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MATHEUS SILVA COSTA

Generalized Toric Varieties, LVMB Manifolds and Lie Groupoids

Variedades tóricas generalizadas, Variedades LVMB e Grupóides de Lie

Campinas

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Matheus Silva Costa

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Variedades tóricas generalizadas, Variedades LVMB e Grupóides de Lie

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- ORCID do autor: https://orcid.org/0000-0002-4100-4046
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Prof(a). Dr(a). LINO ANDERSON DA SILVA GRAMA

Prof(a). Dr(a). VIVIANA JORGELINA DEL BARCO

Prof(a). Dr(a). RICARDO MIRANDA MARTINS

Prof(a). Dr(a). MATHIEU MOLITOR

Prof(a). Dr(a). JOSINEY ALVES DE SOUZA

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Resumo

O objetivo deste trabalho é estudar variedades tóricas generalizadas, explorando a relação entre orbifolds e quasifolds tóricos, de um lado, e grupóides de Lie, do outro. Nós apresentamos uma construção geral que usa o framework matemático das variedades LVMB para relacionar simultaneamente variedades, orbifold e quasifold tóricos, a grupóides de Lie. Como uma aplicação de nossa construção associamos a grupóides de Lie uma família de variedades que incluem o $\mathbb{C}P^d$ e alguns de seus variantes orbifold e quasifold. Em outra aplicação, nós associamos a grupóides de Lie a uma família de variedades que incluem superfícies de Hirzebruch e alguns de seus variantes orbifold e quasifold.

Palavras-chave: quasifolds, orbifolds, variedades tóricas, variedades LVMB, grupóides de Lie.

Abstract

The aim of this work is to study generalized toric varieties, by exploring the relationship between toric orbifolds and quasifolds, on one side, and Lie groupoids, on the other. We present a general construction that uses the mathematical framework of LVMB manifolds to relate simultaneously toric varieties, orbifolds and quasifolds, to Lie groupoids. As an application of our construction we associate to Lie groupoids a family of varieties that include $\mathbb{C}P^d$ and some of its orbifold and quasifold variants. As another application, we associate to Lie grupoids a family of varieties that include Hirzerbruch surfaces and some of its orbifold and quasifold variants.

Keywords: quasifolds, orbifolds, toric varieties, LVMB manifolds, Lie groupoids.

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Introduction

Orbifolds were first introduced by Satake (SATAKE, 1957; SATAKE, 1956) as a generalization of the notion of manifold. While a manifold is locally modelled by open subsets of Euclidean spaces, orbifolds are modelled by the quotient space of such open sets by a finite group action.

More recently, (MOERDIJK; PRONK, 1997; MOERDIJK; PRONK, 1999) a close relation was established between orbifolds and certain kinds of Lie groupoids. More specifically, given an orbifold it is possible to construct a proper effective Lie groupoid. Conversely, a proper effective groupoid yields an effective orbifold (See Section 1.4 for more details).

Subsequently, quasifolds were introduced by Prato (PRATO, 2001) as a generalization to orbifolds. A motivation to this was the correspondence via the convexity theorem of Guillemin-Sternberg and Atiyah (GUILLEMIN; STERNBERG, 1982; ATIYAH, 1982), and results of Delzant (DELZANT, 1988), between toric varieties and certain polytopes. This correspondence was extended by (LERMAN; TOLMAN, 1997) to the case of toric orbifolds. Then (PRATO, 2001; BATTAGLIA; PRATO, 2001) extended it to the case of toric quasifolds.

Toric orbifolds and quasifolds can be considered generalized toric varieties. A recent interest in this type of space comes from mirror symmetry: (AUROUX; KATZARKOV; ORLOV, 2008) proves mirror symmetry for weighted projective planes, which are examples of toric orbifolds, and also for its noncommutative deformations; (KATZARKOV et al., 2021) proposes noncommutative toric varieties as stacks, relating them with LVMB manifolds. For another perspective of toric stacks, see (HOFFMAN; SJAMAAR, 2018).

The main conceptual difference between orbifolds and quasifolds is that the linear finite group action is being replaced by a not necessarily linear action by a discrete group. Another main difference between orbifolds and quasifolds is that quasifolds are usually not Hausdorff. We can see that, up to technical details, the definition of a quasifold generalizes the definition of orbifold. However chart compatibility conditions for quasifolds are more involved than the ones for orbifolds. Although the definition is apparently similar to chart compatibility of manifolds, the requirement that chart domains and also the chart intersections be simply connected implies, in case this is not true, that we must pass to the universal cover and lift the transition maps. This causes practical issues. Another technical difficulty is that the use of discrete group action imposes the absence, in general, of a slice theorem, as is the case for finite actions. As a consequence, a more straightforward generalization of the relationship, as mentioned above, between Lie groupoids and orbifolds, to a relationship between Lie groupoids and quasifolds is made more difficult.

One of the main contributions of this Thesis is to explore the relationship between quasifolds and Lie groupoids. We sought a mathematical framework that would allows us to relate simultaneously toric varieties, orbifolds and quasifolds, to Lie groupoids, and bypass these technical issues. In order to achieve that, we work with LVMB manifolds.

LVMB manifolds are a family of complex, compact foliated manifolds, originally introduced by Lopes de Medrano and Verjovsky (MEDRANO; VERJOVSKY, 1997) and then further generalized by (MEERSSEMAN, 2000) and (BOSIO, 2001). Just like toric varieties, orbifolds and quasifolds can be constructed starting from the combinatorial data that determines their associated polytope, LVMB manifolds can also be constructed from the same data. The work of Battaglia and Zaffran (BATTAGLIA; ZAFFRAN, 2017; BATTAGLIA; ZAFFRAN, 2015) shows how these two types of spaces are linked (see Chapter 2).

In order to relate a toric quasifold with a Lie groupoid, we prove our main result:

Theorem 1. *Consider a triangulated vector configuration* $(V, \mathcal{T})$*. Using this initial data, we can construct an LVMB manifold* N *and a group* $H_{\mathcal{F}}$ *acting on it such that the orbits of the action yield a foliation* $\mathcal{F}$*. There are submanifolds* N_i *of* N *such that each leaf of the foliation intersects at least of the* N_i*, and such that the pullback* $\mathcal{G}$ *of the action groupoid* $H_{\mathcal{F}} \times N$ *via the "inclusion"* $\coprod N_i \to N$ *is a Lie groupoid.*

From $(V, \mathcal{T})$, we can classically obtain a toric variety, orbifold or quasifold. We expect to be able to recover this generalized toric variety from the Lie groupoid $\mathcal{G}$ of the main theorem. With this method, we can put parameters on the combinatorial data and associate a whole family of toric varieties, orbifolds, quasifolds to a corresponding family of Lie groupoids.

We also present direct proofs of two applications of this Theorem.

Theorem 2. *Let*

$$\begin{aligned} v_1 &:= (1, 0, \dots, 0), \\ v_2 &:= (0, 1, 0, \dots, 0), \\ &\vdots \\ v_d &:= (0, \dots, 0, 1), \\ v_{d+1} &:= (-\alpha_1, \dots, -\alpha_d) \end{aligned}$$

be vectors in $\mathbb{R}^d$*. Let* $\Delta \subset \mathbb{R}^d$ *be the fan made out of each proper subset of* $\{v_1, \dots, v_{d+1}\}$*, with* $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$*.*

For each choice of parameters $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$, we construct a Lie groupoid associated to Δ.

We note that for $\alpha_1 = ... = \alpha_d = 1$, this is the fan associated to $\mathbb{C}P^d$, and by varying the parameters $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$, we can classically obtain toric orbifold and quasifold variants of $\mathbb{C}P^d$.

As another application, we prove:

Theorem 3. *Let*

$$\begin{aligned} v_1 &:= (1,0), \\ v_2 &:= (0,1), \\ v_3 &:= (0,-1), \\ v_4 &:= (-1,a) \end{aligned}$$

be vectors in $\mathbb{R}^2$, with $a > 0$.

Let $\Delta \subset \mathbb{R}^2$ be the fan whose higher-dimensional cones are generated by $(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3)$.

For each choice of $a > 0$, we construct a Lie groupoid associated to Δ.

We note that for $a = n \in \mathbb{Z}$ a positive integer, this is the fan associated to the classic Hirzebruch surfaces. $\mathbb{F}_n$, and by varying the parameter $a \in \mathbb{R}_{>0}$, we can classically obtain a family of toric varieties, orbifolds and quasifolds that contain the Hirzebruch surfaces. This family was studied by (BATTAGLIA; PRATO; ZAFFRAN, 2019), in the context of LVMB manifolds.

Now we present a brief description of the contents of this document.

In chapter 1 we talk about smooth toric varieties and we illustrate how they can be constructed from combinatorial objects known as Delzand polytopes. We then mention how this process was generalized, yielding toric orbifolds (LERMAN; TOLMAN, 1997) and toric quasifolds (PRATO, 2001; BATTAGLIA; PRATO, 2001).

We talk about orbifolds, and discuss some of the differences in the statements of the classical definition of orbifolds in the literature. We also expand on some classical proofs, and cite explicitly many of the prerequisite knowledge necessary to understand them. We hope this will be useful to the reader starting to study orbifolds. We also discuss some of the differences between the definition of orbifolds and quasifolds. Finally we touch on how orbifolds and Lie groupoids have been related.

In chapter 2, we describe LVMB manifolds, and mention how the leaf space of these manifold relate to toric varities, orbifolds and quasifolds. We prove the main

Theorem. We exhibit a family of LVMB manifolds associated to n-dimensional complex projective space (and some of its orbifold and quasifold variants) We also exhibit a family of LVMB manifolds related to a family of spaces that contains the Hirzebruch surfaces (and some of its orbifold and quasifold variants). We apply the main Theorem to these two cases, but also present direct proofs.

In the appendix, we collect some basic results that were used, such as the theory of Gale duals, theory of tubes and slices for compact Lie group actions, and some basic manifold theory results. We also collect definitions related to Lie groupoids.

1 Smooth Toric Varieties, Orbifolds and Quasifolds, and Lie groupoids

1.1 Smooth Toric Varieties, Toric Orbifolds and Quasifolds

Toric varieties are objects that can be constructed from combinatorial information and can be studied from the perspective of algebraic geometry or differential geometry.

Definition 1. *(COX; LITTLE; SCHENCK, 2011, p.106) A* ***toric variety*** *is an irreducible variety X containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zarisky open subset such that the action of T_N on itself extends to an algebraic action of T_N on X.*

The toric varieties that appear in this work are projective and compact. Furthermore, they will be either toric manifolds, or toric orbifolds.

Definition 2. *(SILVA, 2001) A* ***toric manifold*** *is a compact connected symplectic manifold (M, ω) equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold, and with a choice of a corresponding moment map μ.*

A reference for toric varieties in algebraic geometry is (COX; LITTLE; SCHENCK, 2011), Chapters 1 to 3 are relevant to this work. In the context of differential geometry, toric manifolds can be constructed, on one hand, from a procedure due to Delzant (DELZANT, 1988), starting from the geometric-combinatorial information of a Delzant polytope.

Definition 3. *A* ***Delzant polytope*** *in $\mathbb{R}^n$ is a polytope that is*

- ***simple,*** *that is, each vertex of the polytope is adjacent to exactly n edges;*
- ***rational,*** *that is, the edges that meet at at vertex p are of the form $p + tu_i$, with $t \geq 0$, where $u_i \in \mathbb{Z}^n$;*
- ***smooth,*** *that is, for each vertex, the corresponding vectors $u_i, \ldots, u_n$ can be chosen to compose a basis for $\mathbb{Z}^n$.*

Theorem 4 (Delzant - Existence)**.** *(DELZANT, 1988) Given a Delzant polytope P, it is possible to construct a toric manifold P.*

Theorem 5 (Delzant - Uniqueness)**.** *(DELZANT, 1988) Let (M_1, μ_1), (M_2, μ_2) be $2n$-dimensional toric manifolds (with $\mathbb{T}^n$ acting on them) such that $\mu_1(M_1) = \mu_2(M_2)$. Then there exists a $\mathbb{T}^n$-equivariant symplectic diffeomorphism $F : M_1 \to M_2$ such that*

$$\mu_1 = \mu_2 \circ F$$

On the other hand, if we restrict the hypothesis of the following Convexity theorem to a toric manifold, the image of the momentum map will be a Delzant polytope.

Theorem 6. *(GUILLEMIN; STERNBERG, 1982; ATIYAH, 1982) Let (M, ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m-torus. Suppose that ψ : $\mathbb{T}^m \times M \to M$ is a Hamiltonian action with momentum map $\mu : M \to \mathbb{R}^m$. Then:*

1. *the levels of μ are connected;*
2. *the image of μ is convex;*
3. *the image of μ is the convex hull of the images of the fixed points of the action. More specifically, the image of μ is a convex polytope.*

The two theorems above, as written, are taken from (SILVA, 2003; SILVA, 2001), where they are discussed.

Now we will show how to construct a smooth toric variety from a Delzant polytope, following the exposition of (GUILLEMIN, 1994).

Let $P \subset \mathbb{R}^n$ be a Delzant polytope. It follows from its definition that P is n-dimensional. We will need the description of P as an intersection of closed half-spaces, which is possible for every polytope (ZIEGLER, 1995):

$$P = \cap_{i=1}^{d} \{x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq \lambda_i\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$, $u_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, and such that $F_i = P \cap H_i$ is a facet of P, where

$$H_i = \{x \in \mathbb{R}^n \mid \langle x, u_i \rangle = \lambda_i\}$$

is the supporting hyperplane of F_i.

Now, it is a property of Delzant polytopes that, in the description above, the vectors u_i can be taken to be in $\mathbb{Z}^n$, and such that they are primitive (this means that there is no positive integer k_i different that one such that $\frac{1}{k_i} u_i$ is also in $\mathbb{Z}^n$). Let us make this choice.

Let $B = (e_1, \ldots, e_d)$ be the standard basis for $\mathbb{R}^d$, and define the map

$$\pi : \mathbb{Z}^d \to \mathbb{Z}^n, \tag{1.2}$$

$$e_i \mapsto u_i \tag{1.3}$$

and its unique extension

$$\pi : \mathbb{R}^d \to \mathbb{R}^n. \tag{1.4}$$

which exists because $\mathbb{Z}^d$ is a free abelian group with basis B.

Since P is smooth, these maps are surjective.

Let $\mathbb{T}^i = \dfrac{\mathbb{R}^i}{\mathbb{Z}^i}$, with $i = d, n$, and with the group quotient given by the standard action (i.e., by translation).

Consider the induced quotient map $\tilde{\pi} : \mathbb{T}^d \to \mathbb{T}^n$ defined by the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{\pi} & \mathbb{R}^n \\ \big\downarrow{\scriptstyle \rho_d} & & \big\downarrow{\scriptstyle \rho_n} \\ \mathbb{T}^d & \xrightarrow{\tilde{\pi}} & \mathbb{T}^n \end{array}$$

where ρ_i is the canonical projection. Since $\rho_n \circ \pi$ is a homomorphism and $\mathbb{Z}^d \subseteq \mathrm{Ker}(\rho_n \circ \pi)$ (it is here that we use that $u_i \in \mathbb{Z}$), the induced map $\tilde{\pi}$ that makes the diagram commutative is well-defined and a homomorphism. Is is smooth, because $\rho_n \circ \pi$ is smooth, and π_d is a smooth submersion. It is surjective because $\rho_n \circ \pi$ is surjective.

From now on, we rename $\tilde{\pi}$ to π.

Let us denote the kernel of $\pi : \mathbb{T}^d \to \mathbb{T}^n$ by N. Since this map is surjective, there is a short exact sequence

$$0 \to N \to \mathbb{T}^d \to \mathbb{T}^n \to 0.$$

Consider $\mathbb{C}^d$ as a symplectic manifold, with the usual symplectic form

$$\frac{i}{2} \sum_{i=1}^{d} dz_i \wedge d\bar{z}_i.$$

$\mathbb{T}^d$ acts on $\mathbb{C}^d$ by the multiplication maping

$$e^{i\theta} z = (e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d)$$

and this action is Hamiltonian with moment map

$$J : \mathbb{C}^d \to \mathbb{R}^d, \tag{1.5}$$

$$z \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + c, \tag{1.6}$$

where $c \in (\mathbb{R}^d)^*$ is an arbitrary constant. We will fix $c = \lambda = (\lambda_1, \ldots, \lambda_d)$ with the λ_i as in (1.1).

By restricting the action of $\mathbb{T}^d$ on $\mathbb{C}^d$ to N, we get a Hamiltonian action of N on $\mathbb{C}^d$ whose moment map is the following: let $\iota : N \to T^d$ be the inclusion map and let

$\mathfrak{n} = \mathrm{Lie}(N)$ be the Lie algebra of N. Let us denote by the same symbol the map $\iota : \mathfrak{n} \to \mathbb{R}^d$, given by the derivative of the inclusion map at the identity element. Let $\iota^* : (\mathbb{R}^d)^* \to \mathfrak{n}^*$ be the transpose of ι. The moment map for the action of N on $\mathbb{C}^d$ is $\iota^* \circ J$

It can be shown (GUILLEMIN, 1994) that

Theorem 7. *$(\iota^* \circ J)^{-1}(0)$ is a compact subset of $\mathbb{C}^d$ and N acts freely on this set.*

Symplectic reduction then gives us that

$$X_P := \frac{(\iota^* \circ J)^{-1}(0)}{N}$$

is a compact symplectic manifold. This manifold is the smooth toric variety associated to the Delzant polytope P.

Example 1. Let P be the interval $[-1, 1] \subset \mathbb{R}$. Then $X_P = \mathbb{C}P^1$.

P is Delzant:

- as can be seen, there is 1 edge meeting in each vertex;
- for the vertex (-1), the edge is of the form $(-1) + tv_1$, where $v_1 = 1 \in \mathbb{Z}^1 \simeq (\mathbb{Z}^1)^*$, and $t \in [0, \infty)$;
- for the vertex (1), the edge is of the form $(1) + tv_1$, where $v_1 = -1 \in \mathbb{Z}^1 \simeq (\mathbb{Z}^1)^*$, and $t \in [0, \infty)$;
- For each vertex, v_1 is a basis for $\mathbb{Z}^1$.

By definition, $X_P = (\iota^* \circ J)^{-1}(0)/N$.

First, we will describe $(\iota^* \circ J)^{-1}(0)$.

Step 1: we write down the supporting hyperplanes for the 0-dimensional faces F of P, which are the vertices (-1) and (1). That is, we need $\lambda_i \in \mathbb{R}, u_i \in \mathbb{Z}^1$ such that $\langle u_i, x\rangle = \lambda_i$, for every $x \in F_i$ and such that

$$P = \cap_{i=1}^{2}\{x \in \mathbb{R} \mid \langle x, u_i\rangle \geq \lambda_i\}.$$

We also want u_i primitive.

For $F_1 = (-1)$, we have $u_1 = 1$ and $\lambda_1 = -1$. Note that for $x = -1$, $\langle 1, -1\rangle = -1 = \lambda_1$, and for $x \in [-1, 1]$, $\langle 1, x\rangle = x \geq -1$.

For $F_2 = (1)$, we have $u_2 = -1$ and $\lambda_2 = -1$. Note that for $x = 1$, $\langle -1, 1\rangle = -1 = \lambda_2$, and for $x \in [-1, 1]$, $\langle -1, x\rangle = -x \geq -1$.

Step 2: Let e_1, e_2 be the standard basis vectors for $\mathbb{R}^2$. Define

$$\begin{aligned}\pi : \mathbb{Z}^2 &\to \mathbb{Z},\\ e_1 \mapsto u_1, &\quad e_2 \mapsto u_2,\end{aligned}$$

and consider its unique extension $\pi : \mathbb{R}^2 \to \mathbb{R}$, and the induced quotient map

$$\begin{aligned}\pi : \mathbb{T}^2 &\to \mathbb{T},\\ [x, y] &\mapsto [x - y].\end{aligned}$$

Step 3: Let $N = \operatorname{Ker} \pi$ We have the exact sequence

$$0 \to N \hookrightarrow \mathbb{T}^2 \to \mathbb{T} \to 0$$

Note that

$$\begin{aligned}[x_1, x_2] \in N &\iff \pi([x_1, x_2]) = 0\\ &\iff [x_1 - x_2] = 0\\ &\iff x_1 - x_2 \in \mathbb{Z}.\end{aligned}$$

Thus $N = \{[x, x] \mid x \in \mathbb{R}\}$.

Step 4: If we write $\mathfrak{n} = \mathrm{Lie}(N) = \{(x, x)\} \mid x \in \mathbb{R}$, then the exponential map $\exp : \mathfrak{n} \to N$ is just the restriction of the exponential map $\exp = \rho_2 : \mathbb{R}^2 \to \mathbb{T}$. Let $\iota : N \to \mathbb{T}^2$ be the inclusion. Then the induced map $\iota : \mathfrak{n} \to \mathbb{R}^2$ is also the inclusion.

We need to compute $\iota^* : (\mathbb{R}^2)^* \to (\mathfrak{n})^*$. Fixing the bases $\{e_1, e_2\}$ for $\mathbb{R}^2$ and $\{(1, 1)\}$ for $\mathfrak{n}$, we use the definition of the transposition map, to see that both $(e_1)^*$ and $(e_2)^*$ are sent by ι^* to $(1, 1)^*$. Now we will use these bases to identify $(\mathbb{R}^2)^* \simeq \mathbb{R}^2$ and $(\mathfrak{n})^* \simeq \mathbb{R}$.

Recall from the general case that the standard action of $\mathbb{T}^2$ on $\mathbb{C}^2$ is Hamiltonian with moment map

$$\begin{aligned}J : \mathbb{C}^2 &\to \mathbb{R}^2,\\ z &\mapsto \frac{1}{2}(|z_1|^2 + \lambda_1, |z_2|^2 + \lambda_2).\end{aligned}$$

Then using the identification above, we have

$$(\iota^* \circ J)(z) = \frac{1}{2}(|z_1|^2 + \lambda_1 + |z_2|^2 + \lambda_2).$$

Step 5: An investigation of the quotient $(\iota^* \circ J)^{-1}(0)/N$, shows that it is a closed disc on $\mathbb{C}$, with the border identified, and so, can be identified with $\mathbb{C}P^1$. □

1.1.1 Toric Orbifolds and Quasifolds

If the smoothness condition in Definition 3 is relaxed, then through a generalization of Delzant's procedure carried out in (LERMAN; TOLMAN, 1997), we obtain a toric orbifold (see Section 1.2 for the definition of orbifold).

A symplectic orbifold is an orbifold equipped with a differential form ω where the representative $\tilde{\omega}$ in each chart $(\tilde{U}, G, \phi)\}$ is a symplectic form on $\tilde{U}$. The definitions of group action, hamiltonian action, momentum map and toric orbifold are analogous to the definitions in the category of smooth manifolds. Analogous results to the convexity theorem and Delzant's classification have also been proved in (LERMAN; TOLMAN, 1997).

Now, if the rationality condition in Definition 3 is relaxed, then through a generalization of Delzant's procedure carried out in (PRATO, 2001), the geometric object obtained is a toric quasifold (See 1.3 for a definition of quasifolds). Just like for toric orbifolds, (PRATO, 2001) has proved analogous results to the convexity and Delzant classification theorems.

It is important to mention that a classification of toric quasifolds via polytopes has also been achieved by (PRATO, 2001). In a subsequent work (BATTAGLIA; PRATO, 2001), another construction for quasifolds in presented, in which they are endowed with a complex structure, compatible with the symplectic one.

1.2 Orbifolds

In this section we collect some of the basic results on orbifolds that are relevant for us. We provide more details to proofs in the literature and also the prerequisite material necessary to understand those proofs.

Orbifolds were originally introduced by Satake (SATAKE, 1956; SATAKE, 1957) (see the introduction of (ADEM; LEIDA; RUAN, 2007) for an account of the history of orbifolds in algebraic and differential geometry). We follow the material as presented in (ADEM; LEIDA; RUAN, 2007; MOERDIJK; PRONK, 1997), which is adapted from (SATAKE, 1957). We point out what are the adaptations.

Definition 4. *Let X be a topological space, and fix $n \geq 0$. An n-dimensional* ***orbifold chart*** *on X is a triple $(\tilde{U}, G, \phi)$, where $\tilde{U} \subset \mathbb{R}^n$ is a connected open subset, G is a finite group acting smoothly on $\tilde{U}$, and $\phi : \tilde{U} \to X$ is G-invariant map which induces a homeomorphism of $\tilde{U}/G$ onto an open subset $U \subset X$, where $\tilde{U}/G$ has the quotient topology.*

Remark 1. The definition of an orbifold chart in (MOERDIJK; PRONK, 1997; ADEM; LEIDA; RUAN, 2007) states, instead of a smooth group action G on $\tilde{U}$, that G is a finite group G of smooth automorphisms of $\tilde{U}$.

Because of this subtle difference, the induced group action of G on $\tilde{U}$ is automatically effective.

In this text, effectivity of the action will be added explicitly.

The definition of an orbifold chart in (SATAKE, 1957), is the same as in (MOERDIJK; PRONK, 1997; ADEM; LEIDA; RUAN, 2007) but with the added hypothesis that the finite group of smooth automophisms G has a set of fixed points of dimension $\leq m-2$.

(SATAKE, 1957) uses this hypothesis to prove a foundational lemma, as we will see. (MOERDIJK; PRONK, 1997) removed this hypothesis, as they were able to prove the same Lemma without it. □

Remark 2. Let $\tilde{\phi}$ be the induced map, and π the canonical quotient map. Then the following diagram is commutative.

$$\begin{array}{ccc} \tilde{U} & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\phi} & \\ \tilde{U}/G & \xrightarrow{\tilde{\phi}} & U \end{array}$$

Recall that π is the quotient map of a continuous action of a topological group, and so it is open (Lemma 29).

Since $\tilde{\phi}$ and π are continuous, surjective, and open and $\phi = \tilde{\phi} \circ \pi$, then so is ϕ (Theorem 19).

On the other hand, if we assume that the G-invariant map ϕ is continuous, and open, then $U \coloneqq \phi(\tilde{U}) \subset X$ is open, and the induced map $\tilde{\phi}$ is continuous (Theorem 20), bijective and open. Thus $\tilde{\phi}$ is a homeomorphism. □

Definition 5. *(MOERDIJK; PRONK, 1997)(ADEM; LEIDA; RUAN, 2007, p. 2) Let X be a topological space, and fix $n \geq 0$. An **embedding** $\lambda : (\tilde{U}, G, \phi) \hookrightarrow (\tilde{V}, H, \psi)$ between two n-dimensional orbifold charts is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ with $\psi \circ \lambda = \phi$.*

Remark 3. Let $\lambda : (\tilde{U}, G, \phi) \hookrightarrow (\tilde{V}, H, \psi)$ be an embedding. By Proposition 8, $\lambda : \tilde{U} \hookrightarrow \lambda(\tilde{U})$ is a diffeomorphism.

Since $\tilde{U}, \tilde{V}$ are both n-dimensional manifolds and $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ is a smooth embedding, then (Proposition 9) $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ is a local diffeomorphism. In particular, it is an open map and $\lambda(\tilde{U}) \subset \tilde{V}$ is open in $\tilde{V}$, and thus in $\mathbb{R}^n$. □

Definition 6. *(MOERDIJK; PRONK, 1997)(ADEM; LEIDA; RUAN, 2007, p. 2) Let X be a topological space, and fix $n \geq 0$.*

1. *An **orbifold atlas** on X is a family $\mathcal{U} = \{(\tilde{U}_i, G_i, \phi_i)\}_{i \in I}$ of orbifold charts, such that $X = \bigcup_{i \in I} \phi_i(\tilde{U}_i)$ and such that each two charts are **locally compatible**: given*

any two charts $(\tilde{U}, G, \phi)$ with $U = \phi(\tilde{U}) \subset X$ and $(\tilde{V}, H, \psi)$ with $V = \psi(\tilde{V}) \subset X$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subset U \cap V$ of x and a chart $(\tilde{W}, K, \mu)$ for W such that there are embeddings $(\tilde{W}, K, \mu) \hookrightarrow (\tilde{U}, G, \phi)$ and $(\tilde{W}, K, \mu) \hookrightarrow (\tilde{V}, H, \psi)$.

2. *An atlas $\mathcal{U}$ **refines**, or is a **refinement** of, another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. Two orbifold atlases are **equivalent** if they have a common refinement.*

Definition 7. *(ADEM; LEIDA; RUAN, 2007, p. 2) An **effective orbifold** $\mathcal{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of an n-dimensional orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i, \phi_i)\}_{i \in I}$, such that the action of G_i on $\tilde{U}_i$ is effective for every $i \in I$.*

Note that the notion of compatibility for orbifold charts is more subtle than the one for manifolds. Instead of asking for smoothness of a change of charts, composing the two partially overlaping chart maps, what is required is a refinement common to both charts. Note however that the existence of this refinement, plus the following lemma ensures that a suitable restriction of one chart embedds into the other.

Lemma 1. *(MOERDIJK; PRONK, 1997, p.4) (ADEM; LEIDA; RUAN, 2007, p.3) Let $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$ be two charts for the same orbifold structure on M. Suppose $\tilde{U}$ is simply connected, and $\phi(\tilde{U}) \subset \psi(\tilde{V})$. Then there exists an embedding $(\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$*

For the notations in the next Lemma, refer to the appendix section on group actions.

Lemma 2. *Let $(\tilde{U}, G, \phi)$ be an orbifold chart.*

1. *(MOERDIJK; PRONK, 1997, p.4) Let $g \in G$, and suppose the action of g on $\tilde{U}$ is not trivial. Then the set S_g of non-fixed points of g is open and dense in $\tilde{U}$.*

2. *(ADEM; LEIDA; RUAN, 2007, p. 2) If the action is effective, it will act freely on a dense open subset, namely, the set S of nonfixed points of the chart.*

3. *The set Z of fixed points of the chart is a finite union of closed submanifolds.*

Proof. (1) follows from Lemma 27 applied to g, which is a nontrivial automorphism on the connected manifold $\tilde{U}$ (Since G is finite, g is of finite order).

If the action is effective, then for any $g \neq 1$, its action on $\tilde{U}$ is not trivial, and we can apply (1). We then observe that G is finite and that a finite intersection of open and dense subsets is open and dense (Lemma 26). Finally we apply Lemma 28.

Let us prove (3). First we note that

$$Z = \{x \in x \mid gx = x \text{ for some } g \neq 1 \in G\}$$
$$= \bigcup_{\{1\} \neq H \leq G} \tilde{U}_H.$$

Since G is finite, so are its subgroups, of which there are finitely many. Now we observe that, by restriction the G-action on $\tilde{U}$, to each H, we can apply Proposition 14, and conclude that each $\tilde{U}_H$ is closed submanifold of $\tilde{U}$. □

Lemma 3. *(MOERDIJK; PRONK, 1997)(ADEM; LEIDA; RUAN, 2007, p. 2) Suppose $\lambda, \mu : (\tilde{U}, G, \phi) \hookrightarrow (\tilde{V}, H, \psi)$ are two embeddings of orbifold charts. Then, there is a unique $h \in H$ such that $\mu = h \cdot \lambda$*

Proof. For the benefit of the reader, we will fill in the details of the proof in (SATAKE, 1956), which works for the case where the set of fixed points in each chart has dimension $\leq m - 2$. The general case is found in the Appendix of (MOERDIJK; PRONK, 1997).

We will apply Theorem 18 to show that the set S of non-fixed points of G' in $\lambda(\tilde{U})$ is connected.

Let Z be the set of fixed points. It is a finite union of closed submanifolds (Lemma 2, (3))

First let us suppose Z is already a closed submanifold.

Let $a, b \in S$, and let $f : [0, 1] \to \lambda(\tilde{U})$ be a smooth curve connecting a and b, with $f(0) = a$ and $f(1) = b$.

In the notation of Theorem 18, $X = [0, 1]$ and $C = \{0, 1\}$. Since $C \subset S$, and $S \cap Z = \emptyset$, the conditions for f are trivially satisfied. Applying the theorem, we get a smooth map $g : X \to \lambda(\tilde{U})$ homotopic to f, such that $g \pitchfork Z, \partial g \pitchfork Z$, and on a neighborhood of C we have $g = f$.

Since Z has dimension $\leq m-2$, and X has dimension 1, transversality between g and Z implies that $g(X) \cap Z = \emptyset$. Therefore S is path-connected, and as a manifold, also connected.

Now let us deal with the case where $Z = Z_1 \cup \cdots \cup Z_r$ is a disjoint union of closed submanifolds of different dimensions. We define $S_1 := \lambda(\tilde{U}) \setminus Z_1$, and we apply the argument above to Z_1 and S_1. Inductively, $S_n = S_{n-1} \setminus Z_{n-1}$. S_n is always open, and therefore a manifold, because there are only finitely many Z_i. At the last step S_N is S, the set of nonfixed points of G'.

We have learned the remainder of this proof from Prof. Elisa Prato (PRATO, 2018). It is the proof of her Orange Lemma, with minor adaptations to the orbifold case.

We will show now that $\lambda^{-1}(S) = \mu^{-1}(S)$. Let $x \in \tilde{U}$ such that $\lambda(x) \in S$. By definition of orbifold embedding

$$\psi(\lambda(x)) = \phi(x) = \psi(\mu(x)), \tag{1.7}$$

which implies there is some $h \in H$ such that

$$h\lambda(x) = \mu(x). \tag{1.8}$$

Since S is H-invariant (Lemma 28), $x \in \mu^{-1}(S)$. The other inclusion is analogous.

Since an orbifold embedding is a diffeomorphism onto its image (Remark 3), and S is connected (as seen above), and open and dense (Lemma 2), then so is $\tilde{U}' := \lambda^{-1}(S) = \mu^{-1}(S)$. We define the continuous mapping

$$F : \tilde{U}' \to S \times S,$$
$$u \mapsto (\lambda(u), \mu(u)).$$

We also define, for each $h \in H$, the set

$$\tilde{V}_h := \{v, h \cdot v \mid v \in S\}$$

This set is the image of the connected set S by the continuous map $S \to S \times S, id \times h$, and is hence also connected. We will see now that if $V_h \cap V_{h'} \neq \emptyset$, then $h = h'$. Suppose $(v, hv) = (v, h'v) \in V_h \cap V_{h'}$. It follows that $(h^{-1}h')v = v$. It is at this point in the proof that we use that the action is free on the set S on nonfixed points (Lemma 28). We then conclude that $h = h'$.

Because of (1.7) and (1.8), $F(\tilde{U}') \subset \cup_{h \in H} \tilde{V}_h$. Since F is continuous, $\tilde{U}'$ is connected, and the $\tilde{V}_h$ are connected and disjoint, there exists a unique $h \in H$ such that $F(\tilde{U}') \subset \tilde{V}_h$. Therefore $\lambda(u) = h\mu(u)$, for every $u \in \tilde{U}'$. Since $\tilde{U}'$ is dense on $\tilde{U}$, by continuity this equality holds on $\tilde{U}$. □

Definition 8. *(ADEM; LEIDA; RUAN, 2007, p. 4) Let G be a compact lie group, M a smooth manifold and $\Psi : G \times M \to M$ an effective, almost free, smooth action. An orbifold given as the quotient space of this action is called an* ***effective quotient orbifold****. The atlas is constructed using the existence of slices for compact Lie group actions.*

For the benefit of the reader, we will provide the details of the atlas construction cited above.

Construction 1 (Effective quotient orbifold)**.** Let G be a compact lie group, M a smooth manifold and $\Psi : G \times M \to M$ an effective, almost free, smooth action.

We will give an orbifold atlas to the topological space X defined by the orbit space of this action

$$X := M/G$$

We denote by $\pi : M \to M/G$ the canonical projection of the orbit space.

By Corollary 1 and Proposition 15, X is Hausdorff. We also note that X is paracompact.

Let $x \in X$. Let $m \in M$ such that $\pi(m) = x$. We will build an orbifold chart for X at x.

Since G is a compact Lie group, this G-action is locally smooth (Lemma 35 and Remark 16). This means there is a linear tube (Definition 36)

$$\varphi : G \times_{G_x} \mathbb{R}^k \to M$$

around the orbit $P \coloneqq G \cdot m$ such that $\varphi[g, 0] = gm$. In particular, there is an orthogonal action of G_x on $\mathbb{R}^k$ for some k. For each $g \in G_x$, its action on $\mathbb{R}^k$ is linear, therefore smooth.

By definition of a tube (Definition 34), φ is a G-equivariant embedding onto an open neighborhood of $\hat{U}$ of P. This implies

$$\dim G \times_{G_x} \mathbb{R}^k = \dim M$$

and since

$$\dim G \times_{G_x} \mathbb{R}^k = \dim G + \dim \mathbb{R}^k - \dim G_x$$

and $\dim G_x = 0$ (because the action is almost free, thus G_x is finite) we have

$$\dim \mathbb{R}^k = \dim M - \dim G \tag{1.9}$$

Therefore the dimension k does not depend on the choice of x.

Now, let $\hat{\varphi}$ be the restriction of φ to the codomain $\hat{U}$.

We will use the embedding (Lemma 34)

$$i_e : \mathbb{R}^k \to G \times_{G_x} \mathbb{R}^k, \quad a \mapsto [e, a]$$

The map of the orbifold chart for $x \in X$ will be the composition

$$\mathbb{R}^k \xrightarrow{i_e} G \times_{G_x} \mathbb{R}^k \xrightarrow{\hat{\varphi}} \hat{U} \hookrightarrow M \xrightarrow{\pi} M/G \tag{1.10}$$

Let $h \in G_x, a \in \mathbb{R}^k$. We observe that

$$\begin{aligned}\Phi : \pi(\varphi(i_e(ha))) &= \pi(\varphi[e, ha]) = \pi(\varphi[eh^{-1}, a]) \\ &= \pi(h^{-1}\varphi[e, a]) = \pi(\varphi[e, a]) = \pi(\varphi(i_e(a)))\end{aligned}$$

which shows that the composition Φ (1.10) is G_x-invariant.

Now we will descend to the orbit spaces

$$\begin{array}{ccccccc}
\mathbb{R}^k & \xrightarrow{\;i_e\;} & G \times_{G_x} \mathbb{R}^k & \xrightarrow{\;\hat{\varphi}\;} & \hat{U} & \hookrightarrow & M \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\
\mathbb{R}^k/G_x & \xrightarrow[\alpha]{\sim} & (G \times_{G_x} \mathbb{R}^k)/G & \xrightarrow[\beta]{\sim} & \hat{U}/G & \hookrightarrow & M/G = X
\end{array}$$

where Proposition 16 justifies that the morphism

$$\alpha : G_x a \mapsto G[e, a]$$

is a homeomorphism. We can also see that the first square in the diagram is commutative.

Since $\hat{\varphi}$ is a G-equivariant homeomorphism we can define a homeomorphism β, also, the diagram commutes (Lemma 30).

Finally we note that $\hat{U}$ is a saturated open set on M with respect to the action, which implies $U_m := \pi(\hat{U}) = \hat{U}/G$ is an open subset of X (because X has the quotient topology).

And since

$$\hat{\varphi}(i_e(0)) = \hat{\varphi}([e, 0]) = m$$

and $\pi(m) = x$, then $x \in U$.

Therefore, given $m \in M$, we have produced a connected open subset of $\tilde{U}_m := \mathbb{R}^k$, and a finite group G_x of smooth automorphisms of $\tilde{U}_m$ (because Ψ is almost free). Moreover, we have a G_x-equivariant map $\phi_m := \Phi : \tilde{U}_m \to X$ which, as we have seen above, induces a homeomorphism of $\tilde{U}_m/G_x$ onto an open subset $U_m \subset X$. This is a k-dimensional orbifold chart $(\tilde{U}_m, G_x, \phi_m)$ at $x \in X$, as intended. We have also observed that k does not depend on x.

From this chart, we will also construct other charts, in order to have an orbifold atlas.

Let $W' \subset U_m \subset X$ be a a connected open subset of U_m. Let $\tilde{W}$ be a connected component of $\phi_m^{-1}(W')$ (the process below is to be repeated for every connected component). Let H be a subgroup of G_x that keeps W stable, that is $hW \subset W$, for all $h \in H$.

Now we define $W := (\beta \circ \alpha)(q(\tilde{W})) \subset W'$ and we restrict the homeomorphism $\beta \circ \alpha$ to

$$(\beta \circ \alpha)|_{q(\tilde{W})} : q(\tilde{W}) \to W. \tag{1.11}$$

This restriction is still a homemorphism. Let $\hat{W} = \pi^{-1}(W)$. By commutativity of the diagram,

$$\begin{array}{ccccccccc}
\tilde{W} & \hookrightarrow & \mathbb{R}^k & \xrightarrow{i_e} & G \times_{G_x} \mathbb{R}^k & \xrightarrow{\hat{\varphi}} & \hat{U} & \hookrightarrow & M \\
\downarrow & & \downarrow q & & \downarrow & & \downarrow & & \downarrow \pi \\
\tilde{W}/H & \hookrightarrow & \mathbb{R}^k/G_x & \xrightarrow[\alpha]{\sim} & (G \times_{G_x} \mathbb{R}^k)/G & \xrightarrow[\beta]{\sim} & \hat{U}/G & \hookrightarrow & M/G = X
\end{array}$$

$\phi_m(\tilde{W}) = \beta \circ \alpha \circ (\tilde{W}) = W$. Then we can also restrict ϕ_m to $\tilde{W} \to W$. This restriction is H-invariant and descends to the homeomorphism (1.11).

We consider the collection $\mathcal{U}$ of orbifold charts tha contains

$$\mathcal{U}_1 = \{(\tilde{U}_m, G_x, \phi_m)\}_{m \in M}$$

and all the orbifold charts constructed via the procedure above. What remains to be shown is that $\mathcal{U}$ defines an orbifold atlas. □

We cite the following result, which shows that every orbifold is isomorphic to a quotient orbifold. As we will see in the next section, this is not true for quasifolds.

Theorem 8. *(ADEM; LEIDA; RUAN, 2007, p. 12) For any given orbifold $\mathcal{X}$, its frame bundle* $\mathrm{Fr}(\mathcal{X})$ *is a smooth manifold with a smooth, effective, and almost free $O(n)$-action. The original orbifold $\mathcal{X}$ is naturally isomorphic to the resulting quotient orbifold* $\mathrm{Fr}(\mathcal{X})/O(n)$.

1.3 Quasifolds

In this section, we follow (PRATO, 2001), the paper where Prato defined the notion of quasifold as a generalization of the notion of orbifold. Our main purpose here is to highlight some of the differences in the definitions of orbifold and quasifold.

(PRATO, 2001) starts by defining the local model for a quasifold, which will be a component of the quasifold chart (the other component being a map).

Definition 9. *(PRATO, 2001, p.963) Let $\tilde{U}$ be a connected, simply-connected manifold of dimension k and let Γ be a discrete group acting smoothly on the manifold $\tilde{U}$ so that the set of points, $\tilde{U}_0$, where the action is free, is connected and dense. Consider the space of orbits, $\tilde{U}/\Gamma$, of the action of the group Γ on the manifold $\tilde{U}$, endowed with the quotient topology, and the canonical projection $p : \tilde{U} \to \tilde{U}/\Gamma$. A **model** of dimension k is a triple $(\tilde{U}/\Gamma, p, \tilde{U})$.*

Remark 4. The fundamental difference between a quasifold and an orbifold is that the group acting on the model space is no longer finite, but discrete. Since a finite group is implicitly considered to be equipped with the discrete topology, this is a straighforward generalization. We will now discuss differences that are more technical in nature.

The first of such differences between a quasifold model and the equivalent model in our definition of orbifold chart, is that here $\tilde{U}$ is a manifold, whereas in the orbifold definition, $\tilde{U}$ is an open subset of some $\mathbb{R}^n$. But we could have also considered $\tilde{U}$ to be a manifold in the orbifold case as well, and obtain an equivalent definition (See the last remark in Section 1 of (SATAKE, 1957)).

The second difference is that $\tilde{U}$ is not only connected, as in the orbifold case, but also simply-connected. This choice is made to simplify subsequent definitions, such as smooth quasifold maps and diffeomorphism. Remark 1.2 in (PRATO, 2001) explains how you can start from an object that is a quasifold model except for the simply-connectedness, and "lift" it to a equivalent model, in a precise sense, but also satisfying this topological condition.

The third difference is the requirement of the existence of a set of points $\tilde{U}_0$ connected and dense, where the action is free. We recall that the orbifold definition in (SATAKE, 1957) required that the set of fixed points of the action had dimension ≤ 2. In the proof of Lemma 3, we show how this condition implies that the complement of the set of fixed points is connected, open, dense, and the action restricted to this complement is free. We have also seen in the same proof the necessity of such a connected, dense set where the action is free. However the fact that the group is finite is used to obtain this result, which may not be true in the infinite case. Hence this may be the reason for this difference in the definitions. □

Definition 10. *A* ***smooth mapping*** *of the models $(\tilde{U}/\Gamma, p, \tilde{U})$ and $(\tilde{V}/\Delta, q, \tilde{V})$ is a mapping $f : \tilde{U}/\Gamma \to \tilde{V}/\Delta$ with the property that there exists a smooth mapping $\tilde{f} : \tilde{U} \to \tilde{V}$ such that $q \circ \tilde{f} = f \circ p$. We say that $\tilde{f}$ is a* ***lift*** *of f. A smooth mapping is a* ***diffeomorphism of models*** *if it is bijective and if the lift $\tilde{f}$ is a diffeomorphism.*

The next lemma is the quasifold analogous of Lemma 3.

Lemma 4. *(PRATO, 2001, p.983). [Orange Lemma] Let $\tilde{U}/\Gamma$ and $\tilde{V}/\Delta$ be two models, and let $f : \tilde{U}/\Gamma \to \tilde{V}/\Delta$ be a diffeomorphism of models. For any two lifts, $\tilde{f}$ and $\bar{f}$, of the diffeomorphism f there exists a unique element $\delta \in \Delta$ such that $\bar{f} = \delta\tilde{f}$.*

Proof. See (PRATO, 2001, p.983). For a bit more details, see the last part of the proof in Lemma 3. □

Definition 11. *(PRATO, 2001, p.964) [Quasifold] A dimension k* ***quasifold structure*** *on a topological space M is the assignment of an* ***atlas****, or collection of charts, $\mathcal{A} = \{U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha \mid \alpha \in A\}$ having the following properties:*

1. *The collection $\{U_\alpha \mid \alpha \in A\}$ is a cover of M*

2. *For each index $\alpha \in A$, the set U_α is open, the space $\tilde{U}_\alpha/\Gamma_\alpha$ defines a model, where the set $\tilde{U}_\alpha$ is an open, connected, and simply connected subset of the space $\mathbb{R}^k$, and the mapping ϕ_α is a homeomorphism of the space $\tilde{U}_\alpha/\Gamma_\alpha$ onto the set U_α.*

3. *For all indices $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the sets $\phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\phi_\beta^{-1}(U_\alpha \cap U_\beta)$ define models and the mapping*

$$g_{\alpha\beta} = \phi_\beta^{-1}\phi_\alpha : \phi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \phi_\beta^{-1}(U_\alpha \cap U_\beta)$$

is a diffeomorphism of models. We then say that the mapping $g_{\alpha\beta}$ is a ***change of charts*** *and that the corresponding charts are* ***compatible****.*

4. *The atlas $\mathcal{A}$ is* ***maximal****, that is: if the triple $(U, \phi, \tilde{U}/\Gamma)$ satisfies property 2, and is compatible with all the charts in $\mathcal{A}$, then $(\tilde{U}, \phi, \tilde{U}/\Gamma)$ belongs to $\mathcal{A}$*

A space M with a quasifold structure is a ***quasifold****.*

Remark 5. Apart from the differences in the local models already mentioned, the difference between a quasifold atlas and an orbifold one is in the notion of compatibility of charts. In the quasifold case, this involves a smoothness requirement for the change of charts. In the orbifold case, this involves a common refinement to charts that partially overlap (see however Lemma 1, and the preceding comment). □

We will mention two more important differences between orbifolds and quasifolds. One is that while every orbifold is isomorphic to a global orbifold (Theorem 8), the same cannot be said about quasifolds ((BATTAGLIA; PRATO, 2010) presents a quasifold that cannot be isomorphic to a quotient of a manifold modulo a smooth discrete group action). The other is that while the underlying topological space of an orbifold is Hausdorff by definition, the underlying space of a quasifold may not be.

Example 2. Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Consider $\mathbb{R}^2 \simeq \mathbb{C}$ with the action

$$\begin{aligned} \mathbb{Z} \times \mathbb{C} &\to \mathbb{C}, \\ (k, z) &\mapsto e^{2\pi i k\alpha} z \end{aligned}$$

This action is free on $\mathbb{C} \setminus \{0\}$, but the isotropy subgroup on $z = 0$ is $\mathbb{Z}$.

The quasifold is the quotient space of this action.

Already in this simple example we see that the action of $\mathbb{Z}$ on $\mathbb{C}$ is not proper, since the isotropy subgroup at $z = 0$ is not compact.

1.4 Lie Groupoids and Orbifolds

Orbifolds can be related with a particular kind of Lie groupoid. In this section we will describe this relation, defining just the necessary material on Lie groupoids, in particular we do not define Lie groupoid morphisms in the text. Our main sources were (ADEM; LEIDA; RUAN, 2007; MOERDIJK; MRcUN, 2003; MACKENZIE, 2005).

Definition 12. *(ADEM; LEIDA; RUAN, 2007, p. 17) A* ***Lie groupoid*** *is a topological groupoid $\mathcal{G}$ where G_0 and G_1 are smooth manifolds, and such that the structure maps s, t, m, u, i are smooth. Furthermore $s, t : G_1 \to G_0$ must be submersions. We assume G_0 and G_1 to be Hausdorff.*

See the Appendix, for a more detailed definition.

Note however that (MOERDIJK; MRcUN, 2003) do not require G_1 to be Hausdorff.

Definition 13. *(ADEM; LEIDA; RUAN, 2007, p. 17) Let $\mathcal{G}$ be a Lie groupoid. Let $x \in G_0$.*

- *The* ***isotropy*** *or* ***local group*** *at x,*

  $$G_x := s^{-1}(x) \cap t^{-1}(x),$$

  is the set of all arrows from x to itself.

- *The* ***orbit*** *of x is the set $t(s^{-1}(x))$ of targets of arrows out of x.*

- *The* ***orbit space*** *$|\mathcal{G}|$ of $\mathcal{G}$ is the quotient space of G_0 under the equivalence relation*

  $$x \sim y \iff x \text{ and } y \text{ are in the same orbit.}$$

  Unless mention of the contrary, we equip $|\mathcal{G}|$ with the quotient topology given by this equivalence relation.

We will see later that the isotropy groups G_x are Lie groups.

Definition 14. *(ADEM; LEIDA; RUAN, 2007, p. 17) Let X be a topological space and $\mathcal{G}$ a Lie groupoid. $\mathcal{G}$ is a* ***groupoid presentation*** *of X if the orbit space of $\mathcal{G}$ is X: $|\mathcal{G}| = X$.*

Remark 6. In Definition 13 note that:

- x and y are in the same orbit if and only if there is an arrow connecting them, that is, iff there exists $g \in G_1$ such that $s(g) = x$ and $t(g) = y$. This follows quickly from the fact that every arrow in a groupoid is invertible.

- $x \sim y$ is an equivalence relation because of the previous point and: of the existence of identity arrows (reflexivity), every arrow is invertible (symmetry), arrows compose (transitivity).

□

Lemma 5. *Let $\mathcal{G}$ be a Lie groupoid. Let $X := |\mathcal{G}|$ be the orbit space and $\pi : G_0 \to X$ be the quotient map, where X is equipped with the quotient topology. Then π is an open map.*

Proof. Let $U \subset G_0$ be a set. We will show first that

$$\pi^{-1}(\pi(U)) = s(t^{-1}(U)).$$

Let $y \in \pi^{-1}(\pi(U))$. Then there exists $x \in U$ such that $\pi(y) = \pi(x)$. By Remark 6, there is an arrow $f : y \to x$. Then $f \in t^{-1}(U)$ and $y = s(f) \in s(t^{-1}(U))$.

Now let $y \in s(t^{-1}(U))$. Then there is $x \in U$ and $f : y \to x$. Then $\pi(y) = \pi(x)$ so $y \in \pi^{-1}(\pi(U))$.

Now we also assume that U is open. Since t is continuous and s is an open map, $s(t^{-1}(U))$ is open (s is an open map because it is a smooth submersion, see Proposition 10).

Since X is equipped with the quotient topology, a subset $V \subset X$ is open if and only if $\pi^{-1}(V)$ is open.

It follows from the above that $\pi(U)$ is open. Therefore, π is an open map. □

In the following we prove some results on the existence of local bisections on Lie groupoids. They will be used to show that the isotropy groups are Lie groups.

Definition 15 (Local Bisections)**.** *(MACKENZIE, 2005, p. 25) Let $\mathcal{G}$ be a Lie groupoid. For $U \subset G_0$ open, a **local bisection** of $\mathcal{G}$ on U is a map $\sigma : U \to G_1$ which is a right inverse to s and for which $t \circ \sigma : U \to (t \circ \sigma)(U)$ is a diffeomorphism from U to the open set $(t \circ \sigma)(U)$ in G_0.*

Definition 16. *(MACKENZIE, 2005, p. 22,25) Let $\mathcal{G}$ be a Lie groupoid, $U \subset G_0$ open, and $\sigma : U \to G_1$ a local bisection. Let $V := (t \circ \sigma)(U)$, which is an open subset of G_0, by definition. The map*

$$L_\sigma : t^{-1}(U) \to t^{-1}(V), \quad g \mapsto m((\sigma \circ t(g)), g)$$

*is the **local left-translation induced by** σ.*

Remark 7. In the context of the previous definition, We will show that the local left-translation induced by σ is a diffeomorphism.

First we will check that this map is well-defined. Since σ is the right inverse of s we have

$$s(\sigma \circ t(g)) = t(g)$$

which is necessary for the multiplication in L_σ to be possible. Also, given $g \in t^{-1}(U)$, we see that

$$t(L_\sigma(g)) = t((\sigma \circ t(g))g) = t((\sigma \circ t(g))) \in (t \circ \sigma)(U) = V,$$

and hence $L_\sigma(t^{-1}(U)) \subset t^{-1}(V)$.

Now we will see that it is smooth because it is a composition of smooth maps:

- $t|_{t^{-1}(U)} : t^{-1}(U) \to U$,
- $\sigma : U \to G_1$,
- $(\sigma \circ t) : t^{-1}(U) \to G_1$,
- $\iota : t^{-1}(U) \to G_1$ (inclusion),
- $(\sigma \circ t) \times \iota : t^{-1}(U) \to G_1 \times G_1$.

We have seen above that the image of the smooth map $(\sigma \circ t) \times \iota$ is contained in $\mathcal{G}^{(2)}$, which is an embedded submanifold. Because of that (See Proposition 12) the restriction

$$J := (\sigma \circ t) \times \iota : t^{-1}(U) \to \mathcal{G}^{(2)}$$

is also smooth. Hence $L_\sigma = m \circ J : t^{-1}(U) \to G_1$ is smooth. We have seen that the image of L_σ is contained in $t^{-1}(V)$, which is an open, thus embedded, submanifold of G_1, and therefore the restriction $L_\sigma : t^{-1}(U) \to t^{-1}(V)$ is also smooth.

The inverse to L_σ is the map

$$(L_\sigma)^{-1} : t^{-1}(V) \to t^{-1}(U), \qquad h \mapsto (\sigma \circ (t \circ \sigma)^{-1}(t(h)))^{-1}h.$$

First we will check that this map is well-defined.

$$s(\sigma \circ (t \circ \sigma)^{-1}(t(h)))^{-1} = t(\sigma \circ (t \circ \sigma)^{-1}(t(h))) = t(h)$$

which is necessary for the multiplication in $(L_\sigma)^{-1}$ to be possible. Also, we see that

$$\begin{aligned} t((L_\sigma)^{-1}(g)) &= t((\sigma \circ (t \circ \sigma)^{-1}(t(h)))^{-1}h) \\ &= t((\sigma \circ (t \circ \sigma)^{-1}(t(h)))^{-1}) \\ &= s((\sigma \circ (t \circ \sigma)^{-1}(t(h)))) \\ &= (t \circ \sigma)^{-1}(t(h)) \in U \end{aligned}$$

The smoothness is verified analogously to the previous case.

Finally we can check that these maps are inverse to each other.

It will be useful to consider the following restriction of $L_\sigma : t^{-1}(U) \to t^{-1}(V)$ to

$$\overline{L_\sigma} : t^{-1}(U) \cap s^{-1}(x) \to t^{-1}(V) \cap s^{-1}(x)$$

where we suppose that $t^{-1}(U) \cap s^{-1}(x) \neq \emptyset$. Since $s(L_\sigma(g)) = s((\sigma \circ t(g))g) = s(g)$, the codomain is correct. Since we are restricting the domain and codomain to submanifolds, the restriction is still smooth (that the submanifold is embedded is important to the codomain restriction, see Propositions 11, 12).

Analogously, we see that

$$(\overline{L_\sigma})^{-1} : t^{-1}(V) \cap s^{-1}(x) \to t^{-1}(U) \cap s^{-1}(x)$$

is the restriction of $(L_\sigma)^{-1}$. In particular it is smooth.

Therefore $\overline{L_\sigma} : t^{-1}(U) \cap s^{-1}(x) \to t^{-1}(V) \cap s^{-1}(x)$ is also a diffeomorphism. □

Lemma 6 (Existence of Local Bisections)**.** *(MACKENZIE, 2005, p. 26) Let $\mathcal{G}$ be a Lie groupoid, and let $g \in G_1$. There exists a local bisection σ with $\sigma \circ s(g) = g$*

Proof. (MACKENZIE, 2005, p. 26) For the benefit of the reader we fill in the details of the proof. Suppose $\dim G_1 = s$ and $\dim G_0 = r$. By definition of Lie groupoids, s, t are submersions, which implies $s \geq r$ and

$$\dim \operatorname{Ker} s_{*_g} = s - r = \dim \operatorname{Ker} t_{*_g}.$$

By Lemma 24, there exists a linear subspace $I \subset T_g G_1$ such that

$$I \oplus \operatorname{Ker} s_{*_g} = I \oplus \operatorname{Ker} t_{*_g} = T_g G_1.$$

Let us apply the Rank Theorem (Theorem 16) to $s : G_1 \to G_0$. Thus we have the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{s} & V \\ \downarrow \varphi & & \downarrow \psi \\ \tilde{U} & \xrightarrow{\tilde{s}} & \tilde{V} \end{array}$$

with $\tilde{U} := \varphi(U), \tilde{V} := \psi(V)$ and

$$\tilde{s}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^s) \mapsto (x^1, \ldots, x^r)$$

Now we take the derivative of $\psi \circ s = \tilde{s} \circ \varphi$ at g:

$$\begin{array}{ccc} T_g G_1 & \xrightarrow{s_{*g}} & T_{s(g)} G_0 \\ \downarrow \varphi_{*g} & & \downarrow \psi_{*_{s(g)}} \\ \mathbb{R}^s & \xrightarrow{P} & \mathbb{R}^r \end{array}$$

where $P := D\tilde{s}(\varphi(g))$. Since $\tilde{s}$ is linear, then

$$P : \mathbb{R}^s \to \mathbb{R}^r$$
$$\tilde{P}(x^1, \dots, x^r, x^{r+1}, \dots, x^s) \mapsto (x^1, \dots, x^r)$$

In particular, $P|_{\tilde{U}} = \tilde{s}$.

Since $\varphi : U \to \tilde{U}$ is a diffeomorphism $\varphi_{*_g} : T_g G_1 \to \mathbb{R}^s$ is a linear isomorphism. It follows that

$$\begin{aligned}\mathbb{R}^s &= \varphi_{*_g}(T_g G_1)\\ &= \varphi_{*_g}(I \oplus \operatorname{Ker} s_{*_g})\\ &= \varphi_{*_g}(I) \oplus \varphi_{*_g}(\operatorname{Ker} s_{*_g})\end{aligned}$$

Let $V := \varphi_{*_g}(I)$. Since $P = \psi_{*_{s(g)}} \circ s_{*_g} \circ (\varphi_{*_g})^{-1}$ and $\psi_{*_{s(g)}}$, φ_{*_g} are isomorphisms, then $\operatorname{Ker} P = \varphi_{*_g}(\operatorname{Ker} s_{*_g})$. Then we have $\mathbb{R}^s = V \oplus \operatorname{Ker} P$.

By Lemma 25, there is a linear map

$$L : \mathbb{R}^r \to \mathbb{R}^s$$

such that $P \circ L(u) = u$ and $L(\mathbb{R}^r) = V$.

L is linear, therefore continuous, and so $L^{-1}(\tilde{U})$ is an open subset of $\mathbb{R}^r$. Define $\hat{V} := \tilde{V} \cap L^{-1}(\tilde{U})$ (this intersection is nonempty because both components contain 0), and define $\tilde{\sigma} : \hat{V} \to \tilde{U}$ by $\tilde{\sigma} := L|_{\hat{V}}$. Finally, on $\psi^{-1}(\hat{V})$ we define $\sigma := \varphi^{-1} \circ \tilde{\sigma} \circ \psi$.

First we will note that $\sigma(s(g)) = g$.

$$\sigma(s(g)) = \varphi^{-1} \circ \tilde{\sigma} \circ \psi(s(g)) = \varphi^{-1} \circ \tilde{\sigma}(0)$$

because (V, ψ) is a chart centered at $s(g)$,

$$= \varphi^{-1}(0) = g$$

because (U, φ) is a chart centered at g.

Now we will see that σ is a right inverse to s. Let $m \in \psi^{-1}(\hat{V})$. Then

$$\begin{aligned}s \circ \sigma(m) &= s \circ \varphi^{-1} \circ \tilde{\sigma} \circ \psi(m)\\ &= \psi^{-1} \circ \psi \circ s \circ \varphi^{-1} \circ \tilde{\sigma} \circ \psi(m)\\ &= \psi^{-1} \circ \tilde{s} \circ \varphi \circ \varphi^{-1} \circ \tilde{\sigma} \circ \psi(m)\\ &= \psi^{-1} \circ \tilde{s} \circ \tilde{\sigma} \circ \psi(m)\\ &= \psi^{-1} \circ P \circ L \circ \psi(m)\\ &= \psi^{-1} \circ \psi(m) = m\end{aligned}$$

Let us compute $(t\circ\sigma)_{*_{s(g)}}(T_{s(g)}G_0)$

$$\begin{aligned}(t\circ\sigma)_{*_{s(g)}}(T_{s(g)}G_0) &= (t\circ\varphi^{-1}\circ\tilde{\sigma}\circ\psi)_{*_{s(g)}}(T_{s(g)}G_0)\\ &= (t\circ\varphi^{-1}\circ\tilde{\sigma})_{*_0}(\mathbb{R}^r)\\ &= (t\circ\varphi^{-1})_{*_0}\circ L(\mathbb{R}^r)\end{aligned}$$

because $\tilde{\sigma}=L$ is linear, so it is its own derivative

$$\begin{aligned}&= (t\circ\varphi^{-1})_{*_0}(V)\\ &= (t)_{*_g}(I)\\ &= (t)_{*_g}(I\oplus \operatorname{Ker} t_{*_g})\\ &= (t)_{*_g}(T_gG_1) = T_{t(g)}M,\end{aligned}$$

where in the last equality we used that t is a submersion.

Therefore $(t\circ\sigma)_{*_{s(g)}}$ is surjective. Since the dimension of the domain $(T_{s(g)}G_0)$ equals the dimension of the codomain $(T_{t(g)}G_0)$, $(t\circ\sigma)_{*_{s(g)}}$ is also injective. Therefore, an isomorphism. By the inverse mapping theorem, we can restrict the open set $\psi^{-1}(\hat{V})$ on which σ is defined to some W, so that $t\circ\sigma|_W$ is a diffeomorphism onto its image, which is an open set in G_0.

The local bisection is the $\sigma|_W$. □

Proposition 1. *(MACKENZIE, 2005, p. 26) Let $\mathcal{G}$ be a Lie groupoid, and let $x\in G_0$. The restriction of t to the source fiber $s^{-1}(x)$ has constant rank.*

Proof. (MACKENZIE, 2005, p. 26) For the benefit of the reader, we fill some of the details of the proof. For readability, let us denote $t|_{s^{-1}(x)}: s^{-1}(x)\to G_0$ by t_x.

Let $g,h\in s^{-1}(x)$. Then $j := gh^{-1}$ is defined and by Lemma 6, there exists a local bisection $\sigma: U\to G_1$ such that $\sigma\circ s(j)=j$.

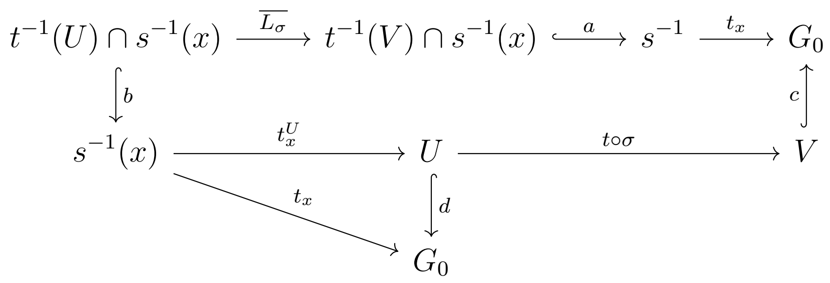
Let $V := (t\circ\sigma)(U)$. Note that $t^{-1}(U)\cap s^{-1}(x)\neq\emptyset$. By Remark 7, there exists a smooth diffeomorphism

$$\begin{aligned}\overline{L_\sigma}: t^{-1}(U)\cap s^{-1}(x) &\to t^{-1}(V)\cap s^{-1}(x)\\ g &\mapsto (\sigma\circ t(g))g\end{aligned}$$

Note that $t(h)=s(gh^{-1})=s(j)\in U, h=j^{-1}g$ and that

$$\begin{aligned}\overline{L_\sigma}(h) &= (\sigma\circ t(h))h\\ &= (\sigma\circ t(j^{-1}g))h\\ &= (\sigma\circ t(j^{-1}))h\\ &= (\sigma\circ s(j))h\\ &= jh = g\end{aligned} \tag{1.12}$$

The following diagram is commutative:



We observe that

- the c, d are inclusions of open subsets into the manifold $s^{-1}(x)$, thus, their derivatives are isomorphisms. In particular, $(t_x^U)_* = (t_x)_* d_*^{-1}$.
- a, b are inclusions of open subsets into the manifold $s^{-1}(x)$, thus, their derivatives are isomorphisms.
- $t \circ \sigma$ and $\overline{L_\sigma}$ are diffeomorphisms.

From the commutative diagram we have

$$c \circ (t \circ \sigma) \circ t_x^U \circ b = t_x \circ a \circ (\overline{L_\sigma})$$

From (1.12) and Taking the derivative at h, we have

$$c_* (t \circ \sigma)_{*_{t(h)}} (t_x^U)_{*_h} b_{*_h} = (t_x)_{*_g} a_{*_g} (\overline{L_\sigma})_{*_h}$$

From the observations above, we conclude that the ranks of t_x at g and h are equal. □

Definition 17. *(ADEM; LEIDA; RUAN, 2007, p. 18) Let $\mathcal{G}$ be a Lie groupoid.*

- *$\mathcal{G}$ is **proper** if $(s, t) : G_1 \to G_0 \times G_0$ is a proper map.*
- *$\mathcal{G}$ is a **foliation groupoid** if each isotropy group G_x is discrete subgroup of G_1.*
- *$\mathcal{G}$ is étale if s and t are local diffeomorphisms. If $\mathcal{G}$ is an étale groupoid, we define its **dimension** $\dim \mathcal{G} := \dim G_1 = \dim G_0$.*

Lemma 7. *(ADEM; LEIDA; RUAN, 2007, p. 18) Let $\mathcal{G}$ be an étale Lie groupoid. Then $\mathcal{G}$ is a foliation groupoid.*

Proof. Let $x \in G_0$. By definition $G_x = s^{-1}(x) \cap t^{-1}(x)$. Let $g \in G_x$. Then $s(g) = x$. Since $\mathcal{G}$ is étale, $s : G_1 \to G_0$ is a local diffeomorphism. It follows that there exist an open subset $U \subset G_1$ such that $U \cap s^{-1}(x) = \{g\}$. Thus $U \cap G_x = \{g\}$. Therefore G_x is discrete. □

Proposition 2. *(ADEM; LEIDA; RUAN, 2007, p. 18) Let $\mathcal{G}$ be a Lie groupoid. Then every isotropy group is a Lie group.*

If $\mathcal{G}$ is also proper, every isotropy group is a compact Lie group. In particular, if $\mathcal{G}$ is a proper foliation groupoid, then all of its isotropy groups are finite.

Proof. (ADEM; LEIDA; RUAN, 2007, p. 18) By definition, $G_x = s^{-1}(x) \cap t^{-1}(x) \subset G_1$. By Proposition 1, the restriction of $t : G_1 \to G_0$ to $s^{-1}(x)$ has constant rank.

By the Constant Rank Level Set Theorem (Theorem 17), G_x is a properly embedded submanifold of G_1.

Now, suppose $\mathcal{G}$ is also proper, that is, the map

$$(s,t) : G_1 \to G_0 \times G_0$$

is a proper map. By definition of a proper map, preimage of compact is compact. Since $G_x = (s,t)^{-1}(x,x)$, G_x is a compact Lie group.

If $\mathcal{G}$ is also a foliation groupoid, G_x is also discrete. A compact discrete Lie group is finite. □

The next definition is analogous to Definition 33.

Definition 18. *Let $\mathcal{G}$ be a Lie groupoid. The* ***orbit relation*** *of this groupoid is the set*

$$\mathcal{O} = (s,t)(G_1) = \{(s(g), t(g)) \in G_0 \times G_0 \mid g \in G_1\}$$

The next Lemma is analogous to Lemma 32.

Lemma 8 (Characterization of Orbit Relation)**.** *Let $\mathcal{G}$ be a Lie groupoid. Let $X := |\mathcal{G}|$ be the orbit space and $\pi : G_0 \to X$ be the quotient map. Let $\mathcal{O}$ be the orbit relation of this action. Then*

$$\mathcal{O} = \{(q,p) \in G_0 \times G_0 \mid \pi(q) = \pi(p)\}$$

Proof. Note that $\pi(q) = \pi(p)$ if and only if there is an arrow connecting p and q (Remark 6). □

Lemma 9. *Let $\mathcal{G}$ be a proper Lie groupoid. Let $X := |\mathcal{G}|$ be the orbit space and $\pi : G_0 \to X$ be the quotient map, where X is equipped with the quotient topology. Then X is Hausdorff.*

Proof. This proof is analogous to the one in Proposition 15.

Since $(s,t) : G_1 \to G_0 \times G_0$ is a proper continuous map, it is closed (Theorem 15). Thus the orbit relation $\mathcal{O} = (s,t)(G_1)$ is a closed subset of $G_0 \times G_0$.

Also note that π is an open map, by Lemma 5.

Then using Lemma 8 and Proposition 7, we have that X is Hausdorff. □

Definition 19 (Orbifold groupoids)**.** *(ADEM; LEIDA; RUAN, 2007, p. 19) An* ***orbifold groupoid*** *is a proper étale Lie groupoid.*

We will also need the following proposition on Lie groupoids, for next chapter:

Proposition 3. *(LERMAN, 2010, p. 324) Let $\mathcal{G}$ be a Lie groupoid and $f : N \to G_0$ a smooth map. Consider the fiber product*

$$N \times_{f,G_0,s} G_1 = \{(x, g) \in (N \times G_1) \mid f(x) = s(g)\}$$

If the map

$$N \times_{f,G_0,s} G_1 \to G_0, \quad (x, g) \mapsto t(g)$$

is a submersion, then the pullback groupoid $f^*\mathcal{G}$ is a Lie groupoid and the functor $\tilde{f} : f^*\mathcal{G} \to \mathcal{G}$ is a smooth functor.

1.4.1 From Orbifolds to Lie groupoids

We will describe three ways to construct a Lie groupoid from an orbifold.

Construction 2 (Lie groupoid from Orbifold - I)**.** (PRONK, 1995) (TOMMASINI, 2012, p. 768)

This construction is presented in (PRONK, 1995), for real orbifolds, and generalized by (TOMMASINI, 2012) for the case of complex orbifolds. We will describe how the space of objects and space of arrows of the groupoid are defined, starting from an orbifold, and refer to (TOMMASINI, 2012, p. 768) for a detailed demonstration of why this yields a Lie groupoid.

Let $\mathcal{U} = \{(\tilde{U}_i, G_i, \phi_i)\}_{i \in I}$ be an orbifold atlas of dimension n on a paracompact and second countable Hausdorff topological space X.

First we define the space of objects G_0 of our future Lie groupoid $\mathcal{G}$.

$$G_0 := \coprod_{i \in I} \tilde{U}_i$$

with the topology of the disjoint union.

Now, given $p \in \tilde{U}_i, q \in \tilde{U}_j$, suppose there is a chart $\tilde{U}_k$ that is a common refinement to both charts, with the embeddings $\mu : \tilde{U}_k \to \tilde{U}_i, \lambda : \tilde{U}_k \to \tilde{U}_j$. We declare that there is an arrow $p \to q$, if $p \in \mu(\tilde{U}_k)$ and $q \in \lambda(\tilde{U}_k)$. □

The other way to construct a Lie groupoid from an orbifold uses pseudogroups.

Definition 20. *(MOERDIJK; MRcUN, 2003, p. 138) Let M be a smooth manifold. A* ***(local) transition*** *on M is a diffeomorphism $f : U \to U'$ between two open subsets of M. We will denote the set of all transitions on M by C^∞_M. A* ***pseudogroup*** *on M is a subset P of transitions on M such that:*

- $\mathrm{id}_U \in P$, *for any open* $U \subset M$,
- *if* $f, f' \in P$, *then* $f' \circ f|_{f^{-1}(\mathrm{dom}(f'))} \in P$, *and* $f^{-1} \in P$,
- *if* f *is a transition on* M *and* (U_i) *is an open cover of* $\mathrm{dom}(f)$ *such that* $f|_{U_i} \in P$ *for any* i, *then* $f \in P$.

Example 3 (Trivial example of pseudogroup)**.** (MOERDIJK; MRcUN, 2003, p. 138) The collection C^∞_M of all transitions on M is a pseudogroup on M. □

Construction 3 (From pseudogroups to effective groupoids)**.** (MOERDIJK; MRcUN, 2003, p. 138)(HAEFLIGER, 1971, p. 136) Let M be a smooth manifold, and P a pseudogroup on M. We will associate an effective groupoid $\Gamma(P)$ over M as follows: for any $x, y \in M$ let

$$\Gamma(P)(x, y) := \{\mathrm{germ}_x f \mid f \in P, x \in \mathrm{dom}(f), f(x) = y\}$$

The multiplication in $\Gamma(P)$ is given by the composition of transitions.

The classical sheaf topology on the space of arrows $\Gamma(P)_1$ is the one generated by the subsets

$$U_f = \{\mathrm{germ}_x f \mid \text{for every } x \in U\}$$

for every $(f : U \to U') \in P$

With the classical sheaf topology $\Gamma(P)_1$ becomes a smooth manifold (which may neither be Hausdorff nor second countable). Also, $\Gamma(P)$ becomes an effective groupoid. □

Construction 4 (Lie groupoid from Orbifold - II)**.** (MOERDIJK; MRcUN, 2003, p. 141) Let Q be an orbifold and $\mathcal{U} = \{(U_i, G_i, \varphi_i)\}_{i \in I}$ be an orbifold atlas of Q. Put $U = \coprod_{i\in I} U_i$ and $\varphi = \varphi_i : U \to Q$. Now let $\Psi(\mathcal{U})$ be the pseudogroup on U of all transition f on U for which $\varphi \circ f = \varphi|_{\mathrm{dom}(f)}$

We define the effective groupoid $\Gamma(\mathcal{U})$ to be the effective groupoid associated to the pseudogroup $\Psi(\mathcal{U})$,

$$\Gamma(\mathcal{U}) = \Gamma(\Psi(\mathcal{U}))$$

□

Proposition 4. *(MOERDIJK; MRcUN, 2003, p. 141) Let* $\mathcal{U}$ *be an orbifold atlas of an orbifold* Q*.*

- $\Gamma(\mathcal{U})$ *is a proper effective groupoid.*
- *If* $\mathcal{U}$ *is an orbifold atlas of an orbifold* Q *and* $\mathcal{U}'$ *is an orbifold atlas of an orbifold* Q'*, then* $\Gamma(\mathcal{U})$ *and* $\Gamma(\mathcal{U}')$ *are weakly equivalent if and only if* Q *and* Q' *are isomorphic.*

Proof. (MOERDIJK; MRcUN, 2003, p. 141) □

Finally, the third way associates the orbifold to the transformation groupoid of the frame bundle (see Theorem 8).

1.4.2 From Lie groupoids to Orbifolds

Proposition 5. *(ADEM; LEIDA; RUAN, 2007, p. 21) Let $\mathcal{G}$ be a proper, effective, étale groupoid. Then its orbit space $X = |\mathcal{G}|$ can be given the structure of an effective orbifold, explicitly constructed from the groupoid $\mathcal{G}$*

Proof. (MOERDIJK; PRONK, 1997, p. 15)(ADEM; LEIDA; RUAN, 2007, p. 21) We will explain some points of the proof, and refer the reader to the sources.

Let $\pi : G_0 \to X$ denote the quotient map associated to the equivalent relation of Definition 13 and Remark 6.

By Lemma 5, π is an open map.

By Lemma 9, X is Hausdorff.

Since $\mathcal{G}$ is étale, it is foliation (Lemma 7). By Proposition 2, G_x is a finite group. □

2 LVMB Manifolds

In this chapter we discuss a family of complex manifolds that are related to toric varieties, orbifolds and quasifolds, called LVMB manifolds (MEDRANO; VERJOVSKY, 1997; MEERSSEMAN, 2000; BOSIO, 2001). They are a family of complex, compacted foliated manifolds that are in general non-Kahler, and are constructed from combinatorial data, much like toric varieties and the generalizations seen above. The construction will be described in the next section, in the context of an example.

The combinatorial data used to construct LVMB manifolds is a triangulated vector configuration (Definition 22). The precise way LVMB manifolds relate to toric varieties, orbifolds and quasifolds was worked out by Battaglia and Zaffran (BATTAGLIA; ZAFFRAN, 2017; BATTAGLIA; ZAFFRAN, 2015). They show that a triangulation vector configuration encodes essentially the same information as a simple polytope (note for example the similarity in the definition of a triangulation and the definition of a fan of a polytope). They also prove the following:

Theorem 9. *(BATTAGLIA; ZAFFRAN, 2017) Let $(V, \mathcal{T})$ be a triangulated vector configuration and N the associated LVMB manifold, with foliation $\mathcal{F}$. If $(V, \mathcal{T})$ encodes the information for the construction of a toric variety, orbifold, or quasifold, then the leaf space $N/\mathcal{F}$ is biholomorphic to X.*

2.1 Construction of LVMB manifolds

In this section we follow closely (BATTAGLIA; ZAFFRAN, 2017).

First we define the terms related to the data that will be used to construct the LVMB manifolds.

Definition 21.

1. *A **configuration of points** $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ in the affine space $\mathbb{C}^m$ is just a finite ordered list with repetitions allowed, such that the affine hull of Λ is the whole space:*

$$\mathrm{Aff}(\Lambda) = \mathbb{C}^m.$$

2. *Each point Λ_j defines a real row vector*

$$\Lambda_j^{\mathbb{R}} := [-\mathrm{Re}(\Lambda_j) - -\mathrm{Im}(\Lambda_j) -] \in \mathbb{R}^{2m}.$$

3. *A **basis** is a subset τ^* of $\{1, \ldots, n\}$, of cardinality $2m+1$, such that the interior $\mathring{C}_\alpha$ of the convex hull $Conv(\{\Lambda_j^{\mathbb{R}}\}_{j\in\tau^*})$ is non empty.*

4. *A* ***virtual chamber*** $\mathcal{T}^*$ *of the configuration* Λ *is a collection of bases* $\{\tau^*_\alpha\}_\alpha$ *satisfying* ***Bosio's conditions*** *(BOSIO, 2001)*

   - $\mathring{C}_\alpha \cap \mathring{C}_\beta \neq \emptyset$, *for every* α, β*;*
   - *for every* $\tau^*_\alpha \in \mathcal{T}^*$ *and every* $i \notin \tau^*_\alpha$, *there exists* $j \in \mathcal{T}^*$ *such that* $(\tau^*_\alpha \backslash \{j\}) \cup \{i\} \in \mathcal{T}^*$

5. *An* ***LVMB datum*** $\{\Lambda, \mathcal{T}^*\}$ *is a configuration* $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ *in* $\mathbb{C}^m$, *with* $n \geq 2m + 1$, *and a virtual chamber of* Λ.

Now we fix an LVMB datum $\{\Lambda, \mathcal{T}^*\}$, and use it to define both a space, and an action on this space. The quotient of this action will be, by definition, the **LVMB manifold** N associated to this datum.

For each $\tau^* \in \mathcal{T}^*$, define

$$U_{\tau^*} := \{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} \mid \forall j \in \tau^*, z_j \neq 0\}, \tag{2.1}$$

$$U(\mathcal{T}^*) := \bigcup_{\tau^* \in \mathcal{T}^*} U_{\tau^*} \tag{2.2}$$

Consider the matrix

$$\begin{bmatrix} -\Lambda_1- \\ \vdots \\ -\Lambda_n- \end{bmatrix} \in \mathbb{C}^{n \times m}$$

and define the subspace $\mathfrak{h} \subset \mathbb{C}^n$ as the span of the m columns of this matrix.

Lemma 10. *(BATTAGLIA; ZAFFRAN, 2017) the subspace* $\mathfrak{h}$ *defined above has dimension* m*.*

Consider the action of $(\mathbb{C}^*)^n$ on $\mathbb{CP}^{n-1}$ by componentwise multiplication, and restrict this action to the subgroup $\exp(\mathfrak{h})$, where

$$\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n, \quad (z_1, \ldots, z_n) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n})$$

This restricted action is free and proper on $U(\mathcal{T}^*)$ (BOSIO, 2001), therefore the quotient

$$N := \frac{U(\mathcal{T}^*)}{\exp(\mathfrak{h})}$$

is a complex manifold, called the **LVMB manifold** associated to the given initial data.

For convenience, let us give an explicit description of this action in terms of the configuration Λ. Let $\{w_1, \ldots, w_m\}$ be the columns of the matrix above. Then for $\xi \in \mathfrak{h}$, $\xi = (\xi_1, \ldots, \xi_n) = \sum_{j=1}^{m} \overline{u}_j w_j$, with $u = (u_1, \ldots, u_m) \in \mathbb{C}^m$ and for $[z] \in U(\mathcal{T}^*)$,

$$\exp(\xi) \cdot [z] = [e^{2\pi i \xi_1} z_1 : \cdots : e^{2\pi i \xi_n} z_n]$$

Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{C}^n$, and let $\langle , \rangle$ be the Hermitian inner product of $\mathbb{C}^n$. Note that

$$\xi_k = \langle \sum_{j=1}^{m} \overline{u}_j w_j, e_k \rangle = \sum_{j=1}^{m} u_j \langle w_j, e_k \rangle = \langle u, \Lambda_k \rangle.$$

We then define another action

$$\begin{aligned} &\Psi : \mathbb{C}^m \times U(\mathcal{T}) \to U(\mathcal{T}), \\ &(u, [z_1, \dots, z_n]) \mapsto [e^{2\pi i \Lambda_1(u)} z_1 : \dots : e^{2\pi i \Lambda_n(u)} z_n], \end{aligned} \tag{2.3}$$

where $\Lambda_j(u)$ is the dot product in $\mathbb{C}^m$ (the one without conjugation). We denote the group of this action as H. This action is is essentially the same as the previous one, and thus it induces the same quotient manifold (the LVMB manifold).

2.1.1 Alternative initial data

We can also give the initial data for the construction of an LVMB manifold in the form of a **triangulated vector configuration**.

Definition 22. *(BATTAGLIA; ZAFFRAN, 2015, p. 11790)*

Let E be a $\mathbb{R}$- vector space of dimension d.

*A **vector configuration** $V = (v_1, \dots, v_n)$ is a finite, ordered list of vectors in E, allowing repetitions, and such that* $\mathrm{Span}_{\mathbb{R}}(V) = E$.

*Let $\tau \subset \{1, \dots, n\}$. The **cone over** τ is the cone*

$$\mathrm{cone}(\tau) = \sum_{j \in \tau} \mathbb{R}_{\geq 0} v_j.$$

By convention, $\mathrm{cone}(\emptyset) = \{0_E\}$.

*τ is a **simplex** if the vectors of V indexed by τ are linearly independent. A **simplicial cone** is a cone over a simplex.*

*A **triangulation** $\mathcal{T}$ of a configuration V is a collection of simplices such that:*

- *if $\tau \in \mathcal{T}$ and $\tau' \subset \tau$, then $\tau' \in \mathcal{T}$.*
- *for all $\tau, \tau' \in \mathcal{T}$,* $\mathrm{cone}(\tau) \cap \mathrm{cone}(\tau') = \mathrm{cone}(\tau \cap \tau')$;
- $\cup_{\tau \in \mathcal{T}} \mathrm{cone}(\tau) \supset \mathrm{cone}(V)$

Definition 23. *(BATTAGLIA; ZAFFRAN, 2015, p. 11792) A **triangulated vector configuration** (TVC) is a pair $(V, \mathcal{T})$, where V is a vector configuration and $\mathcal{T}$ is a triangulation of V. We will also assume that the TVC is **odd**, that is*

$$n - d = 2m + 1, \qquad \textit{with } m \textit{ a positive integer} \tag{2.4}$$

and that it is ***balanced****, that is*

$$\sum_{i=1}^{n} v_i = 0.$$

To see why these extra conditions are necessary, and how a TVC is equivalent to an LVMB datum, check (BATTAGLIA; ZAFFRAN, 2015).

From a triangulation $\mathcal{T}$ in a TVC, we recover a virtual chamber (Definition 22). Let $\{\mathcal{E}_\alpha\}_\alpha$ be the set of maximal simplices of $\mathcal{T}$. Then

$$\mathcal{T}^* := \{\tau_\alpha^* := \{1, \ldots, n\} \setminus \mathcal{E}_\alpha\}_\alpha$$

is a virtual chamber.

Remark 8. We can then rewrite equation (2.1) as

$$U_\alpha := \{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} \mid \forall j \notin \mathcal{E}_\alpha, z_j \neq 0\}. \tag{2.5}$$

We will denote the image of U_α in N by N_α. □

We note some straightforward consequences of the definition of a TVC.

Remark 9.

1. Suppose E is a vector space of dimension $d \geq 1$ (which is any interesting case). Condition (2.4) implies that

$$n = 2m + 1 + d \geq 4$$

2. Since a triangulation is a collection of simplices, and a simplex is a index set of linearly independent vectors, a simplex (in particular, a maximal simplex $\mathcal{E}_\alpha$) must have at most d elements, where d is the dimension of the ambient space.

□

Lemma 11. *At least one maximal simplex in the triangulation has d elements.*

Proof. In a triangulation, the following holds

$$\cup_{\mathcal{E}_\alpha} \operatorname{cone}(\mathcal{E}_\alpha) = \cup_{\tau \in \mathcal{T}} \operatorname{cone}(\tau) \supset \operatorname{cone}(V) = E. \tag{2.6}$$

Note that it is enough to consider the union of cones over maximal simplices $\mathcal{E}_\alpha$.

Suppose every maximal simplex in the triangulation has less than d elements. Then $\operatorname{Span}_{\mathbb{R}}(\mathcal{E}_\alpha)$ is a proper subset of E (therefore, closed and with empty interior). Since the vector configuration has only finitely many vectors, the collection of simplices in the triangulation is finite. By Baire's theorem applied to E, as a a finite dimensional real vector space, a finite union of closed subsets of empty interior has empty interior. Therefore,

$$\cup_{\mathcal{E}_\alpha} \operatorname{span}_{\mathbb{R}}(\mathcal{E}_\alpha) \not\supset E.$$

Since

$$\cup_{\mathcal{E}_\alpha} \operatorname{cone}(\mathcal{E}_\alpha) \subset \cup_{\mathcal{E}_\alpha} \operatorname{span}_{\mathbb{R}}(\mathcal{E}_\alpha)$$

we conclude

$$\cup_{\mathcal{E}_\alpha} \operatorname{cone}(\mathcal{E}_\alpha) \not\supset E,$$

which contradicts equation (2.6). □

Lemma 12. *Let $\mathcal{T}'$ be the set of maximal simplices with d elements. Then*

$$\cup_{\mathcal{E}\in\mathcal{T}'} \operatorname{cone}(\mathcal{E}) \supset \operatorname{cone}(V) = E. \tag{2.7}$$

Proof. Let $\tau \in \mathcal{T}$ be a simplex. Then $\operatorname{cone}(\tau)$ is closed. Since there are only finitely many simplices in a triangulation,

$$\cup_{\mathcal{E}\in\mathcal{T}'} \operatorname{cone}(\mathcal{E}) \tag{2.8}$$

is closed, and

$$M = E \setminus \cup_{\mathcal{E}\in\mathcal{T}'} \operatorname{cone}(\mathcal{E}) \tag{2.9}$$

is open. Again, let $\tau \in \mathcal{T}$, and suppose τ has less than d elements. Since the vectors indexed by τ and E has dimension d, $\operatorname{Span}_{\mathbb{R}}(\tau)$ is a proper and closed subspace of E. Therefore, it has empty interior. Define

$$S_\tau := \operatorname{Span}_{\mathbb{R}}(\tau) \cap M,$$

and suppose it does not have empty interior in M. Since M is open in E, it does not have empty interior in E, thus neither does $\operatorname{Span}_{\mathbb{R}}(\tau)$, contradiction. Therefore S_τ has empty interior in M. Define

$$F_\tau := \operatorname{cone}_{\mathbb{R}}(\tau) \cap M,$$

which is closed in M, because $\operatorname{cone}_{\mathbb{R}}(\tau)$ is closed in E. Since $F_\tau \subset S_\tau$, F_τ also has empty interior in M.

In a triangulation, the following holds

$$\cup_{\tau\in\mathcal{T}} \operatorname{cone}(\tau) \supset \operatorname{cone}(V) = E. \tag{2.10}$$

Let $\mathcal{T}''$ be the set of maximal simplices with less than d elements, and note that it is enough to consider the union of cones over maximal simplices,

$$(\cup_{\mathcal{E}\in\mathcal{T}'} \operatorname{cone}(\tau)) \cup (\cup_{\tau\in\mathcal{T}''} \operatorname{cone}(\tau)) \supset \operatorname{cone}(V) = E. \tag{2.11}$$

Suppose $M \neq \emptyset$. Then M is a non-empty open subset of E, which is a real finite-dimensional vector space and

$$M = \cup_{\tau \in \mathcal{T}''} F_\tau.$$

Thus, M is a finite union of closed subsets with empty interior. By Baire's Theorem, M has empty interior, which is a contradiction. Therefore $M = \emptyset$. Alternatively,

$$\cup_{\mathcal{E} \in \mathcal{T}'} \operatorname{cone}(\mathcal{E}) \supset \operatorname{cone}(V) = E. \tag{2.12}$$

□

Lemma 13. *Every maximal simplex of a triangulation is full-dimensional*

Proof. This is (LOERA; RAMBAU; SANTOS, 2010, Lemma 2.3.4(i)), but we need to use Lemma 12 to ensure that the space is covered by full-dimensional simplexes. □

Let $\operatorname{Rel}(V)$ be the relation space of V, that is,

$$\operatorname{Rel}(V) = \{a \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i v_i = 0\}.$$

Now we choose a basis for $\operatorname{Rel}(V)$, and write down a matrix M whose columns are the vectors of this basis.

Writing

$$M = \begin{bmatrix} \hat{\Lambda}_1^{\mathbb{R}} \\ \vdots \\ \hat{\Lambda}_n^{\mathbb{R}} \end{bmatrix} = \begin{bmatrix} 1 & \Lambda_1^{\mathbb{R}} \\ \vdots & \vdots \\ 1 & \Lambda_n^{\mathbb{R}} \end{bmatrix}$$

and

$$\Lambda_j^{\mathbb{R}} = (a_j^1, \ldots, a_j^{2m}) \in \mathbb{R}^{2m},$$

we define

$$\Lambda_j = (a_j^1 + i a_j^{m+1}, \ldots, a_j^m + i a_j^{2m}) \in \mathbb{C}^m.$$

The list $(\Lambda_1, \ldots, \Lambda_n)$ is the configuration of points used in the construction of the LVMB manifold (see Section 2.1).

Lemma 14. *(BATTAGLIA; ZAFFRAN, 2015, p.11794) Let $\mathcal{E}_\alpha$ be a maximal simplex. It follows that the vectors $\Lambda_j^{\mathbb{R}} - \Lambda_n^{\mathbb{R}}$ with $j \in \mathcal{E}_\alpha^c \setminus \{n\}$ (from the Gale duality construction) are a basis of $\mathbb{R}^{2m}$.*

Proof. Adapted from (BATTAGLIA; ZAFFRAN, 2015, p.11794).

Lemma 22 shows that the construction above yields a Gale Transform where the rows of M are the elements of the transform.

Then Lemma 23 yields that for each maximal simplex (of indices) $\mathcal{E}_\alpha$, $\{\hat{\Lambda}_j^{\mathbb{R}} \mid j \in \mathcal{E}_\alpha^c\}$ is a simplex (of vectors), that is, a linear basis of $\mathbb{R}^{2m+1}$.

Note that every $\hat{\Lambda}_j^{\mathbb{R}}$ has the first component 1 (in the standard basis). Then via the projection onto $\mathbb{R}^{2m}$ where the first component is dropped out, $\{\Lambda_j^{\mathbb{R}} \mid j \in \mathcal{E}_\alpha^c\}$ is an affine basis of $\mathbb{R}^{2n}$.

It follows from a characterization of affine independence that that the vectors $\Lambda_j^{\mathbb{R}} - \Lambda_n^{\mathbb{R}}$ with $j \in \mathcal{E}_\alpha^c \setminus \{n\}$ are a basis of $\mathbb{R}^{2m}$. $\square$

2.2 Foliation of an LVMB manifold and its leaf space

We will also define the action

$$\begin{aligned} \Phi : \mathbb{C}^{2m} \times U(\mathcal{T}) &\to U(\mathcal{T}), \\ (t, [z_1, \ldots, z_n]) &\mapsto [e^{2\pi i \Lambda_1^{\mathbb{R}}(t)} z_1 : \cdots : e^{2\pi i \Lambda_n^{\mathbb{R}}(t)} z_n], \end{aligned} \tag{2.13}$$

where $\Lambda_j^{\mathbb{R}}(t)$ is the dot product in $\mathbb{C}^{2m}$ (the one without conjugation).

Lemma 15. *(BATTAGLIA; ZAFFRAN, 2015) Let $[\boldsymbol{z}] \in U(\mathcal{T})$. The isotropy at $[\boldsymbol{z}]$ is a closed $\mathbb{Z}$-module $L_{\boldsymbol{z}} \subset \mathbb{R}^{2m} \subset \mathbb{C}^{2m}$ of rank at most $2m$.*

The actions Φ and Ψ commute (both are just componentwise multiplication, which is commutative), so Φ descends to N. We also denote by Φ this induced action.

Let

$$H_{\mathcal{F}} = \{t \in \mathbb{C}^{2m} \mid t = \begin{pmatrix} v \\ 0 \end{pmatrix}, v \in \mathbb{C}^m\}$$

Also from the theory, the restriction of Φ to $H_{\mathcal{F}}$ induces a smooth foliation $\mathcal{F}$ of $\dim m$ on N.

$$\begin{aligned} \Phi : H_{\mathcal{F}} \times N &\to N, \\ (t, [z_1, \ldots, z_n]) &\mapsto [e^{2\pi i \Lambda_1^{\mathbb{R}}(t)} z_1 : \cdots : e^{2\pi i \Lambda_n^{\mathbb{R}}(t)} z_n], \end{aligned} \tag{2.14}$$

2.3 Motivation for choice of slices

The remark below is extracted from the proof of a Lemma in (BATTAGLIA; ZAFFRAN, 2015, p. 11803). It serves as a motivation for our choice of "slices" in our construction.

Definition 24. *(BATTAGLIA; ZAFFRAN, 2015, p. 11803). Let $(V, \mathcal{T})$ be a balanced and odd triangulation vector configuration in a vector space of dimension d. Let τ be a subset of a maximal simplex $\mathcal{E}_\alpha$ of $\mathcal{T}$. We define an $\mathcal{F}$-saturated open subset of N, denoted $N_{\alpha,\tau}$, to be N_α if τ is empty (Remark 8), and as the image in N of*

$$U_{\alpha,\tau} := U_\alpha \setminus \{[z_1 : \cdots : z_n] \mid \text{ for all } j \in \tau, z_j = 0\}$$

when τ is nonempty.

Remark 10. Let $(V, \mathcal{T})$ be a balanced and odd triangulation vector configuration in a vector space of dimension d. Let τ be a subset of a maximal simplex $\mathcal{E}_\alpha$ of $\mathcal{T}$. Let $r := \#\tau$.

Suppose that $r > 0$ and assume for simplicity that $\mathcal{E}_\alpha = \{1, \ldots, d\}$ and $\tau = \{1, \ldots, r\}$.

We recall from the definition of a TVC that $n - d = 2m + 1$. By Lemma 14, $\{\Lambda_j^{\mathbb{R}} - \Lambda_n^{\mathbb{R}}\}_{j=d+1,\ldots,n-1}$ is an $\mathbb{R}$-basis of $\mathbb{R}^{2m}$. It is then a $\mathbb{C}$-basis for $\mathbb{C}^{2m}$.

We will show that the following map is surjective.

$$\begin{aligned} g : (\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m} &\to U_{\alpha,\tau} \\ ((z_1, \ldots, z_d), t) &\mapsto \Phi(t, [z_1 : \cdots : z_d : \underbrace{1 : \cdots : 1}_{2m+1}]). \end{aligned}$$

First we note that the domain of g is set so that $z_i \neq 0$, for $i = 1, \ldots, d$, because $[0 : \ldots, 0 : z_{d_1} : \cdots : z_n] \notin U_{\alpha,\tau}$. Thus the map is well-defined. Now we check surjectivity:

$$\begin{aligned} &\Phi(t, [z_1 : \cdots : z_d : \underbrace{1 : \cdots : 1}_{2m+1}]) \\ &\quad = [e^{2\pi i \Lambda_1^{\mathbb{R}}(t)} z_1 : \cdots : e^{2\pi i \Lambda_d^{\mathbb{R}}(t)} z_d : e^{2\pi i \Lambda_{d+1}^{\mathbb{R}}(t)} : \cdots : e^{2\pi i \Lambda_n^{\mathbb{R}}(t)}] \\ &= [e^{2\pi i (\Lambda_1^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t)} z_1 : \cdots : e^{2\pi i (\Lambda_d^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t)} z_d : \\ &\qquad e^{2\pi i (\Lambda_{d+1}^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t)} : \cdots : e^{2\pi i (\Lambda_{n-1}^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t)} : 1] \end{aligned}$$

Let $[w_1 : \cdots : w_{2m} : 1]$ be an arbitrary element of $U_{\alpha,\tau}$. By our fixed choice of α and τ, and the definition of this subset, $w_j \neq 0, j = d + 1, \ldots, n$.

By surjectivity of the exponential $\mathbb{C}^m \to \mathbb{C}^*$, for every $w_j, j \in (d+1, \ldots, n-1)$, there is a u_j such that $e^{u_j} = w_j$. Because row rank equals column ran, in a matrix, and and the observation that $\{\Lambda_j^{\mathbb{R}} - \Lambda_n^{\mathbb{R}}\}_{j=d+1,\ldots,n-1}$ is a $\mathbb{C}$-basis for $\mathbb{C}^{2m}$, there is $t \in \mathbb{C}^{2m}$ such that

$$[2\pi i(\Lambda_{d+1}^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t), \ldots, 2\pi i(\Lambda_{n-1}^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t)] = [u_{d+1}, \ldots, u_{n-1}]$$

Finally, we set

$$z_j = \frac{w_j}{e^{2\pi i (\Lambda_j^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(t)}}, \quad j = 1, \ldots, d.$$

This shows the surjectivity of g. This map induces a homeomorphism

$$(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m}/\Gamma \simeq U_{\alpha,\tau}$$

where Γ is the isotropy subgroup in Lemma 15. □

2.4 Constructing a Lie groupoids from LVMB data

We are interested in constructing Lie groupoids out of LVMB manifolds, in a way that is connected to how Lie groupoids can be built out of orbifolds. Theorem 9 motivates us to first consider the transformation groupoid $H_{\mathcal{F}} \ltimes N$. (LERMAN, 2010) mentions the following result. Let G be a Lie group acting in a free and proper manner on a manifold M. Let Σ be a slice for this action and let $\mathcal{U} = \coprod \Sigma$ be a disjoint union of slices such that every orbit of the action intersects at least one the slices. Then the pullback of the action groupoid $G \rtimes M$ by the "inclusion" $f : \coprod \Sigma \to M$ is a Lie groupoid. This construction relies on the existence of slices, because the action is proper, and on certain properties of slices.

In our case, the action of $H_{\mathcal{F}}$ on N is neither free nor proper, so we cannot rely on this result. Instead we have to be more constructive, and consider particular cases.

The general procedure is:

1. First we consider the action groupoid $\mathcal{G} \coloneqq H_{\mathcal{F}} \times N$.
2. Then we look for submanifolds N_i, on N such that each orbit of the $H_{\mathcal{F}}$ action intersects at least one of the N_i. To stay close to the result above, we use submanifolds with the same dimension a slice would have.
3. We consider the "inclusion" $f : \coprod N_i \to N$ and the pullback groupoid $f^*\mathcal{G}$.
4. We want $f^*\mathcal{G}$ to be a Lie groupoid. From the theory of Lie groupoids, it is sufficient to check that the map

$$\alpha : \coprod N_i \times_{f,G_0,s} G_1 \to G_0, (x, g) \mapsto t(g)$$

is a submersion, with $G_1 = H_{\mathcal{F}} \times N$, for each N_i and $G_0 = N$.

It would be interesting if the submanifolds N_i were slices for the $H_{\mathcal{F}}$ action. However, we know that this cannot happen for the quasifold case.

Our choice for the submanifold N_i are motivated by the remark in the previous section.We will take the submanifolds N_i of the constructions to be the images N_α of the open subsets U_α by the projection map $U(\mathcal{T}) \to N$. The surjective of the map g, shows that condition 2 for the choice of submanifolds in our construction is satisfied.

Now, let us note that

$$\begin{aligned}\coprod N_i \times_{f,G_0,s} G_1 &= \{(n_i,(u,z)) \in \coprod N_i \times G_1 \mid n_i = f(n_i) = s(u,z) = z\} \\ &\simeq \coprod N_i \times H_{\mathcal{F}}.\end{aligned}$$

Using this identification the map α becomes

$$\alpha : \coprod N_i \times H_{\mathcal{F}}. \to G_0, (s_i,u) \mapsto \Phi(u,s_i)$$

Since $\coprod N_i$ is a disjoint union of manifolds, it is enough to show that each restriction $N_i \times H_{\mathcal{F}} \to N$ is a submersion. Also, we note that this map decomposes into

$$N_i \times H_{\mathcal{F}} \to N_i \hookrightarrow N$$

where the last inclusion is already a smooth submersion since N_i is an open subset of N.

Thus we wil show that each $\alpha_i : N_i \times H_{\mathcal{F}} \to N_i, (s_i,u) \mapsto \Phi(u,s_i)$ is a submersion.

We write

$$(\alpha_i)_{*_{(s,u)}}(\xi,\eta) = R_{*_u}\eta + L_{*_s}\xi$$

where

$$R : H_{\mathcal{F}} \to N_i, u' \mapsto \Phi(u',s) \quad L : N_i \to N_i, s' \mapsto \Phi(u,s')$$

Note that R is the orbit map of the action Φ. Therefore is has constant rank $2m$, because every orbit has real dimension $2m$ (see section 2.3 of (BATTAGLIA; ZAFFRAN, 2015)).

The following diagram is commutative,

$$\begin{array}{ccc} U_i & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ N_i & \longrightarrow & N_i \end{array}$$

where π is the restriction of the quotient map $U(\mathcal{T}) \to N$ and

$$T : [z_1 : \cdots : z_n] \mapsto [e^{2\pi i \Lambda_1^{\mathbb{R}}(t)} z_1 : \cdots : e^{2\pi i \Lambda_n^{\mathbb{R}}(t)} z_n],$$

The first row has constant rank $2(n-1)$, since it is just coordinatewise multiplication in a open subset of $\mathbb{C}P^n$. The projection π is a smooth submersion. Therefore L also has constant rank $2(n-1)$.

By dimensionality, $(\alpha_i)_{*_{(s,u)}}$ is surjective.

With this we have proven the following:

Theorem 10. *Consider a triangulated vector configuration $(V, \mathcal{T})$. Using this initial data, we can construct an LVMB manifold N and a group $H_{\mathcal{F}}$ acting on it such that the orbits of the action yield a foliation $\mathcal{F}$. There are submanifolds N_i of N such that each leaf of the foliation intersects at least of the N_i, and such that the pullback of the action groupoid $H_{\mathcal{F}} \times N$ via the "inclusion" $\coprod N_i \to N$ is a Lie groupoid.*

In the remainder of this chapter we will give direct proofs of the following two applications of Theorem 10.

Theorem 11. *Let*

$$\begin{aligned} v_1 &:= (1, 0), \\ v_2 &:= (0, 1), \\ v_3 &:= (0, -1), \\ v_4 &:= (-1, a) \end{aligned}$$

be vectors in $\mathbb{R}^2$, with $a > 0$.

Let $\Delta \subset \mathbb{R}^2$ be the fan whose higher-dimensional cones are generated by $(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3)$.

For each choice of $a > 0$, we construct a Lie groupoid associated to Δ.

Remark 11. We note that for $a = n \in \mathbb{Z}$ a positive integer, this is the fan associated to the classic Hirzebruch surfaces. $\mathbb{F}_n$, and by varying the parameter $a \in \mathbb{R}_{>0}$, we can classically obtain a family of toric varieties, orbifolds and quasifolds that contain the Hirzebruch surfaces.

Theorem 12. *Let*

$$\begin{aligned} v_1 &:= (1, 0, \dots, 0), \\ v_2 &:= (0, 1, 0, \dots, 0), \\ &\vdots \\ v_d &:= (0, \dots, 0, 1), \\ v_{d+1} &:= (-\alpha_1, \dots, -\alpha_d) \end{aligned}$$

be vectors in $\mathbb{R}^d$. Let $\Delta \subset \mathbb{R}^d$ be the fan made out of each proper subset of $\{v_1, \dots, v_{d+1}\}$, with $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$.

For each choice of parameters $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$, we construct a Lie groupoid associated to Δ.

Remark 12. We note that for $\alpha_1 = ... = \alpha_d = 1$, this is the fan associated to $\mathbb{C}P^d$, and by varying the parameters $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$, we can classically obtain toric orbifold and quasifold variants of $\mathbb{C}P^d$.

2.5 Examples of LVMB manifolds

2.5.1 Hirzebruch surfaces in a one-parameter family

The triangulated vector configuration in this example is from (BATTAGLIA; PRATO; ZAFFRAN, 2019).

2.5.1.1 The fan of a generalized Hirzebruch surface

Let

$$\begin{aligned} v_1 &\coloneqq (1,0), \\ v_2 &\coloneqq (0,1), \\ v_3 &\coloneqq (0,-1), \\ v_4 &\coloneqq (-1,a) \end{aligned}$$

be vectors in $\mathbb{R}^2$, with $a > 0$. Let $\Delta \subset \mathbb{R}^2$ be the fan whose higher dimensional cones are generated by $(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3)$. For $a = n \in \mathbb{Z}$ a positive integer, this is the fan associated to the classical Hirzebruch surface $\mathbb{F}_n$.

2.5.1.2 The triangulated configuration associated to the fan

Following the notation of (BATTAGLIA; ZAFFRAN, 2017), Section 1.2, we fix Prato's datum $(\Delta, \{v_1, v_2, v_3, v_4\}, Q)$, where $Q = \text{Span}_{\mathbb{Z}}(v_1, v_2, v_3, v_4) \subset \mathbb{R}^2$ is a (quasi)lattice, and the other objects are from the previous section. If $a \in \mathbb{Z}$ then Q is a lattice.

We want to associate Prato's datum to a triangulated vector configuration $\{V, \mathcal{T}\}$. $V = (v_1, \ldots, v_n)$ is a vector configuration in $\mathbb{R}^2$ such that

- The first 3 vectors are v_1, v_2, v_3 as above;
- $Q = \text{Span}_{\mathbb{Z}}(v_1, \ldots, v_n)$;
- $\sum_{i=1}^{n} v_i = 0$;
- $n - d = 2m + 1, m \in \mathbb{N}$, with $d = 2$.

To satisfy the above, let $n = 5$, with $v_5 = -v_1 - v_2 - v_3 - v_4 = (0, -a)$. Note that $m = 1$.

And $\mathcal{T}$ corresponds to the fan cones. Thus the maximal simplices of $\mathcal{T}$ are

$$\{\mathcal{E}_{12} = \{12\}, \mathcal{E}_{24} = \{24\}, \mathcal{E}_{34} = \{34\}, \mathcal{E}_{14} = \{14\}, \}.$$

2.5.1.3 The dual configuration

Recall that

$$\begin{aligned}\operatorname{Rel}(V) &= \{a \in \mathbb{R}^5 \mid \sum_{i=1}^{5} a_i v_i = 0\} \\ &= \operatorname{Ker}\{T : \mathbb{R}^5 \to \mathbb{R}^2, (a_1, \dots, a_5) \mapsto \sum_{i=1}^{5} a_i v_i = 0\}\end{aligned}$$

Since $\operatorname{Im} T = \mathbb{R}^2$ then $\dim \operatorname{Ker} T = 3$.

We need to choose a M matrix whose columns give a basis for $\operatorname{Rel} V$. One such matrix (from (BATTAGLIA; PRATO; ZAFFRAN, 2019, p. 8) is

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & a \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Writing

$$M = \begin{bmatrix} \hat{\Lambda}_1^{\mathbb{R}} \\ \vdots \\ \hat{\Lambda}_5^{\mathbb{R}} \end{bmatrix} = \begin{bmatrix} 1 & \Lambda_1^{\mathbb{R}} \\ \vdots & \vdots \\ 1 & \Lambda_5^{\mathbb{R}} \end{bmatrix}$$

and

$$\Lambda_j^{\mathbb{R}} = (a_j^1, a_j^2) \in \mathbb{R}^2,$$

we define

$$\Lambda_j = a_j^1 + i a_j^2 \in \mathbb{C}.$$

Then

$$\Lambda = (\Lambda_1, \dots, \Lambda_5) = (i, 1, 1 + ia, i, 0).$$

2.5.1.4 Virtual chamber and $U(\mathcal{T})$

The virtual chamber is

$$\mathcal{E} = \{\{\mathcal{E}_{12}^c = \{345\}\}, \{\mathcal{E}_{24}^c = \{135\}\}, \{\mathcal{E}_{34}^c = \{125\}\}, \{\mathcal{E}_{13}^c = \{245\}\}\}.$$

The corresponding open subsets are

$$\begin{aligned} U_{12} &= \{[\underline{z}] \in \mathbb{C}P^4 \mid z_3 \neq 0, z_4 \neq 0, z_5 \neq 0\}, \\ U_{24} &= \{[\underline{z}] \in \mathbb{C}P^4 \mid z_1 \neq 0, z_3 \neq 0, z_5 \neq 0\}, \\ U_{34} &= \{[\underline{z}] \in \mathbb{C}P^4 \mid z_1 \neq 0, z_2 \neq 0, z_5 \neq 0\} \\ U_{13} &= \{[\underline{z}] \in \mathbb{C}P^4 \mid z_2 \neq 0, z_4 \neq 0, z_5 \neq 0\}. \end{aligned}$$

Then

$$U(\mathcal{T}) = U_{12} \cup U_{24} \cup U_{34} \cup U_{13}$$

2.5.1.5 The $\mathbb{C}$-action and N

We have

$$\Psi : \mathbb{C} \times U(\mathcal{T}) \to U(\mathcal{T}),$$
$$(u, [z_1 : \cdots : z_5]) \mapsto [e^{-2\pi u} z_1 : e^{2\pi i u} z_2 : e^{2\pi i(1+ia)u} z_3 : e^{-2\pi u} z_4 : z_5].$$

We denote $\mathbb{C}$ in this action by H. The LVMB manifold N is the quotient.

$$N = \frac{U(\mathcal{T})}{H}.$$

2.5.1.6 The foliation action

We will apply (2.13) to our particular case, and obtain

$$\Phi : \mathbb{C}^2 \times U(\mathcal{T}) \to U(\mathcal{T}),$$
$$(t, [z_1, \ldots, z_5]) \mapsto [e^{2\pi i t_2} z_1 : e^{2\pi i t_1} z_2 : e^{2\pi i(t_1 + a t_2)} z_3 : e^{2\pi i t_2} z_4 : z_5],$$

where $t = (t_1, t_2)$.

Identifying $H_{\mathcal{F}} \cong \mathbb{C}$, we have the particular case of (2.14)

$$\Phi : \mathbb{C} \times N \to N,$$
$$(v, [z_1, \ldots, z_5]) \mapsto [z_1 : e^{2\pi i v} z_2 : e^{2\pi i v} z_3 : z_4 : z_5],$$

2.5.2 The projective line $\mathbb{C}P^d$ and variants

For the construction of the LVMB associated to $\mathbb{C}P^1$ and orbifold and quasifold variants, see (BATTAGLIA; ZAFFRAN, 2015).

2.5.2.1 The fan of $\mathbb{C}P^d$

Let

$$v_1 := (1, 0, \ldots, 0),$$
$$v_2 := (0, 1, 0, \ldots, 0),$$
$$\vdots$$
$$v_d := (0, \ldots, 0, 1),$$
$$v_{d+1} := (-\alpha_1, \ldots, -\alpha_d)$$

be vectors in $\mathbb{R}^d$. Let $\Delta \subset \mathbb{R}^d$ be the fan composed of every proper subset of $\{v_1, \ldots, v_{d+1}\}$. For $\alpha_1 = \cdots = \alpha_d = 1$, this is the fan associated to $\mathbb{C}P^d$ (See (COX; LITTLE; SCHENCK, 2011, p.86, Exercise 2.3.7)).

2.5.2.2 The triangulated configuration associated to the fan

Following the notation of (BATTAGLIA; ZAFFRAN, 2017), Section 1.2, we fix Prato's datum $(\Delta, \{v_1, \ldots, v_{d+1}\}, Q)$, where $Q = \mathrm{Span}_{\mathbb{Z}}(v_1, \ldots, v_{d+1}) \subset \mathbb{R}^d$ is a (quasi)lattice, and the other objects are from the previous section. If $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ then Q is a lattice.

We want to associate Prato's datum to a triangulated vector configuration $\{V, \mathcal{T}\}$. $V = (v_1, \ldots, v_n)$ is a vector configuration in $\mathbb{R}^d$ such that

- The first $d+1$ vectors are $v_1, \ldots, v_{d+1}$ as above;
- $Q = \mathrm{Span}_{\mathbb{Z}}(v_1, \ldots, v_n)$;
- $\sum_{i=1}^{n} v_i = 0$;
- $n - d = 2m + 1, m \in \mathbb{N}$, with $d = l$.

To satisfy the above, let $m = 1$, and then $n = d + 3$, with $v_{d+2} = \sum_{i=1}^{d+1} -v_i$, and $v_{d+3} = 0$.

And $\mathcal{T}$ corresponds to the fan cones. Thus the maximal simplices of $\mathcal{T}$ are

$$\{\mathcal{E}_i = \{12 \ldots (i-1)\hat{i}(i+1) \ldots (d+1)\}$$

for $i = 1, \ldots d+1$

2.5.2.3 The dual configuration

Recall that

$$\begin{aligned}\mathrm{Rel}(V) &= \{a \in \mathbb{R}^{d+3} \mid \sum_{i=1}^{d+3} a_i v_i = 0\} \\ &= \mathrm{Ker}\{T : \mathbb{R}^{d+3} \to \mathbb{R}^d, (a_1, \ldots, a_{d+3}) \mapsto \sum_{i=1}^{d+3} a_i v_i = 0\}\end{aligned}$$

Since $\mathrm{Im}\, T = \mathbb{R}^d$ then $\dim \mathrm{Ker}\, T = 3$.

We need to choose a M matrix whose columns give a basis for $\mathrm{Rel}\, V$. One such matrix is

$$M = \begin{bmatrix} 1 & \alpha_1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & \alpha_d & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Writing

$$M = \begin{bmatrix} \hat{\Lambda}_1^{\mathbb{R}} \\ \vdots \\ \hat{\Lambda}_{d+3}^{\mathbb{R}} \end{bmatrix} = \begin{bmatrix} 1 & \Lambda_1^{\mathbb{R}} \\ \vdots & \vdots \\ 1 & \Lambda_{d+3}^{\mathbb{R}} \end{bmatrix}$$

and

$$\Lambda_j^{\mathbb{R}} = (a_j^1, a_j^2) \in \mathbb{R}^2,$$

we define

$$\Lambda_j = a_j^1 + i a_j^2 \in \mathbb{C}.$$

Then

$$\Lambda = (\Lambda_1, \ldots, \Lambda_{d+3}) = (\alpha_1, \ldots, \alpha_d, 1, 0, i).$$

2.5.2.4 Virtual chamber and $U(\mathcal{T})$

The virtual chamber is

$$\mathcal{E} = \{\{\mathcal{E}_i^c = \{i (d+2)(d+3)\}\} \mid i = 1, \ldots, d+1, \}.$$

The corresponding open subsets are

$$U_i = \{[\underline{z}] \in \mathbb{C}P^{d+2} \mid z_i \neq 0, z_{d+2} \neq 0, z_{d+3} \neq 0\},$$

for $i = 1, \ldots d+1$. Then

$$U(\mathcal{T}) = \bigcup_{i=1}^{d+1} U_i$$

2.5.2.5 The $\mathbb{C}$-action and N

We have

$$\Psi : \mathbb{C} \times U(\mathcal{T}) \to U(\mathcal{T}),$$
$$(u, [z_1 : \cdots : z_{d+3}]) \mapsto [e^{2\pi i \alpha_1 u} z_1 : \cdots : e^{2\pi i \alpha_d u} z_d : e^{2\pi i u} z_{d+1} : z_{d+2} : e^{-2\pi u} z_{d+3}].$$

We denote $\mathbb{C}$ in this action by H. The LVMB manifold N is the quotient

$$N = \frac{U(\mathcal{T})}{H}.$$

2.5.2.6 The foliation action

We will apply (2.13) to our particular case, and obtain

$$\Phi : \mathbb{C}^2 \times U(\mathcal{T}) \to U(\mathcal{T}),$$
$$(t, [z_1, \ldots, z_{d+3}]) \mapsto [e^{2\pi i \alpha_1 t_1} z_1 : \cdots : e^{2\pi i \alpha_d t_1} z_d : e^{2\pi i t_1} z_{d+1} : z_{d+2} : e^{2\pi i t_2} z_{d+3}],$$

where $t = (t_1, t_2)$.

Identifying $H_{\mathcal{F}} \cong \mathbb{C}$, we have the particular case of (2.14)

$$\Phi : \mathbb{C} \times N \to N,$$
$$(v, [z_1, \ldots, z_{d+3}]) \mapsto [e^{2\pi i \alpha_1 v} z_1 : \cdots : e^{2\pi i \alpha_d v} z_d : e^{2\pi i v} z_{d+1} : z_{d+2} : z_{d+3}],$$

2.6 Charts for LVMB manifolds

In this section, we compute charts for the LVMB manifolds that we described above. These charts will be useful to verify that certain maps that we use are indeed embeddings.

2.6.1 Hirzebruch surfaces in a one-parameter family

2.6.1.1 Derivative of the action

We will need to compute $(\Psi_p)_{*_e}$ where

$$\Psi : \mathbb{C} \times U(\mathcal{T}) \to U(\mathcal{T}),$$
$$(u, [z_1 : \cdots : z_5]) \mapsto [e^{-2\pi u} z_1 : e^{2\pi i u} z_2 : e^{2\pi i(1+ia)u} z_3 : e^{-2\pi u} z_4 : z_5].$$

and $p = [\underline{z}] \in U(\mathcal{T})$. Because $z_5 \neq 0$, we will use the following chart of $\mathbb{C}P^4$:

$$\varphi : U \to \mathbb{C}^4 \tag{2.15}$$
$$[z_1 : \cdots : z_5] \mapsto (\frac{z_1}{z_5}, \frac{z_2}{z_5}, \ldots, \frac{z_4}{z_5}). \tag{2.16}$$

where $U = \{[z_1 : \cdots : z_5] \in \mathbb{C}P^4 \mid z_5 \neq 0\}$.

In this chart

$$(\Psi_p)(u) = \left(e^{-2\pi u} \frac{z_1}{z_5}, e^{2\pi i u} \frac{z_2}{z_5}, e^{2\pi i(1+ia)u} \frac{z_3}{z_5}, e^{-2\pi u} \frac{z_4}{z_5}\right).$$

Then

$$\frac{\partial(\Psi_p)}{\partial u}(u) = \left((-2\pi) e^{-2\pi u} \frac{z_1}{z_5}, (2\pi i) e^{2\pi i u} \frac{z_2}{z_5}, (2\pi i(1+ia)) e^{2\pi i(1+ia)u} \frac{z_3}{z_5}, (-2\pi) e^{-2\pi u} \frac{z_4}{z_5}\right). \tag{2.17}$$

since $p = [\underline{z}] \in U(\mathcal{T})$, $z_5 \neq 0$, we can set $z_5 = 1$.

We evaluate the derivative at $u = 0 = e \in \mathbb{C}$.

$$\frac{\partial(\Psi_p)}{\partial u}(0) = ((-2\pi) z_1, (2\pi i) z_2, (2\pi i(1+ia)) z_3, (-2\pi) z_4).$$

We write $u = a + ib, z_j = x_j + iy_j$, then

$$\frac{\partial(\Psi_p)}{\partial u}(0) = (2\pi(-x_1 - iy_1), 2\pi(-y_2 + ix_2)$$
$$2\pi((-ax_3 - y_3) + i(x_3 - ay_3)), 2\pi(-x_4 - iy_4),).$$

We note that Ψ_p is holomorphic. In particular $\frac{\partial(\Psi_p)}{\partial u} = \frac{\partial(\Psi_p)}{\partial a}$, $\frac{\partial \operatorname{Re}(\Psi_p)}{\partial a} = \frac{\partial \operatorname{Im}(\Psi_p)}{\partial b}$ and $\frac{\partial \operatorname{Re}(\Psi_p)}{\partial b} = \frac{-\partial \operatorname{Im}(\Psi_p)}{\partial a}$.

Hence, the real Jacobian of Ψ_p at $u = 0$, with respect to the basis $\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\right)$ and $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_5},\right)$ is

$$(\Psi_p)_{*_e} = -2\pi \begin{pmatrix} x_1 & -y_1 \\ y_2 & x_2 \\ ax_3 + y_3 & x_3 - ay_3 \\ x_4 & -y_4 \\ y_1 & x_1 \\ -x_2 & y_2 \\ -x_3 + ay_3 & ax_3 + y_3 \\ y_4 & x_4 \end{pmatrix}$$

In the following, we will refer to the columns of this matrix as w_1, w_2 respectively.

2.6.1.2 Checking weak slice condition

We will check now that the following embedded submanifold satisfies (A.5). Let

$$\iota_S : \mathbb{C}^3 \setminus \{0\} \to U(\mathcal{T}), \tag{2.18}$$
$$(z_1, z_2, z_3) \mapsto [z_1 : z_2 : z_3 : 1 : 1]. \tag{2.19}$$

Let $q = (z_1, z_2, z_3)$ and $p = \iota_S(q)$. In the chart (2.15),

$$\iota_S(z_1, z_2, z_3) = (z_1, z_2, z_3, 1) = (x_1, x_2, x_3, 1, y_1, y_2, y_3 0)$$

where $z_k = x_k + iy_k$. Then

$$(\iota_S)_{*_p} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us denote the columns of this matrix by $e_1, e_2, e_3, e_5, e_6, e_7$, using the standard notation for the canonical basis of $\mathbb{R}^8$.

We will see that $\{e_1, e_2, e_3, e_5, e_6, e_7, w_1, w_2$, are linearly independent. Suppose

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7 + \beta_1 w_1 + \beta_2 w_2 = 0$$

In particular

$$\beta_1 x_5 + (-\beta_2 y_5) = 0,$$
$$\beta_1 y_5 + \beta_2 x_5 = 0.$$

Since $z_4 = 1 + i0$, $x_4 = 1$ and $y_4 = 0$. Thus $\beta_1 = \beta_2 = 0$, and it follows that $\alpha_i = 0$.

Thus, $S = \iota_S(\mathbb{C}^3 \setminus \{0\})$ satisfies (A.5), so there exists a chart U around $[z_1 : z_2 : 1 : 1]$ where $S \cap U$ is a weak slice. From the discussion following Lemma 37, we can take this U to be a sufficiently small coordinate ball (i.e. the image of a ball in $\mathbb{C}^3$ by the chart (2.15)). And since S is an embedded submanifold, this means we can just restrict the domain of ι_S to some $\tilde{S} = B_{z_1} \times B_{z_2} \times B_{z_3}$, where $B_k \subseteq \mathbb{C}$ are sufficiently small open balls centered at z_k. Then we rename $S = \iota_S(\tilde{S})$. Also, if $U_S = \pi(S) \subseteq N$, and $\varphi_S : U_S \to S, \varphi_S = (\pi|_S)^{-1}$, then φ_S is a chart for N.

2.6.2 The projective line $\mathbb{C}P^d$ and variants

2.6.2.1 Derivative of the action

We will need to compute $(\Psi_p)_{*_e}$ where

$$\Psi : \mathbb{C} \times U(\mathcal{T}) \to U(\mathcal{T}),$$
$$(u, [z_1 : \cdots : z_{d+3}]) \mapsto [e^{2\pi i \alpha_1 u} z_1 : \ldots | e^{2\pi i \alpha_d u} z_d : e^{2\pi i u} z_{d+1} : z_{d+2} : e^{-2\pi u} z_{d+3}].$$

and $p = [\underline{z}] \in U(\mathcal{T})$. Because $z_{d+2} \neq 0$, we will use the following chart of $\mathbb{C}P^{d+2}$:

$$\varphi : U \to \mathbb{C}^{d+2} \tag{2.20}$$
$$[z_1 : \cdots : z_{d+3}] \mapsto (\frac{z_1}{z_{d+2}}, \frac{z_2}{z_{d+2}}, \ldots, \frac{z_{d+3}}{z_{d+2}}). \tag{2.21}$$

where $U = \{[z_1 : \cdots : z_{d+3}] \in \mathbb{C}P^{d+2} \mid z_{d+2} \neq 0\}$.

In this chart

$$(\Psi_p)(u) = \left(e^{2\pi i \alpha_1 u} \frac{z_1}{z_{d+2}}, \ldots, e^{2\pi i \alpha_d u} \frac{z_d}{z_{d+2}}, e^{2\pi i u} \frac{z_{d+1}}{z_{d+2}}, e^{-2\pi u} \frac{z_{d+3}}{z_{d+2}} \right).$$

Then

$$\frac{\partial(\Psi_p)}{\partial u}(u) = \left((2\pi i \alpha_1) e^{2\pi i \alpha_1 u} \frac{z_1}{z_{d+2}}, \ldots, (2\pi i \alpha_d) e^{2\pi i \alpha_d u} \frac{z_d}{z_{d+2}}, (2\pi i) e^{2\pi i u} \frac{z_{d+1}}{z_{d+2}}, (-2\pi) e^{-2\pi u} \frac{z_{d+3}}{z_{d+2}} \right).$$

since $p = [\underline{z}] \in U(\mathcal{T})$, $z_{d+2} \neq 0$, we can set $z_{d+2} = 1$.

We evaluate the derivative at $u = 0 = e \in \mathbb{C}$.

$$\frac{\partial(\Psi_p)}{\partial u}(0) = ((2\pi i\alpha_1)z_1, \ldots, (2\pi i\alpha_d)z_d, (2\pi i)z_{d+1}, (-2\pi)z_{d+3})\,.$$

We write $u = a + ib, z_j = x_j + iy_j$, then

$$\frac{\partial(\Psi_p)}{\partial u}(0) = (2\pi\alpha_1(-y_1 + ix_1), \ldots, 2\pi\alpha_d(-y_d + ix_d), 2\pi(-y_{d+1} + ix_{d+1}), 2\pi(-x_{d+3} - iy_{d+3}),)\,.$$

We note that Ψ_p is holomorphic. In particular $\frac{\partial(\Psi_p)}{\partial u} = \frac{\partial(\Psi_p)}{\partial a}$, $\frac{\partial \operatorname{Re}(\Psi_p)}{\partial a} = \frac{\partial \operatorname{Im}(\Psi_p)}{\partial b}$ and $\frac{\partial \operatorname{Re}(\Psi_p)}{\partial b} = \frac{-\partial \operatorname{Im}(\Psi_p)}{\partial a}$.

Hence, the real Jacobian of Ψ_p at $u = 0$, with respect to the basis $\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\right)$ and $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial x_{d+1}}, \frac{\partial}{\partial x_{d+3}}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_d}, \frac{\partial}{\partial y_{d+1}}, \frac{\partial}{\partial y_{d+3}},\right)$ is

$$(\Psi_p)_{*_e} = -2\pi \begin{pmatrix} \alpha_1 y_1 & \alpha_1 x_1 \\ \vdots & \vdots \\ \alpha_d y_d & \alpha_d x_d \\ y_{d+1} & x_{d+1} \\ x_{d+3} & -y_{d+3} \\ -\alpha_1 x_1 & \alpha_1 y_1 \\ \vdots & \vdots \\ -\alpha_d x_d & \alpha_d y_d \\ -x_{d+1} & y_{d+1} \\ y_{d+3} & x_{d+3} \end{pmatrix}$$

In the following, we will refer to the columns of this matrix as w_1, w_2 respectively.

2.6.2.2 Checking weak slice condition

We will check now that the following embedded submanifold satisfies (A.5). Let

$$\iota_S : \mathbb{C}^{d+1} \setminus \{0\} \to U(\mathcal{T}), \tag{2.22}$$

$$(z_1, \ldots, z_{d+1}) \mapsto [z_1 : \cdots : z_{d+1} : 1 : 1]. \tag{2.23}$$

Let $q = (z_1, \ldots, z_{d+1})$ and $p = \iota_S(q)$. In the chart (2.20),

$$\iota_S(z_1, \ldots, , z_{d+1}) = (z_1, \ldots, z_{d+1}, 1) = (x_1, \ldots, , x_{d+1}, 1, y_1, \ldots, y_{d+1} 0)$$

where $z_k = x_k + iy_k$. Then

$$(\iota_S)_{*_p} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us denote the columns of this matrix by $e_1, \dots, e_{d+1}, e_{d+3}, \dots, e_{2d+1}$, using the standard notation for the canonical basis of $\mathbb{R}^{2d+2}$.

We will see that $e_1, \dots, e_{d+1}, e_{d+3}, \dots, e_{2d+1}, w_1, w_2$, are linearly independent. Suppose

$$\alpha_1 e_1 + \dots + \alpha_{d+1} e_{d+1} + \alpha_{d+3} e_{d+3} + \dots + \alpha_{2d+1} e_{2d+1} + \beta_1 w_1 + \beta_2 w_2 = 0$$

In particular

$$\beta_1 x_{d+3} + (-\beta_2 y_5) = 0,$$
$$\beta_1 y_{d+3} + \beta_2 x_5 = 0.$$

Since $z_{d+3} = 1 + i0$, $x_{d+3} = 1$ and $y_{d+3} = 0$. Thus $\beta_1 = \beta_2 = 0$, and it follows that $\alpha_i = 0$.

Thus, $S = \iota_S(\mathbb{C}^{d+1} \setminus \{0\})$ satisfies (A.5), so there exists a chart U around $[z_1 : \dots : z_d : 1 : 1]$ where $S \cap U$ is a weak slice. From the discussion following Lemma 37, we can take this U to be a sufficiently small coordinate ball (i.e. the image of a ball in $\mathbb{C}^3$ by the chart (2.20)). And since S is an embedded submanifold, this means we can just restrict the domain of ι_S to some $\tilde{S} = B_{z_1} \times \dots \times B_{z_{d+1}}$, where $B_k \subseteq \mathbb{C}$ are sufficiently small open balls centered at z_k. Then we rename $S = \iota_S(\tilde{S})$. Also, if $U_S = \pi(S) \subseteq N$, and $\varphi_S : U_S \to S, \varphi_S = (\pi|_S)^{-1}$, then φ_S is a chart for N.

2.7 Embedded submanifolds for LVMB manifolds

In this section we exhibit some submanifolds that appear in our construction. The choice of these submanifolds is motivated by Remark 10.

2.7.1 Hirzebruch surfaces in a one-parameter family

Lemma 16. *The map*

$$i : \mathbb{C}^2 \to N,$$
$$(z_1, z_2) \mapsto [z_1, z_2, 1, 1, 1],$$

is locally a smooth embedding.

Proof. We will use the chart $\iota_S^{-1} \circ \varphi_S$ at $(z_1, z_2, 1, 1, 1)$, where $S = \iota_S(\tilde{S})$, $\tilde{S} = (B_{z_1} \times B_{z_2} \times B_1) \setminus \{0\} \subseteq \mathbb{C}^2$ and the maps are as in Subsection 2.6.1.2.

Since $\iota_S^{-1} \circ \varphi_S \circ i(z_1, z_2) = (z_1, z_2, 1)$ is an embedding, we have the result. □

2.7.2 The projective line $\mathbb{C}P^d$ and variants

We will show

Lemma 17. *The map*

$$i : \mathbb{C}^d \to N,$$
$$(z_1, \dots, z_d) \mapsto [z_1, \dots, z_d, 1, 1, 1],$$

is locally a smooth embedding.

Proof. We will use the chart $\iota_S^{-1} \circ \varphi_S$ at $(z_1, \dots, z_d, 1, 1, 1)$, where $S = \iota_S(\tilde{S})$, $\tilde{S} = (B_{z_1} \times \cdots \times B_{z_d} \times B_1) \setminus \{0\} \subseteq \mathbb{C}^d$ and the maps are as in Subsection 2.6.2.2.

Since $\iota_S^{-1} \circ \varphi_S \circ i(z_1, \dots, z_d) = (z_1, \dots, z_d, 1)$ is an embedding, we have the result. □

2.8 Finishing proof of applications

2.8.1 Hirzebruch surfaces in a one-parameter family

In this section, we complete the proof of the following theorem:

Theorem 13. *Let*

$$v_1 := (1, 0),$$
$$v_2 := (0, 1),$$
$$v_3 := (0, -1),$$
$$v_4 := (-1, a)$$

be vectors in $\mathbb{R}^2$, with $a > 0$.

Let $\Delta \subset \mathbb{R}^2$ be the fan whose higher-dimensional cones are generated by $(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3)$.

For each choice of $a > 0$, we construct a Lie groupoid associated to Δ.

We start with

$$i : U \subset \mathbb{C}^2 \to N_1 \subseteq N, (z_1, z_3) \mapsto (z_1, 1, z_3, 1, 1) \tag{2.24}$$

which is an embedding on N (See Lemma 16). with U an open neighborhood of a fixed $(\tilde{z}_1, \tilde{z}_3) \in \mathbb{C}^2$.

In order to prove that the pullback of this action groupoid by the inclusion $f : N_1 \to N$ is a Lie groupoid, we will show that the map

$$\alpha : N_1 \times_{f,G_0,s} G_1 \to G_0, (x, g) \mapsto t(g)$$

is a submersion, with $G_1 = H_{\mathcal{F}} \times N$ and $G_0 = N$.

Since $s : G_1 \to G_0$ is a submersion, then $N_a \times_{f,G_0,s} G_1$, is an embedded submanifold.

Now let $(p, q) \in N_1 \times_{G_0} G_1$, with $q = (u_0, n_0)$ and $p = [z_1, 1, z_3, 1, 1] \in N_1$, and

$$[z, 1, z_3, 1, 1] = f(p) = s(q) = n_0$$

We want to check if $\alpha_{*_{(p,q)}}$ is surjective. For that, let $(\xi, \eta) \in T_{(p,q)}(N_1 \times_{G_0} G_1)$.

First, we note that $(\xi, \eta) \in T_pN_1 \times T_qG_1$ such that $f_{*_p}(\xi) = s_{*_q}\eta$. Then

$$\alpha_{*_{(p,q)}}(\xi, \eta) = \left.\frac{d}{dt}\right|_0 (\alpha \circ \gamma)(t),$$

Where $\gamma : I \to N_1 \times_{G_0} G_1$ is a, yet to be specified, curve satisfying

$$\begin{aligned}
\gamma(0) &= (p, q),\\
\dot{\gamma}(0) &= (\xi, \eta),\\
\gamma(t) &= (\gamma_1(t), \gamma_2(t)) \text{ such that } f(\gamma_1(t)) = s(\gamma_2(t))\\
\gamma_2 &: I \to G_1 = \mathbb{C} \times N, t \mapsto (u(t), n(t))
\end{aligned}$$

With that we have

$$\gamma_1(t) = f(\gamma_1(t)) = s(\gamma_2(t)) = n(t),$$

and

$$\alpha \circ \gamma(t) = \alpha(\gamma_1(t), \gamma_2(t)) = t(\gamma_2(t)) = \Phi(u(t), n(t)) = \Phi(u(t), \gamma_1(t))$$

We need $\gamma_1 : I \to N_1$ with $\gamma_1(0) = p = [z_1, 1, z_3, 1, 1]$ and $\dot{\gamma}_1(0) = \xi$.

Let us consider $\gamma_1(t) = i(z_1 + tv_1, z_3 + tv_3)$ where i is the embedding (2.25) and $i_{*_{(z_1,z_3)}}(v_1, v_3) = \xi$.

We also need $\gamma_2(t) = (u(t), n(t))$ such that $\gamma_2(0) = q = (u_0, n(0)) \in G_1 = \mathbb{C} \times N$ and $\dot{\gamma}_2(0) = \eta$

We also need to satisfy $\gamma_1(t) = n(t)$, for every $t \in I$. In particular $p = \gamma_1(0) = n(0)$.

We also need $f_{*_p}(\xi) = s_{*_q}\eta$. On one hand, we have that $\eta \in T_qG_1 = T_q(\mathbb{C}\times N) = T_{u(0)}\mathbb{C} \times T_pN$. Thus $\eta = (\eta_1, \eta_2)$, and since

$$s : \mathbb{C} \times N \to N,$$
$$(u, n) \mapsto n,$$

is a projection, then $s_{*_q}(\eta_1, \eta_2) = \eta_2$.

On the other hand, $f_{*_p}\xi = \xi$, because f is an embedding. Thus $\eta_2 = \xi$. Therefore $n(t)$ and η_2 are already determined by previous choices. Let us consider $u(t) = u_0 + t\eta_1$, with $\eta_1 \in \mathbb{C}$.

$$\begin{aligned}
&\alpha \circ \gamma(t) \\
&\quad = \Phi(u(t), \gamma_1(t)) \\
&\quad = \Phi(u_0 + t\eta_1, i(z_1 + tv_1, z_3 + tv_3)) \\
&\quad = \Phi(u_0 + t\eta_1, [z_1 + tv_1, 1, z_3 + tv_3, 1, 1]) \\
&\quad = [z_1 + tv_1, e^{2\pi i(u_0+t\eta_1)}, e^{2\pi i(u_0+t\eta_1)}(z_3 + tv_3), 1, 1]
\end{aligned}$$

At $t = 0$:

$$\alpha \circ \gamma(0) = [z_1, e^{2\pi i u_0}, e^{2\pi i u_0} z_3, 1, 1] = [w_1, w_2, w_3, 1, 1]$$

Consider the chart (U_{S_2}, φ_{S_2}) at $[w_1, w_2, w_3, 1, 1]$ given by the slice $S_2 = \iota_S(\tilde{S}_2)$ (on $U(\mathcal{T})$, with $\tilde{S}_2 = B_{a,2} \times B_{b,2} \times B_{c,2}$, where $B_{a,2}$ is centered at w_1 and $B_{b,2}$ is centered at w_2 and $B_{c,2}$ is centered at w_3. Note that $w_2 \neq 0$, so this chart exists (end of section 2.6.1).

By continuity, for $t \in I$ sufficiently close to 0,

$$((z_1 + tv_1), e^{2\pi i(u_0+t\eta_1)}, e^{2\pi i(u_0+t\eta_1)}(z_3 + tv_3)) \in \tilde{S}_2,$$

and in this case $\alpha \circ \gamma(t) \in U_{S_2}$, so

$$\begin{aligned}
&\varphi_{S_2}([(z_1 + tv_1), e^{2\pi i(u_0+t\eta_1)}, e^{2\pi i(u_0+t\eta_1)}(z_3 + tv_3), 1, 1]) \\
&\quad [(z_1 + tv_1) : e^{2\pi i(u_0+t\eta_1)} : e^{2\pi i(u_0+t\eta_1)}(z_3 + tv_3) : 1 : 1] \\
&\quad \in U(\mathcal{T}) \subset \mathbb{C}P^4.
\end{aligned}$$

Using the embedding $\varphi^{-1} := \iota_S$ Subsection 2.6.1.2, with codomain restricted to $\tilde{S}_2$

$$\varphi \circ \varphi_{S_2} \circ \alpha \circ \gamma(t) = (z_1 + tv_1, e^{2\pi i(u_0+t\eta_1)}, e^{2\pi i(u_0+t\eta_1)}(z_3 + tv_3)).$$

Then,

$$\begin{aligned}
&(\varphi \circ \varphi_{S_2})_* \alpha_{*_{(p,q)}}(\xi, \eta) \\
&= \frac{d}{dt}\Big|_0 (\varphi \circ \varphi_{S_2} \circ \alpha \circ \gamma)(t) \\
&= \Big(v_1, e^{2\pi i u_0}(2\pi i \eta_1), e^{2\pi i u_0}(2\pi i \eta_1 z_3 + v_3), \Big)
\end{aligned}$$

For $m_1, m_2, m_3 \in \mathbb{C}$, choose

$$\eta_1 = \frac{m_2}{2\pi i e^{2\pi i \beta u_0}}, \quad v_1 = m_1, v_3 = \frac{m_3}{e^{2\pi i u_0}} - 2\pi i \eta_1 z_3.$$

Which shows that $\alpha_{*_{(p,q)}}$ is surjective.

If we now consider the embeddings

$$i : U \subset \mathbb{C}^2 \to N_2 \subseteq N, (z_1, z_2) \mapsto (z_1, z_2, 1, 1, 1), \tag{2.25}$$

$$i : U \subset \mathbb{C}^2 \to N_3 \subseteq N, (z_2, z_4) \mapsto (1, z_2, 1, z_4, 1), \tag{2.26}$$

$$i : U \subset \mathbb{C}^2 \to N_4 \subseteq N, (z_2, z_4) \mapsto (1, 1, z_3, z_4, 1) \tag{2.27}$$

the computation is analogous.

2.8.2 The projective line $\mathbb{C}P^d$ and variants

In this section, we complete the proof of the following theorem:

Theorem 14. *Let*

$$\begin{aligned}
v_1 &:= (1, 0, \dots, 0), \\
v_2 &:= (0, 1, 0, \dots, 0), \\
&\vdots \\
v_d &:= (0, \dots, 0, 1), \\
v_{d+1} &:= (-\alpha_1, \dots, -\alpha_d)
\end{aligned}$$

be vectors in $\mathbb{R}^d$. *Let* $\Delta \subset \mathbb{R}^d$ *be the fan made out of each proper subset of* $\{v_1, \dots, v_{d+1}\}$, *with* $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$.

For each choice of parameters $\alpha_1, ..., \alpha_d \in \mathbb{R}_{>0}$, *we construct a Lie groupoid associated to* Δ.

We start with

$$i : U \subset \mathbb{C}^d \to N_1 \subseteq N, (z_1, z_3, \dots, z_{d+1}) \mapsto (z_1, 1, z_3, \dots, z_{d+1}, 1, 1) \tag{2.28}$$

which is an embedding on N (See Lemma 17). with U an open neighborhood of a fixed $(\tilde{z}_1, \tilde{z}_d, \dots, \tilde{z}_{d+1}) \in \mathbb{C}^d$.

In order to prove that the pullback of this action groupoid by the inclusion $f : N_1 \to N$ is a Lie groupoid, we will show that the map

$$\alpha : N_1 \times_{f,G_0,s} G_1 \to G_0, (x, g) \mapsto t(g)$$

is a submersion, with $G_1 = H_{\mathcal{F}} \times N$ and $G_0 = N$.

Since $s : G_1 \to G_0$ is a submersion, then $N_a \times_{f,G_0,s} G_1$, is an embedded submanifold.

Now let $(p, q) \in N_1 \times_{G_0} G_1$, with $q = (u_0, n_0)$ and $p = [z_1, 1, z_3, \ldots, z_{d+1}, 1, 1] \in N_1$, and

$$[z, 1, z_3, \ldots, z_{d+1}, 1, 1] = f(p) = s(q) = n_0$$

We want to check if $\alpha_{*_{(p,q)}}$ is surjective. For that, let $(\xi, \eta) \in T_{(p,q)}(N_1 \times_{G_0} G_1)$.

First, we note that $(\xi, \eta) \in T_pN_1 \times T_qG_1$ such that $f_{*_p}(\xi) = s_{*_q}\eta$. Then

$$\alpha_{*_{(p,q)}}(\xi, \eta) = \frac{d}{dt}\bigg|_0 (\alpha \circ \gamma)(t),$$

Where $\gamma : I \to N_1 \times_{G_0} G_1$ is a, yet to be specified, curve satisfying

$$\begin{aligned} \gamma(0) &= (p, q), \\ \dot{\gamma}(0) &= (\xi, \eta), \\ \gamma(t) &= (\gamma_1(t), \gamma_2(t)) \text{ such that } f(\gamma_1(t)) = s(\gamma_2(t)) \\ \gamma_2 &: I \to G_1 = \mathbb{C} \times N, t \mapsto (u(t), n(t)) \end{aligned}$$

With that we have

$$\gamma_1(t) = f(\gamma_1(t)) = s(\gamma_2(t)) = n(t),$$

and

$$\alpha \circ \gamma(t) = \alpha(\gamma_1(t), \gamma_2(t)) = t(\gamma_2(t)) = \Phi(u(t), n(t)) = \Phi(u(t), \gamma_1(t))$$

We need $\gamma_1 : I \to N_1$ with $\gamma_1(0) = p = [z_1, 1, z_3, \ldots, z_{d+1}, 1, 1]$ and $\dot{\gamma}_1(0) = \xi$.

Let us consider $\gamma_1(t) = i(z_1 + tv_1, z_3 + tv_3, \ldots, z_{d+1} + tv_{d+1})$ where i is the embedding (2.25) and $i_{*_{(z_1,z_3,\ldots,z_{d+1})}}(v_1, v_3, \ldots, v_{d+1}) = \xi$.

We also need $\gamma_2(t) = (u(t), n(t))$ such that $\gamma_2(0) = q = (u_0, n(0)) \in G_1 = \mathbb{C} \times N$ and $\dot{\gamma}_2(0) = \eta$

We also need to satisfy $\gamma_1(t) = n(t)$, for every $t \in I$. In particular $p = \gamma_1(0) = n(0)$.

We also need $f_{*_p}(\xi) = s_{*_q}\eta$. On one hand, we have that $\eta \in T_qG_1 = T_q(\mathbb{C}\times N) = T_{u(0)}\mathbb{C} \times T_pN$. Thus $\eta = (\eta_1, \eta_2)$, and since

$$s : \mathbb{C} \times N \to N,$$
$$(u, n) \mapsto n,$$

is a projection, then $s_{*_q}(\eta_1, \eta_2) = \eta_2$.

On the other hand, $f_{*_p}\xi = \xi$, because f is an embedding. Thus $\eta_2 = \xi$. Therefore $n(t)$ and η_2 are already determined by previous choices. Let us consider $u(t) = u_0 + t\eta_1$, with $\eta_1 \in \mathbb{C}$.

$$\begin{aligned}
&\alpha \circ \gamma(t)\\
&= \Phi(u(t), \gamma_1(t))\\
&= \Phi(u_0 + t\eta_1, i(z_1 + tv_1, z_3 + tv_3, \ldots, z_{d+1} + tv_{d+1}))\\
&= \Phi(u_0 + t\eta_1, [z_1 + tv_1, 1, z_3 + tv_3, \ldots, z_{d+1} + tv_{d+1}, 1, 1])\\
&= [e^{2\pi i\alpha_1(u_0+t\eta_1)}(z_1 + tv_1), e^{2\pi i\alpha_2(u_0+t\eta_1)}, e^{2\pi i\alpha_3(u_0+t\eta_1)}(z_3 + tv_3),\\
&\ldots, e^{2\pi i(u_0+t\eta_1)}(z_{d+1} + tv_{d+1}), 1, 1]
\end{aligned}$$

At $t = 0$:

$$\begin{aligned}
\alpha \circ \gamma(0) &= [e^{2\pi i\alpha_1 u_0} z_1, e^{2\pi i\alpha_2 u_0}, e^{2\pi i\alpha_3 u_0} z_3, \ldots, e^{2\pi i u_0} z_{d+1}, 1, 1]\\
&= [w_1, w_2, \ldots, w_{d+1}, 1, 1]
\end{aligned}$$

Consider the chart (U_{S_2}, φ_{S_2}) at $[w_1, w_2, \ldots, w_{d+1}, 1, 1]$ given by the slice $S_2 = \iota_S(\tilde{S}_2)$ (on $U(\mathcal{T})$, with $\tilde{S}_2 = B_{a_1,2} \times B_{a_2,2} \times \cdots \times B_{a_{d+1},2}$, where $B_{a_i,2}$ is centered at w_i. Note that $w_2 \neq 0$, so this chart exists (end of section 2.6.2).

By continuity, for $t \in I$ sufficiently close to 0,

$$\begin{aligned}
&(e^{2\pi i\alpha_1(u_0+t\eta_1)}(z_1 + tv_1), e^{2\pi i\alpha_2(u_0+t\eta_1)}, e^{2\pi i\alpha_3(u_0+t\eta_1)}(z_3 + tv_3),\\
&\qquad \ldots, e^{2\pi i(u_0+t\eta_1)}(z_{d+1} + tv_{d+1})) \in \tilde{S}_2,
\end{aligned}$$

and in this case $\alpha \circ \gamma(t) \in U_{S_2}$, so

$$\begin{aligned}
&\varphi_{S_2}([e^{2\pi i\alpha_1(u_0+t\eta_1)}(z_1 + tv_1), e^{2\pi i\alpha_2(u_0+t\eta_1)}, \ldots e^{2\pi i(u_0+t\eta_1)}(z_{d+1} + tv_{d+1}), 1, 1])\\
&= [e^{2\pi i\alpha_1(u_0+t\eta_1)}(z_1 + tv_1) : e^{2\pi i\alpha_2(u_0+t\eta_1)} : \cdots : e^{2\pi i(u_0+t\eta_1)}(z_{d+1} + tv_{d+1}) : 1 : 1]\\
&\in U(\mathcal{T}) \subset \mathbb{C}P^d.
\end{aligned}$$

Using the embedding $\varphi^{-1} := \iota_S$ Subsection 2.6.2.2, with codomain restricted to $\tilde{S}_2$

$$\begin{aligned}
\varphi \circ \varphi_{S_2} \circ \alpha \circ \gamma(t) = (&e^{2\pi i\alpha_1(u_0+t\eta_1)}(z_1 + tv_1), e^{2\pi i\alpha_2(u_0+t\eta_1)}, e^{2\pi i\alpha_3(u_0+t\eta_1)}(z_3 + tv_3)\\
&\ldots, e^{2\pi i(u_0+t\eta_1)}(z_{d+1} + tv_{d+1})).
\end{aligned}$$

Then,

$$
\begin{aligned}
&(\varphi \circ \varphi_{S_2})_* \alpha_{*_{(p,q)}}(\xi, \eta) \\
&\quad = \frac{d}{dt}\bigg|_0 (\varphi \circ \varphi_{S_2} \circ \alpha \circ \gamma)(t) \\
&\quad = (e^{2\pi i \alpha_1 u_0}(2\pi i \alpha_1 \eta_1 z_1 + v_1), e^{2\pi i \alpha_2 u_0}(2\pi i \alpha_2 \eta_1), e^{2\pi i \alpha_3 u_0}(2\pi i \alpha_3 \eta_1 z_3 + v_3), \\
&\quad \ldots, e^{2\pi i u_0}(2\pi i \eta_1 z_{d+1} + v_{d+1}))
\end{aligned}
$$

For $m_1, \ldots, m_{d+1} \in \mathbb{C}$, choose

$$
\eta_1 = \frac{m_2}{2\pi i \alpha_2 e^{2\pi i \alpha_2 u_0}}, \quad v_l = \frac{m_l}{e^{2\pi i \alpha_l u_0}} - 2\pi i \alpha_l \eta_1 z_l, \quad v_{d+1} = \frac{m_{d+1}}{e^{2\pi i u_0}} - 2\pi i \eta_1 z_{d+1}.
$$

for $l = 1, 3, \ldots, d$. This shows that $\alpha_{*_{(p,q)}}$ is surjective.

If we now consider the embeddings

$$
i : U \subset \mathbb{C}^d \to N_l \subseteq N, (z_1, \ldots, \widehat{z}_l, \ldots, z_{d+1}) \mapsto (z_1, \ldots, 1, \ldots, z_{d+1}, 1, 1) \tag{2.29}
$$

the computation is analogous.

Bibliography

ADEM, A.; LEIDA, J.; RUAN, Y. *Orbifolds and Stringy Topology*. Cambridge University Press, 2007. Disponível em: <https://www-cambridge.ez88.periodicos.capes.gov.br/core/books/orbifolds-and-stringy-topology/677A8058E1D88669C8EE85FF2442CFCD>. Citado 12 vezes nas páginas 19, 20, 21, 22, 23, 26, 29, 35, 36, 37, 39, and 89.

ATIYAH, M. F. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, v. 14, n. 1, p. 1–15, 1982. ISSN 0024-6093. Disponível em: <http://dx.doi.org/10.1112/blms/14.1.1>. Citado 2 vezes nas páginas 10 and 15.

AUROUX, D.; KATZARKOV, L.; ORLOV, D. Mirror symmetry for weighted projective planes and their noncommutative deformations. *Annals of Mathematics. Second Series*, v. 167, n. 3, p. 867–943, 2008. ISSN 0003-486X. Citado na página 10.

BATTAGLIA, F.; PRATO, E. Generalized toric varieties for simple nonrational convex polytopes. *Internat. Math. Res. Notices*, n. 24, p. 1315–1337, 2001. ISSN 1073-7928. Disponível em: <https://doi.org/10.1155/S1073792801000629>. Citado 3 vezes nas páginas 10, 12, and 19.

______. The symplectic penrose kite. *Communications in Mathematical Physics*, v. 299, n. 3, p. 577–601, Nov 2010. ISSN 0010-3616, 1432-0916. Citado na página 28.

BATTAGLIA, F.; PRATO, E.; ZAFFRAN, D. Hirzebruch surfaces in a one-parameter family. *Bollettino dell'Unione Matematica Italiana*, v. 12, n. 1–2, p. 293–305, 2019. ISSN 1972-6724. Citado 3 vezes nas páginas 12, 51, and 52.

BATTAGLIA, F.; ZAFFRAN, D. Foliations modeling nonrational simplicial toric varieties. *Int. Math. Res. Not. IMRN*, n. 22, p. 11785–11815, 2015. ISSN 1073-7928. Citado 10 vezes nas páginas 11, 40, 42, 43, 45, 46, 47, 49, 53, and 75.

______. Simplicial toric varieties as leaf spaces. In: *Special metrics and group actions in geometry*. [S.l.]: Springer, Cham, 2017, (Springer INdAM Ser., v. 23). p. 1–21. Citado 5 vezes nas páginas 11, 40, 41, 51, and 54.

BOSIO, F. Variétés complexes compactes: une généralisation de la construction de Meersseman et López de Medrano-Verjovsky. *Ann. Inst. Fourier (Grenoble)*, v. 51, n. 5, p. 1259–1297, 2001. ISSN 0373-0956. Disponível em: <http://aif.cedram.org/item?id=AIF_2001__51_5_1259_0>. Citado 3 vezes nas páginas 11, 40, and 41.

BREDON, G. E. *Introduction to compact transformation groups*. [S.l.]: Academic Press, New York-London, 1972. xiii+459 p. Pure and Applied Mathematics, Vol. 46. Citado 4 vezes nas páginas 81, 86, 87, and 88.

COX, D. A.; LITTLE, J. B.; SCHENCK, H. K. *Toric varieties*. American Mathematical Society, Providence, RI, 2011. v. 124. xxiv+841 p. (Graduate Studies in Mathematics, v. 124). ISBN 978-0-8218-4819-7. Disponível em: <https://doi.org/10.1090/gsm/124>. Citado 2 vezes nas páginas 14 and 53.

DELZANT, T. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France*, v. 116, n. 3, p. 315–339, 1988. ISSN 0037-9484. Disponível em: <http://www.numdam.org/item?id=BSMF_1988___116_3_315_0>. Citado 3 vezes nas páginas 10, 14, and 15.

DIECK, T. tom. *Transformation groups*. Walter de Gruyter & Co., Berlin, 1987. v. 8. x+312 p. (De Gruyter Studies in Mathematics, v. 8). ISBN 3-11-009745-1. Disponível em: <https://doi.org/10.1515/9783110858372.312>. Citado 4 vezes nas páginas 81, 82, 84, and 88.

GUILLEMIN, V. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*. Birkhäuser Boston, Inc., Boston, MA, 1994. v. 122. (Progress in Mathematics, v. 122). ISBN 978-0-8176-3770-5. Disponível em: <https://mathscinet.ams.org/mathscinet-getitem?mr=1301331>. Citado 2 vezes nas páginas 15 and 17.

GUILLEMIN, V.; POLLACK, A. *Differential Topology*. American Mathematical Society, 2010. ISBN 978-0-8218-5193-7. Disponível em: <http://www.ams.org/chel/370>. Citado na página 80.

GUILLEMIN, V.; STERNBERG, S. Convexity properties of the moment mapping. *Invent. Math.*, v. 67, n. 3, p. 491–513, 1982. ISSN 0020-9910. Disponível em: <http://dx.doi.org/10.1007/BF01398933>. Citado 2 vezes nas páginas 10 and 15.

HAEFLIGER, A. Homotopy and integrability. In: _____. *Manifolds — Amsterdam 1970*. Springer Berlin Heidelberg, 1971. (Lecture Notes in Mathematics, v. 197), p. 133–163. ISBN 978-3-540-05467-2. Disponível em: <http://link.springer.com/10.1007/BFb0068615>. Citado na página 38.

HOFFMAN, B.; SJAMAAR, R. Stacky hamiltonian actions and symplectic reduction. *International Mathematics Research Notices*, 2018. Disponível em: <https://academic-oup-com.ez88.periodicos.capes.gov.br/imrn/advance-article/doi/10.1093/imrn/rnz322/5802880>. Citado na página 10.

KATZARKOV, L.; LUPERCIO, E.; MEERSSEMAN, L.; VERJOVSKY, A. Quantum (non-commutative) toric geometry: Foundations. *Advances in Mathematics*, v. 391, p. 107945, Nov 2021. ISSN 00018708. Citado na página 10.

LEE, J. *Introduction to Smooth Manifolds*. 2. ed. Springer-Verlag, 2012. (Graduate Texts in Mathematics). ISBN 978-1-4419-9981-8. Disponível em: <https://www.springer.com/gp/book/9781441999818>. Citado 7 vezes nas páginas 78, 79, 80, 81, 83, 84, and 85.

LEE, J. M. *Introduction to topological manifolds*. Second. Springer, New York, 2011. v. 202. xviii+433 p. (Graduate Texts in Mathematics, v. 202). ISBN 978-1-4419-7939-1. Disponível em: <https://doi.org/10.1007/978-1-4419-7940-7>. Citado 2 vezes nas páginas 78 and 83.

LERMAN, E. Orbifolds as stacks? *L'Enseignement Mathématique. Revue Internationale. 2e Série*, v. 56, n. 3–4, p. 315–363, 2010. ISSN 0013-8584. Citado 4 vezes nas páginas 37, 48, 89, and 90.

LERMAN, E.; TOLMAN, S. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, v. 349, n. 10, p. 4201–4230, 1997. ISSN 0002-9947.

Disponível em: <https://doi.org/10.1090/S0002-9947-97-01821-7>. Citado 3 vezes nas páginas 10, 12, and 19.

LOERA, J. D.; RAMBAU, J.; SANTOS, F. *Triangulations: Structures for Algorithms and Applications*. Springer-Verlag, 2010. (Algorithms and Computation in Mathematics). ISBN 978-3-642-12970-4. Disponível em: <https://www.springer.com/gp/book/9783642129704>. Citado 6 vezes nas páginas 45, 72, 73, 74, 75, and 76.

MACKENZIE, K. C. H. *General theory of Lie groupoids and Lie algebroids*. Cambridge University Press, Cambridge, 2005. v. 213. (London Mathematical Society Lecture Note Series, v. 213). ISBN 978-0-521-49928-6. Disponível em: <https://mathscinet.ams.org/mathscinet-getitem?mr=2157566>. Citado 4 vezes nas páginas 29, 30, 32, and 34.

MEDRANO, S. López de; VERJOVSKY, A. A new family of complex, compact, non-symplectic manifolds. *Bol. Soc. Brasil. Mat. (N.S.)*, v. 28, n. 2, p. 253–269, 1997. ISSN 0100-3569. Disponível em: <https://doi.org/10.1007/BF01233394>. Citado 2 vezes nas páginas 11 and 40.

MEERSSEMAN, L. A new geometric construction of compact complex manifolds in any dimension. *Math. Ann.*, v. 317, n. 1, p. 79–115, 2000. ISSN 0025-5831. Disponível em: <https://doi.org/10.1007/s002080050360>. Citado 2 vezes nas páginas 11 and 40.

MOERDIJK, I.; MRcUN, J. *Introduction to foliations and Lie groupoids*. Cambridge University Press, Cambridge, 2003. v. 91. x+173 p. (Cambridge Studies in Advanced Mathematics, v. 91). ISBN 0-521-83197-0. Disponível em: <https://doi.org/10.1017/CBO9780511615450>. Citado 3 vezes nas páginas 29, 37, and 38.

MOERDIJK, I.; PRONK, D. A. Orbifolds, sheaves and groupoids. *K-Theory*, v. 12, n. 1, p. 3–21, Jul 1997. ISSN 15730514, 09203036. Citado 6 vezes nas páginas 10, 19, 20, 21, 22, and 39.

______. Simplicial cohomology of orbifolds. *Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae. New Series*, v. 10, n. 2, p. 269–293, 1999. ISSN 0019-3577. Citado na página 10.

MOLITOR, M. *Lecture notes in Symplectic Geometry and Lie groups*. [S.l.]: Universidade Federal da Bahia, 2016. Citado 2 vezes nas páginas 81 and 88.

PETERSEN, P. *Riemannian geometry*. Third. Springer, Cham, 2016. v. 171. (Graduate Texts in Mathematics, v. 171). ISBN 978-3-319-26652-7. Disponível em: <https://mathscinet.ams.org/mathscinet-getitem?mr=3469435>. Citado na página 81.

PRATO, E. Simple non-rational convex polytopes via symplectic geometry. *Topology*, v. 40, n. 5, p. 961–975, 2001. ISSN 0040-9383. Disponível em: <https://doi.org/10.1016/S0040-9383(00)00006-9>. Citado 5 vezes nas páginas 10, 12, 19, 26, and 27.

______. 2018. Personal communication. Citado na página 22.

PRONK, D. *Groupoid representations for sheaves on orbifolds: = Groepoïde representations voor schoven op orbifolds*. [S.l.: s.n.], 1995. ISBN 978-90-393-0620-8. Citado na página 37.

SATAKE, I. On a generalization of the notion of manifold. *Proceedings of the National Academy of Sciences of the United States of America*, v. 42, p. 359–363, 1956. ISSN 0027-8424. Citado 3 vezes nas páginas 10, 19, and 22.

______. The Gauss-Bonnet theorem for V-manifolds. *J. Math. Soc. Japan*, v. 9, p. 464–492, 1957. ISSN 0025-5645. Disponível em: <https://doi.org/10.2969/jmsj/00940464>. Citado 4 vezes nas páginas 10, 19, 20, and 27.

SILVA, A. Cannas da. *Lectures on symplectic geometry*. Springer-Verlag, Berlin, 2001. v. 1764. xii+217 p. (Lecture Notes in Mathematics, v. 1764). ISBN 3-540-42195-5. Disponível em: <http://dx.doi.org/10.1007/978-3-540-45330-7>. Citado 2 vezes nas páginas 14 and 15.

______. Symplectic toric manifolds. In: *Symplectic geometry of integrable Hamiltonian systems (Barcelona, 2001)*. Basel: Birkhäuser, 2003, (Adv. Courses Math. CRM Barcelona). p. 85–173. Citado na página 15.

TOMMASINI, M. Orbifolds and groupoids. *Topology and its Applications*, v. 159, n. 3, p. 756–786, Feb 2012. ISSN 0166-8641. Citado na página 37.

ZIEGLER, G. M. *Lectures on polytopes*. New York: Springer-Verlag, 1995. v. 152. x+370 p. (Graduate Texts in Mathematics, v. 152). ISBN 0-387-94365-X. Disponível em: <http://dx.doi.org/10.1007/978-1-4613-8431-1>. Citado na página 15.

APPENDIX A – Prerequisites

In this chapter we present some aspects of the theory of Gale duality that appears in the study of LVMB manifolds. We also recall some facts about topology, the theory of manifolds, Lie groups and Lie groupoids that we will need.

A.1 Linear algebra and Gale Duality

Our source for Gale duality is (LOERA; RAMBAU; SANTOS, 2010).

Definition 25 (Vector configuration)**.** *(LOERA; RAMBAU; SANTOS, 2010, p.77) Let V be a finite-dimensional real vector space. A **vector configuration** in V is a finite set $\boldsymbol{A} = (v_j \in V \mid v \in J)$ of labeled vectors. J is the **label set**.*

The vectors are labeled just to ensure that vectors of V can be repeated.

*A **vector subconfiguration**, or just a **subconfiguration**, of $\boldsymbol{A}$ is any subset of $\boldsymbol{A}$, indexed by a subset of J.*

*A vector (sub)configuration is **independent** if it does not have repeated vectors and its vectors are linearly independent. It is **dependent** otherwise.*

*The **rank** of a vector (sub)configuration is its rank as a set of vectors of V.*

(The source works with $V = \mathbb{R}^m$, but for convenience we keep V arbitrary).

Definition 26. *(LOERA; RAMBAU; SANTOS, 2010, p.79) A subconfiguration of a vector configuration $\boldsymbol{A}$ is **full-dimensional**, or **maximal**, if it has the same rank as $\boldsymbol{A}$.*

Definition 27. *(LOERA; RAMBAU; SANTOS, 2010, p. 160) [Linear evaluations and dependencies of a vector configuration] Let $\boldsymbol{A} = (v_1, \ldots, v_n) \subset V$ be a vector configuration of rank k. We will define two maps:*

$$\xi_A : \mathbb{R}^n \to V, \qquad [a_1, \ldots, a_n] = \sum_{i=1}^{n} a_i e_i \mapsto \sum_{i=1}^{n} a_i v_i$$

$$\eta_A : V^* \to \mathbb{R}^n, \qquad f \mapsto [f(v_1), \ldots, f(v_n)] = \sum_{i=1}^{n} f(v_i) e_i$$

where (e_i) is the standard basis for $\mathbb{R}^n$.

*The elements of $\operatorname{Rel}(\boldsymbol{A}) := \operatorname{Ker} \xi_A$ are the **linear dependences of $\boldsymbol{A}$**. The elements of $\operatorname{Ev}(\boldsymbol{A}) := \operatorname{Im} \eta_A$. are the **linear evaluations of $\boldsymbol{A}$**.*

The next lemma shows that the maps ξ_A and η_A are, up to an isomorphism, adjoint to each other.

Lemma 18. *Let* $\sigma : \mathbb{R}^n \to (\mathbb{R}^n)^*$ *be the linear isomorphism defined by the choice of the standard basis* (e_i) *for* $\mathbb{R}^n$*, which sends each* e_i *to its dual* $e_i^* \in (\mathbb{R}^n)^*$*. Then the adjoint* $(\xi_A)^* : V^* \to (\mathbb{R}^n)^*$ *of* ξ_A *satisfies*

$$(\xi_A)^* = \sigma \circ \eta_A : V^* \to \mathbb{R}^n \to (\mathbb{R}^n)^*$$

Proof. We need to show that

$$[(\sigma \circ \eta_A)(f)](x) = f(\xi_A(x))$$

for every $f \in V^*$ and $x \in \mathbb{R}^n$.

$$\begin{aligned}
[(\sigma \circ \eta_A)(f)](x) &= [(\sigma(\sum_{i=1}^{n} f(v_i)e_i)](\sum_{j=1}^{n} x_j e_j) \\
&= [\sum_{i=1}^{n} f(v_i)e_i^*](\sum_{j=1}^{n} x_j e_j) \\
&= \sum_{i=1}^{n} f(v_i)x_i \\
&= f(\sum_{i=1}^{n} v_i x_i) \\
&= f(\xi_A(\sum_{i=1}^{n} x_i e_i)) \\
&= f(\xi_A(x))
\end{aligned}$$

□

Lemma 19. *(LOERA; RAMBAU; SANTOS, 2010, p. 160) Let* $\boldsymbol{A} = (v_1, \ldots, v_n) \subset V$ *be a vector configuration of rank* k*. The space of linear evaluations* $\mathrm{Ev}(\boldsymbol{A})$ *and the space of linear dependences* $\mathrm{Rel}(\boldsymbol{A})$ *of* $\boldsymbol{A}$ *form two orthogonal complementary linear subspaces of* $\mathbb{R}^n$ *of dimension* k *and* $n-k$*, respectively.*

Proof. By the Lemma above,

$$(\mathrm{Rel}(\boldsymbol{A}))^0 = (\mathrm{Ker}\,\xi_A)^0 = \mathrm{Im}\,\xi_A^*$$

Applying σ^{-1} to both sides of the equation yields

$$\sigma^{-1}((\mathrm{Rel}(\boldsymbol{A}))^0) = \mathrm{Im}\,\eta_A = \mathrm{Ev}(\boldsymbol{A})$$

Because of the relation between the orthogonal complemente of a linear subspace and its annihilator in the dual vector space,

$$\mathrm{Rel}(\boldsymbol{A}))^\perp = \mathrm{Ev}(\boldsymbol{A})$$

Since $\dim \mathrm{Rel}(\boldsymbol{A}) = \dim \mathrm{Ker}\,\xi_A = n-k$, $\dim \mathrm{Ev}(\boldsymbol{A}) = \dim \mathrm{Rel}(\boldsymbol{A}))^\perp = k$ □

Lemma 20. *(LOERA; RAMBAU; SANTOS, 2010, p. 160) Let* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *be a vector configuration of rank* k*. Linear dependences and linear evaluations of* $\boldsymbol{A}$ *form two orthogonal complementary linear subspaces of* $\mathbb{R}^n$*, of rank* $n - k$ *and* k *respectively.*

Definition 28. *(LOERA; RAMBAU; SANTOS, 2010, p. 160) A* ***Gale transform*** *of a configuration* $\boldsymbol{A}$ *is a configuration* $\boldsymbol{B}$ *such that the linear dependences of* $\boldsymbol{A}$ *are the linear evaluations on* $\boldsymbol{B}$ *and vice versa, that is*

$$\mathrm{Rel}(\boldsymbol{A}) = \mathrm{Ev}(\boldsymbol{B})$$
$$\mathrm{Rel}(\boldsymbol{B}) = \mathrm{Ev}(\boldsymbol{A})$$

We write $\boldsymbol{B} \in \mathrm{Gale}(\boldsymbol{A})$*.*

Remark 13. Both the set of linear evaluations and the set of linear dependences of a vector configuration of n elements are subsets of $\mathbb{R}^n$. Let $\boldsymbol{B}$ be a Gale transform of $\boldsymbol{A}$. Then the linear dependences of $\boldsymbol{A}$ are the linear evaluations on $\boldsymbol{B}$. In particular $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same cardinality. In particular the label set of both are the same, up to isomorphism, and will be identified.

Lemma 21 (Characterization of Gale Duality)**.** *Let* $\boldsymbol{A}\boldsymbol{B}$ *be vector configurations with* n *elements each. The two equalities*

$$\mathrm{Rel}(\boldsymbol{A}) = \mathrm{Ev}(\boldsymbol{B})$$
$$\mathrm{Rel}(\boldsymbol{B}) = \mathrm{Ev}(\boldsymbol{A})$$

are equivalent.

Proof. Suppose $\mathrm{Rel}(\boldsymbol{B}) = \mathrm{Ev}(\boldsymbol{A})$. Then

$$\mathrm{Ev}(\boldsymbol{B}) = \mathrm{Rel}(\boldsymbol{B})^{\perp} = \mathrm{Ev}(\boldsymbol{A})^{\perp} = \mathrm{Rel}(\boldsymbol{A})^{\perp\perp} = \mathrm{Rel}(\boldsymbol{A})$$

where the first and third equalities are from Lemma 19. The other case is analogous. □

Remark 14. Let $\boldsymbol{A}$ a vector configuration with n elements, of rank k Let $\boldsymbol{B}$ be a Gale transform of $\boldsymbol{A}$, thus also with n elements (Remark 13). By Lemma 19, the space of linear dependences of $\boldsymbol{A}$ has rank $n - k$. This is also the space of linear evaluations of $\boldsymbol{B}$. Thus $\boldsymbol{B}$ has rank $n - k$, also by Lemma 19.

Lemma 22. *(LOERA; RAMBAU; SANTOS, 2010, p. 160) Every configuration* $\boldsymbol{A}$ *has at least one Gale transform.*

Proof. We will show that the configuration presented in (LOERA; RAMBAU; SANTOS, 2010, p. 160) is a Gale transform of $\boldsymbol{A}$.

Suppose $\boldsymbol{A} = (v_1, \dots, v_n) \subset V$. Also suppose that $\boldsymbol{A}$ has rank k Let $\operatorname{Rel}(V)$ be the relation space of V, that is,

$$\begin{aligned}\operatorname{Rel}(V) &= \{a \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i v_i = 0\} \\ &= \operatorname{Ker}\{T : \mathbb{R}^n \to B, (a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} a_i v_i\}.\end{aligned}$$

Now we choose a basis for $\operatorname{Rel}(V)$, and write down a matrix M whose columns are the vectors of this basis. (In the source, (LOERA; RAMBAU; SANTOS, 2010, p. 160) used the convention that the rows of M are the vectors of this basis. We however follow the opposite convention, from (BATTAGLIA; ZAFFRAN, 2015)).

Since $\operatorname{Im} T = \operatorname{rank} \boldsymbol{A} = k$ then $\dim \operatorname{Ker} T = n - k$. The rows of M are the elements of a our candidate for the Gale transform $\boldsymbol{B} = (w_1, \dots, w_n) \subset \mathbb{R}^{n-k}$ of $\boldsymbol{A}$. We note that $\operatorname{rank} \boldsymbol{B} = n - k$. The columns of M will be denoted by $(c_1, \dots, c_{n-k})$.

We will show first that every linear dependence of $\boldsymbol{A}$ is a linear evaluation on $\boldsymbol{B}$. Since linear evaluations form a subspace of $\mathbb{R}^n$ (Lemma 19), we will show this for the basis $(c_1, \dots, c_{n-k})$ of $\operatorname{Rel}(V)$. Let (e_s) be the standard basis for $\mathbb{R}^{n-k}$ and (e_s^*) the dual basis on $(\mathbb{R}^{n-k})^*$. For $c_j(e_j)^* : \mathbb{R}^{n-k} \to \mathbb{R}$ satisfies

$$(e_j)^*(w_i) = (c_j)^i$$

which shows that c_j is a linear evaluation of $\boldsymbol{B}$. Therefore every linear dependence of $\boldsymbol{A}$ is a linear evaluation on $\boldsymbol{B}$.

Since $\operatorname{rank} \boldsymbol{B} = n - k$, the space of linear evaluations has dimension $n - k$ (Lemma 19). As mentioned above, this is also the dimension of the space $\operatorname{Rel}(V)$ of linear dependences of $\boldsymbol{A}$. By dimensionality the inclusion above is an equality.

Therefore the space of linear dependence of $\boldsymbol{A}$ is the space of linear evaluations on $\boldsymbol{B}$.

By Lemma 21, we have the result. □

Lemma 23. *(LOERA; RAMBAU; SANTOS, 2010, p. 161) Let* $\boldsymbol{B} \in \operatorname{Gale}(\boldsymbol{A})$. *We assume the label set of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are the same (Remark 13).*

The (label set of) independent subconfigurations of $\boldsymbol{A}$ *are the complements of the (label set of) full-dimensional subconfigurations of* $\boldsymbol{B}$*, and vice versa.*

In particular, the (label set of) full-dimensional independent subconfigurations of $\boldsymbol{A}$ *are the complements of the (label set of) independent full-dimensional subconfigurations of* $\boldsymbol{B}$*, and vice versa.*

Proof. Proof in (LOERA; RAMBAU; SANTOS, 2010, p. 161) relies on previous discussion and results (in the source). Let us see a direct proof.

Let us suppose $\boldsymbol{A} = (v_1, \ldots, v_n) \subset V, \boldsymbol{B} = (w_1, \ldots, w_n) \subset W$. Without loss of generality, let $\boldsymbol{A}' = (v_1, \ldots, v_s)$ be an independent subconfiguration of $\boldsymbol{A}$, that is, $v_1, \ldots, v_s$ are linearly independent. We want to show that $\boldsymbol{B}' = (w_{s+1}, \ldots, w_n)$ s a full-dimensional configuration of $\boldsymbol{B}$, that is, $\boldsymbol{B}$ and $\boldsymbol{B}'$ have the same rank. Since $\boldsymbol{B}'$ is a subset of $\boldsymbol{B}$ this means that any vector generated by $\boldsymbol{B}$ is also generated by $\boldsymbol{B}'$

Let $w = \sum_{i=1}^{n} b_i w_i$. We want to show that there exists $\tilde{b} \in \mathbb{R}^n$ such that $w \sum_{i=1}^{n} \tilde{b}_i w_i$, with $\tilde{b}_i = 0$, for $1 \leq i \leq s$. By contradiction, suppose that for every $\tilde{b} \in \mathbb{R}^n$, with $\tilde{b}_i = 0$, for $1 \leq i \leq s$, we have $w \neq \sum_{i=1}^{n} \tilde{b}_i w_i$. Then

$$\sum_{i=1}^{n} (b_i - \tilde{b}_i) w_i \neq 0$$

In particular for $\tilde{b}_i = b_i$, $s + 1 \leq i \leq n$,

$$\sum_{i=1}^{s} b_i w_i \neq 0.$$

Thus, by definition, $(b_1, \ldots, b_s, 0, \ldots, 0)$ is not a linear dependence of $\boldsymbol{B}$, and by definition of Gale transform, it is not a linear evaluation of $\boldsymbol{A}$.

Since the vectors $\boldsymbol{A}$ are LI, they can be extended to a basis for V. We define a linear functional $f : V \to \mathbb{R}$ that evaluates to b_i on v_i and to zero on the other elements of this basis. But this shows that $(b_1, \ldots, b_s, 0, \ldots, 0)$ is a linear evaluation of $\boldsymbol{A}$. Contradiction.

Since the definition of Gale transform is symmetrical the same proof works for the vice-versa, with $\boldsymbol{A}$ and $\boldsymbol{B}$ interchanged. □

Lemma 24. *Let V be a finite-dimensional vector space. Let X, Y be two subspaces of V of same dimension. There exists a subspace $I \subset V$ such that*

$$V = X \oplus I = Y \oplus I$$

Proof. If $X = Y$ we complete any basis for X and get the result. Thus, let us assume $X \neq Y$.

Let $\mathcal{B}_X = (x_1 \ldots, x_r)$ be a basis for X and let $\mathcal{B}_Y = (y_1 \ldots, y_r)$ be a basis for Y. Note that since $X \neq Y$, and $\dim X = \dim Y$, $X \neq \{0\}$, i.e. $r \geq 1$. W.L.O.G we consider $x_1 \notin Y$.

For the same reason, $X \neq V$. Let $\tilde{u} \in V \setminus X$. Then $(x_1 \ldots, x_r, \tilde{u})$ is LI.

Case $(y_1 \ldots, y_r, \tilde{u})$ is LI. Set $u := \tilde{u}$.

Then $(x_1 \dots, x_r, u)$ and $(y_1 \dots, y_r, u)$ are LI.

Otherwise, $u \in Y$ and we set $u \coloneqq \tilde{u} + x_1$.

We will show than $(x_1 \dots, x_r, u)$ and $(y_1 \dots, y_r, u)$ are LI.

Suppose

$$\sum_{i=1}^{r} a_i x_i + b(\tilde{u} + x_1) = 0$$

Then

$$(a_1 + b)x_1 + \sum_{i=2}^{r} a_i x_i + b\tilde{u} = 0$$

Since $(x_1 \dots, x_r, \tilde{u})$ is LI, all the coefficients in the sum above are zero.

Now, suppose

$$\sum_{i=1}^{r} a_i y_i + b(\tilde{u} + x_1) = 0$$

Then

$$bx_1 = -\sum_{i=1}^{r} a_i y_i + b\tilde{u} \in Y$$

Since $x_1 \notin Y$, $b = 0$. Since $\mathcal{B}_Y = (y_1 \dots, y_r)$ are LI, the coefficients $a_i = 0$.

In both cases we end up with lists $(x_1 \dots, x_r, u)$ and $(y_1 \dots, y_r, u)$, which are are LI. Then we set $u_1 \coloneqq u$

We repeat the procedure to $X' = \operatorname{Span}(x_1 \dots, x_r, u)$ and $Y' = \operatorname{Span}(y_1 \dots, y_r, u)$, first verifying if $X' \neq Y'$. Otherwise we consider the bases $\mathcal{B}_{X'} = (x_1 \dots, x_r, u)$ and $\mathcal{B}_{Y'} = (y_1 \dots, y_r, u)$. After finite s steps, $X' = V$, and $I = \operatorname{Span}(u_1, \dots, u_s)$. □

Lemma 25. *Let $P : \mathbb{R}^s \to \mathbb{R}^r, s \geq r$ the linear map*

$$(x_1, \dots, x_r, x_{r+1}, \dots, x_s) \mapsto (x_1, \dots, x_r)$$

and let $(e_1, \dots, e_s)$ and $(e_1, \dots, e_r)$ be the canonical basis for $\mathbb{R}^s$ and $\mathbb{R}^r$. Then

1. $\operatorname{Ker} P = \operatorname{Span}(e_{r+1}, \dots, e_s)$

2. *Let V be a subspace of $\mathbb{R}^s$ such that $V \oplus \operatorname{Ker} P$. There exists a linear section $\sigma : \mathbb{R}^r \to \mathbb{R}^s$ of P, such that $\sigma(\mathbb{R}^r) = V$.*

Proof. The first item comes from $\operatorname{Span}(e_{r+1}, \dots, e_s) \subset \operatorname{Ker} P$ and $\dim \operatorname{Ker} P = s - r = \dim \operatorname{Span}(e_{r+1}, \dots, e_s)$.

Let $(v_1, \dots, v_r)$ be a basis for V. With respect to the canonical basis of $\mathbb{R}^s$, we have

$$\begin{aligned} v_1 &= a_1^1 e_1 + \dots + a_r^1 e_r + a_{r+1}^1 e_{r+1} + \dots + a_s^1 e_s \\ &\vdots \\ v_r &= a_r^1 e_1 + \dots + a_r^r e_r + a_{r+1}^r e_{r+1} + \dots + a_s^r e_s \end{aligned} \tag{A.1}$$

Recall that there are elementary operations we can apply to (A.1) such that we still have a basis for V, namely, we can replace

$$\begin{aligned} v_i &\rightsquigarrow \alpha v_i, \text{ with } \alpha \in \mathbb{R} \setminus \{0\} \\ v_i &\rightsquigarrow v_i + v_j \\ v_i, v_j &\rightsquigarrow v_j, v_i \end{aligned}$$

These are the operations of Gauss elimination method, which we will apply to (A.1). But before that we note the following: since $\mathbb{R}^s = V \oplus \operatorname{Ker} P = V \oplus \operatorname{Span}(e_{r+1}, \ldots, e_s)$, then for every vector $e_i, 1 \leq i \leq r$ we must have an element of the basis with a nonzero component in the "direction" of e_i.

Then, by Gauss method

$$\begin{aligned} \tilde{v}_1 &= e_1 + 0e_2 \cdots + 0e_r + b^1_{r+1}e_{r+1} + \cdots + b^1_s e_s \\ &\vdots \\ \tilde{v}_r &= 0e_1 + 0e_2 \cdots + 1e_r + b^r_{r+1}e_{r+1} + \cdots + b^r_s e_s \end{aligned} \tag{A.2}$$

we get another basis for V. Note that we have used the operation of interchanging rows so that $\tilde{v}_i$ has a nonzero component in the direction of e_i.

Finally, we define

$$\sigma : \mathbb{R}^r \to \mathbb{R}^s, e_i \mapsto \tilde{v}_i$$

which satisfies the hypotheses. □

A.2 Topology

We collect here some results on topology used in the text. Our main reference is (LEE, 2012).

Lemma 26. *A finite union of open and dense subsets is open and dense.*

Proof. Let $D_1, \ldots, D_n$ be open and dense subsets of X. The intersection $\cap D_i$ is open because finite. Let $V \subset X$ be open and non-empty. We will show that

$$V \cap D_1 \cap \cdots \cap D_n \neq \emptyset$$

Since V is open and D_1 is dense, $V \cap D_1 \neq \emptyset$. Since V and D_1 are open, $V \cap D_1$ is open. We repeat the argument for $(V \cap D_1) \cap D_2$, and so on, finitely many times. □

Proposition 6. *(LEE, 2011, p. 54) A continuous injective map that is either open or closed is a topological embedding.*

Proof. See (LEE, 2011, p. 54). □

Theorem 15 (Proper Continuous Maps Are Closed)**.** *(LEE, 2012, p. 611) Let X be a topological space, Y be a locally compact Hausdorff space, and $F : X \to Y$ a proper continuous map. Then F is closed.*

Proof. (LEE, 2012, p. 611) □

Proposition 7. *(LEE, 2012, p. 696) Suppose $q : X \to Y$ is an open quotient map. Y is Hausdorff if and only if the set*

$$\mathcal{R} = \{(x_1, x_2) \mid q(x_1) = q(x_2)\}$$

is closed in $X \times X$.

A.3 Manifold theory

We collect here some results on manifolds used in the main text. Our main reference is (LEE, 2012).

Proposition 8 (Images of Embeddings as Submanifolds)**.** *(LEE, 2012, p. 99) Let M be a smooth manifold with or without boundary, N a smooth manifold, and $F : N \to M$ a smooth embedding. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.*

Proof. (LEE, 2012, p. 99) □

Proposition 9. *(LEE, 2012, p. 80) Let M and N be smooth manifolds without boundary, and $F : M \to N$ a map.*

1. *F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.*

2. *if $\dim M = \dim N$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.*

Proof. See (LEE, 2012, p. 80). It is a straightforward application of the inverse function theorem for manifolds. □

Theorem 16 (Rank Theorem)**.** *(LEE, 2012, p. 81) Let M and N be smooth manifolds of dimension m and n, respectively. Let $F : M \to N$ be a smooth map with constant rank r. For each $p \in M$ there exists smooth charts (U, φ) for M centered at p and (V, ψ) centered at $F(p)$ such that $F(U) \subset V$, in which F has a coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ of the form*

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0)$$

In particular, if F is a smooth submersion, this becomes

$$\hat{F}(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n)$$

and if F is a smooth immersion, it is

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

(In the source, a chart φ centered at p means that $\varphi(p) = 0$.)

Proof. (LEE, 2012, p. 81) □

Proposition 10. *(LEE, 2012, p. 85) Let M and N be smooth manifolds, and suppose that $\pi : M \to N$ is a smooth submersion. Then π is an open map, and if it is surjective it is a quotient map.*

Proof. (LEE, 2012, p. 85) □

Theorem 17 (Constant-Rank Level Set Theorem)**.** *(LEE, 2012, p. 105) Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M*

Proof. (LEE, 2012, p. 105) □

Proposition 11. *(LEE, 2012, p. 112) If M and N are smooth manifolds with or without boundary, $F : M \to N$ is a smooth map, and $S \subset M$ is an immersed or embedded submanifold, $F|_S : S \to N$ is smooth.*

Proof. See (LEE, 2012, p. 112) □

Proposition 12. *(LEE, 2012, p. 113) Let M be a smooth manifolds and $S \subset M$ be an embedded submanifold. Then every smooth map $F : N \to M$ whose image is contained in S is also smooth as a map from N to S.*

Proof. See (LEE, 2012, p. 113) □

Theorem 18 (Extension theorem)**.** *(GUILLEMIN; POLLACK, 2010, p.70) Let Z be a closed submanifold of Y, both without boundary, and C a closed submanifold of X. Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then there exists a smooth map $g : X \to Y$ homotopic to f, such that $g \pitchfork Z, \partial g \pitchfork Z$, and on a neighborhood of C we have $g = f$.*

A.4 Riemannian Isometries

In this section, we cite a result on Riemannian geometry that is used to study the set of non-fixed points of an orbifold chart. For the definition of the set of non-fixed points of an action, see the next section.

Definition 29. *(PETERSEN, 2016, p.3) A **Riemannian isometry** between Riemannian manifolds (M, g_M) and (N, g_N) is a diffeomorphism $F : M \to N$ such that $F^* g_N = g_M$, that is,*

$$g_N(DF(v), DF(w)) = g_M(v, w)$$

for all tangent vectors $v, w \in T_pM$ and all $p \in M$.

Definition 30. *(PETERSEN, 2016, p.196) A map $F : (M, g_M) \to (N, g_N)$ is a **local Riemannian isometry** if for each $p \in M$ the differential $DF_p : T_pM \to T_{F(p)}N$ is a linear isometry.*

Proposition 13 (Uniqueness of Riemannian Isometries)**.** *(PETERSEN, 2016, p.197) Let $F, G : (M, g_M) \to (N, g_N)$ be two local Riemannian isometries. Suppose M is connected, and suppose there is $p \in M$ such that $F(p) = G(p)$, and $DF_p = DG_p$. Then $F = G$.*

Proof. See (PETERSEN, 2016, p.197). □

Lemma 27. *Let M be a connected manifold, Let $\sigma : M \to M$ be a nontrivial automorphism of finite order (that is, $(\sigma)^n = id_M$ for some $n \in \mathbb{N}$). Then every open subset of U contains a point not fixed by σ. Equivalently, the set of non-fixed points of σ is dense in M.*

Proof. Let G be group of automorphisms of M generated by σ. It is a finite group since σ has finite order.

We can equip M with a G-invariant metric. Let h be a metric on M. A G-invariant metric h_G is

$$h_G(u, v) = \frac{1}{|G|} \sum_{g \in G} g^* h$$

Since $\sigma \in G$, then with respect to h_G, σ is an isometry.

Suppose there is an open set $U \subset M$ such that $\sigma|_U = id_U$. Let $p \in U$. It follows that $\sigma_{*_p} = id_{*_p}$. By the Uniqueness of Riemannian Isometries applied to σ and $id : M \to M$ (Proposition 13), $\sigma = id$. A contradiction, since σ is nontrivial by hypothesis. □

A.5 Topological and Lie group actions

We collect here some results concerning topological and Lie group actions. Our main sources were (LEE, 2012; BREDON, 1972; DIECK, 1987). We have also benefited greatly from (MOLITOR, 2016), especially on proper actions of Lie groups and slices.

We will need to make some definitions regarding fixed and nonfixed points of a group action. This will be applied in particular to orbifold charts.

Definition 31. *(DIECK, 1987, p. 4). Let G be a topological group and X a left G-space. Let H be a subgroup of G. Then*

$$X_H = \{x \in X \mid G_x = H\},$$

$$X_{(H)} = \{x \in X \mid G_x \sim H\},$$

$$X^H = \{x \in X \mid hx = x \textit{ for all } h \in H\} = \{x \in X \mid H \subset G_x\},$$

where $G_x \sim H$ means that these subgroups are conjugate.

Remark 15. Let G be a group acting on a space X.

1. $x \in X$ is a **fixed point** of the action if there is $g \in G$, with $g \neq 1$ such that $g \cdot x = x$
2. Conversely x is a **nonfixed point** if it is not a fixed point that is

$$gx \neq x \text{ for all } g \neq 1 \iff gx = x \text{ implies } g = 1$$

3. If $gx \neq x$, for some $g \in G$ we say that x **is not fixed** g
4. Let $g \in G$. We denote by

$$S_g := \{x \in X \mid g \cdot x \neq x\}$$

the set of points of X not fixed by g.

5. The set of **nonfixed points** of the action is then

$$\begin{aligned} S &= \bigcap_{g \in G, g \neq 1} S_g \\ &= \{x \in X \mid gx \neq x \text{ for every } g \in G, g \neq 1\} \\ &= \{x \in X \mid gx = x \implies g = 1\}. \end{aligned}$$

The last equality shows that this set is the set of points where the action is free.

6. The **set of fixed points** of the action is

$$\begin{aligned} Z &:= \{x \in x \mid gx = x \text{ for some } g \neq 1 \in G\} \\ &= \bigcup_{\{1\} \neq H \leq G} X_H \end{aligned}$$

7. Note that $X = Z \cup S$ form a disjoint union.

Lemma 28. *The set of nonfixed points S of an action is invariant, and it is the largest subset of X, where the restriction of the action of G is free.*

Proof. The second claim is straightforward. Let us prove invariance. Let $x \in S, g \in G, g \neq 1$. We will show that $gx \in S$. Suppose not. Then, for some $h \in G, h \neq 1$ Let $h(gx) = gx$. This implies $g^{-1}hgx = x$. Since $x \in S$, $g^{-1}hg = 1$, which implies $h = 1$. Contradiction. □

Theorem 19. *(LEE, 2011, p.71) [Characteristic Property of the Quotient Topology] Let X and Y be topological spaces and $q : X \to Y$ be a quotient map. Let Z be a topological space. A map $f : Y \to Z$ is continuous if and only if the composite map $f \circ q$ is continuous.*

Proof. See (LEE, 2011, p.71). Just follow the definition of quotient topology. □

Theorem 20 (Passing to the quotient)**.** *(LEE, 2011, p.72) Let $q : X \to Y$ be a quotient map, Z a topological space and $f : X \to Z$ is a continuous map that is constant on the fibers of q (i.e. if $q(x) = q(x')$, then $f(x) = f(x')$). Then there exists a unique continuous map $\tilde{f} : Y \to Z$ such that $f = \tilde{f} \circ q$:*

$$\begin{array}{ccc} X & & \\ {\scriptstyle q}\downarrow & \searrow^{f} & \\ Y & \overset{}{\underset{\tilde{f}}{\dashrightarrow}} & Z \end{array}$$

Proof. See (LEE, 2011, p.72). Continuity follows from the Characteristic Property (Theorem 19). □

Lemma 29. *(LEE, 2012, p. 541) . For any continuous action of a topological group G on a topological space M, the quotient map $\pi : M \to M/G$ is an open map.*

Proof. See (LEE, 2012, p. 541). □

Lemma 30 (Passing to the quotient)**.** *Let G be a group acting continuously on topological spaces X and Y. Let $F : X \to Y$ be a G-equivariant continuous map. Then there is a unique continuous map $\tilde{F} : X/G \to Y/G$ such that the diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X/G & \xrightarrow{\tilde{F}} & Y/G \end{array}$$

where p, q are the respective quotient maps.

If F is a homeomorphism, then so is $\tilde{F}$.

Proof. $p \circ F$ is continuous and constant on the fibers (because F is G-equivariant). Therefore, we can pass to the quotient and obtain $\tilde{F}$ with the required properties (Theorem 20).

Is F is a homemorphism, the analogous argument gives us $\widetilde{F^{-1}}$.

The argument again applied to the identity of X, and then Y, plus unicity shows

$$\widetilde{F^{-1}} = (\tilde{F})^{-1}$$

□

Lemma 31 (Passing to the quotient II)**.** *Let G, H be groups acting continuously on topological spaces X and Y, respectively. Let $\xi : G \to H$ be a homomorphism.*

Let $F : X \to Y$ be a ξ-equivariant continuous map (i.e. $F(gx) = \xi(g)F(x)$. Then there is a unique continuous map $\tilde{F} : X/G \to Y/G$ such that the diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ X/G & \xrightarrow{\tilde{F}} & Y/H \end{array}$$

where p, q are the respective quotient maps.

If F is a homeomorphism, and ξ is an isomorphism, then $\tilde{F}$ is a homeomorphism.

In particular, for $H = G$ and $\xi = id_G$, we have Lemma 30.

Proof. $p \circ F$ is continuous and constant on the fibers (because F is ξ-equivariant):

$$p \circ F(gx) = p(\xi(g)F(x)) = p(F(x)).$$

Therefore, we can pass to the quotient and obtain $\tilde{F}$ with the required properties (Theorem 20).

Is F is a homemorphism, the analogous argument gives us $\widetilde{F^{-1}}$.

The argument again applied to the identity of X, and then Y, plus unicity shows

$$\widetilde{F^{-1}} = (\tilde{F})^{-1}$$

□

Proposition 14. *(DIECK, 1987, p. 42). Let G be a compact Lie group and M a differentiable G-manifold. Let G be any isotropy group of M. Then $M_{(H)} = \{x \in M \mid (G_x) \sim (H)\}$ is a submanifold of M (which may have components of different dimensions). In particular, M^G is always a closed submanifold.*

Proof. (DIECK, 1987, p. 42). □

Definition 32. *(LEE, 2012, p. 542) Let $\Psi : G \times M \to M$ be a continuous left action of a Lie group G on a manifold M. This action is a **proper action** if the map*

$$G \times M \to M \times M, (g, p) \mapsto (\Psi(g, p), p)$$

is a proper map.

Corollary 1. *(LEE, 2012, p. 544) Every continuous action by a compact Lie group on a manifold is proper.*

Proof. See (LEE, 2012, p. 544). □

Definition 33. *(LEE, 2012, p. 543) Let $\Psi : G \times M \to M$ be a continuous left action of a Lie group G on a manifold M. The **orbit relation** of this action is the set*

$$\mathcal{O} = \Theta(G \times M) = \{(\Psi(g,p),p) \in M \times M \mid p \in M, g \in G\}$$

where

$$\Theta : G \times M \to M \times M, (g,p) \mapsto (\Psi(g,p),p)$$

This is called the orbit relation because $(q,p) \in \mathcal{O}$ if and only if p,q are in the same G-orbit.

Lemma 32 (Characterization of Orbit Relation)**.** *Let $\Psi : G \times M \to M$ be a continuous left action of a Lie group G on a manifold M. Let $\pi : M \to M/G$ be the quotient map, and let $\mathcal{O}$ be the orbit relation of this action. Then*

$$\mathcal{O} = \{(q,p) \in M \times M \mid \pi(q) = \pi(p)\}$$

Proof. Note that $\pi(q) = \pi(p)$ if and only if q,p are in the same orbit. □

Proposition 15. *(LEE, 2012, p. 543) Let $\Psi : G \times M \to M$ be a continuous, proper, left action of a Lie group G on a manifold M. Then the orbit space is Hausdorff.*

Proof. (LEE, 2012, p. 543) Let $\Theta : G \times M \to M \times M$ be the proper map $\Theta(g,p) = (\Psi(g,p),p)$, and let $\pi : M \to M/G$ be the quotient map, and $\mathcal{O}$ the orbit relation of the action.

Since Θ is a proper continuous map, it is closed (Theorem 15). Thus $\mathcal{O}$ is a closed subset of $M \times M$.

Also note that π is an open map, by Lemma 29.

Then using Lemma 32 and Proposition 7, we have that M/G is Hausdorff. □

A.5.1 Tubes and Slices

In this section we cite the relevant results and definitions about existence of tubes and slices for compact and proper group actions.

Construction 5 (Twisted Product)**.** (BREDON, 1972, p.46) Let H be a compact subgroup of G and let H act on a space A. We will define the following action of H on $G \times A$:

$$H \times (G \times A) \to (G \times A),$$
$$(h, (g, a)) \mapsto (gh^{-1}, ha),$$

We will denote the orbit space of this H-action by

$$G \times_H A \tag{A.3}$$

and the H-orbit of (g, a) will be denoted by $[g, a]$.

Lemma 33. *(BREDON, 1972, p.46) Let H be a compact subgroup of G and let H act on a space A. The twisted product $G \times_H A$ has the following G-action*

$$g'[g, a] = [g'g, a]$$

Proof. We will apply Lemma 30 to the continuous map

$$F : G \times (G \times A) \to (G \times A),$$
$$(g', (g, a)) \mapsto (g'g, a)$$

F is H equivariant because, for $h \in H$.

$$g'(gh^{-1}, ha) = (g'gh^{-1}, ha) = (g'g, a) = g'(g, a)$$

□

Lemma 34. *(BREDON, 1972, p.46) The map*

$$i_e : A \to G \times_H A,$$
$$a \mapsto [e, a]$$

is a topological embedding (H does not have to be compact for i_e to be an embedding, but compactness of H is used to guarantee that the twisted product is Hausdorff).

Proof. See (BREDON, 1972, p.46).

i_e is closed since it is the composition

$$A \to G \times A \to G \times_H A$$

of closed maps.

The projection map $A \to G \times A \to G \times_H A$ is a closed map because $G \times A$ is an H-space, and H is compact (Theorem in (BREDON, 1972, p. 38)).

The source shows that i_e is injective, continuous and closed. By Proposition 6, it is a topological embedding.

Alternative argument that works for H not necessarily compact: i_e is open since it is the composition

$$A \to G \times A \to G \times_H A$$

of open maps (Lemma 29 for the quotient map).

The source shows that i_e is injective, continuous. Since it is also open, by Proposition 6, it is a topological embedding.

□

Definition 34. *(BREDON, 1972, p. 82) Let X be a G-space, G compact Lie group. Let P be an orbit of type G/H. A* ***tube about*** *P (or a tube around P or a G-tube about (around) P) is a G-equivariant embedding*

$$\varphi : G \times_H A \to X$$

onto an open neighborhood of P in X, where A is some space on which H acts.

Proposition 16. *(BREDON, 1972, p. 81) Let G is a compact group, H is a closed subgroup, and A is a left H-space. (In particular, H is also compact). The inclusion $i_e : A \to G \times_H A$ induces a homeomorphism*

$$A/H \to (G \times_H A)/G,$$
$$H(a) \mapsto G[e, a]$$

Proof. (BREDON, 1972, p. 81) □

Definition 35. *(BREDON, 1972, p. 82) Let $x \in X$, a G-space. Let $x \in S \subset X$ be such that $G_x(S) = S$. Then S is called a* ***slice*** *at x, if the map*

$$G \times_{G_x} S \to X,$$

taking $[g, s] \mapsto g(s)$, is a tube around $G(x)$.

Theorem 21. *(BREDON, 1972, p. 82) Let X be a G-space, let $x \in S \subset X$, and denote $H := G_x$. Then the following statements are equivalent:*

1. *There is a tube $\varphi : G \times_H A \to X$ around $G(x)$ such that $\varphi[e, A] = S$*

2. *S is a slice at x.*

Proof. (BREDON, 1972, p. 83) (1) implies (2) because A can be replaced with S. □

Definition 36. *(BREDON, 1972, p. 170) Let M be a G-space, G compact Lie group. Let P be an orbit of type G/H and let V be a euclidean space on which H operates orthogonally. A **linear tube** about (around) P in M is a tube of the form*

$$\varphi : G \times_H V \to M$$

Definition 37. *(BREDON, 1972, p. 171) A G-space M is **locally smooth** if there exists a linear tube about each orbit.*

Lemma 35. *(BREDON, 1972, p. 308) A smooth action of a compact Lie group is locally smooth.*

Proof. (BREDON, 1972, p. 308) □

Remark 16. About Lemma 35, (DIECK, 1987, p. 40) states it more strongly, that for $m \in M$, we can take $H = G_m$ and $\varphi[g, 0] = gm$. Also, while (BREDON, 1972) considers the tube as a topological embedding, in the result stated in (DIECK, 1987) the tube is also a smooth embedding onto an open subset (thus a diffeomorphism onto its image).

A.5.2 Proper Lie group actions

The source for this section is (MOLITOR, 2016).

We fix a free and proper action $\Phi : G \times M \to M$ of a Lie group G on a manifold M.

Lemma 36. *Given $p \in M$, there exists an embedded submanifold $S \subseteq M$ containing p such that*

$$T_qS \oplus (\Phi_q)_{*e}\mathfrak{g} = T_qM \tag{A.4}$$

for every $q \in S$.

Lemma 37. *Shrinking S if necessary, the map $\Phi|_{G\times S} : G \times S \to M$ is a diffeomorphism onto an open subset of M.*

Looking at the proof of these two lemmas, we see that it is enough to start with an embedded submanifold S' of M containing p and such that

$$T_pS' \oplus (\Phi_p)_{*e}\mathfrak{g} = T_pM. \tag{A.5}$$

and restrict S' enough. That is, we intersect S' with a sufficiently small coordinate open set U of M. Let us call $S := S' \cap U$. By Lemma 36, $\Psi(G \times S)$ is an open embedded submanifold of M, which is also G-invariant.

Then, we write $U_S = \pi(S) \subseteq N$, and $\varphi_S : U_S \to S, \varphi_S = (\pi|_S)^{-1}$. (U_S, φ_S) is a chart for N at $\pi(p) \in N$.

Lemma 38. *Let $\Phi : G \times M \to M$ be a free and proper Lie group action with quotient map $\pi : M \to G\backslash M$. Then for every G-invariant embedded submanifold N of M, the set $\pi(N)$ is an embedded submanifold of $G\backslash M$.*

A.6 Lie groupoids

We collect here the definitions need from the theory of Lie groupoid. Our main sources were (ADEM; LEIDA; RUAN, 2007) and (LERMAN, 2010).

Definition 38. *(ADEM; LEIDA; RUAN, 2007, p. 17) A **groupoid** is a pair of sets (G_0, G_1) equipped with five structure maps, listed below. G_0 is called the space of **objects**, and G_1 is called the space of **arrows**.*

1. *The **source map** $s : G_1 \to G_0$ assigns to each arrow $g \in G_1$ its **source** $s(g)$.*
2. *The **target map** $t : G_1 \to G_0$ assigns to each arrow $g \in G_1$ its **target** $t(g)$. If $s(g) = x$ and $t(g) = y$ we denote $g : x \to y$.*
3. *The **composition map** $m : G_1 \,{}_s\!\times_t G_1 \to G_0$, where*

$$\mathcal{G}^{(2)} := G_1 \,{}_s\!\times_t G_1 = \{(h, g) \in G_1 \times G_1 \mid s(h) = t(g)\}.$$

   We denote $m(h, g) = hg$, and require associativity whenever this operation is defined. We also require that $s(hg) = s(g)$ and $t(hg) = t(h)$.

4. *The **unit** (or **identity**) map $u : G_0 \to G_1$ satisfying*

$$s(u(x)) = x = t(u(x)), \quad g(u(x)) = g = u(y)g$$

   for all $x, y \in G_0$ and $g : x \to y$.

5. *The **inverse map** $i : G_1 \to G_1, (g : x \to y) \mapsto (g^{-1} : y \to x)$ satisfying*

$$g^{-1}g = u(x) \quad gg^{-1} = u(y).$$

*When G_0 and G_1 are topological spaces and the structure maps are continuous we have a **topological groupoid**.*

Definition 39. *(ADEM; LEIDA; RUAN, 2007, p. 17) A **Lie groupoid** is a topological groupoid $\mathcal{G}$ where G_0 and G_1 are smooth manifolds, and such that the structure maps s, t, m, u, i are smooth. Furthermore $s, t : G_1 \to G_0$ must be submersions.*

Definition 40 (Pull-back of a groupoid)**.** *(LERMAN, 2010, p. 324) The **pull-back** of a groupoid $\mathcal{G}$ by a map $f : N \to G_0$ is the groupoid $f^*\mathcal{G}$ with the space of objects N, the space of arrows:*

$$\begin{aligned}(f^*\mathcal{G})_1 &:= (N \times N) \times_{G_0 \times G_0} G_1 \\ &= \{(x, y, g) \in N \times N \times G_1 \mid s(g) = f(x), t(g) = f(y)\}\end{aligned}$$

that is, $(f^*\mathcal{G})_1$ is the pullback in the category of Sets of the diagram

$$\begin{array}{ccc} & & G_1 \\ & & \big\downarrow{\scriptstyle (s,t)} \\ N \times N & \xrightarrow{f \times f} & G_0 \times G_0 \end{array}$$

The source and target maps of the pullback groupoid are

$$s(x, y, g) = x \quad t(x, y, g) = y$$

and multiplication is given by

$$(y, z, h)(x, y, g) = (x, z, hg)$$