in 2.5.1, but in this case this implies that R is a G–invariant morphism. Now, let $V'' \doteq \ker R$, and note that $V = V' + V''$, since if $v \in V$ we can write $v = R(v) + v - R(v)$, as

$$R(v - R(v)) = R(v) - R(R(v)) = R(v) - R(v) = 0.$$

Furthermore, $V' \cap V'' = \varnothing$, since $R\big|_{V'} = \mathrm{Id}_{V'}$ and $V'' = \ker R$, and V'' is a G–invariant space, since it is a kernel of a G–invariant morphism. □

For more on the relation between these conditions, we have the following theorem, which proof is beyond the objectives of this dissertation.

Theorem 2.4.11. *1. Every linearly reductive group is geometrically reductive.*

2. If $\operatorname{char} k = 0$*, every reductive group is linearly reductive.*

3. A smooth affine algebraic group is reductive if and only if it is geometrically reductive.

In particular, for smooth affine group schemes, we have

$$\textit{linearly reductive} \Rightarrow \textit{geometrically reductive} \iff \textit{reductive}$$

and all three notions coincide in characteristic zero.

Proof. (i) follows from 2.4.9. For (ii), we refer to ((MUKAI et al., 2003), Chapter 4), and the proof uses lie algebras and lie groups over $k = \mathbb{C}$. For (iii), $\Rightarrow$ was proved by Nagata in (NAGATA, 1963), and the converse by Haboush in (HABOUSH, 1975). □

Since we are focused in the case when $\operatorname{char} k = 0$, we prove that linearly reductive groups solve Hilbert's fourteenth problem 2.2 in the next section, following (HOSKINS, 2015).

2.5 Hilbert fourteenth problem

Given a finite dimensional representation of a linearly reductive group $\rho : G \to \mathrm{GL}(V)$, we can write $V = V^G \oplus W$, where W is the direct sum of non-trivial irreducible subrepresentations of V. This gives a canonical G–complement of V^G, and a corresponding projection $V \to V^G$.

Definition 2.5.1. Let G be a group acting on a k–algebra A. A linear map $R_A : A \to A^G$ is a **Reynolds Operator** if its a projection onto A^G, and for $a \in A^G$ and $b \in A$, we have

$$R_A(ab) = aR_A(b)$$

Next, we prove that we can construct a Reynolds operator for a finitely generated k–algebra as a colimit of the Reynolds operators in finite-dimensional vector spaces, which exist trivially since G is linearly reductive.

Lemma 2.5.2. Let G be a linearly reductive group acting rationally on a finitely generated k–algebra. Then there exists a Reynolds operator $R_A : A \to A^G$.

Proof. Since A is finitely generated over k, it has a countable basis over k as a vector space, which we denote by $\{a_n\}_{n\in\mathbb{N}}$. Since the action of G is rational, we can let A_n be the finite-dimensional G–invariant subset of A containing $a_1, \cdots, a_n$, whenever $n \geq 1$, such that

$$A = \bigcup_{n\geq 1} A_n,$$

and each A_n is G–invariant. Since G is linearly reductive and each A_n is a finite-dimensional G–representation, we can decompose

$$A_n = A_n^G \oplus A_n',$$

where A_n' is the direct sum of non-trivial irreducible representations of A_n. We let $R_n : A_n \to A_n^G$ be the canonical projection onto the direct factor A_n^G.

Note that, since for $m > n$ there is a commuting square

$$\begin{array}{ccc} A_n & \xrightarrow{R_n} & A_n^G \\ \downarrow & & \downarrow \\ A_m & \xrightarrow{R_m} & A_m^G \end{array}$$

where the vertical maps are inclusion, the map $R_A : A \to A^G$, given by $R_A(x) = R_n(x)$ for $x \in A_n$, is well defined and it is a projection onto A^G.

We only need to check that, for any $a \in A^G$ and $b \in A$, we have $R_A(ab) = aR_A(b)$. We can fix n large enough so that $a, b \in A_n$ and $m \geq n$ such that the multiplication $a(A_n) \subset A_m$. We denote by

$$\begin{aligned} l_a : A_n &\to A_m \\ b &\mapsto a \cdot b. \end{aligned}$$

and $A_n' = W_1 \oplus \ldots \oplus W_r$ the irreducible non-trivial representations of A_n. Since G acts by morphisms of algebras and $a \in A^G$, we have $l_a(A_n^G) \subset A_m^G$. By Schur's lemma 2.3.4, the image of each irreducible component $l_a(W_i)$ is either zero or isomorphic to W_i. Writing $A_n = A_n^G \oplus A_n'$, we have $l_a(W_i) \subset A_m'$.

Thus $l_a(A_n') \subset A_m'$, and if $b = b^G + b'$ for $b^G \in A_n^G$ and $b' \in A_n'$, then

$$ab = l_a(b) = l_a(b^G) + l_a(b'),$$

where $l_a(b^G) = ab^G \in A^G_m$ and $l_a(b') = ab' \in A'_m$. Hence

$$R_A(ab) = ab^G = aR_A(b),$$

as we wanted to prove. □

Corollary 2.5.2.1. Let A, B be k–algebras in which a linearly reductive group G acts rationally, and denote the corresponding Reynolds operators by $R_A : A \to A^G$ and $R_B : B \to B^G$. Then any G–equivariant morphism $h : A \to B$ satisfies the diagram:

$$\begin{array}{ccc} A & \xrightarrow{R_A} & A^G \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^G} \\ B & \xrightarrow{R_B} & B^G \end{array}$$

Proof. As we saw in the last proof, we write $A = A^G \oplus A'$ and $B = B^G \oplus B'$, and since h is G–equivariant, h takes A^G to B^G and A' to B', by Shur's lemma. Thus, if $a = a^G + a'$ is a decomposition of $a \in A$,

$$(R_B \circ h)(a) = R_B(h(a^G) + h(a')) = h(a^G) = (h \circ R_A)(a).$$

□

Finally, to solve Hilbert's Fourteenth problem in this context, we need to prove that the Reynolds operator $R : A \to A^G$ preserves the finite-generality of k–algebras.

Lemma 2.5.3. Let A be a k–algebra with a rational G–action and suppose that A has a Reynolds operator $R_A : A \to A^G$. Then, for any ideal $I \subset A^G$, we have $I \cdot A \cap A^G = I$. More generally, this means that whenever $\{I_j\}_{j \in J}$ is a family of ideals in A^G, we have

$$\left(\sum_{j \in J} I_j A\right) \cap A^G = \left(\sum_{j \in J} I_j\right).$$

In particular, if A is Noetherian, so is A^G.

Proof. Of course we have $I \subset IA \cap A^G$. Let $x \in I \cdot A \cap A^G$. Since $x \in I \cdot A$, we can write

$$x = \sum_{l=1}^{n} i_l x_l$$

with $i_l \in I$ and $x_l \in A$. Since $x \in A^G$ and R_A is a Reynolds operator,

$$x = R_A(x) = \sum_{l=1} i_l R_A(x_l) \in I.$$

For the statement about a family of ideals $\{I_j\}_{j\in J}$, we note:

$$\begin{aligned}\left(\sum_{j\in J} I_j \cdot A\right) \cap A^G &= \langle \bigcup_{j\in J} I_j \cdot A\rangle \cap A^G \\ &= \langle \bigcup_{j\in J} I_j \cdot A \cap A^G\rangle \\ &= \langle \bigcup_{j\in J} I_j\rangle = \sum_{j\in J} I_j.\end{aligned}$$

□

Theorem 2.5.4. *Let G be a linearly reductive group acting rationally on a finitely generated k–algebra. Then A^G is finitely generated.*

Proof. Since A is finitely generated and the action of G is rational, there is a finite dimensional k–linear subspace $V \subset A$ which contains the generators of A and it is G–invariant. Thus, there is a surjective morphism

$$k[x_1, \ldots, x_n] \simeq \mathcal{O}(V^\vee) \twoheadrightarrow A$$

which is G–equivariant, since V has a G–action induced by A. Since G is linearly reductive, both algebra admit a Reynolds operator and, moreover, they commute with this surjection. Therefore, we get an induced surjection

$$\mathcal{O}(V)^G \twoheadrightarrow A^G$$

and so, to prove that A is finitely generated, it suffices to assume that $A = k[x_1, \ldots, x_n] = \mathcal{O}(V)$ and the action of G on $V = k^n$ is linear.

In this case, since $k[x_1, \ldots, x_n]$ is graded and the action is linear, the invariant ring $k[x_1, \ldots, x_n]^G$ is also graded. Since A is Noetherian, by Hilbert's basis theorem, A^G is Noetherian by the previous lemma. Hence, the ideal

$$A^G_+ = \bigoplus_{d>0} k[x_0, \ldots, x_n]^G_d$$

is finitely generated. Since A^G is a graded, it is finitely generated by its positive part A^G_+, hence A^G is finitely generated over k. □

Via 2.4.11, if $\operatorname{char} k = 0$ we also conclude

Theorem 2.5.5 (Nagata)**.** *Let G be a geometrically reductive group acting rationally on a finitely generated k–algebra A. Then the G–invariant subalgebra A^G is finitely generated.*

3 Geometric Invariant Theory

Introduction

In this chapter, we study Mumford's Geometric Invariant Theory, a theory which provides a construction of moduli spaces using the theorems from Algebraic Invariant Theory studied in the previous chapter. We continue using the same notation as before, following (HOSKINS, 2015) notes.

3.1 Categorical Quotients

In this section, we model possible well behaved notions of quotients for G–actions in the category Sch_k, and prove some equivalences between them.

Definition 3.1.1. A **categorical quotient** for an action $\sigma : G \times X \to X$ is a G–invariant morphism $\varphi : X \to Y$ which is universal in the following sense: for each G–invariant morphism $f : X \to Z$ there is a unique morphism $h : Y \to Z$ satisfying the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "h"] \\ & Z \end{tikzcd}$$

Moreover, if $\varphi^{-1}(y)$ is a single orbit whenever $y \in Y(k)$, we say that φ is an **orbit space**. Since φ is constant on orbits, φ is constant on orbits closures, and thus the quotient is an orbit space if and only if the G–action is closed.

The following lemma indicates how categorical quotients can be glued over open coverings:

Lemma 3.1.2. Let G be an algebraic group over k acting algebraically on two k–schemes X, Y, $\varphi : X \to Y$ be a G–invariant morphism and $\{U_i\}_{i \in I}$ be an open cover of Y such that $\varphi_i \doteq \varphi\big|_{V_i} : V_i \to U_i$ is a categorical quotient, where $V_i \doteq \varphi^{-1}(U_i)$. Then φ is a categorical quotient.

Proof. Let $f : X \to Z$ be a G–invariant morphism. By the universal property of categorical quotients, whenever $i \in I$ there is a unique morphism h_i such that

$$\begin{tikzcd} V_i \arrow[r, "\varphi_i"] \arrow[dr, "f|_{V_i}"'] & U_i \arrow[d, dashed, "h_i"] \\ & Z \end{tikzcd}$$

commutes. Moreover, if $i, j \in I$, both morphisms $h_{ij} \doteq h_i|_{U_i \cap U_j}$ and $h_{ji} \doteq h_j|_{U_i \cap U_j}$ satisfy a similar diagram,

$$\begin{tikzcd} V_i \cap V_j \arrow[r, "\varphi_i"] \arrow[dr, "f|_{V_i \cap V_j}"'] & U_i \cap U_j \arrow[d] \\ & Z, \end{tikzcd}$$

which in particular means that $h_{ij} = h_{ji}$ over $U_i \cap U_j$. Since $\mathrm{Hom}_{\mathrm{Sch}_k}(-, Z)$ is a sheaf of sets over Y and sheaves satisfy the gluing lemma, there is a unique morphism $h : Y \to Z$ satisfying the desired diagram. □

Definition 3.1.3. Let G be an affine algebraic group over k acting algebraically on a k–scheme X. A morphism $\varphi : X \to Y$ will be called a **good quotient** if it satisfies the following properties:

1. φ is G–invariant;

2. φ is surjective;

3. If $U \subset Y$ is an open set, the induced morphism

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$$

is an isomorphism onto the subalgebra $\mathcal{O}_X(\varphi^{-1}(U))^G$;

4. If $W \subset X$ is a G–invariant closed subset, $\varphi(W) \subset Y$ is closed;

5. If $W_1, W_2 \subset X$ are disjoint G–invariant closed subsets, $\varphi(W_1) \cap \varphi(W_2) = \varnothing$;

6. φ is an affine map, i.e., the preimage of affine open sets is affine.

If $\varphi^{-1}(y)$ is a single orbit for each $y \in Y$, we say that φ is a **geometric quotient**.

Note that, if $\varphi : X \to Y$ is a good quotient, then for every $U \subset Y$ open set, the restriction $\varphi|_{\varphi^{-1}(U)} : \varphi^1(U) \to U$ is a good quotient. Similarly, if φ is a geometric quotient, every restriction onto an open set is as geometric quotient.

Remark 3.1.4. Assuming (2), the conditions (4) and (5) are equivalent to the following: (5′) If $W_1, W_2 \subset X$ are disjoint G–invariant closed subsets, then the closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

Proposition 3.1.5. Let G be an affine algebraic group over k acting algebraically on a k–scheme X and a morphism $\varphi : X \to Y$ satisfying the properties $(1), (3), (4), (5)$ in the definition of good quotient. Then φ is a categorical quotient.

Proof. Since (1) states that φ is a G–invariant morphism, we only need to prove that φ satisfies the universal property of categorical quotients. To do this, let $f : X \to Z$ be a G–invariant morphism.

Since Z is of finite type over k, we can consider a finite affine open cover $U_1, \dots, U_n$ of Z. We denote $W_i \doteq X \smallsetminus f^{-1}(U_i)$. As f is continuous, and each subset $W_i \subset W$ is closed, and thus G–invariant since f is G–invariant. Using (4), we see that the sets $\varphi(W_i) \subset Y$ are closed. Let $V_i \doteq Y \smallsetminus \varphi(W_i)$ denote the open complement. By construction, $\varphi^{-1}(V_i) \subset f^{-1}(U_i)$, and as the open sets $U_1, \dots, U_n$ cover Z,

$$\bigcap_{i=1}^{n} W_i = \varnothing.$$

We claim that the open sets $V_1, \dots, V_n$ cover Y. To see this, suppose by contradiction that

$$\bigcap_{i=1}^{n} \varphi(W_i) \neq \varnothing.$$

Because Y is of finite type over k, there is a closed point $p \in Y(k)$ in this intersection, and we denote by $W \subset X$ the closed G–orbit in $f^{-1}(p)$. On the other hand, since (5) holds, we must have $W \cap W_i \neq \varnothing$ for each i, since $p \in \varphi(W) \cap \varphi(W_i)$.

Since W is a single orbit and each W_i is G–invariant, we must have $W \subset W_i$ and thus

$$W \subset \bigcap_{i=1}^{n} W_i$$

which contradicts the fact that this intersection is empty.

Since f is G–invariant, the induced morphism

$$\mathcal{O}_Z(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$$

has image in $\mathcal{O}_X(f^{-1}(U_i))^G$. By (4), we also have a corresponding isomorphism

$$\mathcal{O}_Y(V_i) \xrightarrow{\sim} \mathcal{O}_X(\varphi^{-1}(V_i))^G$$

and because $\varphi^{-1}(V_i) \subset f^{-1}(U_i)$, we also have the restriction morphism of the sheaf $\mathcal{O}_X$, which respects the action of G, and thus the diagram

$$\begin{tikzcd} \mathcal{O}_Z(U_i) \arrow[r, "h_i^*"] \arrow[d, "f^*"'] & \mathcal{O}_Y(V_i) \arrow[d, "\varphi^*"] \\ \mathcal{O}_X(f^{-1}(U_i))^G \arrow[r] & \mathcal{O}_X(\varphi^{-1}(V_i))^G \end{tikzcd}$$

where $h_i^* : \mathcal{O}_Z(U_i) \to \mathcal{O}_Y(V_i)$ is the (unique) morphism of k–algebras that makes the diagram commute. Since U_i is affine, h_i^* corresponds to a morphism of schemes $h_i : V_i \to U_i$. By construction, we have

$$f\big|_{\varphi^{-1}(V_i)} = h_i \circ \varphi\big|_{\varphi^{-1}(V_i)},$$

and $h_i = h_j$ on intersections $V_i \cap V_j$ by uniqueness of h_i^*. Therefore, we can glue the morphisms h_i to obtain a unique morphism $h : Y \to Z$ such that $f = h \circ \varphi$. □

Proposition 3.1.6. Let G be an affine algebraic group acting on a scheme X and let $\varphi : X \to Y$ be a good quotient. Then:

(a) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ if and only if $\varphi(x_1) = \varphi(x_2)$

(b) For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed, then φ is a geometric quotient.

Proof. (a). As φ is constant in orbit closures, if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ we get $\varphi(x_1) = \varphi(x_2)$. For the converse, by (5), if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$, $\varphi(\overline{G \cdot x_1}) \cap \varphi(\overline{G \cdot x_2}) = \emptyset$, and in particular $\varphi(x_1) \neq \varphi(x_2)$.

(b). Suppose that W_1 and W_2 are two distinct closed orbits in $\varphi^{-1}(y)$. This contradicts (5), since $\varphi(W_1) = \varphi(W_2) = \{y\}$. □

Lemma 3.1.7. If $\varphi : X \to Y$ is G–invariant and there is an open cover $\{U_i\}_{i\in I}$ of Y such that $\varphi_i \doteq \varphi\big|_{V_i} : V_i \to U_i$ is a good quotient, then φ is a good quotient.

Proof. By hypothesis, φ already satisfies (1). Since each φ_i is surjective and $\{U_i\}_{i\in I}$ covers Y, φ also must satisfy (2).

For the condition (3), whenever $i \in I$ we have the diagram

$$\begin{tikzcd} \mathcal{O}_Y(U_i) \arrow[r, "\varphi_i^*"] \arrow[dr, dashed] & \mathcal{O}_X(\varphi_i^{-1}(U_i)) \\ & \mathcal{O}_X(\varphi_i^{-1}(U_i))^G, \arrow[u, hook] \end{tikzcd}$$

and since φ^* is the gluing φ_i^*, φ^* also satisfies this factorization over any open set U.

If $W \subset X$ is a G–invariant subset which is closed, we can consider the intersection

$$W \cap V_i \subset X$$

which will satisfy $\varphi_i(W \cap V_i) = \varphi(W \cap V_i)$ and because φ_i is a good quotient, $\varphi_i(W \cap V_i) \subset Y$ is a closed set. Moreover,

$$\varphi(W) = \varphi\left(W \cap \bigcup_{i \in I} V_i\right) = \varphi\left(\bigcup_{i \in I} W \cap V_i\right) = \bigcup_{i \in I} \varphi_i(W \cap V_i),$$

which can be rewritten into a finite union because X is quasi-compact, and thus this is closed in Y. This proves condition (4).

Condition (5) can also be proved using the same method since, given $W_1, W_2 \subset X$ closed disjoint G–invariant sets, we can write

$$\begin{aligned}\varphi(W_1) \cap \varphi(W_2) &= \varphi\left(W_1 \cap (\bigcup_{i \in I} V_i)\right) \cap \varphi\left(W_2 \cap (\bigcup_{i \in I} V_i)\right)\\ &= \varphi\left(\bigcup_{i \in I} W_1 \cap V_i\right) \cap \varphi\left(\bigcup_{i \in I} W_2 \cap V_i\right)\\ &= \bigcup_{i \in I} \varphi_i(W_1 \cap V_i) \cap \varphi_i(W_2 \cap V_i),\end{aligned}$$

which must be the empty set because

$$\varphi_i(W_1 \cap V_i) \cap \varphi_i(W_2 \cap V_i) = \varnothing$$

whenever $i \in I$. Finally, by definition affine maps are local on target and it follows again that φ must also be affine, so φ satisfies (6). □

Definition 3.1.8. Let $\mathcal{M}$ be a moduli problem, associated to a functor of families F as in 1.2.1. A family $\mathcal{F}$ over a scheme S has the **local universal property** if, for any other family $\mathcal{G}$ over a scheme T and for any k–point $t \in T(k)$, there exists a neighbourhood U of t in T and a morphism $f : U \to S$ such that

$$\mathcal{G}\big|_U \sim_U f^*\mathcal{F} \doteq \mathcal{M}(f)(\mathcal{F})$$

Proposition 3.1.9. For a moduli problem $\mathcal{M}$, let $\mathcal{F}$ be a family with the local universal property over a scheme S. Furthermore, suppose that there is an algebraic group G acting on S such that two $s, t \in S(k)$ lie in the same orbit if and only if $\mathcal{F}_s \sim \mathcal{F}_t$. Then:

1. Any coarse moduli space is a categorical quotient of the G–action on S.
2. A categorical quotient of the G–action on S is a coarse moduli space if and only if it is an orbit space.

Proof. We first claim that, for any scheme M, the existence of a family with the universal local property over S induces a bijection

$$\begin{aligned}\Phi : \mathrm{Hom}_{\mathrm{Fun}}(\mathcal{M}, h_M) &\to \mathrm{Hom}_G(S, M)\\ \eta &\mapsto \eta_S(\mathcal{F}).\end{aligned}$$

To see that the map Φ is well defined, note that the corresponding morphism $\eta_S(\mathcal{F})$ is G–invariant, since the G–action on S agrees with the equivalence relation in $\mathcal{M}(S)$. Furthermore, we can define an explicit inverse

$$\begin{aligned}\Psi : \mathrm{Hom}_G(S, M) &\to \mathrm{Hom}_{\mathrm{Fun}}(\mathcal{M}, h_M)\\ f &\mapsto \Psi(f),\end{aligned}$$

where, whenever $T \in \mathrm{Sch}_k$, we define the morphism $\Psi(f)_T : \mathcal{M}(T) \to \mathrm{Hom}(T, M)$ as follows: given any family $\mathcal{G} \in \mathcal{M}(T)$, we can cover T by open subsets $\{U_i\}_{i\in I}$ which have associated morphisms $h_i : U_i \to S$ such that $h_i^*\mathcal{F} \sim_{U_i} \mathcal{G}\big|_{U_i}$, by the local universal property. For $u \in U_i \cap U_j$, we can write

$$\mathcal{F}_{h_i(u)} \sim (h_i^*\mathcal{F})_u \sim \mathcal{G}_u \sim (h_j^*\mathcal{F})_u \sim \mathcal{F}_{h_j(u)},$$

so the points $h_i(u), h_j(u) \in S$ must lie in the same G–orbit. Since $f \in \mathrm{Hom}_G(S, M)$, the compositions

$$U_i \xrightarrow{h_i} S \xrightarrow{f} M$$

are compatible over the covering $\{U_i\}_{i\in I}$, so we can glue them to get an element in $\mathrm{Hom}(T, M)$. The gluing does not depend on the choice of open covering, so the morphism $\Psi(f)_T$ is well defined.

To show Ψ is well defined, we only need to show that $\Psi(f)$ is a natural transformation. Given any morphism of schemes $g : Z \to T$ and any element $\mathcal{G} \in \mathcal{M}(T)$, if $\{U_i\}_{i\in I}$ is an open cover of T which satisfies

$$h_i^*\mathcal{F} \sim_{U_i} \mathcal{G}\big|_{U_i}$$

whenever $i \in I$, then the family $\mathcal{M}(\mathcal{G})$ satisfies

$$\mathcal{M}(h_i \circ g) \doteq (h_i \circ g)^*\mathcal{F} = g^*(h_i^*\mathcal{F}) \sim_{U_i} \mathcal{G}\big|_{U_i},$$

and as the operation $h_i \mapsto h_i \circ g$ corresponds to the usual pullback in Sets, the diagram

$$\begin{array}{ccc}\mathcal{M}(T) & \xrightarrow{\Psi(f)_T} & \mathrm{Hom}(T, M)\\ {\scriptstyle \mathcal{M}(g)}\big\downarrow & & \big\downarrow{\scriptstyle g^*}\\ \mathcal{M}(Z) & \xrightarrow{\Psi(f)_Z} & \mathrm{Hom}(Z, M)\end{array}$$

commutes. Whenever $f \in \mathrm{Hom}_G(S, M)$, if we set $\eta \doteq \Psi(f)$, then $\Phi(\eta) = \eta_S(\mathcal{F})$, which by definition can be taken to be the composition

$$S \xrightarrow{\mathrm{Id}_S} S \xrightarrow{f} M,$$

and this implies $(\Phi \circ \Psi)(f) = f$.

On the other hand, if we start with a natural transformation $\eta : \mathcal{M} \to h_M$ and denote by $f \doteq \eta_S(\mathcal{F})$, when applying $\eta' \doteq \Psi(f)$ over an object T to an element $\mathcal{G} \in \mathcal{M}(T)$, this corresponds to the gluing of maps

$$U_i \xrightarrow{h_i} S \xrightarrow{f} M$$

over an open covering $\{U_i\}_{i \in I}$ over T which trivializes $\mathcal{F}$. We have

$$f^*(\mathcal{F}) = \eta_S(f)(\mathcal{F}) \sim_{U_i} \mathcal{G}\big|_{U_i},$$

whenever $i \in I$, so if we compute η' over S, we get

$$\eta'_S(\mathcal{F}) = \eta_S(\mathcal{F}) = f,$$

so $(\Psi \circ \Phi)(f) = f$.

Using the bijection Φ, if the pair $(M, \eta : \mathcal{M} \to h_M)$ is a coarse moduli space, then the family $\eta_S(\mathcal{F}) : S \to M$ is G–invariant and it is universal amongst all G–invariant morphisms from S. This means in particular that the map $\eta_S(\mathcal{F})$ is a categorical quotient for the G–action on S.

Furthermore, a categorical quotient $\eta_S(\mathcal{F}) \in \mathrm{Hom}_G(S, M)$ is an orbit space if and only if the G–action is closed, which in turn happens if and only if the associated morphism

$$\eta_{\mathrm{Spec}\, k} : \mathcal{M}(\mathrm{Spec}\, k) \to M(k)$$

is bijective, so 2 follows. $\square$

3.2 Affine Geometric Invariant Theory

Definition 3.2.1. Let G be an affine reductive group over k acting algebraically on an affine k–scheme X. We saw that this induces an action of G on $\mathcal{O}(X)$, which is finitely generated. By Nagata's theorem, the subalgebra $\mathcal{O}(X)^G$ is finitely generated. We define the **affine GIT quotient** as the morphism

$$\varphi : X \to X//G$$

associated to the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$, where $X//G \doteq \mathrm{Spec}\, \mathcal{O}(X)^G$. In the case when the GIT quotient is also a geometric quotient, we write X/G, as it is an orbit space.

For the rest of this section, we build up to prove that the GIT quotient is a good quotient, in the sense of 3.1.6.

Lemma 3.2.2. Let G be a geometrically reductive group acting on an affine k–scheme X. If W_1 and W_2 are disjoint G–invariant closed subsets of X, then there is an invariant function $f \in \mathcal{O}(X)$ which separates W_1 and W_2, i.e.,

$$f\big|_{W_1} = 0 \text{ and } f\big|_{W_2} = 1.$$

Proof. As W_i are disjoint closed sets,

$$I(\varnothing) = I(W_1 \cap W_2) = I(W_1) + I(W_2),$$

so the corresponding ideals are coprime. Thus, we can write $1 = f_1 + f_2$, where $f_i \in I(W_i)$, and this means $f_1\big|_{W_1} = 0$ and $f_1\big|_{W_2} = 1$. Since the G–action is rational on $\mathcal{O}(X)$ (by 2.2.2), the function f_1 is contained in a finite dimensional G–invariant linear subspace $V \subset \mathcal{O}(X)$. We can choose V to have minimal dimension with respect to this property. Let $\{h_1, \ldots, h_n\} \subset V$ be a basis, and we define a morphism $h : X \to \mathbb{A}^n$ by $h(x) \doteq (h_1(x), \ldots, h_n(x))$.

By the minimality of $\dim V$, each function h_i is a linear combination of translates of f_1, so we have:

$$h_i = \sum_{l=1}^{n_i} c_{il} g_{il} \cdot f_1$$

where $g_{il} \in G$ and $c_{il} \in k$. Then, for $x \in X$,

$$h_i(x) = \sum_{l=1}^{n} c_{il} f_1(g_{il} \cdot x)$$

and as the closed subsets W_i are G–invariant, it follows that

$$h\big|_{W_1} = 0 \text{ and } h\big|_{W_2} \equiv v \in \mathbb{A}^n \setminus \{0\},$$

where

$$v = \sum_{l=1}^{n} \left(\sum_{i=1}^{n} c_{il} \right) e_l.$$

We can write the functions $g \cdot h_i \in V$ as $g \cdot h_i = \sum_{j=1}^{n} a_{ij}(g) h_j$, defining a representation $G \to \mathrm{GL}_n$ given by $g \mapsto (a_{ij}(g))_{i,j}$ such that h is a G–equivariant morphism, considering the G-action on X and the G–action on $\mathbb{A}^n$ via the representation. Therefore $\{v\} = h(W_2)$ is a non-zero G–invariant vector.

Since G is geometrically reductive, there exists a non-constant homogeneous polynomial $P \in k[x_1, \ldots, x_n]^G$ such that $P(v) \neq 0$ and $P(0) = 0$. Thus

$$f \doteq \frac{1}{P(v)} \cdot (P \circ h) : X \to k$$

is the desired invariant function, since $f\big|_{W_1} = 0$ e $f\big|_{W_2} = 1$. ∎

Lemma 3.2.3. Let G be an affine group acting on an affine scheme X. If $f \in \mathcal{O}(X)^G$, then we can induce the action of G into the open set $X_f = X \setminus V(f)$ such that

$$\mathcal{O}(X_f)^G = (\mathcal{O}(X)^G)_f$$

Proof. Since f is G–invariant, both the zero-set $V(f) = \{x \in X : f(x) = 0\}$ and X_f are G–invariant sets. Moreover, since $\mathcal{O}(X_f) \simeq \mathcal{O}(X)_f$, we can write:

$$\begin{aligned} \mathcal{O}(X_f)^G &\simeq \left\{ \frac{h}{f^i} : \frac{h}{f^i}(g \cdot x) = \frac{h}{f^i}(x) \ \forall g \in G, h \in \mathcal{O}(X), i \geq 0 \right\} \\ &= \left\{ \frac{h}{f^i} : h(g \cdot x) = h(x) \ \forall g \in G, h \in \mathcal{O}(X), i \geq 0 \right\} \\ &= (\mathcal{O}(X)^G)_f. \end{aligned}$$

□

Lemma 3.2.4. If $f : X \to Y$ is a morphism of k–schemes which is surjective on closed points, then f is surjective.

Proof. To prove this, we need to use again Chevalley's theorem ((HARTSHORNE, 1977), Exercise II 3.19), to conclude that the image of f is a constructible set, and so it is locally closed.

If we suppose that f is not surjective, then the complement of the image $E \subset Y$ is also a constructible set with no closed points, i.e., $E \cap Y(k) = \varnothing$. We show this leads to a contradiction.

Since Y is noetherian and E is locally closed, we can write

$$E = \bigcup_{i=1}^{n} F_i \cap U_i,$$

where $F_i \subset Y$ are closed and $U_i \subset Y$ are open whenever $i = 1, \ldots, n$. Since E is non-empty, we can choose an index i such that $F_i \cap U_i \neq \varnothing$. Moreover, since Y is noetherian, we can find an affine cover of the open set U_i, and choose an affine open set $V \subset Y$ such that

$$F_i \cap V \subset F_i \cap U_i \subset E$$

and $F_i \cap V \neq \varnothing$. However, since V is an affine scheme and $F_i \cap V \subset V$ is a closed subset, by the Hilbert Nullstellensatz theorem $F_i \cap V \neq \varnothing$ implies that $F_i \cap V$ contains a closed point of V, which must be a k–point and thus a closed point of Y, contradicting $E \cap Y(k) = \varnothing$. □

Theorem 3.2.5. *Let G be a reductive group acting on an affine scheme X. Then the affine G.I.T. quotient $\varphi : X \to X//G$ is a good quotient, and it is an affine scheme.*

Proof. We will present here the proof for $\operatorname{char} k = 0$, and by (2.4.11) we can assume that G is linearly reductive.

By definition, $X//G$ is affine and φ is induced by $\mathcal{O}(X)^G \to \mathcal{O}(X)$, so φ is G–invariant and affine. To prove surjectivity, take $y \in Y(k)$, let m_y be the maximal ideal in $\mathcal{O}(Y) = \mathcal{O}(X)^G$ corresponding to y and choose generators $f_1, \dots, f_m$ of m_y. Since G is linearly reductive, by (2.5.3) we have

$$\left(\sum_{i=1}^{m} f_i \mathcal{O}(X)\right) \cap \mathcal{O}(X)^G = \sum_{i=1}^{m} f_i \mathcal{O}(X)^G \neq \mathcal{O}(X)^G.$$

since $f_1, \dots, f_m$ are generators of a maximal ideal. Since

$$\sum_{i=1}^{m} f_i \cdot \mathcal{O}(X) \neq \mathcal{O}(X),$$

there is a maximal ideal $m_x \subset \mathcal{O}(X)$ containing this ideal, so m_x corresponds to a closed point $x \in X(k)$. In particular, $f_i(x) = 0$ for each $i = 1, \dots m$, and so $\varphi(x) = y$. Therefore, every closed point of Y is in the image of φ and, by the previous lemma, φ is surjective.

The set $\{U = Y_f : f \in \mathcal{O}(X)^G\}$ forms a basis of Y, so that to prove the third property of good quotients it suffices to consider open sets of the form Y_f. We know that $\mathcal{O}_Y(Y_f) = (\mathcal{O}(X)^G)_f$ is the localization, and by 3.2.3,

$$\mathcal{O}_X(\varphi^{-1}(Y_f))^G = \mathcal{O}(X_f)^G = (\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f = \mathcal{O}_Y(Y_f),$$

which proves the third property.

By (3.1.4), as we already know that φ is surjective, to prove that φ is a good quotient we only need to prove the equivalent condition $(5)'$. By the previous lemma, if $W_1, W_2 \subset X$ are disjoint G–invariant closed sets, there exists a regular invariant function $f \in \mathcal{O}(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$. Since $\mathcal{O}(X)^G = \mathcal{O}(Y)$, we can view f as a regular function on Y with $f(\varphi(W_1)) = 0$ and $f(\varphi(W_2)) = 1$. Then it follows that the closures

$$\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \varnothing,$$

which concludes the proof. □

The last theorem enable us to translate the proposition (3.1.6) in the context of reductive groups:

Remark 3.2.6. Let G be a reductive group over k acting algebraically on an affine k–scheme X, with GIT quotient $\varphi : G \to X//G$. Then

$$\varphi(x) = \varphi(x') \iff \overline{G \cdot x} \cap \overline{G \cdot x'} \neq \varnothing.$$

Furthermore, the preimage of each point $y \in Y$ contains a closed orbit. In particular, if the action of G is closed, φ is a geometric quotient.

As we saw before, the action of a reductive group G on a scheme X does not always generate a geometric quotient, since the action is not necessarily closed. Sometimes, however, if we remove the "bad points", as seen in the example 2.2, we get a closed action. We generalize this notion, following (MUMFORD; FOGARTY; KIRWAN, 1994).

Definition 3.2.7. We say that $x \in X$ is **stable** if its orbit is closed in X and $\dim G_x = 0$ (or equivalently, $\dim G \cdot x = \dim G$). We denote by $X^s \subset X$ the set of stable points.

These are the best behaved points of X with respect to our action since, besides the orbits being closed, the points also have a *non-degenerate* orbit, in the sense that $\dim G \cdot x = \dim G$.

Theorem 3.2.8. *Let G be a reductive group acting on an affine scheme X and let $\varphi : X \to X//G$ be the associated affine GIT quotient. Then $X^s \subset X$ is an open G–invariant subset, $Y^s \doteq \varphi(X^s)$ is an open subset of Y such that $X^s = \varphi^{-1}(Y^s)$. Moreover, the restriction $\varphi : X^s \to Y^s$ is a geometric quotient.*

Proof. Let us prove that, whenever $x \in X^s$, there is a neighbourhood of x inside X^s. As we saw in (2.2.12), the dimension of the stabilizer is a upper semi-continuous function, hence the set:

$$X_+ \doteq \{x \in X : \dim G_x \geq 1\}$$

is a closed set. It is also a G–invariant set, because

$$x \in X_+ \iff \dim G \cdot x < \dim G,$$

by (2.2.11). Since $x \in X^s$, the closed invariant sets X_+ and $G \cdot x$ are disjoint in X, and by (3.2.2) there is $f \in \mathcal{O}(X)^G$ such that

$$f(X_+) = 0 \text{ and } f(G \cdot x) = 1.$$

Then $x \in X_f$ and we only need to prove that $X_f \subset X^s$.

Since $f(X_+) = 0$, for all $x \in X_f$ we must have $\dim G_x = 0$. Suppose there is a point $z \in X_f$ with a non-closed orbit. Then there is a point $w \in \overline{G \cdot z} \setminus G \cdot z$, and as f is G–invariant, its constant on orbits, and thus its constant on the closure of orbits, so $w \in X_f$. On the other hand, since the dimension of orbits of boundary points is strictly smaller of the dimension of the orbit, and hence $\dim G \cdot w < \dim G \cdot z = \dim G$. Using (2.2.11) we get $\dim G_w > 0$, a contradiction with $w \in X_f$. This means that the set X^s is open.

Because $\varphi(X_f) = Y_f$ is open in Y and $X_f = \varphi^{-1}(Y_f)$, it follows that $Y^s \subset Y$ is open and $X^s = \varphi^{-1}(Y^s)$. Thus φ is a good quotient, and because the action of G on X^s is closed, φ is also a geometric quotient. □

Note that the requirement of $\dim G_x = 0$ in the definition is used to ensure that the set X^s is an open set, so we can restrict φ to a good quotient.

For the rest of the section, we shall study some examples of affine geometric quotients.

Example 3.1. The permutation groups $G = S_n$ have a natural algebraic group structure over k induced by the inclusion $S_n \subset \mathrm{GL}_n(k)$ as permutation matrices. Moreover, this induces an action of G into $X = \mathbb{A}^n_k$ which is trivially closed and moreover every point is stable. The induced action on polynomials $\mathcal{O}(X) = k[x_1, \dots, x_n]$ is given by

$$\sigma \cdot f(x_1, \dots, x_n) \doteq f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \ \forall \sigma \in S_n \ \forall f \in k[x_0, \dots, x_n],$$

so the ring of invariants $\mathcal{O}(X)^G$ is the ring of symmetric polynomials. We will prove that

$$\mathcal{O}(X)^G = k[x_1, \dots, x_n]^G = k[s_1, \dots, s_n],$$

where $s_i = s_i(x_1, \dots, x_n)$ are the **elementary symmetric polynomials**, given explicitly by the identity

$$\prod_{i=1}^n (X - x_i) = \sum_{i=1}^n (-1)^i s_i(x_1, \dots, x_n) X^{n-i},$$

which gives

$$s_i = \sum_{1 \le j_1 \le \ldots \le n} x_{j_1} \cdot (\cdots) \cdot x_{j_i}.$$

Note that each $s_i \in \mathcal{O}(X)^G$. To show $\mathcal{O}(X)^G = k[s_1, \dots, s_n]$, we use the **lexicographic order** (following (SMITH, 1995), although this is known since Isaac Newton) on monomials

$$x_1^{a_1} \dots x_n^{a_n} \le x_1^{b_1} \dots x_n^{b_n},$$

whenever the first nonzero difference $b_i - a_i$ is positive.

We proceed by induction over the lexicographic order. Let $f \in k[x_1, \dots, x_n]$ be a symmetric polynomial. Since the action of $G = S_n$ preserves degree, it carries homogeneous polynomials to homogeneous polynomials, so we can assume f is homogeneous. Let

$$m = x_1^{a_1} \cdot (\cdots) \cdot x_n^{a_n}$$

be the largest monomial appearing with nonzero coefficient in f. If $a_{i+1} > a_i$, we can use the fact that f is symmetric to choose the transposition $\sigma = (i \ i+1)$ which interchanges i with $i+1$ to act on f, and since f is symmetric, this leads to the monomial

$$x_1^{a_1} \cdot (\cdots) \cdot x_i^{a_{i+1}} \cdot x_{i+1}^{a_i} \cdot (\cdots) \cdot x_n^{a_n}$$

also appearing in f, with the same coefficient as m, but since this is a larger using the lexicographic order, this leads to a contradiction. So we proved that $a_{i+1} \le a_i$ for all i.

On the other hand, the product $s_1^{a_1-a_2} \cdot s_2^{a_2-a_3} \cdot (\cdots) \cdot s_n^{a_n}$ also contains the monomial m as highest monomial, by the choice of indexes. So

$$f - as_1^{a_1-a_2} \cdot s_2^{a_2-a_3} \cdot (\cdots) \cdot s_n^{a_n}$$

is again a symmetric polynomial, which is lower than f in the lexicographical order. Rearranging the resulting equality expresses f as a polynomial in the elementary symmetric polynomials, as we wanted.

Theorem 3.2.9. *The terms $s_1, \dots, s_n$ are algebraically independent over k, i.e., $k[s_1, \dots, s_n] \simeq k[x_1, \dots, x_n]$ and thus*

$$\mathbb{A}_k^n / S_n \simeq \mathbb{A}^n.$$

Proof. Suppose that $f \in k[x_1, \dots, x_n]$ is a polynomial such that

$$f(s_1, \dots, s_n) = 0.$$

We can write f as a sum of monomials, so that each term is of the form

$$x_1^{a_1-a_2} \cdot x_2^{a_2-a_3} \cdot (\cdots) \cdot x_n^{a_n}$$

for integers $a_n \leq \cdots \leq a_1$. If $f \neq 0$, let $ax_1^{a_1-a_2} \cdot x_2^{a_2-a_3} \cdot (\cdots) \cdot x_n^{a_n}$ be the largest monomial in the lexicographical order among all possibilities for $(a_1, \dots, a_n)$ in f such that $a \neq 0$. Then $f(s_1, \dots, s_n)$ as a polynomial in $x_1, \dots, x_n$ would have

$$ax_1^{a_1} \cdot (\cdots) \cdot x_n^{a_n}$$

as its largest monomial lexicographic order, and therefore $a = 0$, which is a contradiction. Thus f must be the zero polynomial, and $s_1, \dots, s_n$ are algebraically independent. □

Example 3.2. Consider $G = \mathrm{GL}_n(k)$ acting on the space of $n \times n$ matrices $M_n(k)$ by conjugation. This conjugation can be viewed as a change of basis on the vector space k^n, so that all the invariants that are preserved by change of basis (or, in other words, that are about the operator itself, not about its matrix representation) are invariant over this action.

If $A \in M_n(k)$, since the characteristic polynomial c_A does not depend on the basis, it will be well defined for each conjugacy class of the action. Thus, each term of the polynomial c_A gives an invariant function of A.

Let $n = 2$ and $\mathcal{O}(M_2(k)) = k[x_1, x_2, x_3, x_4]$ be the coordinate ring of the space of matrices. In this case, we have the following description for the characteristic polynomial

$$c_A(x) = x^2 - \mathrm{tr}(A)x + \det(A)$$

and we can use the theory of Jordan Normal Forms to describe the orbits using the roots of c_A, as follows:

(a) Matrices with characteristic polynomial having distinct roots $\alpha, \beta \in k$ can be represented as diagonal matrices with Jordan form given by

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

We shall prove that the orbits of this type are closed, for every $\alpha, \beta \in k$. Every matrix in the orbit $B \in \mathrm{GL}_2(k) \cdot A$ satisfies $f(B) = 0$ and $g(B) = 0$, where $f, g \in k[x_1, x_2, x_3, x_4]$ are given by

$$f(X) = \det(X) - \det(A) \text{ and } g(X) = \mathrm{tr}(X) - \mathrm{tr}(A),$$

since f and g are $\mathrm{GL}_2(k)$–invariant. Because these conditions are also sufficient to prescribe the characteristic polynomial of a matrix $X \in V(f, g)$, and we have the equality

$$\mathrm{GL}_2(k) \cdot A = V(f, g) \subset M_2(k),$$

thus the orbit is closed. Note that

$$\frac{k[x_1, x_2, x_3, x_4]}{(f, g)} \simeq \frac{k[x_1, x_2, x_3, x_4]}{(\det(X), \mathrm{tr}(X))},$$

and since $\mathrm{tr}(X) = x_1 + x_4$ and $\det(X) = x_1 x_4 - x_2 x_3$,

$$\frac{k[x_1, x_2, x_3, x_4]}{(\det(X), \mathrm{tr}(X))} \simeq \frac{k[x_1, x_2, x_3, x_4]}{(x_1 + x_4, x_1 x_4 - x_2 x_3)} \simeq \frac{k[x_1, x_2, x_3]}{(x_1^2 + x_2 x_3)},$$

and thus the orbits are all isomorphic to $V(x_1^2 + x_2 x_3)$, an algebraic surface on k^3 which is singular at the origin, but smooth everywhere else. Furthermore, by 2.2.11, $\dim G_X = 2$ whenever $X \in M_2(k)$ is a matrix of this type.

(b) Matrices with characteristic polynomial having only one (double) root $\alpha \in k$, and minimal polynomial given by $(x - \alpha)^2$. These matrices are not diagonalizable, since their Jordan form is given by

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

In this case, the orbit is also 2–dimensional, with corresponding polynomials being the trace and determinant conditions as seen in (a). But, because the matrix

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

has the same trace and determinant as the matrices in the orbit $\mathrm{GL}_2(k) \cdot A$, even though $B \notin \mathrm{GL}_2(k) \cdot A$. This means that these orbits are not closed, namely because of the limit point

$$\lim_{t \to 0} A(t) = \lim_{t \to 0} \begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix} = B$$

(c) Matrices with repeated root α for which the minimal polynomial is $(x-\alpha)$. These matrices have Jordan form given by

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

Since they are multiples of the identity, they commute with all matrices, and hence the orbit $\mathrm{GL}_2(k) \cdot A = \{A\}$ is closed and zero dimensional, so that $\dim G_A = 4 = \dim G = 4$.

We note that every orbit of the type (b) contains a type (c) orbit in its closure, and so will be identified in the quotient. From our discussion, the functions det and tr are G–invariant, and so:

$$k[\mathrm{tr}, \det] \subset \mathcal{O}(M_2(k))^{\mathrm{GL}_2}.$$

These are indeed the only generators for the k–algebra of G–invariant functions, and to see this from our discussion about the closed orbits and Jordan forms, every G–invariant function on $M_2(k)$ is completely determined by its values on the diagonal matrices $D_2(k) \subset M_2(k)$. Hence, the ring of GL_2 –invariants on $M_2(k)$ is contained in the ring $\mathcal{O}(D_2(k)) \simeq k[x_1, x_4]$. In fact, using the $\mathrm{GL}_2(k)$–action we can permute the diagonal entries, so that

$$\mathcal{O}(M_2)^{\mathrm{GL}_2} \subset k[x_1, x_4]^{S_2} = k[x_1 + x_4, x_1 x_4],$$

as the symmetric polynomials can be generated by the elementary symmetric polynomials. These polynomials correspond to the trace of determinant of these diagonal matrices, and thus there are no additional invariants, so that $k[\mathrm{tr}, \det] = \mathcal{O}(M_2)^{\mathrm{GL}_2}$ and the affine GIT quotient is given by:

$$\varphi = (\mathrm{tr}, \det) : M_2(k) \to \mathbb{A}^2.$$

Since the 2–dimensional subgroup of G given by multiples of identity fixes every point, there are no stable points for this action.

For the general case, we have a similar theorem

Theorem 3.2.10 ((CONCINI; PROCESI, 2017), Theorem 1.3-1.4)**.** *The ring of invariants*

$$\mathcal{O}(M_n(k))^{\mathrm{GL}_n(k)} \simeq k[x_1, \ldots, x_m]^{S_m}$$

where $m = n^2$, and moreover the restriction of the action to the diagonal matrices gives the isomorphism.

Example 3.3. Let $X = \operatorname{Spec}\mathbb{R}[x]$. The ring $\mathbb{R}[x]$ is a P.I.D., since it is an Euclidean Domain, and thus every maximal ideal is uniquely determined by a irreducible polynomial as generator. Thus, we have two kinds of points:

$$\operatorname{Spec}\mathbb{R}[x] = \{(x-\alpha) : \alpha \in \mathbb{R}\} \cup \{(x^2+bx+c) : b, c \in \mathbb{R}, b^2-4c < 0\},$$

where the first kind corresponds to the usual points on the real line $\mathbb{R}$. The second kind, however, is a more interesting point: it corresponds to a pair of complex conjugated numbers, glued together, so this should correspond to a orbit of the action of the (absolute) Galois group of the real numbers, which in this case is the group $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) = \{z \mapsto z, z \mapsto \overline{z}\} \simeq \mathbb{Z}_2$.

Indeed, whenever the algebraic closure of a field is a finite extension (in this case $[\mathbb{C} = \overline{\mathbb{R}} : \mathbb{R}] = 2$), the associated absolute Galois group is a finite group, which has trivially a closed action and zero-dimensional stabilizer. In a sense, this gives a trivial example of good and geometric GIT quotient, which coincides with the mapping

$$\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R} = \operatorname{Spec}\mathbb{C}^{\operatorname{Gal}(\mathbb{C}/\mathbb{R})}$$

In this case, this looks like a covering map from $\mathbb{C}$ to $\mathbb{R}$, where we fold $\mathbb{C}$ in half over the real line, and glue together the opposite (conjugate) points, which of course leaves the line $\mathbb{R}$ fixed.

More generally:

Theorem 3.2.11. *Let K/F be a finite Galois extension. Then the corresponding morphism*

$$\operatorname{Spec} K \to \operatorname{Spec} F \simeq \operatorname{Spec} K^G,$$

where $G = \operatorname{Gal}(K/F)$ is the corresponding good quotient of the Galois group over K.

The proof given in the particular case works in this case similarly, and these kinds of maps always look like a covering map away from a ramified locus. The analogue in the category of schemes are called **étale morphisms**, and the previous theorem gives a family of examples of them. Moreover, if we are working over $k = \mathbb{C}$ with algebraic varieties, the conditions imply the implicit function theorem, so f is a local isomorphism. For more on étale morphisms, see for example (LIU; ERNE, 2006).

Remark 3.2.12. An important local GIT result, Luna's slice theorem, uses étale morphisms to describe the germ of a GIT quotient. We refer the reader to (DRéZET, 2004), and we only use this on chapter 4 to conclude the smoothness of the moduli space of vector bundles over a smooth curve.

3.3 Projective Geometric Invariant Theory

Sometimes, the space of parameters P for a moduli problem is not affine. For example, the fine moduli space for the grassmanian functor $\mathcal{G}(d,n)$ (see 1.2.2) is a naturally a projective variety, via the Plücker embedding.

In this section, we extend the construction of GIT quotients for projective schemes. First, we fix an embedding $X \subset \mathbb{P}^n$ onto the projective space, with a given G–action on X which extends to a G–action on $\mathbb{P}^n$ using a fixed linear representation $G \to \mathrm{GL}_{n+1}$.

Afterwards, we generalize this, using the fact that every projective scheme over k has a very ample line bundle associated to a specific embedding in the projective space. Using this fixed ample line bundle, we can study the proccess of **linearization** of the G–action to reduce to the previous case.

3.3.1 Linear actions

Definition 3.3.1. Let X be a projective k–scheme with an algebraic action of a reductive group G over k. A **linear G-equivariant projective embedding** of X is a choice of linear representation of $G \to \mathrm{GL}_{n+1}(k)$ together with a G–equivariant projective embedding $X \subset \mathbb{P}^n$ with respect to the induced G–action on $\mathbb{P}^n$.

We shall refer to a linear G–equivariant projective embedding simply as a **linear action** of the algebraic group G on the projective k–scheme $X \subset \mathbb{P}^n$.

Let G and X as above and $I(X)$ be the homogeneous ideal of $k[x_0, \dots, x_n]$ associated with the embedding of $X \subset \mathbb{P}^n$, such that $X = \operatorname{Proj} R(X)$ where $R(X) = k[x_0, \dots, x_n]/I(X)$.

The action of G as a subgroup of GL_{n+1} induces an action on $\mathbb{A}^{n+1}$, and since the embedding $X \subset \mathbb{P}^n$ is G–equivariant, there is also an induced action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over X.

The k–algebras $k[x_0, \dots, x_n]$ and $R(X)$ are graded by homogeneous degree, and as the G–action on $\mathbb{A}^{n+1}$ is linear, it preserves these graded pieces, giving the G–invariant k–algebras also an induced graded structure

$$A(X)^G = \bigoplus_{r \geq 0} k[x_0, \dots, x_n]_r^G \text{ and } R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G.$$

By Nagata's theorem (2.5.5), the k–algebra $R(X)^G$ is also finitely generated. The inclusion $R(X)^G \to R(X)$ induces a rational morphism

$$X \simeq \operatorname{Proj} R(X) \dashrightarrow \operatorname{Proj} R(X)^G,$$

which is well defined as a regular function away from the closed subset $V(R(X)^G_+) \subset X$, where $R(X)^G_+ = \bigoplus_{r>0} R(X)^G_r$ is the irrelevant ideal of $R(X)^G$.

Definition 3.3.2. For a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$, we define the **nullcone** N as the closed subscheme of X defined by the homogeneous ideal $R(X)^G_+$ in $R(X)$. We define the **semistable set** $X^{ss} \doteq X \setminus N$, i.e., a point $x \in X$ is **semistable** whenever there is a $r > 0$ and $f \in R(X)^G_r$ such that $f(x) \neq 0$. Note that X^{ss} is open by construction, and we call the restriction

$$X^{ss} \to X//G \doteq \mathrm{Proj}(R(X)^G)$$

the **GIT quotient** of this action.

Theorem 3.3.3. *For a linear action of a reductive group G on a projective k–scheme $X \subset \mathbb{P}^n$, the GIT quotient $\varphi : X^{ss} \to X//G$ is good quotient of the G–action on the subset X^{ss}. Furthermore, $X//G$ is a projective scheme.*

Proof. First, let us prove that $X//G = \mathrm{Proj}\, R(X)^G$ is projective over k. If $R(X)^G$ is finitely generated by $R(X)^G_1$ as a k–algebra, then this follows from basic scheme theory (see, for example, (HARTSHORNE, 1977) II 5.16 (b)).

If not, since $R(X)^G$ is finitely generated as a k–algebra, we can consider the corresponding sheaf $\mathcal{F}$ over X and applying the Serre Twisting sheaf theorem ((HARTSHORNE, 1977), II, 5.17), there is a number $d \geq 0$ such that the twist $\mathcal{F}(d)$ is generated by global sections, and translating this back into algebra means exactly that the d-twisted graded module

$$(R(X)^G)^{(d)} = \bigoplus_{l \geq 0} R(X)^G_{d+l}$$

is generated by the submodule $((R(X)^G)^{(d)}_1$. Since $X//G = \mathrm{Proj}\, R(X)^G \simeq \mathrm{Proj}(R(X)^G)^{(d)}$ (see (HARTSHORNE, 1977) II Exercise 5.13), it follows that $X//G$ is a projective k–scheme.

Let $\varphi : X^{ss} \to Y \doteq X//G$ denote the projective GIT quotient. Since the set

$$\{Y_f : f \in \mathcal{O}(X)^G\}$$

is a basis for Y, for $f \subset R(X)^G_+ \subset R(X)_+$, we can consider the open affine subset $X_f \subset X$ and, by construction of φ, we have $\varphi^{-1}(Y_f) = X_f$. Let $\tilde{X}_f$ and $\tilde{Y}_f$ denote the affine cones over X_f and Y_f, respectively. Then

$$\mathcal{O}(Y_f) \simeq \mathcal{O}(\tilde{Y}_f)_0 \simeq ((R(X)^G)_f)_0 \simeq ((R(X)_f)_0)^G \simeq (\mathcal{O}(\tilde{X}_f)_0)^G \simeq \mathcal{O}(X_f)^G,$$

where we use again that the G–action respects the grading. Thus, the corresponding morphism of affine schemes given by:

$$\varphi|_{X_f} \doteq \varphi_f : X_f \to Y_f \simeq \mathrm{Spec}\, \mathcal{O}(X_f)^G$$

and this is an affine GIT quotient, which is also a good quotient by (3.2.5). The morphism $\varphi : X^{ss} \to Y$ is thus obtained by gluing the good quotients φ_f, and by 3.1.7 φ is also a good quotient. □

Definition 3.3.4. Consider a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$. Then we say that a point $x \in X$ is **stable** if $\dim G_x = 0$ and there is a G–invariant homogeneous polynomial $f \in R(X)_+^G$ such that $x \in X_f$ and the action of G on X_f is closed. Conversely, we say that x is **unstable** if it is not semistable.

We denote $X^s \subset X$ the set of stable points, and $X^{us} = X \setminus X^{ss}$ the set of unstable points.

Lemma 3.3.5. The sets X^s and X^{ss} are open subsets of X.

Proof. By construction, X^{ss} is an open set, and whenever $x \in X^s$ there is a polynomial $f \in R(X)_+^G$ such that the action of G on X_f is closed. If we denote this set by $\{f_i : i \in I\}$, where I can be taken as finite, since X is quasi-compact, and we can define the open set

$$X_c \doteq \bigcup_{i \in I} X_{f_i},$$

and by definition $X^s \subset X_c$, so that we only need to prove that X^s is open in X_c.

Since the function $x \mapsto \dim G_x$ is an upper semi-continuous function on X, the set of points with zero dimension is an open set, hence X^s is open in X_c. □

Theorem 3.3.6. *For a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$, let $\varphi : X^{ss} \to X//G$ denote the projective GIT quotient. Then there exists an open subscheme $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and such that φ restricts to a geometric quotient $\varphi : X^s \to Y^s$.*

Proof. Using the same notation as before, we can define

$$Y_c \doteq \bigcup_{i \in I} Y_{f_i},$$

so $X_c = \varphi^{-1}(Y_c)$ and the restriction $\varphi_c : X_c \to Y_c$ is constructed by gluing $\varphi_f : X_f \to Y_f$ such that the G–action is closed on X_f.

Since each φ_f is a good quotient and the action on X_f is closed, φ_f is also a geometric quotient, and the gluing φ_c is also a geometric quotient.

By definition, $X^s \subset X_c$ consists of points with zero dimensional stabilizers, and we set $Y^s \doteq \varphi(X^s) \subset Y_c$. As φ_c is a geometric quotient and X^s is a G–invariant subset of X, $\varphi^{-1}(Y^s) = X^s$, so $Y_c \setminus Y^s = \varphi(X_c \setminus X^s)$. Since $X_c \setminus X^s$ is closed, by the property (4) of good quotients, the set $\varphi(X_c \setminus X^s) = Y_c \setminus Y^s$ is closed in Y, so Y^s is open in Y and the geometric quotient φ_c restricts to a geometric quotient $\varphi : X^s \to Y^s$. □

Example 3.4. Consider the linear action of the multiplicative group $G = \mathbb{G}_m$ on the k–scheme $X = \mathbb{P}^n$ determined by

$$t \cdot [x_0 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n],$$

whenever $t \in \mathbb{G}_m$ and $[x_0 : \ldots : x_n] \in \mathbb{P}^n$ using homogeneous coordinates. In this case, $R(X) = k[x_0, \ldots, x_n]$ and it is easy to see that the functions $x_0x_1, \ldots, x_0x_n$ are G–invariant. Moreover, we claim these functions generate the ring of invariants. Indeed, given any $f \in R(X)$ we can write

$$f = \sum_{\underline{m}=(m_0,\ldots,m_n)} a(\underline{m}) x_0^{m_0} \cdots x_n^{m_n}$$

and, for $t \in \mathbb{G}_m$,

$$t \cdot f = \sum_{\underline{m}=(m_0,\ldots,m_n)} a(\underline{m}) t^{m_0 - \sum_{i=1}^n m_i} x_0^{m_0} \cdots x_n^{m_n}.$$

Hence, f is G–invariant if and only if the coefficients $a(\underline{m}) = 0$ whenever

$$m_0 - \sum_{i=1}^{n} \neq 0.$$

If $\underline{m}$ satisfies $m_0 = \sum_{i=1}^{n}$, we can rewrite

$$x_0^{m_0} \cdot x_1^{m_1} \cdots x_n^{m_n} = (x_0x_1)^{m_1} \cdots (x_0x_n)^{m_n},$$

thus $f \in k[x_0x_1, \ldots, x_0x_n]$ and

$$R(X)^G = k[x_0x_1, \ldots, x_0x_n] \simeq k[y_0, \ldots, y_{n-1}].$$

Taking the projective spectrum, we obtain the projective GIT quotient $X//G = \mathbb{P}^{n-1}$. This choice of generators for $R(X)^G$ allows us to write the rational morphism explicitly

$$\begin{aligned} \varphi : \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n-1} \\ [x_0 : \cdots : x_n] &\mapsto [x_0x_1 : \cdots : x_0x_n] \end{aligned}$$

and its clear from this description that the nullcone is

$$N = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 = 0 \text{ or } (x_1, \cdots, x_n) = 0\} \subset V(x_0x_1, \ldots, x_0x_n) \subset \mathbb{P}^n.$$

Moreover,

$$X^{ss} = \bigcup_{i=1}^{n} X_{x_0x_i} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ or } (x_1, \cdots, x_n) \neq 0\} \simeq \mathbb{A}^{n-1} \smallsetminus \{0\},$$

identifying $V(x_0)$ with the corresponding affine chart. Therefore the corresponding map

$$\varphi : \mathbb{A}^{n-1} \smallsetminus \{0\} \dashrightarrow \mathbb{P}^{n-1}$$

is a good quotient for the action on X^{ss}. Since the preimage of each point on $X//G$ is a single orbit, this is also a geometric quotient. Moreover, every semistable point is stable, as the orbit of a point can be written as

$$\mathbb{G}_m \cdot [x_0 : \ldots : x_n] = V(x_0x_1 - x_1x_0, \cdots, x_0x_n - x_nx_0) \subset \mathbb{P}^n,$$

so they correspond to closed sets in $\mathbb{A}^{n-1} \setminus \{0\}$ and have zero dimensional stabilizers.

Before we continue studying the projective quotients, we will need the following algebraic lemma to prove the next proposition:

Lemma 3.3.7. Let G be a geometrically reductive group acting rationally on a finitely generated k–algebra A. For a G–invariant ideal $I \subset A$ and $a \in (A/I)^G$, there is a positive integer $r > 0$ such that

$$a^r \in \frac{A^G}{I \cap A^G}$$

Proof. Let $b \in A$ such that $a = \overline{b} \in (A/I)^G \subset A/I$. We can also suppose $a \neq 0$, otherwise the claim is trivial. If $a \neq 0$, then $b \notin I$ and, since the action of G on A is rational, there is a finite dimensional G–invariant vector space V containing b, which can be spanned by translates of the element b by the group action G.

As a is G–invariant, the condition $g \cdot a = a$ lifts to the condition that

$$g \cdot b = b - x,$$

for some $x \in I$, which implies $g \cdot b - b \in V \cap I$ whenever $g \in G$. Since $b \notin I$, we get $\dim(V) = \dim(V \cap I) + 1$, so every element in V can be uniquely described as $\lambda b + b'$, where $\lambda \in k$ and $b' \in I$.

Let $l : V \to k$ be the projection onto the line spanned by b. Because b is a G–invariant element, l must also be G–invariant.

In terms of the dual representation, the G–invariant projection l corresponds to a G–invariant vector $l^\vee$, and because G is geometrically reductive there is a G–invariant homogeneous function $F \in \mathcal{O}(V^\vee)^G$ of positive degree such that $f(l^\vee) \neq 0$.

We can choose a basis $\{b = v_1, v_2, v_3, \ldots, v_n\}$ such that l is the corresponding first vector in the dual basis in $V^\vee$. Using these coordinates, we can write

$$F(x_1, \ldots, x_n) = \lambda x_1^r + f(x_1, \ldots, x_n)$$

for $\lambda \neq 0$, $r \geq 0$ so f is a polynomial which does not have a factor x_1^r. We define the morphism of k–algebras defined by

$$\begin{aligned} \varphi : k[x_1, \ldots, x_n] = \mathcal{O}(V^\vee) &\to A \\ x_i &\mapsto v_i, \end{aligned}$$

so

$$\varphi(F) = \lambda\varphi(x_1)^r + \varphi \circ f(x_1, \ldots, x_n)$$

where $\varphi \circ f \in I$, and thus

$$\varphi(F) - \lambda b^r \in I,$$

which implies

$$\overline{\varphi(F)} - \lambda a^r = 0 \in A/I,$$

thus $a^r \in A^G/(I \cap A^G)$. □

Lemma 3.3.8. Let G be a reductive group acting on $X \subset \mathbb{P}^n$. A k–point $x \in X(k)$ is stable if and only if x is semistable, the orbit $G \cdot x$ is closed in X^{ss} and the stabilizer G_x is zero dimensional.

Proof. Suppose x is stable and $x' \in \overline{G \cdot x} \cap X^{ss}$. Then $\varphi(x') = \varphi(x)$ and so

$$x' \in \varphi^{-1}(\varphi(x)) \subset \varphi^{-1}(Y^s) \subset X^s.$$

As G acts on X^s with zero dimensional stabilizer, this action must be closed as the boundary of the orbit is a union of orbits of strictly lower dimension. Therefore, $x' \in G \cdot x$ and so the orbit is closed in X^{ss}.

Conversely, suppose that $x \in X^{ss}$ with closed orbit in X^{ss} and zero dimensional stabilizer. As x is semistable, there is a homogeneous polynomial $f \in R(X)_+^G$ such that $x \in X_f$. Since f is G–invariant, X_f is also G–invariant and $G \cdot x \subset X_f$. As $G \cdot x$ is closed in X, it is also closed in the affine open set X_f. By the upper semi-continuity of the dimension of stabilizers, the G–invariant set

$$Z = \{z \in X_f : \dim G_z > 0\}$$

is closed in X_f. Since $\dim G_x = 0$, Z is disjoint from $G \cdot x$, and both are G–invariant closed sets in the affine scheme X_f. By (3.2.2), there is a G–invariant function $h \in \mathcal{O}(X_f)^G$ such that $h(Z) = 0$ and $h(G \cdot x) = 1$.

We now use the Lemma 3.3.7 to conclude. The k–algebra $\mathcal{O}(X_f) = \mathcal{O}(\tilde{X}_f)_0$ is a quotient of $A = ((k[x_0, \ldots, x_n])_f)_0$ by a homogeneous ideal I, so that $h \in (A/I)^G$ and there exists $r > 0$ such that

$$h^r = \frac{h'}{f^s} \in \frac{A^G}{I \cap A^G} = \frac{((k[x_0, \ldots], x_n)_f)_0}{I},$$

for h' homogeneous polynomial and $s > 0$. Note that h' is a G–invariant homogeneous polynomial and $x \in X_{h'f}$, since

$$(h' \cdot f)(x) = (h \cdot f^{s-1})(x) \neq 0$$

and Z is disjoint from $X_{h'f}$, because $h'f = hf^{s-1}$ in X_f and $h(Z) = 0$. Then x has a closed orbit in $X_{h'f}$, and the action of G is closed in $X_{h'f}$ since $\dim G_y = 0$ whenever $y \in X_{h'f}$. □

Definition 3.3.9. Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k–point $x \in X(k)$ is said to be **polystable** if it is semistable and its orbit is closed in X^{ss}. We say that two polystable points x, y are **equivalent** (and write $x \simeq_S y$) if $G \cdot x \cap G \cdot y \neq \emptyset$.

By the previous lemma, every stable point is polystable.

Lemma 3.3.10. Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$ and let $x \in X(k)$ a semistable k–point. Then its orbit closure $\overline{G \cdot x}$ contains a unique polystable orbit. Moreover, if x is semistable but not stable, then this unique polystable orbit is also not stable.

Proof. Each orbit closure $\overline{G \cdot x}$ contains a orbit which is closed, by 2.2.9, and by definition this will be a polystable orbit. Also, if $\varphi : X^{ss} \to X//G$ is the projective GIT quotient, φ is constant in orbit closures and by 3.1.6 the preimage of a k–point under φ contains a unique closed orbit, hence the uniqueness of the polystable orbit.

For the second statement, if the orbit $G \cdot x$ is not closed, then the closed orbit in $\overline{G \cdot x}$ has dimension strictly less than $G \cdot x$ and so cannot be stable. If $\dim G_x > 0$, this implies that $\dim G \cdot x < \dim G$, and the dimension of the polystable orbit will also be strictly less than $\dim G$, so it cannot be stable. □

Corollary 3.3.10.1. Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For two semistable points $x, x' \in X^{ss}$, we have $\varphi(x) = \varphi(x')$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$ and, by definition, this means that $\varphi(x) = \varphi(x')$ if and only if $x \simeq_S x'$. Moreover, there is a bijection of sets:

$$X//G(k) \simeq X^{ps}(k)/G(k) \simeq X^{ss}/\simeq_S,$$

where X^{ps} is the set of polystable k–points.

Proof. The first claim follows from the fact that φ is a good quotient, by 3.1.6. For the second one, note that for every k–point in $X//G$, the preimage by φ contains exactly one polystable orbit, and thus the first bijection holds, since every point in the polystable orbit is a polystable k–point. The first and last sets are also bijective, since the preimage of two k–points is equal if and only if they are S–equivalent. □

As commented at the start of this section, an abstract projective space X does not come with a pre-specified embedding in a projective space. However, every very ample line bundle determines an embedding of X into a projective space.

Let us remember the notion of pullback of vector bundle: If $\pi : L \to X$ is a vector bundle and $\sigma : B \to X$ is a morphism, we can define set:

$$\sigma^*(L) \doteq \{(b, l) \in B \times L : \sigma(b) = \pi(l)\}$$

with the corresponding closed subscheme structure. This also coincides with the categorical fiber product of these two arrows over the base X.

3.3.2 Linearizations and the general case

Let X be a projective k–scheme with an algebraic action of an algebraic group G over k, denoted by $\sigma : G \times X \to X$.

First, we analyze what happens when our projective scheme X is embedded in $\mathbb{P}^n$. In this case, there is a corresponding ample line bundle, obtained as a pullback $L = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ where $i : X \hookrightarrow \mathbb{P}^n$ denotes the embedding. If we denote by $R(X)$ the graded k–algebra such that $\operatorname{Proj} R(X) = X \subset \mathbb{P}^n$, then

$$R(X) = \frac{k[x_0, \dots, x_n]}{I(X)} \simeq \bigoplus_{r \geq 0} \frac{k[x_0, \dots, x_n]_r}{I_r(X)} \simeq \bigoplus_{r \geq 0} H^0(X, L^{\otimes r}).$$

If the G-action is linear, then G respects the grading and induces an action on each piece $H^0(X, L^{\otimes r})$, which in turn induces an action of G in L such that the projection $\pi : L \to X$ is a G–equivariant map.

In the general case, if L is any line bundle over X, we can define the same k–algebra

$$R(X, L) \doteq \bigoplus_{r \geq 0} H^0(X, L^{\otimes r}),$$

and if L is ample, this is generated by degree one elements and we recover the previous case. For more on how the line bundles determine embeddings onto the projective space, see (HARTSHORNE, 1977), chapter II.7.

Definition 3.3.11. A **linearization** of the G–action on X is an invertible sheaf $\mathcal{L}$ on X with an isomorphism of invertible sheaves over $G \times X$:

$$\Phi : \sigma^*\mathcal{L} \xrightarrow{\simeq} \pi_X^*\mathcal{L}$$

which satisfies the following cocycle condition:

$$(m \times \mathrm{Id}_X)^*\Phi = \pi_{23}^*\Phi \circ (\mathrm{Id}_G \times \sigma)^*\Phi$$

where $\pi_{23} : G \times G \times X \to G \times X$ is the projection onto the second and third factors and m denotes the group multiplication. The cocycle condition can be translated as the commutativity of the following diagram of sheaves over $G \times G \times X$:

$$\begin{tikzcd}
{[\sigma \circ (Id_G \times \sigma)]^*\mathcal{L})]} \arrow[rr, "(Id_G \times \sigma)^*\Phi"] \arrow[dd, equal] & & {[\pi_X \circ (Id_G \times \sigma))]^*\mathcal{L}} \\
& {[\sigma \circ \pi_{23}]^*\mathcal{L}} \arrow[r, "\pi_{23}^*\Phi"] & {[\pi_X \circ \pi_{23}]^*\mathcal{L}} \arrow[u, equal] \arrow[d, equal] \\
{[\sigma \circ (m \times \mathrm{Id}_X)]^*\mathcal{L}} \arrow[rr, "(m \times \mathrm{Id}_X)^*\Phi"] & & {[\pi_X \circ (m \times \mathrm{Id}_X)]^*\mathcal{L}}
\end{tikzcd}$$

where the equalities hold by the properties of pullbacks and algebraic group actions.

To see how this definition implies the existence of a G–action on $\mathcal{L}$, we fix a k–point $\alpha \in G(k)$ and denote by T_α the restriction of σ to $\{\alpha\} \times X \to X$. Restricting the isomorphism Φ to $\{\alpha\} \times X$, we get an isomorphism

$$\Phi_\alpha \doteq \Phi\big|_{\{\alpha\}\times X} : T_\alpha^* \mathcal{L} \xrightarrow{\simeq} \left(\pi_X\big|_{\{\alpha\}\times X}\right)^* \mathcal{L} \xrightarrow{\simeq} \mathcal{L}.$$

Then, if we fix $\alpha, \beta \in G(k)$ and consider the cocycle condition restricted to the set $\{\alpha\} \times \{\beta\} \times X \subset G \times G \times X$, we get a commutative diagram

$$\begin{tikzcd} T_{\alpha\beta}^*\mathcal{L} \arrow[r, "T_\beta^*\Phi_\alpha"] \arrow[d, equal] & T_\beta^*\mathcal{L} \arrow[d, "\pi_{23}^*\Phi_{\alpha\beta}=\Phi_\beta"] \\ T_{\alpha\beta}^*\mathcal{L} \arrow[r, "\Phi_{\alpha\beta}"] & \mathcal{L} \end{tikzcd}$$

, as $(\mathrm{Id}_G \times \sigma)^*\Phi$ restricts to $T_\beta^*\Phi_\alpha$ and $(m \times \mathrm{Id}_X)^*\Phi$ restricts to $\Phi_{\alpha\beta}^*$. Thus $\Phi_{\alpha\beta} = \Phi_\beta \circ T_\beta^*\Phi_\alpha$, and in particular we conclude that the procedure is compatible with the multiplication of the group.

Another way to interpret the definition of linearization is to use the language of vector bundles. Let $\pi : L \to X$ be the line bundle associated with the invertible sheaf $\mathcal{L}$. The isomorphism Φ corresponds to an isomorphism of line bundles over $G \times X$:

$$(G \times X) \times_X L \overset{\phi}{\underset{\simeq}{\leftarrow}} (G \times X) \times_X L$$

where the first is the fiber product over X via $\sigma : G \times X \to X$ and the second is the fiber product via $\pi_X : G \times X \to X$. These are, by definition, the pullbacks:

$$\begin{tikzcd} \sigma^* L \arrow[r, "f_2"] \arrow[d, "f_1"] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi_X^* L \arrow[r, "p_2"] \arrow[d, "p_1"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

We note the product $G \times L$ is also a line bundle over $G \times X$, with a projection $(Id_G \times \pi) : G \times L \to G \times X$, since whenever $(g, x) \in G \times X$, $(Id_G \times \pi)^{-1}(g, x) = \{g\} \times L_x \simeq L_x$ has a natural linear structure and, for each point $x \in X$, if U is the trivializing neighbourhood for the line bundle $\pi : L \to X$ around x, then

$$(Id_G \times \pi)^{-1} = \mathrm{Id}_G \times \pi^{-1} : G \times U \xrightarrow{\simeq} G \times U \times \mathbb{A}^1$$

is a trivialization for $G \times L$ over $G \times X$. Forming the pullback diagram

$$\begin{tikzcd} G \times L \arrow[r, "\pi_L"] \arrow[d, "Id_G \times \pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\pi_X"'] & X, \end{tikzcd}$$

by the universal property of the fiber product, there is a unique morphism

$$h : G \times L \to \pi_X^*(L)$$

such that $p_2 \circ h = \pi_L$ and $p_1 \circ h = \mathrm{Id}_G \times \pi$. On each fiber $(g, x) \in G \times X$, the morphism h restricts to $h_{g,x} : \{g\} \times L_x \to \pi_X^*(L)$ such that

$$p_2 \circ h_{g,x}(\{g\} \times L_x) = \pi_L(\{g\} \times L_x) = L_x = \pi^{-1}(x)$$

and since the definition of the fiber $\pi^*(L)_{(g,x)} = p_1^{-1}(g, x) = p_2^{-1} \circ \pi^{-1}(x)$, then the restriction of h respect the fibers: $h_{g,x} : \{g\} \times L_x \to \pi^*(L)_{(g,x)}$. Hence we have the following diagram:

$$\begin{tikzcd} \{g\} \times L \arrow[rr, "h_{(g,x)}"] \arrow[dr, "\pi_L"'] & & (\pi_X^*(L))_{(g,x)} \arrow[dl, "p_2"] \\ & L_x & \end{tikzcd}$$

By construction, the restriction $\pi_L : \{g\} \times L_x \to L_x$ is an isomorphism, so $h_{(g,x)} = \pi_L \circ p_2$ is also an isomorphism. This implies that h is an isomorphism of line bundles. Since the pullback is defined up to isomorphism, we can just identify $\pi_X^* L$ with the line bundle $G \times L$ over $G \times X$, so we also write $\phi : G \times L \to \sigma^* L$ for the identification.

Our goal now is, given a G–linearization of a line bundle L over a scheme X, induce a G–action on L. Using the notation above, follows the commutativity of the diagram

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[dr, "Id_G \times \pi"'] & \sigma^*(L) \arrow[r, "f_2"] \arrow[d, "f_1"] & L \arrow[d, "\pi"] \\ & G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

Thus, if we define the action morphism as $\Sigma \doteq \Phi \circ f_2$, the following square commutes:

$$\begin{tikzcd} G \times L \arrow[r, "\Sigma"] \arrow[d, "Id_G \times \pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

Looking at the fibers over (g,x) and $\sigma(g,x)$, respectively, implies Σ is a bundle isomorphism between the line bundles $G\times L$ over $G\times X$ and L over X. Using this notation, the cocycle condition translates into the commutative cube

$$\begin{tikzcd}
G\times G\times L \arrow[rr, "m\times \mathrm{Id}_L"] \arrow[dr, "\mathrm{Id}_G\times\Sigma"] \arrow[dd, "\mathrm{Id}_G\times \mathrm{Id}_G\times\pi"'] & & G\times L \arrow[dr, "\Sigma"] \arrow[dd] & \\
& G\times L \arrow[rr, "\Sigma"] \arrow[dd] & & L \arrow[dd, "\pi"] \\
G\times G\times X \arrow[rr, "m\times \mathrm{Id}_X"] \arrow[dr, "Id_G\times\sigma"'] & & G\times X \arrow[dr, "\sigma"] & \\
& G\times X \arrow[rr, "\sigma"] & & X,
\end{tikzcd}$$

which implies that Σ is exactly the lifting of the action σ to L via the projection π, and it also makes it G–equivariant. Moreover, these are equivalent conditions (see (MUMFORD; FOGARTY; KIRWAN, 1994)).

Proposition 3.3.12. If G is an affine algebraic group over k acting on a k–scheme X and $\pi_1 : L_1 \to X$, $\pi_2 : L_2 \to X$ are two linearizations for an action $\sigma : G\times X \to X$, then the tensor product

$$\pi_1 \otimes \pi_2 : L_1 \otimes L_2 \to X$$

also defines a linearization for this action.

Proof. Since L_1 and L_2 are linearizations by hypothesis, there are respective bundle isomorphisms

$$G\times L_1 \simeq \sigma^* L_1 \text{ and } G\times L_2 \simeq \sigma^* L_2.$$

The tensor product $L_1\otimes L_2 \to X$ is again a line bundle, with corresponding morphism denoted by $\pi_1\otimes\pi_2$, and the tensor operation commutes with taking pullbacks:

$$\sigma^*(L_1\otimes L_2) \simeq \sigma^* L_1 \otimes \sigma^* L_2$$

over $G\times X$, so the space $G\times_k (L_1\otimes L_2)$ is a line bundle, over $G\times X$, via the map $1_G\times(\pi_1\otimes\pi_2)$, commuting the diagram:

$$\begin{tikzcd}
G\times (L_1\otimes L_2) \arrow[r, "\pi_{L_1\otimes L_2}"] \arrow[d, "Id_G\times(\pi_1\otimes\pi_2)"'] & L_1\otimes L_2 \arrow[d, "\pi_1\otimes\pi_2"] \\
G\times X \arrow[r, "\pi_X"'] & X
\end{tikzcd}$$

where whenever $(g, x) \in G \times X$,

$$Id_G \times (\pi_1 \otimes \pi_2)^{-1}(g, x) = \{g\} \times L_{1,x} \otimes L_{2,x}$$

which will be send isomorphically to $(L_1 \otimes L_2)_x$ by π_X. Then $G \times (L_1 \otimes L_2) \simeq \pi_X^*(L_1 \otimes L_2)$, and we get a chain of bundle isomorphisms over $G \times X$

$$\begin{aligned} G \times (L_1 \otimes L_2) &\simeq \pi_X^*(L_1 \otimes L_2) \\ &\simeq \pi_X^*(L_1) \otimes \pi_X^*(L_1) \\ &\simeq (G \times L_1) \otimes (G \times L_2) \\ &\simeq \sigma^*(L_1) \otimes \sigma^*(L_2) \\ &\simeq \sigma^*(L_1 \otimes L_2) \end{aligned}$$

and this gives a natural G–linearization of $L_1 \otimes L_2$. □

Remark 3.3.13. More concretely, we can translate this argument in terms of the G–actions $\Sigma_1 : G \times L_1 \to L_1$ and $\Sigma_2 : G \times L_2 \to L_2$ induced by these linearizations. If Φ_1 and Φ_2 are the fixed isomorphisms between $G \times L_1 \simeq \sigma^*(L_1)$ and $G \times L_2 \simeq \sigma^*(L_2)$, then

$$G \times (L_1 \otimes L_2) \simeq (G \times L_1) \otimes (G \times L_2) \xrightarrow[\simeq]{\Phi_1 \otimes \Phi_2} \sigma^*(L_1) \otimes \sigma^*(L_2) \simeq \sigma^*(L_1 \otimes L_2).$$

Then the corresponding action Σ commuting the diagram

$$\begin{tikzcd} G \times (L_1 \otimes L_2) \arrow[r, "\Sigma"] \arrow[d, "Id_G \times (\pi_1 \otimes \pi_2)"'] & L_1 \otimes L_2 \arrow[d, "\pi_1 \otimes \pi_2"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

is the composition of the isomorphism $\Phi = \Phi_1 \otimes \Phi_2$ with the projection $f : \sigma^*(L_1 \otimes L_2) \to L_1 \otimes L_2$, which via the isomorphism $\sigma^*(L_1) \otimes \sigma^*(L_2) \simeq \sigma^*(L_1 \otimes L_2)$ corresponds to the tensor product of the maps $f_1 : \sigma^*(L_1) \to L_1$ and $f_2 : \sigma^*(L_2) \to L_2$.

This means we could also define the tensor product of the actions as the tensor product of two morphisms

$$\Sigma_1 \otimes \Sigma_2 = \Phi_1 \otimes \Phi_2 \circ f_1 \otimes f_2.$$

Proposition 3.3.14. If G is an affine algebraic group over k acting algebraically on a k–scheme X via $\sigma : G \times X \to X$ and $\pi : L \to X$ is a linearization for this action, then the dual line bundle

$$\pi^\vee : L^\vee \to X$$

also has an induced linearization for the same action.

Proof. Let L be a a line bundle together with a linearization

$$G \times L \simeq \sigma^*(L).$$

Since taking pullbacks commutes with the dual operation for locally free sheaves of finite rank, we have

$$\sigma^*(L^\vee) \simeq \sigma^*(L)^\vee \text{ and } \pi_X^*(L^\vee) \simeq \pi_X^*(L)^\vee,$$

so the line bundle $G \times L^\vee$ is isomorphic to $\pi_X^*(L^\vee)$, and the dual line bundle has a natural induced G–linearization via

$$\pi_X^*(L^\vee) \simeq \pi_X^*(L)^\vee \simeq \sigma^*(L)^\vee \simeq \sigma^*(L^\vee).$$

□

Remark 3.3.15. The last propositions imply the set all of possible G–linearizations forms a group under the operation of tensor product. We can also consider the notion of an isomorphism of linearizations, meaning an isomorphism of line bundles which is G–equivariant with respect to the corresponding G–actions. This defines a subgroup of the Picard group of the projective variety X, which we will denote by $\mathrm{Pic}^G(X) \subset \mathrm{Pic}(X)$.

Example 3.5. In this example, we explore the relationship between G–linearizations and characters of the algebraic group G.

1. Let $X = \operatorname{Spec} k$, with the trivial G-action. There is only one line bundle $L = \mathbb{A}^1 \to X$ over k, the trivial one, but there are many possible linearizations. If $\mathcal{X} \in X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m(k))$ is a character, we can define a G–action on $\mathbb{A}^1$ by

$$\begin{aligned} G \times \mathbb{A}^1 &\to \mathbb{A}^1 \\ (g, a) &\mapsto \mathcal{X}(g) \cdot a. \end{aligned}$$

   Conversely, every linearization is given by a linear action of G on $\mathbb{A}^1$, that is, a group morphism $\mathcal{X} : G \to \mathrm{GL}_1(k) \simeq \mathbb{G}_m(k)$. Seeing the product of characters as a tensor product of representations, we see that this actually implies the isomorphism of groups $X^*(G) \simeq \mathrm{Pic}(X)^G$.

2. More generally, for any k–scheme X with an algebraic action of an affine algebraic group G over k and a character $\mathcal{X} : G \to \mathbb{G}_m$, we can construct a linearization on the trivial line bundle $L = X \times \mathbb{A}^1 \to X$ using the morphism

$$\begin{aligned} G \times (\mathbb{A}^1 \times X) &\to \mathbb{A}^1 \times X \\ (g, (x, a)) &\mapsto (g \cdot x, \mathcal{X}(g) \cdot a) \end{aligned}$$

3. Not every linearization on a trivial bundle comes from a character. For example, consider the finite group $G \doteq \{\pm 1\}$ acting on $X = \mathbb{A}^1 \setminus \{0\}$ via the rule

$$(1) \cdot x = x$$
$$(-1) \cdot x = x^{-1}.$$

In this case, the linearization on $X \times \mathbb{A}^1$ given by the action

$$G \times X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$$
$$(1) \cdot (x, z) = (x, z)$$
$$(-1) \cdot (x, z) = (x^{-1}, x \cdot z)$$

cannot be isomorphic to a linearization given by a character, as over the fixed points 1 and -1 on X the action of the element $-1 \in G$ on fibers is given by the maps $z \mapsto z$ and $z \mapsto -z$, respectively.

Lemma 3.3.16. Let G be an affine algebraic group over k acting on a k–scheme X via $\sigma : G \times X \to X$ and let $\Sigma : G \times L \to L$ be the corresponding linearization of the action on a line bundle L over X, as constructed above. Then this induces a linear representation:

$$G \to \mathrm{GL}(H^0(X, L))$$

Proof. Note that, via the pullback of σ, we can define a (linear) co-module structure $\varphi : H^0(X, L) \to \mathcal{O}(G) \otimes_k H^0(X, L)$, by the composition:

$$H^0(X, L) \xrightarrow{\sigma^*} H^0(G \times X, \sigma^* L) \simeq H^0(G \times X, G \times L) \simeq H^0(G, \mathcal{O}(G)) \otimes_k H^0(X, L)$$

where the last isomorphism follows from the Künneth formula (see C.2.4) and the middle isomorphism is induced by the linearization $G \times L \simeq \sigma^* L$. This is equivalent to giving a representation $G \to \mathrm{GL}(H^0(X, L))$, for if $s \in H^0(X, L)$ and $\varphi(s) = \sum_i f_i \otimes s_i$, we define the corresponding linear transformation

$$\varphi_g : H^0(X, L) \to H^0(X, L)$$
$$s \mapsto \sum_i f_i(g) \cdot s_i.$$

Thus, we get a linear representation

$$G \to \mathrm{GL}(H^0(X, L))$$
$$g \mapsto \varphi_g,$$

as we wanted. □

Just to see that this machinery is indeed a generalization of the case $X \subset \mathbb{P}^n$, let us recover this case using this representation. If X is a projective k–scheme with a fixed very ample linearization L, then the evaluation map

$$H^0(X, L) \otimes_k \mathcal{O}_X \to L,$$

given over an open set $U \subset X$ as the morphism of $\mathcal{O}_X(U)$–modules

$$\begin{aligned} H^0(X, L) \otimes_k \mathcal{O}_X(U) &\to L(U) \\ s \otimes t &\mapsto t \cdot s\big|_U, \end{aligned}$$

is a G–equivariant surjective map, so it induces a G–equivariant closed embedding

$$X \to \mathbb{P}(H^0(X, L)^*)$$

such that L is isomorphic to the pullback of Serre's twisting sheaf $\mathcal{O}(1)$ over this projective space (as in (HARTSHORNE, 1977), II.7).

Note that this is an embedding of X on a projective space such that the action of G on X comes from a linear action of G on $H^0(X, L)^*$, so this generalizes the setting of G acting linearly on a closed subscheme $X \subset \mathbb{P}^n$.

Now, we are ready to construct a GIT quotient in this context. Let G be a reductive group acting on a projective scheme X and let L be a fixed linearization of the G–action on X. Then consider the graded finitely generated k–algebra:

$$R \doteq R(X, L) \doteq \bigoplus_{r \geq 0} H^0(X, L^{\otimes r}).$$

Since each line bundle $L^{\otimes r}$ has an induced linearization, there is an induced action of G on each space of sections $H^0(X, L^{\otimes r})$, so it amounts to an action of G on the graded algebra which preserves graded pieces. We denote by

$$R^G = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})^G$$

the corresponding graded algebra of G–invariant sections. Note that this is the same setting as in the proof of (3.3.6), and arguing as before we see that R^G is a finitely generated k–algebra and $\operatorname{Proj} R^G$ is projective over the zero-sections $R^G_0 = k^G = k$.

Definition 3.3.17. In this context, a point $x \in X$ is said to be:

- **semistable** with respect to L if there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$.
- **stable** with respect to L if $\dim G \cdot x = \dim G$, there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and the action of G on

$$X_\sigma \doteq \{x \in X : \sigma(x) \neq 0\}$$

  is closed and so X_σ is an affine open set.

We let $X^{ss}(L)$ and $X^{s}(L)$ denote the open sets of semistable and stable points in X, respectively. We define the **projective GIT quotient with respect to** L to be the morphism

$$X^{ss}(L) \to X//_L G \doteq \operatorname{Proj} R(X,L)^G$$

associated with the inclusion $R(X,L)^G \hookrightarrow R(X,L)$.

When we are in the particular case of a linear action of G on $i : X \subset \mathbb{P}^n$, the action can be naturally linearized using the line bundle $L \doteq i^*(\mathcal{O}_{\mathbb{P}^n}(1))$, so that

$$H^0(X, L^{\otimes r})^G = H^0(X, i^*(\mathcal{O}_{\mathbb{P}^n}(r)))^G = R(X)^G_r,$$

whenever $r \geq 0$, using the notation in (3.3.1). Using this, the definitions of (semi)stable reduce to the previous ones.

Theorem 3.3.18. *Let G be a reductive group acting on a projective scheme X and L be an ample linearization of this action. Then the GIT quotient*

$$\varphi : X^{ss}(L) \to X//_L G$$

is a good quotient and $X//_L G$ is a projective scheme with a natural ample line bundle L' such that $\varphi^*(L') = L^{\otimes n}$ for some $n > 0$. Furthermore, there is an open set $Y^s \subset X//_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient.

Proof. As L is ample, for each $\sigma \in R(X,L)^G_+$, the open set X_σ is affine and the above GIT can be obtained by gluing affine GIT quotient, just as in the proof of (3.3.6). □

Theorem 3.3.19. *Let G a reductive group acting on a quasi-projective scheme X and L be a linearization of this action. Then there is a quasi-projective scheme $X//_L G$ and a good quotient $\varphi : X^{ss}(L) \to X//_L G$ of the G–action on $X^{ss}(L)$. Furthermore, there is an open subset $Y^s \subset X//_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient.*

Remark 3.3.20. As we saw in this section, whenever X is a projective k–scheme and G is a reductive group over k acting algebraically, whenever we fix a linearization L for this action, we can construct a GIT-quotient $X//_L G$, which is again a projective k–scheme.

When studying a concrete moduli functor $\mathcal{M}$, as explained in Chapter 1, if X is a k–scheme which can be considered as a space of parameters for $\mathcal{M}$ in Sch_k, such that the G–action induces the same equivalence relation, the space $X//_L G$ could be considered as a fine moduli space for the moduli problem $\mathcal{M}$.

Analyzing how the geometry of the quotient $X//_L G$ changes when we vary linearizations inside $\mathrm{Pic}^G(X)$ for an algebraic G–action on X is a subject called **variation of GIT**, or VGIT (see more about this aspect in (LAZA, 2012)).

3.3.3 Functoriality

In this section, we study the following general situation: if there is a reductive group G over k acting on two k–schemes X and Y, there is a G–equivariant morphism $f : X \to Y$ and a linearization L of this G–action over Y, what is the relation between the sets $X^s(f^*L)$ and $f^{-1}(Y^s(L))$? To answer this, we use Reynolds operators.

Proposition 3.3.21. In the context mentioned above, if moreover the morphism f is quasi-affine, then $f^{-1}(Y^s(L)) \subset X^s(f^*L)$.

Proof. Let $x \in f^{-1}(Y_s(L))$. By definition, this means that there exists a G–invariant section $\sigma \in H^0(Y, L^r)$ such that Y_σ is affine, $f(x) \in Y_\sigma$ and the stabilizers of points of Y_σ are 0-dimensional.

Since f is G–equivariant, the pullback $s \doteq f^*\sigma \in H^0(X, f^*L^r)$ is G–invariant, $x \in X_s$ and all stabilizers are 0–dimensional. We only need to prove that X_s is an affine open set, to get $x \in X^s(f^*L)$.

Note that, since f is quasi-affine and $X_s = f^{-1}(Y_\sigma)$, X_s is quasi-affine, and if we denote by $R = \Gamma(X, \mathcal{O}_X)$ and $\tilde{X} = \operatorname{Spec} R$, we have the following commutative diagram:

$$\begin{tikzcd} X_s \arrow[rr, "I"] \arrow[dr, "f"'] & & \tilde{X} \arrow[dl, "\tilde{f}"] \\ & Y_\sigma & \end{tikzcd}$$

where I is an open immersion and $\tilde{f}$ is the unique extension of the map f to $\tilde{X} = \overline{I(X_s)}$. Moreover, using the dual action of G on R induced by the action on X, we can induce a G–action on $\tilde{X}$ such that $\tilde{f}$ is G–equivariant. It follows that the orbit $G \cdot I(x)$ is closed, as $G \cdot x$ is closed in X_s, and we have two disjoint closed G–invariant sets:

$$Z_1 = \tilde{X} \setminus I(X) \text{ and } Z_2 = G \cdot I(x).$$

By 3.2.2, there is $f \in R$ G–invariant such that $f(Z_1) = 0$ and $f(x) = 1$. This means that $\tilde{X}_f \subset X_s$ and $\tilde{X}_f$ is affine. By (HARTSHORNE, 1977) (II Lemma 5.14), there is an integer k such that the local section $s^k \cdot f$ of the vector bundle f^*L^{rk} over X_s extends to a global section $s' \in H^0(X, f^*L^{rk})$. To get a G–invariant section, we project via the Reynolds operator. Let E be such operator on $H^0(X, f^*L^{rk})$, which exists by 2.5.1. Then $E(s')$ is G–invariant and $E(s')$ still restricts to the invariant section $f^k s$ over the invariant open set X_s.

Furthermore, the section $s \cdot Es'$ is 0 both outside X_s and at points of X_s where $f = 0$ and thus $X_{s \cdot Es'}$ is an affine neighbourhood of x contained in X_s, which implies $x \in X^s(f^*L)$. □

The next question is: when does the equality between these two sets hold? This is answered by the following proposition:

Proposition 3.3.22. Assume f is finite, X is proper over k and L is ample over Y. Then

$$f^{-1}(Y^s(L)) = X^s(f^*L)$$

Proof. Since L is an ample linearization over Y, there is an embedding $i : Y \to \mathbb{P}^n$, a linear action of G on $\mathbb{P}^n$ such that I is G–equivariant and $L \simeq i^*(\mathcal{O}(1))$. Since the sets of stable points will be the same, we only need to prove the linear action case for $\mathbb{P}^n$.

Let us suppose then $Y = \mathbb{P}^n$, $L = \mathcal{O}(1)$, $R = k[x_0, \dots, x_n]$, S denote homogeneous coordinate ring of X and $f : X \to \mathbb{P}^n$ the G–equivariant morphism.

The morphism f induces a graded R–algebra structure on S, and since f is finite, S is a finite R–module. The actions of G on X and on $\mathbb{P}^n$ and the linearizations of $f^*(\mathcal{O}(1))$ and $\mathcal{O}(1)$ define dual actions of G on R and S, compatible with the R–module structure of S induced by f.

Now let E and F be the Reynolds operators on R and S, respectively, and let x be a stable point of X. Then there is an invariant section $s \in H^0(X, f^*(\mathcal{O}_{\mathbb{P}^n}(n)) = S_n$ such that $s(x) \neq 0$, the action of G is closed in X_s and the stabilizer of x is 0-dimensional. Since S is a finite R–module, there is an equation of integral dependence:

$$s^m + \overline{a}_1 s^{m-1} + \cdots + \overline{a}_m = 0$$

where a_i are homogeneous elements of R and $\overline{a}_i$ denotes their image inside S. Applying the Reynolds operator F, we obtain:

$$\begin{aligned} 0 &= F(s^m + \overline{a}_1 s^{m-1} + \cdots + \overline{a}_m) \\ &= s^m + F(\overline{a}_1)s^{m-1} + \cdots + F(\overline{a}_m) \\ &= s^m + \overline{Ea_1} \cdot s^{m-1} + \cdots + \overline{Ea_m}. \end{aligned}$$

Since $s(x) \neq 0$, it follows that for some i, $Ea_i(f(x)) \neq 0$. Therefore, $f(x) \in \mathbb{P}^n_{Ea_i}$. But this implies that:

$$x \in X_{f^*(Ea_i)} = f^{-1}(\mathbb{P}^n_{Ea_i}).$$

Since the orbit of x in $X_{f^*Ea_i}$ is closed and f is proper, the orbit of the k–point $f(x)$ is closed in $\mathbb{P}^n_{Ea_i}$. Since f is finite, the dimension of orbits coincide, so the stabilizer of $f(x)$ is also 0–dimensional, and therefore $f(x)$ is stable. This proves the inclusion:

$$X^s(f^*\mathcal{O}(1)) \subset f^{-1}((\mathbb{P}^n)^s(\mathcal{O}(1)),$$

and the other inclusion holds by the previous proposition. $\square$

Note that, in the proofs of the previous propositions, we also proved the corresponding assertions for the semistable points.

This gives an interesting simplification: if G acts algebraically on X and L is a linearization of this action, there is an induced immersion $X \hookrightarrow \mathbb{P}^n$ and we can compute the set of (semi)stable points of the linear action over $\mathbb{P}^n$.

3.4 Criteria for (semi)stability

In this section, our goal is to give some criteria to better understand the sets of stable and semistable points. We can simplify our situation by supposing that we have a linear action of a reductive algebraic group G on a closed subscheme $X \subset \mathbb{P}^n$.

In this context, as G acts via a linear representation $G \to \mathrm{GL}_{n+1}(k)$, the action of G lifts to the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$. Let $R(X) = \mathcal{O}(\tilde{X})$ denote the graded coordinate ring of X. Looking at the corresponding G–action on the affine cone will give us the first criterion:

Proposition 3.4.1. Let $x \in X(k)$ and choose a lift $\tilde{x} \in \tilde{X}(k)$ of x. Then:

1. x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$.
2. x is stable if and only if $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.

Proof. First, if x is semistable, there is a G–invariant homogeneous polynomial $f \in R(X)^G$ such that $f(x) \neq 0$. We can consider f to be a polynomial over the lift $\tilde{X}$, so that $f(\tilde{x}) \neq 0$.

Since f is G–invariant as a function on $\tilde{X}$, f will be constant in the orbit closure $\overline{G \cdot \tilde{x}}$, and non-zero. But as f is homogeneous, $f(0) = 0$ and then f separates the closed subschemes $\{0\}$ and $\overline{G \cdot \tilde{x}}$, thus they are disjoint in $\tilde{X}$.

For the converse, suppose that $\overline{G \cdot \tilde{x}}$ and 0 are disjoint. Then, as these are both G–invariant closed subsets of $\tilde{X}$ and G is reductive, by (3.2.2) there is a G–invariant polynomial $f \in \mathcal{O}(\tilde{X})^G$ which separates these subsets, such that

$$f(\overline{G \cdot \tilde{x}}) = 1 \text{ and } f(0) = 0.$$

If we decompose $f = f_0 + \ldots + f_r$ into homogeneous polynomials f_i of positive degree, as the G–action is linear, each piece f_i will be G–invariant such that there is at least one that does not vanish on $\overline{G \cdot \tilde{x}}$. Hence, we can choose such $f = f_i \in R(X)^G$ so x must be a semistable point since $f(x) \neq 0$.

For the second statement, if x is a stable point, then $\dim G_x = 0$ and there is a G–invariant homogeneous polynomial $f \in R(X)^G$ such that $x \in X_f$ and $G \cdot x$ is closed in X_f. Since the action is linear, of course we have $G_{\tilde{x}} \subset G_x$ and we conclude that $\dim G_{\tilde{x}} = 0$.

We can again view f as a function on $\tilde{X}$ and consider the closed subscheme:

$$Z = \{z \in \tilde{X} : f(z) = f(\tilde{x})\}$$

on $\tilde{X}$. If we prove that $G \cdot \tilde{x} \subset Z$ is closed, we are done. The projection $\pi : \tilde{X} \setminus \{0\} \to X$ restricts to a surjective finite morphism $\pi : Z \to X_f$, since f is a homogeneous polynomial. The preimage of the closed orbit $G \cdot x$ in X_f under π is closed and G–invariant. As π is finite, it follows that the preimage $\pi^{-1}(G \cdot x)$ is a finite number of G–orbits, which must all have dimension equal to $\dim G \cdot x = \dim G$, so they are closed in the preimage, hence $G \cdot \tilde{x}$ is closed.

Conversely, suppose that $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$. Then $0 \notin \overline{G \cdot \tilde{x}} = G \cdot \tilde{x}$, and x is a semistable point of X by the first part of the proposition. Thus, there is a non-constant homogeneous G–invariant polynomial f such that $f(x) \neq 0$. As above, we consider the finite surjective morphism:

$$\pi : Z \to X_f.$$

As finite morphisms are closed, $\pi(G \cdot \tilde{x}) = G \cdot x$ will be a closed subset of X_f and $\dim G_x = 0$. Since this holds for all f such that $f(x) \neq 0$, it follows that $G \cdot x$ is closed in $X^{ss} = \bigcup_f X_f$. Hence x is stable, since X^{ss} is an open set of X. □

To study the next criterion of stability, we will need to use one-parameter subgroups.

Definition 3.4.2. A **one-parameter subgroup** (over k) of G is a non-trivial morphism of algebraic groups $\lambda : \mathbb{G}_m \to G$ over k.

Fix $x \in X(k)$ and a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$. Then we let

$$\begin{aligned} \lambda_x : \mathbb{G}_m &\to X \\ t &\mapsto \lambda(t) \cdot x \end{aligned}$$

There is a natural embedding $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ given by $t \mapsto [1 : t]$. Since X is projective, it is proper over $\operatorname{Spec} k$ and by the valuative criterion for properness ((HARTSHORNE, 1977) II 4.7), the morphism $\lambda_x : \mathbb{G}_m \to X$ extends uniquely to a morphism $\hat{\lambda}_x : \mathbb{P}^1 \to X$ such that the diagram commutes:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\lambda_x"] \arrow[d] & X \arrow[d] \\ \mathbb{P}^1 \arrow[r] \arrow[ur, dashed, "\hat{\lambda}_x"] & \operatorname{Spec} k. \end{tikzcd}$$

We can use this extension to define the limit points

$$\lim_{t\to 0}\lambda_x(t) \doteq \hat{\lambda}_x([1:0]) \text{ and } \lim_{t\to\infty}\lambda_x(t) \doteq \hat{\lambda}_x([0:1]).$$

Using the usual change of coordinates of $\mathbb{P}^1$ given by $[1:t]=[1/t:1]$ and the fact that λ is a group morphism, it follows that these concepts are dual:

$$\lim_{t\to\infty}\lambda(t)\cdot x = \hat{\lambda}_x([0:1]) = \lim_{t\to 0}\lambda(t^{-1})\cdot x = \lim_{t\to 0}\lambda(t)^{-1}\cdot x,$$

and we could also consider the inverse one-parameter subgroup $\lambda^{-1}:\mathbb{G}_m\to G$.

Proposition 3.4.3. If $\lambda:\mathbb{G}_m\to G$ is a one-parameter subgroup and $x\in X(k)$, then the **limit point**

$$y=\lim_{t\to 0}\lambda(t)\cdot x$$

must be fixed by the subgroup $\lambda(\mathbb{G}_m)$.

Proof. Fix an element $s\in\mathbb{G}_m$. We have

$$\lambda(s)\cdot\hat{\lambda}_x([1:t]) = \lambda(st)\cdot x = \lambda_x([1:st])$$

whenever $t\in\mathbb{G}_m$. We can consider the following diagram:

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "\lambda'_x"] \arrow[d] & X \arrow[d] \\
\mathbb{P}^1 \arrow[r] \arrow[ur, dashed, "\hat{\lambda}'_x"] & \operatorname{Spec} k
\end{tikzcd}$$

where $\lambda'_x(t)\doteq\lambda(st)\cdot x$ and $\hat{\lambda}'_x([1:t])=\hat{\lambda}_x([1:s\cdot t])$ whenever $t\in\mathbb{G}_m$. Thus

$$\lambda(s)\cdot\hat{\lambda}_x([1:t]) = \hat{\lambda}'_x([1:t])$$

whenever $t\neq 0$, and as the set $\{[1:t]\in\mathbb{P}^1 : t\neq 0\}$ is a dense open set in $\mathbb{P}^1$, it follows that $\lambda(s)\cdot\hat{\lambda}_x([1:0])=\hat{\lambda}'_x([1:0])$.

Looking at the affine space $\mathbb{A}^1\hookrightarrow\mathbb{P}^1$ via $t\mapsto[1:t]$, the equality

$$\hat{\lambda}'_x([1:t]) = \hat{\lambda}_x([1:s\cdot t])$$

holds for a dense open subset of $\mathbb{A}^1$, so

$$\hat{\lambda}'_x([1:0]) = \hat{\lambda}_x([1:s\cdot 0]) = \hat{\lambda}_x([1:0])$$

and the result follows. □

Remark 3.4.4. Moreover, on each fiber over y of $\mathcal{O}(1) \doteq \mathcal{O}_{\mathbb{P}^n}(1)\big|_X$, the group $\lambda(\mathbb{G}_m)$ must act by a character $t \mapsto t^r$ of $\mathbb{G}_m$ (see 2.3.3), for $r \in \mathbb{Z}$, as it restricts to a linear action in the trivial line bundle over a point $\{y\} \simeq \operatorname{Spec} k$, as in 3.5.

Definition 3.4.5. Using the previous remark, we define the **Hilbert-Mumford weight** of an action of the one-parameter subgroup λ on $x \in X(k)$ to be

$$\mu^{\mathcal{O}(1)}(x, \lambda) = r$$

where r is the weight of $\lambda(\mathbb{G}_m)$ on the fiber $\mathcal{O}(1)_y$ over $y \doteq \lim_{t \to 0} \lambda(t) \cdot x$

We can translate this definition to the affine cone over the projective variety X, denoted by $\tilde{X}$, as follows. Fix any non-zero lift $\tilde{x} \in \tilde{X}$ of $x \in X$. We can consider the corresponding morphism

$$\lambda_{\tilde{x}} \doteq \lambda(-) : \mathbb{G}_m \to \tilde{X},$$

which may no longer extend to $\mathbb{P}^1$, as $\tilde{X}$ may not be proper over k. If it extends to zero, or infinity, we will use the same limit notation as above.

The action of $\lambda(\mathbb{G}_m)$ is linear on $\mathbb{A}^{n+1}$, and this means (see 2.3.3) that we can choose a basis $e_0, \ldots, e_n$ of k^{n+1} such that $\lambda(t) \cdot e_i = t^{r_i} e_i$ for $r_i \in \mathbb{Z}$. We call the integers r_i the λ–weights for this action on $\mathbb{A}^{n+1}$. For $x \in X(k)$, we can choose a lift $\tilde{x} \in \tilde{X}$ and write it with respect to this basis as

$$\tilde{x} = \sum_{i=0}^{n} x_i e_i.$$

Applying the diagonalization, we get:

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} x_i e_i.$$

Proposition 3.4.6. In the context above, the set $\{r_i : x_i \neq 0\}$ does not depend on the choice of non-zero lift $\tilde{x}$.

Proof. Let $\tilde{x}$ and $\tilde{x}'$ be two possible lifts over x in $\mathbb{A}^{n+1}$. Writing both using the diagonalization coordinates

$$\tilde{x} = \sum_{i=0}^{n} x_i e_i \ , \ \tilde{x}' = \sum_{i=0}^{n} x_i' e_i,$$

this means that $\tilde{x} = \alpha \tilde{x}'$, for a non-zero element $\alpha \in k$, so that

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} x_i e_i = \sum_{i=0}^{n} t^{r_i} (\alpha x_i') e_i = \alpha \lambda(t) \cdot \tilde{x}',$$

and the claim follows. □

Definition 3.4.7. We define the **weight** of a point x with respect to a one-parameter subgroup λ of G to be

$$\mu(x,\lambda) \doteq -\min\{r_i : x_i \neq 0\}.$$

Proposition 3.4.8. The weight of x with respect to a one-parameter subgroup λ satisfies the following properties:

1. $\mu(x,\lambda)$ is the unique integer $\mu \in \mathbb{Z}$ such that $\lim_{t\to 0} t^{\mu}\lambda(t)\cdot\tilde{x}$ exists and is non-zero.
2. $\mu(x,\lambda^n) = n\mu(x,\lambda)$ for $n \geq 1$.
3. For all $g \in G$, $\mu(g\cdot x, g\lambda g^{-1}) = \mu(x,\lambda)$.
4. $\mu(x,\lambda) = \mu(y,\lambda)$, where $y = \lim_{t\to 0}\lambda(t)\cdot x$.

Proof. 1. Using the diagonalization coordinates as before, if $\mu \in \mathbb{Z}$, we can write

$$t^{\mu}\lambda(t)\cdot\tilde{x} = t^{\mu}\cdot\left(\sum_{i=0}^{n} t^{r_i}x_ie_i\right) = \sum_{i=0}^{n} t^{r_i+\mu}x_ie_i$$

whenever $t \in \mathbb{G}_m$. Note that, if there exists at least one r_i such $\mu < -r_i$, $\mu + r_i < 0$ and this means that $\lambda_{\tilde{x}}$ cannot be extended to 0. On the other hand, if $\mu > -r_i$ for all $i = 0,\cdots,n$, all coordinates will have positive powers of t and this only can be the case if the limit is zero. It follows that $\mu = \mu(x,\lambda)$ is the only integer with this property.

2. Using coordinates, its easy to see that:

$$\lambda^n(t)\cdot\tilde{x} = \sum_{i=0}^{n} t^{nr_i}x_ie_i$$

and $\mu(x,\lambda^n) = n\mu(x,\lambda)$ follows.

3. Note that

$$(g\lambda g^{-1})(t)\cdot g\tilde{x} = \sum_{i=0}^{n} t^{r_i}\cdot x_ig(e_i).$$

Using the first property, the claim follows, as g acts on $\mathbb{A}^{n+1}$ by change of basis inside $\mathrm{GL}_{n+1}(k)$.

4. Let $\tilde{x}$ be a non-zero lifting of x on the affine cone. Using the same coordinates, by 1, we have a non-zero limit

$$\tilde{y} = \lim_{t\to 0} t^{\mu(x,\lambda)}\lambda(t)\cdot\tilde{x} = (y_0,\ldots,y_n)$$

where $y_i = x_i$ if $r_i = -\mu(x,\lambda)$ and $y_i = 0$ otherwise. Therefore

$$\lambda(t)\cdot\tilde{y} = t^{-\mu(x,\lambda)}\tilde{y},$$

and $\mu(x,\lambda) = \mu(y,\lambda)$.

□

Proposition 3.4.9. If λ is a one-parameter subgroup of G and $X \subset \mathbb{P}^n$, then we have

$$\mu(x,\lambda) = \mu^{\mathcal{O}(1)}(x,\lambda),$$

where in the right is the Hilbert-Mumford weight associated to the line bundle $L = \mathcal{O}(1)$ over $\mathbb{P}^n$.

Proof. Let $\tilde{y}$ be a point over $y = \lim_{t\to 0} \lambda(t)\cdot x$, so that the $\lambda(t)\cdot\tilde{y} = t^{-\mu(x,\lambda)\tilde{y}}$, as seen in the previous proposition. Using the dual bundle of $L = \mathcal{O}(1)$, denoted by $\mathcal{O}(-1)$ and corresponding to the projection $\mathbb{A}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$, we see that $-\mu(x,\lambda)$ is the weight of the associated $\lambda(\mathbb{G}_m)$–action the fiber $\mathcal{O}(-1)_y$.

Thus the dual action of λ over $\mathcal{O}(1)$ is given by the weight $\mu(x,\lambda)$, which coincides with the Hilbert Mumford weight by definition. $\square$

We can deduce the following lemma, from the properties of the weight proved in 3.4.8.

Lemma 3.4.10. Let λ be a one-parameter subgroup of G and $x \in X(k)$. We diagonalize the $\lambda(\mathbb{G}_m)$–action on the affine cone as above and let $\tilde{x} = (x_0, \ldots, x_n)$ be a non-zero lift of x in these coordinates.

(i) $\mu(x,\lambda) < 0 \iff \tilde{x} = \sum_{r_i>0} x_i e_i \iff \lim_{t\to 0} \lambda(t)\cdot\tilde{x} = 0.$

(ii) $\mu(x,\lambda) = 0 \iff \tilde{x} = \sum_{r_i\geq 0} x_i e_i$ and there exists $r_i = 0$ such that $x_i \neq 0 \iff \lim_{t\to 0} \lambda(t)\cdot\tilde{x} = 0$ exists and is non-zero.

(iii) $\mu(x,\lambda) > 0 \iff \tilde{x} = \sum_{r_i} x_i e_i$ and there exists $r_i < 0$ such that $x_i \neq 0 \iff \lim_{t\to 0} \lambda(t)\cdot\tilde{x} = 0$ does not exist.

We can also use the duality

$$\lim_{t\to 0} \lambda^{-1}(t)\cdot\tilde{x} = \lim_{t\to\infty} \lambda(t)\cdot\tilde{x}$$

to get the corresponding assertions for this case:

(i) $\mu(x,\lambda^{-1}) < 0 \iff \tilde{x} = \sum_{r_i>0} x_i e_i \iff \lim_{t\to\infty} \lambda(t)\cdot\tilde{x} = 0.$

(ii) $\mu(x,\lambda^{-1}) = 0 \iff \tilde{x} = \sum_{r_i\geq 0} x_i e_i$ and there exists $r_i = 0$ such that $x_i \neq 0 \iff \lim_{t\to\infty} \lambda(t)\cdot\tilde{x} = 0$ exists and is non-zero.

(iii) $\mu(x,\lambda^{-1}) > 0 \iff \tilde{x} = \sum_{r_i} x_i e_i$ and there exists $r_i < 0$ such that $x_i \neq 0 \iff \lim_{t\to\infty} \lambda(t)\cdot\tilde{x} = 0$ does not exist.

Locally, any limit point in the closure of the orbit can be identified as a limit of a one-parameter subgroup, by the next theorem.

Theorem 3.4.11. *Let G be a reductive group acting on $\mathbb{A}^n$, $z \in \mathbb{A}^n$ be a closed point. If 0 lies in the orbit of z, then there is a one-parameter subgroup of G*

$$\lambda : \mathbb{G}_m \to G$$

such that $\lim_{t \to 0} \lambda(t) \cdot z = 0.$

Proof. **Step 1.** There is an irreducible curve $C_1 \subset G \cdot z$ which contains 0 in its closure inside $\mathbb{A}^n$.

Consider an embedding $\mathbb{A}^n \subset \mathbb{P}^n$, let $p \in \mathbb{P}^n$ denote the point corresponding to 0 inside $\mathbb{P}^n$ and let Y denote the closure of the orbit $G \cdot z$ inside $\mathbb{P}^n$.

If Y is already a curve, then the claim is trivial, since we only need to remove the points in the boundary $Z \doteq Y \setminus G \cdot z$ to get an irreducible curve inside $G \cdot z$ which contains 0 in its closure inside $\mathbb{A}^n$.

Now, suppose $d \doteq \dim Y > 1$ and $n > 1$. Since hyperplanes in $\mathbb{P}^n$ are given as zero-sets of homogeneous degree one polynomials, we can consider the space

$$\mathcal{H} \doteq \mathbb{P}(k[x_0, \dots, x_n]_1)$$

parametrizing hyperplanes inside $\mathbb{P}^n$, and the subset $\mathcal{H}_p$ of hyperplanes containing the point p, which corresponds to a closed condition and thus it is a closed codimension one subspace inside this space. Moreover, we can define a set

$$\mathcal{H}_0 \doteq \left\{ (H_1, \dots, H_{d-1}) \in \prod_{i=1}^{d-1} \mathcal{H}_p : \dim Y \cap \bigcap_{i=1}^{d-1} H_i = 1 \text{ and } Y \cap \bigcap_{i=1}^{d-1} H_i \cap Y/Z \neq \varnothing \right\}$$

which is open, since both conditions are open, and non-empty, of dimension $(n-1)(d-1) > 0$. Hence we can construct the desired curve C_1 as

$$C_1 \doteq \bigcap_{i=1}^{d-1} H_i \cap Y \cap G \cdot z$$

for any choice of $(H_1, \dots, H_{d-1}) \in \mathcal{H}_0$.

Step 2. There is a smooth projective curve C over k, a rational map $p : C \dashrightarrow G$ and a k–point $c_0 \in C$ such that

$$\lim_{c \to c_0} p(c) \cdot z = 0.$$

To build this curve, we consider the action morphism

$$\begin{aligned} \sigma_z : G &\to \mathbb{A}^n \\ g &\mapsto \sigma(g, z), \end{aligned}$$

and show that there is a curve $C_2 \subset G$ dominating $C_1 \subset G \cdot z$ under the action morphism $\sigma_z : G \to G \cdot z$, so the diagram

$$\begin{tikzcd} C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] \\ G \arrow[r, "\sigma_z"] & G \cdot z \end{tikzcd}$$

commutes.

Let η be the generic point of C_1. We can pick the geometric point $\overline{\eta}$ over η which corresponds to a choice of an algebraically closed finite field extension of $k(C_1) \subset \overline{k(C_1)}$, so we denote by $\sigma_z^{-1}(C_1)_\eta$ and $\sigma_z^{-1}(C_1)_{\overline{\eta}}$ the base changes of C_1 of the preimage of $k(C_1)$ and $\overline{k(C_1)}$, respectively.

By the first step, there is a curve $C_2' \subset \sigma_z^{-1}(C_1)_{\overline{\eta}}$, as this is a closed point over $\overline{k(C_1)}$. The curve C_2' maps to a curve $C_2 \subset \sigma_z^{-1}(C_1)_\eta$ under the finite map

$$\sigma_z^{-1}(C_1)_{\overline{\eta}} \to \sigma_z^{-1}(C_1)_\eta,$$

so, by construction, C_2 is a curve in $\sigma_z^{-1}(C_1) \subset G$ which dominates C_1 under σ_z. Now, let C be a projective completion of the normalization of C_2, with the normalization map

$$C \dashrightarrow \tilde{C}_2 \to C_2.$$

Then the desired rational map $p : C \dashrightarrow G$ is induced by the composition

$$C \dashrightarrow \tilde{C}_2 \to C_2 \to G.$$

As the morphism $\tilde{C}_2 \to C_1$ is a composition of dominant maps, it is also dominant and it can be extended to their smooth projective completions. Moreover, by construction $0 \in \overline{C}_1$ and thus there is a preimage $c_0 \in C$ of zero under this extension, so

$$\lim_{c \to c_0} p(c) \cdot z = \lim_{c \to c_0} \sigma_z(p(c)) = 0,$$

using the base change.

Step 3. Since C is a smooth proper curve over k, the completion of the local ring $\mathcal{O}_{C,c_0}$ is isomorphic to the formal power series ring $k[[t]]$, whose field of fractions we denote by $k((t))$. As the rational map $p : C \dashrightarrow G$ is defined in a punctured neighbourhood of c_0, it induces a morphism

$$q : K \doteq \operatorname{Spec} k((t)) \simeq \operatorname{Spec} \operatorname{Frac} \hat{\mathcal{O}}_{C,c_0} \to \operatorname{Spec} \operatorname{Frac} \mathcal{O}_{C,c_0} \to G,$$

satisfying

$$\lim_{t \to 0} [q(t) \cdot z] = 0.$$

Furthermore, if $R \doteq \operatorname{Spec} k[[t]]$ and $K \doteq \operatorname{Spec} k((t))$, then there is a natural morphism $K \to R$, induced by the localization, which induces an inclusion of groups

$$G(R) = \operatorname{Hom}(R, G) \to \operatorname{Hom}(K, G) = G(K),$$

and the inclusion $k \subset R$ induces a morphism $G(R) \to G(k)$ given by taking the specialization as $t \to 0$.

Let $\rho : K \to \mathbb{G}_m$ be induced by

$$\begin{aligned} k[s, s^{-1}] &\mapsto k((t)) \\ s &\mapsto t, \end{aligned}$$

so that we have the map

$$\begin{aligned} \mathrm{Hom}(\mathbb{G}_m, G) &\to G(K) \\ \lambda &\mapsto \lambda \circ \rho \end{aligned}$$

which takes a one-parameter subgroup λ of G to the element $\langle \lambda \rangle \doteq \lambda \circ \rho$, called the **Laurent series expansion** of λ.

Now, by Iwahori's theorem (see (MUMFORD; FOGARTY; KIRWAN, 1994), 2.1), every double coset of the group $G(K)$ with respect to the subgroup $G(R)$ is represented by a point of the type $\langle \lambda \rangle$ for some one-parameter subgroup $\lambda : \mathbb{G}_m \to G$.

Step 4. Applying Iwahori's theorem for the morphism $q \in G(K)$ constructed in Step 3, there are two elements $l_1, l_2 \in G(R)$ and a one-parameter subgroup λ such that

$$l_1 \cdot q = \langle \lambda \rangle \cdot l_2,$$

and λ must be non-trivial since q cannot be written as a R–valued point of G. Let $g_i \doteq l_i(0)$, so we can write

$$0 = g_1 \cdot 0 = \lim_{t \to 0} l_1(t) \cdot \lim_{q \to 0} (q(t) \cdot z) = \lim_{t \to 0} \left[(\langle \lambda \rangle \cdot l_2(t)) \cdot z \right].$$

Since $l_2 \in G(R)$ and $g_2 = \lim_{t \to 0} l_2(0)$,

$$l_2(t) \cdot z = g_2 \cdot z + \varepsilon(t)$$

where $\varepsilon(t) \in k[t]$ is a polynomial involving only strictly positive powers of t. Using the weight-space decomposition (2.3.3) of the action of λ on $V = \mathbb{A}^n$, we write

$$g_2 \cdot z + \varepsilon(t) = \sum_{r \in \mathbb{Z}} [(g_2 \cdot z)_r + (\varepsilon(t))_r],$$

and since $\lim_{t \to 0} [(\langle \lambda \rangle \cdot l_2)(t) \cdot z] = 0$, it follows that

$$0 = \lim_{t \to 0} [(\langle \lambda \rangle \cdot l_2)(t) \cdot z] = \lim_{t \to 0} \left(\sum_{r \in \mathbb{Z}} a_r t^r (g_2 \cdot z)_r + a_r t^r (\varepsilon(t)) \right),$$

where almost all $a_i = 0$, so $(g_2 \cdot z)_r = 0$ for all $r \leq 0$. Hence

$$\lim_{t \to 0} (\langle \lambda \rangle \cdot g_2 \cdot z) = 0,$$

and the constructed one-parameter subgroup $\lambda' \doteq g_2^{-1} \lambda g_2$ satisfies $\lim \lambda'(t) \cdot z = 0$, as we wanted. $\square$

Proposition 3.4.12. Let G be a reductive group over k acting linearly on a projective scheme $X \subset \mathbb{P}^n$ and let $x \in X(k)$. Then

1. x is semistable $\Rightarrow \mu(x, \lambda) \geq 0$ for every one-parameter subgroup λ.
2. x is stable $\Rightarrow \mu(x, \lambda) > 0$ for every one-parameter subgroup λ.

Proof. Since the limit points for the action are in the closure of the orbit $\overline{\mathbb{G}_m \cdot x}$, this follows from 3.4.10, by the topological criterion given in 3.4.1. If x is semistable, we cannot have $0 \in \overline{G \cdot x}$, and this would be the case only if $\mu(x, \lambda) < 0$ for the one-parameter subgroup λ constructed in the previous theorem.

Moreover, if x is stable and $\mu(x, \lambda) = 0$, the orbit $G \cdot \tilde{x}$ is closed and the limit $y = \lim_{t \to 0} \lambda(t) \cdot \tilde{x}$ will be inside the boundary of $\overline{G \cdot \tilde{x}} = G \cdot \tilde{x}$, but as y is a fixed point for $\lambda(\mathbb{G}_m)$, it will satisfy $\dim G_y \geq \dim \mathbb{G}_m = 1$, and this contradicts the stability of x. □

Lemma 3.4.13. Fix $x \in X$ and let λ some one-parameter subgroup of G. The function

$$\begin{aligned} \mu^{\bullet}(x, \lambda) : \mathrm{Pic}^G(X) &\to \mathbb{Z} \\ L &\mapsto \mu^L(x, \lambda) \end{aligned}$$

defines a morphism of groups.

Proof. To see this, let $L_1, L_2 \in \mathrm{Pic}^G(X)$ be two linearizations for the G–action on X. As seen in 3.3.12, the tensor product $L_1 \otimes L_2$ admits a natural linearization, and using the explicit description in 3.3.13, we get that the linearization is the tensor product of the linearizations and thus, if

$$\mu^{L_1}(x, \lambda) = r_1 \text{ and } \mu^{L_2}(x, \lambda) = r_2,$$

then $\mu^{L_1 \otimes L_2}(x, \lambda) = r_1 \cdot r_2$, since the action of λ on fibers of $L_1 \otimes L_2$ is given as a product of the actions in L_1 and L_2. □

We can use the Hilbert-Mumford weight to characterize the stable and semistable points, by the following theorem:

Theorem 3.4.14. *[Hilbert-Mumford criterion] Let G be a reductive group acting on a proper k–scheme X. Let L be a ample linearization. Then, if $x \in X(k)$,*

$$x \in X^{ss}(L) \iff \mu^L(x, \lambda) \geq 0 \textit{ for every one-parameter subgroup } \lambda$$

and

$$x \in X^{s}(L) \iff \mu^L(x, \lambda) > 0 \textit{ for every one-parameter subgroup } \lambda$$

Proof. As L is ample, there is $n > 0$ such that $L^{\otimes n}$ is very ample. Then, by 3.4.13, we have

$$\mu^{L^{\otimes n}}(x,\lambda) = n \cdot \mu(x,\lambda),$$

so it suffices to prove the statement when L is very ample. In this case, then L induces a G–equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*\mathcal{O}_{\mathbb{P}^n}(1)$, and

$$\mu^L(x,\lambda) = \mu^{\mathcal{O}(1)\big|_X}(x,\lambda) = \mu(x,\lambda),$$

so we can use 3.4.12 for direct implications. For converse implications, if every one-parameter subgroup λ satisfies $\mu^L(x,\lambda) \geq 0$, we can use the theorem 3.4.11 to conclude that x has to be semistable, since it implies $0 \notin \overline{G \cdot x}$.

Moreover, if every one-parameter subgroup λ satisfies $\mu^L(x,\lambda) > 0$, then there is no limit point and thus the orbit must be closed, since any point in the closure $\overline{G \cdot x}$ is a local limit of one-parameter subgroups, by 3.4.11. □

This numerical criterion will be very useful for verifying (semi)stability conditions in examples.

3.5 Applications

In this section, we comment how to apply the methods of GIT to some classical examples of moduli spaces. In the next chapter, we study more closely a specific example, the construction of a moduli space for vector bundles over smooth algebraic curves.

Example 3.6 (Grassmanian)**.** Let $G(r,n)$ be the grassmanian variety of r-dimensional linear subspaces in the affine space $\mathbb{A}^n$. The reductive group $\mathbb{SL}_n(k)$ acts naturally on $G(r,n)$ via its linear representation in the affine n–space.

Given $W \in G(r,n)$, we can choose a basis $(v_1,\cdots,v_r)$ of W to define its **Plucker embedding**:

$$\begin{aligned} P : G(k,n) &\to \mathbb{P}(\bigwedge^k(\mathbb{A}^n)) \\ W = \langle v_1,\ldots,v_r\rangle &\mapsto [v_1 \wedge \cdots \wedge v_r]. \end{aligned}$$

Taking the standard basis of k^n, we can write:

$$v_1 \wedge \ldots \wedge v_r = \sum_{0 \leq i_1 < \cdots < i_r \leq n} p_{i_1,\cdots,i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}$$

where the coefficients $p_{i_1,\cdots,i_r}$ can be taken as the projective coordinates of $G(r,n) \subset \mathbb{P}^N \doteq \mathbb{P}(\bigwedge^k(\mathbb{A}^n))$ via the embedding. It is sometimes convenient to represent W also by a $r \times n$ matrix (associated to the projection onto W)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & \cdots & a_{rn} \end{pmatrix},$$

so the Plucker coordinates $p_{i_1,\cdots,i_r}$ are the maximal minors $A_{i_1,\cdots,i_r}$ of this matrix, formed by choosing the columns $A_{i_1}, \cdots, A_{i_r}$.

Using this representation, two matrices A, A' represent the same subspace if and only if there is a change of coordinates $C \in \mathrm{GL}(r)$ such that $A' = CA$. Note also that the matrix A is of maximal rank, since it's rows are linearly independent, hence we can view elements of $G(r, n)$ as $r \times n$ matrices of rank r, up to the action of $GL(r)$.

Let us consider the space of all $r \times n$ matrices as the affine space $\mathbb{A}^{rn}$ and let $L = \mathbb{A}^{rn} \times \mathbb{A}^1$ be the trivial line bundle. We can define a $\mathrm{GL}_r(k)$–linearization of L explicitly as the action:

$$C \cdot (A, t) \mapsto (CA, \det C \cdot t),$$

where $C \in \mathrm{GL}(r)$, $A \in \mathbb{A}^{rn}$ and $t \in \mathbb{A}^1 = k$.

We can also consider the functions $p_{i_0,\cdots,i_r}$ as invariant sections of this bundle, and this means matrices $A \in \mathbb{A}^{rn}$ of maximal rank are semistable points with respect to this linearization, since there will be a $\mathrm{GL}(r)$–invariant section of L which doesn't vanish at A, and as the group $\mathrm{GL}(r)$ acts freely on the open subset $\{p_{i_1,\cdots,i_r} \neq 0\}$, these are stable points. Thus we can view $G(r, n)$ as an open subvariety of the geometric quotient:

$$G(r, n) \simeq (\mathbb{A}^{rn})^s(L)/\mathrm{GL}(r).$$

Example 3.7. In this example, we consider the classification of projective hypersurfaces of a fixed degree $d \geq 1$ inside the projective space $\mathbb{P}^n$, up to a projective change of coordinates.

Note that we can determine any homogeneous polynomial $f \in k[x_0, \cdots, x_n]_d$ completely by a choice of $N \doteq \binom{n+d}{d}$ parameters, up to multiplication by $k^\times$, so we can consider the parameter space of this moduli problem as the projective space

$$Y_{n,d} \doteq \mathbb{P}_k^{N-1}.$$

Any one-parameter subgroup of $\mathrm{SL}(n+1)$ is conjugated to a one-parameter subgroup of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & & \\ & t^{r_1} & & \\ & & \cdots & \\ & & & t^{r_n} \end{pmatrix}$$

where r_i are integers such that $\sum_i r_i = 0$, and $r_0 \geq r_1 \geq \cdots \geq r_n$. Then the action of λ is diagonal with respect to the basis of the affine cone $\mathbb{A}^N$ over $Y_{d,n}$ given by the monomials

$$x_I = x_0^{i_0} \cdots x_n^{i_n},$$

for $I = (i_0, \cdots, i_n)$ a tuple of non-negative integers which add up to d. Thus, the weight of each monomial x_I for the action λ is given as

$$\mu(x_I, \lambda) = -\sum_{j=0}^{n} r_j i_j.$$

Now, if $F = \sum_I a_I x_I \in k[x_0, \ldots, x_n] \setminus \{0\}$ is a homogeneous degree d polynomial, where $I = (i_0, \ldots, i_n)$ are tuples of non-negative integers which add up to d, we let $p_F \in Y_{d,n}$ be the corresponding equivalence class. Then, we compute

$$\begin{aligned} \mu(p_F, \lambda) &= -\min\left\{ -\sum_{j=0} r_j i_j : I = (i_0, \ldots, i_n) \text{ and } a_I \neq 0 \right\} \\ &= \max\left\{ \sum_{j=0} r_j i_j : I = (i_0, \ldots, i_n) \text{ and } a_I \neq 0 \right\}. \end{aligned}$$

The **resultant polynomial** of a collection of polynomials is a function in the coefficients of these polynomials which vanishes if and only if these polynomials have a common root. For a polynomial $F \in k[x_0, \cdots, x_n]_d$, we define the **discriminant** $\Delta(F)$ of F to be the resultant of the set

$$S = \left\{ \frac{\partial F}{\partial x_i} : 0 \leq i \leq n \right\}.$$

Then, Δ is a homogeneous polynomial in $\mathcal{O}(Y_{n,d})$, and is non-zero at F if and only if F defines a smooth hypersurface.

We also know that F defines a smooth variety if and only if g^*F defines a smooth variety, whenever g is a projective change of coordinates. This means in particular that $\Delta \in \mathcal{O}(Y_{n,d})^{\mathrm{SL}(n+1)}$. If $d = 1$, $Y_{1,n} \simeq \mathbb{P}^n$ and the only SL_{n+1}–invariant homogeneous polynomials are the constant functions, and this also implies that there are no semistable points for the action of SL_{n+1}.

Since $\Delta(F) \neq 0$ if F is smooth, we have the following:

Proposition 3.5.1. For $d > 1$, every smooth degree d hypersurface in $\mathbb{P}^n$ is semistable for the action of SL_{n+1} on $Y_{d,n}$.

For more on the moduli space of hypersurfaces using GIT, we refer the reader to (HOSKINS, 2015) and (DOLGACHEV, 1994).

4 Vector Bundles over a curve

Introduction

In this chapter, we develop the moduli space of μ−(semi)stable vector bundles over a curve, following (HOSKINS, 2015) and (NEWSTEAD, 2012).

We assume familiarity with basic homological algebra of sheaves, and denote by $\mathrm{VectBun}(X)$ the category of vector bundles over a scheme X, by $\mathrm{Loc}(X)$ the category of locally free sheaves over X and by $\mathrm{Coh}(X)$ the category of coherent sheaves over X. We also use the anti-equivalence between the categories $\mathrm{VectBun}(X)$ and $\mathrm{Loc}(X)$ (see, for example, (HARTSHORNE, 1977), Exercise 5.18).

After a review of the basic theory of coherent sheaves over smooth algebraic curves (4.1 and 4.2), we study the concept of **slope** and μ−(semi)stability of a vector bundle (4.3), to introduce the associated moduli problem in 4.4. Afterwards, we use Grothendieck's Quot scheme (see 4.5) as a parameter space, and outline the GIT construction for the moduli space of μ-(semi)stable vector bundles.

4.1 Coherent sheaves over curves

Throughout this section, we use basic constructions in algebraic geometry and homological algebra (see (HARTSHORNE, 1977), Chapter II and III).

Let X be a smooth projective curve over k. If $\mathcal{F}$ is a coherent sheaf over X, its **Euler characteristic** is given by $\mathcal{X}(\mathcal{F}) = h^0(X,\mathcal{F}) - h^1(X,\mathcal{F})$. The **genus** of X is by definition the natural number $g \doteq h^1(X,\mathcal{O}_X)$. As we are working with smooth projective curves, the following groups are isomorphic

$$\mathrm{Pic}(X) \simeq \mathrm{Cl}(X) \simeq \mathrm{CaCl}(X),$$

where we denote by $\mathrm{Pic}(X)$ the Picard group of X, by $\mathrm{Cl}(X)$ the group of classes of Weil divisors and by $\mathrm{CaCl}(X)$ the group of classes of Cartier divisors. We define the **support** of $\mathcal{F}$ as the set

$$\mathrm{supp}\,\mathcal{F} \doteq \{x \in X : \mathcal{F}_x \neq 0\}.$$

Definition 4.1.1. Let $K_0(X)$ be the free group generated by coherent sheaves $[\mathcal{E}]$ on X, subjected to the relations

$$[\mathcal{E}] - [\mathcal{F}] + [\mathcal{G}] = 0$$

whenever there is a short exact sequence of coherent sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0.$$

The group $K_0(X)$ is called the **Grothendieck Group** associated to X.

Theorem 4.1.2 ((HARTSHORNE, 1977), III, Exercise 6.9)**.** *If $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ admits a finite* ***locally free resolution****, i.e., there is an exact sequence*

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

where each sheaf $\mathcal{E}_i$ is a locally free sheaf.

We use the existence of finite locally free resolutions to extend the usual notions for locally free sheaves of rank, denoted by rk, the determinant bundle, denoted by det, to a morphism of groups $(\det, \mathrm{rk}) : K_0(X) \to \mathrm{Pic}(X) \oplus \mathbb{Z}$. Whenever $\mathcal{E}$ is a locally free sheaf over X, we send the class $[\mathcal{E}]$ to the well-defined pair $(\det \mathcal{E}, \mathrm{rk}\, E)$. Using the previous theorem, whenever $\mathcal{F}$ is a coherent sheaf, we can consider a finite locally free resolution of $\mathcal{F}$ and use the relations defining the group $K_0(X)$.

Definition 4.1.3. If D is a divisor on X, the **degree** of the corresponding invertible sheaf is defined as $\deg(\mathcal{O}_X(D)) \doteq \deg D$. If $\mathcal{E}$ is a locally free sheaf over X, we set $\deg \mathcal{E} \doteq \deg(\det \mathcal{E})$, which is well defined since $\det \mathcal{E} \in \mathrm{Pic}(X)$. If $\mathcal{F}$ is a coherent sheaf over X, we set $\deg(\mathcal{F}) \doteq \deg(\det[\mathcal{F}])$ using the map det as defined over $K_0(X)$. Since we define this using the relations inside the group $K_0(X)$, the degree deg is also additive on exact sequences of coherent sheaves.

If D is an effective Weil divisor on X, we can write

$$D = \sum_{i=1}^{n} n_i \cdot x_i$$

for finite points $x_i \in X$. Then, we can consider the corresponding closed (possibly non-reduced) subscheme of X given by the union of each point x_i with its multiplicity n_i, denoted $D \subset X$. We use divisors over X to study some examples of coherent sheaves over X which are not locally free.

Example 4.1. Given any point $x \in X$, we can consider the divisor $D = -x$ and let k_x be the skyscraper sheaf given as

$$k_x(U) \doteq \begin{cases} k, & x \in U \\ 0, & \text{otherwise.} \end{cases}$$

whenever $U \subset X$ is open. Note that the sections of the corresponding invertible sheaf $\mathcal{O}_X(-x)$ are sections of $\mathcal{O}_X$ which vanish at x. Furthermore, we can form the following exact sequence:

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0,$$

and this is a finite locally free resolution of the coherent sheaf k_x, so we can compute

$$\deg(\mathcal{O}_X(-x)) = \deg(D) = -1$$

and $\deg(\mathcal{O}_X) = 0$, thus $\deg(k_x) = 1$. Since $H^0(X, k_x) = \Gamma(X, k_x) = k$ and $H^1(X, k_x) \simeq 0$, by the dimension of $\operatorname{supp} k_x$, we also have $\mathcal{X}(k_x) = 1$.

Example 4.2. Given any invertible sheaf $\mathcal{L}$ over X, and $x \in X$, we can tensor the previous exact sequence by $\mathcal{L}$ and get

$$0 \to \mathcal{L}(-x) \to \mathcal{L} \to k_x \to 0,$$

where $\mathcal{L}(-x)$ is the sheaf of sections of $\mathcal{L}$ vanishing at x. Since the Euler characteristic is additive on exact sequences, in this case we get

$$\mathcal{X}(\mathcal{L}) = \mathcal{X}(\mathcal{L}(-x)) + \mathcal{X}(k_x) = \mathcal{X}(\mathcal{L}(-x)) + 1.$$

Example 4.3. If $D = n \cdot x$, for $n \geq 0$, we can consider the sheaf k_D given by:

$$k_D(U) = \begin{cases} k^n, & x \in U \\ 0, & \text{otherwise.} \end{cases}$$

This coherent sheaf will also satisfy the exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(n \cdot x) \to k_D \to 0,$$

where the corresponding morphism:

$$\mathcal{O}_X \to \mathcal{O}_X(n \cdot x)$$

is given by multiplication by the factor $(t - x)^n$. Taking stalks, we get the exact sequence:

$$0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_x^n \to 0,$$

where the last vector space is n-dimensional over k. Note that, for any divisor $D \geq 0$, we can define k_D as the cokernel:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to k_D \to 0.$$

If $D = n_k x_1 + \cdots n_k x_k$, locally in each stalk $x = x_k$ we have a map $x \mapsto x^{n_k}$, and the resulting sheaf k_D will also be a skyscraper sheaf, with global sections given by

$$H^1(X, k_D) = k^{n_1} \oplus \cdots \oplus k^{n_k} = k^{\deg D}.$$

Theorem 4.1.4 (Riemann Roch, version I)**.** *Let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on X. Then*

$$\mathcal{X}(\mathcal{O}_X(D)) = \mathcal{X}(\mathcal{O}_X) + \deg D.$$

Proof. We can generically write D as a formal sum

$$x_1 + \cdots + x_n - y_1 - \cdots - y_m$$

where $x_i, y_i \in X$. We will prove the theorem by induction on $l = n + m$. If $l = 0$, $D = 0$ and the statement is trivial.

Let us suppose that we proved this for $l = n + m$. If

$$D = x_1 + \cdots + x_n - y_1 - \cdots - y_m + x_{n+1},$$

by the induction hypothesis the divisor $D - x_{n+1}$ satisfies:

$$\mathcal{X}(\mathcal{O}_X(D - x_{n+1})) = \mathcal{X}(\mathcal{O}_X) + \deg(D - x_{n+1}) = \mathcal{X}(\mathcal{O}_X) + \deg(D) - 1,$$

and on the other hand, if $\mathcal{L} = \mathcal{O}_X(D)$, we get

$$\mathcal{X}(\mathcal{O}_X(D - x_{n+1})) = \mathcal{X}(\mathcal{L}(-x_{n+1})) = \mathcal{X}(\mathcal{L}) - 1,$$

by the formula of the previous examples. Thus

$$\mathcal{X}(\mathcal{O}_X(D)) - 1 = \mathcal{X}(\mathcal{O}_X) + \deg(D) - 1.$$

The other possibility is $D = x_1 + \cdots + x_n - y_1 - \cdots - y_m - y_{m+1}$. In this case, we can argue similarly, letting $E = x_1 + \cdots + x_n - y_1 - \cdots - y_m$, so that $D = E - y_{m+1}$ and $\mathcal{O}_X(D) = \mathcal{O}_X(E - y_{m+1})$. Moreover, by the induction hypothesis, we can write

$$\mathcal{X}(\mathcal{O}_X(E)) = \mathcal{X}(\mathcal{O}_X) + \deg(E).$$

On the other hand,

$$\mathcal{X}(\mathcal{O}_X(E)) = \mathcal{X}(\mathcal{O}_X(E - y_{m+1}) + 1 \Rightarrow \mathcal{X}(\mathcal{O}_X(D)) = \mathcal{X}(\mathcal{O}_X(E)) - 1$$

and $\deg(D) = \deg(E) - 1$, so $\mathcal{X}(\mathcal{O}_X(D)) = \mathcal{X}(\mathcal{O}_X) + \deg(D)$, as we wanted to show. □

The sheaf of differentials $\omega_X = \Omega^1_X$ over X is called the **canonical sheaf** over X.

Theorem 4.1.5 (Serre duality for Curves)**.** *Let X be a smooth projective curve of genus and $\mathcal{E}$ a locally free sheaf. There exists a natural perfect pairing:*

$$H^0(X, \mathcal{E}^\vee \otimes \omega_X) \times H^1(X, \mathcal{E}) \to k.$$

Hence, $H^0(X, \mathcal{E}^\vee \otimes \omega_X) = H^1(X, \mathcal{E})^\vee$ and $h^0(X, \mathcal{E}^\vee \otimes \omega_X) = h^1(X, \mathcal{E})$.

For a proof of the Serre duality, see for example ((HARTSHORNE, 1977) III Theorem 7.6).

Remark 4.1.6. We will also need a more homological formulation of Serre duality for Grothendieck's classification of vector bundles over $\mathbb{P}^1$. There are natural isomorphisms

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{G})^\vee \simeq \mathrm{Ext}^0(\mathcal{G},\mathcal{F}\otimes\omega_X) = \mathrm{Hom}(\mathcal{G},\mathcal{F}\otimes\omega_X)$$

whenever $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves over X. The previous version of Serre duality follows from this one, as the functor $\mathrm{Hom}(\mathcal{O}_X,-)$ is naturally isomorphic to $\Gamma(X,-)$.

Theorem 4.1.7 (Riemann-Roch, version II)**.** *Let X be a smooth projective curve of genus g and $\mathcal{L}$ be a degree d invertible sheaf on X. Then:*

$$h^0(X,\mathcal{L}) - h^0(X,\mathcal{L}^\vee\otimes\omega_X) = d+1-g.$$

Proof. First, we can compute:

$$\mathcal{X}(\mathcal{O}_X) = h^0(X,\mathcal{O}_X) - h^1(X,\mathcal{O}_X) = 1-g.$$

Then, by Serre duality and the previous version of Riemann-Roch, it follows that:

$$h^0(X,\mathcal{L}) - h^0(X,\mathcal{L}^\vee\otimes\omega_X) = \mathcal{X}(\mathcal{L}) = d + \mathcal{X}(\mathcal{O}_X) = d+1-g.$$

□

Example 4.4. As an immediate application of the Riemann-Roch theorem, we compute the degree of the canonical bundle on a curve X of genus g:

$$h^0(X,\omega_X) - h^1(X,\omega_X) = g-1 = \deg\omega_X + 1 - g.$$

Thus, $\deg\omega_X = 2g-2$.

Example 4.5. If $g = 1$, then $\deg\omega_X = 0$, and whenever D is an effective divisor over X, we can write $h^0(\mathcal{O}_X(D)) = \deg(D) + h^0(\mathcal{O}_X(-D)$, but as $\deg D > 0$, we conclude $h^0(\mathcal{O}_X(-D)) = 0$, and hence $h^0(\mathcal{O}_X(D)) = \deg D$.

Proposition 4.1.8. Let $\mathcal{E}$ be a locally free sheaf of rank r over X and D an effective divisor with $r\cdot\deg(D) > h^1(\mathcal{E})$. Then the vector bundle $\mathcal{E}(D)$ admits a global section.

Proof. Consider the following exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to k_D \to 0,$$

where k_D is the torsion sheaf defined in 4.1. We can tensor this by the sheaf $\mathcal{E}$, to get the exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(D) \to k_D\otimes\mathcal{E} \to 0.$$

Taking the long exact sequence in cohomology, we get

$$\begin{aligned}0 \to H^0(X,\mathcal{E}) \to H^0(X,\mathcal{E}(D)) \to H^0(X,k_D \otimes \mathcal{E}) \to \\ \to H^1(X,\mathcal{E}) \to H^1(X,\mathcal{E}(D)) \to H^1(X,k_D \otimes \mathcal{E}) \to 0.\end{aligned}$$

Using the description of the sheaf k_D, we have that $H^0(X,\mathcal{E}\otimes k_D) \simeq k^{r\cdot \deg D}$, and $H^1(X,\mathcal{E}\otimes k_D) = 0$. Thus, the above is an exact sequence, and we can use it to compute the Euler characteristic

$$\begin{aligned}0 &= h^0(\mathcal{E}) - h^0(\mathcal{E}(D)) + r\cdot \deg(D) - h^1(\mathcal{E}) + h^1(\mathcal{E}(D)) \\ &= \mathcal{X}(\mathcal{E}) - \mathcal{X}(\mathcal{E}(D)) + r\cdot \deg(D).\end{aligned}$$

To show $\mathcal{E}(D)$ admits a global section, we only need to verify that $h^0(\mathcal{E}(D)) > 0$. Using the hypothesis, we write $\mathcal{X}(\mathcal{E}(D)) - \mathcal{X}(E) = r\cdot\deg(D) > h^1(\mathcal{E})$, so that $h^0(\mathcal{E}(D)) > h^1(\mathcal{E}(D)) + h^0(\mathcal{E}) \geq 0$, and we are done. □

Remark 4.1.9. Using the previous proposition, we conclude the locally free $\mathcal{E}$ has an invertible subsheaf $\mathcal{L}\subset\mathcal{E}$, obtained by tensoring the morphism in $\mathrm{Hom}(\mathcal{O}_X,\mathcal{E}(D))$ corresponding to the global section via the natural isomorphism $\mathrm{Hom}(\mathcal{O}_X,-)\simeq\Gamma(X,-)$.

Proposition 4.1.10. If $\mathcal{L}$ is an invertible sheaf on X with $\deg\mathcal{L} > 2g-2$, then $h^1(\mathcal{L}) = 0$.

Proof. Since $\deg(\omega_X) = 2g-2 \geq 0$ and the degree defines a morphism between $\mathrm{Pic}(X)$ and $\mathbb{Z}$, then

$$\deg(\mathcal{L}^\vee\otimes\omega_X) = \deg(\mathcal{L}^\vee) + \deg(\omega_X) < -(2g-2) + (2g-2) = 0,$$

so the line bundle $\mathcal{L}^\vee\otimes\omega_X$ will not have any global sections over the integral scheme X (see, for example (LIU; ERNE, 2006), Chapter 7, 3.25 (b)), and $h^0(\mathcal{L}^\vee\otimes\omega_X) = 0$, so by Serre duality it follows that $h^1(\mathcal{L}) = 0$. □

Since $h^1(\mathcal{L}) = 0$, such $\mathcal{L}$ is generated by its global sections. In particular, if an invertible sheaf $\mathcal{L}$ on a curve X has sufficiently high degree (depending on the genus) we can compute the dimension of the space of global sections of $\mathcal{L}$ easily using the Riemann-Roch theorem:

$$h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

Using this and the previous results, we get the following corollary:

Corollary 4.1.10.1. If $\mathcal{L}\subset\mathcal{E}$ is an invertible subsheaf of a locally free sheaf of rank r, then the quantity $\deg(\mathcal{L})$ is bounded above.

Proof. Suppose, on the contrary, there is an invertible sheaf $\mathcal{L}$ of degree large enough so $\deg(\mathcal{L}) > 2g-2$ and $r\cdot\deg(\mathcal{L}) > h^1(\mathcal{E})$.

Then, by the previous results, we have the cohomology conditions $h^0(\mathcal{E}) > 0$ and $h^1(\mathcal{L}) = 0$, so $h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$. Rearranging, we get

$$\deg(\mathcal{L}) = h^0(\mathcal{L}) - 1 + g \leq h^0(\mathcal{E}) - 1 + g,$$

so there is an upper bound which does not depend on $\mathcal{L}$, a contradiction on our hypothesis. □

Theorem 4.1.11. *Let $\mathcal{E}$ be a locally free sheaf of rank r over X. Then, there is a short exact sequence of locally free sheaves over X:*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

such that $\mathcal{L}$ is an invertible sheaf and $\mathcal{F}$ has rank $r - 1$.

Proof. Let $\mathcal{L} \subset \mathcal{E}$ be an invertible sheaf of maximal degree in $\mathcal{E}$, and consider the quotient sheaf $\mathcal{F} \doteq \mathcal{E}/\mathcal{L}$. There is an effective divisor D such that $\mathcal{L}(D) \subset \mathcal{E}(D)$ is generated by its global sections, since $h^1(\mathcal{L}(D))$ will eventually be zero by 4.1.10. Thus, we can consider the stalks in

$$\mathcal{F}(D)_x = \frac{\mathcal{E}(D)_x}{\mathcal{L}(D)_x}$$

locally as sections of $\mathcal{E}(D)_x$ without the fixed generating section $\mathcal{L}(D)_x$, which implies that $\mathcal{F}(D)_x$ is a free $\mathcal{O}_{X,x}$–module. Tensoring back by the locally free sheaf $\mathcal{O}_X(-D)$, we get that $\mathcal{F}$ is locally free. □

4.2 Vector bundles and torsion sheaves

The category of locally free sheaves over a scheme X is usually not an abelian category. This can be seen when we try to take (co)kernels of morphisms in this category. For an example, see the cokernel k_D of a morphism between locally free sheaves in 4.1, which is not locally free.

The objective of this section is to manage this problem using the fact that we are over a smooth algebraic curve X over an algebraically closed field k. The solution (see 4.2.8) involves operations on arrows between categories

$$R = K, I : \mathrm{Coh}(X) \to \mathrm{Loc}(X)$$

so the diagram

$$\begin{tikzcd} \mathrm{Loc}(X) \arrow[r, dashed] \arrow[d] & \mathrm{Loc}(X) \arrow[d] \\ \mathrm{Coh}(X) \arrow[r, "(co)kernel"] & \mathrm{Coh}(X) \arrow[u, bend right, "R"'] \end{tikzcd}$$

commutes, where the downwards vertical arrows denote the inclusion $\mathrm{Loc}(X) \hookrightarrow \mathrm{Coh}(X)$ as a full subcategory, so the operations K, I behave like kernels and cokernels, respectively. To better understand the inclusion $\mathrm{Loc}(X) \hookrightarrow \mathrm{Coh}(X)$ in this context, we appeal to the algebra of the corresponding modules.

Definition 4.2.1. If A is a ring, an A–module M is said to be **torsion-free** if, for every $m \in M$ and $r \in A$ regular, $m \cdot r \neq 0$.

Note that, if A is a domain, then we must have $m \cdot r \neq 0$ whenever $m \in M$, for M to be a torsion-free module over A. Moreover, when A is a principal ideal domain, we can use a stronger classification result:

Theorem 4.2.2 ((JACOBSON, 2009), 3.8)**.** *If M is a finitely generated module over a principal ideal domain A, then there is a unique decreasing sequence of proper ideals:*

$$(d_n) \subset (d_{n-1}) \subset \cdots \subset (d_1)$$

such that

$$M \simeq \frac{A}{(d_1)} \oplus \cdots \oplus \frac{A}{(d_n)}.$$

The largest free submodule of M is represented in this composition by choosing factors with $d_i = 0$. Since A is a principal ideal domain, there exists $t \in A$ such that $A/(t)$ is the torsion submodule of M, and we can write

$$M \simeq A/(t) \oplus F,$$

where F is a free module over A. Moreover, we say that a module M is **torsion** if M is isomorphic to its torsion submodule.

We can also interpret this theorem in our geometric context, since when X is an irreducible smooth projective curve, $\mathcal{O}(X)$ is a P.I.D., as every prime ideal must be maximal by the dimension condition.

Proposition 4.2.3. If X is a smooth irreducible projective curve, then

1. Any torsion-free sheaf over X is locally free.
2. Any subsheaf of a locally free sheaf over X is locally free.

Proof. The proof of (1) follows locally from the algebraic discussion above. Moreover, a subsheaf of a locally free sheaf over X cannot have a torsion part, and thus will also be locally free. $\square$

Since we can take a torsion submodule of a general module M, if $\mathcal{F}$ is a sheaf, we define the **torsion-subsheaf** of $\mathcal{F}$ to be the corresponding subsheaf, which we denote by $T(\mathcal{F})$, and when $T(\mathcal{F}) \simeq \mathcal{F}$, we say $\mathcal{F}$ is a **torsion sheaf**.

If $\mathcal{F}$ is a locally free sheaf over X, with an associated vector bundle F and $\mathcal{E} \subset \mathcal{F}$ is a subsheaf, then the inclusion of stalks $\mathcal{E}_x \to \mathcal{F}_x$ is injective whenever $x \in X$. However, the map on the fibers of the associated vector bundles $E_x \to F_x$ is not necessarily injective, as E_x is obtained by tensoring $\mathcal{E}_x$ with the residue field $k(x) \simeq k$, which may not be an exact functor.

Proposition 4.2.4. If $\mathcal{E} \subset \mathcal{F}$ are two locally free sheaves over X such that the quotient $\mathcal{G} \doteq \mathcal{F}/\mathcal{E}$ is torsion free over k, then E is a subbundle of F.

Proof. Since $\mathcal{G}$ is torsion free, it is locally free and thus we can take G as the associated vector bundle. We have the following short exact sequence on stalks:

$$0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0,$$

which can be tensored with the residue field to recover the fibers of the associated vector bundles, and we have the associated long exact sequence

$$\cdots \to \operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{G}_x) \to E_x \to F_x \to G_x \to 0.$$

By hypothesis, $\mathcal{G}_x$ is torsion-free over k, which means $\operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{G}_x) = 0$ and thus the sequence

$$0 \to E_x \to F_x \to G_x \to 0$$

is exact whenever $x \in X$, so that E is a subbundle of F. □

Proposition 4.2.5. Let $\mathcal{T}$ be a torsion coherent sheaf over X. Then $\mathcal{T}$ has finite support, and we set $l(\mathcal{T}) \doteq |\operatorname{supp} \mathcal{T}|$, counted with multiplicity, the **length** of $\mathcal{T}$.

Proof. As freeness of coherent sheaves is local, we conclude that $\mathcal{T}$ must be trivial on an open dense subset of X. Thus, $\operatorname{supp} \mathcal{T}$ is a closed subset of X, and this means $\operatorname{supp} \mathcal{T}$ is finite, as X is Noetherian. □

Definition 4.2.6. For a coherent sheaf $\mathcal{E}$ over a projective scheme Y with a fixed ample invertible sheaf $\mathcal{L}$, the **Hilbert polynomial** of $\mathcal{E}$ with respect to $\mathcal{L}$ is a polynomial $P(\mathcal{E}, \mathcal{L}) \in \mathbb{Q}[t]$ such that for $l \in \mathbb{N}$ sufficiently large,

$$P(\mathcal{E}, \mathcal{L})(l) = \mathcal{X}(\mathcal{E} \otimes \mathcal{L}^{\otimes l}) = \sum_{i \geq 0} (-1)^i \dim H^i(Y, \mathcal{E} \otimes \mathcal{L}^{\otimes l}).$$

By Serre's vanishing cohomology theorem (see (HARTSHORNE, 1977), III, Theorem 5.2), for sufficiently large $l \in \mathbb{N}$, all higher cohomology groups of $\mathcal{E} \otimes \mathcal{L}^{\otimes l}$

vanish. Hence, for l sufficiently large, $P(\mathcal{E},\mathcal{L})(l) = h^0(Y,\mathcal{E}\otimes\mathcal{L}^{\otimes l})$. There is also a more explicit form of the Hilbert polynomial, related to the degree of coherent sheaves (compare (HARDER; DIEDERICH, 2011), p. 157). If we denote by $d \doteq \dim\operatorname{supp}\mathcal{F}$, we can write

$$\mathcal{X}(Y,\mathcal{F}(r)) = \frac{\deg\mathcal{F}}{d(\mathcal{F})} r^{d(\mathcal{F})} + c_1 r^{d(\mathcal{F})-1} + \ldots + c_d(\mathcal{F}),$$

where c_i are rational coefficients varying with $\mathcal{F}$. In particular, when $\mathcal{T}$ is a torsion sheaf on a curve X, then we get equalities $\mathcal{X}(X,\mathcal{T}) = h^0(\mathcal{T}) = \deg(\mathcal{T})$.

Remark 4.2.7. For a proof of the existence of the Hilbert polynomial in the general case, see for example (HUYBRECHTS; LEHN, 2010).

Definition 4.2.8. Let $\mathcal{E}\subset\mathcal{F}$ be locally free sheaves, and E and F denote the corresponding vector bundles. Then the **vector subbundle of** F **generically generated by** E is a vector bundle $\overline{E}$ of F which is the vector bundle associated to the locally free sheaf:

$$\overline{\mathcal{E}} \doteq \pi^{-1}(T(\mathcal{F}/\mathcal{E})),$$

where $\pi : \mathcal{F}\to\mathcal{F}/\mathcal{E}$ denotes the quotient, and this can be done since the sheaf

$$\frac{\mathcal{F}}{\mathcal{E}}/T\left(\frac{\mathcal{F}}{\mathcal{E}}\right)$$

is, by definition, torsion-free. This gives E as a subbundle of F.

As we discussed in 4.2.3, the sheaf $\mathcal{F}/\mathcal{E}$ splits in a direct sum of its torsion part $\mathcal{T} = T(\mathcal{F}/\mathcal{E})$ and its torsion-free part, which we denote by Q. We can form a diagram with exact rows

$$\begin{tikzcd}
 & & & \mathcal{T} \arrow[d] & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, dashed] & \mathcal{F} \arrow[r, "\pi"] \arrow[d, equal] & \mathcal{F}/\mathcal{E} \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r, "q\circ\pi"] & Q \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

where $\mathcal{E}' \doteq \ker(q\circ\pi)$ coincides with $\overline{\mathcal{E}}$, by our construction. Since q is an epimorphism and in the middle we have an equality, by the Snake lemma there is a monomorphism $\mathcal{E}\to\overline{\mathcal{E}}$ completing the diagram, i.e., such that its cokernel is $\mathcal{T}$. Thus, we got the following exact sequence:

$$0\to\mathcal{E}\to\overline{\mathcal{E}}\to\mathcal{T}\to 0,$$

and this means that $\mathcal{X}(\overline{\mathcal{E}}) = \mathcal{X}(\mathcal{E}) + \mathcal{X}(\mathcal{T})$. On the other hand, as $\mathcal{T}$ is a torsion sheaf, it has dimension zero and $h^1(\mathcal{T}) = 0$, thus $\mathcal{X}(\overline{\mathcal{E}}) = \mathcal{X}(\mathcal{E}) + h^0(\mathcal{T})$, and since $\dim X = 1$, $h^0(\mathcal{T})$ gives the number of points in the support of $\mathcal{T}$, counted with multiplicity, so $l(\mathcal{T}) = h^0(\mathcal{T})$.

Remark 4.2.9. When D is a divisor over X and $\mathcal{L} \doteq \mathcal{O}_X(-D) \subset \mathcal{O}_X$ and the associated vector bundles denoted by $L' \subset L$, then the vector subbundle of L generically generated by L' is $\overline{L'} = L$.

Definition 4.2.10. Let $f : E \to F$ be a morphism of vector bundles. We define

1. The vector subbundle $K(f)$ of E, which is generically generated by the kernel $\ker f$ and satisfies $\operatorname{rk} K(f) = \operatorname{rk} \ker f$ and $\deg K(f) \geq \deg \ker f$.
2. The vector subbundle $I(f)$ of F, which is generically generated by the image $\operatorname{Im} f$ and satisfies $\operatorname{rk} I(f) = \operatorname{rk} \operatorname{Im} f$ and $\deg I(f) \geq \deg \operatorname{Im} f$.

For the rest of the section, we derive another Riemann-Roch formula, for locally free sheaves over X, which will be useful for the next sections.

Theorem 4.2.11. *If $\mathcal{E}$ and $\mathcal{F}$ are locally free sheaves over X, then*

$$\deg(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rk} \mathcal{E} \deg \mathcal{F} + \operatorname{rk} \mathcal{F} \deg \mathcal{E}.$$

Proof. We proceed by induction on the rank of $\mathcal{E}$. If $\operatorname{rk} \mathcal{E} = 1$, we can write $\mathcal{E} = \mathcal{O}_X(D)$ for a divisor D over X. When D is effective, we have the short exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to k_D \to 0,$$

as in 4.1, so tensoring with $\mathcal{F}$, we get the exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}(D) \to \mathcal{F} \otimes k_D \to 0,$$

which can also be regarded as a finite locally free resolution for the torsion sheaf $\mathcal{F} \otimes k_D$, so $\deg(\mathcal{F}(D)) = \deg(\mathcal{F} \otimes k_D) + \deg(\mathcal{F})$. On the other hand, $\mathcal{F} \otimes k_D$ is a torsion sheaf, thus $\deg(\mathcal{F} \otimes k_D) = \dim H^0(X, \mathcal{F} \otimes k_D) = \operatorname{rk} \mathcal{F} \cdot \deg D$. If D is a general divisor, we can write $D = D_1 - D_2$ where D_i are effective so that:

$$\deg(\mathcal{F} \otimes \mathcal{O}_X(D_1 - D_2)) = \deg(\mathcal{F}(-D_2)) + \operatorname{rk} \mathcal{F} \cdot \deg D_1,$$

but on the other hand

$$\deg \mathcal{F} = \deg(\mathcal{F}(-D_2) \otimes \mathcal{O}_X(D_2)) = \deg(\mathcal{F}(-D_2)) + \operatorname{rk} \mathcal{F} \cdot \deg(D_2).$$

Rearranging, we conclude that

$$\deg(\mathcal{F} \otimes \mathcal{O}_X(D)) = \deg(\mathcal{F}) + \operatorname{rk} \mathcal{F} \cdot (\deg(D_1) - \deg(D_2)) = \deg(\mathcal{F}) + \operatorname{rk} \mathcal{F} \cdot \deg(D)$$

This concludes the case $\mathrm{rk}\,\mathcal{E} = 1$. Now, we suppose that we proved the theorem for all locally free sheaves $\mathcal{E}, \mathcal{F}$ where $\mathrm{rk}\,\mathcal{E} = n-1$, and then we can do the same trick again, choosing a line bundle $\mathcal{L} \subset \mathcal{E}$ such that

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0$$

is an exact sequence of locally free sheaves and $\mathrm{rk}\,\mathcal{E}/\mathcal{L} = n-1$. Tensoring this sequence with the locally free sheaf $\mathcal{F}$ give us:

$$\deg(\mathcal{F} \otimes \mathcal{E}) = \deg(\mathcal{F} \otimes \mathcal{L}) + \deg(\mathcal{F} \otimes \mathcal{E}/\mathcal{L})$$

and we can apply the induction hypothesis to get the final result. □

Corollary 4.2.11.1 (Riemann-Roch, III)**.** Let X be a smooth projective curve of genus g and $\mathcal{F}$ be a locally free sheaf over X. Then

$$\mathcal{X}(\mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(1-g).$$

Proof. We prove this by induction on $\mathrm{rk}\,\mathcal{F}$ using the previous theorem. If $\mathcal{F}$ is an invertible sheaf, this is the previous form of the Riemann-Roch theorem in 4.1.7. We can then use the same argument of the existence of an invertible sheaf inside $\mathcal{F}$ to get$\mathcal{X}(\mathcal{F}) = \mathcal{X}(\mathcal{L}) + \mathcal{X}(\mathcal{F}/\mathcal{L})$ and applying the induction hypothesis, we conclude

$$\begin{aligned} \mathcal{X}(F) &= \mathcal{X}(\mathcal{L}) + \mathcal{X}(\mathcal{F}/\mathcal{L}) \\ &= \deg(\mathcal{L}) + (1-g) + \deg(\mathcal{F}/\mathcal{L}) + (\mathrm{rk}(\mathcal{F}) - 1)(1-g) \\ &= \deg(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(1-g), \end{aligned}$$

by the previous theorem. □

Remark 4.2.12. When X is a smooth curve and we are dealing with locally free sheaves, we can use the Riemann-Roch theorem to compute the Hilbert polynomial. We fix $\mathcal{L} = \mathcal{O}_X(1)$ an ample invertible sheaf, and whenever $\mathcal{E}$ is a locally free sheaf over X of rank n and degree d, the twist $\mathcal{E}(m)$ has rank n and degree $nm + d$, so we conclude

$$\mathcal{X}(\mathcal{E}(m)) = d + mn + n(1-g)$$

whenever $m \in \mathbb{Z}$. Thus $\mathcal{E}$ has Hilbert polynomial $P(t) = nt + d + n(1-g)$.

4.3 Stability and slope

In this section, we study of the category of vector bundles, following (HOSKINS, 2015), (NEWSTEAD, 2012) and (HUYBRECHTS; LEHN, 2010). The concept of $\mu-$stability introduces a partitioning of this category in a suitable way, so the moduli problem of fixed $\mu-$semistable vector bundles is bounded and it admits a moduli space. As before, we work over a smooth projective curve X over k, and denote its genus by g.

Definition 4.3.1. The **slope** of a coherent sheaf $\mathcal{E}$ over X is defined as the ratio

$$\mu(\mathcal{E}) \doteq \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}} \in \mathbb{Q}.$$

In particular, whenever $\mathcal{E}$ is locally free, and E is the corresponding vector bundle over X, we use the same definition.

Since the degree and the rank are additive on exact sequences of coherent sheaves, the slope is also additive, and whenever

$$0 \to E \to F \to G \to 0$$

is an exact sequence of vector bundles, we conclude that

(a) if two of the vector bundles E, F, G have the same slope, then all three have the same slope;

(b) $\mu(E) < \mu(F) \iff \mu(F) < \mu(G)$;

(c) $\mu(E) > \mu(F) \iff \mu(F) > \mu(G)$.

Definition 4.3.2. A vector bundle E is μ–**stable** (resp. μ–**semistable**) if every proper non-zero (usually called **non-trivial**) vector subbundle $S \subset E$ satisfies:

$$\mu(S) < \mu(E) \quad (\text{resp. } \mu(S) \le \mu(E)).$$

A vector bundle E is μ–**polystable** if it can be written as a direct sum of μ–stable vector bundles of same slope.

Remark 4.3.3. Using the properties of the slope whenever there is an exact sequence of vector bundles, we could also rephrase μ–semistability as follows: a vector bundle E is μ–semistable whenever every quotient bundle F satisfies $\mu(F) \ge \mu(E)$. Indeed, using 4.2.4, we conclude that there must be a non-trivial subbundle $H \subset E$ such that

$$0 \to H \to E \to F \to 0$$

is exact, so $\mu(H) \le \mu(E)$.

Remark 4.3.4. As observed in 4.2.3, every non-trivial subsheaf $\mathcal{F}$ of a locally free sheaf $\mathcal{E}$ over X is also locally free, and in particular corresponds to a non-trivial subbundle of the associated vector bundles $F \subset E$, we could also define μ–(semi)stability in terms of locally free sheaves.

We shall prove later that, whenever we fix suitable pairs (d, n) for degree and rank, respectively, the notion of μ–stability coincides with stability with respect to the GIT problem related to constructing the moduli space of vector bundles over X.

Proposition 4.3.5. Let L be a line bundle, and E be a vector bundle over X. Then L is $\mu-$(semi)stable, and whenever E is $\mu-$(semi)stable, the tensor product $E \otimes L$ is μ-stable.

Proof. If L is a line bundle, it is trivially $\mu-$semistable. For the second statement, let $F' \subset E \otimes L$ be a proper non-trivial subbundle. Then $F \doteq F' \otimes L^{-1}$ is a vector subbundle of E such that $F' = F \otimes L$. By the $\mu-$(semi)stability of E, we have $\mu(F')(\leq)\mu(E)$. On the other hand, using the formula 4.2.11, we can write

$$\mu(F \otimes L) = \frac{\deg(F \otimes L)}{\operatorname{rk} F \otimes L} = \frac{\deg(F) + \deg L \cdot \operatorname{rk} F}{\operatorname{rk} F} = \mu(F) + \deg(L)$$

so that

$$\mu(F') = \mu(F) + \deg(L)(\leq) < \mu(E) + \deg(L) = \frac{\deg(E \otimes L)}{\operatorname{rk} E} = \mu(E \otimes L)$$

and the claim follows. □

Lemma 4.3.6. Let $f : E \to F$ be a non-zero morphism of vector bundles over X. Then:

(a) If E and F are $\mu-$semistable, $\mu(E) \leq \mu(F)$.

(b) If E and F are μ-stable of the same slope, then f is an isomorphism.

(c) If E is μ-stable and F is $\mu-$semistable, $\mu(E) < \mu(F)$.

Proof. Let $G_1 \doteq K(f) \subset E$ and $G_2 \doteq I(f) \subset F$ subbundles generically generated by $\ker f$ and by $\operatorname{Im} f$, respectively, as in 4.2.10. We can write

$$\operatorname{rk} E = \operatorname{rk}(K(f)) + \operatorname{rk}(I(f)) = \operatorname{rk} G_1 + \operatorname{rk} G_2,$$

and because of the induced exact sequence of coherent sheaves

$$0 \to \ker f \to E \to \operatorname{Im} f \to 0,$$

we get

$$\begin{aligned} \deg(E) &= \deg(\ker f) + \deg(\operatorname{Im}(f)) \\ &\leq \deg G_1 + \deg G_2. \end{aligned}$$

Furthermore, if E and F are $\mu-$semistable, then $\mu(G_1) \leq \mu(E)$ and $\mu(G_2) \leq \mu(F)$, which means

$$\begin{aligned} \deg(E) &\leq \deg(G_1) + \deg(G_2) \\ &\leq \operatorname{rk}(G_1) \cdot \mu(E) + \operatorname{rk}(G_2) \cdot \mu(F). \end{aligned}$$

To see how this implies (a), lets suppose that $\mu(E) > \mu(F)$. Then

$$\begin{aligned}\deg(E) &\leq \mathrm{rk}(G_1)\mu(E) + \mathrm{rk}(G_2)\mu(F) \\ &< \mathrm{rk}(G_1)\mu(E) + \mathrm{rk}(G_2)\mu(E) = \deg(E),\end{aligned}$$

which is a contradiction. For (b), we suppose $\mu(E) = \mu(F)$, to get

$$\deg(E) \leq \mathrm{rk}(G_1)\mu(E) + \mathrm{rk}(G_2)\mu(E) = \mu(E) \cdot (\mathrm{rk}\, G_1 + \mathrm{rk}\, G_2) = \deg(E),$$

and equality occurs if and only if $\ker f = 0$ and $\mathrm{Im}\, f = F_2$, so the claim follows.

To prove (c), we proceed as in (a), but in this case as E is μ−stable we have the strict inequality $\mu(G_1) < \mu(E)$, so

$$\begin{aligned}\deg(E) &< \deg(G_1) + \deg(G_2) \\ &< \mathrm{rk}(G_1)\mu(E) + \mathrm{rk}\, G_2 \mu(F).\end{aligned}$$

If we suppose $\mu(F) \leq \mu(E)$, then we arrive at a similar contradiction

$$\deg(E) < \mathrm{rk}(G_1)\mu(E) + \mathrm{rk}(G_2)\mu(E) < \deg(E),$$

so $\mu(E) < \mu(F)$. □

Proposition 4.3.7. If F is a μ−semistable vector bundle over X such that $\mu(\mathcal{F}) < 0$, then $H^0(X, F) = 0$.

Proof. We denote by $\mathcal{F}$ the corresponding locally free sheaf over X, and suppose that $H^0(X, \mathcal{F}) \neq 0$. Then, the existence of a non-zero global section $s \in H^0(X, \mathcal{F})$ implies the existence of a non-zero morphism

$$\mathcal{O}_X \to \mathcal{F},$$

given by the evaluation of this global section. Arguing by dimension, this must be an injection, which gives a non-zero invertible subsheaf $\mathcal{L} \subset \mathcal{F}$, with zero slope. On the other hand, as $\mu(\mathcal{F}) < 0$, and $\mathcal{F}$, we get $\mu(\mathcal{L}) = 0 > \mu(\mathcal{F})$, contradicting the μ−semistability of $\mathcal{F}$. □

Proposition 4.3.8. If there is an exact sequence of vector bundles over X of the form

$$0 \to E \to F \to G \to 0$$

where E and G are μ−semistable vector bundles of same slope, then F must be μ−semistable of the same slope.

Proof. The slope of F must coincide with $\mu \doteq \mu(E) = \mu(G)$, by the additivity of the slope function. To show F is μ–semistable, let $F' \subset F$ be a non-trivial subbundle of F. We let $E' \doteq E \cap F'$, considering E as a subsheaf of F, and denote by G' the image of F' inside G.

Using the exactness of the sequence, if $E' = 0$, then F' can be identified with a subbundle of G, and if $G' = 0$, then F' is contained in E'. In either case, we conclude $\mu(F') \leq \mu$.

Furthermore, as E and G are μ–semistable, $\mu(E') \leq \mu$ and $\mu(G') \leq \mu$, and we can form the following exact sequence

$$0 \to E' \to F' \to G' \to 0,$$

so $\mu(F') \leq \mu$, and the μ–semistability of F follows. □

Proposition 4.3.9. Every stable bundle E satisfies $\mathrm{Hom}(E, E) \simeq k$.

Proof. To see this, if E is stable and $h : E \to E$ is any morphism, we can choose any point $x \in X$ and $\lambda \in k$ an eigenvalue of the linear map $h_x : E_x \to E_x$.

By construction the map $h - \lambda \cdot \mathrm{Id}_E$ is not an isomorphism, so by the previous proposition it must be the zero morphism, and in particular $h = \lambda \cdot \mathrm{Id}_E$. □

Definition 4.3.10. Whenever $\mu \in \mathbb{Q}$, we denote by $\mathcal{C}(\mu)$ the category of μ–semistable vector bundles of slope μ.

Proposition 4.3.11. Whenever $\mu \in \mathbb{Q}$, $\mathcal{C}(\mu)$ is an abelian category.

Proof. First, using 4.3.8, we conclude that $\mathcal{C}(\mu)$ is an additive subcategory of the additive category of vector bundles. Thus, it remains to prove that every map admits a kernel and a cokernel. We let $f : E \to F$ be a morphism between semistable vector bundles of slope μ. If we consider the image of f in the category of coherent sheaves, it needs to be a non-trivial subsheaf of $\mathcal{F}$, the associated sheaf to F. As $\mathcal{F}$ is locally free, then the image of f is also locally free, and thus f needs to have constant rank, so kernels and cokernels are vector bundles. By the additivity of the slope, we conclude $\mu(\ker f) = \mu(\operatorname{coker} f) = \mu$.

Furthermore, as $\ker f \subset E$ is a subbundle, if it had another non-trivial subbundle $H \subset \ker f$ such that $\mu(H) > \mu(\ker f)$, then H would contradict the μ–semistability of E, and thus $\ker f$ must be μ–semistable. Similarly, the vector bundle $\operatorname{coker} f$ must be μ–semistable, since if not there would exist a locally-free quotient of $\operatorname{coker} f$ with slope strictly less then μ, but this would also contradict the semistability of F. □

Proposition 4.3.12. If the degree d and the rank n are coprime, then the concepts of μ–semistability and μ–stability are equivalent.

Proof. We proceed by induction on the rank n. Since every line bundle is trivially μ–semistable and μ–stable, the case $n = 1$ follows trivially. Now, let us suppose that, whenever $k \leq n$ and $d \in \mathbb{Z}$, if $(n, d) = 1$, then the concepts of μ–semistability and μ–stability coincide.

Let E be a μ–semistable vector bundle of rank $n+1$ and degree d, so that $n+1$ and d are coprime. If E is not μ–stable, then there is a non-trivial subbundle $F \subset E$ with $\mu(F) = \mu(E)$. If there is any non-trivial subbundle H of F with $\mu(H) > \mu(F)$, H would be a non-trivial subbundle of E with $\mu(H) > \mu(E)$, contradicting the μ–semistability. Thus, F is also μ–semistable.

As $\mu(F) = \mu(E) \in \mathbb{Q} \setminus \mathbb{Z}$, we conclude $\deg(F)$ and $\operatorname{rk} F$ are also coprime. Thus, by the induction hypothesis we conclude F is μ–stable.

As we proved in 4.3.11, the category $\mathcal{C}(\mu)$ is abelian, the cokernel must be a μ–semistable vector bundle Q with same slope, and we have the exact sequence:

$$0 \to F \to E \to Q \to 0.$$

Denoting by v_F, v_E and by v_Q the vectors of integer entries corresponding to the image of the pair function $(\operatorname{rk}, \deg)$ for each vector bundle E, F, Q, respectively, there must be positive integers $\lambda, \mu > 1$ such that

$$\begin{cases} \lambda v_F = v_E = v_F + v_Q \\ \mu v_Q = v_E = v_F + v_Q, \end{cases}$$

as their slopes coincide and the functions $(\operatorname{rk}, \deg)$ are additive on exact sequences. Thus, we conclude

$$\begin{cases} (\lambda - 1) v_F = v_Q \\ (\mu - 1) v_Q = v_F, \end{cases}$$

so we must have

$$\mu - 1 = \frac{1}{\lambda - 1},$$

contradicting the hypothesis that λ, μ are integers greater than one. □

In the case when the genus of X is zero, we have a simple description of all vector bundles over X, due to Grothendieck.

Theorem 4.3.13 (Grothendieck's Theorem)**.** *Let $\mathcal{E}$ be a locally free sheaf of rank r over $\mathbb{P}^1$. There exists a uniquely determined decreasing sequence of integers*

$$a_r \leq \cdots \leq a_1$$

such that

$$\mathcal{E} \simeq \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r).$$

Proof. We proceed by induction on the rank r. If $r = 1$, then $\mathcal{E}$ is invertible and since $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$, $E \simeq \mathcal{O}(a)$ for some integer $a \in \mathbb{Z}$. Now, we assume the theorem holds for all locally free sheaves of rank strictly less than r, and let E be a vector bundle of rank r.

By 4.1.11, there is an invertible subsheaf $\mathcal{O}(a) \subset \mathcal{E}$ such that the associated cokernel $\mathcal{F} \doteq \mathcal{E}/\mathcal{O}(a)$ is also locally free, of rank $r-1$. Let $a_1 \in \mathbb{Z}$ the largest integer with this property (since the degree of such line bundles is bounded 4.1.10.1, this is well defined). By the induction hypothesis, there is a decomposition

$$\frac{\mathcal{E}}{\mathcal{O}(a_1)} \simeq \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r),$$

where $a_r \leq \cdots \leq a_2$, and we have an exact sequence of locally free sheaves

$$0 \to \mathcal{O}(a_i) \to \mathcal{E} \to \bigoplus_{i=2}^{r} \mathcal{O}(a_i) \to 0.$$

Tensoring with the line bundle $\mathcal{O}(-a_1 - 1)$, we get

$$0 \to \mathcal{O}(-1) \to \mathcal{E}(-a_1 - 1) \to \bigoplus_{i=2}^{r} \mathcal{O}(a_i - a_1 - 1) \to 0.$$

Arguing by contradiction, if the sheaf $\mathcal{E}(-a_1 - 1)$ had a global section, this would induce a non-trivial morphism $\mathcal{O}(1 + a_1) \to \mathcal{E}$, namely the tensor product by this fixed global section. Thus, this gives a line bundle of larger degree inside $\mathcal{E}$, contradicting the maximality hypothesis on the integer $a_1 \in \mathbb{Z}$. Hence, this vector bundle cannot have global sections, and $H^0(X, \mathcal{E}(-1 - a_1)) = 0$.

By basic sheaf projective cohomology (see C.2.1), $H^1(X, \mathcal{O}(-1)) = 0$, and considering the long exact sequence in cohomology we get

$$0 = H^0(X, \mathcal{E}(-a_1 - 1)) \to H^0(X, \bigoplus_{i=2}^{r} \mathcal{O}(a_i - a_1 - 1)) \to H^1(X, \mathcal{O}(-1)) = 0,$$

so that

$$H^0(X, \bigoplus_{i=2}^{r} \mathcal{O}(a_i - a_1 - 1)) \simeq \bigoplus_{i=2}^{r} H^0(X, \mathcal{O}(a_i - a_1 - 1)) = 0.$$

This implies that the degree of each of these line bundles is negative, that is, $a_i < a_1 + 1$, so that $a_1 \geq a_2 \geq \cdots \geq a_r$.

To conclude, it only remains to show that the exact sequence splits. As in this case we have $\omega_X \simeq \mathcal{O}(-2)$, we can apply the Serre duality (4.1.6) in the following way:

$$\begin{aligned}
\mathrm{Ext}^1(\bigoplus_{i \geq 2} \mathcal{O}(a_i), \mathcal{O}(a_1))^\vee &\simeq \mathrm{Hom}(\mathcal{O}(a_1), \bigoplus_{i \geq 2} \mathcal{O}(a_i - 2)) \\
&\simeq \bigoplus_{i \geq 2} \mathrm{Hom}(\mathcal{O}(a_1), \mathcal{O}(a_i - 2)) = 0,
\end{aligned}$$

since $a_1 \geq a_i > a_i - 2$. This implies the splitting, by the properties of the group Ext^1 (see C.1.7), and thus the existence part of the theorem.

Until now, we showed that there are finite dimensional vector spaces V_a such that

$$\mathcal{E} \simeq \bigoplus_{a \in \mathbb{Z}} V_a \otimes_k \mathcal{O}(a)$$

and almost all V_a are zero. Thus, to show the uniqueness of this decomposition, we only need to prove that $\mathcal{E}$ completely determines the dimensions of each vector space V_a. To show this, we define a filtration of $\mathcal{E}$ in the following way: for every integer $b \in \mathbb{Z}$, let

$$H^0(\mathbb{P}^1, \mathcal{E}(b)) \otimes \mathcal{O}(-b) \to E$$

be the evaluation map, and $\mathcal{E}_b$ be its image. By the vanishing theorems over $\mathbb{P}^1$, we know that the sheaf $\mathcal{E}(b)$ has no global sections for negative values of b, and is globally generated for large values of b, so we can construct a finite decreasing filtration:

$$0 = \mathcal{E}_{-1} \subsetneq \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_N = \mathcal{E}$$

where

$$\mathcal{E}_b \simeq \bigoplus_{a \geq -b} V_a \otimes_k \mathcal{O}(a)$$

and this means that the dimension of V_a amounts exactly to the rank of the quotient

$$\dim(V_a) = \mathrm{rk}\left(\frac{\mathcal{E}_{-a}}{\mathcal{E}_{-a-1}}\right),$$

since this will be the only term added in the corresponding direct sums. □

In the following example, we produce two examples of rank 2 vector bundles over a curve of genus $g = 1$ which do not split into sums of line bundles.

Example 4.6. Let us consider the case when $g = 1$, and fix any point $x \in X$. As in 4.1, we consider the exact sequence

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0$$

of coherent sheaves over X, which induces a long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_X(-x)) \to H^0(X, \mathcal{O}_X) \to H^0(X, k_x) \to \\ \to H^1(X, \mathcal{O}_X(-x)) \to H^1(X, \mathcal{O}_X) \to H^1(X, k_x) = 0 \end{aligned}$$

and filling the gaps, as $g = 1$ we get $H^1(X, \mathcal{O}_X) \simeq k$, and can write the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(-x)) = 0 \to k \to k \to H^1(X, \mathcal{O}_X(-x)) \to k \to 0,$$

where the first equality is due to the fact that negative degree line bundles do not admit global sections over a curve. By exactness, we conclude the first non-zero map is an isomorphism, and moreover we completely determine the maps:

$$0 \to k \xrightarrow{\simeq} k \xrightarrow{0} H^1(X, \mathcal{O}_X(-x)) \xrightarrow{\simeq} k \to 0,$$

so in particular $H^1(X, \mathcal{O}_X(-x)) \simeq k$. Using the duality formula for locally free sheaves (see C.1.9.1), we conclude:

$$\operatorname{Ext}^1(\mathcal{O}_X(x), \mathcal{O}_X) \simeq \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}(-x)) \simeq H^1(X, \mathcal{O}(-x)) \simeq k,$$

as the functors $\operatorname{Hom}(\mathcal{O}_X, -)$ and $\Gamma(X, -)$ are naturally isomorphic. Moreover,

$$\operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X) \simeq k,$$

as $h^1(X, \mathcal{O}_X) = g = 1$.

The non-triviality of the groups $\operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$ and $\operatorname{Ext}^1(\mathcal{O}_X(x), \mathcal{O}_X)$ indicates the existence of two non-split exact sequences (see C.1.7), say

$$0 \to \mathcal{O}_X \to V_0 \to \mathcal{O}_X \to 0,$$
$$0 \to \mathcal{O}_X \to V_1 \to \mathcal{O}_X(x) \to 0,$$

and as both groups are isomorphic to k, the sheaves V_0 and V_1 are uniquely determined up to isomorphism. We also observe that both sheaves V_0 and V_1 are locally free, since if we fix a point $x \in X$ and consider the associated long exact sequence in stalks over x, we get an exact fragment:

$$0 = \operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{O}_X) \to \operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, V_{0x}) \to \operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{O}_X) = 0,$$

concluding V_0 must be torsion-free over k, and analogously V_1 is also torsion-free over k.

Using the additivity of the rank and degree on exact sequences, we conclude V_0 and V_1 have rank 2, and $\deg(V_1) = 1$, $\deg(V_0) = 0$. Moreover, by 4.3.8, we get that V_0 is a μ−semistable, and $\mu(V_0) = \mu(\mathcal{O}_X) = 0$. Furthermore, as we have an injection $\mathcal{O}_X \hookrightarrow V_0$, the vector bundle V_0 is not μ−stable.

Let us prove that V_1 is μ−stable. If not, then there is a line bundle $L \hookrightarrow V_1$ with $\mu(V_1) \le \mu(L)$, but as we computed $\mu(V_1)$, and as $\operatorname{rk} L = 1$, we must have that $\mu(L) \in \mathbb{Z}$, so $\mu(L) \ge 1$. Let us consider the induced morphism $\phi : L \to \mathcal{O}(x)$ in the diagram

$$\begin{tikzcd} & & L \arrow[d, hook] \arrow[dr, dashed, "\phi"] & & \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] & V_1 \arrow[r] & \mathcal{O}_X(x) \arrow[r] & 0. \end{tikzcd}$$

If $\mu(L) > 1$, then ϕ is zero, by 4.3.6, so the morphism ϕ must factor through the kernel $\mathcal{O}_X$, which is a contradiction by the same reasoning. On the other hand, if $\mu(L) = 1$, then ϕ must be an isomorphism, and in particular it induces a splitting for this exact sequence, contradicting our assumptions.

Example 4.7. Following the previous example, whenever $g \geq 1$, we conclude

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X) \simeq k^g,$$

so there is at least one non-split sequence of the form

$$0 \to \mathcal{O}_X \to V_0 \to \mathcal{O}_X \to 0,$$

and with the same reasoning we conclude that V_0 must be a μ−semistable, not μ−stable, rank 2 vector bundle over X.

For the rest of this section, we introduce some filtrations of vector bundles which will be used in the construction of the moduli space. For proofs, we refer the reader to (HUYBRECHTS; LEHN, 2010).

Definition 4.3.14. Let E be a vector bundle over X. A **Harder-Narasimhan filtration** of E is a chain of subbundles

$$0 = E^{(0)} \subsetneq E^{(1)} \subsetneq \cdots \subsetneq E^{(s)} = E$$

where the quotients $E_i \doteq E^{(i)}/E^{(i-1)}$ are μ−semistable vector bundles, with slopes

$$\mu(E_s) < \mu(E_{s-1}) < \cdots < \mu(E_1).$$

Theorem 4.3.15 (see (HUYBRECHTS; LEHN, 2010), Theorem 1.3.4)**.** *Let E be a vector bundle. Then E admits a unique Harder-Narasimhan filtration.*

Definition 4.3.16. Let X be a smooth projective curve and let E be a μ−semistable vector bundle of rank d. A **Jordan-Hölder filtration** of E is a chain of subbundles

$$0 = E^{(0)} \subsetneq E^{(1)} \subsetneq \ldots \subsetneq E^{(l)} = E$$

where the quotients $E_i = E^{(i)}/E^{(i-1)}$ are μ−stable vector bundles with same slope as E.

Using the properties of μ−semistability, we see that in particular the subbundles $E^{(i)}$ are also μ−stable, with same slope μ.

Proposition 4.3.17 (see (HUYBRECHTS; LEHN, 2010), Proposition 1.5.2)**.** Let E be a μ−semistable vector bundle over X. Then E admits a Jordan-Hölder filtration.

Unlike Harder-Narasimhan filtrations, Jordan-Hölder filtrations are not unique.

4.4 Study of the moduli functor

To describe the moduli functor for the classification problem of μ–(semi)-stable vector bundles, we fix discrete invariants (n,d) for rank and degree, and consider the functor of families $F^{(s)s}(d,n) : \mathrm{Sch} \to \mathrm{Sets}$, which takes a k–scheme S to the set of all flat coherent sheaves $\mathcal{E} \in \mathrm{Coh}(X \times S)$ such that $\mathcal{E}_s$ is locally free and μ–semistable over X, with $\mu(\mathcal{E}_s) = d/n$, whenever $s \in S$.

In this context, we will say two families $\mathcal{E}$ and $\mathcal{F}$ over S are **equivalent** if and only if there is an invertible sheaf $\mathcal{L}$ over S and an isomorphism $\mathcal{E} \simeq \mathcal{F} \otimes \pi_S^*\mathcal{L}$, where $\pi_S : X \times S \to S$ is the projection.

We denote the moduli functor associated to the functor of families $F^{(s)s}(d,n)$ by $\mathcal{M}^{(s)s}(d,n)$.

Proposition 4.4.1. If there is a semistable vector bundle over X with invariants (n,d) which is not polystable, then the moduli problem of semistable vector bundles $\mathcal{M}^{ss}(n,d)$ does not admit a coarse moduli space.

Proof. The existence of a semistable sheaf $\mathcal{F}$ on X which is not polystable is equivalent to the existence of a non-split short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are semistable vector bundles with the same slope as $\mathcal{F}$. This corresponds to a non-zero element of the group $\mathrm{Ext}^1_X(\mathcal{F}'',\mathcal{F}')$ (see C.1.7).

If we denote this extension as $v \neq 0$, since $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$ is a k–vector space, we could consider the line passing through v and the origin, which in turn corresponds to a family of sheaves $\mathcal{E} \in \mathrm{Coh}(X \times \mathbb{A}^1)$, such that $\mathcal{E}_1 \simeq v$ and $\mathcal{E}_0 \simeq \mathcal{F}'' \oplus \mathcal{F}'$, as the latter corresponds to the origin in $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$. Furthermore, scalar multiplication does not change extension classes inside $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$, so $\mathcal{E}_t \simeq \mathcal{E}_1 \simeq \mathcal{F}$ whenever $t \neq 0$.

Note that $\mathcal{E}$ is a family for the moduli problem $\mathcal{M}^{(s)s}(d,n)$ which exhibits the jump phenomenon (described in 1.2.5), and thus we conclude there is not a coarse moduli space. □

When the notions of μ–semistability and μ–stability coincide (see 4.3.12), this behaviour does not occur, and we are able to construct a coarse moduli space for the case when n and d are coprime. To do this using GIT, we need to find a scheme R of parameters, in which each point corresponds to an isomorphism class of μ–semistable vector bundles with fixed invariants (n,d) over X.

We can assume that the degree of the μ–semistable vector bundle is sufficiently large, as tensoring with line bundles does not change the μ–(semi)stability.

More explicitly, there is a natural isomorphism between the moduli functors:

$$-\otimes\mathcal{L}:\mathcal{M}^{ss}(n,d)\to\mathcal{M}^{ss}(n,d+ne)$$

whenever $\deg(\mathcal{L})=e$. In particular, we assume $d>n(2g-1)$, which is a choice justified by the following proposition.

Lemma 4.4.2. Let $\mathcal{F}$ be a locally free sheaf over X of rank n and degree $d>n(2g-1)$. If the associated vector bundle F is μ–semistable, then the following statements hold:

(a) $H^1(X,\mathcal{F})=0$;

(b) The sheaf $\mathcal{F}$ is generated by its global sections, i.e., the evaluation map

$$ev_{\mathcal{F}}:H^0(X,\mathcal{F})\otimes\mathcal{O}_X\to\mathcal{F}$$

is surjective.

Proof. **(a).** If $H^1(X,\mathcal{F})\neq 0$, by Serre duality (4.1.5), there is a non-zero morphism of sheaves $f:\mathcal{F}\to\omega_X$. Let $K(f)\subset F$ be the vector subbundle generically generated by the kernel of f, which is a vector subbundle of rank $n-1$ satisfying

$$\deg K(f)\geq\deg\ker f\geq\deg\mathcal{F}-\deg\omega_X=d-(2g-2),$$

by the existence of the exact sequence

$$0\to K(f)\to\mathcal{F}\to\omega_X\to 0.$$

Using the μ–semistability of $\mathcal{F}$, we have

$$\frac{d-(2g-2)}{n-1}\leq\mu(K)\leq\mu(F)=\frac{d}{n},$$

so $d\leq n(2g-2)$, contradicting our assumption on the degree of the sheaf $\mathcal{F}$.

(b). Let $x\in X$ and F_x denote the fiber of F at this point. As in 4.1, we can consider the exact sequence

$$0\to\mathcal{O}(-x)\to\mathcal{O}_X\to k_x\to 0.$$

Tensoring with $\mathcal{F}$, we get

$$0\to\mathcal{F}(-x)\to\mathcal{F}\to F_x\to 0,$$

where $F_x=\mathcal{F}\otimes k_x$ is the skyscraper sheaf with support at x with stalk F_x. To show $\mathcal{F}$ is generated by global sections, we only need to show that the induced map $H^0(X,\mathcal{F})\to H^0(X,F_x)=F_x$ is surjective, whenever $x\in X$. To prove this, we use the long exact sequence in cohomology associated with the previous short exact sequence

$$0\to H^0(X,\mathcal{F}(-x))\to H^0(X,\mathcal{F})\to H^0(X,F_x)\to H^1(X,\mathcal{F}(-x))\to\cdots$$

and we only need to show that $H^1(X, \mathcal{F}(-x)) = 0$. Since this is a twist by a line bundle, this does not change the μ–stability, and then $\mathcal{F}(-x)$ is also μ–semistable, of degree

$$\deg(\mathcal{F} \otimes \mathcal{O}(-x)) = d - n \cdot 1 > n(2g - 2),$$

and by **(a)** we have $H^1(X, \mathcal{F}(-x)) = 0$. □

Let $\mathcal{F}$ be a locally free sheaf over X of rank n and degree d satisfying (a) and (b) on the above lemma. By Riemann-Roch (4.2.11.1),

$$\mathcal{X}(\mathcal{F}) = d + n(1 - g) = \dim H^0(X, \mathcal{F}),$$

that is, the dimension of the 0th cohomology is fixed and equal to $N \doteq d + n(1 - g)$. Therefore, we can fix an isomorphism $H^0(X, \mathcal{F}) \simeq k^N$ and combine this with the evaluation map, to produce a surjection:

$$q_{\mathcal{F}} : \mathcal{O}_X^N = k^N \otimes \mathcal{O}_X \to \mathcal{F}$$

from a fixed trivial vector bundle over X. This is the same as choosing a global basis of generators. Motivated by this construction, in the next section we study the Quot scheme, the moduli space of such surjections.

4.5 Quot scheme

For this section, let Y be any projective scheme and $\mathcal{F}$ a coherent sheaf on Y. One can study the moduli problem of classifying quotients of the sheaf $\mathcal{F}$. More precisely, we consider surjective morphisms $q : \mathcal{F} \twoheadrightarrow \mathcal{E}$ up to the equivalence relationship:

$$(\mathcal{F} \xrightarrow{q} \mathcal{E}) \simeq (\mathcal{F} \xrightarrow{q'} \mathcal{E}') \iff \ker q = \ker q'$$

Equivalently, using the snake lemma, $q \simeq q'$ if and only if there is a sheaf isomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ commuting the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "q"] \arrow[d, equal] & \mathcal{E} \arrow[d, "\phi"] \\ \mathcal{F} \arrow[r, "q'"'] & \mathcal{E}' \end{tikzcd}$$

Definition 4.5.1. Let $\mathcal{F}$ be a coherent sheaf on Y. For any scheme S, we let $\mathcal{F}_S \doteq \pi_Y^* \mathcal{F}$ denote the pullback of $\mathcal{F}$ to $Y \times S$ via the projection $\pi_Y : Y \times S \to Y$. A **family of quotients of $\mathcal{F}$ over Y** is a surjective $\mathcal{O}_{Y \times S}$–linear morphism of sheaves over $Y \times S$, denoted by $q_S : \mathcal{F}_S \twoheadrightarrow \mathcal{E}$ such that $\mathcal{E}$ is flat over S. Two families $q_S : \mathcal{F}_S \to \mathcal{E}$ and $q'_S : S \to \mathcal{E}'$ are **equivalent** if $\ker q_S = \ker q'_S$.

Since flatness is preserved under base change, we can also pullback families. The moduli functor associated to this functor of families is called the **Quot functor**, and denoted by

$$\mathcal{Q}_Y(\mathcal{F}) : \mathrm{Sch} \to \mathrm{Set}\,.$$

Note that we can view this as the moduli problem of classifying subsheaves of $\mathcal{F}$ up to equality. We can also fix invariants and restrict the corresponding moduli functor.

Definition 4.5.2. For a fixed ample line bundle L on Y, we have a decomposition

$$\mathcal{Q}_Y(\mathcal{F}) = \bigsqcup_{p \in \mathbb{Q}[t]} \mathcal{Q}_Y^{p,L}(\mathcal{F})$$

into Hilbert polynomials $p \in \mathbb{Q}[t]$ taken with respect to L.

If $Y = X$ is a curve, we can also decompose by degree and rank:

$$\mathcal{Q}_X(\mathcal{F}) = \bigsqcup_{(n,d)} \mathcal{Q}_X^{(n,d)}(\mathcal{F}).$$

Theorem 4.5.3 (Grothendieck)**.** *Let Y be a projective scheme and L an ample invertible sheaf on Y. Then for any coherent sheaf $\mathcal{F}$ over Y and any polynomial $p \in \mathbb{Q}[t]$, the functor $\mathcal{Q}_Y^{p,L}(\mathcal{F})$ is represented by a projective scheme $\mathrm{Quot}_Y^{p,L}(\mathcal{F})$.*

Remark 4.5.4. The full proof of this theorem needs a lot of machinery from algebraic geometry, and can be seen in (FANTECHI; GOTTSCHE; ILLUSIE, 2005). We will sketch some aspects of this construction in this section.

Definition 4.5.5. A **Hilbert scheme** is a Quot scheme of the form $\mathrm{Quot}_Y^p(\mathcal{O}_Y)$, and represents the moduli functor that sends a scheme S to the set of all closed subschemes $Z \subset Y \times S$ that are proper and flat over S with the given Hilbert polynomial $p \in \mathbb{Q}[t]$.

We develop Mumford's theory of $m-$regularity of coherent sheaves, following (FANTECHI; GOTTSCHE; ILLUSIE, 2005), Chapter 2.

Definition 4.5.6. Let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}_k^n$. Given $m \in \mathbb{Z}$, we say $\mathcal{F}$ is $m-$**regular** if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ whenever $i \geq 0$.

Proposition 4.5.7. If $H = Z(l) \subset \mathbb{P}^n$ is a hyperplane not containing any associated points of $\mathcal{F}$, then we have an exact sequence

$$0 \to \mathcal{F}(m-i-1) \xrightarrow{\alpha} \mathcal{F}(m-i) \to \mathcal{F}_H(m-i) \to 0,$$

where α is given locally by the tensor product with the linear section l, whenever $i \geq 0$.

Proof. By hypothesis, as H does not contain any associated points of $\mathcal{F}$, the map α is injective, so taking the cokernel on the abelian category $\mathrm{Coh}(X)$ give us the desired exact sequence. □

Remark 4.5.8. Note that the sheaf $\mathcal{F}_H(m-i)$ restricts to the hyperplane $H \simeq \mathbb{P}^{n-1}$ as a coherent sheaf over $\mathbb{P}^{n-1}$.

Proposition 4.5.9. If $\mathcal{F}$ is $m-$regular over $\mathbb{P}^n$ and $H \subset \mathbb{P}^n$ is a hyperplane which does not contain any associated points for $\mathcal{F}$, then the restriction of $\mathcal{F}_H$ to $H \simeq \mathbb{P}^{n-1}$ is also $m-$regular.

Proof. This follows considering the long exact sequence in cohomology associated to the exact sequence of the previous proposition. We have

$$\cdots \to H^i(\mathbb{P}^n, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n, \mathcal{F}_H(m-i)) \xrightarrow{\delta} H^{i+1}(\mathbb{P}^n, \mathcal{F}(m-i-1)) \to \cdots,$$

and since $\mathcal{F}$ is $m-$regular, the cohomology groups of the left and the right vanish, implying $H^i(\mathbb{P}^n, \mathcal{F}_H(m-i)) = 0$, whenever $i \geq 0$. □

Lemma 4.5.10. Let $\mathcal{F}$ be a $m-$regular sheaf on $\mathbb{P}^n$. Then:

(a) $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ whenever $i \geq 0$ and $r \geq m-i$.

(b) The canonical map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+1))$$

is surjective whenever $r \geq m$.

(c) The sheaf $\mathcal{F}(r)$ is generated by global sections and all its higher cohomology groups vanish for $r \geq m$.

Proof. We proceed by induction on n. The case $n = 0$ is trivial for $(a), (b)$ and (c) as

$$\mathbb{P}^0_k = \operatorname{Proj} k[x_0] \simeq \operatorname{Spec} k.$$

Since k is algebraically closed, it is infinite and there must be a hyperplane $H \subset \mathbb{P}^n$ which does not intersects any associated point of $\mathcal{F}$, since there are only finitely many of these, as $\mathcal{F}$ is coherent over $\mathbb{P}^n$.

By the previous proposition, the restriction of $\mathcal{F}_H$ is also $m-$regular over $H \simeq \mathbb{P}^{n-1}$, and by induction hypothesis we prove it satisfies (a), (b) and (c).

To see (a) when $r = m-i$, $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ by definition of $m-$regularity. Now, we consider an induction on $r \geq m-i+1$. Using the exact sequence

$$H^i(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^i(\mathbb{P}^n, \mathcal{F}(r)) \to H^i(H, \mathcal{F}_H(r))$$

and the vanishing of cohomology groups $H^i(\mathbb{P}^n, \mathcal{F}(r-1)) = 0$ and $H^i(H, \mathcal{F}_H(r)) = 0$ by induction hypothesis on $r-1$ and $n-1$, respectively, we have $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$.

To see (b), consider the commutative diagram

$$\begin{tikzcd} & H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \arrow[r, "\sigma"] \arrow[d, "\mu"] & H^0(H, \mathcal{F}_H(r)) \otimes H^0(H, \mathcal{O}_H(1)) \arrow[d, "\tau"] \\ H^0(\mathbb{P}^n, \mathcal{F}(r)) \arrow[r, "\alpha"] & H^0(\mathbb{P}^n, \mathcal{F}(r+1)) \arrow[r, "\nu_{r+1}"] & H^0(H, \mathcal{F}_H(r+1)) \end{tikzcd}$$

where σ and ν_{r+1} are the morphisms induced by restriction maps, and the vertical maps are induced by the product. Since $\mathcal{F}$ is m−regular, by **(a)**, $H^1(\mathbb{P}^n, \mathcal{F}(r-1)) = 0$ whenever $r \geq m$, so the restriction map

$$\nu_r : H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(H, \mathcal{F}_H(r))$$

is surjective whenever $r \geq m$. The restriction morphism

$$\rho : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(H, \mathcal{O}_H(1))$$

is also surjective, so that the tensor product $\sigma = \nu_r \otimes \rho$ is surjective.

By the induction hypothesis on $n-1$, τ is a surjective map, so that $\tau \circ \sigma$ is surjective, and using the diagram we get that the map $\nu_{r+1} \circ \mu$ is surjective, which implies $H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = \mathrm{Im}(\mu) + \ker(\nu_{r+1})$. Since the bottom row is exact, we have $H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = \mathrm{Im}(\mu) + \mathrm{Im}(\alpha)$, but α is locally the tensor product of global sections of $\mathcal{F}(r)$ by the linear section l, so that $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\mu)$ and thus $H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = \mathrm{Im}(\mu)$, as we wanted to show.

Finally, to see **(c)**, we can iterate **(b)** so that the morphism

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+p))$$

is surjective whenever $r \geq m$ and $p \geq 0$. As we know, for p sufficiently large the sheaf $\mathcal{F}(r+p)$ is generated by its global sections, so that the evaluation morphism

$$H^0(X, \mathcal{F}(r+p)) \otimes \mathcal{O}_X \to \mathcal{F}(r+p)$$

is surjective, and since tensoring with $\mathcal{O}_X$ is exact, we get a surjection

$$H^0(\mathbb{P}^n, \mathcal{F}(r) \otimes \mathcal{O}_X(p)) \otimes \mathcal{O}_X \to \mathcal{F}(r) \otimes \mathcal{O}_X(p)$$

and tensoring back we get that $\mathcal{F}(r)$ is also generated by global sections whenever $r \geq m$.

The higher vanishing condition follows from (b). □

Theorem 4.5.11 (Mumford). *For integers $p, n \geq 0$ there is a polynomial $F_{p,n} \in \mathbb{Z}[t_0, \dots, t_n]$ such that, for any subsheaf*

$$\mathcal{F} \subset \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^n}$$

over $\mathbb{P}^n$, if the Hilbert polynomial of $\mathcal{F}$ is written as

$$\mathcal{X}(\mathcal{F}(r)) = \sum_{i=0}^{n} a_i \binom{r}{i},$$

then $\mathcal{F}$ is $F_{p,n}(a_0, \dots, a_n)$–regular.

Proof. As before, we proceed by induction on n. Let $n \geq 1$, and $H \subset \mathbb{P}^n$ be a hyperplane which does not contain any of the finitely many associated points of the cokernel sheaf

$$\mathcal{E} \doteq \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^n} / \mathcal{F},$$

so the Tor sheaf vanishes $\mathrm{Tor}_1(\mathcal{O}_H, \mathcal{E}) = 0$. Therefore, as the restriction to H can be seen as a base-change, the short exact sequence

$$0 \to \mathcal{F} \to \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E} \to 0$$

restricts to a short exact sequence

$$0 \to \mathcal{F}_H \to \bigoplus_{i=1}^{p} \mathcal{O}_H \to \mathcal{E}_H \to 0$$

over H. This shows that $\mathcal{F}_H$ is isomorphic to a subsheaf of $\bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^{n-1}}$ (under $H \simeq \mathbb{P}^{n-1}$), which is a basic step for the induction.

If $\mathcal{F}$ is non-zero, it is torsion-free and thus we can consider the exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\alpha} \mathcal{F} \to \mathcal{F}_H \to 0$$

where α is locally given by the multiplication by the linear section generating H. From the associated long exact sequence in cohomology, we get

$$\mathcal{X}(\mathcal{F}_H(r)) = \mathcal{X}(\mathcal{F}(r)) - \mathcal{X}(\mathcal{F}(r-1))$$

and we can write explicitly

$$\mathcal{X}(\mathcal{F}_H(r)) = \sum_{i=0}^{n} a_i \binom{r}{i} - \sum_{i=0}^{n} a_i \binom{r-1}{i} = \sum_{i=0}^{n} a_i \binom{r-1}{i-1}$$

so that

$$\mathcal{X}(\mathcal{F}_H(r)) = \sum_{i=0}^{n} a_i \binom{r-1}{i-1} = \sum_{j=0}^{n-1} b_j \binom{r}{j},$$

where $b_0, \dots, b_{n-1}$ have integral expressions over $a_0, \dots, a_n$.

By the induction hypothesis, there is a polynomial $F_{p,n-1}(t_0, \dots, t_{n-1})$ such that $\mathcal{F}_H$ is m_0−regular, where $m_0 = F_{p,n-1}(b_0, \dots, b_{n-1})$. Thus, we can change coordinates to get a polynomial $G \in \mathbb{Z}[t_0, \dots, t_n]$ such that $m_0 = G(a_0, \dots, a_n)$. For $m \geq m_0$, we get the long exact sequence

$$0 \to H^0(\mathcal{F}(m-1)) \to H^0(\mathcal{F}(m)) \xrightarrow{\nu_m} H^0(\mathcal{F}_H(m)) \to H^1(\mathcal{F}(m-1)) \to \cdots$$

which, by the vanishing conditions for m_0−regular sheaves in 4.5.10, induces isomorphisms

$$H^i(\mathcal{F}(m-1)) \xrightarrow{\simeq} H^i(\mathcal{F}(m)),$$

whenever $i \geq 2$. Furthermore, as we also have $H^i(\mathcal{F}(m)) = 0$ for sufficiently large m, these isomorphisms show that $H^i(\mathcal{F}(m)) = 0$ whenever $i \geq 2$ and $m \geq m_0 - 2$. The surjections $H^1(\mathcal{F}(m-1)) \twoheadrightarrow H^1(\mathcal{F}(m))$ show that the function $h^1(\mathcal{F}(m))$ is monotonically decreasing for $m \geq m_0 - 2$. Note that $h^1(\mathcal{F}(m-1)) \geq h^1(\mathcal{F}(m))$ for $m \geq m_0$, and equality holds if and only if the restriction

$$\nu_m : H^0(\mathcal{F}(m)) \to H^0(\mathcal{F}_H(m))$$

is surjective. As the restriction $\mathcal{F}_H$ is also m−regular, it follows from the proof of (c) in 4.5.10 that the restrictions $\nu_j : H^0(\mathcal{F}(j)) \to H^0(\mathcal{F}_H(j))$ are surjective whenever $j \geq m$, so that $h^1(\mathcal{F}(j-1)) = h^1(\mathcal{F}(j))$ for all $j \geq m$, but on the other hand $h^1(\mathcal{F}(j)) = 0$ for sufficiently large j and thus we get that the function $h^1(\mathcal{F}(m))$ is strictly decreasing for $m \geq m_0$, until it reaches zero. This implies that $H^1(\mathcal{F}(m)) = 0$ for $m \geq m_0 + h^1(\mathcal{F}(m_0))$.

To prove the theorem, we only need to find a bound for the quantity $h^1(\mathcal{F}(m_0))$ which does not depend on the sheaf $\mathcal{F}$. For this, we use that $\mathcal{F} \subset \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^n}$ and get

$$h^0(\mathcal{F}(r)) \leq p \cdot h^0(\mathcal{O}_{\mathbb{P}^n}(r)) = p \cdot \binom{n+r}{n},$$

using the usual description for sections of $\mathcal{O}_{\mathbb{P}^n}(r)$ as homogeneous degree r polynomials. Since $h^1(\mathcal{F}(m)) = 0$ for all $i \geq 2$ and $m \geq m_0 - 2$, we now get

$$h^1(\mathcal{F}(m_0)) = h^0(\mathcal{F}(m_0)) - \mathcal{X}(\mathcal{F}(m_0)) = p\binom{n+m_0}{n} - \sum_{i=0}^{n} a_i \binom{m_0}{i}$$

using our previous formulas, and therefore we can find a polynomial $P \in \mathbb{Z}[t_0, \dots, t_n]$ such that $h^1(\mathcal{F}(m_0)) \leq P(a_0, \dots, a_n)$, setting $m_0 = G(a_0, \dots, a_n)$, and therefore P does not depend on the sheaf $\mathcal{F}$ or on the field k. Moreover, since $h^1(\mathcal{F}(m_0)) \geq 0$, as it is the dimension of a k−vector space, we get

$$P(a_0, \dots, a_n) \geq 0$$

and that $H^1(\mathcal{F}(m)) = 0$ for $m \geq G(a_0, \dots, a_n) + P(a_0, \dots, a_n)$, which together with previous observations means that $\mathcal{F}$ is $P_{p,n}(a_0, \dots, a_n)$−regular, where $P_{p,n} = G + P$. □

Now, we give an overview of the steps of the construction of the Quot scheme as given in (FANTECHI; GOTTSCHE; ILLUSIE, 2005):

- Using base-change theorems, the problem is reduced to proving that the functor $\mathcal{Q}^{\Phi,\mathcal{O}(1)}_{\mathbb{P}^n}(\mathcal{O}^p_{\mathbb{P}^n})$ is representable, whenever $n, p \geq 0$.
- By 4.5.11, we can consider an injective natural transformation of functors:

  $$\mathcal{Q}^{\Phi,\mathcal{O}(1)}_{\mathbb{P}^n}(\mathcal{O}^p_{\mathbb{P}^n}) \to \mathcal{G}(V, \Phi(m)),$$

  where $V \doteq k^p \otimes H^0(\mathcal{O}_{\mathbb{P}^n})$ and m depends only on the Hilbert polynomial Φ, since in this case every sheaf is $m-$regular. The strategy for the rest of the proof is to show that this injective natural transformation corresponds to an embedding of the corresponding representing schemes.
- Using flatness, there is a stratification of the functor $\mathcal{Q}$ such that each piece is locally closed inside the grassmanian.
- Using a valuative criterion for properness, these subsets are proved to be projective varieties inside the grassmanian.
- There is a universal family $\mathcal{U}$ over $Q \times X$, and a universal quotient

  $$q_Q : \mathcal{O}_{Q\times X} \twoheadrightarrow \mathcal{U}$$

  so that the quot scheme is a fine moduli space for this moduli problem. Furthermore, this universal family is the pullback of the universal family over the grassmanian variety.

The following proposition uses homological algebra to describe the local behaviour of the Quot scheme Q.

Proposition 4.5.12. For any $k-$point $q : \mathcal{O}^N_X \to \mathcal{F}$ of the quot scheme Q, we have

1. $T_q Q \simeq \mathrm{Hom}(\mathcal{K}, \mathcal{F})$, where $\mathcal{K} \doteq \ker q$.
2. If $\mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$, then Q is smooth in a neighbourhood of q.

Proof. For a proof, see for example (HUYBRECHTS; LEHN, 2010), propositions 2.2.7 and 2.2.8. □

4.6 The GIT setup

Now that we have outlined the construction of the Quot scheme, we are ready to start applying the GIT theory for the construction of the moduli space of μ–semistable vector bundles.

As in 1.2.1, we fix X smooth connected projective curve of genus $g \geq 2$, discrete invariants for rank n degree $d > n(2g-1)$, so by 4.4.2 we can choose an identification

$$H^0(X, \mathcal{E}) \simeq k^N,$$

where $N \doteq d + n(1-g)$ by Riemann-Roch, so the evaluation map

$$H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$$

determines a quotient sheaf $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{E} \in Q(k)$. As we will see, the idea of our GIT setup is to study how GL_N acts as change of coordinates of the vector space of global sections $H^0(X, \mathcal{E})$.

Theorem 4.6.1 ((HUYBRECHTS; LEHN, 2010), 2.3.1)**.** *The following properties of coherent sheaves are open in the Quot scheme Q:*

- *The set of all equivalence classes of μ–(semi)stable sheaves;*
- *The set of all equivalence classes of quotients q such that the induced map $H^0(q)$ is an isomorphism.*

Let $R^{s(s)} \subset Q$ be the open set corresponding to these two properties. As the Quot scheme Q is a fine moduli space, it parametrizes a universal quotient, which we denote by

$$q_Q : \mathcal{O}_{Q \times X}^N \to \mathcal{U},$$

and we can consider the restriction of this family to the open subset $R^{(s)s} \subset Q$, denoted by

$$q^{(s)s} : \mathcal{O}_{R^{(s)s} \times X} \to \mathcal{U}^{(s)s}.$$

The strategy of this section is to use this family to conclude that there is a quotient of $R^{(s)s}$ by a GL_N –action which is a coarse moduli space for the moduli problem of μ–(semi)stable locally free sheaves over X with invariants (n, d).

Lemma 4.6.2. The universal quotient sheaf $\mathcal{U}^{(s)s}$ over $R^{(s)s} \times X$ is a family over $R^{(s)s}$ for the moduli problem of μ–semistable locally free sheaves over X with invariants (n, d) with the local universal property (3.1.8).

Proof. Let $\mathcal{F} \in \mathrm{Coh}(X \times S)$ be a family over a scheme S of μ-(semi)stable locally free sheaves over X with fixed invariants (n, d). By our choice of degree d, for every $s \in S$, we can assume the locally free μ–semistable sheaf $\mathcal{F}_s$ is globally generated and has vanishing first cohomology. Therefore, by the semi-continuity theorem, C.2.3, the sheaf $\pi_{S*}\mathcal{F}$ is also locally free over S, of rank $N = d + n(1 - g)$, since this is the dimension of the first cohomology group of $\mathcal{F}_s$ via the Riemann-Roch theorem.

For each $s \in S$, we need to show that there is an open neighbourhood $U \subset S$ of $s \in S$ and a morphism $f : S \to R^{(s)s}$ such that

$$\mathcal{F}\big|_U \simeq f^*\mathcal{U}^{(s)s}.$$

We choose a neighbourhood U of s over which $\pi_{S*}\mathcal{F}$ is trivial. Thus, we have an isomorphism:

$$\rho : \mathcal{O}_U^N \simeq \pi_{S*}\mathcal{F}\big|_U.$$

Now, consider the pullback of ρ by the projection $\pi_U : U \times X \to U$, and the surjective map of evaluation

$$q_U \doteq \mathcal{O}_{U\times X}^N \xrightarrow{\pi_U^* \circ \rho} \pi_U^*\pi_{S*}\mathcal{F}\big|_U \twoheadrightarrow \mathcal{F}\big|_U$$

By construction, q_U is surjective, and by the universality of $\mathcal{U}$ with respect to surjections, q_U determines a unique morphism $f \in \mathrm{Hom}(U, Q)$ commuting

$$\begin{tikzcd}
\mathcal{O}_{U\times X}^N \arrow[r, "q_U"] \arrow[d, equal] & \mathcal{F}\big|_U \arrow[d, "\simeq"] \\
\mathcal{O}_{U\times X}^N \arrow[r, "f^*q_Q"] & f^*\mathcal{U}
\end{tikzcd}$$

In particular, $\mathcal{F}\big|_U \simeq f^*\mathcal{U}$, and by construction the morphism $f : U \to Q$ factors via the open subset $R^{(s)s}$, as we wanted to show. □

Note that the family $\mathcal{U}^{(s)s}$ over $R^{(s)s}$ is not universal, as the morphism f described in the previous proof is not unique: if we take $S = \mathrm{Spec}\, k$ and $\mathcal{E} \in \mathrm{Loc}(X)$, then different choices of basis on $H^0(X, \mathcal{E})$ will give rise to different points on $R^{(s)s}$. However, any choices of isomorphisms are related by a change of basis on the vector space $H^0(X, \mathcal{E})$. This is the hint to build the GIT quotient for this moduli problem.

Lemma 4.6.3. The natural action of GL_N on Q that acts as change of coordinates on closed points via

$$g \cdot (\mathcal{O}_X^N \xrightarrow{q} \mathcal{E}) \longmapsto (\mathcal{O}_X^N \xrightarrow{g^{-1}} \mathcal{O}_X^N \xrightarrow{q} \mathcal{E})$$

can be extended to an algebraic action $\sigma : \mathrm{GL}_N \times Q \to Q$ such that the orbits in the open set $R^{(s)s}$ are in bijective correspondence with the isomorphism classes of μ-(semi)stable locally free sheaves on X with invariants (n, d).

Proof. To construct this action, we first construct a family over $\mathrm{GL}_N \times Q$ of quotients of $\mathcal{O}_X^N$ with invariants (n,d). The inverse map of the group $i : \mathrm{GL}_N \to \mathrm{GL}_N$ determines a universal inversion

$$\tau : k^N \otimes \mathcal{O}_{\mathrm{GL}_N} \to k^N \otimes \mathcal{O}_{\mathrm{GL}_N},$$

which is an isomorphism of sheaves over GL_N. Let $q_Q : k^N \otimes \mathcal{O}_{Q\times X} \to \mathcal{U}$ denote the universal quotient morphism over $Q \times X$. If we denote the projections by

$$\pi_{\mathrm{GL}_N} : \mathrm{GL}_N \times Q \times X \to \mathrm{GL}_N \ , \ \pi_{Q\times X} : \mathrm{GL}_N \times Q \times X \to Q \times X$$

we consider the composition of pullbacks

$$k^N \otimes \mathcal{O}_{\mathrm{GL}_N \times Q \times X} \xrightarrow{\pi^*_{\mathrm{GL}_N}(\tau)} k^N \otimes \mathcal{O}_{\mathrm{GL}_N \times Q \times X} \xrightarrow{\pi^*_{Q\times X}(q_Q)} \pi^*_{Q\times X}\mathcal{U}.$$

By definition, the resulting $\pi^*_{\mathrm{GL}_N}(\tau) \circ \pi^*_{Q\times X}(q_Q)$ is a family of quotients of $\mathcal{O}_X^N$ over the scheme $\mathrm{GL}_N \times Q$ with fixed invariants (n,d), since these are preserved by pullbacks. Using the universality of $\mathcal{U}$, this determines a morphism

$$\sigma : \mathrm{GL}_N \times Q \to Q$$

such that the pullback of the universal family is the expected one. We just need to prove that the subschemes in $R^{(s)s}$ are preserved by this action, to be able to restrict to a map $\sigma : \mathrm{GL}_N \times R^{(s)s} \to R^{(s)s}$.

To see this, let $q_{\mathcal{E}} : \mathcal{O}_X^N \twoheadrightarrow \mathcal{E}$ and $q_{\mathcal{F}} : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be the corresponding surjections of two k–points in $R^{(s)s}$. If there exists an element $g \in \mathrm{GL}_N(k)$ such that $g \cdot q_{\mathcal{E}} = q_{\mathcal{F}}$, then we can fit these into a commutative square

$$\begin{tikzcd} \mathcal{O}_X^N \arrow[r, "q_{\mathcal{E}}"] \arrow[d, "\simeq"', "g"] & \mathcal{E} \arrow[d] \\ \mathcal{O}_X^N \arrow[r, "q_{\mathcal{F}}"] & \mathcal{F} \end{tikzcd}$$

so g induces an isomorphism between the sheaves $\mathcal{E}$ and $\mathcal{F}$.

Conversely, if $\mathcal{E} \simeq \mathcal{F}$, then there is an induced isomorphism $\phi : H^0(\mathcal{E}) \simeq H^0(\mathcal{F})$, so that we complete the corresponding diagram in cohomology

$$\begin{tikzcd} k^N = H^0(\mathcal{O}_X^N) \arrow[r, "H^0(q_{\mathcal{E}})"] \arrow[d, dashed] & H^0(\mathcal{E}) \arrow[d, "H^0(\phi)"] \\ k^N = H^0(\mathcal{O}_X^N) \arrow[r, "H^0(q_{\mathcal{F}})"] & H^0(\mathcal{F}) \end{tikzcd}$$

with an automorphism, so there is a change of coordinates $g \in \mathrm{GL}_N(k)$ such that $g \cdot q_{\mathcal{E}} = q_{\mathcal{F}}$. □

We will build the moduli space for semistable vector bundles by considering the GIT quotient corresponding to this action of GL_N on $R^{(s)s}$. We should also restrict our action for SL_N, as diagonal matrices inside GL_N act trivially on the Quot scheme.

As sketched in the construction of the Quot scheme (4.5), for a sufficiently large integer m, there is a closed immersion

$$Q = \mathrm{Quot}_X^{(n,d)}(\mathcal{O}_X^N) \to \mathrm{Gr}(H^0(\mathcal{O}_X^N(m)), M) \to \mathbb{P} \doteq \mathbb{P}(\bigwedge^M H^0(\mathcal{O}_X^N(m))^\vee),$$

where $M = mn + d + n(1-g)$, and the latter map is the Plücker embedding. We let $\mathcal{L}_m$ denote the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ to the Quot scheme via this closed immersion. There is a natural linear action of SL_N on the vector space:

$$H^0(\mathcal{O}_X^N(m)) \simeq k^N \otimes H^0(\mathcal{O}_X(m)),$$

which induces a linear action of SL_N on the projective space $\mathbb{P}$. Hence, the invertible sheaf $\mathcal{L}_m$ admits a linearization of the SL_N action on Q, given by the pullback of this action.

We can also describe the sheaf $\mathcal{L}_m$ using the universal quotient sheaf over Q, if we denote the projections $\pi_X : Q \times X \to X$, $\pi_Q : Q \times X \to Q$, using the formula

$$\mathcal{L}_m = \det({\pi_Q}_*(\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(m))),$$

where $\mathcal{U}$ is the universal quotient sheaf over $Q \times X$. Moreover, the universal quotient sheaf $\mathcal{U}$ admits a SL_N –linearization. We denote by $\sigma : \mathrm{SL}_N \times Q \to Q$ the group action, the projections by

$$p_{Q\times X} : \mathrm{SL}_N \times Q \times X \to Q \times X\ ,\ p_{\mathrm{SL}_N} : \mathrm{SL}_N \times Q \times X \to \mathrm{SL}_N$$

and write $q_Q : k^N \otimes \mathcal{O}_{Q\times X} \to \mathcal{U}$ for the universal quotient morphism over $Q \times X$. By construction, there are two equivalent families of quotient sheaves over $\mathrm{SL}_N \times Q$, given respectively as

$$k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q \times X} \xrightarrow{(\sigma\times \mathrm{Id}_X)^* q_Q} (\sigma \times \mathrm{Id}_X)^*(\mathcal{U})$$

and the composition

$$k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q\times X} \xrightarrow{p_{\mathrm{SL}_N}^* \tau} k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q\times X} \xrightarrow{p_{Q\times X}^* q_Q} p_{Q\times X}^* \mathcal{U}.$$

Hence, there is an isomorphism between the pullbacks

$$\Phi : (\sigma \times \mathrm{Id}_X)^* \mathcal{U} \xrightarrow{\simeq} (p_{Q\times X})^* \mathcal{U}$$

satisfying the cocycle condition defining a linearization of the SL_N-action on $\mathcal{U}$ (compare 3.3.11). For m sufficiently large, $\mathcal{L}_m$ is ample and admits a SL_N –linearization, as the construction commutes with base change.

Hence, at any k–point $q : \mathcal{O}_X \twoheadrightarrow \mathcal{F}$ in Q, the fiber of the associated line bundle L_m is naturally isomorphic to an alternating tensor product of exterior products of the cohomology groups of $\mathcal{F}(m)$:

$$L_{m,q} \simeq \det(H^*(X,\mathcal{F}(m))) = \bigotimes_{i\geq 0} \det H^i(X,\mathcal{F}(m))^{\otimes(-1)^i}.$$

By the m–regularity, we have $H^i(X,\mathcal{F}(m)) = 0$ for all $i > 0$ for all points $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q, so for m sufficiently large the fiber at q can be given as $L_{m,q} \simeq \det H^0(X,\mathcal{F}(m))$. We will use this explicit description to apply the Hilbert-Mumford numerical criterion for stability with respect to this linearization in the next section.

4.7 Analysis of stability

In this section, we prove the equivalence between the μ–(semi)stability of vector bundles and the GIT stability for the GIT problem described in the previous section, using the Hilbert-Mumford numerical criterion (3.4.14).

Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a closed point in the Quot scheme Q and $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a one-parameter subgroup. Then the action of $\lambda(t) \cdot q$ is given by the composition

$$\mathcal{O}_X^N \xrightarrow{\lambda^{-1}(t)} \mathcal{O}_X^N \xrightarrow{q} \mathcal{F}.$$

We can decompose the space $V \doteq k^N$ into weight spaces for the action of λ:

$$V = \bigoplus_{r\in\mathbb{Z}} V_r$$

where $V_r = \{v \in V : \lambda^{-1}(t)v = t^r v\}$, and these are zero except for finitely many weights r, satisfying

$$\sum_{r\in\mathbb{Z}} r \cdot \dim V_r = 0.$$

There is an induced ascending filtration of V given by

$$V^{\leq r} \doteq \bigoplus_{s\leq r} V_s,$$

which induces an ascending filtration of $\mathcal{F}$, given by:

$$\mathcal{F}^{\leq r} \doteq q(V^{\leq r} \otimes \mathcal{O}_X)$$

and q restricts to surjections $q_r : V_r \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}_r \doteq \mathcal{F}^{\leq r}/\mathcal{F}^{\leq r-1}$, fitting into the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & V^{\leq r-1} \otimes \mathcal{O}_X \arrow[r] \arrow[d, "q"] & V^{\leq r} \otimes \mathcal{O}_X \arrow[r] \arrow[d, "q"] & V_r \otimes \mathcal{O}_X \arrow[r] \arrow[d, "q_r"] & 0 \\
0 \arrow[r] & \mathcal{F}^{\leq r-1} \arrow[r] & \mathcal{F}^{\leq r} \arrow[r] & \mathcal{F}_r \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Lemma 4.7.1. In these conditions, we have

$$\lim_{t\to 0}\lambda(t)\cdot q = \bigoplus_{r\in\mathbb{Z}} q_r : \bigoplus_{r\in\mathbb{Z}} V_r \otimes \mathcal{O}_X \simeq \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$$

Proof. Since the Quot scheme Q is projective, the limit exists and it is unique. Thus, to show this lemma we only need to construct a family of quotient sheaves of $\mathcal{O}_X^N$ over $\mathbb{A}^1 = \operatorname{Spec} k[t]$, a surjection $\varphi : \mathcal{O}^N_{X\times\mathbb{A}^1} \twoheadrightarrow \mathcal{E}$ such that

$$\varphi_t \doteq \varphi\big|_{X\times\{t\}} = \begin{cases} \lambda(t)\cdot q \ , \ t \neq 0 \\ \bigoplus_{r\in\mathbb{Z}} q_r \ , \ t = 0. \end{cases}$$

To to this, we use the equivalence between the categories of quasi-coherent sheaves over $\mathbb{A}^1$ and $k[t]$–modules. First, let

$$M \doteq \bigoplus_{r\in\mathbb{Z}} (V^{\le r} \otimes_k t^r \cdot k)\,,$$

which has a $k[t]$–module structure defined by the action

$$\begin{aligned} &k[t]\times M \to M \\ &(t, v^{\le r}\otimes t^r) \mapsto v^{\le r}\otimes t^{r+1} \in V^{\le r+1}\otimes t^{r+1}\cdot k. \end{aligned}$$

Since V is finite-dimensional, there is an integer $R \in \mathbb{Z}$ such that $V^{\le r} = 0$ for $r \le R$, and we could consider M as a $k[t]$–submodule of $V \otimes_k t^R \cdot k[t]$, which implies that the corresponding sheaf is coherent over $\mathbb{A}^1$. Using this, the one-parameter subgroup λ induces a sheaf morphism over $\mathbb{A}^1$ that can be translated to a map of $k[t]$–modules$\gamma : V \otimes_k k[t] \to M$ defined as a k–linear map on generators by the rule

$$\gamma(v\otimes t^s) = \gamma\left(\sum_{r\in\mathbb{Z}} v_r \otimes t^s\right) \doteq \sum_{r\in\mathbb{Z}} v_r \otimes t^{r+s},$$

whenever $s \ge 0$. By construction, we have $\gamma\big|_{V_r} = t^r \cdot \mathrm{Id}_{V_r}$. On the other hand, we can define a morphism $\alpha : M \to V \otimes_k k[t]$ on generators by setting

$$\alpha\left(\sum_{r\in\mathbb{Z}} v^{\le r}\otimes a_r t^r\right) \doteq \sum_{r\in\mathbb{Z}} v_{s\le r} t^r = \sum_{r\in\mathbb{Z}} v_{s\le r}\otimes t^{r-s}$$

so that it is well defined since $r - s \ge 0$ whenever $v_s \in V_s \subset V^{\le r}$. Note that, if $s \ge 0$,

$$(\alpha\circ\gamma)(v\otimes t^s) = \alpha(\sum_{r\in\mathbb{Z}} v_r \otimes t^{r+s}) = \sum_{r\in\mathbb{Z}} v_r \otimes t^{r+s-r} = v\otimes t^s,$$

and, for $r \in \mathbb{Z}$ so $V_r \neq 0$, we can write

$$(\gamma\circ\alpha)(v^{\le r}\otimes t^r) = (\gamma\circ\alpha)(v_{s\le r}\otimes t^r) = \gamma(v_{s\le r}\otimes t^{r-s}) = v_{s\le r}\otimes t^{r-s+s} = v^{\le r}\otimes t^r,$$

so $\alpha \circ \gamma = \mathrm{Id}_{V\otimes_k k[t]}$ and $\gamma \circ \alpha = \mathrm{Id}_M$. Using the previous computations, we go back to the category of quasi-coherent sheaves over $X \times \mathbb{A}^1$ using the equivalence with the category of $\mathcal{O}_X \otimes_k k[t]$–modules. Using the filtration of $\mathcal{F}$, we define:

$$\mathcal{E} \doteq \bigoplus_{r\in\mathbb{Z}} \mathcal{F}^{\leq r} \otimes t^r \cdot k \subset \mathcal{F} \otimes t^R k[t],$$

for R as above, so the action of t is identical to the action of t on M. Furthermore, we have the inclusion again, by the same reasoning. In particular, $\mathcal{E}$ is a coherent sheaf on $X \times \mathbb{A}^1$. The surjective morphism $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ induces a surjective morphism of coherent sheaves over $X \times \mathbb{A}^1$:

$$q_{\mathbb{A}^1} : \bigoplus_{r\in\mathbb{Z}} V^{\leq r} \otimes_k t^r k \twoheadrightarrow \mathcal{E},$$

and we can define the family as $\varphi \doteq q_{\mathbb{A}^1} \circ \pi^*_{\mathbb{A}^1}(\tilde{\gamma})$, where $\pi_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection, and the morphism $\tilde{\gamma}$ is the induced map by γ between quasi-coherent sheaves over $\mathbb{A}^1$.

Now, considering values $t \neq 0$ corresponds to inverting the variable t of $k[t]$–modules. In this case, we have the commutative diagram:

$$\begin{tikzcd} \mathcal{O}_X^N \otimes_k k[t,t^{-1}] \arrow[r, "\pi^*|_{\mathbb{A}^1\setminus\{0\}}(\tilde{\gamma})"] \arrow[d, "\varphi|_{\mathbb{A}^1\setminus\{0\}}"] & \mathcal{O}_X^N \otimes_k k[t,t^{-1}] \arrow[d, "q\otimes \mathrm{Id}_{k[t,t^{-1}]}"] \\ \mathcal{E} \otimes_k k[t,t^{-1}] \arrow[r, "\simeq"] & \mathcal{F} \otimes k[t,t^{-1}] \end{tikzcd}$$

and since $\tilde{\gamma}$ corresponds to the action of λ^{-1} by construction, we have $[\varphi_t] = [\lambda(t)\cdot q]$ whenever $t \neq 0$. If we denote by i the inclusion of the origin inside the affine line $\mathbb{A}^1$, the composition $i_* \circ i^*$ kills the action of t on the corresponding $k[t]$–modules. More explicitly,

$$i_* i^*(\mathcal{E}) = \mathcal{E}/t\cdot\mathcal{E} = \left(\bigoplus_{r\geq R} \mathcal{F}^{\leq r} \otimes_k t^r \cdot k\right) \Big/ \left(\bigoplus_{r\geq R} \mathcal{F}^{\leq r} \otimes_k t^{r+1} \cdot k\right) = \bigoplus_{r\in\mathbb{Z}} \mathcal{F}_r \otimes_k t^r \cdot k,$$

with the trivial action by t. Hence, the stalk of the sheaf $\mathcal{E}$ over $0 \in \mathbb{A}^1$ is $\mathcal{E}_0 = \bigoplus_{r\in\mathbb{Z}} \mathcal{F}_r$, which completes the proof. □

Lemma 4.7.2. Using previous notation, we have:

$$\mu^{\mathcal{L}_m}(q,\lambda) = -\sum_{r\in\mathbb{Z}} r\cdot P(\mathcal{F}_r, m) = \sum_{r\in\mathbb{Z}} r\cdot\left(P(\mathcal{F}^{\leq r}, m) - \frac{\dim V^{\leq r}}{N}\cdot P(\mathcal{F}, m)\right).$$

Proof. By definition, the Hilbert-Mumford weight is the opposite of the weight of the action of λ on the fiber of the line bundle L_m over the fixed point $q' \doteq \lim_{t\to 0} \lambda(t)\cdot q$. As we observed before, the fiber over the limit point is given by

$$L_{m,q'} = \bigotimes_{r\in\mathbb{Z}} \det H^*(X, \mathcal{F}_r(m)),$$

where $H^*(X, \mathcal{F}_r(m))$ denotes the complex defining cohomology groups $H^i(X, \mathcal{F}_r(m))$ for $i = 1, 2$. The virtual dimension of $H^*(X, \mathcal{F}_r(m))$ is the alternating sums of the dimensions of the cohomology groups of $\mathcal{F}_r(m)$, and thus it coincides with the Hilbert polynomial $P(\mathcal{F}_r, m)$.

Since λ acts with weight r on each subsheaf $\mathcal{F}_r$, it also acts with weight r on the corresponding k–vector space $H^i(X, \mathcal{F}_r(m))$. Therefore, the weight of λ acting on the line bundle $\det H^*(X, \mathcal{F}_r(m))$ coincides with $r \cdot P(\mathcal{F}_r, m)$. The first equality follows from this, and the definition of Hilbert-Mumford weight.

For the second equality, by construction we have that $\dim V_r = \dim V^{\le r} - \dim V^{\le r-1}$, and considering the long exact sequence in cohomology associated the cokernel exact sequence

$$0 \to \mathcal{F}^{\le r} \to \mathcal{F}^{\le r+1} \to \mathcal{F}_r \to 0$$

implies the identity for Hilbert polynomials $P(\mathcal{F}_r, m) = P(\mathcal{F}^{\le r}, m) - P(\mathcal{F}^{\le r-1}, m)$. On the other hand, since we are acting by $G = \mathrm{SL}_N$, if we write $r_1 < \cdots < r_n = R$ for the weights, we have

$$\sum_{r\in\mathbb{Z}} r \dim V_r = 0 \Rightarrow -R = \frac{1}{N} \cdot \left(\sum_{i=1}^{n-1} (r_i - r_{i+1}) \dim V^{\le r_i} \right),$$

where $N = \dim V = \dim V^{\le R}$. Using the same trick, we write

$$\begin{aligned} -\sum_{r\in\mathbb{Z}} r \cdot P(\mathcal{F}_r) &= -\left(\sum_{r=1}^{n-1} (r_i - r_{i+1}) P(\mathcal{F}^{\le r_i}) - r_n P(\mathcal{F}^{\le r_n}) \right) \\ &= -\left(\sum_{r=1}^{n-1} (r_i - r_{i+1}) P(\mathcal{F}^{\le r_i}) + R \cdot P(\mathcal{F}) \right), \end{aligned}$$

so

$$\begin{aligned} \mu(q, \lambda) &= \sum_{r=1}^{n-1} (r_{i+1} - r_i) P(\mathcal{F}^{\le r_i}) + \frac{(r_i - r_{i+1}) \dim V^{\le r_i}}{N} \cdot P(\mathcal{F}) \\ &= \sum_{r\in\mathbb{Z}} r \cdot \left(P(\mathcal{F}^{\le r}) - \frac{\dim V^{\le r}}{N} P(\mathcal{F}) \right), \end{aligned}$$

as we wanted to prove. □

Remark 4.7.3. We note that the number of distinct weights for the action of λ^{-1} on $V = k^N$ corresponds to the number of jumps in the Harder-Narasimhan filtration of the sheaf $\mathcal{F}$. If we suppose that there are only two weights, $r_1 < r_2$ for λ, we can get a filtration by a single subsheaf:

$$0 = \mathcal{F}^{\le r_1 - 1} \subsetneq \mathcal{F}' \doteq \mathcal{F}^{\le r_1} = \cdots = \mathcal{F}^{\le r_2 - 1} \subsetneq \mathcal{F}^{\le r_2} = \mathcal{F}.$$

Let $V' \doteq V^{\leq r_1}$. Then

$$\mu^{\mathcal{L}_m}(q,\lambda) = (r_2 - r_1)\left(P(\mathcal{F}',m) - \frac{\dim V'}{\dim V}\cdot P(\mathcal{F},m)\right),$$

where $r_2 - r_1 > 0$.

Proposition 4.7.4. Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k–point in Q. Then $q \in Q^{(s)s}(\mathcal{L}_m)$ if and only if for all subspaces $0 \neq V' \subsetneq V = k^N$, we have an inequality:

$$\frac{\dim V'}{P(\mathcal{F}',m)}(\leq) < \frac{\dim V}{P(\mathcal{F},m)}$$

where $\mathcal{F}' \doteq q(V' \otimes \mathcal{O}_X) \subset \mathcal{F}$.

Proof. Suppose the inequality holds for all such subspaces $V' \subset V$. For any one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_N$, there are finitely many weights $r_1 < \cdots < r_s$ for the action of λ^{-1} on $V = k^N$, which induces subspaces $V^{(i)} = V^{\leq r_i} \subset V$ and corresponding subsheaves $\mathcal{F}^{(i)} = \mathcal{F}^{\leq r_i} = q(V^{(i)} \otimes \mathcal{O}_X) \subset \mathcal{F}$. Furthermore, by definition, we have $\mathcal{F}^{\leq n} = \mathcal{F}^{(i)}$ for all $r_i \leq n < r_{i+1}$. Therefore, we can apply the previous formula to compute

$$\mu^{\mathcal{L}_m}(q,\lambda) = \sum_{i=1}^{s-1}(r_{i+1} - r_i)\left(P(\mathcal{F}^{(i)},m) - \frac{\dim V^{(i)}}{\dim V}\cdot P(\mathcal{F},m)\right),$$

but using the inequality of the hypothesis, we get:

$$P(\mathcal{F}^i,m) - \frac{\dim V^{(i)}}{\dim V}\cdot P(\mathcal{F},m) \geq 0.$$

The same proof holds to the strictly stable case, where the inequality we get in the end is strict, so

$$\mu^{\mathcal{L}_m}(q,\lambda)(\geq)0$$

and by the Hilbert-Mumford numerical criterion $q \in Q^{(s)s}(\mathcal{L}_m)$.

Conversely, arguing by contradiction, if there is a subspace $0 \subsetneq V' \subsetneq V$ for which the inequality does not hold (or, in the strictly stable case, holds with equality), then we can define a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_N$ with two weights $r_1 > r_2$, such that $V^{(1)} = V'$ and $V^{(2)} = V$. In this case, we can write:

$$\mu^{\mathcal{L}_m}(q,\lambda) = (r_2 - r_1)\left(P(\mathcal{F}',m) - \frac{\dim V'}{\dim V}\cdot P(\mathcal{F},m)\right),$$

so $\mu^{\mathcal{L}_m}(q,\lambda) < 0$ (or $\mu^{\mathcal{L}_m}(q,\lambda) = 0$, in the strictly stable case), and q is unstable for the SL_N –action with respect to the linearization $\mathcal{L}_m$, by the Hilbert-Mumford criterion. □

Corollary 4.7.4.1. There is an integer $M \in \mathbb{Z}$ such that, whenever $m \geq M$ and $q : \mathcal{O}_X^N \to \mathcal{F}$ corresponds to a closed point in Q, the following are equivalent:

1. q is (semi)stable for the action of GL_N on Q with respect to the ample line bundle $\mathcal{L}_m$.

2. For all subsheaves $\mathcal{F}' \subset \mathcal{F}$ with $V' \doteq H^0(q)^{-1}(H^0(\mathcal{F}')) \neq 0$, we have $\operatorname{rk}\mathcal{F}' > 0$ and

$$\frac{\dim V'}{\operatorname{rk}\mathcal{F}'}(\leq) < \frac{\dim V}{\operatorname{rk}\mathcal{F}},$$

in particular, $\mathcal{F}$ is μ−semistable as defined in 4.3.1.

4.8 Le Potier criterion

In this section, we prove the following result:

Theorem 4.8.1 (Le Potier criterion)**.** *If $\mathcal{F}$ is a locally free sheaf of sufficiently large degree $d = \deg\mathcal{F}$, then*

1. *$\mathcal{F}$ is semistable if and only if whenever $\mathcal{F}' \subset \mathcal{F}$ is a non-trivial subsheaf, then*

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \frac{h^0(X,\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

2. *$\mathcal{F}$ is stable if and only if whenever $\mathcal{F}' \subset \mathcal{F}$ is a non-trivial subsheaf, then*

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} < \frac{h^0(X,\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

For sufficiently large m, the corresponding Hilbert polynomials $P(\mathcal{F}',m)$ and $P(\mathcal{F},m)$ are constant, because both sheaves will be generated by their global sections, and thus $h^1(\mathcal{F}'(m)) = h^1(\mathcal{F}(m)) = 0$. In this case, we can multiply by the denominators and apply the Riemann-Roch theorem 4.2.11.1 to obtain an equivalent inequality:

$$(\dim V' \operatorname{rk}\mathcal{F})m + \dim V'(\deg\mathcal{F} + \operatorname{rk}\mathcal{F}(1-g))$$
$$\leq$$
$$(\dim V \operatorname{rk}\mathcal{F}')m + \dim V(\deg\mathcal{F}' + \operatorname{rk}\mathcal{F}'(1-g))$$

Since this is an inequality of polynomials of same degree in the variable m, it holds if and only if there is a corresponding inequality of their leading terms. If $\operatorname{rk}\mathcal{F}' \neq 0$, then the leading term of the polynomial $P(\mathcal{F}')$ is $\operatorname{rk}\mathcal{F}'$, and otherwise the Hilbert polynomial of $\mathcal{F}'$ is constant.

Therefore, there is an upper bound $M > 0$ (depending on both $\mathcal{F}$ and $\mathcal{F}'$) such that whenever $m \geq M$ we have

$$\operatorname{rk}\mathcal{F}' > 0 \text{ and } \frac{\dim V'}{\operatorname{rk}\mathcal{F}'}(\leq)\frac{\dim V}{\operatorname{rk}\mathcal{F}} > 0 \iff \frac{\dim V'}{P(\mathcal{F}',m)}(\leq)\frac{\dim V}{P(\mathcal{F},m)}.$$

Moreover, as the subspaces $0 \neq V' \subsetneq V = k^N$ form a bounded family (parametrized by a product of Grassmannians) and the quotients $q : \mathcal{O}_X^N \to \mathcal{F}$ form a bounded family (parametrized by the quot scheme), the family of sheaves such that

$$\mathcal{F}' \doteq q(V' \otimes \mathcal{F})$$

is also bounded.

Proposition 4.8.2. Let $n \geq 0, d \in \mathbb{Z}$ be fixed integers such that $d > n^2(2g-2)$. Then a locally free sheaf $\mathcal{F}$ of rank n and degree d is (semi)stable if, for all $\mathcal{F}' \subset \mathcal{F}$ we have

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} (\leq) \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk}\mathcal{F}}.$$

Proof. First, let us suppose $\mathcal{F}$ is not semistable, so there is a non-trivial locally free subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $\mu(\mathcal{F}') > \mu(\mathcal{F})$. We can suppose that $\mathcal{F}'$ is semistable since, if not, there must be another non-trivial locally free subsheaf $\mathcal{F}'' \subset \mathcal{F}'$ with $\mu(\mathcal{F}'') > \mu(\mathcal{F}')$, and we could replace $\mathcal{F}'$ by $\mathcal{F}''$.

Then

$$\deg \mathcal{F}' > \frac{d}{n}\operatorname{rk}\mathcal{F}' > \frac{d}{n} > n(2g-2) > \operatorname{rk}\mathcal{F}'(2g-2),$$

so by 4.4.2 we have $H^1(X,\mathcal{F}') = 0$. However, applying the Riemann-Roch theorem, we can write

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} = \mu(\mathcal{F}') + (1-g) > \mu(\mathcal{F}) + (1-g) = \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk}\mathcal{F}},$$

contradicting the hypothesis.

Furthermore, if the sheaf $\mathcal{F}$ is semistable and the strict inequality holds, we can choose a non-trivial locally free subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $\mu(\mathcal{F}') = \mu(\mathcal{F})$ to get an analogous contradiction. □

Lemma 4.8.3 (Le Potier bounds)**.** For any semistable locally free sheaf $\mathcal{F}$ of rank n and slope $\mu = \mu(\mathcal{F})$, we have

$$\frac{h^0(X,\mathcal{F})}{n} \leq \max\{\mu+1, 0\}$$

Proof. If $\mu < 0$, then $H^0(X,\mathcal{F}) = 0$, by 4.3.7. For $\mu \geq 0$, we proceed by induction on $d = \deg(\mathcal{F})$. If we assume the lemma holds for any degree less than d, we can consider the exact sequence of coherent sheaves (as in 4.1)

$$0 \to \mathcal{F}(-x) \to \mathcal{F} \to F_x \to 0$$

where $x \in X$ is any point. Applying cohomology, we get the long exact sequence and then $h^0(X,\mathcal{F}) \leq h^0(X,\mathcal{F}(-x)) + n$. Since $\mu(\mathcal{F}) = \mu(\mathcal{F}(-x)) + 1$, the result follows by applying the inductive hypothesis to $\mathcal{F}(-x)$. □

The following corollary follows from applying the Le Potier bounds to the Harder-Narasimhan filtration of a locally free sheaf.

Corollary 4.8.3.1. Let $\mathcal{F}$ be a locally free sheaf of rank n and slope μ with a Harder-Narasimhan filtration

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \cdots \subsetneq \mathcal{F}^{(s)} = \mathcal{F},$$

i.e., $\mathcal{F}_i \doteq \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ are semistable and

$$\mu_{\max}(\mathcal{F}) \doteq \mu(\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_s) \doteq \mu_{\min}(\mathcal{F}).$$

Then

$$\begin{aligned} \frac{h^0(X,\mathcal{F})}{n} &\leq \sum_{i=1}^{s} \frac{\operatorname{rk}\mathcal{F}_i}{n} \max\{\mu(\mathcal{F}_i)+1,0\} \\ &\leq \left(1-\frac{1}{n}\right)\max\{\mu+1,0\} + \frac{\max\{\mu_{\min}(\mathcal{F})+1,0\}}{r} \end{aligned}$$

Proposition 4.8.4. Let $n \geq 1, d \in \mathbb{Z}$ fixed such that $d > gn^2 + n(2g-2)$. Let $\mathcal{F}$ be a semistable locally free sheaf over X with rank r and degree d. Then, for all non-zero subsheaves $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$, we have

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

and, if equality holds, then $h^1(X,\mathcal{F}') = 0$ and $\mu(\mathcal{F}') = \mu(\mathcal{F})$.

Proof. Let $\mu = d/n$ denote the slope of $\mathcal{F}$. Then, by hypothesis, $\mu - gn > 2g-2$ and thus there is at least one rational constant C satisfying $\mu - gn > C > 2g-2$. Now, let $\mathcal{F}' \subset \mathcal{F}$ be any subsheaf and consider the Harder-Narasimhan filtration in $\mathcal{F}'$, given by a choice of quotients which we denote by $\{\mathcal{F}'_i\}_{i\in I}$, for a finite set I. From the previous corollary, we have the bound

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \left(1-\frac{1}{n}\right)\max\{\mu+1,0\} + \frac{\max\{\mu_{\min}(\mathcal{F})+1,0\}}{n}$$

We divide the proof into two cases, depending on the value $\mu_{\min}(\mathcal{F}') \in \mathbb{Q}$.

First, if $\mu_{\min}(\mathcal{F}') \leq C$, then we use the inequality defining C to get

$$\begin{aligned} \frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} &\leq \left(1-\frac{1}{n}\right)(\mu+1) + \frac{(C+1)}{n} \\ &< \mu + 1 + g = \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk}\mathcal{F}}, \end{aligned}$$

using the Riemann-Roch theorem. On the other hand, if $\mu_{min}(\mathcal{F}') \leq C$, whenever $i \in I$ we have

$$\mu(\mathcal{F}'_i) \geq \mu_{\min}(\mathcal{F}') > C > 2g-2.$$

Thus

$$\deg \mathcal{F}'_i > \operatorname{rk} \mathcal{F}'_i(2g-2)$$

and, as each $\mathcal{F}'_i$ is semistable, we conclude that $H^1(X, \mathcal{F}'_i) = 0$, using 4.4.2. Furthermore, we have

$$H^1(X, \mathcal{F}') \simeq \bigoplus_{i \in I} H^1(X, \mathcal{F}'_i) = 0.$$

By the semistability of $\mathcal{F}$, we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. Now, using the Riemann-Roch theorem (4.2.11.1) and the fact that $H^1(X, \mathcal{F}') = 0$, we can write

$$\frac{h^0(X, \mathcal{F}')}{\operatorname{rk} \mathcal{F}'} = \mu(\mathcal{F}') + 1 - g \leq \mu(\mathcal{F}) + 1 - g = \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk} \mathcal{F}},$$

with equality only if $\mu(\mathcal{F}') = \mu(\mathcal{F})$. □

4.9 Construction of the moduli space

Theorem 4.9.1. *Let n, d be fixed integers such that*

$$d > \max\{n^2(2g-2), gn^2 + n(2g-2)\}.$$

Then there exists a natural number $M > 0$ such that, for all $m \geq M$, we have

$$Q^{ss}(\mathcal{L}_m) = R^{ss} \text{ and } Q^s(\mathcal{L}_m) = R^s.$$

Proof. We choose the natural M as required so the notions for (semi)stability and μ–(semi)stability coincide, using 4.7.4.1. Since these subschemes are all open, it suffices to check equality on k–points.

First, let $q : \mathcal{O}_X^N \to \mathcal{F}$ be a k–point in R^{ss}, so $\mathcal{F}$ is locally free and $H^0(q)$ is an isomorphism. Using 4.7.4.1, we let $\mathcal{F}' \subset \mathcal{F}$ be a proper non-trivial subsheaf, with $\operatorname{rk} \mathcal{F}' > 0$, and let $V' \doteq H^0(q)^{-1}(H^0(X, \mathcal{F}'))$. As $H^0(q)$ is an isomorphism, we have $\dim V' = h^0(X, \mathcal{F})$, and using the previous simplifications, we have either

1. $h^0(X, \mathcal{F}') < \operatorname{rk} \mathcal{F}' \cdot \dfrac{\mathcal{X}(\mathcal{F})}{\operatorname{rk} \mathcal{F}}$, or
2. $h^1(X, \mathcal{F}') = 0$ and $\mu(F') = \mu(F)$.

In the first case, we can write

$$\frac{\dim V'}{\operatorname{rk} \mathcal{F}'} = \frac{h^0(X, \mathcal{F}')}{\operatorname{rk} \mathcal{F}'} < \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk} \mathcal{F}} = \frac{\dim V}{\operatorname{rk} \mathcal{F}},$$

and in the second case, since $\dim V' = h^0(X, \mathcal{F}') = P(\mathcal{F}')$, we have

$$\frac{\dim V'}{\operatorname{rk} \mathcal{F}'} = \frac{\mathcal{X}(\mathcal{F}')}{\operatorname{rk} \mathcal{F}'} = \frac{\mathcal{X}(\mathcal{F})}{\operatorname{rk} \mathcal{F}} = \frac{\dim V}{\operatorname{rk} \mathcal{F}}$$

so that q must be semistable, using 4.7.4.1.

Furthermore, if $\mathcal{F}$ is a stable locally free sheaf, then q is stable (in the sense of GIT) in Q, since the second behaviour will not occur in this case, and we get the strict inequality. Hence, we have inclusions $R^{(s)s}(k) \subset Q^{(s)s}(\mathcal{L}_m)(k)$.

Let $q : \mathcal{O}_X^N \to \mathcal{F}$ be a k–point of $Q^{(s)s}(\mathcal{L}_m)$. Then, whenever $\mathcal{F}' \subset \mathcal{F}$ is a proper non-trivial subsheaf such that $V' \doteq H^0(q)^{-1}(H^0(X, \mathcal{F}'))$ is nonzero, we have $\operatorname{rk}\mathcal{F}' > 0$ and an inequality

$$\frac{\dim V'}{\operatorname{rk}\mathcal{F}'}(\leq)\frac{\dim V}{\operatorname{rk}\mathcal{F}},$$

by 4.7.4.1. If we prove that $H^0(q)$ is an isomorphism and $\mathcal{F}$ is locally free, we get the desired inclusion.

First, if we denote by $K \subset V$ the kernel of the morphism $H^0(q)$, then $\mathcal{F}' \doteq q(K \otimes \mathcal{O}_X)$ and this means $\mathcal{F}'$ has rank zero, so if K is non-trivial q would not be GIT semistable.

Moreover, if we prove $H^1(X, \mathcal{F}) = 0$, then

$$\dim H^0(X, \mathcal{F}) = \mathcal{X}(\mathcal{F}) = N = \dim V$$

and so we conclude $H^0(q)$ is an isomorphism. If we suppose $H^1(X, \mathcal{F}) \neq 0$, by Serre duality we get a non-zero morphism$\phi : \mathcal{F} \to \omega_X$ whose image $\mathcal{F}'' \doteq \operatorname{Im}\phi \subset \omega_X$ is an invertible sheaf. Using the equivalence between kernels and cokernels in the category $\operatorname{Coh}(X)$, we can equivalently rephrase the GIT (semi)stability of q in terms of quotient sheaves $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ satisfying

$$\frac{\dim V}{n} \leq \frac{\dim V''}{\operatorname{rk}\mathcal{F}''},$$

where V'' is the image of the induced composition

$$V \xrightarrow{H^0(q)} H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'').$$

Moreover, as $V'' \subset H^0(X, \mathcal{F}'') \subset H^0(X, \omega_X)$, we get $\dim V'' \leq g$, so that by GIT semistability we have

$$\frac{d}{n} + (1 - g) \leq g,$$

which contradicts the choice of d. Thus $H^1(X, \mathcal{F}) = 0$, and $H^0(q)$ is an isomorphism.

If $\mathcal{F}' \subset \mathcal{F}$ is a torsion subsheaf, then $\operatorname{rk}\mathcal{F}' = 0$ and since every torsion sheaf admits a section, $H^0(X, \mathcal{F}') \neq 0$ and this contradicts GIT semistability, thus the sheaf $\mathcal{F}$ must be torsion free, and thus locally free over the curve X. □

Theorem 4.9.2. *There is a coarse moduli space $M^s(n, d)$ for the moduli problem of μ-stable vector bundles of rank n and degree d over X, which has a natural projective completion $M^{ss}(n, d)$ whose k–points parametrize polystable vector bundles of rank n and degree d.*

Proof. As pointed out before, we can suppose d is large enough (as in the previous theorem), because we can tensor with line bundles until we reach this degree, and this gives an isomorphism. We linearize the action of SL_N on the Quot scheme Q in the invertible sheaf $\mathcal{L}_m$ with m large enough, as before. Then, $Q^{(s)s}(\mathcal{L}_m) = R^{(s)s}$ and there is a GIT projective quotient

$$\pi : R^{ss} \to Q^{ss}(\mathcal{L}_m) \to Q//_{\mathcal{L}_m} \mathrm{SL}_N,$$

and the latter is by definition $M^{ss}(n,d)$. Moreover, using GIT theory, π restricts to a geometric quotient

$$\pi^s : R^s \to Q^s(\mathcal{L}_m) \to Q^s//_{\mathcal{L}_m} \mathrm{SL}_N \doteq M^s(d,n)$$

As we saw in 4.6.2, $R^{(s)s}$ parametrizes the family $\mathcal{U}^{(s)s}$ which has the local universal property, and by 4.6.3 such that two k–points in $R^{(s)s}$ lie in the same orbit if and only the corresponding vector bundles are isomorphic.

Via the criterion given in 3.1.9, since π^s is both a categorical quotient and an orbit space, $M^s(d,n)$ is a coarse moduli space for stable vector bundles on X of rank n and degree d.

To finish the proof, we only need to show that the orbit of a k–point $q : \mathcal{O}_X \to \mathcal{F}$ in R^{ss} is closed if and only if the sheaf $\mathcal{F}$ is polystable.

If $\mathcal{F}$ is not polystable, then there must be a non-split short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are semistable and $\mu(\mathcal{F}') = \mu(\mathcal{F}'') = \mu(\mathcal{F})$. In this case, as in the proof of 4.4.1 we can find a 1–PS λ such that

$$\lim_{t\to 0} \lambda(t) \cdot [q] = [\mathcal{O}_X^N \twoheadrightarrow \mathcal{F}'' \oplus \mathcal{F}'],$$

which shows that the orbit is not closed.

On the other hand, if $\mathcal{F}$ is a polystable sheaf, so we can write

$$\mathcal{F} = \bigoplus_{i=1}^{l} \mathcal{F}_i^{n_i},$$

for non-isomorphic stable vector bundles $\mathcal{F}_i$ with same slope as $\mathcal{F}$, using a Jordan-Hölder filtration (4.3.16). For any k–point $q' : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}'$ in the closure of the orbit of q, using 3.4.11, there must be a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ such that

$$\lim_{t\to 0} \lambda(t) \cdot q = q'.$$

This corresponds to a family $\mathcal{E} \in \mathrm{Coh}(X \times \mathbb{A}^1)$ of semistable locally free sheaves such that $\mathcal{E}_t \simeq \mathcal{F}_t$ for $t \neq 0$, by flatness, and $\mathcal{E}_0 = \mathcal{F}'$. Since the sheaves $\mathcal{F}_i$ are all stable of same slope, we can use 4.3.6, to conclude that

$$\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_j) \simeq \begin{cases} k, & \text{if } i = j \\ 0, & \text{else,} \end{cases}$$

so $\dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}) = \dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_i^{n_i}) = n_i$. Since $\mathcal{E}$ is flat over $\mathbb{A}^1$, the dimension function is upper semi-continuous and $\dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \doteq m_i \geq n_i$. Also by flatness, it follows that the Hilbert polynomial of fibers must be constant, so $\mu(\mathcal{F}') = \mu(\mathcal{F})$. Now, we consider the natural evaluation maps

$$e_i : \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \otimes \mathcal{F}_i \to \mathcal{F}'$$

which apply morphisms to local sections of each $\mathcal{F}_i$. Since $\deg \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') = 0$ and

$$\mu(\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \otimes \mathcal{F}_i) = \frac{\mathrm{rk}\, \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \cdot \deg \mathcal{F}_i}{\mathrm{rk}\, \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \cdot \mathrm{rk}\, \mathcal{F}_i} = \mu(\mathcal{F}_i),$$

each evaluation map must be injective, as it is a non-zero morphism between μ–stable vector bundles of same slope. On the other hand, as $\mathcal{F}_i \not\simeq \mathcal{F}_j$, the sum

$$\sum_{i=1}^{l} \mathcal{F}_i^{m_i} \subset \mathcal{F}'$$

is direct, so by comparing ranks we get $m_i = n_i$ whenever $i = 1, \ldots, l$ and

$$\mathcal{F}' \simeq \bigoplus_{i=1}^{l} \mathcal{F}_i^{n_i} = \mathcal{F}.$$

In particular, any point in the closure of the orbit of q is inside the orbit, and thus the orbits of polystable sheaves are closed. □

Theorem 4.9.3. *The moduli space $M^s(d, n)$ of stable vector bundles is a smooth quasi-projective variety of dimension $n^2(g-1)+1$.*

Proof. Using the local description of the Quot scheme given in 4.5.12, for $q \in R^s$, we denote by $\mathcal{K} \doteq \ker q$ and apply the functor $\mathrm{Hom}(-, \mathcal{F})$ to the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^N \to \mathcal{F} \to 0,$$

to obtain a long exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{K}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{O}_X^N, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) \to 0.$$

As the hom-functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, -)$ and the functor of global sections $\Gamma_{\mathcal{O}_X}(X, -)$ coincide, we have the isomorphism between derived functors $\mathrm{Ext}^1(\mathcal{O}_X^N, \mathcal{F}) \simeq H^1(X, \mathcal{F})^N$.

On the other hand, by our assumption on the degree, $H^1(X,\mathcal{F})^N = 0$, and by the exact sequence $\mathrm{Ext}^1(\mathcal{K},\mathcal{F}) = 0$, so Q is smooth in a neighbourhood of every point q.

To compute the dimension, we consider the same long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F},\mathcal{F}) \to \mathrm{Hom}(\mathcal{O}_X^N,\mathcal{F}) \to \mathrm{Hom}(\mathcal{K},\mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to 0,$$

and we conclude $\dim \mathrm{Hom}(\mathcal{F},\mathcal{F}) = 1$, as every stable sheaf is simple, and that $\dim \mathrm{Hom}(\mathcal{O}_X^N,\mathcal{F}) = N^2$, since $\mathrm{Ext}^1(\mathcal{O}_X^N,\mathcal{F}) = 0$. Moreover,

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \simeq \mathrm{Ext}^1(\mathcal{O}_X,\mathcal{F}^\vee \otimes \mathcal{F}) \simeq H^1(X,\mathcal{F}^\vee \otimes \mathcal{F}),$$

which has dimension $n^2(g-1)+1$ by the Riemann-Roch formula. Hence, we conclude

$$\dim R^{ss} = \dim T_q Q = \dim \mathrm{Hom}(\mathcal{K},\mathcal{F}) = n^2(g-1)+1+N^2-1 = n^2(g-1)+N^2.$$

Since SL_N acts with finite global stabilizer on the smooth quasi-projective variety R^s and the quotient $R^s \to M^s(d,n)$ is a geometric quotient, it follows from Luna's Slice theorem (see (DRéZET, 2004)) that the moduli space $M^s(d,n)$ is smooth. Furthermore, we can compute the dimension using the formula

$$\dim M^s(d,n) = \dim R^s - \dim \mathrm{SL}_N = \dim R^s - (N^2-1) = n^2(g-1)+1.$$

□

The next proposition and corollary assures that the space $M^s(d,n)$ is a fine moduli space for the moduli problem of vector bundles over a curve when $(n,d) = 1$. For proofs, see (NEWSTEAD, 2012).

Proposition 4.9.4. Let S be a k–scheme. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two families of vector bundles over $S \times X$ such that, whenever $s \in S$, the fiber $(\mathcal{E}_1)_s$ is stable over X and $(\mathcal{E}_1)_s \simeq (\mathcal{E}_2)_s$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent families for the moduli problem of vector bundles over a curve, as defined in 4.4.

Proof. See (NEWSTEAD, 2012), Lemma 5.10. □

Corollary 4.9.4.1. If there is a family $\mathcal{U}$ over $M^s(d,n) \times X$ such that, whenever $s \in M^s(d,n)$, $\mathcal{U}_s$ is the stable vector bundle corresponding to the point s, then $M^s(d,n)$ is a fine moduli space for the moduli problem of stable vector bundles of rank n and degree d over X.

Proof. This follows from the previous proposition and the characterization 1.2.4 for fine moduli spaces. □

For the construction of such family when $(n,d) = 1$, we refer the reader to (NEWSTEAD, 2012), Lemma 5.11.

Remark 4.9.5. The behaviour of the moduli functor when $g = 1$ is somewhat different, and it is studied in detail in (TU, 1993). The moduli space of stable vector bundles of coprime (n,d) over X is in fact isomorphic to the curve X, and the moduli space for semistable vector bundles can be described as a symmetric product over X.

Remark 4.9.6. For explicit descriptions of the moduli space $M^s(d,n)$ when $n = 2$ and $d = 0,1$ when $g = 2$, see (NARASIMHAN; RAMANAN, 1969). In (NARASIMHAN; SESHADRI, 1965), M. S. Narasimhan and C. S. Seshadri use representation theory to describe the stable vector bundles over curves with genus $g \geq 2$.

For the case when X is an algebraic variety of higher dimension, we refer the reader to (HUYBRECHTS; LEHN, 2010).

Bibliography

A'CAMPO, N.; JI, L.; PAPADOPOULOS, A. *On the early history of moduli and Teichmüller spaces*. 2016. Cited on page 10.

BALAJI, T.; NATIONALBIBLIOTHEK, D. *An Introduction to Families, Deformations and Moduli*. Universitätsverlag Göttingen, 2010. (Universitätsdrucke Göttingen). ISBN 9783941875326. Disponível em: <https://books.google.com.br/books?id=Ypp2kZiNXXEC>. Cited on page 15.

CONCINI, C. D.; PROCESI, C. *The Invariant Theory of Matrices*. American Mathematical Society, 2017. (University Lecture Series). ISBN 9781470441876. Disponível em: <https://books.google.com.br/books?id=vSI_DwAAQBAJ>. Cited on page 70.

DOLGACHEV, I. *Introduction to Geometric Invariant Theory*. Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1994. (Lecture Notes Series - Seoul National University, Research Institute of Mathematics, Global Analysis Research Center). Disponível em: <https://books.google.com.br/books?id=T_vuAAAAMAAJ>. Cited on page 102.

DRéZET, J.-M. Luna's slice theorem and applications. In: WISNIEWSKI, J. A. (Ed.). *Algebraic group actions and quotients*. Hindawi Publishing Corporation, 2004. p. 39–90. Disponível em: <https://hal.archives-ouvertes.fr/hal-00742479>. Cited 2 times on pages 71 and 149.

EISENBUD, D.; HARRIS, J.; HARRIS, J. *The Geometry of Schemes*. Springer, 2000. (Graduate Texts in Mathematics). ISBN 9780387986371. Disponível em: <https://books.google.com.br/books?id=BpphspzsasEC>. Cited 2 times on pages 38 and 161.

FANTECHI, B.; GOTTSCHE, L.; ILLUSIE, L. *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*. American Mathematical Society, 2005. (Mathematical surveys and monographs). ISBN 9780821842454. Disponível em: <https://books.google.com.br/books?id=KxH0BwAAQBAJ>. Cited 4 times on pages 19, 25, 127, and 132.

FREUDENBURG, G. A SURVEY OF COUNTEREXAMPLES TO HILBERT'S FOURTEENTH PROBLEM. *Serdica Mathematical Journal*, Bulgarian Academy of Sciences, v. 27, n. 10, p. 171 – 192, 2001. Disponível em: <https://doi.org/>. Cited on page 24.

GÖRTZ, U.; WEDHORN, T. *Algebraic Geometry: Part I: Schemes. With Examples and Exercises*. Vieweg+Teubner Verlag, 2010. (Advanced Lectures in Mathematics). ISBN 9783834897220. Disponível em: <https://books.google.com.br/books?id=XEiLudn6sq4C>. Cited on page 164.

GROTHENDIECK, A. éléments de géométrie algébrique : Iv. étude locale des schémas et des morphismes de schémas, première partie. *Publications Mathématiques de l'IHÉS*, Institut des Hautes Études Scientifiques, v. 20, p. 5–259, 1964. Disponível em: <http://www.numdam.org/item/PMIHES_1964__20__5_0/>. Cited on page 43.

_____. éléments de géométrie algébrique : Iv. étude locale des schémas et des morphismes de schémas, troisième partie. *Publications Mathématiques de l'IHÉS*, Institut des Hautes Études Scientifiques, v. 28, p. 5–255, 1966. Disponível em: <http://www.numdam.org/item/PMIHES_1966__28__5_0/>. Cited on page 163.

HABOUSH, W. J. Reductive groups are geometrically reductive. *Annals of Mathematics*, Annals of Mathematics, v. 102, n. 1, p. 67–83, 1975. ISSN 0003486X. Disponível em: <http://www.jstor.org/stable/1970974>. Cited on page 52.

HARDER, G.; DIEDERICH, K. *Lectures on Algebraic Geometry II: Basic Concepts, Coherent Cohomology, Curves and their Jacobians*. Vieweg+Teubner Verlag, 2011. (Aspects of Mathematics). ISBN 9783834881595. Disponível em: <https://books.google.com.br/books?id=BDlsjgEACAAJ>. Cited on page 112.

HARRIS, J.; MORRISON, I. *Moduli of Curves*. Springer New York, 1998. (Graduate Texts in Mathematics). ISBN 9780387984292. Disponível em: <https://books.google.com.br/books?id=bQ3NDk-i7I8C>. Cited on page 19.

HARTSHORNE, R. *Algebraic Geometry*. [S.l.]: Springer-Verlag New York, 1977. (Graduate Texts in Mathematics). ISBN 978-0-387-90244-9. Cited 15 times on pages 10, 20, 39, 64, 73, 79, 86, 88, 91, 103, 104, 106, 111, 161, and 168.

HILBERT, D. Mathematical problems. *Bulletin of the American Mathematical Society*, American Mathematical Society, v. 8, n. 10, p. 437 – 479, 1902. Disponível em: <https://doi.org/>. Cited on page 24.

HOSKINS, V. *MODULI PROBLEMS AND GEOMETRIC INVARIANT THEORY*. [s.n.], 2015. Disponível em: <https://www.math.ru.nl/~vhoskins/M15_Lecture_notes.pdf>. Cited 10 times on pages 6, 7, 10, 15, 25, 52, 56, 102, 103, and 114.

HUYBRECHTS, D.; LEHN, M. *The Geometry of Moduli Spaces of Sheaves*. Cambridge University Press, 2010. (Cambridge Mathematical Library). ISBN 9781139485821. Disponível em: <https://books.google.com.br/books?id=_mYV1q0RVzIC>. Cited 6 times on pages 112, 114, 123, 132, 133, and 150.

JACOBSON, N. *Basic Algebra I: Second Edition*. Dover Publications, 2009. (Basic Algebra). ISBN 9780486471891. Disponível em: <https://books.google.com.br/books?id=qAg_AwAAQBAJ>. Cited 2 times on pages 33 and 110.

LAZA, R. *GIT and moduli with a twist*. 2012. Cited on page 87.

LEE, D. *The Structure of Complex Lie Groups*. CRC Press, 2001. (Chapman & Hall/CRC Research Notes in Mathematics Series). ISBN 9781420035452. Disponível em: <https://books.google.com.br/books?id=RXbLBQAAQBAJ>. Cited on page 48.

LIU, Q.; ERNE, R. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, 2006. (Oxford Graduate Texts in Mathematics (0-19-961947-6)). ISBN 9780191547805. Disponível em: <https://books.google.com.br/books?id=ePpzBAAAQBAJ>. Cited 4 times on pages 38, 71, 108, and 162.

MILNE, J. *Basic Theory of Affine Groups Schemes*. [s.n.], 2012. Disponível em: <https://www.jmilne.org/math/CourseNotes/AGS.pdf>. Cited 5 times on pages 25, 32, 46, 47, and 48.

_____. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge University Press, 2017. (Cambridge Studies in Advanced Mathematics). ISBN 9781316739150. Disponível em: <https://books.google.com.br/books?id=iD41DwAAQBAJ>. Cited 5 times on pages 25, 43, 46, 47, and 48.

MUKAI, S.; OXBURY, W.; BOLLOBAS, B.; SHIGERU, M.; FULTON, W.; KATOK, A.; KIRWAN, F.; SARNAK, P.; SIMON, B. *An Introduction to Invariants and Moduli*. Cambridge University Press, 2003. (Cambridge Studies in Advanced Mathematics). ISBN 9780521809061. Disponível em: <https://books.google.com.br/books?id=oUcMZbiM7eAC>. Cited 2 times on pages 49 and 52.

MUMFORD, D.; FOGARTY, J.; KIRWAN, F. *Geometric Invariant Theory*. Springer Berlin Heidelberg, 1994. (Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3 Folge/A Series of Modern Surveys in Mathematics Series). ISBN 9783540569633. Disponível em: <https://books.google.com.br/books?id=dFlv3zn_2-gC>. Cited 4 times on pages 10, 66, 82, and 98.

NAGATA, M. Invariants of group in an affine ring. *Journal of Mathematics of Kyoto University*, Duke University Press, v. 3, n. 3, p. 369 – 378, 1963. Disponível em: <https://doi.org/10.1215/kjm/1250524787>. Cited on page 52.

_____. Lectures on the fourteenth problem of hilbert. In: . [S.l.: s.n.], 1965. Cited on page 24.

NARASIMHAN, M. S.; RAMANAN, S. Moduli of vector bundles on a compact riemann surface. *Annals of Mathematics*, Annals of Mathematics, v. 89, n. 1, p. 14–51, 1969. ISSN 0003486X. Disponível em: <http://www.jstor.org/stable/1970807>. Cited on page 150.

NARASIMHAN, M. S.; SESHADRI, C. S. Stable and unitary vector bundles on a compact riemann surface. *Annals of Mathematics*, Annals of Mathematics, v. 82, n. 3, p. 540–567, 1965. ISSN 0003486X. Disponível em: <http://www.jstor.org/stable/1970710>. Cited on page 150.

NEWSTEAD, P. *Introduction to Moduli Problems and Orbit Spaces*. Published for the TIFR (Tata Institute of Fundamental Research), 2012. (Lectures on Mathematics and Physics). ISBN 9788184871623. Disponível em: <https://books.google.com.br/books?id=W7WYuAAACAAJ>. Cited 5 times on pages 10, 103, 114, 149, and 150.

OLSSON, M. *Algebraic Spaces and Stacks*. American Mathematical Society, 2016. (Colloquium Publications). ISBN 9781470427986. Disponível em: <https://books.google.com.br/books?id=TvQrDAAAQBAJ>. Cited on page 19.

REID, M.; SHAFAREVICH, I. *Basic Algebraic Geometry 1*. Springer Berlin Heidelberg, 2013. ISBN 9783642579080. Disponível em: <https://books.google.com.br/books?id=U7zuCAAAQBAJ>. Cited on page 18.

RIEHL, E. *Category Theory in Context*. Dover Publications, 2016. (Aurora: Dover Modern Math Originals). ISBN 9780486809038. Disponível em: <https://books.google.com.br/books?id=Sr09DQAAQBAJ>. Cited 6 times on pages 15, 28, 155, 156, 159, and 160.

ROTMAN, J. *An Introduction to Homological Algebra*. Springer New York, 2008. (Universitext). ISBN 9780387683249. Disponível em: <https://books.google.com.br/books?id=P2HV4f8gyCgC>. Cited on page 167.

SMITH, L. *Polynomial Invariants of Finite Groups*. Taylor & Francis, 1995. (Research Notes in Mathematics). ISBN 9781568810539. Disponível em: <https://books.google.com.br/books?id=6pyFbDZz9AkC>. Cited on page 67.

TU, L. Semistable bundles over an elliptic curve. *Advances in Mathematics*, v. 98, n. 1, p. 1–26, 1993. ISSN 0001-8708. Disponível em: <https://www.sciencedirect.com/science/article/pii/S000187088371011X>. Cited on page 150.

WEIBEL, C. *An Introduction to Homological Algebra*. Cambridge University Press, 1995. (Cambridge Studies in Advanced Mathematics). ISBN 9781139643078. Disponível em: <https://books.google.com.br/books?id=UtIhAwAAQBAJ>. Cited 2 times on pages 159 and 167.

APPENDIX A – Category Theory

In this appendix, we review some of the category theory necessary for this dissertation.

A.1 Category Theory

For the categorical introduction, we will follow (RIEHL, 2016).

Definition A.1.1. A category $\mathcal{C}$ consists of:

- A collection of objects $\mathrm{Ob}(\mathcal{C})$.
- A collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$ whenever A,B are objects in $\mathcal{C}$, which satisfy:
  1. Each morphism f has a specified **domain** $A = \mathrm{dom(f)} \in \mathcal{C}$ and **codomain** $B = \mathrm{codom}(f) \in \mathcal{C}$ objects, and in notation we usually write them
  $$A \xrightarrow{f} B,$$
  whenever $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$.
  2. For any pair of morphisms f,g with the compatibility condition
  $$\mathrm{codom}(f) = \mathrm{dom}(g),$$
  we can consider the **composite morphism**, which we denote by $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, which we usually represent as:
  $$A \xrightarrow{f} B \xrightarrow{g} C.$$
  3. Each object $A \in \mathcal{C}$ comes with a morphism $\mathrm{Id}_A \in \mathcal{C}$, which satisfies the rules of composition
  $$A \xrightarrow{\mathrm{Id}_A} B \xrightarrow{f} C, B \xrightarrow{g} A \xrightarrow{\mathrm{Id}_A} A,$$
  i.e., $f \circ \mathrm{Id}_A = f$ and $\mathrm{Id}_A \circ g$ whenever f,g are compatible as described in the diagram.
  4. The composition operation is also **associative**.

For formal reasons, the word **collection** cannot be replaced by the word **set** always, and there is a lot of work that goes into studying formalisms inside category theory. Since we are using category theory as a tool and as a unifying language, we refer to (RIEHL, 2016) for more comments on this. A **small** category $\mathcal{C}$ is a category where both collections of objects and morphisms are sets, and we say a category is **locally small** if, whenever A, B are objects in $\mathcal{C}$, the collection $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.

Definition A.1.2. A (covariant) **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a transformation $F : \mathcal{C} \to \mathcal{D}$ which consists of the following information:

- An object $F(A) \in \mathrm{Ob}(\mathcal{D})$ for each object $A \in \mathcal{C}$;
- For each pair $A, B \in \mathrm{Ob}(\mathcal{C})$, there is also a mapping

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

  usually also denoted by F, which satisfies:

  1. $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ for $A \in \mathrm{Ob}(\mathcal{C})$.
  2. F preserves commuting diagrams. More explicitly, we only need to ask the following condition: that F takes the (trivially) commuting diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

  in $\mathcal{C}$ into the following diagram $\mathcal{D}$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g \circ f)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd}$$

  such that it also commutes in $\mathcal{D}$, i.e., $F(g \circ f) = F(g) \circ F(f)$.

Example A.1. Let S be a set. A **preorder** $\leq$ on S is a binary relation that is reflexive and transitive, i.e., it satisfies:

- $x \leq x$ whenever $x \in X$;
- If $x \leq y$ and $y \leq z$, then $x \leq z$ whenever $x, y, z \in S$.

The pair $(S, \leq)$ is called a **partially ordered set**, usually abbreviated as **poset**.

If $(S, \leq)$ is a poset, we define the **induced category** of $(S, \leq)$ by the category $\mathcal{C}$ defined as below:

$$\mathrm{Ob}(\mathcal{C}) \doteq S$$

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \doteq \begin{cases} \{*\}, & \text{if } x \leq y \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $\{*\}$ denotes the set of one element. Since S is a set, the category $\mathcal{C}$ is a small category.

Definition A.1.3. Given any category $\mathcal{C}$, we can consider the formal procedure of inverting all arrows in it. This gives rise to the definition of **dual category** $\mathcal{C}^{\text{op}}$ defined as

$$\begin{aligned} \text{Ob}(\mathcal{C}^{\text{op}}) &\doteq \text{Ob}(\mathcal{C}) \\ \text{Hom}_{\mathcal{C}^{\text{op}}}(A,B) &\doteq \text{Hom}_{\mathcal{C}}(B,A) \end{aligned}$$

and, whenever

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a diagram in $\mathcal{C}$, we have the composition

$$C \xrightarrow{g^{\text{op}}} B \xrightarrow{f^{\text{op}}} C.$$

In this case, a (covariant) functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ can also be viewed as a functor which acts on the category $\mathcal{C}$ in a **contravariant** manner, and by this we mean that it inverts all the diagrams which commutes in $\mathcal{C}$. This is also referred as a **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$.

Definition A.1.4. If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, then a **natural transformation** $\eta : F \to G$ is a family of morphisms $\{\eta_A : F(A) \to G(A) : A \in \mathcal{C}_0\}$ in $\mathcal{D}$ such that, whenever $f : A \to B$ is a morphism in $\mathcal{C}$ the diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

commutes.

Definition A.1.5. Given a category $\mathcal{C}$ and a small category I and a functor $F : I \to \mathcal{C}$, we can define a **cone over** F with summit $c \in \text{Ob}(C)$ as a natural transformation

$$\lambda : c(J) \Rightarrow F$$

whose domain is the constant functor $c = c(J)$ defined by

$$\begin{aligned} c(J) : J &\to \mathcal{C} \\ j &\mapsto c \\ (i \to j) &\mapsto \text{Id}_c\,. \end{aligned}$$

More explicitly, we have a family of morphisms $(\lambda_j : c \to F_j)_{i \in J}$ which commutes the diagram

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ F_j \arrow[rr, "F(f)"] & & F_k \end{tikzcd}$$

in $\mathcal{C}$ whenever $f : j \to k$ is a morphism in J.

Dually, a **cone under** F with nadir c is a natural transformation $\lambda : F \Rightarrow c$ which components $(\lambda_j : F_j \to c)_{j \in J}$ commute the dual diagram

$$\begin{tikzcd} F_j \arrow[rr, "F(f)"] \arrow[dr, "\lambda_j"'] & & F_k \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

A **limit** of F is an object $\lim F \in \mathcal{C}$ with a cone above F given by $\lambda : \lim F \Rightarrow F$ which is **universal** among the cones above F, i.e., whenever $\eta : c \to F$ is a cone above F, then there is a unique morphism in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} \lim F \arrow[r, "\lambda"] & F \\ c \arrow[u, dotted] \arrow[ur, "\eta"'] & \end{tikzcd}$$

commutes in the category of functors of $\mathcal{C}$. Dually, a **colimit** of F is an object $\operatorname{colim} F \in \mathcal{C}$ with a cone below F given by $\lambda : F \Rightarrow \operatorname{colim} F$ which is universal among the cones below F in the sense that whenever $\eta : F \Rightarrow c$ is a cone below F.

In practice, the category I can be chosen for each particular problem we are studying in $\mathcal{C}$.

Example A.2. A **terminal object** in a category $\mathcal{C}$ is a limit indexed by an empty category. Dually, a **initial object** in $\mathcal{C}$ is a colimit indexed by an empty category. When these two exist and are isomorphic, we say $\mathcal{C}$ has a **zero object**, and denote it by 0.

This is the case, for example, in the category of R–modules, where R is a commutative ring.

Example A.3. An **equalizer** in a category $\mathcal{C}$ is a limit of a diagram indexed by the **parallel pair**, the category with 2 objects and 2 parallel, non-identity morphisms:

$$\bullet \rightrightarrows \bullet$$

For example, the **kernel** of a morphism $T : V \to W$ of R–modules is the equalizer between T and the zero morphism between V and W.

Dually, a **coequalizer** is a colimit of a diagram indexed by the parallel pair.

For more examples and definitions for (co)limits, see (RIEHL, 2016), Chapter 3.

Definition A.1.6. An **ajunction** consists of a pair of functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ with an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

whenever C is an object of $\mathcal{C}$ and $\mathcal{D}$, which is natural in both variables. In this case, F is called **left adjoint** to G and G is called **right adjoint** to F.

Adjoint functors are a neat way of relating two categories, and they usually are useful since they preserve the universal objects.

Theorem A.1.7 (see (RIEHL, 2016), 4.5)**.** *Right adjoint functors preserve limits. Dually, left adjoint functors preserve colimits.*

A.2 Abelian Categories

For definitions, we follow the appendix $A.4$ on (WEIBEL, 1995).

Definition A.2.1. A category $\mathcal{A}$ is an Ab −**category** if every hom-set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ is given a structure of an abelian group in such a way that composition distributes over addition.

More explicitly, whenever A, B, C, Z are objects in $\mathcal{A}$, and there are diagrams:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & B \arrow[r, "g"] & C \end{tikzcd} \; , \quad \begin{tikzcd} Z \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & B \end{tikzcd} \; ,$$

then $g(f + f') = gf + gf'$ and $h(f + f') = hf + hf'$. Note that, whenever A, B are objects, there is a **zero arrow** $0 \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, the identity of the abelian group structure.

An Ab −category $\mathcal{A}$ is called **additive** if it has a zero object and it has binary products. In and additive category, whenever $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, then the **kernel** of f is the equalizer of f and the zero map. Dually, the **cokernel** of f is the coequalizer of f and the zero map.

An additive category $\mathcal{A}$ is **abelian** if it satisfies:

1. Every map has a kernel and a cokernel.
2. Every monomorphism in $\mathcal{A}$ is the kernel of its cokernel, and
3. Every epimorphism in $\mathcal{A}$ is the cokernel of its kernel.

APPENDIX B – Sheaves and Schemes

Definition B.0.1. Let (X,τ) be a topological space, where we denote by τ its topology, i.e., the set of all open sets of X. Since τ admits a preorder induced by the inclusion, we can consider $(\tau,\subset)$ as a category. A **presheaf** is a functor

$$\mathcal{F}:(\tau^{\mathrm{op}},\leq)\to\mathcal{C},$$

where $\mathcal{C}$ is a concrete category.

Definition B.0.2. A **sheaf of sets** $\mathcal{C}=$ Sets over a topological space (X,τ) is a presheaf $\mathcal{F}:(\tau^{\mathrm{op}},\leq)\to$ Sets which satisfies the following gluing condition: given any open covering $\{U_i\}_{i\in I}$ of X and any collection of sections $s_i\in\mathcal{F}(U_i)$, whenever $i\in I$, such that whenever $i,j\in I$,

$$s_i\big|_{U_i\cap U_j}=s_j\big|_{U_i\cap U_j},$$

then there is a unique section $s\in\mathcal{F}(U)$ such that $s_i=s\big|_{U_i}$.

More generally, if $\mathcal{C}$ is any concrete category, we ask the same condition, since in this case the restriction maps are also morphisms in $\mathcal{C}$. There is also an equivalent categorical definition:

Proposition B.0.3. A presheaf of sets $\mathcal{F}:\tau^{\mathrm{op}}\to$ Sets is a sheaf if and only if it preserves colimits, sending them to limits in Sets.

For proofs and more comments on this relation, see (RIEHL, 2016), chapter 3.

Theorem B.0.4 (Gluing lemma). *[] Let $\{U_i\}_{i\in I}$ be an open cover of a topological space X such that, for each $i\in I$, there is a sheaf $\mathcal{F}_i$ over U_i and, for each $i,j\in I$, there is also an isomorphism of sheaves*

$$\theta_{ij}:\mathcal{F}_j\big|_{U_i\cap U_j}\to\mathcal{F}_i\big|_{U_i\cap U_j}$$

*satisfying the conditions below, which are called **gluing conditions**:*

(i) $\theta_{ii}=\mathrm{Id}_{\mathcal{F}_i|_{U_i\cap U_j}}$

(ii) For all $i,j,k\in I$, the restrictions to $U_i\cap U_j\cap U_k$ satisfy

$$\theta_{ik}=\theta_{ij}\circ\theta_{jk}.$$

Then there is a unique sheaf $\mathcal{F}$ over X and isomorphisms $\eta_i:\mathcal{F}\big|_{U_i}\to\mathcal{F}_i$ such that $\eta_i\circ\eta_j^{-1}=\theta_{ij}$.

Definition B.0.5. Let X, Y be schemes, and $f : U \to Y$, $g : V \to Y$ be scheme morphisms defined on open dense sets $U, V \subset X$. We can define the following equivalence relation:

$$f \simeq g \iff f\big|_W = g\big|_W$$

where $W \subset U \cap V$ is an open dense set of X. A **rational function** $h : X \dashrightarrow Y$ is an equivalence class of this relation.

Definition B.0.6. (HARTSHORNE, 1977) A subset $A \subset X$ is **locally closed** if its a closed subspace of an open subset of X. Explicitly, there exists $U \subset X$ open and $F \subset X$ closed such that $A = U \cap F$.

Note that, if $A \subset X$ is a subset such that $A \subset \overline{A}$ is an open set in $\overline{A}$, then there exists an open set $U \subset X$ such that $A = \overline{A} \cap U$, and this means that A is locally closed. This can be used as an alternative definition, since these facts are equivalent.

Note that, if X is a scheme, a locally closed subspace A can always be seen as an closed subset of the open subscheme $X \setminus \partial A$.

Definition B.0.7. (EISENBUD; HARRIS; HARRIS, 2000) Let $(X, \mathcal{O}_X)$ be a scheme.

1. For each $x \in X$, we define the (algebraic) **dimension of** X **at** x as the krull dimension of the local ring $\mathcal{O}_{X,x}$, and denote it by $\dim(X, x)$. This way we define the (algebraic) dimension of X to be:

$$\dim X = \sup_{x \in X} \dim(X, x)$$

2. For each $x \in X$, we define the **Zariski cotangent space** to X at x to be $m_{X,x}/m_{X,x}^2$, as a vector space over $k(x)$.

3. For each $x \in X$, we say that X is **nonsingular** (or **regular**) at x if

$$\dim(X, x) = \dim_{k(x)} \frac{m_{X,x}}{m_{X,x}^2}$$

4. We define the **topological dimension of** X to be:

$$\dim X = \sup\{l : \exists F_0 \subsetneq \cdots \subsetneq F_l = X \text{ s.t. } F_i \text{ are closed irreducible sets}\}$$

Definition B.0.8. (HARTSHORNE, 1977) Let A be a ring and M be a A–module. We say that M is **flat over** A if the functor $\mathrm{Mod}_A \to \mathrm{Mod}_M$ given by $N \mapsto M \otimes_A N$ is an exact functor. Since the functor $- \otimes_A M$ is right-exact, M is a flat module over A if and only if the functor $- \otimes_A M$ preserves injections.

Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$–module and $y = f(x) \in Y$. We can induce a structure of $\mathcal{O}_{Y,y}$–module in $\mathcal{F}_x$, simply by using the induced morphism

$$f_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$

We say that $\mathcal{F}$ is **flat over** Y **at** y if $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$, and say X **is flat over** Y if $\mathcal{O}_X$ is flat over Y.

Example B.1. 1. (Free Modules) Let $M = \oplus_{e\in E} Ae$. If $f : N \to N'$ is an $A-$module morphism, the map

$$f \otimes Id : N \otimes_A \left(\bigoplus_{e\in E} Ae \right) \to N' \otimes_A \left(\bigoplus_{e\in E} Ae \right)$$

is isomorphic to

$$\oplus f : \bigoplus_{e\in E} Ne \to \bigoplus_{e\in E} N'e.$$

Since the injectivity of f implies the injectivity of $\oplus f$, this means that M is a flat module over A.

2. If k is a field, every $k-$module is free over k, and this means that every $k-$algebra is flat over k.

Lemma B.0.9. 1. If B is a flat $A-$algebra and C is a flat $B-$algebra, then C is a flat $A-$algebra.

2. If M is a flat $A-$algebra and B is an $A-$algebra, $M \otimes_A B$ is flat over B.

3. M is flat over A if and only if M_p is flat over A_p for each $p \in \operatorname{Spec} A$.

4. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A-$modules. If M' and M'' are flat, M is flat. If M and M'' are flat, M' is flat.

This lemma has a version on the category of schemes:

Lemma B.0.10. [(LIU; ERNE, 2006), 4.3.1, Prop 3.3] The following are true:

(a) Open immersions are flat.

(b) Flat morphisms are stable by base change.

(c) The composition of two flat morphisms is flat.

(d) The morphism $A \to B$ is flat if and only if the induced $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

Proposition B.0.11. Let $f : X \to Y$ be a flat morphism of schemes of finite type over k. For any $x \in X$, $y = f(x)$, we have the formula:

$$\dim_x X_y = \dim_x X - \dim_y Y,$$

where $X_y \doteq X \times_Y \operatorname{Spec} k(y)$ is the preimage.

Theorem B.0.12. *[(GROTHENDIECK, 1966), 11.1.1] Let Y be a locally noetherian scheme, $f : X \to Y$ a morphism of finite type and $\mathcal{F}$ an coherent $\mathcal{O}_X$–module. Then the set:*

$$U \doteq \{x \in X : \mathcal{F}_x \text{ is flat over } Y \text{ at } y = f(x)\}$$

is an open dense subset of X.

Lemma B.0.13. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes and $\mathcal{L}$ be an $\mathcal{O}_Y$–module that is locally free of finite rank. Then

$$f^*(\mathcal{L})^\vee \simeq f^*(\mathcal{L}^\vee).$$

Proof. By definition, we have the canonical isomorphism:

$$\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{L}^\vee \xrightarrow{\simeq} \mathcal{O}_Y.$$

Since the pullbacks commute with tensor products, we can pullback this isomorphism to an isomorphism

$$f^*(\mathcal{L}) \otimes_{\mathcal{O}_X} f^*(\mathcal{L}^\vee) \simeq f^*(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{L}^\vee) \to f^*\mathcal{O}_X \simeq \mathcal{O}_Y.$$

And this means that $f^*(\mathcal{L}^\vee)$ is the dual of $f^*(\mathcal{L})$. □

Lemma B.0.14. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and E, F be two locally free $\mathcal{O}_Y$–modules of finite rank. Then there is an isomorphism:

$$f^* \operatorname{Hom}_Y(E, F) \simeq \operatorname{Hom}_X(f^*E, f^*F)$$

Proof. It suffices to construct a map

$$\operatorname{Hom}_Y(E, F) \to f_* \operatorname{Hom}_X(f^*E, f^*F)$$

and use the adjunction of f_* and f^*. Let $U \subset Y$ be open. Then

$$f_* \operatorname{Hom}_X(E, F)(U) = \operatorname{Hom}_X\left(f^*E\big|_{f^{-1}(U)}, f^*F\big|_{f^{-1}(U)}\right).$$

We know that $f^*\big|_{f^{-1}(U)}$ is a functor from sheaves of $\mathcal{O}_U$ modules to sheaves of $\mathcal{O}_{f^{-1}(U)}$ modules. Since

$$f^*E\big|_{f^{-1}(U)} = f^*\big|_{f^{-1}(U)}\left(E\big|_{f^{-1}(U)},\right)$$

from the functoriality, we get a morphism:

$$\begin{aligned}\operatorname{Hom}(E|_U, F|_U) &\to \operatorname{Hom}\left(f^*\big|_{f^{-1}(U)}\left(E\big|_{f^{-1}(U)}\right), f^*\big|_{f^{-1}(U)}\left(F\big|_{f^{-1}(U)}\right)\right) \\ &\to \operatorname{Hom}(f^*E\big|_{f^{-1}(U)}, f^*F\big|_{f^{-1}}(U)).\end{aligned}$$

As every operation respects restrictions, this will induce a map of sheaves

$$\operatorname{Hom}_Y(E, F) \to f_* \operatorname{Hom}_X(f^*E, f^*F).$$

Via the adjunction, we get the corresponding map

$$f^* \operatorname{Hom}_Y(E,F) \to \operatorname{Hom}_X(f^*E, f^*F).$$

To prove that this map is an isomorphism of sheaves, we fix a point $y \in X$ and look at the corresponding morphism of stalks. We have a bijection (see (GÖRTZ; WEDHORN, 2010), Chapter 7, Proposition 7.27)

$$\operatorname{Hom}_Y(E,F)_y \simeq \operatorname{Hom}_{\mathcal{O}_{Y,y}}(E_y, F_y).$$

Therefore, taking stalks, if $f(x) = y$:

$$\operatorname{Hom}_{\mathcal{O}_{Y,y}}(E_y, F_y) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \operatorname{Hom}_{\mathcal{O}_{X,x}}(E_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}, F_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x})$$

and this is an isomorphism by the corresponding result on base change of modules. □

APPENDIX C – Homological algebra

C.1 Exact and Derived Functors

In this appendix, we review some useful constructions and examples of derived functors.

Definition C.1.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. A **cohomological δ-functor** is a collection of functors $F^i : \mathcal{A} \to \mathcal{B}$ such that, whenever we have a short exact sequence on $\mathcal{A}$

$$0 \to A \to B \to C \to 0$$

there is a corresponding long exact sequence on $\mathcal{B}$:

$$0 \to F(A) \to F(B) \to F(C) \to F^1(A) \to F^1(B) \to F^1(C) \to \cdots$$

such that this procedure is functorial in the category of short exact sequences over $\mathcal{A}$.

We can consider a category of such δ–functors of F, where each morphism between δ–functors of F: $\phi : (S^n)_{n\in\mathbb{N}} \to (T^n)_{n\in\mathbb{N}}$ is a family of natural transformations $\phi = (\phi^n : S^n \to T^n)_{n\in\mathbb{N}}$ such that, for every short sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_S^{n-1}"] & S^n(A) \arrow[r] \arrow[d, "\phi_n(A)"'] & S^n(B) \arrow[r] \arrow[d, "\phi_n(B)"] & S^n(C) \arrow[r, "\delta_S^n"] \arrow[d, "\phi_n(C)"] & S^{n+1}(A) \arrow[r] \arrow[d, "\phi_{n+1(A)}"] & \cdots \\
\cdots \arrow[r, "\delta_T^{n-1}"] & T^n(A) \arrow[r] & T^n(B) \arrow[r] & T^n(C) \arrow[r, "\delta_T^n"] & T^{n+1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

on the category of long exact sequences of elements of $\mathcal{B}$.

The terminal object of the category of δ–functors of F is usually written as $(\mathcal{R}^iF)_{i\geq 0}$ and each $\mathcal{R}^iF$ is called the **i-th right-derived functor**

Of course, we could do all of this dually when F is a right-exact functor, with a collection called **homological δ-functor**, and the corresponding **i-th left-derived functors** denoted by $\mathcal{L}_iF$.

We now list some derived functors that exist and some properties of them.

Definition C.1.2 (Sheaf Cohomology)**.** Let X be a topological space. Let $\mathrm{Sh}(X)$ be the category of sheaves of abelian groups on X, and Ab the category of abelian groups. We can consider the **global sections functor**:

$$\Gamma(X, \cdot) : \mathrm{Sh}(X) \to \mathrm{Ab}\,.$$

This is a left-exact functor, and we denote the $i - th$ right derived functor of $\Gamma(X, \cdot)$ by $H^i(X, \cdot)$, and these are called the **i-th sheaf cohomology functor**.

We have the equivalent manner of defining sheaf cohomology on ringed spaces:

Theorem C.1.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of $\Gamma(X, \cdot)$: $\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Ab}$ coincide with the Cech cohomology functors $H^i(X, \cdot)$.*

For more results on Cech Cohomology, see C.2.

Definition C.1.4 (Torsion of modules)**.** Let M be an R–module. Then the functor $M \otimes -$ is right-exact, and it has left-derived functors, which are usually denoted by $\mathrm{Tor}_i^R(M, -)$ and are called **Tor functors**. Since the tensor product is commutative, we can regard Tor as a functor of two variables, and they both carry short exact sequences to corresponding long exact sequences.

It follows immediately by the definition that we have $\mathrm{Tor}_1(M, M) = 0$.

Proposition C.1.5. If $x \in R$ is a non-zero divisor, then we have

$$\mathrm{Tor}_1(R/(x), M) = \{m \in M : xm = 0\},$$

which is called the **tersion submodule** of M for $x \in R$.

Proof. This follows from the derived functor property: If we consider the following exact sequence of R–modules:

$$0 \to (x) \to R \to R/(x) \to 0.$$

Since $- \otimes M$ is right exact, we get the long sequence

$$\mathrm{Tor}_1(M, M) \to \mathrm{Tor}_1(R/(x) \otimes M, M) \to (x) \otimes M \to M \to R/(x) \otimes M \to 0,$$

and since $\mathrm{Tor}_1(M, M) = 0$, we have the exact sequence:

$$0 \to \mathrm{Tor}_1(R/(x) \otimes M, M) \to (x) \otimes M \to M \to R/(x) \otimes M \to 0.$$

Since the first must be a monomorphism, it means that we can see it as a submodule of $(x) \otimes M$. Following the rest of the sequence, since the mapping $M \to R/(x) \otimes M$ is given by $m \mapsto 1 \otimes m$, it follows easily that

$$\mathrm{Tor}_1(R/(x) \otimes M, M) = \{m \in M : xm = 0\}.$$

$\square$

Definition C.1.6 (Ext functors)**.** Let $\mathcal{A}$ be an abelian category. The functor $\mathrm{Hom}(M, -)$ is left-exact, and it has right-derived functors, which are usually denoted by $\mathrm{Ext}^i(M, N)$. If now we want to regard $\mathrm{Hom}(-, N)$, we get the same object $\mathrm{Ext}^i(M, N)$ (see (ROTMAN, 2008), Theorem 7.8).

An **extension** of A by C is an exact sequence in $\mathcal{A}$ of the form:

$$0 \to A \to B \to C \to 0.$$

We say that an exact sequence of this form is **split** if $B \simeq A \oplus C$, where $A \oplus C$ denote the coproduct on $\mathcal{A}$.

Theorem C.1.7. *If* $\mathrm{Ext}^1(C, A) = 0$, *then every extension of A by C is* ***split****.*

Proof. Let

$$0 \to A \xrightarrow{i} B \to C \to 0$$

be any extension of A by C. Applying $\mathrm{Hom}(-, A)$ gives the exactness of

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A) \xrightarrow{i^*} \mathrm{Hom}(A, A) \to \mathrm{Ext}^1(C, A) = 0,$$

which means that i^* is an epimorphism. In particular, this implies that there exists an element $g \in \mathrm{Hom}(B, A)$ such that $1_A = i^*(g) = g \circ i$. $\square$

The procedure in the previous proof defines a map

$$\{\text{Extensions of } C \text{ by } A\} \to Ext^1(C, A),$$

taking any extension $0 \to A \to B \to C \to 0$ to the image of the identity $\mathrm{Id}_A \in \mathrm{Hom}(A, A)$ by the connecting morphism δ in the long exact sequence

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(A, A) \xrightarrow{\delta} \mathrm{Ext}^1(C, A) \to \cdots$$

Theorem C.1.8 ((WEIBEL, 1995), Theorem 3.4.3)**.** *There is a bijection*

$$\{\textit{Extensions of } C \textit{ by } A\} \to \mathrm{Ext}^1(C, A),$$

induced by the previous map.

Theorem C.1.9 ((HARTSHORNE, 1977), III, Prop. 6.7)**.** *Let $\mathcal{L}$ be a locally free sheaf of finite rank, $\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$ be its dual. Then for any $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X)$ we have*

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}).$$

Corollary C.1.9.1. For $\mathcal{E}, \mathcal{F}$ locally free sheaves over X of finite rank, then:

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{E}^\vee \otimes \mathcal{F}) \simeq H^i(X, \mathcal{E}^\vee \otimes \mathcal{F}).$$

C.2 Sheaf Cohomology

We compile here the needed results on sheaf cohomology. For proofs and more details, we refer the reader to (HARTSHORNE, 1977). Fix k field.

Lemma C.2.1 (Cohomology of projective spaces)**.** [(HARTSHORNE, 1977), III, 5.1] Let $X = \mathrm{Proj}\, k[x_0, \ldots, x_d]$. Then, for any $n \in \mathbb{Z}$, we have:

(a) $H^0(X, \mathcal{O}_X(n)) = k[x_0, \ldots, x_d]_n$, that is, the homogeneous polynomials of degree n.

(b) $H^p(X, \mathcal{O}_X(n)) = 0$ if $p \neq 0, p \neq d$.

(c) $H^d(X, \mathcal{O}_X(n)) \simeq H^0(X, \mathcal{O}_X(-n-d-1))^\vee$. In particular, $H^d(X, \mathcal{O}_X(-n-d-1) = 0$ if $n \geq -d$.

Theorem C.2.2 (Serre's Vanishing Theorem)**.** *[(HARTSHORNE, 1977), III, 5.2] Let X be a projective scheme over k and let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(a) For each $i \geq 0$, $H^i(X, \mathcal{F})$ has finite dimension,

(b) There is an integer n_0, depending on $\mathcal{F}$, such that for $i > 0$ and $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.

Theorem C.2.3 ((HARTSHORNE, 1977), III, 12.11)**.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on X that is flat over Y and $y \in Y$ be a point. Then:*

(a) If the natural map:

$$\varphi^i(y) : R^i f_*(\mathcal{F}) \otimes k(y) \to H^i(X_y, \mathcal{F}_y)$$

is surjective, then it is an isomorphism, and the same is true for all y' in a neighbourhood of y.

(b) Assume that $\varphi^i(y)$ is surjective. Then

(i) $\varphi^{i-1}(y)$ is also surjective;

(ii) $R^i f_*(\mathcal{F})$ is locally free in a neighbourhood of y.

Theorem C.2.4 (Kunneth Formula)**.** *Let k be a field, X and Y noetherian k–schemes. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$–module and $\mathcal{G}$ is a $\mathcal{O}_Y$–module, then we have a canonical isomorphism:*

$$H^n(X \times Y, \pi_X^*\mathcal{F} \otimes \pi_Y^*\mathcal{G}) \simeq \bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).$$