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How to Prove Things with Signs: On Frege's Performative Logic

Como Provar Coisas Com Sinais: Sobre a Lógica Performativa de Frege

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How to Prove Things with Signs: On Frege's Performative Logic

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Orientador: Giorgio Venturi

Este exemplar corresponde à versão final da tese defendida pelo aluno João Vitor Schmidt, e orientada pelo Prof. Dr. Giorgio Venturi.

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Prof. Dr. Giorgio Venturi

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Resumo

Esta é uma tese sobre a concepção de lógica de Gottlob Frege. Mais precisamente, é uma leitura do linguagem formal desenvolvida por Frege, a Conceitografia [*Begriffsschrift*], de um ponto de vista performativo. O sentido no qual a Conceitografia pode ser chamada de performativa segue dos três principais componentes do sistema de lógica fregeano, a saber, juízos, definições e inferências. Defendo uma leitura na qual estes componentes são expressos por recursos ilocutórios da linguagem: asserções, declarações e permissões. De modo a desenvolver esta perspectiva, primeiro retomo as motivações fregeanas para considerar a sua lógica como uma linguagem e descrevo a primeira versão da conceitografia no **capítulo 1**. No **capítulo 2**, retomo a evolução do sistema fregeano, que culminaram na versão revisada do sistema em 1893. Nesse mesmo capítulo, mostro finalmente que juízos, inferências e definições são todos representados por sinais de tipos diferentes, isto é, performativos. No **capítulo 3**, resgato o problema do psicologismo que deriva dessa leitura e defendo uma leitura universalista da lógica de Frege. Ela será então utilizada no **capítulo 4**, para defendê-la dos riscos psicologistas. Para tanto, argumento que Frege possui duas concepções de objetividade: uma conteudística e outra ilocutória. A primeira é melhor resumida pela pressuposição fregeana de um terceiro reino de Pensamentos. A segunda deriva do papel normativo que as leis lógicas e regras de inferência possuem. Mesmo que juízos, inferências e definições sejam irredutivelmente atos, eles não estão sujeitos à leituras relativistas que derivem do risco psicologista. A leitura performativa também deixa em aberto uma reflexão sobre a posição de Frege na história da lógica. É o que proponho no **capítulo 5**, onde o foco se dá na herança histórica que juízos possuem. Também argumento que o abandono desta noção, após Frege, marca um processo de despragmatização de sistemas de lógica. Finalmente, no **capítulo 6**, e seguindo o uso pioneiro de Frege dos marcadores de força ilocutória na linguagem, uso a Teoria de Atos de Fala de Searle como um guia para uma leitura não ortodoxa da conceitografia, reforçando o ponto linguístico argumentado nos capítulos anteriores. Como conclusão, aponto que essa leitura performativa possui duas consequências de interesse. Primeiro, ela oferece uma leitura da conceitografia que é historicamente justa ao momento histórico de Frege e seus objetivos, não apenas lendo-a como outra variante de um cálculo de predicados moderno. Segundo, ela ajuda a iluminar perspectivas pragmáticas recentes em filosofia da lógica e filosofia da matemática, onde a lógica de Frege pode ser considerada um interessante caso de uma linguagem lógica orientada à prática.

Palavras-chave: Gottlob Frege, 1848-1925; Lógica Simbólica e Matemática; Atos de Fala; Juízos.

Abstract

This is a thesis about Gottlob Frege's conception of Logic. More precisely, it is a reading of Frege's formal language, the concept-script [*Begriffsschrift*], from a performative viewpoint. The sense in which the concept-script can be called performative follows the three main components of Frege's system of Logic, namely, judgements, definitions and inferences. I defend a reading in which they are expressed through illocutionary devices in the language: assertives, declaratives and permissives. To develop this view, I first go back to Frege's motivation for taking Logic as a language and his first version of the concept-script in **chapter 1**. In **chapter 2**, I track down the evolution of his system culminating in the revised 1893 version and then show that judgements, inferences and definitions are all signs of a different kind, namely, performative signs. In **chapter 3**, I reassess the problem of psychologism and defend a Universalist reading of Frege. This will be used in **chapter 4** in order to defend Frege's Logic from the psychologistic attack. I claim that Frege has two notions of objectivity: one contentual and one illocutionary. The former is described by the third realm of Thoughts and defines what we may call Fregean Semantics. The latter stems from the normative role played by logical laws and inferential rules. Even though judgements, inferences, and definitions are irreducibly acts, they are not subject to relativistic accounts that may derive from psychologistic worries. This reading calls for a reassessment of Frege's position in the history of Logic. I do that in **chapter 5**, where the focus will be the historical heritage coming from the notion of judgements. I also argue that the demise of judgements marked a de-pragmatization of logical systems. Finally, in **chapter 6**, and following Frege's pioneering uses of illocutionary force indicating devices, I use Searle's Speech Act Theory as a guide for a novel reading of the concept-script, strengthening the linguistic point made in previous chapters. In conclusion, I'll point out that this performative reading has two interesting consequences. First, it offers a reading of the concept-script that is historically fair to Frege's time and endeavours, thus not seeing it as another variant of modern predicate calculus. Second, it will shed some light on recent pragmatically oriented perspectives in the philosophy of Logic and Mathematics, where Frege's Logic is an interesting case of a practice-oriented language of Logic.

Keywords: Gottlob Frege, 1848-1925; Logic, Symbolic and mathematical; Speech Acts; Judgments.

List of Abbreviations

- Books and Compilations

  1. *BS* - Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens, translated by Stefan Bauer-Mengelberg as *Begriffsschrift: a formula language, modelled upon that of arithmetic, for pure thought*, (FREGE, 1967)
  2. *GLA* - Die Grundlagen der Arithmetik, translated by John Austin as *The Foundations of Arithmetic*, (FREGE, 1953)
  3. *GGA* - Grundgesetze der Arithmetik, translated by Philip Ebert and Marcus Rossberg as *The Basic Laws of Arithmetic*, (FREGE, 2013)
  4. *PMC* - Wissenschaftlicher Britfwechsel, translated by Felix Klein as *Philosophical and Mathematical Correspondence*, (FREGE, 1980)
  5. *PW* - Nachgelassene Schriften und Wissenschaftlicher Briefwechsel, translated by Peter Long and Roger White as *Posthumous Writings*, (FREGE, 1979)
  6. *CP* - Kleine Schriften, translated by Max Black, V. H. Dudman *et. al.*, edited by Brian McGuinness as *Collected Papers on Mathematics, Logic, and Philosophy*, (FREGE, 1984)

- Papers

  7. *SuB* - Über Sinn und Bedeutung, translated by Max Black as *On Sense and Meaning*, in (FREGE, 1984).
  8. *FuB* - Funktion und Begriff, translated by Peter Geach as *Function and Concept*, in (FREGE, 1984).
  9. *BuG* - Über Begriff und Gegenstand, translated by Peter Geach as *On Concept and Object*, in (FREGE, 1984).
  10. *DG* - Der Gedanke, translated by Peter Geach and R. H. Stoothoff as *Thoughts*, in (FREGE, 1984).
  11. *WBB* - Über die wissenschaftliche Berechtigung einer Begriffsschrift, translated by William T. Bynum as *On the Scientific Justification of a Concept-Script*, (FREGE, 1972).

Contents

Introduction

This is a thesis about Gottlob Frege's conception of Logic. More precisely, it is a reading of Frege's formal language, named concept-script [*Begriffsschrift*], from a performative viewpoint. The sense in which the concept-script can be called performative will be clearer both in going back to its details, but also in comparing this nineteenth-century formalization of logic to nowadays conceptions that are considered anything but performative in the sense Frege's is. This is, at least, what I will try to defend.

For anyone outside the circles of academia, logic is considered the science of the valid forms of reasoning. Its subject is human thinking, and its goal is to describe how one goes from true premises to true conclusions in arguments. If, for instance, someone recognizes that both

(P1) All Humans are Mortal

and

(P2) Socrates is Human

are true, she would be inclined in concluding that

(C) Socrates is Mortal

is also true. But, of course, it is recognizable that this inference is not licensed by the conceptions of humanity, mortality or even if Socrates satisfied such properties. If anything, the inference is valid because of its form (in first-order logic):

$$\text{(FI)}\ \frac{\begin{array}{c}\forall x(H(x) \rightarrow M(x)) \\ H(s)\end{array}}{M(s)}$$

Thus, we say that the inference about Socrates' mortality is valid *because* the formal inference (FI) above is valid, and because (FI) represents somehow the former informal inference. Thus, logic is not about inferences *per se*, but about their formal structures, hence the common recognition that logic is the science of *formal* methods of reasoning.

But to say that Logic is the science of *formal* methods implies that it is so in virtue of being itself expressible in a formal language. By representing $M(s)$ as "Socrates is Mortal" we are taking the formal structure of all predicative sentences, $P(x)$, and giving it an interpretation. The language of logic, however, is not about such interpretations. All we need to know whether from $\forall x(H(x) \rightarrow M(x))$ and $H(s)$ one may get $M(s)$ is their forms, not their intended interpretations. We thus say that this inference is valid because it is truth-preserving: regardless of what $H(x)$, $M(x)$ or the individual constant s denotes, the inference must be formally valid because $M(s)$ will be true under *all* interpretations that we may provide. This is what it is generally said when speaking about formality: that logic is about forms, not about contents.

The conception that logic is a distinctively formal discipline is called by MacFarlane (2000) as *logical hylomorphism*, and there is a variety of different accounts on what logical hylomorphism is about. In MacFarlane's words, logical hylomorphism is a tradition, more than a thesis about logicality. He names at least six senses in which we take logic to be formal. But since his goal is to find what is the best criteria for demarcating logic under the formality accounts, three of these he labels as "decoy" accounts. They are: syntactic formality, when logic is said to be formal in the sense of being about symbols and not their meanings[1]; schematic formality, when it is believed that logic is formal because it is mainly about patterns of reasoning, the schematic forms of inference just as (FI) above; and grammatical formality, where formal concerns to the grammatical structure of a language and its properties. These three senses are very common nowadays, but are not sufficient in demarcating what does actually means to be formal in MacFarlane's analysis[2].

The other three senses are considered more relevant and historically driven. They are:

1-Formality: formal as offering constitutive norms for thought;

2-Formality: formal as being indifferent to the distinguishing features of objects;

3-Formality: formal as abstracting entirely from the content of thoughts.[3]

[1]A position perhaps best exposed by Carnap (1937, p.1), in taking logic to be formal "[...] when no reference is made in it either to the meaning of the symbols (for example, the words) or to the sense of the expressions(e.g. the sentences), but simply and solely to the kinds and order of the symbols from which the expressions are constructed."

[2]"Logic can be treated syntactically — but so can other disciplines. Logical laws are schematic — but so are non-logical laws. Logical validity turns on grammatical form — but only if we set up the grammar to reveal and systematize logical validity." (MACFARLANE, 2000, p.50).

[3]The index numbers are in here representative of the magnitude order of abstraction from content. 1-formality does not abstract at all, whereas 3-formality abstract from all contents.

These three notions are, according to MacFarlane, influential in talking of logical formality since at least Kant. We find Kant's advocating 1- and 3-formality when he says both that logic is necessary for all thought[4] and that it abstracts from all contents[5]. Frege, in contrast, did not accept 2- or 3-formality, and adopted only 1-formality.

Apart from MacFarlane's interests of demarcating logicality through the conception of formality, I want to stress another consequence of logical hylomorphism: its effects on logical practice and on how one perceives and uses a logical system. MacFarlane's distinction is not that suitable for assessing the consequences in how one sees logic as a practice, particularly the ones resulting in the different formalisms that has been presented since Kant. 2- and 3- formality, for instance, seem indifferent to practical concerns. 1-formality does seems to be about practical notions, but it is an obsolete view in today's standards. Macfarlane says that, according to 1-formality, logical "[...] norms are constitutive of concept use as such (as opposed to a particular kind of concept use). 1-formal laws are the norms to which any conceptual activity — asserting, inferring, supposing, judging, and so on — must be held responsible" (MACFARLANE, 2000, p.51). But one will not find talk of assertions, judgements or reasoning in general in any textbook on logic in these days. Neither will one find talk about inferences in the sense of true premises followed by true conclusions. For instance, Mendelson (2015, p.xv) opens his textbook by saying that "The truth or falsity of the particular premises and conclusions is of no concern to logicians. They want to know only whether the premises imply the conclusion." What he's saying is that logic is occupied with the consequence relation. What it takes for such a shift to happen is the formality thesis more generally. If we are concerned with the mere form that valid reasoning takes (syntactically, schematically, or grammatically), it is natural to study such forms as proper objects.

In a similar fashion, (ENDERTON, 2001) calls symbolic logic a model for deductive thought. But not in the sense of modeling deductive reasoning in general. Propositional logic, for example, may be a sufficient model for a good chunk of reasoning processes that permeates our everyday business. But not for all of them. If one wants to offer a good model for the inference from Human mortality to Socrates' mortality, one need's First-Order Logic. If one wants to model reasoning about modalities such as necessity and possibility one needs modal logic. If one wants to reason about epistemic

[4]See, for example, in the Jäsche Logic when Kant defines general logic "[...] a science of the necessary laws of thought, without which no use of the understanding or of reason takes place at al" (KANT, 1992, p.529).

[5]See, for example, in the first *Critique*, where he states that general logic "[...] abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking." (KANT, 1998, B78).

modalities, Epistemic Logic comes to the rescue. For mathematical reasoning, First-Order Logic is the bare minimum.

More interested on how we use formal languages, Dutilh Novaes (2012a) offers a two-fold distinction of formality: formal as de-semantification and formal as computable. On the former,

> to be purely formal amounts to viewing symbols as blueprints (inscriptions) with no meaning at all - as pure mathematical objects and thus no longer as ‘signs’ properly speaking. Indeed, when one speaks of ‘formal systems’, ‘formal theories’, etc., de-semantification is typically an important component. (DUTILH NOVAES, 2012a, p.12)

This conception of formal has its origins in nineteenth century, specially with Hilbert’s program of the 1920’s, but it was certainly present in his *Grundlagen der Geometrie* from 1899, where the model-theoretic perspective was already taking shape. Not only Hilbert and Bernays defended de-semanticized systems, but also Thomae, Frege’s colleague at Jena, defended a formalist view of mathematics even before them. Carnap and Tarski were also examples of de-semantification after Hilbert, according to Dutilh Novaes. On the other hand, the formal as computable is derived from the notion of computability from the 1930’s and is related to how one “performs” in symbolic language. The main idea is that of defining a formal system as a recursively axiomatized theory, in which one defines rules of formation (for generating formulas from the basic terms) and rules of transformation (for generating formulas from formulas).

Notice that in computational terms, “rules” are not exactly read normatively, but mechanically. In stating the rules of formation for formulas, one is not actually generating formulas one by one, but defining the set of formulas at once. We may say that rules in this sense are rules for recognizing if a given formula is, or isn’t, a formula of the intended system. This mode of thinking, together with the de-semantification, let systems of logic became objects of mathematical studies. The formal as computable, as described by Dutilh Novaes, is another evidence that the 1-formality described by MacFarlane is rarely found in logic even today, as formal languages are not thought anymore to include assertions, judgements or inferences[6]. Quite the opposite:

[6] An example is found in Carnap’s *Logical Syntax of Language*: “the development of logic [...] shown clearly that it can only be studied with any degree of accuracy when it is based, not on judgments (thoughts, or the content of thoughts) but rather on linguistic expressions, of which sentences are the most important, because only for them is it possible to lay down sharply defined rules. And actually, in practice, every logician since Aristotle, in laying down rules, has dealt mainly with sentences.” (CARNAP, 1937, p.1). An assessment of this point is found in Sundholm (2008).

> [...] the tendency to objectify language [...] is exacerbated when it comes to thinking about formal languages. This is essentially due to the fact that formal languages are indeed typically construed as mathematical objects, and this is one of the reasons why they fulfil one of the functions they are summoned to perform in logical investigations, namely to allow for meta-theoretical investigations. (DUTILH NOVAES, 2012a, p.52)

In treating formal languages as objects, particularly formal languages of logic, the contemporary scene puts forward another type of abstraction, that is not only semantical, but also pragmatical. This is because most logical systems studied today are not systems to be used. The bulk of studies of modern systems, for instance, are not about usability, but what metatheoretical properties the system satisfy, although some results such as completeness are certainly necessary for any given system to have any practical interest. Even though it can be argued which is the best system for characterizing this or that fragment of human reasoning, in practical terms we are treating the intended system as objects and putting it to the scrutiny of a metalanguage in which we operate. It is not to say that logic became oblivious to applicability, but only that the perspective shifted from *using* a logical language to *studying* it from an advantage viewpoint.

In a sense, if we start from talking about inferences and end up talking about logical consequences, then there is an underlying change of subject. Inferences are acts, while logical consequence is inference objectified as an object of mathematical study. An example is found in John Barwise's on Model-Theoretic logics:

> a logic consists of a collection of mathematical structures, a collection of formal expressions, and a relation of satisfaction between the two. We are primarily interested in logics where the class of structures are those where some important mathematical property is built in, and where the language gives us a convenient way of formalizing the mathematician's talk about the property. We might say, then, that a logic is something we construct to study the logic of some part of mathematics. (BARWISE, 1985, pp.4-5)

Even though the study of logics is motivated in formalizing parts of mathematical reasoning (or reasoning in general), it proceeds from treating them as objects, not as a set of actions.

This anti-practical tendency that is dominant in philosophy of logic today also motivated Dutilh Novaes in developing a practice-based philosophy of logic, as she also argues in (DUTILH NOVAES, 2012b). The idea is that one can study it both from a sociological and psychological level. The latter is of more interest when formal languages are concerned, as functioning as technological devices for reasoning practices. We may say that the first formalizations in the nineteenth-century were developed with that in mind, as kinds of technological devices for overcoming the natural language's imperfections. Boole, Peano, Frege and Russell's systems were all thought as advances from the natural language, and attempts to free the deductive process from its pervasive influence. Frege goes even further in the preface to the *Begriffsschrift* in saying that his concept-script is for natural languages as the microscope is for the human eye: a device for making some parts of reasoning more precise. There is, then, a natural path from the earlier developments of formal languages in logic, in which the tendency was still to overcome natural language and reduce the faulty influences of the subjective logician to a minimum, to the totally de-semanticized and computable systems of logic as objects of today.

But just as Dutilh Novaes is seeking to look at formal systems from the perspective of individual practice, *viz.,* from the viewpoint of those who manipulate the language, earlier formalization were still languages mainly to be used, and designed precisely to be used. In today standards, to say that logic concerns some set of actions might sound uncanny. But historically, this was the case, particularly in the case of Frege's Concep-Script. Much differently from modern formal languages, Frege developed his formalization not as 2- or 3-formality in MacFarlane's sense[7], neither the formal as de-semanticized or computable, but as a language to be used and performed practically by reasoning agents.

Frege published two versions of his system. In 1879 with the publication of a booklet *Begriffsschrift*, and in 1893 with his *magnus opus Grundgesetze der Arithmetik*. His influence for the history of formal logic is impossible to underestimate. Despite the obscurity in his lifetime, there are today no short words for stating his importance in the field. In Quine's famous words, "Logic is an old subject, and since 1879 it has been a great one" (QUINE, 1966, p.vii). The classic historical work of William and Martha Kneale concludes Frege's section by saying that "Frege's work [...] contains all the essentials of modern logic, and it is not unfair either to his predecessors or to his successors to say that 1879 is the most important date in the history of the subject." (KNEALE; KNEALE, 1962, p.511). Jean van Heijenoort's great anthology on the history of mathematical logic, the *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, starts with Frege's *Begriffsschrift* in 1879, when according to the editor, the "great epoch in the history of logic" started (Van HEIJENOORT, 1967a, p.vii).

There are at least two senses in which Frege's system can be thought as per-

[7]He was still adopting 1-formality. MacFarlane dedicates a whole chapter in discussing Frege's position on formality, in contrast to Kant. See chapter 5 in (MACFARLANE, 2000), but also (MACFARLANE, 2002).

formative. The first one is about his design-choices, who were explicitly motivated for practicality. An obvious example is the choice of a two-dimensional notation. We may question whether this choice was actually helpful, given that no-one used Frege's notation besides himself. But his justification in opting for the two-dimensional notation was pragmatically driven[8]. The second one, which will be the main focus of this thesis, it's the presence of illocutionary force indicating devices in his symbolic language. These devices are representative that his concept-script cannot be simply studied as a mathematical object, not without seriously putting aside his motivations, and the philosophical and logical presuppositions behind it. The concept-script is, as I will read it, first and foremost a language to be used.

The illocutionary devices are three: assertions, representing judgements, declaratives, representing definitions, and permissives, representing inferences. All three received particular symbolisms in the concept-script. The first one, assertions, is perhaps the most important or at least the most discussed one, both by Frege and by the literature. Assertions are the manifestation of judgements. And in this regard, Frege is a follower of the logical tradition of taking acts of judgements as the central activity in logical practice. So much so that his name for his formal language could very well be called a *Urtheilsschrift*. In a not so fondly review of the *Begriffsschrift*, Ernst Schröder writed that Frege's logic was not as advanced as Boole's calculus of concepts, and thus did not deserved to be called a concept-script. Instead, he suggested, "The title is incorrect in this respect as well, and actually should have been replaced by 'Judgemental Notation' [*Urtheilsschrift*]" (FREGE, 1972, p.224). I believe that this suggestion can still be accepted, but for other reasons. It is not that Frege's logic does not have a suitable calculus of concepts or that it is only restricted to judgements. But that judgements are the main component of how Frege not only operate his logical system, but are also important in conceptual formation. "I do not proceed from concepts, but from judgements"(FREGE, 1972, p.94), as he declared in multiple occasions. In looking at his formal language from a performative viewpoint, we could take the step in calling Frege's formal language not only a concept-script, but also a judgement-script.

Nonetheless, there are some points of Frege's philosophy to which some care must be taken in reading Frege's logic this way. We should take into account the different illocutionary devices found in Frege's system in the context of the role they play in the formalization, as well as Frege's multiple design choices for his notation that certainly

[8]For a philosophical and pragmatical interpretation of Frege's two-dimensional notation, see section 2.2 in Macbeth (2005).

follows pragmatical reasons. Frege is also famous for his attack on natural language's imprecision. His influence on the search for a suitable formalization of logic is well known. But, his use of illocutionary devices doesn't agree with such criticisms, given that illocutions are not something one found in formal languages, but in ordinary languages instead. This calls for a reassessment of such presupposition in the light of Frege and his contemporaries in the history of logic. Finally, even though Frege's work on logic shows a strong consideration about the practice that was later forgotten, he is a paradigm case of a realist account in logic and mathematics. Isn't treating logic as a practice a contradiction to Frege's realism, given the contingency of the former? The depth of Frege's insights and justification for illocutionary devices should be consistent with his strong positions towards contents, particularly, the postulate of a third realm of abstract entities.

In the light of this points, this work offers an interpretation for Frege's logical system in which the concept-script is regarded as an act-based notation, designed for the practice of logic as a scientific endeavour. The text is structured in three parts, each consisting of two chapters.

1. *Part I* is mostly about the Syntax of Frege's concept-script. The goal is to describe it as a formal language of logic. In **Chapter 1**, I start assessing Frege's main motivation for developing the concepts-script, particularly the deficiencies of natural language in deductive reasoning, and how Frege's logic is still thought as a language in its own right. After this, the 1879 version of the concept-script is presented, but only the primitives that play semantical roles. In **Chapter 2**, I start showing the evolution of the concept-script in 1893, and its main differences to the earlier version, also restricted to its primitives that only play semantical roles. Finally, I discuss the performative signs that are presented in both version of the logic, pertaining to judgements, inferences and definitions. I argue that these can be understood as illocutionary force indicating devices, and should be considered as signs of a different type. The presence of such signs shows already that Frege's concept-script was thought as an instrument for logical and inferential practice.

2. *Part II* is about the Semantics of the concept-script. The term semantics, however, should be read with care, as Frege did not developed formal semantics in the tarskian sense that we know today. The starting point in **Chapter 3** is the problem that a theory involving illocutionary devices could lead into: psychologism. If we take a judgement as an act of judging, definition as an act of declaration and inferences as transitions between judgements licensed by permissive acts, then the

question is on how to take objective claims in such subjectively given setting. In order to explain this, I will assume the universalist interpretation of Frege's conception of logic. Under this reading, Frege's logic is semantical to the extend that it is a totally interpreted system, one that does not admit a richer metatheory and thus lacks a metalanguage. In his case, contents are explained in terms of senses (Thoughts more precisely), which he assumed in a platonistic ontology. Then, in **Chapter 4**, the objectivity of judgements, inferences and definitions are explained against the psychologistic risk. Judgements are said to be objective in two senses: one contentual, in the sense of expressing truths about an immutable and eternally given system of thoughts; and the other illocutionary, following the two conditions under which the judicative act is performed, *i.e.,* (a) judging the axioms, in which Frege's notion of self-evidence plays a role, and (b) in judging from the rules of inference, in which judgements are derived from other judgements following normatively given rules of transition. Finally, the definitional objectivity follows from Frege's conventions for defining expressions, as definitions play no logical role in the objectivity of inferences and judgements. Frege's semantical theory is nothing but the realization of the scientific role of logic, that explains why the concept-script is a legit language and not simply an uninterpreted calculus.

3. In *Part III*, I follow the previous parts in a twofold objective. The first one is to describe Frege's historical heritage. To say that judgements are a key component in logical practice is not to say something new. In **Chapter 5**, I go back to the historical role played by judgements in the logical practice. The interesting point is that the continuous decline of judgements in logical practice coincides with the shift from a logic-as-practice perspective to a logic-as-object, the latter being the dominant trend today. I call this process as the de-pragmatization of logic. Frege's system is considered still pragmatized system, and as such, in **Chapter 6** I advance a reading of the performatives of the concept-script using Searlean Speech Act theory. This is by far the most novel chapter, but my conclusions are the following. First, that not only we can see how Frege's system of logic is a language in the sense of having speech act components, but we can also see how this is consistent with the place of the concept-script in the logical practice: as a tool for scientific discoveries. Second, if we consider the current logical practice, most of the active components relies on informal metalanguages where logicians and mathematicians actually advance their deductive processes. Recent proposals are attempting to read

mathematical language from the speech act theory, in the hopes of describing much of the performative part of mathematical practice. If this can be extended to the logical language and logical practice, then Frege's case can be well accepted as a precursor of this proposals.

If my exposition is clear enough, this performative based reading of Frege's Logic will have two interesting consequences. First, it will offer a reading of the concept-script logical notation that is historically fair to Frege's time and endeavours, thus not seeing it as another variant of modern predicate calculus. Second, it will shed some light into recent pragmatically oriented perspectives of philosophy of logic and mathematics, where Frege's logic, and the philosophy that goes with it, could be thought as another interesting possibility.

Part I

Frege's Logic and its Syntax

Chapter 1

The Path to the Concept-Script

If Kant had lived through the nineteenth century, he would have seen how wrong he was about logic, precisely on his belief that "logic [...] seems to all appearance to be finished and complete" (KANT, 1998, p.106). In a way, Logic has grown more since Kant than it had until his infelicitous remark. But in another, Kant almost predicted the revolution that was about to come:

> if some moderns have thought to enlarge [logic] by interpolating *psychological* [...] or *metaphysical* [...], or *anthropological* chapters [...] then this proceeds only from their ignorance of the peculiar nature of this science, [...] that exhaustively presents and strictly proves nothing but the formal rules of all thinking. (KANT, 1998, p.106-107)

Kant's words have been listened, and logic became more formal than he ever imagined. In fact, the long road taken to contemporary formal logic saw this science gradually detach itself from psychological, metaphysical and anthropological features, as he suggested.

Logic was, of course, already thought to be formal, as Kant's have proclaimed. Even in Aristotle's work on logic one finds schematic letters. But as it came to be recognized by later logicians, is was not formal *enough*. For example, one could say that traditional logic was formal as it studied arbitrary judgements and arbitrary syllogisms formed from them. Kant said that Logic "[...] has only the difference among judgments in regard to their mere form to take into consideration" (KANT, 1992, p.598). But, this was far too limited by grammatical forms of the traditional syllogistic logic. Even de Morgan, in his 1847's *Formal Logic*, important as it was, still employed the classical forms of logic provided by the Aristotelian forms. He states that "logical truth depends upon the structure of the sentence, and not on the particular matters spoken of" (MORGAN, 1847, p.1). This meant that affirmative sentences such as "All men are mortal" has as structure the formal sentence 'All A is B'. This is far too limited, but it took some time to be recognized. De Morgan (1847, p.2), still stuck with the old subject-predicate structure, offers a curious example, by formalizing the sentence "If he should come tomorrow, he will probably stay till Monday" as "The happening of his arrival tomorrow (A) *is* an event from which it may be inferred as probable that he will stay till Monday (B)". Of course, modal logic was still some years away to be developed, but one has to recognize that such a formalization is too confusing.

This shows that it was not enough for logic to be topic-neutral. A better and more abstract notation was to be found, one that wasn't dependent on the grammar of natural languages. Following this path, a common attack on natural language paved the way for Logic to develop. We could say that logic was through a "denaturalization" phase. An incomplete list of key authors, spanning from the nineteenth to the twentieth century, can be provided:

- Augustus de Morgan's *Formal Logic: or The Calculus of Inference, Necessary and Probable* (1847)
- George Boole's *The Mathematical Analysis of Logic* (1847) and *The Laws of Thought* (1854)
- Charles Peirce's papers on the calculus of relatives (1870-)
- Gottlob Frege's *Begriffsschrift* (1879) and *Grundgesetze der Arithmetik* (1883)
- Giuseppe Peano's *Principles of Arithmetic* (1889)
- Ernst Schröder's *Algebra of Logic* (1890)
- Bertrand Russell and Alfred N. Whitehead's *Principia Mathematica* (1910)
- David Hilbert and Wilhelm Ackermann's *The Principles of Mathematical Logic* (1928)

This list of authors is a partial picture of the long road to a complete symbolic, and formalized, language of logic. In De Morgan's work, one still finds the classical judgement structure, but the work is an attempt to refine it in algebraic ways.

Boole's work presented traditional logic in the symbolic language of arithmetic, therefore turning it into a calculus of propositions. His reasons were not explicitly to overcome natural language, but in his terms, to offer a foundation to the science of logic and the 'fundamental laws of those operations of the mind by which reasoning is performed' (BOOLE, 1951, p.1). This endeavour was set not to deny the classical Aristotelian syllogistic, written in natural language, but to offer a more general method in which the former would constitute a part of. In this way, Boole stated that "[...] syllogism, conversion, &c., are not the ultimate processes of Logic, [...] they are founded upon, and are resolvable into, ulterior and more simple processes which constitute the real elements of method in Logic" (BOOLE, 1951, p.10). First, he noted that words are not the only signs which we are capable of employing in representing things or ideas in a given language, 'Arbitrary marks, are equally of the nature of signs, provided that their representative office is defined and understood' (BOOLE, 1951, p.25). Thus, Boole presented just three types of arbitrary marks: Literal symbols $(x, y, ...)$, which denote things; operational symbols $(+, -, \times)$, which denote operations of the mind, and the sign for identity $(=)$.

Again, Boole's purpose was not to eliminate natural language, but only to find the principles in which it is built, therefore given better expression for the so-called laws of thought. A latter point, made about secondary propositions, serves us well here:

> Let us imagine any known or existing language freed from idioms and divested of superfluity, and let us express in that language any given proposition in a manner the most simple and literal,–the most in accordance with those principles of pure and universal thought upon which all languages are founded, of which all bear the manifestation, but from which all have more or less departed. The transition from such a language to the notation of analysis would consist of no more than the substitution of one set of signs for another, without essential change either of form or character. For the elements, whether things or propositions, among which relation is expressed, we should substitute letters; for the disjunctive conjunction we should write $+$; for the connecting copula or sign of relation, we should write $=$. This analogy I need not pursue. Its reality and completeness will be made more apparent from the study of those forms of expression which will present themselves in subsequent applications of the present theory, viewed in more immediate comparison with that imperfect yet noble instrument of thought–the English language. (BOOLE, 1951, p.174)

In the *Mathematical Analysis of Logic*, Boole's justification for such a better language are given in a quotation from Mill, stating that such notation should provide better means to carry out reasoning in a more mechanical fashion (BOOLE, 1948). Thus, we can conclude that for Boole, adopting a mathematical notation was thought as means to improve and overcome the "imperfect yet noble instrument of thought", *i.e.*, natural language.

This trend was continued and prove itself fruitful. Charles S. Peirce have followed Boole's algebra of logic, vindicating his use against the imprecision of ordinary language. He writes about the "[...] deficiency of pronouns [...] in English [...] whenever there is occasion to discourse concerning relations between more than two objects", and the "[...] imperfection of ordinary language" due to "[...] our feeble marks of punctuation" and the fact that ordinary language is more "[...] pictoral than diagrammatic" (PEIRCE, 1976, p.269). The symbolic language of algebra was, for Peirce, the remedy to such vices of the language. Precisely, by employing the quasi-grammatical algebraic signs such as $+$ and $\times$, one finds "[...] a system of abbreviations for invariable significations [...] so chosen that the different relations [...] find their analogues in the relation between the different parts of the expression" (PEIRCE, 1976, p.270). Finding a more rigorous language for logic was not only helpful in avoiding the ambiguities of ordinary language, but also in amplifying the faculties of human reasoning that are relevant in logical practice. In Boole's work, the algebraic signs of $+$ and $\times$ represent perfectly the mental operations in which we form aggregates.

Later authors, from different traditions, expressed similar justifications. Giuseppe Peano, in the 1915 *The importance of symbols in mathematics*, argued both in favour of the economy of a symbolic language, as for its advantages in the reasoning process:

> It thus results that the ideographic symbols are much less numerous than the words that they can express [...] These symbols are not abbreviations of words, but represent ideas. The principal utility of the symbols of logic, however, is that they make reasoning easier. All those who use logical symbolism attest to this. Symbolism [...] allows the construction of a series of arguments in which the imagination would be entirely unable to sustain itself without the aid of symbols. (PEANO, 1973, p.231-32;233)

The same is found in Russell's and Whitehead's *Principia Mathematica*:

> The adaptation of the rules of the symbolism to the processes of deduction aids the intuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed. [...] Ordinary language yields no such help. (RUSSELL; WHITEHEAD, 1910, p.2).

Russell's and Whitehead's point was that symbolic logic has clear advantages in the process of logical reasoning, in aiding the intuition, or in Peano's terms, helping imagination to grasp complex contents. These kinds of justification were common, but one find them even more developed in the works of Gottlob Frege.

In fact, Frege is praised as being the one who freed logic from language's vile influence. In a way, this is justified: Frege's points on the inadequacy of natural language are more incisive than his predecessors, and his logical notation was more radical in detaching itself from the grammar of the ordinary languages. But it is my goal to show that, albeit such radical position on Frege's side, his logical notation did not detach *entirely* from a linguistic perspective. This is supposed to mean that Frege never looked for a computational, or mechanical, view of a formal language. It is as if we had two Frege to discuss: (1) the one who attacked natural languages for its imprecision and inadequacy and (2) the one who praised the linguistic aspect of a symbolic formal language. These two Frege's are not opposed and, as we shall see, it is a result from contemporary, and therefore anachronistic, readings of his notation that tends to overread the first as *the* historical Frege.

1.1 Frege against natural language

Frege's notation, called *Concept-Script*, was first set forward in 1879 book having the same name, *Begriffsschrift*. It is not, as Boole proposed, an algebra of logic, but as the title declared it, a notation based on the language of arithmetic. But as all others, it was designed to be a departure from old traditional logic. Why does Frege needed to invent a formal language for logic? There reasons are many. Some of Frege's remarks on the topic are the following.

1.1.1 Lacking Precision

> I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography [*Begriffsschrift*]. (*BS*, p.5-6)

Simple examples might illustrate the difficulties Frege was having, since any sentence which is not expressed simply by combining terms with a copula as 'A is B' has to be forced into it, if is to be considered at all. The example taken from de Morgan's above might fit.

1.1.2 Ambiguity of Terms

> Language [...] does not even meet the first requirement [...] namely, being unambiguous. (*WBB*, p.84)

Here we can identify two situations in which terms in the ordinary language are ambiguous. First, terms like 'horse', 'man', and so on can represent different things: (a) a concept, as in 'a man', or (b) an individual, as in 'the man'. Likewise, logical particles as the verb 'is' can represent different things: (a) the copula, which affirms the connection between two terms, usually a subject term with a concept term, as in 'A is B', or (b) an identity between object terms, as in 'The morning star is Venus'[1].

1.1.3 Departure from S-P

> [...] logic has hitherto always followed ordinary language and grammar too closely. In particular, I believe that the replacement of the concepts *subject* and *predicate* by *argument* and *function*, respectively, will stand the test of time. (*BS*,p.7)

> In the first draft of my formula language I allowed myself to be misled by the example of ordinary language into constructing judgments out of subject and predicate. But I soon became convinced that this was an obstacle to my specific goal and led only to useless prolixity. (*BS*, p.13).

This transition from the classical structure of judgements to the functional structure is one of Frege's most important contributions to logic. It became a paramount example to the departure from natural languages for formal ones. The use of the classic structure is directly connected with the lack of precision of such languages, as showed in point (1) above. Relational sentences are difficult to come by, as we are obligated in choosing one of the relatas as the subject, and the relation applied to the other as the predicate: 'John loves Mary' has to be read as 'John is a person who is in love with Mary', for example. The other problem is that not every argument is valid by simply applying such translation. Frege was dealing with arithmetical cases. In this case, '$x > y$' has to be rendered as predicating 'being greater than y' to x. Thus, the transitivity of the relation $>$, in which we correctly assert that 'If $x > y$ and $y > z$ then $x > z$' cannot be translated by transforming each sentence in subject-predicate form, without loosing simple facts about the relation. Define Y as 'being greater than y' and Z as 'being greater than z'. In this case, the translation yields the following argument: 'x is Y, y is Z, therefore, x is Z', which clearly is invalid under syllogistic rules of validity. One would need to do violence to the former expressions in order to validate it in the traditional logic.

[1] I here take Frege's example in *Concept and Object*, in (*CP*,p.184). The ambiguity of the copula 'is' is better exposed there, but it is presupposed here as well.

1.1.4 Lack of validity in Logical reasoning

> Language is not governed by logical laws in such a way that mere adherence to grammar would guarantee the formal correctness of thought process. The forms in which inference is expressed are so varied, so loose and vague, that presuppositions can easily slip in unnoticed and then be overlooked when necessary conditions for the conclusion are enumerated. In this way, the conclusion obtains a greater generality than it justifiably deserves. (*WBB*, p.84)

> A strictly defined group of modes of inference is simply not present in language, so that on the basis of linguistic form we cannot distinguish between a "gapless" advance and an omission of connecting links. [...] In language, logical relations are almost always only hinted at - left to guessing, not actually expressed. (*WBB*, p.85)

In the above case, we saw how the traditional structure of judgements cannot yield, necessarily, valid arguments for relational sentences without forcing unwieldy translations on them. If one desires to transform such argument as valid, another presupposition must be provided, that is, that 'All Y's are Z's', which we could read, following the translation suggested above, as 'All numbers that are greater than y are greater than z'. The problem with natural language's structure is that such hidden premises are not always explicit, and the validity of logical reasoning would be in jeopardy: one would conclude more than what he asserted in the premises.

1.1.5 Mixture of Psychological and Logical features

> [...] grammar, which has for speech a significance analogous to that which logic has for thought, is a mixture of the logical and the psychological (*Logik*, in (PW,p.6))

> [...] it is the business of the logician to conduct an unceasing struggle against psychology and those parts of language and grammar which fail to give untrammelled expression to what is logical. (*Logik*, in (PW,p.6))

> [...] we think in some language or other and that grammar, which has a significance for language analogous to that which logic has for judgement, is a mixture of the logical and the psychological. If this were not so, all languages would necessarily have the same grammar. It is true that we can express the same thought in different languages; but the psychological trappings, the clothing of the thought, will often be different. (1897's *Logik*, in (PW,p.142)).

Frege's point here is that natural written languages have a logical structure hidden in a psychological exposition. In a way, this is what Boole's formal language also aims at by employing algebraic operations for representing mental processes. Of course, natural languages have their own merits: it is adaptable and applicable to a large range of cases. But here in Frege's case, it is the logical structure that matters for scientific purposes, and we find a simple argument for it: If our (natural) language were purely logical, there would be only one grammar, the logical one. Since we know by fact that there are multiple grammars, we know that language is not purely logical. But different grammars still share something in common: a logical structure. Finding it, and given a proper expression for it, is the task of the concept-script, one that surely presupposes a sharp distinction between the logical and psychological.

1.1.6 Implicit Assertoric Force

> It seems to me that thought and judgement have not hitherto been adequately distinguished. Perhaps language is misleading. For we have no particular bit of assertoric sentences which corresponds to assertion; that something is being asserted is implicit rather in the assertoric form. (*DG*, p.356n)

A sentence could be uttered in different contexts with different results. Frege states that "When we utter an assertoric sentence, we do not always utter it with assertoric force" (*PW*,p.233). This is because in language, the assertoric force is implicit, to be read out of the context and the speakers intensions. Thus, a sentence uttered by someone speaking seriously would aim at the True, while the same sentence uttered by an actor in a stage would not. It was rather important to have the assertoric force made explicit in the formal language of the concept-script, for Frege. This called for a upgrade on such linguistic device.

More points could be mentioned. But Frege's quarrel against natural language was not simply a quest for eliminating language entirely, but a quest for improving it. It also does not meant for him that the formal language to be presented was a completely computable activity in which deductions were carried over automatically. It was also not a language which we can manipulate from a privileged perspective. Frege defended that Logic is still a language, capable of having proper content and in need for human manipulation. It is precisely because of such assumption that his concept-script was thought to be a helpful tool, one that at the same time should avoid human error or psychological influences in the results achieved.

1.2 Logic as Language

Notwithstanding Frege's criticism on the imperfections of language, he still designed his concept-script as one. First, he recognized the necessity of linguistic means to proceed in logic, and offered ways to achieve logical goal while keeping such inescapable linguistic presupposition. Second, his system of logic is a language in the sense of having the expression of content from the outset.

1.2.1 The Necessity of Language

The first point to be made is that language is the necessary means for thinking, as long as humans are concerned. This implies that if one is to devise a formal language for logical inquiries, it must be designed in conformity to such principle. We have the following quotations of Frege to support this view:

> If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts [...] then my ideography, further developed for these purposes, can become a useful tool for the philosopher. To be sure, it too will fail to reproduce ideas [*Gedanken*] in a pure form [*rein*], and this is probably inevitable when ideas are represented by concrete means. (*BS*,p.7)

Not exactly Frege's own notation, but logic did became a strong tool for philosophers. But there are a few presuppositions here. First, that language is the way by which we express the contents of our mind in a external, *viz.* empirical way. Second, even the concept-script language cannot be precise enough to express ideas, but is helpful enough to avoid a good number of errors. Either way, symbols are that part of language in which we necessarily communicate:

> [...] let no one despise symbols! A great deal depends upon choosing them properly. And their value is not diminished by the fact that, after long practice, we need no longer produce symbols, we need no longer speak out loud in order to think; for we think in words nevertheless, and if not in words, then in mathematical or other symbols.(*WBB*, p.84)

Here we find Frege expanding the previous idea not only to natural written and spoken languages, but to any language that is composed of symbols. For in the natural language, we often think through words, and in mathematical language, through mathematical symbols. From this, one must consider that the concept-script should provide the same: a language to guide thinking, and to think through it. He then continues:

> Also, without symbols we would scarcely lift ourselves to conceptual thinking. Thus, in applying the same symbol to different but similar things, we actually no longer symbolize the individual thing, but rather what have in common: the concept. This concept is first gained by symbolizing it; for since it is, in itself, imperceptible, it requires a perceptible representative in order to appear to us. This does not exhaust the merits of symbols; but it may suffice to demonstrate their indispensability. (*WBB*, p.84)

This is virtually the same idea expressed in the previous quotation: it is through the perceptible symbols of the language that we are able to express, and communicate, abstract things such as concepts. Since we think with concepts, language is essential for thinking. This position is held in two very different points in Frege's career: in the earlier texts quoted above, and in the late 1925 unpublished paper *Erkenntnisquellen der Mathematik und der mathematischen Naturwissenschaften*[2]. There we find the following statements:

> [...] our thinking is closely bound up with language and thereby with the world of the senses. Perhaps our thinking is at first a form of speaking which then becomes an imaging of speech. Silent thinking would in that case be speech which has become noiseless, taking place in the imagination. Now we may of course also think in mathematical signs; yet even then thinking is tied up with what is perceptible to the senses. (*PW*,p.269)

> [...] that a thought of which we are conscious is connected in our mind with some sentence or other is for us men necessary. But that does not lie in the nature of the thought but in our own nature. (*PW*,p.269)

Thus Frege, in his last days, still argued in favour of the indispensability of language for thinking. But now, this is due to our human nature of requiring such medium, not about thoughts themselves, which are something Frege had already defined as abstracts from a third realm. One can find Frege's position about the necessity of language as derived from Kant's classical statement that opens the first *Critique*, that "[...] although all our cognition commences *with* experience, yet it does not on that account all arise *from* experience" (KANT, 1992, B1). Of course, this dictum have traveled a long way to be assumed by Frege in terms of a language.

[2]Frege submitted the paper to the *Wissenschaftliche Grundfragen*, but was asked to enlarge before publication. Frege died three months latter, unable to finish the work.

1.2.2 Language's advantages

Following this principle, Frege thought his formal notation as another language, one that avoided the common problems discussed before. And for that reason, a number of readability principles was assumed in the making of such language. This shows that the concept-script, albeit formal, was still designed for a reasoning agent. It not only avoids the problems of natural written language, but picked up from it's own advantages for the reasoning process. We found such worries in the following quotes:

> definiteness of written symbols will tend to make what is signified also more sharply defined: and just such an effect upon ideas must be asked for the rigour of deduction. This can be achieved however, only if the symbol directly denotes the thing [symbolized] (*WBB*, p.87)

One of the first advantages of the written language of symbols, in comparison with oral languages, is that visual symbols are easily distinguishable from each other. They also have some permanence of reference, which helps out the reasoning process. A formal language should exploit this.

> The spatial relations of written symbols on a two-dimensional writing surface can be employed in far more diverse ways to express inner relationships than [...] in one-dimensional [...] In fact, simple sequential ordering in no way corresponds to the diversity of logical relations through which thoughts are interconnected. (*WBB*, p.87)

This is an important and distinguishable point in Frege's motivation, in comparison to other authors before and after. The visuality of the notation should emulate the logical connections that take place in reasoning, therefore facilitating deductive reasoning by sharply separating contents in their logical relations. Frege also remarks how "the disadvantage of the waste of space [...] is converted into the advantages of perspicuity [...] by allowing the assertible contents to follow one below the other" (*Über den Zweck der Begriffsschrift* in (FREGE, 1972, p.97)). Taking advantage of such auxiliary role of a written language, Frege advises that a proper formal language for logic:

> [...] must have simple modes of expression for the logical relations which, limited to the necessary, can be easily and surely mastered. These forms must be suitable for combining most intimately with a content. Also, such brevity must be sought that the two-dimensionability of the writing surface can be exploited for the sake of perspicuity. (*WBB*, p.88)

Following the preceding principle that language is a necessary means for thinking, here we found Frege exploiting language's advantages in order to facilitate reasoning process. A formal symbolic language should not simply express contents in a precise and definite way, it should also be a tool for carrying over such contents into deductions. This was Frege's motives in the concept-script. Within it, we see that simplicity is required, but not to the point of overcoming readability.

1.2.3 A Lingua Characteristica

In defending his formal system of logic as a language to its own right, Frege used the Leibnizian distinction between a *Lingua Characteristica*, that is, an universal language, and a *Calculus Ratiotinator*, that is, a formal system for carrying calculations. Frege's name for his system of logic, a *Begriffsschrift*, is borrowed from Trendelenburg's paper on the subject that Frege quotes in the preface to the book of the same name. Frege saw himself as Leibniz' follower in making a step towards this goal, as he envisage his concept-script as a language for the advance of science in general. In the 1882 paper *Über den Zweck der Begriffsschrift*, in comparing his concept-script with Boole's logic, Frege writes:

> I did not wish to present an abstract logic in formulas, but to express a content through written symbols in a more precise and perspicuous way than is possible with words. In fact, I wished to produce, not a mere calculus ratiocinator, but a lingua characteristica in the Leibnizian sense. (FREGE, 1972, p.100-1)

The emphasis on content appear also in other places. This paper and the unpublished *Booles rechnende Logik und die Begriffsschrift* are responses to Schröder's review of the *Begriffsschrift*, where Frege's notation was accused of not being a true universal characteristics, and a variation of Boole's already established calculus of judgements (FREGE, 1972, p.221). Responding to it, Frege writes in the mentioned unpublished paper that "I had in mind the expression of a content. What I am striving after is a lingua characterica in the first instance for mathematics, not a calculus restricted to pure logic." (*PW*, p.13). The easier way to read this is that Frege has his logicist goal in mind. For this goal, if logic is simply an uninterpreted calculus of pure logic, then arithmetic is, at best, a formal theory devoided of content. "A lingua characterica ought, as Leibniz says, *peìndre non pas les paroles, maìs les pensées*" (*PW*, p.13), as Frege emphasizes. He also claimed that "Boole's symbolic logic only represents the formal part of a language" (*ibidem*), that is, a *calculus ratiotinator*.

The standard reading for this passages, and the differences between Frege and Boole, comes from Van Heijenoort's seminal paper *Logic as Calculus and Logic as Language* (1967b). For Van Heijenoort, Boole's system of logic represents only the propositional calculus fragment of standard systems of logic of today, while Frege's system is the rightful creator of quantification theory and predicative logic. This is how he see's Frege: predicative logic is what is needed for understanding content beyond unanalysed propositions, as Frege claimed in setting up a lingua characteristica. This interpretation has been contested[3], but another of Van Heijenoort's distinction has been largely adopted: the universalist conception of logic.

Under the universalist conception, logic is not subject to reinterpretations for its variables and quantifiers. The comparison is drawn over Boole's universe class and De Morgan's universe of discourse, which are said to have no ontological import: they can be changed at will. Frege's system, on the other hand, has a fixed domain for interpreting its variables and quantifiers. Van Heijenoort's conclusion is that the Fregean-Boolean divide is representative of two different strands in logic: the universalist conceptions of Frege and Russell, and the model-theoretic conception, or algebraic conception, that of Boole, De Morgan and Schröder's algebra of logic that developed into the model-theoteric perspective of Löwenheim and others.

The universalist conception is the standard interpretation for Frege's claims about a lingua characteristica. He constantly highlights the capabilities of the concept-script in expressing contents thorough a system of signs, and in a sense, the universal conception does exactly that. Once again comparing the concept-script to Booles, he writes in *Über die Begriffsschrift des Herrn Peano und meine eigene* that "Boole's logic is logic and nothing more. It deals solely with logical form, and not at all with the injecting of a content into this form" (*CP*, p.242). But the concept-script is contentful in the sense of being a generally interpreted system of signs. A way to understand this difference, still through the universalist lens, is that for Frege, logic is general in the sense of being about everything. For the model-theoretic perspective that developed from the algebrists, being general is about being nothing at all. This is very well summarized by Goldfarb (2010, p.68):

> On Frege's universalist conception [...] the concern of logic is the articulation and proof of logical laws, which are universal truths. Since they are universal, they are applicable to any subject matter, as application is carried out by instantiation. For Frege, the laws of logic are general, not in being about nothing in particu-

[3]See Peckhaus (2004).

> lar (about forms), but in using topic-universal vocabulary to state truths about everything.

Another important aspect of Frege's conception of logic as a lingua characteristica is how he saw his concept-script system as representing a genuine scientific tool. First and foremost, the motivation was the reduction of arithmetic into logic, already stated in in 1879[4]. Second, Frege's favorite examples for expressing the laws of logic derived in section II of the *Begriffsschrift* is usually imported from other sciences, which speak for their application in other fields. This, of course, was already stated in the preface to that book. In the justifying the concept-script as a continuation of Leibniz ideals, he claims the following:

> It is possible to view the signs of arithmetic, geometry, and chemistry as realizations, for specific fields, of Leibniz's idea. The ideography proposed here adds a new one to these fields, indeed the central one, which borders on all the others. If we take our departure from there, we can with the greatest expectation of success proceed to fill the gaps in the existing formula languages, connect their hitherto separated fields into a single domain, and extend this domain to include fields that up to now have lacked such a language. (*BS*, p.7)

There is a clear defense that the concept-script is suitable for any scientific field, provided the necessary additions to the symbolism are added. Geometry would be Frege's next choice, as it would only demand a few additions in the symbolism. Next, as he expected, mechanics and physics would follow. The gain that logic have in advancing a language that goes beyond ordinary language is also clear in the scientific practice. And for this end, Frege thought that the concept-script would be a tool for making progress in any field, as science in general proceeds dedutively[5]. The paper *Über die wissenschaftliche Berechtigung einer Begriffsschrift* (FREGE, 1972, pp.83-89), the whole point is to argue the gains of the concept-script over ordinary language. He concludes by taking the concept-script as a true instrument for scientific progress. As an instrument of progress, logic must be epistemically driven, which sets Frege in the opposite direction of his predecessors, like Kant, who saw logic as a formal canon of reason, and not a science by

[4] "[...] when I came to consider the question to which of these two kinds the judgments of arithmetic belong, I first had to ascertain how far one could proceed in arithmetic by means of inferences alone [...] I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography." (*BS*, p.5-6)

[5] Contrary to Van Heijenoort's famous interpretation, Korte (2010) argue that this is what Frege meant by saying that his logic is a lingua characteristica: a scientific tool modeled upon aristotelian model of science.

its own right. This includes not only the possibilities of applying the concept-script to other sciences, but also the production of fruitful knowledge by its own. Frege's logicism attempted to do just that. Once the concept-script formalism is presented, I will come back to this topic later, in section 2.3.

1.2.4 The Presence of Performatives

By sketching Frege's praise on the linguistic side of logic we now come to another important feature. We can roughly distinguish two important parts of a language: one descriptive, or referential, and other performative, or pragmatical. Logical systems, as they are defined nowadays, are usually constructed with a syntax and a semantics, and a great deal of the logical work is to find out the relation between them. This includes a presupposition that all terms of the syntax (variables, constants, formulas, sentences) have referential behavior, provided by the interpretation assumed. Constants denote an individual of the chosen domain, variables range over individuals, formulas and sentences describes facts, propositions, relations, etc. There are no pragmatic considerations in logical theories today. They are mainly *descriptive*.

In contrast, a careful reading of Frege's Logic shows that he was not a logician like we now take one to be. Of course, Logic was still referential to a great deal, but by taking his logic as a language to be *used* demanded some non-denotative care. How do we fix the truth-value of a sentence? How can we manipulate the language to introduce new linguistic entities? When are we allowed to form new sentences out of old ones? These demands require some pragmatic conditions on the language. They are about what we *do* with the language, not about what we convey by using it.

In another way, the presence of such performative conditions for the manipulation of the language follows the fact that we often use them for reasoning. It is the task of the formal language do lay down laws in which reasoning is performed. As Frege states:

> In one sense [a law] says what is, in the other it prescribes what ought to be. Only in the latter sense can the logical laws be called laws of thought, in so far as they legislate how one ought to think. Every law stating what is the case can be conceived as prescriptive, one should think in accordance with it, and in that sense it is accordingly a law of thought. (*GGA*, p.XV)

Recall that, as he supposed, language has a logical kernel, and capturing this kernel motivates the de-psychologization of natural languages. It is not that our language constantly guides us to error, if it did, communication itself would be infeasible: we would never speak of the same things, or arrive at the same conclusions. It is always possible to take something as true when in fact it isn't. This is the risk of taking psychological influences in thinking:

> Error and superstition have causes just as much as correct cognition. Whether what you take for true is false or true, your so taking it comes about in accordance with psychological laws. [...] But may not logical laws also have played a part in this mental process? I do not want to dispute this, but if it is a question of truth this possibility is not enough. For it is also possible that something non-logical played a part in the process and made it swerve from the truth. We can decide only after we have come to know the laws of truth; (*DG*, p.351-2)

The problem is that the psychological influence on language and our thinking might guide us to truth, but only in a accidental way. It would never *justifiably* reach the truth, or guide reason properly. This point was also made in the early unpublished paper *Logik*:

> The causes which merely give rise to acts of judgement do so in accordance with psychological laws; they are just as capable of leading to error as of leading to truth; they have no inherent relation to truth whatsoever; they know nothing of the opposition of true and false. (*Logik* in (*PW*,p.2))

The difference between normative and descriptive side of laws of logic correlates with logical and psychological laws which guides our reasoning. Laws of logic are normative. The laws of psychology, only descriptive. Of course, we could read logical laws as stating how things logically are, but this is misleading. They are norms from where "[...] follow prescriptions about asserting, thinking, judging, inferring" (*CP*,p.351).

This sharp distinction between logic and psychology puts some constraints in the connection between such acts of judging, for example, and the things we come to judge. First, we saw that since logic is undeniably a language, it should be rendered feasible for the practice, it should be made psychological friendly, since it is judging agents who actually judge. Second, we cannot be dependent of such tools in justifying our judgements and inferences: their truth cannot be a matter of our taking them as truth, as Frege constantly warned. Nonetheless, since we need such facilitating psychological devices, language can be thought as a ladder, ready to be dropped as soon as we reach its goal, the truth - to use Wittgenstein's metaphor.

Summing up, if the laws of logic are norms that guide our judging, asserting and inferring, *i.e.*, acts that are constitutive of the logical practice, it follows that a formal language such as the concept-script should be able to express those acts and track down their proper logical conditions. In this sense, Frege's logic cannot be properly understood without taking it as an act-based notation as well. Thus, in fixing up Frege's system, one has to keep in mind the different class of signs that the concept-script have.

1.3 The 1879 Version

Frege published two versions of his logical notation, called concept-script. The first in 1879 as a small booklet entitled *Begriffschrift*, and the other supplemented and altered version in 1893's logicist manifesto *Grundgesetze der Arithmetik*. In modern terms, both systems are second-order logics. Its influence was decisive in the history of logic, as we shall see, but the focus of the presentation here taken will be on those elements that has *not* survived the scrutiny of later logicians. In so doing, we shall keep in mind the points above taken: that which we shall overcome about natural languages, and that which we shall keep and improve it. The exposition is organized as follows. We distinguish two classes of signs, according to their respective roles in the formal language of the concept-script: descriptive and performative. In the remainder of this chapter, the 1879 version of the Concept-Script will be presented. The 1893 version and the performative part of the concept-script will be presented in the next chapter.

As Frege declares in the beginning, two kinds of symbols are used: "I [...] divide all the symbols I employ into *those which one can take to signify various things* and *those which have a complete fixed sense*. The first are the letters, and these are to serve mainly for the expression of generality" (*BS*, §1). We thus divide symbols in two classes: the primitive lines called 'strokes', each defining specific logical operators, or relations, and variables of different kinds. The former are the following (separated by commas):

$$— \;,\; {}_{\top} \;,\; \Big| \;,\; \smile \;,\; \equiv \;,\; | \;,\; \|$$

These are, respectively, called the content, negation, conditional, concavity (for generality), identity, judgement and double definitional strokes. They could be said to define a signature in which all other well-formed formulas would be derived, but Frege's strokes are combined with each other in a different way. With these, Frege adds Roman and German letters to express contents. First, he explicitly abandon the subject-copula-predicate structure of judgements, by importing functional symbols from mathematics. Thus, the following letters are added:

1. Lower case Roman letters $a, b, c, ...x, y, z, ...$ for argument variables;

2. Lower case Roman letters $f^n, g^n, h^n, ...$ as n-ary functional variables. In practice, capital F is used for unary function, or concept, and f for binary functions, or relation.

3. Lower case German letters $\mathfrak{a}$,$\mathfrak{b}$,$\mathfrak{c}$... for bounded individual variables;

4. Upper case German letters $\mathfrak{F}^n$, $\mathfrak{G}^n$, $\mathfrak{H}^n$ for bounded functional variables.

This basic distinction between argument and functional variables seems odd for modern eyes. In fact, the role taken by the variables of the type $a, b, c...x, y, z...$ in the concept-script suggest that they can assume any value whatsoever, being that an individual, or what we may call a full proposition. Such behavior suggest also that Frege was already following his metaphysical distinction between saturated and unsaturated entities, since the variable distinctions are only assumed in the level of complete conceptual contents and those incomplete ones (functions). Frege also uses capital Greek letters as expositionary devices, such as $A, B, X, \Lambda, ...$ for propositions and Φ, Ψ for functions. But in his case, they are not part of the language, since they do not have contents assigned to it. Next we discuss the formation rules for the strokes and variables.

1.3.1 Content-Stroke and Assertible Contents

Before presenting which expressions of the concept-script are terms and atomic formulas, we should discuss first his notion of a judgeable, or assertible, content. First, all expressions constructed in the language have a common goal: judgements. Thus, the only contents that matter are those which can be judged as true, or as being ‘a fact’. But recall that Frege abandoned the subject-copula-predicate structure for functions and arguments. The copula’s usual assertoric role is then recaptured in the concept-script language. This was made possible also by separating the *act* from the content of a judgement. We are still to properly address the former, but the latter is capture by the *Content-Stroke*:

$$— A$$

The content-stroke have a double role: First, it denotes the content of what follows to the right of it, it "ties the symbols which follow it into a whole" (*BS*,*S*2). In this case, one reads it as "the content of A". Frege's exact readings are 'the circumstance that A' or 'the proposition that A' (*BS*,§2). Second, the content-stroke is used for distinguishing those contents that *can* be judged, from those that can't: 'Whatever follows the content stroke must always have an assertible content' (*BS*,§2). There are at least four situations where a content must be labeled as unassertible.

1. Functions without arguments: Frege states that "the idea 'house' cannot" become a judgement in (*BS*, §2). Here, instead of simply an idea, 'house' denotes a concept, *i.e.* an unary function. Thus, a single occurrence would be '$h(\quad)$'. Such cannot be judged as being the case.

2. Argument variables with individuals for value: If a variable a is to denote an individual, it cannot render a judgeable content. It make no sense in speaking about 'the circumstance that John', for example.

3. Functions denoting concepts without sharp boundaries, that is, vague concepts: In proving theorem 81 in (*BS*, §27), which is a general principle for induction in second-order logic, Frege declared how the concept 'heap' is not defined for every argument x, since there can be some x for which $heap(x)$ is undetermined. In this case, it is not even assertible.

4. Functions that when saturated yields value for individuals: Frege does not mention explicitly this case, but it's a consequence of the second condition above. For example, let $f(x) = x + 1$. In this case, $f(x)$ is not assertible for values of x.

Cases (2) and (4) define terms. In fact, isolated terms are not assertible in Frege's sense. We could then define them just as those two cases:

Definition 1. *(Terms)*

- *For every argument variable x, if x has an individual as value, then x is a term;*
- *For every function variable f and argument variable x, if $f(x)$ is a term, then $f(x)$ is a term.*

Since Frege uses his variables as ranging over both individuals and propositions, we are left with no choice but to use the notion of assertible content to distinguish the cases which are terms from those who aren't.

Finally, a function f applied to an argument variable x which is a term (in the above sense) can still yield an assertible content, just in case f describes a concept. Thus, we can still define an atomic formula:

Definition 2. *(Atomic Formulas)*

A is an atomic formula if A is some assertible $f(x)$, for some term x. That is,

$$— f(x).$$

Thus, in this reading, fregean conceptual-contents are just those contents that are also judgeable-contents. In the *Begriffsschrift* version of the concept-script, functions are not total just as in *Grundgesetze*. The content-stroke is not applicable to proper names: only to well-formed formulas. This is differently read by Bell (1979). According to his reading, '$\vdash A$' stands for the judgement of A, '— A' stands for the thought that A or the "combination of ideas that A", or the proposition that A, and, finally, 'A' itself represents a conceptual-content. But Frege is never clear about what conceptual-contents are. The usual quoted passage about the subject is the following:

> I remark that the contents of two judgments may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgments, always follow also from the second, when it is combined with these same judgments, or this is not the case. The two propositions 'The Greeks defeated the Persians at Plataea' and 'The Persians were defeated by the Greeks at Plataea' differ in the first way. Even if one can detect a slight difference in meaning, the agreement outweighs it. Now I call that part of the content that is the same in both the conceptual content. (*BS*, §3)

The first, and obvious conclusion of this passage, is that conceptual-contents are not reducible to sentences. The usual terminology for this entities, which are expressed by sentences but not reducible to them, are propositions. The idea is that both sentences, 'The Greeks defeated the Persians at Plataea' and 'The Persians were defeated by the Greeks at Plataea', are to express the same proposition. The second information is that conceptual-contents are that which determines a proposition's possible consequences. Frege's claims are that for two conceptual-contents α and β, $\alpha = \beta$ if, and only if, $\alpha \vdash \Delta \Leftrightarrow \beta \vdash \Delta$ for any Δ. Given that nothing follows from unassertible contents (as specified above by cases 1-4), we must conclude that Frege is also speaking about judgeable contents. Thus, both examples are also obvious cases of judgeable contents[6].

[6] Can an unsaturated concept, such as the example "House" mentioned by Frege, be considered a

If this two expressions are to be expressed in the concept-script language, one way is to choose a ternary relation "x defeated y at z", with "g" as denoting the Greeks, "p" as denoting the Persians, and "q" denoting Plaetea, to yield $— D(g, p, q)$. In contrast, Bell states that "What is that the two sentences express can, without equivocation, be represented by the single sign Δ" (BELL, 1979, p.86). Bell reads it this way because, for him, the content-stroke is a nominalizing operator, one that takes a conceptual-content and transform it in a proposition in the nominalized form "that Δ". But the content-stroke does more than that: it is a marker of well-formed formulas.

From the logical point-of-view, variables range over all kinds of entities, and Frege does use variables as ranging over both objects and propositions. The lower-case Roman letter a does exactly that. If an expression Δ without the content-stroke is already a judgeable content, as Bell takes them to be, Frege could express the other logical connectives directly. But, if Frege had assumed something as $a \rightarrow b$, with a and b ranging over objects, he would include instances such as $1 \rightarrow 2$ in the concept-script. But cases like this do not express a proposition. In order to circumvent such cases, the content-stroke is needed: $— a \rightarrow — b$.

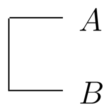
The content-stroke functions as a way to restrict the judgement-stroke to cases that can in fact be judged, and recursively defining the other logical operators in terms of the content-stroke does just that. It restricts the range of the variables in order to prevent ill-formed cases. While a ranges over all objects and propositions, $— a$ is the restriction of a to judgeable cases. To be sure, the sign is in fact read as a nominalization, as a way to express a proposition striped from its assertoric force, and nominalized propositions are judgeable in the same way. But contents without the content-stroke are not necessarily assertible, and to safekeeping well-formedness in the language, the sign is needed.

1.3.2 Conditional

Now that we define atomic formulas in terms of those assertible functions applicable to terms, we can define conditionality. First, if A and B are assertible contents, that is, if both $— A$ and $— B$, then their implication is assertible. First, we connect both

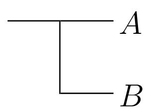
conceptual content, albeit unjudgeable? I take that it can't. When using this example, Frege says that not every *content* [*Inhalt*] can be judgeable. But he never speaks of a *conceptual-content* [*Begrifflichen inhalt*] being unjudgeable.

with a vertical line to represent their implication:



Which allows us to apply the content-stroke to it, as Frege defines it in (*BS*,§5). In a way, this is the same as the following:

Definition 3. *(Conditional) If A and B are formulas, that is, assertible contents, then*

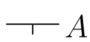


called the implication of B to A, is also a formula, viz. *an assertible content.*

1.3.3 Negation

The same holds for the negation stroke, which Frege defines as a small vertical stroke to the left of the content being denied. Following any formula, which is itself assertible, we can attach to it the negation sign, and the content-stroke for this sign itself. The principle in hand is: the negation of a assertible content is itself assertible. Or, equivalently:

Definition 4. *(Negation) If A is a formula, that is, an assertible content, then*



called the negation of A, is a formula, or an assertible content.

1.3.4 Concavity for Generality

We already saw that every Roman letter denotes generality for the contents being conveyed. In these cases, the scope of the generality range over all the formula: “An Italic letter is always to have as its scope the content of the whole judgement” (*BS*,§11). At this point, Frege’s impact on the history of logic is recognized, since he realized that with such devices one is not able to distinguish between the negation of a generality from the generality of a negation, being both expressed by ‘—┬—A’. For such, the scope of the generality must be delimited: "An italic letter may always be replaced by a German letter which does not yet occur in the judgement" (*idem*). That is, from $\boldsymbol{-\!\!-}\, f(x)$ we can derive

$$-\!\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a}).$$

The reading suggested for such new expression is that "[...] the function is a fact whatever we may take as its argument" (*BS*,§11). Some restrictions are imposed for applying the concavity:

1. Whatever follows the concavity must be assertible, that is, the generality is applicable to any formula;

2. The chosen German letter must not already occur in the content where it is to be substituted;

The interesting cases, as one should expect, are the following:

1. The negation of a generalization: $\top\!\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a})$.

2. The generalization of a negation: $-\!\overset{\mathfrak{a}}{\smile}\!\top\, f(\mathfrak{a})$.

3. And the existential quantifier, derived from both: $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, f(\mathfrak{a})$.

We can now define the rule for applying the concavity:

Definition 5. *(Concavity)*
If $f(x)$ is a formula, that is, an assertible content, the resulting substitution of x with the German letter $\mathfrak{a}$, provided that $\mathfrak{a}$ does not occur already in $f(x)$, with the corresponding concavity in the content-stroke of $f(x)$ with $\mathfrak{a}$, is itself assertible. That is, the resulting expression

$$-\!\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a})$$

is a formula.

1.3.5 Identity of Contents

Finally, Frege introduced a very problematic sign, named the Identity of Contents: $\equiv$. It is problematic because, as he declared, it is defined over the symbols of the language, not to the contents expressed by those symbols. Thus, it does not require content-strokes for the expressions flanking the sign. We can have terms flanking the identity sign, for example. Avoiding further discussions, we can give a simple definition:

Definition 6. *(Identity of Contents) For any A and B terms or formulas, the expression*

$$\text{——}(A \equiv B)$$

is a formula, that is, an assertible content.

The reading is quite odd: "the symbol A and the symbol B have the same conceptual content, so that we can always replace A by B and vice-versa" (*BS*,§8). Given that the identity of contents sign is introducing a bifurcation between use and mention, one would expect Frege to use quotation marks instead. But he is in fact aware of this: "[...] the introduction of a sign for identity of content necessarily produces a bifurcation in the meaning of all signs: they stand at times for their content, at times for themselves" (*BS*, §8). But the adoption of the sign also gets some odd readings. A formula such as '$\text{——}(a \equiv a)$' would be properly read as 'The symbol 'a' has the same content of itself', and not the expected 'a is identical to itself'. We ought not to get into the troubles of such sign. Frege himself abandoned it latter.

If Frege would had a proper syntactical description of its primitive symbols, it would be something very close to what we showed above. Of course, not all symbols were covered up. Earlier, I mentioned how modern logical notation are descriptive by nature: its signs are referential in character. The signs of the concept-script just defined fit this description. In fact, everything follows from the expressive, or denotative, role of the content-stroke, since every other logical operator is defined in terms of the content-stroke (except for the identity of conceptual contents sign, of course). Thus:

- The content-stroke express, or denotes, the content of the symbols that follows immediately after it;
- The conditional-stroke express, or denotes, the material implication between two contents;
- The negation-stroke express, or denotes, the negation of a content;
- The concavity for generality express, or denotes, the generality of a content;
- The identity of contents express the identity between two expressions;

Up to this point, the content-stroke behaves like the usual notion of a well-formed formula, defined metatheoretically, and the operators defined in terms of the content-stroke as those formulas generated recursively from those wff's. But we would mistakenly conclude that Frege's treatment is metatheoretical, as it is not.

Frege's first system of logic is clearly an universal system, following Van Heijenoort's interpretation. And, as described earlier, the expression and calculations of contents are the basic goal of the concept-script, as it is summarized in conceptual and judgeable-contents and the recursively defined expressions in terms of the content-stroke. But the notion of a content-stroke is dependent upon the possibility of it being judged. Frege here follows the tradition: judgements are the cornerstone of logic. He puts a weight on the notion of a judgement, and there are many reasons for him to take it to be important. But judgements have a different role from these descriptive signs, and so we have to threat them differently. But before discussing the normative signs of the language, to which judgements are included, I shall first describe Frege's major changes in the concept-script in the 1890's. Both are matters for the next chapter.

Chapter 2

The Evolution of the Concept-Script

In the 1890's Frege made significant changes in his formalism. He published three papers discussing this changes, *Funktion und Begriff* in 1891, *Über Sinn und Bedeutung* and *Über Begriff und Gegenstand* in 1892. They mostly emphasize two distinctions: the ontological distinction between functions and objects, and the semantical distinction between sense [*Sinn*] and reference [*Bedeutung*][1]. The latter distinction is mostly absent in the *Grundgesetze*, as Frege is mostly dealing with the *Bedeutungs* of the expressions. But the former is a basic presupposition of the formalism, and mostly every symbol employed are either names for objects or names for functions.

The object-function distinction is considered by Frege a fundamental distinction. Functions are essentially incomplete entities, something in need of a supplementation. He calls them "unsaturated". Objects, on the other hand, are complete entities on the opposite spectrum. They are "saturated" entities. Given that both functions and

[1]The translation of the German term *Bedeutung* is one of the most disputed controversy in Frege's scholars. The dispute is, mostly, about two options for translation: "meaning" or "reference". For instance, take Max Black's translation of the *Über Sinn und Bedeutung*, which has appeared in three different collections of Frege's writings, each with a different rendering of the word. First in the Geach and Black *Translations from the Philosophical Writings of Gottlob Frege*, for Basil Blackwell in 1960, translating "*Bedeutung*" as "reference" (FREGE, 1960); then in 1984 for Brian MacGuinness edition of the *Collected Papers of Mathematics, logic and Philosophy*, also for Basil Blackwell, now translating "*Bedeutung*" as "meaning" (FREGE, 1984); and finally, in Michael Beaney's edition *The Frege Reader*, for Blackwell Publishers in 1997, where "*Bedeutung*" is left untranslated. Beaney includes a good discussion of the issue in the introduction, section 4, in (FREGE, 1997). The same dispute appears in translating the *Grundgesetze*, but the complete translation of Philip Ebert and Marcus Rossberg opted for "reference", mostly for three reasons: first, because it is a translation more established in the literature, second, because it sounds less artificial than "denotation", which was Furth's choice for his translation of part I of the book; third, because it fits better in translating the cognates "*gleichbedeuted*" and others. Ebert and Rossberg's work did not face the difficulty of translating the term for papers that span Frege's whole career, as other editors have faced. Here, "reference" is the opted word, even though I use mostly the papers from the MacGuinness' collection. I follow mostly Ebert and Rossberg's first reason for the choice: it seems to be the most adopted term in the literature. When quoting passages where the translation seems problematic, I put the German term in brackets as well.

objects form the basis for Frege's ontology, they cannot be further defined. Answering Kerry's objection that Frege had not provided a sharp definition of the two notions, Frege answered that "What is simple cannot be decomposed, and what is logically simple cannot have a proper definition." So, what is left is to "lead the reader or hearer , by means of hints, to understand the word as is intended"(*BuG*, p.182). This the main goal of the articles in question.

Functions and objects are two distinct kinds of entities which cannot be reduced in terms of the other: "An object is anything that is not a function, so that an expression for it does not contain any empty place." (*FuB*, p.147), as he says. Functions, on the other hand, "[...] by itself must be called incomplete, in need of supplementation, or 'unsaturated'" (*FuB*, p.140). Any expression that does not need supplementation names an object. For example,

$$2^2 + 2$$

names the number 6. We can decompose it in different argument-places, therefore resulting in a unsaturated part, the function, and another saturated part, the argument of the function. Thus, we may obtain

$$(\ )^2 + 2$$

as a function that for argument 2 yields the number 6 as value. These are not the only kinds of complete entities that may be decomposed, yielding functions and arguments. Sentences are also complete in this sense, and thus, are also names for objects. For instance,

$$2^2 + 2 = 6$$

is, for Frege, a name for the True. We may still decompose it, for example, in the right-hand side of the equation,

$$2^2 + 2 = (\ )$$

which is a function that names the True for the argument 6. A function that has a truth-value as value for its arguments is called a concept (*FuB*, p.154).

Decompositions just as these are the proper way to understand the different roles played by functions and arguments. Frege explains that "the decomposition into a saturated and an unsaturated part must be considered a logically primitive phenomenon which must simply be accepted and cannot be reduced to something simpler" (*CP*, p.281). Its primitive character follows the also primitive role played by functions and objects. They are the basic distinction that Frege uses to developed his mature version of the concept-script.

On the other hand, the sense-reference distinction is Frege's development of the less clear notion of a judgeable-content of the *Begriffschrift*. He claims that "Earlier I called it the content-stroke, when I combined under the expression 'judgeable content' that which I now have learnt to distinguish as truth-value and thought" (*GGA*, §5, ft.2). A consequence from his conception of logic as a language, Frege's formalism does not have uninterpreted sentences: every well-formed expression is designed to have a sense and a reference.

The distinction between sense and reference is applicable to exactly three kinds of linguistic expressions: proper names, sentences, and concepts. In the *Sinn und Bedeutung* article, Frege starts re-evaluating his own theory of identity of conceptual-contents as defined in the *Begriffsschrift*, and promptly comes to reject it. Recall that in the 1879 version of the language, Frege defines the identity of contents ($\equiv$) as a relation applicable to the signs of the language, not to their contents (*BS*, §8). The motivation for this choice was simply to explain different modes of designation for contents. He starts the *Über Sinn und Bedeutung* article with the same problem. But now, Frege's solution is another. He returns into accepting identity as a relation between the contents, and in order to explain the cognitive difference between identities $a = b$ and $a = a$, he divides contents between senses and references:

> It is natural, now, to think of there being connected with a sign (name, combination of words, written mark), besides that which the sign designates, which may be called the meaning of the sign, also what I should like to call the sense of the sign, where in the mode of presentation is contained. (*SuB*, p.158)

The sense of a name is the mode of presentation of its reference: how it comes to refer to what it refers. $a = a$ and $a = b$ are informatively different because the senses of a and b are different, even though their reference is the same. In general, "A proper name (word, sign, sign combination, expression) *expresses* its sense, *means* or *designates* its meaning [*bedeutet oder bezeichnet seine Bedeutung*]" (*SuB*, p.161).

The same holds for assertoric sentences and concept-words. On the former, Frege calls the sense of a sentence the *Thought* [*Gedanke*], and its reference a truth-value. He then makes one of his more controversial moves:

> We are therefore driven into accepting the truth-value of a sentence as constituting what it means [*als seine Bedeutung anzuerkennen*]. By the truth-value of a sentence I understand the circumstance that it is true or false. There are no further truth-values. For brevity I

> call the one the True, the other the False. Every assertoric sentence concerned with what its words mean [*in dem es auf die Bedeutung der Wörter ankommt*] is therefore to be regarded as a proper name, and its meaning [*Bedeutung*], if it has one, is either the True or the False. (*SuB*, p.163)

It follows that the sense of sentences is the Thought associated to that sentence. But given that sentences refer, or mean, a truth-value, they behave just as proper names do. And when we judge an assertoric sentence, we go from the Thought associated to its truth-value: "Judgements can be regarded as advances from a thought to a truth-value." (*SuB*, p.164-5) Thus, in the same way the sense of a name is its mode of presentation, that is, the way one goes from the name to the object which it names, the thought is the mode a sentence may refer to the true.

In a letter to Edmund Husserl, dated 24 May 1891 (*PMC*, p.63), Frege detailed his sense-reference distinction to names, sentences and concept-words, in the following way:

Proposition	proper name	concept word		
↓	↓	↓		
sense of the proposition (thought)	sense of the proper name	sense of the concept word		
↓	↓	↓		
meaning (*Bedeutung*) of the proposition (truth-value)	meaning (*Bedeutung*) of the proper name (object)	meaning (*Bedeutung*) of the concept word (concept)	→	object falling under the concept

Frege is motivated in saying that a concept-word relates to an object only indirectly, thus against Husserl's own opinion. And even if they are unsaturated entities themselves, they still have a sense and a reference. Frege held also the compositionality principle for senses and references: that the sense of a sentence (its Thought) is the product of the senses of its parts. For instance, the sense of a saturated functional expression $f(a)$ must be the composite of the sense of the function $f(\ )$ with the sense of the proper name a. This, however, presupposes the existence of unsaturated senses as well:

> not all the parts of a thought can be complete; at least one must be 'unsaturated', or predicative; otherwise they would not hold together. For example, the sense of the phrase 'the number 2' does not hold together with that of the expression 'the concept prime number' without a link. We apply such a link in the sentence 'the number 2 falls under the concept prime number'; it is contained in the words 'falls under', which need to be completed in two ways - by a subject and an accusative; and only because their sense is thus 'unsaturated' are they capable of serving as alink. (*BuG*, p.163).

What Frege holds, then, is that some parts of the senses of sentences (the Thought) are incomplete, or unsaturated. Thus, the sense of a function is unsaturated, and is saturated when completed by the sense of a name (either proper name, or a sentence). For instance, the sense of the unsaturated function

$$(\ ) = (\ )$$

when supplemented by the senses of "$2^2 + 2$" and "6", forms a whole, that is, a Thought (that $2^2 + 2$ and 6 names the same object), and refers to the True[2]. The same goes for the sense of the function

$$(\ )^2 + 2$$

when supplemented by the sense of the proper name "2" yields a sense for another proper name, that is, "$2^2 + 2$".

But what are the senses and references of function and concept names? The reference of such function and concept names are not objects, but functions. It is to no surprise that an unsaturated expression such as the $(\ )^2 + 2$ cannot refer to a saturated entity, and thus, it must refer to another unsaturated entity, *viz.* a function. The senses of such expressions are matter of controversy in interpreting Frege[3]. But one provisional answer is to say that the sense of a function-name, just like senses of names, are the mode of presentation of such function. In the same way $2 + (1 - 1)$ and $2 + 1 \times 0$ are names for the same object (2) but with different senses, so thus the function-names $2 + (x - x)$ and $2 + x \times 0$ refer to the same function, but relates to this function differently, *i.e.*,, their mode of presentation (the senses) are different. This is also what Frege meant in saying, in the diagram above, that concepts (and functions) refer indirectly to objects.

[2]This presupposes a symmetry between the syntactic compositionality (how an expression is composed) and a semantic compositionality. The latter being subdivided between compositionality of senses and references. On this topic, chapter 5 in Landini (2012) offers a detailed and unorthodox reading.

[3]Which I will not venture on.

2.1 The 1893 Version

For what matters the *Grundgesetze der Arithmetik*, Frege take it seriously the distinction between functions and objects, senses and references. And taking them seriously marks the major changes from 1879 to the 1893's version of the concept-script. In comparison to the 1879 version, the mature version is commonly called a terms-logic. This simplify a lot on how to describe the language.

First, all expressions of the language are either names for objects, *object-names*, or names for functions, *function-names*. In (*GGA*, §$1-2$) Frege repeats his theory of functions as unsaturated entities. And in order to express functions and objects in the exposition of the system (what one may consider as the metalanguage), Frege uses Greek letters for different purposes:

1. Upper-case $\Delta, \Gamma, \Theta, \Lambda, ...$ ranging over arbitrary object-names.
2. Upper-case Φ, Ψ as ranging over arbitrary function-names of 1^{st} order.
3. Lower-case ξ, ζ as fixing the argument places for functions. Lower case Greek letters such as these are just used to mark a function that is in need of being complete.

For example, in the function $\Phi(\xi)$, ξ is only fixing the argument-place. In practice, Frege uses this letters when he wants to speak about the function being named. In contrast, in $\Phi(\Delta)$, Δ is an arbitrary object-name, and thus, it names an arbitrary object. In practice, they are used when Frege wants to speak of the value of the function, not the function itself. One may take these as kinds of schematic variables, ranging over arbitrary names. But this is problematic. $\Delta, \Gamma, ...$ do range over arbitrary object-names, but Frege never uses them with this in mind. Rather, they are used as denoting arbitrary objects only.

2.1.1 Primitive Functions

The first assumption to consider is the domain of the concept-script functions. Frege writes that "the domain of what is admissible as an argument has to be expanded and extended to objects in general" (*GGA*, §2). Logical generality, as Frege understands it, demands that the domain of the functions should not be restricted. Concept-script functions are, in this case, total functions.

We saw that functions range over objects to objects. A numerical function such as ξ^2 has a number as value. But the concept-script functions mostly have truth-values as value. It is usual, in expressing one's syntax, to distinguish functions that are term-forming from propositional functions. For instance, we describe for the *Begriffsschrift* two kinds of functions: those that has objects as value from those that has judgeable contents as values, that is, those that could be prefixed with the content-stroke. In that system, Frege did not accept that objects were judgeable. For the *Grundgesetze* system, following his change from judgeable contents to senses and references, all functions are term-forming functions. This is why Frege's system is often called a term-logic[4]. What allowed Frege in taking this step is that the differences between judgeable contents and objects are now explained in terms of their different senses: complete proposition have *Thoughts* as senses, while names for objects have what Frege called the *mode of presentation* of the object named. But from the point-of-view of their reference, they behave in the same manner: proper names have objects as reference, while sentences have truth-values as reference (which are also objects). For distinguishing both cases, Frege said that "The sense of a name of a truth-value I call a *thought*. I say further that a name *expresses* its sense and *refers to* its reference" (*GGA*, §2). Thus, from the point of view of its senses, all functions that has truth-values as value are what we may call sentences, given that the "[...] expression in language for a thought is a sentence." (*PW*, p.174). In general, we have that

$$\text{“}2\text{”}, \text{“}2^2\text{”}, \text{“}2^2 = 4\text{”}$$

are all names for objects,

$$\text{“}\xi^2\text{”}, \text{“}\xi^2 = 4\text{”}, \xi^2 = \zeta$$

are all names for functions. Functions that have a truth-value as reference when saturated are called *concepts*.

Functions can also be distinguished by their arity (*GGA*, §4):

1. Unary functions: $\Phi(\xi)$, with only one argument place;

2. Binary functions: $\Psi(\xi, \zeta)$, with two argument places in need of completion: the ξ-argument and the ζ-argument.

3. n-ary functions, more generally: $X(\xi_1, ..., \xi_n)$. Frege never uses functions with arity higher than 2.

And they can be distinguished by their level:

[4] See, for example, Landini (2012) and Cadet and Panza (2015).

1. 1^{st}-level functions: $\Phi(\xi)$, where ξ is a place-holder for a proper name.

2. 2^{nd}-level functions: $M_\xi(\varphi(\xi))$, where $\varphi(\xi)$ is a 1^{st}-level function.

This could be extended for functions of arity higher than 1. We can visualize this distinctions with the following example. Let ξ^2 be a 1^{st} level function. Then the 2^{nd} level function $\varphi + 2$, when completed with ξ^2 yields $\xi^2 + 2$. Similarly, let $\mu = 6$ be a 3^{rd} level function. It can be saturated with the 2^{nd} level function $\varphi + 2$, which yields $\varphi + 2 = 6$, which is a 2^{nd} level function itself. We can thus complete it with the 1^{st} level function ξ^2 to obtain the also 1^{st} level function $\xi^2 + 2 = 6$. Needless to say, these are all unsaturated functions. Filling 2^{nd} level functions with 1^{st} level functions in its correct argument-places does not yield objects, but functions. In (*GGA*, §23), Frege distinguishes between arguments of first kind (objects), second kind (unary 1^{st} level functions) and third kind (binary 1^{st} level functions). These are to be always matched with, respectively, argument places of first kind, argument places of second kind and argument places of third kind. In the example above, ξ^2 is a function for which ξ marks an argument place of the first kind, which can be saturated by a proper argument of first kind Δ. In $\varphi + 2$, φ marks the argument place of the second kind. Frege rarely uses functions higher than 2^{nd} order, and uses their value-ranges as proxies for reducing all functions to 1^{st} order.

2.1.1.1 The Horizontal (§5)

Given that the *Grundgesetze* functions are total, they can also have truth-values as value. This is the main goal of the *horizontal function*. Frege has abandoned his content-stroke of the 1879 concept-script in favor of the new horizontal function:

> I regard it as a function-name such that
>
> $$— \Delta$$
>
> is the True when Δ is the True, and is the False when Δ is not the True. (*GGA*, §5).

Thus, the horizontal is a function that maps all objects to truth-values, either the True, or the False. Given that it has a truth-value as value, it has a Thought associated to it, and thus names a concept. But Frege's way of defining it in the language is by defining it as a characteristic function of the concept:

$$— \Delta = \begin{cases} \text{the True} & \text{if } \Delta \text{ is the True} \\ \text{the False} & \text{if otherwise} \end{cases}.$$

A natural consequence of this function is that $-\!\!\!-\,\Delta$ is trivial if Δ already names a truth-value. This will motivate Frege in stating a rule of inference for horizontals: the *fusion of horizontals*. The only provision made to the horizontal function is that Δ must always refer to something.

2.1.1.2 Negation (§6)

As is known, Frege only uses one type of illocutionary device for the assertion: that of recognizing something to be True. Frege introduced the Negation function not only for representing the falsity of a content, but also to avoid having another type of illocution for rejecting a content. In this sense, rejecting a content is the same as accepting, of asserting, that its negation is true. The negation is thus introduced as:

> The value of the function
> $$\top\,\xi$$
> is to be the False for every argument for which the value of the function
> $$-\!\!\!-\,\xi$$
> is the True; and it is to be the True for all other arguments. (*GGA*, §6)

As we can see, the negation function is the inverse of the horizontal function. It can also be more simply stated as characteristic function:

$$\top\,\Delta = \begin{cases} \text{the False} & \text{if } \Delta \text{ is the True} \\ \text{the True} & \text{if otherwise} \end{cases}$$

One of the eccentricities of Frege's term-notation follows from such function, as any object-name that names anything other than the False may be judged as being the True. For example, $-\!\!\!-\,2$ is the False, as 2 names a number, not the True. But equally, $\top\,2$ names the True.

2.1.1.3 Equality (§7)

Equality, as the discussion in *Über Sinn und Bedeutung* have already stated, is a relation defined over the references of the expressions.

> '$\Gamma = \Delta$' refers to the True, if Γ is the same as Δ; in all other cases it is to refer to the False. (*GGA*, §7)

The characteristic function is:

$$(\Gamma = \Delta) = \begin{cases} \text{the True} & \text{if } \Gamma \text{ and } \Delta \text{ have the same reference} \\ \text{the False} & \text{if otherwise} \end{cases}$$

2.1.1.4 Generality (§8)

Frege employs two ways of expressing generality in the concept-script, just as he did in 1879: the Roman letters and German letters together with the concavity sign. On the latter, he defines:

> let $\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$ refer to the True if the value of the function $\Phi(\xi)$ is the True for every argument, and otherwise the False; (*GGA*, §8)

Generality through the concavity is a 2^{nd}-level function in which 1^{st}-level functions fall. As a characteristic function, it is:

$$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \begin{cases} \text{the True} & \text{if } \Phi(\xi) \text{ is the True for every argument } \xi \\ \text{the False} & \text{if otherwise} \end{cases}$$

The concavity is used for binding the variable $\mathfrak{a}$ and to specify the scope of the quantification in the formula. Needless to say, in quantifying any function in which a German letter already occurs, the introduction of the concavity must choose a new letter.

Frege also applies the concavity for higher-order quantification. The 3^{rd} level function

$$\overset{\mathfrak{f}}{\smile}\, \Omega_\beta(\mathfrak{f}(\beta))$$

is the True if the value of the 2^{nd}-level function $\Omega_\beta(\Phi(\beta))$ is the True for every fitting argument, that is, for every function $\Phi(\xi)$ that is True whenever ξ names the True.

Once again, the main purpose of the concavity quantification is the ability to distinguish the generality of a negation from the negation of a generality. That is,

$$\overset{\mathfrak{a}}{\smile}\!\!\top\, \Phi(\mathfrak{a})$$

from

$$\top\!\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$$

and, of course, the existential generalization as the negation of the quantification of the negation:

$$\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \Phi(\mathfrak{a})$$

This is not possible with the other source of generality: Roman letters. Frege writes that "the scope of a Roman letter is to include everything that occurs in the proposition apart from the judgement-stroke" (*GGA*, §17). The main reason in having roman letters for generality is to license some transitions from judgements, as with Barbara arguments. Frege's example is that in order to proceed from

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\, \mathfrak{a}^4 = 1\\ \llcorner\, \mathfrak{a}^2 = 1\end{array} \quad \text{and} \quad \vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\, \mathfrak{a}^8 = 1\\ \llcorner\, \mathfrak{a}^4 = 1\end{array}$$

to

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\top\, \mathfrak{a}^8 = 1\\ \llcorner\, \mathfrak{a}^2 = 1\end{array}$$

we have to extend the scope of the generality to the whole judgement. This is achieved by changing both premises to roman letters:

$$\vdash\!\begin{array}{l}\top\, x^4 = 1\\ \llcorner\, x^2 = 1\end{array} \quad \text{and} \quad \vdash\!\begin{array}{l}\top\, x^8 = 1\\ \llcorner\, x^4 = 1\end{array}$$

In order to this be possible, Frege also forces some conditions on the use of Roman letters. First, their scope is to be not restricted to the formula in which they occur. "It thus remains permissible to let the scope extend to multiple propositions so that the Roman letters are suitable to serve in inferences in which the German letters [...] cannot serve" (*GGA*, §17). Second, Roman letters are *always* used in the context of a judgement. They never occur in unasserted formulas, even in the expository part of the language. The same holds for Roman letters for functions, such as $f, g, h, \ldots$.

While object names such as "2^2" or "$2 + 2 = 4$" are said do refer to objects, expressions with the occurrence of roman letter do not *refer* [*bedeuten*] but only to indicate [*andeuten*] an object: "We do not say of a Roman letter that it refers to an object but that it indicates an object" (*GGA*, §17). As a result, any concept-script proposition in which roman letters occur are not names:

> Roman letters, and combinations of signs in which those occur, are thus not *names* as they merely *indicate*. A combination of signs which contains Roman letters, and which always results in a proper name when every Roman letter is replaced by a name, I will call

> a *Roman object-marker*. In addition, a combination of signs which contains Roman letters and which always results in a function-name when every Roman letter is replaced by a name, I will call a *Roman function-marker* or *Roman marker of a function.* (*GGA*, §17)

For that reason, Roman letters cannot be defined as functions in the same sense the concavity can, and so they cannot be defined through characteristic functions.

2.1.1.5 Value-Ranges (§9 – 10)

The controversial value-ranges are different from the functions so far introduced, as the value of the value-range function is not a truth-value:

> let '$\grave{\varepsilon}\Phi(\varepsilon)$' refer to the value-range of the function $\Phi(\xi)$ (*GGA*, §9).

This is a 2^{nd}-level function that maps 1^{st}-level function to value-ranges. A value-range [*Werthverlauf*] can be understood as the graph of the value of the function for each particular argument. In the case of concepts, that is, functions that maps objects to truth-values, the value-range is simply the set of objects that satisfies the concept, what Frege calls the extension of the concept. Given that value-ranges are presumably fully determined, *viz.,* saturated entities, they are also objects.

They are always introduced with lower-case Greek letters, usually ε, with a smooth-breathing ̓ mark to bind the scope of the function, just as the concavity is for generality of German letters.

Frege believed that the transition from an equality of Value-ranges to the generality of an equality, and *vice-versa*, was a logical law. He had reservations upon it, but took this "logical" fact as Basic Law V in the system, which is inconsistent. The inconsistency shattered his belief that we could go from concepts to their extensions without problems, and thus, shattered his attempt to explain numerical judgements into conceptual judgements in terms of their extensions. But besides this failed attempt, Frege also made a pragmatic use of value-ranges in the concept-script, by reducing 2^{nd}-level functions into 1^{st}-level ones with their extensions as proxies, using the Backslash and definitions. Although these are eliminable, the derivations in the concept-script are riddled with value-ranges.

2.1.1.6 The Backslash (§11)

Following the introduction of value-ranges, Frege then introduces the Backslash \ function[5], which can be read as definite article.

Frege's rationale for introducing this function is the following. If we take

$$\dot{\varepsilon}(\Delta = \varepsilon)$$

as the extension of a concept in which only Δ falls, then the function $\backslash\xi$ returns the object that satisfies the concept that ξ is the extension of, if there is any. Otherwise, it returns ξ itself. This is:

$$\backslash\Gamma = \begin{cases} \Delta & \text{if there is a } \Delta \text{ such that } \Gamma = \dot{\varepsilon}(\Delta = \varepsilon) \\ \Gamma & \text{if there is no such } \Delta \end{cases}$$

This definition yields, as Frege explains in (*GGA*, §11), that '$\backslash\dot{\varepsilon}\Phi(\varepsilon)$' refers to the single object that falls under $\Phi(\xi)$ if there is only one such object. If this isn't the case, then $\backslash\dot{\varepsilon}\Phi(\varepsilon) = \dot{\varepsilon}\Phi(\varepsilon)$.

For example, under Frege's definition, $\backslash\dot{\varepsilon}(\varepsilon + 3 = 5)$ behaves as a definite description for the number 2, given that 2 is the only object that returns the True for $(\xi + 3 = 5)$, and thus, it is the only object that falls under the extension of such concept, $\dot{\varepsilon}(\varepsilon + 3 = 5)$.

2.1.1.7 The Conditional (§12 – 13)

We finally reach the conditional function, which is the same as defined in 1879, but now not as a binary function relating judgeable-contents, that is, from sentences to sentences, but a binary function mapping pairs of objects to objects. Frege's definition is:

> I introduce the function with two arguments
>
> $$\begin{array}{l} \top\!\!\!\!\!\!\;\; \zeta \\ \llcorner\, \xi \end{array}$$
>
> by means of the specification that its value shall be the False if the True is taken as the ζ-argument, while any object that is not the True is taken as ξ-argument; that in all other cases the value of the function shall be the True. (*GGA*, 12).

This is more easily written as:

[5] Frege did not name this function as such. The name follows from Roy Cook's appendix A to the translation of the *Grundgesetze*, in (*GGA*).

$$\begin{array}{l}\top\,\Gamma \\ \llcorner\,\Delta\end{array} = \begin{cases} \text{the False} & \text{if } \Delta \text{ is the True and } \Gamma \text{ is the False} \\ \text{the True} & \text{if otherwise} \end{cases}$$

Δ is called the supercomponent, and Γ the subcomponent. From it, we may define other propositional connectives in the following:

Conjunction: $\begin{array}{l}\top\!\!\top\,\Gamma \\ \quad\llcorner\,\Delta\end{array}$

Disjunction: $\begin{array}{l}-\,\Gamma \\ \llcorner\!\top\,\Delta\end{array}$

Neither/Nor: $\begin{array}{l}\top\!-\,\Gamma \\ \quad\llcorner\!\top\,\Delta\end{array}$

It is not hard to see how Frege's logic is a term's logic, as there is no category of formulas. Thus, it is easy to define terms, as all the functions above defined falls under this category:

Definition 7. *(Terms) We start with the most basic cases:*

1. *All object-names are terms;*

2. *For any function-name $\Phi(\xi)$ and object-name Δ, $\Phi(\Delta)$ is a term.*

Then, for Δ, Γ temrs and $\Phi(\xi)$ a function:

3. $-\,\Delta$ *is a term;*

4. $\top\,\Delta$ *is a term;*

5. $\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a})$ *is a term;*

6. $\acute{\varepsilon}\Phi(\varepsilon)$ *is a term;*

7. $\backslash\Gamma$ *is a term;*

8. $\begin{array}{l}\top\,\Gamma \\ \llcorner\,\Delta\end{array}$ *is a term;*

9. *There are no other terms.*

This picture is not complete, as we have not introduced the judgement-stroke $\vdash$, or the definition-stroke $\Vdash$. These are signs of different type, and should be treated separately. The reason being that these signs do not have only syntactic usage. In a sense, none of Frege's primitive functions behaves like a modern system, where one first define formulas, and then proceed in defining its semantics all at once. Frege's functions are defined with an intended semantics. The same goes for the judgement-stroke and the definition-stroke. These, however, are not functions, and although they still are thought with an intended semantical value, they are signs of a different kind. We should also include inferential lines under this same category, thus, rules of inference will be analysed after we better understand the performative signs of Frege's concept-script.

2.1.2 Basic Laws

Frege's six Basic Laws of the *Grundgesetze* version of the Concept-Script are:

$$\text{I)}\ \BGassert\BGconditional{a}{\BGconditional{b}{a}} \qquad \text{II)}\ \BGassert\BGconditional{\BGquant{\mathfrak{a}} f(\mathfrak{a})}{f(a)} \qquad \text{IIb)}\ \BGassert\BGconditional{\BGquant{\mathfrak{f}} M_\beta(\mathfrak{f}(\beta))}{M_\beta(f(\beta))}$$

$$\text{III)}\ \BGassert\BGconditional{g(a=b)}{g\left(\BGquant{\mathfrak{f}}\BGconditional{\mathfrak{f}(b)}{\mathfrak{f}(a)}\right)} \qquad \text{IV)}\ \BGassert\BGconditional{\BGnot(\BGcontent a) = (\BGnot b)}{(\BGcontent a) = (\BGcontent b)}$$

$$\text{V)}\ \vdash (\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} f(\alpha)) = (\BGquant{\mathfrak{a}} f(\mathfrak{a}) = g(\mathfrak{a}))$$

$$\text{VI)}\ \vdash a = \backslash \grave{\varepsilon}(a = \varepsilon)$$

The basic laws may also be taken as laws governing the functions defined in the exposition of the system. Basic Law I governs the use of the conditional function; Basic Law II, governs generality; Basic Law III the equality; Basic Law IV the negation and the horizontal[6]; the infamous Basic Law V asserts a basic law governing the use of value ranges; and finally, Basic Law VI governs a basic fact about the Backslash function.

One may notice that all basic laws are prefixed with the judgement-stroke. They are not simply taken as part of the syntax, as they are asserted as True from the outset. Frege's conception of an axiom does involve a self-evident character: their truth is easily perceived and justified, and thus shall be judged as being true under all circumstances. However, this will be a topic for later.

[6]In a sense Basic Laws I, II, III all have instances of the horizontal function, as Frege considered the conditional, negation and generality functions as being prefixed by horizontals.

2.2 Performative Signs

From what we said, in both versions of the concept-script, logic is mostly about contents and different ways to express it in the multiplicity of logical forms that they can assume. Even though we describe a syntax for both versions, based on Frege's primitives, this procedure is not entirely faithful to Frege's logic. This is because Frege's logic is an universal language, and syntactic elements are included already with intended interpretations. In the *Begriffsschrift* these are judgeable-contents. In the *Grundgesetze* these are senses and, most importantly, references.

What is still missing is means to actually work around such contents to derive theorems and discover truths. For Frege, that is what logic is about. After the development of metamathematics with Hilbert and the works on the semantic truth definition of Tarski, the standard approach became metatheoretical, and the question about which performatives we are going to consider looses its importance. A theory of logic has become simply a set of sentences defined on a signature, which we define from a privileged perspective of the metalanguage, rich enough to give us all the means to precisely describe everything about the object-theory. In other words, what is missing is the formalization, as far as it is possible, of the different actions performed by a reasoning agent.

In the case of the concept-script logic, two are the main actions that an agent should perform on the contents. What it is expected is to agents be able to manipulate the language in a auxiliary way, in order to effectively judge things to be true about the contents expressed in its sentences. The former is the action of *defining* something. The latter, the *judging*, or asserting, something. Asserting also involves another type of non-descriptive sign in the language, the inferential signs for the rules of inference used. Judging, or asserting, is a definitive notion for Logic in Frege's mind, and much of his notation depends on it. We shall start with them.

2.2.1 Judgements

Any modern treatment of logical calculus includes a sign for the derivability of a formula in the system, usually denoted by the turnstile $\vdash$, meaning by

$$\Gamma \vdash \varphi$$

that φ is derivable from (or is a syntactical consequence of) the set Γ of formulas, and by

$$\vdash \varphi$$

that φ is a theorem (a consequence of an empty set of premises). This is known as the consequence relation, defined in the metalanguage of a formal theory.

The turnstile can be traced back to Frege's concept-script. But Frege's usage of the sign is well distant from the nowadays consequence relation of a metalanguage. Russell and Whitehead followed Frege in the *Principia Mathematica* (1910), treating the sign as denoting the assertion of a given proposition. But their use is not exactly the same as Frege's. Given that both Frege's system and the *Principia Mathematica* are not model-theoretic, the assertion-sign has both a syntactic and a semantic role, as this two conceptions were not precisely distinguished. In asserting a content as true, for example, Frege was claiming that the content was true, but at the same time pressuposing that it was *provable* as true, as syntactic provability is one of the conditions for assertions in his system. Thus, if a sentence is asserted, it means both that the content expressed by the sentence is true and that the sentence itself is a theorem. This double role was certainly present in Russell and Whitehead's *Principia*. But already in the 1930's we find the turnstile being used only with the syntactical meaning attached, for example, with Quine (who took the turnstile sign from Russell and Whitehead) defining it as the metamathematical sign for theorem-hood in (QUINE, 1937).

This meaning is, of course, much different from Frege's. When a content prefixed by the content-stroke (either a judgeable content or a sense) is being judged, Frege uses the vertical line on the left, as:

$$\vdash\!\!- A$$

To indicate that the content of A is affirmed or recognized as a fact (in 1879), or as being the True (in 1893). This is known as the *judgement-stroke*, the grandfather of the modern consequence relation symbol.

Judging a content is a inner event: something that happens in the mind of the one who judges. Classically, a judgement is a form of decision: one decides the truth-value of the content being considered, either affirming it or denying it. In Frege's case, this decision is whether to judge it as true, or not to judge at all. Having made a judgement, if one wants to communicate it, he uses the judgement-stroke. Frege distinguishes between *judg-*

ing a content, and *communicating* the judgement, what he calls the Assertion. He draws this distinction in many places throughout his career. For instance, in the unpublished *Logik*, dated around 1879 and 1891, he writes that "[...] *to recognize something as true is to make a judgement*, and to give expression to this judgement is to make an assertion" (*PW*, p.2). Much later, in the *Der Gedanke* paper, he repeat the same distinction:

> Consequently we distinguish:
>
> (1) the grasp of a thought - thinking,
> (2) the acknowledgement of the truth of a thought -judgement,
> (3) the manifestation of this judgement - assertion. (*DG*, p.355-6).

In the *Begriffsshrift*, Judgements are primitives. In fact, Frege first introduced the judgement-stroke (*BS*, §2), to them introduce the content-stroke as the judgement-stroke minus the vertical line occurring on the left. As he says,

> If we omit the small vertical stroke at the left end of the horizontal one, the judgment will be transformed into a mere combination of ideas [*Vorstellungsverbindung*], of which the writer does not state whether he acknowledges it to be true or not. (*BS*, §2)

These combination of ideas is what the content-stroke relates. In fact, they are both presented together in the exposition of the primitives: "The horizontal stroke, which is part of the symbol $\vdash$, ties the symbols that follow it into a whole; and the assertion, which is expressed by means for the vertical stroke at the left end of the horizontal one, relates to this whole" (*idem.*). Moreover, Frege's term for the contents of judgements are conceptual-contents, but he also used another term: judgeable-contents [*beurtheilbare*]. This is because he restrict content in that system to those who can be judged as true or as a fact.

Another point is that his theory of Judgements deviates from the traditional subject-predicate distinction. Under this distinction, a judgement is always expressed through categorical propositions, connecting a subject-term to a predicate-term by means of a copula, the verb "is"[7]. The judgement proper is the affirmation that both subject and predicate are related. Frege's theory includes cases like this, but with a different structure, the functional analysis that we already discussed. This analysis made it possible to separate more properly the assertoric force of a sentence. In the sentence

(1) Snow is white

[7] This will be a topic of discussion for another chapter.

We are indeed predicating whiteness of snow by means of the copula "is", which, by itself, represents the assertoric force. But, in order to express the same idea, we have to use another strategy: the nominalization of the content. Frege uses this strategy when he claims that "When the vertical stroke is omitted, we express ourselves *paraphrastically*, using the words 'the circumstance that' or 'the proposition that'" (*BS*, §3). Bell (1979) reads the content-stroke as being a nominalizing operator, that when applied to (1) yields

(2) That snow is white.

Now, if we want to express that snow is white through (2), we just say:

(3) It is a fact/true that snow is white.

Frege's distinction between the judgement and the conceptual-content is an abandonment of (1) in terms of (2) and (3).

But there is a difference between asserting (2) from just stating (3). It is circular to begin with. (3) is another subject-predicate sentence, one that relies on the copula "is" for having assertoric force. But this is what Frege wants to avoid. (3), thus, cannot be what the proper reading of "$\vdash$ that snow is white".

In *GGA*, where Frege already had distinguished the sense (*Sinn*) of a sign from its reference (*Bedeutung*), the judgement-stroke function as a way to connect both entities of a given expression: linking a sense, the *thought* of a sentence, with a reference, the True. In the seminal article *Über Sinn und Bedeutung*, Frege writes that "Judgements can be regarded as advances from a thought to a truth-value" (*SuB*,p.164-165), and in *GGA*, he reiterates:

> I distinguish the *judgement* from the *thought* in such a way that I understand by a *judgement* the acknowledgement of the truth of a *thought*. The concept-script representation of a judgement by means of the sign '$\vdash$ ' I call a *concept-script proposition* or *proposition* for short. (*GGA*,§5)

Hence, propositions in the concept-script language are those *acknowledged* as being the True[8]. Acknowledgement is an act, and thus, the judgement-stroke cannot be considered as being in the same class of the others. An acknowledgement is the part of a judgement where the judging agent makes a decision. Acknowledging the truth of a Thought is *ipso facto* to just judge the Thought to be true.

[8]Given that the term "proposition" is already well established as being what Frege calls the Thought of a sentence, I will avoid using it for asserted sentences of the concept-script, calling them asserted or judged sentences.

Frege in fact says that the sign for judgements is *sui generis*. It is not a descriptive sign. In the article *Funktion und Begriff*, he states that "The assertion sign (*Urtheilsstrich*) cannot be used to construct a functional expression; for it does not serve, in conjunction with other signs, to designate an object. '$\vdash 2 + 3 = 5$' does not designate anything; it asserts something" (*FuB*,p149n). In the *Grundgesetze*, he states: "I reckon the judgement-stroke to belong neither with the *names* nor with the *markers*; it is a sign of its own kind" (*GGA*,§26).

This is enough evidence for rejecting the idea that the judgement-stroke functions as a sort of truth-predicate. First, because predicating the truth of 2+3=5 would be only expressible through another sentence, which, by itself, denotes a truth-value. If that's the case, the judgement-stroke would denote a Thought, against Frege's claims in *Funktion und Begriff*. Second, taking the judgement-stroke as a truth-predicate is superfluous. If we do accept that the judgement-stroke is a truth-predicate, then judging $2 + 3 = 5$ means:

(4) that $2 + 3 = 5$ is the True.

The judgement of $2 + 3 = 5$ is totally expressed, and explained in terms of (4). Judging, thus, is reduced to grasping another thought. But grasping isn't enough. We need to know that (4) is true, so for the judgement that $2 + 3 = 5$ be true as well. But, if we judge (4), we are just grasping another thought, in the sentence:

(5) that $2 + 3 = 5$ is the True, is the True.

And the same goes for (5). If judging is reduced to predicating truth, we would never in fact reach a true Thought, but only another Thought that may be true itself.

Frege also considered that (4) and (5) express the same thought: that $2+3 = 5$. The truth-predicate is thus superfluous. He explains that we cannot see the relation of a thought to its true as predication.

> The truth claim arises in each case from the form of the assertoric sentence, and when the latter lacks its usual force, *e.g.*, in the mouth of an actor upon the stage, even the sentence 'The thought that 5 is a prime number is true' contains only a thought, and indeed the same thought as the simple '5 is a prime number'. It follows that the relation of the thought to the True may not be compared with that of subject to predicate. (*SuB*, p.164).

What is indeed the important part of an assertoric sentence is the assertoric force, and this cannot be properly found in its syntax, as far as ordinary languages are concerned.

In the sentence (1) above, the copula is the usual carrier of the assertoric force. But as Frege claims, an actor in a play can still express the same sentence without assertoric force. One has to read it from the context. As he writes in the *der Gedanke*,

> we have the case of thoughts being expressed without being actually put forward as true, in spite of the assertoric form of the sentence; [...] Therefore the question still arises, even about what is presented in the assertoric sentence-form, whether it really contains an assertion. And this question must be answered in the negative if the requisite seriousness is lacking. (*DG*, p.356)

The seriousness of a speaker is something to be read out from the context. It pertains to pragmatics, not to semantics, neither to syntax. Frege's judgement-stroke is a force-indicating device: "I have introduced a special sign with assertoric force, the judgement-stroke." (*CP*, p.247), as he claims. In being a illocutionary force-indicating device, Frege's judgement stroke is also a "seriousness operator": it fixes the context of the utterances of the concept-script as being legit assertions.

At least partially. An obvious objection to Frege's pretension is that no indication of context coming from something an agent does is enough, such as an assertion with the judgement-stroke. What does happen if an actor uses Frege's judgement-stroke in a play? I believe this is a challenge to Frege's judgement-stroke, but so it is to any other force indicating-device. In the context of the whole concept-script, an agent using the judgement-stroke is certainly uttering with assertoric force, even if the judgement is faulty. Frege did judged Basic Law V, although being it false. But the question is not if an agent is able to judge and assert in a infallible way, but only if he does it with assertoric force. The relevant uses of the concept-script *with* the judgement-stroke should be considered enough for the seriousness of an agent in asserting a content to be the True.

2.2.1.1 The Essence of Logic

In the article *Der Gedanke*, Frege states the following:

> It seems to me that thought and judgement have not hitherto been adequately distinguished. Perhaps language is misleading. For we have no particular bit of assertoric sentences which corresponds to assertion; that something is being asserted is implicit rather in the assertoric form. (*DG*,p.356n)

What is present in the assertoric form that marks that an assertion is being made is the presence of the *assertoric force*. As Frege intends, a more perfect language must be able to express a thought without asserting it, therefore, separating the sentence from the assertoric force. That's why such a language needs a judgement-stroke sign. In the 1915 unpublished paper *Meine grundlegenden Logischen Einsichten*, he goes even further by characterizing the essence of logic as the presence of assertoric force: "[...] the thing that indicates most clearly the essence of logic is the assertoric force with which a sentence is uttered" (*PW*,p.252).

But Frege also considers that the essence of logic is demarcated by the word *True*. For instance, he opens the *Der Gedanke* stating that

> Just as 'beautiful' points the ways for aesthetics and 'good' for ethics, so do words like 'true' for logic. All sciences have truth as their goal; but logic is also concerned with it in a quite different way [...] To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. (*DG*, 351).

This position was similarly held in the unpublished *Logik*:

> the laws of logic are nothing other than an unfolding of the content of the word 'true'. Anyone who has failed to grasp the meaning of this word [...] cannot attain to any clear idea of what the task of logic is. (*PW*, p.3)

What is the right essence of logic? Truth or the assertoric force? Scholars have attempted to solve this puzzle in different manners. Taschek (2008), Greimann (2014) and Pedriali (2017) are good examples. The debate highlights the important connection between judgements and truth. The assertoric force presented in expressing a judgement is what Frege took to be the proper way to manifest the content of the word "true". Thus, it is in the judicative act that Frege's semantical considerations reside.

The basic idea expressed in the papers above is that the assertoric force does what the truth predicate cannot perform. In the former mentioned article, Frege argues that the truth-predicate "it is true that" is a useless predicate: "[...] the sense of the word 'true' is such that it does not make any essential contribution to the thought. If I assert 'it is true that sea-water is salty', I assert the same thing as if I assert 'sea-water is salty'" (*PW*, p.251). The idea behind this passage, as Greimann (2014) points out, is that predicates only operates within the realm of thoughts, and could never lead to truth-values by their own.

Part of Frege's doctrines says that Thoughts (*Gedanken*) are the senses (*Sinne*) of sentences. The sentence

(6) Sea water is salty.

express a sense, the Thought expressable by the non-assertive nominalized "that sea water is salty." Given that (6) is a sentence, it has a truth-value as reference. If truth is considered as a common predicate, we could predicate (6) with the true:

(7) it is true that sea water is salty.

But, according to Frege, (6) and (7) express the same Thought. The predicate "True" does not alter the sense of (6) at all: "the word 'true' has a sense that contributes nothing to the sense of the whole sentence in which it occurs as a predicate." (*PW*, p.252). Thus, it cannot be used to reach the true of a thought, precisely because true is, in Frege's doctrines, a truth-value. And we also saw above how the truth-predicate is superfluous. What actually contributes to take "the step from the level of thoughts to the level of meaning" (*SuB*, p.164) is the assertoric force of assertions, *viz.*, assertions as presented with the judgement-stroke.

This doesn't mean that the word "true" is pointless. The reason for taking the assertive force over the truth-predicate is simply that one can express the thought that "sea water is salty" is true without being true. Consider the following hypothetical judgement:

(8) if sea water is salty, you should not drink it.

Since (6) and (7) express the same thought, we can take one for the other in (8):

(9) if it is true that sea water is salty, you should not drink it.

In (9), in the context of the hypothetical judgement, the predicate "it is true that" does not confer that the thought that "sea water is salty" is in fact true. No commitment is present. What is missing is the assertoric force. In a hypothetical judgement, an agent does not commit to the true of the antecedent precisely because there is no commitment to the assertion of the antecedent. This helps seeing why one does not commit to the truth of (6) by simply rewriting as (7). Something is missing: the assertoric force.

That "true" is still a required word is quickly recognized when we ask: what does the assertoric force stands for? The truth of what is asserted, is the answer. Truth is what assertoric force aims at, as Frege claims in the following passage:

> When entering upon the study of a science, we need to have some idea, if only a provisional one, of its nature. We want to have in sight a goal to strive towards; we want some point to aim at that will guide our steps in the right direction. The word 'true' can be used to indicate such a goal for logic, just as can 'good' for ethics and 'beautiful' for aesthetics. (*PW*, p.128)

If truth is the goal, the assertoric force is the means, even though we can speak about truth without assertoric force. A similar passage is found in the 1897 *Logik*:

> If I assert that the sum of 2 and 3 is 5, then I thereby assert that it is true that 2 and 3 make 5. [...] Therefore it is really by using the form of an assertoric sentence that we assert truth, and to do this we do not need the word 'true'. Indeed we can say that even where we use the form of expression 'it is true that...' the essential thing is really the assertoric form of the sentence. (*PW*, p.129)

But it is not the case that assertoric force is solely tied up with the true. We do assert in general with assertoric force. But the fact that logic is general shows that we are *always* asserting truth, even when truth is not directly the subject matter. We can make assertive claims about different predicates, as in the following aesthetic judgement:

(10) Villa-Lobos' *Bachiana Brasileira Nº 2* is beautiful.

If I utter (10) with assertoric force, I am making an assertion about the predication of beauty. But as Frege claims, predicating truth is present every time one makes an assertion. Thus, what we actually assert in (10) is

(11) It is true that Villa-Lobos' *Bachiana Brasileira Nº 2* is beautiful.

In the same line of (6) and (7) expressing the same thought, so thus (10) and (11) express the same.

The role of the truth-predicate, albeit not sufficient for making the step from senses to references, is still important in making it explicit when one is allowed to judge. Greimann (2014, p.285) adds that the truth-predicate is just a way of making the assertion explicit. And in this reside part of the its importance. If asserting with the judgement-stroke is the proper way to reach the true in the concept-script, we have to understand under which conditions assertions are warranted. And to this task, one needs a truth-predicate, but at a different level. Still, one must add, it is not a sufficient way of making the assertion. The truth-predicate is not sufficient in making the assertive force explicit by itself (otherwise, we would have to accept the occurrences of the truth-predicate in (7) in (9) as having assertive force, which they do not). Therefore, the judgement-stroke must be taken as irreducibly a illocutionary force indicating device.

In the context of the concept-script, and operating *in* the language, (12) and (13) below are not equivalent[9],

[9] I will take the liberty of abusing the concept-script notation for the next examples.

(12) $\vdash$ sea water is salty

(13) It is true that sea water is salty

(12) has assertive force, while (13) hasn't. To be fair, (13) does have assertive force in normal language situations, as a well constructed English sentence. But, given that the truth-predicate does not confer assertoric force, and given that the nominalized "that sea water is salty" also is striped from its assertoric force, (13) doesn't necessarily have any. (12) is a claim to the truth of "sea water is salty", while (13) is just the expression of the thought that it is true that "sea water is salty", without claiming that in fact is. If (12) is the case, then (13) can be "read out" of the language, but sentences such as (13) never occurs *in* the formalism. But it does have its counterparts in the expository part of the concept-script. If I utter (12) in the concept-script with enough reasons, then I know that it is true that Sea water is salty. I can make use of this information in order to make new judgements. For example, if I have also judged (8) with assertoric force, then I can make use of *modus ponens* and assert

(14) $\vdash$ I should not drink sea water.

If anything, we could say that the difference between the assertive force and the truth-predicate is that the former is meant for *using* true thoughts, while the latter is meant for *mentioning* true thoughts when the former is lacking. We find this kind of distinction in the exposition of the system. In the *Begriffsschrift*, Frege talks about "is affirmed" as proxy for assertions made with the judgement-stroke. In the *Grundgesetze*, Frege uses the predicate "is the True" or "denotes the True" when speaking with quotation marks.

These predicates are not properly speaking assertions, but they are all necessary in expressing the norms for assertions. We can see this in Frege's justification for the Basic Laws. In the mature logic of the *Grundgesetze*, we can ask what is the truth-value of $\begin{array}{l}\top\,\Delta\\ \llcorner\,\Delta\end{array}$. If Δ is the True, then $\begin{array}{l}\top\,\Delta\\ \llcorner\,\Delta\end{array}$ is the True. If Δ is the False, then it is also the True. We are not asserting which is which. But in every case, it would be the True. Given that it is impossible to deny that $\begin{array}{l}\top\,\Delta\\ \llcorner\,\Delta\end{array}$ is the True, asserting it is normatively justified, thus yielding $\begin{array}{l}\vdash\!\top\,a\\ \;\;\llcorner\,a\end{array}$ with assertoric force. That the truth-predicate is incapable of making the step from the thought to a truth-value is precisely to say that it carries no assertive force.

And precisely because of that, it is required for making explicit the normativeness of the laws of logic in the prechamber of the concept-script.

This normativeness is discussed by William Taschek, who reads these passages, on the role of assertoric force, as indicating the normative status that logical laws have in judging, "that the essence of logic is to be found in its normative role *vis-à-vis* judgement and assertion" (TASCHEK, 2008, p.382). On the role of the truth-predicate, he claims that

> we have no independent grip on the notion of truth apart from our appreciation of it as the constitutive aim of judgement. And, for Frege, our appreciation of truth as a norm of correctness for judgement is ultimately bound up with our appreciation of judgements as constitutively subject to normative governance by the logical laws. (*idem*, p.390-1).

Taschek is correct on the normativeness of the assertive force, but it must be added that what makes the normative aspect of the logical laws explicit is not only the assertive force, but also the notion of truth or correlated predicates (the predicate "is affirmed", used in 1879, or the "is the True", in 1893). In order to justify logical laws, or rules of inference, one does not simply *make* assertions, but consider possible assertions as well. The best way to mention assertions is by using their most obvious proxy: the truth-predicate.

In the *Begriffsschrift*, Frege speaks about "being affirmed" or "being denied". The latter is most probably a proxy for the assertion of the negation, as he actually does not have a sign for the illucutionary force of a negation. But the former is clearly a mentioned assertion. One can mention an assertion made such as in

(15) It is asserted that sea water is salty.

His remarks on the 1915 paper cannot be equated to the way he justifies the laws of logic in 1879. For instance, (15) is not the same as (11). The thought related to (15) is not that "sea water is salty", but the fact that someone has asserted it. The predicate "is asserted" does add something to the Thought, differently from "is the true".

But Frege's point on the role of assertoric force in 1915 can still be found in 1879, with some provision. What we can say is that in making inferences, we need assertoric force, but in justifying the laws of logic, we need something akin to a truth-predicate[10]. Greimann (2012, p.81) points out similarly that "we cannot explain what we want to express with '$\vdash$ ' in the formal language without using the word "true" in the metalanguage". And similarly, Pedriali (2017, p.13) says that "Assertoric force is both

[10]This point will be clearer later, when the questions about illocutionary objectivity will be discussed.

essential to inference and inexpressible by language, in exactly the same way that truth is essential to logic but formally undefinable".

If the judgement-stroke is an illocutionary force indicating device, and if judging is primarily a normative feature of the language, when is an agent warranted in asserting something? Judgements in the concept-script always occurs in two contexts: as axioms (the basic laws), or following the rules of inference. In every case, Frege is using a single rationale: that for any Δ, Δ can be asserted if Δ cannot be other than the True. And this, as I take it, can be explained performatively. If in every case Δ is the True, then it is incoherent to deny that Δ is the true. This is also what Frege seem's to imply when stating that axioms self-evidently true, *viz.* those statements that do not require proofs, given that simply grasping the Thought is enough for one to recognize that it is True. In (*GGA*, II, §60), in the context of arithmetic, he states that "we must demand that any assertion that is not completely self-evident [*selbstverständliche*] must actually be proven". In the afterword to the *Grundgesetze*, Frege recognizes its doubts about the Basic Law V. The main difference to the other Basic laws, as he claims, is its lack of obviousness: "I have never concealed from myself that it is not as obvious [*einleuchtend*] as the others nor as obvious as must properly be required as a logical law". Moreover, claims that axioms are truth's that do not require proof is found, for example, in the 1914 unpublished *Logik in der Mathematik*: "The axioms are truths as are the theorems, but they are truths for which no proof can be given in our system, and for which no proof is needed" (*PW*, p.205).

What this all means can be summarized as the following principle:

> **Principle of Coherence for Judgements:** It is incoherent to deny something that one's knows to be True in every scenario.

The principle of coherence is not a condition under which something is True, but only a condition under which one may make judgements. It could be said that this is a judicative consequence of the principle of non-contradiction, that Frege clearly embraced. If a given Thought cannot be both true and false, then it is incoherent to take as false something that cannot be other than the true. Likewise, it is incoherent to take as true something that cannot be other than the false. But since Frege does not have a illocutionary force for negations, this other direction is not needed.

The principle is thus a condition for judging in the sense of warranting an agent to recognize something as true, not in the sense of directing or forcing someone in judging. It warrants judgements both for axioms and for rules of inferences. Grasping the thought of a basic law suffices for knowing that it is true under all circumstances, and thus, cannot be coherently denied, and thus can be judged as such. If I succeed in grasping the sense of $\begin{array}{l}\top\, \Delta \\ \llcorner\, \Delta\end{array}$, and if I succeed in realizing that this function is True in every case, then it is incoherent to deny its truth. For the rules of inference, the recognized truth of the premises shows that denying the conclusion cannot take place. Both cases serve as licenses, from the normativeness of logical laws, for an agent to judge and assert an axiom or the conclusion of a rule of inference.

In the 1897 unpublished paper *Logik* Frege states that "If anyone tried to contradict the statement that what is true is true independently of our recognizing it as such, he would by his very assertion contradict what he had asserted" (*PW*, p.132). Frege is refuting the truth relativist, and the idea that Truth is subjective in judgements just as the predicate "beautiful" in aesthetic judgements are. His conclusion is that "If anyone seriously and sincerely defended the view we are here attacking, we should have no recourse but to assume that he was attaching a different sense to the word 'true'" (*ibidem*)[11]. Similarly, grasping the sense of a logical law and not judging it as true is only possible if true is taken in a entirely different sense.

In the *Grundgesetze*, in his criticism to the psychological logician, Frege writes the following:

> our nature and external circumstances force us to judge, and when we judge we cannot discard this law - of identity, for example - but have to acknowledge it if we do not want to lead our thinking into confusion and in the end abandon judgement altogether. [...] Whoever has once acknowledged a law of being true has thereby also acknowledged a law that prescribes what ought to be judged, wherever, whenever and by whomsoever the judgement may be made (*GGA*, p.XVII).

It is pretty clear from this that (a) it is in human's nature to judge, whenever the circumstance presents itself as such; (b) when we judge, logical laws are necessarily for thinking coherently. If we put both together, we may conclude that, according to Frege, grasping the sense of a logical law not only prevents one to deny it, but it also forces one in judging as true, if it is to judge at all.

I believe that the principle of coherence above stated is Frege's implicit premise for judging the basic laws and in justifying the rules of inference. It is not a principle for their own truths, as this holds from non-contradiction. What Frege need is a justification for going from the truth of the basic laws to the assertion of them as such.

[11]Frege's argument is another example of a performative argument, as Greimann (2015) argues.

In the unpublished *Logische Allgemeinheit*, Frege made a distinction between an Expository Language and an Auxiliary Language, a distinction that the editors of the Frege's *Nachlass* judged to be akin to the object-language/metalanguage. I will argue later that these are not the same distinction. But what we can assume for now is that Frege's concept-script is an Auxiliary Language (in the sense of being a tool for reasoning process), while the Expository language is simply ordinary English, German, or any other language used for communicative purposes. It is the language Frege used for exposing the concept-script and elucidating its primitives. Following this distinction, the relation between the judgement-stroke and the truth-predicate is:

Expository Language:	Auxiliary Language:
Predication:	Assertoric Force:
"It is true that"	$\vdash$

In using, for example, a rule of inference, an agent usually has in his hand some already asserted propositions. Suppose, for instance, that in the course of an inference, he judges a proposition in the form '$\vdash \Delta$'. He does so with assertoric force, and with assertoric force, he commits to the truth of Δ. In that case, he is able to grasp a new Thought: "that Δ is the True". Now, suppose that in the same inference he obtains '$\vdash \genfrac{}{}{0pt}{}{\Gamma}{\Delta}$', judging it with assertoric force. Given that he has advanced from the sense of $\genfrac{}{}{0pt}{}{\Gamma}{\Delta}$ to its truth-value, he has grasped another Thought: "That $\genfrac{}{}{0pt}{}{\Gamma}{\Delta}$ is the True". He can now use both information to the extend of the principle of coherence above described: If both Δ and $\genfrac{}{}{0pt}{}{\Gamma}{\Delta}$ are the True, then it is incoherent to deny that Γ is also the True. And this is enough for an agent to judge it with assertoric force: '$\vdash \Gamma$'. In this rather basic example, an agent may reason from information obtained from previously given assertions. These are made in the level of the auxiliary language, that is, the concept-script. But his reasoning happens in the expository level, and in order to be possible for him to judge Γ, he has to evaluate his information without necessarily having to assert them. In this process he can, for example, calculate what Γ being the False amounts to. But he cannot do that already judging that $\top \Gamma$ is the True. He thus proceed first on the level of Thoughts only. Frege in fact defines the act of reasoning as the grasping of Thoughts. In practice, every assertion made in the auxiliary level derives a 2^{nd}-level Thought that is predicating Truth of the Thought being asserted, which is fundamental in making new assertions, once enough reasons are present. Judging and reasoning can be thus seeing as a back-and-forth transitions from the expository to the auxiliary languages. The same way we cannot count on information that is not asserted as true, we cannot assert something as true without, at some point, evaluate possible truth-values of what follows from what we already has asserted.

In this picture, we can see how the judgement-stroke is read as marking the assertoric force of an assertion, not as a truth-predicate. From the modern speech act terminology, it functions as a illocutionary force indicating device. Up to Frege, the tradition have not clearly distinguished between the act of a judgement and the content of a judgement. The judgement-stroke is thus designed to indicate the *act* of a judgement. In the paper *Über den Zweck der Begriffsschrift* Frege claims: "Through this mode of notation I meant to have a very clear distinction between the act of judging and the formation of a mere assertible content" (FREGE, 1972, p.94).

The importance of judgements in Frege systems is clear. More than once Frege stated that he "start out from judgements and their contents, and not from concepts" (*PW*,p.16). In his review of the *Begriffsschrift*, Schröder writes that "the title [...] should have been replaced by 'Judgemental Notation'", or an *Urtheilsschrift* (*Review of Frege's Conceptual Notation* in (FREGE, 1972, p.224f)). Schröder suggestion, although being made as criticism of Frege's notation, is reasonable for other reasons, as Frege himself confirmed in the article just quoted: "I do not proceed from concepts, but from judgements" (FREGE, 1972, p.94).

The distinction between assertoric force and content of possible assertions has immediate consequences for the traditional division of judgements before Frege: the classification of universal, negative, hypothetical or disjunctive becomes a matter of contents to be judged and not about the judgements themselves (*BS*,§4). Others have recognized more than just one act relative to judgements. Brentano, for example, defended that negation should be included as an act of denial as well. For Frege, this is innocuous, simply because the content of a negation can be rendered from an affirmative assertion: "To hold one thought to be false is to hold a (different) thought to be true—a thought which we call the opposite of the first. [...] the assertion is still conveyed by the indicative form, and has no necessary connection with the word 'not"' (*PW*,p.149). The same point was made in the later article *Die Verneinung* [*Negation*]:

> Perhaps the act of negating [...] is a chimerical construction, formed by a fusion of the act of judging with the negation that I have acknowledged as a possible component of a thought, and to which there corresponds in language the word 'not' as part of the predicate

> - a chimerical construction, because these parts are quite different in kind. (*CP*,p.383).

The lack of distinction between the content to the act of judgement is what creates the impression that negation is another type of act. But this is just a way of mixing together the content of the judgement, which is a propositional negation, with the act of the judgement, which is an illocutionary act of affirmation. As Frege continues, "The act of judging is a psychical process, and as such it needs a judging subject as its owner; negation on the other hand is part of a thought, and as such [...] must not be regarded as a content of a consciousness". (*idem*). Of course, regarding judgements as part of the consciousness of a judging agent asks for better justification on judgements and their independent validity. The first part of the answer comes from the rules of inference, which we now discuss. The second regards the things about we judge, a topic for the next chapter.

2.2.2 Inferential lines

The rightful conditions under which judgements are made, apart from judging the axioms, are given by the *Rules for Inference*, and the main goal of the logical activity is to infer judgements from judgements using such rules. As Frege writes in the earlier unpublished paper *Logik*:

> Logic is concerned only with those grounds of judgement which are truths. To make a judgement because we are cognisant of other truths as providing a justification for it is known as *inferring*. There are laws governing this kind of justification, and to set up these laws of valid inference is the goal of logic. (*PW*,p.3)

The rules of inference are, in a way, permissions that allows us to derive new assertions from old ones, and they can be seen as the unfolding of the coherence principle above described. Thus, rules are required for the transitions between judgements. In this section, the rules will be only described as Frege did. Their correctness, or soundness for using the modern term, will be discussed later in section 4.2.

The concept-script rules of inference are dependent on two kinds of performative signs of the language. The first one, a one may expect, are what he calls the transition-signs in (*GGA*, §26). These are inferential lines. They are used mostly for two reasons: (1) they inform that a transition, or inference, is being performed; (2) they encompass the permission expressed by the inferential rules.

The concept-script was presented with different set of rules in both the 1879 and 1893 version. The former had a more simplified set: *modus ponens* and an unstated rule for substitutions. The 1893 version also included both rules but build up from them. On (*GGA*, §14) Frege defines the *modus ponens rule* in almost the same way he did in 1879 as the following:

Definition 8. *(Modus Ponens:) From '*$\begin{array}{l}\vdash\!\!\top\, \Gamma \\ \quad\llcorner\, \Delta\end{array}$*' and '*$\vdash\!\!-\, \Delta$*' one can infer '*$\vdash\!\!-\, \Gamma$*'.*

Two things must be noticed. The first one is that Frege defines his rule, or at least describes it, with mentioned assertions, that is, concept-script propositions having the judgement-stroke but flanked by quotation-marks. This is consistent with his view that inferences must always proceed from asserted premises to asserted conclusions. This is a recurring theme: all rules of inferences are transitions of judgements. But the presence of the quotation-marks is significant. The difference is that without them, the formulas are asserted on the act. This is not a problem for modern systems, where the *modus ponens* rule is defined in the syntax. Usually, one defines it by saying that ψ is a consequence of φ and $\varphi \to \psi$[12]. Preservation of truth is a topic only to be dealt in the semantics. Frege, of course, wants to have both ways: to express a permission for assertability and, at the same time, justify that the rules are truth-preserving. Nonetheless, the quotation-marks is another evidence for the illucutionary reading of the judgement-stroke. Frege is not simply stating that if $\begin{array}{l}\top\, \Gamma \\ \llcorner\, \Delta\end{array}$ and Δ are the True then Γ must be also the True. He is stating a permission (which is by itself an act) for asserting something (another act). Frege's justification for allowing such rule is semantical: by computing all truth-values of both asserted premises[13], we only leave as an option the truth of the consequent, allowing us to correctly judge it as true[14].

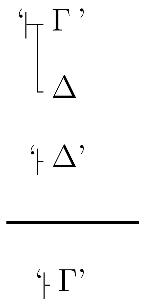
The second point is that all rules of inference are described using the permissive mood, usually with the verb "can" [*kann*] and similar verbs. Expressions such as "it is permissible" [*es ist erlaubt*] are also used in laying down the rules of inference in (*GGA*,§14 − 17). In logical derivations, this is expressed with the horizontal line that separates the asserted premises from the asserted conclusion. The symbol adopted to represent such inferential step is an horizontal line, separating the premises from the

[12] An example is found in Mendelson (2015, p.27), where not even the permissive mood is present.

[13] Frege actually speaks as 'being affirmed' and 'being denied' in the first logical book. This is because Frege adopts a semantics of judgements.

[14] A quick glimpse to the truth-table of the conditional can confirm that.

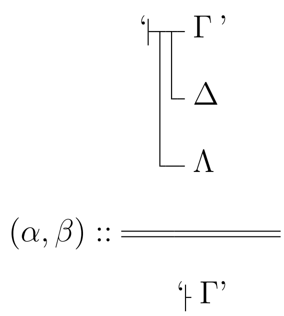
conclusion, as in:



This is an ‘inferential line’ or ‘transition-sign’ as Frege named them. In order to simplify the transitions, Frege adopted some abbreviations in both versions of the concept-script. He mostly simplified inferences by only quoting one of the premises on the top of the inferential line, as the following example shows:

Here, β is a label for ‘$\vdash \Delta$’, and α a label for ‘$\begin{array}{l}\vdash\!\top\ \Gamma\\ \quad\llcorner\ \Delta\end{array}$’. The double colon ‘::’ indicates that the quoted formula is the premise of the conditional being asserted, while single colon ‘:’ indicates that the quoted formula is a conditional for which the asserted premise is the antecedent.

One can also use a double inferential line for stacking inferences with multiple formulas being quoted, as the following example shows:



Here, α denotes '$\vdash \Lambda$', while β denotes '$\vdash \Delta$'. This abbreviation is present in both the versions of the concept-script. It was the only one used in the 1879 version of the logic, but in the *Grundgesetze*, as Frege justified it, more rules were introduced, and with each one, a different inferential line.

It should be mentioned that Frege's version of *modus ponens* and derived rules comes in the generalized versions[15]. First, notice that in $\begin{array}{l} \top\, \Gamma \\ \llcorner\, \Delta \end{array}$, Δ is a name for an object and can in principle be any term of the language (as defined above), no matter how many arguments it may have. More precisely, we can have nested conditionals with any number of subcomponents. This is hard to express typographically. First, we can abbreviate the conditional

$$\begin{array}{l} \top\!\top\, \Gamma \\ \mid\llcorner\, \Delta \\ \llcorner\, \Theta \end{array}$$

as

$$\begin{array}{l} \top\, \Gamma \\ \llcorner\, \{\Theta, \Delta\} \end{array}$$

With this abbreviation, we can express a generalized conditional as: from '$\vdash \Delta_{m \leq n}$' and '$\begin{array}{l} \vdash\!\top\, \Gamma \\ \quad\llcorner\, \{\Delta_1, ..., \Delta_n\} \end{array}$' one may infer

$$\text{`}\begin{array}{l} \vdash\!\top\, \Gamma \\ \quad\llcorner\, \{\Delta_1, ..., \Delta_{m-1}, \Delta_{m+1}, ..., \Delta_n\} \end{array}\text{'.}$$

But in order for this generalized *modus ponens* to be acceptable, Frege adopt another rule, that we shall declare:

Definition 9. *(Permutability of Subcomponents)*

From '$\begin{array}{l} \vdash\!\top\, \Gamma \\ \quad\llcorner\, \{\ldots, \Delta, \Theta, \ldots\} \end{array}$' *one may infer* '$\begin{array}{l} \vdash\!\top\, \Gamma \\ \quad\llcorner\, \{\ldots, \Theta, \Delta, \ldots\} \end{array}$'.

In practice, if we want to infer '$\begin{array}{l} \vdash\!\top\, \Gamma \\ \quad\llcorner\, \Theta \end{array}$' from '$\vdash \Delta$' and '$\begin{array}{l} \vdash\!\top\!\top\, \Gamma \\ \quad\mid\llcorner\, \Delta \\ \quad\llcorner\, \Theta \end{array}$', we first apply the

[15] He did not define it in this way though. I here follow Roy Cook's reading in the appendix about the concept-script in (FREGE, 2013).

Permutability of Subcomponents on the second premise to obtain '$\BGassert\BGconditional{\Delta}{\BGconditional{\Theta}{\Gamma}}$'. Only then we apply *modus ponens*. This is what Frege assumes for every generalized conditional and application of *modus ponens* in any subcomponent of the concept-script proposition.

Next, in (*GGA*, §15) Frege defines a rule for Hypothetical Syllogisms:

Definition 10. *Hypothetical Syllogism From '$\BGassert\BGconditional{\Delta}{\Gamma}$' and '$\BGassert\BGconditional{\Theta}{\Delta}$' one may infer '$\BGassert\BGconditional{\Theta}{\Gamma}$'.*

Just as in the *modus ponens* rule, this is represented by an horizontal, but on this time, dashed line:

$$
\begin{array}{c}
\text{‘}\BGassert\BGconditional{\Delta}{\Gamma}\text{’} \\[1ex]
\text{‘}\BGassert\BGconditional{\Theta}{\Delta}\text{’} \\
\text{– – –– – –} \\
\text{‘}\BGassert\BGconditional{\Theta}{\Gamma}\text{’}
\end{array}
$$

This rule could be derived solely from *modus ponens*, given that the concept-script proposition

$$\BGassert\BGconditional{\BGconditional{a}{b}}{\BGconditional{\BGconditional{b}{c}}{\BGconditional{a}{c}}}$$

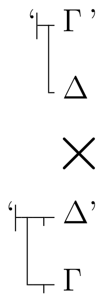
is provable from the basic laws.

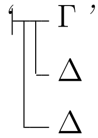
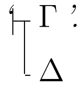
Next is the Contraposition rule:

Definition 11. *(Contraposition) From '$\BGassert\BGconditional{\Delta}{\Gamma}$', one can infer '$\BGassert\BGconditional{\BGnot\Gamma}{\BGnot\Delta}$'.*

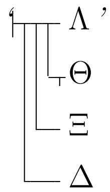
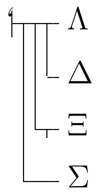
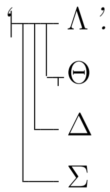
Frege describes this rule as the following: "One may permute a subcomponent with a supercomponent provided one simultaneously *reverses* the truth-value of each"

(*GGA*, §15). The transition sign for the contraposition is not an horizontal line, but the sign $\times$:

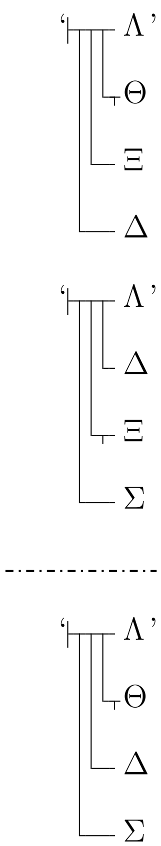


Definition 12. *(Fusion of Subcomponents) From*  *one can infer* 

As Frege explains this rule, "A subcomponent occurring twice need only be written once. We call this the *fusion* of equal subcomponents" (*GGA*, §15). Another version of this rule states that if two judgements agree on the supercomponent, they may be fused into one judgement that does not repeat equal subcomponents.

Definition 13. *(Dilemma) From*  *and*  *one can infer* 

The idea behind this rule, as Frege explains in (*GGA*, §16), is that if two judgements agree in their supercomponent but have contrary subcomponents, one may fuse both judgements with the same supercomponent and all subcomponents except the two contrary ones. If the two judgements have repeating subcomponents, it may be written only once. This rule has the dash-dotted horizontal line as a transition-sign:



Definition 14. *(Introduction of the Concavity) From '$\vdash \Phi(x)$' one may infer '$\vdash\!\overset{\mathfrak{a}}{\smile}\!\Phi(\mathfrak{a})$', provided that $\mathfrak{a}$ does not occurs in $\Phi(x)$.*

Frege writes it as "A proposition with a Roman letter can always be transformed into one with a German letter whose concavity is separated from the judgement-stroke only by a horizontal" (*GGA*, §17). The introduction of the concavity uses the following transition sign: $\smile$.

$$
\begin{array}{c}
`\vdash \Phi(x)\text{'} \\
\smile \\
`\vdash\!\overset{\mathfrak{a}}{\smile}\!\Phi(\mathfrak{a})\text{'}
\end{array}
$$

The last two rules that we can extract from Frege's *Grundgesetze* does not have a transition sign. This is because they are always performed in the context of other inferential rules, as it is the case for the Permutability of Subcomponents. The first one is the Fusion of Horizontals. Frege does not state it as a rule, but mention that since the negation, conditional and concavity functions all are flanked by horizontals, we may fuse exceeding horizontals and still have the same truth-value as reference. This holds even for the judgement-stroke, given that if Δ is the True, then $-\Delta$ also is the True. But, if Δ is anything other than the True, $-\Delta$ is the False. Likewise, if $-\Delta$ is the True, then Δ is the True, and if $-\Delta$ is the False, then Δ is either the False or any other object. A proper definition of this rule would need to specify each possible case in which the horizontal may be fused. Thus, we opt for a more general description of the rule as the following:

Definition 15. *(Fusion of Horizontals) In any judgement where an horizontal function occurs, one may derive a judgement without such horizontal if there is no change in the reference of the object-name or object-marker where the horizontal is applicable.*

Examples are:

1. From '$\vdash (-\Delta)$', infer '$\vdash \Delta$';
2. From '$\vdash - (\top - \Delta)$' infer '$\vdash\!\top\, \Delta$';
3. From '$\vdash\!\!\!\begin{array}{l} \rule{0pt}{0pt}\!\!-\!\!\!\rceil\ (-\Gamma) \\ \quad\ \llcorner (-\Delta)\end{array}$' infer '$\begin{array}{l}\vdash\!\!\rceil\ \Gamma \\ \ \ \llcorner\ \Delta\end{array}$'; and
4. From '$\vdash - (\smile\!\!\!\!^{\mathfrak{a}}\ - \Phi(\mathfrak{a}))$' infer '$\vdash\!\smile\!\!\!\!^{\mathfrak{a}}\ \Phi(\mathfrak{a})$'.

The final rule of inference is one of the most important ones in the concept-script: the rule of uniform substitution. This rule may also be called the rule of Roman letter elimination. Frege calls it as "replacement of Roman letters" in (*GGA*, §48). As he writes, "When citing a proposition by its label, we may effect a simple inference by uniformly replacing a Roman letter within the proposition by the same proper name or the same Roman object-marker". What this means is that in any concept-script proposition we can uniformly substitute a Roman object-marker or function-marker by either 1) a proper name or 2) another Roman object-marker or function-marker of the same type. In other words, this rule is a rule for instantiating the Roman-letter generality device by any other appropriate name or marker. The rule of substitution also has two variants: for German and Greek letters. But these are much simpler to state. We will include them as sub-cases.

This rule is difficult to specify, and thus, it is easier to do as Frege did:

Definition 16. *(Rule of Uniform Substitution) In any judgement, if there is an occurrence of*

1. *a Roman object-marker, then one can derive a new judgement by uniformly substituting it for another Roman object-maker or a proper-name;*

2. *a Roman function-marker, then one can derive a new judgement uniformly substituting it for another Roman function-maker of the same type;*

3. *a German letter on a concavity, then one can derive a new judgement by uniformly substituting it for another German letter in the corresponding concavity and all object-places or function-places where it occurs.*

4. *Greek vowel with smooth-breathing, then one can derive a new judgement by uniformly substituting it for another Greek vowel in all argument places of the function where it occurs.*

The rule of substitution, especially cases 1 and 2 above, are important features of Frege's system. It is akin to the modern use of comprehension axioms in the metalanguage.

We may quote a few examples. Given that Roman letters always occur in the context of judgements, these examples need not include quotation-marks. From $\vdash\!\!\begin{array}{|l} a \\ b \\ a \end{array}$, we may substitute b for a and obtain $\vdash\!\!\begin{array}{|l} a \\ a \\ a \end{array}$. Now from the rule of fusion of subcomponents, we may derive $\vdash\!\!\begin{array}{|l} a \\ a \end{array}$, which is Basic Law Ib.

From the Basic Law II, $\vdash\!\!\begin{array}{|l} f(a) \\ \smile^{\mathfrak{a}}\!\!-\, f(\mathfrak{a}) \end{array}$, we may substitute first $f(a)$ for $a+1>0$ to obtain $\vdash\!\!\begin{array}{|l} a+1>0 \\ \smile^{\mathfrak{a}}\!\!-\, \mathfrak{a}+1>0 \end{array}$. Next, we may substitute a for 0 an obtain $\vdash\!\!\begin{array}{|l} 0+1>0 \\ \smile^{\mathfrak{a}}\!\!-\, \mathfrak{a}+1>0 \end{array}$, *i.e.* the judgement that if it is True that for all objects $\mathfrak{a}$, $\mathfrak{a}+1>0$, then it is True that $0+1>0$.

Rules of inference are indispensable for the practice of logic. It is by means of such rules that judgements are possible, and their usage are clearly non-descriptive. They do not express or denote contents, but give permissions to perform a given action in the language under a precise number of conditions being met. And we saw that for most rules, Frege adopted an inferential line as means for marking the transition, but also to highlight a permissability. These lines are not functions, neither objects. They are unmistakably performative signs.

Even though Frege implicitly used more than one rule of inference, *modus ponens* is the most basic in both logics. In the *Begriffsschrift*, he declared to adopt a parsimony rule: "In logic people enumerate, following Aristotle, a whole series of modes of inference. I use just this one. [...] An inference using any mode of inference can be reduced to our case. Accordingly, since it is possible to manage with a single mode of inference, perspicuity demands that we do so" (*BS*,§6). Of course, we could simply add rules by defining them in terms of multiple occurrences of *modus ponens*. What matters for perspicuity is not only the easiness to conduct inferences, but also keeping track on the validity of inferences. Using just one declared rule makes life easier in this regard. The same point was made in the preface, where we read:

> The restriction to a single mode of inference is justified by the fact that in laying the *foundations* of such a conceptual notation the primitive components must be chosen as simple as possible if perspicuity and order are to be created. This does not preclude that, *later*, transitions from several judgements to a new one, which are possible by this single mode of inference in only an indirect way, be converted into direct ways for the sake of abbreviation. (*BS*,p.107).

Rules of inference have two main objectives: first, they enable us to derive new judgements. Second, they also allow us to keep track on the validity of inferences. These two objectives are present in the above quotation. It is preferable, for keeping derivations gapless, to follow only one mode of inference. But as soon as things get more complicated, abbreviations are welcome. And we saw just that in *Grundgesetze*, where Frege abandoned the parsimony ideal for a more pragmatic concern. As he claimed,

> The demand of scientific parsimony would now usually required [one mode of inference]; but considerations of practicality pull in the opposite direction, and here, [...] an inordinate lengthiness would result if I were not to allow some other modes of inference. (*GGA*, §14).

He was right. Even though the exposition of the system would be much simplified with fewer rules, the already difficult inferences in that book would be certainly even more hard to follow without them.

2.2.3 Definitions

Frege's remarks on Definitions are rather ambiguous. At times, he speaks of stipulative definitions, provided by a special sign in the formalized concept-script language, at others, he speaks of "concept formation" in a way that goes beyond such stipulative cases. Here we discuss only the first usage, leaving the second for later.

A definition is, putting it simply, the act of stipulating that some simple sign of the language have a specific meaning or content. Recall that the concept-script is a language to be used by reasoning agents, and so, must be intuitive for grasping contents and judging through them. Since our capacities of perception are limited, often we need simplifications for the complex formulas in a deductive process. This is Frege's motivation for definitions:

> The only aim of such definitions is to bring about an extrinsic simplification by the establishment of an abbreviation. Besides, they serve to call special attention to a particular combination of symbols from the abundance of the possible ones and thereby obtain a firmer grasp for the imagination. (*BS*,§23)

It should be noticed that logically definitions are superfluous: "[...] nothing follows from it which could not also be inferred without it" (*idem*). Thus, Frege's notion of a definition is constrained by what a language can actually do. We shall discuss this point later. Nevertheless, the justification for definitions in the formal language of the concept-script highlights that the proper role for them is pragmatic: to facilitate the conduction of proofs when complex expressions are used. It has only psychological importance, and no demonstration is in principle altered with the introduction of such new formulae. Thus, "[...] if from a logical point of view definitions are at bottom quite inessential, they are nevertheless of great importance *for thinking as this actually takes place in human beings* [my emphasis]" (*PW*, p.203).

The objective constraints governing stipulative definitions allows one to successfully shrink some complex expression needed in a proof, helping one visualizing the necessary links between the expression and other asserted truths that would be otherwise difficult to reach, and all that without lessening the soundness of the deductions being performed. In other words, definitions make proofs practically feasible, at the same time leaving the proved result independent of the practice.

Definitions are provided by means of the sign $\Vdash$, distinguishing itself from the judgement-stroke by including a second vertical line on the left of the content-stroke, or horizontal. They are always used in the context of the identity of contents sign, or the equality sign. On the former, definitions are provided as

$$\Vdash (\varphi \equiv \psi)$$

Meaning by it that the content of the sign φ is *to be* regarded as the same as the content of the sign ψ. Frege opt to present the defined sign in the right side, the *definiendum*, and the definition in the left, the *definiens*. Definitions thus yielded are declaratives on the form: "Let the sign φ have the same content of the sign ψ". This was, in fact, Frege's own explanation: "It does not say 'The right side of the equation has the same content as the left', but 'It is to have the same content'. Hence this proposition is not a judgment" (*BS*,§24). In the mature logic of the *Grundgesetze*, with the change for the objectual equality sign, definitions stipulate that the signs now share the same sense and reference (*GGA*, §27), thus:

$$\Vdash (\varphi = \psi)$$

The declarative reading thus became: "Let the sign φ have the same sense and reference of the sign ψ". Now, a great care for definitions is given by Frege, in laying down a series of principles governing them. Roughly, that definitions must always have a reference, always be unique and simple, and that any n-ary function must be defined with another n-ary function already given (*GGA*, §33).

There is an important subtlety about the sign for definitions having two vertical lines to the left of the content-stroke for the identity being introduced. First, the presence of the content-stroke indicates that whatever follows the definition, must be an assertible content. But the definition itself does not relate to such content-stroke, nor to the fact that it is an assertible content. This is because the sign for definition is an implementation for the sign of judgements, $\vdash$. Thus, it is the first vertical line in $\Vdash$ (from left to right) that can be regarded as the definition stroke. The core idea is not simply to define an expression, but to put forward a *fact* about such signs. We do not discover that the two expressions share the same meaning, we *make it so*. But, as soon as we transform the language, we can state the very new fact created, thus deriving an analytical judgement about it. This is Frege's strategy, to derive judgements out of definitions:

> [...] once the meaning of the new signs is specified, it must remain fixed from then on; and therefore [the identity proposition] holds also as a judgment, but as an analytic one, since we can only get out what was put into the new symbols. This dual role of the formula is indicated by the doubling of the judgment stroke (*BS*,§26).

In other words, once a definition is successful in establishing the content of the new sign, it is a matter of an assertion to recognize the truth of the identity thus yielded. We therefore conclude that the definition is so designed to set forward a fact that can be asserted, which roughly is the same as putting forward a condition for an assertion.

Frege was highly cautious about the creative powers of definitions. In a way, this is consistent with his ontological perspective about the contents conveyed by the expressions of a logical theory, as we shall see later. In another, it also highlights the active function that definitions have in the concept-script, as reasoners have only the power to define new expressions in the concept-script language within the rules of formation layed down previously. In this setting, there is no risk in wrongly taking something to exist by a simple act of definition, and thus, to wrongly claim any knowledge acquisition by means of definitions alone. This objectivity in definitions is necessary for keeping the inferential process safe, as far as validity is concerned.

In conclusion, just as the judgement-stroke and the transition-signs, the sign for definitions or definition-stroke belongs to a different class: it does not intend to express a content, but to *define* that a given expression have the intended content as a fact, put forward by the very definition. In modern speech act terms, it is an illocutionary force indicating device for a declarative speech act, one that it is restricted only to expressions of the concept-script. The presence of the definition-stroke *in* the language also highlights the performative side of Frege's concept-script, as it is expected from agents to define new expressions in order to make the inferential process more feasible.

2.2.4 Why an Act-based Reading

In a way, the highlight of Frege's illocutionary features is not new, and we have a good measure of literature on Frege's conception of judgements. That both inferences and definitions are illocutionary acts is not a controversial reading by any measure. And even today, inferential rules and definitions are common features of logical practice, albeit in different terms and different forms. The judgement-stroke, on the other hand, divide opinions. It is profitable then to compare the reading here adopted to other attempted readings.

One such alternative reading comes from Landini (2012), who reconstructs Frege's logic as a term-logic, as the concept-script in fact read that way. If we look into Frege's way of reducing sentences to names, one may conclude that he is breaching the distinction between formulas and terms. But Landini's way out of this is to read Frege as taking the judgement-stroke as an operator biding terms and transforming them to well-formed formulas in the language.

We already saw how every function in the concept-script, particularly in the 1893 version, are term-forming functions. If $f(x_1, ..., x_n)$ is a term of the language, then, "To form a *wff* in the formal language, the turnstile sign '$\vdash$' is appended to a term of the language. Thus, for example, $\vdash f(x_1, ...x_n)$ is a open *wff* of the formal language of Frege's *Grundgesetze*" (LANDINI, 2012, p.31). In arguing for his case, Landini quotes Frege's definition of the judgement-stroke in (*GGA*, §2), where Frege explicitly says that the sign is used for making judgements. In order to read away this passage, Landini (2012, p.31) says the following:

> It is certainly one of the consequences of Frege's account of judgment that you cannot succeed in judging a false or truth-valueless thought any more than you can succeed in seeing what is not there or knowing what is false. But Frege only says that attaching a turnstile "aims" to make a judgment. Nothing in the above passage suggests that Frege holds the occult view that the mere syntactic act of attaching the turnstile sign ipso facto succeeds in making a judgment!

But this is doubtful to read out from Frege's words. First, he does not say that his judgement-stroke is to be read so slightly, as just *an attempt* to make a judgement. In attaching the judgement-stroke to a term, one is *ipso facto* making a judgement, even if it fails to reach the True. A failed assertion, for example, is not an assertion that does not name the True, but an assertion that fails in being an speech act. If I assert that "Basic Law V is true", I have still asserted something, albeit falsely. In contrast, a failed assertion is a speech act that failed to fulfill its illocutionary conditions. In Frege's case, an example of a failed assertion is '$\vdash \varnothing$'. This is not an assertion simply because the rules of the concept-script language do not include the symbol '$\varnothing$' as primitive, neither is the symbol '$\varnothing$' previously defined. The syntactic act of attaching the judgement-stroke is the manifestation of a judgement, *viz.,* an assertion. But even this manifestation is more than just the writing of a vertical stroke: it is an illocutionary act. And to be fair, there is nothing wrong in making wrong assertions, as far as the illocutionary act is concerned.

There is also no indication that Frege defended a infallible sense of making assertions. Given that he saw the practice of logic in *a par* with scientific practice, being developed by approximation, we have to believe that he saw the possibility of wrong assertions being made. The concept-script is not designed as an infallible tool, and missing the target in judging is not necessarily a fail in the rules of inference being considered, but a fail in following the rules rigidly.

In the preface to the *Grundgesetze*, and before putting his doubts on Basic Law V, Frege writes that the system is designed in such a form to also facilitate the recognition of an error:

> I have listed everything that can facilitate an assessment whether the chains of inferences are properly connected and the buttresses are solid. If anyone should believe that there is some fault, then he must be able to state precisely where, in his view, the error lies: with the basic laws, with the definitions, or with the rules or a

> specific application of them. If everything is considered to be in good order, one thereby knows precisely the grounds on which each individual theorem rests. (*GGA*, p.VII)

Ironically, Frege was right. The inconsistency was easily identified in following from Basic Law V. Although one may still point the finger at his rule of uniform substitution, an analogue for an unrestricted comprehension principle.

A similar point is found in discussing Russell's paradox and Schoenflies' attempt out of the paradox, in the unpublished 1906 *Über Schoenflies: Die logischen Paradoxien der Mengenlehre*. Frege writes that discovering a paradox is not possible by simply analysing the concepts involved. This, of course, let it open the possibility for mistakes:

> That the concept of a right-angled equilateral pentagon contains a contradiction does not make it inadmissible. For we can see no reason why a man should not be able to say of an object that it is not a right-angled equilateral pentagon, or why he should not be permitted to say there are no right-angled equilateral pentagons. And before he arrives at such judgements, he must consider the matter, and to do that he requires this concept. It is completely wrongheaded to imagine that every contradiction is immediately recognizable; frequently the contradiction lies deeply buried and is only discovered by a lengthy chain of inference (*PW*, p.179)

What Frege is saying is that the discovery of the inconsistency of his Basic Law V, for example, is not something readily visible in the concept of courses-of-value, or extensions of concepts. Before discovering the inconsistency, it is natural that agents will consider such concepts and even judge an object as satisfying it beforehand (as Frege did). Judgements, and by extension assertions, are by no means infallible, even when strictly followed by logical laws. The role of the concept-script, as a language, is to keep the chain of inferences as sound as it is possible. If the language is designed to make the assertions rigidly made, the possibility of error will remain at a minimum. If now we know that Frege's Basic Law V is inconsistent, it does not mean that Frege was not asserting it when he published the *Grundgesetze*. Nor can we say that Frege was lying.

One of the passages ofently used to justify this reading that the judgement-stroke functions as a predicate, thus turning terms into sentences, is found in (*BS*, §3):

> We can imagine a language in which the proposition "Archimedes perished at the capture of Syracuse" would be expressed thus: "The violent death of Archimedes at the capture of Syracuse is a fact". To be sure, one can distinguish between subject and predicate here, too, if one wishes to do so, but the subject contains the whole content, and the predicate serves only to turn the content into a

> judgment. *Such a language would have only a single predicate for all judgments, namely, "is a fact".* We see that there cannot be any question here of subject and predicate in the ordinary sense. *Our ideography is a language of this sort, and in it the sign* $\vdash$ *is the common predicate for all judgments.*

In this passage, Frege seems to be saying exactly that the concept-script is a term-language with a single predicate: the judgement-stroke. All contents then would be rendered as nominalizations, and all judgements as the predication of "is a fact" to that content. This position is clearly not the position that Frege took later. His discussions about the truth-predicate made it impossible to accept that $\vdash$ is a predicate. For "is a fact" is just as insufficient for going from the level of senses to the level of references as the truth-predicate is.

But even for the *Begriffsschrift*, I take that Frege's words should not be taken too seriously. For the goal of this passage is not to offer a definition of the judgement-stroke (if that was the case, Frege would not have called it a judgement-stroke), but to read it on the context of the subject-predicate distinction. "One can distinguish between subject and predicate here, too", he claims, but with the provision "*if one wishes to do so*"[16]. Let's not forget that Frege begins §3 with the words: "A distinction between subject and predicate does not occur in my way of representing a judgment." Taking the above passage as claiming that $\vdash$ is a predicate would contradict this exact claim. And in the remainder of that book, Frege rarely speaks again in the predicate "is a fact", although he is not precise in his wording. In defining the conditional, for example, he speaks of "being affirmed" and "being denied". In other passages, he speaks of a "circumstance occurring", or "a proposition holding". I believe that Frege still hadn't a clear notion of what a judgement aims at: whether the True, a fact, an affirmation, etc. This imprecision makes it even harder to sustain that Frege was, in the quoted passage above, offering a definition for the judgement-stroke as a predicate, if he hasn't a clear understanding of what was being predicated about.

But Frege's logic still is a term's logic. And reading the judgement-stroke as a *wff*-operator seem to make sense for avoiding the conclusion that Frege violated the term-formula distinction in his system. This is Landini's choice. But whether Frege violated the term-formula distinction is only a secondary question. The real question is to ask whether Frege hold the idea that sentences are just names for truth-values. And he clearly adopted such a view. The term-logic in the *Grundgesetze* is consistent with his philosophical view:

[16]The emphasis is mine.

terms for Truth-values expresses Thoughts, and Thoughts are the senses of sentences. Names for truth-values, thus, are representatives of sentences in Frege's system. But the system only operates with truth-values, and the judgement-stroke is not a syntactical tool for making terms into well-formed formulas. The turnstile has a semantical role that must not be ignored: Frege has no metatheory or any model-theoretical means for going from a sentence to its truth-value. Only with the judgement-stroke, as illocutionary device, does one can claim that a name has the True as truth-value.

In their detailed reconstruction of Frege's *Grundgesetze*, Cadet and Panza followed this term-reading. But their reading of the judgement-stroke does not go much further from Frege's own words:

> a statement is obtained by letting a special symbol precede either a name of a truth-value, or an expression obtained by appropriately transforming such a name. This special symbol is '$\vdash$': letting it precede the name of a truth-value results in the statement that the truth-value named by this name is the True; (CADET; PANZA, 2015, p.21)

Frege does reads an expression with a judgement-stroke as a "concept-script proposition" [*Begriffsschriftsatz*], which seem to be what the authors meant by a statement. If that's so, they are certainly right, as Frege does seem to be implying that a concept-script proposition, equipped with the judgement-stroke, as statements, or utterances. If read as simple sentences, then prefixing a term with a judgement-stroke does nothing other than express an assertoric sentence with a truth-predicate, which, as we know, does not contribute for taking the step to truth-values.

Another possible reading for the judgement-stroke is to consider it as being an ambiguous sign. This is David Bell's reading. He claims that "The judgment and the horizontal strokes are, then, systematically ambiguous syncategoremata: they have no independent meaning outside contexts of certain types" (BELL, 1979, p.105). For instance, there is the logical context, in which the judgement-stroke is designed for making it explicit when a proposition is being taken assertively, and when it does not. There is the syntactic distinction, in which $\vdash \Delta$ means that Δ is the True, whereas $— \Delta$ is the nominalization of Δ. But most importantly (at least as I read it), there is the pragmatic context. In this context, the difference between '$\vdash$' and '$—$' is a distinction of two types of acts: the former is an assertion, while the latter is a supposition. In this sense, Bell (1979, p.98) correctly says that "In writing '$\vdash$' before a sentence one *is* asserting it, one is not claiming to assert it. The judgement stroke is, so to speak a *pure performative operator*". Moreover, he also claims that "The entities with which logic deals are neither sentences nor propositions, but bear a striking resemblance to what have lately come to be known as *interiorized speech acts*: mental acts whose linguistic counterpart is the assertive utterance of a declarative sentence" (*ibidem*).

Bell does not see the judgement-stroke as reducible either to one of the other of the different uses he describes. But as I see it, there is some level of dependency of the logical and syntactical to the pragmatic reading. For instance, on the logical distinction, if Frege claims that without assertoric force, one does not assert a proposition, and if it is the case that the judgement-stroke is what confers assertoric force to concept-script propositions, then it is in virtue of being a speech act that the judgement-stroke is capable of distinguishing when a proposition is being asserted and when it is not. In a conditional such as '$\vdash\!\begin{array}{l}\Gamma\\ \Delta\end{array}$', the constituents Δ and Γ are not being asserted simply because the judgement-stroke always binds the whole together and only confers assertoric force to the whole. Since they are not prefixed with the judgement-stroke, they are unasserted components of the conditional. The opposite, however, does not hold. We cannot say that the logical context is supervening on the pragmatic one, as we would be claiming that assertoric force is present whenever a proposition is logically being asserted, as this is a circular definition.

The same holds to the syntactic distinction. If the judgement-stroke is not primarily an assertoric force marker, then we have no reason to take — Δ as the nominalization of Δ. To be sure, there is an obvious syntactic distinction between the sentences

(1) This book is red.

(2) That this book is red.

But if (1) is not uttered with seriousness, as Frege would put it, then (1) and (2) has *only* syntactic differences. And we can thus say that — and $\vdash$— are two different signs, but we want them to differ because the latter has to convey assertoric force, while the former don't. A syntactic distinction with no intended purposes is of no interest to Frege. Thus, I agree with Bell that the judgement-stroke can be read either from a logical, syntactical or pragmatical standpoint, but it seems that both the logical and syntactic contexts of use for the judgement-stroke are dependent on the fact that it is a sign designed for expressing primarily the assertoric force.

A similar distinction is made by Greimann (2012), who distinguishes between a psychological and a logical sense of assertion. Greimann is attributing to Frege an "Assertoric theory of True", that truth is for Frege what is expressed by the form of the assertoric sentence. Ruffino (2006) has objected against such reading, that the assertoric force is nothing but the internal approval of who entertains the thought in question. Greimann's response to Ruffino is that in his understanding, the form of the assertoric force also includes another constituent: the predication of truth.

Consider the following two sentences:

(3) I acknowledge that it is true that sea water is salty.

(4) It is true that sea water is salty.

The psychological sense of the assertion presented in the judgement-stroke is precisely the belief attitude in (3), where the agent is expressing primarily his belief that sea water is salty. The logical sense, on the other hand, is the direct expression of the truth of the Thought presented, as in (4). Thus, in the logical sense, the judgement-stroke functions as a truth-operator, where an agent uttering "sea water is salty" with assertive force is expressing primarily that it is true that sea water is salty.

Greimann takes the logical sense as having priority in Frege's judgement-stroke. In order to avoid the objection that the psychological reading is insufficient, he claims that the form of the assertoric sentence makes an assertoric attribution of truth, and that truth "is the objective part of what is expressed through the form of assertoric sentence, that is, the part we can understand without knowing who makes the assertion" (GREIMANN, 2012, p.78).

In (GREIMANN, 2000), he in fact defend the logical view in full. He argues a position very similar to Landini's. His account is that

> the judgement-stroke is, in the first place, a truth-operator, not an illocutionary force marker. That is, this sign is a device for attributing truth to judgeable contents (or thoughts), not a device for indicating certain propositional attitudes or illocutionary acts: it expresses that something is true, not that the speaker holds something to be true. (GREIMANN, 2000, p.215)

In order to defend this claim, he offers five points that I will briefly mention:

1. Frege in his earlier writings speak of '$\vdash\Delta$' as saying that "Δ is a fact", and that in the later writings he expresses them as "Δ is the True". This suggests that $\vdash$ is making a truth-valuation, not only the speaker's evaluation about the truth.

2. Frege's introduction of the judgement-stroke marks his departure from natural language, as means for expressing a truth-operator that, in natural language, is only expressible through the assertoric force of the sentences.

3. Frege departure from the classical combinatorial view of judgements, such as Kant's theory, in favor of a "holistic" approach in which all judgements are affirmative. Therefore, '$\vdash \Delta$' is the common form of all attributions of truth to a content;

4. In judging something as true with the judgement-stroke we are not predicating truth of a judgeable content, but determining its relation to reality. This, however, corresponds to the form of the assertoric sentence, not to any ordinary predicate.

5. The distinction between the psychological and logical role of the assertion (above discussed), shows that Frege's intentions was to present something as being true, not just that the speaker holds to be true.

I do not wish to go against all this claims, as I agree with most of Greimann's premises. The point, in my view, is that this characterization surpass the fact that Frege wants the assertoric force itself in his concept-script, not just the form of the assertoric sentence. What makes it possible to express that a judgement took place is the form of the assertoric sentence, which needs the assertoric force.

It is also unquestionable that the concept-script is a language to be used, as Frege believed it would be a legit scientific instrument. And he explicitly says in multiple times that judgements are inner events expressible through the judgement-stroke. The sign is defined as the acknowledgement of the truth of a Thought. Had Frege thought that truth-claims were objective in the sense of not requiring a bearer, he would avoid the "acknowledgement" part entirely. As I read it, the judgement-stroke is first and foremost a performative sign of the concept-script language. Greimann is right that an agent asserting through the judgement-stroke is not only expressing his belief, but making a genuine truth-claim. But the latter doesn't go without the former. The judgement-stroke certainly involves a irreducible psychological element, as it expresses the truth of a Thought *through* a psychological attitude, *viz.,* in a judgement that p, p is true *insofar* as it is acknowledged to be true. It does not rule out the fact that we can read "Δ is the True" from '$\vdash \Delta$', if one comes to read it in the *Grundgesetze*, for example. But the former is just a predication of the true of Δ which follows the assertion of Δ with assertive force. If the *Grundgesetze* is readed with the desired seriousness, '$\vdash \Delta$' can be read as "Δ is the True" *because* one has judged it with assertoric force.

In the concept-script, and in the context of inferences, we need assertoric forces as means to assert the true of the thoughts being presented. That a thought is asserted does imply that the thought is true (if, correctly judged). But truth-predication is meaningless without assertoric force, and *a fortiori* the logical sense of an assertion is nothing without the psychological sense (as all utterances are events from the speakers perspective).

Predication of truth is a secondary feature in Frege's logic. It is not that we can go from

(3) I acknowledge that it is true that sea water is salty.

to

(4) It is true that sea water is salty.

or the other way around. In a sense, both are consequences of uttering "Sea Water is Salty" with assertoric force. But none of them represents the assertoric force Frege wants to convey with $\vdash$, if they are not *uttered* with the necessary force. Neither of them would perform the desired step from the level of senses to the level of references. Greimann concludes very similarly from what Landini had much later: that the judgement-stroke functions in the syntax of Frege's language as an operator, taking names of truth-values and transforming them into sentences. But this, I believe, is an incomplete step without the aforementioned assertoric force.

Taking the judgement-stroke as first and foremost an illocutionary act is consistent with the other performatives of Frege's logic. It is unquestionable that Definitions and Inferences are also acts performed by an agent. A definition is a stipulation. It depends on the active side of a speaker to put forward a new meaning for signs that had none. An inference is also an act, where an agent follows some rule and transform a set of premises in a conclusion. And Frege does explain inferences in terms of judgements.

Finally, the main motivation for taking illocutionary forces in the language, particularly for judgements or assertions, is to fill a semantical gap. Their role is much closer to what we now call it semantical than it is syntactical. By taking the judgement-stroke as functioning only syntactically, we bypass Frege's reasons for declaring each and every inferential rule and point of inference as asserted ones. In the case of rules of inference, there is no alternative in Frege's universalist conception of logic other than to define *modus ponens* semantically, and we cannot ask for a soundness proof for it, for soundness is already embedded in it. Frege's concept-script is really a language of it's own. But to properly understand it, we must see the judgement-stroke as Frege himself declared: as a "sign of its own kind" (*GGA*,§26).

2.3 The Informativity of Logic

Before concluding this chapter, we shall go back to one of the key aspects of Frege's conception of logic as a language, now that we discussed most of his primitives. The scientific part of Frege's concept-script, discussed earlier, is related to how it differs to modern logical notations that stem from the model-theoretic perspective, particularly in being also an epistemic tool for discoveries. Instead of studying logical consequence, Frege's logic is a theory of inferences, expressed through judgements. Judgements, are expressed through belief attitudes. Frege's claims that logic can be informative by its own is another important aspect of the performative reading of the concept-script, where judgements, inferences and definitions are foundamental as expressing proper acts.

This epistemic import was applicable to arithmetic, as Frege believed. It meant that we could derive arithmetical truths from logical laws in a analytical, but still informative way. Most of Frege's remarks on this are found in the *Grundlagen*. He questions "[...] how do the empty forms of logic come to disgorge so rich a content?" (*GLA*, §16). The fact that arithmetic is a contentful science is a unquestionable premise for him. If it is possible to show that arithmetic is just a developed logic, then

> [...] the prodigious development of arithmetical studies, with their multitudinous applications, will suffice to put an end to the widespread contempt for analytic judgements and to the legend of the sterility of pure logic. (*GLA*, §17).

Analyticity is a well desired property for logicality, at least if taken in the Kantian sense[17]. If a given argument is deductively valid, it is said to be analytically valid. Deductive, or analytical, form of reasoning is often in contrast to synthetic forms of reasoning. This terminology traces back to Kant. In the first *Critique*, Kant defines analytic judgements in terms of containment between the concepts related: when the concept-predicate is already contained in the subject-concept[18]. In synthetic judgements, on the other hand, this containment does not hold. Moreover, Kant stresses that in analytical judgements we are justified in judging the truth of the connection by non-contradiction: "we must also allow the *principle of contradiction* to count as the universal and completely sufficient *principle of all analytic cognition*" (KANT, 1998, B191).

Analytic judgements, being grounded on logic alone, are not capable of extending knowledge. For that reason, Kant calls them *judgements of clarification*. Even though

[17]The concept of Analyticity became a hot topic in philosophy after Quine's famous criticism in "Two dogmas of empiricism" (QUINE, 1951). But Quine is mostly reacting to the logical empiricist's take on the matter.

[18]Kant's theory of judgements will be discussed in detail later.

formal logic is capable of judging with synthetic judgements, it is incapable of judging beyond them. This is because Kant sees logic as only a formal canon for reasoning: it "abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking" (KANT, 1998, B78)[19]. By being so, formal logic does not have proper objects. It is an uninterpreted formal calculus, and is not a science for the purpose of extending knowledge. It is not "[...] a universal art of discovery [...] and not an organon of truth - not an algebra, with whose help hidden truths can be discovered" (KANT, 1992, p.534), Kant says in the *Jäsche Logic*.

So, for Kant, how can logic disgorge such a rich content? It simply can't. At least not without the aid of intuitions. In Kant's account, it is not possible for a logically valid argument to convey any information that was not already contained in the premises. And given that Kant also accepts that mathematics is a proper science, that is, have a intended interpretation, it follows that mathematics cannot be founded in logic alone. This is the background of Frege's attempt to show that analytic judgements, *contra* Kant, can indeed extend knowledge. Kant had, for him, "[...] underestimated the value of analytic judgements" by "[...] defining them too narrowly" (*GLA*,§88).

Kant's "too narrow view" follows, for Frege, from two fronts: the definition of analyticity and concept-formation. In redefining the former, Frege takes analyticity as being a feature of inferences, not of contents. In (*GLA*, §3) he claim

> The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one.

Thus, a judgement is fregean-analytic if it has a pure logical proof, that is, if it follows from basic logical laws, definitions, and truth-preserving rules of inferences. Within this process, concepts are instantiated and proofs proceed to judge facts about them. The informativity of logic, then, rests upon the process of concept-formation.

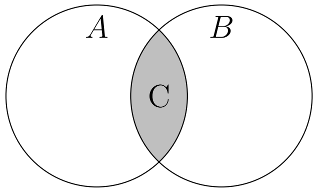
For Frege,

> [Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here we are not simply taking out

[19]This is what Kant calls the *General Logic*. By contrast, it is the Transcendental Logic who deals with the proper connection between concepts and the objects of experience.

> of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. (*GLA*, §88)

Kant's way of defining concept is well represented in boolean algebra. Let C be composed of the concepts A and B. This means that $C = A \cap B$. In this case, the conclusions that 'C is A' or 'C is B' are trivial, and could well be visualized with Venn's Diagrams:



This is what Frege meant by something being contained as "beams are contained in a house". Under this scenario, analytic judgements are trivial, and logic cannot proceed fruitfully. We can trivially form all possible concepts in those same boundary lines[20]. If the concepts A and B are such 'boundary lines', then the set of possible concepts formed from them is defined over all the boolean operations $A \cap B$, $A \cup B$, $A \setminus B$, and so on. As Frege believes, this too narrow way of forming concepts, and their capabilities in a deduction, are the reason for "[...] the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot" (*BLC*, p.34).

In order to advance from such sterile way of defining concepts, Frege defends his "organic" form of definitions. Up to 1884, when much of these comments were made, Frege had one major result to show in which this method was extensively used: his definition of the Ancestral Relation in section III of the *Begriffschrift*. The Ancestral is a definition for the transitive closure of a relation. Briefly, it is a logical analysis for total ordered relations, which by his time was something believed to be grounded in intuitions. The main result achieved, theorem 133, states that if a relation R is functional, then thrichotomy holds for the ancestral of R[21]. From such result, Frege argued, "[...] it can

[20] Frege argues this point also in the unpublished paper about Boole's Logic: "If we look at what we have in the diagrams, we notice that [...] the boundary of the concept, whether it is one formed by logical multiplication or addition is made up of parts of the boundaries of the concepts already given. [...] These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way. (*BLC*, pp.33-34) .

[21] One can imagine how to achieve this intuitively: if no two objects follows from the same object in R, then, for any two objects a and b, we just have to follow the chain of objects related through R and see

be seen that propositions which extend our knowledge can have analytic judgements for their content" (*GLA*, §91). The same point is made in (*BLC*, p.34), but now highlighting the fruitfulness of the definition:

> If we compare what we have here with the definitions contained in our examples [of the Ancestral], [...] totally new boundary lines are drawn by such definitions—and these are the scientifically fruitful ones. Here too, we use old concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation and the conditional.

Frege often speaks of fruitful definitions or fruitful concepts[22]. This feature is, as it seems, pragmatically defined. Something is said to be fruitful if has ampliative consequences. And the way to properly show it is in the context of an inference. The proper role of logic is to draw inferences, and it is by doing so that we can recognize the truth of a proposition. Thus, it is by proving theorems that one is able to evaluate how fruitful, if at all, a definition is.

If fruitfulness is to be taken at centerpiece, then logic is said to be deductively, or analytically, ampliative if it has, or can define, fruitful concepts. We can see what Frege meant in saying that Kantian concepts were not suitable for the task. Every Kantian concept F includes a list of characteristic notes $c_1, ..., c_n$ such that $F = c_1 \cap, ..., \cap c_n$. Conclusions from F, such as 'F is $c_{i \leq n}$' are in fact read '$F \subseteq c_{i \leq n}$', which will holds trivially. Since F is representable in Venn's diagrams, every conclusion is already contained in the definition.

Frege's alternative for concept-formation stem directly from his conception of judgements and the priority given to them. He claims in multiple passages that "I start out from judgements and their contents, and not from concepts. [...] I only allow the formation of concepts to proceed from judgements" (*BLC*, p.16)[23]. This is known as the *Priority Thesis*. Once we get a judgement, we can decompose from its main components. Consider the following Fregean example. From the arithmetical judgement

$$2^4 = 16$$

that either b follows a, a follows b, or a and b are the same. But Frege's proof shows this result without such 'wandering' through the chain of objects. As he claims in (*BS*, §23), we can see in them how "[...] pure thought, irrespective of any content given by the senses or even by an intuition *a priori*, can, solely from the content that results from its own constitution, bring forth judgments that at first sight appear to be possible only on the basis of some intuition".

[22]For example, in (*GLA*,p.xxi), (*GLA*, §70), (*BLC*, p.33) and (*BLC*, p.27).

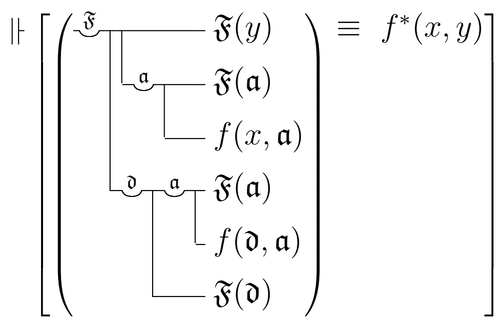
[23]Similarly, in a letter to Anton Marty: "I think of a concept as having arisen by decomposition from a judgeable content" (*PMC*, p.101).

one can decompose in one ore more of the constituents (numerals in object places, or functional symbols for second-level functions) to obtain different functions. Precisely,

$$x^4 = 16, \quad 2^x = 16, \quad 2^4 = x, \quad x^y = 16, \quad ,2^x = y, \quad x^4 = y, \quad x^y = z$$

Each expression above can be regarded as different function or relations obtainable from the original judgement: "4th root of 16", "logarithm of 16 to the base 2", "2 to the power of 4", and so on. Thus, in this procedure, "[...] instead of putting a judgement together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement" (*BLC*, p.17). This procedure was essentially introduced in (*BS*, §9), and was pivotal in proving the basic facts about the ancestral definition.

With the Ancestral, Frege argued for the possibility of proving theorems that, in his words, extend our knowledge. This is where decompositions are most important. Since stipulative definitions involve functional expressions, we can decompose either in the object or the function position and achieve new functions, either first or second-order, provided the priority principle[24]. Consider Frege's definition of the Ancestral in (*BS*,§26)[25]:



The *definiendum* on the right-hand side, $f^*(x, y)$, is read as "x is the ancestral of y", *modulo* the relation f. It is possible, following the above definition, to decompose the *definiendum* in the second variable, acquiring a new function that reads "to be x's descendent", $[z : R^*(x, z)]$, or likewise the first variable to obtain $[z : R^*(z, y)]$, that reads "being y's ancestor".

Clearly, it's easier to proceed with the function derived from the *definiendum*. From it, we can instantiate different axioms, deriving different theorems that are

[24]This is also called by Ruffino (1991) as the epistemological version of the Context Principle.

[25]I will consider Frege's earlier formulation of the Ancestral mostly for its simplicity over the definition of the *Grundgesetze*, which involves value-ranges. I will also simplify the notation for the *definiendum*.

consequences of the definition provided. This was shown by Frege in section III of *Begriffsschrift*. Specially, theorems 98 and 133 are the two main results concerning the definition of the Ancestral. The former is:

$$\vdash\!\!\begin{array}{l} f^*(y,z) \\ f^*(x,y) \\ f^*(x,z) \end{array}$$

which proves that the Ancestral definition is transitive. These results also provides higher-order properties in which the definition is provable to satisfy. Hence, they are theorems *about* the definition, but they are not components of it. This is Frege's point against Kant.

The use of decomposition is closely related to the irrestrict comprehension principles in quantificational logic, in the form:

$$\exists f \forall x (f(x) \leftrightarrow \varphi)^{26}$$

That is, for all x and any expression φ of the language, there is a function $f(x)$ that holds just in case the condition φ holds. Simply put, comprehension asserts that any well-formed expression of the language can define a concept, set, or in Frege's case, a function. We can see how decomposition provides comprehension principles in the language: given any decomposition φ of an expression φ' which is well-formed, there is a function f which holds whenever φ holds. Landini (2012, p.136) states that

> [...] decompositions provides comprehension principles as rich as those of a standard second-order calculus. As is well-known, a second-order calculus is not decidable and not even semantically complete. There can be no question as to the semantic informativeness of its theses. Decompositions is, therefore, all that is required for informativity.

Landini's is certainly right: comprehension is the right path to understand Frege's claims about the informativity of his logic. The difference, also pointed out by Landini, is that Frege does not have comprehension, but decomposition added to a rule of uniform substitution. Frege does not have a metatheory that includes a comprehension schema, but what he has is the presupposition that roman letters, as object and function-markers, denotes generality of contents. If a judgement is true for a generic function or object, it is true for any.

[26] See Boolos (1985) and Landini (1996).

The question about decomposition is also on how to visualize and decide the right decomposition to perform. This, I believe, is motivated by pragmatic reasoning. This point motivates a different perspective on the subject, proposed by Danielle Macbeth, particularly in (2012). In her reading, Frege's concept-script must be read diagrammatically in such a way that the two-dimensional notation is as helpful in visualizing conceptual relations than the diagrams in Euclidean geometry are. As she claims, "Frege's concept-script enables an extension of our knowledge by revealing something new that is achieved by (literally) putting together in joining inferences parts of different wholes into new wholes" (*idem.,* p.311).

Macbeth is correct in stressing the diagrammatically role that Frege's two-dimensional notation has in the practice. The problem of her account, in my opinion, is that it tends to reduce Frege's claims of informativity to a mere psychological realization, if not supplemented. For instance, she consider Frege's definitions on section III of the *Begriffsschrift* crucial for the informativeness of the conclusions about the ancestral relation. She claims that "if [...] we were to replace all the defined signs by their definitions, then we *would* have a mere theorem of logic", but, as she continues, "it would not be a theorem in the theory of sequences" (*idem,* p.306). Frege does claim that the definitions serve only pragmatical functions: the proof's validity does not depend on the use of definitions, as we could achieve the same results solely with the *definiens*. And to be sure, the definitions occurring *in* the concept-script are only abbreviatory devices: $f^*(x, y)$ does not by itself say that x is the ancestral o y, it is just an abbreviation for the more complex function of the *definiens*.

Frege is not always clear in speaking about definitions. There are the abreviatory devices that are used in the concept-script, and there are the so-called "fruitful definitions". The former are certainly not fruitful. I believe that the best interpretation for Frege's claims about the latter is that he is speaking about fruitful definitions *of concepts*, that is, functions of the language that are achieved by decomposition[27]. Macbeth is speaking about definitions as abreviatory devices, and these are wholly eliminable.

She is right in a sense that theorem 133 (or theorem 98 above quoted, for that matter) would be another formula entirely if only the *definiens* were present. But the transformation from one to another does not carry information in the logical sense Frege is speaking of, but only psychological. She claims that "the simple defined signs are needed if what is to be established is to be unequivocally about the concepts of interest. But their

[27]The fact that Frege always speaks of fruitful definitions in the context of rejecting Kant and Boole's definition of concepts speaks favorably to this interpretation.

definitions are needed if anything about those concepts is to be established." (MACBETH, 2012, p.306). But that Frege is offering an analysis for the notion of "following in a sequence" has nothing to do with the choices of abreviatory definitions. What matters is if the results are applicable. For instance, a good analysis of the concept of following in a sequence must show that the provided definition is transitive (theorem 98) and trichotomous (theorem 133). But it is not only the *definiendum* $f^*(x, y)$ that satisfy these properties. The *definiens* satisfies beforehand. "y follows x in the f-sequence" is Frege's chosen analysis for $f^*(x, y)$. But it is only a proper analysis if the *definiens*, that is, "y has all the hereditary-properties of x that all f-successors of x share", is provable as being transitive and trichotomous. And *because* they are, Frege's analysis can be acceptable as sufficient for the desired concepts[28].

On her diagrammatic reading, Macbeth claims that "Much as in reasoning through a diagram in Euclid one *perceptually* joins parts of different wholes to form new wholes, so here we *literally* join, by means of hypothetical syllogisms, parts of different wholes to form a new whole, ultimately, the whole that is theorem 133" (MACBETH, 2012, p.303). What she mentions as the hypothetical syllogisms are Frege's transitions from conditional judgement to conditional judgements by means of substitution and *modus ponens*. What she mentions as "parts of different wholes" are the decomposed functions that one obtains from another judgements. Her reading, nonetheless, offers an important insight: the definitions, much as Frege's notational choices, are indeed designed for visualizing these inferential patterns. The definitions, as abreviatory devices, are indeed helpful in visualizing the correct choices of functions one needs to instantiate in axioms or other judgements previously derived. Macbeth is right on this aspect: "Much as reasoning though a diagram in Euclid does, such a course of reasoning realizes something new that had the potential to be derived but was in no way implicit in the starting point of the derivation" (*idem*, p.307).

To understand it, all decompositions are not performed in the "object-language"[29], they are not, to say it differently, steps in the language that are governed by purely logical laws[30]. To a judgeable expression to be decomposed, and the subsequent instantiation of any axiom with such acquired function, there seems to be a gap: nothing in the original expression says which part is to be decomposed or which function from a multiplicity is to be derived. This is a procedure to be checked and justified, but nonetheless one that

[28]A similar point is made by Blanchette (2012), chapter 1.

[29]Landini (2012) also argues in this direction.

[30]But any result is logically valid, since they are equivalent to a comprehension axiom that states that such acquired function exists.

is not performed entirely in the logical language in question. If decompositions were mechanically and trivially determined, the fruitfulness of a definition wouldn't be a matter of testing one's consequences: "Definitions show their worth by *proving* fruitful" (*GLA*, §70)[31], as Frege declared.

There is no mechanical procedure to decide such decompositions and their fruitfulness. This agrees with the undecidability of second-order logic. Such systems have advanced expressive power, but limited meta-theoretical results. As consequence, second-order logic is undecidable and semantically incomplete, thus, not reducible to a mechanical procedure of discovery. Such meta-theoretical results must, by itself, provide a negative answer whether (second-order) logic is tautological. Propositional logic most certainly is, since truth-tables are mechanical procedures to determine whether a formula is a tautology. But even for propositional logic, the tautology problem is intractable, that is, not solvable in polynomial time[32], meaning that, even though it is in principle computable, it is not practically feasible[33]. Second-order logic, in contrast, is not even in principle computable: there is no algorithm possible to compute, in general, if a second-order formula is a logical truth.

The question then is how, in this scenario, can second-order logic be considered analytical, as Frege would want. Kant would hardly accept Frege's proofs as analytical, since decomposition and comprehension could be taken as intuitive features: what does license us to accept and acknowledge the obvious existence of such functions or concepts declared by such principles, if not by some form of intuition?[34]. Even if some form of intuition is a necessary condition for visualizing the existence of such functions (the ones acquired by decomposition), it is not clear how the logical validity of a proof featuring it would be affected, at least in the Fregean scheme of things. If we do accept the universalist reading of Frege's logic, then the realm of functions (or likewise, concepts) is an objective fact. They are things that we *grasp*, never construct: "[...] a concept is something objective: we do not form it, nor does it form itself in us, but we seek to grasp it" (*CP*, p.133). Therefore, if every step in a proof is correctly judged as true, the instantiation of an axiom or theorem with some acquired functional expression is justified by the general character of such laws, simply because, if the law holds generally, it holds

[31]The emphasis is mine.

[32]See, for example, D'Agostino and Floridi (2008).

[33]The difference is easily explained by D'Agostino (2016, p.176): "An effective procedure or "algorithm" by and large consists in a "mechanical method", *i.e.* one executable in principle by a machine, to solve a given class of problems (answering a certain class of questions). An effective procedure is *feasible* when it can also be carried out in practice, and not only in principle".

[34]Boolos (1985) was the one to mention such Kantian objection.

for a given function.

This answer puts the weight in Frege's assumption about the objectivity of functions. Russell's Paradox, it is well known, is a fair objection that Frege took for granted that such realm is easily obtainable. But at least in Frege's notion of analyticity, the picture is sound. Since in such proofs decompositions are admissible and logically justified (simply because the existence of functions is a feature of generality of logic), analytic judgements can extend our knowledge. Dummett (1991a, p.42) likewise conclude:

> Deductive reasoning is thus in no way mechanical process [...]: it has a creative component, involving the apprehension of patterns[35] within the thoughts expressed [...]. Since it has this creative component, a knowledge of the premises of an inferential step does not entail a knowledge of the conclusion [...] and so deductive reasoning can yield new knowledge. Since the relevant patterns need to be discerned, such reasoning is fruitful; but, since they are there to be discerned, its validity is not called in question.

From all this, it is clear how decomposition, the most important feature of fruitful definitions, helps one yielding new knowledge. The derivations performed within logic are, nonetheless, sound and analytically true in Frege's understanding of the word. Validity is explained not in terms of preservation of truth, but in truth assertability. But still, Frege's logic is something designed for human eyes, meaning that it is a logical language thought to help one carrying the proof over and still avoiding psychological or intuitive reasoning, simply because the assertability condition guarantees the soundness of the deduction[36]. Therefore, even though logic (second-order logic, as Frege understands the matter) is not a computable endeavour (as the undecidability makes it clear), not just the proposition proved, but the whole proof is something new and not easily available in advance. As Frege would put it, "[...] our thinking as a whole can never be coped with by a machine or replaced by purely mechanical activity" (*BLC*, p.35)[37].

But the weak side of this interpretation is that Frege could not be talking about the undecidability of second-order logic, or any limited metatheoretical result, when speaking about the informativity of logic. Landini's interpretation, then, is incomplete. At best, Frege was talking about a feature of second-order logic (decompositions), even though this feature is closely tied with its untractability from the metatheoretical standpoint as we now know (as second-order logic with full comprehension is incomplete).

[35] Such patterns, in Dummett's reading, is those functions that one can acquire by decomposition.

[36] This will be the topic of part II.

[37] Of course, this is his way of contrasting his own logical notation with Boole's and its lack of expression for generality. A similar passage appears in (*GLA*, p.*xvi*): "It is possible, of course, to operate with figures mechanically, just as it is possible to speak like a parrot: but that hardly deserves the name of thought".

There is a missing feature of Frege's arguments that must be highlighted, one that is ignored by Landini and that modern logical systems have long abandoned: judgements. Frege's semantical theory is wholly dependent on the notion of a judgement. They are epistemic attitudes towards a content's truth. It is not enough to derive a function through decomposition, or comprehension schema, of a definition that was not antecipated in the *definiens*. Informativity comes from deriving judgements about the definitions, as there is no such thing such as an information that is false or even undecided upon its truth.

For Frege's scientific goals, there is no information carried by false statements. In a sense, this is strikingly intuitive: one does not get informed about the weather by learning, falsely, that it is raining. Neither does one get informed by not being clear if it is raining or not. In Frege's standards, this is guaranteed by the assertability condition in logic, that inferences are only performed from true premises to reach true conclusions. The scientific practice, that Frege wants the concept-script to be applicable, is rooted in truth: "To discover truths is the task of all sciences" (*CP*, p.351), "the goal of scientific endeavour is truth" (*PW*, p.2). And thus, in logic no step should be taken without the presence of the judgement-stroke, meaning that no deduction should proceed without every point being previously recognized as true: "[...] in presenting an inference, one must utter the premises with assertoric force, for the truth of the premises is essential to the correctness of the inference", Frege writes to Jourdain (*PMC*, p.79).

In conclusion, Frege found a way to form new concepts out of judgements: by decomposition in the functional structure that they express. This far surpasses Kant's notion of concept-formation, and truly expand from what one have started with. But ultimately, informativity is found only in the logical practice: in the inferential practice that includes some normative features, not reducible to a mechanical procedure. These are judgements (and their expression as assertions), inferences and definitions.

2.4 Conclusion of Part I

As we can see, departing from ordinary language does not mean to completely avoid some of it's features. The easiness and perspicuity to manipulate are certainly desirable, given that such formal language is still *a* language. These features are not justified by the precision, but by the readability and practicality required for a proper human manipulation of the now formalized language.

Since then both logical and mathematical languages were developed, and became even more formalized. But we still find use for natural language, given the mixed style that is present in logical and mathematical papers. Thus, one can easily find and argue about the presence and importance of pragmatic features within such mixed language. But for Frege, at least some of those pragmatic elements were present already in the formal language, and even if one eliminates natural language completely, they are still needed in visualizing and conducting proofs. Once again, a formal language is, still, *a* language. And we most importantly find such elements in the illocutionary force indicating devices that he employed in the concept-script.

But we cannot loose track of one of Frege's greatest enemies: psychologism. The task of the concept-script was to sharply separate the question of "[...] how we have gradually arrived at a given proposition" from "[...] how we can finally provide it with the most secure foundation" (*BS*, p.5), and to such task one must avoid ordinary language in carrying over deductions. As he declared in the *Grundlagen*, one should "always to separate sharply the psychological from the logical, the subjective from the objective" (*GLA*, p.xxii). Thus, putting the demands for precision and readability together yields the question: how can logic be designed *for* the practice and at the same time, be independent from it? Even if this task is only approximately feasible, Frege's demands were high, and those illocutionary features were properly constrained by such demands, as we shall see next.

Part II

Frege's Logic and its Semantics

Chapter 3

A Problem and a Pressuposition

As discussed previously, the signs '$\vdash$', '$\Vdash$' and the various types of inferential lines all serve as illocutionary force indicating devices in Frege's logic. Of all three, the judgement-stroke is by far the one who receives the most care by Frege. The reason is the key role that it plays in logic, in modern terms, its semantical role.

We must remember that Frege developed his system of logic without a meta-language. If he had one, he could have a tarskian truth-predicate defined, and thus *every* sentence in the concept-script could be devoided of assertoric force. The truth of each sentence would be achieved all at once, by providing a model. Things were not like that for Frege, mostly for philosophical reasons, as we shall see.

The judgement-stroke is irreducibly an illocutionary force indicating device of an assertion, and we saw that the assertive force is bound to the truth-predicate, although not reducible to it. Frege talks about truth always in the context of Thoughts, which can be considered as fixed interpretations for the concept-script sentences. These are enough reasons to consider that Frege, although not developing formal semantics as we know today, was deeply involved into semantic questions. This is the main purpose of Part II: to spell out this concerns, and their limits.

Following the syntax of Frege's concept-script, as presented in the first chapter, we saw how his logic employed non-descriptive signs, which we labeled 'performative'. Although we presented them as part of the Syntax, the judgement-stroke in particular have semantic functions. The standard division between a syntax and semantics of a calculus is missing in Frege's logic. Both are intertwined in the practice, where judgements, inferences and definitions are all key elements. The problem, which I believe Frege had an answer for, is that all such illocutionary devices are acts, thus, events to be considered from the perspective of an agent. How can we explain these performative signs from such perspective without compromising logical objectivity?

In this chapter and the next, the goal is to answer this question, while providing a broad understanding of Frege's semantic theory. For now, I'll focus on a problem and a presupposition. The problem is the psychologistic risk that specially judgements are bound to. This will be discussed in 3.1. Part of the answer depends on taking one presupposition: the Universalist reading of Frege's logic. This will be presented in 3.2. In the next chapter, the answers for each of the performative primitives will then be given. The case of judgements is more pressing, and it will be discussed in terms of contents in 4.1 and in terms of the conditions for the judicative act in 4.2, provided both for axioms and for rules of inferences. Finally, I will discuss the objectivity of definitional acts in 4.3.

3.1 The Psychologistic risk

In the *Grundgesetze*, Frege is clear in defining a judgement as the "[...] acknowledgement of the truth [*Anerkennung der Wahrheit*] of a *thought*" (*GGA*,§5). This choice of words is a constant in Frege's career. It appeared first in the *Begriffsschrift* (*BS*, §2)[1], and it is still the same in the later paper *Der Gedanke* (*DG*, p.356). Judgements are in fact actions that depends on agents to be performed. An assertion, in the sense used by Frege, is still an act: that of making a judgement explicit. In Frege's sense, a judgement is the mental act of affirming that a certain thought is true, or more precisely, that a Thought has the True as *Bedeutung*. Affirming that a thought has the True as truth-value is a different act from merely designating [*bezeichnen*] a truth-value to a thought. What is missing in the latter is, we already know, the assertive force. But this distinction is not entirely new. Kant had already distinguished a judgement from a mere association of ideas; Brentano also distinguished judgements from presentations in terms of a distinct mental quality: the affirmation. Frege, of course, does not have the combinatorial view of judgements (like in Kant), nor he endorsed the existential view (as in Brentano). But the recognition that a certain Thought has the True as *Bedeutung* is a distinct kind of mental phenomena than a mere designation. Nonetheless, a recognition does depend on some agent in order to be performed, as all judgements do.

For that reason, Frege's conception of judgement was accused of being psychologistic. In a letter to Frege, dated January 15, 1914, Philip Jourdain asked Frege whether he "regard assertion ($\vdash$) as merely psychological" (*PMC*, p.78). This worries

[1]In saying that the content-stroke '— ' does not confer assertoric force, he says that "the writer does not state whether he acknowledges it to be true or not" (*BS*, §52).

had already been pointed out by Russell in the *Principles of Mathematics* (RUSSELL, 1996, §52). Russell separates a psychological from a logical sense of an assertion. For a sentence to be asserted in the logical sense is, in his sense, the same as being true. For a sentence to be asserted in the psychological sense is just the act of taking it to be true (as Frege did). For that reason, Russell takes Frege to be separating truth from assertion[2]. As he claims, "to divorce assertion from truth seems only possible by taking assertion in a psychological sense" (RUSSELL, 1996, §478). Finally, Wittgenstein have repeated the accusation in the *Tractatus*, taking the judgement-stroke as "quite meaningless", as it simply "indicates that these authors hold the propositions marked with this sign to be true"(WITTGENSTEIN, 2002, 4.442). Peano, in his review of the *Grundgesetze* in 1895, also criticized the need for the sign, but on different grounds. He did recognize the need for demarcating the scope of assertions, but did not saw any advantage for the judgement-stroke, as in his logical notation "the particular position a proposition occupies in a given formula shows unequivocally what it is that is being asserted about it in that formula" (PEANO, 1971, p.27).

In a sense, the above accusations are correct: Frege's judgement-stroke does involve a psychological element, as judgements are irreducibly inner events of reasoning agents, and this seems to go against Frege's own anti-psychologism. In the *Grundlagen*, he famously recommended "[...] always to separate sharply the psychological from the logical, the subjective from the objective" (*GLA*, p.xxii). In the foreword to the *Grundgesetze*, this dictum was even further developed, as Frege finds at least three major errors concerning psychologism in logic. These are:

1. To take mental images of things as the things themselves (*GGA*,p.xiv).
2. To take the laws of thinking as descriptions, not prescriptions (*GGA*,p.xv).
3. To take all the objective as actual [*wirklich*], and therefore all the non-actual as subjective (*GGA*,p.xviii).

1) and 3) relates to contents of judgements, while 2), on the other hand, relates to judgements themselves. But all three poses the question whether Frege's adoption of judgements within logic as psychological events isn't a sheer contradiction to his own recommendation.

To take this tension even further, consider the following argument, concerning some of Frege's claims about (2):

[2] As in fact he did, since truth-values were for Frege the *Bedeutungs* of Thoughts, having no dependence on the act of assertion.

1. Logical laws are the justificatory basis for judgements. As he stated, "Whoever has once acknowledged a law of being true has thereby also acknowledged a law that prescribes what ought to be judged" (*GGA*, p.xvii).

2. But these laws are prescriptions on how one ought to judge, not a psychological description of how one happens to judge. These laws are not psychological laws: "By logical laws I do not understand psychological laws of taking to be true, but laws of being true." (*GGA*, p.xvi)

3. Since logical laws are independent from psychology, they are also independent from whether a given agent acknowledges them to be true or not: "being true is thus independent of anyone's acknowledgement" (*ibidem.*).

(C) Therefore, in order for a judgement to be logically true, it must be independent from any psychological attitude towards truth.

But judgements *are* themselves psychological attitudes towards the truth of a thought. To put the conclusion in other words, judgements are psychologically dependent, but the standards for judgements are not. How can this be settled?

I believe this tension can be answered. It's one thing to say that judgements are inner events, but to say that standard of correctness for judgements are psychologically given is entirely another. In fact, one can have the former without the latter. Most of Frege's concerns about psychologism are about the latter, not the former. Nonetheless, there are two main problems in taking judgements psychologically that I will focus on, according to the following table:

If Judgements

(A)	are subjectively justified	$\Rightarrow$	they have relative standards of correctness
(B)	are Inner Events	$\Rightarrow$	they are logically meaningless

First, the psychological reduction can make the reasoning process (thought of as transitions from judgements to judgements) essentially defective. By essentially defective I mean simply the impossibility of having standards of correctness for inferences, as an agent may take anything to be true, following any justification whatsoever. Second, taking assertions to be psychological events can make them logically meaningless (as Wittgenstein accused).

I believe that Frege had an answer to both problems, whilst taking judgements and assertions as mind-dependent events. In other words, he has means to answer that

1) one can have logically sound arguments involving the judgement-stroke; and that 2) assertions and judgements are not meaningless, being essential for the logical practice and its notation. Thus, Frege would accept the implication in (A) but deny that the antecedent holds. He would also deny the implication in (B) altogether.

Answering the first problem requires looking at the three consequences of psychologism that Frege pin down in the foreword to the *Grundgesetze*. They convey Frege's constraints for every performative sign of the language of the concept-script, precisely, of judgements, definitions and inferences. These constraints are of two kinds: a restriction on what a given act may convey and a restriction on how a given act may be performed. In other words, on what one may judge and how one may judge it, and likewise for defining and inferring within the language[3]. To answer the second problem, we shall assume the Universalist reading of Frege's conception of logic, and the fact that the concept-script allows no metatheory in the modern sense of the term. These reasons will provide grounds for understanding Frege's need for the judgement-stroke, and for the other performatives of the concept-script language. For that matter, in the remainder of this chapter, I will provide reasons for accepting the Universalist reading, and will leave the answers to the psychologist risks to the next chapter.

3.2 The Universal Conception of Logic

In chapter 5, I'll discuss two of the decisive moments in the history of twentieth century logic: David Hilbert and Alfred Tarski. Both have introduced, in different manners, an hierarchy of languages. One object-language, the (logical) language that we want to study, and a metalanguage, that is, a language through which we carry such study over. Both Hilbert and Tarski used this method in formalizing two key notions in the logical practice: proof and truth. They were also influential for the development of model theory as an independent field, and shaped most of the logical practice of today.

Is Frege's logic model-theoretic? Is there any metalanguage in which we may conduct metatheoretical results concerning the concept-script? Is Frege's logic semantically complete? These are all common questions to ask about any system of logic today. But at Frege's time, these were unknown. One way to not answer this questions is to accept that Frege adopted an universal conception of logic. This conception, or thesis, has two basic implications:

[3]In chapter 6, I will take these considerations to a linguistic level, in describing how assertions, definitions and inferences may be understood as proper speech acts.

Universality of Logic:

1) There is one ‘correct’ logic, that cannot have a richer metatheory

2) Logic has a single domain of interpretation, the *universal* domain.

The thesis is also known in the literature as the Van Heijenoort-Dreben thesis. It follows Van Heijenoort (1967b) now classical distinction, presented in the paper *Logic as Language and Logic as Calculus*[4], between the uninterpreted algebraic systems of Boole, Schröder, de Morgan, and the universalist logics of Frege and Russell. In the former (logic as calculus), the universe class (like in Boole) or the universe of discourse (in De Morgan), denoted by 1 and 0, have no ontological import: one may take them to consist of any domain of objects whatsoever. For the latter (logic as language), the quantifiers range over all there is: one can restrict the quantifiers using conditional clauses, but ultimately, they range over all there is, in the most inclusive sense.

In the logic as language trend there was no conception of a logical interpretation for variables and predicates. As Van Heijenoort (1967b, p.325) states, “[...] it cannot be a question of changing universes. One could not even say that he restricts himself to one universe. His universe is the universe.” Frege is the most representative of this conception, as his “[...] universe consists of all that there is, and it is fixed.” (*ibidem*).

The universal conception of logic is clearly the opposite view of the model-theoretic conception. It also goes against the hierarchy of different logical languages, for “nothing can be, or has to be, said outside of the system” (Van HEIJENOORT, 1967b, p.325), since this would imply the existence of different levels of logicality, which is clearly denied. Metatheoretical concerns are thus out of the question, which also impose a limit for formalizations of core concepts. The universality of logic also demands for the explication or clarification needed for the primitives that cannot be further formalized[5].

And since there is no way of treating logic from a superior perspective (for the lack of a metalanguage), the universal conception takes logic as a practice (as Frege certainly did). In the logic as calculus side, which would become the model-theoretical perspective, logic can be treated as a genuine object of investigation. But for the universalist, reasoning occurs only from within. For instance, Van Heijenoort and Dreben

[4] Already discussed in the first chapter.

[5] Van Heijenoort (1967b, p.325) continues: “Since logic is a language, that language has to be learned [...] by suggestions and clues. Frege repeatedly states, when introducing his system, that he is giving hints’ to the reader, that the reader has to meet him halfway and should not begrudge him a share of ’good will’. The problem is to bring the reader to ’catch on’; he has to get into the language”.

(1986, pp.44-45) argues that under the universalist “We are within logic and cannot look at it from outside. [...] The only way to approach the problem of what a formal system can do is to derive theorems.”

The Universalist reading is a well accepted interpretation for Frege’s philosophy of logic. For instance, we can mention Shapiro (1991), Hintikka and Sandu (1994), Goldfarb (2010) and Ricketts (1986) as followers of Van Heijenoort and Dreben (1986). They mostly agree on the impossibility of reading Frege’s logic from metatheoretical lens:

> the logicists,including Frege, did not develop model-theoretic semantics, partly because their systems were fully interpreted. There was no non-logical terminology whose referents would vary from model to model. (SHAPIRO, 1991, p.11)
>
> Frege could not develop (or even understand) any model theory [Given his] one-world ontology with one fixed universe of discourse. (HINTIKKA; SANDU, 1994, p.279)
>
> Nor is it possible [...] to read the contemporary view back into Frege. For the contemporary view requires the ineliminable use of a truth predicate. Such a use is antithetical to Frege’s conception of judgment. This conception of judgment precludes any serious metalogical perspective and hence anything properly labeled a semantic theory. (RICKETTS, 1986, p.76)

In this respect, Frege definitely assumed logic to be universal. It comes to no surprise given that the philosophical tradition also took logic in universal terms. Even if we can find in Frege questions and worries that pertain to meta-theoterical level, he never crossed the line to fully distinguish levels of logicality. He does at some point distinguish between an expository language and a auxiliary language. And anyone who reads the *Grundgesetze* readily grasps the difference between the constructive and the expository parts of that book. But, as we shall still see, both languages are just different aspects of the same, fully interpreted, language — the universal one. For this reason, the Universality of Logic thesis is a good measure to read Frege’s logic and its heritage.

I’ll argue that the universality reading is the correct one for Frege. I shall make my point in a few topics: the transcendental role that logic plays for reasoning (meaning that to reason is *ipso facto* to follow logical rules); the Frege-Hilbert controversy, which is set on the background of the model-theoretic approach; the scope of quantifiers and finally the absence of a metalanguage in the modern sense for the concept-script. In this last point, I shall depart from some scholars that saw glimpses of metatheory in Frege’s logical practice.

3.2.1 The Transcendentality of Logic

On of the fregean themes that seems to endorse the universalist reading is what I may call the transcendentality of logic. By transcendental, I'm implying the thesis that logic is a necessary condition for all thinking as such[6]. For instance, in the *Grundlagen*, Frege uses the following argument for the logicallity of arithmetic:

> [...] we have only to try denying any one of them [the fundamental propositions of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible. [...] The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. (*GLA*, §14)

The argument is straightforward, and could be recast as follows:

(P1) Logical truths are necessary for thinking consistently

(P2) Any arithmetical truth, if denied, implies contradiction

(C) Arithmetical truths are logical truths.

The presupposition of a "widest domain of all" is another evidence for the universalist reading here. Frege's wants to show that arithmetical truths are logical truths since they have the same range of applicability. And he does that by arguing that both cannot be denied safely from contradictions. I'm not interested in the argument *per se*, but only with the presupposition in premise (P1).

Frege assumes that logical truths are normative laws for thinking. In the *foreword* to *Grundgesetze* this is strongly defended against psychologism, which in his words, would yield such laws as descriptions of how humans happens to think. As he claims, "In one sense [a law] says what is, in the other it prescribes what ought to be. Only in the latter sense can the logical laws be called laws of thought, in so far as they legislate how one ought to think" (*GGA*, *foreword*, p.xv). Why the laws of logic must be taken as normative guidance for thought? Because if they were only descriptions, we would have different laws for different thinkers. With different thinkers being regulated by different laws, we would never reach agreement, or even succeed in understanding

[6]Authors such as Sluga (1980) defended the view that Frege was a transcendental idealist, and not a realist in the platonistic sense, as argued by Dummett. Here, I favor Dummett's reading. Nonetheless, the transcendentality of logic here defended is not inconsistent with this reading, as it is not a thesis about thoughts and its objectivity, but a thesis about what constitutes inferential reasoning. Moreover, we may count Thoughts as elements of logical reasoning, and thus, part of what constitutes the transcendentality of logic, as the illogical thinking would be an incorrect mirroring of the realm of thoughts.

the reasons for an agent asserting $—\Delta$ and other asserting $_\top\Delta$, without ever realizing any contradiction. Other minds would be, as Frege says, "a hitherto unknown kind of madness" (*GGA*, *foreword*, p.xvi), a logical alien [7].

The impossibility of mutual understanding is not just an impossibility between us and the logical aliens, but it reveals the impossibility of one's understanding *beyond* the laws one is abiding to. Either we subject the logical alien's assertions under our laws, or we simply cannot understand them.

> Stepping outside logic, one can say: our nature and external circumstances force us to judge, and when we judge we cannot discard this law - of identity, for example - but have to acknowledge it if we do not want to lead our thinking into confusion and in the end abandon judgement altogether. [...] If others dare in the same breath to both acknowledge a law and doubt it, then that seems to me to be an attempt to jump out of one's own skin against which I can only urgently warn. Whoever has once acknowledged a law of being true has thereby also acknowledged a law that prescribes what ought to be judged, wherever, whenever and by whomsoever the judgement may be made. (*GGA*, p.XVII)

The implication is that we cannot "jump out of our own skin" in order to evaluate the laws we accept as true, without abiding by such rules. The metaphor also holds for different levels of logicality in an hierarchy of languages. At best, they would be different applications of the same logical rules, and would be hardly different logics at all. Returning to the logical alien, the only possibility for understanding them is to deny our own rules, which yields inconsistency. As stated above, logic is necessary for thinking as long as logical laws are prescriptions on how we ought to think.

The same holds, *mutatis mutandis*, with different hierarchies of logicallity. If one accepts the model-theoretic approach, we clearly has ways to step outside the scope of the intended logic, and study it from a higher perspective. Thus, the transcendentality of logic and the universality thesis goes hand-in-hand.

[7] It is not the existence of a logical alien that Frege is opposing to, but the possibility of mutual understanding:

> this impossibility, to which we are subject, of rejecting the law does not prevent us from supposing beings who do so; but it does prevent us from supposing that such beings do so rightly; and it prevents us, moreover, from doubting whether it is we or they who are right. (*GGA*, *foreword*, p.xvii)

3.2.2 Frege-Hilbert Controversy

This position is also found in the now famous Frege-Hilbert controversy over the role of axioms and definitions in mathematics. The controversy took place in a series of letters from 1899-1900, after Frege and Hilbert met at Göttingen and following Hilbert's publication of the *Grundlagen*. Frege also published a series of papers on the subject, which Hilbert fail to respond: these are *Über die Grundlagen der Geometrie* first and second series, published in 1903 and 1906[8]. The dispute is also a good assessment of Hilbert's innovative axiomatic method in comparisson to Frege's own conception of logic.

Recall first that Hilbert's proof of independence of the axioms of geometry relies on the construction of different *ad hoc* geometries, each serving as a model for the different combinations of the five groups of axioms. Also, Hilbert's proof of the relative consistency relied on the interpretation of the axioms in a model of algebraic numbers. These are model-theoretic procedures. Hilbert did say that the axioms were divided following geometric intuitions, but his method showed that the axioms were quite independent from such source of knowledge. Frege thought otherwise: geometry was inherently founded on intuition, and thus, did not accept Hilbert's method, neither its motivation for them. Given that truths of geometry are intuitively founded, there's no need for any consistency proof:

> I call axioms propositions that are true but are not proved because our knowledge of them flows from a source very different from the logical source, a source which might be called spatial intuition. From the truth of the axioms it follows that they do not contradict one another. There is therefore no need for a further proof. (*Letter to Hilbert, 27.12.1899* in (*PMC*,p.37))[9]

Given that the truths of geometry are warranted by the spacial (geometrical) intuition, there's no need for a consistency proof. Consistency follows from truth, not the other way around. Hilbert, on the other hand, believed in the opposite. In a letter to Frege, dated December 29th, 1899, he writes: "if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist" (*PMC*, p.39). It is not only Frege's position towards geometry that prevents him to realize what Hilbert was doing. Since he was adopting the universalist conception, Frege was oblivious to the model-theoretic perspective that Hilbert was employing.

[8]Translated in *CP* as *On the Foundations of Geometry.*

[9]The same point is made in the paper *On the Foundations of Geometry: First series*, in (*CP*, p.275): "Axioms do not contradict one another, since they are true; this does not stand in need of proof."

This can be seen from another point: Frege's quarrel against Hilbert's definitions. In the *Grundlagen*, Hilbert says that the axioms "define" relations such as between, congruence, etc. In his terms, it makes total sense, as each set of axioms will have different models, and hence, differente realizations. But for Frege, it didn't. His criticisms were based on illocutionary considerations:

> To definitions that *stipulate* something, I opposed principles [*Grundsätzen*] and theorems that *assert* something. The former contain a sign (word, expression) that is intended to receive a meaning by their agency; the latter contain no such sign. [...] what is an axiom really supposed to do? Is it supposed to assert something, or is it supposed to stipulate something? (*Foundations of Geometry, second Series*, in *CP*, p.294)

We can say that Frege saw assertions and definitions as speech acts: an assertive and a stipulative speech act respectively. Hilbert's position could also be read that way, if considered in the proper model-theoretic context. Just take axioms to be collectively declaratives and assertives individually. But following Frege's universalist conception, it could not: either the axioms are assertions about previously given domain of interpretation (the geometric one)[10], or they are not axioms at all. Definitions and axioms are two different kinds of illocutionary acts: definitions are not supposed to assert something in the same way axioms are not supposed to define anything. But if we read the dispute as a universalist v. model-theoretic conception of logic, the dispute between them become clearer. Hilbert, for instance, stated that

> to try to give a definition of a point in three lines is to my mind an impossibility, for only the whole structure of axioms yields a complete definition. Every axiom contributes something to the definition, and hence every new axiom changes the concept. A 'point' in Euclidean, non-Euclidean, Archimedean and non-Archimedean geometry is something different in each case. (Letter to Frege, December 29th, 1899, in *PMC*, p.40).

This passage is strikingly model-theoretic. It is clear how Hilbert was speaking in capturing different models from different sets of axioms. Seeing the issue through the lens of the universalist conception, Frege could see only an illocutionary mistake.

[10]Notice that this does not contradict the unversality thesis about the domain of quantification considered, as one can just specify in the axioms the right objects in the domain with a conditional clauses.

3.2.3 The Scope of Quantification

Another reason to take Frege as a universalist is that the conception of a variable domain of quantification is entirely missing in his works[11]. Moreover, he explicitly employs means for quantification without any previously given restriction, as one would expected today. Right from the beginning of the *Begriffsschrift*, he emphasizes to "[...] adopt this fundamental idea of distinguishing two kinds of symbols [...] in order to use it for *the more inclusive domain of pure thought in general*." (*BS*, §1)[12]. Notice how Frege is not defining an universe of objects for first-order variables to denote, but actually starting with the more general notion of a 'thought'. By 'thoughts', at this stage of his philosophy, he is implying the idea of thinkable conceptual contents. And by taking the domain to be the realm of 'pure thought', he's just assuming it unrestrictedly: everything that can be thought conceptually is considered within the range of the variables. This is everything that can be prefixed by the content-stroke, as defined earlier in section 1.3.1.

With the subsequent change of primitives in the *Grundgesetze*, the scope of the horizontal line changed, but the universal character was kept, although in different terms. The horizontal '— ' is now a total function over objects. For any object Δ, — Δ returns a truth-value: either the True if Δ denotes the True, or the False otherwise. As a total function, there is no restriction to the kinds of objects that Δ may refer to. If it had, Frege wouldn't have to consider '— ' as a total function. Since it is a total function, the presupposition is also that first-order variables, what Frege calls object-markers, range over any object whatsoever. This is what Frege explains in the following:

> the domain of what is admissible as an argument has to be expanded and extended to objects in general. Objects stand opposed to functions. Accordingly, I count as an object everything that is not a function, e.g., numbers, truth-values and the value-ranges introduced below. (*GGA*, p.7).

Thus, in both Frege's early logic and the matured concept-script of the *Grundgesetze*, variables are said to vary unrestrictedly, *modulo* the category of entities considered, *i.e.*, conceptual-contents or objects.

It should be also mentioned that variables in the *begriffsschrit* are not schematic. In today's standards, we may simply adopt axioms, for example,

(1) $p \to (q \to p)$

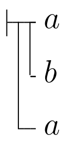
[11]See, as a comparison, Boole's universe of discourse in (BOOLE, 1951, *chap III*, §4, 5).
[12]My emphasis.

and treat each variable p, q as schematic letters. In this sense, we may adopt any axiom that preserve the schematic structure, as for example,

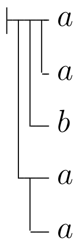
(1b) $(p \rightarrow p) \rightarrow (q \rightarrow (p \rightarrow p))$.

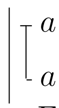
(1) is said to include all instances such as (1b).

In contrast, Frege takes the axiom

(F1) 

as not schematic. It is a real axiom in which a, b are universally quantified variables. In the *Grundgesetze*, these are roman object-markers. They do not refer but only indicate possible objects in its range. The concept-script analog of (1b),

(F1b) 

is not just an instance of (F1), but it is derivable from (F1) with Frege's rule of uniform substitution. In the first version of the concept-script, this is an unstated rule, but Frege uses it oftenly in section II and III of the book, usually by stating the substitutions being performed. In the transition from (F1) to (F1b) above, a  is the only substitution.

In the *Grundgesetze*, the same rule is used, but Frege doesn't bother stating each substitution being made. He would simply quote (F1b) above as (F1). But he does mention substitutions as inferential rules in (*GGA*, §48). These involve the replacement of Roman, German and Greek letters, and the corresponding conditions, as already discussed in section 2.2.2.

The difference between having axioms as schema and axioms as generalized expressions needing a rule of substitution may sound trivial, but it lies on the thesis that logic is universal. First and foremost, schema are not formulas, as they are devoided of content: they encompass infinitely many instances which are formulas themselves. Axioms schemata are then not formulas of the object-language, but of the metalanguage. Frege would never accept schemas as proper formulas.

A possible objection for this conclusion would be to consider Frege's notion of indication [*andeuten*] as applicable to schema. Frege claims that propositions expressing generality such as those with roman letters 'x', 'y', 'z' and those with German letters '$\mathfrak{a}$', '$\mathfrak{b}$', '$\mathfrak{c}$' do not refer, as they do not behave as proper names. What he claims is that "We do not say of a Roman letter that it *refers to* [*bedeuten*] an object but that it *indicates* [*andeuten*] an object" (*GGA*, §17). Couldn't we likewise conclude that axiom schemas only indicate while its instances refer? This wouldn't make sense without clarifying first what are formulas referring or indicating to. In Frege's case, roman letters indicate objects, and thus, generalized expressions containing them are indications of truth-values. But an axiom schema, as we understand it today, is at best only indicative of formulas, not objects nor propositions. This goes because without an interpretation, formulas are devoided of content. Frege, on the other hand, stipulates that roman letters are indicative of objects: it is clear that he was speaking about objects as their intended interpretation. And given that he was not restricting the domain of objects, but taking objects in general, he was presupposing an universal conception of logic.

In sum, in any application of the Horizontal, we have generally that the judgement

$$`\vdash \Delta'$$

asserts that, whatever object Δ may indicate, — Δ is the True. And given that it is the True regardless of what Δ indicates, it allows another judgement to be made when Δ is changed by any other expression that, at least, refers or indicates an object. We can see that by following (F1) above. Regardless of the referents of Δ and Γ[13] , the corresponding horizontals — Δ and — Γ both refer or indicate a truth-value. And regardless to which truth-value they indicate, (F1) will always denotes the True. Following this fact, it doesn't matter what Δ and Γ indicates, and so, they may be substituted by any roman object-marker or proper name of any complexity, provided they are non-empty, in making judgement. Thus, the judgement of (F1b) is allowed in terms of (F1) by substitution.

It is clear that Frege's usage of substitution is not a schematic tool for deriving instances of syntactic formulas as modern systems of logic. It is a rule also embedded with semantics, as its main goal is to allow new judgements to occur within an inference. This is a difference with respect to a specific part of modern logical practice: the notion of a variable domain of interpretation. But the fact that judgements having general

[13] I'm here taking the roman object-markers a, b, that occurs in (F1) above, by their representative upper case Greeks Δ, Γ, following Frege's recommendation that Roman object-markers only appears in the context of a judgement-stroke.

contents are not schematic shows also that Frege's logic lacks another modern feature: a metalanguage.

3.2.4 The absence of a logical Metalanguage

Does Frege has a metalanguage in his concept-script? This matter is far from being settled, and there has been debates on the literature on how much does Frege employ any metatheoretical reasoning and how far can we take him into accepting a metalanguage[14]. I believe that, apart from the fact that Frege has obvious semantical concerns in his system, he does not have a metalanguage. Part of the reason for rejecting this possibility comes from judgements and their illocutionary role within the concept-script. Nonetheless, there are a few topics concerning Frege's possibility of a metalanguage that are discussed in the literature that we must read away.

3.2.4.1 The Use-Mention distinction:

Frege is considered the first logician to stress the importance of quotation-marks for separating the use from the mention of a name. In the following examples,

(a) Frege has five letters.

(b) "Frege" has five letters.

The sentence (a) is ambiguous between making an statement about Frege, the logician, and the name of Frege as a syntactic item. To disambiguate, quotation-marks are introduced, as in (b). Thus, names without quotation-marks are said to be in use, as they speak about the meaning of the name; whereas names with quotation-marks are said to be mentioned, as they speak about the name itself. Frege clarifies this in (*GGA*, p.4): "Someone may perhaps wonder about the frequent use of quotation marks. It is by this means that I distinguish cases in which I speak of the sign itself from cases in which I speak of its reference."

Why would Frege need that? When justifying the employment of quotation marks, he writes that "[...] people have managed to regard number-signs as numbers, the name as what is named, the mere auxiliary means as the object of arithmetic itself" (*ibidem*). The first worry is that the absence of the use-mention distinction would drive

[14]For example, Heck (2012a), Antonelli and May (2000), Stanley (1996), Tappenden (1997) and Greimann (2008) can be mentioned as defending that we can, partially or completely, read a metatheory back to Frege, while Ricketts (1997, 1986), Blanchette (2012), Weiner (1990, 2005, 2008) and Goldfarb (2010) argue in the opposite direction, that we cannot read modern logic back to him.

the derivations into some kind of formalism, where derivations are about signs and not their contents. Another worry is that of confusing auxiliary means of expression with the expressed.

3.2.4.2 Auxiliary and Expository Language:

Quotation marks are mostly used for explanatory purposes. In proving theorems in the *Grundgesetze*, Frege separates a constructive part (where proofs are actually conducted) from an analysis of the derivation. It is in the analysis that the use-mention distinction is put to use, mostly because they are *not* written with the concept-script language. The presence of quotations inevitably shows two distinct languages: the concept-script and a natural informal language used as an exposition device. In the unpublished *Logische Allgemeinheit*, Frege explicitly distinguishes between an expository language [*Darlegungsprache*] and an auxiliary language [*Hilfssprache*] in explaining the use of quotations. Trying to analyse the role of variables in expressing general contents (the main topic of the paper), he describes a written language that is able to express thoughts directly through names (that refer) and letters (that indicate)[15]. He then writes the following:

> The language we have just indicated, which I will call the *auxiliary language*[16] is to serve for us as a bridge from the perceptible to the imperceptible. It contains two different constituents: those with the form of words and the individual letters. The former correspond to words of the spoken language, the latter have an indefinitely indicating role. This auxiliary language is to be distinguished from the language in which I conduct my train of thought. That is the usual written or printed German, my expository language. But the sentences of the object-language are the objects to be talked about in my expository language. [...] As such proper names of the sentences of the auxiliary language I use these very sentences, but enclosed in quotation marks. Moreover it follows from this that the sentences of the auxiliary language are never given assertoric force. (*PW*, p.260)

Frege's way of describing the auxiliary language does not equate simply to an object-language, as we understand it today. Of course, the role of the expository language

[15] There is no explicit indication that Frege is here speaking about the concept-script. But the similarities are abundant, and It is clear that the concept-script is itself an auxiliary language.

[16] This is the *Hilfssprache*, which the translators of the Posthumous Writings opt in translating as *object-language*, and *Darlegungsprache* as *meta-language*. They do so giving the resemblance to Tarski's use of the terms. They do not hold that both are the same, and the conclusion, as they claim, is left for the reader. I find both inadequate for reasons to be still showed, and so I took the liberty of changing each term's translation for auxiliary language and expository language, respectively. The rest of the quotation is left unaltered.

resembles the metalanguage: the language in which we talk about the auxiliary language. But the latter is not just an object of mathematical study, and it is called auxiliary precisely because it is how we grasp the imperceptible (the senses of sentences, *viz.,* Thoughts) in the perceptible (the written words and signs). It is auxiliary not in the sense of being helpful in understanding another language, but auxiliary in grasping senses[17]. It is a language in its own right, not an uninterpreted set of sentences. And given that it does not need one, the role played by the expository language differs from a metalanguage. It is not its task to prove theorems about the auxiliary language, but solely that of making it more perspicuous and handable. The expository language is a tool for learning another language, not a tool for studying it thoroughly. For this reason only, crediting Frege's *hilfssprache-darlegungssprache* as an anticipation of Tarski's object language-metalanguage distinction is a too strong claim to make.

We have only to make it precise in what sense Frege's expository language differ from a metalanguage. It's not that the expository language isn't capable of studying the auxiliary language, but that it cannot be done logically. The notions that the concept-script are supposed to formalize cannot be reduced to further notions: they are already in use in the expository language, albeit confusedly. What is left to do is just elucidations or explications [*Erläuterung*] and hints [*Winke*]. This corresponds to the part I of *Begriffsschrift*, called "Definition of the Symbols" [*Erklärung der Bezeichnungen*], and part I of *Grundgesetze*, the "Exposition of the Concept-script" [*Darlegung der Begriffsschrift*].

Frege speaks about elucidations in a number of topics. The issue is most prominently discussed in the *Über die Grundlagen der Geometrie*.

> We must admit logically primitive elements that are indefinable. [...] Since definitions are not possible for primitive elements, something else must enter in. I call it explication [*Erläuterung*]. It is this, therefore, that serves the purpose of mutual understanding among investigators, as well as of the communication of the science to others. We may relegate it to a propaedeutic. It has no place in the system of a science; in the latter, no conclusions are based on it. Someone who pursued research only by himself would not need it. The purpose of explications is a pragmatic one; and once it is achieved, we must be satisfied with them. [...] Since mutual cooperation in a science is impossible without mutual understanding of the investigators, we must have confidence that such an under-

[17] Sundholm (2018, p.553) informs us that the expression *hilfssprache* is a "German rendering of the French *langue auxiliaire*, which term stands for the artificial languages that were considered in the artificial languages movement, of which Frege's correspondents Couturat and Peano were prominent members." Frege's *begriffsschrift* is a *hilfssprache* in the same sense of Esperanto or the *Latine sine flexione*, albeit a formal one.

> standing can be reached through explication, although theoretically the contrary is not excluded. (*CP*, p.300-301)

There are limitations concerning the expository part of a formal language. As the passage above shows, the role of explications is not to offer proper definitions, neither to conduct proofs. Explications can only show (to use Wittgenstein's later term) what one means by showing examples and making things more explicit. As Frege claims above, "no conclusions are base on it". Thus, agreeing with Weiner (2005), Frege's elucidations cannot be taken as methatheoretical proofs.

The logic behind the auxiliary language and the expository language are one and the same. The concept-script is not just a system of second-order logic, it is logic made formal[18]. To this extend, justifying the laws of logic in the expository language would demand the same logical laws. Frege seem to imply that even natural languages have a logical core embedded in it, although in a confusing manner. For that reason, logically simple notions are always in use. The primitives of the concept-script require explanations because they are the expression of such primitive logical notions[19]. Therefore, the concept-script is an auxiliary language in terms of being a tool for making the logical core precise. But it has its own limits of formalization, as Frege summarized in (*GGA*, p.3-4):

> even after the concepts are sharply circumscribed, it would be hard, almost impossible, to satisfy the demands necessarily imposed here on the conduct of proof without special auxiliary means. Such an auxiliary means is my concept-script, whose exposition will be my first task. The following may be noted in advance. It will not always be possible to give a regular definition of everything, simply because our ambition has to be to reduce matters to what is logically simple, and this as such allows of no proper definition. In such a case, I have to make do with gesturing at what I mean.

Sundholm (2018) indicates that Frege's expository language is much closer to Curry's U-language than it is to modern metalanguages. For instance, Curry's (1977) U-language is the language being used for communicative purposes (U stands for "Use"). As such, is not possible to avoid it, or to describe it exhaustively: "All we can say is that it contains the totality of linguistic conventions which, at the moment, we understand. This

[18]In terms of the ongoing debate about logical pluralism, Frege would see himself as a monist.

[19]For instance, section II of the *Begriffsschrift*, where Frege introduces axioms (or laws of pure thought) and some derivations in the system, begins with the following claim: "We have already introduced a number of fundamental principles of thought in the first chapter in order to transform them into rules for the use of our signs. These rules and the laws whose transforms they are cannot be expressed in the ideography because they form its basis." (*BS*, §13)

may seen vague, but in that vagueness we are no worse off than in any other field of study." (*idem*, p.28). On vagueness, recall that Frege had used a similar metaphor to describe the advantages of the concept-script, the eye-microscope comparison in the preface to the *Begriffsschrift*. From the point of view of precision and sharpness, the eye, just as the ordinary language, is a vague instrument. But, while it lacks precision, it excels in versatility and adaptability[20]. Frege's expository language is an ordinary language, while the concept-script is an auxiliary language much as the microscope is as an auxiliary tool for optic precision. As an ordinary language, the expository language could be German, English, or any language that makes communication possible, or as Frege also calls it, the meeting of minds[21].

Another point of agreement between Curry's U-language and Frege's expository language is the presupposition that language is the necessary mean for reasoning's communication. Thus, both languages are inescapable instruments of exposition. Finally, Curry (1977, p.29) takes four basic features of the U-language that could be taken as features of Frege's expository language as well:

1. The U-language is specific, meaning that even if we speak about multiple U-languages, we would choose one as means for communication, making this choice the U-language and all others just object-languages;

2. The U-language may contain technical terms and other devices;

3. The U-language is mutable, in continuous growing and changing;

4. The U-language, although intrinsically vague, can still obtain some level of precision.

Curry's U-language is much akin to Frege's expository language. Both are inescapable tools for communication. But in Curry's case, an object-language is still an uninterpreted object. For Frege, the auxiliary language cannot be taken as such. Both are interpreted languages, and the interpretation of both are one and the same, the universal one. The auxiliary language just happens to be the logical microscope to the expository language's eye.

[20]Frege's exact words are: "Because of the range of its possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. Considered as an optical instrument, to be sure, it exhibits many imperfections, which ordinarily remain unnoticed only on account of its intimate connection with our mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others" (*BS*, p.6).

[21]The "meeting of minds" metaphor is used constantly by Frege when indicating the elucidatory role of the expository language. See, for example, (*PW*, p.207), (*PW*, p.235), (*PW*, p.259), (*PW*, p.271).

3.2.4.3 *GGA* sections §§29 – 31 and the "New Science":

To have metatheoretical interests does not equate in having a metatheory. In Frege's case, this is an impossibility. But there are still two points that we must argue away, in which Frege seem to take the expository part to another level. This are section §§29 – 31 of the *Grundgesetze*, and a brief part of the *Über die Grundlagen der Geometrie*, in which Frege speaks of a *new science*.

In (*GGA*, §§29 – 31), he tries to show that every well formed name of the concept-script refers. His argument follows from induction on the referentiality of each primitive function of the language previously defined. §31 offers the base case for the induction, that is, Frege argues that all the primitive functions

$$
\begin{array}{c}
`\!\!-\!\!-\,\xi\text{'}, \quad `\!\!\top\,\xi\text{'}, \quad `\backslash\xi\text{'}, \\
`\begin{array}{l}\top\,\zeta\\ \llcorner\,\xi\end{array}\text{'}, \quad `\xi = \zeta\text{'}, \\
`\smile^{\mathfrak{a}}\,\varphi(\mathfrak{a})\text{'}, \quad `\dot{\varepsilon}\varphi(\varepsilon)\text{'}, \\
`\smile^{\mathfrak{f}}\,\mu_{\beta}(\mathfrak{f}(\beta))\text{'}, \quad `\smile^{\mathfrak{f}}\,\mu_{\beta\gamma}(\mathfrak{f}(\beta,\gamma))\text{'}
\end{array}
$$

all refer, provided that ξ, ζ, $\varphi(\xi)$ refer to something. The induction step is in fact made in the previous section §30. There, Frege argues that every function that is build from the primitives are also referential, provided that the base case have reference as well. To take a simple example, if we want to form the name

$$
\begin{array}{l}
\top\!\!\top\,\Delta\\
\,\,\llcorner\,\Gamma\\
\llcorner\,\Delta
\end{array}
$$

we first start with the binary primitive function

$$
\begin{array}{l}
\top\,\zeta\\
\llcorner\,\xi
\end{array}
$$

and take the proper name Δ for ζ and Γ for ξ. Given that Δ and Γ have reference (by assumption), so thus $\begin{array}{l}\top\,\Delta\\ \llcorner\,\Gamma\end{array}$ (from the base case). Now, we just take the same binary primitive function and simply take Δ for ξ and $\begin{array}{l}\top\,\Delta\\ \llcorner\,\Gamma\end{array}$ for ζ, acquiring what we wanted, which also has a reference following that Δ refers (by assumption) and $\begin{array}{l}\top\, \Delta \\ \llcorner\, \Gamma\end{array}$ also (as shown). Other cases follow similarly. Frege wants to argue inductively that every such possible expression-formation has reference as well[22]. Having made this two points, he concludes:

> Thus it is shown that our eight primitive names have a reference and thereby that the same holds of all names correctly formed out of them. However, not only a reference but also a sense belongs to all names correctly formed from our signs. Every such name of a truth-value *expresses* a sense, a *thought*. (*GGA*, §32)

It is safe to say that in the arguments in sections §§$29-32$, Frege shows both semantical and metatheoretical concerns. It is semantical given that what he settles to show is that all concept-script expressions refer. It is metatheoretical because, ultimately, it states something about the language, the auxiliary language to be precise. But at the same time, we cannot say, as Heck (2012a), that Frege is presenting a *proof*, but only an informal argument by induction. In a sense, he isn't presenting a proof because the whole procedure is ultimately flawed: expressions formed with the smooth breathing does not always refer, following Russell's paradox. But the point is not that the proof is flawed, but what Frege intended to do in the first place. The expository language is not a language for conducting proofs, but to set the stage where proofs are conducted, *i.e.*, the auxiliary language. Agreeing with Weiner (1990, p.243), "most of the discursive writing that seems as if it might express informal proof [...] should be regarded as elucidatory". The arguments in sections §§$29-32$ are elucidatory in such a sense: they provide reasons for the user of the concept-script that referentiality is preserved through the formation of proper names and function names.

If Frege wanted to provide a rigorous proof for such a fact *in* the concept-script, more would have to be included. Frege comes close in stipulating the conditions for the concept-script to prove facts about itself, in discussing independency results with Hilbert in the paper *Über Grundlagen der Geometrie*. Recall first that Hilbert proves independence results model-theoreticaly, by offering different geometries as interpretation for the axioms, showing that for each, at least on axiom of the groups does not hold.

To start, Frege does not accept this model-theoretical background. He is clear in distancing himself from Hilbert: we should speak of independence not of axioms as uninterpreted sentences but of thoughts, given that "A real proposition, however, expresses

[22]He mentions some ways in which proper names and function names can be formed out of the primitive ones, like the example showed. (HECK, 2012b) provides a summary and a throughout discussion on them.

a thought, and the latter is either true or false; *tertium non datur*" (*CP*, p.334). Also, given that thoughts are True or false, it cannot be the case that a thought is a derived from a group of thoughts without being itself true. The key notion here is inference, not logical consequence. If there is an inference from a group of thoughts to another, then each thought is itself true, as only true thoughts can occur as premises.

How, then, can a Thought be independent from another? Frege's answer is only a sketch. First, we divide each level of the vocabulary (of the same language) in disjoint sets: two disjoint sets of proper names, two disjoint sets of concept-words (for 1st level), two disjoint sets of concept-words (for 2nd level), and so on. We also consider a bijection between each set, such that for every element of the first with determinate sense, matches a single element of the second with a different sense. Now, for a proposition occurring in the first set (that is, having a thought expressible through elements of the first set), does another proposition is obtained by translating it through the bijection defined? Frege's answer is yes, provided that we fix the logical vocabulary and assume a "logical perfect language" in which every possible translation has a thought associated.

If this setting is feasible, we can translate inferences from a set to another, by taking inference as sequences of true thoughts. Then, Frege sketches a "new law" for stating the independence from thoughts:

> Let us now consider whether a thought G is dependent upon a group of thoughts Ω. We can give a negative answer to this question if, according to our vocabulary, to the thoughts of group Ω there corresponds a group of true thoughts Ω', while to the thought G there corresponds a false thought G'. For if G were dependent upon Ω, then, since the thoughts of Ω' are true, G' would also have to be dependent upon Ω' and consequently G' would be true. (*CP*, p.339)

In other words, a thought G is independent from Ω if, and only if, for the bijection π above described, $\pi(G) = G'$ and $\pi(\Omega) = \Omega'$[23], and $\vdash \Omega'$ but $\vdash\!\!\top G'$[24].

Antonelli and May (2000) argued that, despite Frege's own reservations to this new law and new science, it is possible to derive a Fregean metatheory. The basic reason being that, if logic is universal for Frege, it should include logic itself in its range. Frege does say that mathematics is carried in Thoughts, but are not *about* thoughts (*CP*, p.336).

[23]More precisely, for each thought $T \in \Omega$, $\pi(T) = T'$ for $T' \in \Omega'$. Frege says that "we can think of a group of thoughts of one thought constituted out of other thoughts", that is, connected with the conjunction "and" (*CP*, p.334). Both ways are equivalent though.

[24]Again, by '$\vdash \Omega$' I mean the assertion of each thought in Ω as true. Also, given that Frege does not have an illocutionry rejection as primitive, to say that a thought is false means to assert its negation as true: $\vdash\!\!\top G'$.

Should we say the same for logic? Yes and no. In a sense, we can make logically sound arguments about Thoughts:

1. The thought that “sea water is salty” is the same as the thought that “*Meerwasser ist salzig*”.

2. The thought that “*Meerwasser ist salzig*” is the same as the thought that “*a água do mar é salgada*”.

3. Therefore, the thought that “sea water is salty” is the same as the thought that “*a água do mar é salgada*”.

If we can express such Thoughts and the relations between then, it must be possible to express the independence between Thoughts too. But as far as the concept-script is concerned, Thoughts cannot be mentioned within the system, as Frege did not include a primitive sign for them. Even if thoughts are in the range of Roman quantifiers (if we treat them as objects), there would be no use for them. To express the above argument, we would need an operator such as $\dot{\varphi}$ as denoting the sense of the sentence φ[25]. Thus, in order to formalize the inference above, we would need something such as

$$\begin{array}{l} \top\, \dot{q} \\ \llcorner\, \dot{p} \end{array}$$

But, if $\dot{p}$ names an object other than the True (which it does), — $\dot{p}$ names the False, following the definition of the horizontal. And following the definition of the conditional stroke, $\begin{array}{l} \top\, \dot{q} \\ \llcorner\, \dot{p} \end{array}$ names the True, given that both — $\dot{p}$ and — $\dot{q}$ names the False. Of course, this would still trivially validate the inference above. But so it would in any case involving reference to Thoughts. For this reason, it cannot be the case that Frege’s “new science” can “arises from the old [the concept-script] by the addition o new axioms”, such as the new law described above, as Antonelli and May (2000, pp.255-6) describes. It would, at least, require some crucial changes in the concept-script system.

What we know is that Frege did not pursue this new science after 1906. As he believed, “we are far from having a more precise execution of this. In particular, we will find that this final basic law which I have attempted to elucidate by means of the above-mentioned vocabulary still needs more precise formulation, and that to give this will not be easy” (*CP*, p.339). Because of the way this new law was sketched, Frege believed to be

[25] I borrow the dot-notation from Landini (2012).

crucial to determine "[...] what counts as a logical inference and what is proper to logic" (*ibidem*), given that the permutation between thoughts of the language cannot change between logical constants. It needs a precise formulation of the logical vocabulary. As Antonelli and May (2000, p.257) puts it, "we must find some way of knowing that these logical terms are *all* the logical terms", for otherwise, the permutation might give false positives.

Frege was an universalist and could not accept such procedure easily. For Antonelli and May, the universalist conception is not antithetical to developing a metatheory in the terms sketched, but not pursued, by Frege[26]. That the task is feasible does not explain why Frege did not adventure in developing it. I believe that the answer is simple: Frege's concept-script did not have ways of mentioning thoughts, inferences or truth in such a way to make it possible the expression of the new law into the formalism. These are only dealt in the expository language informally. Frege would need to define a new language for this "science of thoughts". But just as the concept-script (as an auxiliary language) is just a logic-made-precise, so will be this new science of thoughts. It would inherit the same problems in defining logical primitives, which prevents Frege from developing a metatheory in the first place.

The concept-script, as we saw, was designed as a scientific tool, and as such, its purpose was to be a language to be used and studied only practically. This, alongside Frege's philosophical motives for rejecting a higher standpoint such as a metatheory, leads to the conclusion that conducting logical investigation could only be done by proving things *in* the language. In a paper about Peano's logic, concerning the completeness of the axioms (as he understands it), Frege claimed the following: "In order to test whether a list of axioms is complete, we have to try and derive from them all the proofs of the branch of learning to which they relate." (*CP*, p.235) It is clear that no metatheoretical proof is even conceivable for Frege. Completeness, as he understood the notion, could only be achieved practically.

[26]Differently, Rickett's (1997) argued that the reason for Frege abandon this new science is simply that he cannot have such a precise demarcation of the logical. To accept a new law such as the one sketched would made Frege "invoke the concept of *thought that is a logical law*", but "Frege would be unwilling to identify a thought as a logical law just in case it is expressed by a sentence derivable from such and so axioms by such and so inference rules" (RICKETTS, 1997, p.184).

Chapter 4

Semantics and Objectivity

In the previous chapter, the question was raised whether Frege's conception of judgements, inferences and definitions can avoid the psychologistic problem. In this chapter, I shall give what I believe to be Frege's answer. The answer is dependent on the universalist reading of the concept-script above described, which yields that no metalogical justification could be provided, as one would do in modern logical systems. Rather, the justification for judgements, inferences and definitions are given in terms of restricting the conditions under which one is entitled to assert, infer and define within the limits of the concept-script.

The way out of the psychologistic dilemma is to recognize that the items in the first column on lines (A) and (B) in section 3.1 are not necessarily connected: being an inner event does not mean to be subjectively justified. If we consider the judgement-stroke $\vdash$ as meaning the first-person belief that the content is true (or is a fact), then the judgement

(a) $\vdash$ sea water is salty

can be read as

(b) I acknowledge/believe that sea water is salty

As we saw, (b) expresses an agent's attitude towards the saltiness of sea water, and he may well have different reasons for taking sea water to be salty. He may have faulty reasons, a false belief. But having beliefs can be equally justified in a logical manner. Thus, to consider judgements as inner events does not mean that no proper logical correctness can be given.

Nonetheless, once we accept that judgements are primarily belief attitudes, the psychologism problem kicks in. The problem can also be extended to Frege's other performatives of the language: definitions and inferences, as an agent can make empty definitions and unsound inferences. Frege's conception of judgements, definitions and inferences are all designed to avoid such cases[1]. Even though they all express an agents attitude, they still convey objectivity.

In an unsent letter to Jourdain[2], answering his previous question whether the judgement-stroke is merely psychological, Frege says the following:

> Judging (or recognizing as true) is certainly an inner mental process; but that something is true is independent of the recognizing subject; it is objective. If I assert something as true I do not want to talk about myself, about a process in my mind. And in order to understand it one does not need to know who asserted it. Whoever understands a proposition uttered with assertoric force adds to it his recognition of the truth. (*PMC*,p.78-79).

Here, Frege recognizes that judgements do involve inner mental processes, but that these processes do not contribute to a judgements' truth. A judgement is not about one's acknowledgement, it does not have the judging agent as part of its content, and neither is about his mental ideas. It seems that this passage should be enough to reject (b) above as a possible reading for the judgement-stroke, and to reject the psychological reading. But Frege's point is entirely about a judgements' content. What he means is that taking a judgement as an inner event does not equate in saying that the content is also inner events or entities. And precisely because of that, recognition of truth is not reducible to an inner event either. Thus, we shouldn't take a judgement such as (a) above as meaning

(d) sea water^{my} is salty^{my}[3].

i.e., that my idea of sea water falls within my idea of being salty. If this were the case, as Frege complains, every judgement would only be understandable by knowing who assert it in the first place. This kind of confusion, as I shall call it, is **contentual**. It takes the contents of judgements as subjective because judgement are themselves subjective.

The other mistake that we may commit in taking judgements as inner processes is not contentual, but rather **illocutionary**. One might take judgement's content to be objective Thoughts, but still undermine the idea that we can have correct judgements

[1]They are not however bullet-proof, since as we know Frege's axiom system was inconsistent.

[2]As the editors explain, this is a draft to a letter Frege sent to Jordain on January 28th, 1914. Frege's answer about judgements is not present in the official letter, however.

[3]The supperscript *my* here is simply meant as an indexical operator, denoting the corresponding agent's idea of the concepts in place.

from acts of acknowledgement alone, for there is no guarantee that subjective attitudes can necessarily reach true Thoughts only. In a sense, this worry is correct. Inner acts of acknowledgement are not perfect in reaching the True[4]. But they can still hit the target. The goal of the scientific language of the concept-script is to guarantee the standard of correctness for judgements in an inferential manner, making it possible for judgements to reach true thoughts, albeit being inner acts of acknowledgements.

In avoiding the two mistakes, Frege is endorsing two kinds of objectivity: one contentual, which will be discussed in 4.1, and other illocutionary. Illocutionary objectivity involves the two conditions under which judgements are properly made, in judging axioms and judging from rules of inferences. These will be dealt in 4.2

4.1 Contentual Objectivity

It is easy to see how Frege endorsed contentual objectivity when confronted with his platonistic metaphysics of Thoughts, that is, his defense of a third realm of abstract entities as exposed in the later phase of his philosophy, most prominently in the article *Der Gedanke*. However, this position is not found in earlier writings, and the best we may extract from them is the anti-psychologistic attitude concerning contents of judgements[5].

In the preface to the *Begriffsschrift*, Frege begins with the famous distinction between the genetic approach and the justificational approach towards truth[6]. This is, the distinction between the psychological apprehension of some truth, which may be different from person to person, from its justification for being true, which is solely dependent on the content, the "inner nature of the proposition considered" (*BS*, p.5). Justificatory basis for a judgement can be given either from sense experience, if it is fact dependent, or logically, if can be proved by means of logic alone. Either way, the justification for a given judgement is considered as mind-independent, and thus independent from the genetic origin of one's mental apprehension[7].

[4]As a matter of fact, Frege did judge Basic Law V as true in the *GGA*.

[5]A realist position is anti-psychologistic, but not every anti-psychologistic position is necessarily a realist one.

[6]In the preface to the *Grundgesetze* a similar distinction is presented between *being* true and *taking* to be true.

[7]The distinction is perhaps derived from Kant's distinction between *quid juris* and *quid factis* in the first *Critique*, and it is still a common practical assumption in analytic philosophy nowadays. Beaney (2020) argues that the genetic v. justification distinction is a dogma of analytic philosophy, as the practice took it as a cleavage between both perspectives (Russell's book on Leibniz being the case in point). Frege is not speaking about philosophical practice, but as I see it, even in his eyes the distinction is not a cleavage. The concept-script goal is to provide logical justification for its judgements, but even

Frege is certainly more interested in those judgements that can be logically justified, as the logicist analysis of Arithmetic was already his main goal with the development of the concept-script language. And those judgements, as he claim, are "[...] surely compatible with the fact that it could nevertheless not have come to consciousness in a human mind without any activity of the senses." (*BS*, p.5). The interesting point comes from a footnote to this passage, where he gives the following addendum: "Since without sensory experience no mental development is possible in the beings known to us, that holds of all judgments." (*ibidem.*). Two important pieces of information can be extracted. First, there is the recognition that the genetic approach cannot be avoided as far as humans are concerned, as all judgements are first and foremost known as mental products. Second, Frege adds that, to what concerns justification, *all* judgements are mind-independent. That is, a judgement's truth cannot be settle solely from the genetic approach, as he concludes: "it is not the psychological genesis but the best method of proof that is at the basis of the classification." (*ibidem.*). Therefore, in the *Begriffsschrift* we found that judgements, albeit being mental acts, are justified in terms of their contents, more precisely, their conceptual-contents.

In the early logic, contents are *judgeable contents* or content of possible judgements (*BS*, §2), roughly, contents that may be true or false. These contents are marked with the content-stroke "— ". But Frege does not yet have a definite way of treating them, offering different terms for what follows the content-stroke: "a mere combination of ideas", "a circumstance", "a proposition" or "a thought". Be that as it may, a judgement is only possible when (1) the complex totality that corresponds to the content, and marked with the content-stroke, is judgeable, and (2) when the content is "a fact", "affirmed" or "true" as recognized by the agent. Both are conditions on the contents of judgements, and both are not dependent on the relation with an agent. Even though Frege is not yet making the step to a third-realm of mind independent entities, it is pretty clear that these contents are not psychological in what truth is concerned.

We find a clear statement towards this position also in the unpublished paper *Logik*, written around 1879 and 1891, thus a representative of the early logic. Frege states:

> What is true is true independently of the person who recognizes it as true. What is true is therefore not a product of a mental process or inner act; [...] If the content of the sentence $2 + 3 = 5$ is exactly the same, in the strictest sense, for all those who recognize it to be

judgements are still mental apprehensions of some kind. The question, which is the goal of this chapter, is precisely on how to go from psychological origin to a logical justification, to which the concept-script comes to the rescue.

> true, this means that it is not a product of the mind of this person and a product of the mind of that person, but that it is grasped and recognized as true by both equally. Even if subjective elements are a necessary part and parcel of this grasping of a content, we shall not include them in what we call 'true'. (*PW*, p.3-4)

Frege's argument can be break down in the following way:

(P1) If the contents of judgements were inner/mental entities, they would be incommunicable.

(P2) The contents of our judgements *are* communicable (*e.g.* that 2+3=5 is true is a piece of common arithmetical knowledge)

(C) Therefore, they are not inner/mental entities.

Which follows simply from *modus tollens.* With such argument, Frege reaches the conclusion, still crudely putted, that the contents of our judgements are things we grasp equally by different agents. No precise formulation about what those contents are, or how are we able to grasp them, is yet offered.

The anti-psychological point is also made in the *Grundlagen.* Frege states the following:

> Never let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we become conscious of a proposition for a proof of it. A proposition may be thought, and again it may be true; let us never confuse these two things. (*GLA*, p.xviii)

Again, the genetic v. justification distinction is in place: a proposition [*satz*] should not be taken as true simply because one takes it to be true, or because one is conscious of it[8]. And repeating virtually the same argument from the preface to the *Begriffsschrift*, Frege concludes that "[...] a proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes." (*idem*). The whole business of the *Grundlagen* was, of course, to reduce Arithmetic to Logic, and thus to reduce arithmetical truth into logical truth. But in the outset of this goal, Frege was also endorsing contentual objectivity more generally. Even if he is not completely clear in the earlier writings concerning the contents of judgements, there is evidence that contentual objectivity is a thesis that he held. What justifies us in making a true judgement rests on what the judgement is about, its content, not on the judging subject.

[8]In this case, one would collapse assertoric force with entertaining a proposition or Thought, as it was the case with the combinatorial view on judgements.

But the objectivity held in the *Grundlagen* is a different kind to the platonic realism that would come latter. Frege still had no sharpen conception of contents. This would change with the major updates in his concept-script in the 1890's. What followed is what Dummett (1991b) called the fregean mythological theory of a third realm of senses.

First, a quick recap. In the early logic, signs expresses what Frege called conceptual-contents [*begrifflichen Inhalt*]. Starting in the paper *Funktion und Begriff* in 1891, but more throughfully in *Über Sinn und Bedeutung* in 1892, conceptual-contents were split up in two: a sense [*Sinn*] and a reference or meaning [*Bedeutung*]. In the latter, Frege says that "A proper name (word, sign, sign combination, expression) *expresses* its sense, *means* or *designates* its meaning. By employing a sign we express its sense and designate its meaning". (*SuB*, p.161). Proper names, concept names and sentences all have both a sense and a reference. The sense of sentences is what Frege calls *Thoughts* [*Gedanken*]. The reference of sentences is identified by Frege as a truth-value, here taken as proper objects. True sentences have the True as reference. False sentences have the False as reference.

As Frege explains, there is nothing preventing a sentence of having a sense but not a reference, "In grasping a sense, one is not certainly assured of meaning anything" (*SuB*, p.159). Thus, from a sense one does not necessarily reach a truth-value. But when judgement is concerned, one starts with senses to then reach a reference. In judging an assertoric sentences P, an agent is recognizing that P has the True as reference, and he does so in virtue of P's sense, its Thought. As he claims, "Judgements can be regarded as advances from a thought to a truth-value" (*idem.,* p.164-5). Of course, one can still make faulty judgements by missing the target. But correct judgements are to be regarded as true in virtue of the sense of the expression being judged, its thought. And this goes according to Frege's conception of senses and thoughts as being, once again, different from mental entities.

Both the sense and reference of a sign are not mental entities, and cannot be confused with the ideas (in the psychological sense) that they may elude in a speaker. It is not hard to see how this goes for references. By using the proper name "Moon" one is not referring to the idea of the Moon that one may have in their mind, but to the physical object. Senses, on the other hand, are not that easy to distinguish from mental ideas. But Frege is clear in taking them as objective as references are. The argument for it, in *Über Sinn und Bedeutung*, is virtually the same already present in the unpublished paper above quoted: if they were ideas, they would not be communicable, which they clearly are[9]. As

[9]First, Frege assumes the communicability premise: "For one can hardly deny that mankind has a

he concludes, "By a thought I understand not the subjective performance of thinking but its objective content, which is capable of being the common property of several thinkers" (*idem,* p.162, ft.7).

The point against the psychologistic reduction of contents is also made in the preface to the *Grundgesetze*. There, the accusation is that psychologistic authors such as Benno Erdmann (Frege's choosen target)[10], take the non-actual [*nichtwirkliche*] to be necessarily subjective, to which Frege blatantly reject:

> I acknowledge a realm of the objective, non-actual, while the psychological logicians take the non-actual to be subjective without further ado. Yet it is utterly incomprehensible why something that has being independently of the judging subject has to be actual, i.e., has to be capable of acting, directly or indirectly, upon the senses. (*GGA*, p.XVIII)

Frege's point is that a realm of non-actual but objectively given objects exists. This is what nowadays is called the realm of abstract objects. In Frege's case, it includes numbers, concepts, value-ranges, and most importantly, senses. The following table illustrate the possibilities that Frege is speaking of:

	non-actual	actual
objective	abstract objects	external (sensible) objects
subjective	mental objects	x

If, as Frege claims, the psychologistic author rejects the possibility of objective non-actual objects, there is nothing left but to take all objective as actual, and all non-actual as subjective. But, as his argument goes, the objective [*objektive*] and the actual [*wirklich*] must be distinguished.

This distinction was already hinted in the *Grundlagen*. Frege's worries in 1884 was precisely about the content of arithmetical judgements. There, he claims the following:

> I distinguish what I call objective [*Objective*] from what is handleable [*Handgreiflichen*] or spatial [*Räumlichen*] or actual [*Wirklichen*]. The axis of the earth is objective, so is the centre of mass of the solar system, but I should not call them actual in the way the earth itself is so. We often speak of the equator as an imaginary line; but it would be wrong to call it an imaginary line in

common store of thoughts which is transmitted from one generation to another". Then, he claims that ideas cannot be shared, while senses can: "[men] are not prevented from grasping the same sense; but they cannot have the same idea." (*SuB*, p.160).

[10] I wish not to evaluate if Frege is fair in his criticisms of Erdmann psychologism, as it will be beside the point.

> the dyslogistic sense; it is not a creature of thought, the product of a psychological process, but is only recognized or apprehended by thought. If to be recognized were to be created, then we should be able to say nothing positive about the equator for any period earlier than the date of its alleged creation. (*GLA*, §26)

The point being made, just as in the preface to the *Grundgesetze*, is to detach objectivity from actuality. The equator does not have actuality in the sense employed by Frege, as it is incapable of acting on our senses. In (*GLA*, §85), actuality is described in terms of "[...] what acts on our senses or at least produces effects which may cause sense-perceptions as near or remote consequences", which is an expected description, as *wirklich* is that which is capable of acting [*wirken*]. Nonetheless, this is a property that the Earth itself has, but not the equator. The passage also alerts, albeit not explicitly, not to take what is non-actual as psychological, as from the fact that the equator is imaginary do not follows that it is a product of our minds. In the end of the passage, the same communicability argument is repeated: nothing could be said about the equator if we take it as a mental creation, besides the trivial facts about the creation itself.

4.1.1 Dummett, Sluga, Lotze on Objectivity

Communicability is the main mark of the objective as Frege points out in the earlier phase of his philosophy. But communicability does not requires a strong ontological stance. Why then is Frege called a platonic realist? This point, and the distinction between objectivity and actuality, was one of the topics fiercely debated between Michael Dummett and Hans Sluga in the seventies and eighties. The debate took on a variety of topics about Frege's philosophy, but mostly it was famous for exposing two different perspectives on the analytic philosophy. Sluga complained that analytic philosophy became too much anti-historical[11], and emphasized the historical background of some of Frege's thesis and its importance in shaping his philosophy in general. Dummett, on the other hand, took a more analytic approach in interpreting Frege's philosophy, to which Sluga strongly disagreed[12]. But besides this interesting methodological battle, they also debated

[11] "Granted that Frege is a seminal figure in analytic philosophy and that he has a significant place in both the history of logic and the history of philosophy of the last hundred years, we may ask: why has it taken so long to this fact to be recognized? Why is it even now not fully acknowledged? Among the reasons for this curious neglect is the analytic tradition's lack of interest in historical questions — even in the question of its own roots. Anti-historicism has been part of the baggage of the tradition since Frege." (SLUGA, 1980, p.2)

[12] "Michael Dummett's extensive discussion of Frege and the philosophy of language can serve as a paradigm for the failure of analytic philosophers to come to grips with the actual, historical Frege." (SLUGA, 1980, p.3)

on the best interpretation for Frege's alleged realism. More precisely, on the distinction between objectivity and actuality.

The question was whether Frege's concept of objectivity should be read ontologically or epistemically and how far should be read Frege's platonism. Sluga (1980) argued that Frege's objectivity was only epistemically driven, following Lotze's influence. He also pushed for an interpretation of Frege as a transcendental idealist, rather than a realist as Dummett did, in (SLUGA, 1977). Dummett concurred, arguing both that Sluga's reading of Lotze missed the point and that Frege's conception of objectivity was in fact an ontological conception. Far from exhausting this debate, I want to restate Lotze's claims on the matter and assume the best of both worlds. Sluga is right in assessing Frege's historical heritage. Frege took a course on religion with Lotze at Göttingen, and it's a fair assumption to assume that he knew Lotze's work on logic, not only because Frege was his student, but because most of the German philosophers at the time read Lotze's logic anyway. Lotze's *Logik* was a commonly known work in academic philosophy, as Gabriel (2002) mentions. Lotze's influence was decisive for what would become the analytic and continental philosophies and also on neo-Kantianism, particularly with the concept of Validity. The genetic-justificational approach distinction present in Frege, discussed above, owns much to Lotze's notion of validity. But apart from Sluga's correct assessment of the importance of looking Frege's historical heritages, his assessment of Lotze's influence is not at all clear, and so Dummett's point should not be discarded too quickly.

Sluga summarize Frege's conception of objectivity under three theses:

> 1. The objective is that which can be grasped by more than one human (rational) being. The objective, in other words, is the intersubjective. (SLUGA, 1980, p.117)
> 2. The objective is that which does not require a bearer. (*ibidem.*)
> 3. The objective must be distinguished from that which is *wirklich*, *i.e.*, actual or real. (*idem.*, p.118)

At first sight, they seem all acceptable. Communicability is a common feature of the objective, and therefore, it is reasonable to take it as intersubjectivity. Moreover, Frege also did take the subjective as that which requires a bearer, and thus, the objective as independent from having one. That the objective is distinguished from the actual is also a clear thesis that Frege defended, as we already mentioned. But for Sluga, this conception of objectivity is imported from Lotze, which according to him is not an ontological concept. As he claims, Lotze "is an epistemological rather than an ontological Platonist" (SLUGA, 1980, p.119), and since Frege took the notion from Lotze, "Frege's doctrine of objectivity, like Lotze's, was intended as an epistemological thesis and that he was a critical rather than a dogmatic thinker" (*idem.,* p.120).

In fact, Frege's distinction between objectivity and actuality is found in Lotze's work. The latter appears in book III of Lotze's *Logik*, particularly in chapter II, in which he discusses Plato's conception of ideas. Lotze's opinion is that Plato never intended for Ideas to exist in the same sense as things are said to exist. For it, Lotze (1884, §317) introduces what he meant by reality (or actuality) [*wirklichkeit*] and their different manifestations. Different ontological categories are said to have reality or actuality: "[...] when we call anything Real, we mean always to *affirm* it, though in different senses according to the different forms which it assumes" (*idem.,* §316). There are at least four modes of reality identified by Lotze:

1. Of things, when they exist,

2. Of events, when they occur,

3. Of relations, when they obtain,

4. Of propositions [*sätze*], when they hold, or have validity [*geltung*]

Moreover, this distinctions are for Lotze mutually exclusive, in the sense of not being reducible to others. The interesting case is, of course, the reality of propositions. As Lotze claims,

> Ideas, in so far as they are present in our minds, possess reality in the sense of an Event,— they *occur* in us: for as utterances of an activity of presentation they are never a Being at rest but a continual Becoming; their content on the other hand, so far as we regard it in abstraction from the mental activity which we direct to it, can no longer be said to occur, though neither again does it exist as things exist, we can only say that it possesses Validity. (*ibidem.*)

That ideas as events in our minds are continual becoming follows from the Heraclitean thesis that everything in the world is in continual flux of change. Plato's presupposition, restated by Lotze, is that such flux demands an "unchangeable system of thought"[13]. But, as he claims, platonic ideas, here taken as the contents of our minds, do not exist as later

[13] The whole passage reads: "Whatever mutability the things may display, that which they are at each moment they are by a transient participation in conception which are not transient but for ever identical and constant, and which taken together constitute an unchangeable system of thought, and form the first adequate and solid beginnings of a permanent knowledge." (LOTZE, 1884, §314).

Realist accounts read it from Plato, simply because they do not possess reality in terms of existence, but only in terms of validity. Lotze's reading, be it a correct interpretation or not, implies that Plato never intended to hypostasize ideas.

Lotze's conception of objectivity is found in Book I, chapter I of *Logik*, concerning the formation of ideas. The first, and most basic of the functions of the mind is by him called *objectification of the subjective* [*Objectivirung des Subjectiven*] (LOTZE, 1884, §3). Such objectification is what Lotze takes as the "conversion of an impression into an idea", that is, "as something which has its being and its meaning in itself, and which continues to be what it is and to mean what it means whether we are conscious of it or not" (*idem.,* §2). The conversion follows from the linguistic act of giving a name to the content of sensible impressions.

Lotze then makes the distinction between objectivity and reality (actuality) [*wirklichkeit*]. As he claims, ideas do not necessarily have a reality attached to it:

> Such objectivity [*Objectivität*], therefore, does not entirely coincide with the reality [*wirklichkeit*] which belongs to things as such; it is only the fact of their claiming such a reality, on the ground of the distinctive peculiarity of their real nature, which language has met and expressed in their names. (*idem.*, §3)
>
> The logical objectification, then, which the creation of a name implies, does not give an external reality [*äußere Wirklichkeit*] to the matter named; the common world, in which others are expected to recognise what we point to, is, speaking generally, only the world of thought; (*ibidem.*)

Of course, the objectivity found in such names can still have *wirklichkeit.* But both are ultimately not the same: named ideas have objectivity insofar as they can be communicated and grasped equally by other persons, regardless whether its content exists, occurs, obtains or have real validity[14]. It is for that reason that Lotze takes the common world, that is, the world shared with such ideas, as being the world of thought. This world of thought is the same for all thinking subjects, but it is indifferent whether such common ideas "[...] indicate something which has [...] an independent reality outside the thinking minds" (*ibidem.*). For that reason, Sluga's reading that for Lotze objectivity is intersubjectivity is correct. All that it takes for something to be objective is to belong to such a realm of thoughts which need not be *wirklich* in regard to its contents.

In defining what he meant by being objective, Lotze does speaks of ideas being independent of our consciousness. But if the objective is the intersubjective, independence

[14]Whereas the content of an idea may have no reality, the mental counterpart must always have reality as it is an event in the mind of the one who grasps it.

is still only relative to a shareable language. True independence, such as the one found in Frege's conception of Thoughts, Lotze found in the real validity:

> we all feel certain in the moment in which we think any truth[15], that we have not created it for the first time but merely recognised it; it was valid before we thought about it and will continue so without regard to any existence of whatever kind, of things or of us, whether or not it ever finds manifestation in the reality of Existence, or a place as an object of knowledge in the reality of Thought. [...] the truth which is never apprehended by us is valid no whit less than that small fraction of it which finds its way into our intelligence. (LOTZE, 1884, §318)

There are strong echoes of Frege's own conception of objectivity here. But Lotze's is speaking about the validity of ideas. As he explains earlier, "[...] the reality of a proposition means that it holds or is valid and that its opposite does not hold" (*idem.,* §316). If a proposition holds, that is, if it has real validity, it holds independently from other ways in which it may have reality. Precisely, it holds independently of *occurring* in our minds as an event. Thus, independence follows from *wirklich* as validity. The ideas that we may communicate do not exhaust the ideas that may have validity. In other words, the world of thought (in the psychologistic sense) is not the whole world of valid propositions.

Intersubjectivity is certainly a weaker concept than full objectivity. The latter implies the former, but not the other way around. Lotze's conception leave it open the possibility that what is objective (in the intersubjective sense he employs) can still be real or actual[16], but it need not to be in order to be objective. When an idea does happen to possess real validity (as a form of *wirklichkeit*), it is also objective in the stronger sense: something which has real validity is not only common to all minds, but fully independent from them. This imply that in Lotze we may find two notions of objectivity: intersubjectivity, *i.e.*, ideas that are shareable and communicable (but still dependent on events in the mind), and objectivity proper, *i.e.,* which valid ideas (as *wirklich*) have.

This conception of objectivity as the intersubjective does find its place in Frege's *Grundlagen.* In the section §26, quoted before, Frege uses the example of the equator line as a non-actual entity which is still objective. What is striking in the remainder of the passage is how structuralist it sounds today. Frege asks us to image two individuals A and B which can only intuit projective geometrical properties. But, what

[15]Notice that Lotze is speaking in truths, not propositions. Thus, he is speaking about real validity in the *wirklich* sense.

[16]In a sense, it does have *wirklichkeit*, as it depends on language's shareability as events in our minds.

A sees as a line, B sees its dual, a point[17]. It follows from the duality principle in Projective Geometry[18] that for every intuition that individual A have, individual B have an equivalent true, and dual, intuition. They would never agree on each others intuition, but they would still be able to communicate[19]. Frege takes the example to show that

> With the word "point", for example, one would connect one intuition and the other another. We can therefore still say that this word has for them an objective meaning, provided only that by this meaning we do not understand any of the peculiarities of their respective intuitions. And in this sense the axis of the earth too is objective. (*GLA*, §26).

This is a striking passage to find in Frege. First, it is far from what he would claim later. Second, the duality principle, that Frege knew very well following his training in projective geometry, is much more related to nowadays model-theoretic approach to logic and mathematics, than it was to Frege's universal conception[20]. Nonetheless, this passage makes the case for Sluga's interpretation. The conception of objectivity expressed is that of Lotze's, as intersubjectivity. Communicability is all Frege demands for objectivity, as the example in projective geometry clearly shows.

But despite of it, a different interpretation of Frege's conception of objectivity comes from Dummett (1991c). He claims that

> Frege admits no category of the intersubjective intermediate between what is private to some individual subject and what is independent of all subjects. Lotze's '*wirklich*' corresponds to Frege's 'objective': but Frege has no notion corresponding to 'objective' understood in Lotze's restricted sense. (DUMMETT, 1991c, p.117)

To be fair, Dummett's point is about Frege's later philosophy of logic, after the sense-reference split, relying heavily on *Grundgesetze* and *Der Gedanke*. On this phase of Frege's thought, Dummett's interpretation seems correct.

Sluga's interpretation of the objectivity as intersubjectivity seems fine for the Frege of the *Grundlagen*. The later Frege, on the other hand, depends on the stronger

[17] Frege renders the example slightly different, but to the same effect.

[18] The Duality Principle states, roughly, that theorems in plane geometry containing points or lines have a dual theorem. That is, for any given theorem in plane geometry, if we simply change occurrences of "line" to "point", and vice-versa, we obtain a new theorem.

[19] The underlying assumption here seem's to be that communicability does not require sameness of meaning, but only an isomorphism between each person's private language. Even if A and B do not agree on each word's meaning, they would still understand each other if the same conversion of meaning is preserved throughout a conversation.

[20] It could be made the case that Frege was a disguised structuralist in this passage, but I do not endeavour to explore this possibility. About the principle of duality and its role in the development of model theory and its relation to Frege's metatheoretical concerns, see Tappenden (1997).

notion of objectivity for his refutation of psychologism. If ultimately Thoughts are objective only in the intersubjective sense, they are not independent from events and not fully immune to the psychologism. They are, at best, what is *common* to all consciousness. Frege also cannot rely on Lotze's conception of *wirklich* for solving such deficit, given that he (Frege) explicitly says that Thoughts are non-actual. It also goes along this lines that Frege's conception of what is *wirklich* seem to be narrower than that of Lotze. As mentioned before, Frege takes being capable of acting upon the senses as a mark of the *wirklich*. This cannot be a characteristic of *wirklich* as validity in Lotze's terms. This might be fine for the Frege of the *Grundlagen*, since there he distinguishes the objective from the *wirklich* just as Lotze did.

For this reason, Dummett's conclusion that Frege's conception of objectivity corresponds to Lotze's *wirklich* as validity seems more acceptable, if contrasted with his mature philosophy of logic. Thus, he goes against Sluga's interpretation that Frege's conception of objectivity is epistemically driven. If the objective is that which is independent of all thinking subjects (in such a stronger sense), and independence speaks about modes of reality, then objectivity must be an ontological classification. But given that both were resting their arguments in two distinct phases of Frege's thought, I follow Michael Resnik (1980, p.170) in concluding that "[...] we get a much clearer and simpler interpretation of Frege if we grant that he changed his mind and admitted an abstract world that exists independently of all thinking and thinkers".

But is Frege a platonist in the sense of also being a realist? Is he claiming that senses exist just as things are said to exist? Focusing on the later Frege, one of the ways one could argue in favour of this interpretation is how Frege often speaks about grasping [*fassen*] Thoughts. Rejecting psychologism implies that the contents of our thoughts and judgements are not creations, neither psychological entities. If they are not, one is tempted to consider them as things that reasoning subjects equally grasp. And naturally, the grasping presupposes something to be grasped:

> If we ever want to get past the subjective, then we have to think of cognition as an activity that does not create what is cognised, but grasps what is already there. [...] It is essential to grasping that there is something [*etwas da ist*] which is grasped; the inner changes alone are not the grasping. Similarly, what we mentally apprehend has being [*besteht auch das*] independently of this activity, of the ideas and their changes that are part of or accompany the apprehension; it is neither the sum of these processes nor is it created as part of our mental life. (*GGA*, p.XXIV)

Frege is always careful in saying that from the fact that Thoughts are grasped does not follows that they do so just as physical objects. We thus grasp thoughts in a metaphorical sense only. The point is repeated in the unpublished papers *Logik*, from 1897, and *Kurze Übersicht meiner logischen Lehren?*, from 1906 (*PW*, p.198). In the former, we find the following interesting point.

> What is grasped, taken hold of, is already there and all we do is take possession of it. Likewise, what we see into or single out from amongst other things is already there and does not come into existence as a result of these activities. Of course all metaphors go lame at some point. We are inclined to regard what is independent of our mental processes as something spatial or material, and the words that we have just listed make it look as if this is what a thought actually is. But this is not where the point of the comparison lies. What is independent of our mental processes, what is objective, does not have to be spatial or material or actual. (*idem.,* p.137)

Frege is then speaking on a metaphor when he says that we grasp Thoughts. But the metaphor, as he made clear in this passage, lies on the fact that grasping is usually employed for physical things, whereas Thoughts are neither spatial, material or actual. In fact, just like physical objects are independent from the grasping — if I hold a pencil in my hand, the pencil exists independently from the holding — Thoughts are said to be independent from the grasping as well. And this is the precise reason why Frege choose this metaphor to begin with. Even more, objectivity follows from the fact that Thoughts are independent from our mental processes.

The metaphor of grasping and the claim that Thoughts are non-actual is more thoroughly discussed in the paper *der Gedanke*. First, the choice of the word "grasp" is justified precisely in the same way as other writings: a metaphor for the act of holding a content.

> We are not owners of thoughts as we are owners of our ideas. [...] So it is advisable to choose especial expression; the word 'grasp' suggests itself for the purpose. (*DG*, p.368)
> The expression 'grasp' is as metaphorical as 'content of consciousness'. The nature of language does not permit anything else. What I hold in my hand can certainly be regarded as the content of my hand; but all the same it is the content of my hand in quite another and a more extraneous way than are the bones and muscles of which the hand consists or again the tensions these undergo. (*idem.*, ft.8)

Sluga claims that this passage is evidence on how Frege cannot be taken as a platonist on Thoughts. He claims that this passage "[...] seems to imply that Frege does not hold that thoughts are in the mind as the bird is in the hand, but rather as the muscles and bones are in the hand" (SLUGA, 1980, p.121). His intended conclusion is that Thoughts are objective in the sense of being "[...] not something alien or external to the mind, but constitutive of it" (*ibidem*). But Sluga missed the point that Frege is making with the grasping metaphor, as he is reading the mature Frege as the *Grundlagen* Frege. Thoughts, as he claims in the above passage, are not something we own such as we own ideas. Similarly, "content of consciousness" must be taken metaphorically given that the content is *not* in my consciousness as my ideas are. A Thought is in my consciousness in the same sense as the bird is in my hand, *i.e.*, metaphorically. Therefore, to keep the analogy together, the bones and muscles are *in* my hand just as ideas are *in* my consciousness: they are constitutive of it. Thoughts are not constitutive of my consciousness, as Sluga is concluding. It is ideas, as events that possess *wirklichkeit*, that are[21].

If thoughts' objectivity were constitutive of my consciousness, as Sluga claims, Frege would not take pains in trying to explain how can we be conscious of Thoughts in the first place. It is undeniable that the grasping also presupposes a consciousness in which the thought is grasped (in the same obvious sense in which grasping the bird with my hand presupposes a hand). And, as he explains, "Although the thought does not belong with the contents of the thinker's consciousness, there must be something in his consciousness that is aimed at the thought. But this should not be confused with the thought itself." (*DG*, p.369). The problem, then, is to explain how a consciousness is "aimed at" a Thought, if they are non-actual.

The best we can extract from Frege on this issue is that language is the way we may grasp thoughts. It is "wrapped up in a perceptible linguistic form" (*DG*, p.360 ft.6) that we came to grasp and interact with them. "The expression in language for a thought is a sentence." (*PW*, p.174), as he assume in *17 Kernsätze zur Logik*. "A thought cannot be perceived by the senses, but in the sentence it is represented by what can be heard or seen", in the unpublished *Logik in der Mathematik* (*PW*, p.206). We should not however expected this to be a solved issue, as neither did Frege believed that it was the purpose of a logician such as himself to solve it. He claimed that the grasping of a thought is in fact a mental process, but

> a process which takes place on the very confines of the mental and which for that reason cannot be completely understood from a purely psychological standpoint. For in grasping the law something comes into view whose nature is no longer mental in the proper sense, namely the thought; and this process is perhaps the

[21] Dummett made a similar comment on Sluga's conclusion in (DUMMETT, 1991c, p.123, ft.77).

> most mysterious of all. [...] It is enough for us that we can grasp thoughts and recognize them to be true; how this takes place is a question in its own right. (*PW*, p.145).

Frege is already convinced that Thoughts are not mental entities, given their non-actuality. Thus, even if grasping thoughts involve mental processes of some kind, it will remain mysterious how the actual does succeed in grasping the non-actual, as far as Frege's claims are concerned. All we need to know is that Thoughts are non-actual objective entities, the senses of assertoric sentences which we are able to judge and communicate through language.

Communication here does not mean that one person is able to transfer his ideas to another, only that he provides enough hints for others to grasp what he has grasped, to emulate what he has himself thought[22]. This ability to communicate thoughts (although not directly) presupposes the fact that Thoughts do act on what is actual, although they are not actual themselves. In a passage that much reassemble Lotze, Frege claims that "Thoughts are not wholly unactual but their actuality is quite different from the actuality of things." (*ibidem*). We should not jump to the conclusion that Frege is inconsistent with his classification of Thoughts as non-actual, though. Thoughts are not effected by being grasped, only the one who grasps it is effected, and even then, in a indirect manner.

In fact, a Thought that is grasped "[...] remains untouched in the core of its essence" (*ibidem*). Its core, as Frege considers, is "timeless, eternal, unvarying" (*idem.,* p.370). This are characteristics of the Thought irrespective if it is true or not. Nonetheless, if a Thought is true, it is timelessly and eternally true. If it is false, it is likewise timelessly and eternally false: "Almost everything that we have said about the predicate true holds for the predicate false as well. In the strict sense it applies only to thoughts" (*PW*, p.138).

To all that has been said, it is hard to separate Frege from a platonistic account of senses. We cannot assume that senses exists in the same way things are said to exist in the world (as Lotze would already claim), but still, the independence from the realm of physical and mental worlds (which Frege defines as being timeless, unchanged and non-actual) are not just an epistemic feature, but should be read as an ontological characterization of such abstract entities[23]. For such reasons, "A third realm must be

[22] As Frege claims, "People communicate thoughts. How do they do this? They bring about changes in the common external world, and these are meant to be perceived by someone else, and so give him a chance to grasp a thought and take it to be true." (*DG*, p.371)

[23] It is precisely because Frege sees objectivity of Thoughts similar to how Lotze saw ideas as having real validity that Dummett (1991c, p.120) concludes, against Sluga, that "[...] objectivity is in Frege's philosophy a type of ontological status, that of independence from any conscious subject. We may therefore look to Frege's assignment of objectivity to ascertain his ontological views: since thoughts,

recognized" (*DG*, p.363), as Frege himself did.

To what matters here, both the conceptions of objectivity, as intersubjective or as ontological independence, answers the psychologistic problem. If the contents of judgements are somehow subject to the mind of the one who grasps and judges it, then not even intersubjectivity can be reclaimed. But, if either of this positions holds, then the contents of judgements cannot be reduced the mind who grasps it. Young Frege seem to endorse the former, while mature Frege was a platonistic realist concerning such contents. In either case, Frege fully endorsed contentual objectivity.

4.1.2 Priority of Judgements or Priority of Thoughts?

Contentual objectivity is, simply put, the thesis that the contents of our judgements are objective. The sense in which they are said to be objective in Frege's mature philosophy must be read ontologically: thoughts are objective in having an ontological independence from the realm of the sensible and the realm of the psychological.

There is an ensuing dilemma concerning this account of objectivity. Should we explain the objectivity of judgements from the objectivity of contents? Are judgements objective because they have objective contents as constituents, or, on the contrary, are thoughts objective because they are the contents of objectively made judgements? This is a dilemma about the order of explanation between judgements and their contents[24]. The ontological picture painted in the previous section seems to imply the priority of thoughts, as objects of a Third Realm.

Both orders of priority can be explained in two senses of platonism, following Reck (2005a). Platonist A starts with a definite sense of objecthood, postulating abstract objects as residing in some platonic heaven. Then, from the objectivity of such realm, one derives the objectivity of judgements about them. Platonist B on the other hand, starts from judgements and their objectivity given in logical terms, to then derive the objectivity of objects occurring as their contents.

The standard account of Frege's philosophy is to take him as a pure platonist, thus taking him as a Platonist A. This is at least the intuitive conclusion following all the ontological passages provided above. However, a different reading comes from Erich Reck (2005a, 2007) and Thomas Ricketts (1986): that Frege is actually a platonist B.

According to Reck (2005a), there are a few problems in claiming that Frege

logical objects, and concepts are all of them objective, they can none of them be identified with ideas, and so must form distinct ontological categories".

[24] The dilemma is presented in Reck (2005b).

was a Platonist of the type A. The first one is that making our judgements objective by postulating a realm of objects is an appeal to a correspondence theory of true, that Frege explicitly rejects. The second problem, also found in Ricketts (1986), is that the priority of explanation provided by Platonist *A* seems to underline Frege's emphasis on judgements and their role within logic. I want to answer both problems, and say that Reck and Ricketts position must be taken only epistemically. Moreoever, I want to argue that there is an ontological order from the objectivity of Thoughts to objectivity of judgements, but an epistemic order of correctly made judgements to correctly depictions of the realm of Thoughts.

There are passages in the *Grundlagen* where objectivity is closely tied with reasoning and the capacity to judge, which is a perfectly reasonable thesis to hold in the *Grundlagen*, given that at that point Frege was not yet the full platonist that he would become. Nonetheless, the main passage is the same section §26 quoted before. After discussing what he means by being objective, Frege concludes the following:

> What is objective in it is what is subject to laws, what can be conceived and judged,what is expressible in words. [...] I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of the reason, — for what are things independent of the reason? To answer that would be as much as to judge without judging, or to wash the fur without wetting it. (*GLA*, §26).

As expected, differently from the passages of the later period, Frege is here emphasizing that objectivity is not independent from reasoning and judging itself, something that does not fit well with Platonist A. There are, of course, other issues that any Platonist A must solve in order to be reliable, for example, explaining how we come to access such realm of abstract non-actual entities.

Nonetheless, Reck's answer is simply that Frege is not a platonist of type A, since he "[...] starts directly from the objectivity of our logical judgements and inferences", where such objectivity is "a basic assumption; it is fixed and not further reducible." (RECK, 2005a, p.32). Similarly, Ricketts claims that "ontological categories are supervenient on logical ones" (RICKETTS, 1986, p.89), where "our grasp of the notion of an object is exhausted by the apprehension of inference patterns and the recognition of the truth of the basic logical laws in which these variables figure" (*idem,* p.84). For example, Frege's two ontological categories, objects and functions, are defined as supervenient to logical patterns involving first and second-order quantifiers.

In both cases, Frege's logicism comes to mind. The objectivity, and objecthood, that comes from arithmetic is tied up with the objectivity of purely logical judgements governing extensions of concepts. In the *Grundlagen*, Frege's context principle[25] plays a significant role on how it is possible to have access to numbers as objects.

> How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. (*GLA*, §62).

Ultimately, this means to reduce judgements about numerical identity into logical judgements, and further into logical laws and definitions. For the Platonist B, which both Reck and Ricketts agree to be the correct assessment of Frege, the objecthood of numbers is derivative of the objectivity of judgements in which numerical names occur. This is also what both authors understand by objectivity as dependent on reasoning. The follow passage is also mentioned as indicating this dependence:

> In arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it. (*GLA*, §105)

Reck and Ricketts' metaphysics of judgement does agree with Sluga's thesis that Frege is not a platonistic realist, only a transcendental idealist[26] And, in fact, their assessment is based more on passages from Frege's *Grundlagen*, where Frege's realism was not yet developed in full. Thus, although I agree with Reck and Ricketts' regarding this period, there is a basic issue that, to my mind, does not solve the psychological problem that Frege was so focused on later. Given that judgements are still recognition that a thought is true, they are irreducibly mental phenomena. In Lotze's terms, they are events in someone's mind. Even assertions, the manifestation of judgements in language, would not go much further from objectivity in Lotzean terms, *viz.,* intersubjectivity. If Thoughts and Numbers are objective only in this sense, we are not speaking of platonism anymore.

We saw before how Frege is not a objectivist in the weaker sense of intersubjectivity. A passage from the unpublished *17 Kernsätze zur Logik* make the point even stronger:

> '2 times 2 is 4' is true and will continue to be so even if, as a result of Darwinian evolution, human beings were to come to assert that

[25]"never to ask for the meaning of a word in isolation, but only in the context of a proposition" (*GLA*, p.xxii).

[26]A conclusion that Reck tries to avoid, however.

> 2 times 2 is 5. Every truth is eternal and independent of being thought by anyone and of the psychological make-up of anyone thinking it.

One possible reading of this passage is to take what "human beings come to assert" as what they are able to communicate mutually. If that's the case, then we should take him as saying that arithmetical truth are true independent not only from the "psychological make-up", but also from what is communicable, and *a fortiori*, from reasoning subjects. Compare the example of geometrical intuitions in (*GLA*, §26), where Frege drawed such a structuralist position. Here, even if everyone agrees that 2+2=5, that would not make it true.

Given the close proximity of Platonist B to what Sluga calls a transcendental idealist, in what sense is Platonist B still a platonist? Reck (2005a, p.35) claims that "existence is [...] understood in terms of the objective determinateness of the corresponding judgments, specially existentially quantified judgments". By determinateness he means the "corresponding judgements and inferences that are true or valid" concerning the objects in question. If those judgements are objective, so does the objects determined by them. If those judgements include words for such objects that behave as proper names, they are then self-subsisting objects.

But we are far from taking such existential claims as a form of conventionalism. The difference between both platonists A and B seems not to be ontological, but only epistemic, as far as existential claims are concerned. The determinateness of logical judgements concerning extensions of concepts, for example, warrants us in claiming that numbers exists as objects in their own right. If the reduction of numbers to extensions of concepts is logically proved, then we are not simply making one reduction out of many, but showing that numbers *are* essentially given in extensions of concepts. For instance, Reck (2005a, p.36) takes that the priority of judgements implies that

> the way in which number are understood to be logical objects in Platonist B is not necessarily connected with Frege's specific reduction of numbers to extensions of concepts. His reduction is, rather, just one of potentially many ways in which arithmetic judgments and inferences can be determinate with respect to their truth value.

But I believe this paints Frege as a structuralist, similarly to the geometrical example in the *Grundlagen* §26, which is not entirely consistent with Frege's mature platonistic philosophy. The ways in which Frege saw his reduction of arithmetic into extensions was not only "one out of many". As far as Frege's opinion is concerned, it was the *only* one available. In the afterword to the *Grundgesetze*, and even after knowing that Basic Law

V was inconsistent, Frege said in hindsight that “Even now, I do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not — at least conditionally — permissible to pass from a concept to its extension” (*GGA*, p.253). Not only Frege still believed that numbers were objects, but saw no alternative for apprehend them other than from the passage of concepts to their extensions. It should also be mentioned that Frege did not accepted Hume’s principle as a sufficient axiom in terms of not being about logical objects. Even if all judgements concerning HP are consistent and a coherent depictions of numbers, something essential would be missing. Taking HP as foundational, as Wright and Boolos’ neo-logicism took it, would in Frege’s mind “[...] depend on blindly accepting numbers as logical objects, without any reduction to entities that were referred to in an essential way in logical theory as it is the case with extensions (or value-ranges)”, as argued by Ruffino (2003, p.71). It was only after 1924 that Frege took another route: the geometrical source of knowledge. Frege believed that only through Basic Law V that numbers could be defined logically. Basic Law V was not only a failed route from logic to numbers, it was the failure of all routes, as Frege concluded that from the falsity of Basic Law V logic “on its own [...] cannot yield us any objects” (*PW*, p.279). Thus, Frege’s reduction of arithmetical judgements to logical judgements governing extensions has a scientific rationale of clarifying the ontological features of numbers. Judgements, in this sense, are also means of discovery what is there: “the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name“ (*GLA*, §96), as Frege would say against the formalists.

Frege’s attempted reduction also shows another problem in deriving existential claims from objective judgements. They may warrant us in claiming the existence of objects that occur in them, but given that judgements are irreducibly acts of acknowledgments (belief attitudes), an agent may still miss the target. An obvious example is the inconsistency of Basic Law V. As it seems, Frege was justified (logic-wise) in taking value-ranges as legit objects following the (alleged) determinateness of Basic Law V and the logically justified judgements concerning the smooth-breathing operator (‘,’). But, as we know, this is not the case. Reck of course explains that no judgement can be considered determinate if an inconsistency is present, given that Basic Law V renders some judgements, in which value-ranges occur, that are neither true or false. But before knowing the contradiction, Frege took the occurrences of the smooth-breathing as determined, governed by Basic Law V anyway.

Frege’s error, and the possibility of making mistakes, are nothing but evidence that Frege saw his system of logic as a scientific tool for discoveries, in which judgements play a crucial role (as the platonist B rightfully stress). As such, the logical practice of judging and inferring is also prone to error. Reck (2007) claims that the scientific (logical) inquiry in Frege's understanding "[...] involves more than the piece-by-piece derivation of one thought from another. It also brings with a more global, holistic desiderata, including [...] commitment to coherence and consistency", and thus, an intrinsic justification for the basic laws may be the right understanding, not a "external structure 'out there'" (*ibidem*). He is mostly correct in this reading, but we should not be so fast in dismissing the important role played by the "external structure 'out there'". Once again, Frege rejected Hume's Principle as sufficient for deriving numbers as proper objects, which would be a perfect coherent and consistent way of proving the basic laws of cardinal numbers. Nonetheless, if platonist B is to be a platonist at all, the "external structure" is an inescapable assumption. What the determinateness of our judgements (if consistent and coherent) claims is precisely the objectivity (and objecthood) of those entities that depicts this external structure. In this sense, I agree with (BURGE, 2005) in understanding Frege's claims of objectivity not in an idealist way, such as in Sluga or similarly as Ricketts and Reck, but as a platonist in the strong sense of the term (what they call Platonist A). He adds that , for Frege, "assuming the mind-independence of all thought contents, concepts, and logical objects, is necessary to understanding the objectivity of scientific practice and the universal applicability of logic and mathematics", given that "Logical and ontological notions are interrelated for Frege" Burge (2005, p.311, ft.15).

But I don't want to rule out judgements entirely. Burge claims that "Frege was interested in judgment as a norm-yielding form, not in judgment as a human activity" (BURGE, 2005, p.308, ft.12), and that "he makes it very clear that his logical theory is not about practices, and does not take its authority from such practices. They are not what ground the normative structures that logic articulates" (*idem,* p.309). What he means by this claims is that Frege does not take judgements to have any ontological priority simply because, as Frege claims specially in the foreword to the *Grundgesetze*, the laws of logic are prescriptions, not descriptions about how we happens to judge. This is correct, but it doesn't follow that Frege was not interested in judgements as a human activity at all. From the epistemic perspective, judgements, as an human activity, are necessary as far as the logical and scientific practice is concerned. And Frege does *define* judgements as the recognition of the True of a Thought. From what has been said so far, I believe that Platonist A is the correct ontological thesis concerning Thoughts, but Platonist B is its complementary epistemic thesis, that it is through the judicative practice that true Thoughts are given to us. Judgements are means for knowing and understanding Thoughts *as* independent entities.

Frege's platonism is not restricted to numbers or extensions of concepts as objects. Thoughts are the platonistic objects *par excellence*, and it is to their objectivity that we must focus on, given that to every judgement there is a thought correlated to it. The thought relates to a particular truth-value independently of being grasped or judged. Given that thoughts seem to be postulated in advance, Reck (2005a) claims that Platonist A does not avoid something akin to a correspondence theory of truth, which is not consistent with Frege's own claims against it, particularly in (*DG*, p.352-3). In this reading, a judgement is true if, and only if, it correctly depicts a particular state of affairs obtainable from the objective realm of abstract entities being postulated.

But this is not how the grasping and judging of thoughts happens. A Thought is not true *because* it depicts a state of affairs. The Thought is true *simpliciter*. Recall that Frege includes all Thoughts as being timeless, in the sense of including the time of the utterance in order to avoid ambiguities: "If a time-indication is conveyed by the present tense one must know when the sentence was uttered in order to grasp the thought correctly. Therefore the time of utterance is part of the expression of the thought" (*DG*, p.358). If someone asserts that "Snow is White", one is actually asserting "At t, Snow is White"[27], where t is the time of the utterance. What this shows is that the Thought does not changes truth-value if, for whatever reason, Snow ceases to be white. What changes is the Thought associated. As a consequence, Thoughts are true regardless of their ability to depict some state of affairs. It happens that some Thoughts are true, and some are not, and to judge correctly is simply to grasp and recognize that a Thought is True.

Our grasping and judging are not ways in which a given thought can be made true by correspondence with a realm of abstract entities. Neither are thoughts true in virtue of correspondence with such realm: they *constitute* the realm. Grasping and judging thoughts are ways of *mirroring* this realm, that is, the structure of thoughts. This means that our reasoning activities, which includes grasping and judging, and implicitly the notion of Truth, are objective as long as they can emulate the same dependence relations between thoughts. This point, together with an attempted answer to the access problem, is found in the unpublished 1923 paper *Logische Allgemeinheit*, which Frege intended as a fourth piece of the *Logische Untersuchungen*:

> If a thought cannot be perceived by the senses, it is not to be expected that its generality can be. [...] Language may appear to

[27]Although Frege did not specify this, Thoughts may also be local-sensitive: "At t, in l, Snow is White".

> offer a way out, for, on the one hand, its sentences can be perceived by the senses, and, on the other, they express thoughts. As a vehicle for the expression of thoughts, language must model itself upon what happens at the level of thought. So we may hope that we can use it as a bridge from the perceptible to the imperceptible. Once we have come to an understanding about what happens at the linguistic level, we may find it easier to go on and apply what we have understood to what holds at the level of thought — to what is mirrored in language. [...] Of course for this we have to reckon upon a meeting of minds between ourselves and others, and here we may be disappointed. Also, the use of language requires caution. We should not overlook the deep gulf that yet separates the level of language from that of the thought, and which imposes certain limits on the mutual correspondence of the two levels. (*PW*, 259)

This passage shows a good summary of Frege's doctrines. We find the following thesis:

1. Sentences, written or spoken, are sensible (actual in the sense of *wirklich*) objects. Sentences are also the expressions of thoughts. Thus, we grasp thoughts through sentences.

2. A (formal) language must express the relations between sentences in such a way to emulate the same relations between thoughts (that most certainly obtains or not independently from language). Putting in more modern terms, there should be an isomorphism between (formal) language and the realm of thoughts, *i.e.* for each sentence there must corresponds a definite thought; for each constituent of a sentence (concept-words, proper names), a definite constituent of a thought (function, object), and so on.

3. Finally, Frege recognize that a full and precise matching between language and the realm of thoughts is impossible (there is a "deep gulf" between them). Even though language is the medium through which we know thoughts and their relations and properties, the task of the logician is that of a scientist: to make such mirroring more a more coherent. We must also content ourselves with a "meeting of minds", giving that language is the way we grasp and communicate thoughts.

This considerations are, I believe, also indicatives that Frege would not accept the Platonist B label so easily. First, that a "meeting of minds" may disappoint us is indicative that communicability is a necessary, but not sufficient, condition for objectivity[28]. Second,

[28] This is consistent with the fact that objectivity in the strong sense implies, but is not restricted to, intersubjectivity. Physical objects are both objective and intersubjective. But some institutions (language being the obvious example) are intersubjective, but not objective in the same sense as the Sun or the Moon.

since logic is still a linguistic device, it is performed in the level of what is actual (in the *wirklich* sense), precisely as events within reasoning agents' minds. But objectivity is, as Frege constantly remarks, what is independent from the grasping and judging subjects in this precise way. Even though the concept-script is made precise enough in reducing the mistakes to a minimum, we can still miss the target entirely, as Frege himself missed with Basic Law V. For that reason, it is the framework of the realm of thoughts that we must mirror our linguistic practices, not the linguistic practices themselves.

I believe that Platonist A, despite Reck's critics, is still Frege's intended reading of Thoughts. But Platonist B can be taken as a correct epistemic reading of our judging and inferential practices. As already known, Frege treats logical primitives as indefinables. The best we can do is to elucidate them. Reck (2005a) cleverly put that Frege's notion of elucidation involves an hermeneutic triangle concerning thoughts, truth and judgements, in the following way:

1. Judgements are the recognition of the truth of a thought;

2. Truth is what we acknowledge about thoughts in judging;

3. Thoughts are what we acknowledge as true in judging.

This is a fair assumption on how these notions come to be used and kwown in the practice. But it is not a sufficient ontological depiction of Frege's intentions, as thoughts cannot be ontologically derivative of objectively made judgements, as the latter are irreducibly acts of recognition. It also adds that Frege recognize all Thoughts as being objectively true or false, but not every thought can be objectively made in a logical sense. Judgements relating sense perceptions, or geometric intuitions, cannot be called objective in these terms, as they do not follow solely from logical laws. And still, if true, the corresponding thoughts are true in the absolute sense.

Reck's hermeneutical triangle is, however, a good assessment on how we manage to understand this notions in the practice, in the sense that each can only be grasped in a meaningful way by taking the whole elucidatory circle as a whole. As far as linguistics is concerned, this is the best reasoning agents can do. The whole hermeneutic triangle is representative of the structure of thoughts. But in order for the whole picture to be consistent, the logical, mathematical and scientific practice demands for Thoughts to be objective in the stronger platonistic sense: we have to postulate them in order to safeguard objectivity in the stronger sense. This practice cannot disregard the fundamental role played by judgements: it is by means of correctly, and logically objective made judgements that agents are able to correctly mirror the realm of Thoughts within the realm of language. If we can take the analogy further, the mirror (the concept-script) is a necessary condition for the mirroring (objectively made judgements in the concept-script, following the normative rules of logic). But the objects being mirrored (true thoughts) are not dependent on the mirror neither on the mirroring.

The goal so far was to show that Frege's conception of judgement is objective enough for logical purposes, albeit the psychologism problem. In his mature philosophy (at least after 1890's) this contentual objectivity met a clear platonistic background. How, then, reasoning agents are able to go from inner belief attitudes (judgements as acknowledgements of true thoughts) to objective true contents? To make the long story short, by mirroring the realm of thoughts within formal language of logic[29]. The best one can achieve in mirroring such a realm of thoughts is in pure logic, where judgements are logically and generally justified. That can be no perfect and complete mirroring is both a limitation and a motivation, as it is what strives the scientific goal of truth further.

4.2 Illocutionary Objectivity

So far, contentual objectivity means that if judgements are correctly made, they express a true Thought that is objective in Frege's standards. What is left to argue is how a judgement is correctly and objectively made. I call this kind of objectivity as *illocutionary objectivity*, mostly because it pertains to how one can have objectively made actions, both of judging and inferring. The latter means that the transition from judgements to judgements, if made in a logically valid way, mirrors the transition from true thoughts to true thoughts. But not every act of judgement follows from another judgements in inferring. This is the case of the Basic laws. Apart from Basic Law V, their general contents provide enough justification for judging them as being true, following Frege's ideas about axioms being self-evidently true. The illocutionary objectivity as a whole is another name for the *Soundness* of Frege's logical system. But one thing is missing in Frege's logic that prevents us from simply proving its soundness as it is commonly done in today's logical text-books. Frege has no underlying conception of a metalanguage in which we can carry over such proof. Even if we can separate an expository from an auxiliary language, his universal conception of logic takes the latter as a fragment of the former.

[29]If Frege's answer is philosophically sound is another story entirely.

4.2.1 Soundness as Elucidations

In the standard logical practice, any axiomatic system must meet the demands of being sound, *i.e.*, that every deduction that is obtained under it leads to sound conclusions. This means to be impossible to start with true premises and advance to false conclusions. Given that the whole point of having deductions and inferences is to progress from truths to truths, soundness is one of the most basic conditions for any logical calculus. In model theoretic standards, soundness is expressed as the conditional between two metalogical relations:

$$\text{if} \vdash \varphi \text{ then} \vDash \varphi$$

where $\vdash$ is the relation of syntactical consequence, and $\vDash$ is the relation of semantical consequence. The proof of soundness of a given axiomatic system usually falls into three basic steps:

1. One shows that all axioms are true (semantic-wise);

2. Then, shows that the rules of inference are truth-preserving;

3. Finally, one proves by induction that every possible deduction of a formula φ ($\vdash \varphi$) is also valid ($\vDash \varphi$), given that it will be an instance of an axiom (case 1) or will be derivable from axioms following the truth-preserving rules of inference (case 2).

The strong soundness of a calculus follows when $\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$, meaning that if a set of sentences Γ derives φ, then φ is logical consequence of Γ. In the model-theoretic setting, this means that every model of Γ is also a model of φ, provided that φ is derivable from Γ on the calculus. The complexity of the proof depends on the system being considered, whether propositional, first-order, and so on.

There are great obstacles in reproducing such a proof in Frege's system. In fact, this is not just difficult, but impossible if we follow Frege's concerns rigidly[30]. Besides the universalist conception of logic, Frege's system of logic deviates significantly in the following three points: (1) the judgement-stroke is not a sign of a metalanguage; (2) there is no conception of interpretation of sentences in Frege's universal logic, and (3) a tarskian truth-schema is not expressible in Frege's logic. I'll first detail these three points before going to Frege's elucidatory way of justifying its Basic Laws and Rules:

[30]There is some consensus from the "no-metatheory" scholars of Frege that such metatheoretical results are not translatable into Frege's conception of logic. See, for example, Ricketts (1997), Blanchette (2012), Weiner (2005).

(1): Frege's judgement-stroke $\vdash$ is not a sign of what we would call the metalanguage, or in Frege's case, the expository language[31]. If read carefully, the judgement-stroke includes both a syntactic and a semantical reading. Given that proofs in the concept-script are inferences, syntactic consequence is thought in terms of assertability. And since assertions are belief attitudes towards a Thought's truth, we can also find a semantic consequence in it as well. It makes sense from our point-of-view to take both relations as included in Frege's judgement-stroke, but it would sound odd for Frege to think that both could be treated separately. Thus, soundness cannot be stated as a relation between syntactic and semantic notions, because these two notions cannot be sharply distinguished in his formalism.

(2): Frege's system is universal in terms of being unconditionally interpreted. An inference is not the transition from sentences to sentences, but from Thoughts to Thoughts, and this admits no exception. In discussing Hilbert's "pseudo-propositions", as he called, Frege rejected the possibility of treating deductions from hypothetical premises:

> surely one can make deductions from certain thoughts purely hypothetically without adjudging the truth of the latter. [...] But then it is not these thoughts that are the premises of such inferences. Rather, the premises are certain hypothetical thoughts that contain the thoughts in question as antecedents. Even in the final result, the thoughts in question must occur as conditions; whence it follows that they were not used as premises, for otherwise they would have disappeared in the final result. (*CP*, p.335).

Frege could not accept deductions from unasserted sentences simply because, in his system, Thoughts are the objects of inferences, not sentences. If we make the hypothetical assumption that φ, and derive ψ, it is not the implication $\varphi \rightarrow \psi$ that is derived, but the hypothesis-that-φ $\rightarrow$ the hypothesis-that-ψ. Thus, if we start with the hypothesis that φ, we cannot conclude something of φ itself. This may sound odd, given that Frege himself has separated the assertive force from the content. Why not formalizing an hypothetical proof from unasserted sentences? Moreover, Frege seems to imply this when he says that

[31] To be fair, the sign is also used in the expository language. Both in the *Begriffsschrift* and in the *Grundgesetze*, judgements are presented in the exposition of the system in a careless way. In the latter, Frege sometimes uses quotation marks to mention an asserted formula, sometimes he just assert it directly. For example, in stating the inferential rule *modus ponens*, asserted formulas are only mentioned (*GGA*, §14), but in justifying axioms, formulas are asserted directly (*GGA*, §18). In the *Begriffsschrift* Frege is even careless, where the use-mention distinction is entirely missing. But the fact that it is mainly designed for the auxiliary language makes the point.

> This separation of the act from the subject matter of judgement seems to be indispensable; for otherwise we could not express a mere supposition [*Annahme*] — the putting of a case without a simultaneous judgement as to its arising or not. (*FuB*, p.149)

But differently from the supposition laid down by an unasserted sentence, Frege seems to be speaking about what he called "hypothetical compound thoughts", in the paper *Gedankegefüge* (*CP*, p.399-). In fact, unasserted sentences never occur isolated in the concept-script. One of the ways in which an unasserted sentence may occur is as the antecedent of a conditional sentence, that expresses hypothetical compound thoughts. In the inference from

$$\text{`}\begin{array}{l}\vdash\!\!\!\top\,\Gamma\text{'}\\ \quad\llcorner\,\Delta\end{array} \quad \text{and} \quad \text{`}\begin{array}{l}\vdash\!\!\!\top\,\Theta\text{'}\\ \quad\llcorner\,\Gamma\end{array}$$

to

$$\text{`}\begin{array}{l}\vdash\!\!\!\top\,\Theta\text{'}\\ \quad\llcorner\,\Delta,\end{array}$$

all propositions are hypothetical compound thoughts as Frege would say. And we could say that in such inference, Δ occurs as an hypothesis. But the conclusion is not about Δ, but Δ *modulo* antecedent of a conditional[32].

The interesting point is that unasserted sentences never occur in the auxiliary language, but they may occur in the expository one, as ways of grasping and thinking about Thoughts before the judicative activity of judging and asserting them as true. Frege's discussion of Russell's paradox in the appendix to *Grundgesetze* is another hint for such a fact. Frege omits the judgement-stroke in deriving the contradiction from Basic Law Vb, but he also does not derive it in the concept-script proper. The derivations are informal, in the expository level. But when Frege wants to prove the falsity of Basic Law Vb, he does use the judgement-stroke and proceeds formally with inferential lines, just as the constructive parts of that book.

He never venture in adding a sign for the illocutionary act of putting a sentence as an isolated hypothesis in the concept-script, something like

$$\vdash_{?} \Delta$$

If he had treat it in the same level of illocutions, just as the judgement-stroke, he would not be at odds in explaining hypothetical judgements, as evident in quotation above.

[32] To be fair, the hypothetical rule of implication introduction, the conclusion is not about the hypothesis either.

What his insistence that hypothetical reasoning is not possible in the terms advanced by Hilbert is that in the concept-script *all* sentences are interpreted sentences, expressing definite Thoughts. Thus, deriving a sentence φ is *ipso facto* to show that φ express a true Thought. Of course Soundness is something that Frege was worried about, as we shall see. But a proof in the model-theoretical level would not be feasible.

(3): The model-theoretic soundness proof requires a Tarskian truth-predicate in order to be carried over. The semantic consequence relation $\vDash$ is defined for truth-assignments in the vocabulary of the calculus. We say that $\Gamma \vDash \varphi$ holds if every truth-assignment that makes each $\gamma \in \Gamma$ true, also makes φ true (the $\vDash \varphi$ case is just the same, if considered as $\varnothing \vDash \varphi$). A truth-assignment is defined over interpretations in the model-theoretic way. A model is simply an interpretation in which every sentence is true under truth-assignments defined over it. It all depends on taking the predicate "is the true" as definable in the metalanguage for sentences of the object-language.

Does Frege had a tarskian truth-predicate? This is a controversial topic. Weiner (2005 2008) argues that Frege does not have and neither needs a truth-predicate for the justification of its laws and rules of inference. Weiner argument runs over some topics already discussed here, particularly on Frege's no-metatheory perspective, and the absence of schematic letters in the language. But the core of her argument is the fact that Frege's treatment of truth does not include a truth-predicate at all, but truth as objects. Frege, as we know, adopted in the *Grundgesetze* the strategy of taking truth as objects of denotation for sentences. Since he saw concepts as functions from objects to truth-values, a saturated concept, *viz.* a sentence, must have a truth-value as its value. According to Weiner's reading, Frege does not use the locution "denotes the True [*bedeute das Wahre*]" in his justification of logical laws and inferences, but "is the True [*dar Wahre ist*]". But, as she claims, "is the True" is not a truth-predicate at all, given that it only holds true of a single object, *viz.,* the True. A truth-predicate must be applicable to every true sentence, not only to a single object.

A different approach comes from Greimann (2008), who objects to Weiner's conclusion. According to him, Frege does need and use a truth-predicate in justifying his basic laws and rules of inference. Frege needs a truth-predicate because justification of basic laws is obtained by making it explicit the self-evident character of its senses. And in doing so, we predicate truth of them. Greimann also objects to Weiner that "is the True" is in fact a truth-predicate, or at least it can be adapted as a tarskian one. This is achieved by reading "Δ is the True" as "the truth-value that: Δ, is the True", which satisfy a truth-schema such as "the truth-value of 'p' is the True if, and only if, p".[33].

I want to partially agree with both perspectives in the following way. First, agreeing with Weiner, that Frege does not have a metatheoretical soundness proof, mostly because a tarskian truth-schema is not expressible in Frege's logic. But, I also agree with Greimann that the truth-predicate does play a role in justifying the Basic Laws. But this role, as I read it, is elucidatory.

What does such a tarskian truth-schema would represents in Frege's logic? Can we adapt it to a fregean "metatheory"? What does the right-hand side of the schema (T) means? Is p being asserted? The question is not only if "is the True" can be cast or not as a truth-predicate (it certainly does), but how a truth-schema could fit to Frege's system. Frege's expository language cannot be said to be essentially richer than the auxiliary language: they are both at the same level.

If we want to express the schema (T) in Frege's expository language, we would need some changes. First, notice that in (T), the right-hand side has p being used, not mentioned. But in Frege's case, this means that p is naming its truth-value. This is not enough, for (T) in this case express:

the truth-value of 'p' is the True if, and only if, the truth-value of p.

Now, in order for such schema to be useful, one must *assert* p in the right-hand side. That is, p has the True as truth-value just in case p is somewhat asserted as being the case. In the usual (T) schema, p is being used in the sense of being asserted in the metalanguage. If p obtains in the conditions of the metalanguage, then 'p' is true in the object-language. In Frege's case, this cannot be expressed without the judgement-stroke. Thus, it should be include in the right-hand side[34]:

the truth-value of 'p' is the True if, and only if, $\vdash p$.

Given that we can change the locution "the truth-value of 'p'" for the horizontal function, a simplified fregean version of the (T) schema is:

$-\, p$ is the True if, and only if, $\vdash p$.

The problem is about what is happening in the right-hand side of the biconditional. If we are *ipso facto* asserting p already, then the whole truth-schema is pointless, not to

[33]This position is more thoroughly presented in Greimann (2000).

[34]Frege does use the judgement-stroke in both "levels" of his language. He does assert the true of the axioms, for example. Thus, it is safe to say that the judgement-stroke can occur in the expository language.

mention, circular. If the truth-conditions of p is given in terms of (T), and (T) already contains an assertion that p, then an assertion that p cannot be warranted in terms of (T). Thus the assertion should be only mentioned. Also, Frege never uses the judgement-stroke in a subformula: assertoric force is only present in the context of the whole formula. In this case, (T) should be striped out of any assertoric force:

$$-\!\!-\, p \text{ is the True if, and only if, '} \vdash p\text{'.}$$

The problem here is how can we read '$\vdash p$'? Greimann seems to read it as a assertoric sentence, "p is the True" *simpliciter*. If that's the case, (T) becomes

$$-\!\!-\, p \text{ is the True if, and only if, } p \text{ is the True.}$$

Which is trivial. If the judgement-stroke is an illocutionary act, as I read it, the mention of an expression containing it reads differently: "that — p is asserted as being the True". (T) thus reads it as

$$-\!\!-\, p \text{ is the True if, and only if, } -\!\!-\, p \text{ is asserted as True.}$$

The problem is that this is also circular, because the conditions for the assertability of p cannot be given in terms of (T) if (T) depends on the assertability of p.

Another problem is that "is asserted as the True" seem not to be suitable for the task at all. If we want to give truth-conditions for $\top \Delta$, we would first say

$$\top \Delta \text{ is the True if, and only if, } \top \Delta \text{ is asserted as True.}$$

but, $\top \Delta$ is the True if the Δ is the false. But we cannot express being the False in terms of assertability other than saying that the negation of Δ is the True, which is what the left-hand side of (T) is already stating. Moreover, the negation of "is asserted as True" does not mean that Δ is the False, but rather that Δ is simply not asserted at all.

The whole point is the following: reading the judgement-stroke as an illocutionary act means that mentioned assertions must be read as "it is asserted that Δ is the True", rather than "Δ is the True". But this reading is not suitable for (T) if (T) should explain how our assertions are warrant. Of course, (T) explains the truth-conditions, but truth-conditions are the justificatory means for making assertions. To offer a proper soundness proof would require a (T) schema in order to explain why axioms are true and rules of inference truth-preserving, in order to say when one is entitled to assert them. This is not feasible if (T) includes the conditions as being asserted beforehand. (T) should not include assertions at all. But, if we read the mentioned assertions as simply "Δ is the

True", then (T) becomes a triviality. To say that "Δ is the True if, and only if, Δ is the True" is not a warranting condition for asserting that Δ is the True.

Where does it all go wrong? I believe that even if "is the True" can be cast as a tarskian truth-predicate, it would be of no use in Frege's case, where no sharp distinction between languages is available. The expository and the auxiliary language are in the same level truth-wise. This is what Frege assumes, for instance, in his famous argument against the correspondence theory of truth in *Der Gedanke*. The whole passage reads as following:

> But could we not maintain that there is truth when there is correspondence in a certain respect? But which respect? For in that case what ought we to do so as to decide whether something is true? We should have to inquire whether it is true that an idea and a reality, say, correspond in the specified respect. And then we should be confronted by a question of the same kind, and the game could begin again. So the attempted explanation of truth as correspondence breaks down. And any other attempt to define truth also breaks down. For in a definition certain characteristics would have to be specified. And in application to any particular case the question would always arise whether it were true that the characteristics were present. So we should be going round in a circle. So it seems likely that the content of the word 'true' is *sui generis* and indefinable. (*DG*, p.353)

We are only going in circles in trying to specify truth-conditions in this way if no proper distinction between languages is made in advance. The circularity objection that Frege made is simply that the truth-predicate being defined must also be used in specifying the conditions under which it could be applied. This, I believe, attests for the above argument that no proper truth-schema in the tarskian form is applicable to Frege's logic, simply because no sharp distinction between languages is assumed by Frege. Truth is the same in both expository and auxiliary language, which is another hint for why one needs a judgement-stroke as a assertoric force indicating device in the auxiliary language.

These three points make it hard to believe that Frege was providing proper soundness proofs for his Basic Laws and rules of inferences. Given that there's no proper way to prove the soundness, justification is provided as elucidatory remarks in the expository language instead. For this expositions, a truth-predicate finds its way in making it explicit the self-evident character of the laws.

For instance, Greimann reads assertions such as "$\vdash \Delta$" as assertoric sentences. He quotes Frege's justification for *modus ponens* in (*GGA*, §18), and reads it in the following way:

> According to the semantic interpretation of the conditional, the truth-conditions of the sentence '$\begin{array}{l}\vdash\!\!\top\ \Gamma\\ \quad\llcorner\ \Delta\end{array}$'[35] are that it is false if Δ is the True and Γ is the not the True, and that it is true in all other cases. Hence, if Γ is not the True, '$\begin{array}{l}\vdash\!\!\top\ \Gamma\\ \quad\llcorner\ \Delta\end{array}$' can be true only if Δ is not the True. But in this case '$\vdash \Delta$' is not the true. Consequently, the truth-conditions of '$\begin{array}{l}\vdash\!\!\top\ \Gamma\\ \quad\llcorner\ \Delta\end{array}$' already rule out that the premises are true and the conclusion is false. (GREIMANN, 2008, p.408)

He then continues saying that "it is obvious that in order to elaborate the sketch of his justification of Basic Law I and *modus ponens*, Frege needs a Tarskian truth-predicate" (*ibidem*). This shows that something akin to a truth-predicate does find its use in justifying the basic laws. But as I read it, Frege offers performative arguments as elucidations for accepting the truth of the Basic Laws and the rules of inference. Justification for the basic laws and rules of inference does proceed from making explicit the senses of the expressions considered, and we do use the "is the True" predicate. But I do not consider expressions containing the judgement-stroke as sentences. When we flank a sentence (or proper name in Frege's case) with the judgement-stroke, we *say* that it names the True.

The relation between what is happening in the expository language and in the auxiliary language is the following: in justifying its basic laws and rules of inference, Frege uses proxies for possible assertions ("is affirmed" in 1879; "is the True" in 1893). These proxies are needed for elucidating basic laws and rules of inference in such a way that any agent will be inclined into judging it as true. The task is not to predicate true for the basic laws or the conclusion of the rules, but to warrant the assertion of them.

A usual justification for *modus ponens* runs simply by checking the truth-table for φ, $\varphi \to \psi$ and ψ. In this case, we use metatheoretical variables and predicate truth of then in the metatheory. We would simply say that if φ is true and $\varphi \to \psi$ is true under a generic assignment, then ψ is also true in this assignment. Given that we are talking about the justification of a rule of inference, the conclusion must be cast as a sort of permission: since such and such is the case, one is allowed to derive ψ under these conditions. The rule, however, is defined in the syntax, and so the justification is a semantical argument for a syntactical procedure. I believe that Frege's justification for *modus ponens* does that, but in a single elucidatory argument. His argument is twofold: it provides a semantical justification in terms of denotation, to then conclude a permission

[35] I took the liberty of translating the usual symbol for implication $\to$, defined over sentences, to Frege's own notation, defined over names. The choice of doing so was, however, only visual. There is no loss in the interpretation of Greimann's because of it.

for assertability. First, he justify *modus ponens* in the following:

> From the propositions '$\vdash\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$' and '$\vdash \Delta$' one can infer: '$\vdash \Gamma$'; for if Γ were not the True, then, since Δ is the True, $\top\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$ would be the False. (*GGA*, §14)

Recall first that Frege names expressions with the judgement-stroke as concept-script propositions [*Begriffsschriftsatz*] or simply propositions[36]. Notice also that Frege opts in enclosing each concept-script proposition with quotation marks. Following his use-mention distinction, is safe to say that Frege is not asserting anything, but quoting possible assertions. Greimann, on the other hand, calls such expressions sentences, and speaks about the truth-conditions of "$\vdash\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$".

But as I take it, these are not sentences, but mentioned assertions, as they do not carry any assertoric force. Truth-conditions for '$\vdash\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$' are not the same as the truth-conditions for $\top\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$. The first is an expression for an assertion, while the latter names a truth-value. Truth-conditions should be read differently for each. In Frege's case, the latter asks in which conditions does the function $\top\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$ denotes the True: this is exactly what the definition of the conditional function provides, and exactly what Frege is arguing next. The former asks under which condition the assertion that the latter denotes the truth is warranted, or to put it more precisely under Frege's words, when the acknowledgement of the true of a Thought hits the target. As I read it, we should not read '$\vdash\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$' as being true or false, but as being warranted or not[37]. What is the True or the False is the object named by $\top\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$.

Frege does not separate syntax from semantics. For this reason, his justification for *modus ponens* can be read as being both: a semantical argument, given in terms of the denotations of the primitives, followed by a syntactical conclusion, providing means

[36]"The concept-script representation of a judgement by means of the sign '$\vdash$' I call a concept-script proposition or proposition for short" (*GGA*, §9)

[37]As far I know, Frege does not speaks about asserted expressions having Thoughts assigned to them. But it could be made the case that quoted assertions does have thoughts. In this sense, an expression such as "$\vdash\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$" could be said to be true or false, but the Thought expressed would be different. We would talk about the Thought that "$\top\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$ is asserted", rather than the thought of "$\top\!\!\begin{array}{l}\Gamma\\ \Delta\end{array}$" *simpliciter*.

for asserting a proposition in the calculus. As exposed in the beginning of the chapter, the "is the True" predicate functions as a proxy for assertions. Given that the role of the assertion is to recognize that a name has the True as reference, one is warrant to make an assertion when it is somewhat obvious that a name has the True as reference in all possible cases. But saying that a name names the True does not suffice: one has to assert it. Thus, truth plays a significant semantical role in justifying the rules of inference and basic laws, but this occurs as elucidations in the expository part of the language. Once they are made, an agent is warrant in asserting a basic law, or the consequence of a rule, in the auxiliary language, but now with assertoric force, not a truth-predicate.

We also know that Frege's adoption of the judgement-stroke as a sign for assertions is designed primarily for being used in the auxiliary language (or object-language). This, added the presence of the quotation-marks, hints to the following: Frege is justifying when an agent is allowed to make an assertion *in* the auxiliary language, and the reason he offers comes from the definition of the conditional function and the horizontal function in terms of the denotation of its subcomponents. If both $—\Delta$ and $\begin{array}{l}\top\!\!-\Gamma\\ \llcorner\!\!-\Delta\end{array}$ denotes the True[38], there is no other option for an agent other than recognize, following the appropriate definitions and corresponding judgements, that Γ must also denote the True. The recognition that it denotes the True is what Frege calls an acknowledgement, *viz.*, a judgement, expressed by means of an assertion. Such assertion is warranted by the aforementioned coherence principle for judgements: that it is incoherent to deny that Γ is the True under the conditions that both premises in *modus ponens* name the True.

A similar stance could be made on how Frege justifies his basic Laws. The difference is that in justifying them in the *Grundgesetze*, Frege is stating assertability conditions directly. He is not just quoting possible assertions in the auxiliary language, but asserting propositions directly in the expository language. But the reasoning is the same. Basic Law I, for instance, reads:

> According to §12
>
> $$\begin{array}{l}\top\!\!\top\!\!-\Gamma\\ \mid\;\llcorner\!\!-\Delta\\ \llcorner\!\!-\Gamma\end{array}$$
>
> would be the False only if Γ and Δ were the True while Γ was not

[38] In the auxiliary language, both are known to denote the True if prefixed with the judgement-stroke. We can talk about 'Δ' denoting the True as a provisional in the expository language, but in the auxiliary language, the assertive force of the judgement-stroke is the appropriate way of recognizing such a fact.

the True. This is impossible; accordingly,

$$
\begin{array}{l}
\vdash\!\top\!\!\top\; a\\
\;\;\;\;\llcorner\; b\\
\;\;\llcorner\; a.
\end{array}
$$

(*GGA*, §18)

According to Frege, what allows one to assert Basic Law I is the definition of the conditional function in (*GGA*, §12). The definition shows that $\begin{array}{l}\top\!\!\top\,\Gamma\\ \;\;\llcorner\,\Delta\\ \llcorner\,\Gamma\end{array}$, which is a name of a truth-value, cannot name anything other than the True. It is a name following Frege's stipulation for uppercase Greek variables: "I here use the capital Greek letters as if they were names referring to something, without stating their reference." (*GGA*, §5, ft.3). And given that it is a name that necessarily names the True, we are allowed under assertability conditions to acknowledge it as denoting the True, *viz.*, to assert it.

Frege's conclusion is that an acknowledgement of truth is permitted. The difference with the Tarskian treatment is clear: while in modern model-theoretic justification, we conclude that the rule of *modus ponens* is truth-preserving, Frege's argument takes the truth-preserving property (in terms of denotation) to then say that the assertion of the conclusion is permissible.

Greimann points out correctly that the role of these arguments is to make it explicit the sense of the expressions. The self-evident character of the Basic Laws is what warrant one's assertions, or as Frege puts it, the advance from sense to a reference. The truth-predicate, or similar versions such as "is affirmed" or "is the true", are needed for seeing the normative function that logical laws and rules of inference play in their generality. The two-step procedure is this: (1) one recognizes that a logical law denotes the True generally, and thus (2) one sees that it is incoherent not to assert it as such. Likewise for rules of inference: (1) one recognizes that the truth-value of the conclusion is the True if the truth-value of the premises are also the True, thus (2), it is incoherent not to assert the true of the conclusion. Both steps occurs in the expository language. But given that logical inferences are made in the auxiliary language, the truth-predicate is not enough: the agent must assert it. In order to visualize how Frege proceeds in such manner, I will now provide a few examples from both logics.

4.2.2 Judging the Axioms

Given that Frege does not have a rigorous metatheory, there can be no formal way to prove the soundness of its logical inferences. But the justification provided for axioms both in the *Begriffsschrift* and *Grundgesetze* are typical semantical arguments nonetheless, provided by the proxies of assertions, "is affirmed" and "is the true". The same goes for the justification of its most basic rules of inference. In all cases, it is possible that besides preservation of truth, there is a underlying performative argument for accepting both axioms as denoting the True and the inferential rules as truth-preserving.

The case is even clearer in the *Begriffsschrift*, where Frege had not yet introduced True and False as objects. Rather than saying that thoughts are true, Frege says that they are *affirmed*. The idea is that rejecting the axioms, or the conclusion of an inference following *modus ponens*, is incoherent. Rejecting it would yield that the agent is affirming and denying something simultaneously.

But before going to the axioms, one last thing should be said about judgements. All the Basic Laws in the *Begriffsschrift* and the *Grundgesetze* are expressed using the Roman generality device. As we saw, any concept-script expression involving a Roman letter is an Roman object marker, where an object is only indicated, not named. If judgements are the recognition that a Thought is true, then only proper names of the True shall be judged. But in this case, we do not have proper names, but object-markers. What, then, are we doing when judging the axioms?

As Frege claims in (*GGA*, §26), not only Roman letters in general but German letters that occur outside the concavity and Greek letters that are used for the expression of value-ranges are also not names. They are only indicative of a truth-value. For instance, while $\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$ is a name, for it stands for the True, individual occurrences of $\mathfrak{a}$ are not. However, any expression that Roman letters occur are not names at all, while expressions such as the quantified self-identity above, where German letters are used, still names a truth-value.

But Frege still calls the application of the judgement-stroke to a Roman Object-marker as a judgement. In his words,

> A sign which consists of a judgement-stroke and a name of a truth-value with a prefixed horizontal, I call a *concept-script proposition*, or *proposition*, where there can be no doubt. Likewise, I call a *concept-script proposition* (or *proposition*) a sign which consists of a judgement-stroke and a Roman marker of a truth-value with a prefixed horizontal. (*ibidem*).

What is then the difference between

$$\vdash\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a} \quad \text{and} \quad \vdash a = a?$$

If judgements are the recognition of the True, what are we recognizing in a judgement to which no name is available, such as $\vdash a = a$?

If we accept that judgements about Roman object markers are still the recognition of the True of a Thought, then we must recognize that the thing being judged is a name, not a Roman object marker as Frege defines it. This, I believe, is a inescapable dilemma. Either we say that expressions containing Roman letters behave like proper names, or there can be no judgements about them. I believe the best is to interpreted Frege as saying the former, otherwise, we would have to accept that judging the Basic Laws are not judgements at all.

This problem can be explained in a number of ways. A first solution is to argue that judgements such as $\vdash a = a$ are indirect judgements. Thus, an agent asserting it is not making an actual assertion, but putting forward a conjunction of assertions such as $\vdash (\Delta_1 = \Delta_2) \quad \wedge, ..., \wedge \quad \vdash (\Delta_n = \Delta_n)$, for each of the possible instances with proper names for Roman object markers. This solution, however, must be rejected as it goes against Frege's theory of judgements. Not only the Basic Laws would end up to be no actual judgements, but we cannot make infinite judgements all at once. To be sure, we can make judgements with infinite content, but cannot make infinite judgements, as judgements are acts. A second attempted solution would be precisely to render $\vdash a = a$ as a judgement with an infinite conjunction, such as $\vdash ((\Delta_1 = \Delta_1)\wedge, ..., \wedge(\Delta_n = \Delta_n))$[39]. This is a better attempt, but still unsatisfactory, as it would demand an infinitary language. We most certainly do not have names for every possible object in order to write such a conjunction. I will assume a third, but still problematic, solution: that an assertion of $\vdash a = a$ must be read as an assertion about possible proper names that the placeholder a is here representing.

This reading stems from Heck (2012c). In his reading, Frege's use of Roman object markers is similar to truth-assignments. In fact, $\vdash a = a$ should not state that $a = a$ is true, but that it is true under any assignment[40]. But, as he claims, "Frege does

[39]In this case, I took the liberty of using the more easily expressible $\wedge$ sign for conjunction, instead of Frege's $\top\!\!\top$.

[40]Frege seem to have this in mind when he says, in (*GGA*, §17), that for any given judgement, if "signs are put which do not refer to an object [...] by merely containing Roman letters, then [the judgement] still holds if for each Roman letter a name is put, whichever name it may be, and thus it holds generally".

not use (or have) the notion of an assignment; he speaks instead of substituting names which refer to objects for Roman letters which merely indicate them." (HECK, 2012c, p.65). What he means is roughly that instead of assuming Tarskian assignments, Frege uses auxiliary names with uppercase Greek letters such as Δ, Γ as representing an object, to then substitute the Roman-object markers for having proper names. Frege's usual explanation for $\vdash a = a$ is that $\Delta = \Delta$ is the True for any proper name 'Δ'. For instance, Frege uses this explanation in the following passage:

> [...] an equation such as
>
> $$`\Phi(x) = \Psi(x)'$$
>
> always yields a name for the True, whatever proper name we might insert for 'x', provided only that this really refers to an object. (*GGA*, §8)

But Heck is not worried about what one is judging when using Roman letters, and so we must take this idea further in order to explain what one is judging with Roman object markers.

Going back to both expressions for generality, we can grasp the difference between the judgements $\vdash\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$ and $\vdash a = a$ by grasping what are the Thoughts being judged as True in both cases (assuming both are genuine judgements in Frege's sense). In the former case, '$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$' express the following Thought:

That $\xi = \xi$ is the True for every argument ξ,

This Thought follows from the definition of the concavity generality in (*GGA*, §8). Since this is in fact the True, the Thought can be itself judged. But the Thought associated with the judgement $\vdash a = a$ cannot be primarily about the function $\xi = \xi$, for the Roman letters are ranging over proper names, not over objects: this is what Frege meant in saying that they only indicate an object.

Recall that Roman object markers are never used outside the scope of a judgement. And in judging the axioms, Frege often goes from upper-case Greek letters Δ, Γ as possible proper names, for judging generally over such proper names using the Roman letters. For example, he goes from $\begin{array}{l}\top\, \Delta \\ \llcorner\, \Delta\end{array}$ to judging $\begin{array}{l}\vdash\!\top\, a \\ \quad\llcorner\, a\end{array}$. The employment of Roman generality in judging generalizes over proper names, not over objects. We can talk about what the expression $\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$ has as reference because it is a name of a truth-value following the definition of the concavity. But to talk about what a Roman object-marker refers to, we would have to talk about its possible instances. Properly speaking, the Thought that $a = a$ is something like:

> That for any proper name 'Δ', $\Delta = \Delta$ names the True.

Notice that we cannot even talk about the Thought of this judgement using Roman letters, as this is restricted only to judging. Thus, we may say that the difference between the Thoughts associate with the judgements $\vdash\!\overset{\mathfrak{a}}{\smile}\!-\, \mathfrak{a} = \mathfrak{a}$ and $\vdash a = a$ is that the former is a judgement about the concavity function, while the latter is a judgement about proper names.

This reading have the undesired consequence that many of the concept-script judgements are not properly about objects, but about names for objects. But, I think that this is consistent with how Frege reads Roman object Markers. It also offers a solution for why Frege calls the attachment of the judgement-stroke for expressions with Roman object markers also judgements. $\vdash a = a$ is not saying that $a = a$ names the True, but it is saying that $\Delta = \Delta$ names the True for any proper name 'Δ'. And this is also consistent with how one may go from one generality to the other, given that $\Delta = \Delta$ holds for every 'Δ' if, and only if, $\xi = \xi$ is the True for every argument ξ[41].

Finally, Frege has no rule of inference for Universal Instantiation for the Concavity generality. Instead, if one wants to go from

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!-\, \mathfrak{a} = \mathfrak{a}$$

to

$$\vdash 2 = 2$$

one has to first use an instance of Basic Law II

$$\begin{array}{l} \vdash\!\!\top\!\!-\, a = a \\ \;\;\llcorner\!\overset{\mathfrak{a}}{\smile}\!-\, \mathfrak{a} = \mathfrak{a} \end{array}$$

and then use the rule for replacement of Roman Letters to obtain

$$\begin{array}{l} \vdash\!\!\top\!\!-\, 2 = 2. \\ \;\;\llcorner\!\overset{\mathfrak{a}}{\smile}\!-\, \mathfrak{a} = \mathfrak{a} \end{array}$$

[41] Another obvious problem with this reading is that it seems to bring back Frege's old identity-of-content sign $\equiv$ of the *Begriffsschrift*, that he rejected in the opening of *Sinn und Bedeutung*, as it was defined over expressions, not contents. But in this case, we would have to read $\vdash a = a$ as "That the name '$a = a$' names the True", which is not the reading here proposed.

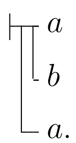
One can then have universal instantiation from *modus ponens*, deriving the wanted judgement that $2 = 2$. Clearly, Frege never goes from a concavity generality to any instances without appealing Roman object-markers as representative of all possible names for objects.

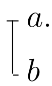
This solution is still not entirely satisfactory, but as far as I know, it is the best reading for keeping Frege's theory of judgements consistent. From this reading, we can see that judgements are still the recognition that a Thought is True, even in cases where Roman Object Markers are being used. There is no change in the form of judgements, only in the Thought being judged. And there is no loss of applicability, given that any Thought such as "that $\Delta = \Delta$ is the True for every name 'Δ'" implies Thoughts such as "that 2 is equal to 2".

Having made this provision, we can proceed in showing the elucidatory explanations Frege uses in justifying his axioms and rules of inferences.

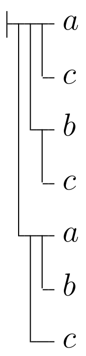
4.2.2.1 Axioms in *Begriffsschrift*

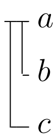
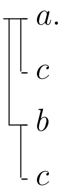
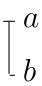
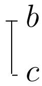
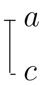
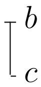
The first axiom, or "law of thought" in the 1879 version of the concept-script is the following:

(1) 

Frege states, in justifying (1), that "This is evident, since a cannot at the same time be denied and affirmed" (*BS*, §14). What he meant is more precisely the following reasoning by *reductio ad absurdum*: if (1) is denied, following the definition of the conditional stroke, then we must affirm a and deny  But to deny the latter is to affirm b and deny a. Thus, rejecting (1) implies affirming and denying a, which cannot be the case. Thus, at risk of being incoherent, (1) must be affirmed.

The same line of reasoning is provided for the next axiom:

(2) 

Once again, the task is to argue that it is not possible to deny (2) safely from incoherence. Denying (2) means to affirm the antecedent subformula , and deny the consequent subformula . Thus, we must check what does it mean in both cases. In the first case, affirming the antecedent subformula means precisely that it is not possible to affirm c and deny . And since we cannot deny the latter under the condition that c is affirmed, b cannot be affirmed and a denied. We only get so far a general condition: a cannot be denied if both b and c are affirmed. For the second part, we consider the denial of the consequent subformula. This means that we affirm  but deny . Denying the latter means that c is affirmed and a is denied. But given that  is affirmed, b cannot be denied in the condition where c is affirmed. And given that c is affirmed, so thus b must be affirmed. And as we consider in the antecedent formula, if b and c are affirmed, a must also be. Thus, a is both affirmed and denied, which cannot happen. Again, coherence state that (2) can be affirmed under any circumstance, as its negation is not possible to be affirmed.

This reasoning, albeit not that complex, it still too much clumsy to be followed textually. Modern truth-tables are much more precise in stating exactly the same. There is no inclusion of a truth-predicate yet, only the correlate “is affirmed” and “is denied”. But there is bivalence principle implicit for judgements here, which I called the coherence principle for judgements: that as far as logic is concerned, one cannot affirm and deny the same proposition safe from contradiction. Either the affirmation or the denial shall be judged, but not both.

Axioms (8) and (28) holds from very much the same reasoning, and shall be ignored. Axioms (31) and (41) are even simpler:

$$(31)\quad \BGassert\BGconditional{a}{\BGnot\BGnot a}$$

In Frege's terms, "$\BGnot\BGnot a$ means the denial of the denial, hence the affirmation of a" (*BS*, §18). The same goes for the other direction in the axiom (41).

Axioms (52) and (54) are judgements about the controversial identity-of-contents sign:

$$(52)\quad \BGassert\BGconditional{\BGconditional{f(d)}{f(c)}}{(c \equiv d)}$$

Given that c and d have the same content, substitution *salva veritate* may be performed. Thus, it cannot be the case that $(c \equiv d)$ and $f(c)$ are affirmed and $f(d)$ is denied, given that denying $f(d)$ would forces one to deny $f(c)$ if $(c \equiv d)$ is affirmed.

Avoiding axiom (54), we finally have:

$$(58)\quad \BGassert\BGconditional{f(a)}{\BGquant{\mathfrak{a}} f(\mathfrak{a})}$$

Given that the generality by means of the concavity states that $f(\mathfrak{a})$ holds for any $\mathfrak{a}$, $f(a)$ can be affirmed on the condition that $\BGquant{\mathfrak{a}} f(\mathfrak{a})$ is. Denying (58) would be to affirm $\BGquant{\mathfrak{a}} f(\mathfrak{a})$ but deny $f(a)$. But affirming the first means to also affirm $f(a)$, an instance of the concavity. Thus, denying (58) means to affirm and deny $f(a)$. Hence, (58) cannot be denied.

Frege's analysis of the assertability of each axiom follows from performative reasoning. It is not possible to deny each axiom without, at the same time, being incoherent in affirming and denying something simultaneously. In the expository part of the concept-script, Frege speaks "is a fact" instead of the predicate "is the true". But in justifying each of its basic laws, only "is affirmed" or "is denied" occurs as predicates. At that stage of his thinking, no quotation marks are used and no distinction between using and mentioning formulas is in place. But if we extend Frege's auxiliary-expository language distinction back to the *Begriffsschrift*, "is affirmed" is a convenient way to express judgements in expository language without carrying the assertive force. That A is affirmed represents A's being asserted in the auxiliary language, *viz.*, $\vdash A$. This allows

Frege to express conditional judgements in the expository language. Expressions such as "if a is affirmed then b is affirmed" never occur in the concept-script. But it is what Frege needs to justify when judgements are properly made in the auxiliary language: one is entitled to judge when it is incoherent to deny it.

4.2.2.2 Axioms in *Grundgesetze*

In *Grundgesetze* there is a shift of talking about affirmation and denial, to denoting the True or the False in justifying the basic laws. The reasoning, however, is fairly similar, and a performative argument is still present, albeit less explicit than before.

The justification of Basic Law I was already discussed above, but it is worth repeating it:

> According to §12
>
> $$\begin{array}{l} \top\!\!\top\, \Gamma \\ \quad \llcorner\, \Delta \\ \;\llcorner\, \Gamma \end{array}$$
>
> would be the False only if Γ and Δ were the True while Γ was not the True. This is impossible; accordingly,
>
> $$\begin{array}{l} \vdash\!\top\!\!\top\, a \\ \quad\;\; \llcorner\, b \\ \quad \llcorner\, a. \end{array}$$
>
> (*GGA*, §18)

As argued, Frege's definition of a judgement as the acknowledgement of the truth of a thought is in use here. The analysis of the conditions for denotation of expressions governing the conditional function make it the case that the name $\begin{array}{l} \top\!\!\top\, \Gamma \\ \quad \llcorner\, \Delta \\ \;\llcorner\, \Gamma \end{array}$ cannot have another denotation other than the True. It is a triviality then to acknowledge that it names the True given that it cannot name anything else. This is the justification for the judgement of the Basic Law I. If we read it similarly as for the same basic law in the *Begriffsschrift*, we would say that it is not possible to deny Basic Law I without taking Γ to denote the True and the False. Recall that Frege saw the laws of logic as being normative for thought in general:

> The ambiguity of the word "law" here is fatal. In one sense it says what is, in the other it prescribes what ought to be. Only in the latter sense can the logical laws be called laws of thought, in so far as they *legislate* how one ought to think. Every law stating

> what is the case can be conceived as prescriptive, one should think in accordance with it, and in that sense it is accordingly a law of thought. (*GGA*, p.XV)[42].

Under such conditions, Basic Law I forces itself into judging it as True. Given that Frege is not simply arguing that the name $\begin{array}{l} \top\!\!\!-\, \Gamma \\ \big|\llcorner\, \Delta \\ \llcorner\, \Gamma \end{array}$ has necessarily the True as value, but taking this fact as a normative guidance for judging, he is also offering a performative justification for Basic Law I, *viz.*, offering reasons for judging it.

The justification for other Basic Laws follows in a similar fashion, with suitable complexities. In justifying Basic Law II (*GGA*, §20) Frege first argue that the name

$$\begin{array}{l} \top\!\!\!-\!\!-\!\!-\, \Phi(\Gamma) \\ \llcorner\overset{\mathfrak{a}}{\smile}\!-\, \Phi(\mathfrak{a}) \end{array}$$

is always the True regardless of the function $\Phi(\xi)$. This fact is expressed by the following function

$$\begin{array}{l} -\overset{\mathfrak{f}}{\smile}\!\top\!\!\!-\!\!-\, \mathfrak{f}(\Gamma) \\ \qquad\llcorner\overset{\mathfrak{a}}{\smile}\!-\, \mathfrak{f}(\mathfrak{a}) \end{array}$$

which, following the definition of concavity, as Frege reads, is the True regardless of the value of Γ. Given that it is always the True, it can thus be judged:

$$\begin{array}{l} \vdash\!\overset{\mathfrak{f}}{\smile}\!\top\!\!\!-\!\!-\, \mathfrak{f}(a) \\ \qquad\llcorner\overset{\mathfrak{a}}{\smile}\!-\, \mathfrak{f}(\mathfrak{a}) \end{array}$$

And since the concavity in $\mathfrak{f}$ is on the broader scope possible on the formula, it may be omitted:

$$\begin{array}{l} \vdash\!\!\top\!\!\!-\!\!-\!\!-\, f(a) \\ \quad\llcorner\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a}) \end{array}$$

Frege's reasoning can be simplified in the following manner: if $\overset{\mathfrak{a}}{\smile}\!-\, \Phi(\mathfrak{a})$ is the True for every $\mathfrak{a}$ and function Φ, then $\Phi(\Gamma)$ is the True. Following the definition of the conditional, Basic Law II must name the True, and thus, can be judged as such. As Frege reads it, "what holds of all objects, also holds of any". Similarly to axiom (58) of the *Begriffsschrift*, judging $\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a})$ as the True means that $f(a)$ must also name the True, following the rules concerning German and Roman letters for generality. Denying Basic Law II would mean

[42]The emphasis is mine

that ther former is the True while the latter is the False, which cannot happen. And since it cannot happen, it cannot be denied without being incoherent.

Basic Law IV proceeds more easily (*GGA*, §18). $-\Delta$ and $\top\Delta$ are always different in truth-values. In comparison, $-\Gamma$ must be either equal to $-\Delta$ or to $\top\Delta$ (*tertium non datur*). Likewise, $(-\Gamma) = (-\Delta)$ cannot have the same value as $(-\Gamma) = (\top\Delta)$. Hence,

$$\begin{array}{l} \ulcorner\!-\, (-\Gamma) = (-\Delta) \\ \llcorner\!\top\, (-\Gamma) = (\top\Delta) \end{array}$$

is always the True, and thus can be judged as such:

$$\begin{array}{l} \vdash\!\!\ulcorner\!-\, (-a) = (-b) \\ \quad\llcorner\!\top\, (-a) = (\top b) \end{array}$$

We could say, more simply, that the truth-value of Γ must either be equal to the truth-value of Δ or the negation of Δ. Otherwise, Γ would denote the True and the False. In all cases, Basic Law IV denotes the True and thus can be judged as such, for its negation is incoherent and thus cannot be judged.

Frege's justificatory basis for Basic Law V was, as known, flawed. In (*GGA*, §20), he makes reference to §3 and §9 as sufficient reasons for judging it as true. Particularly, in (*GGA*, §9), he states that "we can convert the generality of an equality into a value-range equality and vice versa. This possibility must be regarded as a logical law of which, incidentally, use has always been made, even if tacitly, whenever extensions of concepts were mentioned". In the foreword, however, a different tone is expressed: "a dispute can arise only concerning my basic law of value-ranges (V), which perhaps has not yet been explicitly formulated by logicians" (*GGA*, p.VII). But that Frege was able to make such a crucial mistake does not speak about the analysis here presented. Quite the opposite. Justificatory basis for axioms are given in performative reasoning, following the definitions offered by the formalism. Frege's mistake is not in judging Basic Law V itself, but on the definitions provided for the smooth-breathing. The definition provided allowed him to judge precisely what he thought was true: the transition from the equality of value ranges to the generality of an equality (Basic Law Vb). That Russell's paradox follows speaks about his definition of value-ranges and the rules for using them in the expository language, not the standards for judgements in the auxiliary language.

4.2.3 Judging from Rules of Inference

What allows us to accept a given rule of inference is truth-preservation: that such rule will not lead us from truths to falsities. This principle is certainly present in Frege's justification for *modus ponens* and other derived rules. But they are not dependent on a truth-predicate. In the *Begriffsschrift*, Frege speaks of being affirmed or denied. In the *Grundgesetze*, he based his justification for *modus ponens* rather on preservation of denotation. In both cases, a performative argument is provided.

4.2.3.1 Rules in *Begriffsschrift*

The only rule of inference mentioned by Frege in the *Begriffsschrift* is *modus ponens*. It follows from the definition of the conditional stroke, which is provided in terms of affirmation and denial:

> If A and B stand for contents that can become judgments (§2), there are the following four possibilities:
>
> 1. A is affirmed and B is affirmed;
> 2. A is affirmed and B is denied;
> 3. A is denied and B is affirmed;
> 4. A is denied and B is denied
>
> Now
>
> $$\begin{array}{l}\vdash\!\!\!\top A\\ \quad\llcorner B\end{array}$$
>
> stands for the judgment that *the third of these possibilities does not take place, but one of the three others does.* (*BS*, §5)

Frege describes the rule *modus ponens*, and its justification, as follows:

> The definition given in §5 makes it apparent that from the two judgments
>
> $$\begin{array}{l}\vdash\!\!\!\top A\\ \quad\llcorner B\end{array} \quad \text{and} \quad \vdash B$$
>
> the new judgement
>
> $$\vdash A$$
>
> follows. Of the four cases enumerated above, the third is excluded by
>
> $$\begin{array}{l}\vdash\!\!\!\top A\\ \quad\llcorner B\end{array}$$
>
> and the second and fourth by
>
> $$\vdash B,$$

> so that only the first remains. (*BS*, §6).

Frege speaks of possibilities and judgements taking place as events. Affirmation and denial, as acts, are also events. The underlying principle is that no two contradictory events can take place at once: one cannot affirm and deny something simultaneously. This helps on seeing why *modus ponens* must be admited as a rule: if A is denied, then necessarily either B is denied or $\begin{array}{l}\top A\\ \llcorner B\end{array}$ is denied. Both cannot be denied on the condition of A being affirmed. Thus, if both are judged, we are entitled to judge A, that is, $\vdash A$. Otherwise, we would both affirm and deny B or $\begin{array}{l}\top A.\\ \llcorner B\end{array}$

It is incoherent for those who accept the rules of logic to both affirm and deny the same content. This can also be explained, following Searle and Vanderveken (1985, p.25-26), as an illocutionary inconsistency. An illocutionary inconsistency yields whenever a speaker tries to perform an illocutionary act and its denegation. If one accepts the rules of logic, such as excluded middle, then one accepts that no proposition can be both true and false. In this sense, simultaneously affirming and denying p would contradict this acceptance. While accepting it, either one asserts p or denies p, but not both. The rules of inference, thus, are permissions for inference: if both premises are judged, the conclusion can be judged, for it is not possible to affirm the conclusion incoherently.

4.2.3.2 Rules in *Grundgesetze*

As mentioned, the mature Frege adopted truth-values as objects of denotation for names and sentences, which alter the way rules were justified. It is still, I hold, a performative argument just as those in the *Begriffsschrift*. It is worth quoting again the *modus ponens* rule in *Grundgesetze*:

> From the propositions '$\begin{array}{l}\vdash\!\top \Gamma\\ \quad\llcorner \Delta\end{array}$' and '$\vdash \Delta$' one can infer: '$\vdash \Gamma$'; for if Γ were not the True, then, since Δ is the True, $\begin{array}{l}\top \Gamma\\ \llcorner \Delta\end{array}$ would be the False. (*GGA*, §14)

The idea behind the rule is that judging Δ (premise 1) means that Δ is the True, and judging $\begin{array}{l}\top \Gamma\\ \llcorner \Delta\end{array}$ (premise 2) means that the implication also is the True. These two facts combined yields that Γ cannot be other than the True. If so, both judged premises are enough reasons for judging Γ, *viz.,* '$\vdash \Gamma$'. But this depends on taking "is the True" as proxy for the assertion of something as true. If we assert that '$\vdash\!\!-\,\Delta$', we can then claim in the expository language that Δ is the True. It means that from judgements one derives elucidatory information: if for example $\vdash\!\!\!\begin{array}{l} \top\, a \\ \llcorner\, a \end{array}$, then we know that the function $\begin{array}{l} \top\, \Delta \\ \llcorner\, \Delta \end{array}$ names the True whatever Δ is. The same goes for judging that '$\vdash\!\!\!\begin{array}{l} \top\, \Gamma \\ \llcorner\, \Delta \end{array}$': it can be expressed using the proxy "is the True". Given these conditions, we obtain first the condition that Γ cannot name other than the True. This is enough reasons for judging it, otherwise, it would be incoherent to judge its negation, given that we would made contradictory judgements at once. And precisely for this reason, one can make a transition by using the horizontal transition-sign for *modus ponens*, therefore uttering a permissive. Given that denying the conclusion is incoherent assertion-wise, one is allowed to assert it as True.

Frege uses quotation marks to mention the asserted propositions that occurs in the auxiliary language, and we should never loose sight that *in* the auxiliary language, judgements are acts of acknowledgements, expressed through the judgement-stroke with assertive force. Hence, the fact that both premises shows that necessarily Γ denotes the True also shows that an agent can objectively assert Γ with assertive force.

There are other rules in the *Grundgesetze* that governs the use of the different primitives of the language, and as *modus ponens* above, they all occur in the context of judgements: if the premises were judged, another judgement that satisfy the conclusion may be derived. A few more examples are sufficient in strengthening the point.

The *fusion of horizontals* rule, for instance, reads: "If as argument of the function $-\,\xi$ there occurs the value of this same function for some argument, then the horizontals may be fused" (*GGA*, §48). Recall that the negation, conditional and concavity function all are enclosed by horizontals themselves. The rule, thus, is applicable to any duplication of the horizontal occurring between these functions. For instance, from the judgement

$$\text{‘}\vdash\!\!-\,(-\,\Delta)\text{’}$$

one may infer the simplified judgement

$$\text{‘}\vdash\!\!-\,\Delta\text{’}$$

given that if $-\,\Delta$ is the True, Δ is the true by the definition of the horizontal. The same holds for other functions. If

$$\text{‘}\vdash\!\!\!\top\,(-\,\Delta)\text{’}$$

then $\top(-\Delta)$ is the True, meaning that $-\Delta$ is the False. In this case, Δ is the false and so $\top\Delta$ is the True. Thus, one may infer

$$\text{‘}\vdash\!\top\,\Delta\text{’}$$

The same goes for the conditional stroke. If

$$\text{‘}\vdash\!\!\begin{array}{l} \top(-\Gamma)\text{ ’} \\ \llcorner\,(-\Delta) \end{array}$$

then either $-\Delta$ is the False or $-\Gamma$ is the True. This means that either Δ is the False or Γ is the True. From it, we can judge

$$\text{‘}\vdash\!\!\begin{array}{l} \top\,\Gamma\text{ ’} \\ \llcorner\,\Delta \end{array}$$

All such cases have the same reasoning: if the denotation of the subcomponents remains the same, we are free to judge the same formula minus the exceeding horizontals. In fact, we are licensed to judge it because there is no other possibility other that judging it as True, following that from the assertion of the premises, we obtain the information that the corresponding functions denotes the true.

Another example is the rule for introducing the concavity for the German generality letters. The rule is fairly simple: "A roman letter may be replace wherever it occurs in a proposition by one and the same German letter" (*GGA*, §17), provided, of course that such a German letter is placed above a concavity marking the scope of the substitution. The idea is that if we have a judgement such as

$$\vdash\!\!\begin{array}{l} \top\,a \\ \llcorner\,a \end{array}$$

we may derive another judgement with German letters substituted uniformly:

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l} \top\,\mathfrak{a} \\ \llcorner\,\mathfrak{a} \end{array}$$

We know that the former names the true since the function $\begin{array}{l} \top\,\xi \\ \llcorner\,\xi \end{array}$ names the True for any value of ξ. Now, this is the same as the function $\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\ \mathfrak{a}\\ \llcorner\ \mathfrak{a}\end{array}$, which returns the True for every value of $\mathfrak{a}$. Thus, we are entitled to judge it accordingly.

Other rules are likewise justified. As already mentioned, Frege is providing a sort of semantical background in order to safeguard a syntactical step. But since he operates with assertions, that is, the illocutionary device of the judgement-stroke, which expresses as he calls the "acknowledgement of True", the argument can be called performative: according to some semantical facts (mostly denotative), it would be nonsense to judge that certain expressions are other than the True. Given that we cannot judge it differently safe from being incoherent in our judgements, the assertion is warranted. If we don't judge it, the information provided by the truth-conditions in terms of denotation never takes the necessary "step from senses to references", as Frege would say in justifying the need for the judgement-stroke. The conditions for Basic Laws forces one to judge it. In the case of the rules of inference, the judgement of the premises derives elucidatory information that forces one to judge the conclusion, thus obtaining an inferential permission.

Under this kind of elucidatory justification, we can see that illocutionary objectivity is obtained if judgements and inferences are coherently made. The problem of psychologism is that an agent could make false judgements or invalid inferential steps. The justificatory basis in terms of coherence prevents this from happening, given that judging is only allowed when truth-preservation is in place. But since we are not simply making truth predications, coherence is the proper way to explain this step in terms of judgements: an agent making judgements that are not truth-preserving is, at least, incoherent in doing so. Therefore, the normative status of logical laws (for example, the *tertium non datur* that Frege often quotes) prescribes how one ought to judge coherently, and this is enough for ruling out most of the relevant psychological problems that stems from taking judgements as acts of recognition.

4.3 Definitional Objectivity

A final topic of his chapter pertains to definitions. Definitions are for Frege stipulations. They involve some active part of the agent, as they are illocutionary devices as well. As such, they are also prone to the psychologistic problem: how can we avoid creative definitions?

Frege's solution is simply to restrict definitions to a minimum. First, the domain in which the sign '$\Vdash$' has its declarative powers is, of course, the own concept-script language. More precisely, definitions are only performed *in* and *about* the auxiliary language. The only thing properly "created" is the sense and reference of a sign that previously had none: "[...] a property which a thing just does not have cannot be magically attached to it by mere definition, except for the property of now being called by the name that one has given to it" (*GGA*, p.xiii). Thus, Frege's notion of definitions is limited on its creative powers:

> Definitions themselves are not creative, and in my view must not be; they merely introduce abbreviative notations (names), which could be dispensed with were it not for the insurmountable external difficulties that the resulting prolixity would cause (*GGA*, p.iv)

Also in the *Grundgesetze* we find Frege's famous metaphor:

> Often one seems to credit [definitions] with a creative power, although in truth nothing takes place except to make something prominent by demarcation and designate it with a name. Just as the geographer does not create a sea when he draws borderlines and says: the part of the water surface bordered by these lines I will call Yellow Sea, so too the mathematician cannot properly create anything by his definitions. (*GGA*, p.xiii)

What we read here is Frege's realist position blocking any definition from falsely taking something to be true in the world. Since definitions only operate in the language, nothing in the world could be changed by simply defining it as possessing this or that property. In fact, Frege's dispute with Hilbert on the proper role of axioms within a system relies on this point, as he accused Hilbert of confusing axioms with definitions. As he argues in the 1906 paper *Über die Grundlagen der Geometrie*, Hilbert's axioms fail in grasping the difference between the two kinds of acts here discussed: the assertive character of axioms and the stipulative character of definitions[43].

Since definitions are not creative, they should not be responsible for any epistemic gain:

> For to begin with, they are arbitrary stipulations and thus differ from all assertoric propositions. [...] By defining, no knowledge is engendered; [...] No definition extends our knowledge. It is only a means for collecting a manifold content into a brief word or sign, thereby making it easier for us to handle. This and this alone is the use of definitions in mathematics. (*CP*,p.274)

[43]In Frege's own, not so kind choice of, words: "Mr. Hilbert chops both definitions and axioms very fine, carefully blends them, and makes a sausage out of this. " (*CP*,p.297)

A definition, thus, do not simply create or predicate something of an object by stipulation. That an object has such property, or even that a defined concept is instantiated or not, "[...] first requires an investigation" (*GGA*, p.xiv). In the context of the formal language, however, defined signs always have a sense and reference that are provided by the sense and reference of the signs used in the *definiens*, and by those principles that govern definitions in Frege's system. Every discovery is always given with assertions. In a way, and as far as logic is concerned, Frege's world is fixed. All we have is assertions in order to discover it.

But in order to restrict definitions to the language, a few provisions are necessary. And Frege offers careful principles for definitions in the *Grundgesetze*. The idea is not only to block cases in which definitions do more than the intended, but also to keep the functionality and expressability of the concept-script intact.

In (*GGA*, §33), he stablishes seven principles that govern the institution and use of definitions within the system:

1. The first principle is the referentiality: "Every name correctly formed from the defined names must have a reference" (*ibidem*). If a definition is provided in such a way that no reference is available, at some point, all inferences following it would be empty. In order to achieve this goal of referentiality, it suffices that the *definiens* is composed of primitive signs that all refer, a goal that Frege tried to argued in (*GGA*, §32).

2. The second is the unicity principle: "the same must never be defined twice, since it would remain in doubt whether these definitions were in harmony with one another" (*ibidem*).

3. The third principle is the simplicity: "A defined name must be simple; i.e., it must not be composed of names known or still to be explained" (*ibidem.*).

4. The fourth principle is that of originality. As Frege explains, if the *definiens* is composed solely from the primitives, "then this will always have a reference and we can put on the right a simple, hitherto unused sign which is now introduced by our definition as a co-referential proper name" (*ibidem*).

5. The fifth principle is that of arity equality for unary functions. "A name that is introduced for a first-level function with one argument may only contain a single argument place" (*ibidem*), and likewise for names of functions for other arities. This

also requires that each side of the definition the same roman object-letter is used, and that the values of the first for a given argument of object name or marker is the same as the argument of the second for the same arguments.

6. The sixth principle is the arity equality for binary functions. “A name that is introduced for a first-level function with two arguments must contain two and no more than two argument places” (*ibidem*). And the same care is given as for unary functions.

7. Finally, the seventh principle we may call as the roman-letter mirroring. “A Roman letter must accordingly never occur on one side of such a definitional equation which does not also occur on the other.” (*ibidem*).

If all such conditions are met, Frege is confident that every definition in the concept-script has a reference and is suitable for being used in inferences. The conditions above also guarantees that they are only auxiliary devices and nothing is added to them that is not already fixed in the *definiens* in the first place. They are ultimately not needed for the correctness of the inferences, but are needed for the conduction of them. They are designed for easing the human eye into complex derivations, but are objectively conducted in such a way that no psychological worries derives from them.

4.4 Conclusion of Part II

Taking judgements in the foreground of logic has its risks. The fact that judgements are acts of acknowledgements tend to be seen as a psychological step, and all the relativistic risks come about. As I’ve tried to show, this is not a problem for Frege. The concept-script is as objective as any system of logic can be.

First, we saw that, for what concerns contents, Frege endorsed contentual objectivity through and through. In the later part of his logic and philosophy, this was a strong position: the senses of sentences, the Thoughts, are objective in a strong realistic sense. And the truths they express are objective to the same extend. But, of course, the way we can actually judge Thoughts to be True is still dependent on certain acts. These acts, as I argue, are objectively enough for the scientific standards.

For both the Basic Laws and the Rules of Inferences, as we saw, the senses make themselves evident in such a way that to judge them follows from a simple act of analysis. As Frege puts it, “Every such name of a truth-value *expresses* a sense, a *thought*.

For owing to our stipulations, it is determined under which conditions it refers to the True. The sense of this name, the thought, is: that these conditions are fulfilled." (*GGA*, §32).

Agreeing with Greimann (2008, p.406), "If the truth of a logical law is evident from the sense of its expression, then in order to justify it, we must make the sense of the sentence by which it is expressed explicit". But I would add that the assertability of a logical law is evident from the sense of its expression, and in order to justify it, we make it explicit the conditions under which the expression is a possible name for the True. Once this is done, in a self-evident manner or not, the Basic Law, or the inferential step of a rule, is correctly and objectively judged as True. Thus, truth-conditions do not only say that the Basic Law cannot be the False, but it says that the Basic law is *judgeable* as the True.

A similar problem can be stated about the inferential rules. They are illocutions just as judgements and assertions are. Rules are described as permissions for an agent to perform an action under certain conditions. As such, they are still subject to psychology. But the harm is only apparent. Given that inferences are transitions from judgements to judgements, any psychologistic problem is solved if, in principle, judgements are objectively made already. Even more, the justification for each inferential rule follows the same reasoning of the justification for the basic laws be judged as True, as showed.

If Frege had means to adopt a comprehensive metalanguage in his system, he could have very much abandoned judgements, and justify every axiom and rule of inference in the metalanguage. *A fortiori*, he would have justified every theorem of the system. I believe that his universalist conception explains why this wasn't the case. And for this, judgements are a central feature of Frege's rejection. An illocutionary device such as assertions, as means to express judgements, is the classical solution to keep track, in the context of an inference, of the propositions recognized as True by virtue of proof.

Definitions, on the other hand, offered a more simplified case. Given that Frege saw them as purely pragmatical, operating only in the level of the expressions of the language, they are not responsible for "creating" thoughts. They have no significant contribution to the correctness of the inferences, and thus their objectivity is completely obtained from the rules established. If followed strictly, only by making elementary mistakes on the manipulation of the language one would have creative definitions. In conclusion, Frege's signs for judgements, inferences and definitions all include illocutionary force indicating devices. Although depending on the agent's performance, psychologism is not a risk.

Part III

Judgements: Then and Now

Chapter 5

Historical Perspectives

If my reading of Frege's logic is so far acceptable, it is inevitable to consider his position in the context of the history of logic. This is the main purpose of the present chapter. Contrary to Michael Dummett's belief that the concept-script logic was "born from Frege's brain unfertilized by external influences," (DUMMETT, 1973, p.xvii), one has to accept the fact that even Frege, isolated as he was in Jena, still had a rich philosophical background that we have to account for. One of such influences we already uncover in discussing Hermann Lotze. But, the prominent role played by judgements in his logic must be placed within both the context of the Aristotelian tradition and more recently from the nineteenth-century German philosophy, to which the act-content distinction and anti-psychologism were already a common place for philosophers. This is *history up to Frege*.

The chapter will also takes on Frege's influence nowadays. His influence is undeniable, but should not be overstated. The performative reading of the previous chapters shall be taken as evidence that, for better or worse, contemporary logic is non-Fregean. The main reason for it, as I believe, is the influence of the metatheoretical perspective. Given the growing study of logical systems as mathematical objects, pragmatic features were quickly eliminated, or, as I shall conclude, ignored. For such assessment, we must review *history from Frege*.

Moreover, the standard interpretation of Frege's logic here admitted (the Universalist reading) states that his conception is devoided of a metatheory. Also, Frege followed the tradition in employing illocutions, specifically judgements, to deal with the absence of a truth-predicate. Both points made Frege's logic more philosophical than mathematical, and both would not survive the posterity. The birth of the metatheory saw the downfall of such universalist conception, and the abandonment of the use of judgements and assertions. Ironically, Frege's criticism against psychologism in logic was pivotal in taking this step, further away from his own conceptions. Thus, fregean logic can be seen as judgement's last stand against a new conception of logic, the methateoretical one. My conclusion is that Frege's actual influence in the history of Logic was only local, but not general: contemporary logic is not, and never was, completely Fregean.

To assess logic before and after Frege, I'll argue that a process of de-pragmatization took place after Frege, that saw a substantial change from a logic-as-practice to a logic-as-object perspective. The main tenets of the de-pragmatization is described in section 5.1. Then, I'll reassess the logic-as-practice in section 5.2, following the tradition started with Aristotle. In section 5.3, the shift to a logic-as-object is sketched, following Hilbert and Tarski's contribution.

5.1 Logical de-Pragmatization

As it is common today, unaware philosophy students usually start an introduction to logic course expecting to learn the tool that will guide philosophical discussions in the future. But as the course goes on, the expectation changes, as soundness, completeness and other complex meta-results are presented in details. The student may expect to learn to use a tool, a *know-how* process, but what he gets is mostly information about the tool's capabilities, a *know-what*.

In this process, students usually have a hard time understanding the difference between, for example, the propositional or first-order logic that he's learning about, from the logic hidden in the metatheory that he is using to reason about them. The difficulty is on grasping the difference between what we will consider logic as a tool for reasoning, from logic as an object of mathematical studies. In these days, logic is mostly about the latter than it is about the former.

This choice of method has its reasons and origins. The problem that it seeks to solve is present, either explicitly or implicitly, in every modern logical textbook. As simply putted by Kleene in the sixties, "how can we treat logic mathematically (or in any systematic way) without using logic in the treatment?" (KLEENE, 1967, p.3). The question was not new, as it can be traced back to Frege, Russell, Peano, and even further to Aristotle.

When formalizations started to rise in late nineteenth century, the pressing issue was that not everything could be formalized. This problem was clear enough for Frege concerning many primitive logical notions that could not be further analysed. As such, Frege treated them as indefinables. Examples are the notion of *concept* (*BuG*, p.182-183), *object* (*FuB*,p.147), *judgements* (*DV*, p.381), *truth* (*DG*,p.353) and also the saturated and unsaturated parts of a proposition (FREGE, 1984, p.281). In the article *Über die Grundlagen der Geometrie*, he states that "Since definitions are not possible for primitive elements, something else must enter in. I call it explication." (*CP*, p.300). The point taken is that primitives are too simple to be analysed further without some circularity. It will be difficult to offer a definition of object or concept without having some object or concept involved. The same goes for Truth, given that

> in a definition certain characteristics would have to be specified. And in application to any particular case the question would always arise whether it were true that the characteristics were present. (*DG*,p.353)

Frege thought that we have no choice other than take them as primitive and indefinable notions. The need for an explication [*erläuterung*] comes from pragmatic reasons: "mutual cooperation in a science is impossible without mutual understanding of the investigators, [thus] we must have confidence that such an understanding can be reached through explication" (FREGE, 1984, p.301). And the same need holds for learning logic as a scientific tool: we would simply use ordinary language as means for understanding logic as an object, exactly how first-year students expect to happen.

What motivates this notion of explication is a foundational problem: that of reducing informal logical notions in formal languages. For sure, Frege's answer relies on accepting that the chain of definitions has a breakpoint, rather than finding a way around the issue. This answer is Aristotelian in at least two points. First, it resembles Aristotle solution for the regress problem of the deductive sciences in the *Posterior Analytics* (ARISTOTLE, 1984c, I.2): we simply have to take a starting point for the reasoning process. In Aristotle's case, those were named Axioms. But the matter here is more basic, as it pertains the primitives of the language, not starting points for deductions. Nevertheless, there is a second point of comparison with Aristotle: by rejecting that we can have knowledge of the first principles deductively, Aristotle hinted, also in the *Posterior Analytics* (ARISTOTLE, 1984c, II.19), that knowledge of the first principles must be achieved by comprehension [*nous*] instead[1].

Be that as it may, other important figures in the late nineteenth-century

[1]A similar point is made in the *Nichomachean Ethics*: "[...] comprehension is concerned with the ultimates in both directions; for both the primary definitions and the ultimates are objects of comprehension and not of argument" (ARISTOTLE, 1984b, Book VI,11). I'm not implying that Aristotle's first principles and Frege's primitive notions are the same, though.

drawed similar conclusions. Peano stated that

> with opportune definitions, one may reduce the ideas of logic to a smaller and smaller number of fundamental ideas, or primitive ideas, which must be expressed in ordinary language, or clarified with examples, but which cannot be expressed symbolically using others which are simpler. [...] the added clarifications and observations, in ordinary language, will facilitate its understanding. (PEANO, 1973, p.192)

Peano speaks about expressing and clarifying such ideas in ordinary language, which is on a par with Frege's Explication. The goal is the same: to facilitate understanding and to grasp the primitive notions that could not be otherwise defined. Russell and Whitehead, claiming to have followed Peano, stated in the *Principia Mathematica* that

> The primitive ideas are explained by means of descriptions intended to point out to the reader what is meant; but the explanations do not constitute definitions, because they really involve the ideas they explain. (RUSSELL; WHITEHEAD, 1910, p.95)

And much as Frege's explications or Peano's clarifications, here Russell and Whitehead speaks about ostensibly pointing out the primitive ideas in order to hint its original meaning. As Frege would say it, "there is nothing for it but to lead the reader or hearer, by means of hints, to understand the word as is intended." (*BuG*, p.182-183)

The interesting point here is that Frege, Peano and Russell's answer are very different from the answer one finds in logical textbooks today: the distinction between object-language and metalanguage. At the time when Kleene asked the question that we quoted above, in 1967, the distinction was already standard in logic textbooks[2]. But its development can be traced back to two pivotal moments: Hilbert's metamathematics, and Tarski's semantic conception of truth[3]. In tracing the distinction back to them, I will focus on one particular consequence that such innovation brought to logical formalisms, one that we may call the *de-pragmatization* of logic.

By a de-pragmatization, I refer to the process of formalization of notions that were considered primitive, or that were at least tacitly used without much precision. Given the assumed impossibility of treating them as proper *objects* of mathematical or logical study, they were mostly notions to be *used*. One way to define it is following the semiotic distinction between syntax, semantics and pragmatics. Carnap, who not only de-pragmatized but also de-semanticized logic, defined in the following way:

[2]So much so to the point of Kleene suggesting to anyone who does not grasp it to "[...] close the book now, and pick some other subject instead, such as acrostics or beekeeping." (KLEENE, 1967, p.4)

[3]These are, at least, the moments I'm going to focus. I'm not claiming this to be the full story.

> an investigation which refers explicitly to the speaker of the language - no matter whether other factors are drawn in or not - falls in the region of *pragmatics*. If the investigation ignores the speaker, but concentrates on the expressions of the language and their designata, then the investigation belongs to the province of *semantics*. Finally, an investigation which makes no reference either to the speaker or to the designata of the expressions, but attends strictly to the expressions and their forms [...], is said to be a formal or syntactical investigation and is counted as belonging to the province of (logical) *syntax*. (CARNAP, 1958, p.79)

Carnap's definition is general, but is suitable. The de-pragmatization is simply the disregard of the speaker or user of the language. If a logical notion is treated as indefinable, it cannot be treated as an object and therefore must be considered implicitly within the context of the user, as far as it is necessary. To make matters a bit more precise, there is two ways for a given primitive notion to be considered pragmatically. Either (1) the notion is undefined and simply considered *as it is used* or, and even strongly, (2) it is essentially a practical notion. The notions of concept, function, set or truth all fall into the first category. Judgements and inferences seem to fall into the second, as they are kinds of actions. More precisely, they are kinds of utterances.

A language is pragmatical in (1) if it makes indirect reference to the user, as means to capture the meaning of an informal, primitive, notion. A language is pragmatical in (2) if it makes direct reference to the user, as means to perform some required action. On the other hand, a language can be de-pragmatized in a strong sense, if it can avoid (1), and in a weak sense, if can avoid (2). The reason being that it's questionable if one can completely de-pragmatize a given language by formalizing every notion employed. Nonetheless, a weak de-pragmatization is not only possible, but as we shall see, is the dominant trend in logic nowadays.

This distinction between a de-pragmatized from a pragmatized language is related to many other distinctions in the literature. In a sense, it all started with Van Heijenoort's (1967b) distinction between a logic as calculus and a logic as language, thought to model the Leibnizian distinctions, well discussed by Frege himself, between a *lingua characteristica* and a *calculus ratiocinator*. Hintikka (1997) offered similar distinction between a language as the universal medium, or universal language, and a language as calculus, the latter being representative of the model-theoretic perspective. Hintikka's extension was also followed by Kusch (1989). Sundholm (2001) distinguished a logical atavism, meaning the contentful logic of our ancestors, from the present-day uninterpreted metalogical investigations. Finally, Dutilh Novaes (2012a, 2.1.4.) separates two ways of treating formal languages: as objects of mathematical studies, labeled language-as-objects, and as a practice embedded in human actions and contexts, labeled language-as-practice. The distinctions above are closely related with the de-pragmatized vs. pragmatized separation here taken. Nonetheless, following Dutilh Novaes, a pragmatical language of logic in any of the senses above can be named as *logic-as-practice*, whereas a de-pragmatized language of logic in the weaker sense will be called *logic-as-object*.

In today standards, some notions were successfully de-pragmatized, while others are at best debatable, if not still primitives. But in the beginning of formalization programs of the late nineteenth century and beginning of the twentieth, a lot of work was still to be done. Two major moments in the history of logic are key in turning the logic-as-object the standard view: the formalization of Proof with David Hilbert and Truth with Alfred Tarski. Their work successfully dethroned two key notions of the logical tradition: Judgements and Inferences. With them, the whole logic-as-practice perspective became obsolete. This shift is a perfect assessment of the history of logic before and after Frege's conception. In order to better understand it, we shall look at the tradition first.

5.2 The tradition: Logic-as-Practice

The logic-as-practice perspective can be considered the dominant view from Aristotle to Frege. Its influence still resonates today in informal definitions of logic. Oxford English Dictionary, for example, defines it as "the science of thinking about or explaining the reason for something using formal methods". Thinking and reasoning are human activities, and logic was then defined in terms of other human or intellectual acts. Traditionally, the structure of such logical activities includes the apprehension of concepts, the judging of propositions, and the transition from judgements to judgements, *viz.,* inferences. Following Sundholm (2008), each activity was expressed in the tripartite relation

$$\frac{\text{act}}{\quad} \quad \text{object} \leftrightarrow \text{sign}$$

Generally, this means that concepts were thought as the objects of acts of apprehension, judgements were the objects of acts of judging, and so on. Of course, different authors may relate differently to such tripartite relation, but the underlying structure was dominating from Aristotle to Frege. In Frege's case, much emphasis was made to judgements: not only judgements have priority over concept formation or apprehension, but inferences are defined as transitions between them[4]. In the paper *Ûber den Zweck der Begriffsschrift*[5], Frege says that

> [...] it is one of the most important differences between my mode of interpretation and the Boolean mode —- and indeed I can add the Aristotelian mode — that I do not proceed from concepts, but from judgements. (FREGE, 1972, p.94)

Thus, given that in the tradition of logical practice, and in Frege's concept-script, most of the logician activities boils down into making judgements, the focus will be exclusively on the evolution of such notion.

5.2.1 From Aristotle to Port-Royal

It is without of question that judgements, up to Frege, were structured following the Aristotle's works. But the history of the term has received different meanings. We begin, nonetheless, with Aristotle.

The term *judgement* does not occur in Aristotle's works on logic, the *Órganon*. Aristotelian logic is about syllogisms [$\sigma\upsilon\lambda\lambda o\gamma\iota\sigma\mu\acute{o}\varsigma$], that is, inferences occurring between two premises and a conclusion. In *Prior Analytics*, these are what Aristotle called *prótasis* [$\pi\rho\acute{o}\tau\alpha\sigma\iota\varsigma$], which translates to *proposition*[6], but not in modern sense. In the *Prior Analytics*, he defines:

> A proposition [$\pi\rho\acute{o}\tau\alpha\sigma\iota\varsigma$], then, is a statement [$\lambda o\gamma o\varsigma$] affirming or denying something of something; and this is either universal or particular or indefinite. (ARISTOTLE, 1984d, p.103)

Similarly, in the *De Interpreatione*, the affirming and denying is thus described as:

> An affirmation is a statement [$\alpha\pi\acute{o}\phi\alpha\upsilon\sigma\iota\varsigma$] affirming something of something, a negation is a statement denying something of something. (ARISTOTLE, 1984a, p.76)

Which only reiterates the previous definition. Notice that in *De Interpretatione*, "statement" translates *apophansis* and not *lógos* as in the *Prior Analytics*. *Apophansis* is very

[4]Generally, as explained by Sundholm (2008), the product of an inference, thought as an act, is the judgement to which the inference is aimed at. As he explains, "An inference act is a *mediate* act of judgment, in which one judgment, the *conclusion*, is known on the basis of certain other judgments, the *premises*, being known. Thus, an act of inference is a particular kind of judging, whence its (mental) product is a judgment made" (SUNDHOLM, 2008, p.267).

[5]Translated as *The aim of 'Concept-Script'* by Terrel W. Bynum in (FREGE, 1972).

[6]This is Barne's choice in (ARISTOTLE, 1984a). Łukasiewicz translate as "premiss', in (ŁUKASIEWICZ, 1957), although still mentioning that they are all propositions just like the conclusion. There is a dispute over this point, which I avoid, opting with Barne's translation. On this, see (CHARLES; CRIVELLI, 2011).

close in usage to the *prótasis*. In *De Interpretatione*, *lógos* is translated as *sentence*. Aristotle defines it in the following way:

> A sentence [$\lambda o\gamma o\varsigma$] is a significant spoken sound some part of which is significant in separation — as an expression, not as an affirmation. (ARISTOTLE, 1984a, p.74)

Lógos, *próstasis* and *apophansis* are all difficult terms to translate. But they all seem to include an assertive component. They cannot be read syntactically or semantically, as we understand syntax and semantics today. Aristotle's propositions are not what the modern term suggest. A proposition is, in today's meaning, much more closer to the 'thing' being affirmed or denied than its utterance.

But there's a difference, considered by Aristotle, between *lógos* and *apophansis*. Propositions or statements (in the *apophansis-próstatis* sense) are those sentences of speech (*lógos*) that can be uttered as true or false:

> Every sentence is significant (not as a tool but, as we said, by convention), but not every sentence is a statement-making sentence, but only those in which there is truth or falsity. There is not truth or falsity in all sentences: a prayer is a sentence but is neither true or false. The present investigation deals with the statement making sentence; the others we can dismiss, since consideration of them belongs rather to the study of rhetoric or poetry. (ARISTOTLE, 1984a, p.75)

This is a quite a modern passage. By "statement-making", Aristotle is considering those sentences (*lógos*) that are capable of being affirmed or denied as true or false. Prayers are still utterances, although not capable of being true or false.

Nonetheless, this begs the question about what are those things that are capable of being true or false in a proposition or statement, *viz.* the things being affirmed or denied. Aristotle does have an analysis of them: these are relations between terms. Back in the *Prior Analytics* we have the following definition:

> I call a term that into which the proposition is resolved, i.e. both the predicate and that of which it is predicated, 'is' or 'is not' being added. (ARISTOTLE, 1984d, p.104)

These are the subject and predicate terms, *viz.* the things that are being related. Here Aristotle also states that such affirmations and negations are made in terms of the verbs 'is' and 'is not', and, as the definition of proposition showed, includes quantification over the subject term. Thus, propositions have the following structure:

$$\text{Proposition} = \begin{cases} \text{quantifier (all, some)} \\ + \text{ subject term (S)} \\ + \text{ copula (is, is not)} \\ + \text{ predicate term (P)} \end{cases}$$

This structure represents every proposition of the Aristotelian syllogistic that is well known and studied in logical courses even today. These propositions are not simply sentences in modern terms, nor are they simply the things denoted by sentences. As *lógos*, they are utterances, and as *prótasis*, or equivalently *apophansis*, those utterances that are capable of being affirmed or denied. Therefore, the Aristotelian propositions already included the components that latter will be called Judgements.

As we saw, the term “Judgement” does not occur in Aristotle’s works on logic. How we end up with the term is a long story. The English word “Judgement” is derived from the French “*jugement*” (HOAD, 1996), which in turn comes from the Latin “*judicium*”. The German word for it, “*Urteil*”, has roots in old english “*ordal*” (WALSHE, 1951), which also derived the English word “ordeal”.

The mystery, then, is how from *apophansis* we ended up with the Latin *judicium*[7]. In the sixth century translation of *De Interpretatione* to Latin, Boethius translated the term as “*enuntiatio*”, meaning “enunciation”. But apart from Boethius few translations, most of Aristotle’s work were lost to the west. They were survived in the arabic world, where arabic and syriac translations were made. Particularly, we mention the Abu Bishr (870-890) translation of the *Posterior Analytics* from the Syriac to Arabic. There, as Per Martin-Löf claims, *apophansis* was translated by the arabic word “*ḥukm*”, that is undisputedly translated to the now usual “judgement”. Another point of evidence is Al-Farabi’s (c.872 - 951) comments to Aristotle’s *De Interpretatione*, where *apophansis* is rendered as “*qaḍiyya*”, which also translates as “judgement”.

Finally, it was with Gerard of Cremona’s (c.1114 - 1187) translation of the *Posterior Analytics* from Arabic to Latin that this judicative meaning made to Europe. Gerard of Cremona was one of the main translators of the Toledo School, and allegedly used an annonymous arabic version of the text which also used “*ḥukm*” for “*apophansis*”. It was in Cremona’s translation that, according to Martin-Löf, “*judicium*” made it to the

[7]Here I will follow almost exclusively Per Martin-Löf’s 2011 lecture *How did ’judgement’ come to be a term of logic?* in (MARTIN-LÖF, 2011).

logical vocabulary in the twelfth century Europe.

The employment of the term "judgement" also introduced a new meaning for *apophansis*. A judgement is usually an act of decision that follows a period of careful thought and deliberation. A judge, in this sense, is anyone qualified for having such careful deliberation. This act of decision is mostly imported from legal contexts, as both words have its roots in the latin "*jus*", meaning "law", also found in words such as "justice". Nonetheless, this might be one of the reasons for medieval logic to have departed from a pure linguistic analysis to a more mentalist perspective. As Aho and Yrjönsuuri (2009, p.17) argue,

> In the Middle Ages, the art of logic was not taken to be concerned with abstract structures in the way modern logic and modern mathematics are, but with actual linguistic practices of reasoning. It was generally accepted that logic is, at least in some sense, a practical science giving advice on how to understand and make assertive statements and how to argue and reason in an inferential manner.

This trend shifted in the thirteenth century after the Arabic influence, as they not only saw logic as a "science of words", but also as a "science of reasoning". As Aho and Yrjönsuuri (2009, p.18) continue:

> According to al-Farabi, the *logos* (in Arabic, *al-nutq*) discussed in logic occurs on two levels, one inscribed in the mind, and the other existing externally in spoken sounds. [...] Avicenna was also influenced by al-Farabi's discussion, and gave even further impetus to the idea that logic is concerned with intellectual structures other than with what we do in spoken discourse. Thus, logic should be called "a science of reason" (*scientia rationis*), as the Latin world translated the idea.

The idea that logic is as much about linguistic structures as it is about mental structures was fully adopted in the latin world, being influential to the Scholasticism movement that followed, and even after in the Renaissance and Modern philosophy. Thomas Aquinas (1224?–1274) followed this idea and systematized logic as consisting in three different mental activities: formation of concepts, judgements and inferences. This division, that has its roots in neo-platonistic commentators of Aristotle, was the basis for every logic textbook until Frege and the development of formal logic took over in the nineteenth century. In fact, in Aquinas, "making a judgment is not primarily to be understood as a speech act but as a mental act", given that "externally expressed linguistic structures should be seen as results and representations of intellectual acts, and only in this intermediate way does logic come to be concerned with linguistic structures", as Aho and Yrjönsuuri (2009, p.19) explains.

In conclusion, the term “judgement” seem to have been adopted at the same time medieval logic, from the influence of Arabic logicians, shifted from a purely linguistic to a rational perspective, where the focus on mental activities became standard. This mental activity is present in the change from “*apophansis*“ as “*enuntiatio*” (as in Boethius translation) to “*apophansis*“ as “*judicium*” (as in Gerard of Cremonas’s translation). Moreover, it also reflected in a distinction that was still in use by Frege much later: between the mental act of a judgement and its linguistic expression, an assertion.

In modern philosophy, judgements continued to be regarded as basic activities of the mind in which logic was founded. One of the key moments is the 1662 influential *La Logique ou l’art de penser*, or the *Port-Royal Logic* as it is commonly known. Written by Antoine Arnauld and Pierre Nicole (ARNAULD; NICOLE, 1996), the treatise covers a great deal of topics from Logic, Philosophy, Grammar and Language, and it is highly inspired in the Cartesian rationalism.

As expected, the thesis that logic is about certain operations is followed in Port-Royal Logic. They define logic as “[...] four principal operations of the mind: conceiving, judging, reasoning, and ordering”, where “Judging is the action in which the mind, bringing together different ideas, affirms of one that it is the other, or denies of one that it is the other” (ARNAULD; NICOLE, 1996, p.23). In the case of Port-Royal, Descartes influence was decisive in this recognition of logic to mental acts. This is clear in two points, according to Buroker (2017): first, they held the view that thought is prior to language. Second, the expository structure of the *L’art de Penser* followed cartesianism directly in the four orderly operations of the mind.

The first point is clear when the authors states that words are just “[...] distinct and articulated sounds that people have made into signs to indicate what takes place in the mind” (ARNAULD; NICOLE, 1996, p.74). Language is thus only a medium for mental operations. Three kinds of words are important for judgements: nouns, pronouns and verbs. All judgements are made by connecting these three elements linguistically. For example, in connecting the “Men”, “Mortal” with the copula “are” one obtains the form “Men are mortal”.

As it is expected, the verb “is/are”, as the copula, offers the necessary illocution for judging. The verb is

> “[...] nothing other than a word whose principal function is to signify an affirmation, that is, to indicate that the discourse where this word is employed is the discourse of a person who not only conceives things, but who judges and makes affirmations about them. (ARNAULD; NICOLE, 1996, p.79).

The verb also represents the mental act responsible in making a judgement. In modern terms, it marks the illocutionary part of the judgement, the affirmation or negation. As they confirm, "[...] the word "is" indicates the action of the mind that affirms, that is, that connects the two ideas [...] as belonging together" (ARNAULD; NICOLE, 1996, p.82). One of the problems of having the copula to mark the assertive force (as the mental act), is that we have no way to distinguish judging from thinking a content. As Buroker (2017) explains it, "every time one connects a subject and a predicate one is *ipso facto* judging. Thus there is no room for thinking propositions while suspending judgment".

For what concerns us, the Port-Royal Logic does not offer too much progress from Aristotle: it still builds from categorical propositions, adding all the mental influences from cartesianism. Nonetheless, in Port-Royal Logic, the systematization of logic in term of judgements is an important moment for the history of the notion.

5.2.2 Immanuel Kant, and After

The tradition from Aristotle to Port-Royal logic have already established the tripartite structure for the logical practice. First, one conceives ideas. Then, judges these ideas to finally draw inferences. This agenda became standard in every text-book on logic, containing a theory of concepts, a theory of judgements and a theory of inferences.

One of the philosophers to greatly influence how judgements and logic were constructed within this triad structure was Immanuel Kant[8]. But differently from Port-Royal, Kant's theory of judgment is deeply rooted in his transcendental metaphysics.

Judgements are still considered from the perspective of a subject and predicate, and in Kant's transcendental philosophy, these are all mental representations. In the *Jäsche Logic*, Kant defines:

> A judgment is the representation of the unity of the consciousness of various representations, or the representation of their relation insofar as they constitute a concept. (KANT, 1992, p.597)

Broadly speaking, a judgement is the synthesis of both subject and predicate representations into a single representation. Whereas a concept is the representation of some

[8]His *Jäsche Logic* text-book is constructed under this division. Moreover, Kant believe this to be an essential division for logic in general, based on the cognitive faculties of human reason:

> General logic is constructed on a plan that corresponds quite precisely with the division of the higher faculties of cognition. These are: **understanding, the power of judgment**, and **reason**. In its analytic that doctrine accordingly deals with **concepts, judgments**, and **inferences**, corresponding exactly to the functions and the order of those powers of mind, which are comprehended under the broad designation of under standing in general. (KANT, 1998, B170)

characteristic marks, *viz.,* other concepts, a judgement is the subsumption of an object under a concept, or a concept under another concept, if mediated by an object[9]. For example, the judgement "All Bodies are Heavy" is the cognitive act of combining both concepts of "Body" and "Heavy". The unity is marked by the copula "are".

Judgements are also considered the most important function of the faculty of understanding, as concepts have no other purpose other than form judgements: "[...] understanding can make no other use of these concepts than that of judging by means of them", as Kant (1998, B93) argues. He saw a prominent role for judgements in his theory of knowledge, given that it is through judgements that we acquire knowledge[10]. This faculty for judging is an active human condition (its dual is the passive faculty of the sensibility), where judgements are performed by the *a priori* conditions of the human mind, and explained by a proper action of synthesis.

The most basic condition in judging is a conscious subject. Kant calls it the *unity of apperception*[11]:

> I find that a judgment is nothing other than the way to bring given cognitions to the **objective** unity of apperception. That is the aim of the copula **is** in them. (KANT, 1998, B142)

This is what gives a judgement its validity, as it differs from a mere association of ideas in the mind:

> a *judgment*, *i.e.*, a relation that is *objectively valid*, and that is sufficiently distinguished from the relation of these same representations in which there would be only subjective validity, *e.g.*, in accordance with laws of association. (KANT, 1998, B142)

Validity here is not to be confused with logical validity. A judgement is objectively valid if it is applicable to some object, provided by an intuition. And it is within transcendental logic that its truth is verifiable. Following Kant's example, I may feel the heaviness of a body, but to *judge* that the body is heavy requires more than just a feeling: it requires an active and conscious unity of mind in order to combine both representations into one. Only

[9] This goes even with analytic judgements. See (MACFARLANE, 2000, p.122).

[10] The following passages reiterates this point:

> All judgments are accordingly functions of unity among our representations. (KANT, 1998, B94)
>
> We can, however, trace all actions of the understanding back to judgments, so that the understanding in general can be represented as a faculty for judging. (KANT, 1998, B94)

[11] "The transcendental unity of apperception is that unity through which all of the manifold given in an intuition is united in a concept of the object." (KANT, 1998, B140).

then truth can arise: "truth [...] are to be found only in judgments, *i.e.*, only in the relation of the object to our understanding" (KANT, 1998, B350). Moreover, judgements relates to truth as much as they can relate to possible experiences, *viz.*, intuitions. A judgement is therefore true "only insofar as it contains nothing more than what is necessary for the synthetic unity of experience in general" (KANT, 1998, B197), where by "synthetic unit of experience" we read a truthful representation of reality[12] in terms of objects and concepts.

There may be different ways to combine representations in a judgement, following their different logical forms. Kant uses them a starting point for discovering the categories, or pure concepts of the understanding. These judgements are further divided in four blocks, with twelve in total, according to the following table:

Quantity	**Quality**
Universal:	Affirmative:
All F's are G's	F's are G's
Particular:	Negative:
Some F's are G's	No F's are G's
Singular:	Infinite:
The F is G	F's are non-G's

Relation	**Modality**
Categorical:	Necessity:
F's are G's	Necessarily, F's are G's
Hypothetical:	Possibility:
If F's are G's, then H's are I's	Possibly, F's are G's
Disjunctive:	Assertoric:
F's are either G's or H's	Actually, F's are G's

Most of them are just the classical subject-predicate statements, others are composed by other judgements, such as the hypothetical and disjunctive ones (conditionals and disjunctions in propositional terms). But by extending the forms of judgements, Kant had also extended the logical constants contained in them. For example, the definite description "the" for particular judgements, the "if-then" and "or" connectives for the hypothetical and disjunctive judgements, and the modalities "necessarily", "possibly", "actually".

[12]Here, by reality one should read the manifold of sensible experience made intelligible by understanding.

Kant is also famous for having standardized the analytical-synthetical distinction. The distinction is about judgements and their contents. The contents are relevant for justifying the judgement's truth. Hence, judgements are true in virtue of the meaning of its constituents. These are, as we know, relations between representations (either concepts or objects) that are constituted from other representations (as concepts are). If we say that "All Bodies are Heavy", we are not saying that the class of body-things is a subset of the class of heavy-things, but asserting that "Heavy" is one of the characteristic marks of the concept of being a Body. Thus, we can describe the general judgement "A *is* B" as $B \in_R A$: the containment of the representation B in the representation A. The analytic-synthetic distinction then becomes:

$$B \in_R A \text{ is } \begin{cases} \text{Analytic, if } A = \{r_1, ..., r_n, B\} \\ \text{Synthetic, otherwise.} \end{cases}$$

Here, $r_1, ..., r_n$ is a list of representations that are parts of the complex A. The truth of analytic judgments follows from non-contradiction: there is no way in which $B \notin_R \{r_1, ..., r_n, B\}$ can be true safe from contradiction (KANT, 1998, B190-191). For synthetic judgements, on the other hand, one has to relate both representations to verify its validity. And such is only possible through the medium of the apperception, as we stated above[13].

There is too much depth in Kant's theory of judgement to be digged here. Even though Kantian judgements are based on certain mental acts, his theory still goes far beyond the psychologistic theories of his posterity[14]. They certainly are products of mental activities, but since they are rooted in *a priori* conditions, they still enjoy enough objectivity. As Hanna (2018) states, Kantian judgements are "intersubjectively shareable, rationally communicable, cognitively-generated mental-act structures".

Kant's theory is still the continuity of the Aristotelian tradition. The basic S *is* P structure of judgements is still a constant. One major shift, however, is that the prominent role played by Judgements in Kant's epistemology marks a change in the order of dependence between each of the foundational acts (apprehending, judging, inferring),

[13] Kant also add that

> The synthesis of representations rests on the imagination, but their synthetic unity (which is requisite for judgment), on the unity of apperception. (KANT, 1998, B193)

Which states that, in order to unify representations, something must be already given in cognition in a unified manner: consciousness.

[14] In a nutshell, Kant's judgements are "neither merely psychological objects or processes, nor are they essentially mind-independent, abstract objects, nor again are they inherently assertoric takings of propositions to be true" (HANNA, 2018).

as argued by Sundholm (2008), given that the faculty of judging have priority over the apprehension of concepts. This shift will be followed later by Frege, albeit in different terms. But has Kant advanced a clear-cut distinction between the content and the act of a judgement? This is up for discussion. Martin-Löf (1996) claims that "with Kant, the term judgement became ambiguous between the act of judging and that which is judged, or the judgement made, if you prefer." But one can find such distinction, for example, in the distinction between assertory judgements (A is B) and problematic judgements (Possibly, A is B) in Kant's table. In the *Jäsche Logic*, Kant claims the following:

> On the distinction between problematic and assertoric judgments rests the true distinction between judgments [*Urtheilen*] and *propositions* [*Sätzen*], which is customarily placed, wrongly, in the mere expression through words, without which one simply could not judge at all. In judgment the relation of various representations to the unity of consciousness is thought merely as problematic, but in a proposition as assertoric. [...] Before I have a proposition I must first judge; and I judge about much that I cannot decide, which I must do, however, as soon as I determine a judgment as a proposition. (KANT, 1992, p.605)

This distinction is also similar to Frege's own between the judgement, as a mental act, and the assertion, as the manifestation of such judgement. Either way, in a problematic judgement, as Kant explains, one can in fact entertain a content as possible without, at the same time, judging it as being true or not, as in the assertory ones.

Nevertheless, Kant's theory did offered some insights. Not only the mental act contained in the copula "is"/"are" received a transcendental foundation, but its relata as conceptual containment received some now well known truth-conditions: Truth in virtue of meaning for analytic judgements, truth in virtue of being an objective representation of possible experience for synthetic judgements (in verificationist terms), whatever cumbersome this latter part may seem today.

Kant's theory of judgements and the Port-Royal logicians had a common principle, that Rojszczak and Smith (2003, p.157) called the *combinational theory of judgements*: the idea that the activity of judging is "conceived as a process of combining or separating certain mental units called 'concepts', 'presentations', or 'ideas' ". This combinatorial theory was so much successful that lasted until the second half of the nineteenth century.

In the earlier stages of such combinational theory, the philosophical background was still given in transcendent terms: even though judgements were mental actions of combining different ideas, concepts or representations, truth was still dependent of certain facts of the world, *viz.* a background ontology. Thus, “All Bodies are Heavy” is true whether there are bodies which are all heavy. This reading was well established until Kant. Kant’s transcendental answer to judgement’s validity is not the same as transcendent theories that postulate an independent and objective ontology in order to validate the mental acts that define judgements. After Kant, with the birth of idealism, combinatorial theories shifted from transcendent background to a *Immanentistic* background “according to which the process of judging is to be understood entirely from the perspective of what takes place within the mind or consciousness of the judging subject” (ROJSZCZAK; SMITH, 2003, p.157-8).

Rojszczak and Smith (2003) found three problems with the combinational theory under immanentistic background: First, on how to explain existential judgements, given that there seems to be no combination in them, whereas only a single member is in question. Second, the theory seems not capable of distinguishing the “moment of affirmation”, which latter will become in Frege’s terms, the assertoric force. Finally, the theory also fails in providing a clear, transcendent and objective, standard for truth. But there is another problem, this time not with combinational theory *per se*, but with its new immanentistic characterstic: *psychologism*.

Psychologism was a dispute that took over Germany’s philosophy in the second half of the nineteenth to the beginning of the twentieth century. Broadly speaking, psychologism refers to the attempt to reduce some non-psychological entity within the framework of psychology. It has many associated terminology, involving many subfields. But basically, the main arena of the dispute was over whether logic was to be reduced to the newly established science of psychology.

Although the dispute took place in the nineteenth century, psychologism has in its roots a principle much older, what Porta (2017, p.137) called the *Immanence Principle* (IP): “the fundamental Cartesian conviction, made explicit in its classical form by Locke, that the only possible objects of my knowledge are contents of my mind”. However, the immanence principle was not subject of controversy until after Hegelianism has found its decline. After Hegel, Germany saw the birth of many new ideas and discussions about the place of philosophy under the rapid development of the new empirical sciences, such as psychology. To no surprise, the immanence principle was retaken and reinterpreted.

One of the crucial battlefields of the *Psychologismus-Streit* was in Logic and its attempted reduction, with logical validity, to psychology. At the psychologistic side, names such as Wilhelm Wundt, Christoph Sigwart, Benno Erdmann and Theodor Lipps are usually mentioned, while at the anti-psychologistic side we may mention a few names that, although not directly involved in the dispute, played a role in it, such as Bolzano, Hermann Lotze and some early neo-kantians such as Johann Herbart. But more directly involved were neo-kantians like Hermann Cohen, Wilhelm Windelband, Paul Natorp, and the two big names mostly consider: Edmund Husserl and Frege.

The dispute also took the form of another, as Porta (2017) summarize: objectivism versus relativism. The anti-psychologists, more than just fight for logical objectivity, also showed a complete lack of interests on the subjective conditions for knowledge[15]. On the one hand, psychologists insisted that objectivism could not successfully explain how a real subject could grasp ideal, or abstract, entities[16]. On the other hand, objectivists have mostly accused psychologists of three things: (1) of being relativistic, given that deriving the ideal from the real could not offer the necessary objectivity for logic; (2) of reducing the normative status of logical laws into descriptive features of the human mind; and (3) of confusing the *genesis* of our representation of a judgement's truth with its proper logical *justification*.

It is also to no surprise that the debate around logic and psychology focused on judgements. In the anti-psychologistic side, the worry was on how to provide objective validity for judgements if, following the immanence principle defended by the psychologistic philosophers, every judgement is fully determined by subjective conditions.

5.2.3 Psychologism: those in favour

The birth of psychologism is difficult to locate precisely. But some key psychologistic statements are found in of John Stuart Mill's texts, although debatable whether he was fully committed to it. An example is the following excerpt from the 1865 *Examination of Sir William Hamilton's Philosophy*:

> [logic] is not a Science distinct from, and coordinate with, Psychology. So far as it is a science at all, it is a part, or branch, of Psychology; differing from it, on the one hand as a part differs from the whole, and on the other, as an Art differs from a Science. Its theoretic grounds are wholly borrowed from Psychology, and include as much of that science as is required to justify the rules of the art. [...] A consequence of this is, that the Necessary Laws of Thought [...] are precisely those with which Logic has least to do, and which belong the most exclusively to Psychology. (MILL, 1979, p.360)

[15]Perhaps a good example of this stance was Kerry's accusation that Frege suffered from a *horror subjectivi*. On this, see Porta (2014).

[16]A problem that philosophers of mathematics in the twentieth century completely forgot until Benacerraf (1973) revived in new terms in the seventies.

This fairly psychologistic passage, in which logic is just a branch of psychology and having his laws borrowed from it, does not settles the question for Mill. In fact, we could very much put him in the anti-psychologistic side by considering his theory of judgements in the 1843 *A System of Logic*. His definition of propositions, rather than judgements, has its familiarity with Aristotle's original definition: "A proposition [...] is a portion of discourse in which a predicate is affirmed or denied of a subject." (MILL, 1974, Book I, ch5, §1), hence, the expression of an act. But Mill also separates the active mental attitude that we take towards some content from the content itself. A theory of propositions must either focus on one or the other: on judgements or the contents of judgements. But most surprisingly, the study of such propositional attitudes are not on the scope of logic, for Mill: "Logic [...] has no concern with the nature of the act of judging or believing; the consideration of that act, as a phenomenon of the mind, belongs to another science" (MILL, 1974, book I, ch.V, §1). And so, Mill has in mind a distinction between the locutionary and illocutionary parts of a proposition.

The content of a judgement (the subject-predicate relation) is denotative to some state of affairs. Ultimately, it is such state of affairs that matters for turning a judgement true or not. And thus, the strikingly anti-psychologistic part is that the truth of a proposition turns out to be fully independent from the mental part that is performed with the contents: "propositions [...] are not assertions respecting our ideas of things, but assertions respecting the things themselves" (*ibidem.*). This, of course, is made under Mill's empiricist background. Mill has an interesting view on judgements, and a hint for the more precise act-content distinction that is still to come. Nonetheless, his psychologism is, at best, accidental to his whole philosophy which I don't endeavor to read it further[17].

Going back to Germany, where psychologism really developed, Stelzner (2003) provides a good summary of the psychologistic authors and their psychologistic thesis[18]. For instance, Jakob Friedrich Fries (1773-1843), who developed an anthropological view on Logic; Friedrich Eduard Beneke (1798-1854), with perhaps the first empiricistic psychological conception of logic, defended that psychology should provide the scientific basis for logic, and that logic "should include the analysis of things in real connection with our imagination" (STELZNER, 2003, p.87), but not simply in themselves; Benno Erdmann (1851-1921), Frege's choice of target in the *Grundgesetze*, defended a implicit founda-

[17]For a summary of the dispute over Mill's psychologism, see (KUSCH, 2020).

[18]The summary here presented is nowhere near the desired attention that these authors and themes deserve to receive. For the lack of time and space, however, their names and philosophical accounts will be only briefly mentioned.

tional psychologism in logic and gave a definition and criteria for the soundness of logical laws in terms of empirical *common agreement*. He also considered logical necessity as not absolute, but relative to the psychological organization of the inhabitants of the world; Theodor Lipps (1851-1914) defended an unlimited foundational logical psychologism, a physics of the thought: "[...] logic is a psychological discipline, as certain as the cognition occurs only in the psyche, and the thinking, which completes itself in the cognition, is a psychical event" (In his 1893 *Logik*, quoted in (STELZNER, 2003, p.89)); Christoph Sigwart (1830-1904), in his *Logik*, defended that judgements were mental states, and thus had "real existence only in active judging, in the mental act of a thinking individual which takes place at a given moment" (SIGWART, 1895, p.25). This, of course, does not constitute by itself a defense of psychologism. But Sigwart also defended that "the truth is not a possible feature of the content (which is given by a sentence) of a judgment in separation from any judgment" (STELZNER, 2003, p.90), therefore taking truth to be dependent not only on content, but on the mental activities of the judging agent.

Franz Brentano: There is another key author for the evolution of Judgements that may or may not be included in the psychologistic side. That is Franz Brentano. Brentano's philosophy does have many psychologistic tendencies, and he has been accused of psychologism by his student Edmund Husserl, an accusation that he rejected. His influence will be felt later in the polish logical scene, indirectly leading to Tarski's interest in the truth-predicate[19].

The publication of *Psychology from an Empirical Standpoint* in 1874, a major contribution to Psychology as an emerging science, is an important mark for the notion of a judgement. Brentano famously reintroduced the notion of intentionality, that is, the property of "referring to something" that every mental phenomena satisfies. By being intentional, a mental phenomena has, or is about, an object, and Brentano includes "Every idea or presentation [*Vorstellung*] which we acquire either through sense perception or imagination". This is broad enough to include the thinking of a concept, judgements and also inferences as mental phenomena that are intentional (BRENTANO, 2009, p.60). More precisely, there are three main classes of mental phenomena: presentations, judgements and emotions (BRENTANO, 2009, p.152).

We saw that the difference between a judgement and a mere association of ideas were already distinguished in Kant. We also saw that Mill distinguished between such associations and the act of combining them, which is the psychological state of a

[19]On this influence, see (WOLEŃSKI; SIMONS, 1989).

belief. Brentano follows a similar distinction, as presentations are different kinds of mental phenomena than judgements. I may have the presentation of a green Tree in my mind, but to judge that “The Tree is Green”, as Brentano says, is an instrinsically different kind of mental act.

Judgements are intentional mental phenomena just as presentations are. But the relation of a judgement to its object includes the quality of acceptance or denial which presentations lack. As Brentano continues, “By ‘judgement’ we mean, in accordance with common philosophical usage, acceptance (as true) or rejection (as false).” (BRENTANO, 2009, p.153). But against the tradition, he rejected that such acts of acceptance or rejection are combination of different presentations. His theory is, thus, a break from the classical combinatorial view that saw a judgement as acts of predication between two terms.

Brentano first notice that all categorical judgements can be converted to pure existential judgements. From “A man is sick“ one obtains “A sick man exists“. From “No stone is living” one obtains “A living stone does not exists”. From “All men are mortal” one obtains “An immortal man does not exists”. Finally, from “Some man is not learned” one obtains “An unlearned man exists” (BRENTANO, 2009, p.165-6). This is a hint that acts of affirmation and negation are in fact acts of acceptance and denial of existence.

Usual existential judgements are hard to account within the combinatorial view. There is no subject-term in cases like “There are horses”. The combinatorial view must then include existence as predicate, and “horses” as the subject (yielding the judgement “Horses has existence”). But to take existence as genuine predicate also runs into problems. In this account, “Pegasus don’t exist” is true if there is some object such as “Pegasus” that does not satisfy the predicate of existence, which is contradictory. Brentano’s option is to follow Kant’s view that existence is not a real predicate.

His argument is that existence is a redundant predicate (BRENTANO, 2009, p.161). It is based in the two following principles:

(AW) The affirmation of a whole is the affirmation of its parts

(DW) The denial of a whole is not the denial of each of its parts.

Following (AW), when we affirm “A exists”, in which the ideas “A” and “existence” are combined, we are also affirming both: the existence of the presentation “man” and the existence of the presentation “existence”. Since the latter is already asserted in the former, the judgements “A exists” and “A” are one and the same judgement. Negative existential judgements, such as "*A* does not exists" is also the denial of the existence of *A*. To deny "*A*" and deny that "*A* exists" are the same judgement. When we judge, we *ipso facto* made existential claims, and this is enough evidence for Brentano to take existential judgements as the primitive judicative act[20].

There are, of course, a few problems with this account. How to assess judgements such as the denial that "*A* exists"? Following (DW), we are not denying "*A*" when we deny *A*'s existence. But what are we denying anyway? This calls for a definition of existence applicable to judgements. Given that, for Brentano, all judgements are existential, we must know what judgements are about. In a debate with Mill, Brentano tells us that the judgement "Centaur is a fiction" "[...] does not presuppose that there is a centaur, but only that there is an imagined centaur" (BRENTANO, 2009, p.170). The implicit existential claim occurring in "Centaur is a fiction" does not entails the *real* existence of centaurs, but simply the mental presentation that the judgement is about, its immanent content: "Nothing remains, then, but to think of the distinctive feature of judgement as a particular kind of relation to the immanent object", as he claims (BRENTANO, 2009, p.172).

Thus, judgements are mainly about immanent objects[21]. When we judge, we affirm or deny existence in the sense that the immanent object may or may not have a real counterpart. But Brentano's theory of judgements does not yield a correspondence theory of truth, between one's immanent object with transcendent ones. Instead, Brentano came to develop a theory of truth based on the concept of evidence, in which a judgement is true if can be judged with enough evidence. Following the distinction between presentations and judgements, one may describe Brentano's notion of evidence as the relation between a primary psychological event, in which one intentionally presents a given immanent object (which may or may not have real existence) and a secondary psychological event, a judgement, in which one affirms or denies the reality of the primary intentional presentation (if it has real existence or not)[22]. This relation is not a simple correspondence relation, but one of harmony as Brentano explains in the 1889 lecture *On the Concept of Truth*: "To correspond does not mean to be the same or to be similar; but it does mean to be adequate, to fit, to be in agreement with, to be in harmony with, or whatever equivalent expressions one may choose to apply" (BRENTANO, 1966, p.14).

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[20] There is a similarity between Brentano's argument for the redundancy of existential claim in judging, to Frege's later argument for the redundancy of the truth-predicate in judging.

[21] This is also named as the non-relational reading of Brentano's intentionality, *viz.,* that intentionality is not directed to external object directly, but to mental contents, the immanent objects, as explained by Rojszczak and Smith (2003).

[22] On this, see Brito (2014).

Given that a judgement is, in fact, a kind of second-order mental phenomena, it is clear that a correspondence relation is not suitable, as one judges correctly if enough evidence is provided from the immanent object to one's judgement about it. Thus, to have an adequate, fit or harmonic relation is to have evidence for one's judgement's truth.

This, of course, puts some restrains in one's hability to judge. The fact is that the only kind of judgements that can be judged with absolute evidence are those about inner perceptions alone, as Rojszczak and Smith (2003) explains. Judgements about outer perceptions (in which our immanent objects are said to correspond to external ones), although they can be true in their own right, "do not belong to our knowledge in the strict sense" (*ibidem.*), given that no sufficient evidence is available to us.

In conclusion, in Brentano's theory, judgements are dependent on the conception of intentionality, as they are mainly about immanent object, and only secondarily about real external entities. Given this reliance on the immanence principle, logic is highly dependent on psychology, which made Brentano an easy target of psychologism[23]. Brentano's reform on logic, albeit changing the tone on the kind of acts judgements are, still maintained the performativeness of logic. In fact, even more prominence was given for the judging subject. Acts of affirmation and denial, although not acts of combination, are still acts that are taken from the perspective of a real subject.

5.2.4 Psychologism: those against it

The heavy number of authors and logical books with psychologistic flavour above did receive a number of answers, criticisms and alternative proposals. The responses on the anti-psychologistic side focused on a number of strategies. For instance, one might reject the immanence principle, as neo-Kantians and others did, and rethink subjectivity in transcedental terms. But one can also focus on distinguishing, in judgements, the content and the act in such a way that content itself can be objectified. Both are not isolated strategies. Even objectifying content, as we saw, leave open the problem of grasping them, and so an analysis of the subjectivity must be provided. For that reason, even transcendent authors such as Frege had still included judgements as acts in his logic. Nonetheless, Frege has followed, directly or indirectly, other anti-psychologistic philosophers, such as Herbart, Bolzano and Lotze and Windelband[24].

[23]Brentano defended himself of such charge, but it is not of my interested here to dwelve into such details or give an account on the matter.

[24]Others could be mentioned. I have no intent to offer an exhaustive list of influences.

Johann Herbart: An earlier representative of this position is found in Johann Friedrich Herbart (1776-1841). Herbart lived much earlier than the dispute, but represent some of the anti-psychologistic thesis. A post-Kantian, Herbart was one of the philosopher who tried to reinterpret Kant's classification of judgements, by taking the quality (affirmation, negation) as its basis, therefore taking judgements to be essentially acts of affirmation or negation (recall that for Kant, judgements were rather acts of unification). Herbart was a strong anti-psychologistic, according to Beiser (2013, p.180):

> It was chiefly Herbart who [...] had stressed the autonomy of logic. He had made a sharp distinction between the activity of thinking, which is the subject of psychology, and the content of thought, which is the subject of logic. Logic would be utterly ruined, Herbart warned, if one brought into it enquiries into the origin of logical forms. So determined was Herbart on keeping psychology out of logic that he even questioned whether the content of thought, in the strict logical sense, ever appears to the mind.

Bernard Bolzano: After Herbart, and still before the dispute, an even stronger stance against psychologism was taken by Bernard Bolzano's (1781-1848) *Theory of Science*, published in 1837. Bolzano is best known for the concept of a *proposition-in-itself* [*Satz an sich*], or simply, proposition. This single concept is perhaps the most influential movement for theories of judgements, given that, as it will be shown for the case of Frege and subsequent authors, in the long run it helped logic get free of such notion.

Bolzano is a prime example of the objectification of content in a judgement. First, there is a clear distinction, discussed in the first volume of the *Theory of Science*, (BOLZANO, 2014a, §19), between what he called a "spoken propositions", which are utterances, from the "thought propositions", the thing being uttered, the propositions-in-themselves. These are not ideas, beliefs or judgements which take place in some human mind, but what we now call abstract entities:

> one must not ascribe being (existence or actuality) to propositions in themselves. Only the thought or claimed proposition, i.e., the thought of a proposition, likewise the judgement which contains a given proposition, has existence in the mind of the being that thinks the thought or makes the judgement. (BOLZANO, 2014a, §19)

Moreover, bolzanian judgements are complex combinations of objects and predicates, just like the traditional "A is B". But Bolzano changes the copula for the verb "to has". For an object A and a predicate b, the classical "A is b" becomes "A has b" (BOLZANO, 2014b, §127). Following Rojszczak and Smith (2003) assessment, Bolzano still offered a combinatorial theory of judgements, but clearly from a platonistic kind. They are acts of combination that refer to spoken propositions which denotes propositions-in-themselves. These are abstract entities which are true or false independently.

Thus, propositions-in-themselves are the referents of spoken or written propositions, sentences. Given that they are abstract entities that have no existence, they cannot be mind-dependent objects. And since propositions are objectively given, part of them are also truths-in-themselves, *i.e.,* they are absolute, timelessly and independently true (while others are simply false), not depend on any consciousness or cognition. A given true proposition is true regardless of any epistemic attitude that we may or may not have on it. We may think about propositions, or we may judge propositions to be true. Either way, both require grasping propositions. But in the case of judgements, *judging* a proposition to be true is nothing more than *recognizing* that it is true. As he says, "Every judgement contains a proposition which is either true or false" (BOLZANO, 2014a, §34). Notice then that Bolzano does not speak of true judgements, but of true propositions. This is because a judgement does not make a proposition true, but only recognize it as such. It may well misfire, what Bolzano call errors, but only true judgements are properly *cognitions*, or knowledge proper. (BOLZANO, 2014a, §36).

Finally, a judgement is wholly dependent on the proposition-in-itself and in the relation between the ideas-in-themselves that are its subparts. As he explains, the only influence that the human will can have on judging is on the choice of which judgement one wants to perform. Under such restrictions, it is clear that, although too early to joing the dispute, Bolzano was a strong name against psychologism.

Hermann Lotze: Another important figure is Hermann Lotze (1817 - 1881). A post-Hegelian idealist, Lotze rise as a philosopher against the growth of naturalism in 1830's and 1840's, mostly due to the birth of the new empirical sciences such as biology and, one may guess, psychology. Lotze's goal, as Beiser (2013, pp.127-131) tells us, was to redeem value for the world among naturalism and its ugly child, materialism. And in this quest, he anticipated much of the distinctions that influenced contemporary philosophers, such as between validity and existence, normativity and nature, act and content. I'll focus on three main contributions that Lotze gave to posteriority, mostly on Frege: the anti-psychologism stance, the introduction of the term "*sachverhalt*", and the functional analysis on the contents of judgements.

In Lotze's philosophy, judgements are still based on subject-predicate distinctions of the tradition. But he advanced a new interpretation on concepts and concept-formation. Traditionally, a concept is a list of characteristic marks

$$C = a_1, a_2, ..., a_n$$

For Lotze (1884, §28), there seems to be a "[...] delusion that the elements of a concept are universally of equal value, connected in the same way each with the whole and each with each". Roughly, a concept is not a mere list of uncoordinated marks, but requires a rule for combining them in specific orders, given that each concept may have different internal relations between each characteristic mark. This is argued in the following manner:

> an appropriate symbol for the structure of a concept is not the equation $S = a + b + c + d, etc.$, but such an expression as $S = F(a, b, c, etc.)$, indicating merely that, in order to give the value of S, $a, b, c, etc.$, must be combined in a manner definable precisely in each particular case, but extremely variable when taken generally. (*ibidem*).

Thus, the concept S is obtained by the function, or rule, of coordination of each characteristic mark. This is not yet the functional analysis that Frege will offer, much more mathematically oriented, but it has some of its insights[25].

Aside from this new conceptual analysis, Lotze's theory of judgement is mostly borrowed from Kant's table. Judgements still are provided by the subject-copula-predicate structure. But Lotze also add the following provision:

> Every judgment formed in the natural exercise of thought is intended to express a relation between the matters of two ideas, not a relation of the two ideas themselves. Of course some sort of relation between the ideas follows inevitably from the objective relation in the matter which they represent; but it is not this indispensable relation in the mental media through which we endeavour to grasp the matter of fact, but this matter of fact itself, which is the essential meaning of the act of judgement. (LOTZE, 1884, §36)

Later, Lotze will use another term for the matter of judgements: state of affair [*sachverhalt*]. This is a very similar idea to Bolzano's propositions-in-themselves. Lotze is recognizing a realm of mental phenomena and another one, which later Frege will call senses. The mental counterpart of a judgement is the linguistic entity in which we gain access to its content. Moreover, it is the content of a judgement, the state of affair, that provides the objective validity *i.e.,* its reference to an actual fact, and therefore, its truth. Lotze is, therefore, objectifying content in order to safeguard judgements objectivity, while still considering it a mental act of some sort.

[25]For a better comparison between both, see Heiss (2013).

The story is of course longer and more detailed. But Lotze has the traits to be in the anti-psychologistic side. In fact, he is emphatic in claiming logic as independent from psychology. Under such background, his theory of judgments, albeit not that far from Kant's influence in keeping the combinatorial form, did not obliged to any immanence principle, founding his validity outside the realm of psychology. In conclusion, I leave the following passage from Lotze:

> [...] Logic cannot derive any serious advantage from a discussion of the conditions under which thought as a psychical process comes about. The significance of logical forms is to be found in the meaning and purport of the connexions into which the content of our world of ideas ought to be brought; that is to say, in the utterances of thought or the laws which it imposes, after or during the act of thinking, not in those productive conditions of thought itself which lie behind. (LOTZE, 1884, §332)

Wilhelm Windelband: From Lotze, we jump to the neo-Kantians. An author, famous on the anti-psychologistic side, is the neo-Kantian from the Baden Scholl Wilhelm Windelband (1848-1918). Windelband gave substantial contribution to Herbart's first declared focus on the quality of judgements (the acts of affirmation and negation), as Gabriel (2013) tells us. For Windelband, every judgement is twofold: there is a theoretical judgement (the content, or sense of the propositional content being judged) and a practical judgement (a practical "act" that decides - either affirming or denying - the truth-value).

Windelband saw in affirmation and negation the foundation of logic:

> The doctrine of the quality of judgments leads us necessarily to the norms of affirmation and negation, which, under the name of laws of thought, are known as the most general logical principles. (WINDELBAND, 1961, p.32)

He also defended the normative character of logic, against psychologism's tendency of reducing norms into descriptions, and very much anticipating Frege's own words: "[Logic] must not be supposed to teach how people actually think, but how they should think if they want to think rightly" (WINDELBAND, 1961, p.23).

Another anti-psychologistic theme, that Windelband anticipate Frege, is the distinction between genesis and validity of truths:

> logic is concerned not with the origin but with the validity or truth of ideas. Logic is interested in these psychogenetic inquiries only in so far as they are necessary or fitted to make the different types

> of presentation-processes clear and distinct through their interrelations. [...] There are logical principles of Psychology (as of every science), but there are no psychological principles of Logic. (WINDELBAND, 1961, p.9)

Which is a similar stance to Lotze's.

All such authors help explaining the path that leads the traditional judicative logics to Frege's own conception. Even if Frege is celebrated as the one to start the mathematization of logic, he was still a full fledged philosopher logic-wise. "[...] a mathematician with no element of philosophy in him is only half a mathematician." (*PW*, p.273) as he claimed later in the unpublished paper *Erkenntnisquellen der Mathematik und der mathematischen Naturwissenschaften*.

5.2.5 Judgements last stance

The influence of anti-psychologism for the theories of judgements, and logic itself, was decisive. But this influence would eventually be the demise of judgements. Throughout the history, Logic, as the science of thinking, was centered on three stages: the apprehension or formation of concepts, the act of judging and the act of drawing inferences. This division was first drawed by medieval logicians (under the influence of Aristotle's logical texts), standardized in Port-Royal Logic, and only abandoned after the birth of mathematical logic. Given that, until this demise, logic was centered on different mind-dependent activities (apprehending, judging, inferring), the tradition saw logic as inherently practical, *viz.*, they saw logic-as-practice.

The final years of the logic-as-practice view started with Bolzano, Lotze and Frege, although none had completely rejected the view. Sundholm (2013, 2008, 2018) emphasises Bolzano's role and influence in the shift from judgements to propositions. He goes even further in rephrasing Quine, saying that modern logic started in 1834, the year when Bolzano's *Theory of Science* came out, not 1879, the year when Frege's *Concept-script* was published[26]. Logic was, as Sundholm describes it, a theory of judgements and inferences rather than a theory propositions and consequences. As a theory of inferences, logical practice was an epistemic activity. To judge that P is, *ipso facto*, to known that P is true. To infer P from Q is to know Q as true in virtue of P's being true.

Many different accounts on the correctness of judgements as epistemic acts were offered, but ultimately, the anti-psychologistic spirit, even before the dispute was

[26]The first words in Quine's *Methods of Logic* reads "Logic is an old subject, and since 1879 it has been a great one" (QUINE, 1966, p.vii).

explicitly taken, led to the common solution of taking such correctness as dependent on content entirely. As Sundholm puts it, “Frege, and before him Bolzano, secures the metaphysical role of truth, namely, that of providing the notion of rightness for epistemic acts, via the bivalence of truth for judgmental contents” (SUNDHOLM, 2008, p.279). Within this background in the late nineteenth-century, de-pragmatization have started to take shape.

As we saw before, all mental acts were structure under the tripartite relation of an act, a content and a sign or linguistic expression of the act. Thus, we had that judgements were generally expressed within three components:

(a) The mental act: judging.

(b) The expression of such an act: an utterance in the assertive form.

(c) The content of the act: the thing judged.

Each component may or may not be explicitly exposed in each of the authors above discussed. Aristotle’s “propositions”, for instance, seem to include both (b) and (c). Port-Royal Logic did distinguished between (a) and (b), but (c), albeit still expressible, were not clearly distinguished from (b), as there was no way to entertain a proposition without judging it. Kant seemed to merge (b) and (c) as well, while offering a different account for (a). Later, psychologism relativized partially or completely (c) under (a). It was the anti-psychologists who, in order to avoid it, sharply distinguished between (a), (b) and (c), in order to objectify (c) in transcendent terms, keeping the validity of the whole enterprise.

Nonetheless, the view that logic included pragmatic elements, in the sense of presupposing an active agent, was standard. Sometimes, it became too strong. As we saw, psychologistic theories put too much weight on the mental conditions under which one asserts, judges, denies, infer, and so on. That is, they tried to reduce judgements validity to the mental conditions under which one judges. Since this were, for anti-psychologists, a descriptive notion, it turned logic into a empirical science, to their denial. But how can one keep the standards for correctness of the judging acts? The common solution included making the content independent from the act, as it was the case with Bolzano, Frege and Lotze, who shared a common platonistic flavour. However, one of the consequences was that ultimately, contents became prior to judgements as acts. We can already found this faith in authors of the early analytic philosophy: Moore, Russell and Wittgenstein.

G.E. Moore and early Russell adopted a realist account for propositions, now the standard English word for judgements. In Moore's account, exposed in *The Nature of Judgment*, and against Bradley's idealism in which judgements are thought as relations between ideas (mental states that compose the meanings of the signs expressed in a judgement), a judgement, or a proposition by his choice of words, "[...] is composed not of words, nor yet of thoughts, but of concepts" (MOORE, 1899, p.179). Concepts, as he explains, although being possible objects of our thoughts, are entities that are independent from a thinking subject. Thus, a judgement is about nothing more than a connection between concepts. Its truth or falsity depends on such connection, not on the relation between ideas and reality, as thought by Bradley.

Moore's position influenced Russell's analysis of propositions in the *Principles of Mathematics.* Russell equally criticized Bradley's import of meanings as mental products, and saw a proposition as being not simply a compound of concepts (as in Moore), but between terms[27]. And continuing Moore's realist account of concepts, Russell takes terms as immutable and indestructible: "What a term is, it is, and no change can be conceived in it which would not destroy its identity and make it another term" (RUSSELL, 1996, §47). Thus, terms are the constituents of propositions, which are the contents of assertions. "In every proposition, [...] we may make an analysis into something asserted and something about which the assertion is made" (*idem*, §46). But in Russell's early analysis, what we have is a subject term and something that is asserted about it, which Russell calls "the assertion". In "Socrates is a man", "Socrates" is the subject-term, while "is a man" is the assertion. The mark of the assertion is present in what Russell calls "the verb" of the proposition: the "is" in "is a man", which confers the connection between the subject term "Socrates" with the predicate "a man"[28].

The relation between the assertion and truth is something which Russell is not sufficient clear. He distinguishes between a psychological from a logical sense of assertion. The psychological sense is simply the performative act of asserting a proposition either as true or false. Moreover, the distinction between asserted and unasserted propositions is, for Russell, about the psychological sense only (*idem.,* §52). But a proposition's truth

[27]"Whatever may be an object of thought, or may occur in any true or false proposition, or can be counted as *one*, I call a *term*" (RUSSELL, 1996, §47). Generally, a term is any constituent that can be the subject of a proposition, either a thing or a concept, as Russell divides them in (RUSSELL, 1996, §48).

[28]That the verb confers such unity, what we may call the assertive force of the assertion, is clearer if the verb is transformed to a verbal noun: in "Socrates is a man", truth is asserted while in "the mankind of Socrates", no assertion is made. "There appears to be an ultimate notion of assertion, given by the verb, which is lost as soon as we substitute a verbal noun, and is lost when the proposition in question is made the subject of some other proposition" (RUSSELL, 1996, §52).

holds according to the relation between its terms[29], which Russell takes it as the logical sense of an assertion:

> when a proposition happens to be true, it has a further quality, over and above that which it shares with false propositions, and it is this further quality which is what I mean by assertion in a logical as opposed to a psychological sense. (*idem.*,§52)

Russell is far from being clear in this passage. But what the logical sense of assertion seems to represent the actual connection between the terms of a proposition, which obtains regardless if the proposition is or isn't linguistically asserted (in the psychological sense).

Similar conclusion was held in the 1905 paper *The Nature of Truth*, where Russell takes truth not to be the correspondence between ideas and facts, but to be facts themselves. In this account, judgements are not the truth-bearers. Instead, truth is taken as "[...] a property of the objects of judgments, *i.e.* of what we may call facts" (RUSSELL, 1994, p.495). Moreover, these facts *are* the true propositions[30]. Then, propositions exists, or have being as he claims, independently from judgements. Also, true propositions are not true because someone happens to judge or assert it as such. Rather, true propositions are said to share a property that false propositions lacks. This is why, in the *Principles of Mathematics*, Russell takes a true proposition and an asserted proposition (in the logical sense) to be the same. Against Frege's option of taking truth-values as the *Bedeutung* of sentences and the assertion as the recognition that a sentence has the True as referent, Russell states that "It is also almost impossible [...] to divorce assertion from truth, as Frege does. An asserted proposition, it would seem, must be the same as a true proposition". (RUSSELL, 1996, §478.). Moreover, this allows Russell to consider that every false proposition is always unasserted (*idem.*,§479).

We may conclude that Russell saw logical assertion *on a par* with the truth-predicate: something that is contained in an asserted proposition but that is not itself

[29]The proposition, expressed linguistically, is literally composed by the entities denoted by the terms: "a proposition [...] does not contain words: it contains the entities indicated by words" (RUSSELL, 1996, §51). Russell would express this same idea in a letter to Frege dated December 12th, 1904:

> I believe that in spite of all its snowfields Mont Blanc itself is a component part of what is actually asserted in the proposition "Mont Blanc is more than 4000 metres high". We do not assert the thought, for this is a private psychological matter: we assert the object of the thought, and this is, to my mind, a certain complex (an objective proposition, one might say) in which Mont Blanc is itself a component part. (*PMC*, p.169)

[30]As he puts it, "People imagine that if A exists, A is a fact; but really the fact is "A's existence" or "that A exists". Things of this sort, i.e. "that A exists", do not themselves exist: it is things of this sort that I call *propositions*, and it is things of this sort that are called *facts* when they happen to be true". (RUSSELL, 1994, p.492).

a constituent of the proposition. And given that Russell concluded that "to divorce assertion from truth seems only to be possible by taking assertion in a psychological sense" (*idem.,*§478), it is clear that he also saw Frege's judgement-stroke as having only psychological importance. For this reason, Russell complains that Frege's employment of the assertion sign as the recognition of the truth of a thought is a psychological move, and thus, unecessary logic-wise. Taking Russell's cryptic account of assertion aside, it is clear that Moore and Russell's realist theory of propositions (at least in the *Principles*) marks the objectificational perspective above discussed, and another key moment for the understanding of the term "proposition" as such.

But Russell (with Alfred Whitehead) came to adopt Frege's judgement-stroke in the *Principia Mathematica*, re-labeling it as the Assertion-sign. What made Russell change his mind was, perhaps, his rethinking about the notion of proposition and truth. In the 1910 paper *On the Nature of Truth and Falsehood*, Russell goes back to consider the correspondence theory of truth. First, he change his previous view in saying that "the things that are true or false [...] are statements, and beliefs or judgments" (RUSSELL, 1992, p.117). In the previous account, an assertion, or judgement, was simply the relation of the mind to a single entity, the proposition taken as an objective entity. Truth could not come out from such a relation, since truth was already a property of the proposition. Now, Russell takes a judgement to be the relation between the mind and multiple entities (what he calls the "multiple relation"), the "[...] various other terms with which the judgment is concerned" (*idem.*, p.122). The judgement "Socrates is a man" or "Plato was Socrates' student" cannot be taken as an assertion of the verbal nouns "that Socrates is a man" or "that Plato was Socrates' student". Rather, they must be taken as the assertion that Socrates has the property of being a man, and that Socrates and Plato stands in the relation of "being the student of", which are both true just in case both relations hold, or false otherwise. Thus, Russell came to conclude that "truth and falsehood are primarily properties of judgments" (*idem.,* p.124).

This account is also present in the *Principia Mathematica*, which helps explaining the presence of the assertion-sign. First, the authors consider a background ontology composed of simple and complex objects. That Plato was Socrates' student corresponds, in this universe, to a complex which may be referred as "Plato-as-Socrates's-Student". This complex, however, cannot be judge without being break up in its parts[31], given that "[...] a judgment does not have a single object, namely the proposition, but has several

[31]Differently from the earlier account, each part can only be perceived as one, and thus cannot be judged.

interrelated objects.” (RUSSELL; WHITEHEAD, 1910, p.46), *i.e.,* Plato, Socrates, the relation of being the student of, and the judging agent. A judgement is true when there is a complex that corresponds to the relation, and is false when there is none[32].

Given that a judgement relates not to a single object, the linguistic expression used to express their relata (the proposition) is not considered a complete entity. Russell and Whitehead opted for this solution in order to explain the difference between “Plato is Socrates’ Student” from “’Plato is Socrates’ Student’ is a proposition”, given that the corresponding complex is asserted in the first but not in the second, albeit having the same linguistic expression in both cases. The same kind of distinction is made in justifying the notion of an assertion (*idem.,* p.96), and the introduction of the assertion-sign. Thus, the assertion-sign not only confers what we may consider the assertive force, it also confers unity to the constituent of the proposition, making it a proper judgement, and not just a list of constituents with no truth commitment[33].

Even after adopting the Fregean sign for assertion, Russell and Whitehead have conferred a different use. Their use of the sign is already pointing out for a metatheoretical reading, although there is no distinction of levels of language in the Tarskian sense in the *Principia.* In Frege, the assertion-sign is completely *sui generis*: it is non-denotative, neither predicative. Since it is an illocutionary act, in Frege’s concept-script, the sign is never included as sub-formula of a concept-script formula. A concept-script proposition such as

$$\vdash A$$

cannot be read as “A denotes the True”, since this would imply the inclusion of the predicate “denotes the True” in the auxiliary language, which goes against Frege’s recommendations. This proposition can only properly be read outside the object-language, within the level of explications, since Frege does not have a formalized metalanguage. For that reason, expressions such as

$$\begin{array}{l} \top\!\vdash B \\ \llcorner A \end{array}$$

[32]“[...] we may define truth, where such judgments are concerned, consisting in the fact that there is a complex corresponding to the discursive thought which is the judgment.”, as they say. (RUSSELL; WHITEHEAD, 1910, p.46)

[33]Logically, the inclusion of the sign is also justified for distinguishing cases such as

$$\vdash p$$

from

$$\vdash p \supset p$$

where p is asserted individually in the first case, but not in the second.

represents a sort of clash of levels in the concept-script: the implication from A to the assertion of B, which never occurs in the auxiliary language.

In the concept-script, inferences are taken as sequences of asserted concept-script sentences. The expression of an inference must then occur in the same level of the assertions being made. Inferences such as "A, therefore, B" cannot be expressed within the language, such as in

$$\begin{array}{l} \top\!\vdash B \\ \llcorner\!\vdash A \end{array}$$

Instead, the inference from A to B (more precisely, from the assertion of A to the assertion of B) is performed using the inferential line:

$$\frac{\vdash A}{\vdash B}$$

All this goes to say that, from Russell and Whitehead's perspective, the assertion-sign can occur as the subformula of another formula, which marks a complete different reading. In their reading (RUSSELL; WHITEHEAD, 1910, p.9), an inference does occur from asserted propositions to asserted propositions. But in the *Principia*, inferences, such as

$$\text{From} \vdash A \text{ and} \vdash A \supset B, \text{ therefore} \vdash B$$

can be abbreviate as:

$$\vdash A \supset\vdash B.$$

The same point is latter made in explaining the different scope of assertions, using the dots notation (*idem.*, p.96). Whereas in

$$\vdash: p. \supset .q$$

one reads "it is true that p implies q", in

$$\vdash .p \supset . \vdash .q$$

one reads "p is true; therefore q is true"[34]. This duality of the implication sign, as operating both with propositions and assertions (and allowing the assertion-sign to occur in a subformula), is nowhere to be found in Frege's original use, as we saw. In fact, this

[34]The choice of taking the assertion sign as a truth-predicate here seems to be solely a *façon de parler*, as the authors explains.

dual meaning for the implication may be taken as two different implications: one within the object-language, from propositions to propositions, and another in a "metalanguage", from assertions to assertions. It is, of course, a mere abbreviation with no real logical meaning. But practically, it takes implication to another level.

There is another point in the *Principia* where the authors also allow assertions to occur in the scope of other assertions. The authors distinguish between "elementary judgements", judgements of elementary propositions (those with no variables) such as

$$\vdash \Phi(a),$$

from "general judgements", that is, judgements of propositional functions. Truth-conditions for elementary judgements are straight-forward: they are true whenever the complex fact denote by the sentence "occurs in the universe" (RUSSELL; WHITEHEAD, 1910, p.47).

General judgements, on the other hand, are more tricky. One strategy, that Russell and Whitehead quickly dismiss, is to read general judgements such as "all men are mortal" as the conjunction of elementary judgements "Plato is mortal", "Socrates is mortal", and so on, which would be impossible to verify[35]. The strategy chosen was to distinguish levels of truth. Truth-conditions for elementary judgements above define "elementary truths". The truth-conditions for general judgements are now to be define in terms the elementary truth of its constituents. For example, the judgement

$$\vdash (x).\Phi(x)$$

is true whenever Φx has elementary truth for the values of x. This is truth of second-order. Thus, $(x).\Phi x$ express a general judgement, asserting all elementary judgements of the form Φx. The expedient is not problematic in the *Principia* hierarchy of types, as "no such judgment can include itself in its own scope, since such a judgment is always of higher order than the judgments to which it refers." (RUSSELL; WHITEHEAD, 1910, p.49). Thus, in *Principia*,

$$\vdash \Phi(a) \qquad \text{and} \qquad \vdash (x).\Phi(x)$$

are different kinds of judgements entirely, conferring different senses to the assertion sign. Again, this expedient is rejected in Frege's original use of the sign, as only judgeable

[35]It would also made Socrates, Plato, Aristotle, and an infinite number of other objects as the constituents of the original judgement, which is also nonsensical.

contents (in the early logic) or truth-value names and markers can occur within the scope of an assertion.

Nonetheless, for Russell and Whitehead, the act of judgement is still a mental event. Russell clearly took a different perspective than the earlier propositionalism, where he rejected any psychological event in logic. The change of mind and the subsequent adoption of the assertion-sign was later criticized by Wittgenstein in the *Tractatus*. Wittgenstein considered that assertion was only psychologically relevant, and thus we should stick with propositions alone.

> Frege's 'judgement-stroke' '$\vdash$' is logically quite meaningless: in the works of Frege (and Russell) it simply indicates that these authors hold the propositions marked with this sign to be true. Thus '$\vdash$' is no more a component part of a proposition than is, for instance, the proposition's number. It is quite impossible for a proposition to state that it itself is true. (WITTGENSTEIN, 2002, 4.442)

First, Wittgenstein seems to posit that the assertion-sign functions as a truth-predicate. If this is the case, he missed the point. In Russell, truth is a property of judgements, not propositions, and so the assertion-sign confers unity to the proposition, it does not make truth a component or constituent of the proposition. It may make sense for Frege, where truth-values are the referents of concept-script names (in mature concept-script), but the judgement-stroke is not simply a truth-predicate, but an illocutionary force indicating device. Moreover, it is doubtful that the assertive force is a part of a proposition. Even if this position was held in the tradition, where there was no proper way to express a proposition without asserting it, the sole purpose of the judgement-stroke was precisely to make such distinction between a content, in Frege's case the nominalization in $—\Delta$, and its assertion as True, "$\vdash \Delta$".

Regardless, Wittgenstein is able to held such position following his picture theory in the *Tractatus*. Roughly, for Wittgenstein, propositions are able to express (more precisely, to project) facts of the world directly[36]. Thus, a proposition is made true according to the obtaining of the states of affairs [*sachverhalt*] that the proposition represents, or is a picture of. Moreover, to grasp the truth of a proposition is simply to grasp its sense, that is, to know whether it obtains or not. For that reason, Wittgenstein thought that the assertion was only psychologically relevant[37].

[36]The relation between the linguistic structure build in the *Tractatus* and its ontology is thought to be that of an isomorphism: every proposition with sense is the picture of a given state of affairs, and every state of affairs is expressed precisely by a single proposition.

[37]This point is also found in the section 4.063 of the *Tractatus*. See Proops (1997) on this topic.

According to Sundholm (2001, p.283-4), Wittgenstein is making an ontological reduction of rightness of acts of reasoning, and for that reason, he denied that “taking something to be true” had any place in such acts. Similar point was held by early Russell and Moore. Once we objectify contents and dissociate the asserted proposition from the assertive force, and if we can express a content’s truth directly, one might start wondering whether we really need the assertive force at all. De-pragmatization was taking shape.

In the Polish tradition of logic, judgements were also a tendency to be overcome. In the Lwów–Warsaw school, the influence of Brentano, through his student Kazimierz Twardowski (1866-1938), made judgements central for polish logicians. In fact, the polish tradition started with judgements and inferences, and ended up with interpreted sentences and logical consequences with Alfred Tarski, were formal semantics were finally developed in the 1930’s.

We can see this change happening directly with Jan Łukasiewicz. In the 1921 paper *Two-Valued Logic*, he still employed symbols for assertion and negation, influenced by Frege’s and Brentano’s theories (as he declared). But he does not seem to connect his assertions with any theory of judgements. Influenced by Frege, Bolzano and Brentano, Łukasiewicz also advocated against psychologism, and rejected judgements on these terms, claiming that logic “[...] is not a science of judgements or convictions, since that belongs to psychology” (ŁUKASIEWICZ, 1970, p.90).

What Łukasiewicz truly believed was that logic was “[...] the science of objects of a specific kind, namely a science of logical values” (*ibidem.*). He imported from Frege the notion that True and False were objects denoted by true and false sentences (truth as values), and also adopted a sign for “affirmation”. He also claimed to import from Brentano the idea of “rejection”, for denying truth. He clarified that :

> by assertion and rejection I mean the ways of behaviour with respect to the logical values, the ways known to everyone from his own experience. I wish to assert truth and only truth, and to reject falsehood and only falsehood. (ŁUKASIEWICZ, 1970, p.91)

Formally, with 1 and 0 denoting truth and falsity respectively (inspired by Schröder), affirmation and rejection were:

$$U : 1 \text{ (I assert Truth)}$$

$$N : 0 \text{ (I reject falsehood)}$$

Although Łukasiewicz took judgements to have only psychological interest, his early logic did used assertions in order to predicate truth for sentences. Here, Łukasiewicz logic is still pragmatized in our terms.

But the process of de-pragmatization was completed not much after. Subsequent logical works by Łukasiewicz did not use any sign for assertions or rejections. In 1928 paper *Elements of Mathematical Logic*, the influence of anti-psychologism was decisive, as he declared:

> Philosophical logic is not a homogeneous discipline, but contains issues of diverse content. In particular, it encroaches upon psychology when it refers not only to sentences in the logical sense of the word, but also to those psychic phenomena which correspond to sentences and which are called 'judgements' or 'propositions'. Combining logic with psychology is a result of an erroneous interpretation of the subject matter of logical research. (ŁUKASIEWICZ, 1963, p.9)

In his account, judgements and psychologism goes hand to hand, and should, therefore, be eliminated from logical investigations. Needless to say, not so long after that, Tarski would complete this process, following Łukasiewicz influence, and finally developing a mathematical treatment to semantics and the conception of truth.

5.3 The Shift to Logic-as-Object

What the anti-psychologistic authors helped to achieve was the common belief that both a proposition and its truth-conditions may be given prior, or regardless, of any epistemic attitude towards it. But this belief is not sufficient for making judgements and inferences entirely obsolete. Frege held similar position: the sense of a sentence, its Thought, is independent from judging agents and, just as in Bolzano, obtains or not independently. But linguistically, Frege and others saw no option for expressing Thoughts or Propositions as true without *asserting* it as such (more precisely, expressing the judgment that such Thought is true with the aid of a linguistic assertion). Much of the tradition was in fact unable to entertain a thought or propositions without asserting it in the first place, a problem avoided by Frege with the aid of the judgement-stroke. But the judgement-stroke is just a convenient way for making the assertive force explicit: any language with such sign is still making assertions. The other solution for the problem above is to avoid assertions entirely: every proposition is, linguistically, only entertained. If we want to still be able to separate true propositions from false ones, this must be done on another level, or on another language entirely: we mention propositions in one language, and predicate truth of them in another. What is needed then is a language in which we speak *on* (one where our assertions, but of different kind, may still occur), and one in which we want to speak *of*. The former is called a metalanguage, while the latter is the object-language.

The development and adoption of such distinction is a key moment in the history of logic. It was central in developing two major fields of research: proof theory and model theory. The aftermath of this revolution was the demise of judgements, inferences and the logic-as-practice view. The two key authors in this revolution was David Hilbert and Alfred Tarski.

5.3.1 Hilbert's Axiomatics and Metamathematics

David Hilbert's *Grundlagen der Geometrie* in 1899 (HILBERT, 1902b) is one of the pivotal moments for the metatheoretical perspective on mathematics. This is because Hilbert's axioms are not assertions about a given geometrical intuition, but sentences that, at least in principle, are devoided of meaning. To be fair, Hilbert does quote Kant in saying that all knowledge begins with intuitions (HILBERT, 1902b, p.1), but the five groups of axioms presented (connection, order, parallels, congruence and continuity) are just means to *express* [*ausdrücken*] "[...] certain related fundamental facts of our intuition." (HILBERT, 1902b, p.3) One should not take Hilbert's wording too strongly, though. There is no ontological presupposition in taking axioms to *express* something, as they are not suppose to be fixed into any domain of entities in advance. In fact, axioms are only capable of grasping or characterizing some domain if the they satisfy some structural properties. For instance, Hilbert begins his analysis in the following way:

> Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters $A, B, C, ...$; those of the second, we will call straight lines and designate them by the letters $a, b, c, ...$; and those of the third system, we will call planes and designate them by the Greek letters $\alpha, \beta, \gamma, ...$ (HILBERT, 1902a, p.3)

Thus, axioms will be statements about *systems of things*. The things considered are not, necessarily, the objects given by our geometric intuition, although we may call them as points, lines and planes as Hilbert did. In a letter to Frege dated from December 1899, Hilbert states that "instead of points, think of a system of love, law, chimney-sweep ... which satisfies all axioms" (*PMC*, p.42)[38]. He then continues, saying that

[38]The mathematician Blumenthal tells a similar story, that in a train to Berlin, Hilbert stated that "one must be able to say 'tables, chairs, beer-mugs' each time in place of 'points, lines, planes"'. (GRATTAN-

> Any theory can always be applied to infinitely many systems of basic elements. For one only needs to apply a reversible one-one transformation and then lay it down that the axioms shall be correspondingly the same for the transformed things (as illustrated in the principle of duality and by my independence proofs). (*PMC*, p.42)

In principle, the axioms are devoided of interpretation, and it does not matter if we use them to express points or chimney-sweeps. What does matter is what we can say about the *whole system*. It is only by proving facts about the whole system of axioms that we can say whether they successfully characterize the intended domain of entities.

In other words, Hilbert was inaugurating a model-theoretic approach to axiomatics. And so, the chosen axioms had to completely capture the relations between the intended *things*, by taking them to represent points, lines and planes[39]. To achieve this goal, the axioms had to be proven consistent as a set and independent from one another. To prove the consistency of the five groups of axioms, Hilbert constructed a model based on a set Ω of algebraic numbers. He then proceeded by interpreting points, lines and planes into numbers in such a way that all axioms were satisfiable[40]. Then, as Hilbert stated, "[...] it follows that every contradiction resulting from our system of axioms must also appear in the arithmetic related to the domain Ω" (HILBERT, 1902b, p.29). Given Ω's alleged consistency, the five groups of axioms are also consistent. Independency results followed from Hilbert defining five different geometries, each excluding one of the axiom groups, and then constructing a model for them[41]. The existence of a model attests for the consistency of each defined subgroup, as they define a geometry, or domain of objects, that satisfy the axioms. If all the axiom groups are disjoint on this terms, they are then independent from each other.

Hilbert's axiomatic method was so important that it paved the way for model theory to dominate mathematics and, specially, logic[42]. But it also paved the way for the distinction between a meta and an object-language. Notice that the main results presented are not geometrical. They are not about points, lines or planes, but about the system of axioms itself[43].

GUINNESS, 2000, p.208)

[39]And so the relations in questions were interpreted as 'are situated,' 'between,' 'parallel,' and so on.

[40]To satisfy all five groups of axioms, Ω must be defined as a set of real numbers. But Hilbert opted on the former for simplicity.

[41]A brief survey of Hilbert's strategy is found in (KNEEBONE, 2001), chapter 7.

[42]The idea, as Grattan-Guinness argues, was already present and used in non-euclidean geometries, but also on Boole's algebra of logic (GRATTAN-GUINNESS, 2000, p.211). But its development was largely improved by Hilbert.

[43]The Completeness Axiom, that Hilbert added to the second edition, is another example: it is meta-

Even if the distinction is not clearly posited in the *Grundlagen*, it certainly was in the following years. For the next three decades after, Hilbert kept the axiomatic investigation as one of the main centers of his work, which would result later in the Hilbert Program: the program of reducing all mathematics into axiomatic form. It was mostly carried out in the 1920's with the aid of Bernays, Ackermann, among others. Even though the program mostly ended after Kurt Gödel's incompleteness theorem, it has been influential enough for a number of subsequent developments, such as proof theory. One of the main goals of the program, derived directly from the *no ignorabimus* motto of the famous 1900's lecture[44], was the decidability problem, *i.e.,* whether there exists a finitary and effective method for deciding the truth-value of any mathematical problem. In order to achieve an answer, Hilbert thought that the concept of a mathematical proof should be investigated as well.

This lead Hilbert into the development of a metamathematics. The idea is the following. First, the language of mathematics, or what Hilbert calls "mathematics proper", is strictly formalized such that it "becomes a stock of provable formulae." (HILBERT, 1996, §79)[45]. It is within this stock of formulas that proofs are defined and investigated. Hilbert's definition is still how logicians define it today, and is worth quoting:

> A *proof* [...] consists of inferences according to the schema
>
> $$\frac{\begin{matrix}\mathfrak{S}\\ \mathfrak{S}\to\mathfrak{T}\end{matrix}}{\mathfrak{T}}$$
>
> where at each stage each of the premises [...] is either an axiom, or results directly from an axiom by substitution, or agrees with the end-formula $\mathfrak{T}$ of an inference that occurs earlier in the proof, or results from such an end-formula by substitution. (HILBERT, 1996, §57)

This is the now usual syntactic definition of a proof as a finite sequence of formulas. It states that any formula is provable, in the context of a given axiomatic system, if it is either an axiom or results from the application of any inference rule to other provable formulas of the system. This definition made it possible to treat proofs as objects of study, the basic assumption for proof theory.

theoretical axiom, as it is not about the system of objects of the theory, but about the other axioms.

[44]"This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus" (HILBERT, 1902a, p.445).

[45]For the assessment Hilbert's metamathematics, I will follow mostly the 1922 report *The New Grounding of Mathematics*, based on Hilbert's lectures in that year.

In order to study such proofs, Hilbert defines a second mathematics, or metamathematics. Proposing it to be a contentual [*inhaltliche*] number theory, the metamathematics is only composed of concrete signs, or number tokens, such as

$$1, 11, 111, 1111, 11111$$

and so on. This is the finitary point of view: no infinitary assumption is made in the metamatemathics. In fact, no assumption whatsoever is made about them, besides the fact that they are primitive signs, "extra-logical discrete objects, which exist intuitively as immediate experience before all thought" (HILBERT, 1996, §25). The flavour here is clearly Kantian, and Hilbert in fact assumed this background[46].

The objects of the metamathematics are the signs themselves, and thus, no ontological pressuposition is ever made, as they have no meaning [*Bedeutung*] (HILBERT, 1996, §29). Hilbert also added that other signs can be defined for purposes of communication, such as $2, +, =, <$ and so on. These are only abreviatory devices that ultimately refer to the primitive sign-tokens. The actual theory consists essentially of these primitive undefined signs. In conclusion, in the contentual number theory, which Hilbert's takes as the metamathematics, "there are no axioms, and no contradictions of any sort are possible. We simply have concrete signs as objects, we operate with them, and we make contentual statements about them" (HILBERT, 1996, §31).

It is in the context of the contentual number theory (the metamathematics) that the proofs of the mathematics can be studied and, as Hilbert hoped, proved to be consistent. Even though the project failed, it brought about proof theory as a direct result. But it also solidified the metatheoretical perspective, as many important results followed from the direct influence of Hilbert's metamathematics. For example, Bernays 1918 proof for the semantical completeness of the propositional calculus of the *Principia Mathematica*, Gödel's proof of the completeness of first-order logic in 1929, and for better or worse, Turing and Church's negative answer to the decision problem and Gödel's incompleteness theorem in the 1930's[47].

In the tradition of logic-as-practice, the notion of judgement is central. It is through judgements that both truth and inference enter the logical practice. A true proposition is just an asserted proposition (or at least known as true following an assertion). An inference is a sequence of assertions. A deduction, under this tradition, is not

[46]Which is more clearly assumed in, for example, the 1925 lecture *On the Infinite* (HILBERT, 1967).
[47]For more on Hilbert's program, see (ZACH, 2007).

detached from the epistemic role of assertions. Hilbert's metamathematics, on the other hand, is a direct formalization of deductions without assertions or judgements. His formal approach allowed to treat deductions as formal transitions irrespective of its semantical value. It is the de-pragmatization of the first aspect in which judgements are mainly employed: inferences.

This de-pragmatization, however, follows the metalanguage - object-language distinction. Ultimately, it is possible to formalize proofs in the object-language given that reasoning is operated within the metalanguage. In Hilbert's case, the metamathematics. Thus, there are some interesting assumptions about Hilbert's de-pragmatization that we shall consider.

Reasoning occurs mostly in the metalanguage: As we saw, the purpose of the metamathematics is to warrant the deductive process that takes place in mathematics proper. Hilbert's goal was "[...] to eliminate, once and for all, the general doubt about the reliability of mathematical inference" (HILBERT, 1996, §1). His choice of metamathematics, the contentual number theory of primitive tokens, is a simplified theory. But the reasoning performed within such theory is not completely formalized, given its primitive character. In fact, Hilbert demanded that

> [...] these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else (HILBERT, 1996, §25)

As a result, the most basic manipulation is the "[...] construction and deconstruction of number-signs" (HILBERT, 1996, §31), or the concatenation of tokens. Since this procedure is too simple for achieving higher results, the metamathematics is further enhance with abbreviations and other signs that ultimately refer to the primitives. Nonetheless, the reason for taking the metamathematics as such is to reduce any informality of the object-language, the mathematics proper, to a minimum, taking away much of the pragmatic features. However, the metamathematics, being a primitive and intuitive realm, is not de-pragmatized.

Proofs became purely syntactical elements: The most important formalization performed is on proofs. By considering a proof as a sequence of formulas of the language, one is disregarding the fact that proofs are actions[48]. Nothing is preventing one to operate within the object-language, but ultimately, if we can analyse the proofs in the metalanguage, it means that their realization is only a secondary feature. Thus, proofs became primarily objects of a formal language and only secondarily an human activity[49].

Recall also that Hilbert's definition of proof follows the inference schema, the *modus ponens* rule,

$$\frac{\begin{array}{c}\mathfrak{S}\\ \mathfrak{S} \to \mathfrak{T}\end{array}}{\mathfrak{T}}$$

With this schema, no actual interpretation is provided for the horizontal line, usually taken as an indicator for the inferential step. Here, the German letter $\mathfrak{S}$ and $\mathfrak{T}$ are meta-variables for formulas. Given that in the metalanguage one is able to quantify over formulas, the inference scheme is, in fact, an infinite conjunction of sequence of formulas in the inferential structure defined. Thus, there is no need for taking the inference as a rule, given that all such instances are available by defining it as a schema. Generally, by taking proofs as "something concrete and displayable" (HILBERT, 1996, §59) Hilbert is objectifying something that, at least up to Frege, was still an activity warranted by normative rules of reasoning.

These two points made it clear how, in the context of the hilbertian metamathematics v. mathematics proper, the pragmatic phenomenon remained in the metamathematics (at least most of it), while yelding a de-pragmatization of the language of mathematics as an object-language.

The acomplishments of Hilbert's metamathematics was possible following his model-theoretic approach already developed in 1899 with the *Grundlagen der Geometrie*, a completely different approach from that o Frege: with no metatheory (and no metalanguage in the rigorous sense), Frege saw no option than to take the proper object-language as a performative language. Hilbert, on the other hand, keep the metamathematics enough pragmatized in order to objectify and study one of the most active components of the logical practice: proofs. In a sense, Frege was doing epistemology *in* the object-language, while Hilbert was doing epistemology *about* the object-language.

[48]I am, of course, referring to proofs *in* the object-language, that from the metamathematical standpoint are just sequences of formulas.

[49]This very idea became subject of criticism much later with Lakatos' revitalization of the mathematical practice as a proper subject of philosophy of mathematics, in (LAKATOS, 1976).

5.3.2 Tarski's Conception of Truth

Hilbert formalized the concept of proof, resulting in a de-pragmatization of that notion in the object-language. The other important concept, *i.e.,* truth, was formalized by Alfred Tarski's semantical conception in the 1930's. Tarski's conception is more detailed in the 1933 article *The Concept of Truth in Formalized Languages*, but a summary was later published in 1944 as *The Semantic Conception of Truth: and the Foundations of Semantics*[50].

Following Hilbert's metamathematics, Tarski's semantic definition of truth highly depends on the distinction between an object-language and a metalanguage. This nomenclature is mostly due to Tarski's papers, but it also made other logical tools much more usual, *e.g.* the use of quotations, the use-mention distinction, among others[51].

From a formal point of view, the truth predicate offers some tricky situations to be dealt with. Frege had already recognized that the predicate cannot be defined without circularity, as we already saw. But Tarski is also worried about the paradoxes that such circularity may produce. For instance, let's assume two things: First, that our language, formally or not, is embedded with the truth-predicate. Second, that all true sentences are defined through the truth-schema (T'):

(T') 'p' is true if, and only if, p.

wheres p is a sentence. (T') is an equivalence relation that obtains between the sentence p and its name, 'p'. Here, the quotation marks are used to distinguish between asserting a sentence, as in p, from mentioning a sentence, as in 'p'. Given that truth is something predicated of sentences, we must refer to the name of a sentence in order for truth to be predicated of that sentence (TARSKI, 1944, §4). Also, given that Tarski (1944, §3) is assuming the Aristotelian correspondence theory of truth, truth is obtained through an equivalence between a mentioned sentence, 'p', and the state of affairs that it refers, that is, p.

Having these conditions established, we can derive the Liars Paradox, as Tarski (1944, §7) did, but with a slight difference on the presentation. First, consider the sentence:

[50] In (TARSKI, 1944). My assessment will be mainly from this source.

[51] To be fair, the use-mention distinction and the use o quotations for such purpose were introduced by Frege in (*PW*, p.260). His original purpose was to use such a tool in the expository language in order to mention formulas and not commit to his assertion. Tarski's usage is very similar, as means to mention formulas of the object-language. But Tarski's metalanguage and Frege's expository language are not exactly the same. Their employment of the use-mention distinction, thus, deviates on purposes.

(1) The sentence printed in the line 24 of the page 227 is not true.

Now, let this sentence have the name 's' for short. We then have that

(2) The sentence printed in the line 24 of the page 227 is 's'.

Following the schema (T'), we have

(T') 's' is true if, and only if, the sentence printed in the line 24 of the page 227 is not true.

Finally, from the equality on (2), we conclude that:

(3) The sentence printed in the line 24 of the page 227 is true if, and only if, the sentence printed in the line 24 of the page 227 is not true,

which is a contradiction.

The main problem in the assumptions above is what Tarski (1944, §8) called a *semantic closed language*, that is, a language that is closed to the predicate "is true". This is an obvious assumption for ordinary languages, but one that must be abandoned in order to avoid the paradox in formalized ones. Tarski's solution followed Hilbert: a distinction between an object-language and a metalanguage, that is, the language that is "talked about" and the language we use to "talk about", respectively (TARSKI, 1944, §9).

The idea is to take the truth-schema (T') as a sentence of the metalanguage, say $\mathcal{L}^*$, and 's' a sentence of the object-language $\mathcal{L}$. Thus, the truth-predicate does not occur in the object-language, only in the metalanguage, while being defined only for sentences of the former. It follows that the object-language is necessarily contained in the metalanguage, or it is at least translatable, as every sentence 's' of $\mathcal{L}$ must also be a sentence of $\mathcal{L}^*$ or it must have an interpretation. Finally, (T') is weakened such that:

(T) 'p' is true in $\mathcal{L}$ if, and only if, p.

Which blocks the liars paradox.

The precise definition for 'p' being true in $\mathcal{L}$ is still missing[52]. Tarski defines it in terms of *satisfiability* between sentencial functions and objects in a recursive manner. The resulting definition is the usual model-theoretic definition still in use today[53]. The first step is to define satisfiability for sentential functions with arbitrary number of free variables. First, let us consider the simplest case. For sentential functions with one variable, the following schema suffices (TARSKI, 1956, p.190)[54]:

[52]We will now follow Tarski's main article on the subject above mentioned.
[53]For example, Mendelson (2015) follows Tarski's definition directly.
[54]With a slight modification for Tarski's own presentation

> for all a, a satisfies the sentential function "$f(x)$" if, and only if, $f(a)$.

Nothing difficult here. Things however get more complex when we have to consider sentences of arbitrary number of free variables. For it, instead of saying that a given object a satisfies a given sentential function, Tarski (1956, p.191) now consider "a given infinite sequence of objects [that] satisfies a given sentential function." The idea is fairly simple: we consider infinite sequences of objects $(a_1, a_2, ...)$ and for each free variable in the sentential function $f(x_1, ..., x_n)$ considered, we match each $a_{i \leq n}$ to each $x_{i \leq n}$ with the same index. Thus, as in (TARSKI, 1956, p.192), we have that:

> a sequence $(a_1, a_2, ...)$ satisfies the sentential function "$f(x_1, ..., x_n)$" if, and only if, $f(a_1, ..., a_n)$.

Where $f(a_1, ..., a_n)$ is the sentential function "$f(x_1, ..., x_n)$" interpreted in the metalanguage. Notice that we only has to consider the first n elements of the sequence, given that only they contribute in making the interpreted sentence true or not. The extreme case, as Tarski (1956, p.194) explains, is sentences that has no free variables. In this case, a given sentence 's' is either satisfied by every sequence or no sequence at all.

If we consider these sentential functions to be what we now call atomic formulas, we would continue to define the notion of satisfatiability recursively for the remaining complex formulas, depending on the language $\mathcal{L}$ considered. For example, if $\mathcal{L}$ has $\{\rightarrow, \neg\}$ as signature, and for s a sequence, we have:

1. a sequence s satisfies $\neg\varphi$ if, and only if, it does not satisfy φ.

2. a sequence s satisfies $\varphi \rightarrow \psi$ if, and only if, it either satisfy $\neg\varphi$ or satisfies ψ.

3. a sequence s satisfies $\forall x_i \varphi$ if, and only if, every sequence that differs in the ith position from s satisfies φ.

Having the notion of satisfiability of sentences formally given, Tarski (1956, p.195) finally defines truth as the following:

> p is a true sentence if, and only if, p is satisfied by every infinite sequence of objects.

This solution is clearly what we now call a model-theoretical one[55], as it will depend on the domain set of objects that one considers as an interpretation for the language

[55]Nowhere I intended to offer a thoroughly account of Tarski's semantic conception from a technical standpoint. But since his conception is still influential in modern-day textbooks, the reader may consider Mendelson (2015) for a technical updated version.

$\mathcal{L}$[56]. Nowadays, one would first define a syntax, where the object-language is rigorously defined, to then define a semantics, where the goal is to study the models of the theory, sets of first-order sentences. A model is, simply put, an interpretation that makes every sentence true. An interpretation is a pair $\langle D, \cdot \rangle$ where D is a domain of objects and $\cdot$ an interpretation function that assign for each individual constant an object of the domain, and for each predicate a subset of the domain. The rest is fairly tarskian, as we would simply add that 'p' is a true sentence in given model M, formally $\vDash_M p$.

For our purposes, Tarski also contributed to the de-pragmatization of logic, just as Hilbert did, but in his case on the concept of Truth.

De-pragmatization of truth in the object-language: From the standpoint of, and relative to any object-language, the notion of Truth is completely de-pragmatized with Tarski, given that every true sentence would be decided formally according to the interpretation provided in the metalanguage. Here I'm mostly referring to abandonment the act-based solutions for the notion of Truth. In Frege's case, the assertability of a given content is strictly related to its provability, and provability was still explained in terms of assertability, which relies on the pragmatic features of the language. With Tarski, none of that is required: one can simply predicate truth of a sentence according to higher syntax of the metalanguage. This is possible following the complete disassociation of contents from assertive force in the object-language. We can just have a formal language and study its models from a higher stand-point. Nothing is judged *in* the object language, as truth is only predicated from the outside. With a tarskian consequence, one is much more concerned with forms of valid reasoning than it is with sound arguments, a position that is very well summarized in the very beginning of Mendelson's textbook:

> The truth or falsity of the particular premises and conclusions is of no concern to logicians. They want to know only whether the premises imply the conclusion. The systematic formalization and cataloguing of valid methods of reasoning are a main task of logicians. (MENDELSON, 2015, p.xv)

Tarski's definition was pivotal in turning logic from a reasoning device to an object of mathematical studies. And this change includes the complete detachment of judgements from the object-language. Once again, if compared with Frege's own conception, Tarski formalized the truth-predicate outside the object-language, while Frege, to whom this option was not available (following his universal conception of logic), had to treat it as

[56] To be fair, in both articles here considered, Tarski does not speak about "truth in a model" or "truth in a structure", although coming close to it. For an account of this missing notion, see Hodges (1986).

an illocutionary device *in* the object-language (or the auxiliary language, as he called the concept-script).

The relativity of Tarski's formal definition: We saw earlier how Frege, Peano and Russell, not operating with a clear object-language v. metalanguage distinction, had to assume that certain basic primitive concepts were introduced without a clear-cut definition, which forces some to be treated pragmatically. Tarski's solution for such problem cannot be thought to be absolute, but only relative to the languages in question. This follows from two presuppositions. First, the primary condition for a metalanguage is to be "essentially richer" (TARSKI, 1944, §10) than the object-language. If it isn't richer, then, as Tarski argues:

> with any given term of the metalanguage a well-determined term of the object-language can be correlated in such a way that the assertible sentences of the one language turn out to be correlated with assertible sentences of the other. As a result of this interpretation, the hypothesis that a satisfactory definition of truth has been formulated in the meta-language turns out to imply the possibility of reconstructing in that language the antinomy of the liar; and this in turn forces us to reject the hypothesis in question. (TARSKI, 1944, §10)

As Tarski concludes above, without an "essentially richer" metalanguage, one cannot assume that the truth-predicate is well defined for the object-language without importing the Liar's Paradox in the metalanguage as well. That is, even if the metalanguage is not a semantically closed language *per se*, without "essential richness", both languages will compose a semantically closed pair of languages. This reasoning, together with the fact that one cannot rely on semantically closed languages for any formal definition of truth, yields the second point as a conclusion: the hierarchy of languages.

Tarski (1944, §9) do recognize the existence of an hierarchy of languages, but only loosely. What we are concluding is that such hierarchy is inevitable if any formal correct and materially adequate conception of truth can be achieved for a given object-language without the risks of antinomies. Any cross-interpretation between languages would have semantic-closeness as consequence. Thus, one would never get an absolute definition for the truth-predicate in this way.

Given that an infinite number of languages is required for keeping Tarski's solutions consistent, not resulting in a chain of semantically closed languages, it is questionable whether at some point, no primitives are invoked. But this is certainly the case in regular logical practices on these days. We saw that the truth-schema depends on Tarski's definition of satisfiability. In the simplest case:

a satisfies '$f(x)$' if, and only if, $f(a)$.

Questions about the truth-conditions for the bicondicional in the metalanguage, for example, are hardly presented. Moreover, we would also need a proper understanding of the meta-satisfiability for the right-side of the bicondicional, where "$f(a)$" is being asserted as true. This, of course, would be answered in a richer metametalanguage, particulary one of higher order. But in practice, we simply tend to stop the chain of justification. It is worth repeat a quotation from Frege concerning the undefinability of the truth-predicate:

> in a definition certain characteristics would have to be specified. And in application to any particular case the question would always arise whether it were true that the characteristics were present. So we should be going round in a circle. (*DG*,p.353)

Here we can see how Frege's worries can be directly aplied to Tarski's solution. Where Frege writes about characteristics to be specified, one reads Tarski's bicondicional in the metalanguage or the meta-satisfiability that the (T) schema depends on. Where Frege writes "going round in circles", we would read Tarski's infinite hierarchy of languages. Of course, Tarski's solution is not circular. But the chain of languages would never end, and thus, never result in a complete definition for *the* concept of truth, just as a circular definition would.

Nonetheless, and in conclusion, Tarski's semantic conception of truth was, and still is, influential. The whole agenda for logical research became tarskian, based upon the object-language/metalanguage distinction and his semantic conception of truth. It helped growing and turning model theory into a substantial and independent mathematical and logical field, continuing the steps made by Hilbert before.

5.3.3 De-Pragmatization Completed (?)

We come full circle in realizing where, and how, de-pragmatization took place. First, we saw that logical tradition was highly pragmatized, defined in terms of mental activities with the correspondent linguistic expressions. Logic was primarily an activity. Second, we saw that gradually the content of such acts became objectified and the act of judgement only secondary. Frege is the key logician in this moment. Finally, Hilbert's metamathematics and Tarski's development of formal semantics made it possible to take interpreted sentences as the main logical objects and to take logical consequence over inferences as representatives of the logical activities. This was only possible in adopting a metalanguage as means to treat sentences without having to necessarily asserting them.

This shift meant roughly the following transition:

(1) S is P

(2) $\vdash F(a)$

(3) $\models_{\mathcal{M}} F(a)$

(1) is simply any judgement in ordinary language having the structure that "S is P", with the copula expressing the unity, or the assertive force, of the judgement. (2) is roughly the Fregean-Russell rendering of the same propositional content, with the function-argument structure. But now the assertion-sign confers the assertive force, sharply separating asserted from unassarted contents. Finally, (3) is the metatheoretical assertion that $F(a)$ is true in the model $\mathcal{M}$, as it is usually taken today. Notice that both (1) and (2) are assertions taken in the object-language. (3), on the other hand, is an assertion in the metalanguage affirming the satisfiability of $F(a)$ under model $\mathcal{M}$. The shift of levels in (3) made it possible to treat logic as a mathematical object, precisely because the de-pragmatization of the object-language. But assertions, and other pragmatical phenomena, were not fully eliminated, as they are now being used in the metalanguage.

Frege's participation on this shift is notorious. He shared with Bolzano the objectification of contents, which at first sight would demise judgements altogether. But his conception of logic could not allow him to make the step further: from his universalist conception of logic, and the lack of a formalized metatheory, there was no way to predicate truth of Thoughts directly. Hence, Frege's decision was to keep it in the level of illocutions only, adopting the illucutionary indicating device for an assertion, *i.e.,* the judgement-stroke. This puts Frege right in the middle of the de-pragmatization of logic shift. His logical system was caught between a dying tradition and the new mathematical logic. But since he played on both fronts, there is no other option than conferring to him part of the blame.

But more than just taking Frege as a curious case of this historical path, I believe there is more to be learned from it. As claimed above, even in Hilbert's metamathematics and Tarski's hierarchy of languages, a logical language is still something to be manipulated and used by an user. Textbooks on logic are usually written using a hybrid metalanguage that includes both English (or any other spoken languages) and a handful of formal devices. Usually, the metatheory is defined using set theory as background: proofs are ordered sets of sentences; models are sets of individuals equipped with an interpretation function. But even this doesn't prevent the language to be enriched with pragmatical phenomena, as the metalanguage contains assertions, definitions, inferential steps, and everything needed for logical practice. The same goes for the mathematical language, as Ruffino, San Mauro and Venturi (2020, 2021) showed. The authors conclusion, that the language used in mathematics has many pragmatical features and that cannot be reduced to literal meanings, can be applied, *mutatis mutandis*, to the logical language as well. But the interesting case is that Frege's conception of logic is already a pragmatical, *viz.,* performative, conception. In the next chapter, the goal is to analyze Frege's concept-script logic linguistically. Particularly, the aim will be to read it with the aid of Speech Act theory.

Nonetheless, the goal of this chapter was twofold. First, to show the depragmatization of logic through its history, by tracing its most basic component: judgements. Second, to place Frege in this history. Frege was an interesting case in depragmatization: he was one of the last logicians to truly adopt judgements within logic. It is without any doubt that Frege's logic was *sui generis* at his time, and still is from our perspective. The fact that the concept-script was not a metatheoretical perspective was not only a consequence of his universalist conception of logic, but also the pride of place that judgements had in the concept-script, or as we may say, the *Urteilsschrift*, to use Schröder recommended name.

The moral of the story is that Frege's employment of illocutions were clearly connected with his own conception of logic. The very fact that logicians forgot about such phenomena can be explained with the birth of the metatheory in the twenties (and the death of a purely philosophical approach to logic). Even though Frege is highly influential in the development of mathematical logic, his system was also philosophical in the logic-as-practice manner. Without any doubt, contemporary logic is not Fregean in this sense.

Chapter 6

A Logic of Speech Acts

In the previous chapters, it was argued that Frege's logic, the concept-script, is not a purely descriptive language. It includes non-descriptive signs, the judgement-stroke, the definition-stroke and finally, the inferential lines and other signs for making transitions from judgements to judgements. These are all performative signs, they do not refer or denote objects, but express actions.

But having such performative features is not an obvious condition to accept for formal languages. If we recall Frege's own worries, and even the worries of other logicians at the time (Boole, Peano, Russell), the idea of having a formal language was precisely to overcome the deficiencies of spoken language. Frege himself seems to rule out all such features in saying in the *Begriffsschrift* that

> [...] all those peculiarities of ordinary language that result only from the interaction of speaker and listener — as when, for example, the speaker takes the expectations of the listener into account and seeks to put them on the right track even before the complete sentence is enunciated — have nothing that answers to them in my formula language, since in a judgment I consider only that which influences its *possible consequences.* (*BS*, p.12)

But this passage only shows that Frege was concerned with what a speaker may consider in judging. And his answer is that judgements are only about conceptual-contents, where these contents are only individuated in the context of logical consequences. Judging is still an act that logic cannot (in his perspective) discharge with.

Nonetheless, this idea that all features concerning the speaker or the listener are irrelevant in formal languages does find its place. We saw, for example, Carnap's syntactic approach as "an investigation which makes no reference either to the speaker or to the designata of the expressions" (CARNAP, 1958, p.79). The idea of a formal language as formal includes a de-pragmatization as means to cancel out our faulty human influence on logical reasoning.

Even in formal languages in the de-semantification sense, the purposes of its symbols are, ultimately, to be embedded with a semantics. We could say that formal languages of logic are, in this sense, *descriptive* by nature, as their most important symbolic elements are denotative. Whether one wants to proceed from syntax to semantics or not, the main structure of languages is that they are concerned primarily with the expression of contents. Logic is such a language, and this is what is meant in saying that it has a descriptive nature. But there are other important linguistic functions that goes in the development of a formal language, if it is thought as a language at all. And if it is thought to be used, it will include different aspects and conditions about the manipulation of its symbols and their communicative role, such as utterances, propositional attitudes, implicatures, and much more.

A similar situation happens with the language of mathematics. As it is usually perceived by its practitioners, mathematics is a descriptive science. It aims to describe what mathematical objects are, how they behave, what holds for them and what doesn't, through a language of written signs. As G. H. Hardy famously said,

> I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations. (HARDY, 1992, §22)

In this perspective, there is a risk in taking pragmatic phenomena in the language which we use to 'discover' such entities. Mathematics offers a restrict case ontology-wise, as it is commonly believed that its objects must be invariant and independent of the mathematician actions that guides into discovering theorems. If we are to take such account seriously, there should be no relevant normative role in the language of mathematics. This can be also said, *mutatis mutantis*, to formal logic as no one expects that the correctness of a deduction to be by any means influenced by reasoners actions. Whatever is responsible for the soundness of a inference, it must be reasoner-invariant.

But this is not enough for ruling out all all pragmatically driven features of a formal language. A logician might guide his choice of symbols purely in pragmatic terms, or a system of axioms by their advantage in one's goal, or prove a lemma for the perspicuity that it provides in visualizing the proof of the main theorem. This is even more striking in mathematics, where we found mathematicians acting freely upon a given subject by postulating new entities and finding creative ways in solving problems.

The problem then, is on how to address these seemingly incompatible views. In the case of mathematics, by asserting the existence of an independent and objective mathematical reality, mathematicians creativity is highly constrained, and their practice very much diminished. But at the same time, the objectiveness of their results, their validity and universal applicability requires a non-relativistic reading. In the case of logical languages, the demands for objective correctness of the deductive reasoning circumvents any pragmatical condition. Soundness cannot be something dependent on what an agent does. In both cases, a compatibility problem ensues.

With the development of formal languages, the problem of compatibility was commonly overlooked, not only in regard to the role of a formal language, but in the philosophy of logic and mathematics as well. Pragmatic features, such as the mathematicians creativity, were taken as secondary, usually not to be accounted for. Often they were considered only of heuristic interests, being dwelled in psychology[1]. For example, the problem whether logic and mathematics are informative seemed to be common up to Frege, but was latter forgotten after being accused by Wittgenstein and the Vienna Circle as only psychological curiosities, as they were inclined in taking logic and mathematics as analytic, and therefore, limited to tautologies. The risk in putting logic and psychology close together is an old tale: that any result followed could only be accommodated with a relativistic epistemology and ontology, *i.e.*, a relativistic notion of truth and existence in logic and mathematics, whereas both would be empirically subject to mathematicians will.

In general, logic and mathematics have forgot about the practice. In the case of mathematics, Mancosu (2008) describes such neglect as a feature of the foundational programs in late nineteenth and early twentieth century. Here, philosophy of mathematics were predominantly *foundations* of mathematics, and within it, very little was left for the practice itself. One could take logicism (in its many variants) as the position mostly prone to such neglect, given that mathematical logic was, and still is, a canon for doing philosophy of mathematics. It was only in the 60's, starting with Lakatos and others, that a "Maverick" position started to account the practice as equally important. It started with the recognition that, as Mancosu (2008, p.4) describes,

> Mathematical logic, which had been essential in the development of the foundationalist programs, was seen as ineffective in dealing with the questions concerning the dynamics of mathematical discovery

[1]Again, Carnap can be mentioned: "the words and expressions of a language have a close relation to actions and perceptions, and in that connection they are the objects of psychological study." (CARNAP, 1937, p.5).

and the historical development of mathematics itself.

This is certainly true. But the mathematical logic dominant today is not simply the same as those systems of logic that started the formalization program, such as Boole, Frege, Peano, Russell and others. It is not at all clear that, from the beginning, formal logic was entirely oblivious to mathematical discoveries. Frege, as we saw, was a clear exception.

Either way, that mathematics is anything but practice-oriented is not a rare opinion to find. It was to this tendency that the Mavericks in the philosophy of mathematics had rise against. Some authors went even further in deriving a philosophical position where the ontology of mathematics is dependent on the practice. This are the social-constructivists such as Hersh (1997), Ernest (1998). But adding to this trend, recent works has been challenging the idea that the language of mathematics is purely descriptive or literal, such as Ruffino, San Mauro and Venturi (2020). They argue that the mathematical language is embedded with pragmatic phenomenon. This is trivially true in the case of mathematical communication in general, where we often find different pragmatic features — *e.g.* metaphors, rhetorical figures, irony, etc. — that are employed in lectures, conversations, the explanation of ideas, and many other mathematical activities. But the authors also argue that even written mathematical texts often contain expressions that are hard to account without any appeal to pragmatics. This goes for simple auxiliary phrases that encode propositional attitudes (*e.g.* "we believe that"), directions (*e.g.* "suppose that"), assertions (*e.g.* "it is the case that"), among others, but also for formal devices. Even quantifiers, as they argue, seem to be context-sensitive. There is substantial evidence that "[...] literal meaning is simply not sufficiently fine-grained to encode all possible shades of meaning provided by different mathematical context" (RUFFINO; SAN MAURO; VENTURI, 2020, p.5), they conclude.

Some developments in philosophy of mathematics more focused in the practice are also considering the importance of Speech Acts in mathematics. Something that is obviously present in the practice, speech act components can be used in explaining those pragmatic features of mathematical language, either formal or informal. For instance, Cole (2013) offers a version of social constructivist that incorporate speech acts, specially declarations, in order to defend the thesis that mathematical domains are institutional entities, much in the sense of Searle (1995). In this account, one consider the mathematical practice as substantially important in creating such entities. Cole argues that such an account is able to maintain the objectivity and necessity of mathematical truths[2], thus reconciliating mathematical practice with a weakened form of realism, the Practice-Dependent Realism. It maintains that mathematical objects are not only existent abstract entities, but have objective properties and epistemic independence from the practice, even though it is not ontologically independent from it. Regardless if his social constructivism is able to accommodate mathematical ontology or not, Cole had the interesting insight of looking to mathematical practice from the perspective of speech act theory, but he comes short of analysing also mathematical language under such lens, an analysis found in Ruffino, San Mauro and Venturi (2021). Their idea is to read the mathematical language with Searle and Vanderveken's taxonomy of illocutionary acts, following Searle and Vanderveken (1985) and Searle (1979).

The core idea is that there are hidden illocutionary force indicating devices in mathematical texts that are important features of how mathematicians not only communicate their findings, but also structure their reasoning. This is clear if one considers that

1. they are commonly built upon affirmative statements (theorems, lemmas, corollaries) that can be analysed from a variety of assertive illocutionary forces;

2. they also made extensive use of definitions that are uttered with a declarative illocutionary force, and that

3. reasoning is performed using inferential rules, which can be read as having a directive illocutionary force in play.

Surprisingly enough, this same structure is found *mutatis mutandis* in Frege's concept-script, as Frege already had recognized and implemented illocutonary devices in the late nineteenth century. The authors did recognize Frege's influence, but it is the task of this chapter to build from their claims, and to develop how Frege's logic can be read on Speech Act terms.

Frege is well accepted as a major figure in the rise of philosophy of language as a mature field of philosophical inquiry. His semantic theory and the famous sense-reference distinction had an enormous impact in studies of language phenomenon. But the performative reading here undertaken shows another part of his influence, in showing how language, aside from its descriptive function, can be used to perform different sorts of

[2]The compatibility problem between the practice and mathematical ontology is his starting point in (COLE, 2009).

actions. But Frege's strong platonism oftenly overshadows this aspect of his formalism[3]. That he has defended a strong platonism towards arithmetic does not take away the fact that the concept-script was deeply practice-oriented. And this, I believe, is a testament for the fact that taking the language of logic or mathematics as embedded in illocutionary devices has nothing to do with one's ontological commitments[4]. Another consequence is that Frege's concept-script cannot be read as a formal language in the de-pragmatized sense described earlier. Therefore, Lakatos' claims that mathematics became too crudely formal, in the sense established by Hilbert's Program, cannot be read back to Frege[5].

We already saw in previous chapters how Frege's logic is based on three types of acts, that received special symbols in the concept-script for expressing the illocutionary forces relative to them. In this sense, the concept-script can be read as a performative language, and its illocutionary devices can be further analyzed. In discussing the possibility of reading Frege's logic as a "science of the assertoric force", Ruffino (2006, p.39) claims not to be clear what such a science would amount to. But he raised the possibility of such a science being "perhaps something like a speech act theory". If this chapter is successful, this possibility will be clearer.

For that matter, I will first introduce Speech Act Theory in three parts. In 6.1, on how its development was a answer to the dominant descriptivism in philosophy of the Vienna circle. Then, in 6.2, I will quickly restate John Austin's seminal work in response to descriptivism. Then, in 6.3, I will detail Searle and Vanderveken's taxonomy of Speech Acts and its components. They will be further used in explaining how Frege's logic can be read following their taxonomy, in 6.4. Particularly, and as expected, I will read assertions, definitions and inferences using the taxonomy of Searle and Vanderveken. This will be mostly a novel reading, and highly inspired on Ruffino, San Mauro and Venturi's (2021) analysis of speech acts in the language of mathematics.

6.1 Departing from Descriptions

Frege's insight went unnoticed and philosophy of language developed itself giving privileged position to descriptive statements. This was the case in Wittgenstein's *Tractatus Logico-Philosophicus* (2002), as he rendered language entirely out of propo-

[3]In the eyes of the Mavericks, such as Hersh (1997, p.143), Frege was the grandfather of the mainstream, which he certainly was.

[4]Contrary to Cole's account of a practice-oriented ontology as in (COLE, 2013). For instance, the speech act analysis in Ruffino, San Mauro and Venturi (2021) is not ontologically oriented. Similarly, Schmidt and Venturi (2021) also developed an ontologically neutral reading of axioms as speech acts.

[5]See, for example, the introduction to (LAKATOS, 1976).

sitions and its descriptive nature. We can grasp Wittgenstein's case in the following passages:

> A logical picture of facts is a thought. (3)
>
> In a proposition a thought finds an expression that can be perceived by the senses (3.1)
>
> A proposition is a picture of reality. (4.01)
>
> The totality of propositions is language. (4.001)

In Wittgenstein's first theory of language, the proper function for propositions is to denote state of affairs, and language is thought simply as the totality of propositions. Since propositions and facts match together from their logical structure, so does the language and the world.

We already mention Carnap's emphasis on syntax. We can also mention his verificationism as another key moment for such neglect of language's performatives. In the *Elimination of Metaphysics Through Logical Analysis of Language*, and following Wittgenstein, a strong restriction upon meaningful propositions was employed: "A statement asserts only so much as is verifiable with respect to it. Therefore a sentence can be used only to assert an empirical proposition, if indeed it is used to assert anything at all" (CARNAP, 1959, p.76). Thus, in Carnap's logical analysis, being meaningful equates in being verifiable, which in turn equates in having a distinct truth-value. This is too strong. In one sense, a lot of the linguistic phenomenon is left behind by not being capable of having truth-values. On another, entire fields in which those non-descriptive statements are relevant would be simply ignored, for they are not simply verifiable in Carnap's terms. For example:

> the same [...] must be passed on all philosophy of norms, or philosophy of value, on any ethics or esthetics as a normative discipline. For the objective validity of a value or norm is [...] not empirically verifiable nor deducible from empirical statements; hence it cannot be asserted (in a meaningful statement) at all. (CARNAP, 1959, p.77)

Thus, in Carnap's eyes, philosophy could not theorize about normative sciences such as ethics and aesthetics, for they are meaningless in not being able to verify statements in which values are predicated - not to mention those statements in which nothing is ever predicated, as in giving an order or in stating a delightful experience with an interjection.

Logical positivism, here represented by Carnap, could not survive later scrutiny in its idealized version of a language. Wittgenstein himself became disillusioned by such perspective, as he turned to be his own critic. The now labeled ‘late Wittgenstein’ offered another theory of meaning, in which propositions became meaningful by their *use* in what he called *language-games*[6]. Thus, meaningful sentences were not just those capable of being true or false, but those useful in the many different contexts of human interaction. Philosophy of language started to look more closely to ordinary language, not just an idealized version of it, and to the diversity of different meaningful uses found within it.

6.2 Austin’s *How to do Things with Words*

But aside from Wittgenstein’s change of mind, it was in Oxford with John Austin in the fifties that a better grasp of this diversity started taking shape. It started with the groundbreaking *How to do things with Words*, published in 1962. The enemy, once again, was logical positivism and the verificationism dogma. Precisely, Austin’s starting point was the recognition that utterances are not necessarily nonsensical and can contain those words accused of not being in the range of the verificationism principle. As he stated, these utterances

> do not ‘describe’ or ‘report’ or constate anything at all, are not ‘true or false’; and the uttering of the sentence is, or is a part of, the doing of an action, which again would not normally be described as saying something (AUSTIN, 1962, p.5)

Common examples are utterances such as ‘I name this ship *Queen Elizabeth*’ (a christening), or ‘I bet you that it will rain tomorrow’ (a bet), which are not aimed to describe what is the case, but to make something the case. These kinds of utterances are named *Performatives* by Austin. They are so named in contrast to those we here called descriptive, or ‘constatives’ in Austin’s choice of words, that only describes state of affairs. This doesn’t imply that utterances can be simply divided in constatives and performatives, in Austin’s mind. In fact, the core idea is that even when our only purpose is to inform or say something, we are actually *doing* something regardless. For that matter, Austin famously introduced three kinds of acts: locutionary, illocutionary and perlocutionary.

A locution is that part of an utterance that carries the meaning, or the content, being informed. In this sense, an locutionary act is the act of saying something. But things might be said differently depending on speakers intentions. As Austin describe it:

[6]Wittgenstein main theory - if we can call it one - was published posthumously in *Philosophical Investigations* (1958).

> When we perform a locutionary act, we use speech: but in what way precisely are we using it on this occasion? For there are very numerous functions of or ways in which we use speech, and it makes a great difference to our act in some sense [...] in which way and which sense we were on this occasion 'using' it. (AUSTIN, 1962, p.99).

Here enters *illocutionary acts*, *i.e.* that part of an utterance that determines for which purpose the utterance is being made. Quoting Austin once again:

> To perform a locutionary act is in general, we may say, also and *eo ipso* to perform an illocutionary act, as I propose to call it. To determine what illocutionary act is so performed we must determine in what way we are using the locution: asking or answering a question, giving some information or an assurance or a warning, announcing a verdict or an intention, pronouncing sentence, making an appointment or an appeal or a criticism, making an identification or giving a description, and the numerous like. (AUSTIN, 1962, p.98)

Each of such way of making an utterance correspond, roughly, to some feature of language. From the illocutionary point of view, an utterance can be said to be felitious or infelitious, rather then true or false, depending whether its goal is properly met. There is, finally, a final kind of act performed, according to Austin's analysis. An utterance might "[...] produce certain consequential effects upon the feelings, thoughts, or actions of the audience, or of the speaker, or of other persons" (AUSTIN, 1962, p.101). These are called perlocutionary acts. Summing all up with an example, by uttering "It is chilling in here", one is saying something (that the room temperature is low), with the intent of making an request (to the hearer to close the windows), with or without the desired consequence (the windows being closed or not). These three aspects are the locutionary, illocutionary and perlocutionary acts, respectively.

The motivation for studying speech acts lies in the illocutionary part. Giving any content, we can act differently towards it depending on the effect desired and for that reason a variety of different illocutionary acts can be recognized. This variety is marked in what Austin called the *illocutionary force*, that is, the part of language that points the particular illocutionary act being performed. From this, Austin (1962, p.150) offered a list of classes of forces for the great variety found in the language: the (1) verdictives (those where some verdict is given), (2) exercitives (the exercising of some power, right or influence), (3) commissives (those where one commits in doing something), (4) behabitives (those related to some social behavior) and (5) expositives (those that explain how our utterances fits in a conversation or argument).

6.3 Searle and Vanderveken Taxonomy

I shall not discuss Austin's taxonomy of speech acts, but instead present another: John Searle's taxonomy, who developed from Austin's ideas. In (SEARLE, 1979) and (SEARLE; VANDERVEKEN, 1985), jointly with Daniel Vanderveken, the goal was to offer a more consistent principle for classifying the different illocutionary forces in speech acts, thus avoiding some of the problems of Austin's classification[7]. These principles correspond to different ways, or dimensions, in which illocutionary acts can be performed and said to be successful. The basic form of every speech act is

$$F(p),$$

where F is the illocutionary force and p the propositional content. The diversity of F's and their different successful conditions are analysed in term of some basic components. In (SEARLE, 1979), twelve of them are mentioned, but Searle and Vanderveken (1985) narrow it down to just seven[8]:

i. *Illocutionary Point*: This is the most basic component in specifying the type of illocutionary force. It's the point or purpose of a giving utterance, its perlocutionary intent. If the point is obtained, then the act is said to be successful. *e.g.*, the point of an assertion is to tell how things are, the point of an order is to make the hearer do something.

ii. *Degree of strength of the Illocutionary Point*: the different strengths in which an illocutionary point can be made in a speech act. For example, an insistence is stronger than an suggestion, but the illocutionary point is the same in both: to make the hearer do something. Searle and Vanderveken (1985, p.98-99) summarized the different degrees to only three: strong (+1), intermediate (0) and weak (-1) degrees of strength. For example, a suggestion or hypothesis are assertive speech acts, but with a weak degree os strength. A claim or a report have intermediate degrees of strength. Finally, an swear or insistence have all strong degrees of strength of their illocutionary point.

[7]Searle summarized in the following way: "there are (at least) six related difficulties with Austin's taxonomy; in ascending order of importance: there is a persistent confusion between verbs and acts, not all the verbs are illocutionary verbs, there is too much overlap of the categories, too much heterogeneity within the categories, many of the verbs listed in the categories don't satisfy the definition given for the category and, most important, there is no consistent principle of classification" (SEARLE, 1979, p.11-2)

[8]In what follows, I'll discuss them using both papers, since they do intersect each other in the components recognized.

iii. *Mode of achievement*: Illocutionary acts can have different ways of achieving the illocutionary point. Commanding and requesting are different ways to perform the same point. Some acts might demand special modes, in this case, a *characteristic mode of achievement*.

iv. *Propositional content conditions*: some illocutionary forces F might restrict the kind of propositional contents p in an illocutionary act. One cannot promise something about the past, or apologize something about the future, for example.

v. *Preparatory conditions*: For an illocutionary act to be successful and nondefective[9], some conditions have to be met prior. A promise might be successful but defective if the speaker does not intend to fulfill the propositional content promised.

vi. *Sincerity Conditions*: the performance of an speech act usually includes a psychological state that is expressed by the act. For example, in asserting p one express the belief that p. Some illocutionary acts can be successful even if the speaker is insincere. If one makes a promise without the intent to realize it, he still have made a promise: he would have expressed the sincerity conditions nonetheless. These intentions are represented formally by the capitalized initial letter of each intention-word[10].

vii. *Degree of strength of the sincerity conditions*: This is analogous to the degree of strength of the illocutionary point. Sincerity conditions can vary: a speaker would express a stronger psychological state by begging, than he would by simply requesting something for a hearer.

These classes are ways in which illocutionary force can be specified and be called successful in a speech act. But one more piece of detail is necessary before giving Searle's and Vanderveken's taxonomy. As they describe:

> The illocutionary point of an illocutionary force always relates the propositional content of that illocutionary force to the world of the utterance, and there are a limited number of ways that propositional contents can be related to a world of utterance. (SEARLE; VANDERVEKEN, 1985, p.52)

[9]Searle and Vanderveken explains that acts can be successful but defective, as in an statement lacking evidence. The conditions here discussed are required for an illocutionary act to be both successful and non-defective (SEARLE; VANDERVEKEN, 1985, p.13).

[10]'B' for beliefs, 'W' for wanting, and so on.

This is the *direction of fit*, the way in which a content is related to the world. There are four cases:

1. A speaker might want to just describe the world, as in making a statement. This is the *word-to-world* direction of fit, represented by the downward arrow: $\downarrow$;

2. A speaker might want, for example, to alter the world to match his words, as in an order. This is the *world-to-word* direction of fit, represented by the upward arrow: $\uparrow$;

3. A speaker might want to alter the world by means of the speech act itself, therefore changing the world automatically, as in a declaration. This is the *double* direction of fit, represented by a double arrow: $\updownarrow$;

4. A speaker might simply express it's attitude, therefore not directing any correspondence between world and word. This is the *null* direction of fit, represented by the null-sign: $\varnothing$.

Following the seven components of illocutionary forces and the direction of fit between the illocutionary point and the propositional content, Searle (1979) offered a basic structure for his list of five classes of illocutionary acts. In this taxonomy, the illocutionary point, the direction of fit, the sincerity condition and the propositional content are more basic. They are the following:

a. *Assertives*: The illocutionary point of an assertion is to "[...] commit the speaker (in varying degrees) to something's being the case, to the truth of the expressed proposition" (SEARLE, 1979, p.12). There is a variety of cases, with different degrees of strength, but the main dimension of an assertion is to be true or false. The illocutionary point is represented by Searle with Frege's assertion sign $\vdash$. The direction of fit is word-to-world, and the sincerity condition is a belief, here represented by the letter B. Thus, the structure of an assertive is

$$\vdash\downarrow B(p)$$

b. *Directives*: The point of directives is "[...] attempts by the speaker to get the hearer to do something" (SEARLE, 1979, p.13). The formalization is:

$$! \uparrow W(H \text{ does } A),$$

Where '!' is the directive point, the direction of fit is world-to-word, 'W' is for the sincerity condition (meaning 'want'), and 'H does A' for the content of hearer performing the action A.

c. *Commissives*: the commissive point is "[...] to commit the speaker to some future course of action' (SEARLE, 1979, p.14). The formalization is:

$$C \uparrow I(S \text{ does } A)$$

Where C is the commissive point, the direction of fit is world-to-word, the sincerity condition is I of 'intend', and 'S does A' for the content of the speaker performing the action A;

d. *Expressives*: the illocutionary point of an expressive is "[...] to express the psychological state specified in the sincerity condition about a state of affairs specified in the propositional content" (SEARLE, 1979, p.15). The formalization is:

$$E\varnothing(P)(S/H + \text{Property})$$

Where E is the illocutionary point, the direction of fit is null, P is a variable ranging over psychological states, and the propositional content is some attribution of property to either the speaker or the hearer.

e. *Declarations*: This is a *sui generis* class of illocutionary acts. Its point is that "[...] the successful performance of one of its members brings about the correspondence between the propositional content and reality, successful performance guarantees that the propositional content corresponds to the world" (SEARLE, 1979, p.16-17). This distinguishes declarations from the other four classes, since it alters the world simply by the fact of being a successful performance[11]. Depending on the case, it could demand some preparatory conditions, for example, some position in an extra-linguistic institution. The basic structure of such declarations are:

$$D \updownarrow d/b(P)$$

Where 'D' is the illocutionary point for declaration, the direction of fit is double, and 'd' and 'b' is the desire and belief as sincerity conditions, and 'P' is the propositional

[11] In comparison, a directive can be successful (if the speaker manages to give an order) but defective (when the hearer fails to proceed accordingly).

content. In (SEARLE, 1979), declarations are said to have no sincerity conditions. But, in (SEARLE; VANDERVEKEN, 1985, p.57), they are said to express both belief and desire as psychological states: the speaker desires to bring about the state of affairs that the propositional content represents, and the belief that his utterance is successful in doing that. However, insincere declarations are rare and usually a speaker that utter some declaration is already in the condition of believing and desire of making so the case.

One last important point should be mentioned. There's a difference between a speech act being successful from being non-defective. In Searle's and Vanderveken's words: "a speech act may be unsuccessful, it may be successful but defective, and it may be successful and nondefective" (SEARLE; VANDERVEKEN, 1985, p.12). We have to distinguish these three possibilities from each other:

1. If one of the various conditions for the performance of an speech act is not satisfied, then the act is unsuccessful. For example, if a soldier utters for his superior "I command you to leave", he would not have made a successful directive speech act, for he is not in the institutional position to do so.

2. If a speech act is successful but the desired effect isn't met, then the act is defective, albeit successful. For example, if I utter "It is twelve o'clock" when asked for the time, I would still made a successful assertion even if the time was not actually twelve o'clock. In this case, I have a successful but defective speech act.

3. Finally, if both the conditions for the correct performance of the speech act and the desired effect are met, then the performance is said to be both successful and non-defective[12].

Whether Searle's taxonomy is successful and comprehensive enough for the vast variety of speech acts, it's up to debate. What I shall focus is on how Frege's logic can be read in the light of this five classes and the seven dimensions presented. At least three of them are relevant: assertives, directives and declarations. Since Frege intention was to provide a formalized language, the conditions for success are giving within such artificial context. This is what I'll try to analyse now.

[12]A more comprehensive list of conditions under which a performance is both successful and non-defective is found in (SEARLE; VANDERVEKEN, 1985, p.21-2).

6.4 From Speech Act theory to Frege's Logic

Up to now, we can say that Frege's logic is at least a logic *with* illocutionary force indicating devices. In contrast, Searle's and Vanderveken's work is a logic *of* illocutionary devices, which in (SEARLE; VANDERVEKEN, 1985) is developed as a modal logic. Frege's logic is not intended to study the structure of illocutionary acts, but to implement them as they were necessary for the manipulation of the formalism. In this sense, the concept-script is an *illocutionary logic*, while Searle and Vanderveken's is a *logic of the illocutionary devices*. It is not my goal to say that the latter has foundational priority over the former, but only to analyse the concept-script with the aid of speech act theory as developed by Searle and Vanderveken.

The three main acts in the concept-script are judgements, definitions and inferences. These are expressed with illocutionary components such as the judgement-stroke, the definitional-stroke and the transition-signs. These are illocutionary force indicating devices for assertives, declaratives and permissives speech acts, respectively. We can thus analyse these illocutionary forces with the main dimensions described in (SEARLE, 1979) and (SEARLE; VANDERVEKEN, 1985): the illocutionary point and its degree of strength, mode of achievement, propositional content and preparatory conditions, sincerity conditions and its degree of strength, and finally, the direction of fit.

One thing should be noticed prior. It is one thing to consider illocutionary acts in the context of language, in which multiple agents are present, but it is another to consider it in the context of a formal language, artificially defined. In order to follow strictly the conditions of natural language, Searle and Vanderveken formalize illocutionary logic in terms of a *world of utterance*, in which at least a speaker, a hearer, a time and place relevant to the utterance are considered (SEARLE; VANDERVEKEN, 1985, p.27-8). But in an artificial language, we don't needs a hearer, but just the speaker: the reasoning agent that performs deductions and manipulate the language. This should not be a problem. Assertions and definitions do not require any relevant hearer-position. Of course, in the case of assertions, an assertion made will be pointed at a possible hearer or reader. But the conditions for having sucessful assertions can be described regardless. The problematic case is for inferential rules, and we shall define them as some kind of directive, which it does need to consider a hearer. But even these cases do not pose a problem: it is possible for an speaker to utter some directive silently to himself. Consider, for example the uttering of: 'I should exercise more often'. This is a perfectly reasonable directive, that it is only successful when uttered for the speaker to himself. Thus, in the more problematic cases, the hearer and speaker in the context of utterances will be represented solely by the agent that happens to manipulate the formal language.

In a sense, the reading here proposed takes the concept-script as a language to be manipulated, and thus readed in the first-person perspective. This idea was argued first by Maria Van der Schaar (2017). The distinction between being truth and holding something to be true can be viewed, she argues, as a distinction between first person and third person views in judging. While judgements in the first-person perspective is logical, in the third-person they are empirical. This is provided by the epistemic reading of the judgement-stroke, in which only by *knowing* the truth of its premises, one is capable of asserting something based on such premises. Thus, it is only by acknowledging to himself that an agent is capable of rightfully judge something to be true. In this reading, one is able to maintain how judging is both agent-dependent as an *act* but independent in regard to it's product, since it's from the perspective of the agent that the laws of logic, and their normative status, can yield correct (that is, sound) deductions. It is a distinction between what happens in the agent's mind, and what happens in the written language. The latter is an expression of the former. We should then read the former *out* of the language. The reading to be developed in this chapter is consistent with Van der Schaar's account: that the proper manipulation of the concept-script language happens in a single-agent scenario. This will be clear precisely in the case of rules of inference: they will be described as directives from the speaker to himself, provided the rightful conditions given by logical laws.

Since we are dealing with a logical system, the expectation is to have no successful illocutionary acts that are defective. In a sense, all speech acts must be of an ideal nature. If the speaker asserts P successfully, it must assert rightfully, that is, P must be true regardless of his beliefs. This, however, is only an expectation. To require all assertions to be successful and non-defective is an impossible condition to have, as there is no such thing as an irrefutable belief.

Moreoever, knowing that an assertion was non-defectively made requires a good deal of epistemic work. If I assert that "it is twelve o'clock" in order to assert that it is lunch time, I made a successful assertion if it is indeed twelve o'clock. But to know whether it was indeed twelve o'clock requires some investigatory work, for example, checking if my watch was representing the right time for my time-zone. One way to maximally reduce such cases is to have such investigatory process already in the preparatory conditions for the speech act. This is already the case for scientific discourse. A scientist is expected to have more than simply a clue in order to assert her findings. She cannot assert in the same way she claims to be twelve o'clock. Thus, scientific claims have stronger preparatory conditions. For example, a mathematician cannot made a successful assertion if not backed up by the corresponding proof[13]. In Ruffino, San Mauro and Venturi's analysis (2021) analysis, "An assertion without a backing proof is just a mock mathematical assertion, not to be taken seriously within mathematics, although, as we said, it may be regarded as an ordinary assertion in other contexts." The assertions being performed in the concept-script must be read this way, as all we can verify in order for an assertion to be non-defective must be presented already in the preparatory conditions. Is this enough for having non-defective and successful assertions only? No, as Frege himself asserted Basic Law V within the concept-script: a successfull, but still defective assertion.

Similar conditions follow for declaratives. If the speaker declares something successfully, he should bring about the very thing being declared as well. Of course, in the case of declarations, there can be no successfully defective acts. Therefore, it is enough to specify the necessary conditions for a successful declaration in order to guarantee its non-defectiveness. Finally, in the case of inferences, for which permissive speech acts occur, if the speaker allows himself successfully into some course of actions, he must be able to fully act in terms of his permission. If the five classes of speech acts provided by Searle and Vanderveken taxonomy have their behavior and successful conditions from natural language, being essentially a linguistic phenomenon, we'll see that in a formal language such as Frege's concept-script, their function will correspond only to a subpart of it. But these conditions also describes a performative language, albeit a more restrict and constrained one.

6.4.1 Assertions

As we already saw, Frege's sign for judgements — the judgement-stroke — is the same sign adopted by Searle for the assertive illocutionary force: the turnstile $\vdash$, also named the assertion sign[14]. There are only two distinct cases where the assertion sign is allowed to be attached in Frege's logic: in axioms and what we may call theorems, following rules of inference. But the different assertives that one finds in mathematical (and logical) texts is not present in Frege's concept-script. That is, theorems, propositions, lemmas, corollaries and the like do not occur. These are called *In-Block Mathematical*

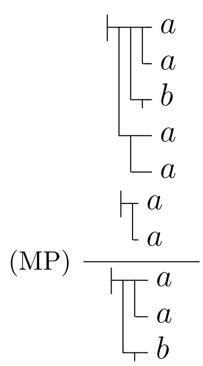
[13]But, as said, there's no absolute sense in which the speaker believes in the theorem's truth even if asserted after the corresponding proof. The proof may be flawed or made unknown presuppositions.

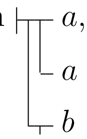
[14]I'll employ the turnstile with a longer horizontal line to distinguish Frege's content-stroke, or horizontal, from the usual turnstile: $\vdash$.

Assertions, or simply IBMAs, by Ruffino, San Mauro and Venturi (2021. The sheer presence of IBMAs in mathematical language is a good example of its pragmatic features, as IBMAs serve only explanatory functions.

On the other hand, the derivations in the concept-script are purely transitions between judgements. But, in *Grundgesetze* Part II of volume I and II, the expository part have two hierarchies that much resembles the IBMAs. The proof of the basic laws of cardinal numbers are divided in sections marked with capital Greek letter $A, B, \Gamma, \Delta, ...$, marking the basic results, and then with subsections marked with lower Roman letters $a, b, c, d, ...$ These are preliminary results, used for achieving the main ones marked with upper Greek letters. In a sense, the former are theorems while the latter are lemmas. Frege also adds an appendix to volume I with a list of the most important results. But these, as he sees it, are not part of the derivations. The division is mostly used for the exposition and readability of the results.

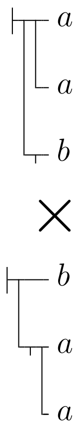
This distinction does not occur in the constructive part of the derivations, where simple transitions from judgements to judgements are performed directly. In it, every application of a rule of inference resulting in a successfully made judgement already counts as a proof. In the context of what happens in the constructive parts, which counts as being *the* proofs of the concept-script, no such hierarchy is found. To be fair, we could define such hierarchies, but not much would be gained. If a judgement follows from the application of a rule of inference on two other judgements, there is a sense in which the former depends on the latter to be judged. For instance, in the inference

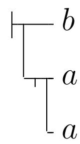
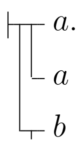


the derived judgement is only achievable following the two instances of Basic Law I and *modus ponens*. But if in any sense such premises are lemmas to the main theorem 

than *all* premises are lemmas for every other judgement that follows from it.

There would also be a problem in explaining what a corollary means in the concept-script proof structure. Consider the following inference by contraposition, derived from the previous acquired judgement:



If we accept that a corollary is an immediate consequence of a given theorem, then  is a corollary of . But, we may also say that the latter is a theorem following a proof by contraposition from the former. In the judgement-to-judgement proof structure of the constructive parts of the concept-script, all preceding judgements of a particular judgement would be lemmas, whereas all succeeding judgements would be theorems or corollaries. The distinction between different kinds of assertability is to no use in the concept-script.

Frege's explanations in the expository part of the language (the *analysis* parts preceding each constructive derivation in Part II of the *Grundgesetze*) do have important illocutionary components, and much could be said about the performative part of the expository language. But in Frege's case, it is the performative part of the auxiliary language that is of much interest. In modern logical practice, it is quite clear that the metalanguage is embedded in performatives, as it is the language logicians use for proving results about the object-language. But as we saw, Frege's expository language is no metalanguage. Nothing is properly proved in it. And the auxiliary language is no object-language either, as proofs occurring in the auxiliary language are not formal sequences of formulas, but sequences of judgements, *i.e.,* assertions.

Giving that judgements are, for Frege, the acknowledgement that a Thought is the True, and since assertions are the manifestation of judgements, assertions are then the expressions of such acknowledgment, or, the expression of a belief that the thought is the True. From this, the illocutionary dimensions of the judgement-stroke $\vdash\!\!\!-$ as an illocutionary force indicating device for assertions are the following:

1. *Illocutionary Point*: The point of an assertion is to put forward something as being true. Frege defines a judgement in terms of an acknowledgement of the True of a thought, and uses the judgement-stroke as means for expressing such acknowledgement, that is, making an assertion. In the *Begriffsschrift*,

   $$`\vdash\!\!\!-\ \Delta'$$

   makes the assertion that Δ is the case, while in the *Grundgesetze* it makes the assertion that Δ has the True as reference. The $\vdash\!\!\!-$ sign fixes the illocutionary point by convention. By prefixing Δ with the judgement-stroke, one is then making an assertion and succeeds in making the illoctionary point of an assertion if, in the context of the concept-script, he does express the belief that Δ is the case, *i.e.*, if the assertion corresponds to the judgement made. The context of the concept-script must be read broadly. It includes the use of the formalism in real scientific contexts. In an absolute sense, there is no way to guarantee that the illocutionary point of an assertion is always achieved whenever $\vdash\!\!\!-$ is being used, given that one cannot fix every context by convention for every case. An actor may use the concept-script and the judgement-stroke in a play, but he is not making assertions as the main illocutionary point will not be achieved. But, of course, an actor in a play is not using the formalism in the context of a scientific research. The fact that we cannot fix the context in advance is not only a restriction for the concept-script, but for every language.

2. *Degree of strength of the illocutionary point*: It might seem that there's nothing special in the degree of strength for an assertion in Frege's case. But there's a catch. By using the turnstile '$\vdash$' following Frege's assertion sign, Searle is denoting the whole class of assertive illocutionary acts. Thus, '$\vdash\downarrow B(P)$' can be used to express different strengths of illocutionary point, such as the usual assertion, a conjecture, an hypothesis, a suggestion, and many others. In Frege's case, the sign '$\vdash\!\!\!-$ ' is only used for assertion in the usual sense. By writing

   $$`\vdash\!\!\!-\ \Delta'$$

one is actually saying that Δ *is* the case.

But we cannot settle in saying that the strength of the illocutionary point in this case is the same as regular assertions. Recall above that Searle and Vanderveken narrow the strengths to just strong, intermediate and weak. We could also say that Frege was advocating for a strong sense of Assertion, in which an assertion that Δ is only allowed if the agent has means for ensuring that his assertion will be both successful and non-defective. The problem is that even in the concept-script, there's no secure and *a priori* way to guarantee that all successful assertions are non-defective. In fact, even with some caveat, Frege asserted Basic Law V in *Grundgesetze*, a disastrous mistake.

Both a sincere speaker and a logician with a rock-solid proof asserts in the same way, provided that they both have same sincerity conditions: that both express the belief in the truth of the content being asserted. The difference is that, in the concept-script, there's a principle to always assert when proof have been already provided. Thus, we not only have the usual sense of an assertion (excluding cases such as hypothesis, conjectures, and the like) but an stronger sense. For this reason, I distinguish such usual cases from Fregean-assertions, or assertions^F. The degree of strength of the illocutionary point, in the case of assertions^F, is maximal: a speaker that asserts^F Δ, asserts it with enough logical reasons for Δ being true. Again, there's no *a priori* guarantee that this will be enough. But even if the psychological state of a logician and a speaker are virtually the same in every assertion, we have enough reasons to believe that the logician is doing *more* to secure his belief. The illocutionary point of the logician is much stronger.

3. *Mode of Achievement*: Usually, assertions doesn't have any special mode of achievement. There's no restriction in asserting something as being the case, even if one has no clear justification for it. But in Frege's case, assertions^F require a special mode of achievement: the concept-script language. This might sound odd in considering that the concept-script is a language. It would be something like saying that English is the mode of achievement for assertions that are expressed in English. But, the concept-script is not an ordinary language, and it is designed for, among other things, making assertions more precise. Assertions^F are not just assertions written in concept-script, but assertions with maximized degree of strength. Thus, in the same sense a military order depends on having a particular mode of achievement for fulfilling its strong degree of strength (*e.g.* having the necessary authority), so thus

the concept-script is the mode of achievement for making assertions^F. Thus, if one make's an assertion outside the rules provided by the concept-script, or even without the formal language, then one has not made an assertion^F. More specifically, all assertions^F are made with the judgement-stroke $\vdash$ and none are made without it.

4. *Propositional Content Conditions*: There are different kinds of propositional content conditions for different kinds of assertions. A prediction is an assertion about the future. A report is an assertion about the past. But ultimately, all assertions have a common propositional content condition, *viz.* that whatever is being asserted has a possible truth-value. One cannot assert, in any way, a promise, a question, or even ill-formed expressions. Assertability in the concept-script fulfils only this general propositional content conditions. But, we find at least two versions explicitly in Frege's texts, relative to both versions of the language:

    a. *Assertability*: One of the most discussed dimensions of Frege's earlier logic, the propositional content conditions are basically exposed in what he called the content-stroke and the assertability function that it plays on that version of the concept-script. Following the discussion already made here, the content-stroke '$—$' marks the conditions for well-formed formulas in the language. It reflects the condition that assertions^F (and the usual assertions as well) are only successful if the content being asserted is capable of being true or false. For that matter, the content-stroke is defined as restricting the range of the variables in such a way that by $— \Delta$ in that logic one reads the restriction of Δ to only truth-functional contents. This is why every other connective of the language includes itself a content-stroke and is only well-formed when every subcomponent already contains their own content-stroke. The implication of Δ to Γ is not judgeable if Δ or Γ are unjudgeable, as for example if one attempts to assert that "If it rains, then blue". Likewise for the other connectives. The negation of Δ is only judgeable if Δ is itself judgeable. One cannot assert something like "it is not the case that house". In this sense, $\begin{array}{l}\top\!\!\!-\Gamma\\ \llcorner\,\Delta\end{array}$ and $\top\!\!\!-\Delta$ both reflect the judgeability that is marked by the content-stroke. It guarantees that Γ and Δ are restricted to judgeable-contents themselves. It is clear that in the *Begriffschrift*, the content-stroke marks the propositional content conditions for the judgement-stroke to be applicable, that is, to assertions^F be made.

b. *Referentiability*: In *Grundgesetze*, Frege changes the assertability requirement in terms of referentiability. The first condition is that all names of the *Grundgesetze* must be non-empty: "Names without reference must not occur in concept-script. The stipulation is made such that under all circumstances '— Δ' refers to something, provided only that 'Δ' refers to something" (*GGA*,§5, f3).

The second condition is fulfilled by the horizontal function. First, Frege's universalist reading of logic still requires total functions, as all functions in the 1893 version are total. But, judgements, as we recall, are not any kind of belief attitude, but precisely the belief that a given thought is the True, and Frege defines a judgement as the "advance from the thought to its truth-value" (*CP*,p.177). Thus, in order to a judgement to be the recognition that Δ is the True, Δ itself must be a name of a truth-value. This is where the horizontal function comes in. It maps objects to truth-values. If Δ refers to an object, — Δ names a truth-value: the True if Δ is itself the True, and the False otherwise. All judgement conditions in the *Grundgesetze* includes a horizontal function. For instance, the negation $\top$, the conditional $\lceil$ and the concavity $\smile$ all are flanked by horizontals.

The other functions do not necessarily involve the horizontal. For instance, the equality is a function mapping pairs of objects to truth-values, which is not problematic. The remaining two functions are not functions to truth-values. The backslash \ maps object to objects, and the smooth-breathing ʽ maps functions to objects. But in practice, a judgement never occurs without an horizontal, simply because the judgement-stroke itself is composed out of an horizontal. Frege claims the following: "I regard '$\vdash$' as composed of the vertical stroke, which I call the *judgement-stroke*, and the horizontal stroke, which I now propose to be label simply the *horizontal*" (*GGA*, §5). It never occurs for the judgement-stroke to occur in isolation. Even for functions that are not defined with horizontals, such as the equality, It never occurs something like

$$|(\Delta = \Gamma),$$

but only

$$\vdash (\Delta = \Gamma)$$

instead, when Δ and Γ name the same object.

Likewise, Frege never judges value-ranges or the backslash outside the context of an equality, therefore still judging in terms of truth-values. In this sense, both

$$\text{`}\vdash \dot{\varepsilon}\Phi(\varepsilon)\text{'} \quad \text{and} \quad \text{`}\vdash \backslash\Gamma\text{'}$$

are faulty judgements, in the same sense '$\vdash 2$' is, given that $— \dot{\varepsilon}\Phi(\varepsilon)$, $— \backslash\Gamma$ and $— 2$ are not names for the True. On the other hand, $\vdash\!\!\top \dot{\varepsilon}\Phi(\varepsilon)$ or $\vdash\!\!\top \backslash\Gamma$ are correctly made assertions, given that $— \dot{\varepsilon}\Phi(\varepsilon)$ and $— \backslash\Gamma$ are names for the False, following the horizontal function.

Thus, in the *Grundgesetze*, judgements require an specific propositional content condition: that the content being judged is a name of the True. In order to be a name of the True, the horizontal maps objects to truth-values in general, realizing such conditions.

These are the two general propositional content conditions presented in the concept-script for assertions^F. There are some content differences that, at first sight, seem to relate to different conditions for assertions^F. For instance, Frege has two different modes for expressing generality, the German letters with the concavity sign, and the Roman letters. As he defined, Roman letters are Roman Object Markers (*GGA*, §17), and therefore, are not proper names for truth-values. This seems to imply a different mode of assertions^F being performed especifically for Roman Object Markers, as they are said to only *indicate* an truth-value. In section 4.2.2, I tried to rule out this possibility. It seems more plausible to render Frege's conception of judgements as general as possible: all judgements are acts of recognition of the True of a Thought. Roman Object Markers, therefore, are differences on the contents of assertions^F, not on the assertions^F themselves. In this sense, asserting^F $\Phi(a)$, where $\Phi(a)$ is a Roman Object Marker, has the same general propositional content-conditions of all other assertions^F: assertability and referentiality.

5. *Preparatory Conditions*: As we already discussed, the usual mathematical language distinguishes, for pragmatical reasons, different levels of assertibility: theorems, lemmas, corollaries, etc. They seem to presuppose some peculiar way in which assertions are taken in mathematical speech. Ruffino, San Mauro and Venturi (2021) argue that such different roles can be explained by means of an hybrid class of illocutionary acts: one that includes the assertive component of mathematical assertions,

and a declarative component responsible for fixing some position in a structured hierarchy, or dependency between assertions. In Frege's case, no hybrid kind of illocutionary act is necessary, simply because the distinction between theorems, lemmas and corollaries collapses into a single mode of assertion: the assertion^F. In the former case, to prove a theorem, a mathematician/logician tends to break down some other facts necessary for the main proof. Some of them are proved in the theorem itself, others are proved prior in lemmas. In it, multiple occurrences of rules of inference happen in the same block, but the only place where the illocutionary point of an assertion appears is in the main theorem. The preparatory conditions for these kinds of assertions might simply suppose that, if a theorem is asserted, a proof must follow. But of course, nothing inside the proof is itself asserted with illocutionary force either (except, of course, other $IBMA$s explicitly declared).

In contrast, in Frege's concept-script, theorems are reduced to each assertion^F. And since Frege's logic is a logic of judgements, every derived sentence is uttered with illocutionary force of assertion. Thus, instead of asking for a proof following the assertion of a theorem, the correct preparatory conditions for assertions^F are conditions *prior* the assertion[15]. It turns out to be a difference in exposition: Frege's concept-script lay down the preparatory conditions before an assertion, in order to be able to assert every single sentence in a inference. In the current standard logical and mathematical sense, the assertive part of a theorem block seems to have only pragmatical purpose, while in Frege's case, the assertive have logical importance, as it is used to track down each inferential point for soundness[16].

Assertions^F can be divided into axioms and theorems. There's no precise and definitive way to distinguish axioms from theorems, aside from the fact that axioms are assertions^F that doesn't follow from inferential rules, while theorems are assertions^F that do. Besides that, we could use the same reasoning in justifying the assertion^F of axioms to also justify some theorems, avoiding inferential rules. Thus, this distinction is not of nature but only of choice. Frege explicits this in the 1914 unpublished paper *Logik in der Mathematik*: "Not every truth for which no proof is required is an axiom, for such a truth might still be proved in our system. Whether a truth is an axiom depends therefore on the system, and it is possible for a truth to be

[15]Of course, if one does not provide a proof of a theorem, there's no theorem in the $IBMA$ sense either.

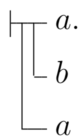
[16]We may contrast both cases in saying that assertions^F are just inferential steps in a logically regimented calculus, just as in natural deduction systems, where one always make a step following other already made steps. The written mathematical and logical texts, on the other hand, usually declare theorems by following it a proof that is likewise informally written.

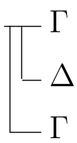
an axiom in one system and not in another" (*PW*, p.205). Moreover, one of the desired properties for an axiom system is actually a pragmatical one: simplicity. In *Begriffsschrift*, Frege states that:

> It seems natural to deduce the more complex of [...] judgements from the simpler ones - not to make them more certain, which generally would be unnecessary, but to bring out the relations of the judgements to one another [...] In this way, we obtain a small numbers of laws in which is included, though in embryonic form, the content of all of them. (*BS*, §13)

Frege is here talking about axioms. To be an axiom is, virtually, to be taken as a simple starting point for an inference. In sum, axioms are those assertions^F that are simple and evident in such a way that no proof is required, understanding 'required' in a pragmatical sense. Theorems are to be regarded as those assertions^F that require proof, or in the case of the concept-script, those following rules of inference. This is enough for giving the preparatory conditions for both cases:

(a) *Axioms*: One can assert^F an axiom if it can be made in a successful and non-defective way. This means that axioms can be asserted^F simply because they cannot be denied. To the exception of Basic Law V, Frege uses this kind of justification in both versions of the concept-script. In the *Begriffsschrift*, this is done in terms of 'affirmation' and 'denial'. For example, the axiom



One can assert^F it because denying it would be to assert^F the antecedent but not the consequent. To deny the consequent is then to deny a, which was the antecedent asserted^F. In Frege's words, this must hold because "a cannot be denied and affirmed at the same time" (*BS*, §14). In *Grundgesetze*, when Frege changes his words, the reasoning is virtually the same. He states that 

would denote the False only if Γ would denote both the *True* and the *False*, which cannot happen (*GGA*,§18).

All this rationale depends on the meaning of the primitive functions and the underlying logical assumption that referentiality is a single-valued relation between a name and a object. Given that no name can have different referents, there is no other option other than assert^F the Basic Laws. We can summarize the case of axioms in the following way: an assertion^F of an axiom A is successful and non-defective if there's no way to assert^F A wrongly. In 4.2.2, I provided more examples of this exact justificatory procedure that Frege uses for judging its Basic Laws as being true.

(b) *Theorems*: Virtually any propositional tautology can be asserted as an axiom, following the preparatory conditions just mentioned. We cannot take only self-evidency as the basis for accepting a given judgement as an axiom, given that most of theorems Frege proves in section II of the *Begriffsschrift* can be claimed to be self-evident as well. Given that they are all propositional tautologies, any reasonable analysis of its constituents will result in the realization that it cannot be other than the True. The reason why Frege choose this or that set of axioms is also pragmatic, perhaps in the same sense why current logical and mathematical practice relies on $IBMA$'s. Be that as it may, what are the preparatory conditions for theorems? We can mentioned the following: an assertion^F of A is successful and non-defective if A is the consequence of the successful and non-defective use of a rule of inference. Therefore, the case for theorems will only be completed when discussing which type of illocutionary act is performed with the application of a rule of inference. Thus, the preparatory conditions or asserting^F something as a theorem is the rules of inferences.

(c) *Definitions*: There is a final preparatory condition for assertions that has not been mentioned yet. Definitions are still to be analyzed in terms of declarative speech acts, but one of the consequences of a successful and non-defective declarative speech act is to perform a change in the world in representing it so changed. Since the world being considered by Frege restricts itself only to the expressions of the concept-script language, a successful declaration alters the language in fixing the content or sense and reference of a new sign. But once this happens, a new fact is derived. As Frege claimed, "once the meaning of the new sign is specified, it must remain fixed, and therefore [...] also holds as a judgements" (*BS*, §24). In the *Grundgesetze*, he claims the following:

> The new sign thereby becomes co-referential with the explaining sign; the definition thus immediately turns into a proposition. Accordingly, we are allowed to cite a definition just like a proposition replacing the definition-stroke by a judgement-stroke. (*GGA*, §27)

If I successfully define that Δ is the same as Γ, an immediate fact follows: that Δ is the same as Γ. This is what Frege claims, allowing an assertion^F to follows every successfully made definition. Thus, we may say more simply that an assertion^F can be made following any successfully and non-defective use of the definition-stroke $\Vdash$, by simply changing it for the judgement-stroke $\vdash$.

6. *Sincerity Conditions and Degree of Strength*: in usual assertions, a speaker who asserts P expresses the belief that P. If a speaker conjectures that P, he still expresses a belief, but with a lesser degree of strength. Since assertions^F are distinguished from conjectures, hypothesis, suggestions, and the like, a speaker who asserts^F P expresses the belief that P, as the usual standard case, but with maximal degree of strength[17]. This is a consequence of the mode of achievement: by asserting^F that P in the concept-script language, and following the preparatory conditions just mentioned, a speaker have the maximal degree of strength in his belief that P, since in principle he has a proof for it.

   But why do we set the degree of strength of the illocutionary point of assertion^F as maximal, and not just the strength of the sincerity conditions? Because the degree of strength of the illocutionary point determines the mode of achievement of a illocutionary act, and asserting^F in the concept-script must be somehow different from simply asserting something out of plain conviction. By analogy, if someone asserts "I'm sick!" from simply believing to be, he might not be that much convincing. But if he asserts "I'm sick!" showing a medical diagnosis, he has asserted more strongly than before. It is not just his degree of belief that changed (if it changed at all - he could have already known the diagnosis before showing it to others), it was the degree of the illocutionary point that changed as well. Thus, we need to differentiate a speaker that asserts P with full belief but no proof, from those cases where the speaker has full belief and reasons for it. These are the assertions^F cases.

7. *Direction of Fit*: the direction of fit is the same for all cases of the illocutionary point of an assertion. If a speaker asserts, conjectures, hypothesize, suggest or swears that P, he is trying to make his words fit the world. This is a general way of saying that assertions are the attempt to match the words to the some state of affairs in the world, taking by "world" the relevant set of circumstances that P ranges over. In

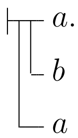
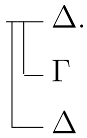
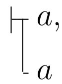
[17] This goes against Frege's assertion of Basic Law V, as he indeed claimed to have doubts in its logicality in (*GGA*, p.VII). But instead of taking this fact as evidence that Frege took assertions within the concept-script loosely, it is best to consider that the assertion of Basic Law V was Frege's own misread of his own logical recommendations.

the *Begriffsschrift*, Frege speaks about a content "being the case" and correlate expressions, in order to express what it is being affirmed in a judgement. In this case, assertions^F made with the judgement-stroke have the word-to-world direction of fit.

This, of course, must be supplemented for the mature Frege of the *Grundgesetze*. To begin with, a simple word-to-world explanation would turn all assertions in the concept-script true by correspondence, which Frege rejected. We must then restrict the world here considered as the *world of Thoughts*, following Frege's platonistic setting. As we saw, Thoughts are in Frege's standards abstract entities that have an independent ontology. And part of this ontology includes its relation to a truth-value. All true Thoughts are true in a independently and eternally way, as all false Thoughts are false in a independently and eternally way. If someone makes an assertion^F that is true, it happens that such assertion rightfully represents the Thought associated as being the True. The matching, thus, is of a different kind.

Let D^T be the set of true Thoughts. In this sense, to an assertion^F be successfully and non-defectively made, it must not be an assertion *about* D^T, but it must grasp an element of D^T rightfully. Of course one can still make an assertion about D^T, but in this case, it would grasp a thought of D^T, since D^T is closed under all true thoughts. We may call these thoughts that are about other thoughts as second-level thoughts. It happens that these second-level thoughts are never asserted in the context of the auxiliary language of the concept-script, as there is no sign for having thoughts as references[18].

[18] As I believe it, there are good reasons for not having them, as cross-referencing thoughts does yield inconsistencies. Given that the realm of Thoughts is all-inclusive, one has to consider that the Thought expressed by the sentence "This Thought is not true" must be included, as it is a sentence to which the question whether it is true or not arises. Thus, the set D^T must be taken only informally, as it is not well-defined at all. Consider the thought expressed by the sentence "This is not a Thought from D^T". Let the Thought expressed by this sentence be P. Then, if $P \in D^T$, then P is a true Thought. If it is a true Thought, then it must not be a Thought from D^T. But if $P \notin D^T$, then P is a false Thought. If it is false, then it is false that it does not belong to D^T, and so $P \in D^T$. Obviously, $P \in D^T \Leftrightarrow P \notin D^T$. As far as I know, Frege never talks about Thoughts with undetermined truth-values, as P above. Thus, we may say that some thoughts are true, some are false, and some are undetermined. This imply that we may have vague thoughts as a simple case of semantic vagueness (On the distinction between semantic, epistemic and ontological vagueness in Frege, see Ruffino (2003)). The problem is that such thoughts with undeterminate truth-values are not simply consequences of vague concepts. We may say that the concept "The thought that x is False" is not vague as it depends simply on recognizing that x express a thought. And a thought is easily achieved whenever the question of its truth can be raised in advance. For instance, "the thought that 2+2=5 is false" is a perfectly well formed expression that has a (second-level) true thought. But, "The thought that this sentence is false" is well-formed but has an undeterminate truth-value nonetheless. The problem is to suppose that one does have D^T defined from the outset. That some thoughts have undeterminate truth-values speaks for the fact that one may never claim whether it belongs to D^T or not by a simple act of judgement, as no judgement can be made concerning it.

> In making an ordinary assertion, one is attempting to express through words what the world is about. Similarly, in asserting^F, one is attempting to express what the set D^T is about. By asserting^F A, one is simply grasping the thought that A and asserting^F that A is the True, or simply, claiming that $A \in D^T$. For example, in asserting^F that
>
> 
>
> one is grasping the thought that
>
> 
>
> and claiming that it is the True, and that it falls under D^T. If such a thought falls in D^T, then by the rules defined in the concept-script, one may derive  which claims that  also has its associated thought as a member of D^T. In a sense, in every made assertion^F, one has a clearer grasp of the realm of thoughts, as one proceeds between the structured relation between them. Then, the so-called direction of fit for assertions^F is from words to the world of thoughts, ore more precisely, *word-to-*D^T.

As we saw from the seven dimensions above, the assertions made with the judgement-stroke, or assertions^F, can be perfectly explained from the perspective of Speech Act Theory. By making $\vdash$ an illocutionary force indicating device, Frege was also defining the conditions that, putted together, yields assertions^F as proper speech acts. I do not claim, however, that this what Frege was ultimately doing. Only that it is perfectly consistent with what he has done.

6.4.2 Inferential Rules

Frege's inferential rules are all described with the permissive mood, usually with verbs such as "may", "can", and the like. They are permissives, given that they set that a given action (a judgement) is allowed to be performed under certain conditions being met (other judgements). It is not straight-forward how this permission should be described. But the analysis here taken, in a nutshell, is that to follow an inferential rule is to give itself the permission to assert something under certain specified conditions.

As said before, in the context of the concept-script language we don't need more than a speaker, or reasoning agent, to perform the acts required for logical reasoning. Thus, it might look from this that an inferential rule is some form of commissive speech act in which the speaker commits himself in doing something. This is misleading. Logical laws have a normative status: they prescribe the way reasoning must be performed in order to be sound and reach the truth. A comissive seems too weak for such strong purposes.

A second alternative is to consider inferential rules as directives from the logical laws to the hearer. In such case, an inferential step is a speech act performed from logical laws to the hearer. More precisely, that under certain conditions, the logical laws (here in the position of a speaker) *allows*, or *permit* the reasoning agent (or the hearer) to perform some action. However, we could ask the simple question: who is performing the directive? It cannot be the logical laws. What it seems to be happening in the performance of a inferential step is the speaker (or reasoning agent) directing himself in acting in some specific way.

The case for the inferential rules highlights the fact that all speech acts being performed in the concept-script falls in the category of written acts. An agent asserting^F with the judgement-stroke is asserting^F by writing something. But there is no difficulty in the case of assertions. A written sentence can be simply taken as an assertion by taking writing as its mode of achievement. The speaker is simply the one who has written it, as for example all assertions in a book are assertions made by its author. In this case, we have a fixed speaker, and unspecified listener. Things are a bit more difficult when considering written directives. Simply put, a written directive is a directive with a specific mode of achievement. But speech acts are events, and a written sentence is not.

When a medical doctor writes down a prescription for her patient to take a pill every twelve hours for two days, she is making a directive speech act, either an order (you should take the pill now) or a permission (you may take the pill now). But what she provides is not one directive for every pill the patient must take. It is not that the doctor is making a particular directive every twelve hours. The directive can in fact be break in four others, but this, as I take it, are issued from the patient to himself. Following the general expression of a directive, the medical doctor's directive is something like:

$$!(\uparrow W(\text{1 pill will be taken every twelve hours, for two days}))$$

There are multiple ways to break this directive up. One may reduce it to a directive of a conjunction such as $!(\uparrow W(P\wedge,...,\wedge P))$, or a conditional directive such as $T \rightarrow !(\uparrow W(P))$, where T marks the temporal condition for taking the pills. Nonetheless, in order for the desired changes be fulfilled, the patient must himself break this down into another four distinct acts of the kind:

$$!(\uparrow W(P))$$

If he does so every twelve hours for two days, then he has bring about the state of affairs directed by the doctor.

I don't want to get into the specifics of this case. What this seems to imply, that interests the analysis of the inferential rules here, is that a written directive like this seem to depend on other instances of speech acts in order to be fulfilled. A written law, just like the doctor's prescription above, seem to depend on instantiations in other speech acts in order to be realized in particular cases. If the law states that one shall have its tax returns once a year, the law is not uttering a distinct directive for every qualified citizen each year. Instead, these are the conditions for the fulfillment of the law read as a directive, just as taking the pill four times in two days falls into the success conditions for the directive given by the doctor.

What we want to say is that inferential rules behave just as written laws or prescriptions above. Frege does not specify the rules of the concept-script schematically, but his approach is similar, as it is not specified in a infinitary way. The rule of *modus ponens*, for instance, gives a general schema stating that any set of formulas satisfying the general condition of the premises grants one the permission for judging a formula that satisfy the general conditions of the conclusion. What in practice happens is that an agent performing an instance of *modus ponens* by writing down the horizontal transition line issues a permissive speech act to himself, satisfying the specified conditions of the general rule (just as the patient let himself take the pill if twelve hours has passed). All this is just for saying that a rule of inference works as a permission issued by the user of the concept-script language to himself, as the agent using the formalism is both the speaker and the listener, following the specified conditions given by Frege. We may also take that the former permissive is issued by Frege himself, or those responsible for the realization of the formal language. It is not that the rule is uttered by any logical law (logical laws are not agents and cannot utter commands). The rule is declared *according* to the laws of logic. But in practice, it is up to the agents to realize that each particular case fulfills the original conditions of the rule, and he does so in the context of the written concept-script by means of self-proclaimed speech acts.

In a sense, there are two different acts of permission in place. There is the former permission, issued by who designed the language in the first place. This is Frege. We can say that in defining *modus ponens* as a rule for the concept-script, Frege was uttering a permissive speech act but only schematically[19]: from every two pairs of judgements in the form ‘$\begin{array}{l}\vdash\!\!\top\, \Gamma \\ \quad \llcorner\, \Delta\end{array}$’ and ‘$\vdash \Delta$’, one may derive the judgement form ‘$\vdash \Gamma$’. This rule is provided in the expository language. And even being schematic, it is a singular speech act. But an agent using the *modus ponens* rule is using it in the auxiliary language. And using it there, she is making particular cases of the general rule stated. What she does, then, is to update somehow the original rule. If, for instance, she has in a derivation both $\begin{array}{l}\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\top\, a = a \\ \qquad \llcorner\, \mathfrak{a} = \mathfrak{a}\end{array}$ and $\vdash\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$, she will realizes that both premises have the same structure as ‘$\begin{array}{l}\vdash\!\!\top\, \Gamma \\ \quad \llcorner\, \Delta\end{array}$’ and ‘$\vdash \Delta$’. Then, she also realize that, following the original permissive, she may conclude something in the form ‘$\vdash \Gamma$’, following the interpretations for Δ and Γ in place. But since Γ here interpreted is just $a = a$, she thus writes down the horizontal line, making a new permissive speech act, that allows the judging of the conclusion: $\vdash a = a$. Of course, one may object that just understanding the applicability of the original permissive does not imply a new speech act, as the agent may simply act without any speech act being in place. However, the agent is written down something (the transition sign) and as such, an speech act is being performed, even if it is for himself[20]. In what follows, the analysis will be about this second permissive, the ones being performed in the auxiliary language by writing down something.

1. *Illocutionary Point*: The point or purpose of a directive is basically to attempt the hearer to perform some act. In the case of inferential rules, there is an obvious restriction to which kind of act one is able to perform in the formal language. Precisely, there is only one such action, to assert^F something. We already saw how it makes little sense to consider an inferential rule as giving the permission by itself, since this presupposes the performance of an action. It is more productive to think in terms of the speaker giving himself some directions, or more precisely, given himself the permission for acting in such and such way. This is basically done

[19]It is perhaps best to understand Frege's uttering of *modus ponens* as an elucidation. But from the perspective of Speech Act Theory this is not relevant.

[20]I may written down in a paper a directive for myself such as “You should exercise more often”. There's no problem in a speech act being directed to the speaker.

with the multiple transition signs relative to the different rules (if there is one). The most basic one is the *modus ponens* horizontal line: ———, which is used as an illocutionary force indicating device for the point of a permission. But permissions are just kinds of directive speech acts, thus, the illocutionary point in this case is that of a directive, where uttering something with a transition sign the agent is expressing the desire for some course of action to be performed.

2. *Degree of Strength of the Illocutionary Point*: It might seem unnatural to think of a permission in terms of an attempt to make a hearer to do something. This is certainly the case in a order, or request, but not in a permission. But we can make good sense of a permission following Searle and Vanderveken's idea that permissions are, in fact, denegations of prohibitions. We have first to introduce the notion of a denegation illocutionary act:

> Another type of complex illocutionary act involves the negation of the illocutionary force, and we will call these acts of *illocutionary denegation*. It is essential to distinguish between acts of illocutionary denegation and illocutionary acts with a negative propositional content (SEARLE; VANDERVEKEN, 1985, p.4).

A denegation is the negation of an illocutionary force. In order to express a denegation, the usual sign of negation $\neg$ is used, and contrasted with the $\sim$ sign for the propositional negation. The idea is that one still performs an act by negating the illocutionary force. For example, "I do not promise to come" and "I promise not to come" are different acts. The former negates the illocutionary force of the commissive, that is

$$\neg C(\uparrow I(\text{I will come})),$$

therefore a denegation, while the latter is an usual commissive with a negative content, that is,

$$C(\uparrow I \sim (\text{I will come})).$$

Following this condition, to give a hearer the permission for doing A is to denegate the prohibition for the hearer to do A, which in turn is the denegation of the order to the hearer for not doing A. Simply put, 'I allow you to A' is the same as 'I do not order you to not A', this is

$$\neg!(\uparrow W \sim (A))$$

Granting that permissions are the denegation of directives, it is best to consider them as a particular type of directive speech acts. In this sense, while ordering is a directive with high degree of strength of the illocutionary point and suggesting is a directive with a low degree, a permission seems to have the lowest possible degree of strength for the directive point. Searle and Vanderveken do no specify what this degree of strength is, but it is safe to assume that this is the lowest a directive can get, given that it is a way of making an action possible without even demanding that it should be performed. For that reason, the degree of strength of a rule of inference, read as a permission, is the possible minimum.

3. *Mode of Achievement*: As was the case for assertions^F, the performance of an inferential step within the concept-script has the concept-script as its mode of achievement, which is an obvious conclusion to have. But since we're actually talking about a directive, it is included in the mode of achievement that the speaker is actually giving itself the permission for making the inferential step. Thus, under certain conditions, the speaker denegates the order to not act in such and such way. Of course, the speaker will not be successful if the preparatory conditions are not met, so there's no problem in saying that he can give himself the permission[21]. Therefore, an inferential step in the concept-script is achieved using the transition-signs that are specifically defined within the language.

4. *Propositional Content Conditions*: The propositional content to be considered in directives is that an speaker S directs a hearer H to perform some action A. The propositional condition in this case is specified by A being performed on the future. It makes no sense to direct something to be performed that was already performed or cannot be performed in the future. In the case of a permissive speech act, this is basically the same. The only difference is that we are equating H as S himself. In the case of rules of inference, the only permission being granted is that an assertion^F can be made. Nothing else. Thus, the propositional content conditions must be consistent with the conditions for assertions^F to be made. We can specify at least three:

   (a) *Action Restriction*: If someone allows you to perform some action A, it is A, and nothing else, that you have permission to perform. In the case here

[21] Similarly, If I am not the owner of the store, I cannot give myself the permission to take a product off the shelf free of charge.

considered, the only action permitted from inferential rules are assertions^F, and assertions^F only. Rules of inference are rules for making new assertions^F.

(b) *Possibility*: By granting the permission for making an inferential step, it is required for the speaker to be capable of making an assertion^F, *i.e.* there are means that make it possible for the speaker to assert^F something. It would be senseless to grant someone the permission for doing something for which he's incapable of doing in the first place. These conditions are the same conditions laid down for assertions, for instance, assertability and referentiability. If none of these conditions are in place (depending on which version of the concept-script one is considering), then no permission for assertability can be given from rules of inference.

(c) *Non-forbideness*: Obviously, to permit for a hearer to do A, it is presupposed that he is not prohibited to do A. This is not a problem, since by definition, the permission for doing A is the denegation of the prohibition of A. In this case, the denegation makes it possible for an assertion^F to be made.

5. *Preparatory Conditions*: What the reasoning agent have to do in performing inferences is to make assertions^F. In a sense, every assertion^F is by default prohibited, except when told it otherwise[22]. This is what the inferential rules do: it informs us the right situations in which a denegation for the prohibition can be uttered, granting the permission for a new assertion^F. Thus, different inferential rules will describe different preparatory conditions. In general, a permission for taking an inferential step is successful and non-defective following the conditions provided by each rule:

   (a) *Modus Ponens*: The usual description for the *modus ponens* rule is that from both a conditional $\Delta \rightarrow \Gamma$ and its antecedent Δ one can infer the consequent Γ. Everything that comes before the verb 'can' describes the preparatory conditions for the inference itself. In the case of the concept-script, these conditions are virtually the same, with the exception that everything should be asserted^F prior. Thus, in order for an inferential step to be successful and non defective, that is, for the speaker to successfully permit himself into making an assertion^F, it must:

      i. Have asserted^F some content — Δ, that is, '$\vdash \Delta$'

[22]With the clear exception of the Axioms, which can be asserted without such permissions.

ii. Have asserted^F some conditional content $\begin{array}{|l}\!\!-\,\Gamma\\ \!\!\lfloor\,\Delta\end{array}$, that is, '$\vdash\!\begin{array}{|l}\!\!-\,\Gamma\\ \!\!\lfloor\,\Delta\end{array}$'

iii. One of the asserted^F contents must be the antecedent sub-part of the other asserted^F content, just as i. is the the antecedent part of ii.

If all these conditions are met, then the agent have permission to assert^F no other than the consequent sub-part of the conditional proposition. To mark the utterance of this permission, the agent draws the horizontal line ——— successfully. If one of the conditions i, ii or iii is missing, and the speaker draws the line regardless, he has not made a successful utterance of a permission, thus, he has not used the inferential rule correctly.

Notice that the preparatory conditions for *modus ponens* describes the successful and non-defective conditions for someone to be *permitted* in using the rule. If the conditions are met, then the utterance of the rule (identified by the emission of a horizontal inferential line) is successful in *making a permission* by denegating a previous prohibition, not in making an assertion^F: I can successfully and non-defectively utter a permission without making the allowed assertion^F. We are here describing the conditions for a permission, not the conditions for an assertion^F. Moreover, the permission for asserting^F something can be successful and non-defective, while the assertion^F itself is not: if one of the premises (conditions i. or ii.) is mistakenly taken as an assertion^F, I'm still successful in permitting myself to assert^F something, even if the assertion^F itself is infelicitous[23].

(b) *Other rules*: The other rules, that are justified in a similar way, and their corresponding transition signs are:

i. *Hypothetical Syllogism*: a dashed line, – – – – – –;

ii. *Contraposition*: a cross, ╳;

iii. *Dilemma*: a dot-dashed line, -·-·-·-·-·-·-·-;

iv. *Concavity Introduction*: a turned bracket, ⌣.

All these signs are illocutionary force indicating devices, just as the continuous horizontal line for the *modus ponens*. Of course, each have its own preparatory conditions, but the reasoning is the same: if assertions^F of some specific types of formulas had been made, then an agent is allowed to perform an assertion^F of another formula of specific type related to the other.

[23] As it was the case for every consequence of Basic Law V.

The remaining rules do not have transition signs. These are:

v. *Permutability of Subcomponents*;

vi. *Fusion of Subcomponents*;

vii. *Fusion of Horizontals*;

viii. *Rule of Uniform Substitution*;

And given that they do not have transition signs, they are always performed simultaneously with another rule that has one. Thus, these rules have another separate preparatory condition: that one is only allowed to perform any of the rules v-viii if any transition sign is present. The other conditions are standard and very similar to the others.

In conclusion, all these rules of inference have preparatory conditions specified in such a way to determine when a successful and non-defective permissive speech act is being performed. Needless to say, these conditions all respect preservation of truth, as expected from any rule of inference in logic.

6. *Sincerity Conditions and Degree of Strength*: The sincerity condition, and the degree of strength of the sincerity condition is the same for usual directives. A speaker who permits successfully a hearer in acting in some way is expressing the desire or intend to allowing him to do so. Likewise, when drawing any inferential sign and uttering the permission to make an assertion^F, the speaker expresses the desire in permitting the assertion^F to be made.

7. *Direction of Fit*: In uttering a directive, a speaker is trying to make the world to fit his words, by attempting the hearer to perform some action. This might sound different when the illocutionary point is a permission, but it is not. When a speaker permits a hearer into doing some action A, he is denegating any order for not doing A. He is not just describing something, but revoking an order, to put it in other words. He is still aiming at some action. Hence, the direction of fit is still *world-to-word*. But, notice, that the world here considered is not the same as the world of thoughts in the direction o fit of assertions^F.

The multiple signs for inferences are all illocutionary force indicating devices in this sense. They all have the force of directives, particularly, permissions. All have the same mode of achievement, but they have different preparatory conditions, as each inferential rule is itself applicable in different situations. Since the inferential rules restricts themselves into permitting an agent to perform only one action (assertions^F), we can see how the rules are means for the manipulation of the concept-script language. It means that from a written transition sign in the concept-script, we allow the user into making new written signs, assertions^F.

6.4.3 Definitions

As previously stated, Frege discuss two kinds of definitions. There is what he calls stipulative definitions, and the fruitful definitions discussed in papers before the *Grundgesetze*. The latter is mostly related to the rule of uniform substitution and decomposition of functions. The former is the only one present formally in the language. Stipulative definitions are expressed with the aid of the $\Vdash$ sign, which also is an illocutionary force indicating device. In contrast, we usually find another type of definitions in the practice, what Ruffino, San mauro and Venturi (2021) labelled *contentual* definitions. These are the kind which attempts to not only stipulate the meaning of a new sign but to introduce some analysis of an ordinary notion in logical terminology. In Frege's case, these are the fruitfull definitions, but they do not occur in Frege's concept-script directly[24], thus, we only have to focus on the first kind, marked by $\Vdash$. The stipulative definitions has a single goal in mind: to fix the sense and reference of an new symbol introduced in the language, by means of stipulation. These are declarative speech acts restricted to a single domain: the concept-script language. The seven dimensions related to the definition sign are the following:

1. *Illocutionary Point*: the point of a definition is to fix the content, or sense and reference, of a newly introduced sign by means of a declaration. Therefore, the illocutionary point of all such definitions are the same: the declarative point. The declarative point is the speaker' attempt to perform a change in the world (relative to the domain in question) in virtue of her words and the institutional position that she represents, if this is relevant for the type of change desired. Differently from the other points, which still requires some sort of confirmation for its effects (an assertion depends on the states of affairs in order to be true; a directive depends on the action of the hearer in order to be fullfiled), the declarative speech act has immediate effects if it is a successful speech act. The same holds for the definition sign in the concept-script: if used correctly, the user achieves the desired changes in

[24]A different view is found in Macbeth (2012), as she sees Frege's stipulative definitions as also fruitful ones.

the formalism by the simple act of declaring it so changed.

2. *Degree of Strength of the Illocutionary Point*: In Searle and Vanderveken's taxonomy, the degree of strength of the illocutionary point of all declaratives is null: "one either does it or one does not" declare something (SEARLE; VANDERVEKEN, 1985, p.57), regardless of the chosen form. This means that there is no such thing as a stronger or weaker declaration.

3. *Mode of Achievement*: Depending on the specific kind of declaratives one may find different ways, or modes, to perform a declaration. Roughly, choosing the name of a child and officially registering the name in the notary's office have the same declarative point, but are achieved in different ways. Also, declarations usually requires more. As Searle and Vanderveken (1985, p.57) claims,

   > All declarative illocutionary forces have the mode of achievement that the speaker invokes his power or authority to perform the declaration and the general preparatory condition that the speaker has that power or authority to change the world by the performance of the appropriate utterance act.

   For example, only judges and priests can perform a marriage, and only the approved members of the *PhD* committee have the power to grant the title. Should it be the same for the concept-script? I think not. As far as how Frege designed the language, there's no previous deontic conditions for an agent to successfully and non-defectivelly perform declarations in the language, as long as all the preparatory and propositional conditions are fulfilled, as it is the case for any competent speaker of any language.

   Nonetheless, a declarative definition performed in the concept-script language have its own mode of achievement. Some restrictions and constraints are expected that often we don't find in the ordinary declaratives of language. Thus, as was the case for the other two illocutionary acts, definitions in the concept-script language are *sui generis*. We shall say *definitions*^{F} to denote such definitions, in contrast with the more ordinary notion of a declaration, which does not have a characteristic mode of achievement. Clearly, definitions^F are performed with the double vertical line for definitions: $\Vdash$. The sign can be read as "I declare that ...", following the propositional contents that we shall next specify.

4. *Propositional Content Conditions*: The range of declarations in general goes way beyond those of the concept-script. Searle and Vanderveken (1985, p.57) claims that

"there are no general propositional content conditions on declaratives, though the range of the subject matter over which declaratives can operate is obviously very restricted". Indeed, the cases in the concept-script are very restricted, and even in these cases, there are some propositional content conditions. These are what Searle (1979, p.18) called "declarations that concern language itself, as for example, when one says, 'I define, abbreviate, name, call or dub'". The general propositional for a definition^F that concerns the concept-script itself are given in the basic identity form in which one have a *definiens*, a *definiendum* and an identity signs relating both. Since we have two versions of the concept-script, there are two different versions of definitions^F. Taking Δ as the *definiens* and Γ as the *definiendum*, we have the following two cases:

(a) $\Vdash \Delta \equiv \Gamma$, as in the *Begriffsschrfit*. Here, the declarative is to be read 'let the sign Γ have the same conceptual-content than the sign Δ', using the identity of contents $\equiv$. Recall that this sign introduces a bifurcation in the language, as now the Δ and Γ are being mentioned, not used. Also, given that none of the flanking expressions Δ and Γ have corresponding content-strokes, they range over both propositional contents and objects.

(b) $\Vdash \Delta = \Gamma$, as in the *Grundgesetze*. Here, the declarative is to be read 'let Γ have the same sense and reference than Δ'. Obviously, these signs are being used, not mentioned as the previous sign for identity of contents. Thus, this definition^F is operating not on the expressions, but on the references and senses of the expressions. In terms of definitions^F, the effect is the same. It is only with respect to the identity of contents signs that motivated Frege into changing it to the usual identity sign.

Although the illocutionary point is the same in both cases, the propositional contents are not. There is another kind of propositional content condition that is applicable to both cases. For example, one would fail to name an object if what is given is not a name. One the concept-script case, we have the simple condition:

(a) *Contentual Conditions*: the sign Δ should range only over conceptual-contents or objects in the case of the *Begriffsschrift*, and it should range only over senses and references in the case of the *Grundgesetze*. Notice that this is not a condition for non-emptiness of the *definiendum* sign. It may be non-empty, but it would be worthless if it didn't expresses the expected content types.

The next conditions may sound about propositional conditions as well, but they do not refer to what kind of content the definition^F may accept, but the conditions for the act of definition^F to be successfully and non-defectively performed in the concept-script language.

5. *Preparatory Conditions*: In order to a definition^F be successful, and hence non-defective, it must follows some preparatory conditions. These preparatory conditions relates to the manipulation of the language in question. These are[25]:

   (a) *Structure*: Every definition^F is provided in a identity sentence (either with *Begriffsschrift*'s identity of content-sign $\equiv$, or the usual identity sign $=$ of *Grundgesetze*).

   (b) *Referentiability*: It could happen that a definition is successful in attaching a sense to a new expression, but not a reference. A empty-name, for example. This cases must be blocked. For that matter, every definition^F must always refer to something (*GGA*,§28).

   (c) *Unicity*: A name, or expression more generally, should never be defined^F twice.

   (d) *Simplicity*: The introduced sign, or name, must be simple. It must not have another defined^F or introduced name or expression.

   (e) *Newness of the Definiendum*: The introduced sign must not have a previously attached content (in either form). Otherwise, it cannot be uniquely defined^F to designate the content of the another sign;

   (f) *Non-emptiness of the Definiens*: The sign used for defining^F the new one should already have a fixed conceptual content or object (for the *Begriffsschrift*), and express a sense and refer to an object (for the *Grundgesetze*).

   (g) *Arity equality*: If a name is given for a $n-$ary function, it must have exactly n variables. Moreover, the defined^F name must preserve the same unbounded variable letters as in the *definiens*.

If these conditions are all met, than a successful definition^F is performed. And as it is the case for all definitions in general, if it is successful, it is non-defective, meaning that an agent uttering with the $\Vdash$, and fulfilling these conditions, is successful and non-defective in introducing a new sign in the language.

[25]Those are mostly found in (*GGA*,§33).

6. *Sincerity Conditions and Degree of Strength*: As we defined above, in Searle's and Vanderveken's taxonomy, there are two sincerity conditions for declarations: belief and desire. Likewise, someone defining^F in the concept-script expresses the desire that the *definiendum* has the same content as the *definiens*, and the belief that this act is successful in doing so. But these two play no role whatsoever. Even if an agent is insincere, but fulfill every condition for making a definition^F, he still has made a successful and non-defective declarative. For that matter, the Degree of Strength of the sincerity conditions is irrelevant either: if the conditions are all realized, nothing changes in believing or desiring harder.

7. *Direction of Fit*: Declarations have the interesting effect that, if successful, they automatically change the world to fit the propositional content being declared. For the definitions^F case this also holds but with some restrictions. First, definitions^F are constrained to the concept-script language. This is *the* world being considered here. The same is the case for directives in play with inferential rules. Even though directives have the world-to-word direction of fit, the ultimate goal of the inferential rules is to allow the agent to perform some changes in the concept-script by asserting^F something. In the same sense, a definition^F is by itself a change in the language. In contrast, assertions^F have the word-world direction of fit with 'world' being the world of thoughts, in the case of the mature *Grundgesetze* logic.

The case of definitions^F shows how the $\Vdash\!\!-$ sign is also an illocutionary force indicating device, used for putting forward declarations concerning the language of the concept-script. It is a restricted sign, given that it has no creative power other than choosing a new sign for the definition^F. Another piece of evidence that Frege saw his definition sign as such is that, as he realized, every definition has immediate and trivial consequences. If one define^F that $\Delta = \Gamma$, then one can immediately assert^F that $\Delta = \Gamma$ in virtue of the declaration. "Saying make it so", as Searle and Vanderveken (1985, p.57) summarize it.

6.5 Conclusion of Part III

As we can see, Frege's logic can be praised as one of the first (if not *the* first) to point out the importance of illocutionary acts in language. Not just natural language, but formal ones too. We have two lines of evidence for this: first, Frege's own remarks, and second, the consistency of reading the concept-script illocutionary force indicating devices (Assertions, Definitions, Inferential Lines) with modern Speech Act theory, as proposed by Searle and Vanderveken's taxonomy. This should also highlight even more the linguistic features of the concept-script, that we mentioned in the beginning.

Frege did started his career as mathematician. His doctoral dissertation, named *Über eine geometrische Darstellung der imaginären Gebilde in der Ebene* in 1873, and his dissertation for the *Habilitationsschrift* in Jena in 1874, *Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen*[26], were both written as any mathematical and logical text is written nowadays: with an hybrid language, partly mathematical, partly German. And as such, it made constant use of illocutionary devices, consistent with Ruffino, San Mauro and Venturi's (2021) analysis, apart from the fact that there is no IBMA's in Frege's texts.

One has to consider that when Frege made the point that ordinary languages are unfeasible for the precision required in a purely logical investigation, it is precisely this type of texts that he was referring to. Moreover, the basic point of developing a concept-script was to made the inferential process more precise, or as he claimed back in 1879, "to keep the chain of inferences free of gaps". And as he complained, "no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required" (*BS*, p.6). A similar complaint occurs in a undated letter to Edward Huntington, where he claimed to "first tried to carry out the proofs in the German language, but I soon became convinced that, carried out in this way, the proofs grew to an enormous length" (*PMC*, p.57).

The point is also more detailed discussed in the 1882 *Über die wissenschaftliche Berechtigung einer Begriffsschrift*, where Frege puts even more emphasis on the fact that ordinary language is not capable of expressing logical relations in full. He claims for example that "A strictly defined group of modes of inference is simply not present in [ordinary] language, so that on the basis of linguistic form we cannot distinguish between a 'gapless' advance and an omission of connecting links" (FREGE, 1972, p.85). And this is most certainly the case for the hybrid language of mathematics, which is part formal, part informal. Apart from having a few perspicuous way of expressing inferences, the mathematical notation is capable of an insufficient precision, but "[...] where the logical progression is different, it is generally necessary to express it in words. Thus, the arithmetic language of formulas lacks expressions for logical connections; and, therefore,

[26]Both are translated in *CP* as *On a Geometrical Representation of the Imaginary Forms in the Plane* and *Methods of Calculation based on an Extension of the Concept of Quantity*, respectively.

it does not merit the name of conceptual notation in full sense" (*idem.,* p.88). This is how Frege learned to write mathematical texts, as both his dissertations show, which he later deemed insufficient for a precise logical analysis of mathematical concepts.

All this goes into saying that Frege's invention of a concept-script is not simply a language entirely different from ordinary languages, as it still carries some of its features. The logic behind the ordinary language and the concept-script is the same, but it happens that the former includes the "psychological trappings, the clothing of the thought" (*PW*, p.142) that makes the logical analysis difficult to perform. But once adjustments are made and one is able to precisely express the logical laws under which all inferential steps are made, one achieves a language adequate for making judgements and inferences. This, I believe, was made clearer in reading the concepts-script with the aid of Speech Act Theory. Is not that such reading turns Frege's logic into an ordinary language, but that Frege's logic can be explained as a more precise and restrict language that happens to still be an illocutionary one.

This reading also highlights the place of Frege's in the logical tradition, and how he still define what we called a pragmatized system of logic. Furthermore, it is clear that modern day systems are not entirely free from pragmatical phenomena, as Ruffino, San Mauro and Venturi (2021) showed, which makes a *zurück zu Frege* even more important for better understanding of what we do in logic and mathematical practice today.

Conclusion

If what have being said here is plausible, then it is possible to read Frege's system of logic, the concept-script, as a performative language. My point in defending this reading is that the concept-script includes some performative signs, that, from the modern speech act point of view, can be read as illocutionary force indicating devices. I'm not claiming that Frege was developing a logic of speech acts in the strong sense, but only that such reading is consistent with how Frege developed his own logic at the time. This, I believe, is possible because some features of the concept-script foresaw the development of speech act theory. Of course, it would be anachronistic to view Frege as developing a logic of speech acts, but it is not anachronistic to say that Frege's logic can, at least, be read that way. My argument proceeds by building three main arguments about: Frege's syntax, its intended semantics, and finally, his position in the history of logic. While the first shows how Frege's logic manifest the performative features here defended, the second tries to established how such features are consistent with the remainder of his philosophy. The third, finally, assess the significance of the performative reading in the history of logic. Each point served the following goals.

Frege's syntax of the concept-script can be divided in two kinds of signs: descriptives, and performatives. The updates that his language received in the 1890's were mostly about the former, while the latter remained the same, apart from a few additions of rules of inferences. The presence of performative signs, and the needs for having them, follow from the fact that the concept-script is not a formal language in the modern sense, but a fragment of ordinary language, one that expresses more precisely its logical kernel. As showed in chapter 1, Frege's motivation meant only a partial departure from ordinary language. By being only partial, the concept-script was meant to have both parts of Leibniz' project: a *lingua characterica* and a *calculus ratiotinator*. Both points yields the concept-script as a scientific tool for discoveries. As such, both versions of the language can be found in their basic syntax, following the distinction between descriptive and performative signs. The class of performative signs is discussed in chapter 2. Particularly, they are of three kinds: the judgement-stroke, the definition-stroke and the transition signs for inferential rules. All three are better understood as signs representing actions needed for the conducting of proofs, fulfilling the idea of the concept-script as a calculus. The most discussed, and perhaps most controversial of the three is the judgement-stroke. I tried to show that reading it as something akin to an illocutionary device, representing a belief attitude, is consistent with Frege's comments that the essence of logic is fulfilled by the assertoric force of a sentence.

However, reading the judgement-stroke as a belief attitude seem to go against Frege's anti-psychologism. The same goes for the other two performatives. An agent could make false assertions, invalid inferences or even empty definitions. I tried to show that, despite these performatives, Frege has an answer against these problems. In chapter 3, the problem of psychologism is briefly sketched. But explaining how Frege is able to keep the soundness of deductions using judgements and inferences from an agent's perspective cannot rely on a formal proof, as modern logical systems do. This is because of the universalist conception that Frege adopts. From taking the concept-script as a fragment of language in general, it follows that there can be no advantage standpoint for studying it. Similarly, soundness can be only elucidated. But Frege did defended that judgements, inferences and definitions are objectively made. I argue in chapter 4 that Frege has two senses of objectivity for judgements. One of them is contentual, as it pertains to what judgements are about. The other is inferential, as it pertains to the conditions under which we judge. The contentual objectivity follows from Frege's strong platonist claims, specially in the mature part of his logic and philosophy. The inferential part stems from the normativity of logic. In this sense, I argue that Frege's motivation for judging the axioms and judging from rules of inference has a performative reasoning: that judgements are correctly made when it is not possible to deny the Truth of the content being judged. Finally, under normal circumstances, the contentual and the inferential objectivity come together, as from judgements to judgements one is advancing in the realm of thoughts and their relation to the True. The definitional objectivity completes the point of the chapter, in showing that Frege's demands for definitions are restricted to the concept-script language, operating solely in the possible manipulation of the expressions of the formalism.

If my arguments in the previous sections are at least plausible, then what should we take from it? I believe that Frege's insistence of taking the judgement-stroke as an assertoric force in particular has some historical significance. Historically, the fact that Frege's logic adopted a sign for making assertions is at odds with the current logical practice. Of course, current logical practice still made assertions, but as far as the study of logical systems is concerned, assertions and judgements are nowhere to be found. I argue in chapter 5 that this is due to a de-pragmatization of logical languages. Judgements had a brief existence in formal systems of logic, which made Frege's system even more uncanny to modern eyes. But ultimately, the presence of judgements in the concept-script is on a par with the fact that logical and also mathematical written languages still have assertions. Given that Frege's logic was a tool for scientific inquiries, what in modern eyes happens in the metalanguage, was being done in Frege's auxiliary one. The fact that the illocutionary portion found in modern day metalanguages is on a par with Frege's use of illocutionary devices in the auxiliary language can be compared from another perspective. This was done in chapter 6, where I described what the illocutionary dimensions of each performative sign of Frege's logic would be from the perspective of Searlean Speech Act Theory. The analysis there undertaken may sound *ad hoc* from a linguistic perspective, but so does the concept-script. If we want to design a strict illocutionary writen language for the practice of logic, relative to just three acts and similar to Frege's own written language, I believe that this chapter gives a sketch of what it would look like. But the analysis can also be taken retroactively. The fact that one can describe assertions, inferences and definitions in that way also shows that the path from Frege's use of such signs to Speech Act Theory is not entirely implausible. Finally, the fact that much of the current mathematical and logical practice is still made through assertions, definitions, and inferences in a informal and hybrid written language attests to the actuality of Frege's system of logic.

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