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Instituto de Filosofia e Ciências Humanas

DANIEL SANTIAGO JOCKWICH MARTINEZ

Models of non classical set theory

Modelos da teoria de conjuntos não clássicas

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Daniel Santiago Jockwich Martinez

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Orientador: Prof. Dr. Giorgio Venturi
Coorientador: Prof. Dr. Rodolfo Ertola Biraben

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Presidente da Comissão Julgadora

Dr. Sourav Tarafder
Universidade Estadual de Campinas/Campinas

Dr. Graham George Priest
University of Melbourne/Melbourne

Dr. Rodrigo de Alvarenga Freire
Universidade de Brasília /Brasília

Dra. Ekaterina Kubyshkina
Universidade Estadual de Campinas/Campinas

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São duas, Avé Marias
Dum rosário d'amarguras
Que eu rezo todos os dias.

Quando a dor me amargurar,
Quando sentir penas duras,
Só me podem consolar
Teus olhos, contas escuras.

Deles só brotam amores;
Não há sombras d'ironias;
Esses olhos sedutores
São duas Avé Marias.

Mas se a ira os vem turvar
Fazem-me sofrer torturas
E as contas todas rezar
Dum rosário d'amarguras.

Ou se os alaga a aflição
Peço p'ra ti alegrias
Numa fervente oração
Que rezo todos os dias.

(Poesia 1902-1917, Fernando Pessoa)

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Resumo

O problema essencial das teorias de conjuntos não clássicas é a falta de modelos naturais. Em particular, não temos modelos que sejam matematicamente expressivos. Nesta tese, pretendemos abordar este problema para várias classes de teorias de conjuntos não clássicas. Nós fornecemos modelos de teorias de conjuntos paraconsistentes e paracompletos na forma de modelos com valores algébricos. Mais especificamente, construímos uma classe de modelos paraconsistentes do fragmento livre de negação de ZF e construímos uma classe de modelos não clássicos de ZF que não são paraconsistentes nem paracompletos. Em seguida, exploramos duas extensões diferentes deste trabalho: (1) expandindo a linguagem da álgebra subjacente com diferentes operadores e (2) modificando a interpretação da pertinência e igualdade do conjunto em nossos modelos de valores algébricos. Isso dá origem a várias classes de modelos paraconsistentes da teoria dos conjuntos e a uma classe de modelos paracompletos da teoria dos conjuntos. Além disso, mostramos que esses modelos não satisfazem as mesmas sentenças da linguagem da teoria dos conjuntos e que podemos construir um modelo paraconsistente de ZFC baseado na Lógica do Paradoxo de Priest. Acreditamos que isso sugere que as teorias de conjuntos não clássicas e, em particular, as teorias de conjuntos paraconsistentes podem capturar uma quantidade razoável de matemática clássica.

Palavras-chave: Modelos algébricos da teoria dos conjuntos, Fundamentos da matemática, Teorias dos conjuntos não clássicas e Lógicas não clássicas.

Abstract

The essential weakness of non-classical set theories is their lack of natural models. In particular, we lack models that are mathematically expressive. In this thesis, we aim to tackle this problem for several classes of non-classical set theories. We provide models of paraconsistent and paracomplete set theories in the form of algebra-valued models. More especifically, we construct a class of paraconsistent models of the negation-free fragment of ZF and we build a class of non-classical models of ZF which are neither paraconsistent nor paracomplete. Then, we explore two different extensions of this work: (1) expanding the language of the underlying algebra with different operators and (2) modifying the interpretation of set membership and equality in our algebra-valued models. This gives rise to a several classes of paraconsistent models of set theory and to a class paracomplete models of set theory. Moreover, we show that these models are different from each other and that we can construct paraconsistent model of ZFC based on Priest's Logic of Paradox. We believe that this suggests that non-classical set theories and, in particular, paraconsistent set theories can capture a reasonable amount of standard mathematics.

Keywords: Algebra-valued models of Set Theory, Foundation of Mathematics, Non-classical Set Theories and Non-classical logics.

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Chapter 1

Introduction

In this thesis, we try to convince the reader that we can construct interesting models of non-classical set theories, in particular, paraconsistent set theories. However, we will also construct non-classical models of set theory which are not only paraconsistent but also paracomplete, as well as, non-classical models which are neither paraconsistent nor paracomplete. And most importantly, we will propose new model constructions which can be extended (in principle) to any non-classical logic with algebraic semantics. Moreover, we show that the resulting models are mathematically expressive enough to carry out a reasonable amount of classical set theory and that they are coherent with respect to their philosophical motivation.

In general, we believe that most non-classical set theories suffer from a lack of natural models or have models which are very weak. Consider, for instance, Quine's New Foundation or Restall's LP-set theory. Both set theories, New Foundation and LP-set theory, are without a doubt, interesting and well-motivated set theories, however, New Foundation has no natural model at all (PRIEST, 2006, p. 31) and the best that LP-set theory (à la Restall) can do is a single-element model (THOMAS, 2014, Section 3 & 4). Thus, these set theories are usually considered pointless (at least for mathematical practice).

Now, let us consider the case of intutionistic and constructive set theory. We argue that both of these set theories have gained legitimacy within the community of (classical) set theorists and logician mainly due to the model-theoretic advances for these set theories. In particular, we claim that it is precisely because of the existence of mathematically expressive models that these set theories seem appealing.

For instance, it is a well-known fact that it is consistent with intutionistic set theory that the real numbers are *subcountable* (we say that a set X is subcountable if it is the range of a function defined on a subset of the natural numbers) and *uncountable*, at the same time. However, this fact is simply due to the failure of a syntactic rule and remains simply a "strange" and meaningless derivation from a particular axiom system. But does there exist also a model where this strange fact really occurs ? We want to see it! And indeed, it has been shown that we can construct a Heyting-valued model of intutionistic set theory where the real numbers are subcountable and uncountable (BELL, 2014, p. 81). Thus, we have now gained an intuitive representation of this fact.

We believe that the situation is analogous with paraconsistent set theories. Until we manage to construct models for these set theories they remain simply abstract curiosities. In particular, to convince the classical set-theorist we need to construct models of paraconsistent set theories that are mathematically expressive. This constitutes the **main goal** of this thesis. So, on one side, we want to get models that resemble closely the cumulative hierarchy and, on the other side, we want models that validate some large fragment of ZF and its theorems. Thus, the project that we pursue in this thesis is a model-theoretic one.

We want to make two remarks on the importance of this project. First of all, we are showing how close paraconsistent set theories can get to classical set theory. We will show that the dividing line between classical and paraconsistent set theory is much finer than expected. Thus, we propose a new view on paraconsistent set theory. On the other hand, we will show that once that we assume that ZF is consistent we can do as much in paraconsistent ZF as in classical ZF. This result suggests that ZF is compatible with different logical environments and that non of these environments should be prioritized above another one. So, we endorse a pluralism with respect to the logical axioms that compose ZF. This renders evidence to the claim that ZF is not intrinsically classical. Moreover, we believe that this last point is a novel contribution to the philosophy of set theory.

Our main result is to have used algebra-valued model constructions to show that we can obtain infinitely many models of paraconsistent set theory which validate the negation-free fragment of ZF. Moreover, it was shown in (TARAFDER, 2021) that these models can reproduce cardinal and ordinal arithmetic in the same fashion just as classical models of set theory. Thus, we know that these models are mathematically expressive. Besides, we argue that our models are coherent to their underlying conception of set: the iterative conception of set. Moreover, we also show that we can modify the underlying algebraic structures of our models in such a way that we obtain infinitely many new models of paraconsistent and paracomplete set theory.

These results were improved recently in (JOCKWICH-MARTINEZ; TARAFDER; VENTURI, 2021c) where we show that the same algebraic structures that we explored in this thesis can give rise to paraconsistent models of full ZF. Thus, providing for the first time paraconsistent models of ZF where the ZF axioms hold consistently. Moreover, a possible criticism against our models could consist in arguing that the underlying algebraic structures of our models are too simple for mathematical practice. However, in (JOCKWICH-MARTINEZ; TARAFDER; VENTURI, 2021b) we demonstrate that we can construct paraconsistent models of full ZF based on a much wider class of algebras. Finally, in (TARAFDER; VENTURI, 2021) the authors provide the first applications of our models to the forcing method, therefore, also enriching classical set theory.

This thesis is structured as follows. In Chapter 2, we give all the basic definitions that we use throughout the thesis. We introduce some algebraic notions and some basic set theory. Then we review the topic of Boolean-valued models and the generalization of Boolean-valued models to non-classical contexts, which is due to (LÖWE; TARAFDER, 2015).

In Chapter 3, we give an overview of paraconsistent set theories that we can find in the literature. We present the five main proposals for developing a paraconsistent set theory: (1) the material approach, (2) the relevant approach, (3) the model-theoretic approach, (4) the da Costa approach, and (5) LFI-set theories. We argue that all of these approaches are unfeasible and that they all have considerable drawbacks. Moreover, we distinguish between two general approaches to paraconsistent set theory; the *naïve* paraconsistent set theory and the *iterative* paraconsistent set theory. Notice that the concept of an iterative paraconsistent set theory constitutes a novelty and has been introduced in (JOCKWICH-MARTINEZ; TARAFDER; VENTURI, 2021a). We argue that the iterative approach to paraconsistent set theory seems to be more promising than the naïve approach.

In Chapter 4, we try to extend the validity of the results of (LÖWE; TARAFDER, 2015) to a whole class of algebras. In particular, we explore two different classes of lattices suitable for this task: join complemented lattices and meet complemented lattices. We show that we can find a large class of meet complemented lattices that give rise to paraconsistent models of set theory, if we choose a suitable negation. In case that we choose an intutionistic negation for this class of lattices we manage to construct models of ZF, whose internal logic is neither classical, nor intuitionistic, nor paraconsistent. We apply these models to give an independence proof of $\mathsf{Foundation}$ from ZF.

In Chapter 5, we expand the underlying algebraic structures of our paraconsistent models of set theory with different unary operators which interpret the negation symbol in the language of set theory. As a result, we show that we can obtain infinitely many non-$\in$-elementarily equivalent models of set theory. So, we get *different* models of paraconsistent set theory. Moreover, we point out that there exists a trade-off between the regularity properties that a negation fulfills and the expressivity of our language. Finally, we give a philosophical account of negation, inspired by the algebraic framework we work in.

In Chapter 6, we explore two different ways of twisting algebra-valued models which give rise to more paraconsistent models of set theory, and also to paracomplete models of set theory. The first one consists in expanding totally-ordered Heyting algebras with a particular unary operator. We obtain a class of paraconsistent models of set theory which are non-$\in$-elementarily equivalent from all the paraconsistent models introduced previously. Moreover, we also get an infinite class of paracomplete models of set theory. The second one consists in modifying the interpretation map for membership and identity. So, we propose a new interpretation map for algebra-valued models. This interpretation map allows us to construct a model of ZFC that has as internal logic Priest's logic of Paradox. To end, we explore the mathematical tractability of the new interpretation map.

In Chapter 7, we summarize our main results and state some directions for future research.

Chapter 2

Technical Preliminaries

2.1 Basic Set Theory

The main axiom system used in this dissertation is the classical Zermelo-Fraenkel (ZF) axiom system, in the language of set theory $\mathcal{L}_\in$, which is displayed in Figure 2.1. In the schemes, φ is a formula with $n+2$ free variables. This formulation follows closely (BELL, 2005). Moreover, this definition of ZF is equivalent to intutionistic ZF, i.e., IZF and is chosen to simplify the task of checking the validity of the axioms in algebra-valued models. Additionally, we will denote with ZF^-, ZF minus the Foundation axiom scheme ($\mathsf{Foundation}_\varphi$) and with ZFC, ZF plus the Axiom of Choice (AC) .

Moreover, we will use the following abbreviations to state AC:

(i) $z = \{x\} =_{df.} \exists y(y \in z) \land \forall y(y \in z \to y = x)$,

(ii) $z = \{x, y\} =_{df.} \exists s(z \in z \land s = x) \land \exists t(t \in z \land t = y) \land \forall w(w \in z \to w = x \lor w = y)$,

(iii) $\mathsf{Pair}(z;\ x, y) =_{df.} \exists s\Big(s \in z \land (s = \{x\})\Big) \land \exists t\Big(t \in z \land (t = \{x, y\})\Big) \land$
$\forall w\Big(w \in z \to (w = \{x\}) \lor (w = \{x, y\})\Big)$,

(iv) $\mathsf{Func}(f) =_{df.} \forall x\Big(x \in f \to \exists s \exists t \mathsf{Pair}(x;\ s, t)\Big) \land$
$\forall x \forall y \forall s \forall t \forall w \forall v\Big((x \in f \land y \in f \land \mathsf{Pair}(x; w, s) \land \mathsf{Pair}(y;\ v, t) \land w = v) \to s = t\Big)$,

(v) $\mathsf{Dom}(f;\ x) =_{df.} \forall y\Big(y \in x \to \exists w \exists z\Big(w \in f \land \mathsf{Pair}(w;\ y, z)\Big)\Big) \land$
$\forall w\Big(w \in f \to \exists y \exists z \mathsf{Pair}(w;\ y, z) \land z \in x\Big)$.

$$\forall x \forall y \Big(\forall z (z \in x \leftrightarrow z \in y) \to x = y \Big) \qquad \text{(Extensionality)}$$
$$\forall x \forall y \exists z \forall w \Big(w \in z \leftrightarrow (w = x \vee w = y) \Big) \qquad \text{(Pairing)}$$
$$\exists x \Big(\exists y (\forall z\ \neg (z \in y) \wedge y \in x) \wedge \forall w (w \in x \to \exists u (u \in x \wedge w \in u)) \Big) \qquad \text{(Infinity)}$$
$$\forall x \exists y \forall z \Big(z \in y \leftrightarrow \exists w (w \in x \wedge z \in w) \Big) \qquad \text{(Union)}$$
$$\forall x \exists y \forall z \Big(z \in y \leftrightarrow \forall w (w \in z \to w \in x) \Big) \qquad \text{(Power Set)}$$
$$\forall p_0 \cdots \forall p_n \forall x \exists y \forall z \Big(z \in y \leftrightarrow (z \in x \wedge \varphi(z, p_0, \ldots, p_n)) \Big) \qquad (\mathsf{Separation}_\varphi)$$
$$\forall p_0 \cdots \forall p_{n-1} \forall x \Big(\forall y (y \in x \to \exists z \varphi(y, z, p_0, \ldots, p_{n-1})) \\ \to \exists w \forall v (v \in x \to \exists u (u \in w \wedge \varphi(v, u, p_0, \ldots, p_{n-1}))) \Big) \qquad (\mathsf{Collection}_\varphi)$$
$$\forall p_0 \cdots \forall p_n \forall x \Big((\forall y (y \in x \to \varphi(y, p_0, \ldots, p_n)) \to \varphi(x, p_0, \ldots, p_n)) \\ \to \forall z \varphi(z, p_0, \ldots, p_n) \Big) \qquad (\mathsf{Foundation}_\varphi)$$
$$\forall u \Big(\neg (u = \varnothing) \to \exists f \Big(\mathsf{Func}(f) \wedge \mathsf{Dom}(f; u) \wedge \forall x \Big(x \in u \wedge \neg (x = \varnothing) \\ \to \exists z \exists y (\mathsf{Pair}(z;\ x, y) \wedge z \in f \wedge y \in x) \Big) \Big) \Big) \qquad \text{(AC)}$$

Figure 2.1: The axioms of ZFC.

2.2 Algebraic Considerations

We use the following basic algebraic definitions.

Definition 2.2.1. *We call a poset* $\langle \mathbf{A}; \leq \rangle$ *a* meet semilattice *if every two elements* $x, y \in \mathbf{A}$ *have an infimum, denoted by* $x \wedge y$. *If there also exists a supremum,* $x \vee y$, *for any two elements* $x, y \in \mathbf{A}$, *then* $\langle \mathbf{A}; \leq \rangle$ *is a* lattice. *We say that* $\langle \mathbf{A}; \leq \rangle$ *is a* bounded lattice *if it is a lattice that has a greatest element* $\mathbf{1}$ *and a least element* $\mathbf{0}$. *A lattice* $\langle \mathbf{A}; \leq \rangle$ *is* complete *if the supremum* $\bigvee X$ *and the infimum* $\bigwedge X$ *exist for every* $X \subseteq \mathbf{A}$. *A lattice is called* distributive *if it satisfies the distributivity law, that is,* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *for all* x, y, z *in its universe.*

We are now in a position to define Boolean algebras and Heyting algebras.

Definition 2.2.2. *The structure* $\langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$ *is said to be a Boolean algebra* $\mathbb{B}$ *if*

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ *is a bounded distributive lattice,*

(ii) for every element $a \in \mathbf{A}$, $a \wedge a^* = \mathbf{0}$ *and* $a \vee a^* = \mathbf{1}$ *and*

(ii) for any two elements $a, b \in \mathbf{A}$, $a \Rightarrow b = (a^* \vee b)$.

Notice that, by definition any Boolean algebra is also a Heyting algebra.

Definition 2.2.3. *The structure* $\langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$ *is said to be a Heyting algebra* $\mathbb{H}$ *if*

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ *is a bounded distributive lattice,*

(ii) for any pair of elements $a, b \in \mathbf{A}$*, the set* $\{x \in \mathbf{A} : x \wedge a \leq b\}$ *has a largest element, which is* $a \Rightarrow b$ *and*

(iii) the unary operation * *is defined, for every* $a \in \mathbf{A}$*, as* $a^* = (a \Rightarrow \mathbf{0})$*.*

Moreover, we will also use the notion of filter and ideal.

Definition 2.2.4. *We say that a set* $G \subseteq \mathbf{A}$*, where* $\mathbf{A}$ *is the universe of the bounded distributive lattice* $\langle \mathbf{A}; \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$*, is a filter on* $\mathbf{A}$ *if*

(i) $\mathbf{1} \in G$*, but* $\mathbf{0} \notin G$*,*

(ii) if $x \in G$ *and* $x \leq y$*, then* $y \in G$*, and*

(iii) for any $x, y \in G$*,* $x \wedge y \in G$*.*

Definition 2.2.5. *We say that a set* $I \subseteq \mathbf{A}$*, where* $\mathbf{A}$ *is the universe of the bounded distributive lattice* $\langle \mathbf{A}; \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$*, is an ideal on* $\mathbf{A}$ *if*

(i) $\mathbf{0} \in I$*, but* $\mathbf{1} \notin I$*,*

(ii) if $y \in I$ *and* $y \leq x$*, then* $y \in I$*, and*

(iii) for any $x, y \in I$*,* $x \vee y \in I$*.*

Finally, we use the following notation. Let $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ be any bounded distributive lattice. Then, $Pos_{(\mathbb{A})} = \{x \in \mathbf{A} : x \neq \mathbf{0}\}$ and $Neg_{(\mathbb{A})} = \{x \in \mathbf{A} : x \neq \mathbf{1}\}$.

2.3 Boolean-valued Models of Set Theory

Boolean-valued models were introduced by Dana Scott, Robert M. Solovay, and Petr Vopenka and represent an algebraic presentation of the forcing method. Forcing was invented by Paul Cohen (COHEN, 1963) and is now considered the cornerstone of contemporary set theory. It is a tool for producing relative consistency results that have been used to prove the independence of the Continuum Hypothesis, i.e., the assertion that $2^{\aleph_0} = \aleph_1$, from the axioms of ZFC.

The intuitive idea behind Boolean-valued models is to replace every set x of the cumulative hierarchy $\mathbf{V}$ with its characteristic function c_x, i.e., the function c_x which has the Boolean algebra $2 = \{\mathbf{0}, \mathbf{1}\}$ as range such that $x \subseteq \mathrm{dom}(c_x)$ and for every $y \in \mathrm{dom}(c_x)$ we have:

$$\begin{aligned} c_x(y) &= \mathbf{1} \text{ if } y \in x \\ &= \mathbf{0}, \text{ otherwise.} \end{aligned}$$

Furthermore, identifying each $x \in \mathbf{V}$ with c_x, we can think of $\mathbf{V}$ as a class of two-valued functions. The problem with this procedure is that c_x fails to be *homogeneous* since the domain of c_x is in general not a set of two-valued functions. To fix this problem, (BELL, 2005) proposes the following definition by transfinite recursion.

$$\begin{aligned} \mathbf{V}^{(2)}_\alpha = \{x :\ & x \text{ is a function and } \mathrm{ran}(x) \subseteq 2 \\ & \text{and there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathbf{V}^{(2)}_\xi\} \text{ and} \\ \mathbf{V}^{(2)} = \{x :\ & \exists \alpha (x \in \mathbf{V}^{(2)}_\alpha)\}. \end{aligned}$$

We shall call $\mathbf{V}^{(2)}$ the *universe* of *two-valued homogeneous* functions or simply the *universe* of *two-valued* sets. Moreover, we observe that each *two-valued* set is a *two-valued* function whose domain is also a set of *two-valued* functions.

Now, we can generalize this procedure, so c_x takes values in any complete Boolean algebra $\mathbb{B}$, which gives rise to the *universe* of $\mathbb{B}$-*valued sets*. Given a complete Boolean algebra $\mathbb{B}$, a Boolean-valued model is defined by transfinite recursion:

$$\begin{aligned} \mathbf{V}^{(\mathbb{B})}_\alpha = \{x :\ & x \text{ is a function and } \mathrm{ran}(x) \subseteq \mathbb{B} \\ & \text{and there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathbf{V}^{(\mathbb{B})}_\xi\} \text{ and} \\ \mathbf{V}^{(\mathbb{B})} = \{x :\ & \exists \alpha (x \in \mathbf{V}^{(\mathbb{B})}_\alpha\}. \end{aligned}$$

Extending the language of set theory $\mathcal{L}_\in$ with constants for every element of $\mathbf{V}^{(\mathbb{B})}$ (which we call $\mathbb{B}$-names), we can define a new language $\mathcal{L}^{\mathbb{B}}_\in$, that allows one to talk about the Boolean-valued universe $\mathbf{V}^{(\mathbb{B})}$ by means of what are called $\mathbb{B}$-sentences (where with $\mathbb{B}$-sentences and $\mathbb{B}$-formulas we shall mean $\mathcal{L}_{\in}^{\mathbb{B}}$-sentences and $\mathcal{L}_{\in}^{\mathbb{B}}$-formulas respectively). Moreover, it is possible to define a $\mathbb{B}$-evaluation function $[\![\cdot]\!]^{\mathbb{B}}$ that assigns to every $\mathbb{B}$-sentence σ its Boolean truth value $[\![\sigma]\!]^{\mathbb{B}} \in \mathbf{A}$, where $\mathbf{A}$ is the universe of a complete Boolean algebra $\mathbb{B}$.[1] Hence, whenever $[\![\sigma]\!]^{\mathbb{B}} = \mathbf{1}_{\mathbb{B}}$, then we say that σ is *valid* in $\mathbf{V}^{(\mathbb{B})}$.

Notice that the definition of the $\mathbb{B}$-evaluation function cannot be fully defined within the language of set theory, because, the collection of all ordered pairs $\langle \sigma, [\![\sigma]\!]^{\mathbb{B}} \rangle$ is not a definable class in ZFC. Thus, we can think of this function as being defined meta linguistically.

The interpretations of the basic notions of equality and membership are defined by recursion on a particular well-founded relation. So, define for $x, y, u, v \in \mathbf{V}^{(\mathbb{B})}$

$$\langle x, y \rangle < \langle u, v \rangle \text{ iff either } (x \in \operatorname{dom}(u) \text{ and } y = v) \text{ or } \Big(x = u \text{ and } y \in \operatorname{dom}(u)\Big).$$

Then $<$ is a well-founded relation on the class $\mathbf{V}^{(\mathbb{B})} \times \mathbf{V}^{(\mathbb{B})} = \{\langle x, y \rangle : x \in \mathbf{V}^{(\mathbb{B})} \wedge y \in \mathbf{V}^{(\mathbb{B})}\}$. If we now fix for $u, v \in \mathbf{V}^{(\mathbb{B})}$:

$$G(\langle u, v \rangle) = \langle [\![u \in v]\!]^{\mathbb{B}}, [\![v \in u]\!]^{\mathbb{B}}, [\![u = v]\!]^{\mathbb{B}}, [\![v = u]\!]^{\mathbb{B}} \rangle,$$

then $[\![\cdot \in \cdot]\!]^{\mathbb{B}}$ and $[\![\cdot = \cdot]\!]^{\mathbb{B}}$ may be written for some class function F,

$$G(\langle u, v \rangle) = F(u, v, G : \{\langle x, y \rangle : \langle x, y \rangle < \langle u, v \rangle\}).$$

This constitutes a definition of G by recursion on $<$, and from G we can obtain the value of $[\![\cdot = \cdot]\!]^{\mathbb{B}}$ and $[\![\cdot \in \cdot]\!]^{\mathbb{B}}$. Given a complete Boolean algebra $\mathbb{B}$ and $u, v \in \mathbf{V}^{(\mathbb{B})}$,

$$\begin{aligned}
[\![u \in v]\!]^{\mathbb{B}} &= \bigvee_{x \in \operatorname{dom}(v)} \Big(v(x) \wedge [\![x = u]\!]^{\mathbb{B}}\Big), \\
[\![u = v]\!]^{\mathbb{B}} &= \bigwedge_{x \in \operatorname{dom}(u)} \Big(u(x) \Rightarrow [\![x \in v]\!]^{\mathbb{B}}\Big) \wedge \bigwedge_{y \in \operatorname{dom}(v)} \Big(v(y) \Rightarrow [\![y \in u]\!]^{\mathbb{B}}\Big).
\end{aligned}$$

Notice that the above cases of the evaluation function are inter-defined by simultaneous recursion on the hierarchy of $\mathbb{B}$-names. Moreover, the logical connectives

[1] $0_{\mathbb{B}}$ and $1_{\mathbb{B}}$, the smallest and largest elements of a Boolean algebra $\mathbb{B}$, respectively, may be interpreted as falsehood and truth. The other elements of the algebra have the intuitive meaning of intermediate truth values.

and the quantifiers are interpreted by recursion on the complexity of φ,

$$
\begin{aligned}
[\![\neg\varphi]\!]^{\mathbb{B}} &= ([\![\varphi]\!]^{\mathbb{B}})^*,\\
[\![\varphi \wedge \psi]\!]^{\mathbb{B}} &= [\![\varphi]\!]^{\mathbb{B}} \wedge_{\mathbb{B}} [\![\psi]\!]^{\mathbb{B}},\\
[\![\varphi \vee \psi]\!]^{\mathbb{B}} &= [\![\varphi]\!]^{\mathbb{B}} \vee_{\mathbb{B}} [\![\psi]\!]^{\mathbb{B}},\\
[\![\varphi \to \psi]\!]^{\mathbb{B}} &= [\![\varphi]\!]^{\mathbb{B}} \Rightarrow_{\mathbb{B}} [\![\psi]\!]^{\mathbb{B}},\\
[\![\exists x \varphi(x)]\!]^{\mathbb{B}} &= \bigvee_{u \in V^{(\mathbb{B})}} [\![\varphi(u)]\!]^{\mathbb{B}}, \text{ and}\\
[\![\forall x \varphi(x)]\!]^{\mathbb{B}} &= \bigwedge_{u \in V^{(\mathbb{B})}} [\![\varphi(u)]\!]^{\mathbb{B}},
\end{aligned}
$$

where on the left-hand side of the equality logical connectives are displayed, while on the right-hand side the operations of the complete Boolean algebra $\mathbb{B}$.

The interest of these construction lays in the fact that, all axioms of ZFC are valid in $\mathbf{V}^{(\mathbb{B})}$, i.e., if σ is an axiom of ZFC, then $[\![\sigma]\!]^{\mathbb{B}} = \mathbf{1}_{\mathbb{B}}$. On the other hand, if $[\![\sigma]\!]^{\mathbb{B}} < \mathbf{1}_{\mathbb{B}}$, we can say that σ is not consistent with ZFC. From now on, for the sake of readability, we will drop the superscript $\mathbb{B}$ from the evaluation map $[\![\cdot]\!]^{\mathbb{B}}$ (as well as the subscript from elements of the universe of $\mathbb{B}$), whenever it is clear from the context to which algebra we are referring.

Moreover, notice that:

Theorem 2.3.1. (BELL, 2005, Thm. 1.17). *All the axioms of the first-order predicate calculus with equality are true in $\mathbf{V}^{(\mathbb{B})}$, and all its rules of inference are valid in $\mathbf{V}^{(\mathbb{B})}$. In particular, we have:*

(i) $[\![u = u]\!] = \mathbf{1}$,

(ii) $u(x) \leq [\![x \in u]\!]$ *for every* $x \in \operatorname{dom}(u)$,

(iii) $[\![u = v]\!] = [\![v = u]\!]$,

(iv) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$,

(v) $[\![u = v]\!] \wedge [\![v \in w]\!] \leq [\![u \in w]\!]$,

(vi) $[\![u = v]\!] \wedge [\![u \in v]\!] \leq [\![u \in w]\!]$,

(vii) $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]$, *for any* $\mathbb{B}$*-formula.*

Furthermore, the following theorem specifies the laws governing the assignment of Boolean values to bounded quantifiers, where we abbreviate as usual $\exists x\big(x \in u \wedge \varphi(x)\big)$ as $\exists x \in u\ \varphi(x)$ and $\forall x\big(x \in u \to \varphi(x)\big)$ as $\forall x \in u\ \varphi(x)$. As will be shown later bounded quantifiers will play an essential role in the construction of algebra-valued models for set theory. In particular, all the ZFC axioms contain bounded quantifiers and any Δ_0-formula does *only* contain bounded quantifiers.

Theorem 2.3.2. (BELL, 2005, Thm. 1.18). *For any $\mathbb{B}$-formula $\varphi(x)$ with one free variable x, and all $u \in \mathbf{V}^{(\mathbb{B})}$,*

$$[\![\exists x \in u\ \varphi(x)]\!] = \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![\varphi(x)]\!]) \text{ and}$$
$$[\![\forall x \in u\ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

Furthermore, we can show that $\mathbf{V}^{(2)}$ is isomorphic to the standard universe $\mathbf{V}$. For this we will need the notion of *canonical* name.

Definition 2.3.3. (BELL, 2005, Def. 1.22). *For each $x \in \mathbf{V}$ we define*

$$\check{x} = \{\langle \check{y}, 1\rangle : y \in x\}.$$

Notice that for every $x \in \mathbf{V}$ we have $\check{x} \in \mathbf{V}^{(2)} \subseteq \mathbf{V}^{(\mathbb{B})}$. In particular, we can think of $\check{x}$ as a natural representative for each set $x \in \mathbf{V}$. Then, we have the following theorem where item (*iii*), (*iv*) and (*v*) indeed show that $\mathbf{V}^{(2)}$ is isomorphic to the standard universe $\mathbf{V}$. In particular, item (*v*) shows that that $\mathbf{V}$ and $\mathbf{V}^{(2)}$ validate the same sentences in the languages of set theory.

Theorem 2.3.4. (BELL, 2005, Thm. 1.23) .

(i) For any $x \in \mathbf{V}$, for each $u \in \mathbf{V}^{(\mathbb{B})}$ we have

$$[\![u \in \check{x}]\!] = \bigvee_{y \in x} \Big([\![u = \check{y}]\!]\Big).$$

(ii) For any $x, y \in \mathbf{V}$,

$$x \in y \leftrightarrow \mathbf{V}^{(2)} \models \check{x} \in \check{y} \text{ and } x = y \leftrightarrow \mathbf{V}^{(2)} \models \check{x} = \check{y}.$$

(iii) *The map* $x \mapsto \check{x}$ *is a one-one from* $\mathbf{V}$ *into* $\mathbf{V}^{(2)}$.

(iv) *For each* $u \in \mathbf{V}^{(2)}$ *there is a unique* $x \in \mathbf{V}$ *such that* $\mathbf{V}^{(\mathbb{B})} \models u = \check{x}$

(v) *For any formula* $\varphi(x_1, ..., x_n)$ *and* $x_1, ..., x_n \in \mathbf{V}$,

$$\varphi(x_1, ..., x_n) \leftrightarrow \mathbf{V}^{(2)} \models \varphi(\check{x_1}, ..., \check{x_n})$$

and for any Δ_0*-formula* $\psi(x_1, ..., x_n)$ *then*

$$\psi(x_1, ..., x_n) \leftrightarrow \mathbf{V}^{(\mathbb{B})} \models \psi(\check{x_1}, ..., \check{x_n}).$$

Notice that item (v) of Theorem 2.3.4 is quiet useful since we know that any Δ_0-formula will be valid in any Boolean-valued model. For instance, the formula $\mathsf{Ord}(x)$, which can be read as *x is an ordinal*, is a Δ_0-formula and then due to item (v) of Theorem 2.3.4 we get that $[\![\mathsf{Ord}(\check{\alpha})]\!] = \mathbf{1}$ for any ordinal α.

Furthermore, as we already pointed out we have the following well-known result.

Theorem 2.3.5. (BELL, 2005, Thm. 1.33) *Let* $\langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ *be any complete Boolean algebra* $\mathbb{B}$ *, then* $\mathbf{V}^{(\mathbb{B})} \models \mathsf{ZFC}$.

Notice that we can define analogously for every Heyting algebra $\mathbb{H}$, the Heyting-valued model $\mathbf{V}^{(\mathbb{H})}$ and we have a similar result concerning the validity of IZF. This result was originally proved by (GRAYSON, 1977).

Theorem 2.3.6. (GRAYSON, 1977, p. 410) *Let* $\langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ *be any complete Heyting algebra* $\mathbb{H}$*, then we have* $\mathbf{V}^{(\mathbb{H})} \models \mathsf{IZF}$.

Remember that the particular choice of the ZF axiom system that we use throughout this thesis is equivalent in strength to IZF, So we can also resume the two previous theorems in the following way.

Theorem 2.3.7. (BELL, 2005, Section 8) *Let* $\langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ *be any complete Heyting algebra* $\mathbb{H}$*, then we have* $\mathbf{V}^{(\mathbb{H})} \models \mathsf{ZF}$.

In the following section we will study further generalizations of these results.

2.4 Generalized algebra-valued Models of Set Theory

There exist several approaches to replacing Heyting algebras with different lattices to build non-classical models of set theory. In particular, this idea was implemented by (TITANI; KOZAWA, 2003), (TITANI, 1999), (TAKEUTI; TITANI, 1992) and (OZAWA, 2007) who replaced complete Heyting algebras $\mathbb{H}$ by lattices of a certain kind that allowed them to construct models of *quantum* and *fuzzy* set theory. In this thesis, our focus will be mainly on other non-classical set theories, such as *paraconsistent* and *paracomplete* set theories.

A first step in the construction of algebra-valued models for paraconsistent set theory has been undertaken in (LÖWE; TARAFDER, 2015), where the authors individuated a specific class of algebras suited for this purpose, which are called *deductive reasonable implication algebras* (DRI-algebras henceforth).

Definition 2.4.1. *The structure $\langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0}\rangle$ is said to be a deductive reasonable implication algebra if $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0}\rangle$ is a bounded distributive lattice and the binary operation $\Rightarrow$ satisfies the following properties: for any a, b, c in $\mathbf{A}$,*

$$(a \wedge b) \leq c \textit{ implies } a \leq (b \Rightarrow c), \tag{P1}$$

$$b \leq c \textit{ implies } (a \Rightarrow b) \leq (a \Rightarrow c), \tag{P2}$$

$$b \leq c \textit{ implies } (c \Rightarrow a) \leq (b \Rightarrow a), \tag{P3}$$

$$((a \wedge b) \Rightarrow c = a \Rightarrow (b \Rightarrow c). \tag{P4}$$

Remark 2.4.2. *Notice that the properties* (P1-P4) *only depend on the meet and the implication. Therefore, we will slightly abuse notation calling DRI-algebras also the structures $\langle \mathbf{A}; \wedge, \Rightarrow, ^*, \mathbf{0}\rangle$ where $\langle \mathbf{A}; \wedge, \mathbf{0}\rangle$ is a semilattice and for every a in $\mathbf{A}$, there exists its meet complement $a^* = \mathsf{max}\{b \in \mathbf{A} : a \wedge b = \mathbf{0}\}$ and for every a, b, c in $\mathbf{A}$, the binary operation $\Rightarrow$ satisfies properties* (P1-P4).

However, a further schema of conditions is required in order to validate the axioms of ZF, i.e.,

$$[\![\forall x \in u\ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x)\ \Rightarrow [\![\varphi(x)]\!]), \tag{$\mathcal{BQ}_\varphi$}$$

Table 2.1: Operations for $\mathbb{L}_3$

$\Rightarrow$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$	$\vee$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$	$\wedge$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

where $\varphi \in \mathcal{L}_{\in}^{\mathbb{A}}$. We will refer to this property as the bounded quantification property and in the case that we only allow negation-free formulas (NFF) as instances of this schema we refer to it as the NFF-bounded quantification property. This property, together with (P1-P4), is sufficient to show the validity of ZF in a DRI-algebra-valued model.

Moreover, we have the following result for formulas that contain bounded existential quantifiers.

Lemma 2.4.3. (LÖWE; TARAFDER, 2015, Prop. 3.2) *Let* $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ *be a DRI-algebra,* $\varphi(x)$ *an* $\mathbb{A}$*-formula with one free variable* x*, and* $u \in \mathbf{V}^{(\mathbb{A})}$*, then*

$$[\![\exists x \in u\ \varphi(x)]\!] \geq \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![\varphi(x)]\!]).$$

□

The situation is different for formulas that contain bounded universal quantifiers. In particular, we can show that there exist DRI-algebras for which neither the NFF-bounded quantification property nor the bounded quantification hold. Consider, for instance, the DRI-algebra $\mathbb{L}_3$ (depicted in Table 1.1), the formula $\varphi(x) =_{df.} (p_{\mathbf{0}} = x)$, and the $\mathbb{L}_3$-names

$$p_{\mathbf{0}} = \{\langle \varnothing, \mathbf{0} \rangle\}, p_{\frac{1}{2}} = \{\langle \varnothing, \frac{1}{2} \rangle\}, p_{\mathbf{1}} = \{\langle \varnothing, \mathbf{1} \rangle\} \text{ and } u = \{p_{\frac{1}{2}}, \frac{1}{2}\}.$$

Then we can calculate readily:

$$\frac{1}{2} = [\![\forall x \in u\ \varphi(x)]\!] < \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \ \Rightarrow [\![\varphi(x)]\!]) = \mathbf{1}.$$

Finally, as in the case of Boolean-valued models we can define a consequence relation that induces a notion of validity in DRI-valued models. Let $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ be a DRI-algebra. We say that a formula $\varphi \in \mathcal{L}_{\in}^{\mathbb{A}}$ is *F-valid* if $[\![\varphi]\!] \in F$. We will use the notation $\mathbf{V}^{(\mathbb{A})} \models_F \varphi$.

The crux of the approach of (LÖWE; TARAFDER, 2015) was to show that any DRI-algebra-valued model that satisfies the NFF-bounded quantification property is a model of NFF-ZF.

Theorem 2.4.4. (LÖWE; TARAFDER, 2015, Thms. 3 and 4) *Let $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ be a DRI-algebra such that $\mathbf{V}^{(\mathbb{A})}$ satisfies the NFF-bounded quantification property, and let F be any filter on $\mathbb{A}$. Then Extensionality, Pairing, Infinity, Union and NFF-Replacement, Power set and NFF-Separation are F-valid in $\mathbf{V}^{(\mathbb{A})}$; in fact, they all get the value $\mathbf{1}$.*

In particular, (LÖWE; TARAFDER, 2015) found a three-valued DRI-algebra, called $\mathbb{PS}_3$ (depicted in Table 2.2), for which the NFF-bounded quantification property holds in $\mathbf{V}^{(\mathbb{PS}_3)}$.

Theorem 2.4.5. (LÖWE; TARAFDER, 2015, Theorem 9)*. The DRI-algebra-valued model $\mathbf{V}^{(\mathbb{PS}_3)}$ has the NFF-bounded quantification property.*

Therefore, by Theorem 2.4.4 we get immediately that $\mathbf{V}^{(\mathbb{PS}_3)}$ is a model of NFF-ZF^{-} and it is proved separately that Foundation is valid, as well, in $\mathbf{V}^{(\mathbb{PS}_3)}$. Thus, $\mathbf{V}^{(\mathbb{PS}_3)} \models_F$ NFF-ZF. Then (LÖWE; TARAFDER, 2015) showed that it is possible to add the unary operator * to the signature of $\mathbb{PS}_3$ and to obtain the algebra, $(\mathbb{PS}_3,^*)$. Moreover, the propositional logic associated to $(\mathbb{PS}_3,^*)$ modulo the filter $Pos_{(\mathbb{PS}_3,^*)}$ is paraconsistent (see Definition 3.3.1).

Theorem 2.4.6. (LÖWE; TARAFDER, 2015, Theorem 10)*. There exists a sentence $\varphi \in \mathcal{L}_{\in}$ such that $\mathbf{V}^{(\mathbb{PS}_3,^*)} \models_{Pos_{(\mathbb{PS}_3,^*)}} \varphi$ and $\mathbf{V}^{(\mathbb{PS}_3,^*)} \models_{Pos_{(\mathbb{PS}_3,^*)}} \neg\varphi$.*

The resulting model $\mathbf{V}^{(\mathbb{PS}_3,^*)}$ modulo the filter $Pos_{(\mathbb{PS}_3,^*)}$ is the first example of an algebra-valued model that validates NFF-ZF and $\varphi \wedge \neg\varphi$ for some $\varphi \in \mathcal{L}_{\in}$ without trivializing our model, i.e., there exists a $\psi \in \mathcal{L}_{\in}$ such that $\mathbf{V}^{(\mathbb{PS}_3,^*)} \not\models_{Pos_{(\mathbb{PS}_3,^*)}} \psi$. In other words, $\mathbf{V}^{(\mathbb{PS}_3,^*)}$ modulo the filter $Pos_{(\mathbb{PS}_3,^*)}$ is a model of a paraconsistent set theory.

Table 2.2: Operations for $(\mathbb{PS}_3, ^*)$

$\Rightarrow$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\vee$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$

$\wedge$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

x	x^*
$\mathbf{1}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$

2.5 The Propositional Logic of $(\mathbb{A}, F)$ and $(\mathbf{V}^{(\mathbb{A})}, F)$

In this section, we will briefly talk about the propositional logic that corresponds to the complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ modulo a filter F and the underlying propositional logic of the respective $\mathbb{A}$-valued model $\mathbf{V}^{(\mathbb{A})}$ modulo a filter F. The relations between these two kinds of logics have been first studied in (LÖWE; PASSMANN; TARAFDER, 2021).

We denote by $\mathcal{L}_{Prop}$ a propositional language with countably many propositional letters. Given a complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$, by an $\mathbb{A}$-assignment we mean an homomorphism $\iota : \mathcal{L}_{Prop} \to \mathbf{A}$. We define the propositional logic of $(\mathbb{A}, F)$ as follows.

Definition 2.5.1. *Let $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ be a complete bounded distributive lattice and F a filter on $\mathbf{A}$, the propositional logic $\mathbf{L}(\mathbb{A}, F)$ is defined as*

$$\mathbf{L}(\mathbb{A}, F) = \{\varphi \in \mathcal{L}_{Prop} : \iota(\varphi) \in F \textit{ for all assignments } \iota\}.$$

This definition allows us to introduce the consequence relation $\models_{(F,\ \mathbb{A})}$, where $\psi \models_{(F,\ \mathbb{A})} \varphi$ if and only if for every $\mathbb{A}$-assignment ι we have, if $\iota(\psi) \in F$, then $\iota(\varphi) \in F$. Notice that for the sake of readability we will simply write $\varphi \models_F \psi$. Moreover, we can also define the the propositional logic of $(\mathbf{V}^{(\mathbb{A})}, F)$, by treating sentences in the language of set theory as propositional variables. Let $\mathsf{Sent}_{\in}$ be the class of sentences in the language $\mathcal{L}_{\in}$. By an $\in$-*translation* we mean an homomorphism $\mathsf{T} : \mathcal{L}_{Prop} \to \mathsf{Sent}_{\in}$.

Definition 2.5.2. (LÖWE; PASSMANN; TARAFDER, 2021). *Given a complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ and F a filter on $\mathbf{A}$, the propositional logic of $\mathbf{L}(\mathbf{V}^{(\mathbb{A})}, F)$ is defined as*

$$\mathbf{L}(\mathbf{V}^{(\mathbb{A})}, F) = \{\varphi \in \mathcal{L}_{Prop} : [\![\mathsf{T}(\varphi)]\!] \in F \textit{ for all S-translations } \mathsf{T}\}.$$

Furthermore, (PASSMANN, 2018) introduced the concepts of *loyalty* and *faithfulness*, which are used in (LÖWE; PASSMANN; TARAFDER, 2021) to study the relation between these two kinds of logics. In particular, we say that a model $\mathbf{V}^{(\mathbb{A})}$ is *loyal* to $(\mathbb{A}, F)$, if the propositional logic of $\mathbf{V}^{(\mathbb{A})}$ modulo a filter F matches the propositional logic associated to $(\mathbb{A}, F)$, whereas $\mathbf{V}^{(\mathbb{A})}$ is *faithful* to $\mathbb{A}$, if every element $a \in \mathbf{A}$ is the truth value of at least one sentence in the language of set theory. Let us introduce the following definition.

Definition 2.5.3. (LÖWE; PASSMANN; TARAFDER, 2021) *An $\mathbb{A}$-valued model $\mathbf{V}^{(\mathbb{A})}$ is called loyal to $(\mathbb{A}, F)$ if*

$$\mathbf{L}(\mathbb{A}, F) = \mathbf{L}(\mathbf{V}^{(\mathbb{A})}, F)$$

and faithfull to $\mathbb{A}$ if for every $a \in \mathbf{A}$ there exists a $\varphi \in \mathsf{Sent}_\in$ such that $[\![\varphi]\!]^{\mathbb{A}} = a$.

Furthermore, notice that faithfulness implies loyality.

Lemma 2.5.4. (LÖWE; PASSMANN; TARAFDER, 2021, Lemma 2.1). *Let $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ be a complete bounded distributive lattice and let $\mathbf{V}^{(\mathbb{A})}$ be the respective $\mathbb{A}$-valued model. Then if $\mathbf{V}^{(\mathbb{A})}$ is faithfull to $\mathbb{A}$, then it is loyal to $(\mathbb{A}, F)$ for any filter F.*

Moreover, we want also to be able to distinguish two algebra-valued models. In particular, we will say that two algebra-valued models are non-$\in$-elementarily equivalent with each other in the case that there exists a sentence in the language of set theory that is valid in one model, however, fails in the other.

Definition 2.5.5. *Let $\mathbb{A}_1 = \langle \mathbf{A}_1; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ and $\mathbb{A}_2 = \langle \mathbf{A}_2; \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0}\rangle$ be two complete bounded distributive lattices. Then we say that $\mathbf{V}^{(\mathbb{A}_1)}$ and $\mathbf{V}^{(\mathbb{A}_2)}$ are non-$\in$-elementarily equivalent with respect to a filter F_1 on $\mathbb{A}_1$ and a filter F_2 on $\mathbb{A}_2$, and write*

$$(\mathbf{V}^{(\mathbb{A}_1)}, F_1) \not\equiv_\in (\mathbf{V}^{(\mathbb{A}_2)}, F_2)$$

whenever $\mathbf{V}^{(\mathbb{A}_1)} \nvDash_{F_1} \varphi$ and $\mathbf{V}^{(\mathbb{A}_2)} \vDash_{F_2} \varphi$, for some $\varphi \in \mathsf{Sent}_\in$.

2.6 Classical and Intuitionistic Propositional Logic

In this section, we will specify the set of axioms that we will use to identify classical and intuitionistic propositional logic.

Definition 2.6.1. *Classical propositional logic* (CPL) *is defined over* $\mathcal{L}_{Prop}$ *by the following axiom schemes and rules.*

Axiom schemes:

(i) $\varphi \to (\psi \to \varphi)$,

(ii) $\big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big)$,

(iii) $\varphi \to (\varphi \vee \chi)$,

(iv) $\varphi \to (\chi \vee \varphi)$,

(v) $(\varphi \to \chi) \to \big((\psi \to \chi) \to \big((\varphi \vee \psi) \to \chi\big)\big)$,

(vi) $(\varphi \wedge \psi) \to \varphi$,

(vii) $(\varphi \wedge \psi) \to \psi$,

(viii) $\varphi \to \big(\psi \to (\varphi \wedge \psi)\big)$,

(ix) $\neg\neg\varphi \to \varphi$.

Inference rule:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,(\mathsf{MP}).$$

Notice that we can characterize intuitionistic propositional logic (IPL) with the same list of axiom schemes and inference rules, if we replace double negation elimination $(\neg\neg\varphi \to \varphi)$ for $\neg\varphi \to (\varphi \to \psi)$ and add the schema $(\varphi \to \psi) \to \big((\varphi \to \neg\psi) \to \neg\varphi\big)$.

Chapter 3

A Taxonomy of Paraconsistent Set Theories

Summary

In this chapter we present an overview of paraconsistent set theories that we can find in the literature. We start by giving a historic introduction to naïve set theory. In particular, we present the *pathological* sets that gave rise to the inconsistency of naïve set theory and show how they have motivated paraconsistent set theories. We go on to distinguish two possible approaches to paraconsistent set theory; *naïve paraconsistent set theory* and *iterative paraconsistent set theory*. Whereas the first approach is widely known, the second approach constitutes a novelty that has only been studied recently. Unlike the first approach, the latter opts for developing paraconsistent set theory taking ZF as logical axioms instead of Comprehension and Extensionality.

3.1 Cantor's Paradise

> Quando malsucedidos, os melhores projetos parecem estúpidos!
>
> (DOSTOIÉVSKI, 2016)

The birth of set theory can be traced back to the early work of Georg Cantor in the field of functional analysis, in particular, to the study of trigonometric series and their representations. In (CANTOR et al., 1932, pp. 92-102), he begins to study infinite sets of discontinuities, which lead him to develop his account of the real numbers in terms of *Cauchy-sequences* but also to introduce one of the first cornerstones of set theory, the *ordinals*.

Then two years later, Cantor publishes (CANTOR et al., 1932, pp. 115-118), where he unfolds the second fundamental notion of set theory, *cardinality*. More specifically, he proves that there exists no bijection between the set of natural numbers and the set of real numbers. We can grasp the infinite! In one of the most beautiful and intellectually deepest turn of events in modern history, Cantor creates a mathematical theory able to capture hierarchies in the infinite.

The two underlying (non-logical) axioms of this theory, also known as *naïve* set theory, are Extensionality, which states that two sets are identical if they have the same members, i.e.,

$$\forall x \forall y \Big(\forall z (z \in x \leftrightarrow z \in y) \rightarrow (x = y)\Big) \qquad \text{(Extensionality)}$$

and Unrestricted Comprehension, which reflects the intuition that every well-defined formula of the language of set theory constitutes a set, so if y does not occur free in φ, then

$$\exists y \forall x \Big(x \in y \leftrightarrow \varphi(x)\Big). \qquad \text{(Unrestricted Comprehension)}$$

3.2 Trouble in Heaven

Even though the axiom schemes of naïve set theory might seem very appealing, given their intuitiveness and elegant presentation, we have that naïve set theory is inconsistent.

Among the many inconsistencies (or *pathological* sets) that do arise from naïve set theory we have, for instance, the Russell set, i.e., the set of non-self membered sets (FREGE, 1979, p. 133), the universal set, i.e., the set of sets (ZERMELO, 1932, pp. 448-449) and the set of (von Neumann) ordinals (BURALI-FORTI, 1897).

Each of these sets highlights some “problematic” aspect of naïve set theory. In the case of the Russell set, we have a tension between the $\in$-relation and unrestricted set formation and in the case of the universal set we have a similar conflict between the openendedness of the set-theoretic universe and the sets we can instantiate by unrestricted comprehension. Similar considerations hold for the set of (von Neumann) ordinals.

Notice that the following proofs are carried out within ZF and are thus announced as theorems of ZF. Moreover, the next proof was discovered by Russell and communicated to Frege in a letter that we can find in (FREGE, 1979, p. 130).

Theorem 3.2.1. (FREGE, 1979). *There is no Russell set.*

Proof. Suppose there exists the Russel Set, call it R. Now, suppose $R \in R$. Then by definition of R, $R \notin R$, a contradiction. So, $R \notin R$. Then, again by definition of R, $R \in R$, a contradiction. So, R does not exist. $\square$

The following proof was first discovered by Cantor and then promptly communicated to Dedekind in (ZERMELO, 1932, p. 448). However, we give the following proof by Zermelo.

Definition 3.2.2. *A set X is* transitive*, if for all $x \in X$ and for all $y \in x$ we have $y \in X$.*

Theorem 3.2.3. (ZERMELO, 1932). *There is no universal set.*

Proof. Suppose there exists the *universal set*, say U. Then we construct the power set $P(U)$ of U. Applying Cantor's theorem it follows that $|U| < |P(U)|$. But U was supposed to be the set of all sets, so $P(U)$ is an element of U. Hence, $P(U) \subseteq U$ since U is transitive. Therefore, $|P(U)| \leq |U|$, which delivers us the desired contradiction. $\square$

The next set that we introduce is a little bit more complex, since it relies on the definition of (von Neumann) ordinals. Therefore, we will introduce some basic definitions regarding order types in set theory, which will allow us to state this set within ZF.

Definition 3.2.4. *A* partial ordering *of a set X is a binary relation $\leq$ on X such that*

$$\text{for any } x \in X, x \leq x, \tag{reflexivity}$$

$$\text{for any } x, y, z \in X, \text{ if } x \leq y \text{ and } y \leq z \text{ then } x \leq z, \tag{transitivity}$$

$$\text{for any } x, y \in X, \text{ if } x \leq y \text{ and } y \leq x \text{ then } y = x. \tag{antisymmetry}$$

Definition 3.2.5. *A poset $(X, \leq)$ consists of a set X with a partial ordering $\leq$ on X. A poset $(X, \leq)$ is said to be* well-founded *if for any non-empty $x \subseteq X$, x has a minimal element k, i.e., $k \in x$ and for all $y \in X$, if $y < k$ then $y \notin x$. A totally ordered set $(X, \leq)$ is a poset such that*

$$\text{for any } x, y \in X \ (x \leq y \text{ or } y \leq x). \qquad \text{(connectedness)}$$

Definition 3.2.6. *A* well-ordered set $(X, \leq)$ *is a totally ordered and well-founded poset.*

Definition 3.2.7. *A strict ordering of a set X is a binary relation $<$ such that $<$ is transitive, antisymmetric and*

$$\text{for any } x \in X \ (x \not< x). \qquad \text{(irreflexivity)}$$

Definition 3.2.8. *A* von Neumann ordinal α *is a transitive set which is strictly well-ordered by $\in$.*

Theorem 3.2.9. (BURALI-FORTI, 1897). *There is no set of (von Neumann) ordinals.*

Proof. Suppose there exists the set of ordinals and denote it by Ω. We want to show that Ω is transitive and well-ordered by $\in$. For any $\alpha \in \Omega$, every $\gamma < \alpha$ is contained in Ω given that each $\gamma < \alpha$ is an ordinal itself. So Ω is transitive. But we also know that Ω is well-ordered by $\in$, since every $\alpha \in \Omega$ is well-ordered by $\in$. Thus; Ω is an ordinal and so $\Omega \in \Omega$. But by the definition of strict well-order, $\Omega \notin \Omega$, a contradiction. □

Despite being *pathological* sets, in the sense that these sets witness the inconsistency of naïve set theory, they might be insightful for studying models of set theory. This concerns, in particular, non-classical models of set theory, where we can accommodate these sets.

> However, it is always a mistake to think of anything in mathematics as a mere pathology, for there are no such things in mathematics. (...) One should think of the paradoxes as supernatural creatures, oracles, minor demons, etc. (or perhaps the Aleph in the eponymous story by Borges)-on whom one should keep a weather eye in case they make prophecies or by some other means inadvertently divulge information from another world not normally obtainable otherwise. (FORSTER, 1995)

3.3 Naïve and Iterative Paraconsistent Set Theory

Naïve paraconsistent set theory is the attempt of maintaining some *pathological* sets or inconsistencies, within a non-trivial set theory. The crux of this proposal is to weaken the logical axioms of naïve set theory and to keep the non-logical axioms. This theoretical move is in direct opposition to the more orthodox strategy of the iterative conception of set, which proposes to restrict the non-logical axioms of naïve set theory (specifically Unrestricted Comprehension) and to maintain the same logical axioms, i.e., classical logic.

- Naïve paraconsistent set theory: The *logical axioms* of naïve set theory are wrong.
- The iterative conception of set: The *non-logical axioms* of naïve set theory are wrong and the *logical axioms* are classical.

For a seminal exposition of the iterative conception of sets, consider (BOOLOS, 1971) and (BOOLOS, 1989).

Let us turn to the first view point. The main claim of the paraconsistent logician is that the logical axioms of set theory are not classical, but paraconsistent. Moreover, we say that a logic is *paraconsistent*, if the existence of a contradiction does not trivialize our logic. Formally:

Definition 3.3.1. *A propositional logic* $\mathbf{L}$ *is paraconsistent, if the law of explosion fails for some formulas* φ, ψ *in our language, i.e.,*

$$\textit{there exist } \varphi, \psi \in \mathcal{L}_{Prop} \textit{ such that } (\varphi \wedge \neg\varphi) \nvdash \psi. \tag{ECQ}$$

The original motivation of this proposal goes along the following lines; the cost of removing *pathological* sets of our universe and maintaining the underlying logic is higher than the cost of weakening our underlying logic and accepting them.

Notice, however, that we are not necessarily bounded to the naïve conception of set in order to pursue a paraconsistent set theory. We propose that we can also build paraconsistent set theories grounded on the iterative conception. Therefore, we can identify two classes of paraconsistent set theories:

- Naïve paraconsistent set theory: The *true* non-logical axioms of paraconsistent set theory are Extensionality and Unrestricted Comprehension.

- Iterative paraconsistent set theory: The *true* non-logical axioms of paraconsistent set theory are ZF-like.

So, even though both classes of paraconsistent set theories agree on the fact that the underlying logic of set theory should be paraconsistent, they disagree concerning the right non-logical axioms of set theory (which again will make them differ in their respective choice of underlying logic). Furthermore, this last choice is heavily restrained by two criteria: *non-triviality* and logical *strength*. In more intuitive terms, we neither want that our set theory proves too much, nor too little.

In the case of naïve paraconsistent set theory, we are worried about the fact that Unrestricted Comprehension without ECQ, can still generate a trivial system. In particular, this concerns *Curry's paradox* and its set-theoretical variations.

Theorem 3.3.2. *Let* **T** *be a theory which validates* Unrestricted Comprehension, MP *and* Contraction. *Then* $T \vdash \varphi$, *for any* φ.

Proof. Let φ be any formula.

1. $\exists y \forall x \big(x \in y \leftrightarrow (x \in x \to \varphi)\big)$ (Unrestricted Comprehension)
2. $\forall x \big(x \in a \leftrightarrow (x \in x \to \varphi)\big)$ (Supposition)
3. $a \in a \leftrightarrow (a \in a \to \varphi)$
4. $a \in a \to (a \in a \to \varphi)$ (Simplification)
5. $a \in a \to \varphi$ (Contraction)
6. $a \in a$ (MP 3,5)
7. φ (MP 5,6)
8. φ ($\exists$ elim. 1,2-7)

$\square$

Notice that we still get the *Curry* paradox if we just have Unrestricted Comprehension, MP and the *modus ponens axiom* (MPA), i.e.,

$$\big(\varphi \wedge (\varphi \to \psi)\big) \to \psi. \tag{MPA}$$

The fact that Unrestricted Comprehension together with this minimal set of logical axioms trivializes our set theory rules out already many paraconsistent calculi, such as W (the logic of relevant entailment) and R (the logic of relevant implication), for the construction of a naïve paraconsistent set theory.

Furthermore, in the case of iterative paraconsistent set theory, this constraint is usually given for granted, since we have adopted ZF-like axioms, which block out the paradoxes and ensure thus non-triviality.

The second constraint, i.e., logical *strength*, applies equally to both strategies. Any non-classical set theory should be able to reproduce a *reasonable* amount of standard set theory. But how much is *enough*? It seems difficult to give an exact cut-off point here since set-theoretical strength is an inherently *vague* notion within set theory. However, we believe that we can not talk about sets without models, and secondly, the final word on how much is *enough* should belong to set-theoretical practice. Thus, we emphasize specifically the *model-theoretic* properties of a set theory. As a consequence, it is an imperative requirement for any (naïve or iterative) paraconsistent set theory to have a natural model, which is equipped with a rich ontology. That is, it should have an ontology which allows the existence of objects such as singletons, pairs, non-identical sets, linear orders, ordinals, etc.

So despite not having an exact cut-off point for this criteria, we can still make the request for a natural model where we can carry out basic set-theoretical arguments. However, the ideal scenario would consist in having a fully-fledged set-theoretical model that allows us to carry out contemporary set-theoretical techniques such as forcing and where we can accommodate as many sets as possible.

> "We might not require everything; we might be prepared to write off various results concerning large cardinality, or peculiar consequences of the Axiom of Choice. But if we lose too much, set theory is voided of both its use and interest." (PRIEST, 2006, p. 248)

Similarly, (INCURVATI, 2020, p. 104) goes on:

> "If it is too weak, set theory is deprived of the role, with which it has traditionally been conferred, of giving a foundation for mathematics and of

providing a tool for understanding the infinite. We need to preserve at least certain crucial results of set theory."

Notice that in the case of the naïve paraconsistent strategy there exists a clear tension between these two desiderata. The first criterion demands that we should choose a logic that is relatively weak concerning its conditional since we either lose MP or Contraction, which are arguably both fundamental properties that we would like to attribute to a conditional. Now, the second criterion, on the other hand, pressures us to focus on calculi which are as close as possible to classical logic, so they remain sufficiently expressive. Hence, the first criterion pushes us away from classicality, whereas the second one demands us to get closer to it.

Moreover, given the granted non-triviality of set theories following the iterative paraconsistent strategy, these do not face the same dilemma. Though, they seem to evidence a similar trade-off between non-classicality and logical strength, when it comes to the treatment of identity. In particular, in (almost) all the algebra-valued models for paraconsistent set theories that we will develop in the following chapters we have instances of Leibniz's law of indiscernibility of identicals (LL) and Separation that will fail. This issue has been discussed extensively in (JOCKWICH-MARTINEZ; TARAFDER; VENTURI, 2021a).

In the following sections of this chapter, we review the naïve and iterative paraconsistent set theories that we can find in the literature. We start by giving an overview of the different approaches following the naïve paraconsistent strategy. As already mentioned, due to the scope of the *Curry paradox* we have to weaken the implication of our language, so we can either block MP (*material* approach) or Contraction and MPA (*relevant* approach). Moreover, we also review a third approach; the *model-theoretic* approach. The crux of this approach is, roughly speaking, to show that there exists a set-theoretic model that validates both ZF and Unrestricted Comprehension and that contains some large fragment of the cumulative hierarchy as consistent inner model. We will follow the discussion in (INCURVATI, 2020) and argue that all these approaches face philosophical and technical objections.

Then, we will consider a fourth approach, the *da-Costa* approach, which does not fit in as nicely as the other approaches in the dichotomy described above. The reason is the following. Instead of using the non-logical axioms of naïve set theory, this approach consists of combining a modified version of Stratified Comprehension and Extensionality with the logical axioms of some particular paraconsistent logic (the C-systems). We will argue that this approach is unfeasible, as well, given that it is not coherent with its motivation.

Finally, we have the iterative paraconsistent strategy. The heart of this approach consists of constructing set theories where the non-logical axioms are ZF-like axiom systems and the logical axioms are those of a paraconsistent logic. In particular, the set theory developed in (LÖWE; TARAFDER, 2015) is a paradigmatic example of this approach (see Chapter 2.4). This is due to the reason that the mentioned set theory takes NFF-ZF as non-logical axioms and $\mathbf{L}\Big((\mathbb{PS}_3,^*), Pos_{(\mathbb{PS}_3,^*)}\Big)$ as logical axioms. Moreover, the only existing paraconsistent set theories that we could find in the literature that follow this approach are LFI-set theories, i.e., ZF-like set theories where the logical axioms are determined by a logic of formal inconsistency (LFI).

We will show, however, that this class of set theories also face philosophical objections and point out diverse difficulties that one encounters when constructing algebra-valued models for these set theories (see Chapter 4.5).

Notice that all the models of paraconsistent set theory, presented in the latter chapters, follow the iterative paraconsistent strategy and are constructed in the form of algebra-valued models.

We summarize:

Naïve paraconsistent strategy:

- The *material* approach: The underlying logic of a paraconsistent set theory should interpret $\rightarrow$ as material conditional.

- The *relevant* approach: The underlying logic of a paraconsistent set theory should interpret $\rightarrow$ as relevant conditional.

- The *model-theoretic* approach: There exist models that validates both ZFC and the non-logical axioms of of naïve set theory. These models are constructed via two possible equivalence relations: the type-lift and the Hamkins-type-lift. In the case of the type-lift, we start with a model of ZF with two inaccessible cardinals, say, κ_1 and κ_2. While everything is preserved below V_{κ_1}, on the other hand, everything is collapsed between V_{κ_1} and V_{κ_2}. The resulting object of such a collapse, call it a, is what witnesses the paraconsistency of the model. Instead, the standard hierarchy below κ_1 is responsible for the validity of ZF in a cumulative hierarchy.

- The *da-Costa* approach: The logical axioms of naïve paraconsistent set theory should be provided by a C-system, combined with (a modified version of) the non-logical axioms of *New Foundation* (NF).

Iterative paraconsistent strategy:

- The non-logical axioms of a paraconsistent set theory should be ZF-like and the logical axioms are given by a paraconsistent logic. In particular, we will explore LFI-set theories.

3.4 The Material Approach

Following the material approach, the underlying logic of a naïve paraconsistent set theory should invalidate MP and define the materical conditional as follows:

$$\varphi \to \psi =_{df.} \neg\varphi \vee \psi.$$

Notice that the biconditional $\leftrightarrow$ is defined as usual.

Moreover, Graham Priest has proposed to apply the three-valued logic LP for this enterprise. (PRIEST, 2006, p. 248). The truth values of LP are $\mathbf{1}$ (true), $\mathbf{0}$ (false) and $\frac{1}{2}$ (both true and false), where $\mathbf{1}$ are $\frac{1}{2}$ act as designated values. We will use D to denote this set, so $D = \{\mathbf{1}, \frac{1}{2}\}$. The truth tables of the logical connectives of LP are depicted in Table 2.1. Furthermore, let Var be an infinite set of countably many propositional variables and R_n a set of relation symbols, where n indicates the arity. Then we can define a model for LP as $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{D}$ is a domain of objects and $\mathcal{I}$ an interpretation function.

Furthermore, we say that a function $S : \mathsf{Var} \to \mathcal{D}$ is a variable assignment for $\mathcal{M}$ and $S[x/a]$ denotes the same variable assignment where we uniformly substitute x with a. Then, given a model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ and a variable assignment S, we define an

Table 3.1: Operations for LP

$\rightarrow$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\vee$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$

$\wedge$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

x	$\neg x$
$\mathbf{1}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$

evaluation function $v_S^{\mathcal{M}}$ as follows. If φ is an atomic formula, then

$$v_S^{\mathcal{M}}\Big(R\langle x_1, ..., x_n\rangle\Big) = \mathcal{I}\Big(R\langle x_1, ..., x_n\rangle\Big).$$

If φ is a formula of the form $\neg\psi$, $\chi \wedge \psi$, $\chi \vee \psi$, $\chi \rightarrow \psi$, then $v_S^{\mathcal{M}}$ assigns truth values to φ according to Table 3.1. Furthermore, if φ is of the form $\forall x\psi(x)$ we have

$$v_S^{\mathcal{M}}\Big(\forall x\psi(x)\Big) = \mathsf{min}\{v_S^{\mathcal{M}}\Big(S_{[x/d]}\psi(x)\Big)\}_{d\in\mathcal{D}},$$

where $\mathbf{0} < \frac{1}{2} < \mathbf{1}$. Finally, to conclude the presentation of the semantics of LP, we define a consequence relation. As usual, this relation is given in terms of the preservation of designated truth values. Morevoer, we say that a evaluation function satisfies a sentence φ, if $v_S^{\mathcal{M}}(\varphi) \in D$. So, let Σ be a set of sentences and φ a sentence. Then,

$$\Sigma \models \varphi \text{ iff for every } \mathcal{M} \text{ and } S, \text{ whenever } v_S^{\mathcal{M}} \text{ satisfies } \Sigma, \text{ then } v_S^{\mathcal{M}} \text{ satisfies } \varphi.$$

Now, we are in the position to state the most salient logical properties of LP. Let (TB) denote the inference rule known as *transitivity of the biconditional* and (DS) the inference rule, known as *disjunctive syllogism*:

$$\frac{\neg\varphi \qquad \varphi \vee \psi}{\psi}\,(\mathsf{DS}).$$

Notice that ECQ, DS and TB fail within LP. For the failure of ECQ and DS, simply consider a valuation $v_S^{\mathcal{M}}$ and two sentences φ, ψ such that $v_S^{\mathcal{M}}(\varphi) = \frac{1}{2}$ and $v_S^{\mathcal{M}}(\psi) = \mathbf{0}$. Moreover, for the failure of TB, consider a valuation $v_S^{\mathcal{M}}$ and three sentences φ, ψ, χ such that we have $v_S^{\mathcal{M}}(\varphi) = \mathbf{1}, v_S^{\mathcal{M}}(\psi) = \frac{1}{2}$, and $v_S^{\mathcal{M}}(\chi) = \mathbf{0}$.

So far we had a quick look at LP. Moreover, we can add a binary predicate $=$ to LP, where $x = y$ receives value $\mathbf{1}$ or $\frac{1}{2}$ in the case that $x = y$ in our meta-theory. The result is first-order LP with identity ($\mathsf{LP}_=$). We say that a model $\mathcal{M}$ has a *standard* identity if for any $x, y \in \mathsf{Var}$ we have $\mathcal{I}(x = y) \in \{\mathbf{1}, \mathbf{0}\}$. Otherwise, we say that $\mathcal{M}$ has a *glutty* identity, i.e., there exist $x, y \in \mathsf{Var}$ such that $\mathcal{I}(x = y) = \frac{1}{2}$. This means that in the latter case there will also exist identity statements within the language, which are mapped to the intermediate value $\frac{1}{2}$.

Now, we will present two naïve set theories. The first one takes $\mathsf{LP}_=$ as logical axioms and the non-logical axioms are those of naïve set theory. We will denote this set theory as $\mathsf{NLP}_=$.

Definition 3.4.1. *The theory* $\mathsf{NLP}_=$ *consists of the axioms and inference rules of* $\mathsf{LP}_=$ *for the language* $\mathcal{L}_\in$ *extended by* Extensionality *and* Unrestricted Comprehension*.*

The second naïve set theory is a variation of $\mathsf{NLP}_=$ and is due to (RESTALL, 1992b). The main modification is that identity is not added to the language, $x = y$ will be used as an abbreviation of the formula $\forall z(x \in z \leftrightarrow y \in z)$. We will denote this set theory as NLP. Notice that the logical axioms of NLP are those of LP and the non-logical axioms are those of naïve set theory.

Definition 3.4.2. *The theory* NLP *consists of the axioms and inference rules of* LP *for the language* $\mathcal{L}_\in$ *extended by* Extensionality *and* Unrestricted Comprehension*.*

The following result shows that NLP is mathematically expressive.

Theorem 3.4.3. (RESTALL, 1992a, Lemma 3) $\mathsf{NLP} \models \mathsf{ZF}^-$. $\square$

We also know that NLP is non-trivial given that Foundation fails within NLP. Moreover, due to the failure of TB, many standard proofs of set theory, such as the one of *Cantor's theorem*, do not go through anymore. Though this does not exclude the possibility of giving an alternative proof for Cantor's theorem[1] within NLP, it highlights

[1]We denote by $\mid a \mid$ the cardinality of a and by $P(a)$ the power set of a. Suppose towards contradiction that we have a surjective function from a to $P(a)$. So we can define

$$b = \{x \in a \mid x \notin f(x)\} = f(c)$$

for some $c \in a$. Then $c \in b \leftrightarrow c \in f(c) \leftrightarrow c \notin b$. But since TB fails, this does not give rise to any contradiction.

that the choice of a material conditional comes with a price. NLP seems to be proof-theoretically poor!

A more serious drawback of NLP is the treatment of identity. The original proposal of (RESTALL, 1992b) uses the Russell Set R as witness for the statement that there exist at least two different elements in NLP. This forces us to have a *glutty* identity in all models for NLP since the formulas $\mathsf{R} = \mathsf{R}$ and $\mathsf{R} \neq \mathsf{R}$ always receive value $\frac{1}{2}$. Moreover, (WEIR, 2004, p. 393) has shown that, if equality is introduced as primitive as suggested by (PRIEST, 2006), then there exist models of $\mathsf{NLP}_=$ that do not grant the existence of two different elements anymore, i.e., the formula $\exists x \exists y (x \neq y)$ receives value $\mathbf{0}$ in these models. Therefore, NLP is committed exclusively to models where we have a *glutty* identity, if we want to be able to express the existence of two different elements.

This move entails two problems. On one side, it seems *ad-hoc* to only allow models of NLP with a *glutty* identity. Why should we *a piori* exclude any model of NLP with a *standard* identity ? This seems to enter in conflict with our intuition that in our (classical) meta-theory we use a standard identity (WEIR, 2004, cfr. p. 397). On the other side, *Leibniz law of indiscernibility of identicals* fails in NLP. More specifically, this means that the following formula

$$\forall x \forall y \Big((x = y \wedge \varphi(x)) \to \varphi(y) \Big) \tag{LL}$$

fails in NLP. This principle states that identical objects share the same properties, which is the uncontroversial side of the *Law of Leibniz* and constitutes a widely accepted property of identity. This is of course also a desirable property for technical reasons since it allows us to build quotient models (where we have a standard identity) out of our algebra-valued models. Thus, we have philosophical and technical motivations for accepting this principle.

Theorem 3.4.4. LL *fails within* NLP.

Proof. Let $\varphi(u) = u \in w$ and consider a variable assignment S such that $S(u) = a$ and $S(z) = b$. Furthermore, let $v_S^{\mathcal{M}}(u \in w) = \frac{1}{2}$ for any $S[w/x]$ and $v_S^{\mathcal{M}}(v \in w) = \mathbf{0}$, when $S(w) = c$. Let $S(w) = c$, then given that $v_S^{\mathcal{M}}(u \in w) = \frac{1}{2}$ for any $S[w/x]$, it follows that $v_S^{\mathcal{M}}\Big(\forall w (u \in w \leftrightarrow z \in w)\Big)$. So, $v_S^{\mathcal{M}}(u = z) = \frac{1}{2}$. Therefore, we have $v_S^{\mathcal{M}}\Big(u = z \wedge \varphi(u)\Big) \in D$, however, $v_S^{\mathcal{M}}\Big(\varphi(z)\Big) \notin D$. $\square$

Consider (INCURVATI, 2020, pp. 108-109) and (WEIR, 2004, pp. 397-398) for a similar discussion on the status of LL in NLP. Even though there exist arguments supporting the failure of LL in a paraconsistent setting, this seems to point towards a trade-off between the intuitiveness of Unrestricted Comprehension and the one of LL.

> Dialetheists claim that we should stick to the Naïve Comprehension Schema because of its intuitiveness. But to do this, we are asked to reject another principle which is hardly less intuitive than the Comprehension Schema itself. As a result, the initial appeal of a paraconsistent set theory based on the naïve conception seems seriously diminished. (INCURVATI, 2020, p. 109)

Now, we will address the drawbacks of $\mathsf{NLP}_=$. Broadly speaking, it is model-theoretically very weak. As (THOMAS, 2014, Theorem 3.4.) has shown, it is not possible to define basic set-theoretic objects such as singletons, cartesian pairs or ascending linear orders in $\mathsf{NLP}_=$. A set-theoretic model without natural numbers or transfinite ordinals seems to reflect inadequately the set theoretic universe, besides being much to weak for set-theoretic practice. Thus, $\mathsf{NLP}_=$ does not fulfill the requisite that any non-classical set theory should allow us to develop a reasonable amount of classical set theory.

Let us now consider some variations of LP, which might offer semantically richer set-theories. The model-theoretic weakness of $\mathsf{NLP}_=$ is mainly due to the unrestricted occurrence of gluts. So let us introduce a cousin of LP, *minimally inconsistent* LP (LPm), which manages to restrain the occurrence of gluts. The essential idea behind LPm is to introduce an ordering over the interpretations of $\mathsf{LP}_=$ depending on the number of atomic formulas that receive value $\frac{1}{2}$ in each interpretation. Then LPm corresponds to those interpretations which are minimal with respect to this ordering. Following the presentation of (THOMAS, 2014, p. 3), we define models for LPm as follows.

Definition 3.4.5. *Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ be a model for $\mathsf{LP}_=$ and let $R\langle x_1, ..., x_n \rangle$ be an n-ary relation symbol and $x_1, ..., x_n \in \mathcal{D}$, then we define:*

$$\mathcal{M}! = \{R\langle x_1, ..., x_n \rangle : \mathcal{I}\Big(R\langle x_1, ..., x_n \rangle\Big) = \frac{1}{2}\}.$$

We have $\mathcal{M} \prec \mathcal{M}'$ iff $\mathcal{M}! \subset \mathcal{M}'!$. Let T *be a theory, then we denote with $\mathcal{M}$ an* LPm *model of* T *iff $\mathcal{M}, S \models \mathsf{T}$ for some variable assignment S and for all models M', if $M' \prec M$, then we have $M', S' \not\models \mathsf{T}$ for every variable assignment S'.*

Then we can define a consequence relation for the logic LPm as follows:

Definition 3.4.6. *Moreover, we say that $\mathsf{T} \models \varphi$ iff for all* LPm *models $\mathcal{M}$ of T and all variable assignments S, if $\mathcal{M}, S \models \mathsf{T}$, then $\mathcal{M}, S \models \varphi$.*

Notice that every model of $\mathsf{LP}_=$ is also an LPm-model, but not the other way around, since LPm will also have classical models in the absence of inconsistent premises (PRIEST, 2006, p. 226, Fact 1). In particular, we will have models of LPm with a *standard* identity. So we might get, as intended, a richer ontology. Then we can define the naïve set theory that takes as logical axioms LPm as follows.

Definition 3.4.7. *The theory* NLPm *consists of the axioms and inference rules of* LPm *for the language $\mathcal{L}_\in$ extended by* Extensionality *and* Unrestricted Comprehension*.*

Let us remark a particular feature of LPm. Namely, if a theory based on $\mathsf{LP}_=$ is non-trivial, then so is the same theory based on LPm (PRIEST, 2006, p. 227). Moreover, Restall's proof of the non-triviality of NLP carries over to $\mathsf{NLP}_=$. Thus, NLPm is non-trivial. Nevertheless, it is model-theoretically still very weak. In particular, (THOMAS, 2014, Lemma 4.4) has shown that any NLPm-model, with more than one element in its domain and a *standard* identity, is not a model of Unrestricted Comprehension anymore. Therefore, the only model of NLPm consists of single element, which is *glutty* with respect to its own membership relation. As a consequence, we have that every element is *glutty* with respect to every membership relation. In other words, applying the terminology of (INCURVATI, 2020) the only model of NLPm is *almost trivial*, i.e., the formula

$$\forall x \forall y (x \in y \land x \notin y) \tag{AT}$$

holds in NLPm. Again, this clashes not only with our intuition that there exist elements in our meta-theory for which membership behaves *standard* but also with our intuition that there should exist more than a single element in our set-theoretic universe. In conclusion, the one element NLPm-model is untenable.

Furthermore, the three other variations of NLPm; $\mathsf{NLP}_{\equiv}$, $\mathsf{NLP}_{\subseteq}$ and $\mathsf{NLP}_{\supseteq}$, introduced by (CRABBÉ, 2011) face similar problems. $\mathsf{NLP}_{\subseteq}$ has as in the case of NLPm, only a one-element model (THOMAS, 2014, Theorem 4.5), whereas $\mathsf{NLP}_{\supseteq}$ and $\mathsf{NLP}_{\equiv}$ suffer from the same weakness as $\mathsf{NLP}_{=}$, i.e. we are unable to define singletons, cartesian pairs or ascending linear orders. Thus, we have exhausted the resources of the material strategy and showed that each set theory following this paradigm fails to deliver what it promises. As (INCURVATI, 2020) suggests, we might have to change strategy.

"I conclude that the prospects for developing a naive set theory by pursuing the material strategy look rather dim, and those for developing it on the basis of LP and cognate systems even dimmer." (INCURVATI, 2020, p. 111)

3.5 The Relevant Approach

Let us now turn to the relevant approach. The general procedure of this approach is similar to the material approach; first we introduce a family of non-classical logics which are based on a weak conditional (in this case a *relevant* one) and then we use these logics to interpret the non-logical axioms of naïve set theory. Thus, a naïve paraconsistent set theory based on the relevant approach consists of the non-logical axioms of naïve set theory and the logical axioms of a relevant logic. Moreover, given the Curry-Paradox this strategy is bounded to a relevant conditional, for which Contraction and MPA fails.

The intuitive idea behind a relevant conditional is that its antecedent and consequent have to be "on the same topic", i.e., a conditional is relevant in the case that, whenever $\varphi \to \psi$ is valid, then φ and ψ share at least one propositional variable. This constraint is also known as the *variable sharing principle* (BRADY, 2006, p. 5).

Now, we will give a sketch of the Routley star semantics. In particular, we will use Kripke frames with impossible worlds, where the accessibility relation is given in terms of a ternary relation between worlds. The choice of a ternary accessibility relation is due to the reason that the semantic interpretation of the strict conditional, where we have a binary accessibility relation, validates vacuously $\varphi \to (\psi \to \psi)$, where the antecedent and consequent are completely disconnected.

Notice that the set of worlds of our frames is partitioned, into possible worlds and impossible ones, where classical tautologies such as $\varphi \to \varphi$ may fail. Now, the truth of a formula $\varphi \to \psi$ at a possible world w, requires the preservation of truth at *all* worlds it accesses, including impossible ones. Notice that we can explain the failure of Contraction along these lines. Simply consider a possible world w which accesses an impossible world v, where MP fails and so φ and $\varphi \to \psi$ hold there, and ψ fails. Then we have that the premise of Contraction (i.e., $\varphi \to (\varphi \to \psi)$) holds at w, but its conclusion (i.e., $\varphi \to \psi$) does not. In general, the Routley star semantics allows us to successfully invalidate irrelevant formulas.

Definition 3.5.1. *A Routley star interpretation is a structure $\langle W, N, R, {}^*, v\rangle$, where W is a set of worlds, $N \subseteq W$ (the set of possible worlds), $R \subseteq (W \times W \times W)$, * is a unary function on W, and v assigns a truth value to every propositional variable at every world.*

(i) $v_w(\neg\varphi) = \mathbf{1}$ *iff* $v_{w^*}(\varphi) = \mathbf{0}$,

(ii) $v_w(\varphi \to \psi)$ *(where w is an impossible world) iff for all $u, v \in W$ such that $Rwuv$, if* $v_u(\varphi) = \mathbf{1}$, *then* $v_v(\varphi) = \mathbf{1}$.

Notice, that $\varphi \to \psi$ is evaluated as usually in the case that w is a possible world. Moreover, the unary function * is known as the *Routley Star* and was introduced originally by (ROUTLEY; ROUTLEY, 1972). The intuitive idea is that every world w will have a twin world w^* assigned and that $\neg\varphi$ holds at w if and only if φ does not hold at w^*. So the relevant negation will not be evaluated at w itself, but at its twin world w^*. Normally, it is assumed that the Routley Star satisfies the following constraints $w = w^{**}$ and $R(w, w_1, w_2) \to R(w, w_2^*, w_1^*)$, to ensure that the relevant negation has more inferential features, such as double negation introduction (DNI) and contraposition (CP).

Next, we can show that the relevant negation give rise to a paraconsistent logic. Consider a Routley star interpretation where

$$W = \{u, w\}, v_u(p) = \mathbf{1}, v_u(q) = \mathbf{0}, v_w(p) = \mathbf{0} \text{ and } u^* = w.$$

Then p holds at world u, as well as, $\neg p$ holds at world u (since $v_{u^*}(p) = \mathbf{0}$). However, q does not hold at u, so ECQ fails.

We will analyze two naïve paraconsistent set theories that follow the relevant approach. Notice that there may exist other interesting relevant set theories (BRADY, 2006, Chapter 5), however, we argue that the criticism that will be put forward against the intensional character of the relevant conditional in these set theories can be extended to all relevant systems. Thus, these two set theories will serve us as paradigmatic cases to cover entirely the relevant approach.

We begin by presenting the relevant set theory NDKQ which was proposed initially by (ROUTLEY, 1979). First, we will cover the logical axioms of this set theory. More specifically, NDKQ is build upon the first-order relevant logic DKQ. We go on to present an Hilbert-axioms system for DKQ following (ISTRE; MCKUBRE-JORDENS, 2019).

Definition 3.5.2. DKQ *is defined over* $\mathcal{L}_{Fol}$ *by the following axiom schemes and inferences rules.*

Axiom schemes:

$$\varphi \to \varphi \tag{Ax$_1$}$$

$$\Big((\varphi \to \psi) \wedge (\psi \to \gamma)\Big) \to (\varphi \to \gamma) \tag{Ax$_2$}$$

$$\psi \to (\varphi \vee \psi) \tag{Ax$_3$}$$

$$\varphi \to (\varphi \vee \psi) \tag{Ax$_4$}$$

$$(\varphi \wedge \psi) \to \varphi \tag{Ax$_5$}$$

$$(\varphi \wedge \psi) \to \psi \tag{Ax$_6$}$$

$$\Big((\varphi \to \psi) \wedge (\chi \to \psi)\Big) \to \Big((\varphi \vee \chi) \to \psi\Big) \tag{Ax$_7$}$$

$$\Big((\varphi \to \psi) \wedge (\varphi \to \gamma)\Big) \to \Big(\varphi \to (\psi \wedge \gamma)\Big) \tag{Ax$_8$}$$

$$\Big(\varphi \wedge (\psi \vee \gamma)\Big) \to \Big((\varphi \wedge \psi) \vee (\varphi \wedge \gamma)\Big) \tag{Ax$_9$}$$

$$\neg\neg\varphi \to \varphi \tag{Ax$_{10}$}$$

$$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi) \tag{Ax$_{11}$}$$

$$\varphi \vee \neg\varphi \tag{Ax$_{12}$}$$

$$\forall x\varphi \to \varphi[y/x],\ \textit{where } y \textit{ is free for } x \textit{ in } \varphi. \tag{Ax$_{13}$}$$

$$\forall x\Big(\varphi \to \psi\Big) \to \Big(\varphi \to \forall x\psi\Big),\ \textit{where } x \textit{ is not free in } \varphi. \tag{Ax$_{14}$}$$

$$\forall x(\varphi \vee \psi) \to (\varphi \vee \forall x\psi),\ \textit{where } x \textit{ is not free in } \varphi. \tag{Ax$_{15}$}$$

$$\forall x\Big(\varphi \to \psi\Big) \to \Big(\exists x\varphi \to \psi\Big),\ \textit{where } x \textit{ is not free in } \psi. \tag{Ax$_{16}$}$$

$$\varphi[y/x] \to \exists x\varphi,\ \textit{where } x \textit{ is not free in } \varphi. \tag{AX$_{17}$}$$

$$(\varphi \wedge \exists x\psi) \to \exists x(\varphi \wedge \psi)\ \textit{where } x \textit{ is not free in } \varphi. \tag{AX$_{18}$}$$

Inference rules:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,(R_1)$$

$$\frac{\varphi, \psi}{\varphi \wedge \psi}\,(R_2)$$

$$\frac{\varphi \to \psi, \chi \to \gamma}{(\psi \to \chi) \to (\varphi \to \gamma)}\,(R_3)$$

$$\frac{x = y}{\varphi(x) \to \varphi(y)}\,(R_4)$$

$$\frac{\varphi}{\forall x \varphi}\,(R_5)$$

Meta-inference rules:

$$\frac{\varphi \vdash \psi}{\varphi \vee \chi \vdash \psi \vee \chi}\,(MR_1)$$

$$\frac{\varphi \vdash \psi}{\exists x \varphi \vdash \exists x \psi}\,(MR_2)$$

Additionally, it is imposed that the meta-rules can not be applied on formulas of the form $\varphi \vdash \psi$ if universal quantifier bounds a free variable in φ (BRADY, 2006). Now, we can define the set theory NDKQ, which takes DKQ as logical axioms and the axioms of naïve set theory as non-logical axioms. However, Unrestricted Comprehension is modified slightly, since now the set that is being defined $\{x : \varphi(x)\}$ may occur freely within $\varphi(x)$. In particular, we will use the following variation of Unrestricted Comprehension,

$$\exists y \forall x \Big(x \in y \leftrightarrow \varphi(x)\Big), \text{ where y can occur free in } \varphi(x). \qquad \text{(Modified Comprehension)}$$

Then, we define:

Definition 3.5.3. *The Theory* NDKQ *consists of the axioms and inference rules of* DKQ *for the language* $\mathcal{L}_\in$ *extended by* *Extensionality* *and* *Modified Comprehension.*

As a consequence of Modified Comprehension, NDKQ allows us to instantiate besides the Russel Set, also non-well-founded sets, as for example, the set that instantiates the formula $\exists y \forall x(x \in y \leftrightarrow x \notin y)$. More importantly, (BRADY, 1989) showed that NDKQ is non-trivial.

There exist many problems with NDKQ. The authors of (ISTRE; MCKUBRE-JORDENS, 2019), for instance, have criticized the failure of weakening and the deduction theorem. Another problem concerns the existence of certain sets within NDKQ . From the usual definition of the empty set as $\{x : x \neq x\}$ we can not deduce that the empty set is a subset of every set. This would constitute a case of irrelevance since we can not prove in general $(x \neq x) \to \varphi$ in the underlying logic (BERTO, 2007, p. 250). Dual considerations hold for the universal set $\mathbf{V}$, which seems to suggest that the meaning of the subset relation is quiet different from the usual one in set theory. Moreover, (ROUTLEY, 1979) has proposed alternative definitions of $\varnothing$ and $\mathbf{V}$ which avoid this problem, however, if we accept these surrogates we end up with various sets that can be interpreted as $\varnothing$ and $\mathbf{V}$ and (DUNN, 1988) has shown that if we force the uniqueness of these sets then we end up with classical logic.

To sum up NDKQ seems to be an ideal case of a naïve paraconsistent set theory since we do not only have inconsistent but non-well founded sets in our universe. However, it remains unclear how much of standard set theory may be carried out in this framework as pointed out in (ISTRE; MCKUBRE-JORDENS, 2019, Section 17.5) and (PRIEST, 2006, p. 253). To overcome this problem, (WEBER, 2012) has proposed to extend the logical axioms of NDKQ with the Counterexample rule. The result is the logic DLQ.

Definition 3.5.4. *We obtain the logic* DLQ*, by adding to* DKQ *the following inference rule;*

$$\frac{\varphi, \neg\psi}{\neg(\varphi \Rightarrow \psi)} \text{ Counterexample Rule.}$$

Then we can define:

Definition 3.5.5. *The Theory* NDLQ *consists of the axioms and inference rules of* DLQ *for the language* $\mathcal{L}_\in$ *extended by* *Extensionality* *and* *Modified Comprehension.*

Given that NDLQ is an extension of NDKQ, the proof of non-triviality of NDKQ, by (BRADY, 1989), carries over to NDLQ. Furthermore, NDLQ has very strong model-theoretic properties and seems to be the most promising candidate of the relevant approach (INCURVATI, 2020, cfr., p. 117). First, we examine whether this set theory enables us to carry out a reasonable amount of standard set theory, and then we consider the possible disadvantages.

It was shown in (WEBER, 2012) and (WEBER, 2010), that it is possible to define ordinals in NDLQ and that the collection of all ordinals is a *set*. This set can be well-ordered. So it follows that there exists a choice function for every non-empty set, that is, NDLQ proves AC. Then cardinals are defined using AC and the *von Neumann* cardinal assignment, which basically consists in taking particular ordinals as cardinals (WEBER, 2012, p. 278). Furthermore, it is shown that essential results of standard cardinal arithmetic, such as the *Cantor-Schröder-Bernstein Theorem*, the *Pigeonhole principle* and *Cantors Theorem*, all hold. Thus, it is possible to develop basic cardinal and ordinal arithmetic within NDLQ. Moreover, (WEBER, 2012, Theorem 6.6) has proved that the Continuum hypothesis fails in NDLQ and that NDLQ is compatible with the existence of large cardinals such as Inaccessible, Mahlo and Measurable cardinals.

This renders evidence for the claim that NDLQ is indeed sufficiently strong for mathematical practice. Hence, NDLQ is non-trivial and is expressive enough to carry out a reasonable amount of standard set theory. Nevertheless, we will point out two disadvantages of this set theory.

Notice that we obtain $\neg(\varphi \to \varphi)$, if we apply this rule to the premises $\varphi, \neg\varphi$. Thus, the Counterexample Rule has quiet unintuitive consequences in the context of dialetheias. So as in the case of LP-set theories and LL; here we have a similar trade-off between the intuitiveness of Unrestricted Comprehension and the logical properties of the relevant conditional. On one hand, we are keeping Unrestricted Comprehension because of its intuitive character and on the other hand, we are forced to accept very unintuitive consequences.

Moreover, (WEBER, 2010) has argued that the logical axioms of NDLQ are justified, given that DLQ is *the* strongest possible logic that does not trivialize our set theory in presence of Unrestricted Comprehension. But then we have a problem of demarcation, since there exist various logical calculi which give rise to set theories that are proof-theoretically equally strong, however, they invalidate different logical principles (FIELD; LEDERMAN; ØGAARD, 2017). Which logical axioms should we then choose for our set theory ?

> What we end up with is a number of different logics, each of which enables us to prove certain results, and for none of which we seem to have a good motivation: any attractions of the paraconsistent solution to the set-theoretic paradoxes seem seriously undermined. (INCURVATI, 2020, p. 119)

Secondly, as pointed out by (INCURVATI, 2020, Ibid.) there exists a more fundamental problem with the inhabitant of the models of NDLQ. As in the case of NDKQ, we have duplicates of the empty set, i.e., the empty set can not be defined uniquely (PRIEST, 2006, p. 253). But it gets even worse, we have duplicates for every set in our universe (WEBER, 2010, p. 88). To be precise, let φ be any true sentence. Then the left-to-right conditional of

$$x \in y \leftrightarrow (x \in y \wedge \varphi)$$

is generally relevantly not valid. So, even though y and $\{x : x \in y \wedge \varphi\}$ have the same members we can not conclude that $y = \{x : x \in y \wedge \varphi\}$, due to the relevant conditional. This problem undermines seriously the concept of *set* in relevant set theories. Although $\mathsf{Extensionality}$ holds, that does not guarantee that sets are extensional entities in NDLQ, which is arguably the most essential property of a set, as stated in the following lines:

> But a theory that did not affirm that the objects with which it dealt were identical if they had the same members would only by charity be called a theory of *sets* alone. (BOOLOS, 1971, p. 27)

In view of this Priest proposes two possible solution attempts. The first one is to replace the biconditional in $\mathsf{Extensionality}$ by the material biconditional. But this strategy is unfeasible. Let $\varphi(x)$ be a formula which receives value $\frac{1}{2}$ under a particular interpretation function. Then, given the particular semantics of the material biconditional, we have for any set y,

$$\forall x\Big(x \in y \leftrightarrow \varphi(x)\Big).$$

By $\mathsf{Extensionality}$, we conclude that there exists only a single set in our universe, i.e., the set $y = \{x : \varphi(x)\}$. This picture seems not to capture adequately the set-theoretic universe, since we would like to have more than a single set in our universe. Notice that we do not have this problem in the case of the material approach, since the conditional that acts in $\mathsf{Extensionality}$ is the *material* one, and thus fails to detach. So we do no get the identity between y and $\{x : \varphi(x)\}$.

The second proposal by Priest consists in replacing the biconditional in $\mathsf{Extensionality}$ by an enthymematic biconditional $\rightleftharpoons$. Let τ be a logical constant which can be thought of as the conjunction of all logical truths. Indeed, τ validates the following inference rules:

$$\frac{}{\tau}\,,$$

$$\frac{\varphi}{\tau \Rightarrow \varphi}\,.$$

Then we can define the enthymematic conditional as follows;

$$\varphi \rightharpoondown \psi =_{df.} (\tau \wedge \varphi) \Rightarrow \psi.$$

Moreover, the enthymematic biconditional $\rightleftharpoons$ is defined as usual. The idea is then to reformulate $\mathsf{Extensionality}$ as $\forall x\Big(x \in y \rightleftharpoons x \in z\Big) \Rightarrow y = z$. It is easy to see that this proposal solves the problem of duplication. For let φ be any truth. Then by definition of the enthymematic conditional we have $x \in y \rightharpoondown (x \in y \wedge \varphi)$ and obviously $(x \in y \wedge \varphi) \rightharpoondown x \in y$. Hence we can conclude that $y = \{x : x \in y \wedge \varphi\}$. Nevertheless, (WEBER, 2010, Section 6) has shown that this formulation of $\mathsf{Extensionality}$ leads to triviality.

As in the case of the material approach, we have exhausted the resources of the relevant approach. Moreover, we conclude that this approach to naïve paraconsistent set theory is unfeasible as well. To sum up:

> (...) the relevant strategy, at least as developed by Weber, enables the paraconsistent set theorist to make some progress over the material strategy, since NDLQ has the resources to carry out some reasonable amount of set theory. However, the theory has a background logic which is poorly motivated. And, on pain of triviality, the current attempts to provide it with

a genuine principle of extensionality fail, so that it cannot be regarded as a *set* theory. (INCURVATI, 2020, pp. 120-121)

3.6 The Model-theoretic Approach

Let us now explore the model-theoretic approach to naïve paraconsistent set theory. The model-theoretic approach of Priest can be found in (PRIEST, 2006, Section 18.4) and (PRIEST, 2017, Section 11). The main idea is to build a model of $\mathsf{NLP}_=$ and ZF (which contains a large fragment of the cumulative hierarchy). As a consequence, we can regard the axioms of ZF and all the theorems that we can derive classically from ZF as true within a paraconsistent framework.

We follow closely the exposition of (INCURVATI, 2020, Chapter 4.5). Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ be a model , where v is a valuation under $\mathcal{I}$ and $\sim$ an equivalence relation on $\mathcal{D}$, such that for any $d \in \mathcal{D}$, $[d]$ is the equivalence class induced by $\sim$.

Definition 3.6.1. *We say that $\mathcal{M}^\sim = \langle \mathcal{D}^\sim, \mathcal{I}^\sim \rangle$ is the collapsed model of $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$, when $\mathcal{D}^\sim = \{[d] : d \in \mathcal{D}\}$ and $\mathcal{I}^\sim$ is fixed as follows:*

(i) If c is a constant which denotes d in $\mathcal{M}$, then c denotes $[d]$ in $\mathcal{M}^\sim$.

(ii) If f is an n-place function symbol which denotes d in $\mathcal{M}$ when taking as arguments terms denoting $d_1, ..., d_n$, then f denotes $[d]$ in $\mathcal{M}^\sim$ when taking as arguments terms denoting $[d_1], ..., [d_n]$.

(iii) If P is an n-place predicate, we let $a_1, ..., a_n$ be in P's positive (negative) extension in $\mathcal{M}^\sim$ iff $\exists x_1 \in a_1, ..., \exists x_n \in a_n$ an such that $\langle x_1, ..., x_n \rangle$ is in the positive (negative) extension of P in $\mathcal{M}$.

We say that $\mathcal{M}^\sim$ is a *collapsed* model of $\mathcal{M}$, because in the collapsed model $\mathcal{M}^\sim$ we are dealing "only" with the equivalence classes that we obtained from the original model $\mathcal{M}$. Next, we have the following lemma which guarantees that the value of a formula in a given model propagates through to the collapsed model. Notice, however, that a formula might get mapped to an additional value.

Lemma 3.6.2. (PRIEST, 2006, Lemma 4) *For every formula φ in the language of $\mathcal{M}$, we have $v(\varphi) \subseteq v^\sim(\varphi)$.* $\square$

Priest has presented two different equivalence relation, which succeed in collapsing a model of ZF into a model of $\mathsf{NLP}_{=} + \mathsf{ZF}$. The first one is known as the *type-lift* and consists in dividing the set-theoretic universe into two layers. The intuitive idea is that every set below a fixed ordinal is preserved as usual and every set above this ordinal is collapsed to a single glutty object $[a]$ which is designated by a such that $x \in a$ receives value $\frac{1}{2}$ for any x (where x refers to x in $\mathcal{M}$ and $[x]$ in $\mathcal{M}^{\sim}$). Then, by the semantics of the material biconditional it follows that $\forall x\big(x \in a \leftrightarrow \varphi(x)\big)$ receives values $\frac{1}{2}$ and thus Unrestricted Comprehension will hold in our collapsed model. So, let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ be a model of ZF, α an ordinal in $\mathcal{M}$ and a be V_α (the sets of rank less then α). Then we define the following equivalence relation $\sim$ on $\mathcal{D}$:

$$\big(x \text{ and } y \text{ are in } a \text{ and } x = y \text{ (in } \mathcal{M})\big) \text{ or } \big(x \text{ and } y \text{ are not in } a \text{ (in } \mathcal{M})\big).$$

As desired, this equivalence relation leaves every set of rank less than α alone and collapses the rest to the equivalence class $[a]$ which is designated by a. In particular, every equivalence class in our collapsed model is an element of a.

Lemma 3.6.3. (PRIEST, 2006) *For any $b \in D^{\sim}$ we have $v^{\sim}(b \in a) = \frac{1}{2}$.*

Thus, using a as witness for the existential quantifier within Unrestricted Comprehension, it follows that $\mathcal{M}^{\sim}$ is a model of Unrestricted Comprehension. Moreover, since ZF holds in the original model, by applying Lemma 3.6.2 we get that $\mathcal{M}^{\sim}$ is a model of ZF and $\mathsf{NLP}_{=}$.

Now, suppose that the original model $\mathcal{M}$ besides being a model of ZF, contains two inaccessible cardinals κ_1 and κ_2 such that $\kappa_2 > \kappa_1$ and fix $\alpha = \kappa_2$. Then, our collapsed model $\mathcal{M}^{\sim}$ is a model of ZF and $\mathsf{NLP}_{=}$ and contains the cumulative hierarchy up to κ_1 as inner model. Formally:

Theorem 3.6.4. (PRIEST, 2017). *Suppose that $\mathcal{M}$ is a classical model of ZF containing two inaccessible cardinals κ_1 and κ_2. Then there exists a collapsed model $\mathcal{M}^{\sim} = \langle \mathcal{D}^{\sim}, \mathcal{I}^{\sim} \rangle$ such that:*

(i) $\mathcal{M}^{\sim} \models \mathsf{ZF} + \mathsf{NLP}_{=}$ *and*

(ii) $\mathcal{M}^{\sim}$ *contains a model $\mathcal{M}'$, where $\mathcal{M}'$ is a classical model of* ZF.

A remarkable feature of the model-theoretic approach is the fact that we do not only get the validity of ZF but also all the theorems of ZF. For Priest, this suggests that we can embrace the theorems of ZF from a non-classical perspective.

> Since the universe of sets is a model of ZF (as well as naïve set theory), these hold in it. We may therefore establish things in ZF in the standard classical way, knowing that they are perfectly acceptable from a paraconsistent perspective. (PRIEST, 2006, p. 257)

Nevertheless, we can raise the following two objections against the *type-lift*:

(1) $\mathcal{M}^{\sim}$ is constructed by leaving alone *only* a proper fragment of the original model.

(2) A single set a is the witness for all the instances of Unrestricted Comprehension.

To avoid these problems, Priest has used a second construction proposed in (PRIEST, 2017, p. 101), known as the *Hamkins-type lift.* Rather than partitioning the universe, this construction uses a covering that preserves the set-theoretic universe but adds some inconsistent sets to it. Without going too much into technical details, this construction has one salient advantage over the first construction. If the original model has cardinality κ then the collapsed model (by the partition) is of cardinality less or equal to κ, but by the covering we obtain models of cardinality less or equal to 2^{κ}. So potentially we might get models which reflect more accurately the set-theoretic universe.

Moreover, the *Hamkins type-lift* fixes partially problem (1), given that the partition construction is produced by leaving alone the *entire* original model. Furthermore, using the type-lift we obtain models where different sets witness different instances of Unrestricted Comprehension, thus providing more discriminating models. Thus, this construction also provides a solution to problem (2).

But we have a decisive drawback in the case of the *Hamkins-type lift*: as in the case of NLP, we are forced to define identity, since $\mathcal{M}^{\sim}$ does not necessarily interpret " $=$ " as the identity relation. So, this construction faces the same criticism as NLP: we lose LL.

> The first is that in the collapsed model the denotation of $=$ may not be the identity relation. However, as far as constructing models of set theory goes, we

> can ignore this, since we do not need to assume that the language contains the identity predicate. We can just define $x = y$ as $\forall z(z \in x \leftrightarrow z \in y) \wedge \forall z(x \in z \leftrightarrow y \in z)$. In ZF this delivers the substitutivity of identicals. In naïve set theory, as we are understanding it here, it does not. So identity will behave in an unusual way in such a theory. But since the name of the game at this point is recapturing the theorems of ZF, this does not matter. (PRIEST, 2017, p. 102)

But then again, if we can ignore the loss of such intuitive principles in favor of fruitful derivations, why should we not simply choose to start with (classical) ZF in the first place ? It seems therefore that this particular argument by Priest supports more the (classical) iterative conception of set rather than the naïve paraconsistent strategy.

However, there exists a deeper problem that affects both equivalence relations proposed by Priest. It concerns the philosophical plausibility of his model-theoretic approach in general. As noticed by (MEADOWS, 2015), if the purpose of Priests model-theoretic strategy is to justify the axioms and theorems of ZF, then we have a vicious circle, since the collapsed models are carried out within ZF itself. Nevertheless, Priest argues that the criticism put forward by (MEADOWS, 2015) gets off on the wrong foot since the model-theoretic approach is not intended to *justify* ZF but to show that it is possible to have a model of both ZF and $\mathsf{NLP}_=$.

Unfortunately, this argument presupposes that ZF is consistent, which seems to be an assumption that is unavailable for advocates of the naïve paraconsistent approach. As we have already seen any argument that appeals to the fruitfulness of this assumption seems to point away from the motivation of naïve paraconsistent set theory.

On the other hand, we could argue that this assumption is grounded on the belief that we have an intuitive model of ZF within the cumulative hierarchy. Unfortunately, this argument seems to presuppose that the existence of such a model implies syntactic consistency, however, in a paraconsistent setting this is not the case. Actually, this fact constitutes one of the main motivations of Priest's model-theoretic approach. Moreover, as noticed in (PRIEST, 2006, p. 98), the axioms of ZF are not *consistently* true, since both $\mathsf{Separation}_\varphi$ and its negation hold. So, even the validity of the ZF axioms in the collapsed models, can not support the assumption that ZF is consistent.

Finally, we conclude that the model-theoretic approach is also unsatisfactory, as stated as follows:

> (...) the model-theoretic strategy, consisted in arguing that the universe of sets models the ZF axioms and properly extends the cumulative hierarchy of sets. I argued that this strategy has several problems, chief among which is the fact that it ultimately needs to assume that the cumulative hierarchy is a consistent domain. I argued, however, that the dialetheist does not seem to have any reason to believe that the cumulative hierarchy is a consistent domain. (INCURVATI, 2020, p. 127)

3.7 The da Costa approach

In this section, we introduce another family of naïve paraconsistent set theories: the NF_n- set theories or simply da Costa set theories. These set theories are obtained by combining the non-logical axioms of *New Foundation* (NF) with the first-order logic $C_n^=$ which was introduced by da Costa in (COSTA, 1986) and (COSTA, 1974). Let us first discuss the logical axioms.

The philosophical motivation of the C-systems is to preserve as much as possible from classical propositional logic and to obtain a paraconsistent logic by modifying adequately the negation. The C-systems were intended to offer a logical framework for inconsistent theories in which "there are *good* theorems, whose negation are not provable and *bad* theorems whose negation are provable." (COSTA, 1986, p. 498) Furthermore, da Costa proposed the following maxims that each of his paraconsistent calculi should satisfy:

(1) The law of non-contradiction, i.e., $\neg(\varphi \wedge \neg\varphi)$ is not a valid scheme.

(2) There is a set of formulas Γ and formulas φ and ψ such that $\Gamma \cup \{\varphi, \neg\varphi\} \nvdash \psi$.

(3) It must be *simple* to extend the calculi C_n to its corresponding first-order calculi C_n^*.

(4) The calculi C_n should validate as many rules and theorems as possible of CPL (C_0) without violating (1) and (2).

Having these criteria in mind, we will start to build up the C-systems as introduced in (MARCOS, n.d.) and start with an axiomatic presentation of C_{min}, the minimal paraconsistent logic:

Definition 3.7.1. C_{min} *is defined over* $\mathcal{L}_{Prop}$ *by the following axiom schemes and inferences rules.*
Axiom schemes:

$$\varphi \to (\psi \to \varphi) \tag{Min_1}$$

$$\big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big) \tag{Min_2}$$

$$\varphi \to \big(\psi \to (\varphi \wedge \psi)\big) \tag{Min_3}$$

$$(\varphi \wedge \psi) \to \varphi \tag{Min_4}$$

$$(\varphi \wedge \psi) \to \psi \tag{Min_5}$$

$$\varphi \to (\varphi \vee \psi) \tag{Min_6}$$

$$\psi \to (\varphi \vee \psi) \tag{Min_7}$$

$$\big((\varphi \to \psi) \to (\psi \to \chi)\big) \to \big((\varphi \vee \psi) \to \chi)\big) \tag{Min_8}$$

$$\varphi \vee (\varphi \to \psi) \tag{Min_9}$$

$$\varphi \vee \neg\varphi \tag{Min_{10}}$$

$$\neg\neg\varphi \to \varphi \tag{Min_{11}}$$

Inference rule:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,(\mathsf{MP}).$$

Notice that because of Min_1, the logic C_{min} and all its extensions are incompatible with the relevant approach. This is, in particular, due to the reason that Min_1 does not satisfy the variable sharing property (see section 3.5). Also C_{min} is maximally consistent with respect to classical propositional logic. Moreover, the deduction theorem holds, so we can obtain the rest of C-systems and LFI's (see section 3.8) by adding new inferences rules.

Following (MARCOS, n.d., p. 50) we extend our language by adding the unary operators " $\bullet$ " and " $\circ$ ", which respectively represent the metalinguistic property of *inconsistency* and *consistency*. We denote this extended propositional language by $\mathcal{L}^{\{\bullet,\circ\}}_{Prop}$. Then we can define the *basic logic of inconsistency* (bC) as follows:

Definition 3.7.2. *We obtain the logic* bC *which is defined over the language* $\mathcal{L}^{\{\bullet,\circ\}}_{Prop}$*, by adding to* C_{min} *the following inference rule;*

$$\frac{\circ\varphi, \varphi, \neg\varphi}{\psi}\ (\textsf{gentle law of explosion}).$$

The gentle law of explosion can be intuitively read as *if* φ *is consistent and contradictory, then it explodes.* The motivation behind this rule is maxim (4.) of da Costa's criteria, which suggests that the law of explosion should hold at least for the consistent fragment of each C-system. Additionally, using the consistency operator we can define a strong negation $\sim \varphi =_{df.} \neg\varphi \land \circ\varphi$ that behaves as the classical negation in all C-systems.

Notice that CP does not hold for da Costa's weak negation, see (MARCOS, n.d., Thrm. 3.20), because of Min_1 which coupled with CP, restores the law of explosion, i.e., $\varphi \to (\neg\varphi \to \psi)$. Now, we can advance to some extensions of bC, namely bbC and bbbC by adding inferences rules that ensure us that $\bullet$ and $\circ$ become duals.

Definition 3.7.3. *We obtain the logic* bbC*, by adding to* bC *the following inference rules;*

$$\frac{\bullet\neg\varphi}{\circ\varphi}\ (\mathsf{bbC}_1),$$

$$\frac{\circ\neg\varphi}{\bullet\varphi}\ (\mathsf{bbC}_2).$$

Moreover, we have:

Definition 3.7.4. *We obtain the logic* bbbC*, by adding to* bbC *the following inference rules;*

$$\frac{\bullet\varphi}{\circ\neg\varphi}\ (\mathsf{bbbC}_1),$$

$$\frac{\circ\varphi}{\bullet\neg\varphi}\ (\mathsf{bbbC}_2).$$

In bbbC we have $\varphi \land \neg\varphi \vdash \bullet\varphi$ but not the other way other around, i.e., $\bullet\varphi \vdash \varphi \land \neg\varphi$ does not hold. Thus we can add precisely this desiderata as an inference rule and define a logic, where a contradiction matches the notion of *inconsistency.*

Definition 3.7.5. *We obtain the logic* Ci*, by adding to* bbbC *the following inference rules;*

$$\frac{\bullet\varphi}{\varphi \wedge \neg\varphi}\ (\mathsf{Ci}).$$

At this point, we are able to prove that $\bullet\varphi \dashv\vdash \varphi \wedge \neg\varphi$, but on the other hand, $\circ\varphi \dashv\vdash \neg(\varphi \wedge \neg\varphi)$ does not hold. Nevertheless, we can again add precisely this desiderata as inference rule and define the following logic:

Definition 3.7.6. *We obtain the logic* Cil*, by adding to* Ci *the following inference rules;*

$$\frac{\neg(\varphi \wedge \neg\varphi)}{\circ\varphi}\ (\mathsf{cl}).$$

Notice that no extension of Cil can have $\neg(\varphi \wedge \neg\varphi)$ as a theorem since cl would render a consistent system. From this logic on, we get da Costa's C - systems. These form an n-ary hierarchy ($0 \leq n \leq \omega$) of propositional calculi C_n (where C_0 corresponds to the negation-free fragment of classical propositional logic), first-order calculi C_n^* and first-order calculi with identity $\mathsf{C}_n^=$. The intuitive idea behind these calculi is that consistency propagates from atomic formulas to more complex formulas.

Thus we define C_1 as follows:

Definition 3.7.7. *We obtain the logic* C_1*, by adding to* Cil *the following inference rules;*

$$\frac{\circ\varphi \wedge \circ\psi}{\circ(\varphi \wedge \psi)}\ (\wedge^\circ),$$

$$\frac{\circ\varphi \vee \circ\psi}{\circ(\varphi \vee \psi)}\ (\vee^\circ),$$

$$\frac{\circ\varphi \to \circ\psi}{\circ(\varphi \to \psi)}\ (\to^\circ).$$

Each of da Costa's systems has the same axioms, however, each system modifies the meaning of the formula $\circ\varphi$. In the case of C_1 we have already seen that $\circ\varphi$ abbreviates the formula $\neg(\varphi \wedge \neg\varphi)$. Then, in the case of C_2, $\circ\varphi$ abbreviates the formula $\circ\varphi \wedge \circ\circ\varphi$, and for C_3, $\circ\varphi$ abbreviates $\circ\varphi \wedge \circ\circ\varphi \wedge \circ\circ\circ\varphi$ and so on. The seminal intuition here is that we have to add more and more premises to each calculi in order to guarantee consistency (MARCOS, n.d., p. 76). Furthermore, each system C_{n+1} is strictly weaker than the calculus C_n, which makes C_ω the weakest calculus of the hierarchy.

Now, we will turn our attention to Quine's set theory NF, as introduced in (MENDELSON, 2015, pp. 300-303). For the original exposition of Quine, consider (QUINE, 1937). Quine's main motivation behind NF, was to provide an alternative way of dealing with the pathological sets of naïve set theory. His proposal: a *type theory without types.*

First of all, notice that $x = y$ abbreviates the formula $\forall z(x \in z \leftrightarrow y \in z)$ in NF. Next, we go on to define the notion of *stratification*.

Definition 3.7.8. *A well formed formula φ is said to be* stratified *if we can assign integers to the variables of φ such that:*

(i) All occurrences of the same free variable are assigned the same integer.

(ii) All occurrences of a variable bound by the same quantifier are assigned the same integer.

(iii) Any subformula of the form $x \in y$ of φ, is such that the number assigned to y is the successor of that assigned to x.

Notice that the formula $x \in x$ is not stratified, since x would have to receive an integer n and $n + 1$ at the same time, contradicting condition (i) of Definition 3.7.8. As a consequence, the Russel-set does not exist in NF. On the other hand, the formula $\exists y(x \in y) \wedge \exists y(y \in x)$ is stratified given the assignment: $\exists y_2(x_1 \in y_2) \wedge \exists y_0(y_0 \in x_1)$. Now, we can define $\mathsf{Stratified\ Comprehension}$ as follows.

Definition 3.7.9. *For any stratified formula $\varphi(x)$, we have*

$$\exists y \forall x \Big(x \in y \leftrightarrow \varphi(x)\Big). \qquad \text{(Stratified Comprehension)}$$

Definition 3.7.10. *The Theory* NF *consists of the axioms and inference rules of classical first-order logic (without identity) for the language $\mathcal{L}_\in$ extended by Extensionality and Stratified Comprehension.*

A particular nice property of NF at the meta-theoretic level is the fact that it is finitely axiomatizable, as proven by (HAILPERIN, 1944). Furthermore, NF is a set theory with an universal set.

Lemma 3.7.11. *$V \in V$ holds in* NF*.*

Proof. We obtain V by instantiating the formula $(\exists x)(\forall y)(y \in x \leftrightarrow y = y)$. Then by definition of V we get by Stratified Comprehension that $\forall x(x \in V \leftrightarrow x = x)$. Furthermore, given that $V = V$, we conclude $V \in V$. □

Interestingly, NF is not a paraconsistent set theory given that all the remaining pathological sets and set-theoretical paradoxes of naïve paraconsistent set theory that we introduced in Chapter 3.2 are blocked (FORSTER, 1974, p. 24). For instance, *Cantor's paradox* does not go through due to the fact that we are unable to prove Cantor's theorem. Let us have a closer look at the original proof and why it fails within NF.

The original proof of Cantor's theorem consists in showing by reductio that there exists no surjection $f : X \to P(X)$. In particular, we need to construct the diagonal set $\{x \in X : x \notin f(x)\}$, however, this object is a set only in the case that the formula $x \in X \wedge x \notin f(x) \wedge f : X \to P(X)$ is stratified. At the same time, this formula depends on the formula $\exists y\big(y \in P(X) \wedge \langle y, x\rangle \in f \wedge f : X \to P(X)\big)$ being stratified. Notice that due to $\langle y, x\rangle \in f$ we get that x and y will be given the same type, however, due to $f : X \to P(X)$ and, in particular, due to the the subformulas $x \in X$ and $y \subseteq X$, y will receive one type higher than x. Again, as in the case of NLP there could exist an alternative proof, nevertheless, until now there has not been found any successful proof-strategy.

Although we do not have Cantor's theorem in its original form, NF can still prove some weaker analog: the cardinality of singleton subsets of a set is smaller than the cardinality of its subsets. In particular, let $S(A)$ abbreviate $\{x \mid \exists u(u \in A \wedge x = \{u\})\}$. Then, we can show in NF that for any set A we have

$$\mid S(A) \mid < \mid \mathbf{V} \mid .$$

So, $\mid S(\mathbf{V}) \mid < \mid \mathbf{V} \mid$, which means that $\mathbf{V}$ is not equinumerous to the set of singletons of its elements.

Another remarkable fact about NF is that we can derive large parts of standard set theory and mathematics from it (ROSSER, 1953). On the "bad" side we have that NF disproves AC, the domain of application of Cantor's theorem is considerably reduced and mathematical induction holds only for stratified properties. Finally, in a recent surprising turn of events, (HOLMES, 2015) claims to have proven that NF is consistent. Another interesting feature of NF is that a considerable part of category theory can be carried out within NF as demonstrated by (MACLANE, 1971).

Now, in order to obtain the NF_n-hierarchy, we take as non-logical axioms Extensionality and a slightly modified version of Stratified Comprehension and as logical axioms we choose one of da Costa's C-systems. Moreover, we say that a formula φ is *normal*, if φ is stratified or, if φ unstratified and neither the strong negation nor the conditional occur in it. Thus, the crux of da Costa's approach is that Modified Stratified Comprehension is restricted to normal formulas.

Definition 3.7.12. *For any normal formula $\varphi(x)$,*

$$\exists y \forall x \Big(x \in y \leftrightarrow \varphi(x) \Big). \qquad \text{(Modified Stratified Comprehension)}$$

Then we define NF_n where $n \geq 1$ as follows:

Definition 3.7.13. *The Theory NF_n where $n \geq 1$ consists of the axioms and inference rules of $\mathsf{C}_n^{=}$ for the language $\mathcal{L}_\in$ extended by Extensionality and Modified Stratified Comprehension.*

It has been shown that each NF_n set theory is a paraconsistent set theory. The only exception is NF_0 which is Quine's original NF.

Theorem 3.7.14. (COSTA, 1986, Thm. 7). *NF_n where $n \geq 1$ is paraconsistent.*

Furthermore, assuming that NF is consistent we can show that each NF_n set theory is non-trivial.

Theorem 3.7.15. (COSTA, 1986, Thm. 9). *If NF is consistent, NF_n where $n \geq 1$ is non-trivial.*

Let us now address a major drawback of these set theories. We argue that they are unfaithful to their philosophical motivation. As already pointed out: the guiding idea behind a naïve paraconsistent set theory is that the correct non-logical axioms of set theory are Unrestricted Comprehension and Extensionality. In particular, this implies that our comprehension schema should be able to instantiate as many sets as possible. Nevertheless, in each NF_n set theory we have to restrict the formulas occurring in Stratified

Comprehension to normal formulas, i.e., implication and strong negation-free formulas. Notice that only in the case of NF_0 we are allowing Stratified Comprehension to instantiate every stratified formula, however, NF_0 is not a paraconsistent set theory. Now, in the case that we do not apply this restriction, then every NF_n set theory, where $n > 1$, is trivial via a variation of the Curry-paradox (COSTA, 1974, p. 507, Remark 4). Hence, we are forced to give up on a considerable amount of instances of the comprehension schema in NF_n in order to grant the non-triviality of these set theories. This seems to go against our desiderata that we want to accommodate as many sets as possible in our set-theoretic universe. More importantly, there exists no convincing reason of why a set-theorist should embrace Stratified Comprehension restricted to normal formulas instead of Separation and the classical iterative conception of set. We conclude that this approach is unfeasible as well.

> There should be more sets around than are dreamt of in a consistentist's philosophy. Therefore, the fact that such set-theoretic constructions can not avoid *all* the limitations of set building speaks against them. (BERTO, 2007, p. 248)

3.8 LFI - Set Theories

The LFIs form a family of paraconsistent logics that extend Da Costa's C-systems. These logics where first introduced by João Marcos in his doctoral thesis (MARCOS, n.d.).[2] Formally, we define these logics as those which satisfy the following two conditions:

1. There is a set of formulas Γ and formulas φ and ψ such that $\Gamma \cup \{\varphi, \neg\varphi\} \nvdash \psi$.

2. Given a formula φ, there is a set of formulas $\circ(\varphi)$, uniquely determined by φ, such that for any set of formulas Γ and for any formula ψ, $\Gamma \cup \circ(\varphi) \cup \{\varphi, \neg\varphi\} \vdash \psi$.

Principle (1.) implies that LFIs are paraconsistent logics and (2.) imposes that negation has only to be explosive with respect to consistent formulas (which is exactly the idea behind the gentle law of explosion introduced in the previous section). The *weakest* LFI is called mbC and consists of positive classical propositional logic (CPL^+), to which a

[2] For a state-of-the-art presentation of LFIs consider (CARNIELLI; CONIGLIO, 2016a).

paraconsistent negation $\neg$ and a consistency operator $\circ$ are added. We will denote this expanded language as $\mathcal{L}_{Prop}^{\{\circ,\neg\}}$.

Moreover, the classical negation can be defined within mbC as

$$\sim \varphi =_{df.} \varphi \to (\circ\varphi \wedge \varphi \wedge \neg\varphi).$$

Also notice that mbC constitutes a conservative extension of CPL^+, since every theorem of CPL^+ can be recovered in mbC.

Definition 3.8.1. *The logic* mbC *is defined over* $\mathcal{L}_{Prop}^{\{\circ,\neg\}}$ *and is axiomatized by the following axiom schemes and inference rules.*

Axiom schemes:

$$\varphi \to (\psi \to \varphi) \tag{Ax_1}$$

$$\big(\varphi \to (\psi \to \gamma)\big) \to \big((\varphi \to \psi) \to (\varphi \to \gamma)\big) \tag{Ax_2}$$

$$\varphi \to \big(\psi \to (\varphi \wedge \psi)\big) \tag{Ax_3}$$

$$(\varphi \wedge \psi) \to \varphi \tag{Ax_4}$$

$$(\varphi \wedge \psi) \to \psi \tag{Ax_5}$$

$$\varphi \to (\varphi \vee \psi) \tag{Ax_6}$$

$$\psi \to (\varphi \vee \psi) \tag{Ax_7}$$

$$(\varphi \to \gamma) \to \big((\psi \to \gamma) \to ((\varphi \vee \psi) \to \gamma)\big) \tag{Ax_8}$$

$$\varphi \vee \neg\varphi \tag{Ax_9}$$

$$\circ\,\varphi \to \big(\varphi \to (\neg\varphi \to \psi)\big) \tag{Ax_{10}}$$

Inference rule:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,(\mathsf{MP})$$

It is noticeable that the axioms schemes ($\mathsf{Ax}_1 - \mathsf{Ax}_9$) plus the inference rule MP, defined over the same language constitute CPL^+. It follows that the only axiom scheme that differentiates mbC from CPL^+ is (Ax_{10}), which is meant to capture the content of principle (2). Thus, (Ax_{10}) characterizes primarily mbC as an LFI.

Next, we present $\mathsf{Qmbc}_{\approx}$, which is the first-order logic with identity of mbC. Then, let $\mathcal{L}_{Fol}^{\approx}$ denote the first-order language with identity that extends the propositional language $\mathcal{L}_{Prop}^{\{\circ,\neg\}}$.

Definition 3.8.2. *We obtain the logic* $\mathsf{Qmbc}_{\approx}$ *which is defined over* $\mathcal{L}_{Fol}^{\approx}$*, by adding to* mbC *the following axiom schemes and inference rules.*

Axiom schemes:

$$\varphi[x/t] \to \exists x\varphi, \; \textit{if } t \textit{ is a term free for } x \textit{ in } \varphi. \tag{Ax_{11}}$$

$$\forall x\varphi \to \varphi[x/t], \; \textit{if } t \textit{ is a term free for } x \textit{ in } \varphi. \tag{Ax_{12}}$$

$$\varphi \to \psi, \; \textit{whenever } \varphi \textit{ is a variant of } \psi. \tag{Ax_{13}}$$

$$\forall x(x \approx x) \tag{Ax_{14}}$$

$$\forall x\forall y\Big((x \approx y) \to (\varphi \to \varphi[x/y])\Big), \; \textit{if } y \textit{ is a variable free for } x \textit{ in } \varphi. \tag{Ax_{15}}$$

Inference rules:

$$\frac{\varphi \to \psi}{\varphi \to \forall x\psi}\;, \textit{ if } x \textit{ is not free in } \varphi. \tag{$\forall_{\mathsf{In}}$}$$

$$\frac{\varphi \to \psi}{\exists x\varphi \to \psi}\;, \textit{ if } x \textit{ is not free in } \psi \tag{$\exists_{\mathsf{In}}$}$$

The first iterative paraconsistent set theory (based on a LFI) that we would like to introduce here is ZFmbC (see Definition 3.8.3). Moreover, the logical axioms of ZFmbC are given by $\mathsf{Qmbc}_{\approx}$ and the non-logical axioms of ZFmbC are ZF-like axioms.

Now, let us have a look at the non-logical axioms of ZFmbC. Observe that within the axiom of infinity $\varnothing^*$ stands for the strong empty set which is defined as $\varnothing^* =_{df.} \{x : \sim (x \approx x)\}$. For the replacement schema let $\varphi(x, y)$ be a formula with two free variables. Then FUN_{φ} denotes the following formula:

$$FUN_{\varphi} =_{df.} \forall x\forall y\forall z\Big(\varphi(x,y) \wedge \varphi(x,z) \to (y \approx z)\Big).$$

Moreover, let $\mathcal{L}_{\in}^{*}$ denote the language of set theory which contains, besides the equality predicate $\approx$ and binary predicate $\in$, a further unary predicate C, which stands for the consistency of sets.

Definition 3.8.3. *The theory* ZFmbC *consists of the axiom schemes and inference rules of* $\mathsf{Qmbc}_{\approx}$ *for the language* $\mathcal{L}^*_{\in}$ *extended by the following set-theoretic schemes:*

$$\forall x \forall y \Big(\forall z (z \in y \leftrightarrow z \in x) \to (x = y) \Big) \qquad \text{(Extensionality)}$$

$$\forall x \exists y \forall z \Big(z \in y \leftrightarrow \forall w \in z (w \in x) \Big) \qquad \text{(Power Set)}$$

$$FUN_{\varphi} \to \exists b \forall y \Big(y \in b \wedge \exists x (x \in a \wedge \varphi(x, y)) \Big) \qquad \text{(Replacement}_{\varphi}\text{)}$$

$$\forall x \exists y \forall z \Big(z \in y \leftrightarrow \exists w \in x (z \in x) \Big) \qquad \text{(Union)}$$

$$\forall x \forall y \exists z \forall w \Big(w \in x \leftrightarrow (w \approx x \vee w \approx y) \Big) \qquad \text{(Pairing)}$$

$$\exists w \Big((\varnothing^* \in w) \wedge \forall x (x \in w \to x \cup \{x\} \in w) \Big) \qquad \text{(Infinity)}$$

$$\forall x \exists y \forall z \Big((z \in y) \leftrightarrow ((z \in x) \wedge \varphi(x)) \Big) \qquad \text{(Separation}_{\varphi}\text{)}$$

$$C(x) \to \Big(\exists y (y \in x) \to \exists y \Big((y \in x) \wedge \sim \exists z (z \in x \wedge z \in y) \Big) \Big) \qquad \text{(Weak regularity)}$$

$$(x \not\approx y) \leftrightarrow \exists z \Big((z \in x) \wedge (z \in y) \Big) \vee \exists z \Big((z \in y) \wedge (z \in x) \Big) \qquad \text{(Unextensionality)}$$

$$\forall x \Big(x \in y \to \Big(C(x) \to C(y) \Big) \Big) \qquad \text{(Con}_0\text{)}$$

$$\forall x \Big(C(x) \to \circ (x = x) \Big) \qquad \text{(Con}_1\text{)}$$

$$\forall x \Big(\neg \circ (x = x) \to \neg C(x) \Big) \qquad \text{(Con}_2\text{)}$$

Notice that in ZFmbC we have a classical negation, $\sim$ and a paraconsistent negation $\neg$ at disposal (since we have these two negations in the underlying logic $\mathsf{Qmbc}_{\approx}$). Moreover, the first six axioms of Definition 3.8.3 together with $\mathsf{Foundation}_{\varphi}$ constitute the standard ZF axiom system. The non-standard axioms of ZFmbC are thus Weak regularity, Unextensionality, and the axioms governing the consistency predicate. Moreover, in (CARNIELLI; CONIGLIO, 2016b) it is proved that ZFmbC is non-trivial provided that ZF is consistent.

Notice that ZFmbC is too weak to define inconsistent sets or the inconsistency operator. However, the main motivation behind LFI-set theories is (1.) to have inconsistent sets, as well as, consistent sets in our ontology and (b) the ability to talk about consistent and inconsistent formulas via operators in the object language. Thus, to overcome this weakness, we can consider extensions of ZFmbC, defined by taking stronger LFIs and appropriate axioms for the consistency predicate. The first set theory that is strong enough to support both (a) and (b), is based on the logic mCi.

Definition 3.8.4. *The theory* ZFmCi *is obtained from* ZFmbC *by adding the following axiom schemes, for* $n \geq 0$:

$$\neg \circ \varphi \rightarrow (\varphi \wedge \neg\varphi) \tag{ci}$$

$$\neg^{n+2} \circ \varphi \rightarrow \neg^{n} \circ \varphi \tag{$\neg^n$}$$

$$\forall x(\neg C(x) \rightarrow \neg \circ (x \approx x) \tag{Con_3}$$

$$\forall x(\neg C(x) \rightarrow \neg \circ (x \in x) \tag{Con_4}$$

The first two axioms (ci) and ($\neg^n$) transform the underlying logic mbC into the stronger logic mCi, in which the inconsistency operator can be defined as $\bullet\varphi =_{df.} \neg \circ \varphi$ and inconsistent sets as the dual of consistents sets, so $I(x) =_{df.} \neg\, C(x)$.

Finally, we have two more extensions of mCi which are strong enough to satisfy (a) and (b).

Definition 3.8.5. *The theory* ZFCi *is obtained from* ZFmCi *by adding the following axiom schema:*

$$\neg\neg\varphi \rightarrow \varphi \tag{cf}$$

And secondly, we have:

Definition 3.8.6. *The theory* ZFCil *is obtained from* ZFCi *by adding the following axiom schemes:*

$$\neg(\varphi \wedge \neg\varphi) \rightarrow \circ\varphi, \tag{cl}$$

$$\forall x(\neg(x \approx x) \wedge (x \not\approx x)) \rightarrow C(x) \tag{Con_5}$$

$$\forall x((\neg(x \in x) \wedge (x \notin x)) \rightarrow C(x)). \tag{Con_6}$$

A quick word on the motivation of the consistency and inconsistency predicate. The possibility to distinguish between inconsistent and consistent sets is intended to capture the intuition of Cantor, that also *inconsistent totalities* may exist. The authors of (CARNIELLI; CONIGLIO, 2016a), go one and cite Cantor's well-known argument, where he uses inconsistent totalities in an ad-absurdum proof, to show that the collection of cardinals is totally ordered, see (NILSON; MESCHKOWSKI, 1991, p. 410). Nevertheless, this motivation seems unsatisfactory, since it does not follow from Cantors argument that also *inconsistent sets* exist.

Thus, one feasible interpretation of inconsistent sets could be in term of proper classes and consistent sets in terms of sets. But these seems problematic, as well, since this enters in conflict with axiom con_0 which says that consistency is a property that is preserved $\in$-upwards. So even proper classes would become consistent sets and the whole proper class/set distinction would dissolve. We will address this issue in the context of algebra-valued models in Chapter 4.5 (pp. 105-106).

Nevertheless, we believe that this objection against the consistency and inconsistency predicate is not enough to discard LFI-set theories. Especially, since we are dealing with iterative paraconsistent set theories the deciding criteria should be the existence of natural models with a rich ontology. Thus, we will try to construct algebra-valued models for LFI-set theories

Chapter 4

A Class of Models for Non-classical Set Theories

Summary

In this chapter, our main goal was to find a wider class of DRI-algebras that give rise to non-classical models of set theory. First, we show that join complemented lattices equipped with a suitable binary operation $\Rightarrow$ are indeed DRI-algebras, however, these join complemented lattice-valued models fail to be expressive enough for the construction of set-theoretic models. Then, we show that meet complemented meet semilattices equipped with a binary operation $\Rightarrow$ are as well DRI-algebras and each of this meet complemented lattice-valued models validates the negation-free fragment of ZF. Then, we show that each totally ordered meet complemented lattice equipped with the same binary operation $\Rightarrow$ allows us to build algebra-valued models of full ZF, whose internal logic is neither paraconistent, nor intuitionistic, nor classical. We apply these models to give an independence proof of Foundation from ZF. Finally, we attempt to build algebra-valued models for LFI-set theories and point out that we face several philosophical and technical difficulties.

4.1 A Wider Class of DRI-algebras

In this section, we define two classes of DRI-algebras. While the first does not give rise to models of NFF-ZF, the second one does. The former attempt constitutes a negative result. However, on the one hand, it helps in understanding better the class of DRI-algebras, and, on the other hand, it justifies the introduction of the latter.

4.2 Join Complemented Lattices

We are searching for DRI-algebras $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$ such that $\mathbb{A}$ is not a Heyting algebra and $\mathbf{V}^{(\mathbb{A})} \models_F \mathsf{ZF}$ for some filter F on $\mathbb{A}$. The first class that we will consider here that of lattices expanded with the join complement. These algebras are the duals of meet complemented lattices (GOODMAN, 1981). As the latter do not always verify the equality $x \vee x^{*_p} = \mathbf{1}$, for every element x, where x^{*_p} indicates the meet complement of x, the former do not verify $x \wedge x^{*_d} = \mathbf{0}$, where x^{*_d} is the join complement of x. Consequently, the join complement would provide a paraconsistent negation. Moreover, in this section, we will consider lattices expanded with join complements.

Definition 4.2.1. *We call a structure $\langle \mathbf{A}, \wedge, \vee, ^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ a join complemented lattice, if*

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a complete bounded distributive lattice and

(ii) the unary operator ^{*_d} is defined for every $x \in \mathbf{A}$ as $x^{*_d} = \mathsf{min}\{y \in \mathbf{A} : x \vee y = \mathbf{1}\}$.

For matters of space, we will not give a detailed presentation of join complemented lattices. For more details see (URBAS, 1996) and (GOODMAN, 1981). However, we now state the main properties of the join complement, i.e.,

$$x \vee x^{*_d} = \mathbf{1} \tag{DI}$$

$$\text{if } x \vee y = \mathbf{1}, \text{then } x^{*_d} \leq y. \tag{DE}$$

It will also be useful to bear in mind the following two lemmas.

Lemma 4.2.2. *Let $\langle \mathbf{A}, \wedge, \vee, ^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ be a join complemented lattice and $x, y \in \mathbf{A}$. If $y \leq x$, then $x^{*_d} \leq y^{*_d}$.*

Proof. Since $y \vee y^{*_d} = \mathbf{1}$ and $y \leq x$, we get $x \vee y^{*_d} = \mathbf{1}$. Therefore, by (DE), $x^{*_d} \leq y^{*_d}$. □

Lemma 4.2.3. *Let $\langle \mathbf{A}, \wedge, \vee, ^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ be a join complemented lattice. For any $x, y \in \mathbf{A}$ we have:*

$$(x \wedge y)^{*_d} = x^{*_d} \vee y^{*_d} \tag{DM$_\wedge$}$$

Table 4.1: Operations for $\mathbb{A}_3$

$\Rightarrow_a$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\vee$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$

$\wedge$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

x	x^*
$\mathbf{1}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$

Proof. As $x \wedge y \leq x$, then, using Lemma 4.2.2, $x^{*_d} \leq (x \wedge y)^{*_d}$. Analogously, $y^{*_d} \leq (x \wedge y)^{*_d}$. So, $x^{*_d} \vee y^{*_d} \leq (x \wedge y)^{*_d}$. For the other inequality, note that, using (DI) and (DE) we have both $(x^{*_d} \vee y^{*_d})^{*_d} \leq y$. So, $(x^{*_d} \vee y^{*_d})^{*_d} \leq x \wedge y$. Then, using Lemma 4.2.2, we get $(x \wedge y)^{*_d} \leq (x^{*_d} \vee y^{*_d})^{*_d *_d} \leq x^{*_d} \vee y^{*_d}$, where the last inequality follows from the fact that $x^{*_d *_d} \leq x$. □

4.2.1 The $\Rightarrow_a$-operator

Notice that so far we have no conditional in the language of our join complemented lattices. Thus, we will now define a binary operator $\Rightarrow_a$ for our join complemented lattices that mimics the definition of the material conditional.

Definition 4.2.4. *We call a structure $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_a, ^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ an implicative join complemented lattice, if*

(i) $\langle \mathbf{A}, \wedge, \vee, ^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ is a join complemented lattice and

(ii) the binary operation $\Rightarrow_a$ is defined for any $x, y \in \mathbf{A}$ as

$$x \Rightarrow_a y =_{df.} x^{*_d} \vee y.$$

We have depicted the operations of the three-element implicative join complemented lattice, which we will call $\mathbb{A}_3$, in Table 3.1 above. Next, we show that every implicative join complemented lattice is a DRI-algebra.

Theorem 4.2.5. *Every implicative join complemented lattice $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_a, ^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ is a* DRI-*algebra.*

Proof. **(P1)**. Suppose $x \wedge y \leq z$. Then, by $\vee$-monotonicity, $y^{*_d} \vee (x \wedge y) \leq y^{*_d} \vee z$. By distributivity, it follows that $(y^{*_d} \vee x) \wedge (y^{*_d} \vee y) \leq y^{*_d} \vee z$. Finally, as $y^{*_d} \vee y = \mathbf{1}$ and $x \leq y^{*_d} \vee x$, it follows that $x \leq y^{*_d} \vee z$. That is, **P1** holds.
(P2). Suppose $y \leq z$. Then, by $\vee$-monotonicity, $x^{*_d} \vee y \leq x^{*_d} \vee z$. That is, **P2** holds.
(P3). Suppose $y \leq z$. Then, by ^{*_d}-antimonotonicity, $z^{*_d} \leq y^{*_d}$. So, by $\vee$-monotonicity, $z^{*_d} \vee x \leq y^{*_d} \vee x$. That is, **P3** holds.
(P4). In order to see that **P4** holds, on the one hand, by ($\mathbf{DM}_\wedge$), we have that $(x \wedge y)^{*_d} \vee z \leq (x^{*_d} \vee y^{*_d}) \vee z$. So, by $\vee$-associativity, it follows that $(x \wedge y)^{*_d} \vee z \leq x^{*_d} \vee (y^{*_d} \vee z)$. On the other hand, as $x \wedge y \leq x$, by ^{*_d}-antimonotonicity, we have that $x^{*_d} \leq (x \wedge y)^{*_d}$. Analogously, we get $y^{*_d} \leq (x \wedge y)^{*_d}$. Then, $x^{*_d} \vee (y^{*_d} \vee z) \leq (x \wedge y)^{*_d} \vee z$. So, **P4** holds. □

Although Theorem 4.2.5 shows that any implicative join complemented lattice $\mathbb{A}$ is a DRI-algebra, to demonstrate that we can build lattice-valued models of set theory on top of these structures we still need to check that the $\mathcal{BQ}_\varphi$ property, for negation free φ in $\mathbf{V}^{(\mathbb{A})}$. That is,

$$[\![\forall x \in u\ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \ \Rightarrow_a [\![\varphi(x)]\!]),$$

should hold for any $\varphi \in \mathsf{NFF}\text{-}\mathcal{L}_\in^\mathbb{A}$. Unfortunately, this is not the case, since we will prove that there exists a negation-free formula $\varphi \in \mathcal{L}_\in^\mathbb{A}$ for $\mathcal{BQ}_\varphi$ fails.

Lemma 4.2.6. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_a, {}^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ be an implicative join complemented lattice with more than two elements and let $\mathbf{V}^{(\mathbb{A})}$ be the corresponding $\mathbb{A}$-valued model. Then, for any $u, v, w \in \mathbf{V}^{(\mathbb{A})}$ the following claims do not hold:*

(i) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$,

(ii) $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$,

(iii) $\Big([\![u = v]\!] \Rightarrow_a [\![u \in w]\!]\Big) = \Big([\![u = v]\!] \Rightarrow_a [\![v \in w]\!]\Big)$.

Proof. (i) Consider $\mathbb{A}_3$ (see Table 3.1) and the elements $p_\mathbf{0}, p_{\frac{1}{2}}, p_\mathbf{1} \in \mathbf{V}^{(\mathbb{A}_3)}$ defined as $p_\mathbf{0} = \{\langle \varnothing, \mathbf{0} \rangle\}$, $p_{\frac{1}{2}} = \{\langle \varnothing, \frac{1}{2} \rangle\}$, and $p_\mathbf{1} = \{\langle \varnothing, \mathbf{1} \rangle\}$. Then, we have

$$\Big([\![p_\mathbf{0} = p_{\frac{1}{2}}]\!] \wedge [\![p_{\frac{1}{2}} = p_\mathbf{1}]\!]\Big) = \frac{1}{2} > [\![p_0 = p_1]\!] = \mathbf{0}.$$

(ii) Consider again $\mathbb{A}_3$ and the elements $p_{\mathbf{0}}$, $p_{\frac{1}{2}}, p_{\mathbf{1}} \in \mathbf{V}^{(\mathbb{A}_3)}$ and additionally let $u = \{\langle p_{\mathbf{0}}, \frac{1}{2}\rangle\}$. Then,

$$[\![p_{\frac{1}{2}} \in u]\!] = (u(p_{\mathbf{0}}) \wedge [\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!]) = \frac{1}{2}$$

and

$$[\![p_{\mathbf{1}} \in u]\!] = (u(p_{\mathbf{0}}) \wedge [\![p_{\mathbf{0}} = p_{\mathbf{1}}]\!]) = \mathbf{0}.$$

Hence, $\left([\![p_{\frac{1}{2}} = p_{\mathbf{1}}]\!] \wedge [\![p_{\frac{1}{2}} \in u]\!]\right) = \frac{1}{2} > [\![p_{\mathbf{1}} \in u]\!] = \mathbf{0}$.

(iii) Consider $\mathbb{A}_3$ and the elements $p_{\mathbf{0}}$, $p_{\frac{1}{2}}, p_{\mathbf{1}} \in \mathbf{V}^{(\mathbb{A}_3)}$ and let $v = \{\langle p_{\mathbf{1}}, \mathbf{1}\rangle\}$. Then,

$$[\![p_{\frac{1}{2}} \in v]\!] = (v(p_{\mathbf{1}}) \wedge [\![p_{\frac{1}{2}} = p_{\mathbf{1}}]\!]) = \frac{1}{2}.$$

Hence,

$$[\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!] \Rightarrow_a [\![p_{\mathbf{0}} \in v]\!] = \mathbf{0}$$

and

$$[\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!] \Rightarrow_a [\![p_{\frac{1}{2}} \in v]\!] = \frac{1}{2}.$$

Therefore, $\left([\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!] \Rightarrow_a [\![p_{\frac{1}{2}} \in v]\!]\right) \neq \left([\![p_{\mathbf{0}} = p_{\frac{1}{2}}]\!] \Rightarrow_a [\![p_{\mathbf{0}} \in v]\!]\right)$. □

This gives us immediately the following result.

Theorem 4.2.7. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_a, ^{*d}, \mathbf{1}, \mathbf{0}\rangle$ be an implicative join complemented lattice with more than two elements. Then $\mathcal{BQ}_\varphi$ does not hold in $\mathbf{V}^{(\mathbb{A})}$.*

Proof. We use $p_0, v, p_{\frac{1}{2}} \in \mathbf{V}^{\mathbb{A}_3}$ (where v is defined as in the previous Lemma) and the formula $\varphi(x) = (x \in v)$. Define the name $u = \{\langle p_{\frac{1}{2}}, \mathbf{1}\rangle\}$. Then,

$$\begin{aligned}
&\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_a [\![\varphi(x)]\!]) \\
&= (u(p_{\frac{1}{2}}) \Rightarrow_a [\![\varphi(p_{\frac{1}{2}})]\!]) \\
&= (\mathbf{1} \Rightarrow_a \frac{1}{2}) \\
&= \frac{1}{2}.
\end{aligned}$$

However, on the other hand,

$$\begin{aligned}
&\bigwedge_{x\in\mathbf{V}^{(\mathbb{A}_3)}}\Big(\bigvee_{y\in\mathrm{dom}(u)}(u(y)\wedge[\![y=x]\!])\Rightarrow_a[\![\varphi(x)]\!]\Big)\\
&\leq (u(p_{\frac{1}{2}})\wedge[\![p_0=p_{\frac{1}{2}}]\!])\Rightarrow_a[\![\varphi(p_0)]\!]\\
&=(\mathbf{1}\Rightarrow_a\mathbf{0})\\
&=\mathbf{0}.
\end{aligned}$$

This concludes our proof. $\square$

Theorem 4.2.7 shows that taking an implicative join complemented lattice of the form $\mathbb{A}=\langle\mathbf{A},\wedge,\vee,\Rightarrow_a,^{*_d},\mathbf{1},\mathbf{0}\rangle$ is unfortunately not enough to validate the negation-free fragment of ZF.

4.2.2 The $\Rightarrow_b$-operator

Before giving up on this approach, we can consider another class of implicative join complemented lattices which we obtain by defining another binary operation on join complemented lattices.

Definition 4.2.8. *We call a structure $\langle\mathbf{A},\wedge,\vee,\Rightarrow_b,^{*_d},\mathbf{1},\mathbf{0}\rangle$ an implicative join complemented lattice, if*

(i) $\langle\mathbf{A},\wedge,\vee,^{*_d},\mathbf{1},\mathbf{0}\rangle$ is a join complemented lattice and

(ii) the binary operation $\Rightarrow_b$ is defined for any $x,y\in\mathbf{A}$ as

$$x\Rightarrow_b y=_{df.}(x\wedge y^{*_d})^{*_d}.$$

Lemma 4.2.9. *Every implicative join complemented lattice $\langle\mathbf{A},\wedge,\vee,\Rightarrow_b,^{*_d},\mathbf{1},\mathbf{0}\rangle$ satisfies* **P2**-**P4**.

Proof. (**P2**). Suppose $y\leq z$. By ^{*_d}-antimonotonicity, it follows that $z^{*_d}\leq y^{*_d}$. Then, by $\wedge$-monotonicity, we get $x\wedge z^{*_d}\leq x\wedge y^{*_d}$. Again by ^{*_d}-antimonotonicity we conclude $(x\wedge y^{*_d})^{*_d}\leq(x\wedge z^{*_d})^{*_d}$. Therefore, $x\Rightarrow_b y\leq x\Rightarrow_b z$. So, **P2** holds.
(**P3**). Suppose $y\leq z$. By $\wedge$-monotonicity it follows that $y\wedge x^{*_d}\leq z\wedge x^{*_d}$. Now, by

^{*_d}-antimonotonicity, we conclude $(z \wedge x^{*_d})^{*_d} \leq (y \wedge x^{*_d})^{*_d}$. Therefore $z \Rightarrow_b x \leq y \Rightarrow_b x$. So, **P3** holds.

(**P4**). We have to show that $\Big((x \wedge y) \wedge z^{*_d}\Big)^{*_d} = \Big(x \wedge (y \wedge z^{*_d})^{*_d *_d}\Big)^{*_d}$. Now, on one hand, using ($\mathsf{DM}_\wedge$), the left-hand side equals $(x \wedge y)^{*_d} \vee z^{*_d *_d}$, which, using ($\mathsf{DM}_\wedge$) again, equals $(x^{*_d} \vee y^{*_d}) \vee z^{*_d *_d}$. On the other hand, also using ($\mathsf{DM}_\wedge$), the right-hand side equals $x^{*_d} \vee (y \wedge z^{*_d})^{*_d *_d *_d}$, which, given that ${}^{*_d *_d *_d} = {}^{*_d}$ and using ($\mathsf{DM}_\wedge$) again, yields $x^{*_d} \vee (y^{*_d} \vee z^{*_d *_d})$. Using $\vee$-associativity, we reach our goal. So, **P4** holds. □

However, property **P1** does not hold for the implicative join complemented lattices $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_b, {}^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ with more than two elements. Indeed, consider the three-element join complemented lattice $\mathbb{A}_3 = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_b, {}^{*_d}, \mathbf{1}, \mathbf{0} \rangle$. Let $y = \mathbf{1}$ and $x = z = \frac{1}{2}$. Then,

$$(x \wedge y) = z = \frac{1}{2},$$

thus verifying the antecedent of **P1**. However, $y = z^{*_d} = (y \wedge z^{*_d}) = \mathbf{1}$ and so,

$$(y \wedge z^{*_d})^{*_d} = \mathbf{0},$$

contradicting the consequent of **P1**. Thus, every implicative join complemented lattices $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_b, {}^{*_d}, \mathbf{1}, \mathbf{0} \rangle$ with more than two elements is not a DRI-algebra.

4.3 Meet Complemented Lattices

In this section, we explore a second class of lattices for the construction of algebra-valued models of set theory. We will show that every meet complemented semilattice that is equipped with a particular binary operation $\Rightarrow$ is indeed a DRI-algebra.

Definition 4.3.1. *We call a structure $\langle \mathbf{A}, \wedge, {}^{*_p}, \mathbf{0} \rangle$ a meet complemented meet semilattice, if*

(i) $\langle \mathbf{A}, \wedge, \mathbf{0} \rangle$ is a meet semilattice and

(ii) the unary operation ^{*_p} is defined for any $x \in \mathbf{A}$ as

$$x^{*_p} = \mathsf{max}\{y \in \mathbf{A} : (x \wedge y) = \mathbf{0}\}.$$

Moreover, the main properties of the meet complement that we will be using are the following:

$$(x \wedge x^{*_p}) = \mathbf{0}, \tag{PI}$$

$$\text{if } (x \wedge y) = \mathbf{0}, \text{ then } y \leq x^{*_p}. \tag{PE}$$

Then we can show that the meet complement has further properties:

Lemma 4.3.2. *Let $\langle \mathbf{A}, \wedge, ^{*_p}, \mathbf{0} \rangle$ be a meet complemented meet semilattice. Then for any $x, y \in \mathbf{A}$ we have:*

(i) $x \leq x^{*_p *_p}$,

(ii) if $x \leq y$, *then* $y^{*_p} \leq x^{*_H}$,

(iii) $x^{*_p *_p *_p} = x^{*_p}$,

(iv) $(x \wedge y)^{*_p} \wedge x \leq y^{*_x}$,

(v) $(x \wedge y)^{*_p *_p} = x^{*_p *_p} \wedge y^{*_p *_p}$.

Proof. (i) By (PI) and $\wedge$-commutativity we get $(x^{*_p} \wedge x) = \mathbf{0}$. Then using (PE), $x \leq x^{*_p *_p}$.
(ii) From $x \leq y$ we get $(x \wedge y) = x$. Then using (PI) and a property of $\wedge$ we get $x \wedge (y \wedge y^{*_p}) = \mathbf{0}$, which using $\wedge$-associativity implies $(x \wedge y) \wedge y^{*_p} = \mathbf{0}$. So $x \wedge y^{*_p} = \mathbf{0}$. Then by (PE), $y^{*_p} \leq x^{*_p}$.
(iii) By part (i) we get $x^{*_p} \leq x^{*_p *_p *_p}$. Now, by part (i) and part (ii) we get $x^{*_p *_p *_p} \leq x^{*_p}$.
(iv) By (PI) we have $(x \wedge y) \wedge (x \wedge y)^{*_p} = \mathbf{0}$, which, using $\wedge$-commutativity and $\wedge$-associativity, implies $y \wedge \big((x \wedge y)^{*_p} \wedge x\big) = \mathbf{0}$. Our goal follows using (PE).
(v) Applying part (ii) twice to $x \wedge y \leq x$ we get $(x \wedge y)^{*_p *_p} \leq x^{*_p *_p}$. Analogously $(x \wedge y)^{*_p *_p} \leq y^{*_p *_p}$. So, $(x \wedge y)^{*_p *_p} \leq y^{*_p *_p} \wedge x^{*_p *_p}$. For the other inequality proceed as follows :

(1) $y \wedge (x \wedge y)^{*_p} \leq x^{*_p}$ (by part (iv))
(2) $x^{*_p *_p} \wedge y \wedge (x \wedge y)^{*_p} \leq x^{*_p}$ (by (1) and a property of $\wedge$)
(3) $x^{*_p *_p} \wedge y \wedge (x \wedge y)^{*_p} \leq x^{*_p *_p}$ (by a property of $\wedge$)
(4) $x^{*_p *_p} \wedge y \wedge (x \wedge y)^{*_p} = \mathbf{0}$ (by (2) and (3))
(5) $y \wedge (x^{*_p *_p} \wedge (x \wedge y)^{*_p}) = \mathbf{0}$ (from (4) by properties of $\wedge$)

(6) $x^{*_p*_p} \wedge (x \wedge y)^{*_p} \leq y^{*_p}$ (from (5) by (PE))

(7) $y^{*_p*_p} \wedge x^{*_p*_p} \wedge (x \wedge y)^{*_p} \leq y^{*_p}$ (from (6) by a property of $\wedge$)

(8) $y^{*_p*_p} \wedge x^{*_p*_p} \wedge (x \wedge y)^{*_p} \leq y^{*_p*_p}$ (by a property of $\wedge$)

(9) $y^{*_p*_p} \wedge x^{*_p*_p} \wedge (x \wedge y)^{*_p} = \mathbf{0}$ (by (7) and (8))

(10) $(x \wedge y)^{*_p} \wedge (y^{*_p*_p} \wedge x^{*_p*_p}) = \mathbf{0}$ (from (8) and properties of $\wedge$)

(11) $y^{*_p*_p} \wedge x^{*_p*_p} \leq (x \wedge y)^{*_p*_p}$ (from (10) and (PE)).

$\square$

Now, we can define the binary operation $\Rightarrow_t$ within our meet complemented meet semilattices. We will call the resulting structures *implicative* meet complemented semilattices.

Definition 4.3.3. *We call a structure $\langle \mathbf{A}, \wedge, \Rightarrow_t, ^{*_p}, \mathbf{0} \rangle$ an implicative meet complemented meet semilattice, if*

(i) $\langle \mathbf{A}, \wedge, ^{*_p}, \mathbf{0} \rangle$ is a meet complemented meet semilattice and

(ii) the binary operation $\Rightarrow_t$ is defined for any $x, y \in \mathbf{A}$ as

$$x \Rightarrow_t y =_{df.} (x \wedge y^{*_p})^{*_p}.$$

We go on to show that every implicative meet complemented meet semilattice is a DRI-algebra.

Theorem 4.3.4. *Any implicative meet complemented meet semilattice $\langle \mathbf{A}, \wedge, \Rightarrow_t, ^{*_p}, \mathbf{0} \rangle$ is a* DRI-*algebra.*

Proof. **(P1.)** Assume $x \wedge y \leq z$, that is, $(x \wedge y) \wedge z = x \wedge y$. By $\wedge$-monotonicity it follows that $\big((x \wedge y) \wedge z\big) \wedge z^{*_p} = (x \wedge y) \wedge z^{*_p}$, whose left-hand side equals $\mathbf{0}$. Then, $x \wedge (y \wedge z^{*_p}) = \mathbf{0}$, which implies $x \leq (y \wedge z^{*_p})^{*_p}$. So, **P1** holds.

(P2.) Suppose $y \leq z$. By Lemma 4.3.2(ii), it follows that $z^{*_p} \leq y^{*_p}$, which, using $\wedge$-monotonicity, implies $x \wedge z^{*_p} \leq x \wedge y^{*_p}$. Using Lemma 4.3.2(ii) again, it follows that $(x \wedge y^{*_p})^{*_p} \leq (x \wedge z^{*_p})^{*_p}$. So, **P2** holds.

(P3.) Assume $y \leq z$. Using $\wedge$-monotonicity, it follows that $y \wedge x^{*_p} \leq z \wedge x^{*_p}$, which, by Lemma 4.3.2(ii), implies $(z \wedge x^{*_p})^{*_p} \leq (y \wedge x^{*_p})^{*_p}$. So, **P3** holds.

(P4.) For **P4** we have

$$\Big((x \wedge y) \wedge z^{*_p}\Big)^{*_p} \leq \Big(x \wedge (y \wedge z^{*_p})^{*_p *_p}\Big)^{*_p}$$

if and only if

$$\Big((x \wedge y) \wedge z^{*_p}\Big)^{*_p} \wedge \big(x \wedge (y \wedge z^{*_p})^{*_p *_p}\big) = \mathbf{0}$$

if and only if (by Lemma 4.3.2(iii) and Lemma 4.3.2(v))

$$\Big((x \wedge y) \wedge z^{*_p}\Big)^{*_p} \wedge \Big(x \wedge (y^{*_p *_p} \wedge z^{*_p})\Big) = \mathbf{0}.$$

Now, the left-hand side of this equation is dominated by y^{*_p} and $y^{*_p *_p}$ at the same time and we are done. On the other hand, as $y \leq y^{*_p *_p}$, by $\wedge$-monotonicity, it follows that

$$(x \wedge y) \wedge z^{*_p} \leq (x \wedge y^{*_p *_p}) \wedge z^{*_p},$$

if and only if (by Lemma 4.3.2(iii) and Lemma 4.3.2(v))

$$(x \wedge y) \wedge z^{*_p} \leq x \wedge (y \wedge z^{*_p})^{*_p *_p}.$$

Finally, using Lemma 4.3.2(ii), it follows that

$$\Big(x \wedge (y \wedge \neg z)^{*_p *_p}\Big)^{*_p} \leq \Big((x \wedge y) \wedge z^{*_p}\Big)^{*_p}.$$

Then, **P4** holds. □

Notice that the property of being a DRI-algebra depends only on the binary operation $\Rightarrow$ and the meet $\wedge$, so we can generalize the result of Theorem 4.3.4 to any complete bounded distributive lattices that is equipped with the binary operation $\Rightarrow_t$. As before, to construct lattice-valued models of NFF-ZF, we still need to check that the property $\mathcal{BQ}_\varphi$ holds for negation-free φ. We show in the next section that this is indeed the case for any totally ordered implicative meet complemented lattice.

4.4 Totally Ordered Lattice-valued Models

Definition 4.4.1. *A poset* $\langle \mathbf{A}; \leq \rangle$ *is totally ordered (or a chain) iff for all* $x, y \in \mathbf{A}$ *it holds that either* $x \leq y$ *or* $y \leq x$*.*

Definition 4.4.2. *By a totally ordered implicative meet complemented lattice and well ordered implicative meet complemented lattice we mean a complete bounded distributive implicative meet complemented lattice whose underlying poset is totally ordered and a complete bounded distributive implicative meet complemented lattice whose underlying poset is well ordered, respectively.*

(i) *By* $\mathcal{T}$ *we indicate the class of totally ordered implicative meet complemented lattices of the form* $\mathbb{T} = \langle \mathbf{A}, \wedge, \vee, ^{*_p}, \Rightarrow_t, \mathbf{0}, \mathbf{1} \rangle$*. By an implicative meet complemented* $\mathcal{T}$*-lattice we mean a member of* $\mathcal{T}$*.*

(ii) *By* $\mathcal{W}$ *we indicate the class of well ordered implicative meet complemented lattices of the form* $\mathbb{T} = \langle \mathbf{A}, \wedge, \vee, ^{*_p}, \Rightarrow_t, \mathbf{0}, \mathbf{1} \rangle$*. By an implicative meet complemented* $\mathcal{W}$*-lattice we mean a member of* $\mathcal{W}$*.*

(iii) *By* $\mathcal{W}_{\mathcal{F}}$ *we indicate the class of well ordered implicative meet complemented lattices of the form* $\mathbb{T} = \langle \mathbf{A}, \wedge, \vee, ^{*_p}, \Rightarrow_t, \mathbf{0}, \mathbf{1} \rangle$*, where the underlying poset is of finite size. By an implicative meet complemented* $\mathcal{W}_{\mathcal{F}}$*-lattice we mean a member of* $\mathcal{W}_{\mathcal{F}}$*.*

Notice that even though we have the meet complement in the signature of each meet complemented $\mathcal{T}$-lattice, we have not fixed a negation yet. This will be done only later. Moreover, due to Theorem 4.3.4, we know that every meet complemented $\mathcal{T}$-lattice is a DRI-algebra.

We now show that each implicative meet complemented lattice that belongs to (i) of Definition 4.4.2 generates a $\mathbb{T}$-valued model of NFF-ZF^- and that each meet complemented lattice of (ii) of Definition 4.4.2 generates a $\mathbb{T}$-valued model of NFF-ZF.

Lemma 4.4.3. *Let* $\langle \mathbf{A}, \wedge, \Rightarrow_t, ^{*_p}, \mathbf{0} \rangle$ *be an implicative meet complemented meet semilattice and take any pair* $x, y \in \mathbf{A}$*. Then, we have:*

(i) $x^{*_p} = \mathbf{1}$ *iff* $x = \mathbf{0}$*,*

(ii) $x^{*_p} = \mathbf{0}$ *or* $x^{*_p} = \mathbf{1}$*,*

(iii) $x^{*_p} = \mathbf{0}$ *iff* $x \neq \mathbf{0}$*,*

(iv) $(x \Rightarrow_t y) = \mathbf{0}$ *iff* $x \neq \mathbf{0}$ *and* $y = \mathbf{0}$*,*

(v) $(x \Rightarrow_t y) = \mathbf{1}$ *or* $(x \Rightarrow_t y) = \mathbf{0}$*.*

Proof. (i) Suppose $x^{*_p} = \mathbf{1}$. Then, $\mathbf{0} = (x \wedge x^{*_p}) = (x \wedge \mathbf{1}) = x$. For the other direction, it is clear that $\mathbf{1}^{*_p} = \mathbf{0}$.

(ii) We either have $x \leq x^{*_p}$ or $x^{*_p} \leq x$. In the first case, we have $\mathbf{0} = (x \wedge x^{*_p}) = x$, and so, $x^{*_p} = \mathbf{1}$. In the second case, we have $\mathbf{0} = (x \wedge x^{*_p}) = x^{*_p}$, i.e., $x^{*_p} = \mathbf{0}$.

(iii) Suppose $x^{*_p} = \mathbf{0}$ and $x = \mathbf{0}$, a contradiction, as $\mathbf{0}^{*_p} = \mathbf{1}$. For the other direction, suppose $x \neq \mathbf{0}$. Now, suppose $x^{*_p} = \mathbf{1}$. Then, $\mathbf{0} = (x \wedge x^{*_p}) = x$, a contradiction. So, $x^{*_p} \neq \mathbf{1}$. Using part (ii), it follows that $x^{*_p} = \mathbf{0}$.

(iv) Suppose $x \Rightarrow_t y = \mathbf{0}$, i.e., $(x \wedge y^{*_p})^{*_p} = \mathbf{0}$. Firstly, suppose $x = \mathbf{0}$. Then, $(x \wedge y^{*_p})^{*_p} = \mathbf{1}$, a contradiction. So, $x \neq \mathbf{0}$. Secondly, it similarly follows that $y^{*_p} \neq \mathbf{0}$. Then, using part (ii), $y^{*_p} = \mathbf{1}$. So, $\mathbf{0} = (y \wedge y^{*_p}) = (y \wedge \mathbf{1}) = y$, i.e., $y = \mathbf{0}$, as desired. For the other direction, suppose $x \neq \mathbf{0}$ and $y = \mathbf{0}$. Then, using part (iii), $x \neq \mathbf{0}$ implies that $x^{*_p} = \mathbf{0}$. On the other hand, $y = \mathbf{0}$ implies that $(x \wedge y^{*_p}) = x$. Hence, $(x \wedge y^{*_p})^{*_p} = \mathbf{0}$.

(v) This follows directly from the definition of $\Rightarrow_t$ and part (ii). $\square$

Corollary 4.4.4. *For any* $\mathbb{T} \in \mathcal{T}$ *and for any* $x, y \in \mathbf{V}^{(\mathbb{T})}$ *the following holds:*

(i) $[\![\varphi(x) \to \psi(y)]\!] = \mathbf{0}$ *if and only if* $[\![\varphi(x)]\!] \neq \mathbf{0}$ *and* $[\![\psi(y)]\!] = \mathbf{0}$*.*

(ii) $[\![\varphi(x) \to \psi(y)]\!] = \mathbf{0}$ *or* $[\![\varphi(x) \to \psi(y)]\!] = \mathbf{1}$*,*

(iii) $[\![x = y]\!] = \mathbf{0}$ *or* $[\![x = y]\!] = \mathbf{1}$*.* $\square$

Notice that the $\mathbb{T}$-value of atomic formulas of the form $(u \in v)$ can range over all elements of the universe of $\mathbb{T}$. This is due to the fact that $[\![u \in v]\!]$ is equal to $\bigvee_{x \in \mathrm{dom}(u)} u(x) \wedge [\![x = v]\!]$, where the value of $u(x)$ can be any possible element of $\mathbb{T}$. For instance, if u is an $\mathbb{T}$-name, then $v = \{\langle u, a \rangle\}$ (where $a \in \mathbb{T}$) is also an $\mathbb{T}$-name and $[\![u \in v]\!] = a$.

Now, hereafter we follow very closely the proof strategy of (LÖWE; TARAFDER, 2015). Although the proofs of Lemma 4.4.5 (including Claim 4.7), Lemma 5.4.3, Theorem 4.4.7, Theorem 4.4.9 and Theorem 4.4.10 are essentially the same, in this case, the conclusion holds for a whole class of DRI-algebras and not only for $\mathbb{PS}_3$. What allows this generalization is the abstract definition of $\Rightarrow_t$, which, in the three-element case, coincides with the implication of $\mathbb{PS}_3$.

Lemma 4.4.5. *Let $\mathbb{T} \in \mathcal{T}$. Then, for any three elements $u, v, w \in \mathbf{V}^{(\mathbb{T})}$, we have:*

(i) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$,

(ii) $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$.

Proof. (i) Consider any $\mathbb{T}$ and apply induction on w. Assume that for all $z \in \mathrm{dom}(w)$ we have:

$$[\![u = v]\!] \wedge [\![v = x]\!] \leq [\![u = z]\!].$$

Due to Corollary 4.4.4(iii), it is enough to consider the case $[\![u = w]\!] = \mathbf{0}$. Therefore, suppose:

$$[\![u = w]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![x \in w]\!]) \wedge \bigwedge_{z \in \mathrm{dom}(w)} (w(z) \Rightarrow_t [\![z \in u]\!]) = \mathbf{0}.$$

Given that $\mathbb{T}$ is totally ordered, one of the conjuncts must get value $\mathbf{0}$. So, we do the following case distinction:

Case 1. Suppose $\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![x \in w]\!]) = \mathbf{0}$. Then, by Lemma 4.3.2(v) there exists a x_0 such that $\Big(u(x_0) \Rightarrow_t [\![x_0 \in w]\!]\Big) = \mathbf{0}$. By Corollary 4.4.4(i), this can only be the case if $u(x_0) \neq \mathbf{0}$ and $[\![x_0 \in w]\!] = \mathbf{0}$.

Claim 4.7 For any $y_0 \in \mathrm{dom}(v)$ with $v(y_0) \neq \mathbf{0}$ we have that either $[\![y_0 \in w]\!] = \mathbf{0}$ or $[\![x_0 = y_0]\!] = \mathbf{0}$.

Proof. If $[\![y_0 \in w]\!] = \bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![y_0 = z]\!]) \neq \mathbf{0}$, then there exists a $z_0 \in \mathrm{dom}(w)$ such that $w(z_0) \neq \mathbf{0}$ and $[\![y_0 = z]\!] \neq \mathbf{0}$. Since $w(z_0) \neq \mathbf{0}$,

$$[\![x_0 \in w]\!] = \bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![x_0 = z]\!]) = \mathbf{0}$$

yields $[\![x_0 = z_0]\!] = \mathbf{0}$. Now, by induction hypothesis $[\![x_0 = y_0]\!] \wedge [\![y_0 = z_0]\!] \leq [\![x_0 = z_0]\!]$. Hence, $[\![x_0 = y_0]\!] = \mathbf{0}$. ☐

Using Claim 4.7, either there is some $y_0 \in \mathrm{dom}(v)$ with $v(y_0) \neq \mathbf{0}$ and $[\![y_0 \in w]\!] = \mathbf{0}$, but then $(v(y_0) \Rightarrow_t [\![y_0 \in w]\!]) = \mathbf{0}$ (by Corollary 4.4.4(i)), thus:

$$[\![v = w]\!] = \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow_t [\![y \in w]\!]) \wedge \bigwedge_{z \in \mathrm{dom}(w)} (w(z) \Rightarrow_t [\![z \in v]\!]) = \mathbf{0}$$

or for all such y_0 we have $[\![x_0 = y_0]\!] = \mathbf{0}$, so

$$[\![x_0 \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![x_0 = y]\!]) = \mathbf{0}$$

and therefore $(u(x_0) \Rightarrow_t [\![x_0 \in v]\!]) = \mathbf{0}$ (by Corollary 4.4.4(i)), hence

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow_t [\![y \in u]\!]) = \mathbf{0}.$$

Case 2. Suppose $\bigwedge_{z \in \mathrm{dom}(w)} (w(z) \Rightarrow_t [\![z \in u]\!]) = \mathbf{0}$. This case is proved analogously.

(ii) This claim follows immediately from (i):

$$\begin{aligned}
[\![u = v]\!] \wedge [\![u \in w]\!] &= [\![u = v]\!] \wedge \bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![z = u]\!]) \\
&= \bigvee_{z \in \mathrm{dom}(w)} w(z) \wedge ([\![z = u]\!] \wedge [\![u = v]\!]) \\
&\leq \bigvee_{z \in \mathrm{dom}(w)} w(z) \wedge ([\![z = v]\!]) \\
&= [\![v \in w]\!]
\end{aligned}$$

□

Lemma 4.4.6. *Let $\mathbb{T} \in \mathcal{T}$. Then, for any three elements $u, v, w \in \mathbf{V}^{(\mathbb{T})}$, we have*

(i) $([\![u = v]\!] \Rightarrow_t [\![u = w]\!]) = ([\![u = v]\!] \Rightarrow_t [\![v = w]\!])$,

(ii) $([\![u = v]\!] \Rightarrow_t [\![u \in w]\!]) = ([\![u = v]\!] \Rightarrow_t [\![v \in w]\!])$,

(iii) $([\![u = v]\!] \Rightarrow_t [\![w \in u]\!]) = ([\![u = v]\!] \Rightarrow_t [\![w \in v]\!])$.

Proof. (i) By part (i) of the previous lemma, we have that

$$[\![u = v]\!] \wedge [\![u = w]\!] \leq [\![v = w]\!].$$

By applying **P2**, we get

$$[\![u = v]\!] \Rightarrow_t ([\![u = v]\!] \wedge [\![u = w]\!]) \leq [\![u = v]\!] \Rightarrow_t [\![v = w]\!].$$

Since, for every $\mathbb{T} \in \mathcal{T}$, it holds that $x \Rightarrow (x \wedge y) = x \Rightarrow y$ (use **P2**), it immediately follows that

$$[\![u = v]\!] \Rightarrow_t [\![u = w]\!] \leq [\![u = v]\!] \Rightarrow_t [\![v = w]\!].$$

We establish the other direction by symmetry.

(ii) Similar proof but uses part (ii), instead of part (i) of the previous lemma.

(iii) Given Corollary 4.4.4(iii), we may assume $[\![u = v]\!] = \mathbf{1}$ and $[\![w \in u]\!] = \mathbf{0}$. So,

$$(') \; [\![w \in u]\!] = \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![w = x]\!]) = \mathbf{0}$$

and

$$('') \; [\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow_t [\![y \in u]\!]) = \mathbf{1}.$$

If for all $y \in \mathrm{dom}(v)$ we have $v(y) = \mathbf{0}$, then $[\![w \in v]\!] = \mathbf{0}$ and the desiderata follows. Thus, let us assume that there exists some y_0 such that $v(y_0) \neq \mathbf{0}$. Then, using $('')$ and $v(y_0) \neq \mathbf{0}$, it follows that

$$[\![y_0 \in u]\!] = \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![y_0 = x]\!]) \neq \mathbf{0}.$$

Thus, there exists $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \neq \mathbf{0} \neq [\![y_0 = x_0]\!]$, which implies that $[\![w = x_0]\!] = \mathbf{0}$ via our first assumption $(')$. Then, due to part (i), we get

$$[\![w = y_0]\!] \wedge [\![y_0 = x_0]\!] \leq [\![w = x_0]\!],$$

so $[\![w = y_0]\!] = \mathbf{0}$. Finally, this gives us

$$[\![w \in v]\!] = \bigvee_{y_0 \in \mathrm{dom}(v)} (v(y_0) \wedge [\![y_0 = w]\!]) = \mathbf{0}.$$

$\square$

Theorem 4.4.7. *Let $\varphi \in \mathsf{NFF}\text{-}\mathcal{L}_{\in}^{\mathbb{T}}$ and let $\mathbb{T} \in \mathcal{T}$. Then, for every $u, v \in \mathbf{V}^{(\mathbb{T})}$, we have*

$$([\![u = v]\!] \Rightarrow_t [\![\varphi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\varphi(v)]\!]).$$

Proof. By induction on the complexity of φ. Atomic cases are provided by Lemma 5.4.3. For the inductive steps it is enough to consider the two cases $[\![\varphi]\!] = \mathbf{0}$ and $[\![\varphi]\!] \neq \mathbf{0}$. Due to Corollary 4.4.4(iii) we assume $[\![u = v]\!] \neq \mathbf{0}$, otherwise the desiderata follows trivially.

Case 1. $\varphi = \psi \wedge \chi$.

(i) Let $[\![\psi(u) \wedge \chi(u)]\!] = \mathbf{0}$. Since any meet complemented $\mathcal{T}$-lattice is totally ordered, we assume $[\![\psi(u)]\!] = \mathbf{0}$. By induction hypothesis, we get

$$([\![u = v]\!] \Rightarrow_t [\![\psi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v)]\!]).$$

Thus, by Corollary 4.4.4(i) the left-hand side of our equation will receive value $\mathbf{0}$ and therefore $[\![\psi(v)]\!] = \mathbf{0}$. Then, we obtain immediately that

$$([\![u = v]\!] \Rightarrow_t [\![\psi(u) \wedge \chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v) \wedge \chi(v)]\!]).$$

(ii) Now let $[\![\psi(u) \wedge \chi(u)]\!] \neq \mathbf{0}$. Thus, $[\![\psi(u)]\!] \neq \mathbf{0}$ and $[\![\chi(u)]\!] \neq \mathbf{0}$. By induction hypothesis we get $([\![u = v]\!] \Rightarrow_t [\![\psi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v)]\!])$ and as well we have $([\![u = v]\!] \Rightarrow_t [\![\chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\chi(v)]\!])$. Then, we immediately have that both $[\![u = v]\!] \Rightarrow_t [\![\psi(u) \wedge \chi(u)]\!]$ and $[\![u = v]\!] \Rightarrow_t [\![\psi(v) \wedge \chi(v)]\!]$ get value $\mathbf{1}$, since the consequent of our implication does not receive value $\mathbf{0}$.

Case 2. $\varphi = \psi \vee \chi$.

(i) Let $[\![\psi(u) \vee \chi(u)]\!] = \mathbf{0}$. Therefore, $[\![\psi(u)]\!] = \mathbf{0}$ and $[\![\chi(u)]\!] = \mathbf{0}$. By induction hypothesis we have $([\![u = v]\!] \Rightarrow_t [\![\psi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v)]\!])$ and we get that $([\![u = v]\!] \Rightarrow_t [\![\chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\chi(v)]\!])$. By Corollary 4.4.4(i) we get $\mathbf{0}$ on both sides of these equations. Hence, $[\![\psi(v)]\!] = \mathbf{0}$ and $[\![\chi(v)]\!] = \mathbf{0}$. From this we can conclude that $[\![\psi(v) \vee \chi(v)]\!] = \mathbf{0}$ and thus

$$([\![u = v]\!] \Rightarrow_t [\![\psi(u) \vee \chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v) \vee \chi(v)]\!]).$$

(ii) Let $[\![\psi(u) \vee \chi(u)]\!] \neq \mathbf{0}$. Assume that $[\![\psi(u)]\!] \neq \mathbf{0}$. By induction hypothesis we get $([\![u = v]\!] \Rightarrow_t [\![\psi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v)]\!])$. Since both sides of this equation receive value $\mathbf{1}$, it follows that $[\![\psi(v)]\!] \neq \mathbf{0}$. Thus $[\![\psi(v) \vee \chi(v)]\!] \neq \mathbf{0}$ and therefore $([\![u = v]\!] \Rightarrow_t [\![\psi(u) \vee \chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v) \vee \chi(v)]\!])$.

Case 3. $\varphi = \psi \to \chi$.

(i) Let $[\![\psi(u) \to \chi(u)]\!] = \mathbf{0}$. Then by Corollary 4.4.4(i) this can only be the case if $[\![\psi(u)]\!] \neq \mathbf{0}$ and $[\![\chi(u)]\!] = \mathbf{0}$. By induction hypothesis we get

$$([\![u = v]\!] \Rightarrow_t [\![\psi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v)]\!])$$

and $([\![u = v]\!] \Rightarrow_t [\![\chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\chi(v)]\!])$. Given that the first equation receives value $\mathbf{1}$ and the second equation receives value $\mathbf{0}$, we know that $[\![\psi(v)]\!] \neq \mathbf{0}$ and $[\![\chi(v)]\!] = \mathbf{0}$. By Corollary 4.4.4(i) we have that $[\![\psi(v) \to \chi(v)]\!] = \mathbf{0}$. Finally, we get

$$([\![u = v]\!] \Rightarrow_t [\![\psi(u) \to \chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v) \to \chi(v)]\!]).$$

(ii) Let $[\![\psi(u) \to \chi(u)]\!] \neq \mathbf{0}$. This can only be the case if either (1) $[\![\psi(u)]\!] \neq \mathbf{0}$ and $[\![\chi(u)]\!] \neq \mathbf{0}$, or (2) the antecedent receives value $\mathbf{0}$, i.e., $[\![\psi(u)]\!] = \mathbf{0}$. For (1) we get via our induction hypothesis that $[\![\psi(v)]\!] \neq \mathbf{0}$ and $[\![\chi(v)]\!] \neq \mathbf{0}$. Hence, $[\![\psi(v) \to \chi(v)]\!] \neq \mathbf{0}$ and therefore

$$([\![u = v]\!] \Rightarrow_t [\![\psi(u) \to \chi(u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\psi(v) \to \chi(v)]\!]).$$

Similarly for (2) we get $[\![\psi(v)]\!] = \mathbf{0}$. Thus, $[\![\psi(v) \to \chi(v)]\!] \neq \mathbf{0}$. The desiderata follows immediately.

Case 4. $\varphi = \exists x \varphi(x, u)$. Suppose that φ is of the form $[\![\exists x \varphi(x, u)]\!]$ and notice that

$$[\![\exists x \varphi(x, u)]\!] = \bigvee_{z \in \mathbf{V}^{(\mathbb{T})}} [\![\varphi(z, u)]\!].$$

By induction hypothesis, for all $z \in \mathbf{V}^{(\mathbb{T})}$ we have

$$([\![u = v]\!] \Rightarrow_t [\![\varphi(z, u)]\!]) = ([\![u = v]\!] \Rightarrow_t [\![\varphi(z, v)]\!]).$$

If there is a $z \in \mathbf{V}^{(\mathbb{T})}$ such that $[\![\varphi(z,u)]\!] \neq \mathbf{0}$, then $\bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,u)]\!] \neq \mathbf{0}$. Therefore we have $([\![u=v]\!] \Rightarrow_t [\![\varphi(z,u)]\!]) = \mathbf{1}$ and $([\![u=v]\!] \Rightarrow_t \bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,u)]\!]) = \mathbf{1}$. But by Inductive Hypothesis we get that $([\![u=v]\!] \Rightarrow_t [\![\varphi(z,v)]\!]) = \mathbf{1}$, which implies $[\![\varphi(z,v)]\!] \neq \mathbf{0}$. Thus $\bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,v)]\!] \neq 0$ and therefore

$$([\![u=v]\!] \Rightarrow_t \bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,v)]\!]) = \mathbf{1}.$$

On the other hand, if for all $z \in \mathbf{V}^{(\mathbb{T})}$ we have $[\![\varphi(z,u)]\!] = \mathbf{0}$, then by the inductive hypothesis we get $([\![u=v]\!] \Rightarrow_t [\![\varphi(z,u)]\!]) = \mathbf{0} = ([\![u=v]\!] \Rightarrow_t [\![\varphi(z,v)]\!])$. But this means that for all $z \in \mathbf{V}^{(\mathbb{T})}$ we have $[\![\varphi(z,v)]\!] = \mathbf{0}$. Therefore we get that

$$([\![u=v]\!] \Rightarrow_t \bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,u)]\!]) = \mathbf{0} = ([\![u=v]\!] \Rightarrow_t \bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,v)]\!]).$$

In conclusion,

$$([\![u=v]\!] \Rightarrow_t \bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,u)]\!]) = ([\![u=v]\!] \Rightarrow_t \bigvee_{z\in\mathbf{V}^{(\mathbb{T})}} [\![\varphi(z,v)]\!]).$$

Case 5. $\varphi = \forall u\varphi(u)$. Similar to the previous case. □

Lemma 4.4.8. *Let $\mathbb{T} \in \mathcal{T}$ and take any pair $x, y \in \mathbb{T}$. Then, we have:*

$$(\bigvee_{i\in I} x_i) \Rightarrow_t y = \bigwedge_{i\in I}(x_i \Rightarrow_t y). \tag{†}$$

Proof. For one inequality, it is enough to prove that

$$\Big((\bigvee_{i\in I} x_i) \wedge y^{*_p}\Big)^{*_p} \leq (x_i \wedge y^{*_p})^{*_p},$$

for any $i \in I$. In fact, we have $x_i \wedge y^{*_p} \leq (\bigvee_{i\in I} x_i) \wedge y^{*_p}$. For the other inequality, for each $i \in I$ we have

$$(x_i \wedge y^{*_p})^{*_p} \wedge \bigvee_{i\in I}(x_i \wedge y^{*_p}) \leq (x_i \wedge y^{*_p})^{*_p} \wedge x_i,$$

by Lemma 4.3.2(iv)

$$(x_i \wedge y^{*_p})^{*_p} \wedge x_i \leq y^{*_p *_p}.$$

On the other hand, clearly

$$(x_i \wedge y^{*_p})^{*_p} \wedge \bigvee_{i \in I} (x_i \wedge y^{*_p}) \leq y^{*_p}.$$

The desiderata follows immediately. $\square$

Theorem 4.4.9. *If $\mathbb{T} \in \mathcal{T}$, then $\mathbf{V}^{(\mathbb{T})}$ satisfies $\mathcal{BQ}_\varphi$, for φ a negation-free formula.*

Proof. We want to show that for any $u \in \mathbf{V}^{(\mathbb{T})}$ we have:

$$[\![\forall x(x \in u) \to \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![\varphi(x)]\!]).$$

Notice that,

$$\begin{aligned}
[\![\forall x(x \in u) \to \varphi(x)]\!] &= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} ([\![y \in u]\!] \Rightarrow_t [\![\varphi(y)]\!]) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} (\bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![y = x]\!]) \Rightarrow_t [\![\varphi(y)]\!]) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big((u(x) \wedge [\![x = y]\!]) \Rightarrow_t [\![\varphi(y)]\!]\Big) && (\dagger) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t ([\![x = y]\!] \Rightarrow_t [\![\varphi(y)]\!])\Big) && (\mathbf{P4}) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t ([\![x = y]\!] \Rightarrow_t [\![\varphi(x)]\!])\Big) && (\textbf{Theorem 4.4.7}) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big((u(x) \wedge [\![x = y]\!]) \Rightarrow_t [\![\varphi(x)]\!]\Big). && (\mathbf{P4})
\end{aligned}$$

Moreover,

$$\begin{aligned}
\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![\varphi(x)]\!]) &= \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![\varphi(x)]\!]) \\
&\leq \bigwedge_{y \in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![x = y]\!]) \Rightarrow_t [\![\varphi(x)]\!].
\end{aligned}$$

For other direction take any $x \in \mathrm{dom}(u)$ and use Lemma 4.4.5 to obtain

$$\bigwedge_{y\in \mathbf{V}^{(\mathbb{T})}} (u(x) \wedge [\![x = y]\!]) \Rightarrow_t [\![\varphi(x)]\!] \leq (u(x) \wedge [\![x = x]\!]) \Rightarrow_t [\![\varphi(x)]\!]$$

$$= u(x) \Rightarrow_t \varphi(x),$$

and hence,

$$\bigwedge_{y\in \mathbf{V}^{(\mathbb{T})}} \bigwedge_{x\in \mathrm{dom}(u)} (u(x) \wedge [\![x = y]\!]) \Rightarrow_t [\![\varphi(x)]\!] \leq \bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow_t [\![\varphi(x)]\!]).$$

□

Corollary 4.4.10. *Let $\mathbb{T} \in \mathcal{T}$ and let F be a filter on $\mathbb{T}$. Then,* $\mathbf{V}^{(\mathbb{T})} \models_F \mathrm{NFF}\text{-}\mathsf{ZF}^-$. □

Proof. Every $\mathbb{T} \in \mathcal{T}$ is a DRI-algebra and every $\mathbf{V}^{(\mathbb{T})}$ satisfies $\mathcal{BQ}_\varphi$ for every negation-free formula φ. Now we apply Theorem 2.4.4 we immediately obtain the desiderata. □

In the following proof of Theorem 4.4.11, is the only place where we use essentially the well-order of $\mathbb{T}$. All the remaining results hold for any implicative meet complemented $\mathcal{T}$-lattice.

Theorem 4.4.11. *If $\mathbb{T} \in \mathcal{W}$, then the $\mathsf{NFF}\text{-}\mathsf{Foundation}_\varphi$ holds in $\mathbf{V}^{(\mathbb{T})}$.*

Proof. We want to show that for every negation-free formula φ we have that

$$[\![\forall x\Big((\forall y \in x\varphi(y)) \to \varphi(x)\Big) \to \forall x\varphi(x)]\!] = \mathbf{1}.$$

Consider the following two cases.

(i) Let $[\![\varphi(x)]\!] \neq \mathbf{0}$ for every $x \in \mathbf{V}^{(\mathbb{T})}$. Then since $\mathbb{T}$ is well-ordered we have $[\![\forall x\varphi(x)]\!] \neq \mathbf{0}$. Which, given Lemma 4.4.3(iii), implies that $[\![\Big(\forall x\varphi(x)\Big)^{*_p}]\!] = \mathbf{0}$. Therefore, we readily calculate that $[\![\Big(\forall x\Big((\forall y \in x\varphi(y)) \to \varphi(x)\Big) \wedge \Big((\forall x\varphi(x)\Big)^{*_p}\Big)^{*_p}]\!] = \mathbf{1}$.

(ii) Let $[\![\varphi(x)]\!] = \mathbf{0}$ for some $x \in \mathbf{V}^{(\mathbb{T})}$. Then take a minimal $u \in \mathbf{V}^{(\mathbb{T})}$ such that $[\![\varphi(u)]\!] = \mathbf{0}$ and such that for any $y \in \mathrm{dom}(u), [\![\varphi(y)]\!] \neq \mathbf{0}$. Since there exists a $x \in \mathbf{V}^{(\mathbb{T})}$ such that $[\![\varphi(x)]\!] = \mathbf{0}$ we get $[\![\forall x\varphi(x)]\!] = \mathbf{0}$. Moreover, Corollary 4.4.4(i) implies that

$$[\![\forall x\Big((\forall y \in x\varphi(y)) \to \varphi(x)\Big)]\!] \leq [\![(\forall y \in u\ \varphi(y)) \to \varphi(u)]\!] = \mathbf{0}.$$

Hence,

$$[\![\forall x\Big((\forall y \in x\varphi(y)) \to \varphi(x)\Big) \to \forall x\varphi(x)]\!] = \mathbf{1}.$$

$\square$

Corollary 4.4.12. *Let $\mathbb{T} \in \mathcal{W}$ and let F be any filter on $\mathbb{T}$. Then,* $\mathbf{V}^{(\mathbb{T})} \models_F$ NFF-ZF. $\square$

4.4.1 Adding a Negation

In this section, we extend the results of (LÖWE; TARAFDER, 2015) showing how to give an abstract definition not only of the implication but also of the negation of $(\mathbb{PS}_3, *)$. Since the aim of (LÖWE; TARAFDER, 2015) was to construct a paraconsistent model of a sufficiently large fragment of ZF, we now define a negation that coincides with that of $(\mathbb{PS}_3, *)$, when considered in the context of the three-element bicomplemented $\mathcal{W}_{\mathcal{F}}$-lattice (see Definition 4.4.13), but that will also give rise to paraconsistent models of NFF-ZF, when evaluated in bicomplemented $\mathcal{W}_{\mathcal{F}}$-lattices.

Towards this goal, we expand the language of an implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice to a richer language containing a new operator. This will not change the structures which interpret this new language, since the construction of our models depends only on the base set of $\mathbb{T}$, and not on its logical operations. So, any expansion of the logical operation will give the same universe of names.

The construction of the following paraconsistent negation takes place in a bicomplemented setting since both the meet complement ^{*_p} and the join complement ^{*_d} are needed.

Definition 4.4.13. *We call a structure $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_t,^{*_p},^{*_d}, \mathbf{0}, \mathbf{1}, \rangle$ a bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice if*

(i) $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_t,^{*_p}, \mathbf{0}, \mathbf{1}, \rangle$ is an implicative $\mathcal{W}_{\mathcal{F}}$-lattice and

(ii) the unary operator ^{*_d} is defined for every $x \in \mathbf{A}$ as

$$x^{*_d} = \mathsf{min}\{y \in \mathbf{A} : x \vee y = \mathbf{1}\}.$$

Definition 4.4.14. *We call a structure $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_t,^{*_p},^{*_d},^{*_r} \mathbf{0}, \mathbf{1}, \rangle$ a reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice if* *(i)* $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_t, ^{*p}, ^{*d}, \mathbf{0}, \mathbf{1}, \rangle$ *is a bicomplemented implicative* $\mathcal{W}_{\mathcal{F}}$*-lattice*

(ii) *the unary operation* ^{*r} *is defined for every* $x \in \mathbf{A}$ *as*

$$x^{*r} = x^{*d} \wedge (x \vee x^{*p}).$$

We will denote by $(\mathbb{T}, ^{*r})$ *a reflexive bicomplemented implicative* $\mathcal{W}_{\mathcal{F}}$*-lattice, where we interpret* ^{*r} *as negation.*

Lemma 4.4.15. *Let* $(\mathbb{T}, ^{*r})$ *be a reflexive bicomplemented implicative* $\mathcal{W}_{\mathcal{F}}$*-lattice and take any* $x \in \mathbb{T}$*. Then the following holds:*

(i) $x^{*r} = \mathbf{0}$ *iff* $x = \mathbf{1}$*,*

(ii) $x^{*r} = \mathbf{1}$ *iff* $x = \mathbf{0}$*,*

(iii) $x^{*r} = x$ *when* $\mathbf{1} \neq x \neq \mathbf{0}$*.* □

Proof. (i) For the first direction, suppose $x^{*r} = \mathbf{0} = x^{*d} \wedge (x \vee x^{*p})$. Hence, $x^{*d} = \mathbf{0}$ and so $x = \mathbf{1}$. For the other direction, let $x = \mathbf{1}$. Then $x^{*r} = \mathbf{1}^{*d} \wedge (\mathbf{1} \vee \mathbf{1}^{*p}) = \mathbf{0}$, since $1^{*d} = \mathbf{0}$.
(ii) Let $x^{*r} = \mathbf{1} = x^{*d} \wedge (x \vee x^{*p})$. Hence, both $x^{*d} = \mathbf{1}$ and $(x \vee x^{*p}) = \mathbf{1}$. From $(x \vee x^{*p}) = \mathbf{1}$ we get that x is either $\mathbf{0}$ or $\mathbf{1}$, but if $x = \mathbf{1}$, then $x^{*d} = \mathbf{0}$. Thus $x = \mathbf{0}$. For the other side, let $x = \mathbf{0}$. Then $x^{*r} = \mathbf{0}^{*d} \wedge (\mathbf{0} \vee \mathbf{0}^{*p}) = (\mathbf{1} \wedge \mathbf{1}) = \mathbf{1}$.
(iii) Let $\mathbf{1} \neq x \neq 0$. So $x^{*r} = x^{*d} \wedge (x \vee x^{*p}) = (\mathbf{1} \wedge x) = x$. □

Moreover, in this section we will use ^{*r} and $\Rightarrow_t$ to interpret negation and implication, respectively, when evaluating $(\mathbb{T}, ^{*r})$-sentences in $\mathbf{V}^{(\mathbb{T}, ^{*r})}$. Slightly abusing notation we will use the symbol ^{*r} for both the syntactic negation and the algebraic operator which will interpret it.

We now show that every reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice with strictly more then two elements can engender a paraconsistent consequence relation. Interestingly, the same sentence that was used to show this property in (LÖWE; TARAFDER, 2015) works here. This suggests that the implication $\Rightarrow_t$ and the negation ^{*r} are the correct generalizations of the corresponding operations of $(\mathbb{PS}_3, ^{*})$.

Corollary 4.4.16. *Let* $\mathbb{T} \in \mathcal{W}_{\mathcal{F}}$ *and let* F *be any filter on* $\mathbb{T}$*. Then, we have* $\mathbf{V}^{(\mathbb{T}, ^{*r})} \models_F$ NFF-ZF. □

Theorem 4.4.17. *For any $\mathbb{T} \in \mathcal{W}_{\mathcal{F}}$ with more than two elements there exists a sentence $\sigma \in \mathcal{L}_{\in}$ and a filter F, such that $\mathbf{V}^{(\mathbb{T},^{*r})} \models_F \sigma$ and $\mathbf{V}^{(\mathbb{T},^{*r})} \models_F \neg\sigma$.*

Proof. Take any $\mathbb{T}$ with strictly more than two elements and consider the following three $\mathbb{T}$-names $u, v, w \in \mathbf{V}^{(\mathbb{T},^{*r})}$ such that $\mathrm{dom}(u) = \mathrm{dom}(v) = \{w\}$, where $u(w) = \mathbf{1}$ and $v(w) = a$ (where a is the co-atom of $\mathbb{T}$). Then we can define the sentence

$$\sigma = \exists xyz(x = y \wedge z \in x \wedge z \notin y)$$

The three names we just defined witness that:

$$\bigvee_{u,v,w \in \mathbf{V}^{(\mathbb{T},^{*r})}} (\llbracket u = v \rrbracket \wedge \llbracket w \in u \rrbracket \wedge \llbracket w \notin v \rrbracket) \geq a.$$

Furthermore, for any $u, v, w \in \mathbf{V}^{(\mathbb{T},^{*r})}$, we have that if $\llbracket u = v \rrbracket = \mathbf{1} = \llbracket w \in u \rrbracket$, then $\llbracket w \in v \rrbracket \neq \mathbf{0}$. Suppose $\llbracket u = v \rrbracket = \mathbf{1} = \llbracket w \in u \rrbracket$, so for every $x \in \mathrm{dom}(u)$ we have $u(x) = \mathbf{1} = \llbracket x = w \rrbracket$. Now, since $\llbracket u = v \rrbracket = \mathbf{1}$ and $u(x) = \mathbf{1}$, then $\llbracket x \in v \rrbracket \neq \mathbf{1}$. That is, for every $x \in \mathrm{dom}(u)$,

$$\llbracket x \in v \rrbracket = \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge \llbracket x = y \rrbracket) \neq \mathbf{0}.$$

So for every $y \in \mathrm{dom}(v)$ we have $v(y) \neq \mathbf{0}$ and $\llbracket x = y \rrbracket \neq \mathbf{0}$. Therefore, by Lemma 4.4.15(i) we get $\llbracket x = w \rrbracket \wedge \llbracket x = y \rrbracket \leq \llbracket y = w \rrbracket \neq \mathbf{0}$, for each $y \in \mathrm{dom}(v)$. Hence, also $\llbracket w \in v \rrbracket \neq \mathbf{0}$. Consequently, there are no $u, v, w \in \mathbf{V}^{(\mathbb{T},^{*r})}$ such that

$$\Big(\llbracket u = v \rrbracket \wedge \llbracket w \in u \rrbracket \wedge \llbracket w \notin v \rrbracket\Big) = \mathbf{1}.$$

Therefore $\mathbf{1} > \llbracket \sigma \rrbracket > \mathbf{0}$. Then, by Lemma 4.4.15(iii) we get $\llbracket \sigma \rrbracket = \llbracket \sigma \rrbracket^{*r} = \llbracket \neg\sigma \rrbracket$. Hence, $\mathbf{V}^{(\mathbb{T},^{*r})} \models_{Pos_{(\mathbb{T})}} \sigma$ and $\mathbf{V}^{(\mathbb{T},^{*r})} \models_{Pos_{(\mathbb{T})}} \neg\sigma$. □

Therefore, every reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice, with more than two elements can give rise to a paraconsistent model of NFF-ZF. Furthermore, the interest in bicomplemented $(\mathbb{T},^{*r})$-lattices does not only rely upon the possibility of generalizing $(\mathbb{PS}_3,^{*})$, but also in explaining the relation of $(\mathbb{PS}_3,^{*})$ with the classical case. Indeed, the two-element bicomplemented $(\mathbb{T},^{*r})$-lattice coincides with the two-element Boolean algebra.

4.4.2 Non-classical Models of ZF

In this section we build lattice-valued models of full ZF, where the internal logic of this structure is neither classical nor intuitionistic. In other words, we generalise Theorem 2.3.7.

We will denote by $(\mathbb{T}, ^{*_p})$ an implicative meet complemented $\mathcal{T}$-lattice, where we interpret ^{*_p} as negation. Now, we construct $\mathbb{T}$-valued models where the meet complement ^{*_p} and $\Rightarrow_t$ are interpreted as negation and implication, respectively when evaluating $(\mathbb{T}, ^{*_p})$-sentences in $\mathbf{V}^{(\mathbb{T}, ^{*_p})}$. As before, notice that the change of interpretation for negation does not change the underlying structure $\mathbf{V}^{(\mathbb{T})}$. We now show that every $\mathbf{V}^{(\mathbb{T}, ^{*_p})}$ can indeed be a model of full ZF^-. To do so, we need to show that $\mathcal{BQ}_\varphi$ holds for every $\varphi \in \mathcal{L}_\in$.

Lemma 4.4.18. *Let $\varphi \in \mathcal{L}_\in^{\mathbb{T}}$ and let $\mathbb{T} \in \mathcal{T}$. Then, for any three elements $u, v, w \in \mathbf{V}^{(\mathbb{T}, ^{*_p})}$, we have*

$$(\llbracket u = v \rrbracket \Rightarrow_t \llbracket \varphi(u) \rrbracket) = (\llbracket u = v \rrbracket \Rightarrow_t \llbracket \varphi(v) \rrbracket).$$

Proof. The atomic cases and the steps $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$ are dealt with in Theorem 4.4.7. We only need to deal with the case $\neg$. Given Corollary 4.4.4 (iii), we will assume that $\llbracket u = v \rrbracket = \mathbf{1}$. We have two cases:

(i) Let $\llbracket \neg\varphi(u) \rrbracket = \llbracket \varphi(u) \rrbracket^{*_p} = \mathbf{1}$. Then, by definition of ^{*_p}, $\llbracket \varphi(u) \rrbracket = \mathbf{0}$. Thus, we have $(\llbracket u = v \rrbracket \Rightarrow_t \llbracket \varphi(u) \rrbracket) = \mathbf{0}$. By inductive hypothesis

$$(\llbracket u = v \rrbracket \Rightarrow_t \llbracket \varphi(u) \rrbracket) = (\llbracket u = v \rrbracket \Rightarrow_t \llbracket \varphi(v) \rrbracket)$$

holds and therefore $\llbracket \varphi(v) \rrbracket = \mathbf{0}$. Consequently, $\llbracket \varphi(u) \rrbracket^{*_p} = \llbracket \neg\varphi(v) \rrbracket = \mathbf{1}$ and then

$$\llbracket u = v \rrbracket \Rightarrow_t \llbracket \neg\varphi(v) \rrbracket) = \mathbf{1} = (\llbracket u = v \rrbracket \Rightarrow_t \llbracket \neg\varphi(u) \rrbracket).$$

(ii) Let $\llbracket \neg\varphi(u) \rrbracket = \llbracket \varphi(u) \rrbracket^{*_p} = \mathbf{0}$, then $\llbracket \varphi(u) \rrbracket \neq \mathbf{0}$. This implies $(\llbracket u = v \rrbracket \Rightarrow_t \llbracket \varphi(u) \rrbracket) \neq \mathbf{0}$. As before, the only way to match the induction hypothesis is by assuming that we have $\llbracket \varphi(v) \rrbracket \neq \mathbf{0}$. Consequently, $\llbracket \varphi(u) \rrbracket^{*_p} = \llbracket \neg\varphi(v) \rrbracket = \mathbf{0}$ and then

$$(\llbracket u = v \rrbracket \Rightarrow_t \llbracket \neg\varphi(v) \rrbracket) = \mathbf{0} = (\llbracket u = v \rrbracket \Rightarrow_t \llbracket \neg\varphi(u) \rrbracket).$$

$\square$

Therefore we get the following.

Theorem 4.4.19. *Let* $\varphi \in \mathcal{L}_{\in}^{(\mathbb{T},^{*p})}$ *and let* $\mathbb{T} \in \mathcal{T}$*, then* $\mathbf{V}^{(\mathbb{T},^{*p})}$ *satisfies* $\mathcal{BQ}_{\varphi}$.

Proof. Similar to the proof of Theorem 4.4.9 (we simply use Theorem 4.2 instead of Theorem 3.6). □

Moreover, essentially the same proofs of (BELL, 2005) show that ZF^{-} holds in any $(\mathbb{T},^{*p})$-valued model that is build on an implicative meet complemented lattice belonging to class (i) of Definition 4.4.2, and that ZF holds in any $(\mathbb{T},^{*p})$-valued model that is build on an implicative meet complemented lattice belonging to class (ii) of Definition 4.4.2.

Theorem 4.4.20. *Let* $\mathbb{T} \in \mathcal{T}$*, then* $\mathbf{V}^{(\mathbb{T},^{*p})}$ *validates* ZF^{-}.

Proof. Given that every implicative meet complemented $\mathcal{T}$-lattice is a DRI-algebra and that any $\mathbf{V}^{(\mathbb{T},^{*p})}$ satisfies $\mathcal{BQ}_{\varphi}$ for $\varphi \in \mathcal{L}_{\in}^{(\mathbb{T},^{*p})}$, we apply Theorem 2.4.4. Thus every $\mathbf{V}^{(\mathbb{T},^{*p})}$ is a model of NFF-ZF. Furthermore, in order to show that full ZF^{-} holds, we need to prove that $\mathsf{Separation}_{\varphi}$ and $\mathsf{Collection}_{\varphi}$ hold. Notice that this time we are also considering schemata, where $\varphi(x)$ is a formula that contains negation.

$\mathsf{Separation}_{\varphi}$: We want to show that $[\![\forall z\big(z \in y \leftrightarrow \big(z \in x \wedge \varphi(z)\big)\big)]\!] = \mathbf{1}$ where $\varphi \in \mathcal{L}_{\in}^{(\mathbb{T},^{*p})}$. Let $x \in \mathbf{V}^{(\mathbb{T},^{*p})}$ and define y by setting $\mathrm{dom}(x) = \mathrm{dom}(y)$ and for any $z \in \mathbf{V}^{(\mathbb{T},^{*p})}$ let $y(z) = \big(x(z) \wedge [\![\varphi(z)]\!]\big)$. First we want to show that $[\![\forall z\big(z \in y \rightarrow \big(z \in x \wedge \varphi(z)\big)\big)]\!] = \mathbf{1}$. Using $\mathcal{BQ}_{\varphi}$ we get the following equation:

$$[\![\forall z\big(z \in y \rightarrow \big(z \in x \wedge \varphi(z)\big)\big)]\!] = \bigwedge_{z \in \mathrm{dom}(y)} \big(y(z) \Rightarrow_t ([\![z \in x]\!] \wedge [\![\varphi(z)]\!])\big).$$

Then due to $y(z) = \big(x(z) \wedge [\![\varphi(z)]\!]\big)$ and given that for every $u \in \mathbf{V}^{(\mathbb{T},\neg)}$ we have that $u(x) \leq [\![x \in u]\!]$ for any $x \in \mathrm{dom}(u)$, we get:

$$\begin{aligned}
&\bigwedge_{z \in \mathrm{dom}(y)} \big((x(z) \wedge [\![\varphi(z)]\!]) \Rightarrow_t ([\![z \in x]\!] \wedge [\![\varphi(z)]\!])\big) \\
&\leq \bigwedge_{z \in \mathrm{dom}(y)} \big(([\![z \in x]\!] \wedge [\![\varphi(z)]\!]) \Rightarrow_t ([\![z \in x]\!] \wedge [\![\varphi(z)]\!])\big) \\
&= \mathbf{1}.
\end{aligned}$$

We now need to show $[\![\forall z \in x\ (\varphi(z) \to z \in y)]\!] = \mathbf{1}$. Using $\mathcal{BQ}_\varphi$ we get:

$$[\![\forall z \in x\ (\varphi(z) \to z \in y)]\!] = \bigwedge_{z \in \mathrm{dom}(x)} \Big(x(z) \Rightarrow_t ([\![\varphi(z)]\!] \Rightarrow_t [\![z \in y]\!]) \Big).$$

Now, if $x(z) = \mathbf{0}$ or $[\![\varphi(z)]\!] = \mathbf{0}$ we are done, so assume $x(z) \neq \mathbf{0}$ and $[\![\varphi(z)]\!] \neq \mathbf{0}$. Remember that for every $z \in \mathbf{V}^{(\mathbb{T},*p)}$, $y(z) = \Big(x(z) \wedge [\![\varphi(z)]\!]\Big)$, thus $\mathbf{0} \neq y(z) \leq [\![z \in y]\!]$. Hence, $[\![\varphi(z) \to z \in y]\!] = \mathbf{1}$ and therefore:

$$\bigwedge_{z \in \mathrm{dom}(x)} \Big(x(z) \Rightarrow_t ([\![\varphi(z)]\!] \Rightarrow_t [\![z \in y]\!]) \Big) = \mathbf{1}.$$

$\mathsf{Collection}_\varphi$: We want to show that

$$[\![\forall u \Big(\forall x \in u \exists y \varphi(x,y) \to \exists v \forall x \in u \exists y \in v \varphi(x,y) \Big)]\!] = \mathbf{1},$$

where $\varphi \in \mathcal{L}_\in^{(\mathbb{T},*p)}$. Let $u \in \mathbf{V}^{(\mathbb{T},*p)}$. Applying $\mathcal{BQ}_\varphi$ we get the following equality:

$$[\![\forall x \in u \exists y \varphi(x,y)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t \bigvee_{y \in \mathbf{V}^{(\mathbb{T},*p)}} [\![\varphi(x,y)]\!] \Big). \tag{1}$$

We know that $\mathbf{A}$ is a set, so $\mathbf{A} \in \mathbf{V}$. Moreover, for any $x \in \mathrm{dom}(u)$ we can define $\mathbf{A}_x$ where $\mathbf{A}_x = \{[\![\varphi(x,y)]\!] : y \in \mathbf{V}^{(\mathbb{T},\neg)}\} \subseteq \mathbf{A}$. Thus $\mathbf{A}_x \in \mathbf{V}$. Hence, for every $a \in \mathbf{A}_x$ there exists an ordinal α such that $[\![\varphi(x,y)]\!] = a$ and $y \in \mathbf{V}_\alpha^{(\mathbb{T},*p)}$. Furthermore, we can use the Replacement axiom in $\mathbf{V}$ to obtain a map $x \mapsto \alpha_x$, with domain $\mathrm{dom}(u)$ and range a set of ordinals such that, for each $x \in \mathrm{dom}(u)$:

$$\bigvee_{y \in \mathbf{V}^{(\mathbb{T},*p)}} [\![\varphi(x,y)]\!] = \bigvee_{y \in \mathbf{V}_{\alpha_x}^{(\mathbb{T},*p)}} [\![\varphi(x,y)]\!] \tag{2}$$

Now we use Union in $\mathbf{V}$ to define $\alpha = \bigcup\{\alpha_x : \ x \in \mathrm{dom}(u)\}$. Then by (2) we get:

$$\begin{aligned} \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t \bigvee_{y \in \mathbf{V}^{(\mathbb{T},*p)}} [\![\varphi(x,y)]\!] \Big) &= \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t \bigvee_{y \in \mathbf{V}_{\alpha_x}^{(\mathbb{T},*p)}} [\![\varphi(x,y)]\!] \Big) \\ &\leq \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t \bigvee_{y \in \mathbf{V}_{\alpha}^{(\mathbb{T},*p)}} [\![\varphi(x,y)]\!] \Big) \qquad (3) \end{aligned}$$

We define v as $\mathrm{dom}(v) = \mathbf{V}_\alpha^{(\mathbb{T},^{*p})}$ and for every $y \in \mathrm{dom}(v), v(y) = \mathbf{1}$. Now we claim that

$$\bigvee_{y \in \mathbf{V}_\alpha^{(\mathbb{T},^{*p})}} [\![\varphi(x,y)]\!] \leq [\![\exists y \in v\varphi(x,y)]\!].$$

Indeed, since $v(y) = \mathbf{1}$ and given that the bounded quantification property gives us an inequality in the case of the existential quantifier, we have the following:

$$\begin{aligned}\bigvee_{y \in \mathbf{V}_\alpha^{(\mathbb{T},^{*p})}} [\![\varphi(x,y))]\!] &= \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![\varphi(x,y))]\!]) \\ &\leq [\![\exists y \in v\varphi(x,y)]\!].\end{aligned}$$

And since $a \Rightarrow_t b \leq a \Rightarrow_t c$, whenever $b \leq c$, then by (1) and (3) we have:

$$\begin{aligned}[\![\forall x \in u\exists y\varphi(x,y)]\!] &\leq \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow_t [\![\exists y \in v\varphi(x,y)]\!]\Big) \\ &= [\![\forall u \in x\exists y \in v\varphi(x,y)]\!].\end{aligned}$$

□

Theorem 4.4.21. *Let $\mathbb{T} \in \mathcal{W}$ and let F be any filter on $\mathbb{T}$. Then,* $\mathbf{V}^{(\mathbb{T},^{*p})} \models_F \mathsf{ZF}$.

Proof. As corollary of Theorem 4.4.20 we have already that for any $\mathbb{T} \in \mathcal{W}$, $\mathbf{V}^{(\mathbb{T},^{*p})} \models \mathsf{ZF}^-$. Thus we only need to show that $\mathsf{Foundation}_\varphi$ holds. But this proof is exactly the same as of Theorem 4.4.11. Only this time we are considering any $\varphi \in \mathcal{L}_\in^{(\mathbb{T},^{*p})}$. □

4.4.3 The Logics of $\mathbf{V}^{(\mathbb{T},^{*p})}$

In this section, we show that the internal logic of $(\mathbb{T},^{*p})$-valued models is neither classical nor intuitionistic when we use the filter $\{1\}$.

Theorem 4.4.22. *The internal logic of the structure $\mathbf{V}^{(\mathbb{T},^{*p})}$, given the filter $\{1\}$ is neither classical, i.e., $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, \{1\}) \neq \mathsf{CPL}$, nor intuitionistic; i.e, $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, \{1\}) \neq \mathsf{IPL}$.*

Proof. Fix an arbitrary $\mathbf{V}^{(\mathbb{T},^{*p})}$. Consider the following sentence:

$$\varphi = \exists y\forall x\Big((y \in x) \vee (y \notin x)\Big).$$

We show that $\forall x\Big((y \in x) \vee (y \notin x)\Big)$ always gets as value the minimal positive element of $\mathbf{A}$. Let u and v be two elements of $\mathbf{V}^{(\mathbb{T},^{*p})}$. Now, if $[\![u \in v]\!] = \mathbf{0}$, then $[\![(u \in v) \vee (u \notin v)]\!] = \mathbf{1}$. Otherwise $[\![u \in v]\!] = a$, with $a \neq \mathbf{0}$, but then $[\![(u \in v) \vee (u \notin v)]\!] = a$. Now notice that, given a $\bar{u} \in \mathbf{V}^{(\mathbb{T},^{*p})}$ and a $\bar{a} \in \mathbf{A}$ different from $\mathbf{0}$, we can always find a $\bar{v} \in \mathbf{V}^{(\mathbb{T},^{*p})}$ such that $[\![\bar{u} \in \bar{v}]\!] = \bar{a}$; namely $\bar{v} = \{\langle \bar{u}, \bar{a} \rangle\}$. Therefore, the minimal value that we obtain by choosing different elements $v \in \mathbf{V}^{(\mathbb{T},^{*p})}$, in the formula $\Big(u \in v \vee u \notin v\Big)$, is the atom of $\mathbf{A}$: i.e., the second least element of $\mathbf{A}$, that we may call b. Consequently, $[\![\forall v\Big(u \in v \vee u \notin v\Big)]\!] = b$. Notice that the above argument does not depend on the the choice of u and so $[\![\varphi]\!] = b$. Thus, if $\mathbf{A}$ has at least three elements, then $[\![\varphi \vee \neg\varphi]\!] \notin \{1\}$ and so the Law of Excluded Middle (LEM) fails in $\mathbf{V}^{(\mathbb{T},^{*_l})}$.

In order to show that also intuitionistic logic fails we proof that MP does not hold. Let $\psi = \exists x(x = x)$ and notice that $[\![\psi \to \varphi]\!] = \mathbf{1}$ and $[\![\psi]\!] = \mathbf{1}$. However, as we have shown $[\![\varphi]\!] = b$. Therefore, MP fails. □

On the other hand, if we choose the filter $Pos_{(\mathbb{T})}$ on $\mathbb{T}$, then we can show that $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, Pos_{(\mathbb{T})})$ is classical.

Theorem 4.4.23. $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, Pos_{(\mathbb{T})}) = \mathsf{CPL}$.

Proof. It can be easily checked that $(\mathbb{T},^{*p})/Pos_{(\mathbb{T})}$ is the two-valued Boolean algebra $2 = \{0, 1\}$. So, $\mathbf{L}\Big((\mathbb{T},^{*p}), Pos_{(\mathbb{T})}\Big) = \mathbf{L}(\{0,1\}, 1)$. Furthermore, given that for any Boolean algebra $\mathbb{B}$ and filter F we have $\mathbf{L}(\mathbb{B}, F) = \mathbf{L}(\mathbf{V}^{(\mathbb{B})}, F)$, it follows immediately that $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, Pos_{(\mathbb{T})}) = \mathsf{CPL}$. □

Even if we do not have an axiomatisation of $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, Pos_{(\mathbb{T})})$ we know that it is not much different from intuitionistic logic.

Theorem 4.4.24. *The propositional logic* $\mathbf{L}(\mathbf{V}^{(\mathbb{T},^{*p})}, \{1\})$ *validates all axioms of* IPL.

Proof. For this proof we assume that the $(\mathbb{T},^{*p})$-value of the antecedent of each axiom is not equal to $\mathbf{0}$. So if φ is such an antecedent, then we suppose $[\![\varphi]\!] \neq \mathbf{0}$. Otherwise, the desiderata follows trivially.

(i) $\varphi \to (\psi \to \varphi)$. Suppose $[\![\varphi]\!] \neq 0$, then by Corollary 4.4.4(i),(ii) we get immediately that $[\![\varphi \to (\psi \to \varphi)]\!] = \mathbf{1}$.

(ii) $\big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big)$. Suppose we have $[\![\varphi \to (\psi \to \chi)]\!] \neq \mathbf{0}$, then we have two cases. Either $[\![\varphi]\!] = \mathbf{0}$ or $[\![\varphi]\!] \neq \mathbf{0}$. In the first case we get by Corollary 4.4.4(i),(ii) that $[\![\varphi \to \psi]\!] \neq \mathbf{0}$, as well as $[\![\varphi \to \chi]\!] \neq \mathbf{0}$. Thus we calculate readily $[\![\big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big)]\!] = \mathbf{1}$. In the second case given the initial assumption we have $[\![\psi \to \chi]\!] \neq \mathbf{0}$, so it is **not** the case that $[\![\psi]\!] \neq \mathbf{0}$ and $[\![\chi]\!] = \mathbf{0}$, so by Corollary 4.4.4(i),(ii) $[\![\big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big)]\!] = \mathbf{1}$.

(iii) $\varphi \to (\varphi \vee \chi)$. Suppose $[\![\varphi]\!] \neq \mathbf{0}$, then $[\![\varphi \vee \chi]\!] \neq \mathbf{0}$. Therefore, by Corollary 4.4.4(i),(ii) we have $[\![\varphi \to (\varphi \vee \chi)]\!] = \mathbf{1}$.

(iv) $\chi \to (\varphi \vee \chi)$. Similar to proof of item (iii).

(v) $(\varphi \to \chi) \to \big((\psi \to \chi) \to (\varphi \vee \psi \to \chi)\big)$. Suppose $[\![\varphi \to \chi]\!] \neq \mathbf{0}$ and $[\![\psi \to \chi]\!] \neq \mathbf{0}$. Now, if $[\![\varphi \vee \psi]\!] = \mathbf{0}$, then we are done. So let $[\![\varphi \vee \psi]\!] \neq \mathbf{0}$. We know that $[\![\varphi]\!] \neq \mathbf{0}$ and $[\![\psi]\!] \neq \mathbf{0}$, so in order to match our initial assumption we need to have $[\![\chi]\!] \neq \mathbf{0}$. By Corollary 4.4.4(i),(ii) we get $[\![(\varphi \to \chi) \to \big((\psi \to \chi) \to (\varphi \vee \psi \to \chi)\big)]\!] = \mathbf{1}$.

(vi) $(\varphi \wedge \psi) \to \varphi$. Suppose $[\![\varphi \wedge \psi]\!] \neq \mathbf{0}$ then $[\![\varphi]\!] \neq \mathbf{0}$, so by Corollary 4.4.4(i),(ii) we get $[\![(\varphi \wedge \psi) \to \varphi]\!] = \mathbf{1}$.

(vii) $(\varphi \wedge \psi) \to \psi$. Similar to proof of item (vi).

(viii) $\varphi \to \big(\psi \to (\varphi \wedge \psi)\big)$. Suppose $[\![\varphi]\!] \neq \mathbf{0}$ and $[\![\psi]\!] \neq \mathbf{0}$, then $[\![\varphi \wedge \psi]\!] \neq \mathbf{0}$. Then by Corollary 4.4.4(i),(ii) we get $[\![\varphi \to \big(\psi \to (\varphi \wedge \psi)\big)]\!] = \mathbf{1}$.

(ix) $\neg\varphi \to (\varphi \to \psi)$. Suppose $[\![\neg\varphi]\!] = [\![\varphi]\!]^{*_p} \neq \mathbf{0}$. Then we have $[\![\varphi]\!] = \mathbf{0}$, so by Corollary 4.4.4(i),(ii) we have $[\![\varphi \to \psi]\!] = \mathbf{1}$ and therefore we get $[\![\neg\varphi \to (\varphi \to \psi)]\!] = \mathbf{1}$.

(x) $(\varphi \to \psi) \to \big((\varphi \to \neg\psi) \to \neg\varphi\big)$. Suppose $[\![\varphi \to \psi]\!] \neq \mathbf{0}$ and $[\![\varphi \to \neg\psi]\!] \neq \mathbf{0}$. We know that either $[\![\psi]\!] = \mathbf{0}$ or $[\![\neg\psi]\!] = \mathbf{0}$, so in order to match our initial assumption we need to have $[\![\varphi]\!] = \mathbf{0}$. Therefore, $[\![\varphi]\!]^{*_p} = [\![\neg\varphi]\!] = \mathbf{1}$ and we are done. $\square$

4.4.4 A Non-well-founded Model of ZF

We now offer an application of $(\mathbb{T}, ^{*_p})$-valued models. Remember that by ZF^- we denote ZF minus $\mathsf{Foundation}_\varphi$. Then by Theorem 4.4.21 we know that for any $\mathbf{V}^{(\mathbb{T}, ^{*_p})}$ (where $\mathbb{T} \in \mathcal{W}$) and for any filter F on $\mathbb{T}$ we have

$$\mathbf{V}^{(\mathbb{T}, ^{*_p})} \models_F \mathsf{ZF}^- + \mathsf{Foundation}_\varphi.$$

We now show that we can also construct models of the negation of $\mathsf{Foundation}_\varphi$. To do so we consider an implicative meet complemented $(\mathbb{T},^{*_p})$-lattice where the underlying base set is non-well founded. Specifically, we will consider

$$\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)} \in \mathcal{T},$$

where $(\omega+1)^*$ is an ordered set whose order is the reverse of that of $\omega+1$. Thus we define the set $((\omega+1)^* \cup \ (\omega+1), <)$ as an order-isomorphic copy of $\mathbb{Z}$ with a top and bottom element added. Before starting, notice that $\mathbf{V}^{(\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)},^{*p})}$ is a model of ZF^- by Theorem 4.4.20. Indeed, all results of the paper, except Theorem 4.4.11 which depends on well-foundedness, hold for any implicative meet complemented $\mathcal{T}$-lattice.

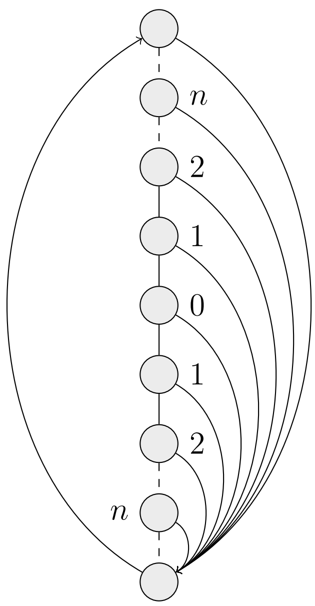


Figure 1. $\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)}$.

Theorem 4.4.25. *Fix $\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)}$ and let F be any filter on $\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)}$. Then we have $\mathbf{V}^{(\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)},^{*p})} \models_F \mathsf{ZF}^- + \neg\, \mathsf{Foundation}_\varphi$.*

Proof. It is enough to show that there is a formula for which the Foundation schema does not hold. Let $\varphi(x) := \exists y(y \in x \vee y \notin x)$. Notice that $\varphi(x)$ never gets value zero for any $x \in \mathbf{V}^{(\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)},\neg)}$. Indeed, let u and v be two elements of $\mathbf{V}^{(\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)},^{*p})}$. Now, if $[\![u \in v]\!] = \mathbf{0}$, then $[\![(u \in v) \vee (u \notin v)]\!] = \mathbf{1}$. Otherwise $[\![u \in v]\!] = a$, with $a \neq \mathbf{0}$, but then $[\![(u \in v) \vee (u \notin v)]\!] = a$. Thus $[\![\varphi(u)]\!] \neq \mathbf{0}$, for all $u \in \mathbf{V}^{(\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)},^{*p})}$.

Now, consider the name $x_a = \{\langle \varnothing, a \rangle\}$, for $a \in \mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)}$. Then, for every $a \in \mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)}$, there is an u such that $[\![\varphi(u)]\!] = a$. This means that the values of $[\![\varphi(x)]\!]$ are unbounded in $(\omega+1)^*$. This implies the following equality

$$[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in \mathbf{V}^{(\mathbb{T}_{(\omega+1)^* \,\cup\, (\omega+1)}, {}^{*p})}} [\![\varphi(u)]\!] = \mathbf{0}.$$

Moreover, since $[\![\varphi(u)]\!] \neq \mathbf{0}$ for all names u's, we get that $[\![\forall x \Big(\forall y \in x \varphi(y) \to \varphi(x)\Big)]\!] = \mathbf{1}$. Hence,

$$\begin{aligned} & [\![\forall x \Big(\forall y \in x \varphi(y) \to \varphi(x)\Big)]\!] \Rightarrow_t [\![\forall x \varphi(x)]\!] \\ = {} & [\![\forall x \Big(\forall y \in x \varphi(y) \to \varphi(x)\Big) \to \forall x \varphi(x)]\!] \\ = {} & \mathbf{0}. \end{aligned}$$

□

4.5 Algebra-valued Models for LFI-Set Theories

In this section, we will show that the iterative paraconsistent set theories ZFmbC, ZFmCi, ZFCi, and ZFCil are not valid in any $(\mathbb{T}, {}^{*r})$-valued model. This result is counterintuitive since the three-element reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice (or simply three-element $(\mathbb{T}, {}^{*r})$-lattice) is nothing else than $(\mathbb{PS}_3, {}^*)$, which is an LFI when we expand its signature with the consistency operator $\circ$, as noted in (CONIGLIO; SILVESTRINI, 2014). Moreover, we will show that we have problems in interpreting the ZF-like axioms of LFI-set theories and that we have even more problems when it comes to the axioms that govern the consistency predicate. Finally, we conclude that it seems unfeasible to construct algebra-valued models for LFI-set theories, at least, as they are presented in (CARNIELLI; CONIGLIO, 2016a).

4.5.1 The ZF-like Axioms of LFI-Set Theories

To check whether the axioms of an LFI-set theory are valid in $(\mathbb{T}, {}^{*r})$-valued models, we first need to specify the semantic interpretation of the $\circ$-operator (see Chapter 3.7 and Chapter 3.8).

Table 4.2: The $\circ$-operator in the three-element $(\mathbb{T},{}^{*r})$-lattice

φ	$\circ\varphi$
1	**1**
$\frac{1}{2}$	**0**
0	**1**

Now, given that we can define this operator in terms of other logical connectives, we can extend the evaluation map $[\![\cdot]\!]$ with the following clause

$$[\![\circ\varphi]\!] = [\![(\varphi \to \bot) \vee (\neg\varphi \to \bot)]\!],$$

which allows us to evaluate formulas of the form of $\circ\varphi$ in $(\mathbb{T},{}^{*r})$-valued models. Notice that the formula on the right-hand side of the equality is a formula in the language of set theory and thus $\neg$ denotes the syntactic negation of the language of set theory. Intuitively, this semantic interpretation of the consistency operator says that the formula φ is consistent if either φ or its negation is false. In the three-element $(\mathbb{T},{}^{*r})$-lattice, this gives rise to Table 3.2.

We now check the validity of the axioms of ZFmbC in $(\mathbb{T},{}^{*r})$-valued models. By Corollary 4.4.16 we know that NFF-ZF is valid in any $(\mathbb{T},{}^{*r})$-valued model. Therefore, any $(\mathbb{T},{}^{*r})$-valued model verifies Union, Extensionality, Power set, Separation$_\varphi$, Foundation$_\varphi$, and Replacement$_\varphi$, where φ is negation-free formula.

On the other hand, Infinity and Weak regularity, as presented in (CARNIELLI; CONIGLIO, 2016a), are not valid in any $(\mathbb{T},{}^{*r})$-valued model. Nonetheless, by a suitable modification, we can obtain their validity, without distorting the spirit of ZFmbC. The issue with Infinity consists in the use of the *strong* empty set $\varnothing^*$ which cannot be defined in $(\mathbb{T},{}^{*r})$-valued models. In (CARNIELLI; CONIGLIO, 2016a), the authors define the strong empty set using Separation$_\varphi$ and Extensionality and the fact that

$$\mathsf{ZFmbC} \vdash \Big((x \in a) \wedge \sim (x = x)\Big) \Leftrightarrow \sim (x = x).$$

However, this can not be done in $(\mathbb{T}, {}^{*r})$-valued models, since the formula that defines the strong empty set is not a negation-free instance of $\mathsf{Separation}_{\varphi}$. But this is not dramatic, since we can use a different, negation-free, formulation of $\mathsf{Infinity}$ that suits better an algebra-valued model treatment of set theory: [1]

$$\exists x\Big(\exists y(\forall z(z \in y \to \bot) \wedge y \in x) \wedge \forall w(w \in x \to \exists u(u \in x \wedge (w \in u)))\Big).$$

Moreover, $\mathsf{Weak\ regularity}$ states that only consistent sets can not be elements of themselves. So according to this view there exist inconsistent sets in our set-theoretic universe, which are ill-founded. However, notice that $\mathsf{NFF\text{-}Foundation}_{\varphi}$ is valid in $(\mathbb{T}, {}^{*r})$-valued models and thus we can easily show that $[\![x \in x]\!] = \mathbf{0}$, for any $x \in \mathbf{V}^{(\mathbb{T}, {}^{*r})}$. Therefore, *every* set that inhabits $\mathbf{V}^{(\mathbb{T}, {}^{*r})}$ is well-founded. This seems to suggest, on one hand, that non-well-foundedness is not a property of inconsistent sets in $(\mathbb{T}, {}^{*r})$-valued models and on the other, that we should discard $\mathsf{Weak\ regularity}$ in favor of $\mathsf{NFF\text{-}Foundation}_{\varphi}$. Notice that, even though we need to adjust $\mathsf{Weak\ regularity}$ and $\mathsf{Infinity}$, we can until this point interpret all the ZF axioms of ZFmbC in $(\mathbb{T}, {}^{*r})$-valued models. For what concerns $\mathsf{Unextensionality}$ this is unfortunately not the case.

Theorem 4.5.1. *Let $(\mathbb{T}, {}^{*r})$ be a reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice with more than two elements and F any filter, then $\mathbf{V}^{(\mathbb{T}, {}^{*r})} \not\models_F \mathsf{Unextensionality}$.*

Proof. Fix any $(\mathbb{T}, {}^{*r})$-valued model, where $\mathbf{A}$ has more than two elements. Then consider the following names as witnesses: $u = \{\langle \varnothing, a\rangle\}, v = \{\langle u, 1\rangle\}, w = \{\langle u, a\rangle\}$, where $a \neq \mathbf{1}, \mathbf{0}$. Now, we simply calculate $[\![\mathsf{Unextensionality}]\!]$ as follows:

$$\begin{aligned}
&[\![\mathsf{Unextensionality}]\!] \\
&= [\![\exists z\big((z \in x) \wedge (z \notin y)\big) \vee \exists z\big((z \in y) \wedge (z \notin x)\big) \leftrightarrow (x \not\approx y)]\!] \\
&\leq [\![\exists z\big((z \in x) \wedge (z \notin y)\big) \vee \exists z\big((z \in y) \wedge (z \notin x)\big) \to (x \not\approx y)]\!] \\
&= \Big((\mathbf{1} \wedge \frac{1}{2}) \vee (\mathbf{0} \wedge \frac{1}{2})\Big) \Rightarrow_t \mathbf{0} \\
&= \frac{1}{2} \Rightarrow_t \mathbf{0} \\
&= \mathbf{0} \notin F.
\end{aligned}$$

[1] This definition of the axiom of infinity was already proposed in (BELL, 2005) and is displayed in Figure 2.1.

$\square$

Notice that, since $[\![\mathsf{Unextensionality}]\!] = \mathbf{0}$, these arguments hold for any choice of filter. This is already troubling, but a possible strategy would consist of biting the bullet and just to eliminate this axiom (even though this axiom was given a particular emphasis in the motivation of LFI-set theories). However, we will show that the fundamental issue with LFI-set theories lays much deeper; it concerns the interpretation of inconsistent sets.

4.5.2 The C-axioms of LFI-Set Theories

In (CARNIELLI; CONIGLIO, 2016b) the definition of inconsistency is given in terms of the violation of the basic logical operations of set theory: equality and membership.

$$\neg C(x) =_{df.} (x \in x) \tag{i}$$

$$\neg C(x) =_{df.} \neg \circ (x \in x) \tag{ii}$$

$$\neg C(x) =_{df.} (x \not\approx x) \tag{iii}$$

$$\neg C(x) =_{df.} \neg \circ (x \not\approx x) \tag{iv}$$

We now show that these definitions cannot capture the notion of inconsistency in any $(\mathbb{T}, ^{*_r})$-valued model.

Lemma 4.5.2. *Let* $(\mathbb{T}, ^{*_r})$ *be any reflexive bicomplemented implicative* $\mathcal{W}_{\mathcal{F}}$*-lattice. Then, for any* $x \in \mathbf{V}^{(\mathbb{T}, ^{*_r})}$ *we have:*

(i) $[\![x \in x]\!] = \mathbf{0}$,

(ii) $[\![\neg \circ (x \in x)]\!] = \mathbf{0}$,

(iii) $[\![x \not\approx x]\!] = \mathbf{0}$,

(iv) $[\![\neg \circ (x \not\approx x)]\!] = \mathbf{0}$.

Proof. (i) Notice that $\mathsf{Foundation}_{\varphi}$, for φ negation-free, yields $[\![x \in x]\!] = \mathbf{0}$. This establishes (i).
(ii) From (i) it follows that $[\![\circ(x \in x)]\!] = \mathbf{1}$. Then by Lemma 4.4.15 we have

$[\![\neg \circ (x \in x)]\!] = \mathbf{0}$, establishing (ii).
(iii) We know that for every $x \in \mathbf{V}^{(\mathbb{T},*_r)}$ we have $[\![x \approx x]\!] = \mathbf{1}$, thus $[\![x \not\approx x]\!] = \mathbf{0}$. So (iii) holds as well.
(iv) Furthermore, given that $[\![x \not\approx x]\!] \in \{\mathbf{1}, \mathbf{0}\}$ it follows that $[\![\circ(x \not\approx x)]\!] = \mathbf{1}$. Finally, given Lemma 4.4.15 we can conclude $[\![\neg \circ (x \not\approx x)]\!] = \mathbf{0}$, which settles (iv). □

This proposition shows clearly that none of the above interpretations of inconsistency can ever be realized in any $(\mathbb{T},^{*_r})$-valued model. Notice that this does not imply that it is not possible to define inconsistent sets in $(\mathbb{T},^{*_r})$-valued models in general. There might be other ways to capture these ideal objects, but again, this would transcend the core ideas and tools of LFI-set theories as conceived in (CARNIELLI; CONIGLIO, 2016b).[2]

Nevertheless, the failure of the characteristic axioms of these LFI-set theories, i.e., the C-axioms, together with the impossibility to define inconsistency in the way proposed in (CARNIELLI; CONIGLIO, 2016b), leaves us with two options. Either we assume that, for all $x \in \mathbf{V}^{(\mathbb{T},*_r)}$ we have $[\![\neg C(x)]\!] = \mathbf{0}$, therefore assuming that every set is consistent, or we can try to modify the interpretation of the consistency predicate in order to provide examples of inconsistent sets in $(\mathbb{T},^{*_r})$-valued models. We argue that both strategies are unfeasible.

Let us make explicit the two horns of this dilemma:

Horn 1: The first strategy consists in eliminating inconsistent totalities from the picture. This option is, as a matter of fact, not very far from Cantor's ideas. Indeed in several letters to Hilbert and Jourdain (NILSON; MESCHKOWSKI, 1991, pp. 425–435), Cantor identified sets and consistent totalities, arguing that the universe of set was the collection of all totalities that did not lead to any contradiction. On this basis we can therefore postulate an axiom, that we can call Cantorian Axiom, that expresses this intuition.

$$C(x) \leftrightarrow Set(x) \qquad \text{(Cantorian Axiom)}$$

[2]On a possible fix. (1) Try to find some LFI that can be represented as a complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$, such that $[\![u = u]\!] = a$, where $a \neq \{\mathbf{0}, \mathbf{1}\}$, for some $u \in \mathbf{V}^{(\mathbb{A},^*)}$. Then if a is a designated element and given a suitable choice of negation; we will get $[\![\neg(x = x)]\!] = a$. Nevertheless, it is unclear whether the respective structure is still a model of set theory. (2) Try to use a non-well founded algebra $\mathbf{A}$ as underlying algebra, something order-isomorphic to $\mathbb{Q}$, then we could get $[\![u \notin u]\!] \neq \mathbf{0}$ for some $u \in \mathbf{V}^{(\mathbb{A},^*)}$. A particularly interesting solution would consist of constructing an algebra-valued model where both (1) and (2) are realized.

Now, if we extend ZFmbC with the Cantorian Axiom we obtain that all $(\mathbb{T}, {}^{*_r})$-valued models validate ZFmbC + Cantorian Axiom and all its extensions. Indeed these LFI-set theories would all collapse to ZF, since we have just removed all inconsistent sets. However, this move would totally trivialize the main motivation of using a consistency predicate in dealing with an inconsistent set theory, as clearly expressed in (CARNIELLI; CONIGLIO, 2016a, p. 366):

> The main idea is to assume that not only sentences can be taken to be consistent or inconsistent, but also that sets themselves can be thought to be consistent or inconsistent. We establish the basis for new paraconsistent set-theories (such as ZFmbC and ZFCil) under this perspective and establish their non-triviality, provided that ZF is consistent.

Horn 2: On the other hand, we could try to modify the interpretation of the (in)consistency predicate, in order to have at least one set $x \in \mathbf{V}^{(\mathbb{T}, {}^{*}\searrow)}$ such that $[\![\neg C(x)]\!] \geq a$, where $a \neq \mathbf{1}$ and a is a designated element. So suppose that there exists at least one inconsistent set in our ontology. However, also this strategy is doomed to fail, since no $(\mathbb{T}, {}^{*_r})$-valued model, would be able to validate ZFmCi, or ZFCi, or ZFCil.

Lemma 4.5.3. *Let $(\mathbb{T}, {}^{*_r})$ be any reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice and F any filter on $\mathbb{T}$ such that $a \in F$. Suppose that there exists a $x \in \mathbf{V}^{(\mathbb{T}, {}^{*_r})}$ such that $[\![\neg C(x)]\!] \geq a$. Then, we have $[\![\neg C(x) \to \neg \circ (x \approx x)]\!] = \mathbf{0}$.*

Proof. We know that for any $x \in \mathbf{V}^{(\mathbb{T}, {}^{*_r})}$ we have $[\![x \approx x]\!] = \mathbf{1}$ and thus $[\![\circ(x \approx x)]\!] = \mathbf{1}$. Therefore by Lemma 4.4.15 we get $[\![\neg \circ (x \approx x)]\!] = \mathbf{0}$. Then;

$$
\begin{aligned}
&[\![\neg C(u) \to \neg \circ (u \approx u)]\!] \\
&= (a \Rightarrow_t \mathbf{0}) \\
&= \mathbf{0} \notin F.
\end{aligned}
$$

$\square$

Notice that in this case, we would validate vacuously all the axioms of ZFmbC that govern the consistency predicate, with exception of $\mathbf{con_0}$. This is because either $\neg C(x)$ occurs in the consequent of an axiom, so that the antecedent is always false, as it is the case with $\mathbf{con}_2$, or $C(x)$ occurs in the antecedent, so that the consequent will be trivially true, as in $\mathbf{con}_1$.

Besides the formal problems that arise from accepting the existence of inconsistent totalities, there is also a deeper conceptual issue. A charitable interpretation of the inconsistency predicate—that seems faithful to the history of set theory—would consist of equating inconsistent objects and proper classes. However, at a closer look, the axioms of ZFmbC force us to discard this possibility, since the axiom $\mathbf{con}_0$ would imply that the universal class is a set and not a proper class. Indeed, it says that the property of being a set is propagated $\in$-upward, which clearly cannot be the case since all elements of $\mathbf{V}$ are, by definitions, sets.

We can therefore conclude, from **Horn 1** and **Horn 2**, that LFI-set theories are not valid in any $(\mathbb{T},^{*_r})$-valued model. This is even more belittling, considering that the internal logic of $\mathbf{V}^{(\mathbb{PS}_3,^*)}$ is an LFI. Not only we need to give up the characteristic axiom $\mathsf{Unextensionality}$, but **Horn 2**, together with the charitable interpretation of inconsistent totalities as classes, shows that not even the ZFmbC-axioms that govern the C-predicate can hold in any $(\mathbb{T},^{*_r})$-valued model. We believe that the fact that inconsistent totalities do not find a place in these LFI-set theories is a serious betrayal of their original motivation, which therefore suggest a fresh new start. The failure of this attempt leaves open the challenging problem of making inconsistent totalities compatible with an iterative paraconsistent set theory.

Chapter 5

On Negation for Non-classical Set Theories

Summary

In this chapter, we will apply algebra-valued models to enrich the philosophical debate on what constitutes a negation and evaluate whether it is possible to ground negation on more fundamental notions. In particular, we present a case study for the debate between the American and the Australian plans, analyzing a crucial aspect of negation: expressivity within a theory. We discuss the case of non-classical set theories, presenting four different negations and testing their expressivity within algebra-valued models for iterative paraconsistent set theories. Finally, we give a minimal account of negation, inspired by the algebraic framework we work in. We will propose minimal regularity properties for negation and argue that the essential intuition behind negation is the linguistic ability *to tell things apart*. Moreover, we conclude by pointing out that logical properties, such as paraconsistency, might not be predicated *only* to negation and that, thus, both plans fall short in what they have set out to accomplish.

5.1 Playing with Negation

One of the main issues with the models of non-classical set theory, introduced in Chapter 4.4, is that it is hard to show that these models are non-$\in$-elementarily equivalent with each other, at least, in the case of $(\mathbb{T},^{*_p})$-valued and $(\mathbb{T},^{*_r})$-valued models. Thus, even though we have found an infinite class of non-classical models of set theory, we are unable to show that we have *different* models.

One possible way of tackling this problem is to increase the complexity of our language by twisting the negation. In particular, we will treat the unary operator in the signature of our lattices (that is used to interpret the negation of the language of set theory) as a further parameter. So, in principle, any unary function on an implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice that fulfills some minimal constraints is a suitable candidate for negation. In more simple words, we will *play* around with negation.

This chapter is structured as follows. First, we will summarize briefly the current philosophical debate on whether it is possible to ground negation on more fundamental notions. Thus we present two views that have been put forward; the *American* Plan, which we can find in (DE; OMORI, 2018), (WANSING, 2008), (DUNN, 1999) and *Australian* Plan, advocated by (BERTO; RESTALL, 2019), (RESTALL, 2013), (MEYER; MARTIN, 1986), (BERTO, 2015). Moreover, we complement this presentation with a formal account of each of these plans in the context of algebra-valued models of set theory.

Secondly, we introduce four different unary operators defined within the structure of implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice and discuss whether these operators can be considered genuine negations. This analysis will show that the more we relax the requirements on the regular behavior of these negations, the more non-elementarily equivalent models we will find. In other terms, we will find an inverse proportionality between the number of conditions we impose on negation and the number of incompatible sentences we can express using such a negation. Finally, we present our own minimal account of negation, the algebraic account of negation, and conclude that both the American and Australian Plan fall short in capturing certain aspects of negation.

5.2 The Australian Plan

Among the supporters of the Australian Plan we find authors such as Berto, Meyer, Martin and Restall, (BERTO; RESTALL, 2019), (RESTALL, 2013), (MEYER; MARTIN, 1986), (BERTO, 2015), who present the basic tenants of this viewpoint as:

1. Negation is a device meant to capture a notion of *exclusion*, and

2. negation has a *modal character* grounded on the concept of incompatibility.

Usually, we can find the following characterization of negation by generalized Kripke semantics; a sentence $\neg\varphi$ is said to be true at a world w if all worlds v that are compatible with w (here compatibility is understood in terms of the accessibility relation) do not validate φ.

Moreover, the authors of (BERTO; RESTALL, 2019) have put forward two criteria that a genuine negation has to satisfy: contraposition (CP) and double negation introduction (DNI). We believe that it is important to make a distinction between the *algebraic* version of these rules —independent from any choice of truth and falsity—and their *logical* version— given in terms of a consequence relation. Take for instance the following passage from (LÖWE; PASSMANN; TARAFDER, 2021, p. 8).

> When twisting the negation, we need to pay attention to the fact that not every unary function on an implication algebra is a sensible negation. In his survey of varieties of negation, Dunn (1995) lists Hazen's *subminimal negation* as the bottom of his *Kite of Negations*: only the rule of contraposition, i.e., $a \leq b$ implies $\neg b \leq \neg a$, is required. In the following, we shall use this as a necessary requirement to be a reasonable candidate for negation.

On this account, given a lattice $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$, an operation * counts as a legitimate negation, if CP holds in $\mathbb{A}$ and not necessarily in $\mathbf{L}(\mathbb{A}, F)$. Hence, we can evaluate CP and DNI *algebraically.*

Definition 5.2.1. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ be a complete bounded distributive lattice. Then we say that CP and DNI hold in $\mathbb{A}$ if for any $x, y \in \mathbf{A}$ the following holds:*

(i) $x \leq y$ implies $y^* \leq x^*$ and

(ii) $x \leq x^{**}$.

This gives us the following definition.

Definition 5.2.2. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ be a complete bounded distributive lattice. We say that an algebraic operator *, interpreting the negation in the language of set theory in an $\mathbb{A}$-valued model $\mathbf{V}^{(\mathbb{A})}$ satisfies the Australian Plan algebraically whenever CP and DNI hold in $\mathbb{A}$.*

Table 5.1: Operations for $\mathbb{K}_3$

$\Rightarrow$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\vee$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$

$\wedge$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

x	x^*
$\mathbf{1}$	$\mathbf{0}$
$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{0}$	$\mathbf{1}$

This definition has two main motivations. Firstly, the account of (BERTO; RESTALL, 2019) can be formulated entirely in algebraic terms, see (DUNN, 1999, pp. 30-34), and secondly, the $\leq$-relation offers a purely syntactical perspective on the inferential features of a logic obtained from an algebra.

Nonetheless, since (BERTO; RESTALL, 2019) is concerned with defining regularity properties for negation within a logical calculus, we can interpret CP and DNI also *logically*.

Definition 5.2.3. *Let* $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ *be a complete bounded distributive lattice and* F *a filter on* $\mathbf{A}$. *We say that* CP *and* DNI *hold in* $\mathbf{L}(\mathbb{A}, F)$ *if for any* $\varphi, \psi \in \mathcal{L}_{Prop}$ *the following holds:*

(i) $\varphi \models_F \psi$ implies $\neg\psi \models_F \neg\varphi$ *and*

(ii) $\varphi \models_F \neg\neg\varphi$.

Definition 5.2.4. *Let* $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ *be a complete bounded distributive lattice. We say that an algebraic operator* **, interpreting the negation in the language of set theory in an* $\mathbb{A}$*-valued model* $\mathbf{V}^{(\mathbb{A})}$ *satisfies the Australian Plan logically whenever* CP *and* DNI *hold in* $\mathbf{L}(\mathbb{A}, F)$.

Indeed, there exist very basic examples which show that the two mentioned forms of evaluating inferential rules may have incompatible results. For instance, consider the following complete bounded distributive lattice $\mathbb{K}_3$ (see Table 4.1). The operations of Table 4.1 can be associated both to the connectives of Kleene's logic (K_3), as well as, to the connectives of LP (compare to Table 2.1). Indeed, if we consider the filter $\{\mathbf{1}\}$ on $\mathbb{K}_3$, then $\mathbf{L}(\mathbb{K}_3, \{\mathbf{1}\}) = \mathsf{K}_3$, but if we consider the filter $\{\mathbf{1}, \frac{1}{2}\}$, then $\mathbf{L}(\mathbb{K}_3, \{\mathbf{1}, \frac{1}{2}\}) = \mathsf{LP}$.

Moreover, K_3 admits a proper negation for the Australian Plan, via the Routley star semantics (DE; OMORI, 2018, p.14). Indeed, we can obtain the Routley star semantics for K_3 by adding a further condition to the Routley star semantics for First Degree Entailment (FDE).[1]

On the other hand, the supporters of the Australian Plan maintain that LP has no negation since CP fails. However, since LP is obtainable from K_3 by taking also the intermediate value of K_3 to be designated, we have a case that evidences the relativity of the conditions proposed by the Australian Plan. Indeed, the choice of an appropriate filter on a given lattice can engender a logic with a proper "Australian" negation, i.e., K_3, while it is also possible to choose another filter which makes the same negation unacceptable, i.e., LP.

This line of criticism was already raised by the American Plan, as the next quotation shows.

> Berto's account of negation seems therefore sensitive to which values are taken as designated. We find it curious that one and the same operator should and should not count as a negation depending on which values are taken to be truth-like, a curiosity that does not arise on our account. (DE; OMORI, 2018, p. 297).

We can summarize this situation, using the notation introduced in this section in the following way.

Corollary 5.2.5. *The interpretation of the negation $\neg$ from the logic $\mathbf{L}\Big(\mathbb{K}_3, \{\mathbf{1}\}\Big)$ satisfies the Australian Plan logically in $\mathbf{V}^{(\mathbb{K}_3)}$.*

Corollary 5.2.6. *The interpretation of the negation $\neg$ from the logic $\mathbf{L}\Big((\mathbb{K}_3, \{\mathbf{1}, \frac{1}{2}\}\Big)$ does not satisfy the Australian Plan logically in $\mathbf{V}^{(\mathbb{K}_3)}$.*

5.3 The American Plan

The advocates of the American Plan, like Wansing, Omori, and De (DE; OMORI, 2018), (WANSING, 2008), (DUNN, 1999) give a *non-modal* account of negation. They interpret negation in terms of an intuitive switch operator between truth

[1]For more details see Chapter 8 of (PRIEST, 2008).

and falsity. Concretely, one can say that a logical connective $\neg$ is a negation if and only if $\neg$ is a contradictory-forming operator: i.e., for any formula φ we have that

(1) φ is true iff $\neg\varphi$ is false and

(2) φ is false iff $\neg\varphi$ is true.

On this account, negation is grounded on the primitive *sui generis* notions of truth and falsity. Now, notice that there are several legitimate relations between these two notions. For instance, if they are both exhaustive and exclusive, then only the Boolean negation will be considered a genuine negation (SLATER, 1995). However, given that we can make the assumption that truth and falsity are not exclusive, a paraconsistent negation can also be considered a contradictory-forming operator.

Let us now turn our attention to the regularity properties of the American Plan. Also here we need to adapt the criteria for being a negation to an algebraic context. In doing so we will slightly improve these criteria, spelling out necessary and sufficient conditions. What motivates the following definition is the consideration that filters and ideals give algebraic representations of truth and falsity.

Definition 5.3.1. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ be a complete bounded distributive lattice, F a filter on $\mathbf{A}$, I an ideal on $\mathbf{A}$, and let ι be an evaluation function from $\mathcal{L}_{Prop}$ into $\mathbf{A}$. We say that a negation $\neg$ is a* contradictory-forming operator *in $\mathbf{L}(\mathbb{A}, F)$ if there exist an ideal $I \subseteq A$ such that for any assignment ι and every formula $\varphi \in \mathcal{L}_{Prop}$ we have*

$$(1)\ \iota(\neg\varphi) \in F \textit{ iff } \iota(\varphi) \in I \textit{ and } (2)\ \iota(\neg\varphi) \in I \textit{ iff } \iota(\varphi) \in F.$$

Notice that we are not imposing any restriction on the exclusivity or exhaustivity of F and I. Then, for example, we can evaluate the criteria of the American Plan with respect to a paraconsistent negation just by assuming that $G \cap I \neq \varnothing$. Moreover, Definition 5.3.1 offers sufficient conditions that perfectly match the necessary ones we find in the literature (where ML denotes the following inference rule: for some formulas φ, ψ we have $\varphi \nvDash \neg\varphi$ and $\neg\psi \nvDash \psi$).

Corollary 5.3.2. *If $\neg$ is a contradictory-forming operator in $\mathbf{L}(\mathbb{A}, F)$, then* DNI*,* DNE*, De Morgan laws and* ML *hold in $\mathbf{L}(\mathbb{A}, F)$.*

Definition 5.3.3. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ be a complete bounded distributive lattice. We say that an algebraic operator *, interpreting the negation in the language of set theory* *in an* $\mathbb{A}$*-valued model* $\mathbf{V}^{(\mathbb{A})}$, satisfies the American Plan *whenever it is a contradictory-forming operator in* $\mathbf{L}(\mathbb{A}, F)$.

A final comment on the algebraic rendering we offered of the conditions of the American Plan. Being a contradictory-forming operator an American negation depends chiefly on the choice of a filter and on that of an ideal. This causes no harm for a supporter of the American Plan, since the notion of negation is rooted on the primitive notions of truth and falsity, which, in turn, are formalized by filters and ideals.

5.4 The Reflexive Operator

The first connective we will study is what we call the *reflexive* operator. This operator is not only definable as an equational class for the algebras under consideration, but the resulting set-theoretical models also extend the minimal Boolean-valued model $\mathbf{V}^{(2)}$ and the $(\mathbb{PS}_3,^*)$-valued model $\mathbf{V}^{(\mathbb{PS}_3,^*)}$. We will show that although this connective is not acceptable for the followers of the Australian Plan, however, it constitutes a genuine negation from an American perspective.

We introduced the *reflexive* operator ^{*_r} for the first time in Definition 4.4.14 in the context of relexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattices. Moreover, by $(\mathbb{T}_n,^{*_r})$ we will denote the reflexive bicomplemented implicative $\mathcal{W}_{\mathcal{F}}$-lattice (whose universe has n-many elements) and where we interpret ^{*_r} as negation.

Lemma 5.4.1. *We have that* $\mathbf{V}^{(\mathbb{T}_2,^{*_r})} = \mathbf{V}^{(\mathbb{B}_2)}$*, where* $\mathbb{B}_2$ *is the two-valued Boolean algebra, while when* $n = 3$ *we get that* $\mathbf{V}^{(\mathbb{T}_3,^{*_r})} = \mathbf{V}^{(\mathbb{PS}_3,^*)}$. □

By Theorem 4.4.12 and Theorem 4.4.17, we know that $(\mathbb{T}_n,^{*_r})$-valued models are an infinite class of paraconsistent models of set theory that extend $\mathbf{V}^{(\mathbb{B}_2)}$ and $\mathbf{V}^{(\mathbb{PS}_3,*)}$. However, it is still an open question whether we can prove that the $(\mathbb{T}_n,^{*_r})$-valued models are non-$\in$-elementarily equivalent with each other. We now evaluate whether the reflexive operator is suitable for either the Australian or the American Plan.

Lemma 5.4.2. DNI *holds for any* $(\mathbb{T}_n,^{*_r})$*, but* CP *fails for any* $(\mathbb{T}_n,^{*_r})$ *with* $n \geq 4$.

Proof. The first part follows directly from Lemma 4.4.15 and for the second part consider the algebra $(\mathbb{T}_4,^{*_r})$ where $\mathbf{A} = \{\mathbf{1}, 2, 3, \mathbf{0}\}$ such that $\mathbf{0} \leq 3 \leq 2 \leq \mathbf{1}$, then $3 \leq 2$ but $2^{*_r} \not\leq 3^{*_r}$. □

Lemma 5.4.3. DNI *holds for any* $\mathbf{L}\Big((\mathbb{T}_n, ^{*_r}), Pos_{(\mathbb{T}_n)}\Big)$*, but* CP *fails for any* $\mathbf{L}\Big((\mathbb{T}_n, ^{*_r}), Pos_{(\mathbb{T}_n)}\Big)$ *with* $n \geq 3$.

Proof. Consider $\mathbf{L}\Big((\mathbb{T}_3, ^{*_r}), Pos_{(\mathbb{T}_3)}\Big)$ where $\mathbf{A} = \{\mathbf{1}, 2, \mathbf{0}\}$ such that $\mathbf{0} \leq 2 \leq \mathbf{1}$ and the assignment ι such that $\iota(\psi) = 2$ and $\iota(\varphi) = \mathbf{1}$, where $\varphi, \psi \in \mathcal{L}_{Prop}$. Then $\varphi \models_{Pos_{(\mathbb{T}_3)}} \psi$, but $\neg\psi \not\models_{Pos_{(\mathbb{T}_3)}} \neg\varphi$. $\square$

Corollary 5.4.4. *When the operator* ^{*_r} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, ^{*_r})}$*, then it does not satisfy the Australian Plan algebraically, for* $n \geq 4$. *Moreover, when* ^{*_r} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, ^{*_r})}$*, then it does not satisfy the Australian Plan logically, for* $n \geq 3$.

Thus, the choice between evaluating CP algebraically or logically does make a difference in the case of $(\mathbb{T}_3, ^{*_r})$.

Lemma 5.4.5. ^{*_r} *is a contradictory-forming operator in* $\mathbf{L}\Big((\mathbb{T}_n, ^{*_r}), Pos_{(\mathbb{T}_n)}\Big)$ *with* $n \geq 2$.

Proof. Fix $(\mathbb{T}_n, ^{*_r})$, where $\{\mathbf{0}, \mathbf{1}, \ldots, n-1\}$ such that $\mathbf{0} < n-1 < n-2 < \ldots < 2 < \mathbf{1}$, with $n \geq 2$. In the case of $n = 2$, we get the Boolean negation, since $\mathbf{L}\Big((\mathbb{T}_2, ^{*_r}), Pos_{(\mathbb{T}_2)}\Big) = \mathsf{CPL}$. Clearly, ^{*_r} satisfies Definition 5.3.1 in the case of $(\mathbb{T}_2, ^{*_r})$.

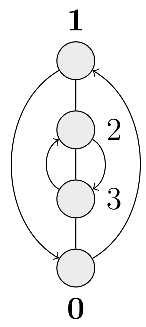
So, consider $n > 2$. Then define the filter $F = Pos_{\mathbb{T}_n}$ and the ideal $I = Neg_{\mathbb{T}_n}$. Clearly, $G \cap I \neq \varnothing$. For (1) suppose there exists an assignment ι such that $\iota(\neg\varphi) \in G$, so either $\iota(\neg\varphi) = \mathbf{1}$ or $\iota(\neg\varphi) = x$ such that $x \neq \mathbf{1}$. By Lemma 4.4.15 we get immediately that $\iota(\varphi) \in I$. The other side follows by symmetry. For (2) suppose we have an assignment ι such that $\iota(\neg\varphi) \in I$. So, either $\iota({\sim}\varphi) = \mathbf{0}$ or $\iota({\sim}\varphi) = x$ such that $x \neq \mathbf{0}$, analogous to (1) we get in both cases that $\iota(\varphi) \in F$. By symmetry we complete the proof. $\square$

Corollary 5.4.6. *When the operator* ^{*_r} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, ^{*_r})}$*, then it satisfies the American Plan, whenever* $n \geq 3$.

5.5 The Symmetric Operator

The second connective we introduce is what we call the *symmetric* operator ^{*_s}, which behaves as the negation in a finite Łukasiweicz logic. The n-valued Łukasiweicz logic can be semantically defined as $\mathbf{L}(\mathbb{LV}_n, \{\mathbf{1}\})$ where we have $\mathbb{LV}_n = \langle \mathbf{A}, \wedge, \vee, ^*, \Rightarrow \rangle$, where $\mathbf{A} = \{\mathbf{0}, \frac{1}{n}, ..., \frac{n-1}{n}, \mathbf{1}\}$ is a total order and for any $x \in \mathbf{A}$ we have $x^* = \mathbf{1} - x$. It

Figure 5.1: Lattice $(\mathbb{T}_4, ^{*_s})$



is easy to check that ^{*_s} and * are extensionally equivalent. Intuitively, the ^{*_s}-operator switches elements which are opposite with respect to the order of the lattice.

Definition 5.5.1. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1}\rangle$ be a complete bounded distributive lattice. Then, $\mathbb{A} \oplus \top$ stands for $\mathbb{A}$ with a new top element, above all others, and $\bot \oplus \mathbb{A}$ stands for $\mathbb{A}$ with a new bottom element, below all others.*

Definition 5.5.2. *We define the symmetric operator ^{*_s} for implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattices where the underlying universe is $\{\mathbf{0}, \mathbf{1}, \ldots, n-1\}$ such that $\mathbf{0} < n-1 < n-2 < \ldots < 2 < \mathbf{1}$, inductively as follows:*

(i) Case n even. If $n = 2$, then $\mathbf{0}^{*_s} = \mathbf{1}$ and $\mathbf{1}^{*_s} = \mathbf{0}$. If $n \geq 2$, the ^{*_s}-operator is defined in $\mathbb{T}_{n+2}$ as follows. Using that $\mathbb{T}_{n+2}$ isomorphic to $\bot \oplus \mathbb{T}_n \oplus \top$, we extend the definition of ^{*_s} in $\mathbb{T}_n$ by setting $\bot^{*_s} = \top$ and $\top^{*_s} = \bot$.

(ii) Case n odd. If $n = 3$, so $\mathbb{T}_3 = \{\mathbf{0}, 2, \mathbf{1}\}$ with $(\mathbf{0} < 2 < \mathbf{1})$, then $\mathbf{0}^{*_s} = \mathbf{1}$, $2^{*_s} = 2$ and $\mathbf{1}^{*_s} = \mathbf{0}$. If $n \geq 3$, the ^{*_s}-operator is defined in $\mathbb{T}_{n+2}$ as follows. Using that $\mathcal{T}_{n+2}$ isomorphic to $\bot \oplus \mathcal{T}_n \oplus \top$, we extend the definition of ^{*_s} in $\mathcal{T}_n$ by setting $\bot^{*_s} = \top$ and $\top^{*_s} = \bot$.

By $(\mathbb{T}_n, ^{*_s})$ we will denote the implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice (whose universe has n many elements) and where we interpret ^{*_s} as negation. Notice that in any $(\mathbb{T}_n, ^{*_s})$-valued model we have $[\![\neg\varphi]\!] = [\![\varphi]\!]^{*_s}$, i.e., ^{*_s} is interpreted as negation when evaluating $(\mathbb{T}_n, ^{*_s})$-sentences in $\mathbf{V}^{(\mathbb{T}_n, ^{*_s})}$.

Theorem 5.5.3. *For any $(\mathbb{T}_n, ^{*_s})$-lattice with $n > 2$, there is a formula $\varphi \in \mathcal{L}_{\in}$ and a filter F, such that $\mathbf{V}^{(\mathbb{T}_n, ^{*_s})} \models_G \varphi$ and $\mathbf{V}^{(\mathbb{T}_n, ^{*_s})} \models_F \neg\varphi$.*

Proof. Fix $(\mathbb{T}_n, ^{*_s})$ with $n > 2$ and consider the sentence φ defined in the proof of Theorem 4.4.17 and the filter $Pos_{(\mathbb{T}_n)}$. $\square$

This theorem shows that every $(\mathbb{T}_n, ^{*_s})$-lattice, with more than two elements, gives rise to a paraconsistent model of NFF-ZF.

Lemma 5.5.4. *We have that* $\mathbf{V}^{(\mathbb{T}_2, ^{*_s})} = \mathbf{V}^{(\mathbb{B}_2)}$, *where* $\mathbb{B}_2$ *is the two-valued Boolean algebra, while when* $n = 3$ *we get that* $\mathbf{V}^{(\mathbb{T}_3, ^{*_s})} = \mathbf{V}^{(\mathbb{PS}_3, ^*)}$.

Lemma 5.5.5. DNI *and* CP *hold for any* $(\mathbb{T}_n, ^{*_s})$. $\square$

Corollary 5.5.6. *When the operator* ^{*_s} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, ^{*_s})}$, *then it satisfies the Australian Plan algebraically.* $\square$

Lemma 5.5.7. DNI *holds for any* $\mathbf{L}\Big((\mathbb{T}_n, \flat), Pos_n\Big)$, *but* CP *fails for any* $\mathbf{L}\Big((\mathbb{T}_n, ^{*_s}), Pos_{(\mathbb{T}_n)}\Big)$ *with* $n \geq 3$.

Proof. Similar to the proof of Lemma 5.4.3. $\square$

Corollary 5.5.8. *When the operator* ^{*_s} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, ^{*_s})}$, *then it does not satisfy the Australian Plan logically, for* $n \geq 3$. $\square$

We can conclude that, from the perspective of the Australian Plan, the operator ^{*_s} is indeed a negation in every $(\mathbb{T}_n, ^{*_s})$-valued model, if we choose to interpret DNI and CP algebraically. On the other hand, if we interpret these rules logically, the operator ^{*_s} fails to be a legitimate negation.

Lemma 5.5.9. ^{*_s} *is a contradictory-forming operator in* $\mathbf{L}\Big((\mathbb{T}_n, ^{*_s}), Pos_{(\mathbb{T}_n)}\Big)$.

Proof. Fix any $(\mathbb{T}_n, ^{*_s})$. Then define the filter $F = Pos_{(\mathbb{T}_n)}$ and the ideal $I = Neg_{(\mathbb{T}_n)}$. For (1) suppose we have an ι such that $\iota(\neg\varphi) \in F$. So, either $\iota(\neg\varphi) = \mathbf{1}$ or $\iota(\neg\varphi) = x$ such that $x \neq \mathbf{0}$. By Definition 5.5.2, in both cases $\iota(\varphi) \in I$. Thus, $\iota(\varphi) \in I$. The other side follows by symmetry. For (2) suppose we have an ι such that $\iota(\neg\varphi) \in I$ so either $\iota(\neg\varphi) = \mathbf{0}$ or $\iota(\neg\varphi) = x$ such that $x \neq \mathbf{1}$, analogous to (1) we get in both cases that $\iota(\varphi) \in F$. By symmetry we complete the proof. $\square$

Corollary 5.5.10. *When the operator* ^{*_s} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, ^{*_s})}$, *then it satisfies the American Plan.*

Thus, the American and Australian Plan disagree again, but they could be reconciled if we choose the right interpretation of the conditions of the Australian Plan. Moreover, on the model-theoretic side, the ^{*_s}-operator is more expressive than the ^{*_r}-operator, in the context of $\mathbb{T}$-lattices.

Theorem 5.5.11. *The model* $\mathbf{V}^{(\mathbb{T}_4,{}^{*_s})}$ *is faithful to* $(\mathbb{T}_4,{}^{*_s})$ *and hence loyal to* $\Big((\mathbb{T}_4,{}^{*_s}), F\Big)$*, for any filter* F*.*

Proof. Fix $(\mathbb{T}_4,{}^{*_s})$ and let $\sigma = \exists xy(x \in y \wedge x \notin y)$. We use the names $u, v \in \mathbf{V}^{(\mathbb{T}_4,{}^{*_s})}$, where $u = \{\langle \varnothing, \mathbf{1}\rangle\}$ and $v = \{\langle\langle \varnothing, \mathbf{1}\rangle, 2\rangle\}$, to witness that the sentence σ receives value 3 in $\mathbf{V}^{(\mathbb{T}_4,{}^{*_s})}$. So $[\![\sigma]\!] = (2 \wedge 2^{*_s}) = 3$. Then, by symmetry of the negation, $[\![\neg\sigma]\!] = [\![\sigma]\!]^{*_s} = 2$. □

Theorem 5.5.12. *The model* $\mathbf{V}^{(\mathbb{T}_6,{}^{*_s})}$ *is faithful to* $(\mathbb{T}_6,{}^{*_s})$ *and hence loyal to* $\Big((\mathbb{T}_6,{}^{*_s}), F\Big)$*, for any filter* F*.*

Proof. Fix $(\mathbb{T}_6,{}^{*_s})$. Then, it is easy to calculate that $[\![\sigma]\!] = 4$ and $[\![\neg\sigma]\!] = [\![\sigma]\!]^{*_s} = 3$, whereas $[\![\varphi]\!] = 2$, while $[\![\neg\varphi]\!] = [\![\varphi]\!]^{*_s} = 5$. □

These results suggest that the ^{*_s}-operator is expressive enough to match the logical expressivity of the logic of a $(\mathbb{T}_n,{}^{*_s})$-lattice with that of the set theory based on it. But we can do better, showing that we are able to obtain non-$\in$-elementarily equivalent models of set theory.

Theorem 5.5.13. *There are filters* F_1 *and* F_2 *such that*

$$(\mathbf{V}^{(\mathbb{T}_6,{}^{*_s})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{PS}_3,{}^{*})}, F_2).$$

Proof. Consider again the sentence φ as in the proof of Theorem 4.4.17 and let $F_2 = Pos_{(\mathbb{T}_3)}$. Notice that $\mathbf{V}^{(\mathbb{PS}_3,{}^{*})} \vDash_{F_2} \neg\varphi$. On the other hand, if we fix $\mathbf{V}^{(\mathbb{T}_6,{}^{*_s})}$ and consider the filter $F_1 = \{4, 3, 2, \mathbf{1}\}$, then $[\![\varphi]\!]^{*_s} = 5$. Thus, $\mathbf{V}^{(\mathbb{T}_6,{}^{*_s})} \nvDash_{F_1} \neg\varphi$. □

Notice that $\mathbf{V}^{(\mathbb{T}_6,{}^{*_s})}$ coupled with the filter $F_1 = \{4, 3, 2, \mathbf{1}\}$ is still paraconsistent, since both $[\![\sigma]\!] \in F_1$ and $[\![\neg\sigma]\!] = [\![\sigma]\!]^{*_s} \in F_1$. Hence, $\mathbf{V}^{(\mathbb{T}_6,{}^{*_s})}$ is a paraconsistent model of set theory that validates NFF-ZF, which is non-$\in$-elementarily equivalent from $\mathbf{V}^{(\mathbb{PS}_3,{}^{*})}$.

This shows that we have succeeded (at least to some degree) in finding non-$\in$-elementarily equivalent models of paraconsistent set theory, by twisting the unary operator, that interprets negation, in the signature of implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattices. Furthermore, in the next section, we will show that we can go a step further and produce an entire hierarchy of non-$\in$-elementarily equivalent models of paraconsistent set theory.

5.6 The Predecessor

In this section, we introduce the ^{*_e}-operator which we call *predecessor*. This operator is similar to that of a n-valued Post algebra. A n-valued Post algebra is a complete bounded distributive lattice $\mathbb{A} = \langle\{\mathbf{0}, \mathbf{1}, \ldots, n-1\}, \wedge, \vee, ^*, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ where $\{\mathbf{0}, \mathbf{1}, \ldots, n-1\}$ is a total order with $\mathbf{0} < n-1 < n-2 < \ldots < 2 < \mathbf{1}$ and $\mathbf{1}^* = 2$, $2^* = 3$, ... , $(n-1)^* = \mathbf{0}$, $\mathbf{0}^* = \mathbf{1}$. See (BURRIS; SANKAPPANAVAR, 1981, p. 29), for further technical details.

Definition 5.6.1. *We can define the* predecessor ^{*_e} *for any implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice where the underlying universe is $\{\mathbf{0}, \mathbf{1}, \ldots, n-1\}$ such that $\mathbf{0} < n-1 < n-2 < \ldots < 2 < \mathbf{1}$, recursively as follows. Let $\mathbf{0}^{*_e} = \mathbf{1}$, $\mathbf{1}^{*_e} = \mathbf{0}$ and for any $x \in \mathcal{W}_{\mathcal{F}}$ such that $\mathbf{0} \neq x \neq \mathbf{1}$ we define x^{*_e} to be the predecessor of x in the total order of $\mathcal{W}_{\mathcal{F}}$.*

By $(\mathbb{T}_n, ^{*_e})$ we will denote the implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice (whose universe has n many elements) and where we interpret ^{*_e} as negation. Notice that in any $(\mathbb{T}_n, ^{*_e})$-valued model we have $[\![\neg\varphi]\!] = [\![\varphi]\!]^{*_e}$, i.e., ^{*_e} is interpreted as negation when evaluating $(\mathbb{T}_n, ^{*_e})$-sentences in $\mathbf{V}^{(\mathbb{T}_n, ^{*_e})}$.

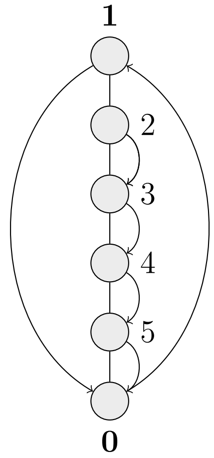
Even though $\mathbf{V}^{(\mathbb{T}_2, ^{*_e})} = \mathbf{V}^{(\mathbb{B}_2)}$, where $\mathbb{B}_2$, is the two-valued Boolean algebra, it is hard to consider the ^{*_e}-operator as a proper negation. We will show that almost every regularity property (of both plans) fails for the ^{*_e}-operator.

Lemma 5.6.2. DNI *and* CP *fail for any $(\mathbb{T}_n, ^{*_e})$, with $n \geq 4$.*

Proof. Fix any $(\mathbb{T}_n, ^{*_e})$ with $n \geq 4$. In order to see that DNI fails, just notice that $2 \not\leq 2^{*_e *_e}$. For the failure of CP, let x and y be 2 and 3, respectively. We get $y \leq x$, however, $x^{*_e} \not\leq y^{*_e}$. $\square$

Corollary 5.6.3. *When the operator ^{*_e} interprets the negation of the language of set theory in $\mathbf{V}^{(\mathbb{T}_n, ^{*_e})}$, then it does not satisfy the Australian Plan algebraically, for $n \geq 4$.*

Figure 5.2: Lattice $(\mathbb{T}_6, {}^{*_e})$



Lemma 5.6.4. DNI *and* CP *fail for any* $\mathbf{L}\Big((\mathbb{T}_n, {}^{*_e}), Pos_{(\mathbb{T}_n)}\Big)$ *with* $n \geq 4$.

Proof. Fix any $(\mathbb{T}_n, {}^{*_e})$ with $n \geq 4$. Consider an assignment ι such that $\iota(\varphi) = n - 2$ and $\iota(\psi) = n - 1$. Then $\psi \models_{Pos_{(\mathbb{T}_n)}} \varphi$, however, $\neg\varphi \not\models_{Pos_{(\mathbb{T}_n)}} \neg\psi$. Thus CP fails. Moreover, consider the same assignment ι, so $\iota(\varphi) = n - 2$, then $\varphi \not\models_{P} os_{(\mathbb{T}_n)} \neg\neg\varphi$. Hence, DNI fails as well. □

Corollary 5.6.5. *When the operator* ^{*_e} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, {}^{*_e})}$*, then it does not satisfy the Australian Plan logically, for* $n \geq 4$.

This time the logical and algebraic interpretation of the Australian Plan coincide in rejecting the predecessor as a negation for any $(\mathbb{T}_n, {}^{*_e})$-valued model with $n \geq 4$. Only $(\mathbb{T}_3, {}^{*_e})$ and $(\mathbb{T}_2, {}^{*_e})$ have a legitimate negation on the Australian Plan.

But also the American Plan agrees in rejecting the operator ^{*_e} as a genuine negation for any $(\mathbb{T}_n, {}^{*_e})$-valued model with $n > 3$, due to the failure of DNI and one of the de Morgan laws: $\neg\varphi \vee \neg\psi \models_{Pos_{(\mathbb{T}_n)}} \neg(\varphi \wedge \psi)$. Furthermore, if we suppose that truth and falsity are not exclusive, then the operator ^{*_e} fails to satisfy the conditions of Definition 5.3.1, even when $n = 3$.

Lemma 5.6.6. ^{*_e} *is not a contradictory-forming operator in* $\mathbf{L}\Big((\mathbb{T}_n, {}^{*_e}), Pos_{(\mathbb{T}_n)}\Big)$*, with* $n \geq 3$*, whenever we represent falsity by an ideal* I *such that* $I \cap Pos_{(\mathbb{T}_n)} \neq \varnothing$.

Proof. Fix a $\mathbf{L}\Big((\mathbb{T}_n, {}^{*_e}), Pos_{(\mathbb{T}_n)}\Big)$ and define the filter $F = Pos_{(\mathbb{T}_n)}$. Moreover, let I be such that $Pos_{(\mathbb{T}_n)} \cap I \neq \varnothing$. We will show that there exists an assignment ι and a formula

$\varphi \in \mathcal{L}_{Prop}$ such that (1) of Definition 5.3.1 does not hold. Indeed, consider a formula $\varphi \in \mathcal{L}_{Prop}$ and an assignment ι such that $\iota(\varphi) = n-1$, so $\iota(\neg\varphi) = \mathbf{0}$, by definition of the ^{*e}-operator. But then $\iota(\varphi) \in I$, which implies $\iota(\neg\varphi) \notin F$. $\square$

However, the American and the Australian Plan can find here a common ground, if we assume that truth and falsity are both exclusive and exhaustive. Indeed, in this case, the ^{*e}-operator becomes a contradictory-forming operator in $\mathbf{L}\big((\mathbb{T}_3, {}^{*e}), Pos_{(\mathbb{T}_3)}\big)$. But this is not the case anymore for any $\mathbf{L}\big((\mathbb{T}_n, {}^{*e}), Pos_{(\mathbb{T}_n)}\big)$ with $n \geq 4$, since ^{*e} would fail to satisfy (2) of Definition 5.3.1. Therefore, ^{*e} is acceptable for the American Plan, when it interprets a negation in $\mathbf{V}^{(\mathbb{T}_2, {}^{*e})}$ and $\mathbf{V}^{(\mathbb{T}_3, {}^{*e})}$.

Although the ^{*e}-operator can hardly satisfy any of the conditions imposed by both plans, it allows us, however, to obtain infinitely many non-$\in$-elementarily equivalent models of NFF-ZF.

Theorem 5.6.7. *Any $(\mathbb{T}_n, {}^{*e})$-valued model with $n \geq 4$ is non-$\in$-elementarily equivalent from each other.*

Proof. Take any $\mathbf{V}^{(\mathbb{T}_n, {}^{*e})}$ and $\mathbf{V}^{(\mathbb{T}_m, {}^{*e})}$ with $m > n > 3$. Now, consider again φ as defined in the proof of Theorem 4.4.17 and let ${}^{*e}_{k}$ stand for a sequence of k-many symbols ^{*e}. The same convention can be set up for the negation of the language of set theory: by $\neg_k$ we mean a sequence of k-many symbols $\neg$. Then we can calculate that $(\llbracket\varphi\rrbracket^{*e}_{n-3})^{(\mathbb{T}_n, {}^{*e})} = \mathbf{0}$, however, $(\llbracket\varphi\rrbracket^{*e}_{n-3})^{(\mathbb{T}_m, {}^{*e})} \neq \mathbf{0}$. Hence, $\mathbf{V}^{(\mathbb{T}_n, {}^{*e})} \not\models_{Pos_{(\mathbb{T}_n)}} \neg_{n-3}\varphi$ and $\mathbf{V}^{(\mathbb{T}_m, {}^{*e})} \models_{Pos_{(\mathbb{T}_m)}} \neg_{n-3}\varphi$. $\square$

We now show that every $(\mathbb{T}_n, {}^{*e})$-valued model with more than four elements is a paraconsistent model of set theory. We will use the following abbreviation: $\perp =_{df.} x \neq x$. Notice that the formula $\perp$ does not have parameters and is thus a legitimate formula in the language of set theory $\mathcal{L}_{\in}$.

Theorem 5.6.8. *For any $(\mathbb{T}_n, {}^{*e})$ with $n \geq 4$, there exists a $\varphi \in \mathcal{L}_{\in}$ and a filter F, such that $\mathbf{V}^{(\mathbb{T}_n, {}^{*e})} \models_F \varphi$ and $\mathbf{V}^{(\mathbb{T}_n, {}^{*e})} \models_F \neg\varphi$.*

Proof. Fix any $(\mathbb{T}_n, {}^{*e})$ with $n \geq 4$. Then we can define the following sentence in the language of set theory;

$$\gamma = \exists x \exists y \big((x \in y \vee x \notin y) \wedge \neg(x \in y \to \perp \vee x \notin y \to \perp)\big).$$

It is readily shown that $[\![\gamma]\!] = 2$, where 2 is the co-atom of $(\mathbb{T}_n, ^{*e})$. Consider an arbitrary name u and $v = \{\langle u, 2\rangle\}$ as witnesses. Then:

$$\begin{aligned}[\![\gamma]\!] &\geq (2 \vee 3) \wedge (2 \Rightarrow_t \mathbf{0} \vee 3 \Rightarrow_t \mathbf{0})^{*e} \\ &= 2 \wedge (\mathbf{0} \vee \mathbf{0})^{*e} \\ &= 2.\end{aligned}$$

Suppose towards contradiction that $[\![\gamma]\!] = \mathbf{1}$. Hence, we there exist two $u, v \in \mathbf{V}^{(\mathbb{T}_n, ^{*e})}$ such that $[\![u \in v \vee u \notin v]\!] = \mathbf{1} = [\![\neg(u \in v \to \bot \vee u \notin v \to \bot)]\!]$. So either $[\![u \in v]\!] = \mathbf{1}$ or $[\![u \notin v]\!] = \mathbf{1}$. Suppose the former, thus $[\![u \notin v]\!] = \mathbf{0}$. Then;

$$\begin{aligned}&[\![(x \in y \vee x \notin y) \wedge \neg(x \in y \to \bot \vee x \notin y \to \bot)]\!] \\ &\qquad = (\mathbf{1} \vee \mathbf{0}) \wedge (\mathbf{1} \Rightarrow_t \mathbf{0} \vee \mathbf{0} \Rightarrow_t \mathbf{0})^{*e} \\ &\qquad = \mathbf{1} \wedge (\mathbf{0} \vee \mathbf{1})^{*e} \\ &\qquad = \mathbf{0}.\end{aligned}$$

Therefore, there are no $u, v \in \mathbf{V}^{(\mathbb{T}_n, ^{*e})}$ such that

$$[\![(u \in v \vee u \notin v)]\!] = \mathbf{1} = [\![\neg(u \in v \to \bot \vee u \notin v \to \bot)]\!].$$

The second case follows by an analogous argument. Hence, $[\![\gamma]\!]^{(\mathbb{T}_n, ^{*e})} \neq \mathbf{1}$. We may conclude $[\![\gamma]\!] = 2$. Moreover, $[\![\gamma]\!]^{*e}$ equals 3, and thus $[\![\gamma]\!]^{*e} \neq \mathbf{0}$. Hence, $\mathbf{V}^{(\mathbb{T}_n, ^{*e})} \models_{Pos_{(\mathbb{T}_n)}} \varphi$ and $\mathbf{V}^{(\mathbb{T}_n, ^{*e})} \models_{Pos_{(\mathbb{T}_n)}} \neg\varphi$. □

Theorem 5.6.8 and Theorem 5.6.7 show that every $(\mathbb{T}_n, ^{*e})$-valued model, with more than three elements, is a paraconsistent model of NFF-ZF and that each of these models is non-∈-elementarily equivalent from each other.

In the case of $\mathbf{V}^{(\mathbb{T}_3, ^{*e})}$ we do not obtain a paraconsistent consequence relation since the corresponding propositional logic is some fragment of IPL. Actually, $\mathbf{L}(\mathbf{V}^{(\mathbb{T}_3, ^{*e})}, \{1\}) = \mathbf{L}(\mathbf{V}^{(\mathbb{T}_3, ^{*p})}, \{1\})$. So as shown in Chapter 4.4.3 we know that all the axioms of IPL hold in $\mathbf{L}(\mathbf{V}^{(\mathbb{T}_3, ^{*e})}, \{1\})$, but MP fails. Moreover, by Theorem 4.4.23, we know that the corresponding propositional logic of $\mathbf{V}^{(\mathbb{T}_3, ^{*e})}$, given the positive filter on $\mathbb{T}_3$, is classical i.e., $\mathbf{L}(\mathbf{V}^{(\mathbb{T}_3, ^{*e})}, Pos_{(\mathbb{T}_3)}) = \mathsf{CPL}$.

Theorem 5.6.9. *The model* $\mathbf{V}^{(\mathbb{T}_n,^{*e})}$ *with* $n \geq 2$ *is faithful to* $(\mathbb{T}_n,^{*e})$ *and hence loyal to* $\left((\mathbb{T}_n,^{*e}), F\right)$*, for any filter* F.

Proof. Let $n = 2$. Then it is easy to see that $(\mathbb{T}_2,^{*e})/Pos_{(\mathbb{T}_2)}$ is the two-valued Boolean algebra. Hence, $\mathbf{V}^{(\mathbb{T}_2,^{*e})}$ is trivially faithfull to $(\mathbb{T}_2,^{*e})$.
Let $n > 2$. Fix a $(\mathbb{T}_n,^{*e})$. We know that $[\![\mathsf{Extensionality}]\!] = \mathbf{1}$ and similarly we get $[\![\mathsf{Extensionality}]\!]^{*e} = \mathbf{0}$. So, we are done if we can find a sentence $\varphi \in \mathcal{L}_\in$ such that $[\![\varphi]\!] = 2$, since we can access the remaining elements of the universe of $(\mathbb{T}_n,^{*e})$ by negating this sentence iteratively. We use the same sentence γ of the proof of Theorem 5.6.8. Clearly, $[\![\gamma]\!] = 2$. Moreover, for every $a \in \mathbf{A}$ such that $a \neq \mathbf{1}, 2, \mathbf{0}$ we have that $[\![\gamma]\!]^{*e}_k = a$, for some $k < (n-2)$. □

This result shows that indeed every $(\mathbb{T}_n,^{*e})$-valued model is faithful to $(\mathbb{T}_n,^{*e})$ and that the corresponding propositional logic of the set-theoretic models matches the logic of the underlying lattices. So from a model-theoretic perspective, the ^{*e}-operator performs ideally.

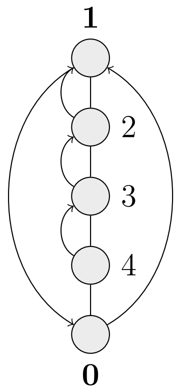
Nevertheless, we would not associate the ^{*e}-operator intuitively with a negation-like operator since almost every regularity property is violated.[2] But we believe that precisely for this reason, the ^{*e}-operator is capable of providing enough complexity to our language so that we can define an entire hierarchy of non-$\in$-elementarily equivalent models for non-classical set theories. In conclusion, it seems that the complexity of the ^{*e}-operator stems from the fact that; by applying the predecessor to the co-atom of $(\mathbb{T}_n,^{*e})$ can access every element in the universe of the lattice. We want to explore this fact even further in the following section, by introducing an operator that behaves complementary to the predecessor.

5.7 The Successor

In this section, we introduce the ^{*c}-operator, which we call *successor*. This negation is defined as the complement of the predecessor. So, intuitively speaking, instead of bringing a sentence closer to falsity (the bottom element of a lattice) by repeatedly

[2]To be fair with the intuitiveness of the predecessor. In the case of the two-element and three-element $(\mathbb{T},^{*e})$-lattice it seems that the ^{*e}-operator behaves as a genuine negation, given that ^{*e} imitates the behavior of the classical negation in the two-element lattice and the meet-complement in the case of the three-element lattice.

Figure 5.3: Lattice $(\mathbb{T}_5,^{*_c})$.



negating a sentence, this operator does the inverse, i.e., by negating a sentence we bring it closer to truth (top element of the lattice).

Definition 5.7.1. *We can define the* successor ^{*_c} *for any implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice where the underlying universe is $\{\mathbf{0}, \mathbf{1}, \ldots, n-1\}$ such that $\mathbf{0} < n-1 < n-2 < \ldots < 2 < \mathbf{1}$, recursively as follows. Let $\mathbf{0}^{*_c} = \mathbf{1}$, $\mathbf{1}^{*_c} = \mathbf{0}$ and for any $x \in \mathcal{W}_{\mathcal{F}}$ such that $\mathbf{0} \neq x \neq \mathbf{1}$ we define x^{*_c} to be the successor of x in the total order of $\mathcal{W}_{\mathcal{F}}$.*

By $(\mathbb{T}_n,^{*_c})$ we will denote the implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattice (whose universe has n many elements) and where we interpret ^{*_c} as negation. Notice tha in any $(\mathbb{T}_n,^{*_c})$-valued model we have $[\![\neg\varphi]\!] = [\![\varphi]\!]^{*_c}$, i.e., ^{*_c} is interpreted as negation when evaluating $(\mathbb{T}_n,^{*_c})$-sentences in $\mathbf{V}^{(\mathbb{T}_n,^{*_c})}$.

Notice that $\mathbf{V}^{(\mathbb{T}_2,^{*_c})} = \mathbf{V}^{(\mathbb{B}_2)}$, where $\mathbb{B}_2$, is the two-valued Boolean algebra. As in the case of the ^{*_e}-operator it is hard to consider the ^{*_c}-operator as a proper negation. We will show that every regularity property of both plans fails for this operator.

Lemma 5.7.2. DNI *fails for any $(\mathbb{T}_n,^{*_c})$, with $n \geq 3$ and* CP *fails for any $(\mathbb{T}_n,^{*_c})$, with $n \geq 4$.*

Proof. Fix any $(\mathbb{T}_n,^{*_c})$ with $n \geq 3$. In order to see that DNI fails, just notice that $2 \not\leq 2^{*_c*_c}$, since $2^{*_c*_c} = \mathbf{0}$. For the failure of CP, let x and y be 2 and 3, respectively. We get $y \leq x$, however, $x^{*_c} \not\leq y^{*_c}$. $\square$

Corollary 5.7.3. *When the operator ^{*_c} interprets the negation of the language of set theory in $\mathbf{V}^{(\mathbb{T}_n,^{*_c})}$, then it does not satisfy the Australian Plan algebraically, for $n \geq 3$.*

Lemma 5.7.4. DNI *and* CP *fail for any* $\mathbf{L}\Big((\mathbb{T}_n, {}^{*c}), Pos_{(\mathbb{T}_n)}\Big)$*, with* $n \geq 3$.

Proof. Fix any $(\mathbb{T}_n, {}^{*c})$ with $n \geq 3$. In order to see that DNI fails, consider an assignment ι such that $\iota(\varphi) = 2$. Then $\varphi \not\models_{Pos_{(\mathbb{T}_n)}} \neg\neg\varphi$. For the failure of CP, take an assignment ι such that $\iota(\varphi) = \mathbf{1}$ and $\iota(\psi) = 2$. Then, applying Definition 5.7.1 we have $\varphi \models_{Pos_{(\mathbb{T}_n)}} \psi$, however, $\neg\psi \not\models_{Pos_{(\mathbb{T}_n)}} \neg\varphi$. $\square$

Corollary 5.7.5. *When the operator* ^{*c} *interprets the negation of the language of set theory in* $\mathbf{V}^{(\mathbb{T}_n, {}^{*c})}$*, then it does not satisfy the Australian Plan logically, for* $n \geq 3$.

The successor, unsurprisingly, does neither satisfy the logical nor the algebraic interpretation of the Australian Plan. Both interpretations coincide in rejecting the successor as a negation for any $(\mathbb{T}_n, {}^{*c})$-valued model with $n \geq 3$. Only the $(\mathbb{T}_2, {}^{*c})$-valued model (which is the two-valued Boolean valued model) has a legitimate negation on the Australian and American Plan. Moreover, the American Plan agrees in rejecting the ^{*c}-operator as legitimate negation in any $(\mathbb{T}_n, {}^{*c})$-valued model with $n \geq 3$, even supposing that truth and falsity are not exclusive.

Lemma 5.7.6. ^{*c} *is not a contradictory-forming operator in* $\mathbf{L}\Big((\mathbb{T}_n, {}^{*c}), F\Big)$*, with* $n \geq 3$*, whenever we represent falsity by an ideal* I *and truth by a filter* F *such that* $I \cap F \neq \varnothing$.

Proof. Fix a $\mathbf{L}\Big((\mathbb{T}_n, {}^{*c}), F\Big)$ where I and F are such that $I \cap F \neq \varnothing$. We will show that there exists an assignment ι and a formula $\varphi \in \mathcal{L}_{Prop}$ such that (2) of Definition 5.3.1 does not hold. Indeed, consider a formula $\varphi \in \mathcal{L}_{Prop}$ and an assignment ι such that $\iota(\varphi) = 2$, so $\iota(\neg\varphi) = \mathbf{1}$, by definition of the ^{*c}-operator. Notice that we have $\iota(\varphi) \in F$, since otherwise $F \cap I = \varnothing$. But then $\iota(\neg\varphi) \in F$, which implies $\iota(\neg\varphi) \notin I$. $\square$

This time, the American and the Australian Plan match perfectly. Both plans accept the ^{*c}-operator as legitimate negation in the case of $\mathbf{V}^{(\mathbb{T}_2, {}^{*c})}$ and reject the ^{*c}-operator in the case of the remaining $(\mathbb{T}_n, {}^{*c})$-valued models.

We go on to show that the ^{*c}-operator allows us to obtain infinitely many paraconsistent models of NFF-ZF.

Theorem 5.7.7. *For any* $(\mathbb{T}_n, {}^{*c})$ *with* $n \geq 3$*, there exists a* $\varphi \in \mathcal{L}_{\in}$ *and a filter* F*, such that* $\mathbf{V}^{(\mathbb{T}_n, {}^{*c})} \models_F \varphi$ *and* $\mathbf{V}^{(\mathbb{T}_n, {}^{*c})} \models_F \neg\varphi$.

Proof. Fix a $(\mathbb{T}_n, ^{*c})$ with $n \geq 3$. Consider the sentence σ of the proof of Theorem 5.5.11. We can readily calculate that $[\![\sigma]\!] = 2$. Hence, $[\![\neg\sigma]\!] = [\![\sigma]\!]^{*c} = \mathbf{1}$, by the Definition 5.7.1. Therefore, $\mathbf{V}^{(\mathbb{T}_n, ^{*c})} \models_{Pos_{(\mathbb{T}_n)}} \sigma$ and $\mathbf{V}^{(\mathbb{T}_n, ^{*c})} \models_{Pos_{(\mathbb{T}_n)}} \neg\sigma$. □

Furthermore, we can show that any $(\mathbb{T}_n, ^{*c})$-valued model is non-$\in$-elementarily equivalent from each other. For the following proof we will use the following sentences;

$$\begin{aligned}
&\tau_0 = \exists xy\Big(x \in y \wedge x \notin y \wedge \neg(x \notin y)\Big),\\
&\tau_1 = \exists xy\Big(x \in y \wedge x \notin y \wedge \neg(x \notin y) \wedge \neg\neg(x \notin y)\Big),\\
&\tau_2 = \exists xy\Big(x \in y \wedge x \notin y \wedge \neg(x \notin y) \wedge \neg\neg(x \notin y) \wedge \neg\neg\neg(x \notin y)\Big),\\
&\ldots
\end{aligned}$$

Notice that we can resume these sentences in the form of the schema of sentences τ_n as follows;

$$\tau_n = \exists xy\Big(x \in y \wedge x \notin y \wedge ... \wedge \neg_{n+1}(x \notin y)\Big).$$

Theorem 5.7.8. *There are filters F_1 and F_2 such that*

$$(\mathbf{V}^{(\mathbb{T}_n, ^{*c})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{T}_m, ^{*c})}, F_2),$$

where $m > n \geq 2$.

Proof. Suppose $n = 2$ and consider the sentence σ of the proof of Theorem 5.5.11. Then $[\![\sigma]\!]^{(\mathbb{T}_n, ^{*c})} = \mathbf{0}$, however, for any $\mathbf{V}^{(\mathbb{T}_m, ^{*c})}$ with $m > 2$, we have $[\![\sigma]\!]^{(\mathbb{T}_m, ^{*c})} = 2$. So, $\mathbf{V}^{(\mathbb{T}_m, ^{*c})} \models_{Pos_{(\mathbb{T}_m)}} \sigma$, however, we get $\mathbf{V}^{(\mathbb{T}_2, ^{*c})} \not\models_{Pos_{(\mathbb{T}_2)}} \sigma$. So we know that $\mathbf{V}^{(\mathbb{T}_2, ^{*c})}$ is non-$\in$-elementairly equivalent from each $\mathbf{V}^{(\mathbb{T}_m, ^{*c})}$ where $m > 2$. Now, suppose $n > 2$. Then we can readily calculate that $[\![\tau_{n-3}]\!]^{(\mathbb{T}_n, ^{*c})} = \mathbf{0}$ and $[\![\tau_{n-3}]\!]^{(\mathbb{T}_m, ^{*c})} \neq \mathbf{0}$. Hence, $\mathbf{V}^{(\mathbb{T}_n, ^{*c})} \not\models_{Pos_{(\mathbb{T}_n)}} \tau_{n-3}$, however, $\mathbf{V}^{(\mathbb{T}_m, ^{*c})} \models_{Pos_{(\mathbb{T}_m)}} \tau_{n-3}$. □

Theorem 5.7.7 and Theorem 5.7.8 show that every $(\mathbb{T}_n, ^{*c})$-valued model, with more than two elements, is a paraconsistent model of NFF-ZF and that each of these models is non-$\in$-elementarily equivalent from each other. Moreover, we can show, as well, that each $(\mathbb{T}_n, ^{*c})$-valued model is non-$\in$-elementarily equivalent from each $(\mathbb{T}_n, ^{*x})$-valued model of the same cardinality, where ^{*_x} is a placeholder for any of the first three operators introduced in this chapter, i.e., ${}^{*_x} \in \{{}^{*_r}, {}^{*_e}, {}^{*_s}\}$.

Theorem 5.7.9. *There are filters F_1 and F_2 such that*

$$(\mathbf{V}^{(\mathbb{T}_n, {}^{*_c})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{T}_n, {}^{*_x})}, F_2),$$

where $n > 2$.

Proof. **Case 1:** let $n = 3$. Consider the positive filter on $\mathbb{T}_3$ and the sentence σ (defined in Theorem 5.5.11). Then we have $[\![\sigma]\!]^{(\mathbb{T}_3, {}^{*_c})} \neq \mathbf{0}$ and $[\![\sigma]\!]^{(\mathbb{T}_3, {}^{*_e})} = \mathbf{0}$. Now, consider the sentence $\neg\neg\sigma$. Then we get $[\![\neg\neg\sigma]\!]^{(\mathbb{T}_3, {}^{*_c})} = \mathbf{0}$, however, $[\![\neg\neg\sigma]\!]^{(\mathbb{T}_3, {}^{*_s})} = [\![\neg\neg\sigma]\!]^{(\mathbb{T}_3, {}^{*_r})} \neq \mathbf{0}$.
Case 2: let $n > 3$ and consider the positive filter on $\mathbb{T}_n$. Then we calculate $[\![\tau_{n-4}]\!]^{(\mathbb{T}_n, {}^{*_c})} = n - 1$ $\Big($where $n - 1$ is the atom of $(\mathbb{T}_n, {}^{*_c})\Big)$ and $[\![\tau_{n-4}]\!]^{(\mathbb{T}_n, {}^{*_e})} = \mathbf{0}$. Now we will address the remaining negations. By Theorem 5.7.8 we know that $[\![\tau_{n-3}]\!]^{(\mathbb{T}_n, {}^{*_c})} = \mathbf{0}$. Furthermore, we readily calculate that $[\![\tau_{n-3}]\!]^{(\mathbb{T}_n, {}^{*_s})} = [\![\tau_{n-3}]\!]^{(\mathbb{T}_n, {}^{*_r})} \neq \mathbf{0}$. The desiderata follows immediately. $\square$

We conclude this section by pointing out that, as in the case of the predecessor, we have a perfect match between the propositional logic of each set-theoretical model with the propositional logic of the underlying lattice.

Theorem 5.7.10. *The model $\mathbf{V}^{(\mathbb{T}_n, {}^{*_c})}$ with $n \geq 2$ is faithful to $(\mathbb{T}_n, {}^{*_c})$ and hence loyal to $\Big((\mathbb{T}_n, {}^{*_c}), F\Big)$, for any filter F.*

Proof. Notice that for any $(\mathbb{T}_n, {}^{*_c})$-valued model we have $[\![\mathsf{Extensionality}]\!] = \mathbf{1}$ and $[\![\mathsf{Extensionality}]\!]^{*_c} = \mathbf{0}$. Let $n = 3$. Then we are done, since for sentence σ (defined in Theorem 5.5.11), we have $[\![\sigma]\!] = 2$. So let $n > 3$. Then we can readily calculate that $[\![\tau_{n-4}]\!] = n - 1$. Now, for any $a \in \mathbf{A}/\{\mathbf{1}, \mathbf{0}, n-1\}$ we have $[\![\tau_{n-4}]\!]_k^{*_c} = a$, for some $k < (n-2)$. $\square$

5.8 A Minimal Account of Negation

In this section, we will give a summary of the previous section. We have depicted the main results of the previous section in Table 4.2. Moreovoer, we have shown that the ^{*_r}-operator is not a negation according to the Australian Plan, while it is for the

Table 5.2: Operators and their respective regularity properties

Operators	American Plan	Australian Plan (algebra)	Australian Plan (logic)
Reflexive ($*^r$)	✓	$\times$ (for $n \geq 3$)	$\times$ (for $n \geq 4$)
Symmetric ($*^s$)	✓	$\times$ (for $n \geq 3$)	✓
Predecessor ($*^e$)	$\times$ (for $n \geq 4$)	$\times$ (for $n \geq 4$)	$\times$ (for $n \geq 4$)
Successor ($*^c$)	$\times$ (for $n \geq 3$)	$\times$ (for $n \geq 3$)	$\times$ (for $n \geq 3$)

American Plan. Regarding the $*^s$-operator, both plans agree in considering it a negation (at least if we consider the logical version of the Australian Plan). Finally, the $*^e$-operator and $*^c$-operator are not a negation for neither the American nor the Australian Plan.

On the other hand, $*^e$ and $*^c$, were the only operators that allowed us to obtain enough expressive power to be able to distinguish infinitely many paraconsistent models of NFF-ZF. Moreover, Theorem 5.5.13, Theorem 5.6.7 and Theorem 5.7.8 offer the first examples of independence proofs within ZF-like paraconsistent set theory. Indeed, we have been able to show that there are two paraconsistent models of the form $\mathbf{V}^{(\mathbb{A})}$ and $\mathbf{V}^{(\mathbb{B})}$, each validating the negation-free fragment of ZF, and a formula $\varphi \in \mathcal{L}_\in$ such that $\mathbf{V}^{(\mathbb{A})} \vDash_F \varphi$ and $\mathbf{V}^{(\mathbb{B})} \nvDash_F \varphi$, for some filter F. For a much more sophisticated and state-of-the-art elaboration of this topic, i.e., independence proofs in the context of paraconsistent models of set theory, consider (TARAFDER; VENTURI, 2021).

Let us now turn to our own account of negation, inspired by the algebraic framework discussed. Firstly, we have noticed a trade-off between expressivity of the language and regularity properties that our operators satisfy, which makes it difficult to pin down objective criteria for being a negation. Secondly, we claim that a negation, at least in the algebraic context, is simply a unary operator defined on a lattice, which has the ability to separate the elements which belong to the universe of a given lattice. Thus more generally, a negation *tells apart points in a relational structure*.

On this account, negation is a device which linguistically allows us to distinguish semantic situations from each other, without committing to their validity.

This notion is of course weaker than the idea of negation as a switch operator between truth and falsity (like the American Plan), since the separation offered by negation might not commit to any change in truth-value. Moreover, this notion is also weaker than a conception of negation as a ruling out operator (like the Australian Plan), since, again, if telling things apart does not commit us to their validity, then we can separate semantic facts linguistically, without excluding any of them. From this perspective, it is then easy to see that $*_r$ cannot be considered a negation, while $*_s$, $*_e$ and $*_c$ in principle, can.

The ability to tell things apart is a minimal requirement on negation that may offer a common ground for both plans. Indeed, they both base their analysis of negation on more primitive concepts, which express the difficulty of considering together things that are told apart from negation. On the American side we find contradictoriness, logical in nature, while on the Australian side incompatibility, that, in turn can be understood epistemically or metaphysically. It is interesting to notice that, as it is the case for both plans, also this minimal requirement yields a classical negation when restricted to the classical case. In this context, this means that when we consider the two-element algebra, the ability to tell things apart forces us to switch the two elements of the algebra: i.e., truth and falsity. We call this the minimality priniciple.

So, let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ be a complete bounded distributive lattice. We say that * satisfies the minimality priniciple, if

$$\mathbf{0}^* = \mathbf{1} \text{ and } \mathbf{1}^* = \mathbf{0}.$$

This leads us to the following definition which is the core of our algebraic account of negation:

Definition 5.8.1. *Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1} \rangle$ be a complete bounded distributive lattice. We say that * is a legitimate negation on the algebraic account, if*

(i) * satisfies the minimality priniciple and

(ii) for any $a \in \mathbf{A}$ there exists some $b \in \mathbf{A}$ such that $a^* = b$ where $a \neq b$.

Furthermore, we believe that it might be useful to distinguish between a *strict* formulation of the algebraic account and a *tolerant* formulation of the algebraic account.

Where the *strict* formulation is equivalent to Definition 5.8.1 and the *tolerant* formulation considers the minimality priniciple the only requirement that a unary operator has to satisfy to be considered a legitimate negation. So on the tolerant account, we can consider the reflexive operator $*^r$ a negation, whereas, on the strict account we can not.

The *algebraic* account, therefore, is meant to offer a more fundamental take on negation. But the broader the look is, the less precise the criteria we can propose are; and indeed, the only negation that we can discard in this chapter is the reflexive one (at least from the perspective of the *strict* formulation of the algebraic account). However, on the positive side, we believe that the algebraic account and in general the algebraic perspective can offer a more fine-grained analysis of logical properties; as it has been the case in refining the conditions imposed by both plans.

One may object that an algebraic context, however, is too wide to be able to capture essential features of the logical vocabulary. In other terms, the relativity engendered by the plethora of algebraic structures at disposal gets to the point where no regularity properties can be ascribed anymore to a logical constant. We believe that this is not the case. On the contrary, in purely algebraic terms we can offer a general distinction between different kinds of negation; for example between a *classical-like* negation and a *intuitionistic-like* negation. The former is expressed by a negation that is cyclical: not only does it form loops between truth-values, but it also allows us to come back to every element of the loop; while the latter is a negation that does not have this cyclical property. The classicality is given here by the exhaustiveness of a negation, and therefore an intuitionistic negation is just one which allows the logical space not to be uniformly covered by the act of negation. As an example, notice that the $*^s$-operator is a classical-like negation, while the $*^e$-operator and the $*^c$ is an intuitionistic-like negation.

Because of the minimal requirements we impose on negation, our perspective is quite comprehensive. For this reason, there exist very interesting connections with our account of negation and logical pluralism. Consider (JOCKWICH MARTINEZ; VENTURI, 2021, Section 4.3) for a more detailed discussion of this topic.

Now, a word on why we believe that both plans fall short in describing what a negation really is. Take for instance, the case of paraconsistency, which is supposed to be an important case study for both plans. Indeed, it seems difficult to express a form of paraconsistency by structural properties of negation and this is, we believe, an intrinsic difficulty of both the American and Australian Plan: the need to account, through negation, for logical phenomena which, unfortunately, are hardly expressible without resorting to inferential aspects of a logical system. This observation is not surprising, since the main tenant of paraconsistency is the refutation of ECQ, which does not involve only negation but also implication and the consequence relation.

But does the invalidity of $(\varphi \wedge \neg\varphi) \to \psi$ depend necessarily on the properties of the negation involved? We believe not. Indeed, it is possible to construct an example where $\mathbf{L}_0$ is a logic in the same signature of CPL, with all connectives defined as in CPL except implication, which behaves like $\perp$: for all formulas φ and ψ, $\varphi \to \psi$ is always false. By definition $\mathbf{L}_0$ is paraconsistent, but there is no formula φ such that $\varphi \wedge \neg\varphi$ is true. But then, how is it possible to capture the paraconsistency of the logic $\mathbf{L}_0$ in terms of specific properties of the negation? We believe that this is impossible because in $\mathbf{L}_0$ the negation *is* the classical negation. In other terms, the stronger claim that we are supporting here – suggested by an algebraic viewpoint – is that paraconsistency is a logical property which can not be predicated only of negation but of a logical system in its entirety. Therefore a paraconsistent setting might not offer the right logical environment where to test properties meant to capture the nature of negation.

Chapter 6

An Extended Class of Models for Non-classical Set Theories

Summary

In this chapter, we will present two new methods of constructing algebra-valued models for non-classical set theories. On the one hand, we show that we can expand the signature of well-known DRI-algebras with some unary operator and that these algebras give rise to non-classical models of set theory. In particular, we will use totally-ordered Heyting algebras and expand these algebras with the reflexive operator. We show that the resulting algebras give rise to paraconsistent and paracomplete models of set theory. On the other hand, we show that we can construct algebra-valued for non-classical set theories by modifying the interpretation map for membership and identity. In particular, we will build an LP-model of ZFC. Finally, we explore the mathematical tractability of the modified interpretation map.

6.1 Totally-ordered Heyting-valued Models

In this section, we explore a broader class of models for non-classical set theories. In particular, we build algebra-valued models for non-classical set theories which are paraconsistent, paracomplete or both. Our constructions will be based on the class of totally ordered complete bounded distributive lattices of the form $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_h, \mathbf{0}, \mathbf{1} \rangle$ where $\mathbf{A}$ is of finite size and where we interpret the binary operator $\Rightarrow_h$ as an Heyting conditional. We will refer to the class of such lattices as *Heyting-implication* lattices.

The crux of this approach is that we know that these lattices are DRI-algebras (remember that the properties that characterize a DRI-algebra depend only on the binary operator $\Rightarrow$ and the meet $\wedge$) and that the respective algebra-valued models $\mathbf{V}^{(\mathbb{A})}$ will satisfy $\mathcal{BQ}_\varphi$ for any $\varphi \in \mathsf{NFF\text{-}ZF}$ (given that these lattices are a subclass of Heyting algebras). Then we can expand these lattices with a unary operator that gives rise to a non-classical logic and consequently we build non-classical models of set theory.

We define these lattices as follows:

Definition 6.1.1. *We call a structure $\langle \mathbf{A}, \wedge, \vee, \Rightarrow_h, \mathbf{0}, \mathbf{1}\rangle$ a Heyting-implication lattice if*

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ *is a complete bounded distributive lattice,*

(ii) $\mathbf{A}$ *is totally ordered and finite,*

(iii) the binary operator $\Rightarrow_h$ is defined for any $a, b \in \mathbf{A}$ as:

$$a \Rightarrow_h b = \begin{cases} \mathbf{1}, & \text{if } a \leq b; \\ \\ b, & \text{if } a > b. \end{cases}$$

It follows immediately that;

Theorem 6.1.2. *Let $\mathbb{A}$ be a Heyting-implication lattice. Then we have $\mathbf{V}^{(\mathbb{A})} \models_D$ NFF-*ZF *for any filter D.*

Proof. It is easy to check that every Heyting-implication lattice $\mathbb{A}$ is a DRI-algebra and it is a fact that $\mathcal{BQ}_\varphi$ holds for every negation-free formula φ . The desiderata follows immediately by application of Theorem 2.4.4. □

Now, we go on to expand the signature of Heyting-implication lattices with a unary operator * that we will be interpreted as negation when we construct algebra-valued models on top. Of course, we are free in principle to explore different unary operator to construct different non-classical models of set theory. However, having in mind Chapter 5.8, the only requirement that we make on our choice of the unary operator * is that it has to satisfy the minimality principle. Moreover, we will show that any Heyting-implication lattice that is expanded with the ^{*_r}-operator gives rise to paraconsistent and paracomplete logics (depending on the set of designated values) and thus to models of non-classical set theory.

6.1.1 Models of Paraconsistent Set Theory

In this subsection, we will construct models of paraconsistent set theory on top of Heyting-implication lattices. In particular, we will expand our Heyting-implication lattices with the ^{*_r}-operator. Moreover, we refer to these expanded lattices as *reflexive* Heyting-implication lattices.

Definition 6.1.3. *We call a structure* $\langle \mathbf{A}; \wedge, \vee, \Rightarrow_h, ^{*_r}, \mathbf{0}, \mathbf{1} \rangle$ *a reflexive Heyting-implication lattice, if:*

(i) $\langle \mathbf{A}; \wedge, \vee, \Rightarrow_h, \mathbf{0}, \mathbf{1} \rangle$ *is a Heyting-implication lattice,*

(ii) the algebraic operator ^{*_r} *is defined as follows:*

$$a^{*_r} = \begin{cases} \mathbf{0}, & \text{if } a = \mathbf{1}; \\ a, & \text{if } a \in A \setminus \{\mathbf{1}, \mathbf{0}\}; \\ \mathbf{1}, & \text{if } a = \mathbf{0}. \end{cases}$$

By $(\mathbb{A}, ^{*_r})$ we will denote a reflexive Heyting-implication lattice where we interpret ^{*_r} as negation. Notice, that the reflexive Heyting-implication-negation lattice $(\mathbb{A}, ^{*_r})$ with two elements modulo the top filter coincides with the two-valued Boolean algebra. So;

Lemma 6.1.4. *Let* $(\mathbb{A}_2, ^{*_r})$ *be the reflexive Heyting-implication lattice with two elements. Then we have* $\mathbf{L}\Big((\mathbb{A}_2, ^{*_r}), \{\mathbf{1}\}\Big) = \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A}_2, ^{*_r})}, \{\mathbf{1}\}\Big) = \mathsf{CPL}$.

Proof. This follows immediately from the fact that $(\mathbb{A}_2, ^{*_r})/\{\mathbf{1}\}$ is the two-valued Boolean algebra $2 = \{\mathbf{0}, \mathbf{1}\}$. □

We go on to show that every reflexive Heyting-implication lattice $(\mathbb{A}, ^{*_r})$, with more than two elements in its universe, is indeed a non-classical model of set theory.

Remember that by Lemma 6.1.2 we already know that every $\mathbf{V}^{(\mathbb{A},^{*r})}$ is a model of NFF-ZF.

Theorem 6.1.5. *Let* $(\mathbb{A},^{*r})$ *be a reflexive Heyting-implication lattice with more than two elements. Then there exists a sentence* $\varphi \in \mathcal{L}_\in$ *and a filter* F*, such that* $\mathbf{V}^{(\mathbb{A},^{*r})} \models_F \varphi$ *and* $\mathbf{V}^{(\mathbb{A},^{*r})} \models_F \neg\varphi$.

Proof. Fix any $(\mathbb{A},^{*r})$ with more than two elements. Now take the $\mathbb{A}$-names $u, v \in \mathbf{V}^{(\mathbb{A},^{*r})}$ such that $u = \{\langle v, a\rangle\}$, where v is arbitrary and a is the co-atom of A. Then we use the sentence

$$\varphi = \exists xy(x \in y \wedge x \notin y)$$

and the positive filter $Pos_{(\mathbb{A})}$. The two names we just defined witness that:

$$\bigvee_{u,v\in\mathbf{V}^{(\mathbb{A},\sim)}} (\llbracket v \in u\rrbracket \wedge \llbracket v \notin u\rrbracket) \geq a.$$

We go on to show that $\llbracket\varphi\rrbracket < \mathbf{1}$. Suppose $\llbracket u \in v\rrbracket = \mathbf{1}$, then by Definition 6.1.3 we have $\llbracket u \notin v\rrbracket = \llbracket u \in v\rrbracket^{*r} = \mathbf{0}$. Similarly, if $\llbracket u \notin v\rrbracket = \mathbf{1}$ we get $\llbracket u \in v\rrbracket = \mathbf{0}$. Therefore, $\llbracket\varphi\rrbracket < \mathbf{1}$ and thus $\llbracket\varphi\rrbracket = a$. By Definition 6.1.3 we get $\llbracket\neg\varphi\rrbracket = \llbracket\varphi\rrbracket^{*r} = a$. Hence, $\mathbf{V}^{(\mathbb{A},^{*r})} \models_{Pos_{(\mathbb{A})}} \varphi$ and $\mathbf{V}^{(\mathbb{A},^{*r})} \models_{Pos_{(\mathbb{A})}} \neg\varphi$. □

In fact, this theorem witnesses that ECQ fails in every $(\mathbb{A},^{*r})$-valued model modulo the positive filter, where $(\mathbb{A},^{*r})$ has more than two elements, since we know that we can always find sentences in the language of set theory that will receive value $\mathbf{0}$ in our algebra-valued model. Thus, putting together Lemma 6.1.2 and Theorem 6.1.5, we know that the mentioned $(\mathbb{A},^{*r})$-valued models modulo the positive filter are paraconsistent models of set theory that validate NFF-ZF.

Moreover, we can easily show that neither $\mathbf{L}\Big((\mathbb{A},^{*r}), Pos_{(\mathbb{A})}\Big)$ nor $\mathbf{L}\Big((\mathbf{V}^{(\mathbb{A},^{*r})}, Pos_{(\mathbb{A})}\Big)$ are paracomplete.

Lemma 6.1.6. *Let* $(\mathbb{A},^{*r})$ *be a reflexive Heyting-implication lattice. Then we have that*

$$\mathsf{LEM} \in \mathbf{L}\Big((\mathbb{A},^{*r}), Pos_{(\mathbb{A})}\Big) \textit{ and } \mathsf{LEM} \in \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*r})}, Pos_{(\mathbb{A})}\Big).$$

Proof. Consider an ι-assignment and a formula $\varphi \in \mathcal{L}_{Prop}$ such that $\iota(\varphi) = \mathbf{0}$ or $\iota(\varphi) = \mathbf{1}$. Then we get immediately that $\iota(\varphi \vee \neg\varphi) = \mathbf{1} \in Pos_{(\mathbb{A})}$. Otherwise, take an ι-assignment and a formula $\varphi \in \mathcal{L}_{Prop}$ such that $\iota(\varphi) = a$ where $a \in A \setminus \{\mathbf{1}, \mathbf{0}\}$, then we have that $\iota(\neg\varphi) = a$ and thus $\iota(\varphi \ \vee \neg\varphi) = a \in Pos_{(\mathbb{A})}$. Therefore, for any assignment ι we have $\iota(\varphi \ \vee \neg\varphi) \in Pos_{(\mathbb{A})}$. So;

$$\mathsf{LEM} \in \mathbf{L}\Big((\mathbb{A},^{*_r}), Pos_{(\mathbb{A})}\Big)$$

Then we use the fact that we have

$$\mathbf{L}(\mathbb{A}, F) \subseteq \mathbf{L}(\mathbf{V}^{(\mathbb{A})}, F), \tag{‡}$$

for any complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow,^{*}, \mathbf{0}, \mathbf{1}\rangle$ and filter F on $\mathbb{A}$. Thus;

$$\mathsf{LEM} \in \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, Pos_{(\mathbb{A})}\Big).$$

$\square$

6.1.2 Models of Paracomplete Set Theory

In this subsection, we will show how to build models of paracomplete set theory based on reflexive Heyting-implication lattices $(\mathbb{A},^{*_r})$ if we choose a filter F on $\mathbb{A}$ which is not the positive filter. So we will pick a filter F on $\mathbb{A}$ such that there exists an element a in the universe of $\mathbb{A}$ such that $a \neq \mathbf{0}$ and $a \notin F$.

Lemma 6.1.7. *Let $(\mathbb{A},^{*_r})$ be a complemented Heyting-implication lattice and F a filter on $\mathbb{A}$ such that $F \subset Pos_{(\mathbb{A})}$. Then $\mathsf{LEM} \notin \mathbf{L}\Big((\mathbb{A},^{*_r}), F\Big)$.*

Proof. Consider an ι-assignment and formula $\varphi \in \mathcal{L}_{Prop}$ such that $\iota(\varphi) = a$ where a is the atom of $\mathbb{A}$. Then we have $\iota(\varphi \ \vee \neg\varphi) = a$. Since we know that F is a proper subset of $Pos_{(\mathbb{A})}$, it follows immediately that $a \notin F$ and thus $\iota(\varphi \ \vee \neg\varphi) \notin F$. $\square$

Let us now turn to the propositional logic of the respective set-theoretical models. We will show that we get models that are paracomplete and that the respective logic of these models is different from IPL given to the validity of DNE.

Lemma 6.1.8. *Let $(\mathbb{A},^{*_r})$ be a reflexive Heyting-implication lattice with more than two elements and F a filter on $\mathbb{A}$ such that $F \subset Pos_{(\mathbb{A})}$. Then $\mathsf{LEM} \notin \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, F\Big)$.*

Proof. Fix a $(\mathbb{A},^{*_r})$ with more than two elements and consider the sentence

$$\varphi = \forall x \forall y (x \in y \vee x \notin y).$$

Notice that for any $u, v \in \mathbf{V}^{(\mathbb{A},^{*_r})}$ we have $[\![u \in v \vee u \notin v]\!] \neq \mathbf{0}$. Moreover, let a be the atom of the universe of $(\mathbb{A},^{*_r})$ and let $v = \{\langle u, a\rangle\}$ where u is an arbitrary $(\mathbb{A},^{*_r})$-name. Then we have $[\![u \in v]\!] = a$ and $[\![u \notin v]\!] = [\![u \in v]\!]^{*_r} = a$. Therefore, $[\![u \in v \;\vee\; u \notin v]\!] = a \notin F$. $\square$

Then, we can show that:

Lemma 6.1.9. *Let $(\mathbb{A},^{*_r})$ be a reflexive Heyting-implication lattice with more than two elements and F a filter on $\mathbb{A}$ such that $F \subset Pos_{(\mathbb{A})}$. Then we have:*

$$\mathbf{L}\Big((\mathbb{A},^{*_r}), F\Big) \neq \mathsf{IPL} \text{ and } \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, F\Big) \neq \mathsf{IPL}.$$

Proof. Consider any ι-assignment and a formula $\varphi \in \mathcal{L}_{Prop}$ such that $\iota(\neg\neg\varphi) = a$ where $a \in F$. In the case that $a = \mathbf{1}$ we get $\iota(\varphi) = \mathbf{1}$ and if $a \neq \mathbf{1}$, we get $\iota(\varphi) = a \in F$. Therefore;

$$\mathsf{DNE} \in \mathbf{L}\Big((\mathbb{A},^{*_r}), F\Big).$$

So our first desiderata holds. Then by (‡);

$$\mathsf{DNE} \in \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, F\Big).$$

This establishes our second desiderata. $\square$

Notice that we can show that every $\mathbb{A}$-valued model modulo the top filter is not paraconsistent and that the axiom schema $(\varphi \to \psi) \to \Big((\varphi \to \neg\psi) \to \neg\varphi\Big)$ of IPL does not hold for any $\mathbb{A}$-valued model where $\mathbb{A}$ has more than three elements.

Lemma 6.1.10. *Let $(\mathbb{A},^{*_r})$ be a reflexive Heyting-implication lattice with more than two elements. Then we have*

$$\mathsf{ECQ} \in \mathbf{L}\Big((\mathbb{A},^{*_r}), \{\mathbf{1}\}\Big) \text{ and } \mathsf{ECQ} \in \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, \{\mathbf{1}\}\Big).$$

Proof. Consider any ι-assignment and a formula $\varphi \in \mathcal{L}_{Prop}$ such that $\iota(\varphi) \in \{\mathbf{1}, \mathbf{0}\}$, then we get $\iota(\varphi \wedge \neg\varphi) = \mathbf{0} \notin \{\mathbf{1}\}$ and the desiderata follows vacuously. On the other hand, suppose that $\iota(\varphi) = a \in \mathbf{A} \setminus \{\mathbf{1}, \mathbf{0}\}$, then $\iota(\varphi \wedge \neg\varphi) = a \notin \{\mathbf{1}\}$ and again the desiderata follows vacuously. Thus;

$$\mathsf{ECQ} \in \mathbf{L}\Big((\mathbb{A},^{*_r}), \{\mathbf{1}\}\Big).$$

Then by (‡):

$$\mathsf{ECQ} \in \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},*_r)}, \{\mathbf{1}\}\Big).$$

. $\square$

Moreover, we have:

Lemma 6.1.11. *Let* $(\mathbb{A},^{*_r})$ *be a reflexive Heyting-implication lattice with more than three elements. Then we have* $(\varphi \to \psi) \to \Big((\varphi \to \neg\psi) \to \neg\varphi\Big) \notin \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},*_r)}, \{\mathbf{1}\}\Big)$.

Proof. Fix any $(\mathbb{A},^{*_r})$ be a reflexive Heyting-implication lattice with more than three elements and consider the sentence $\varphi \in \mathcal{L}_\in$ of Theorem 6.1.8. We immediately get that $[\![\varphi]\!] = a$ where a is the atom of $\mathbb{A}$. Now, consider the sentence $\psi = \exists x \exists y(x \in y \wedge x \in y)$, then we get $[\![\psi]\!] = b$ where b is the co-atom of $\mathbb{A}$.

$$\begin{aligned}
[\![(\varphi \to \psi) \to \Big((\varphi \to \neg\psi) \to \neg\varphi\Big)]\!] &= (a \Rightarrow b) \Rightarrow \Big((a \Rightarrow b) \Rightarrow a\Big) \\
&= \mathbf{1} \Rightarrow (\mathbf{1} \Rightarrow a) \\
&= (\mathbf{1} \Rightarrow a) \\
&\notin \{\mathbf{1}\}.
\end{aligned}$$

$\square$

It is interesting to notice that Theorem 6.1.9 shows that we have theorems in $\mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},*_r)}, \{\mathbf{1}\}\Big)$ which are not theorems of IPL and Theorem 6.1.11, on the other hand, shows that there exist theorems in IPL that do not hold $\mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},*_r)}, \{\mathbf{1}\}\Big)$.

6.1.3 Models of Paraconsistent and Paracomplete Set Theory

In this subsection, we show that we can also build models of paraconsistent and paracomplete set theory on top of reflexive Heyting-implication algebras, if we choose a suitable filter. In particular, we show that for any filter F on $\mathbb{A}$ which contains the co-atom of $\mathbb{A}$ and where F is a proper subset of the positive filter, then $\mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},*_r)}, F\Big)$ is paraconsistent and paracomplete. Formally:

Lemma 6.1.12. *Let $(\mathbb{A},^{*_r})$ be a reflexive Heyting-implication lattice with more then three elements and let F be a filter such that $\{\mathbf{1}\} \subset F \subset Pos_{(\mathbb{A})}$. Then*

$$\mathsf{ECQ} \notin \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, F\Big) \text{ and } \mathsf{LEM} \notin \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, F\Big).$$

Proof. Due to Lemma 6.1.5 we know that $[\![\varphi]\!] = [\![\exists x \exists y(x \in y \wedge x \notin y)]\!] = a$ where a is the co-atom of $\mathbb{A}$. Thus $[\![\varphi \wedge \neg\varphi]\!] = a \in F$. We know that $a \in F$ given that $\{\mathbf{1}\}$ is a proper subset of F. Then let $\psi = \forall x \forall y(x = y)$ and consider the $\mathbb{A}$-names $\{\langle\varnothing, \mathbf{0}\rangle\}$ and $\{\langle\varnothing, \mathbf{1}\rangle\}$. We readily calculate $[\![\{\langle\varnothing, \mathbf{1}\} = \{\langle\varnothing, \mathbf{0}\rangle\}]\!] = \mathbf{0}$ and thus $[\![\psi]\!] = \mathbf{0}$. Thus;

$$\mathsf{ECQ} \notin \mathbf{L}\Big(\mathbf{V}^{(\mathbb{A},^{*_r})}, F\Big).$$

The second desiderata follows immediately by Lemma 6.1.8. $\square$

Thus, we have found a class of algebras that gives rise to non-classical models of paraconsistent and paracomplete set theory. We go on to show that these models are non-$\in$-elementarily equivalent from $(\mathbb{T},^{*_r})$-valued models and $(\mathbb{T},^{*_p})$-valued models that we introduced in Chapter 4.4. In particular, we compare the propositional logic associated to these models with the logic of the totally ordered lattice-valued models.

Theorem 6.1.13. *Let $\mathbf{V}^{(\mathbb{A},^{*_r})}$ be any $(\mathbb{A},^{*_r})$-valued model where $(\mathbb{A},^{*_r})$ has more than two elements and $\mathbf{V}^{(\mathbb{T},^{*})}$ be any $(\mathbb{T},^{*})$-valued model where $(\mathbb{T},^{*})$ has more than two elements and $^{*} \in \{^{*_r},^{*_p}\}$. Moreover, let F_1 be a filter on $\mathbb{A}$ and F_2 a filter on $\mathbb{T}$, then*

$$(\mathbf{V}^{(\mathbb{A},^{*_r})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{T},^{*})}, F_2).$$

Proof. Consider the following sentence:

$$\chi = \exists xy\Big(x = y \wedge x \neq y\Big).$$

Fix any $(\mathbb{T},^{*})$ with more than two elements and let us evaluate this sentence in $\mathbf{V}^{(\mathbb{T},^{*})}$. Notice that for any $u, v \in \mathbf{V}^{(\mathbb{T},^{*})}$ we have either $[\![u = v]\!]^{(\mathbb{T},^{*})} = \mathbf{1}$ or $[\![u = v]\!]^{(\mathbb{T},^{*})} = \mathbf{0}$. Thus, for any $u, v \in \mathbf{V}^{(\mathbb{T},^{*})}$ we have $[\![u = v \wedge u \neq v]\!]^{(\mathbb{T},^{*})} = \mathbf{0}$ and hence $[\![\chi]\!]^{(\mathbb{T},^{*})} = \mathbf{0} \notin F_2$, for any filter F_2 on $\mathbb{T}$. Therefore, $\mathbf{V}^{(\mathbb{T},^{*})} \not\models_{F_2} \chi$. Now, fix any $(\mathbb{A},^{*_r})$ with more than two elements and consider $\{\langle\varnothing, \mathbf{1}\rangle\}$ and $\{\langle\varnothing, a\rangle\}$ where a is the co-atom of $(\mathbb{A},^{*_r})$. Let $F_1 = \{\mathbf{1}, a\}$.

Then, we calculate readily;

$$\begin{aligned}
[\![\{\langle\varnothing,\mathbf{1}\rangle\}=\{\langle\varnothing,a\rangle\}]\!] &= (1\Rightarrow a)\wedge(a\Rightarrow 1)\\
&= a\\
&\in F_1.
\end{aligned}$$

Therefore, $\mathbf{V}^{(\mathbb{A},*_r)}\models_{F_1}\chi$. $\square$

Now we will show that we can find $(\mathbb{A},^{*_r})$-valued models which are non-$\in$-elementarily equivalent from each other. Moreover, we make use of the following abbreviation:

$$\delta(\varphi) =_{df.} \Big((\varphi\to\bot)\ \vee(\neg\varphi\to\bot)\Big)$$

where $\bot$ abbreviates $\exists x(x\neq x)$ and φ is any formula in the language of set theory. Similarly, we will use the abbreviation :

$$\epsilon(\varphi) =_{df.} \neg\Big((\varphi\to\bot)\ \vee(\neg\varphi\to\bot)\Big).$$

Intuitively, the idea behind these schemata is that they allow us to "fix" the truth values of certain formulas. For instance; $\delta(\varphi)$ will only be true if φ receives a *classical* value, i.e., $[\![\varphi]\!]\in\{\mathbf{1},\mathbf{0}\}$ and $\delta(\varphi)$ will only be false if φ receives a *non-classical* value, i.e., $[\![\varphi]\!]\in\mathbb{A}\setminus\{\mathbf{1},\mathbf{0}\}$. Exactly the opposite is the case for $\epsilon(\varphi)$. Formally;

Lemma 6.1.14. *Let $\mathbf{V}^{(\mathbb{A},*_r)}$ be any $(\mathbb{A},^{*_r})$-valued model where $(\mathbb{A},^{*_r})$ has more than two elements and $\varphi\in\mathcal{L}_\in$. Then:*

(i) $[\![\delta(\varphi)]\!]=\mathbf{1}$ *iff* $[\![\varphi]\!]\in\{\mathbf{1},\mathbf{0}\}$ *and* $[\![\delta(\varphi)]\!]=\mathbf{0}$ *iff* $[\![\varphi]\!]\in\mathbb{A}\setminus\{\mathbf{1},\mathbf{0}\}$,

(ii) $[\![\epsilon(\varphi)]\!]=\mathbf{1}$ *iff* $[\![\varphi]\!]\in\mathbb{A}\setminus\{\mathbf{1},\mathbf{0}\}$ *and* $[\![\epsilon(\varphi)]\!]=\mathbf{0}$ *iff* $[\![\varphi]\!]\in\{\mathbf{1},\mathbf{0}\}$.

Proof. (i) For the first conjunct we notice:

$$[\![\delta(\varphi)]\!]=\mathbf{1}\text{ iff }[\![\varphi\to\bot]\!]=\mathbf{1}\text{ or }[\![\neg\varphi\to\bot]\!]=\mathbf{1}.$$

Moreover,

$$[\![\varphi\to\bot]\!]=\mathbf{1}\text{ iff }[\![\varphi]\!]=\mathbf{0}\text{ and }[\![\neg\varphi\to\bot]\!]=\mathbf{1}\text{ iff }[\![\varphi]\!]=\mathbf{1}.$$

For the second conjunct we notice:

$$[\![\delta(\varphi)]\!] = \mathbf{0} \text{ iff } [\![\varphi \to \bot]\!] = \mathbf{0} \text{ and } [\![\neg\varphi \to \bot]\!] = \mathbf{0}.$$

However, this can only be the case, if $[\![\varphi]\!] = a \in \mathbb{A} \setminus \{\mathbf{1}, \mathbf{0}\}$ since

$$(a \Rightarrow [\![\bot]\!]) = \mathbf{0} \text{ and } (a^{*_r} \Rightarrow [\![\bot]\!]) = \mathbf{0}.$$

(ii) Follows immediately by item (i) and Definition 6.1.3(ii). □

Then we can show that we have non-$\in$-elementarily equivalent models of set theory. Moreover, we will need two more formulas which were first introduced in a slightly different presentation in (TARAFDER, 2015). The first formula $\mathsf{Nat}_{\mathbf{0}}$ is only satisfied by $\mathbf{0}$-like elements and is meant to define the natural number $\mathbf{0}$ in $\mathbf{V}^{(\mathbb{A},{}^{*_r})}$. The second formula $\mathsf{Nat}_{\mathbf{1}}$, on the other, hand is meant to define the natural number $\mathbf{1}$ in $\mathbf{V}^{(\mathbb{A},{}^{*_r})}$, i.e., the element that contains only a $\mathbf{0}$-like element in its domain. Moreover, due to the second conjunct of this formula, we know that only $\mathbf{1}$-like elements with a non-classical range will satisfy this formula, i.e., only names of the form $\{\langle x, a\rangle\}$ where x is a $\mathbf{0}$-like element and $a \in \mathbf{A} \setminus \{\mathbf{1}, \mathbf{0}\}$ will satisfy this formula.

$$\mathsf{Nat}_{\mathbf{0}}(x) =_{df.} \forall y \forall z (x = y \to z \notin y)$$

and

$$\mathsf{Nat}_{\mathbf{1}}(x) =_{df.} \exists z \Big(\mathsf{Nat}_{\mathbf{0}}(z) \wedge \epsilon(z \in x) \wedge \forall y \Big(y \in x \to (y = z)\Big)\Big).$$

Then we can show that:

Lemma 6.1.15. *Let $(\mathbb{A}_3, {}^{*_r})$ and $(\mathbb{A}_4, {}^{*_r})$ be reflexive Heyting-implication lattices. Moreover, let F_1 be a filter on $\mathbb{A}_3$ and F_2 a filter on $\mathbb{A}_4$, then we have*

$$(\mathbf{V}^{(\mathbb{A}_3,{}^{*_r})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{A}_4,{}^{*_r})}, F_2).$$

Proof. Consider the following sentence:

$$\gamma =_{df.} \exists x_1 \exists x_2 \Big(\mathsf{Nat}_{\mathbf{1}}(x_1) \wedge \mathsf{Nat}_{\mathbf{1}}(x_2) \wedge (x_1 \neq x_2)\Big).$$

Notice that only $\mathbf{1}$-like names of the form $\{\langle x, a\rangle\}$ where x is a $\mathbf{0}$-like element and $a \in \mathbf{A} \setminus \{\mathbf{1}, \mathbf{0}\}$ satisfy the first two conjuncts. Nevertheless, in the case of $(\mathbb{A}_3, ^{*_r})$ we only can define one such $\mathbf{1}$-like name, i.e., $u = \{\langle x, a\rangle\}$ where $a \in \mathbf{A}$ and $\mathbf{0} < a < \mathbf{1}$. Therefore $[\![u \neq u]\!] = \mathbf{0}$ and thus $[\![\gamma]\!] = \mathbf{0} \notin F_1$ for any filter F_1 on $\mathbb{A}_3$. Now, consider $(\mathbb{A}_4, ^{*_r})$ and fix $F_2 = Pos_{(\mathbb{A}_4)}$. In this case we can define two $\mathbf{1}$-like name, i.e., $u = \{\langle x, a\rangle\}$ and $v = \{\langle x, b\rangle\}$ where $a, b \in \mathbf{A}$ and $\mathbf{0} < a < b < \mathbf{1}$. Then;

$$\begin{aligned}
[\![\mathsf{Nat}_{\mathbf{1}}(u) \wedge \mathsf{Nat}_{\mathbf{1}}(v) \wedge (u \neq v)]\!] &= (a \wedge b \wedge a) \\
&= a \\
&\in F_2.
\end{aligned}$$

□

Notice that we can use the sentence γ and the same proof strategy to show that $\mathbf{V}^{(\mathbb{A}_3, ^{*_r})}$ is non-$\in$-elementarily equivalent from any other $\mathbf{V}^{(\mathbb{A}_n, ^{*_r})}$ where $n > 3$. Now, we can generalize this procedure and use the following schema of sentences to show that the remaining $(\mathbb{A}, ^{*_r})$-valued models are non-$\in$-elementarily equivalent with each other. Given a natural number $n \geq 3$, let γ_n be the following sentence in the language of set theory:

$$\begin{aligned}
\gamma_n =& \exists x_1, ..., \exists x_n \Big(\mathsf{Nat}_{\mathbf{1}}(x_1) \wedge ... \wedge \mathsf{Nat}_{\mathbf{1}}(x_n) \wedge \Big((x_1 \neq x_2 \wedge .... \wedge x_1 \neq x_n) \\
&\wedge (x_2 \neq x_3 \wedge .... \wedge x_2 \neq x_n) \wedge ... \wedge (x_{n-1} \neq x_n)\Big)\Big).
\end{aligned}$$

Theorem 6.1.16. *Let $(\mathbb{A}_n, ^{*_r})$ and $(\mathbb{A}_m, ^{*_r})$ be two arbitrary Heyting-implication-negation lattices, where $n, m \geq 4$ and $n \neq m$. Moreover, let F_1 be a filter on $\mathbb{A}_n$ and F_2 a filter on $\mathbb{A}_m$, then we have*

$$(\mathbf{V}^{(\mathbb{A}_n, ^{*_r})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{A}_m, ^{*_r})}, F_2).$$

Proof. Fix two arbitrary $(\mathbb{A}_n, ^{*_r})$ and $(\mathbb{A}_m, ^{*_r})$ where $4 \leq n < m$. Notice that to validate γ_{n-3} we need $(n-1)$-many $\mathbf{1}$-like elements which are all different from each other, however, we only have $(n-2)$-many different $\mathbf{1}$-like elements available in $\mathbf{V}^{(\mathbb{A}_m, ^{*_r})}$. Thus, we can readily calculate that $\mathbf{V}^{(\mathbb{A}_n, ^{*_r})} \not\models_{F_1} \gamma_{n-3}$ and $\mathbf{V}^{(\mathbb{A}_m, ^{*_r})} \models_{F_2} \gamma_{n-3}$ where $F_2 = Pos_{(\mathbb{A}_m)}$. □

Furthermore, we conclude this section by showing that all the models that we explored in this section are faithful.

Theorem 6.1.17. *The model* $\mathbf{V}^{(\mathbb{A}_n, {}^{*r})}$ *is faithful to* $\mathbb{A}$ *and hence loyal to* $\Big((\mathbb{A}_n, {}^{*r}), F\Big)$*, for any filter* F*.*

Proof. Fix an arbitrary $\mathbf{V}^{(\mathbb{A}_n, {}^{*r})}$. We know that $[\![\mathsf{Extensionality}]\!] = \mathbf{1}$ and $[\![\neg\mathsf{Extensionality}]\!] = [\![\mathsf{Extensionality}]\!]^{*r} = \mathbf{0}$. Moreover, we also know that for the sentence φ of Theorem 6.1.5 we have $[\![\varphi]\!] = d$ where d is the co-atom of $\mathbb{A}_n$. So without loss of generality suppose that $n \leq 4$ and consider the sentence γ_{n-3}. We calculate readily that $[\![\gamma_{n-3}[\![= a$ where a is the atom of $\mathbb{A}_n$. In the case that the sentence γ_{n-3} is the sentence γ_1 we are done. If not, we use the sentence γ_{n-4} to show that $[\![\gamma_{n-4}[\![= b$ where b is the successor of a in the total order of $\mathbb{A}_n$ and repeat this procedure until we reach the sentence γ_1. Thus for every $a \in \mathbb{A}_n \setminus \{1, \mathbf{0}, d\}$ there exists a γ_n such that $[\![\gamma_n]\!] = a$. □

In conclusion, we have shown that $(\mathbb{A}, {}^{*r})$-valued models have several advantages over $(\mathbb{T}, {}^{*r})$-valued models and $(\mathbb{T}, {}^{*p})$-valued models. It boils down to the choice of the binary operation $\Rightarrow$ that will interpret the conditional in the language of set theory. The binary operation $\Rightarrow_h$ of Heyting-implication lattices is much more expressive than the $\Rightarrow_t$-operator in the case of the $\mathbb{T}$-lattices. In particular, the $\Rightarrow_h$-operator allows us to differentiate all the elements of the universe of our lattices. As a consequence of this, we get, as shown by Theorem 6.1.16 and Theorem 6.1.17 that every $\mathbb{A}$ -valued model is non-$\in$-elementarily equivalent from each other and every $\mathbf{V}^{\mathbb{A}}$ is faithful to $\mathbb{A}$. So we have found an infinite class of non-$\in$-elementarily equivalent models of non-classical set theories where we have a perfect match between the propositional logic of reflexive Heyting-implication lattices and the propositional logic of the set theories build on top of them.

6.2 Algebra-valued models for LP-Set Theory

In this section, we build algebra-valued models of set theory based on Priest's Logic of Paradox. We show that we can build a non-classical model of ZF which has as underlying logic Priest's Logic of Paradox and validates Leibniz's law of indiscernibility of identicals. This is achieved by modifying the interpretation map for $\in$ and $=$ in our algebra-valued model.

6.2.1 The model $\mathbf{V}^{(\mathbb{LP})}$

Let us consider the lattice $\mathbb{LP} = (A; \wedge, \vee, \Rightarrow, ^*, 1, \mathbf{0})$, where the algebraic operations of $\mathbb{LP}$ correspond extensionally to the truth tables of the logical connectives of LP (depicted in Table 3.1). Furthermore, we take our universe to be $A = \{1, \frac{1}{2}, \mathbf{0}\}$, where $\mathbf{0} < \frac{1}{2} < 1$ and the filter $F = \{1, \frac{1}{2}\}$ acts as the set of designated values. Then we can define the $\mathbb{LP}$-valued universe as follows:

Definition 6.2.1. *We define by transfinite recursion the set-theoretic universe $\mathbf{V}^{(\mathbb{LP})}$.*

$$\begin{aligned} \mathbf{V}^{(\mathbb{LP})}_\alpha = \{x\,;\; & x \textit{ is a function and } \operatorname{ran}(x) \subseteq A \\ & \textit{and there is } \xi < \alpha \textit{ with } \operatorname{dom}(x) \subseteq \mathbf{V}^{(\mathbb{LP})}_\xi)\} \textit{ and} \\ \mathbf{V}^{(\mathbb{LP})} = \{x\,;\; & \exists\alpha(x \in \mathbf{V}^{(\mathbb{LP})}_\alpha)\}. \end{aligned}$$

Let $\mathcal{L}_{\mathbb{LP}}$ be the extended language of $\mathcal{L}_\in$, which we obtain by adding constant symbols for every element in $\mathbf{V}^{(\mathbb{LP})}$. Moreover, to increase the readability, the name corresponding to each $u \in \mathbf{V}^{(\mathbb{LP})}$ will be denoted by the symbol u in the extended language $\mathcal{L}_{\mathbb{LP}}$. A mapping $[\![\cdot]\!]$ is recursively defined from the collection of all closed well-formed formulas in $\mathcal{L}_{\mathbb{LP}}$ to the complete bounded distributive lattice $\mathbb{LP}$ as follows (cf. Bell 2005).

Definition 6.2.2. *For any pair of elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$,*

$$\begin{aligned} [\![u \in v]\!] &= \bigvee_{x \in \operatorname{dom}(v)} \Big(v(x) \wedge [\![x = u]\!] \Big), \\ [\![u = v]\!] &= \bigwedge_{x \in \operatorname{dom}(u)} \Big(u(x) \Rightarrow [\![x \in v]\!] \Big) \wedge \bigwedge_{y \in \operatorname{dom}(v)} \Big(v(y) \Rightarrow [\![y \in u]\!] \Big). \end{aligned}$$

Then, we can extend the map $[\![.]\!]_{\mathbb{LP}}$ to non-atomic formulas: for any two closed well-formed formulas φ and ψ,

$$\begin{aligned} [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \wedge [\![\psi]\!], \\ [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \vee [\![\psi]\!], \\ [\![\varphi \to \psi]\!] &= [\![\varphi]\!] \Rightarrow [\![\psi]\!], \\ [\![\neg\varphi]\!] &= [\![\varphi]\!]^*, \end{aligned}$$

$$\llbracket \forall x \varphi(x) \rrbracket = \bigwedge_{u \in \mathbf{V}^{(\mathbb{LP})}} \llbracket \varphi(u) \rrbracket, \textit{ and}$$

$$\llbracket \exists x \varphi(x) \rrbracket = \bigvee_{u \in \mathbf{V}^{(\mathbb{LP})}} \llbracket \varphi(u) \rrbracket.$$

Definition 6.2.3. *A formula φ is said to be valid in $\mathbf{V}^{(\mathbb{LP})}$, which is denoted by $\mathbf{V}^{(\mathbb{LP})} \models_F \varphi$, whenever $\llbracket \varphi \rrbracket \in F$.*

We show in the following lemma, that due to the $\Rightarrow$ operation of $\mathbb{LP}$, the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$ is too weak to validate certain set-theoretic properties which hold in the case of Boolean and Heyting-valued models. As a consequence, we lose some properties which would be helpful in further calculations and many standard arguments that we use generally in algebra-valued models break down. To worsen the situation many calculations are blocked in $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$, given the failure of the transitivity of the conditional and the lack of modus ponens.

Lemma 6.2.4. *For any $u, v, w \in \mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$ the following claims do not hold in general:*

(i) $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F u = v \wedge v = w$ *implies* $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F u = w$,

(ii) $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F u = v \wedge u \in w$ *implies* $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F v \in w$,

(iii) $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F u = v \wedge w \in u$ *implies* $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F w \in v$.

Proof. (i) Consider the elements $u, v, w \in \mathbf{V}^{(\mathbb{LP})}$ defined as $p_{\mathbf{0}} = \{\langle \varnothing, \mathbf{0} \rangle\}$, $p_{\frac{1}{2}} = \{\langle \varnothing, \frac{1}{2} \rangle\}$, and $p_{\mathbf{1}} = \{\langle \varnothing, \mathbf{1} \rangle\}$. Then we calculate readily

$$\left(\llbracket p_{\mathbf{0}} = p_{\frac{1}{2}} \rrbracket \wedge \llbracket p_{\frac{1}{2}} = p_{\mathbf{1}} \rrbracket \right) \in F,$$

however, $\llbracket p_{\mathbf{0}} = p_{\mathbf{1}} \rrbracket = \mathbf{0}$.

(ii) Consider the $\mathbb{LP}$ name $z = \{\langle p_{\mathbf{1}}, \mathbf{1} \rangle\}$. Then we calculate

$$\left(\llbracket p_{\mathbf{0}} = p_{\frac{1}{2}} \rrbracket \wedge \llbracket p_{\frac{1}{2}} \in z \rrbracket \right) \in F$$

and $\llbracket p_{\mathbf{0}} \in z \rrbracket = \mathbf{0}$.

(iii) Consider the $\mathbb{LP}$-names $r = \{\langle p_{\mathbf{0}}, \frac{1}{2}\rangle\}$ and $q = \{\langle p_{\mathbf{0}}, \mathbf{0}\rangle\}$. Then we have

$$\Big(\llbracket r = q \rrbracket \wedge \llbracket p_{\mathbf{0}} \in r \rrbracket\Big) \in F$$

and $\llbracket p_{\mathbf{0}} \in q \rrbracket = \mathbf{0}$. □

In particular, we get:

Corollary 6.2.5. *For any $u, v \in \mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$ and any formula $\varphi(x)$ in $\mathcal{L}_{\mathbb{LP}}$ having one free variable x it is generally not the case that, if $\Big(\llbracket u = v \rrbracket \wedge \llbracket \varphi(u) \rrbracket\Big) \in F$ then $\llbracket \varphi(v) \rrbracket \in F$.* □

Therefore, we can raise the first line of criticism against the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$. In particular, Corollary 6.2.5 shows that the Leibniz's law of indiscernibility of identicals fails within $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$. On the one side, since we can not build equivalence classes we are unable to define natural numbers and other basic kinds of sets in $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$. So we are also unable to quotient down our algebra-valued model and to build a model of set theory with a *proper* notion of identity. On the other side, we have a conceptual problem given that we are dealing with an uncontroversial and widely accepted property of equality (INCURVATI, 2020, pp. 108–109). It is not clear why we should abandon such an intuitive principle regarding equality within a paraconsistent set theory.

Thus, we believe that the failure of Leibniz's law of indiscernibility of identicals constitutes a serious challenge for the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$. Notice that (RESTALL, 1992a) has shown that Leibniz's law of indiscernibility of identicals fails, as well, in NLP.

We go on to show that in $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)}$ we have non-well-founded sets.

Lemma 6.2.6. $\mathbf{V}^{(\mathbb{LP},\ \llbracket\cdot\rrbracket)} \models_F \exists x (x \in x)$.

Proof. Consider the $\mathbb{LP}$-name $p_{\frac{1}{2}}$ (as defined in Lemma 6.2.4). Then we can readily calculate that

$$\begin{aligned}
\llbracket p_{\frac{1}{2}} \in p_{\frac{1}{2}} \rrbracket &= \Big(p_{\frac{1}{2}}(\varnothing) \wedge \llbracket \varnothing = p_{\frac{1}{2}} \rrbracket\Big) \\
&= \Big(\frac{1}{2} \wedge \frac{1}{2}\Big) \\
&= \frac{1}{2} \in F.
\end{aligned}$$

$\square$

We go on to point out the second issue of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. We will need the definition of $\frac{1}{2}$-*like* elements (these names are constituted just as canonical names with the only difference that the range of these names is $\frac{1}{2}$ instead of $\mathbf{1}$): for any $x \in \mathbf{V}$ let

$$x^{\circ} = \{\langle y^{\circ}, \frac{1}{2}\rangle : y \in x\}.$$

It is easily observable that every $\frac{1}{2}$-*like* element u° is a non-well-founded set in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$, i.e., $[\![u^{\circ} \in u^{\circ}]\!] \neq \mathbf{0}$. We believe that the existence of these sets is not problematic by itself, however, it seems unsatisfactory that every $\frac{1}{2}$-*like* element is identical in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. In other words, every $\frac{1}{2}$-*like* element collapses to a single element from the perspective of our algebra-valued model.

Lemma 6.2.7. *For any $u^{\circ}, v^{\circ} \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ we have $[\![u^{\circ} = v^{\circ}]\!] \in F$.*

Proof. Fix any two $\frac{1}{2}$-*like* $\mathbb{LP}$-names u° and v°. Then:

$$\begin{aligned} [\![u^{\circ} = v^{\circ}]\!] &= \big((\frac{1}{2} \Rightarrow \frac{1}{2}) \wedge (\frac{1}{2} \Rightarrow \frac{1}{2})\big) \\ &= \frac{1}{2} \in F. \end{aligned}$$

$\square$

The situation is even worse since not only is every $\frac{1}{2}$-*like* element identical in our algebra-valued model, but every $\frac{1}{2}$-*like* element is, as well, identical to any $\mathbf{0}$-*like* element. We call an $\mathbb{LP}$-name u a $\mathbf{0}$-*like* element whenever $u = \varnothing$ or for any $x \in \mathrm{dom}(u)$ we have $u(x) = \mathbf{0}$, i.e. we can think of $\mathbf{0}$-like elements as representatives of the empty set $\varnothing$ in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$. Thus every $\frac{1}{2}$-*like* and $\mathbf{0}$-*like* element collapses to a single element from the perspective of our model, i.e., the empty set $\varnothing$. Hence, we believe that in the case that of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ we have a case of an *excessive* duplication of $\mathbb{LP}$-names.

Moreover, it was observed by (WEIR, 2004, pp. 393-395) that in the case of $\mathsf{NLP}_{=}$ we have also problems regarding identity. More specifically, there exists an $\mathsf{NLP}_{=}$-model where the formula $\exists x \exists y (x \neq y)$ does not hold. In the case of NLP, on the other hand, it is possible to find two sets x and y such that $x \neq y$ holds (RESTALL, 1992a, Theorem 7). However, NLP is still unable to prove that there exist two sets x and y such that $x = y$ does not hold. This is due to fact that in Restall's NLP-model every formula receives value $\frac{1}{2}$. The moral that we can draw from this, is that a non-classical notion of identity is problematic for $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$, (the models of) NLP and (the models of) $\mathsf{NLP}_{=}$.

Finally, we conclude that the algebra-valued model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ does not seem very fruitful. On one side, it is unclear how much set theory we can derive since various basic set-theoretic properties are blocked and due to the lack of basic inferential features of $\Rightarrow$. This carries over to $[\![\cdot = \cdot]\!]$, i.e., the interpretation of identity in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ since $[\![\cdot = \cdot]\!]$ is interpreted as the conjunction of conditional statements. As a consequence, Leibniz's law of indiscernibility of identicals fails in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ and every $\frac{1}{2}$-*like* and $\mathbf{0}$-*like* element collapses

6.2.2 The Model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$

It has been claimed by (PRIEST, 2006) that the key problem of a set theory based on LP is the weak conditional. His solution consisted of supplying LP with a stronger conditional or modifying the consequence relation. This gave rise to many variations of LP such as *multiple conclusion* LP originally introduced in (BEALL, 2011) or *minimally inconsistent* LP originally introduced in (PRIEST, 1991). We want to explore another possibility of constructing an LP-set theory without distorting the spirit of LP. So, we will build an $\mathbb{LP}$-valued model where we retain the conditional, however, we modify the interpretation map of the algebra-valued model. In particular, we propose to define a new interpretation map, denoted by $[\![\cdot]\!]_{IN}$, which does not allow for glutty identity statements anymore.

Definition 6.2.8. *For any pair of elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$;*

$$[\![u \in v]\!]_{IN} = \bigvee_{x \in \mathrm{dom}(v)} \Big(v(x) \wedge [\![x = u]\!]_{IN}\Big),$$

$$[\![u = v]\!]_{IN} = \mathbf{0} \textit{ iff}$$

there exists a $x \in \mathrm{dom}(u)$ such that $u(x) > [\![x \in v]\!]_{IN}$,

or there exists a $y \in \mathrm{dom}(v)$ such that $v(y) > [\![y \in u]\!]_{IN}$.

Otherwise; $[\![u = v]\!]_{IN} = \mathbf{1}$.

Then, we extend the map $[\![\cdot]\!]_{IN}$ to non-atomic formulas as in definition 6.2.2.

Definition 6.2.9. *Let $\mathbf{V}^{(\mathbb{LP})}$ be the universe of $\mathbb{LP}$-valued functions. Then we denote with $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ the $\mathbb{LP}$-valued model that we obtain by using $[\![\cdot]\!]_{IN}$ as interpretation map.*

Definition 6.2.10. *A formula $\varphi \in \mathcal{L}_{\mathbb{LP}}$ is said to be valid in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ given a designated set D, whenever $[\![\varphi]\!]_{IN} \in F$. We denote this fact by $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F \varphi$.*

Notice that for any $u, v \in \mathbf{V}^{(\mathbb{LP})}$ we have either

$$[\![u = v]\!]_{IN} = \mathbf{1} \text{ or } [\![u = v]\!]_{IN} = \mathbf{0}.$$

In other words, the range of the modified interpretation map of identity is $\{\mathbf{0}, \mathbf{1}\}$, whereas the modified interpretation map of membership can range, as in the case of the usual interpretation map, over all the elements of the universe of $\mathbb{LP}$. For instance, if u is an $\mathbb{LP}$-name, then $v = \{\langle u, a\rangle\}$ (where $a \in \mathbb{LP}$) is also an $\mathbb{LP}$-name and $[\![u \in v]\!] = a$. Moreover, every time we want to prove that $[\![u = v]\!]_{IN} \in F$ it is enough to show that for any $x \in \mathrm{dom}(u)$ it is the case that $u(x) \leq [\![x \in u]\!]_{IN}$ and similarly for the elements of the domain of v.

Theorem 6.2.11. *Consider any two elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$. Then, $[\![u = v]\!]_{IN} \in F$ if and only if both of the following hold:*

(i) if $u(x) = \mathbf{1}$ then there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ and $[\![x = y]\!]_{IN} \in F$, and vice-versa; and

(ii) if $u(x) = \frac{1}{2}$ then there exists a $y \in \mathrm{dom}(v)$ such that $v(y) \geq \frac{1}{2}$ and $[\![x = y]\!]_{IN} \in F$, and vice-versa.

Proof. Let us consider two elements $u, v \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![u = v]\!]_{IN} \in F$.

For (i), suppose there exists an element $u(x) = \mathbf{1}$. We want to show that $[\![x \in v]\!]_{IN} = \mathbf{1}$. Suppose otherwise, so either $[\![x \in v]\!]_{IN} = \mathbf{0}$ or $[\![x \in v]\!]_{IN} = \frac{1}{2}$. In both cases we have a $x \in \mathrm{dom}(u)$ such that $u(x) > [\![x \in v]\!]_{IN}$, so by Definition 6.2.8 we get $[\![u = v]\!]_{IN} = \mathbf{0}$. Thus in both cases we are contradicting our initial assumption. Hence, we must have $[\![x \in v]\!]_{IN} = \mathbf{1}$, i.e., there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ such that $[\![x = y]\!]_{IN} = \mathbf{1}$. Similarly, if there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ then there also exists a $x \in \mathrm{dom}(u)$ such that $u(x) = \mathbf{1}$ and $[\![x = y]\!]_{IN} \in F$, otherwise $[\![u = v]\!]_{IN} = \mathbf{0}$, and hence our assumption fails.

For (ii), let there be a $x \in \mathrm{dom}(u)$ such that $u(x) = \frac{1}{2}$. If there is no $y \in \mathrm{dom}(v)$ such that $v(y) \in \{\mathbf{1}, \frac{1}{2}\}$ and $[\![x = y]\!]_{IN} \in F$ we must have $[\![x \in v]\!]_{IN} = \mathbf{0}$. So there exists a $x \in \mathrm{dom}(u)$ such that $u(x) > [\![x \in v]\!]_{IN}$. Then by Definition 6.2.8 we get $[\![u = v]\!]_{IN} = \mathbf{0}$, which contradicts our initial assumption. It follows immediately that there exists a $y \in \mathrm{dom}(v)$ such that $v(y) \in \{\mathbf{1}, \frac{1}{2}\}$ and $[\![x = y]\!]_{IN} \in F$. Similarly, if there exists $y \in \mathrm{dom}(v)$ such that $v(y) = \frac{1}{2}$ and there does not exist any $x \in \mathrm{dom}(u)$, then $[\![u = v]\!]_{IN} = \mathbf{0}$, leads to a contradiction.

Conversely, let (i) and (ii) hold. Suppose that $u(x) = \mathbf{1}$. By (i) we have $[\![x \in v]\!]_{IN} = \mathbf{1}$, so $u(x) \leq [\![x \in v]\!]_{IN}$. Similarly, if $u(x) = \frac{1}{2}$ we get by (ii) that $[\![x \in v]\!]_{IN} \in \{\mathbf{1}, \frac{1}{2}\}$, so again we have $u(x) \leq [\![x \in v]\!]_{IN}$. We proceed analogously for the elements of the domain of v. This leads to the fact that, $[\![u = v]\!]_{IN} \in F$. □

Lemma 6.2.12. *For any $u, v, w \in \mathbf{V}^{(\mathbb{LP})}$ the following hold:*

(i) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F u = u$,

(ii) for any $x \in \mathrm{dom}(u)$, $u(x) \in F$ implies $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F x \in u$,

(iii) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F u = v \wedge v = w$ *implies* $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F u = w$.

Proof. (i) Consider any $x \in \mathrm{dom}(u)$ such that $u(x) \in F$. Suppose $u(x) = \mathbf{1}$, then by (i) of Theorem 6.2.11 we have $[\![x \in u]\!]_{IN} = \mathbf{1}$. Hence, $u(x) \leq [\![x \in u]\!]_{IN}$. Similarly, if $u(x) = \frac{1}{2}$ then by (ii) of Theorem 6.2.11 we get $[\![x \in u]\!]_{IN} \in \{\mathbf{1}, \frac{1}{2}\}$. This means that for any $x \in \mathrm{dom}(u)$ we have $u(x) \leq [\![x \in u]\!]_{IN}$. We may conclude $[\![u = u]\!]_{IN} \in F$ for any $u \in \mathbf{V}^{(\mathbb{LP})}$.
(ii) Let $u(x) \in F$, so we have $[\![x \in u]\!]_{IN} \geq (u(x) \wedge [\![x = x]\!]_{IN}) \in F$, since $[\![x = x]\!]_{IN} \in F$ by item (i).
(iii) By induction on the domain of w. Assume that for all $z \in \mathrm{dom}(w)$ we have:

$$\Big([\![u = v]\!]_{IN} \wedge [\![v = z]\!]_{IN}\Big) \in F \text{ implies } [\![u = z]\!]_{IN} \in F.$$

Take any $x \in \mathrm{dom}(u)$ such that $u(x) \in F$. We want to show that $u(x) \leq [\![x \in w]\!]_{IN}$. If $u(x) = \mathbf{1}$, then since $[\![u = v]\!]_{IN} \in F$ by item (i) of Theorem 6.2.11 we have $[\![x \in v]\!]_{IN} = \mathbf{1}$, i.e., there exists a $y \in \mathrm{dom}(v)$ such that $v(y) = \mathbf{1}$ and $[\![x = y]\!]_{IN} \in F$. Now, since $[\![v = w]\!]_{IN} \in F$ and $v(y) = \mathbf{1}$ we can apply the same argument again, so $[\![y \in w]\!]_{IN} = \mathbf{1}$, i.e., there exists a $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$ and $[\![z = y]\!]_{IN} \in F$. Then by induction hypothesis: $\big([\![x = y]\!]_{IN} \wedge [\![y = z]\!]_{IN}\big) \in F$ implies $[\![x = z]\!]_{IN} \in F$. Hence, there exists a $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$ and $[\![x = z]\!]_{IN} \in F$, i.e., $[\![x \in w]\!]_{IN} = \mathbf{1}$. We can proceed similar for any $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$. Moreover, if $u(x) = \frac{1}{2}$ we simply apply (ii) of Theorem 6.2.11 instead of (i) and proceed similarly as in the previous case. Likewise, for any $z \in \mathrm{dom}(w)$ such that $w(z) = \frac{1}{2}$ we can show that $w(z) \leq [\![z \in u]\!]_{IN}$. Hence, for any $x \in \mathrm{dom}(u)$ we have $u(x) \leq [\![x \in w]\!]_{IN}$ and for any $z \in \mathrm{dom}(w)$ we have $w(z) \leq [\![z \in u]\!]_{IN}$. Hence, we may conclude $[\![u = w]\!]_{IN} \in F$. □

Lemma 6.2.13. *For any $u, v \in \mathbf{V}^{(\mathbb{LP})}$ and any formula $\varphi(x) \in \mathcal{L}_{\mathbb{LP}}$, if $[\![u = v]\!]_{IN} \in F$ then the following hold:*

(i) if $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$ then $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$,

(ii) if $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$ then $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Proof. By induction on the complexity of φ.

Base case (I). (i) Let $\varphi(x) := w = x$, where $w \in \mathbf{V}^{(\mathbb{LP})}$. If $[\![u = w]\!]_{IN} = \mathbf{1}$, then by Lemma 6.2.12$(iii)$ we have that $[\![v = w]\!]_{IN} = \mathbf{1}$. (ii) Follows vacuously, since we have either $[\![u = v]\!]_{IN} = \mathbf{1}$ or $[\![u = v]\!]_{IN} = \mathbf{0}$ for every $u, v \in \mathbf{V}^{(\mathbb{LP})}$.

Base case (II). (i) Let $\varphi(x) := w \in x$, where $w \in \mathbf{V}^{(\mathbb{LP})}$. Suppose $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$. Then, there exists a $p \in \mathrm{dom}(u)$ such that $u(p) = \mathbf{1}$ and $[\![p = w]\!]_{IN} = \mathbf{1}$. Since $[\![u = v]\!]_{IN} \in F$, by item (i) of Theorem 6.2.11, there exists $q \in \mathrm{dom}(v)$ satisfying $v(q) = \mathbf{1}$ and $[\![p = q]\!]_{IN} \in F$. By Lemma 6.2.12(iii), $[\![q = w]\!]_{IN} \in F$, i.e., $[\![q = w]\!]_{IN} = \mathbf{1}$. So there exists a $q \in \mathrm{dom}(v)$ such that $v(q) = \mathbf{1}$ and $[\![q = w]\!]_{IN} = \mathbf{1}$, i.e., $[\![w \in v]\!]_{IN} = \mathbf{1}$. Hence $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$.
(ii) Now suppose $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$. Then, there exists $p \in \mathrm{dom}(u)$ such that $u(p) = \frac{1}{2}$ and $[\![p = w]\!]_{IN} \in F$. At the same time there does also not exist any $s \in \mathrm{dom}(u)$ such that $u(s) = \mathbf{1}$ and $[\![s = w]\!]_{IN} \in F$. Since, it is given that $[\![u = v]\!]_{IN} \in F$, Theorem 6.2.11 ensures the existence of $q \in \mathrm{dom}(v)$ satisfying $v(q) = \frac{1}{2}$ and $[\![p = q]\!]_{IN} \in F$, in addition, there does not exist any $t \in \mathrm{dom}(v)$ such that $v(t) = \mathbf{1}$ and $[\![t = w]\!]_{IN} \in F$. By Lemma 6.2.12 (iii), we have $[\![q = w]\!]_{IN} \in F$. Hence $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Base case (III). Let $\varphi(x) := x \in w$, where $w \in \mathbf{V}^{(\mathbb{LP})}$. (i) Let $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$, i.e.,

$$\bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![u = z]\!]_{IN}) = \mathbf{1}$$

So there exists a $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \mathbf{1} = [\![z_1 = u]\!]_{IN}$. Thus, we have that $[\![u = v]\!]_{IN} \wedge [\![z_1 = u]\!]_{IN} \in F$ and by Lemma 3.5(iii), $[\![z_1 = v]\!]_{IN} \in F$. So there exists a $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \mathbf{1} = [\![z_1 = v]\!]_{IN}$, i.e., $[\![v \in w]\!]_{IN} = \mathbf{1}$. (ii) Now, suppose; $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$, i.e.,

$$\bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![u = z]\!]_{IN}) = \frac{1}{2}$$

This can only be the case if;

1. There exists $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \frac{1}{2}$ and $[\![z_1 = u]\!]_{IN} = \mathbf{1}$.

2. For any $z \in \mathrm{dom}(w)$, if $w(z) = \mathbf{1}$ then $[\![z = u]\!]_{IN} = \mathbf{0}$.

By Lemma 6.2.12(iii) we have $[\![z_1 = v]\!]_{IN} \in F$. So there exists a $z_1 \in \mathrm{dom}(w)$ such that $w(z_1) = \frac{1}{2}$ and $[\![z_1 = v]\!]_{IN} \in F$, i.e., $[\![v \in w]\!]_{IN} \in F$. We shall now prove that we have $[\![v \in w]\!]_{IN} < \mathbf{1}$. Suppose otherwise, so there exists a $z_2 \in \mathrm{dom}(w)$ such that $w(z_2) = \mathbf{1} = [\![z_2 = v]\!]_{IN}$. Since $[\![u = v]\!]_{IN} = \mathbf{1}$ by Lemma 6.2.12(iii), we have $[\![z_2 = u]\!]_{IN} = \mathbf{1}$. So there exists a $z \in \mathrm{dom}(w)$ such that $w(z) = \mathbf{1}$ and $[\![z = u]\!]_{IN} = \mathbf{1}$.This contradicts that $[\![u \in w]\!]_{IN} = \frac{1}{2}$. Hence we get, $[\![v \in w]\!]_{IN} = \frac{1}{2}$.

Induction step:

Case (I). Let $\varphi(x) := \psi(x) \wedge \gamma(x)$. (i) If $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$ then both of $[\![\psi(u)]\!]_{IN}$ and $[\![\gamma(u)]\!]_{IN}$ get value $\mathbf{1}$. By the induction hypothesis, $[\![\psi(v)]\!]_{IN}$ and $[\![\gamma(v)]\!]_{IN}$ are $\mathbf{1}$, as well. Hence $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$. (ii) Now, if $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$ holds, then we have $[\![\psi(u)]\!]_{IN} = \frac{1}{2}$ or $[\![\gamma(u)]\!]_{IN} = \frac{1}{2}$. Again, by the induction hypothesis it can be concluded that $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Similarly, Case II, Case III and Case IV can also be proved.

Case (II). Let $\varphi(x) := \psi(x) \vee \gamma(x)$.

Case (III). Let $\varphi(x) := \psi(x) \rightarrow \gamma(x)$.

Case (IV). Let $\varphi(x) := \neg\, \psi(x)$.

Case (V). Let $\varphi(x) := \exists y\, \psi(y, x)$. (i) Suppose $[\![\varphi(u)]\!]_{IN} = \mathbf{1}$. So there exists $p \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\psi(p, u)]\!]_{IN} = \mathbf{1}$. Therefore, $[\![\psi(p, v)]\!]_{IN} = \mathbf{1}$, by the induction hypothesis. Hence $[\![\varphi(v)]\!]_{IN} = \mathbf{1}$. (ii) Let $[\![\varphi(u)]\!]_{IN} = \frac{1}{2}$. Then, there exists $p \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\psi(p, u)]\!]_{IN} = \frac{1}{2}$ and there does not exist any $q \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\psi(q, u)]\!]_{IN} = \mathbf{1}$. The induction hypothesis ensures that $[\![\psi(p, v)]\!]_{IN} = \frac{1}{2}$ and $[\![\psi(q, u)]\!]_{IN} \neq \mathbf{1}$, for all $q \in \mathbf{V}^{(\mathbb{LP})}$. Finally, $[\![\varphi(v)]\!]_{IN} = \frac{1}{2}$.

Case (VI). Let $\varphi(x) := \forall y\, \psi(y, x)$. By an immediate application of the induction hypothesis, both (i) and (ii) can be proved in this case also. $\square$

Hence, we obtain as corollary the validity of Leibniz's law of indiscernibility of identicals in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$.

Corollary 6.2.14. *For any $u, v \in \mathbf{V}^{(\mathbb{LP})}$ and any formula $\varphi(x)$ in $\mathcal{L}_{\mathbb{LP}}$ having one free variable x, if $\Big([\![u = v]\!]_{IN} \wedge [\![\varphi(u)]\!]_{IN}\Big) \in F$ then $[\![\varphi(v)]\!]_{IN} \in F$.*

Lemma 6.2.15. *For any $u \in \mathbf{V}^{(\mathbb{LP})}$, and a formula $\varphi(x)$, having one free variable x, in $\mathcal{L}_{\mathbb{LP}}$,*

$$[\![\forall x\Big(x \in u \to \varphi(x)\Big)]\!]_{IN} = \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow [\![\varphi(x)]\!]_{IN}\Big). \qquad (\mathcal{BQ}_\varphi)$$

Proof. By the definition of the assignment function $[\![\cdot]\!]_{IN}$,

$$\begin{aligned}
&[\![\forall x\Big(x \in u \to \varphi(x)\Big)]\!]_{IN}\\
&\qquad = \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} [\![y \in u \to \varphi(y)]\!]_{IN}\\
&\qquad = \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \Big(\bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![y = x]\!]_{IN}) \Rightarrow [\![\varphi(y)]\!]_{IN}\Big)\\
&\qquad = \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(y)]\!]_{IN}\Big), \text{ by } (\dagger)\\
&\qquad = \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow ([\![x = y]\!]_{IN} \Rightarrow [\![\varphi(y)]\!]_{IN})\Big), \text{by } (\mathbf{P4})\\
&\qquad = \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow ([\![x = y]\!]_{IN} \Rightarrow [\![\varphi(x)]\!]_{IN})\Big), \text{ by Corollary 6.2.14}\\
&\qquad = \bigwedge_{y \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{x \in \mathrm{dom}(u)} \Big((u(x) \wedge [\![x = y]\!]_{IN}) \Rightarrow [\![\varphi(x)]\!]_{IN}\Big), \text{by } (\mathbf{P4}).
\end{aligned}$$

Moreover, by (**P3**) we conclude that,

$$
\begin{aligned}
\bigwedge_{x\in\mathrm{dom}(u)}\Big(u(x)\Rightarrow[\![\varphi(x)]\!]_{IN}\Big) &= \bigwedge_{y\in\mathbf{V}^{(\mathbb{LP})}}\bigwedge_{x\in\mathrm{dom}(u)}\Big(u(x)\Rightarrow[\![\varphi(x)]\!]_{IN}\Big)\\
&\leq \bigwedge_{y\in\mathbf{V}^{(\mathbb{LP})}}\bigwedge_{x\in\mathrm{dom}(u)}\Big((u(x)\wedge[\![x=y]\!]_{IN})\Rightarrow[\![\varphi(x)]\!]_{IN}\Big).
\end{aligned}
$$

On the other hand, for any $x\in\mathrm{dom}(u)$,

$$
\begin{aligned}
\bigwedge_{y\in\mathbf{V}^{(\mathbb{LP})}}\Big((u(x)\wedge[\![x=y]\!]_{IN})\Rightarrow[\![\varphi(x)]\!]_{IN}\Big) &\leq (u(x)\wedge[\![x=x]\!]_{IN})\Rightarrow[\![\varphi(x)]\!]_{IN}\\
&= u(x)\Rightarrow\varphi(x),\ \text{ using Lemma 6.2.12}(ii),
\end{aligned}
$$

which implies,

$$
\bigwedge_{y\in\mathbf{V}^{(\mathbb{LP})}}\bigwedge_{x\in\mathrm{dom}(u)}\Big((u(x)\wedge[\![x=y]\!]_{IN})\Rightarrow[\![\varphi(x)]\!]_{IN}\Big)\leq\bigwedge_{x\in\mathrm{dom}(u)}\Big(u(x)\Rightarrow\varphi(x)\Big).
$$

Hence,

$$
\bigwedge_{y\in\mathbf{V}^{(\mathbb{LP})}}\bigwedge_{x\in\mathrm{dom}(u)}\Big((u(x)\wedge[\![x=y]\!]_{IN})\Rightarrow[\![\varphi(x)]\!]_{IN}\Big)=\bigwedge_{x\in\mathrm{dom}(u)}\Big(u(x)\Rightarrow\varphi(x)\Big),
$$

and as a conclusion,

$$
[\![\forall x\Big(x\in u\to\varphi(x)\Big)]\!]_{IN}=\bigwedge_{x\in\mathrm{dom}(u)}\Big(u(x)\Rightarrow\varphi(x)\Big).
$$

□

We will use the following definitions to show the validity of choice in our model. Moreover, this proof follows closely the proof of choice in (TARAFDER, 2021). However, here we are considering another interpretation map and algebraic structure.

Definition 6.2.16. *Let* $u\in\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. *Then we can define the subset* $\mathrm{dom}_{pos}(u)$ *of* $\mathrm{dom}(u)$ *as*

$$
\mathrm{dom}_{pos}(u)=\{x\in\mathrm{dom}(u):u(x)\neq\mathbf{0}\}.
$$

Definition 6.2.17. *We define* $\mathrm{dom}_{pos}(u)/\sim$ *as the partition of* $\mathrm{dom}_{pos}(u)$ *by* $\sim$ *where for any* $u, v \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$

$$u \sim v \ \textit{iff}\ \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F u = v.$$

It is easy to check that $\sim$ is indeed an equivalence relation. Moreover, we denote with $[x] = \{v \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} : \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F x = v\}$ the elements of $\mathrm{dom}_{pos}(u)/\sim$ where $x \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Now, we are in a position to show that ZFC holds in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Moreover, notice that the following proof is a modification of the proof of Theorem 3.13 of (JOCKWICH-MARTINEZ; TARAFDER; VENTURI, 2021c).

Theorem 6.2.18. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F \mathsf{ZF}$.

Proof. $\mathsf{Extensionality}$: We want to show that for any $x, y \in \mathbf{V}^{(\mathbb{LP})}$ we have

$$\begin{aligned}
&[\![\forall z(z \in x \leftrightarrow z \in y) \Rightarrow x = y]\!]_{IN}\\
&= [\![\forall z(z \in x \leftrightarrow z \in y)]\!]_{IN} \Rightarrow [\![x = y]\!]_{IN}\\
&\in F.
\end{aligned}$$

Suppose that $[\![\forall z(z \in x \leftrightarrow z \in y)]\!]_{IN} \leq \frac{1}{2}$, then due to the definition of the $\Rightarrow$ operator in $\mathbb{LP}$ we have $[\![\mathsf{Extensionality}]\!]_{IN} \in F$. So let $[\![\forall z(z \in x \leftrightarrow z \in y)]\!]_{IN} = \mathbf{1}$, i.e.,

$$\bigwedge_{z \in \mathrm{dom}(x)} [\![(x(z) \rightarrow z \in y)]\!]_{IN} \wedge \bigwedge_{z \in \mathrm{dom}(y)} [\![(y(z) \rightarrow z \in x)]\!]_{IN} = \mathbf{1}$$

Notice however that this can only be the case when for every $z \in \mathrm{dom}(x)$ we have that $x(z) \leq [\![z \in y]\!]_{IN}$ and that for every for every $z \in \mathrm{dom}(y)$ we have $y(z) \leq [\![z \in x]\!]_{IN}$. Then by Definition 6.2.8 we get $[\![x = y]\!]_{IN} = \mathbf{1}$. Therefore, $[\![\mathsf{Extensionality}]\!]_{IN} \in F$.

$\mathsf{Pairing}$: We show that for any two $x, y \in \mathbf{V}^{(\mathbb{LP})}$ there exists a $z \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall w\Big(w \in z \rightarrow (w = x \vee w = y)\Big)]\!]_{IN} \wedge [\![\forall w\Big((w = x \vee w = y) \rightarrow w \in z\Big)]\!]_{IN} \in F.$$

We begin by showing that,

$$[\![\forall w\Big(w \in z \rightarrow (w = x \vee w = y)\Big)]\!]_{IN} \in F.$$

Fix two arbitrary $x, y \in \mathbf{V}^{(\mathbb{LP})}$. Let z be such that $\mathrm{dom}(z) = \{x, y\}$ and $\mathrm{ran}(z) = \{\mathbf{1}\}$. By applying $\mathcal{BQ}_\varphi$ we get:

$$[\![\forall w\Big(w \in z \to (w = x \vee w = y)\Big)]\!]_{IN} = \bigwedge_{w \in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![(w = x \vee w = y)]\!]_{IN}\Big).$$

Now take any $w \in \mathrm{dom}(z)$ such that $z(w) \in F$, then due to the construction of z we have $[\![w_0 = x \vee w_0 = y]\!]_{IN} = \mathbf{1}$. Therefore,

$$[\![\forall w\Big(w \in z \to (w = x \vee w = y)\Big)]\!]_{IN} \in F.$$

Now we will show that this is also the case for the second conjunct of $\mathsf{Pairing}$, i.e., we show,

$$[\![\forall w\Big((w = x \vee w = y) \to w \in z\Big)]\!]_{IN} = \bigwedge_{w \in \mathbf{V}^{(\mathbb{LP})}} \Big([\![(w = x \vee w = y)]\!]_{IN} \Rightarrow [\![w \in z]\!]_{IN}\Big) \in F.$$

Take any $w_0 \in \mathbf{V}^{(\mathbb{LP})}$ and suppose that $[\![w_0 = x \vee w_0 = y]\!]_{IN} \in F$. Then, by the construction of z, we get immediately that $[\![w_0 \in z]\!]_{IN} \in F$. Therefore,

$$[\![\forall w\Big((w = x \vee w = y) \to w \in z\Big)]\!]_{IN} \in F.$$

Combining the previous results we conclude that that $[\![\mathsf{Pairing}]\!]_{IN} \in F$.

$\mathsf{Infinity}$: Define for each $x \in \mathbf{V}$, where $\mathbf{V}$ is the ground model, $\check{x} = \{\langle \check{y}, 1\rangle : y \in x\}$. For any $x \in \mathbf{V}$, it can be observed that $\check{x} \in \mathbf{V}^{(\mathbb{LP})}$. It is enough to show that,

$$[\![\forall z \neg(z \in \varnothing)]\!]_{IN} \wedge [\![\varnothing \in \check{\omega}]\!]_{IN} \wedge [\![\forall w\Big(w \in \check{\omega} \to \exists u(u \in \check{\omega} \wedge w \in u)\Big)]\!]_{IN} \in F,$$

where $\varnothing$ is the empty function in $\mathbf{V}^{(\mathbb{LP})}$ and ω is the collection of all natural numbers in $\mathbf{V}$. The image of the first two conjuncts of $\mathsf{Infinity}$ is clearly in F under the interpretation map $[\![\cdot]\!]_{IN}$. We show that,

$$[\![\forall w\Big(w \in \check{\omega} \to \exists u(u \in \check{\omega} \wedge w \in u)\Big)]\!]_{IN} \in F.$$

By applying $\mathcal{BQ}_\varphi$ we have:

$$[\![\forall w\big(w \in \check{\omega} \to \exists u(u \in \check{\omega} \wedge w \in u)\big)]\!]_{IN} = \bigwedge_{w \in \mathrm{dom}(\check{\omega})} \big(\check{\omega}(w) \Rightarrow [\![\exists u(u \in \check{\omega} \wedge w \in u)]\!]_{IN}\big).$$

Now take any $\check{w}_0 \in \mathrm{dom}(\check{\omega})$. By the definition of $\check{\omega}$, we have $\check{\omega}(\check{w}_0) = \mathbf{1}$. Therefore, $\check{w}_0 \in \omega$ holds in $\mathbf{V}$. Now due to Infinity in $\mathbf{V}$ we know that there exists a $u_0 \in \mathbf{V}$ (the successor of w_0) such that $u_0 \in \omega$ and $w_0 \in u_0$ holds in $\mathbf{V}$. Thus $\check{u}_0 \in \mathbf{V}^{(\mathbb{LP})}$. We can check readily that $[\![\check{u}_0 \in \check{\omega}]\!]_{IN} = \mathbf{1}$ and $[\![w_0 \in u_0]\!]_{IN} = \mathbf{1}$. Therefore, $[\![\exists u(u \in \check{\omega} \wedge \check{w}_0 \in u)]\!]_{IN} \in F$. Furthermore, since the choice of $\check{w}_0$ was arbitrary we have

$$[\![\forall w\big(w \in \check{\omega} \to \exists u(u \in \check{\omega} \wedge w \in u)\big)]\!]_{IN} \in F.$$

Union: We have to prove, for any $u \in \mathbf{V}^{(\mathbb{LP})}$, there exists an element $v \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\forall x\big(x \in v \leftrightarrow \exists y(y \in u \wedge x \in y)\big)]\!]_{IN} \in F$. Indeed, it is sufficient to show that,

$$[\![\forall x\big(x \in v \to \exists y(y \in u \wedge x \in y)\big)]\!]_{IN} \wedge [\![\forall x\big(\exists y(y \in u \wedge x \in y) \to x \in v\big)]\!]_{IN} \in F.$$

Take any $u \in \mathbf{V}^{(\mathbb{LP})}$ and define $v \in \mathbf{V}^{(\mathbb{LP})}$, as follows:

$$\mathrm{dom}(v) = \bigcup\{\mathrm{dom}(y) \mid y \in \mathrm{dom}(u)\} \text{ and } v(x) = [\![\exists y(y \in u \wedge x \in y)]\!]_{IN}, \text{ for } x \in \mathrm{dom}(v).$$

We first prove that,

$$[\![\forall x\big(x \in v \to \exists y(y \in u \wedge x \in y)\big)]\!]_{IN} \in F,$$

By using $\mathcal{BQ}_\varphi$, we have

$$\begin{aligned}
&[\![\forall x\big(x \in v \to \exists y(y \in u \wedge x \in y)\big)]\!]_{IN} \\
&= \bigwedge_{x \in \mathrm{dom}(v)} \big(v(x) \Rightarrow [\![\exists y(y \in u \wedge x \in y)]\!]_{IN}\big) \\
&= \bigwedge_{x \in \mathrm{dom}(v)} \big([\![\exists y(y \in u \wedge x \in y)]\!]_{IN} \Rightarrow [\![\exists y(y \in u \wedge x \in y)]\!]_{IN}\big) \\
&\in F, \text{ since } a \Rightarrow a \in F, \text{ for any element } a \in \mathbb{LP}.
\end{aligned}$$

We shall now show that $[\![\forall x\Big(\exists y(y \in u \wedge x \in y) \to x \in v\Big)]\!]_{IN} \in F$. Let $x_0 \in \mathbf{V}^{(\mathbb{LP})}$ be an element such that $[\![\exists y(y \in u \wedge x_0 \in y)]\!]_{IN} \in F$. The proof will be completed if it can be derived that $[\![x_0 \in v]\!]_{IN} \in F$ as well. By definition, $[\![\exists y(y \in u \wedge x_0 \in y)]\!]_{IN} \in F$ implies that, there exists $y_0 \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![y_0 \in u \ \wedge \ x_0 \in y_0]\!]_{IN} \in F$. Now, $[\![y_0 \in u]\!]_{IN} \in F$ guarantees an element $y_1 \in \mathrm{dom}(u)$ such that $u(y_1) \in F$ and $[\![y_1 = y_0]\!]_{IN} \in F$. So, we have, $[\![y_1 = y_0 \ \wedge \ x_0 \in y_0]\!]_{IN} \in F$ and by using Lemma 6.2.12(v) it can be concluded that $[\![x_0 \in y_1]\!]_{IN} \in F$. Hence, there exists $x_1 \in \mathrm{dom}(y_1)$ satisfying $y_1(x_1) \in F$ and $[\![x_0 = x_1]\!]_{IN} \in F$. Since, by our assumption, $u(y_1) \in F$ and $y_1(x_1) \in F$, applying Lemma 6.2.12(ii) both $[\![y_1 \in u]\!]_{IN} \in F$ and $[\![x_1 \in y_1]\!]_{IN} \in F$ hold. Hence, $[\![y_1 \in u \ \wedge \ x_1 \in y_1]\!]_{IN} \in F$, which leads to the fact that $[\![\exists y(y \in u \ \wedge \ x_1 \in y)]\!]_{IN} \in F$, i.e., $v(x_1) \in F$. So, we have derived that, $[\![x_0 = x_1]\!]_{IN},\ v(x_1) \in F$. Hence, $[\![x_0 \in v]\!]_{IN} \in F$.

Therefore, we can conclude $[\![\mathsf{Union}]\!]_{IN} \in F$.

Power Set: We have to prove that for any $x \in \mathbf{V}^{(\mathbb{LP})}$ there exists a $y \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\forall z\Big(z \in y \leftrightarrow \forall w(w \in z \to w \in x)\Big)]\!]_{IN} \in F$. It is enough to show that

$$[\![\forall z\Big(z \in y \to \forall w(w \in z \to w \in x)\Big)]\!]_{IN} \wedge [\![\forall z\Big(\forall w(w \in z \to w \in x) \to z \in y\Big)]\!]_{IN} \in F.$$

We begin by showing that,

$$[\![\forall z\Big(z \in y \to \forall w(w \in z \to w \in x)\Big)]\!]_{IN} \in F.$$

Take any $x \in \mathbf{V}^{(\mathbb{LP})}$ and define y such that

$$\mathrm{dom}(y) = \mathbb{LP}^{\mathrm{dom}(x)} \text{ and for any } z \in \mathrm{dom}(y), y(z) = [\![\forall w(w \in z \to w \in x)]\!]_{IN}.$$

We may apply $\mathcal{BQ}_\varphi$ on the first conjunct of Power Set,

$$\begin{aligned}
&[\![\forall z(z \in y \to \forall w(w \in z \to w \in x))]\!]_{IN} \\
&= \bigwedge_{z\in \mathrm{dom}(y)} \Big(y(z) \Rightarrow [\![\forall w(w \in z \to w \in x)]\!]_{IN} \Big) \\
&= \bigwedge_{z\in \mathrm{dom}(y)} \Big([\![\forall w(w \in z \to w \in x)]\!]_{IN} \Rightarrow [\![\forall w(w \in z \to w \in x)]\!]_{IN} \Big) \\
&\in F, \text{ since } a \Rightarrow a \in F, \text{ for any element } a \in \mathbb{LP}.
\end{aligned}$$

For the second conjunct of Power Set, fix an arbitrary $z \in \mathbf{V}^{(\mathbb{LP})}$. Then,

$$\begin{aligned}
&[\![\forall w(w \in z \to w \in x)]\!]_{IN} \Rightarrow [\![z \in y]\!]_{IN} \\
&= \bigwedge_{w\in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![w \in x]\!]_{IN} \Big) \Rightarrow \bigvee_{q\in \mathrm{dom}(y)} \Big(y(q) \wedge [\![z = q]\!]_{IN} \Big), \text{ by } \mathcal{BQ}_\varphi \\
&= \bigwedge_{w\in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![w \in x]\!]_{IN} \Big) \Rightarrow \bigvee_{q\in \mathrm{dom}(y)} \Big(\bigwedge_{p\in \mathrm{dom}(q)} \Big(q(p) \Rightarrow [\![p \in x]\!]_{IN} \Big) \wedge \\
&\qquad \bigwedge_{w\in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![w \in q]\!]_{IN} \Big) \wedge \bigwedge_{p\in \mathrm{dom}(q)} \Big(q(p) \Rightarrow [\![p \in z]\!]_{IN} \Big)\Big).
\end{aligned}$$

Let us assume that,

$$\bigwedge_{w\in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![w \in x]\!]_{IN} \Big) \in F.$$

Then it is enough to show that there exists a $q' \in \mathrm{dom}(y)$ for which,

$$\bigwedge_{p\in \mathrm{dom}(q')} \Big(q'(p) \Rightarrow [\![p \in x]\!]_{IN} \Big) \wedge \bigwedge_{w\in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![w \in q']\!]_{IN} \Big) \wedge \bigwedge_{p\in \mathrm{dom}(q')} \Big(q'(p) \Rightarrow [\![p \in z]\!]_{IN} \Big) \in F.$$

Notice that, for any $q \in \mathrm{dom}(y)$, we have $\mathrm{dom}(q) = \mathrm{dom}(x)$. Fix $q' \in \mathrm{dom}(y)$ such that $q'(p) = x(p) \wedge [\![p \in z]\!]_{IN}$, for any $p \in \mathrm{dom}(q')$. The third conjunct follows immediately and the first one by the Lemma 6.2.12(ii). For the second conjunct, suppose $w \in \mathrm{dom}(z)$ is such that $z(w) \in F$. By our assumption, $[\![w \in x]\!]_{IN} \in F$, i.e.,

$$\bigvee_{p\in \mathrm{dom}(x)} (x(p) \wedge [\![p = w]\!]_{IN}) \in F.$$

Hence, there exists $p' \in \mathrm{dom}(x)$ such that $(x(p') \wedge [\![p' = w]\!]_{IN}) \in F$. Now, we notice that, $[\![p' \in z]\!]_{IN} \geq z(w) \wedge [\![p' = w]\!]_{IN} \in F$, by our assumptions. Hence, we get the following,

$$\begin{aligned}
[\![w \in q']\!]_{IN} &= \bigvee_{p \in \mathrm{dom}(q')} (q'(p) \wedge [\![p = w]\!]_{IN}) \\
&\geq q'(p') \wedge [\![p' = w]\!]_{IN}, \text{ since } p' \in \mathrm{dom}(q') \text{ as well} \\
&= x(p') \wedge [\![p' \in z]\!]_{IN} \wedge [\![p' = w]\!]_{IN} \\
&\in F.
\end{aligned}$$

Hence, $\bigwedge_{w \in \mathrm{dom}(z)} \Big(z(w) \Rightarrow [\![w \in q']\!]_{IN} \Big) \in F$.

Combining all the above results we conclude that $[\![\textsf{Power Set}]\!]_{IN} \in F$.

Separation: Let $\varphi(x)$ be any formula in $\mathcal{L}_{\mathbb{LP}}$, where x is the only free variable. We want to show that for any $x \in \mathbf{V}^{(\mathbb{LP})}$ there exists a $y \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall z \Big(z \in y \leftrightarrow (z \in x \wedge \varphi(z)) \Big)]\!]_{IN} \in F.$$

It is sufficient to show that:

$$[\![\forall z \Big(z \in y \rightarrow (z \in x \wedge \varphi(z)) \Big)]\!]_{IN} \wedge [\![\forall z \Big((z \in x \wedge \varphi(z)) \rightarrow z \in y \Big)]\!]_{IN} \in F.$$

For any $x \in \mathbf{V}^{(\mathbb{LP})}$ define $y \in \mathbf{V}^{(\mathbb{LP})}$ as follows:

$$\mathrm{dom}(y) = \mathrm{dom}(x) \text{ and for any } z \in \mathrm{dom}(y) \text{ let } y(z) = x(z) \wedge [\![\varphi(z)]\!]_{IN}.$$

We first prove that,

$$[\![\forall z \Big(z \in y \rightarrow (z \in x \wedge \varphi(z)) \Big)]\!]_{IN} \in F.$$

By using $\mathcal{BQ}_{\varphi}$, we have

$$
\begin{aligned}
&[\![\forall z\Big(z \in y \to (z \in x \wedge \varphi(z))\Big)]\!]_{IN} \\
&= \bigwedge_{z \in \mathrm{dom}(y)} \Big(y(z) \Rightarrow [\![z \in x \wedge \varphi(z)]\!]_{IN}\Big) \\
&= \bigwedge_{z \in \mathrm{dom}(y)} \Big((x(z) \wedge [\![\varphi(z)]\!]_{IN}) \Rightarrow ([\![z \in x]\!]_{IN} \wedge [\![\varphi(z)]\!]_{IN})\Big) \\
&\in F, \text{ by Lemma 6.2.12(ii).}
\end{aligned}
$$

Now we show that the second conjunct of $\mathsf{Separation}$ holds as well. Since, for any a, b, c in $\mathbb{LP}$, $(a \wedge b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$, we have

$$
\begin{aligned}
\bigwedge_{z \in \mathbf{V}^{(\mathbb{LP})}} [\![(z \in x \wedge \varphi(z)) \to z \in y]\!]_{IN} &= \bigwedge_{z \in \mathbf{V}^{(\mathbb{LP})}} \Big([\![z \in x]\!]_{IN} \Rightarrow ([\![\varphi(z)]\!]_{IN} \Rightarrow [\![z \in y]\!]_{IN})\Big) \\
&= [\![\forall z\Big(z \in x \to (\varphi(z) \to z \in y)\Big)]\!]_{IN} \\
&= \bigwedge_{z \in \mathrm{dom}(x)} \Big(x(z) \Rightarrow [\![(\varphi(z) \Rightarrow z \in y)]\!]_{IN}\Big).
\end{aligned}
$$

For any $z_0 \in \mathrm{dom}(x)$, suppose $x(z_0), [\![\varphi(z_0)]\!]_{IN} \in D$. Then by construction of y we have $y(z_0) \in F$ and by lemma 6.2.12(ii) $[\![z_0 \in y]\!]_{IN} \in F$. Therefore, we can conclude that for any $z_0 \in \mathrm{dom}(x)$, $\Big(x(z_0) \Rightarrow [\![(\varphi(z_0) \Rightarrow z_0 \in y)]\!]_{IN}\Big) \in F$. Hence,

$$
[\![\forall z\Big((z \in x \wedge \varphi(z)) \to (z \in y)\Big)]\!]_{IN} \in F.
$$

Since the images of both the conjuncts of $\mathsf{Separation}$ are in F under the interpretation map $[\![\cdot]\!]_{IN}$, we conclude that $[\![\mathsf{Separation}]\!]_{IN} \in F$.

$\mathsf{Collection}$: Let $\varphi(x, y)$ be any formula in the language of set theory with two free variables. We want to proof that for every $u \in \mathbf{V}^{(\mathbb{LP})}$ we have

$$
[\![\forall x\Big(x \in u \to \exists y \varphi(x, y)\Big) \to \exists v \forall x\Big(x \in u \to \exists y(y \in v \wedge \varphi(x, y))\Big)]\!]_{IN} \in F.
$$

Take any $u \in \mathbf{V}^{(\mathbb{LP})}$ and assume the antecedent holds, so $[\![\forall x\Big(x \in u \to \exists y \varphi(x, y)\Big)]\!]_{IN} \in F$. In particular, by using $\mathcal{BQ}_{\varphi}$, we have;

$$
[\![\forall x\Big(x \in u \to \exists y \varphi(x, y)\Big)]\!]_{IN} = \bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}} [\![\varphi(x, y)]\!]_{IN}\Big) \in F. \tag{1}
$$

Now we will show that there exists a $v \in \mathbf{V}^{(\mathbb{LP})}$ such that

$$[\![\forall x\Big(x \in u \to \exists y(y \in v \wedge \varphi(x,y))\Big)]\!]_{IN} \in F,$$

We know that $\mathbb{LP}$ is a set, so $\mathbb{LP} \in \mathbf{V}$. Thus, we may apply $\mathsf{Collection}$ in $\mathbf{V}$ so that for any $x \in \mathrm{dom}(u)$ we obtain an ordinal α_x such that

$$\bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}} [\![\varphi(x,y)]\!]_{IN} = \bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}_{\alpha_x}} [\![\varphi(x,y)]\!]_{IN}.$$

So we have

$$\bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathbb{LP})}_{\alpha_x}} [\![\varphi(x,y)]\!]_{IN}\Big) \in F. \tag{2}$$

We apply the union axiom in $\mathbf{V}$ to define $\alpha = \bigcup\{\alpha_x : x \in \mathrm{dom}(u)\}$. We define the element $v \in \mathbf{V}^{(\mathbb{LP})}$ as $\mathrm{dom}(v) = \mathbf{V}^{(\mathbb{LP})}_{\alpha}$ and for every $y \in \mathrm{dom}(v), v(y) = 1$. We move on to show,

$$\bigwedge_{x \in \mathrm{dom}(u)} \Big(u(x) \Rightarrow [\![\exists y\Big(y \in v \wedge \varphi(x,y)\Big)]\!]_{IN}\Big) \in F.$$

Take any $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \in F$. By (2) we have $y_0 \in \mathbf{V}^{(\mathbb{T})}_{\alpha_{x_0}}$ such that $[\![\varphi(x_0,y_0)]\!]_{IN} \in F$. By our construction $y_0 \in \mathrm{dom}(v)$ and $v(y_0) = \mathbf{1}$. Then, it follows by Lemma 6.2.12(ii) that $[\![y_0 \in v]\!]_{IN} \in F$. Therefore, $[\![y_0 \in v \wedge \varphi(x_0,y_0)]\!]_{IN} \in F$ and thus,

$$[\![\exists y\Big(y \in v \wedge \varphi(x_0,y)\Big)]\!]_{IN} \in F.$$

Since the choice of x_0 is arbitrary we have,

$$[\![\forall x\Big(x \in u \to \exists y(y \in v \wedge \varphi(x,y))\Big)]\!]_{IN} \in F.$$

Therefore, we conclude that $[\![\mathsf{Collection}]\!]_{IN} \in F$.

$\mathsf{Foundation}$: We want to show that

$$[\![\forall x\Big(\forall y(y \in x \to \varphi(y)) \to \varphi(x)\Big) \to \forall x \varphi(x)]\!]_{IN} \in F.$$

Thus take any $x \in \mathbf{V}^{(\mathbb{LP})}$ and consider the following two cases:

(i) Let $[\![\varphi(x)]\!]_{IN} \in F$ for every $x \in \mathbf{V}^{(\mathbb{LP})}$, which implies $[\![\forall x\varphi(x)]\!]_{IN} \in F$. Therefore, we get immediately that

$$[\![\forall x\Big((\forall y(y \in x \to \varphi(y))) \to \varphi(x)\Big) \to \forall x\varphi(x)]\!]_{IN} \in F.$$

(ii) Let $[\![\varphi(x)]\!]_{IN} \notin F$ for some $x \in \mathbf{V}^{(\mathbb{LP})}$. Then, take a minimal $u \in \mathbf{V}^{(\mathbb{LP})}$ such that $[\![\varphi(u)]\!]_{IN} \notin F$ and for any $v \in \mathrm{dom}(u)$, $[\![\varphi(v)]\!]_{IN} \in F$. For this u, we claim that

$$[\![\forall y(y \in u \to \varphi(y))]\!]_{IN} \in F.$$

Using $\mathcal{BQ}_\varphi$ and our assumption, it is immediate that $[\![\forall y(y \in u \to \varphi(y))]\!]_{IN} \in F$. Now we have two cases:

$$[\![\forall y(y \in u \to \varphi(y))]\!]_{IN} = \frac{1}{2} \text{ or } [\![\forall y(y \in u \to \varphi(y))]\!]_{IN} = \mathbf{1}.$$

Then we have either

$$[\![\forall y(y \in u \to \varphi(y)) \to \varphi(u)]\!]_{IN} = \frac{1}{2} \text{ or } [\![\forall y(y \in u \to \varphi(y)) \to \varphi(u)]\!]_{IN} = \mathbf{0}.$$

Thus

$$[\![\forall x\Big(\forall y(y \in x \to \varphi(y)) \to \varphi(x)\Big)]\!]_{IN} \leq \frac{1}{2}.$$

Hence, the antecedent of $\mathsf{Foundation}$ receives a value less or equal to $\frac{1}{2}$. Thus we have $[\![\mathsf{Foundation}]\!]_{IN} \in F$.

Choice: Fix an arbitrary non-empty $u \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$, i.e.,

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F \neg(u = \varnothing).$$

In particular, this means $\mathrm{dom}_{pos}(u) \neq \varnothing$. Now, take any $[x] \in \mathrm{dom}_{pos}(u)/\sim$ and consider the following two cases.

Case (I): Suppose that $[x]$ does not contain any $\mathbf{0}$-like element. Fix an element $s_{[x]} \in [x]$. By our assumption, we know that $\mathrm{dom}_{pos}(s_{[x]}) \neq \varnothing$. Moreover, choose a $t_{[x]} \in \mathrm{dom}_{pos}(s_{[x]})$. Then we define three elements $p_{[x]}, q_{[x]}, w_{[x]} \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ such that

$$p_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle\}, q_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle, \langle t_{[x]}, \mathbf{1}\rangle\} \text{ and } w_{[x]} = \{\langle p_{[x]}, \mathbf{1}\rangle, \langle q_{[x]}, \mathbf{1}\rangle\}.$$

Case (II): Suppose that $[x]$ is the class of $\mathbf{0}$-like elements in $\mathrm{dom}_{pos}(u)$. Let us arbitrarily fix any two $\mathbf{0}$-like elements $s, t \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Following the same construction as in Case I, we define three elements $p_{[x]}, q_{[x]}, w_{[x]} \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ such that

$$p_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle\}, q_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle, \langle t_{[x]}, \mathbf{1}\rangle\} \text{ and } w_{[x]} = \{\langle p_{[x]}, \mathbf{1}\rangle, \langle q_{[x]}, \mathbf{1}\rangle\}.$$

Then consider an element f such that

$$f = \{\langle w_{[x]}, \mathbf{1}\rangle : [x] \in \mathrm{dom}(u)/\sim\}.$$

The existence of f in $\mathbf{V}$ follows by the fact that Choice holds in $\mathbf{V}$. Then, by the construction $f \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. Furthermore, it can be shown readily that

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F \mathsf{Func}(f) \wedge \mathsf{Dom}(f;\ u).$$

We are done if we prove that

$$\forall x\Big(x \in u \wedge \neg(x = \varnothing) \to \exists z \exists y(\mathsf{Pair}(z;\ x, y) \wedge z \in f \wedge y \in x)\Big).$$

Consider any $v \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ such that

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F v \in u \wedge \neg(v = \varnothing).$$

Then there exists an element $x \in \mathrm{dom}_{pos}(u)$ such that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F v = x$, and x is not $\mathbf{0}$-like. Consider the equivalence class $[x]$ containing x in $\mathrm{dom}_{pos}(u)/\sim$. By the construction of f, there exists $w_{[x]} \in \mathrm{dom}(f)$ which is of the form $\{\langle p_{[x]}, \mathbf{1}\rangle, \langle q_{[x]}, \mathbf{1}\rangle\}$, where

$$p_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle\}, q_{[x]} = \{\langle s_{[x]}, \mathbf{1}\rangle, \langle t_{[x]}, \mathbf{1}\rangle\}, s_{[x]} \in [x] \text{ and } t_{[x]} \in \mathrm{dom}_{pos}(s_{[x]}).$$

Since $s_{[x]} \in [x]$, we get that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F s_{[x]} = x$, which implies $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F s_{[x]} = v$. Hence, we can derive that

$$\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F \mathsf{Pair}(w_{[x]};\ v, t_{[x]}) \wedge w_{[x]} \in f \wedge t_{[x]} \in v.$$

Thus, we can finally conclude that $[\![\mathsf{Choice}]\!] \in F$. $\square$

Moreover, we can go on to show that the model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ modulo the filter F is paraconsistent.

Lemma 6.2.19. *There exists a formula $\varphi \in \mathsf{Sent}_{\in}$ such that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_F \varphi \wedge \neg\varphi$.*

Proof. Consider the following sentence: $\varphi := \exists x \exists y (x \in y \wedge x \notin y)$. Now, simply consider the $\mathbb{LP}$-name $u = \{\langle v, \frac{1}{2}\rangle\}$ where v is an arbitrary $\mathbb{LP}$-name. We readily calculate that $[\![u \in v]\!]_{IN} = \frac{1}{2}$, as well as $[\![u \notin v]\!]_{IN} = [\![u \in v]\!]^*_{IN} = \frac{1}{2}^* = \frac{1}{2}$. Hence,

$$[\![\varphi]\!]_{IN} = \frac{1}{2} = \frac{1}{2}^* = [\![\neg\varphi]\!]_{IN} \in F,$$

which completes the proof. $\square$

Then, we can readily show that:

Lemma 6.2.20. *For any $u \in \mathbf{V}^{(\mathbb{LP})}$ we have $[\![u \in u]\!]_{IN} = \mathbf{0}$ and thus we have that $[\![\exists y \forall x (x \in y)]\!]_{IN} = \mathbf{0}$.*

Proof. Suppose we have a minimal counterexample to the claim. So there exists a u such that $[\![u \in u]\!]_{IN} \neq \mathbf{0}$ (†) , however, for every $x \in \mathrm{dom}(u)$ we have $[\![x \in x]\!]_{IN} = \mathbf{0}$. Due to (†) we know that there exists a $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \neq \mathbf{0}$ and $[\![u = x_0]\!]_{IN} \neq \mathbf{0}$. In particular, $[\![u = x_0]\!]_{IN} = \mathbf{1}$, so for every $x \in \mathrm{dom}(u)$ we have $u(x) \leq [\![x \in x_0]\!]_{IN}$. Moreover, given that $u(x_0) \neq \mathbf{0}$ we have $[\![x_0 \in x_0]\!]_{IN} \neq \mathbf{0}$ which delivers us the desired contradiction. Hence, for any $u \in \mathbf{V}^{(\mathbb{LP})}$ we have $[\![u \in u]\!]_{IN} = \mathbf{0}$ and thus we have that $[\![\exists y \forall x (x \in y)]\!]_{IN} = \mathbf{0}$. $\square$

This allows us to show that $\mathsf{Comprehension}$ fails in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$.

Lemma 6.2.21. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \not\models_F \mathsf{Comprehension}_{\varphi}$.

Proof. Consider $\varphi =_{df.} y \notin y$. Due to Lemma 6.2.20, for every $u \in \mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$, we have $[\![u \in u]\!]_{IN} = \mathbf{0}$ and thus $[\![u \notin u]\!]_{IN} = [\![u \in u]\!]^*_{IN} = \mathbf{1}$. Then we get:

$$[\![u \in u \leftrightarrow u \notin u]\!]_{IN} = \Big((\mathbf{0} \Rightarrow \mathbf{1}) \wedge (\mathbf{1} \Rightarrow \mathbf{0})\Big) = \mathbf{0}.$$

$\square$

6.2.3 The logics of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$

We go on to show that the model $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is indeed faithful to the lattice $\mathbb{LP}$.

Theorem 6.2.22. *The model* $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ *is faithful to* $\mathbb{LP}$ *and hence loyal to* $(\mathbb{LP}, F)$*, for any filter* F*.*

Proof. We know that the sentence $\forall x(x = x)$ receives value $\mathbf{1}$, i.e., $[\![\forall x(x = x)]\!]_{IN} = \mathbf{1}$ and hence we have $[\![\neg\forall x(x = x)]\!]_{IN} = \mathbf{0}$. So we are done in the case that we can find a sentence $\varphi \in \mathsf{Sent}_{\in}$ such that $[\![\varphi]\!]_{IN} = \frac{1}{2}$. Simply consider sentence φ of Lemma 6.2.19. $\square$

Moreover, it is a well-known fact that the propositional logic associated to the lattice $\mathbb{LP}$ modulo the filter $F = \{\mathbf{1}, \frac{1}{2}\})$ is LP and that the propositional logic associated to the same lattice given the set of designated values that contains only the top element, i.e., $\{\mathbf{1}\}$, is Kleene's Logic K_3. Thus we get:

Corollary 6.2.23. $\mathbf{L}\Big(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}, \frac{1}{2}\}\Big) = \mathrm{LP}.$ $\square$

Corollary 6.2.24. $\mathbf{L}\Big(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}\}\Big) = \mathrm{K}_3.$ $\square$

It is easy to notice that Theorem 6.2.11, Lemma 6.2.12, Lemma 6.2.13, Corollary 3.6 and Lemma 6.2.15 are still valid in $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ given the top filter. Introspection of the relevant proofs shows that exactly the same calculations work for this case. The validity of ZF, however, does not extend to this model due to the failure of $\mathsf{Extensionality}$.

Theorem 6.2.25. $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \not\models_{\{\mathbf{1}\}} \mathsf{Extensionality}$

Proof. Consider the $\mathbb{LP}$-names $p_{\frac{1}{2}}$ and $p_{\mathbf{1}}$. Then we calculate readily:

$$
\begin{aligned}
&[\![\mathsf{Extensionality}]\!]_{IN}\\
&= [\![\forall x \forall y \big(\forall z(z \in x \leftrightarrow z \in y) \to x = y\big)]\!]_{IN}.\\
&= \bigwedge_{u \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{v \in \mathbf{V}^{(\mathbb{LP})}} \Big([\![\forall z(z \in u \leftrightarrow z \in v)]\!]_{IN} \Rightarrow [\![u = v]\!]_{IN}\Big)\\
&= \bigwedge_{u \in \mathbf{V}^{(\mathbb{LP})}} \bigwedge_{v \in \mathbf{V}^{(\mathbb{LP})}} \Big(\Big(\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]_{IN}) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]_{IN})\Big) \Rightarrow [\![u = v]\!]_{IN}\Big)\\
&\leq \Big((p_{\mathbf{1}}(\varnothing) \Rightarrow [\![\varnothing \in p_{\frac{1}{2}}]\!]_{IN}) \wedge (p_{\frac{1}{2}}(\varnothing) \Rightarrow [\![\varnothing \in p_{\mathbf{1}}]\!]_{IN})\Big) \Rightarrow [\![u = v]\!]_{IN}\\
&= \Big((\mathbf{1} \Rightarrow \frac{1}{2}) \wedge (\frac{1}{2} \Rightarrow \mathbf{1})\Big) \Rightarrow \mathbf{0}\\
&= \frac{1}{2}\\
&\notin \{\mathbf{1}\}.
\end{aligned}
$$

□

The failure of Extensionality shows that if we choose a more classical set of designated values, i.e., $\{\mathbf{1}\}$, on $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ we end up with a model which is properly speaking not a model of set theory anymore. What do we learn from this? On the one hand, it shows that the choice of the set of designated values is relevant, unlike in the case of Boolean-valued models and the standard interpretation map $[\![\cdot]\!]$. Thus, even though the $[\![\cdot]\!]_{IN}$-interpretation map allowed us to build a non-classical model of ZFC based on LP, it comes with a price: we are bounded to one particular choice of designated values. Moreover, given that Extensionality is a sentence in the language of set theory we can show that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ and $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ are non-$\in$-elementarily equivalent with each other.

Corollary 6.2.26. *We have*

$$(\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}, \frac{1}{2}\}) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}\}).$$

□

Moreover, we can show as well that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ modulo the positive filter is non-$\in$-elementarily equivalent with each $(\mathbb{T}, ^*)$-valued model (where $^* \in \{^{*_p}, ^{*_r}, ^{*_s}, ^{*_c}, ^{*_e}\}$) and each $(\mathbb{A}, ^{*_r})$-valued model.

Theorem 6.2.27. *Let* $(\mathbb{T},^*)$ *be an implicative meet complemented (or bicomplemented)* $\mathcal{W}_{\mathcal{F}}$*-lattice where* $^* \in \{^{*_p}, ^{*_r}, ^{*_s}, ^{*_c}, ^{*_e}\}$ *with more than two elements and* $(\mathbb{A},^{*_r})$ *be any reflexive Heyting-implication lattice with more than two elements. Moreover, let* F_1 *be a filter on* $\mathbb{T}$ *and* F_2 *a filter on* $\mathbb{A}$*. Then we have;*

$$(\mathbf{V}^{(\mathbb{T},^*)}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, Pos_{(\mathbb{LP})}) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{A},^{*_r})}, F_2).$$

Proof. Consider the following sentence $\psi = \forall w\Big(\exists x \exists y(x \in y \wedge x \notin y) \rightarrow (w \neq w)\Big)$. We know by Lemma 6.2.12(i) that $[\![\forall w(w = w)]\!]^{\mathbb{LP}}_{IN} = \mathbf{1}$. Thus we get

$$[\![\neg\forall x(x = x)]\!]^{\mathbb{LP}}_{IN} = \Big([\![\forall x(x = x)]\!]^{\mathbb{LP}}_{IN}\Big)^* = \mathbf{1}^* = \mathbf{0}.$$

Moreover, due to Lemma 6.2.19 we get

$$[\![\psi]\!]^{\mathbb{LP}}_{IN} = (\frac{1}{2} \Rightarrow \mathbf{0}) = \frac{1}{2} \in Pos_{(\mathbb{LP})}.$$

Then, we calculate readily:

$$[\![\neg\forall w(w = w)]\!]^{(\mathbb{T},^*)} = [\![\neg\forall w(w = w)]\!]^{(\mathbb{A},^{*_r})} = \mathbf{0}$$

and we know that

$$[\![\exists x \exists y(x \in y \wedge x \notin y)]\!]^{(\mathbb{T},^*)} = [\![\exists xy(x \in y \wedge x \notin y)]\!]^{(\mathbb{A},^{*_r})} = a,$$

where a is the co-atom of the universe of $\mathbb{T}$ and $\mathbb{A}$. We conclude

$$[\![\psi]\!]^{(\mathbb{T},^*)} = [\![\psi]\!]^{(\mathbb{A},^{*_r})} = (a \Rightarrow \mathbf{0}) = \mathbf{0}.$$

This is not the case for the three-valued $(\mathbb{A},^{*_e})$-lattice since $[\![\exists x \exists y(x \in y \wedge x \notin y)]\!]^{(\mathbb{A}_3,^{*_e})} = \mathbf{0}$. However, for this model we can use the sentence

$$\psi' = \forall w\Big(\forall x \forall y(x \in y \vee x \notin y) \rightarrow (w \neq w)\Big).$$

We calculate readily that $[\![\psi']\!]^{(\mathbb{A}_3,^{*_e})} = \mathbf{0}$, however, $[\![\psi']\!]^{\mathbb{LP}}_{IN} \in Pos_{(\mathbb{LP})}$. $\square$

Similarly, we can show as well that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ modulo the top filter is non-$\in$-elementarily equivalent with each $(\mathbb{T},^{*})$-valued model (where $^{*} \in \{^{*_p}, ^{*_r}, ^{*_s}, ^{*_c}, ^{*_e}\}$) and each $(\mathbb{A},^{*_r})$-valued model.

Theorem 6.2.28. *Let* $(\mathbb{T},^{*})$ *be an implicative meet complemented (or bicomplemented)* $\mathcal{W}_{\mathcal{F}}$*-lattice where* $^{*} \in \{^{*_p}, ^{*_r}, ^{*_s}, ^{*_c}, ^{*_e}\}$ *with more than two elements and* $(\mathbb{A},^{*_r})$ *be any reflexive Heyting-implication lattice with more than two elements. Moreover, let* F_1 *be a filter on* $\mathbb{T}$ *and* F_2 *a filter on* $\mathbb{A}$*. Then we have;*

$$(\mathbf{V}^{(\mathbb{T},^{*})}, F_1) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}, \{\mathbf{1}\}) \not\equiv_{\in} (\mathbf{V}^{(\mathbb{A},^{*_r})}, F_2).$$

Proof. Consider the sentence $\sigma := \psi \to \psi$, where $\psi = \exists x \exists y(x \in y \wedge x \notin y)$. We calculate readily that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \not\models_{\{\mathbf{1}\}} \sigma$. However, we have $\mathbf{V}^{(\mathbb{T},^{*})} \models_{\{\mathbf{1}\}} \sigma$ and $\mathbf{V}^{(\mathbb{A},^{*_r})} \models_{\{\mathbf{1}\}} \sigma$. $\square$

We have shown that we can build an algebra-valued model that validates ZFC, which has as internal logic LP and which preserves all the intuitive properties we would like to attribute to identity. Moreover, we observe that we have two fundamental differences in our model construction.

(a) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is **not** a model of NLP,

(b) $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ does **not** allow us to derive all the theorems of ZFC.

Notice that (a) is due to Lemma 6.2.21. Without going into technical details we believe that it might be possible to extend our algebra-valued model with class functions that might be used to interpret the universal set. Nevertheless, it is unclear how exactly our underlying model $\mathbf{V}$ has to look like. What we can say here is that $\mathbf{V}$ should be a model of a class theory, so we can talk about class-functions in our extended algebra-valued model and that the resulting model should avoid that a single set becomes the whiteness of all the instances of Comprehension.

Moreover, (b) is a drawback compared to Priest's model construction since it is unclear how many theorems of ZFC we can derive in our model. At the same time, this is also a distinctive feature of our approach, i.e., the possibility of determining the set of valid theorems of ZFC within a non-classical model of ZFC. We leave this task for future work. However, we believe that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is an excellent candidate for this enterprise due to the validity of Leibniz's law of indiscernibility of identicals. So we can actually build a quotient model out of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ where we have a reasonable ontology.

Finally, instead of arguing that one model construction is preferable over another one, we simply acknowledge that two different games are played. Whereas Priest is concerned in showing that we can make sense of the classical theorems of ZFC from a paraconsistent perspective, we are concerned in finding out how much set theory we can obtain in a paraconsistent model of ZFC. It seems that a moral that we can draw from this is that we can not have the whole cake. There exists a trade-off between the validity of ZFC (and its theorems) and desirable model-theoretic properties. In the case of Priest we get the validity of ZFC and all the theorems of ZFC, however, we lose model-theoretic properties. In our case, we get a model of the theory axioms of ZFC and all the desirable model-theoretic properties, but we do not get all the theorems of ZFC.

Chapter 7

Conclusions

In this thesis, we have demonstrated that it is possible to build many different non-classical models of set theory applying algebra-valued model constructions. In particular, we have shown that we can construct models of $\mathsf{NFF\text{-}ZF}$, ZF, and ZFC that are compatible with non-classical logics, including paraconsistent logics, paracomplete logics, and logics that are both paraconsistent and paracomplete. Moreover, we have also shown that these models are mathematically expressive.

7.1 What we have done

Now, we state the main results of this thesis. In Chapter 3, we gave an overview of paraconsistent set theories that we can find in the literature. We divided these into two classes: naïve and iterative paraconsistent set theories. This new conceptual distinction allowed us to rethink the notion and extent of paraconsistent set theories. In particular, we showed that we can pursue paraconsistent set theories outside of a dialethist framework. Moreover, we argued that a big advantage for paraconsistent set theories constructed on algebra-valued models is that these models resemble closely the cumulative hierarchy. So, from a classical perspective these models provide the best approach to paraconsistent set theories.

In Chapter 4, we searched for DRI-algebras that give rise to paraconsistent models of set theory. This was done in two steps. Firstly, we searched for implicative complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$ and showed that these lattices are DRI-algebras. Then, we checked that the corresponding algebra-valued model validates $\mathcal{BQ}_\varphi$ for negation-free formulas. So, we get that $\mathbf{V}^{(\mathbb{A})} \models_F$ NFF-ZF for any filter F on $\mathbb{A}$. Secondly, we would expand the language of our lattices with a suitable unary operator * that will be interpreted as the negation in the language of set theory. As a result, we were able to generalize $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$ to the class of $(\mathbb{T},{}^{*_r})$-valued models. The novelty displayed here consisted in giving an abstract algebraic definition for the $\Rightarrow$ operator and * operator of $(\mathbb{PS}_3,{}^*)$ which allowed us to obtain infinitely many paraconsistent models of set theory.

To sum up,

(1) Consider an implicative complete bounded distributive lattice $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$.

(2) Check if $\mathbb{A}$ is a DRI-algebra.

(3) Check if $\mathbf{V}^{(\mathbb{A})} \models_F \mathcal{BQ}_\varphi$, where $\varphi \in \mathsf{NFF}\text{-}\mathcal{L}_\in^{\mathbb{A}}$ and F is a filter on $\mathbb{A}$.

(4) Expand the signature of $\mathbb{A}$ with an unary operator * that we interpret as negation in $\mathbf{V}^{(\mathbb{A},{}^*)}$ such that for some sentence $\varphi \in \mathcal{L}_\in$ we have $\mathbf{V}^{(\mathbb{A},{}^*)} \models_F \varphi \wedge \neg\varphi$, for some filter F on $\mathbb{A}$.

Moreover, we showed that by a suitable choice of * we can build also non-classical models of ZF, i.e., $(\mathbb{T},{}^{*_p})$-valued models. Interestingly, the propositional logic of $\mathbf{V}^{(\mathbb{T},{}^{*_p})}/\{\mathbf{1}\}$ is neither intuitionistic, nor classical, nor paraconsistent. These models were the first algebra-valued models of full ZF that were built on algebras that are not Heyting. Furthermore, we applied these models to give an independence proof of $\mathsf{Foundation}$ from ZF.

In Chapter 5, we showed that we can find infinitely many non-$\in$-elementarily equivalent models of paraconsistent set theory. In particular, these models offer the first examples of independence results for non-classical set theories built using algebra-valued models. We have been able to show that there exist paraconsistent algebra-valued models $\mathbf{V}^{(\mathbb{A}_1)}$ and $\mathbf{V}^{(\mathbb{A}_2)}$, each validating NFF-$\mathsf{ZF}$, and a formula $\varphi \in \mathcal{L}_\in$ such that $\mathbf{V}^{(\mathbb{A}_1)} \models_F \varphi$ and $\mathbf{V}^{(\mathbb{A}_2)} \not\models_F \varphi$, for some filter F. Moreover, we also found two new classes of paraconsistent set theories which are faithful to their underlying lattice. We achieved this result, by expanding the signature of implicative meet complemented $\mathcal{W}_{\mathcal{F}}$-lattices with different unary operations * that interpret the negation in the language of set theory.

Again, to sum up,

(1) Expand the signature of $\mathbb{T} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_t, ^{*_p}, \mathbf{0}, \mathbf{1} \rangle$ with an unary operator * that we interpret as negation in $\mathbf{V}^{(\mathbb{T},^*)}$ such that for some sentence $\varphi \in \mathcal{L}_\in$ we have $\mathbf{V}^{(\mathbb{T},^*)} \models_F \varphi \wedge \neg\varphi$, for some filter F on $\mathbb{T}$.

(2) Define a set of sentences Γ in the language of set theory such that for each $(\mathbb{T}_n,^*)$ and $(\mathbb{T}_m,^*)$, where $n, m \in \mathbb{N}$ and $n \neq m$, there exists a $\varphi \in \Gamma$ such that $\mathbf{V}^{(\mathbb{T}_n,^*)} \models_{F_1} \varphi$ and $\mathbf{V}^{(\mathbb{T}_m,^*)} \not\models_{F_2} \varphi$, for some filter F_1 on $\mathbb{T}_n$ and some filter F_2 on $\mathbb{T}_m$.

In chapter 6, we proposed two new strategies to build non-classical models of set theory. The first strategy gave rise to models of NFF-ZF that are paraconsistent, paracomplete, or both paraconsistent and paracomplete. In particular, we showed that each of these models is non-$\in$-elementarily equivalent from each other and faithful to its underlying lattice. The second strategy allowed us to construct a model of ZFC which validates Leibniz's law of indiscernibility of identicals and has as internal logic Priest's logic of paradox.

We begin by describing the first strategy. Our starting point were the lattices $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_h, \mathbf{0}, \mathbf{1} \rangle$ where the binary operation $\Rightarrow_h$ is a Heyting condtional. Moreovoer, the base set $\mathbf{A}$ is finite and totally-ordered. Then we expanded the signature of $\mathbb{A}$ with the unary operator ^{*_r} and observed that depending on our choice of the filter F on $\mathbb{A}$ we obtain non-classical models of set theory with very different logical properties. In particular, we showed that:

1. The propositional logic of $\mathbf{V}^{(\mathbb{A},^{*_r})}/\{\mathbf{1}\}$ is paracomplete but different from intuitionistic logic.

2. The propositional logic of $\mathbf{V}^{(\mathbb{A},^{*_r})}/F$ where F is any filter that extends the top element is paraconsistent.

3. The propositional logic of $\mathbf{V}^{(\mathbb{A},^{*_r})}/F$ where F is any filter that extends the top element and which does not contain the co-atom of $\mathbb{A}$ is both paraconsistent and paracomplete.

As before, we can summarize as follows:

(1) Consider the lattices $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow_h, \mathbf{0}, \mathbf{1} \rangle$ where the binary operation $\Rightarrow_h$ is a Heyting condtional and $\mathbf{A}$ is totally ordered and finite. Expand the signature of $\mathbb{A}$ with the ^{*r}-operator that we interpret as negation in $\mathbf{V}^{(\mathbb{A},^{*r})}$ and choose a suitable filter F on $\mathbb{A}$.

(2) Define a set of sentences Γ in the language of set theory such that for each $(\mathbb{A}_n,^{*r})$ and $(\mathbb{A}_m,^{*r})$ there exists a $\varphi \in \Gamma$ such that $\mathbf{V}^{(\mathbb{A}_n,^{*r})} \models_{F_1} \varphi$ and $\mathbf{V}^{(\mathbb{A}_m,^{*r})} \not\models_{F_2} \varphi$, for some filter F_1 on $\mathbb{A}_n$ and some filter F_2 on $\mathbb{A}_m$.

Let us now describe the second strategy. The crux of the second strategy consisted in modifying adequately the interpretation of set-membership and identity in our algebra-valued model. We called the resulting the modified interpretation map $[\![\cdot]\!]_{IN}$. This made it possible to build an algebra-valued model based on the lattice $\mathbb{LP}$ (defined on p. 145), i.e., $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$. This model was particularly intestersting given that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})} \models_{Pos_{(\mathbb{LP})}}$ ZFC and because the propositional logic of $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}/Pos_{(\mathbb{LP})}$ is Priest's logic of paradox. Finally, we showed that $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!]_{IN})}$ is non-$\in$-elementarily equivalent from all the models introduced previously in this thesis.

7.2 On Pluralism

In this section, we want to highlight one important philosophical implication of this thesis. In particular, we want to discuss the relation between our results and set-theoretic pluralism. Set-theoretic pluralism is the view (opposed to set-theoretic universalism) that there exists more than one intended model of the universe of sets and that each of these models exists independently of the others., i.e., they have the same ontological status. In some of these models, certain set-theoretic statements such as the continuum hypothesis will hold and in some others, they will be false. The collection of these models is called the pluriverse of sets.

We claim that the pluriverse is much wider than assumed given that ZFC and ZF can also be realized by non-classical models of set theory. In particular, we showed that there exist paraconsistent models of ZFC and ZF. Moreover, these models bear a great resemblance to the cumulative hierarchy and are as Boolean-valued models simply a collection of functions. So, from the perspective of $\mathbf{V}$, there exists absolutely no difference between a Boolean-valued model and our paraconsistent models of set theory.

Notice also that our models fit neatly into Priest's perspectivalism:

> Thus, one might have a universe of predicatively definable sets; a universe of sets with the set theoretic axioms of ZFC, but the underlying logic of which is intuitionist logic; or a set-theory based on the intuitionistic notion of a spread; or a fuzzy set-theory based on, say, Lukasiewicz continuum-valued logic; or a set theory based on a relevant logic; or one based on quantum logic. (PRIEST, 2020, p. 11)

We believe that, in principle, we can build a model of full ZF (or some classically equivalent axiom system) based on any non-classical logic with algebraic semantics. However, we need to find the right interpretation of set membership and identity for our algebra-valued models. This is a non-trivial task since there exists a trade-off between the strength of the notion of identity within our models and the validity of Extensionality. Notice, however, that we need that the binary operation $\Rightarrow$ and unary operator * which will interpret the implication and negation in the language of set theory, respectively, satisfy some minimal constraints to pull through the proof of the ZF axioms.

Thus, we have the following hypothesis:

Hypothesis: It is possible to tailor an interpretation map $[\![\cdot]\!]_X$ for any non-classical logic that is representable as a lattice of the form $\mathbb{A} = \langle \mathbf{A}; \wedge, \vee, \Rightarrow, ^*, \mathbf{0}, \mathbf{1}\rangle$ and that full fills some minimal inferential features, such that $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!]_X)} \models_F \mathsf{ZF}$ for some filter F on $\mathbb{A}$ and where the underlying logic of $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!]_X)}$ modulo F is non-classical.

Thus, we think (similarly as Priest) that we should generalize the pluriverse consisting of classical models of ZFC and ZF to a pluriverse containing models of ZFC and ZF based on all kinds of non-classical logics. Moreover, all these non-classical models are based on the iterative conception of set and mimic the cumulative hierarchy. Therefore, we believe that the iterative conception of set is compatible with many (if not all) logical environments. So, nowadays the iterative conception might be associated exclusively to classical set theory, however, we have shown that this seems to be a contingency. Finally, we argue that set-theoretic pluralism does reach even deeper as assumed: we are not only faced with the choice between classical models of ZFC and ZF, but also with the choice of the right non-logical axioms that compose ZFC and ZF.

7.3 What still has to be done

We conclude by pointing out some directions for future research. Especially, Chapter 6 offers two new roads that could be explored and lead to further generalizations. The first road is intended to extend our model constructions to complete bounded distributive lattices where the underlying universe is not totally-ordered. The starting point is to consider any complete bounded distributive lattice (with an atom and co-atom) equipped with a Boolean conditional. Then, we could expand these lattices with diverse unary operators * and explore the resulting algebra-valued models. This method could give rise to some new classes of non-classical models of set theory. Moreover, the second road that we outlined in Chapter 6 is to consider the interpretation map $[\![\cdot]\!]$ as a new constraint in the construction of algebra-valued model.

Moreover, the following questions have remained unanswered:

1. Determine the class of complete bounded distributive lattices $\mathbb{A}$ such that $\mathbf{L}\Big(\mathbf{V}^{(\mathbb{A})}, Pos_{(\mathbb{A})}\Big) = \mathbf{L}\mathbb{PS}_3$.

2. Determine the class of DRI-algebras $\mathbb{A}$ such that we have $\mathbf{V}^{(\mathbb{A})} \models_F \mathcal{BQ}_\varphi$ for some filter F on $\mathbb{A}$.

3. Does there exist a complete bounded lattice $\mathbb{A}$ such that $\mathbf{V}^{(\mathbb{A},\ [\![\cdot]\!])} \models_F \mathsf{ZFC}$ where $\mathbb{A}$ is not a Boolean Algebra, $[\![\cdot]\!]$ is the standard interpretation map and F is any filter on $\mathbb{A}$?

4. Does there exist a unary opertor * that satisfies either the American or the Australian Plan (algebraically or logically) and that allows us to show that every $(\mathbb{T},^*)$-valued model is non-$\in$-elementarily equivalent ?

5. Which are the propositional logics that correspond to $\mathbf{V}^{(\mathbb{T},^{*c})}/F$ where F is any filter on $\mathbb{A}$?

6. Which are the propositional logics that correspond to $\mathbf{V}^{(\mathbb{T},^{*e})}/F$ where F is any filter on $\mathbb{A}$?

7. Elaborate the minimal account of negation in more detail. Which further *types* of negations can we distinguish on behalf of this account ?

8. Build quotient models on top of reflexive Heyting-implication lattice-valued models. Are these models full/well-defined?

9. Which are the propositional logics that correspond to $\mathbf{V}^{(\mathbb{A}, *r)}/F$ where F is any filter on $\mathbb{A}$?

10. Is $\mathbf{V}^{(\mathbb{LP},\ [\![\cdot]\!])}$ a model of NLP or $\mathsf{NLP}_=$?

11. Does any complete De Morgan algebra give rise to models of ZFC under the $[\![\cdot]\!]_{IN}$-interpretation map?

12. Does any complete Heyting algebra give rise to models of ZFC under the $[\![\cdot]\!]_{IN}$-interpretation map?

We hope that these questions open the door for new and exciting philosophical discussions.

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