UNIVERSIDADE ESTADUAL DE
CAMPINAS

Instituto de Matemática, Estatística e
Computação Científica

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Local and Asymptotic Bridgeland stability

Estabilidade de Bridgeland assintótica e local

Campinas

2021

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics, and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Este trabalho corresponde à versão final da Tese defendida pelo aluno Victor do Valle Pretti e orientada pelo Prof. Dr. Marcos Benevenuto Jardim.

Campinas

2021

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

P927L
Pretti, Victor do Valle, 1993-
Local and asymptotic Bridgeland stability / Victor do Valle Pretti. – Campinas, SP : [s.n.], 2021.

Orientador: Marcos Benevenuto Jardim.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Estabilidade de Bridgeland. 2. Regiões de quivers (Matemática). 3. Instantons. 4. Estabilidade assintótica. I. Jardim, Marcos Benevenuto, 1973-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Estabilidade de Bridgeland assintótica e local
Palavras-chave em inglês:
Bridgeland stability
Quiver regions (Mathematics)
Instantons
Asymptotic stability
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Marcos Benevenuto Jardim [Orientador]
Abdelmoubine Amar Henni
Daniele Faenzi
Antony Maciocia
Cristian Maurício Martínez Esparza
Data de defesa: 20-09-2021
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
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- Currículo Lattes do autor: http://lattes.cnpq.br/8984076070044830

Tese de Doutorado defendida em 20 de setembro de 2021 e aprovada pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Acknowledgements

Gostaria de aqui agradecer a todos que participaram do meu desenvolvimento durante esse período do doutorado.

Primeiramente agradeço a minha familia por me dar todo o apoio necessário durante estes quase dez anos de estudo, culminando nesta tese de doutorado. Sem o suporte de vocês isto seria apenas um sonho distante.

À minha namorada, Luana Carletto de Queiroz, por estar sempre ao meu lado. Ela esteve presente desde a minha qualificação e sempre fez tudo o que pode para me ajudar. Obrigado!

Em seguida, aos meus amigos, que estiveram presentes mesmo quando estivemos longe. Obrigado por tudo e espero encontra-los logo!

Agradeço também a Universidade de Edimburgo pela hospitalidade e apoio durante o período que passei na universidade. Foi uma experiência imprescindível para o desenvolvimento da minha pesquisa.

Aos meus orientadores, Marcos Benevenuto Jardim e Antony Maciocia, agradeço pela paciência, apoio, ajuda e por todas oportunidades que me proporcionaram ao longo deste doutorado. Sem a orientação de vocês seria impossível concluir esta etapa da minha formação e seguir para a próxima.

Finalizo agradecendo a Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) por fomentar o meu estudo acadêmico desde a graduação, neste caso com a bolsa integral de doutorado através do processo 2016/25249-0 e a bolsa de estágio de pesquisa no exterior do processo 2019/05207-9. Esta oportunidade foi de incrível significância para a minha carreira acadêmica.

Victor

Resumo

Neste trabalho de conclusão de doutorado exploramos aspectos locais e assintóticos da teoria de estabilidade de Bridgeland. A primeira parte da tese está concentrada em dois capítulos: o primeiro referente a notação da teoria de esquemas, categorias derivadas e caracteres de Chern. O segundo estabelece os fundamentos da teoria de estabilidade de Bridgeland que serão usados nos capítulos seguintes, com uma breve discussão sobre coleções excepcionais ao final.

A seguir, apresentamos os principais resultados obtidos ao longo do doutorado. O capítulo de estabilidade assintótica mostra resultados analogos aos obtidos por Jardim–Maciocia, agora para objetos com caracter de Chern zero igual a zero. O último capítulo diz respeito ao estudo local da estabilidade de Bridgeland, descrevendo as regiões de aljavas no semi-plano superior de estabilidade $\mathbb{H}$ e com isso provando a estabilidade dos instantons nestas regiões.

Além disso, a tese possui um apêndice e anexo a respeito de alguns resultados teoricos e computacionais produzidos com o intuito de obter exemplos concretos de regiões de aljavas e seus respectivos usos.

Palavras-chave: Estabilidade de Bridgeland. Classes excepcionais. Estabilidade assintótica de Bridgeland. Regiões de aljavas.

Abstract

In this Ph.D. thesis, we explore local and asymptotic aspects of the theory of Bridgeland stability. The first part of the thesis is concentrated in two chapters: the first one referring to a brief discussion on the theory of schemes, derived categories and Chern characters. The second one establishes the fundamental theory of Bridgeland stability necessary for the following chapters, with a brief discussion of exceptional collections towards its end.

Next, we present the main results obtained during the Ph.D. The chapter on asymptotic stability shows results analogous to the ones obtained by Jardim–Maciocia, now for objects with zero Chern character equal to zero. The last chapter is concerned with the local behavior of Bridgeland stability, describing the quiver regions on the upper-half plane of stability conditions $\mathbb{H}$ and with that proving the stability of the instantons in these regions.

The thesis has an appendix and annex that discuss some intermediate theoretical and computational results obtained during the search for examples of quiver regions and their uses.

Keywords: Bridgeland stability. Exceptional Classes. Asymptotic Bridgeland stability. Quiver regions.

Contents

Introduction

The classification problem in mathematics is as ancient as the subject in itself. In modern algebraic geometry, it is mostly expressed through a *moduli problem*, that is, determining if a given functor that describes the space of solutions of a given classifying problem is representable or almost representable. The notion of stability comes to hand at this point as a given functor may be non-representable but a restricted version of it is, this situation is common when dealing with GIT problems.

Moreover, stability conditions can also be useful in understanding the whole classification problem, as they can be used as building blocks of the non-stable objects we were previously interested in, for example, via the Harder-Narasimhan filtration of slope-stability. In that way, Bridgeland's notion of stability can be used in both situations and here we will focus more on the first approach, determining the stable objects of a given stability condition to classify them.

Introduced in (BRIDGELAND, 2007) as a way of defining a stability condition on triangulated categories, formalizing the notion of Π-stability of Douglas's, Bridgeland proposed two definitions of this new stability condition, one using slices and one using hearts of bounded t-structures. The former can be seen as a special case of Gorodentsev–Kuleshov–Rudakov's (GORODENTSEV; KULESHOV; RUDAKOV, 2004) t-stability considering only the partially ordered set $\mathcal{R}$ indexing the slope of the semistable objects, and the latter can be seen as a generalization of Mumford's notion of μ-stability where we associate a degree and a rank function to objects in a heart of a bounded t-structure.

One of the achievements of this first paper was establishing the existence of a complex manifold structure of the space of Bridgeland stability conditions through the application of the support property. This space can be seen as an avatar where the group of automorphisms of the derived category acted on and was easier to study than the whole of the derived category, as expressed by Prof. Bertram at a presentation in the Mathematical Conference of the Americas.

In a subsequent paper, (BRIDGELAND, 2008), Bridgeland investigates this relation of the space of stability conditions and the automorphism group of the derived category in the case of the $K3$ surface, a two-dimensional Calabi–Yau variety. He also proves the existence of a wall-and-chamber structure of the space of stability conditions and a large volume limit theorem, that relates this new notion of stability to Gieseker's stability at infinity.

Research over their existence and a description of the space of stability conditions is still active. Some notable contributions include:

- (MACRì, 2007): Macrì's paper on the space of stability of curves where he describes the space of stability conditions over smooth curves of genus $g \geqslant 1$ and relates Bridgeland's stability condition to exceptional collections;
- (ARCARA; BERTRAM, 2013): Arcara–Bertram explore Bridgeland's approach to providing examples of stability condition over the $K3$ surface and prove their existence over any smooth projective surface;
- (BAYER; MACRÌ; TODA, 2014): In this paper, the authors introduce the second-tilt to a heart of bounded t-structure in an attempt to conjecture the existence of a Bridgeland stability condition over threefolds, while also discussing large-volume limit results and a generalized version of the Bogomolov inequality;
- (MACRÌ, 2014; SCHMIDT, 2020): Both authors employ the same technique of using exceptional collections to prove the existence of a Bridgeland stability condition over a small portion of the upper-half plane of stability conditions and then extend the result to the whole $\mathbb{H}$, for the projective space $\mathbb{P}^3$ and the smooth quadric Q_3, respectively;
- (BAYER; MACRÌ; STELLARI, 2016; MACIOCIA; PIYARATNE, 2016): With different techniques, the authors in these papers prove the existence of Bridgeland stability conditions of abelian threefolds;
- (LI, 2019b; BERNARDARA et al., 2017): In these papers, the authors completely prove the existence of Bridgeland stability conditions over Fano threefolds, with the first applying a technique that was later going to be used for the quintic threefold;
- (LI, 2019a): In a more recent paper, Li proves the existence of a Bridgeland stability condition over the quintic threefold, one of the first examples of a Bridgeland stability condition on a Calabi–Yau threefold.

The theory then grown to include applications to many branches of algebraic geometry including quiver stability, minimal model problem, Brill–Noether theory, and others, see for example (ARCARA et al., 2013; MU, 2020),(BAYER; MACRì, 2014), (BAYER; LI, 2017) and (BAYER et al., 2020) respectively.

Now, we discuss the structure of the thesis. It is divided into two parts: the first is background material and the second one deals with the main results obtained during the years of this Ph.D.

The first chapter *Schemes* has three different sections, the beginning deals with notation and basic concepts related to schemes using mostly (EISENBUD; HARRIS, 2000). This is then applied in the next section to the formalism of derived categories and functors, where we give a brief overview of the theory, the main references here are (GELFAND;

MANIN, 2003; HUYBRECHTS, 2006). To close this chapter, we discuss the theory of λ-rings as in (FULTON; LANG, 1985) in an attempt to give a concise summary of the concepts related to Chern characters and their properties, a tool used extensively in the theory of Bridgeland stability conditions.

In the next chapter, we discuss the fundamentals of Bridgeland stability, with a focus on the properties needed in our results later in this thesis. We start by giving some necessary results on sheaf stability focusing on a variation of Gieseker-Simpson stability, one of the prototypes for the definition of Bridgeland stability. The main references here are (HUYBRECHTS, 2006) and for notation (JARDIM; MACIOCIA, 2019; SCHMIDT; SUNG, 2018).

Continuing in this chapter we introduce the two main definitions of weak and Bridgeland stability conditions and their general consequences, such as the existence of a stability manifold $\mathrm{Stab}(X)$ and a wall-and-chamber decomposition, both results obtained by Bridgeland in consecutive papers (BRIDGELAND, 2007; BRIDGELAND, 2008). Then, we give a more in-depth discussion on the known results about the stability conditions defined over smooth projective varieties of dimension less or equal to 3, with a focus on the threefold case where our results lie.

To close the first part, we define the exceptional collections and some of their basic properties but most importantly we give a definition of the upper-half plane condition for a given Bridgeland stability condition, the main reference here is (MACRì, 2007). This condition is the necessary condition used in Chapter 4 to define the quiver regions and therefore prove some of the main results in the thesis.

One of my main goals while writing the first part was to create a roadmap of the main references that helped me understand Bridgeland's stability, which can be used later for other people interested in learning this rich subject.

The main results are divided into two independent chapters: The first deals with the asymptotic behavior of Bridgeland stability following (JARDIM; MACIOCIA, 2019) and the second deals with quiver regions and the stability of instantons sheaves, inspired by (MU, 2020).

In Chapter 3 we obtain our results complementing the results obtained in (JARDIM; MACIOCIA, 2019) about the nature of the objects asymptotic (semi)stable, where we consider only those with zero Chern character equal to zero. Our results are analogous to the ones obtained by Jardim–Maciocia as we prove these objects are either the Gieseker-(semi)stable objects or the derived dual of Gieseker-(semi)stable objects, depending on either going to the left-hand infinity or the right-hand infinity in H, respectively.

In their work, Jardim–Maciocia introduce the concept of Θ^{-}-curves which in here we use a weaker condition on the curves $\gamma(t) = (\beta(t), \alpha(t))$ by asking that

$$c_\gamma := \lim_{t\to+\infty} \frac{\alpha^2(t)}{\beta^2(t)} < 1. \tag{1}$$

This condition is important because it is the one that guarantees that whenever $E \in \mathrm{Coh}(X)$ we have that $\lim_{t\to+\infty} \nu_{\gamma(t)}(E) > 0$ (see equation (3.7)), in geometric terms it means that for every $E \in \mathrm{Coh}(X)$ there is some $t_0 \in \mathbb{R}$ such that when $t > t_0$, $\gamma(t) \in \Theta_E^+$ holds.

We start by proving the asymptotic tilt-stability of the objects with zero Chern character equal to zero. Differently than in (JARDIM; MACIOCIA, 2019), we do not need to consider the direction of the unbounded curve as there is curve $\mu_\beta = 0$.

Proposition. *Let $E \in \mathrm{D}^{\mathrm{b}}(X)$ be an object with $\mathrm{ch}_0(E) = 0$ and $\mathrm{ch}_1(E) \neq 0$. Then E is asymptotic ν_γ-(semi)stable if and only if it is GS_1-(semi)stable.*

The version of this proposition related to the case $\mathrm{ch}_0(E) = \mathrm{ch}_1(E) = 0$ is realized by knowing that $\Im(Z^t_{\beta,\alpha}(E)) = \mathrm{ch}_1^\beta(E) = 0$ for all $(\beta, \alpha) \in \mathbb{H}$, so that $E \in \mathcal{B}^{\beta,\alpha}$ for all $(\beta, \alpha) \in \mathbb{H}$ and $\nu_{\beta,\alpha}(F) = +\infty$ when $F \hookrightarrow E$ in $\mathcal{B}^{\beta(t)}$.

Next, we consider the asymptotic Bridgeland stability to the left-hand side of the upper-half plane of stability conditions.

Main Theorem 1. Let $\lim_{t\to+\infty} \beta(t) = -\infty$ and $c_\gamma < 1$. Suppose that $E \in \mathrm{D}^{\mathrm{b}}(X)$ with $\mathrm{ch}_0(E) = 0$, for E to be asymptotic $\lambda_{\gamma,s}$-(semi)stable it is necessary and sufficient that $E \in \mathrm{Coh}(X)$ is a Gieseker-(semi)stable sheaf.

In contrast to (JARDIM; MACIOCIA, 2019), we provide a more hands-on approach to proving the right-hand side of the theorem by proving a series of lemmas and proving directly the result. One key lemma in the proof is the reduction to the possibility of only testing the stability of an object on its quotients that are derived duals of a given pure sheaf, just as in the Gieseker-stability case with pure sheaves.

Main Theorem 2. Let $\lim_{t\to+\infty} \beta(t) = +\infty$ and $c_\gamma < 1$. An object $E \in \mathrm{D}^{\mathrm{b}}(X)$ with $\mathrm{ch}_0(E) = 0$ is asymptotic λ_γ-(semi)stable if and only if it is the dual of a Gieseker-(semi)stable sheaf.

In our next chapter, our approach to the study of quiver regions is to provide a systematic framework on which we can make concrete calculations with them. This is first expressed in the proposition used to determine the quiver regions.

Proposition. *For a full Ext-exceptional collection $\mathcal{E} = \{E_0, ..., E_3\}$ in $\mathrm{D}^{\mathrm{b}}(X)$ such that $E_i \in \langle \mathcal{A}^{\beta,\alpha}, \mathcal{A}^{\beta,\alpha}[1] \rangle$, for some $(\beta, \alpha) \in \mathbb{H}$ and a positive $s \in \mathbb{R}$. Then $\mathcal{E}$ satisfies the* *upper-half plane condition with respect to* $(Z_{\beta,\alpha,s}, \mathcal{A}^{\beta,\alpha})$ *if and only if there exists a* i *such that:*

(∗) (β, α) *is inside the walls* $\Upsilon_{E_i[-1],E_j[-1],s}$, *for all* $j \in \{j|E_j \in \mathcal{A}^{\beta,\alpha}[1]\}$, *and outside the walls* $\Upsilon_{E_i[-1],E_k,s}$ *for* $k \in \{k|E_k \in \mathcal{A}^{\beta,\alpha}\}$.

Once inside of a quiver region, we need tools for determining which are the stable objects and which are not. These are given by the lemma defining the *determinant condition* and the proposition determining the subobjects of a given linear complex, as follows.

Lemma. *Let* K *be an object in* $\langle S(\mathcal{E})\rangle$ *for an quiver region* $R_{\mathcal{E}}$ *with respect to a strong exceptional collection* $\mathcal{E}$ *and dimension vector* $\mathrm{ch}(K) = (-1)^i a\,\mathrm{ch}(E_i) - (-1)^i b\,\mathrm{ch}(E_{i+1})$. *If there exists a* λ*-wall in* $\mathcal{A}^{\beta,\alpha}$, *for some* $(\beta, \alpha) \in R_{\mathcal{E}}$, *defined by*

$$0 \to F \to K \to G \to 0, \tag{2}$$

then $\Upsilon_{F,K,s} = \Upsilon_{E_i,E_{i+1},s}$. *Furthermore, if every subobject* F *of* K *in* $\langle S(\mathcal{E})\rangle$ *with* $\mathrm{ch}(F) = (-1)^i c\,\mathrm{ch}(E_i) - (-1)^i d\,\mathrm{ch}(E_{i+1})$ *satisfies* $(a \cdot d - b \cdot c)(\geqslant) > 0$ *then* K *is Bridgeland (semi)stable outside the curve determined by* $\Upsilon_s(E_i, E_{i+1})$ *and inside the region where* $K \in \langle S(\mathcal{E})\rangle \cap \mathcal{A}^{\beta,\alpha}$. *In this case, we will say that* K *satisfies the (semi)-determinant condition.*

We do not provide an example of the application of this proposition but it can be used with concrete matrix examples to compute whether a 2-step complex is stable or not.

Proposition. *Let* $\mathcal{E} = \{E_0, ..., E_3\}$ *be a strong exceptional collection of sheaves,* $S(\mathcal{E})$ *its shift. Let*

$$K \simeq (V \otimes E_i \xrightarrow{T} W \otimes E_j)$$

be a 2-step complex in $\langle S(\mathcal{E})\rangle$, $\{\gamma_0, ..., \gamma_k\}$ *a base for the* k*-vector space* $\mathrm{Hom}(E_i, E_j)$. *For any subspace* $I \hookrightarrow V$, *the subcomplexes of* K *of the form* $I \otimes E_i \xrightarrow{S} J \otimes E_j$ *satisfies* $J_T^I \subset J$. *Furthermore,* $I \otimes E_i \xrightarrow{T|_I} J_T^I \otimes E_j$ *is a subobject of* K *in* $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$.

With these results at hand and by applying (ANCONA; OTTAVIANI, 1994, Proposition 2.8) we can achieve the stability of the instanton sheaves shifted by [1] for every charge $c > 0$.

Proposition. *Let*

$$\mathcal{I}(c) = \{I \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle |\ \dim(I) = [0, c, 2c + 2, c],\ \mathcal{H}^{-2}(I) = 0 \text{ and } \mathcal{H}^0(I) = 0\}.$$

Then any object in $\mathcal{I}(c)$ *is* $\lambda_{\beta,\alpha}$*-stable, for every* $(\beta, \alpha) \in \tilde{R}_-$ *outside of* $\tilde{\Upsilon}_1$.

That brought into the setting of moduli spaces by (ARCARA et al., 2013, Theorem 8.1) can be stated in a more interesting way as:

Main Theorem 3. For any $(\beta, \alpha) \in \tilde{R}_{-}$ outside and sufficiently close to the λ-wall $\tilde{\Upsilon}_1$, inside the Bridgeland moduli space

$$\mathcal{M}_{\beta,\alpha}(c) = \{E \in \mathrm{D^b}(\mathbb{P}^3) | E \text{ is } \lambda_{\beta,\alpha,1/3}\text{-stable with } \mathrm{ch}(E) = (-2, 0, c, 0)\}$$

we have the set

$$\mathcal{N}(c) := \bigcup_{T \in \mathcal{K}} \{F \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-1)^c[2], T) | \text{ with } \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1)[2], F) = 0\}$$

where $\mathcal{K} = \{T \in \langle \mathrm{S}(\tilde{\mathcal{E}}) \rangle | \dim(T) = [0, 0, 2c + 2, c]$ and T is $\lambda_{\beta,\alpha,1/3}$-stable $\}$.

If the charge c is odd then $\mathcal{N}(c)$ is equal to $\mathcal{M}_{\beta,\alpha}(c)$ and $\mathcal{M}_{\beta,\alpha}(c)$ is a projective space. Otherwise, if c is even, $\mathcal{M}_{\beta,\alpha}(c)$ is a quasi-projective variety. In both cases $\mathcal{I}(c)$ is a subset of $\mathcal{M}_{\beta,\alpha}(c)$.

The thesis ends with an Appendix regarding some computational observations made during the course of researching the quiver regions, and their applications. The important results obtained there are the algorithm, which is later given explicitly in the Annex, for finding and testing exceptional collections satisfying the upper-half plane condition and a proposition for geometrically determining the existence/or not of real walls for a given object inside of a quiver region.

1 Schemes

The first chapter deals with the standard theory of schemes, derived categories and λ-rings. The goal here is to understand why we need the conditions necessary to develop Bridgeland's stability. To that extent, and to try to keep the text as self-contained as possible, we will start with the definition of a scheme and walk through their basic notions in the first section. Next, we define the derived category of coherent sheaves over a scheme and some of their derived functors used in this thesis.

To close this chapter we discuss the λ-rings, an abstract notion introduced in (BERTHELOT; GROTHENDIECK; ILLUSIE, 1971) that summarizes the necessary conditions to defining a Chern class/character and, in the case of regular schemes, prove a version of the Grothendieck–Riemann–Roch. In this thesis, the λ-ring formalism will substitute the need to develop the Chow ring and their intersection theory.

We will not define the basic notions of category theory, sheaf theory and triangulated category necessary for this project but it can be found in (GELFAND; MANIN, 2003, Chp. 2) and (NEEMAN, 2001, Chp 1).

1.1 Geometry of schemes

We start by defining our most fundamental notion, the concept of a scheme. It was first introduced by Grothendieck to generalize the classical notion of varieties in an attempt to study more general behavior that the classical theory was not able to handle well. One simple example of this is the study of nonreduced schemes, that is, schemes with "fat" points, see (EISENBUD; HARRIS, 2000, Section II.3).

The construction is analogous to the construction of a differentiable manifold with the euclidean coordinates as its local model. In this case, the local model of a scheme is given by the affine scheme $(\mathrm{Spec}(R), \mathcal{O}_R)$ where: R is a commutative ring, $\mathrm{Spec}(R)$ is a topological space with the prime ideals of R as its points, and a sheaf over $\mathrm{Spec}(R)$ $\mathcal{O}_R$ such that if $f \in R$ and $D(f) = \{[p] \in \mathrm{Spec}(R) | f \notin p\}$ then $\mathcal{O}_R(D(f)) = R_f$, the localization of R with respect to $S = \{f^k\}_{k \in \mathbb{Z}_{\geqslant 0}}$. The open subsets $\{D(f)\}_{f \in R}$ form a base for the topology in $\mathrm{Spec}(R)$, see (EISENBUD; HARRIS, 2000, Section I.1 and I.2)

Definition 1.1.1. (EISENBUD; HARRIS, 2000, Section I.3) A scheme X consists of two data: a topological space $|X|$ and sheaf $\mathcal{O}_X$ called the sheaf of regular functions on X. These objects are locally affine, that is, $|X|$ can be covered by open sets U_i such that $U_i \simeq \mathrm{Spec}(R_i)$ and $\mathcal{O}_X|_{U_i} \simeq \mathcal{O}_{R_i}$.

For any point x in a scheme X we can define the germ of the regular functions on X over x by $\mathcal{O}_{X,x} = \varinjlim_{V \ni x} \mathcal{O}_X(V)$ which has the same role as the germ of regular functions over a smooth manifold, leading to the following notion of morphism between schemes. The ring $\mathcal{O}_{X,x}$ is a local ring with maximal ideal denoted by $m_{X,x}$.

Definition 1.1.2. (EISENBUD; HARRIS, 2000, Definition I.39) A morphism between schemes X and Y is a pair $(\psi, \psi^\sharp)$ where $\psi : |X| \to |Y|$ is a continuous map of topologica spaces and $\psi^\sharp : \mathcal{O}_Y \to \psi_* \mathcal{O}_X$ is a morphism of sheaves in Y where $\psi^\sharp_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,p}$ takes the maximal ideal $m_{Y,y}$ into $m_{X,p}$, with $y = \psi(p)$ and $p \in X$.

To simplify the notation, we usually denote by $\psi : X \to Y$ for the morphism $(\psi, \psi^\sharp)$ of the schemes X and Y. This defines the category of schemes Sch. There is also a relative notion of schemes to a given scheme S, the so-called S-schemes, where the objects are all morphisms of schemes $X \to S$ and the morphism in this category of the objects $X \to S$ and $Y \to S$ is any scheme morphism $X \to Y$ making the following diagram commute

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[dl] \\ S. & \end{tikzcd}$$

Denote by Sch_S the category of S-schemes with their respective morphisms. When $S = \mathrm{Spec}(R)$ for some ring R we say that an S-scheme is an R-scheme or a scheme over R. In the scheme category Sch_S it is possible to glue schemes so that Sch_S has a well-defined product, the fiber product $X \times_S Y$, see (EISENBUD; HARRIS, 2000, Sec. I.2.4). An open subscheme of a scheme X is an open subset U of X with its sheaf of regular function induced by the restriction of the regular functions over X.

From now we will always assume the schemes to be noetherian, these are the ones that can be covered by a finite number of open $\mathrm{Spec}(R_i)$ with R_i noetherian rings. The next example is a resume of the Proj construction and can be found more explicitly in (EISENBUD; HARRIS, 2000, Section III.2.1).

Example 1.1.3. [The affine space] If $R = A[x_1, ..., x_n]$ for some commutative ring A then $\mathrm{Spec}(R)$ is always denoted by $\mathbb{A}^n_A$, which we sometimes denote by just $\mathbb{A}^n$ when dealing with a fixed field k. It is also possible to define $\mathbb{A}^n_V$ with V being an A-scheme as $\mathbb{A}^n_V = \mathbb{A}^n_A \times_A V$.

Example 1.1.4. [The projective space] Let A be a ring and $\mathcal{S}$ a positively graded A-algebra, that is $S = \bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} S_k$ such that $S_v S_k \subset S_{v+k}$ and $S_0 = A$. Assume that S is generated by S_1 as an A-algebra. We can define a scheme called the $\mathrm{Proj}(S)$ over $\mathrm{Spec}(A)$ by having its topological space $|\mathrm{Proj}(S)|$ as the set of all homogeneous prime ideals of $\mathcal{S}$ not containing the irrelevant ideal $S^+ = \bigoplus_{k \in \mathbb{Z}_{>0}} S_k$, and its structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ is obtained by gluing $\mathcal{O}_{S[f^{-1}]_0}$ over the affine subsets $D_+(f) = \{[p] \in |\operatorname{Proj}(S)||f \notin p\}$ for each homogeneous element f of degree 1 in S, where $S[f^{-1}]_0$ is the ring obtained from S by localizing it by the powers of f and taking the elements with degree zero(with the degree induced by the quotient degree).

Let A be a ring and $S = A[X_0, ..., X_n]$ the graded polynomial ring then $\mathbb{P}^n_A = \operatorname{Proj}(S)$ and if V is any A-scheme we can define $\mathbb{P}^n_V = \mathbb{P}^n_A \times V$, the fiber product in the category of A-schemes.

Grothendieck‘s scheme theory can be seen as a way to generalize the objects in commutative algebra to a geometric stage. With this, the general scheme behaves as the rings in the theory and the quasi-coherent sheaves are the modules, they allow for us to transport the notions of homological algebra and some other universal constructions in this setting. To define them, we will need the $\mathcal{O}_X$-modules.

Definition 1.1.5. Let X be a scheme and $\mathcal{F}$ a sheaf over X. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module if for every open subset of X, $\mathcal{F}(U)$ is $\mathcal{O}_X(U)$-module and the restriction maps of $\mathcal{F}$ is natural for the module operations.

Given a ring R and a R-module M, it is possible to construct a sheaf of $\mathcal{O}_R$-modules $\widetilde{M}$ over $\operatorname{Spec}(R)$ such that on each $D(f)$, $\widetilde{M}(D(f)) = M_f$ the localization of M with respect to $S = \{f^k\}_{k \in \mathbb{Z}_{\geqslant 0}}$. For a general scheme we can consider the $\mathcal{O}_X$-modules which can be expressed locally as $\widetilde{M_i}$, these $\mathcal{O}_X$-modules are known as the *quasi-coherent sheaves*. If we can take the M_i to be finitely generated then these are *coherent sheaves.*

Remark 1.1.6. The quasi-coherent and coherent sheaves define abelian categories. That is, if we consider $\operatorname{QCoh}(X)$ and $\operatorname{Coh}(X)$ the full subcategory of the category of sheaves over X with objects as the quasi-coherent and coherent sheaves, respectively, then these categories are abelian because their defining property of being locally modules induces well-defined notions of the kernel, image, product and so on. This is one instance where X being noetherian is important so that the submodule of a finitely generated module is finitely generated.

One important class of quasi-coherent sheaves is the ideal sheaves associated with a closed subscheme. Let X be a scheme, we will say that $\phi : Y \to X$ is a closed subscheme of X if there exists a quasi-coherent sheaf $\mathcal{I}_{Y/X}$ such that $\phi_*(\mathcal{O}_Y)$ can be obtained by the exact sequence

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{O}_X \to \phi_*(\mathcal{O}_Y) \to 0.$$

The sheaf $\mathcal{I}_{Y/X}$ is called the sheaf of ideals determining Y.

Another important class of $\mathcal{O}_X$-modules are the locally free sheaves. A $\mathcal{O}_X$-module $\mathcal{F}$ to be called locally free is necessary that we can cover X by open subsets U_i

such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$. We will sometimes call the locally free sheaves as vector bundles, as there exists a $1-1$ correspondence between them (HARTSHORNE, 1978, Ex. 5.18, Chp. 2).

To further develop this correspondence between rings/modules and schemes/quasi-coherent sheaves we will overview the construction of the Global Proj, as in (EISENBUD; HARRIS, 2000, Section III.2.3). Let $\mathcal{F}$ be a positively graded quasi-coherent $\mathcal{O}_X$-algebra, these are quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{F} = \bigoplus_{k\in\mathbb{Z}_{\geqslant 0}} \mathcal{F}_k$ with $\mathcal{F}_v\mathcal{F}_k \subset \mathcal{F}_{v+k}$ and $\mathcal{F}_0 = \mathcal{O}_X$. Assume that $\mathcal{F}$ is generated by $\mathcal{F}_1$, we have a surjective morphism $\mathrm{Sym}(\mathcal{F}) \to \mathcal{F}$. Then we can define $\mathrm{Proj}(\mathcal{F})$ as the gluing of $\mathrm{Proj}\,\mathcal{F}(U_i)$, where $X = \bigcup_i U_i$ with each U_i being affine schemes and $\mathcal{F}(U_i) = \widetilde{M}_i$.

Inside the projective spaces, it is possible to construct a sheaf associated with a graded module, just as in the construction we have done before, with the only difference being that now we would have to take care of the gradings, as in the definition of the Proj. These can be done on either the local or global Proj, wherein the latter we would need a graded $\mathcal{O}_X$-module to do it. This construction is done properly in (EISENBUD; HARRIS, 2000, Section III.2.6). The notation for this is the same as in the non-projective case.

One special example of it is when, given a positively graded $\mathcal{O}_X$-algebra $\mathcal{F}$ we define $\mathcal{F}(k)$ as the shift by k of the grading of $\mathcal{F}$ and $\mathcal{O}_{\mathrm{Proj}(\mathcal{F})}(k) = \widetilde{\mathcal{F}(k)}$. This collection of sheaves are really important to the projective geometry of these spaces. One of their properties is that they are always line bundles.

Example 1.1.7. If $\mathcal{F}$ is an $\mathcal{O}_X$-module we can construct its symmetric algebra $\mathrm{Sym}(\mathcal{F})$ so that we define $\mathbb{P}(\mathcal{F}) := \mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$. When $\mathcal{F}$ is locally free, we will say that $\mathbb{P}(\mathcal{F})$ is a projective vector bundle over X.

We are now ready to define a sequence of morphism properties that will allow us to understand the regular maps, which are morphisms that can be factored in using projective bundles.

Definition 1.1.8. A morphism $f : X \to Y$ is called flat if for every $x \in X$, the induced morphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ with $f(x) = y$ is a flat morphism of rings, that is, $\mathcal{O}_{X,x}$ can be seen as a flat $\mathcal{O}_{Y,y}$-module.

Although this definition can be seen as a rather abstract notion in algebraic geometry, it has a very concrete geometric consequence as it can be seen in (EISENBUD; HARRIS, 2000, Lemma II-30, Chp 2), where the notion of flatness at a point p in $\mathbb{A}^n_Y$ over a reduced scheme Y is proven to be equivalent to the uniqueness of the limit of the fibers $f^{-1}(g(t))$ whenever you consider g to be a morphism parametrizing a non-singular curve in Y passing through the point $f(p)$. It is also a necessary condition imposed in many moduli space definitions because of this property of making limits well-defined.

Definition 1.1.9. A morphism $f : X \to Y$ is étale if it is flat and for every point $y = f(x)$ the induced map $f^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ satisfies $f^\sharp(m_{Y,y})\mathcal{O}_{X,x} = m_{X,x}$ and $k(y) := \mathcal{O}_{Y,y}/m_{Y,y}$ is a finite separable extension of $k(x)$.

The étale morphisms are a class of morphisms analogous to the local diffeomorphisms when considering differentiable manifolds. There are several ways of defining smooth morphisms, we choose to use (FULTON; LANG, 1985) as it was the one used to learn the information in the section about Chern characters. Different approaches include imposing the smooth map is flat and the cotangent bundle $\Omega^1_{X/Y}$ is locally free when X is integral, (HARTSHORNE, 1978, Section 10, Chp.3).

Definition 1.1.10. A morphism $f : X \to Y$ is smooth if it can be decomposed as a composition $X \to \mathbb{A}^n_Y \to Y$, where $X \to \mathbb{A}^n_Y$ is an étale morphism and $\mathbb{A}^n_Y \to Y$ is the natural projection.

In (FULTON; LANG, 1985, IV.2 and IV.3), the authors define a regular sequence in a ring A to be any sequence $(a_1, ..., a_k)$ such that the ideal generated by this sequence is non-trivial and a_i is not a zero divisor in $A/(a_1, ..., a_{i-1})$ for all i.

Definition 1.1.11. A closed subscheme $i : Y \to X$ is a regular imbedding if for every point of X there exists an affine open subset $\mathrm{Spec}(A)$ of X where the ideal associated to Y, $\mathcal{I}_{Y/X}$, can be generated by a regular sequence of elements in A.

At last, the definition of the regular morphisms. From now on, a scheme X over a field k is said to be $*$ if the structure morphism $X \to \mathrm{Spec}(k)$ satisfies $*$, where $*$ = smooth, étale, regular, flat, regular respectively.

Definition 1.1.12. A morphism $f : Y \to X$ is called a regular morphism if it can be factored by $Y \xrightarrow{i} \mathbb{P}(\mathcal{E}) \xrightarrow{p} X$, where i is a regular imbedding and p is a bundle projection.

1.2 Derived categories

The derived categories were first introduced by Verdier in his master's dissertation as a way of extending Serre's duality to more general varieties. To achieve this he used the construction of localization over the rings and applied it to the category of complexes over an abelian category. As a consequence, he arrived at a category not abelian but triangulated, a concept used throughout modern-day algebraic geometry. In this subsection, we will look at what are the derived categories of an abelian category and many tools that come with it.

Given any abelian category $\mathcal{A}$ it is possible to construct a new abelian category $\mathrm{Kom}(\mathcal{A})$ the category of complexes with objects in $\mathcal{A}$ and the morphisms are morphisms of complexes, see (HUYBRECHTS, 2006, Proposition 2.3, Chp.2). We will use the superscript notation to denote our complexes as in $(A^\bullet, d^\bullet)$, so that we can define a natural cohomology functor $\mathcal{H}^i_{\mathcal{A}} : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}$ given by $\mathcal{H}^i_{\mathcal{A}}(A^\bullet) = \ker d^i / \operatorname{Im} d^{i-1}$. Whenever clear from the context, we will omit the subscript while expressing the cohomology functor.

A complex is called bounded below(resp. above) if $\mathcal{A}^i = 0$ if $i << 0$(resp. $i >> 0$) and it is bounded if it is bounded above and below. There are variants of the category $\mathrm{Kom}(\mathcal{A})$ where we change the types of complexes considered, we will express them by $\mathrm{Kom}^*(\mathcal{A})$ with $*$ being either one of $b, +, -$ or empty representing the bounded, bounded below, bounded above or not necessarily bounded complexes, respectively.

It is also possible to define the homotopy category of complexes $\mathcal{K}^*(\mathcal{A})$, where the objects are the same as in $\mathrm{Kom}^*(\mathcal{A})$ but the morphisms are equivalent classes under the homotopy equivalence, see (GELFAND; MANIN, 2003, Lemma 2, Sec. 1, Chp.3). These will be important in the following subsection.

One important class of morphism of complexes is the quasi-isomorphism class, the morphisms $f^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ where $\mathcal{H}^i(f^\bullet) : \mathcal{H}^i(\mathcal{A}^\bullet) \to \mathcal{H}^i(\mathcal{B}^\bullet)$ is an isomoprhism for every i. These will be the morphisms we are interested in localizing(e.g. inverting).

Theorem 1.2.1. *(GELFAND; MANIN, 2003, Theorem 1, Sec.2 and Chp. 3) Let $\mathcal{A}$ be an abelian category,* $\mathrm{Kom}^*(\mathcal{A})$ *the category of $*$-complexes over $\mathcal{A}$. Then there exists a triangulated category* $\mathrm{D}^*(\mathcal{A})$ *and a functor $Q : \mathrm{Kom}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{A})$ such that:*

- *Q takes quasi-isomorphisms into isomorphisms.*
- *Any functor $F : \mathrm{Kom}^*(\mathcal{A}) \to \mathcal{D}$ that takes quasi-isomorphisms to isomorphism can be factored by Q.*

The construction of $\mathrm{D}^*(\mathcal{A})$ is very similar to the construction of the localization of a ring. Its objects are exactly the same as the objects in $\mathrm{Kom}^*(\mathcal{A})$ and the morphisms are determined by *roofs*. Given complexes $A^\bullet$ and $B^\bullet$, a roof is a collection of a complex $C^\bullet$ with two morphisms

$$\begin{tikzcd} & C^\bullet \arrow[dl] \arrow[dr, "q-iso"] & \\ A^\bullet & & B^\bullet. \end{tikzcd}$$

satisfying the equivalence and composition conditions in (GELFAND; MANIN, 2003, Lemma 8, Sec. 2 and Chp.3).

Remark 1.2.2. (HUYBRECHTS, 2006, Proposition 2.30, Chp. 2) The natural functor $\mathrm{D}^*(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful and identifies the category $\mathrm{D}^*(\mathcal{A})$ with the subcategory of $D(\mathcal{A})$ of complexes $A^\bullet$ with $\mathcal{H}^i(A^\bullet) = 0$ for $|i| >> 0$, $i >> 0$, $i << 0$ if $* = b, -, +$ respectively. Furthermore, there exists a fully faithfull natural functor $\mathcal{A} \to \mathrm{D}^*(\mathcal{A})$.

When we consider the abelian category $\mathcal{A}$ to being equal to the coherent sheaves over a scheme X we make a slight change of notation to $\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$

Next, we focus on t-structures and their hearts, a notion first introduced in (DELIGNE; BEILINSON; BERNSTEIN, 1983) in the context of *perverse sheaves*. These are responsible for finding abelian categories inside a given derived category $\mathrm{D}^*(X)$ that can generate it. In some sense, the t-structure can be seen as an alternative complex structure for the objects in $\mathrm{D}^*(X)$.

Definition 1.2.3. (BRIDGELAND, 2007, Definition 3.1) A t-structure in $\mathrm{D}^*(X)$ is a full subcategory $\mathcal{F}$ of $\mathrm{D}^*(X)$ such that the objects $\mathcal{F}[1]$ are in $\mathcal{F}$ and if we define $\mathcal{F}^\perp = \{G \in \mathrm{D}^*(X) | \operatorname{Hom}(F, G) = 0 \text{ for all } F \in \mathcal{F}\}$ so that any object $E \in \mathrm{D}^*(X)$ can be written as

$$F \to E \to G \to F[1]$$

with $F \in \mathcal{F}$ and $G \in \mathcal{F}^\perp$.

The category $\mathcal{A} = \mathcal{F} \cap \mathcal{F}^\perp[1]$ is called the core or heart of this t-structure and it defines an abelian category, see (GELFAND; MANIN, 2003, Theorem 4, Sec. 4 and Chp.4). Furthermore, for every heart of a t-structure it is possible to define a cohomology functor $\mathcal{H}^i : \mathrm{D}^*(X) \to \mathcal{A}$, in the sense that it takes distinguished triangles into long exact sequences, see (GELFAND; MANIN, 2003, IV.4).

One variant of the notion of a t-structure is that of a bounded t-structure, where we assume the objects in $\mathcal{F}$ and $\mathcal{F}^\perp$ have $\mathcal{H}^i = 0$ for all but a finite number of i's. An equivalent characterization of this notion can be found in (BRIDGELAND, 2007, Lemma 3.2), which we are going to state here due to its connection with the theory developed by Bridgeland.

Lemma 1.2.4. *(BRIDGELAND, 2007, Lemma 3.2) Let $\mathcal{A}$ be a full additive subcategory of $\mathrm{D}^*(X)$. Then $\mathcal{A}$ is a heart of a bounded t-structure if and only if the following two conditions hold:*

- *If k_1 and k_2 are integers with $k_1 > k_2$ then $\operatorname{Hom}(A[k_1], B[k_2]) = 0$ for all A, B in $\mathcal{A}$.*
- *For every object $D \in \mathrm{D}^*(X)$, there are a finite sequence of integer $k_1 > \dots > k_n$ where the object D can be decomposed by the diagram:*

$$\begin{tikzcd}
0 = D_0 \arrow[r] & D_1 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & \dots \arrow[r] & D_{n-1} \arrow[r] \arrow[d] & D_n = D \arrow[d] \\
& A_1[k_1] \arrow[ul, dotted] & A_2[k_2] \arrow[ul, dotted] & & A_{n-1}[k_{n-1}] \arrow[ul, dotted] & A_n[k_n] \arrow[ul, dotted]
\end{tikzcd}$$

where each $A_i \in \mathcal{A}$ and the triangles represent distinguished triangles in $\mathrm{D}^*(X)$.

The statement in Lemma 1.2.4 is responsible for the interpretation of the bounded t-structure as a new complex structure for the objects in $\mathrm{D}^*(X)$, where the objects $A_i[k_i]$ are the $\mathcal{H}_{\mathcal{A}}^{-k_i}$-cohomology of the object D.

In the theory of hearts of bounded t-structures there exists a construction responsible for creating most of the examples of hearts that are used in this text, this technique is called *tilting*. The idea behind it is to determine a torsion pair, with properties analogous to torsion and torsion-free sheaves, inside of a heart of a bounded t-structure and splitting the object by its torsion and torsion-free parts. This achieves an abelian category that is also a heart of bounded t-structure. Moreover, by using the Harder-Narasimhan filtration, it is possible to use this construction to create a family of hearts inside the bounded derived category of coherent sheaves.

Definition 1.2.5. Let $\mathcal{A}$ be a heart of a bounded t-structure, a torsion pair in $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of sub-additive categories of $\mathcal{A}$ satisfying:

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- Every object E in $\mathcal{A}$ can be decomposed in

  $$0 \to T \to E \to F \to 0,$$

  for some $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The objects in $\mathcal{T}$ and $\mathcal{F}$ are said to be the torsion and torsion-free objects for the torsion pair $(\mathcal{T}, \mathcal{F})$, respectively.

Theorem 1.2.6. *(HAPPEL; REITEN; SMALO, 1996, Theorem 2.1) For a given torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category then*

$$\mathcal{A}^{\sharp} = \left\{ E \in \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \;\middle|\; \begin{array}{l} \mathcal{H}^i(E) = 0 \text{ for all } i \neq 0, -1 \\ \mathcal{H}^0(E) \in \mathcal{T} \\ \mathcal{H}^{-1}(E) \in \mathcal{F} \end{array} \right\}.$$

defines a heart of a bounded t-structure in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$.

For every heart of a bounded t-structure $\mathcal{A}$ in $\mathrm{D}^*(X)$ we have a cohomology functor $\mathcal{H}^i_{\mathcal{A}} : \mathrm{D}^*(X) \to \mathcal{A}$ which, to the case of $\mathcal{A}^{\sharp}$, can be used to describe the objects in $\mathcal{A}^{\sharp}$ by the exact sequencein $\mathcal{A}^{\sharp}$

$$0 \to \mathcal{H}^{-1}_{\mathcal{A}}(E)[1] \to E \to \mathcal{H}^0_{\mathcal{A}}(E) \to 0$$

for every $E \in \mathcal{A}^{\sharp}$. This decomposition of the objects in a tilted category is of much use when trying to find possible destabilizers when dealing with Bridgeland stability conditions.

1.2.1 Derived functors

One of the goals of the formalism of derived categories defined by Verdier was to properly study derived functors in algebraic geometry. They are a way of circumventing the non-exactness of an almost-exact additive functor(i.e. left or right exact functors) F by defining a new functor RF or LF that measures its non-exactness. This construction has its limitations as it requires a special class of objects inside our category, known as the F-adapted objects, which could be seen as the "control set" where the functor is exact.

The existence of F-adapted objects is one of the reasons why, in this context, it is important to work with quasi-coherent sheaves even if we are only interested in the coherent ones, as they always have F-adapted objects for left exact functors and the coherent may not have, see (HUYBRECHTS, 2006, Section 3.1).

Definition 1.2.7. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is left exact if the image of an exact sequence

$$0 \to A \to B \to C \to 0$$

by F is exact to the left, that is,

$$0 \to F(A) \to F(B) \to F(C)$$

is exact.

There is also a notion of right exact functors, which assumes the image of the functor takes exact sequence into right exact sequences. The results are completely analogous and therefore we will only state the left exact versions. An important right exact functor that is used later in the text is the derived tensor product. To see proof of its existence see (HUYBRECHTS, 2006, Section 3.3).

Definition 1.2.8. Let $F : \mathcal{A} \to \mathcal{B}$ be left exact functor. A class of objects $\mathcal{I}_F$ in $\mathcal{A}$ is called F-adapted if it satisfies

(a) If $A^\bullet \in K^+(\mathcal{A})$ is such that $\mathcal{H}^i(A^\bullet) = 0$ and $A^i \in \mathcal{I}_F$ then $F(A^\bullet)$ also satisfies $\mathcal{H}^i(F(A^\bullet)) = 0$.

(b) Every object $A^\bullet \in K^+(\mathcal{A})$ can be monomorphically mapped into an object in $\mathcal{I}_F$.

Example 1.2.9. One of the most important classes of F-adapted objects for left exact functors is the injective objects, they have their properties analogous to the injective modules. Therefore, an object I of an abelian category $\mathcal{A}$ is said to be injective if the contravariant functor $\mathrm{Hom}(-, I)$ takes exact sequences into exact sequences. These are known as exact functors. The injective modules satisfy condition (a) for any left exact functor by (GELFAND; MANIN, 2003, Theorem 12, Sec. 6 and Chp.3), making it so that we just have to check (b).

The abelian category $\mathrm{QCoh}(X)$ is known to have the class of injective modules satisfying (b). Because for any given quasi-coherent $\mathcal{F}$ and we can inject the stalk of F at any point $x \in X$ into an injective $\mathcal{O}_{X,x}$-module and then make a direct product of these injective modules to obtain an injective quasi-coherent sheaf, see (GELFAND; MANIN, 2003, Proposition 1, Sect. 8 and Chp. 3).

Whenever the class of injective modules in an abelian category $\mathcal{A}$ satisfy condition (b), we will say that $\mathcal{A}$ has enough injectives. One other way of describing the existence of enough injectives is expressed in the next proposition, where we have an equivalence of the derived category with its morphisms defined as equivalence classes of roofs and in the homotopic category the morphisms are equivalent classes of morphisms of complexes, a much simpler description.

Proposition 1.2.10. *(HUYBRECHTS, 2006, Proposition 2.40, Chp. 2) Let $\mathcal{A}$ be an abelian category with enough injectives. Then $i : K^+(\mathcal{I}) \hookrightarrow \mathrm{D}^+(\mathcal{A})$ is an equivalence of categories.*

Definition 1.2.11. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. The right derived functor $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ is defined by $RF := Q_{\mathcal{B}} \circ K(F) \circ i^{-1}$. In this situation, $R^iF(A^\bullet) := \mathcal{H}^i(RF(A^\bullet))$.

The inverse of $i : K^+(\mathcal{I}) \hookrightarrow \mathrm{D}^+(\mathcal{A})$ is not uniquely defined but Definition 1.2.11 is only one representation of the derived functor, instead we could define it by a universal property as in (GELFAND; MANIN, 2003, Definition 6, Sec.6 and Chp. 3) and since the functor defined in Definition 1.2.11 satisfies this universal property by (HUYBRECHTS, 2006, Proposition 2.47) it is therefore unique.

Remark 1.2.12. (HUYBRECHTS, 2006, Proposition 3.5) Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh(X)}}(\mathrm{QCoh}(X))$ be the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{QCoh}(X))$ of the complexes $A^\bullet$ with cohomology $\mathcal{H}^i(A^\bullet)$ in $\mathrm{Coh}(X)$ for all i. The natural inclusion

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{Coh(X)}}(\mathrm{QCoh}(X))$$

is an equivalence. So when trying to define a derived functor in $RF : \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(Y))$ we first define it over $RF : \mathrm{D}^{\mathrm{b}}(\mathrm{QCoh}(X)) \to \mathrm{D}^{\mathrm{b}}(\mathrm{QCoh}(Y))$ and then analyze what conditions are necessary for the image of the objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh(X)}}(\mathrm{QCoh}(X))$ to be in $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh(X)}}(\mathrm{QCoh}(Y))$, descending the derived functor of F to the coherent sheaves.

Example 1.2.13. (HUYBRECHTS, 2006, Section 3.3) There is a great number of functors in algebraic geometry where we apply this construction of derived functor, in here we will present just a few that are used in this text. Suppose X is a scheme over $\mathrm{Spec}(k)$,

- Let $\Gamma : \mathrm{QCoh}(X) \to \mathrm{Vect}(k)$ be the functor that takes quasi-coherent sheaves to their respective k-vector space of global sections. It is known that Γ is left exact and the cohomology of the derived functor $R\Gamma$, $R^i\Gamma(\mathcal{F})$, is usually denoted by $H^i(X, \mathcal{F})$.

- There is a generalization of this functor to any pushforward, where a projective map $f : X \to Y$ is taken into a $Rf_* : \mathrm{D^b}(\mathrm{Coh}(X)) \to \mathrm{D^b}(\mathrm{Coh}(Y))$.

- Given a (quasi-)coherent sheaf $\mathcal{F}$, there exists $\mathcal{H}om(\mathcal{F}, -) : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ that takes coherent sheaves into coherent sheaves. This functor is left exact and the i-cohomology of its derived functor is denoted by $\mathcal{E}xt^i(\mathcal{F}, -)$.

- Its also possible to define the right derived functor of $\mathcal{H}om(-, \mathcal{F})$ but for this we need further conditions on X such as smoothness or regularity. These will be defined in the next section.

- The sheaf pullback of a scheme map $f : X \to Y$ is given by $f^*(\mathcal{F}) = f^{-1}(\mathcal{F}) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$. This functor is easily seen as a composition of the functors f^{-1} and $\otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$, where the former is an exact functor and the later is right exact. Therefore, the derived functor of f^* is a left derived functor and it is contained in the bounded derived category when X is smooth.

1.3 Grothendick's group and Chern character

There are several ways of defining both the Grothendieck group and the Chern characters, which can range from complete geometric construction such as a complex version of Stiefel-Whitney classes or a more abstract version using λ-rings and γ-filtrations. In this chapter, we will be following (FULTON; LANG, 1985) account of the λ-rings to study the Grothendieck group of the coherent sheaves over a regular variety as this can be readily applied to our case of interest and makes it a clear construction for these classes.

1.3.1 General λ-rings

To start, we will define the notion of λ-ring and a general Chern character for it. The Grothendieck group for locally free sheaves and its λ-ring structure are defined to provide concrete examples for these notions.

For a given collection of functions $f^i : K \to K$ over a ring K for every $i \in \mathbb{Z}_{\geqslant 0}$, we can define the formal series $f_t(x) = \sum_{i \in \mathbb{Z}_{\geqslant 0}} f^i(x) t^i$.

Definition 1.3.1. In a ring K, a structure of λ-ring on K is determined by a collection of functions $\lambda^i : K \to K$, for every $i \in \mathbb{Z}_{\geqslant 0}$, and an augmentation function $\epsilon : K \to \mathbb{Z}$ satisfying:

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for every $x \in K$.
- The map $\lambda_t : K \to K[[t]]$ given by $x \mapsto \lambda_t(x)$ is a homomorphism from the additive structure of K to the multiplicative structure of $K[[t]]$.
- ϵ is a surjective ring homomorphism.

The next example is the prototype for the previous definition.

Example 1.3.2. Let X be an integral scheme. We denote by $\mathrm{K}_0(X)$ the Grothendieck group of X as the quotient of the free abelian group generated by the formal objects $[\mathcal{E}]$, for each locally free sheaf $\mathcal{E}$ in X, by the subgroup generated by the elements

$$[\mathcal{E}] - [\mathcal{F}] - [\mathcal{G}],$$

where $\mathcal{E}$ is an extension of $\mathcal{F}$ and $\mathcal{G}$. It has a multiplicative structure given by the tensor product of sheaves. The λ-ring structure on $\mathrm{K}_0(X)$ is determined by the external product, that is

$$\lambda^i(\mathcal{E}) := [\Lambda^i(\mathcal{E})],$$

with the augmentation homomorphism $\epsilon : \mathrm{K}_0(X) \to \mathbb{Z}$ given by the free abelian extension of the rule $\epsilon(\mathcal{E}) = \mathrm{rk}(\mathcal{E})$.

Example 1.3.3. We can also define the Grothendieck group of a triangulated category $\mathcal{T}$. To define it we use the same rule as in the case of $\mathrm{K}_0(X)$ where we take its Grothendieck group $\mathrm{K}(\mathcal{T})$ as the quotient of the free abelian group generated by the formal objects $[E]$, for each object E in $\mathcal{T}$, by the subgroup generated by the elements

$$[E] - [F] - [G],$$

where E is an extension of F and G in $\mathcal{T}$. It does not have a natural λ-ring structure. Moreover, in the same it is possible to define the Grothendieck group of an abelian category $\mathcal{A}$ and if $\mathcal{A}$ is the heart of a bounded t-structure in $\mathcal{T}$ then $\mathrm{K}(\mathcal{A}) = \mathrm{K}(\mathcal{T})$.

With this example in mind, we can define the notion of positive elements and a subgroup of it of the line elements. As in the study of sheaf theory, understanding the vector bundles and especially the line bundles can give information about the general behavior of a non-locally free sheaf.

Definition 1.3.4. The set of positive elements E in a λ-ring K is defined by the following conditions:

- $E \cup \{0\}$ is an additive subgroup;
- $\mathbb{Z}^+$ is in E;

- The product of elements in E is in E;
- Every object in K can be written as difference of elements in E;
- If $e \in \mathcal{E}$ and $\epsilon(e) = r$ then $\lambda^i(e) = 0$ for $i > r$ and $\lambda^r(e)$ is a unit in K.

Furthermore, the objects e in E with $\epsilon(e) = 1$ are said to be the line elements. They form a multiplicative subgroup of the semigroup $K \backslash \{0\}$.

One of the most important properties an λ-ring can possess is the splitting principle, it is a useful tool when trying to define properties of the Chern characters and other calculations.

Definition 1.3.5 (Splitting Principle)**.** For any positive element e in a λ-ring K, there exists a λ-ring extension K' of K such that e can be written as a finite sum of line elements in K'.

To obtain the splitting principle in $\mathrm{K}_0(X)$ for a given vector bundle $e = [\mathcal{E}]$ we would need to apply the same procedure as the one employed to finding roots for a polynomial over a non-algebraic closed field, that is, by extending the field of constants. The details can be found in (FULTON; LANG, 1985, Theorem 2.3, Ch. 5; Theorem 2.1, Ch. 1).

We use the Grothendieck group K_0 of the projective bundle $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$, which can be seen, just as in the polynomial case, as a quotient of $\mathrm{K}_0(X)$ by the defining polynomial $\lambda_t(e) \in \mathrm{K}_0(X)[t]$. Hence we obtain $[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)]$ as one of the line elements in the splitting decomposition of e in $\mathrm{K}_0(\mathbb{P}(\mathcal{E}))$ and we can proceed by induction due to the universal exact sequence

$$0 \to \mathcal{H} \to p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0.$$

1.3.2 Chern class and character

In the setting of λ-rings, it is possible to define the properties a Chern class and character have to satisfy, to be an honest representation of these concepts, that is, an invariant measuring how much a vector bundle(or sheaf) is different from the trivial bundle. The Chern class and character can be seen as a map from the chosen λ-ring to a graded ring, which can be thought of as the analogous of the Chow ring with its usual grading. In a given graded ring $A = \bigoplus_{i=1}^{+\infty} A_i$ we can construct the multiplicative group

$$\Lambda(A) = \{1 + \sum_{i \in \mathbb{Z}_{\geqslant 1}} a_i t^i | a_i \in A_i\}.$$

Definition 1.3.6. Let K be a λ-ring so that the Chern class is a homomorphism of abelian groups $c_t : K \to \Lambda(A)$ satisfying:

- **Normalization:** For any linear element u in K, $c_t(u) = 1 + c^1(u)t$.
- **Multiplication:** Given two linear elements u, v in K, $c^1(uv) = c^1(u) + c^1(v)$.
- **Finiteness:** $c_t(v)$ is a polynomial, not only a power series.

By using the universal symmetric polynomials (FULTON; LANG, 1985, Section 1, Ch. 1) it is possible to define a λ-ring structure on $\Lambda(A)$ such that the Chern class is a λ-ring homomorphism, hence $c_t(\lambda^n(x)) = \lambda^n(c_t(x))$ for any $x \in K$. Denote by $c^i(x)$ the coefficient of t^i in $c_t(x)$ for any $x \in K$.

The Chern character is more of a smart modification of the Chern class by an appropriate function than a new invariant, in our case the exponential function. Let e be a positive element in a λ-ring K, $e = u_1 + ... + u_n$ be a splitting of e in some extension K' of K and $a_i = c^1(u_i) \in A_i$ for some graded ring A where we have a Chern class. If we assume that A is $\mathbb{Q}$-algebra, we could also tensor an existing commutative algebra by $\mathbb{Q}$, then we can define

$$\mathrm{ch}(e) = \sum_{i=1}^{n} \exp(a_i).$$

This expression, while it seems to depend on the choice of the roots, it actually is well-defined. This is a consequence that we can describe this sum by using the associated Hirzebruch polynomials $H_j^{\exp}$ such as

$$\mathrm{ch}(e) = \sum_{j=1}^{n} H_j^{\exp}(c^1(e), ..., c^n(e)).$$

To extend this definition to all elements in K we just have to remember that every object in K can be written as a difference of positive elements so that we get a ring homomorphism $\mathrm{ch} : K \to A$, see (FULTON; LANG, 1985, Prop. 4.1, Ch. 1).

Now we approach the question of existence of a Chern class in a general λ-ring K, which as an example proves the existence of a Chern class for the Grothendieck group. We start by defining a new λ-ring structure on any given λ-ring by setting

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = \sum_{i \in \mathbb{Z}_{\geqslant 0}} \gamma^i(x)t^i.$$

This defines a different λ-ring structure on any λ-ring K because the transformation $s = \dfrac{t}{1-t}$ determines an isomorphism $K[[s]] = K[[t]]$. In such a way that we can define the γ-filtration of K with $F^1K = \ker(\epsilon)$ and

$$F^j(K) = \langle \gamma^{i_1}(x_1)...\gamma^{i_k}(x_k) |\ x_i \in F^1(K) \text{ and } i_1 + ... + i_k \geqslant j \rangle,$$

where we consider the objects as generated using a $\mathbb{Z}$-additive structure. This defines a decreasing filtration in K and we conveniently introduce the notation $F^lK = K$ if $l \leqslant 0$.

The γ-filtration, as any ring filtration, can be used to define a graded ring $\mathrm{Gr}(K) = \bigoplus_{i \in \mathbb{Z}} F^i K / F^{i+1} K$ so that we have a Chern class for any λ-ring determined by $c^j(e) = \gamma^i(e - \epsilon(e)) \in F^i(K)/F^{i-1}(K)$, for any positive element e and extended to any element in K by the homomorphism properties of the γ function. It comes directly from the definition of the γ-filtration that the Chern class is well-defined if $F^i(K) = 0$ for i sufficiently large which will be the case in our examples.

In the same way, we can define the Chern character as a ring homomorphism $\mathrm{ch} : K \to \mathbb{Q} \otimes \mathrm{Gr}(K)$. One important property is the λ-ring version to the relation between Cartier and Weil's divisor given in the next proposition.

Proposition 1.3.7. *(FULTON; LANG, 1985, Theorem 1.7,Ch. 2) Let L be the multiplicative group of line elements in a given λ-ring K. Then the map $c^1 : L \to F^1(K)/F^2(K)$ is an isomorphism with inverse* $\det : F^1(K)/F^2(K) \to L$.

Throughout the text, we will call the objects in $L \simeq F^1(K)/F^2(K)$ as the divisors of X. Now that we dealt with the general construction of the Chern class and character for a given λ-ring, we will focus on how to link these notions with geometric constructions. This is done by defining different types of filtration over the Grothendieck group, now with a more geometric flavor to them. To start, we will have to enlarge the Grothendieck group to include sheaves, that way we will be able to use it to describe lower dimensional structures in a variety X.

Let $\mathrm{K}^0(X)$ be the quotient of the free abelian group generated by the formal objects $[\mathcal{E}]$, for each coherent sheaf $\mathcal{E}$ in X, moded out by the subgroup generated by the elements $[\mathcal{E}] - [\mathcal{F}] - [\mathcal{G}]$, where $\mathcal{E}$ is an extension of $\mathcal{F}$ and $\mathcal{G}$.

Theorem 1.3.8. *(FULTON; LANG, 1985, Proposition 3.1, Ch.6) If X is regular then the inclusion $i : \mathrm{K}_0(X) \to \mathrm{K}^0(X)$ is an isomorphism.*

When X is regular we will be able to define $K(X) = \mathrm{K}_0(X) = \mathrm{K}^0(X)$. In $K(X)$ let $F_m K(X)$ be the descending filtration where $x \in K(X)$ is in $F_m K(X)$ if $x = [\mathcal{F}_1] - [\mathcal{F}_2]$ with $\dim(\mathrm{Supp}(\mathcal{F}_i)) \leqslant m$ for both $i = 1, 2$. Which can be described geometrically by the following proposition:

Proposition 1.3.9. *(FULTON; LANG, 1985, Proposition 5.1, Ch. 6) The subgroup $F_m K(X)$ is generated by the classes $[\mathcal{O}_V]$ of the structure sheaf of integral closed subschemes of X with dimension less or equal to m.*

The filtration $F_m K(X)$ is usually denoted as the contravariant topological filtration and it can be seen as a link to the Chow ring. These filtrations do not correspond to the same information but they do so when we consider them with rational coefficients, as expressed in the next proposition.

Proposition 1.3.10. *(FULTON; LANG, 1985, Proposition 5.5, Ch. 6) When X is an n-dimensional regular scheme then $F^jK(X) \subset F_{n-j}K(X)$ and these subsets are equal when tensored by $\mathbb{Q}$ in $\mathbb{Q} \otimes K(X)$.*

With this, we conclude that our definition of Chern class satisfies the finiteness condition in Definition 1.3.6 because $F^lK(X) = 0$ when $l > n$ and we can state one of the versions of the Grothendieck-Riemann-Roch:

Theorem 1.3.11. *(FULTON; LANG, 1985, Corollar 3.3 and Theorem 3.5, Ch. 3) For each $k \in \mathbb{Z}_{\leqslant n}$ it is possible to define subspaces $V(k)$ such that*

$$\mathbb{Q} \otimes F^lK(X) \simeq \bigoplus_{k=l}^{n} V(k).$$

Moreover, The Chern character $\mathrm{ch} : \mathbb{Q} \otimes K(X) \to \mathbb{Q} \otimes \mathrm{Gr}\, K(X)$ is an isomorphism and it is an isomorphism for each grade as in $\mathrm{ch} : V(m) \to \mathbb{Q} \otimes \mathrm{Gr}^m K(X)$.

Another version of this theorem, which is referred to as the Hirzebruch–Riemann–Roch theorem can also be obtained via the formalism of λ-rings. To achieve it, we would have to describe many other tools, which is beyond the scope of this presentation. Moreover, we will introduce some of the notation to state it, as this will be used throughout the text. These notations and concepts can be found in the discussion before (FULTON; LANG, 1985, Corollary 7.4).

Let X be a smooth projective scheme over $\mathrm{Spec}(k)$ and define the Euler characteristic of a sheaf $\mathcal{E}$ by $\chi(X, \mathcal{E}) = \Sigma_{i=0}^{\dim(X)}(-1)^i H^i(X, \mathcal{E})$. Next we need the degree function

$$\int_X : \mathbb{Q} \otimes \mathrm{Gr}(K(X)) \to \mathbb{Q} \otimes \mathrm{Gr}(K(\mathrm{Spec}(k)) = \mathbb{Q},$$

which can be seen to have the expected geometric interpretation in the discussion after (FULTON; LANG, 1985, Corollary 5.4, Chp. 6).

In the next theorem we will need the Todd class associated to an object in $K(X)$. It is defined analogously as the Chern character but instead of using the exponential and the summation, we use

$$Q(x) = \frac{x}{1 - \exp(-x)}$$

and the product. Making the Todd class to be $td(E) = \prod Q(\alpha_i)$, where each $\alpha_i = c^1(e_i)$ and $E = e_1 + \ldots + e_k$ for some splitting of E in K'. This construction is also independent of the choice of the splitting.

Theorem 1.3.12. ***(Hizerbruch–Riemann–Roch Theorem)*** *In the above situation, $\chi(X, \mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E}) td(\mathcal{T}_X)$, with $td(\mathcal{T}_X)$ the Todd class of the tangent bundle $\mathcal{T}_X$ of X.*

This construction of Chern character could be done for the Chow ring $\mathrm{CH}(X)$, see (RYAN, 2015), and reach the same conclusion, implying that we have an isomorphism $\mathbb{Q} \otimes \mathrm{CH}(X) \simeq \mathbb{Q} \otimes \mathrm{Gr}(K(X))$ respecting the grading.

Definition 1.3.13. Let K be λ-ring. A line element $u \in L$ is said to be ample if given $x \in K$ there exists an integer $m(x)$ such that for all $n \geqslant m(x)$, $u^n x = e - m$ for some positive element e and some integer m.

The ampleness condition in the theory of λ-ring is related to the ampleness of a line bundle by (FULTON; LANG, 1985, Lemma 3.1, Chp. 5). Usually, when dealing with concrete examples of Bridgeland stability conditions, we will be only concerned with the degree of the variety, not the whole structure in the Chow ring. For that, we will be applying the theorem by Nakai–Moishezon–Kleiman.

Theorem 1.3.14. *(LAZARSFELD, 2004, Theorem 1.2.23) Let L be a line bundle in a projective scheme X. Then L is ample if and only if*

$$\int_V c_1(L)^{\dim(V)} > 0 \tag{1.1}$$

for every positive-dimensional irreducible subvariety V of X

To see how this is used in our case we just have to translate the inequality 1.1. The Chern class $c_1(L)$ is the image of the map in 1.3.7 in the graded Grothendieck ring(or Chow ring), the power $()^{\dim(V)}$ is to represent the multiplication of $c_1(L)$ to adjust the codimension and the last piece is the integral with limit V, which is defined as the degree of the product(i.e. intersection) of $c_1(L)^{\dim(V)}$ with the class of the structure map of V in X $[\mathcal{O}_V]$.

Lemma 1.3.15. *(FULTON; LANG, 1985, Proposition 3.6, Ch.3) If $\mathrm{ch}_i(x) = 0$ for all $i < d$ then $\mathrm{ch}_d(x) = x$.*

Example 1.3.16. Let $\mathcal{F}$ be a sheaf of codimension $d \neq 0$ on a n-dimensional projective regular scheme X. The first thing to note is that $\mathcal{F} \in \mathbb{Q} \otimes F_{n-d}K(X) = \mathbb{Q} \otimes F^d K(X)$ and we can decompose $Q \otimes F^d K(X)$ into the direct sum $\bigoplus_{i \geqslant d}^{n} V(i)$ and in each of these $V(k)$ the Chern character is an isomorphism taking an element in $V(k)$ to ch_k of the said element, by Theorem 1.3.11. Hence, we can conclude $\mathrm{ch}_i(\mathcal{F}) = 0$ for $i < d$ and by Lemma 1.3.15 we know that $\mathrm{ch}_d(\mathcal{F}) = \mathcal{F} \in F^d K(X)/F^{d+1}K(X)$ which is non zero by Proposition 1.3.9.

The last part of this section is dedicated to introducing the twisted version of the Chern characters, which is used when dealing with the geometric stability conditions. Let B be a divisor, E an element of $K(X)$ then we define the B-twisted Chern character of E by the formula $\mathrm{ch}^B(E) = \mathrm{ch}(E) \cdot e^{-B}$. Explicitly, this means for example:

$$\begin{aligned}
\mathrm{ch}_0^B(E) &= \mathrm{ch}_0(E) \\
\mathrm{ch}_1^B(E) &= \mathrm{ch}_1(E) - B\,\mathrm{ch}_0(E) \\
\mathrm{ch}_2^B(E) &= \mathrm{ch}_2(E) - B\,\mathrm{ch}_1(E) + \frac{1}{2}B^2\,\mathrm{ch}_0(E) \\
\mathrm{ch}_3^B(E) &= \mathrm{ch}_3(E) - B\,\mathrm{ch}_2(E) + \frac{1}{2}B^2\,\mathrm{ch}_1(E) - \frac{1}{6}B^3\,\mathrm{ch}_0(E).
\end{aligned}$$

2 Bridgeland stability

In this chapter, we will discuss the fundamentals of Bridgeland stability necessary for the development of the main results obtained. Before going to the general theory, we need a brief review of sheaf stability considering both Gieseker-Simpson‘s and Mumford‘s stability, stating important definitions and theorems that we are going to use throughout the thesis. These can be seen as models for the general theory.

Then we turn to the general definition of weak and Bridgeland stability conditions, with a discussion of the support property and some of its consequences such as the deformation theorem and the existence of a wall-and-chamber decomposition to the space of stability conditions. Following this, we study the known cases of Bridgeland stability conditions for smooth schemes with dimensions less or equal to 3 and their distinguished curves.

To close the chapter we make the connection of the notion of exceptional collections inside the derived category and Bridgeland stability conditions, characterized by the definition of the quiver regions.

2.1 Sheaf stability

We start this section by establishing the notion of Gieseker-Simpson (semi)stability and related variations as defined in both (SCHMIDT; SUNG, 2018) and (JARDIM; MACIOCIA, 2019). These are going to be the base stability condition on which we will model our later definitions.

We will be using the nomenclature found in (*HUYBRECHTS*; *LEHN*, 2010) : let $E \in \mathrm{Coh}(X)$ with dimension d in a smooth projective scheme X of dimension n, E is called a pure sheaf if $\dim(F) = d$ for all subsheaves F of E, similarly a torsion subsheaf of E is any subsheaf F with $\dim(F) < d$. In this situation, consider $E^D := \mathcal{E}xt^{n-d}(E, \mathcal{O}_{\mathbb{P}^3})$ such that E is said to be reflexive if $E \simeq E^{DD}$.

Definition 2.1.1. Let E be a coherent sheaf over X of dimension d and k be an integer $1 \leqslant k \leqslant d$. Let $P_E(n) = \chi(E \otimes \mathcal{O}_{\mathbb{P}^3}(n)) = \sum_{i=0}^{d} \alpha_i n^i$ be the Hilbert polynomial associated to E with leading coefficient α_d then define

$$p_{E,k}(t) = \sum_{i=d-k}^{d} (\alpha_i/\alpha_d) t^i$$

We say that E is GS_k-(semi)stable if E is a pure sheaf and for every non zero subsheaf $A \hookrightarrow E$ we have $p_{E,k}(m) < (\leqslant) p_{E/A,k}(m)$ for $m \gg 0$. If $k = \dim(E)$, a GS_k-(semi)stable E is called Gieseker-(semi)stable and denote $p_{E,d}$ by p_E.

It was observed in (SCHMIDT; SUNG, 2018) GS_k-stability implies GS_{k-1}-stability and also GS_k-semistability implies GS_{k-1}-semistability, for all $k \leqslant \dim(E)$. There is another, more useful in our situation, way of seeing GS_k-(semi)stability as a inequality involving the notation introduced in (JARDIM; MACIOCIA, 2019) of the $\delta_{i,j}(E,F)$. Let $E, F \in K(X)$ then

$$\delta_{ij}(E,F) = \mathrm{ch}_i(E)\,\mathrm{ch}_j(F) - \mathrm{ch}_j(E)\,\mathrm{ch}_i(F).$$

When the objects are known from the context we will denote $\delta_{ij}(E,F)$ by δ_{ij}.

Remark 2.1.2. We can use a factorization of $p_E(t)$ to establish an equivalent definition for GS_k-(semi)stability. By Grothendieck-Riemann-Roch we establish that for a 2-dimensional pure sheaf E and F a subsheaf of E we have

$$p_E(t) - p_F(t) = \frac{1}{\mathrm{ch}_1(F)\,\mathrm{ch}_1(E)}(\delta_{21}(E,F)x_1(t) + \delta_{31}(E,F)),$$

for some $x_1(t)$ linear polynomial. This implies that a 2-dimensional pure sheaf E is Gieseker-(semi)stable sheaf if and only if $(\delta_{21}(E,F), \delta_{31}(E,F)) > (\geqslant)(0,0)$ in the lexicographical order. Similarly, E is GS_1-(semi)stable if and only if $\delta_{21}(E,F) > (\geqslant)0$.

Remark 2.1.3. One special case of stability is GS_1 for sheaves with dimension $d = \dim(X)$, which is usually known as Mumford's μ-stability. Moreover, using the same argument as in Remark 2.1.2 we can describe this stability as $\delta_{10} > 0$. Another way to describe this stability is by defining the μ-slope with respect to a divisor H of a d-dimensional sheaf E

$$\mu_H(E) = \frac{\mathrm{ch}_1(E)H^{d-1}}{\mathrm{ch}_0(E)H^n}.$$

When clear from the context, we will denote $\mu_H(E)$ by just $\mu(E)$.

Theorem 2.1.4. *Let E be a coherent sheaf over a scheme X and k an integer less or equal to* $\dim(E)$*. There exists a filtration*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E,$$

such that $\dim(E_1) < \dim(E)$*, if E is not pure, and $F_i = E_i/E_{i-1}$ are* GS_k*-semistable sheaves satisfying $p_{F_i,k} > p_{F_{i+1},k}$, for all $i > 1$.*

One example we will use in Chapter 3 is the structure sheaf of smooth subvarieties S of $\mathbb{P}^3$. Let $i : S \hookrightarrow \mathbb{P}^3$ a smooth irreducible closed subscheme and $i_*\mathcal{O}_S$ the image of its structure sheaf in $\mathrm{Coh}(X)$. We will show that $i_*\mathcal{O}_S$ is always Gieseker-stable. Before we start proving this, we will need to do a few observations about i_* and i^*. Through the next remark define $E^\vee = R\mathcal{H}om(E, \mathcal{O}_{\mathbb{P}^3})[2]$ for any $E \in \mathrm{D^b}(\mathrm{Coh}(X))$.

Remark 2.1.5. Since S is a closed subscheme, then i_* is an exact functor satisfying $i^*(i_*(E)) = E$ for all sheaves E in $\mathrm{Coh}(S)$. The functor i^* is also exact as a consequence of S being a smooth subscheme and, applying (HUYBRECHTS, 2006, Section 3.4), we can define

$$i^!(E) := i^*(E) \otimes \omega_i,$$

where $\omega_i = \omega_S \otimes i^*\omega_{\mathbb{P}^3}[-c]$, with $c = \mathrm{codim}(S)$, $E \in \mathrm{D^b}(X)$, ω_S and $\omega_{\mathbb{P}^3}$ are the dualizing bundles of S and $\mathbb{P}^3$, respectively. The last calculation we will need is

$$\begin{aligned}
i_*(i^*(E)) &= i_*(i^!(E) \otimes \omega_i^\vee[-2]) && (\omega_i \otimes \omega_i^\vee = \mathcal{O}_S[2])\\
&= i_*(i^!(E) \otimes i^*(i_*(\omega_i^\vee))[-2]) && (i^*i_* = Id)\\
&= i_*(i^!(E)) \otimes i_*(\omega_i^\vee)[-2] && \text{(Projection Formula)}\\
&= E \otimes i_*(\omega_i) \otimes i_*(\omega_i^\vee)[-2] && (\text{Def. } i^! \text{ and Proj. Formula})\\
&= E \otimes i_*\mathcal{O}_S && (i_*(\omega_i) \otimes i_*(\omega_i^\vee) = i_*\mathcal{O}_S[2])
\end{aligned}$$

Let $F \overset{f}{\hookrightarrow} i_*\mathcal{O}_S$, $L = \mathrm{coker}(f)$ in $\mathrm{Coh}(X)$ and consider the exact sequence, in $\mathrm{Coh}(S)$,

$$0 \to i^*(F) \to \mathcal{O}_S \to i^*(L) \to 0.$$

We have that $i^*(L) = \mathcal{O}_C$, for some subvariety C of S, and since S is irreducible we can see that $\dim(C) < \dim(S)$. Moreover, due to Remark 2.1.5, this implies that $\mathrm{ch}_k(L) = 0$, for $k = \mathrm{codim}(S)$, concluding our reasoning, because this implies that every subsheaf of $i_*\mathcal{O}_S$ satisfy the inequality in Remark 2.1.2.

At last, let us remind a result proved in (HUYBRECHTS; LEHN, 2010, Lemma 1.7.9) and referenced as *Grothendieck's Theorem*. It tells us about the boundedness of families of pure quotients of a given sheaf, and this will be important when proving the existence of certain limits in Chapter 3.

Theorem 2.1.6 (Grothendieck Theorem)**.** *Let $E \in \mathrm{Coh}(X)$ be a d-dimensional sheaf with $d > 0$ over a projective scheme X of dimension 3, Hilbert Polynomial P and Mumford–Castelnuovo regularity $\mathrm{reg}(E) = p$. There exists C, depending only on P and p, such that for every purely d-dimensional quotient Q then $\mathrm{ch}^C_{4-d}(Q) \geqslant 0$. Moreover, the family of purely d-dimensional quotients Q with $\mathrm{ch}^C_{4-d}(Q)$ bounded from above is a bounded family.*

2.2 General theory

In his original paper (BRIDGELAND, 2007), Bridgeland proposed two definitions to this new stability condition, one using *slices* and one using *hearts of bounded t-structures*. The former can be seen as a special case of Gorodentsev–Rudakov–Kuleshov's (GORODENTSEV; KULESHOV; RUDAKOV, 2004) t-stability where we consider only the partially ordered set $\mathbb{R}$ indexing the slope of the semistable objects, and the latter can be seen as a generalization of Mumford's notion of μ-stability where we associate a degree and a rank function to objects in a heart of a bounded t-structure.

Throughout this section, let $\mathcal{T}$ be a triangulated category. The next definition is inspired by the property of the hearts of bounded t-structure expressed in 1.2.4.

Definition 2.2.1. A slice $\mathcal{P}$ in $\mathcal{T}$ is a collection of subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for every $\phi \in \mathbb{R}$ satisfying:

- $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$,
- $\mathrm{Hom}(A, B) = 0$ if $A \in \mathcal{P}(\phi_1)$ and $B \in \mathcal{P}(\phi_2)$ with $\phi_1 > \phi_2$,
- For every object $E \in \mathcal{T}$ we have a decomposition

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \dots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n = E \arrow[d] \\
& A_1 \arrow[ul, dotted] & A_2 \arrow[ul, dotted] & & A_{n-1} \arrow[ul, dotted] & A_n \arrow[ul, dotted]
\end{tikzcd}$$

  such that $A_i \in \mathcal{P}(\phi_i)$ and $\phi_i > \phi_{i+1}$, for every i.

The *slices* encode the information on semistable objects concerning the following notion of stability condition. The technical difficulty that comes from using the slices is that defining a priori all the semistable objects for a stability condition can be hard so that this definition is not the one usually employed when constructing examples. Next, we describe the concept of weak stability conditions, one of the building blocks used to construct a Bridgeland stability condition over a threefold.

Definition 2.2.2. [Slice version] A weak stability condition is a pair $\sigma = (Z, \mathcal{P})$ of a group homomorphism $Z : \mathrm{K}(\mathcal{T}) \to \mathbb{C}$, known as the stability function or central charge, and a slice $\mathcal{P}$ satisfying the following conditions:

- **Weak-Positivity:** For every $E \in \mathcal{P}(\phi)$, $Z(E) = p.e^{i\phi\pi}$ for some $p \in \mathbb{R}_{\geqslant 0}$.
- **Finiteness:** The homomorphism Z can be factored by a finite free $\mathbb{Z}$-lattice surjective map $v : \mathrm{K}(\mathcal{T}) \to \Lambda$.

For a given weak stability condition $\sigma = (Z, \mathcal{P})$, E is said to be σ-semistable if it is in $\mathcal{P}(\bar{\phi}_\sigma(E))$ for some $\bar{\phi}_\sigma(E) \in \mathbb{R}$.

Definition 2.2.3. [Slice version] A Bridgeland stability condition is a weak stability condition $\sigma = (Z, \mathcal{P})$ such that

- **Positivity:** For every $E \in \mathcal{P}(\phi)$, $Z(E) = p.e^{i\phi\pi}$ for some $p \in \mathbb{R}_{>0}$.
- **Support Property:** Let $|| \cdot ||_{\mathbb{R}}$ be a norm in $\Lambda \otimes \mathbb{R}$. Then

$$C_\sigma := \inf\left\{\frac{|Z(E)|}{||v(E)||} : E \in \mathcal{P}(\phi) \text{ and } \phi \in \mathbb{R}\right\} > 0.$$

In the same way, it is possible to define a weak/Bridgeland stability condition using hearts of bounded t-structures.

Definition 2.2.4. [Heart version] A weak stability condition is a pair $\sigma = (Z, \mathcal{A})$, where $Z : K(X) \to \mathbb{C}$ is a group homomorphism and $\mathcal{A}$ is a heart of a bounded t-structure satisfying:

- **Weak-Positivity:** For every $E \in \mathcal{A}$ we have $Z(E) \in \mathbb{R}_{\geqslant 0} \cdot e^{i\pi\phi}$ for some $\phi \in [0, 1)$. The phase of E is defined as $\phi_\sigma(E) := -\Re(Z(E))/\Im(Z(E))$ and $\phi_\sigma(E) = +\infty$ if $\Im(Z(E)) = 0$.

  An object $A \in \mathcal{A}$ is called σ-(semi)stable if $\phi_\sigma(A) > (\geqslant)\phi_\sigma(F)$ for every non-zero subobject F of A in $\mathcal{A}$.

- **Harder–Narasimhan filtration:** Let $E \in \mathcal{A}$, then there exists $n \in \mathbb{Z}_{>0}$ and $E_0, ..., E_n \in \mathcal{A}$ such that

$$E_0 = 0 \subset E_1 \subset E_2 \subset ... \subset E_{n-1} \subset E = E_n$$

  with $F_i = E_i/E_{i-1}$ σ-semistable objects and $\phi_\sigma(F_i) > \phi_\sigma(F_{i+1})$ for all $1 \leqslant i \leqslant n$.

- **Finiteness:** The homomorphism Z can be factored by a finite free $\mathbb{Z}$-lattice surjective map $v : \mathrm{K}(\mathcal{T}) \to \Lambda$.

Furthermore, a notion of Bridgeland stability can also be translated into this setup.

Definition 2.2.5. [Heart version] A weak stability condition $\sigma = (Z, \mathcal{A})$ is a Bridgeland stability condition if satisfies:

(a) **Positivity:** There is no non-zero object E in $\mathcal{A}$ such that $Z(E) = 0$.

(b) **Support Property:** Let $|| \cdot ||_\mathbb{R}$ be a norm in $\Gamma_\mathbb{R}$. Then

$$C_\sigma := \inf\left\{\frac{|Z(E)|}{||v(E)||} : E \neq 0 \text{ and } \sigma\text{-semistable}\right\} > 0.$$

Theorem 2.2.6. *(BRIDGELAND, 2007, Proposition 5.3) The definitions 2.2.5 and 2.2.3 are equivalent.*

Sketch of Proof. Let $\sigma = (Z, \mathcal{P})$ be a stability condition defined with a slice $\mathcal{P}$, then we obtain a stability condition $(Z, \mathcal{A})$ with a heart of a bounded t-structure $\mathcal{A}$ such as

$$\mathcal{A} = \mathcal{P}((\psi, \psi + 1]) := \langle E | E \in \mathcal{P}(\phi) \text{ with } \phi \in (0, 1] \rangle.$$

For the reverse construction, given a stability condition $\sigma = (Z, \mathcal{A})$ with a heart of a bounded t-structure $\mathcal{A}$ we can obtain a slice by setting

$$\mathcal{P}(\phi) = \{E \in \mathcal{A}[k] |\ k \in \mathbb{Z} \text{ such that } \phi - k \in (0, 1] \text{ and } E[-k] \text{ is } \sigma\text{-semistable } \}.$$

□

As shown in (BRIDGELAND, 2007), it is possible to define a natural topology in the space of stability conditions. This topology is given by generalized metric that measures the difference in slope of all the objetcs in $\mathcal{T}$ with respect to two different stability conditions. Let $\mathrm{Stab}_\Lambda(\mathcal{T})$ be the space of stability conditions with respect to a given finite rank lattice $v : \mathrm{K}(\mathcal{T}) \to \Lambda$, and a generalized metric $d : \mathrm{Stab}(\mathcal{T}) \times \mathrm{Stab}(\mathcal{T}) \to [0, +\infty]$ given by:

$$d(\sigma, \sigma') = \sup_{E \in \mathcal{T}} \{|\phi^+_\sigma(E) - \phi^-_{\sigma'}(E)|, |\phi^-_\sigma(E) - \phi^-_{\sigma'}(E)|, ||Z - Z'||_\Lambda\},$$

where Z and Z' are the respective stability fuctions of σ and σ', and we are taking the norm to be in the finite dimensional vector space $\mathrm{Hom}(\Lambda, \mathbb{C})$.

Theorem 2.2.7. *(BRIDGELAND, 2007, Theorem 1.2) The projection of the first coordinate*

$$\mathrm{Stab}_\Lambda(\mathcal{T}) \to \mathrm{Hom}(\Lambda, \mathcal{C})$$

induces a complex manifold $\mathrm{rk}(\Lambda)$*-dimensional structure on* $\mathrm{Stab}_\Lambda(\mathcal{T})$*.*

The support property is a crucial ingredient in this proof, as it is the condition that guarantees that small deformations of the stability condition still gives a stability condition, see (BRIDGELAND, 2007, Theorem 7.1). This formulation of the support property was later substituted by requiring the existence of a quadratic form, a first instance of this new formulation can be found in (KONTSEVICH; SOIBELMAN, 2008, Section 2.1). In their first remark, the authors in (KONTSEVICH; SOIBELMAN, 2008, Section 1) suggest a motivation for the support property relating it to the large volume limit inside the derived Fukaya category.

Theorem 2.2.8. *(BAYER; MACRÌ; STELLARI, 2016, Lemma A.4, Appendix A) Let* $\sigma = (Z, \mathcal{A})$ *be a weak stability condition with respect to* $v : \mathrm{K}(\mathcal{A}) \to \Lambda$*, then* σ *satisfies the support property if and only if there exists a quadractic form* $Q : \Lambda \otimes \mathbb{R} \to \mathbb{R}$ *satisfying:*

- *All semistable objects* $E \in \mathcal{A}$ *satisfy* $Q(v(E), v(E)) \geqslant 0$*,*

- *All non-zero vectors $v \in \Lambda \otimes \mathbb{R}$ with $Z(v) = 0$ satisfy $Q(v, v) < 0$.*

Another direct application of the support property can be found in (BRIDGELAND, 2008, Proposition 9.3). In the aforementioned proposition, Bridgeland proves the wall and chamber structure of the space of stability conditions where, for a given $v \in \Lambda$, it is possible to decompose the space $\mathrm{Stab}_\Lambda(\mathrm{D^b}(\mathrm{Coh}(X)))$ into chambers separated by $2\,\mathrm{rk}(\Lambda) - 1$-real manifolds called the walls, on which the stability of any object $E \in \mathrm{D^b}(\mathrm{Coh}(X))$ with $v(E) = v$ is preserved. Moreover, he proved that if we were to take a compact subset of $\mathrm{Stab}_\Lambda(\mathrm{D^b}(\mathrm{Coh}(X)))$ then the number of walls intersecting this compact would be finite.

To close this subsection we will just briefly discuss the two natural actions in $\mathrm{Stab}_\Lambda(\mathcal{T})$. The first is by the group of automorphisms that respect the morphism $v : \mathrm{K}(\mathcal{T}) \to \Lambda$, $\mathrm{Aut}_\Lambda(\mathcal{T})$, where an automorphism $\phi : \mathcal{T} \to \mathcal{T}$ with its action ϕ_* in $\mathrm{K}(\mathcal{T})$ can be seen to act on $\mathrm{Stab}_\Lambda(\mathcal{T})$ by letting $\phi \cdot (Z, \mathcal{P}) = (Z \circ \phi_*, \phi(\mathcal{P}))$, and $\phi(\mathcal{P})(\theta) = \phi(\mathcal{P}(\theta))$.

The other action is done by the universal cover of the topological group $\mathrm{Gl}_2^+(\mathbb{R})$, $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$, which can be described as in (MACRÌ; SCHMIDT, 2019, Remark 5.4) by the set of pairs (T, f) where $T \in \mathrm{Gl}^+(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(\phi + 1) = f(\phi) + 1$ and $f|_{\mathbb{R}/2\mathbb{Z}} = T|_{S^1}$. These pairs act on a stability function by $(T, f) \cdot (Z, \mathcal{P}) = (T^{-1} \cdot Z, \mathcal{P}(f))$. A way to see this action is by realizing T as a deformation of the image of the stability function in the complex plane and the function f is what keeps track of the slope the image of these deformations will have.

2.3 Curves and surfaces

The space of Bridgeland conditions was first studied in the case of curves by Bridgeland in his first paper on the subject (BRIDGELAND, 2007). In that instance he proved for an elliptic curve C that $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$ acts transitively over the space of stability conditions and the canonical heart $(Z, \mathrm{Coh}(C))$ with $Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$ is a Bridgeland stability condition. Later, this result was extended to all smooth curves with genus greater or equal than 1 by Macrì in (MACRÌ, 2014).

To complete the description of smooth curves it was only left to prove a description of the space of stability conditions for $C = \mathbb{P}^1$, which was done by Okada in (OKADA, 2006) by proving that $\mathrm{Stab}(\mathrm{D^b}(\mathbb{P}^1))$ is isomorphic to $\mathbb{C}^2$ as a complex manifold. This proof uses an action of $\mathbb{C} \times \mathbb{Z}$ to simplify $\mathrm{Stab}(\mathrm{D^b}(\mathbb{P}^1))$. Also in (MACRÌ, 2014), he is not able to prove a full description of $\mathrm{Stab}(\mathbb{P}^1)$ but he introduces much of the techniques used while relating Bridgeland stability conditions and exceptional objects.

The surface case on the other hand is much more complicated to describe. To start, the heart Coh(S) for any smooth projective variety with $\dim(S) \geqslant 2$ cannot be a heart of a Bridgeland stability condition, see (TODA, 2009a, Lemma 2.7). Also, differently from the one-dimensional case, a complete description of the space of Bridgeland stability conditions is much harder to achieve.

Although at the time was not yet a problem, in his article (BRIDGELAND, 2008) Bridgeland proposes a way of constructing hearts of bounded t-structures in a stability condition by applying Theorem 1.2.6 to a torsion pair in Coh(K), where K is a $K3$-surface. The stability function, in this particular situation, is defined by applying the natural Mukai bilinear form defined over $\Lambda = H^*(K, \mathbb{Z})$ to the Mukai vector of an object $E \in \mathrm{D^b}(K)$ and a numerical vector in $\mathcal{N}(K) = \mathbb{Z} \oplus NS(K) \oplus \mathbb{Z}$, where $NS(K)$ is defined as the Nerón-Severi group of K, that is, the group obtained by identifying any two divisors whose intersection with any curve yield the same degree.

This construction of Bridgeland stability conditions over surfaces by using Theorem 1.2.6 continued to be used by many authors, to cite some (ARCARA; BERTRAM, 2013; BAYER; MACRÌ, 2011; MACIOCIA, 2014), but the stability function needed to follow an equivalent definition, also given in (BRIDGELAND, 2008), that uses the Euler form and the Riemann-Roch theorem. In (MACRÌ; SCHMIDT, 2019), the authors provide a systematic way of defining a Bridgeland stability condition for the surface case and we will review some of the aspects of it here for completeness.

We start by the construction of the heart of a bounded t-structure. We will fix S to be a smooth projective surface over an algebraic closed field k, an ample class $\omega \in K(S)$ and a divisor $D \in \mathbb{R} \otimes F^1(X)$. For any pair ω, B it is possible to define the pair

$$\mathcal{T}_{\omega,B} := \{E \in Coh(S)|\text{ for every } E \twoheadrightarrow Q,\ \mu_\omega(Q) > \frac{\omega \cdot B}{\omega^2}\},$$

$$\mathcal{F}_{\omega,B} := \{E \in Coh(S)|\text{ for every } F \hookrightarrow E,\ \mu_\omega(F) \leqslant \frac{\omega \cdot B}{\omega^2}\}.$$

The pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ defines a torsion pair in Coh(S). To see why this is true we just need to apply the Harder-Narasimhan filtration associated with μ_ω-stability, 2.1.4. Moreover, the heart obtained by applying Theorem 1.2.6 is denoted by $\mathcal{B}^{\omega,B}$. The stability function, in this case is defined by the simple formula

$$Z_{\omega,B}(E) = \int e^{i\omega} \cdot \mathrm{ch}^B(E),$$

which can be explicitly expressed by

$$Z_{\omega,B} = -(\mathrm{ch}_2^B(E) - \frac{\omega^2}{2}\mathrm{ch}_0(E)) + i\omega(\mathrm{ch}_1(E) - B\,\mathrm{ch}_0(E)). \tag{2.1}$$

For the pair $(Z_{\omega,B}, \mathcal{B}^{\omega,B})$ needs to satisfy certain conditions for it to be a Bridgeland stability condition. The proof of positivity can be found in (ARCARA; BERTRAM, 2013, Corollary 2.1), it depends heavily on arguments using 1.3.16, the Hodge index theorem and the Bogomolov-inequality, only the latter is explicitly expressed here.

Theorem 2.3.1. *(HUYBRECHTS; LEHN, 2010, Theorem 3.4.1) If F is a μ-semistable torsion free sheaf over S then*

$$\Delta_{\omega,B}(F) = (\omega\,\mathrm{ch}_1(F))^2 - 2\omega^2\,\mathrm{ch}_0(F)\,\mathrm{ch}_2(F) \geqslant 0$$

To prove the existence of a Harder-Narasimhan filtration involves first proving the existence of the Harder-Narasimhan filtration when $B \in \mathbb{Q} \otimes F^1(S)$ because in this situation the image of $Z_{\omega,B}$ is a discrete subset of $\mathbb{H}$ since the elements in its definition are either in $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$ and only products with ω can be in $\mathbb{R}$. The other necessary condition in this proof is that $\mathcal{B}^{\omega,B}$ be noetherian, proven in (PIYARATNE; TODA, 2019, Lemma 2.17). This is a consequence of (MACRÌ; SCHMIDT, 2019, Theorem 4.10).

To extend this result to real divisors in $\mathbb{R} \otimes F^1(S)$ we would need to deform the stability conditions obtained with $B \in \mathbb{Q} \otimes F^1(S)$ using a small deformation of the Bogomolov-inequality which is also the support property of these stability conditions, as in (MACRÌ; SCHMIDT, 2019, Lemma 6.20), leading to the following theorem.

Theorem 2.3.2. *(MACRÌ; SCHMIDT, 2019, Theorem 6.10) Let S be a smooth projective surface. The pair $\sigma_{\omega,B} = (Z_{\omega,B}, \mathcal{B}^{\omega,B})$ gives a Bridgeland stability condition on S. The map $Amp(S) \times F^1(S) \otimes \mathbb{R} \to \mathrm{Stab}_\Lambda(\mathrm{D^b}(S))$ that takes (ω, B) into $\sigma_{\omega,B}$ is a continuos embedding.*

Example 2.3.3. When dealing with examples we usually make some restrictions that make the calculations easier such as require that ω and B are linearly dependent, that is, $\omega = \alpha H$ and $B = \beta H$ for some ample divisor H, $\beta, \alpha \in \mathbb{R}$ and $\alpha > 0$. In this case we change the notation of $\sigma_{\omega,B} = (Z_{\omega,B}, \mathcal{B}^{\omega,B})$ to $\sigma_{\beta,\alpha} = (Z_{\beta,\alpha}, \mathcal{B}^{\beta})$, while it is important to note that $\mathcal{B}^{\omega,B}$ may depend on ω but does not change for positive multiples of ω. Theorem 2.3.2 allows for us to define the upper-half plane of stability conditions $\mathbb{H}$.

2.4 Threefolds

Since its introduction, one of the main goals of the theory of Bridgeland stability conditions was to study the behavior of stable of objects over Calabi-Yau varieties and, for a long time, in the case of threefolds, it was not known if they even had any stability function, to begin with. This question leads to the search of the so-called *generalized Bogomolov-type inequalities*, a quadratic form satisfying the conditions to be a support property for these threefolds. The existence of a Bridgeland stability condition on a Calabi-Yau was first proved by Li in (LI, 2019b) for the quintic threefold hypersurface in $\mathbb{P}^4$.

In this section, we will not focus on the Calabi-Yau case, as the main results obtained do not deal with them. Instead, we will focus on constructing what is known as the *geometric Bridgeland stability conditions* as in (BAYER; MACRÌ; TODA, 2014) and see a few examples of how they appear in the literature. Furthermore, the notation used is going to be the same as in Example 2.3.3. For that fix the ample class H, $\mathrm{ch}^{\beta}(E) = \mathrm{ch}^{\beta H}(E)$ and $\mathrm{ch}_i(E) = \mathrm{ch}_i(E) \cdot H^{3-i}$.

The construction of the geometric Bridgeland stability conditions by the authors in (BAYER; MACRÌ; TODA, 2014) uses the same approach as when we constructed a Bridgeland stability for the surface case, by tilting a known heart of a bounded t-structure and taking the stability function as a top degree function of the intersection of the $\exp iH$ and the twisted Chern character. Only this time we have to do it twice! In this case, the stability obtained with $Z_{\beta,\alpha}$ as in equation (2.1) is a weak stability, not a Bridgeland stability because its stability function has non-trivial objects in the heart that have zero image by $Z_{\alpha,\beta}$, namely the skyscraper sheaves.

To start, let X be a smooth projective threefold over an algebraically closed field k and $\sigma_{\beta,\alpha} = (Z_{\alpha,\beta}, \mathcal{B}^{\beta})$ be the weak stability obtained by tilting as in the construction done in the previous Section. This is indeed a weak stability by a composition of results: (BAYER; MACRÌ; STELLARI, 2016, Appendix B) proves the deformation of the Harder-Narasimhan filtration for the weak stability condition; (BAYER; MACRÌ; TODA, 2014, 7.3.2) proves the Bogomolov-inequality for this situation, proving, therefore, the support property, the weak-positivity is the same proof as in the surface case. This stability condition is denoted by tilt stability.

The next step is defining a torsion pair in $\mathcal{B}^{\beta}$:

$$\mathcal{T}_{\beta,\alpha} := \{E \in \mathcal{B}^{\beta} |\ \text{for every } E \twoheadrightarrow Q,\ \nu_{\beta,\alpha}(Q) > 0\},$$

$$\mathcal{F}_{\beta,\alpha} := \{E \in \mathcal{B}^{\beta} |\ \text{for every } F \hookrightarrow E,\ \nu_{\beta,\alpha}(F) \leqslant 0\}.$$

Applying Theorem 1.2.6 to the torsion pair $(\mathcal{T}_{\beta,\alpha}, \mathcal{F}_{\beta,\alpha})$ in $\mathcal{B}^{\beta}$ we obtain the heart of bounded t-structure $\mathcal{A}^{\beta,\alpha}$. The stability function is defined in the same as in the surface case, but this time there will be more terms because we are considering intersections of dimension i and $3-i$, for all $i \leqslant 3$.

$$Z_{\beta,\alpha,s}(E) = \left(-\mathrm{ch}_3^{\beta}(E) + \left(s + \frac{1}{6}\right)\mathrm{ch}_1^{\beta}(E)\right) + i\left(\mathrm{ch}_2^{\beta}(E) - \frac{1}{2}\alpha^2 \mathrm{ch}_0(E)\right), \tag{2.2}$$

for $E \in \mathrm{D^b}(X)$ so that $\sigma_{\beta,\alpha,s} = (Z_{\beta,\alpha,s}, \mathcal{A}^{\beta,\alpha})$ is the geometric stability condition, for every $s \in \mathbb{R}_{>0}$. The proof of the positivity of $\sigma_{\beta,\alpha,s}$ is done in (BAYER; MACRÌ; TODA, 2014, Lemma 3.2.1). The abelian categories $\mathcal{A}^{\beta,\alpha}$ are noetherian by (BAYER; MACRÌ; TODA, 2014, Theorem 5.2.2), so that we would just need a form of the support property to prove the existence of a Harder-Narasimhan filtration.

The support property in the threefold case is usually given by the generalized Bogomolov inequality, as introduced in (BAYER; MACRÌ; TODA, 2014). Many authors proved the existence of some kind of generalized Bogomolov inequality for threefolds, see (PIYARATNE; TODA, 2019; MACRì, 2007; SCHMIDT, 2020; BAYER; MACRÌ; TODA, 2014; LI, 2019b; LI, 2019a; BERNARDARA et al., 2017; BAYER; MACRÌ; STELLARI, 2016). In this collection, the many authors prove the existence of Bridgeland stability conditions over the abelian threefolds, the Fano threefolds and the quintic threefold. The techniques employed in these papers are different from one another, some used exceptional collections and extended the result outside of their quiver regions, some used Fourier–Mukai transforms to achieve the expected result and others utilized a generalized version of the geometric stability condition to change the setting and prove different, but equivalent, version of the inequality.

We express the generalized Bogomolov inequality as it was presented in (SCHMIDT, 2020; JARDIM; MACIOCIA, 2019), as a quadratic form involving the third Chern character. In this work, we will only need the generalized Bogomolov inequality for the threefolds $X = \mathbb{P}^3$ and the smooth quadric $X = Q_3$, proved by (MACRì, 2007) and (SCHMIDT, 2014) respectively.

Theorem 2.4.1. *Let X being either the 3-dimensional projective space or the smooth quadric threefold and E be $\nu_{\beta,\alpha}$-semistable object in $\mathcal{B}^\beta$. Then E satisfy the generalized Bogomolov inequality:*

$$Q_{\beta,\alpha}(E) := (\mathrm{ch}_1(E)^2 - 2\cdot \mathrm{ch}_2(E)\,\mathrm{ch}_0(E))\alpha^2 + 4(\mathrm{ch}_2^\beta(E))^2 - 6\,\mathrm{ch}_1^\beta(E)\,\mathrm{ch}_3^\beta(E) \geqslant 0.$$

2.4.1 Distinguished curves

We will now discuss the natural curves that appear when trying to understand the geometry of the walls inside the upper-half plane of stability conditions $\mathbb{H}$. In the surface case, these were characterized in (MACIOCIA, 2014) as non-intersecting semicircles centered in the β-axis, which was later translated to the case of tilt stability in (SCHMIDT, 2020). In the threefold case, an extensive discussion can be found in (JARDIM; MACIOCIA, 2019). Here we will discuss the most important aspects of the theory of distinguished curves that are going to be used in the later parts of the thesis.

The first definition is regarding *the walls*. These are important concepts in any theory of stability as it is the structure determining when an object becomes stable or unstable.

Definition 2.4.2. A *numerical wall* inside the space of (weak)Bridgeland stability conditions with respect to an element $w \in \Lambda$ is the subset of stability conditions $\sigma = (Z, \mathcal{A})$ with non trivial solutions to the equation $\bar{\phi_\sigma}(w) = \bar{\phi_\sigma}(u)$ for some fixed $u \in \Lambda$. Denote the numerical with respect to w and u in Λ by $(\bar{\Sigma}_{w,u})$ $\Upsilon_{w,u}$.

A subset of a numerical wall is called an *actual wall* if, for each point $\sigma = (\mathcal{A}, Z)$ in this subset, there is a sequence of σ-semistable objects $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $v(B) = w$ and $\bar{\phi_\sigma}(A) = \bar{\phi_\sigma}(B) = \bar{\phi_\sigma}(C)$.

We can use that the stability function has image in the complex numbers to show that a Bridgeland stability $\sigma = (Z, \mathcal{P})$ is in $\Upsilon_{w,u}$ if and only if it satisfies the equation

$$f_{w,u}(\beta, \alpha) = \Re(Z(v(u)))\Im(Z(v(w))) - \Re(Z(v(w)))\Im(Z(v(u))) = 0.$$

The previous definition is for the whole space of stability conditions, we will be mostly concerned with two types of numerical walls: the tilt walls or ν-walls $\Sigma_{E,F}$ and the λ-walls $\Upsilon_{E,F,s}$, for given $E, F \in K(X)$ and $s \in \mathbb{R}_{>0}$. That is, let $s \in \mathbb{R}_{>0}$ then $\Upsilon_{E,F,s}$ is the $\mathbb{H}$-section of the numerical wall for $E \in \mathrm{K}_0(X)$ with respect to F and the Bridgeland stability conditions $\sigma_{\beta,\alpha,s}$. The points $(\beta, \alpha) \in \Upsilon_{E,F,s}$ satisfy

$$f_{E,F,s}(\beta, \alpha) := \tau_{\beta,\alpha,s}(E)\rho_{\beta,\alpha}(F) - \tau_{\beta,\alpha,s}(F)\rho_{\beta,\alpha}(E) = 0, \tag{2.3}$$

where $\tau_{\beta,\alpha,s}(E) = -\Re(Z_{\beta,\alpha,s}(E))$ and $\rho_{\beta,\alpha}(E) = \Im(Z_{\beta,\alpha,s}(E))$.

It will be useful to expand the expression in equation (2.3) so that we get the following degree 4 polynomial:

$$f_{E,F,s}(\beta, \alpha) = \frac{6s+1}{12}\delta_{10}\alpha^4 +$$
$$\left(\frac{3s-1}{6}\delta_{10}\beta^2 + \frac{1-3s}{3}\delta_{20}\beta + \frac{6s+1}{6}\delta_{21} - \frac{1}{2}\delta_{30}\right)\alpha^2 +$$
$$\left(\frac{1}{12}\delta_{10}\beta^4 - \frac{1}{3}\delta_{20}\beta^3 + \frac{\delta_{30}+\delta_{21}}{2}\beta^2 - \delta_{31}\beta + \delta_{32}\right).$$

The tilt-wall is defined similarly but instead of using Bridgeland stability condition, it is used the tilt stability function. The numerical and actual walls can be tricky to work in the case of Bridgeland stability conditions $\sigma_{\beta,\alpha,s}$ as they are plane curves in degree 4 but they have nice behavior along some notable curves as proven in (JARDIM; MACIOCIA, 2019, Section 4 and 6) and (SCHMIDT, 2020).

Definition 2.4.3. Let $w, v \in \Lambda$ and define the following curves:

- The numerical wall for v and w in $\mathbb{H}$ for $\sigma_{\beta,\alpha}$ is denoted by $\Sigma_{v,w}$, known as the ν-wall associated with v and w.
- Let s be a positive real number, the numerical wall for v and w in $\mathbb{H}$ for $\sigma_{\beta,\alpha,s}$ is denoted by $\Upsilon_{v,w,s}$, known as the λ-wall associated with v and w.
- $L_w := \{(\beta, \alpha) \in \mathbb{H} | \operatorname{ch}_1^\beta(w) = 0\}$. The space to the left of the line L_w will be denoted by $L_w^+ := \{\operatorname{ch}_1^\beta(w) > 0\}$ and respectively the right-hand side will be $L_w^- := \{\operatorname{ch}_1^\beta(w) < 0\}$.

- $\Theta_w := \{(\beta,\alpha) \in \mathbb{H} | \Re(Z^t_{\beta,\alpha}(w)) = 0\}$. Theta may divide the plane in two regions $\Theta_w^+ := \{\Re(Z_{\beta,\alpha}(w)) > 0\}$ and $\Theta_w^- := \{\Re(Z_{\beta,\alpha}(w)) < 0\}$.

- $\Gamma_w := \{(\beta,\alpha) \in \mathbb{H} | \Re(Z_{\beta,\alpha,s}(w)) = 0\}$.

Example 2.4.4. Let E be a μ-semistable sheaf and consider what is necessary for $(\beta,\alpha) \in \mathbb{H}$ to imply either that $E \in \mathcal{B}^\beta$ or $E \in \mathcal{A}^{\beta,\alpha}$. The first observation is that $E \in \mathcal{B}^\beta$ whenever $\beta < \mu(E)$, to the left of L_E in $\mathbb{H}$, and $E \in \mathcal{B}^\beta[-1]$ otherwise. If E is $\nu_{\beta,\alpha}$-semistable then there are 3 possiblities:

- $E \in \mathcal{A}^{\beta,\alpha}$ if $(\beta,\alpha) \in L_w^+ \cap \Theta_w^+$

- $E \in \mathcal{A}^{\beta,\alpha}[-1]$ if $(\beta,\alpha) \in (L_w^- \cap \Theta_w^-) \cup (L_w^+ \cap \Theta_w^-) \backslash \Theta_w$ or

- $E \in \mathcal{A}^{\beta,\alpha}[-2]$ if $(\beta,\alpha) \in L_w^- \cap \Theta_w^+$.

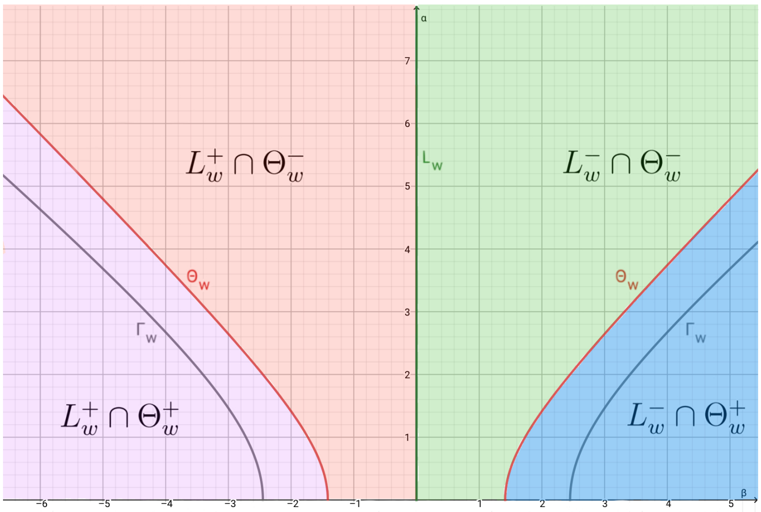

Figure 2.4.4.1 – Distinguished curves related to $w = (2,0,-2,0) = \mathrm{ch}(I)$, where I is an Instanton sheaf of charge 2.

Next, we describe the interaction of these curves with the ν-walls for an object w in Λ. The interaction between the curve $\Re(Z(w)) = 0$ and the ν-walls will be analogous in the Bridgeland stability case.

Theorem 2.4.5. *(SCHMIDT, 2020; MACIOCIA, 2014) Fix a vector $w = (R,C,D,E) \in \Lambda$. The walls are with respect to w.*

(a) Numerical ν-walls are of the form

$$x\alpha^2 + x\beta^2 + y\beta + z = 0$$

for $x = Rc - Cr$, $y = 2(Dr - Rd)$ and $z = 2(Cd - Dc)$. In particular, they are all semicircles with center at the β-axis or vertical rays.

(b) Two numerical ν-walls intersect if and only if they are identical.

(c) If $R \neq 0$, the curve Θ_w is given by the hyperbola

$$(\beta - C/R)^2 - \alpha^2 = \frac{\Delta(w)}{R^2}$$

(d) If a numerical ν-wall is an actual ν-wall for some point then it is an actual ν-wall at every point.

(e) If $\Sigma_{w,v}$ is a numerical ν-wall then $\Sigma_{w,v} \cap \Theta_w = \Sigma_{w,v} \cap \Theta_v = \Theta_w \cap \Theta_v = \{p\}$ is the only point in the semi-circle $\Sigma_{w,v}$ with horizontal tangent space.

For the latter parts of the paper it will be important to keep track of the orientation of the numerical walls. This means keeping track of which points in $\mathbb{H}$ an object F destabilizes(numerically) another object E or if it does not affect its stability. We will provide a general definition of orientation as it can provide an useful generalization.

Definition 2.4.6. Let $w, u \in \Lambda$. We will define the *inside* of the numerical wall $\Upsilon_{w,u}$ as the subset of stability conditions $\sigma = (Z, \mathcal{P})$ in $\mathrm{Stab}_\lambda(\mathrm{D^b}(X))$ such that $f_{w,u} < 0$. Moreover, the outside of the numerical wall is the subset where $f_{w,u} > 0$.

This can be translated to the geometric stability conditions by saying that if E, F are in $\mathcal{A}^{\beta,\alpha}$ then: (β, α) is inside(outside) of $\Upsilon_{E,F,s}$ if and only if $\lambda_{\beta,\alpha,s}(E) < (>)\lambda_{\beta,\alpha,s}(F)$.

Example 2.4.7. Let X be the smooth quadric Q_3. This threefold was shown to satisfy the generalized Bogomolov inequality by Schmidt (SCHMIDT, 2014). Lets fix an ample generator for $\mathrm{Pic}(Q_3)$ as $\mathcal{O}_{Q_3}(H)$ such that $H^3 = 2$ in the Chow ring. In this situation we will also use the notation $\mathcal{O}(k)$ for $\mathcal{O}_{Q_3}(kH)$, whenever clear from the context.

Using the results as in (OTTAVIANI, 1988), we can define the spinor bundle S over Q_3 as the pullback of the universal bundle of the Grassmannian by the natural map $s : Q_3 \to Gr(2^3 - 1, 2^4 - 1)$. This bundle is almost self-dual satisfying $S^* = S(1)$.

Comparing this to (SCHMIDT, 2014) we can see that he is using S^* as his Spinor bundle when defining the exceptional region, so it is important to keep this in mind when our results here and the ones cited. In (OTTAVIANI, 1988, Theorem 2.1), he proves that spinor bundles are μ-stable and in (SCHMIDT, 2014, Lemma 4.6) he proves that $S^*(-1)[1]$ is tilt stable at the line $\beta = 0$ of the upper-half plane $\mathbb{H}$ of tilt stability conditions. To our use here, we will need to translate the upper half-plane by tensoring by $\mathcal{O}_{Q_3}(-1)$ so that we will study the tilt stability of $S^*(-2)[1]$.

Moreover by applying the numerical conditions of Theorem 2.4.5 we can see that every numerical tilt wall for the object $S^*(-2)[1]$ has to cross the hyperbola

$\rho_{\beta,\alpha}(S^*(-2)[1]) = 0$ at their apex and can never cross the numerical wall $\beta = \mu(S^*(-2)[1])$. So that we have the following Picture 2.4.7.1:

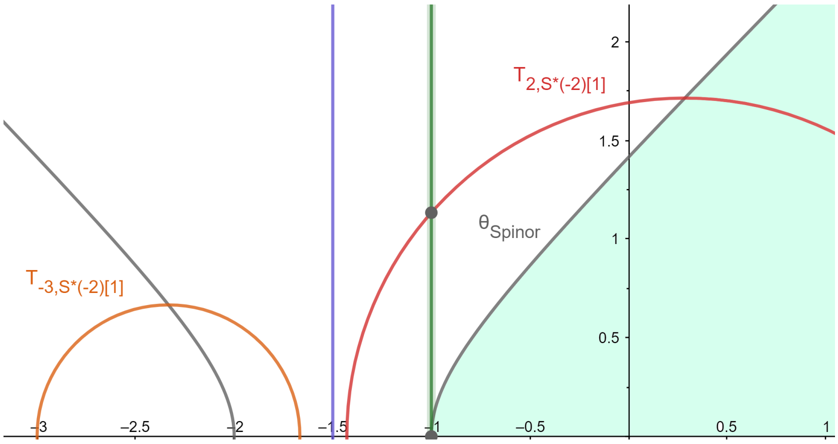

Figure 2.4.7.1 – An example of two numerical tilt-walls with respect to $S^*(-2)[1]$ and the region, in green, representing the points where $S^*(-2)[2] \in \mathcal{A}^{\beta,\alpha}$.

Since $S^*(-2)[1]$ is tilt stable at $\beta = -1$, we can see that no numerical wall to the right-hand side of $\beta = -3/2 = \mu(S^*(-2))$ can be an actual destabilizing tiltd wall. So that $S^*(-2)[1] \in \mathcal{F}_{\beta,\alpha}$ if (β, α) satisfies $p_{\beta,\alpha}(S^*(-2)[1]) \leqslant 0$, i.e. the light green region in Picture 2.4.7.1.

In the last chapter, we will reduce to the case of $s = 1/3$, we can do this because of the following result about the non-existence of new walls when we increase s beyond $1/3$. Hence, to study the existence of walls and therefore the stability of objects, we can consider $s = 1/3$.

Theorem 2.4.8. *(JARDIM; MACIOCIA, 2019, Theorem 6.12) Let $\Upsilon_{E,F,s}$ be the λ-wall for two objects $E, F \in \Lambda$ and any $s, s' \geqslant 1/3$. Then $\Upsilon_{E,F,s} \neq \varnothing$ if and only if $\Upsilon_{E,F,s'} \neq \varnothing$. If $s < 1/3$ and $\Upsilon_{E,F,s} \neq 0$ then $\Upsilon_{E,F,s'} \neq \varnothing$ for every $0 < s' < s$.*

2.5 Exceptional Collections

We will follow the notation used in (MACRì, 2007) for exceptional collections and related concepts. We can define a structure of linear triangulated category in $\mathrm{D^b}(X)$ by defining for any $A, B \in \mathrm{D^b}(X)$ the $\mathbb{Z}$-graded vector space

$$\mathrm{Hom}^\bullet(A,B) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(A,B) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(A, B[i]).$$

For any $\mathbb{Z}$-graded vector space $V^\bullet$ and $E \in \mathrm{D^b}(X)$, let $V^\bullet \otimes E := \bigoplus_{i\in\mathbb{Z}} V^i \otimes E$, where $V^i \otimes E$ is the direct sum of E $\dim(V^i)$-times. The dual of a $\mathbb{Z}$-graded vector space $V^\bullet$ is $V^{\bullet *}$ defined as $(V^{\bullet *})^i := (V^{-i})^*$ so that the dual of an object $V^\bullet \otimes E \in \mathrm{D^b}(X)$ is $(V^{\bullet *}[2]) \otimes E^\vee$, with $E^\vee = R\,\mathrm{Hom}(E, \mathcal{O}_X)[2]$.

Definition 2.5.1. An object E in $\mathrm{D^b}(X)$ is called exceptional if $\mathrm{Hom}^\bullet(E, E) = \mathbb{C}$. A collection $\mathcal{E} := \{E_0, ..., E_n\}$ of exceptional objects is called exceptional if they satisfy $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ if $i > j$. An exceptional collection can have other properties such as

- Strong: if $\mathrm{Hom}^k(E_i, E_j) = 0$ for all i, j and $k \neq 0$,
- Ext: if $\mathrm{Hom}^{\leqslant 0}(E_i, E_j) = 0$ for all $i \neq j$,
- Full: if the category generated by $\mathcal{E}$ via shifts and extensions is $\mathrm{D^b}(X)$.

It is easy to construct an Ext-exceptional collection from a strong exceptional collection by doing a trick: If we start with a strong exceptional collection $\mathcal{E} = \{E_0, E_1, E_2, E_3\}$ then $S(E) := \{E_0[3], E_1[2], E_2[1], E_3\}$ is an Ext-exceptional collection. This allows us to reduce the search for possible Ext-exceptional collections by using strong exceptional collections.

Theorem 2.5.2. *(MACRì, 2007, Lemma 3.4 and Lemma 3.6) Let $\{E_0, ..., E_n\}$ be a full Ext-exceptional collection in $\mathrm{D^b}(X)$ then the category generated by extensions $\langle E_0, ..., E_n\rangle$ is a heart of a bounded t-structure. Assume that $(Z, \mathcal{P})$ is a stability condition and $E_0, ..., E_n$ are all in $\mathcal{P}((\phi, \phi + 1])$ for some $\phi \in \mathbb{R}$, then $\langle E_0, ..., E_n\rangle = \mathcal{P}((\phi, \phi + 1])$ and E_i are stable, for all i.*

Theorem 2.5.2 leads to the following definition used to determine the regions of $\mathbb{H}$ we are interested in.

Definition 2.5.3. A full Ext-exceptional collection $\{E_0, ..., E_n\}$ satisfies the upper-half plane condition for a stability condition $(Z, \mathcal{P})$ if there exists a $\phi \in \mathbb{R}$ such that $\langle E_0, ..., E_n\rangle = \mathcal{P}((\phi, \phi + 1])$.

As a consequence of Theorem 2.5.2, Definition 2.5.3 is equivalent to the notion of σ-exceptional collection defined in (DIMITROV; KATZARKOV, 2016, Definition 3.19).

Example 2.5.4. Let $\mathcal{E} = \{E_0, ..., E_3\}$ be a complete Ext-exceptional collection and $\mathcal{C} = \langle\mathcal{E}\rangle$ the heart of bounded t-structure generated by $\mathcal{E}$. If $\sigma_{\beta,\alpha,s} = (Z_{\beta,\alpha,s}, \mathcal{A}^{\beta,\alpha})$ is a stability condition in X such that $\mathcal{C} \subset D^{\beta,\alpha} := \langle\mathcal{A}^{\beta,\alpha}, \mathcal{A}^{\beta,\alpha}[1]\rangle$ then we can define the $\mathbb{C}$-slope of a semistable object $E \in D^{\beta,\alpha}$ as the unique $\psi \in (0, 2]$ such that $Z_{\beta,\alpha,s}(E) = r \cdot e^{\pi\psi i}$ with $r \in \mathbb{R}_{>0}$. Moreover, we can rephrase Definition 2.5.3 as the existence of a $\phi \in (0, 1]$ where the upper-half plane $\overline{\mathbb{H}}$ rotated by $(\phi\pi)$-degrees contains all the complex numbers $Z_{\beta,\alpha,s}(E_i)$, for all i.

We will denote by $\overline{\mathbb{H}}_\phi$ the upper-half plane obtained from rotating $\overline{\mathbb{H}}$ by $(\phi\pi)$-degrees. In the previous example it is clear that if E is $\sigma_{\beta,\alpha,s}$-semistable with $\mathbb{C}$-slope $\phi \in (0, 2]$ then either E or $E[1]$ is in C.

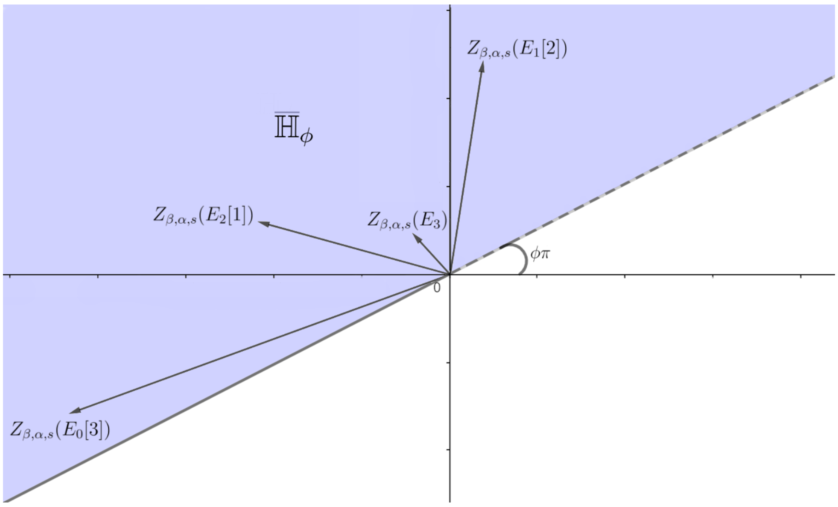

Figure 2.5.4.1 – The purple region determines the region determined by $\overline{\mathbb{H}}_\phi$.

Remark 2.5.5. It is important to note that the choice of ϕ in Example 2.5.4 is not usually canonical and we can have an interval of angles satisfying this condition. Even more, assume $\mathcal{E}$ to be a complete Ext-exceptional collection satisfying the upper-half plane condition for all $\phi \in (\psi, \psi') \subset (0, 1]$ and fix $(\beta, \alpha) \in \mathbb{H}$, then there is no semistable object $E \in \mathcal{A}^{\beta,\alpha}$ such that $Z_{\beta,\alpha,s}(E) = r_E \cdot e^{i\pi\gamma}$ with $\gamma \in (\psi, \psi')$, because if it were to exist such an E then $E \in \mathcal{P}_{\beta,\alpha}((\psi, \psi + 1])$ and not in $\mathcal{P}_{\beta,\alpha}((\psi', \psi' + 1])$ but both of these categories are equal to $\mathcal{C}$.

One way to produce numerical λ-walls for the objects $F \in \langle\mathcal{E}\rangle$ is by utilizing truncation functors. Let $\mathcal{E} = \{E_0[3], E_1[2], E_2[1], E_3\}$ be a full Ext-exceptional collection with $E_i \in \mathrm{Coh}(X)$, for all i, satisfying the upper-half plane condition then every object F is quasi-isomorphic to a complex

$$E_0^{\oplus a_0} \to E_1^{\oplus a_1} \to E_2^{\oplus a_2} \to E_3^{\oplus a_3}$$

with a_i uniquely determined so that we have the stupid truncation functors $t_{\leqslant k}$ and $t_{\geqslant l}$ with respect to the heart $\mathrm{Coh}(X)$ of $\mathrm{D^b}(X)$, leading to the functorial exact sequences for $E \in \langle\mathcal{E}\rangle$:

$$0 \to t_{>k}E \to E \to t_{\leqslant k}E \to 0 \tag{2.4}$$

3 Asymptotic stability

The study of asymptotic Bridgeland stability, or large volume limit, has been a major topic of research inside the field of Bridgeland stability, as it is both of interest to physicists and mathematicians. The advantage of this approach to stability is that usually, the asymptotic behavior makes things simpler. It was first introduced in (BRIDGELAND, 2008) as a way of relating Bridgeland stability with slope stability for the case of the $K3$ surfaces.

Later, many authors approached the problem of determining the large volume limit to the Bridgeland stability conditions, see (TODA, 2009b; BAYER, 2009; JARDIM; MACIOCIA, 2019; BAYER; MACRÌ; TODA, 2014). We will focus on the approach defined by Jardim–Maciocia, which studies an asymptotic stability condition using unbounded curves in the upper-half plane of stability conditions $\mathbb{H}$ to simplify the limits to one parameter spaces and then employ analysis techniques.

In this chapter, we examine the conditions that an object with zero Chern character equal to zero has to satisfy for it to be asymptotically (semi)stable with respect to weak or Bridgeland stability conditions. When considering weak stability, the objects asymptotically weak (semi)stable does not depend on the curve chosen, as long as the curve is not asymptotically vertical. This is due to the non-existence of a vertical tilt wall associated with the μ-slope of sheaves with zero Chern character zero.

For curves γ going asymptotically to the left-hand side of the upper-half plane of stability conditions $\mathbb{H}$, the asymptotically Bridgeland (semi)stable objects are the Gieseker-Simpson (semi)stable sheaves. On the other hand, if we focus on curves asymptotically going to the right-hand side of $\mathbb{H}$ then the asymptotically Bridgeland (semi)stable are the derived dual of Gieseker-Simpson (semi)stable sheaves.

The results in this chapter were first proven for the case of the threefold $X = \mathbb{P}^3$ and after submitting the paper the referee pointed out that this restriction was unnecessary, that the argument works just as well for any smooth projective threefold X whenever the geometric stability conditions $\sigma_{\beta,\alpha,s} = (Z_{\beta,\alpha,s}, \mathcal{A}^{\beta,\alpha})$ exist.

3.1 Definitions

Throughout the chapter we will fix that X is a smooth projective variety over an algebraic closed field k having geometric stability conditions of the form $\sigma_{\beta,\alpha,s} = (Z_{\beta,\alpha,s}, \mathcal{A}^{\beta,\alpha})$ for every $(\beta, \alpha) \in \mathbb{H}$ and $s > 0$, and the notation that γ is an unbounded curve in $\mathbb{H}$ parametrized by $\gamma(t) := (\beta(t), \alpha(t))$ satisfying

$$c_\gamma := \lim_{t\to+\infty} \frac{\alpha^2(t)}{\beta^2(t)} < 1. \tag{3.1}$$

This condition is important because it is the one that guarantees that whenever $E \in \mathrm{Coh}(X)$ we have that $\lim\limits_{t\to+\infty} \nu_{\gamma(t)}(E) > 0$ (see equation (3.7)), in geometric terms it means that for every $E \in \mathrm{Coh}(X)$ there is some $t_0 \in \mathbb{R}$ such that when $t > t_0$, $\gamma(t) \in \Theta_E^+$ holds. In (*JARDIM; MACIOCIA*, 2019) these curves are called Θ^--curves.

As seen in the previous subsection, the geometry and structure of the walls for tilt stability in threefolds are described by Lemma 2.4.5, but not much is known about the behavior of these walls for Bridgeland stability conditions. This is due to their definition, is done so by zeros of a degree 4 polynomial, and also that $\mathcal{A}^{\beta,\alpha}$ is much more complicated than $\mathcal{B}^\beta$, being a tilt of $\mathcal{B}^\beta$ it may have 3-step object which does not occur in $\mathcal{B}^\beta$.

One way to study the stability of objects in $\mathrm{D^b}(X)$, circumventing this difficulty, is proposed in (JARDIM; MACIOCIA, 2019) by trying to understand how objects behave asymptotically at infinity. It turned out to be closely related to sheaf Gieseker-stability.

Definition 3.1.1. Let $\sigma_{\beta,\alpha} = (Z_{\beta,\alpha}, C^{\beta,\alpha})$ be a familiy of weak stability condition parametrized by $(\beta, \alpha) \in \mathbb{H}$ and $\phi_{\beta,\alpha}$ its slope. For an object $A \in \mathrm{D^b}(X)$ to be asymptotic ϕ_γ-(semi)stable it has to satisfy the following conditions:

(a) There exists $t_0 \in \mathbb{R}$ such that $A \in C^{\gamma(t)}$ for $t > t_0$,

(b) Suppose that there exists $t_1 \geqslant t_0 \in \mathbb{R}$ such that $F \overset{f}{\hookrightarrow} A \in C^{\gamma(t)}$ for some $F \in \mathrm{D^b}(X)$ and every $t > t_1$, then exists $t_2 \in \mathbb{R}$ with $\phi_{\gamma(t)}(F) < (\leqslant) \phi_{\gamma(t)}(A)$ whenever $t > t_2$.

The cases we will be applying this definition are when $\sigma_{\beta,\alpha}$ are either tilt or Bridgeland stability conditions obtained by the tilt algorithm we described in the previous subsection, and $\phi_{\beta,\alpha} = \nu_{\beta,\alpha}$ or $\phi_{\beta,\alpha} = \lambda_{\beta,\alpha,s}$, for a fixed s, respectively.

It is important to note that this definition is equivalent to using quotients instead of subobjects in item (b), because of the way exact sequences are defined in hearts of t-structures. Using the notation of Definition 3.1.1, the short exact sequences in $C^{\beta,\alpha}$ correspond to distinguished triangles in $\mathrm{D^b}(X)$. Therefore, since we are fixing the map f, we know that f is monomorphic in $C^{\beta,\alpha}$ if and only if its quotient in $C^{\beta,\alpha}$ is the cone $C(f) \in \mathrm{D^b}(X)$.

One technique that we are going to use extensively in this chapter is to relate the cohomologies $\mathcal{H}^i_{\mathcal{B}^\beta}$ and $\mathcal{H}^i_{\mathcal{A}^{\beta,\alpha}}$, which for simplicity are going to be denoted by $\mathcal{H}^i_\beta$ and $\mathcal{H}^i_{\beta,\alpha}$ respectively. This relation is discussed in (JARDIM; MACIOCIA, 2019, Section 2.3), here we will use the following exact sequences for every object $E \in \mathcal{A}^{\beta,\alpha}$:

$$0 \to \mathcal{H}^{-1}_\beta(E)[1] \to E \to \mathcal{H}^0_\beta(E) \to 0 \tag{3.2}$$

$$0 \to \mathcal{H}^0(\mathcal{H}^{-1}_\beta(E)) \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(\mathcal{H}^0_\beta(E)) \to 0 \tag{3.3}$$

with $\mathcal{H}^{-2}(E) = \mathcal{H}^{-1}(\mathcal{H}^{-1}_\beta(E))$, $\mathcal{H}^0(E) = \mathcal{H}^0(\mathcal{H}^0_\beta(E))$, sequence (3.2) being exact in $\mathcal{A}^{\beta,\alpha}$ and sequence (3.3) being exact in $\mathrm{Coh}(X)$.

Remark 3.1.2. Let E be an object in $\mathcal{A}^{\beta,\alpha}$ and also in $\mathcal{A}^{\beta',\alpha'}$, where $\beta \neq \beta'$ then $\mathcal{H}^i(E)$ does not vary with β but $\mathcal{H}^i_\beta(E)$ is not necessarily equal to $\mathcal{H}^i_{\beta'}(E)$, thus $\mathrm{ch}_k(\mathcal{H}^i_\beta(E))$ may vary whenever we change β. One of the technical difficulties we will have to address is to prove that $\lim\limits_{t\to+\infty} \mathrm{ch}_i(\mathcal{H}^i_{\beta(t)}(E))$ exists, whenever $\lim\limits_{t\to+\infty} \beta(t) = \pm\infty$.

To exemplify this consider the sheaf $\mathcal{O}_{\mathbb{P}^3}$. Since $\mathcal{O}_{\mathbb{P}^3}$ is a μ-stable sheaf we can see that $\mathcal{O}_{\mathbb{P}^3} \in \mathcal{B}^\beta$ when $\beta < 0$ and $\mathcal{O}_{\mathbb{P}^3}[1] \in \mathcal{B}^\beta$, otherwise. Therefore, for negative β we have $\mathcal{H}^{-1}_\beta(\mathcal{O}_{\mathbb{P}^3}[1]) = \mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{H}^0_\beta(\mathcal{O}_{\mathbb{P}^3}[1]) = 0$, conversely for positive β we have $\mathcal{H}^{-1}_\beta(\mathcal{O}_{\mathbb{P}^3}[1]) = 0$ and $\mathcal{H}^0_\beta(\mathcal{O}_{\mathbb{P}^3}[1]) = \mathcal{O}_{\mathbb{P}^3}[1]$. It is important to note that if $-\beta < \alpha$ and $\alpha > \beta$ then $\mathcal{O}_{\mathbb{P}^3}[1] \in \mathcal{A}^{\beta,\alpha}$, for either case of β positive or negative.

Lemma 3.1.3. *(JARDIM; MACIOCIA, 2019, Lemma 2.8) If A is an object in $\mathcal{A}^{\beta,\alpha}$, then $\mathcal{H}^{-2}(A)$ is a reflexive sheaf of dimension* 3.

3.2 Asymptotic ν_γ-stability

Due to its construction, $\sigma_{\beta,\alpha} = (Z_{\beta,\alpha}, \mathcal{B}^\beta)$ is much simpler than its Bridgeland counterpart, making it a great starting point to the study of asymptotic (semi)stable objects. For $\mathrm{ch}_0(E) = 0$ objects we have to consider two cases: Either $\dim(\mathrm{Supp}(E)) = 2$ or $\dim(\mathrm{Supp}(E)) \leqslant 1$, in the first case ν_γ-stability will be equivalent to GS_1-stability and on the latter case, every sheaf is ν_γ-semistable. This is because $\nu_{\beta,\alpha}$ does not take into account ch_3, making it a bad stability condition to distinguish low-dimensional sheaves.

When we consider objects E with $\mathrm{ch}_0(E) = 0$ we are avoiding the existence of the canonical vertical wall $\{\beta = \mu(E)\}$, such wall is responsible for separating the regions $\{E \in \mathcal{B}^{\beta,\alpha}\}$ and $\{E \in \mathcal{B}^{\beta',\alpha'}[-1]\}$, if E is a μ-stable sheaf for example. This is the reason we are able to prove the same theorem for unbounded curves going either to the right or the left. We assume in this section that γ is an unbounded curve satisfying $\lim\limits_{t\to+\infty} |\beta(t)| = +\infty$.

Proposition 3.2.1. *Let $E \in \mathrm{D}^{\mathrm{b}}(X)$ be an object with $\mathrm{ch}_0(E) = 0$ and $\mathrm{ch}_1(E) \neq 0$. Then E is asymptotic ν_γ-(semi)stable if and only if it is GS_1-(semi)stable.*

The version of this proposition related to the case $\mathrm{ch}_0(E) = \mathrm{ch}_1(E) = 0$ is realized by knowing that $\Im(Z_{\beta,\alpha}(E)) = \mathrm{ch}^\beta_1(E) = 0$ for all $(\beta,\alpha) \in \mathbb{H}$, so that $E \in \mathcal{B}^{\beta,\alpha}$ for all $(\beta,\alpha) \in \mathbb{H}$ and $\nu_{\beta,\alpha}(F) = +\infty$ when $F \hookrightarrow E$ in $\mathcal{B}^{\beta(t)}$

Proof. To begin, assume that E is asymptotic ν_γ-(semi)stable. This implies a few properties about E: $E \in \mathcal{B}^{\beta(t)}$ for all t greater than some t_0; $\mathcal{H}^i(E) = 0$ for all $i \neq 0, -1$; $\mathrm{ch}_0(\mathcal{H}^{-1}(E)) = \mathrm{ch}_0(\mathcal{H}^0(E))$ and $\mathrm{ch}_1(\mathcal{H}^{-1}(E)) < \mathrm{ch}_1(\mathcal{H}^0(E))$. Let us prove that $\mathrm{ch}_0(\mathcal{H}^{-1}(E)) = \mathrm{ch}_0(\mathcal{H}^0(E)) = 0$, if this was not the case then we would have that $\mathcal{H}^{-1}(E)[1]$ and $\mathcal{H}^0(E)$ are both objects in $\mathcal{B}^{\beta(t)}$, for all t sufficiently large, with finite μ-slope which is a contradiction to $\lim\limits_{t\to+\infty}\beta(t) = -\infty$ and $\lim\limits_{t\to+\infty}\beta(t) = +\infty$, respectively. Moreover, if non zero, then $\mathcal{H}^{-1}(E)$ is a torsion-free sheaf because $E \in \mathcal{B}^{\beta(t)}$ for some t and this implies that $\mathcal{H}^{-1}(E) = 0$, in either direction of γ, concluding that E is a sheaf.

The next step is to prove its sheaf stability. Let

$$0 \to F \to E \to G \to 0 \tag{3.4}$$

be an exact sequence in $\mathrm{Coh}(X)$ and we can see that $\mathrm{ch}_0(F) = \mathrm{ch}_0(E) = \mathrm{ch}_0(G) = 0$, implying that $F, E, G \in \mathcal{B}^{\beta(t)}$ and that sequence (3.4) is also an exact in $\mathcal{B}^{\beta(t)}$, for all $t \in \mathbb{R}$. Therefore

$$\nu_{\gamma(t)}(E) - \nu_{\gamma(t)}(F) = \frac{\delta_{12}(E,F)}{\mathrm{ch}_1(F)\,\mathrm{ch}_1(E)}, \tag{3.5}$$

which is not dependent of t, proving the GS_1-(semi)stability of E.

Assume now that E is a 2-dimensional sheaf GS_1-(semi)stable. Being a 2-dimensional implies that $E \in \mathcal{B}^{\beta(t)}$ for all t, since every quotient of E in $\mathrm{Coh}(X)$ is also a 2-dimensional sheaf and therefore have infinite μ-slope. Assume now that we have a sequence as (3.4) but in $\mathcal{B}^{\beta(t)}$ such that $F, E, G \in \mathcal{B}^{\beta(t)}$ for all t greater than some t_0, by applying the argument we started the proof it is clear that both F and G are sheaves and from (3.5) we conclude asymptotic ν_γ-(semi)stability.

□

3.3 Stability at $-\infty$

We start the study of asymptotic Bridgeland stability by analyzing the left-hand side of $\mathbb{H}$. It turns out that asymptotic stability is much simpler at this side of the half-plane because objects in $\mathcal{A}^{\gamma(t)}$, for t sufficiently large, are coherent sheaves and asymptotic stability is equivalent to GS_k-stability. Throughout this section we will be studying the unbounded curves established in (3.1) but with a new condition: $\lim\limits_{t\to+\infty}\beta(t) = -\infty$.

The first result we prove is related to what kind of object can appear at infinity when considering $\mathcal{A}^{\gamma(t)}$, for t sufficiently large. Turns out the large volume limit objects are exactly sheaves, and their (semi)stable objects are the Gieseker-(semi)stable ones.

Proposition 3.3.1. *An object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is in $\mathcal{A}^{\gamma(t)}$ for every t sufficiently large if and only if $E \in \mathrm{Coh}(X)$.*

Proof. Let us start assuming that $E \in \mathcal{A}^{\gamma(t)}$ for t sufficiently large. We know that $\mathcal{H}^{-j}(E) = 0$ for all $j \neq 0, 1, 2$ and suppose that $\mathcal{H}^{-2}(E) \neq 0$, by Lemma 3.1.3 we know that $\mathrm{ch}_0(\mathcal{H}^{-2}(E)) > 0$ but that would be impossible because $\mathcal{H}^{-2}(E) \in \mathcal{F}_{\beta(t)}$, whenever $E \in \mathcal{A}^{\gamma(t)}$, an absurd because it implies that

$$0 \geqslant \lim_{t\to+\infty} \mathrm{ch}_1^{\beta(t)}(\mathcal{H}^{-2}(E)) = \lim_{t\to+\infty} \mathrm{ch}_1(\mathcal{H}^{-2}(E)) - \beta(t)\cdot \mathrm{ch}_0(\mathcal{H}^{-2}(E)) = +\infty.$$

Therefore $\mathcal{H}^{-2}(E) = 0$. Moreover, consider $\mathcal{H}^{-1}(E)$ decomposed by the following exact sequence in $\mathrm{Coh}(X)$

$$0 \to \mathcal{H}^0(\mathcal{H}^{-1}_{\beta(t)}(E)) \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E)) \to 0, \tag{3.6}$$

which can change for each value of t. Again, applying Grothendieck's Theorem we have that exists $C \in \mathbb{R}$ such that $\mathrm{ch}_1^C(Q) > 0$ for all quotients Q of $\mathcal{H}^{-1}(E)$. If $\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E)) \neq 0$, because it is in $\mathcal{F}_{\beta(t)}$, it needed to satisfy $\mathrm{ch}_1^{\beta(t)}(\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E))) \leqslant 0$ for sufficiently large t, which is impossible. Concluding that $\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E)) = 0$.

It is clear now that $\mathcal{H}^{-j}_{\beta(t)}(E)$ are, eventually, constant with respect to t. Fixing t_0 as the value for which $\mathcal{H}^{-j}_{\beta(t)}(E)$ are constant and $E \in \mathcal{A}^{\gamma(t)}$ for every $t > t_0$. In this ray, if $\mathcal{H}^{-1}_{\beta(t)}(E) = \mathcal{H}^0(\mathcal{H}^{-1}_{\beta(t)}(E))$ is non zero we would have

$$\lim_{t\to+\infty} \frac{1}{(-\beta(t))} \nu_{\gamma(t)}(\mathcal{H}^{-1}_{\beta(t)}(E)) = \begin{cases} (1-c_\gamma) & \text{if } \mathrm{ch}_0(\mathcal{H}^{-1}_{\beta(t)}(E)) \neq 0, \\ 1 & \text{if } \mathrm{ch}_0(\mathcal{H}^{-1}_{\beta(t)}(E)) = 0,\ \mathrm{ch}_1 \neq 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

and all possible results contradict $\mathcal{H}^{-1}_{\beta(t)}(E) \in \mathcal{F}_{\gamma(t)}$ for $t > t_0$, because $c_\gamma < 1$.

Now we turn to the case $E \in \mathrm{Coh}(X)$. Using the Harder–Narasimhan filtration for coherent sheaves we obtain a filtration

$$E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E$$

where E_0 is the maximal torsion subsheaf of E, $\tilde{E}_i := E_i/E_{i-1}$ are Gieseker-semistable sheaves with reduced Hilbert polynomial p_i and they satisfy $p_i > p_{i+1}$, for $i > 0$. If $\mathrm{ch}_1(E_0) \neq 0$ we can determine a Harder–Narasimhan filtration for E_0 as $(E_0)_i$, satisfying the same conditions with $\widetilde{(E_0)}_i$ as its Gieseker-semistable sheaves, and in this case the dimension for the maximal torsion sheaf $(E_0)_0$ is at most 1, therefore $(E_0)_0$ is in $\mathcal{A}^{\gamma(t)}$ for all t. Using the equalities in (3.7) and Proposition 3.2.1 we can conclude that all $\widetilde{(E_0)}_i$ are in $\mathcal{A}^{\gamma(t)}$ for $t > t_0$ and $i > 0$, for some t_0.

Inductively, all $(E_0)_i$ are in $\mathcal{A}^{\gamma(t)}$ as they are extensions of $(E_0)_{i-1}$ and $\widetilde{(E_0)}_i$. Applying the same argument to $\tilde{E}_i$ and E_i we conclude that $E \in \mathcal{A}^{\gamma(t)}$. If $\mathrm{ch}_1(E_0) = 0$ then $E_0 \in \mathcal{A}^{\gamma(t)}$ and we can skip using its Harder–Narasimhan filtration to prove that $E \in \mathcal{A}^{\gamma(t)}$.

□

As in the previous section, we can provide a characterization of asymptotic stability but now in the case of $\phi_\gamma = \lambda_\gamma$. It is important to note that the proof only relies on $c_\gamma < 1$ when we apply Proposition 3.3.1, implying that this condition is necessary for controlling the structure of the objects and not their λ-slope.

Main Theorem 1. *Suppose that $E \in \mathrm{D}^{\mathrm{b}}(X)$ with $\mathrm{ch}_0(E) = 0$, for E to be asymptotic $\lambda_{\gamma,s}$-(semi)stable it is necessary and sufficient that $E \in \mathrm{Coh}(X)$ is a Gieseker-(semi)stable sheaf.*

Proof. From Proposition 3.3.1 it is known that $E \in \mathrm{Coh}(X)$ is equivalent to $E \in \mathcal{A}^{\gamma(t)}$ for all t bigger than some t_0. Therefore, $F \hookrightarrow E \twoheadrightarrow G$ is an exact sequence in $\mathcal{A}^{\gamma(t)}$ for all t sufficiently large if and only if it is an exact sequence in $\mathrm{Coh}(X)$. So, in either one of the implications of the theorem we know that $E \in \mathrm{Coh}(X)$ and let t_0 be such that $E \in \mathcal{A}^{\gamma(t)}$ for all $t > t_0$.

Suppose that E has a torsion subsheaf F, by the previous observation it is clear that F is also a subobject of E in $\mathcal{A}^{\gamma(t)}$ for all $t > t_1 \geqslant t_0$, for some t_1, and if $\mathrm{ch}_2(F) \neq 0$ then

$$\lim_{t\to+\infty} \frac{1}{(-\beta(t))} (\lambda_{\gamma(t),s}(E) - \lambda_{\gamma(t),s}(F)) = (-1)\left(\left(s + \frac{1}{6}\right) c_\gamma + \frac{1}{2}\right) < 0,$$

which would contradict asymptotic $\lambda_{\gamma,s}$-(semi)stability. The case where $\mathrm{ch}_2(F) = 0$ would also contradict because $\lambda_{\gamma(t),s}(F) = +\infty$ for all t.

Now in both implications of the theorem, $E \in \mathrm{Coh}(X)$ and is a pure sheaf. We just have to compare their stabilities. This is done by the following inequalities for the case $\mathrm{ch}_1(E) \neq 0$:

If $\delta_{21}(E, F) \neq 0$:

$$\lim_{t\to+\infty} (\lambda_{\gamma(t),s}(E) - \lambda_{\gamma(t),s}(F)) = \delta_{12} \frac{\left(s + \frac{1}{6}\right) c_\gamma + \frac{1}{2}}{\mathrm{ch}_1(E)\,\mathrm{ch}_1(F)} \geqslant 0.$$

If $\delta_{21}(E, F) = 0$ and $\delta_{31}(E, F) \neq 0$:

$$\lim_{t\to+\infty} (-\beta(t)) \cdot (\lambda_{\gamma(t),s}(E) - \lambda_{\gamma(t),s}(F)) = \frac{\delta_{31}}{\mathrm{ch}_1(E)\,\mathrm{ch}_1(F)} \geqslant 0.$$

If both δ_{21} and δ_{31} are zero then $\lambda_{\gamma(t),s}(E) = \lambda_{\gamma(t),s}(F)$ for all t.

For the case where $\mathrm{ch}_1(E) = 0$ and $\mathrm{ch}_2(E) \neq 0$ we have:

$$\lim_{t\to+\infty} \frac{1}{(-\beta(t))} (\lambda_{\gamma(t),s}(E) - \lambda_{\gamma(t),s}(F)) = \frac{\delta_{31}}{\mathrm{ch}_2(E)\,\mathrm{ch}_2(F)} \geqslant 0.$$

The equivalence is proved using the above inequalities and Remark 2.1.2.

□

Example 3.3.2. By the discussion after Remark 2.1.2, we know that $i_*\mathcal{O}_S$ is Gieseker-stable for any $i : S \to \mathbb{P}^3$ smooth subvariety of $\mathbb{P}^3$. Therefore, using Main Theorem 1, it is clear that $i_*\mathcal{O}_S$ is asymptotic λ_γ-stable. If $S = H$ is a hyperplane in $\mathbb{P}^3$ we have the defining distinguished triangle

$$\mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_H \to \mathcal{O}_{\mathbb{P}^3}(-1)[1] \to \mathcal{O}_{\mathbb{P}^3}[1] \tag{3.8}$$

and since $\mathcal{O}(k)[i]$ are both Tilt and Bridgeland stable by (SCHMIDT, 2020, Proposition 4.1) whenever they are in the correct space according to Example 2.4.4, we can determine actual walls for $i_*\mathcal{O}_H$:

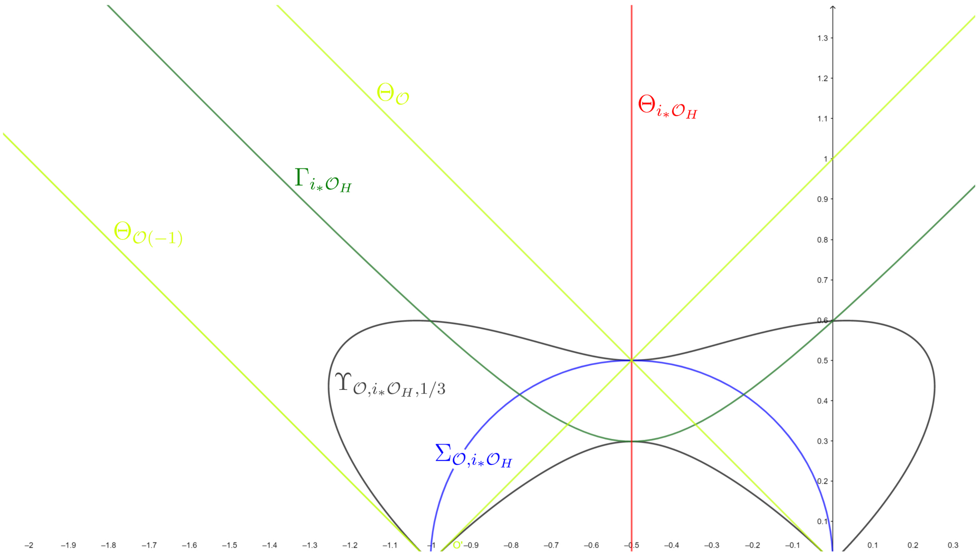

Figure 3.3.2.1 – Distinguished curves and walls related to $i_*\mathcal{O}_H$ and triangle (3.8), in both tilt and Bridgeland stability, for $s = 1/3$.

From this we conclude that $i_*\mathcal{O}_H$ is stable right after crossing $\Gamma_{\mathcal{O}_{\mathbb{P}^3},i_*\mathcal{O}_H,s}$ for every $s > 0$. This reasoning also works for any hypersurface of degree d. We do not know if there are other actual walls destabilizing and stabilizing $i_*\mathcal{O}_H$, despite knowing that if you go further left enough $i_*\mathcal{O}_H$ will be stable because it is asymptotic λ-stable.

3.4 Stability at $+\infty$

We turn our attention to the right-hand side of the half-plane $\mathbb{H}$, for that assume the unbounded curve γ satisfies: $c_\gamma < 1$ and $\lim_{t\to+\infty} \beta(t) = +\infty$.

This problem seems to be already solved by (BAYER; MACRÌ; TODA, 2014, Section 4.4), where the authors state the existence of a duality between $\sigma_{\beta,\alpha,s}$ and $\sigma_{-\beta,\alpha,s}$. But, in that form, this is not possible to be true in our setting.

Take, for example, a μ-stable and $\nu_{\beta,\alpha}$-stable self-dual vector bundle E (i.e. $\mathcal{O}_X$), and a point $(\beta, \alpha) \in \Theta_E$ so that $E[2] \in \mathcal{A}^{\beta,\alpha}$. By dualizing this object, we would obtain $E \in \mathcal{A}^{-\beta,\alpha}$, which is not possible due to the difference in the inequalities defining $\mathcal{F}_{\beta,\alpha}$ and $\mathcal{T}_{\beta,\alpha}$, one being strict and the other being non-strict.

Applying the same techniques to the asymptotic λ_γ-stability as in the left-hand side makes the argument more involved, because the objects $E \in \mathcal{A}^{\gamma(t)}$, for $t \gg 0$, are not sheaves but can be factored by derived duals of sheaves in $\mathcal{A}^{\gamma(t)}$. This will be enough to prove a relation between asymptotic λ_γ-stability with Gieseker-stability.

We use spectral sequences to find conditions for when an object in $\mathrm{D^b}(X)$ comes from the derived dual functor $(-)^\vee := R\mathcal{H}om(-, \mathcal{O}_X)[2]$ of a pure sheaf, making the calculations more elaborate. The case where $\mathrm{ch}_0(E) \neq 0$ was already described by Jardim and Maciocia with conditions when this happens in (JARDIM; MACIOCIA, 2019, Main Theorem 3).

We will provide these conditions for the case $\mathrm{ch}_0(E) = 0$ and $\mathrm{ch}_1(E) \neq 0$. If both $\mathrm{ch}_0(E) = \mathrm{ch}_1(E) = 0$, the derived dual applied to a pure sheaf E satisfying these conditions is equal to $E^D := \mathcal{E}xt^d(E, \mathcal{O}_X)$, where $d = \mathrm{codim}(E)$, and as we will see the left-hand and right-hand side of the upper half-plane, in this case, have the same asymptotic λ_γ-(semi)stable objects.

Lemma 3.4.1. *Let $E \in \mathrm{D^b}(X)$ satisfying:*

(a) $\mathcal{H}^i(E) = 0$ *if* $i \neq 1, 0$*;*

(b) $F = \mathcal{H}^{-1}(E)$ *is a reflexive sheaf of dimension* 2*;*

(c) $G = \mathcal{H}^0(E)$ *is a dimension* 0 *sheaf;*

(d) The natural map $f : \mathcal{E}xt^1(F, \mathcal{O}_X) \to \mathcal{E}xt^3(G, \mathcal{O}_X)$ *is an epimorphism.*

if and only if $E^\vee = \ker(f)$ *is a pure sheaf of dimension* 2*.*

Proof. As in (JARDIM; MACIOCIA, 2019, Lemma 2.14), assuming E satisfy the conditions described, we can decompose E in the distinguished triangle

$$F[1] \to E \to G \to F[2]. \tag{3.9}$$

Applying the cohomological functor $R^0\mathcal{H}om(-, \mathcal{O}_X)$ to (3.9) we find that $E^\vee = \ker(f)$. To see that $E^\vee = \ker(f)$ imply these properties for A we just have to dualize $\ker(f)$ and see that $\mathcal{H}^i(E) = \mathcal{H}^i((\ker(f))^\vee) = \mathcal{E}xt^{i+2}(\ker(f), \mathcal{O}_X)$ satisfy (a),(b) and (c) because $\ker(f)$ is a pure 2-dimensional sheaf. To see property (d) we apply the spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{H}^{-q}(E), \mathcal{O}_X) \implies \mathcal{H}^{p+q-2}(\ker(f))$$

and for this convergence to happen we need the map f to be an epimorphism. ∎

Remark 3.4.2. The case where $\mathrm{ch}_0(E) = \mathrm{ch}_1(E) = 0$ is easier to see that $E = A^\vee$, for some pure sheaf A with $\dim(A) \leqslant 1$, if and only if $\mathcal{H}^i(E) = 0$ for $i \neq 0$ and $\mathcal{H}^0(E)$ is a pure sheaf with dimension less or equal to 1. This is because in dimension less or equal to 1, being a pure sheaf is the same as being reflexive, see (HUYBRECHTS; LEHN, 2010, Proposition 1.1.10).

Next, we find conditions every object in $\mathrm{D^b}(X)$ has to satisfy to be in $\mathcal{A}^{\gamma(t)}$ for every $t \gg 0$. We will need a "right-hand side" version of the equality in-display (3.7), consider $F \in \mathrm{D^b}(X)$ and we would have

$$\lim_{t\to+\infty} \frac{1}{\beta(t)} \nu_{\gamma(t)}(F) = \begin{cases} (c_\gamma - 1) & \text{if } \mathrm{ch}_0(F) \neq 0, \\ -1 & \text{if } \mathrm{ch}_0 = 0, \mathrm{ch}_1(F) \neq 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.10}$$

This equation, for the case $\mathrm{ch}_0 \neq 0$, is what justifies the need for the condition $c_\gamma < 1$.

Lemma 3.4.3. *Suppose $E \in \mathrm{D^b}(X)$ is in $\mathcal{A}^{\gamma(t)}$ for all t sufficiently high. Therefore,*

- $\mathcal{H}^{-2}(E) = 0$,
- $\mathcal{H}^{-1}(E) = \mathcal{H}^0(\mathcal{H}^{-1}_{\beta(t)}(E))$ *is either a pure 2-dimensional sheaf or zero,*
- $\mathcal{H}^0(E) = \mathcal{H}^0_{\beta(t)}(E)$ *with* $\dim(\mathcal{H}^0(E)) \leqslant 1$.

Proof. As in the case where $-\infty$, we do not know a priori that $\mathcal{H}^i_{\beta(t)}(E)$ eventually becomes constant. To deal with this technical problem we start by considering that $E \in \mathcal{A}^{\gamma(t)}$ implies $\Im(Z_{\gamma(t),s}(E)) = \mathrm{ch}_2(E) - \beta(t)\,\mathrm{ch}_1(E) \geqslant 0$, for $t > t_0$. This is equivalent to $\mathrm{ch}_1(E) \leqslant 0$.

Also, it is known that $\mathcal{H}^0(E) \in \mathcal{T}_{\beta(t)}$ for $t > t_0$ and therefore $\mathrm{ch}_0(\mathcal{H}^0(E)) = 0$, implying also that $\mathrm{ch}_0(\mathcal{H}^{-1}(E)) = \mathrm{ch}_0(\mathcal{H}^{-2}(E))$. Suppose $\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E)) \neq 0$ and examine the exact sequence

$$0 \to \mathcal{H}^0(\mathcal{H}^{-1}_{\beta(t)}(E)) \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E)) \to 0 \tag{3.11}$$

and let $K = \mu(\mathcal{H}^{-1}(E)) < \infty$. We can apply Grothendieck's theorem once again to show that

$$\mathcal{S} := \left\{ p(Q) | \mathcal{H}^{-1}(E) \twoheadrightarrow Q : \begin{array}{l} Q \text{ is torsion free,} \\ \mu(Q) \leqslant K \end{array} \right\}$$

is a finite set, where $p(Q)$ is the Hilbert polynomial of the sheaf Q. Moreover,

$$\tilde{\mathcal{S}} = \left\{ p(F) | f : F \hookrightarrow \mathcal{H}^{-1}(E) : \begin{array}{l} \mathrm{coker}(f) \text{ is torsion free,} \\ \mu(F) \geqslant K \end{array} \right\}.$$

is also finite

Now, applying this to the fact $\mathcal{H}^0(\mathcal{H}^{-1}_{\beta}(t)(E)) \in \mathcal{T}_{\beta(t)}$ for $t > t_0$, while using that $\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E)) \in \mathcal{F}_{\beta(t)}$, we can conclude that $\mathrm{ch}_0(\mathcal{H}^0(\mathcal{H}^{-1}_{\beta(t)}(E))) = 0$ for t sufficiently large. Actually, we can also conclude that $\mathcal{H}^0(\mathcal{H}^{-1}_{\beta(t)}(E))$ is the maximal torsion subsheaf in $\mathcal{H}^{-1}(E)$ because its quotient is torsion free and the maximal torsion subsheaf is unique. This uniqueness implies that $\mathrm{ch}_k(\mathcal{H}^i(\mathcal{H}^{-j}_{\beta(t)}(E)))$ is fixed for all i, j, k and t sufficiently large.

With this we can apply the equality in display (3.10) to $\mathcal{H}^0_{\beta(t)}(E)$ in order to prove that $\mathrm{ch}_0(\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E))) = 0$. Since both $\mathcal{H}^{-1}(\mathcal{H}^0_{\beta(t)}(E))$ and $\mathcal{H}^{-2}(E)$ are torsion free sheaves with the same ch_0 character we can conclude that both are zero.

We only have to prove that $\mathcal{H}^0(E)$ is not 2-dimensional and that $\mathcal{H}^{-1}(E)$ is either pure or zero. The first assertion comes from equality (3.10) applied to $\mathcal{H}^0(E) \in \mathcal{T}_{\gamma(t)}$. For the second one assume that $\mathcal{H}^{-1}(E)$ is a non-pure 2-dimensional sheaf then exist a subsheaf $T \hookrightarrow \mathcal{H}^{-1}(E)$ with $\dim(T) \leqslant 1$, but this is also a subobject of $\mathcal{H}^{-1}(E)$ in $\mathcal{B}^{\beta(t)}$ which is impossible because $\mathcal{H}^{-1}(E) = \mathcal{H}^{-1}_{\beta(t)}(E) \in \mathcal{F}_{\gamma(t)}$ for $t \gg 0$ and $\nu_{\beta,\alpha}(T) = +\infty$. From this argument, if $\mathcal{H}^{-1}(E)$ had dimension less than 2 we would conclude that $\mathcal{H}^{-1}(E) = 0$. $\square$

The last lemma is a reduction in the kind of object we need to test for asymptotic stability.

Lemma 3.4.4. *Suppose Q is an object in $\mathcal{A}^{\gamma(t)}$ for $t \gg 0$ with $\mathrm{ch}_0(Q) = 0$, $\mathrm{ch}_1(Q) \neq 0$ and $\dim(\mathcal{H}^0(Q)) = 0$. Then there is a 2-dimensional pure sheaf K and a 0-dimensional sheaf L satisfying the exact sequence*

$$0 \to L \to Q \to K^{\vee} \to 0$$

in $\mathcal{A}^{\gamma(t)}$ for $t \gg 0$.

Proof. By Lemma 3.4.3 we know that $\mathcal{H}^{-2}(Q) = 0$, $Q_1 := \mathcal{H}^{-1}(Q)$ is pure 2-dimensional sheaf and $Q_0 = \mathcal{H}^0(Q)$ a 0-dimensional sheaf. Suppose that Q_1 is not reflexive and we can use the exact sequence

$$0 \to Q_1 \to Q_1^{DD} \to L \to 0,$$

where L is the 0-dimensional singularity sheaf associated with Q_1. This is an exact sequence in $\mathrm{Coh}(X)$ and $\mathcal{B}^{\beta(t)}$ for all t, we just have to consider the $\nu_{\gamma(t)}$-slope of subobjects F of Q_1^{DD} in $\mathcal{B}^{\beta(t)}$. This is done via the following diagram

$$\begin{tikzcd} F' \arrow[r, hook] \arrow[d, hook] & F \arrow[r, two heads] \arrow[d, hook] & I \arrow[d, hook] \\ Q_1 \arrow[r, hook] & Q_1^{DD} \arrow[r, two heads] & L \end{tikzcd}$$

proving that $Q_1^{DD} \in \mathcal{F}_{\gamma(t)}$ whenever $Q_1 \in \mathcal{F}_{\gamma(t)}$, because $I \hookrightarrow L$ is a 0-dimensional sheaf and $Z_{\beta,\alpha}(I) = 0$ for all $(\beta, \alpha) \in \mathbb{H}$ making $Z_{\gamma(t)}(F) = Z_{\gamma(t)}(F')$. Leaving us with the exact sequence

$$0 \to L \to Q_1[1] \to Q_1^{DD}[1] \to 0$$

in $\mathcal{A}^{\gamma(t)}$. Let W be the cokernel of the composition $L \hookrightarrow Q_1[1]$ and $Q_1[1] \hookrightarrow Q$ in $\mathcal{A}^{\gamma(t)}$. By the construction of L we know that W satisfy conditions (a),(b),(c) in Lemma 3.4.1, if W were to satisfy condition (d) we would conclude the proof. Fix $W_1 = \mathcal{H}^{-1}(W)$ and $W_0 = \mathcal{H}^0(W)$.

Assume instead that $f : \mathcal{E}xt^1(W_1, \mathcal{O}_X) \to \mathcal{E}xt^3(W_0, \mathcal{O}_X)$ is not surjective and let $\tilde{P} = \mathrm{coker}(f)$, $\tilde{L} = \ker(f)$, $\tilde{I} = \mathrm{Im}(f)$ such that they are defined by the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \tilde{L} \arrow[r] & W_1^D \arrow[rr, "f"] \arrow[dr] & & W_0^D \arrow[r] & \tilde{P} \arrow[r] & 0 \\
& & & \tilde{I} \arrow[ur] & & &
\end{tikzcd}$$

Now, in $\mathrm{D}^{\mathrm{b}}(X)$, we can dualize the exact sequence defining $\tilde{P}$ as a cokernel in $\mathrm{Coh}(X)$, keeping in mind that every 0-dimensional sheaf is reflexive, and compose with the distinguished triangle defining W in $\mathcal{A}^{\gamma(t)}$ to obtain

$$\begin{tikzcd}
& & \tilde{P}^D \arrow[d] \arrow[dr, "h"] & \\
W_1[1] \arrow[r] & W \arrow[r] & W_0 \arrow[r] \arrow[d] & W_1[2] \\
& & \tilde{I}^D. &
\end{tikzcd}$$

After applying the dualization functor to the right part of the diagram we see that we have the commutative diagramm:

$$\begin{tikzcd}
\tilde{P} & \\
W_0^D \arrow[u] & W_1^D \arrow[l, "f"] \arrow[ul, "h^\vee"']
\end{tikzcd}$$

implying that $h^\vee$ is the composition of f with its cokernel, making $h^\vee = 0$. As a consequence, the map $\tilde{P}^D \to W_0$ lifts to $g : \tilde{P}^\vee \to W$

Moreover, due to $\tilde{P}$ being a 0-dimensional sheaf, it is clear that the cone of g in $\mathrm{D}^{\mathrm{b}}(X)$, $C(g)$, is also in $\mathcal{A}^{\gamma(t)}$ whenever $W \in \mathcal{A}^{\gamma(t)}$, making $\tilde{P}$ a subobject of W in $\mathcal{A}^{\gamma(t)}$ such that $C(g)$ satisfy all conditions in Lemma 3.4.1 making it a dual of a pure 2-dimensional sheaf K. Since the kernels of both maps $Q \to W$ and $W \to C(g)$ are 0-dimensional sheaves, it is clear that $L := \ker(Q \to C(g))$ is a 0-dimensional concluding the proof. □

Remark 3.4.5. One application of the previous Lemma is that we can verify asymptotic $\lambda_{\gamma(t)}$-(semi)stability only by considering quotients which are duals of 2-dimensional pure sheaves because

$$\lim_{t\to+\infty}\beta^l(\lambda_{\gamma(t),s}(Q)-\lambda_{\gamma(t),s}(K^\vee)) = \begin{cases} \dfrac{\mathrm{ch}_3(L)}{\mathrm{ch}_1(K^D)} \geqslant 0 & \text{if } l=1,\\ 0 & \text{if } l=0, \end{cases} \tag{3.12}$$

such that $\lambda_{\gamma(t),s}(E) \leqslant \lambda_{\gamma(t),s}(Q)$ if and only if $\lambda_{\gamma(t),s}(E) \leqslant \lambda_{\gamma(t),s}(K^\vee)$ for t sufficiently large. This is the Bridgeland stability equivalent to (HUYBRECHTS; LEHN, 2010, Proposition 1.2.6), where it is shown that we can test Gieseker-(semi)stability only using pure quotients.

Main Theorem 2. *An object $E \in \mathrm{D^b}(X)$ with $\mathrm{ch}_0(E) = 0$ is asymptotic λ_γ-(semi)stable if and only if it is the dual of a Gieseker-(semi)stable sheaf.*

Proof. Case $\mathrm{ch}_1(E) \neq 0$*:* We start by assuming that E is asymptotic λ_γ-(semi)stable and we already have information on the cohomology of E given by Lemma 3.4.3. Suppose that $\mathcal{H}^0(E)$ is a 1-dimensional sheaf and observe that

$$\lim_{t\to+\infty}\frac{1}{\beta(t)}\lambda_{\gamma(t),s}(\mathcal{H}^0(E)) = -1.$$

Not only $\mathcal{H}^0(E)$, in this case, is a quotient of E but also

$$\lim_{t\to+\infty}\frac{1}{\beta(t)}(\lambda_{\gamma(t),s}(E)-\lambda_{\gamma(t),s}(\mathcal{H}^0(E))) = c_\gamma\left(s+\frac{1}{6}\right)+\frac{1}{2} > 0,$$

making this a contradiction to E's asymptotic stability. Now we can apply Lemma 3.4.4 to find a 0-dimensional sheaf as a subobject of E and suppose E is not a dual of a sheaf, this subobject would contradict the asymptotic λ_γ-(semi)stability of E, making $K = E^\vee$ a pure sheaf of dimension 2. Let Q be a pure 2-dimensional quotient of K with kernel $F \in \mathrm{Coh}(X)$. Dualizing the exact sequence in $\mathrm{Coh}(X)$ determined by K, Q and F we obtain the distinguished triangle

$$Q^\vee \to E \to F^\vee \to G^\vee[1] \tag{3.13}$$

in $\mathrm{D^b}(X)$. Now, $Q^\vee$ is in $\mathcal{A}^{\gamma(t)}$ whenever E is in $\mathcal{A}^{\gamma(t)}$ because $\mathcal{H}^{-1}(Q^\vee) \hookrightarrow \mathcal{H}^{-1}(E) \in \mathcal{F}_{\gamma(t)}$. To see that $F^\vee \in \mathcal{A}^{\gamma(t)}$ we just apply the cohomology functor to (3.13) and study the subobjects V of F^D in $\mathcal{B}^{\beta(t)}$ using the diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & V \arrow[r, two heads] \arrow[d, hook] & I \arrow[d, hook] \\ K^D \arrow[r] & F^D \arrow[r] & \mathcal{E}xt^2(Q,\mathcal{O}_X) \end{tikzcd}$$

with $\dim(\mathcal{E}xt^2(Q,\mathcal{O}_X)) = 0$ to conclude that $F^D \in \mathcal{F}_{\gamma(t)}$ whenever $K^D \in \mathcal{F}_{\gamma(t)}$. To finish, we just have to look at the limits

$$\lim_{t\to+\infty}(\lambda_{\gamma(t),s}(E)-\lambda_{\gamma(t),s}(Q^\vee)) = (-1)\delta_{21}\frac{\left(s+\frac{1}{6}\right)c_\gamma+\frac{1}{2}}{\mathrm{ch}_1(E)\,\mathrm{ch}_1(Q^D)} \leqslant 0, \tag{3.14}$$

if $\delta_{1,2}(E, F^\vee) \neq 0$ and on the contrary we have

$$\lim_{t \to +\infty} \beta(t) \cdot (\lambda_{\gamma(t),s}(E) - \lambda_{\gamma(t),s}(Q^\vee)) = (-1)\frac{\delta_{31}}{\mathrm{ch}_1(E)\,\mathrm{ch}_1(Q^D)} \leqslant 0, \tag{3.15}$$

concluding that $K = E^\vee$ is a Gieseker (semi)stable object.

Now let $K = E^\vee$ be Gieseker (semi)stable sheaf and remember that $K^D = \mathcal{H}^{-1}(E)$ is a GS_1-(semi)stable sheaf implying, by Proposition 3.2.1, that K^D is asymptotic ν_γ-(semi)stable. One consequence of this fact is that $K^D \in \mathcal{F}_{\gamma(t)}$ for $t \gg 0$ because $\lim_{t \to +\infty} \nu_{\gamma(t)}(K^D) = -\infty$, therefore $E \in \mathcal{A}^{\gamma(t)}$ for t sufficiently large. Consider now the quotient $Q = \tilde{Q}^\vee$ of E in $\mathcal{A}^{\gamma(t)}$ for $t \gg 0$, such that $\tilde{Q}$ is a 2-dimensional pure sheaf, and the exact sequence in $\mathcal{A}^{\gamma(t)}$

$$0 \to F \to E \to \tilde{Q}^\vee \to 0. \tag{3.16}$$

Since $F \in \mathcal{A}^{\gamma(t)}$ whenever Q is a quotient of E, we see that F satisfies the properties described in Lemma 3.4.3. Furthermore, we can dualize the sequence (3.16) to obtain that $\mathcal{H}^{-i}(F^\vee) = 0$ whenever $i \neq 1, 0$, and by applying the dualizing functor to the sequence decomposing F in $\mathcal{A}^{\gamma(t)}$ we can conclude that $F^\vee = \tilde{F}$ is a sheaf. In this case, we have

$$0 \to \tilde{Q} \to K \to \tilde{F} \to 0$$

and we only have to apply equations (3.14) and (3.15) to finish the proof.

Case $\mathrm{ch}_1(E) = 0$*:* Considering $\mathrm{ch}_2(E) \neq 0$ we first assume that E is λ_γ-(semi)stable and use Lemma 3.4.3 to see that $E \in \mathrm{Coh}(X)$ and E is pure because a 0-dimensional sheaf would destabilize E. If $F \in \mathrm{Coh}(X)$ is a sheaf of dimension at most 1 then $F \in \mathcal{A}^{\gamma(t)}$ for $t \gg 0$, consequence of $\mu(F) = +\infty$ and $\mathrm{ch}_1^\beta(F) = 0$. Now we just have to consider the equality for a subsheaf F of E

$$\lambda_{\gamma(t),s}(E) - \lambda_{\gamma(t),s}(F) = \frac{\delta_{32}(E, F)}{\mathrm{ch}_2(F)\,\mathrm{ch}_2(E)} \tag{3.17}$$

to see that E is Gieseker-(semi)stable. Conversely, if $F \hookrightarrow E$ in $\mathcal{A}^{\gamma(t)}$ for all t sufficiently large then $\mathrm{ch}_0(F) = \mathrm{ch}_1(F) = 0$ because otherwise $\Im(\mathrm{coker}(F \hookrightarrow E))$ would be negative for some $t \gg 0$. Therefore, by Lemma 3.4.3 again, $F \in \mathrm{Coh}(X)$ and the same is true for its quotient in $\mathcal{A}^{\gamma(t)}$, and applying the same equality in display (3.17) to prove that E is λ_γ-(semi)stable.

If $\mathrm{ch}_2(E) = 0$, by applying Lemma 3.4.3 we conclude that E is both a Gieseker-semistable sheaf and $\lambda_{\beta,\alpha,s}$-semistable for all $(\beta, \alpha) \in \mathbb{H}$ and $s > 0$, in either case of the theorem.

□

We finish this section with an example to illustrate our Main Theorem 2.

Let $i : S \hookrightarrow \mathbb{P}^3$ be a smooth subvariety of codimension $c \leqslant 2$ with structure sheaf $i_*\mathcal{O}_S$. As discussed in Remark 2.1.5, we know that $i_*(\mathcal{O}_S)$ is Gieseker-stable and by applying our Main Theorem 2 we can conclude that $(i_*\mathcal{O}_S)^\vee$ is asymptotic λ_γ-stable. This sheaf is described splicitly in (HUYBRECHTS, 2006, Corollary 3.40) as

$$(i_*\mathcal{O}_S)^\vee \simeq i_*\omega_S \otimes \omega^*_{\mathbb{P}^3}[2-c],$$

where ω_S and $\omega_{\mathbb{P}^3}$ are the dualizing bundles of S and $\mathbb{P}^3$, respectively. The left-hand side of this isomorphism is the i_*-image of the relative dualizing bundle with respect to i.

Example 3.4.6. If C is a curve over $\mathbb{P}^3$ then $(i_*\mathcal{O}_C)^\vee = (i_*\mathcal{O}_C(d))^D$ is asymptotic λ_γ-stable. When C is complete intersection between hypersurfaces of degree f and g, then we can find a resolution

$$0 \to \mathcal{O} \to \mathcal{O}(f) \oplus \mathcal{O}(g) \to \mathcal{O}(f+g) \to (i_*\mathcal{O}_C)^D \to 0. \tag{3.18}$$

Moreover, we know that $i_*\mathcal{O}_C$ is a pure sheaf making $(i_*\mathcal{O}_C)^\vee = (i_*\mathcal{O}_C)^D$ an asymptotic λ_γ-stable sheaf. Using equation (3.18) we can determine the walls $\Upsilon_{\mathcal{O},\mathcal{O}(f)\oplus\mathcal{O}(g),s}$ and $\Upsilon_{\mathcal{O}(f+g),(i_*\mathcal{O}_C)^D,s}$, for a fixed $s > 0$, these can be described by Figure 3.

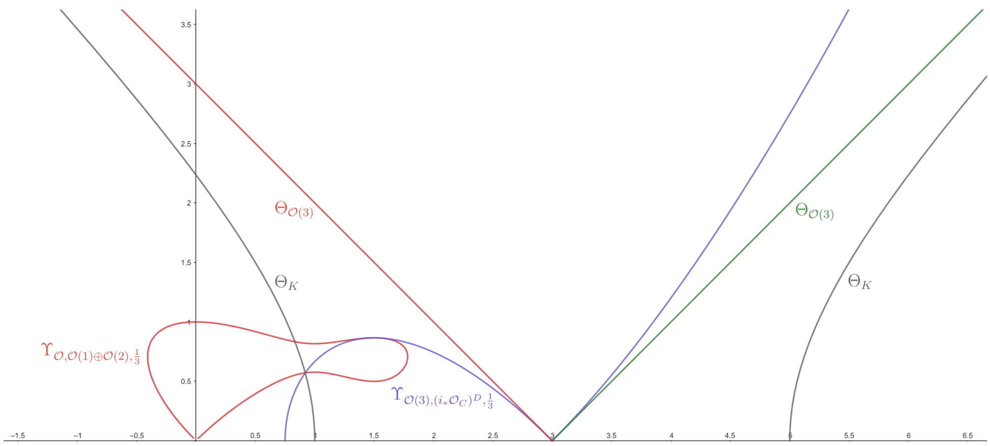

Figure 3.4.6.1 – Distinguished curves and walls related to $(i_*\mathcal{O}_C)^D$ when $f = 1$, $g = 2$ and $s = 1/3$.

Let $(\tilde{\beta}, \tilde{\alpha})$ be the point in $\Upsilon_{\mathcal{O},\mathcal{O}(1)\oplus\mathcal{O}(2),\frac{1}{3}} \cap \Upsilon_{\mathcal{O}(3),(i_*\mathcal{O}_C)^D,\frac{1}{3}}$ with $\tilde{\beta} = 1.5$ and $K = \ker(\mathcal{O}(3) \to (i_*\mathcal{O}_C)^D)$ in $\mathrm{Coh}(X)$. We have the exact sequences

$$0 \to \mathcal{O}(1) \oplus \mathcal{O}(2)[1] \to K[1] \to \mathcal{O}[2] \to 0$$

$$0 \to \mathcal{O}(3) \to (i_*\mathcal{O}_C)^D \to K[1] \to 0$$

in $\mathcal{A}^{\tilde{\beta},\tilde{\alpha}}$ such that $(i_*\mathcal{O}_C)^D$ is Bridgeland stable right after crossing the intersection of these walls, towards $\beta \to +\infty$, but it is still not clear how to conclude that there is no other wall destabilizing $(i_*\mathcal{O}_C)^D$ going further from the origin.

4 Quiver regions

The relation between quivers and Bridgeland stability is known to be of great importance since the early days of Bridgleand stability. Some of the fundamental results about these relations are proved in (MACRì, 2007), a paper where Macrì uses that some stability conditions have exceptional collections associated with them to prove topological properties of the space of Bridgeland stability conditions.

This relation was further explored by other authors since then, see for instance (ARCARA et al., 2013; DIMITROV; KATZARKOV, 2016; RUAN S.AND WANG, 2021; MU, 2020). It was also used in proving the existence of a generalized Bogomolov inequality by (MACRÌ, 2014) and (SCHMIDT, 2020) for the projective space $\mathbb{P}^3$ and the smooth quadric Q_3, respectively.

The purpose of the chapter is to give a systematic way of calculating the quiver regions inside the upper-half plane of Bridgeland stability conditions and use them to prove the stability of the instanton sheaves(shifted by [1]). To do that we use the concept of *determinant conditions*, where we further approximate Bridgeland stability to quiver stability, by giving it an equivalent stability that defined by determinants.

The main results were obtained in the case of the projective space $\mathbb{P}^3$ and the smooth quadric Q_3 as it is needed an exceptional collection satisfying the upper-half plane condition and a version of (ANCONA; OTTAVIANI, 1994, Theorem 2.8). Furthermore, Section 4.2 does not rely on these results and can be seen to apply in greater generality.

4.1 Definitions

The goal of this section is to lay the ground floor on which we will work when dealing with quiver regions. We start by defining what is meant to be and prove a result that provides a simple way of calculating them. This allows for us to then define the specific quiver regions on which we will work in this chapter.

In this section we assume X to be a smooth projective variety where $\mathrm{Stab}_\lambda(\mathrm{D^b}(X))$ is non empty.

Definition 4.1.1. Fix $s > 0$. A subset $\tilde{R_{\mathcal{E}}}$ of Stab is called a quiver region of a full Ext-exceptional collection $\mathcal{E} = \{E_0, ..., E_n\}$ if $\mathcal{E}$ satisfies the upper-half plane condition (see Definition 2.5.3 for every stability condition $(Z, \mathcal{P}) \in \tilde{R_{\mathcal{E}}}$.

To simplify the notation for threefolds, we denote by $R_{\mathcal{E}}$ the intersection of quiver region associated to $S(\mathcal{E})$, $\tilde{R}_{S(\mathcal{E})} \in$ Stab, with the upper-half plane of stability condition $\mathbb{H}$, where $\mathcal{E}$ is a strong exceptional collection of $\mathrm{D^b}(X)$.

Remark 4.1.2. The number $\phi_{\beta,\alpha} \in (0,1]$ such that $\mathcal{E} \in \mathcal{P}_{\beta,\alpha}((\phi_{\beta,\alpha}, \phi_{\beta,\alpha}+1]))$ can vary when choosing $(\beta,\alpha) \in R_{\mathcal{E}}$. The quiver region we will work may not be maximal, in the sense that we are not assuming that these are the only points in $\mathbb{H}$ where satisfying $\mathcal{E} \in \mathcal{P}_{\beta,\alpha}((\phi_{\beta,\alpha}, \phi_{\beta,\alpha}+1]))$ for some $\phi_{\beta,\alpha} \in (0,1]$.

Moreover, due to the existence of translation by twists, that is, $(Z_{\beta+1,\alpha,s}, \mathcal{A}^{\beta+1,\alpha}) = (Z_{\beta,\alpha,s}, \mathcal{A}^{\beta,\alpha} \otimes \mathcal{O}_X(1))$, we can define $R_{\mathcal{E}}[n] = \{(\beta+n,\alpha) | (\beta,\alpha) \in R_{\mathcal{E}}\} = R_{\mathcal{E}\otimes\mathcal{O}_X(n)}$

Conceptually, the existence of non-empty quiver regions for a full Ext-exceptional collection can be regarded as a discretization of the categories $\mathcal{A}^{\beta,\alpha}$, which vary continuously. Following Mu's definition (MU, 2020), and the relation between the category generated by exceptional collections and quivers expressed in (MACRì, 2007) and (ARCARA et al., 2013), we define a dimension in the category generated by a strong exceptional collection of sheaves.

Definition 4.1.3. Let $\mathcal{E} = \{E_0, ..., E_n\}$ be a full strong exceptional collection of sheaves in $\mathrm{D^b}(X)$. A dimension vector for $F \in \langle S(\mathcal{E})\rangle$ is $\dim_{\mathcal{E}}(F) = [a_0, ..., a_n]$, where $\mathrm{ch}(F) = \Sigma_{i=0}^{n}(-1)^{n+1-i} a_i \,\mathrm{ch}(E_i)$ and $a_i \in \mathbb{Z}$.

We will define a partial ordering in $\mathrm{D^b}(X)$ given by $\dim_{\mathcal{E}}(F) = [b_0, ..., b_n] \leqslant \dim_{\mathcal{E}}(E) = [a_0, ..., a_n]$ if and only if $b_i \leqslant a_i$, for each i. When clear from the context, we will omit the subscript $\mathcal{E}$.

Lemma 4.1.4. *Let $\mathcal{E} = \{E_0, ..., E_n\}$ be a strong exceptional collection of sheaves in $\mathrm{D^b}(X)$ and $S(\mathcal{E})$ its shift then for $E \in \langle S(\mathcal{E})\rangle$:*

(a) $\dim_{\mathcal{E}}(E) = [a_0, ..., a_n]$ *with* $a_i \geqslant 0$ *for all* i,

(b) If

$$0 \to F \to E \to G \to 0$$

is an exact sequence in $\langle S(\mathcal{E})\rangle$ *then* $\dim_{\mathcal{E}}(F), \dim_{\mathcal{E}}(G) \leqslant \dim_{\mathcal{E}}(E)$.

Proof. The first item in the Lemma is clear from the definition, as objects in $\langle S(\mathcal{E})\rangle$ are obtained by isomorphism classes of extensions of the objects in $S(\mathcal{E})$. Item (b) is a consequence of the additivity of the Chern character in $\mathrm{K}_0(X)$ and item (a). □

The next proposition is responsible for calculating quiver regions when applied to $R_{\mathcal{E}}$ in $\mathbb{H}$ for a fixed $s > 0$. We will describe the regions we will work within the following examples.

Proposition 4.1.5. *For a full Ext-exceptional collection $\mathcal{E} = \{E_0, ..., E_3\}$ in $\mathrm{D^b}(X)$ such that $E_i \in \mathcal{P}((\phi, \phi+2])$, for some $\phi \in \mathbb{R}$ for some slice $\mathcal{P}$. Then $\mathcal{E}$ satisfies the upper-half plane condition with respect to a Bridgeland stability condition $\sigma = (Z, \mathcal{P})$ if and only if there exists a i such that:*

($*$) $(Z, \mathcal{P})$ is inside the walls $\Upsilon_{E_i[-1], E_j[-1]}$, for all $j \in \{j | E_j \in \mathcal{P}((\phi+1, \phi+2])\}$, and outside the walls $\Upsilon_{E_i[-1], E_k}$ for all $k \in \{k | E_k \in \mathcal{P}((\phi, \phi+1])\}$.

In this proof, the notion of slope used is the $\mathbb{C}$-slope defined in Example 2.5.4.

Proof. We start by assuming that $\mathcal{E}$ satisfies the upper half-plane condition for some $\phi \in [0, 1)$. If $\phi = 0$ this would imply that $E_i \in \mathcal{P}((0, 1])$ for all i and we can always choose k such that $\phi_\sigma(E_k)$ is the smallest slope within the set $\{\phi_\sigma(E_i)\}$, this will satisfy condition $(*)$.

Assume now that $\phi \neq 0$ and that there exists $E_i \in \mathcal{P}((\phi+1, \phi+2])$, if this was not the case then we could apply the previous case. Ordering the $\mathbb{C}$-slopes of $Z(E_i)$ in $\mathbb{C}$ we can choose k such that $Z(E_k)$ has a slope greater than or equal to all of the other $Z(E_i)$. Therefore, k must satisfy the condition $(*)$ in the theorem as the $Z(E_i)$ are all constrained by a upper-plane rotated $\phi \cdot \pi$-degrees.

For the reverse implication let k be such that satisfies condition $(*)$ for $\sigma = (Z, \mathcal{P})$. Then we can consider ϕ as the $\mathbb{C}$-slope of $Z(E_k[-1])$ and prove that $\mathcal{E}$ satisfies the upper half-plane condition for ϕ. This is done by observing that we can place every exceptional object in $\mathcal{E}$ into $\mathcal{P}((\phi, \phi+1])$, either it already is in $mathcalP((\phi, \phi+1])$ or it will be by shifting by $[-1]$, and divide the upper half-plane $\overline{\mathbb{H}}$ by the vector $Z(E_k[-1])$ making it such that the right-hand side we have the vectors $Z(E_k[-1])$ with slope less than the slope of $Z(E_i[-1])$, similarly for the vectors $Z(E_k)$ the slope is greater than the slope of $Z(E_i[-1])$. In other words, the vectors $Z(E_i)$, for all i, are bounded by the upper half-plane defined by $Z(E_k)$.

$Z(E_k[-1])$ with $(Z, \mathcal{P})$ inside the wall $\Upsilon_{E_i[-1], E_j[-1]}$, similarly for the vectors $Z(E_k)$ when $(Z, \mathcal{P})$ is outside the walls $\Upsilon_{E_i[-1], E_k}$. In other words, the vectors $Z(E_i)$, for all i, are bounded by the upper half-plane defined by $Z(E_k)$. □

For the threefold case we can replace the numerical condition $(*)$. It is equivalent, assuming the conditions in Proposition 4.1.5, to assuming that there exists a k such that (β, α) is inside all the numerical walls $\Upsilon_{E_k, E_i, s}$. This equivalence is important to produce the numerical results we are about to present, because we do not even have to calculate whether the objects are in $\mathcal{A}^{\beta,\alpha}$ or $\mathcal{A}^{\beta,\alpha}[1]$.

Example 4.1.6. Let $X = \mathbb{P}^3$ and $S(\mathcal{E}_1) = \{\mathcal{O}_{\mathbb{P}^3}(-2)[3], \mathcal{O}_{\mathbb{P}^3}(-1)[2], \mathcal{O}_{\mathbb{P}^3}[1], \mathcal{O}_{\mathbb{P}^3}(1)\}$ a strip of the canonical helix of $\mathbb{P}^3$ shifted by the necessary degrees so that $S(\mathcal{E}_1)$ is a full

Ext-exceptional collection. We will fix $s = 1/3$, since $s > 1/3$ does not change the existence of walls in $\mathbb{H}$, it just dilates the wall in the α direction (JARDIM; MACIOCIA, 2019). The region of the points $(\beta, \alpha) \in \mathbb{H}$ where $\mathcal{E}$ is in $\langle \mathcal{A}^{\beta,\alpha}, \mathcal{A}^{\beta,\alpha}[1] \rangle$ is described in Image 4.1.6.1.

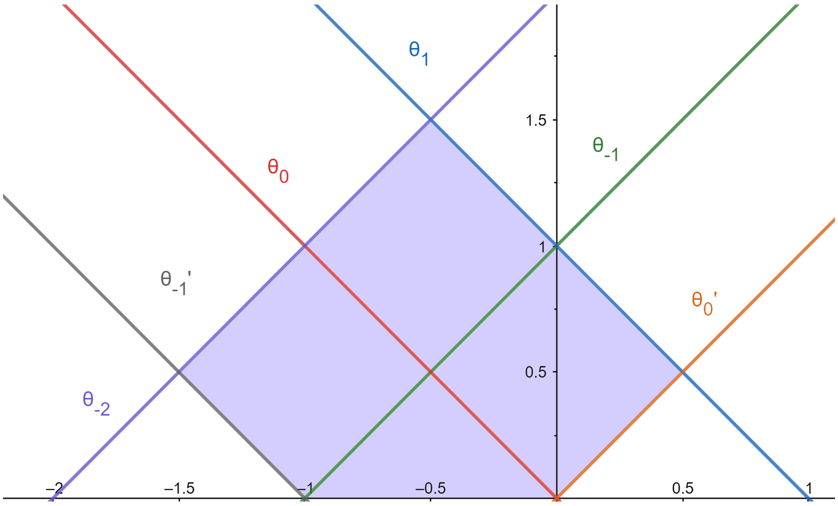

Figure 4.1.6.1 – Region of $\mathbb{H}$ where the exceptional objects $E_{2-k} = \mathcal{O}_{\mathbb{P}^3}(-k)[k+1]$ satisfy $E_i \in \langle \mathcal{A}^{\beta,\alpha}, \mathcal{A}^{\beta,\alpha}[1] \rangle$ and $\theta_i = \theta_{\mathcal{O}_{\mathbb{P}^3}(i)}$

To find a region where $\mathcal{E}$ satisfies the upper-half plane condition we will use Proposition 4.1.5 to obtain Image 4.1.6.2.

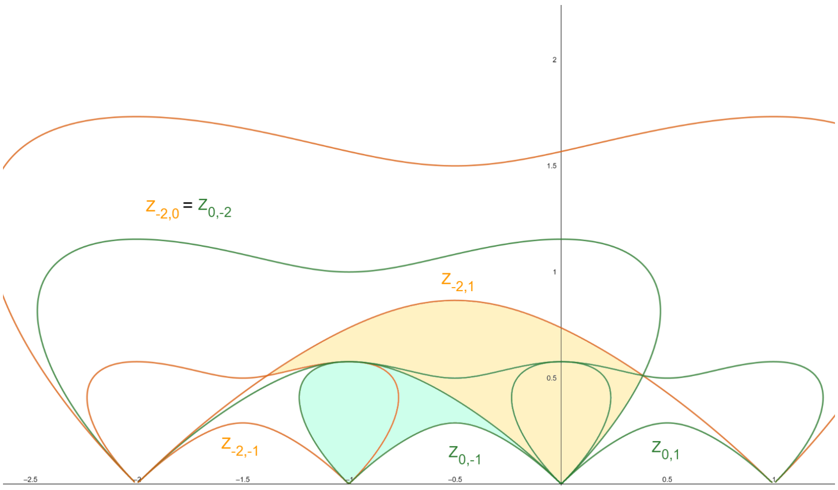

Figure 4.1.6.2 – The wall $Z_{i,j}$ represents $\Upsilon_{\mathcal{O}_{\mathbb{P}^3}(i)[1-i],\mathcal{O}_{\mathbb{P}^3}(j)[i-j],\frac{1}{3}}$, with the yellow and green walls representing the application of Proposition 4.1.5 to $i = -2$ and $i = 0$, respectively.

To conclude, we can determine the region R_1 as the intersection of the regions obtained by Images 4.1.6.1 and 4.1.6.2.

Example 4.1.7. Consider now $X = Q_3$ the smooth quadric. Combining images 4.1.6.1 and 2.4.7.1, we can see a region of $\mathbb{H}$ where the exceptional collection

$$S(\mathcal{E}_2) = \{S^*(-2)[3], \mathcal{O}_{Q_3}(-1)[2], \mathcal{O}_{Q_3}[1], \mathcal{O}_{Q_3}(1)\}$$

is contained in $\langle \mathcal{A}^{\beta,\alpha}, \mathcal{A}^{\beta,\alpha}[1]\rangle$, denote it by P. To apply Proposition 4.1.5 we just need to analyze the λ-walls defined by the exceptional objects, this is done in Image 4.1.7.1.

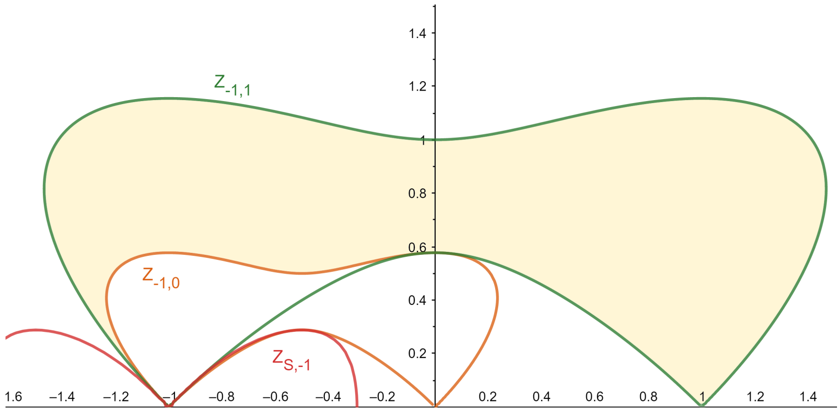

Figure 4.1.7.1 – Same notation as the one used in Image 4.1.6.2 and $Z_{S,-1} = \Upsilon_{S^*(-2)[3],\mathcal{O}_{Q_3}(-1)[2],\frac{1}{3}}$

We define the region R_2 as the intersection of the regions P and the yellow region in Image 4.1.7.1.

4.2 Technical results and some linear algebra

In this section we will explore a few technical results that will be used further into the chapter, these are important tools to deal with semistable objects inside the quiver region.

We will keep using that X is a projective smooth variety with non empty space of Bridgeland stability conditions and we will fix $\mathcal{E}$ to be a full Ext-exceptional collection, $\bar{R}_{\mathcal{E}}$ its quiver region in $\mathrm{Stab}_{\Lambda}(\mathrm{D}^{\mathrm{b}}(X))$ and $R_{\mathcal{E}} = \bar{R}_{\mathcal{E}} \cap \mathbb{H}$, the section of the geometric stability conditions with respect to some $s > 0$.

The next lemma determines what are the walls for a 2-step complex and defines the determinant condition. This condition is responsible for relating Mumford's μ-stability with Bridgeland stability and consequently proves the stability of the instanton sheaves in the next sections.

Lemma 4.2.1. *Let K be an object in $\langle S(\mathcal{E})\rangle$ and dimension vector $\mathrm{ch}(K) = (-1)^i a\,\mathrm{ch}(E_i) - (-1)^i b\,\mathrm{ch}(E_{i+1})$. If there exists an actual Bridgeland wall with respect to $\sigma = (Z, \mathcal{P})$ defined by*

$$0 \to F \to K \to G \to 0, \tag{4.1}$$

then $\Upsilon_{F,K} = \Upsilon_{E_i,E_{i+1}}$. Furthermore, if every subobject F of K in $\langle S(\mathcal{E})\rangle$ with $\mathrm{ch}(F) = (-1)^i c\,\mathrm{ch}(E_i) - (-1)^i d\,\mathrm{ch}(E_{i+1})$ satisfies $(a \cdot d - b \cdot c)(\geqslant) > 0$ then K is Bridgeland *(semi)stable outside the curve determined by $\Upsilon_{E_i,E_{i+1}}$ and inside of $\bar{R_{\mathcal{E}}}$. In this case, we will say that K satisfies the (semi)-determinant condition.*

Proof. Assume that we have the actual determined by equation (4.1) so that F, K, G are all objects of $\mathcal{P}(\phi)$, and by our hypothesis $K \in \langle S(\mathcal{E})\rangle$, therefore all of them are in $\langle S(\mathcal{E})\rangle$. Furthermore, since $\langle S(\mathcal{E})\rangle = \mathcal{P}(\psi, \psi + 1])$ because we are in the quiver region we can use their $\mathbb{C}$-slope to calculate their stability with respect to $\sigma = (Z, \mathcal{P})$. This also implies the restrictions to the dimension vector of the subobject F by Lemma 4.1.4.

Now, in order to prove the uniqueness and establish the orientation of the actual wall for K we have to consider the defining equation of the actual wall and use the fact that the stability function is a group homomorphism Z in Λ. The defining equation for the wall $\Upsilon_{F,K}$ is

$$\phi_\sigma(F) = \phi_\sigma(K),$$

which is the same as

$$\frac{c\cdot(-\Re(Z(v(E_i)))) - d\cdot(-\Re(Z(v(E_{i+1}))))}{c\cdot\Im(Z(v(E_i))) - d\cdot\Im(Z(v(E_{i+1})))} = \frac{a\cdot(-\Re(Z(v(E_i)))) - b\cdot(-\Re(Z(v(E_{i+1}))))}{a\cdot(\Im(Z(v(E_i)))) - b\cdot(\Im(Z(v(E_{i+1}))))}$$

and this equation is the equivalent to

$$(a\cdot d - b\cdot c)f_{E_i,E_{i+1}}(\beta,\alpha) = 0. \tag{4.2}$$

The positivity (negativity) of the determinant $(a\cdot d - b\cdot c)$ provides the orientation of the wall $\Upsilon_{F,K}$, whether it will make $\phi_\sigma(F) > \phi_\sigma(K)$ or $\phi_\sigma(F) < \phi_\sigma(K)$. □

Corollary 4.2.2. *Inside an exceptional region $R_{\mathcal{E}}$ with respect to a strong exceptional collection $\mathcal{E}$, there exists only one possible numerical λ-wall for a given 2-complex object $K \in \langle S(\mathcal{E})\rangle$. This λ-wall either stabilizes K or destabilizes K depending on the sign of the determinant.*

Since we are mostly working with 3-step complexes, it will be useful to generalize the calculations done in Lemma 4.2.1.

Lemma 4.2.3. *For any 3-step complex, $E \in \langle \mathrm{S}(\mathcal{E})\rangle$ with dimension vector $\mathrm{ch}(E) = -(-1)^i a\,\mathrm{ch}(E_{i=1}) + (-1)^i b\,\mathrm{ch}(E_i) - (-1)^i c\,\mathrm{ch}(E_{i+1})$, and every numerical wall for E defined by an object in $\langle \mathrm{S}(\mathcal{E})\rangle$ goes through the points where $\phi_\sigma(E_i) = \phi_\sigma(E_{i-1}) = \phi_\sigma(E_{i+1})$ if it exists.*

Proof. The proof is just an observation that for any $F \in \langle \mathrm{S}(\mathcal{E})\rangle$ subobject of E with $\mathrm{ch}(F) = -(-1)^i a'\,\mathrm{ch}(E_{i=1}) + (-1)^i b'\,\mathrm{ch}(E_i) - (-1)^i c'\,\mathrm{ch}(E_{i+1})$, we have a linear description of the defining equation for the λ-wall as

$$f_{F,E} = \begin{vmatrix} f_{E_2,E_1} & f_{E_2,E_0} & f_{E_1,E_0} \\ a & b & c \\ a' & b' & c' \end{vmatrix} \tag{4.3}$$

Therefore if $\sigma = (Z, \mathcal{P}$ satisfies $\phi_\sigma(E_i) = \phi_\sigma(E_{i-1}) = \phi_\sigma(E_{i+1})$ then $f_{F,E}(\beta, \alpha) = 0$. $\square$

When clear, for the threefold case, we will suppress the notation $f_{E,E_1[2],s}, f_{E,E_2[1],s}, f_{E,E_3,s}$ by using f_1, f_2 and f_3, respectively, for the equations defining the canonical λ-walls obtained by comparing the slopes of a fixed object E in $\langle \mathrm{S}(\mathcal{E})\rangle$ to the slope of the generating objects of $\langle S(\mathcal{E})\rangle$. The λ-walls will be denoted as Υ_1, Υ_2, Υ_3 respectively.

Remark 4.2.4. Consider $\mathcal{E} = \{E_0, ..., E_3\}$ to be a strong exceptional collection, $S(\mathcal{E})$ its shift and $R_{\mathcal{E}}$ the quiver region associated to $\langle S(\mathcal{E})\rangle$ and F be an object in $\langle S(\mathcal{E})\rangle$ with $\dim(F) = [0, a, b, c]$. The λ-walls obtained by the truncation functors $\tau_{\leqslant -1}$ and $\tau_{\geqslant -1}$ are vanishing walls because every 3-step complex can be decomposed in these natural exact sequences (2.4).

Proposition 4.2.5. *Suppose that $\{E_0, ..., E_n\}$ is a strong exceptional collection of μ-stable sheaves. For a coherent sheaf K with an exact resolution $0 \to E_i^{\oplus a} \to E_{i+1}^{\oplus b} \to K \to 0$ for some $i \in \{0, ..., n-1\}$ to satisfy the (semi) determinant condition it is sufficient that K is a μ-(semi)stable sheaf.*

Proof. By the definition of $\langle S(\mathcal{E})\rangle$, it is clear that $K[i] \in \langle S(\mathcal{E})\rangle$ and suppose that $F[i]$ is a subobject of $K[i]$ in $\langle S(\mathcal{E})\rangle$. Using Lemma 4.1.4 we know that $F[i]$ is quasi-isomorphic to $(E_i^{\oplus c} \to E_{i+1}^{\oplus d})[i]$ and from applying the cohomology functor $\mathcal{H}^0_{\mathrm{Coh}(X)}$ to

$$0 \to F[i] \to K[i] \to Q[i] \to 0 \tag{4.4}$$

we conclude that $F[i]$ is a sheaf shifted by $[i]$. Therefore, we can view the exact sequence (4.4) as the following exact diagram in $\mathrm{Coh}(X)$:

$$\begin{tikzcd}
 & & & \mathcal{H}^{-1}(Q) \arrow[d, hook] & \\
0 \arrow[r] & E_i^{\oplus c} \arrow[r] \arrow[d, hook] & E_{i+1}^{\oplus d} \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_i^{\oplus a} \arrow[r] \arrow[d, two heads] & E_{i+1}^{\oplus b} \arrow[r] \arrow[d, two heads] & K \arrow[r] \arrow[d, two heads] & 0 \\
\mathcal{H}^{-1}(Q) \arrow[r, hook] & E_i^{\oplus a-c} \arrow[r] & E_{i+1}^{\oplus b-d} \arrow[r] & \mathcal{H}^0(Q) \arrow[r] & 0
\end{tikzcd}$$

We have that $\mu(\mathcal{H}^{-1}(Q)) \leqslant \mu(E_i)$ and $\mu(F) \geqslant \mu(E_{i+1}) > \mu(E_i)$, the latter inequality is a consequence of Remark A.0.6. Hence the condition $\mu(F) < \mu(K)$ is equivalent to $(a \cdot d - c \cdot d)\delta_{01}(E_i, E_{i+1}) > 0$, that is, if the determinant condition is satisfied. $\square$

Remark 4.2.6. Analogously, one can prove the same result for sheaves that are a kernel of a surjective morphism $E_{i+1}^{\oplus a} \to E_i^{\oplus b}$. We will use both versions of the previous proposition in the following sections.

Next, we find a numerical criterion to find regions where every 3-step complex of dimension $[0, a, b, c]$ is in $\mathcal{A}^{\beta,\alpha}$. We will need this to prove the uniqueness of the λ-walls for the instantons.

Lemma 4.2.7. *Consider $\mathcal{E} = \{E_0, ..., E_3\}$ to be a strong exceptional collection, $S(\mathcal{E})$ its shift and $R_{\mathcal{E}}$ the quiver region associated to $\langle S(\mathcal{E})\rangle$. Let $Q \in \langle S(\mathcal{E})\rangle$ be an object with $dim_{\mathcal{E}}(Q) = [a_0, a_1, a_2, a_3]$ then $Q \in \mathcal{A}^{\beta,\alpha}$ when $(\beta, \alpha) \in \tilde{R} \cap R_{\mathcal{E}}$, where $\tilde{R} = \{(\beta, \alpha)|E_i[n - i] \in \mathcal{A}^{\beta,\alpha}$ for every i with $a_i \neq 0\}$.*

Proof. This Lemma is a consequence of $\mathcal{A}^{\beta,\alpha}$ being closed under extensions and that we can decompose Q using the stupid truncation into the following, possibly trivial, exact sequences:

$$0 \to E_3^{\oplus a_3} \to Q \to \tau_{\leqslant -1}Q \to 0$$

$$0 \to E_2^{\oplus a_2} \to \tau_{\leqslant -1}Q \to \tau_{\leqslant -2}Q \to 0$$

$$0 \to E_1^{\oplus a_1} \to \tau_{\leqslant -2}Q \to E_0^{\oplus a_0} \to 0.$$

□

Example 4.2.8. One example we will use throughout the chapter is that, for $X = \mathbb{P}^3$ or $X = Q_3$, any object with dimension $[0, a, b, c]$ with respect to the exceptional collections in Example 4.1.7 and 4.1.6 is in $\mathcal{A}^{\beta,\alpha}$ for (β, α) in the intersection of the exceptional regions obtained in the aforementioned examples and the region displayed in Image 4.2.8.1.

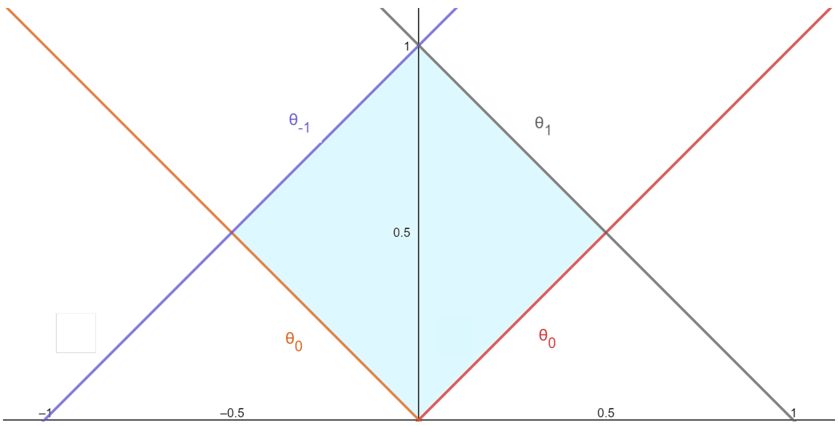

Figure 4.2.8.1

In the case of concrete examples of linear complexes it is possible to calculate all of its linear subcomplexes. We start by an observation of the well established natural isomorphism for $F, E \in \mathrm{D^b}(X)$ and V, W vector spaces

$$\mathrm{Hom}(V \otimes F, W \otimes E) = \mathrm{Hom}(V, W) \otimes \mathrm{Hom}(F, E).$$

So that when we fix a base $\{\gamma_0, ..., \gamma_k\}$ for the vector space $\mathrm{Hom}(F, E)$ we can describe $\phi \in \mathrm{Hom}(V \otimes F, W \otimes E)$ as

$$\phi = \Sigma_i \gamma_i \phi_i,$$

where $\phi_i \in \mathrm{Hom}(V, W)$ are linear transformations. Define $J^I_\phi = \bigoplus_{l=0}^{k} Im(\phi_l|_I)$, where I is a subspace of V and $Im(\phi_l|_I)$ is image of the linear transformation ϕ_l restricted to the subspace I.

Proposition 4.2.9. *Let $\mathcal{E} = \{E_0, ..., E_n\}$ be a strong exceptional collection of sheaves, $S(\mathcal{E})$ its shift. Let*

$$K \simeq (V \otimes E_i \xrightarrow{T} W \otimes E_{i+1})$$

be a 2-step complex in $\langle S(\mathcal{E})\rangle$, $\{\gamma_0, ..., \gamma_l\}$ a base for the k-vector space $\mathrm{Hom}(E_i, E_{i+1})$. For any subspace $I \hookrightarrow V$, the subcomplexes of K of the form $I \otimes E_i \xrightarrow{S} J \otimes E_{i+1}$ satisfies $J^I_T \subset J$. Furthermore, $I \otimes E_i \xrightarrow{T|_I} J^I_T \otimes E_j$ is a subobject of K in $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$.

Proof. As previously observed, given $K \simeq (V \otimes E_i \xrightarrow{T} W \otimes E_{i+1})$ and a base $\{\gamma_0, ..., \gamma_l\}$ for $\mathrm{Hom}(E_i, E_{i+1})$ then T can be factored into $\Sigma_j T_j \otimes \gamma_j$ so that composing with $I \otimes E_i \hookrightarrow V \otimes E_i$, the morphism induced by the inclusion $I \hookrightarrow V$, we obtain a natural commutative diagram

$$\begin{tikzcd} I \otimes E_i \arrow[r, "T|_I"] \arrow[d, hook] & J^I_T \otimes E_{i+1} \arrow[d, hook] \\ V \otimes E_i \arrow[r, "T"] & W \otimes E_{i+1}. \end{tikzcd}$$

The map $T|_I$ has a well-defined image in $J^I_T \otimes E_{i+1}$ due to the property that every $T_l|_I$ has its image inside J^I_T. The same argument proves that every sub-object of K with $I \otimes E_i$ as its E_i coordinate has to factorize □

4.3 Instanton sheaves

We are ready to apply the notion of quiver regions to study the stability of *instanton sheaves* of rank 2 over some Fano threefolds and the rank 0 instanton in $\mathbb{P}^3$. These are really important objects inside the moduli space of Gieseker-semistable sheaves, being linear sheaves and for their nice cohomological properties. Another reason to apply the methods described in the previous section is that instantons can be defined as cohomology of linear monads of exceptional objects, see (JARDIM; MAICAN; TIKHOMIROV, 2017; FAENZI, 2013; KUZNETZOV, 2012; COSTA; MIRó-ROIG, 2009).

Recall that a monad is a sequence of coherent sheaves

$$0 \to A \to B \to C \to 0$$

such that $A \to B$ and $B \to C$ are a monomorphism and an epimorphism, respectively. Let X be either Q_3 or $\mathbb{P}^3$, the rank 2 instantons we will work with are cohomology of monads of the form

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{f} \mathcal{O}_X^{\oplus 2c+2} \xrightarrow{g} \mathcal{O}_X(1)^{\oplus c} \to 0. \quad (4.5)$$

The positive integer c is called the charge of the instanton $I = Ker(g)/Im(f)$. For the case $X = \mathbb{P}^3$, there exists a notion of rank 0 instanton Q defined by the exact resolution

$$0 \to \mathcal{O}(-1)^{\oplus d} \to \mathcal{O}^{\oplus 2d} \to \mathcal{O}(1)^{\oplus d} \to Q \to 0, \quad (4.6)$$

as can be seen in (JARDIM; MAICAN; TIKHOMIROV, 2017, Proposition 3). Both of these notions are related by their cohomology conditions, as any instanton E satisfies $h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0$. The Chern characters $\mathrm{ch}(I) = (2, 0, -c, 0)$ and $\mathrm{ch}(Q) = (0, 0, d, 0)$ are associated to the rank 2 and 0 instanton sheaves, respectively.

It is important to make a distinction between the instantons obtained in (FAENZI, 2013) and (COSTA; MIRó-ROIG, 2009). Both papers treat the case of instantons over smooth quadrics, but Faenzi's definition deals with the case of odd instantons (i.e. those instantons sheaves with $c_1(F_{\mathrm{norm}}) = -1$) while Costa–Miro-Roig works with even instantons, these can be defined as in Monad (4.5). The instantons defined by Faenzi can also be defined as cohomology of a monad but involving more complex sheaves, in the case of the smooth quadric we need the spinor bundle.

Now we define a few other notations that are going to be used in this chapter.

- Fix X to be equal to $\mathbb{P}^3$ or Q_3.
- $\langle \mathrm{S}(\tilde{\mathcal{E}}) \rangle$ means either $S(\mathcal{E}_1)$ or $S(\mathcal{E}_2)$, as defined in Examples 4.1.7 and 4.1.6, this is because we are mainly interested in the objects of these exceptional collections which are $\mathcal{O}_X(i)$ with $i = -1, 0, 1$.
- Let us denote by $\tilde{R}$ the region obtained by Lemma 4.2.7 with respect to $\langle \mathrm{S}(\tilde{\mathcal{E}}) \rangle$ and $\tilde{R}_+, \tilde{R}_-$ the points in $\tilde{R}$ with $\beta > 0$ and $\beta \leqslant 0$, respectively.
- The segments of $\Upsilon_1, \Upsilon_2, \Upsilon_3$ that can be actual λ-walls will be denoted by $\tilde{\Upsilon}_1, \tilde{\Upsilon}_2, \tilde{\Upsilon}_3$. These are respectively $\Upsilon_1 \cap \tilde{R}_-$, $\Upsilon_2 \cap \{(0, \alpha) \in R_1 | \alpha^2 \geqslant \dfrac{1}{3}\}$, $\Upsilon_3 \cap \tilde{R}_+$.

- Fix $s = 1/3$ for the rest of the chapter and suppress the subscript s from the definitions of walls, stability function and etc. Also, fix $k \geqslant 2$ and $c \geqslant 1$ integers.

We start by proving a complement of (ANCONA; OTTAVIANI, 1994, Proposition 2.8) to apply also to instanton sheaves, instead of only instanton bundles. When $X = Q_3$, the analogous of (ANCONA; OTTAVIANI, 1994, Proposition 2.8) was proved in (COSTA; MIRó-ROIG, 2009, Proposition 3.3). Combining these results with Proposition 4.2.5 will be responsible for the existence of the actual λ-walls for the instantons.

Lemma 4.3.1. *Let*

$$0 \to \mathcal{O}_X(-1)^{\oplus b} \to \mathcal{O}_X^{\oplus 2b+2} \to \mathcal{O}_X(1)^{\oplus b} \to 0 \tag{4.7}$$

be a monad over X with middle cohomology a torsion free sheaf E. Then we can decompose (4.7) *in two ways:*

$$0 \to \mathcal{O}_X(-1)^{\oplus b} \to \mathcal{O}_X^{\oplus 2b+2} \to K_1 \to 0 \tag{4.8}$$

$$0 \to E \to K_1 \to \mathcal{O}_X(1)^{\oplus b} \to 0 \tag{4.9}$$

and also,

$$0 \to K_2 \to \mathcal{O}_X^{\oplus 2b+2} \to \mathcal{O}_X(1)^{\oplus b} \to 0 \tag{4.10}$$

$$0 \to \mathcal{O}_X(-1)^{\oplus b} \to K_2 \to E \to 0 \tag{4.11}$$

Such that K_2 is always a μ-stable bundle and K_1 is a μ-stable bundle if E is also a vector bundle.

Proof. The case where E is a vector bundle is a direct application of (ANCONA; OTTAVIANI, 1994, Proposition 2.8) and (COSTA; MIRó-ROIG, 2009, Proposition 3.3) to the exact sequence (4.8) and (4.9), and by dualizing (4.10) and (4.11). For the case where E is not locally free, notice that K_2 is a locally free sheaf by being a kernel of a map between vector bundles, and one can dualize sequence (4.11) to obtain

$$0 \to E^* \to K^* \to \mathcal{O}_X(1)^{\oplus b} \to \mathcal{E}xt^1(E, \mathcal{O}_X) \to 0. \tag{4.12}$$

Let L be the kernel of $\mathcal{O}_X(1)^{\oplus b} \to \mathcal{E}xt^1(E, \mathcal{O}_X)$ and $S = \mathrm{Supp}(\mathcal{E}xt^1(E, \mathcal{O}_X))$, S is also the singular set of the instanton sheaf E. We know from (HUYBRECHTS; LEHN, 2010, Proposition 1.1.10) that $\dim(S) \leqslant 2$ because E is torsion free and therefore we can consider the sequence (4.12) over the open subset $U = X \backslash S$.

In this situation, we obtain that $L = \mathcal{O}_X(1)^{\oplus b}|_U$, hence we are in place to apply the same argument as in (ANCONA; OTTAVIANI, 1994, Proposition 2.8) with the caveat that K being locally free K implies that $\wedge^q K(l)$ is also normal for any q and l, in the sense of (OKONEK; SPINDLER, 1986, Definition 1.1.11), making it so that $h^0(\wedge^q K(l)|_U) = h^0(\wedge^q K(l))$. $\square$

Using this technique we are not able to prove that K_1 is also μ-stable in the non-locally free case because there is no way to establish that K_1 is a vector bundle, being a cokernel of a map between vector bundle and an extension of a torsion-free sheaf and a vector bundle. Furthermore, if the cohomology of the monad is a non-locally free sheaf then K_1 will never be μ-stable, this will be clear in the following results.

Lemma 4.3.2. *The $\tau_{\geqslant -1}(F)$ satisfies the (semi)determinant condition if and only if the $\tau_{\leqslant -1}(F^\vee)$ also satisfies the (semi)determinant condition, for any $F \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ with* $\dim(F) = [0, a, b, c]$.

Proof. This is a direct consequence of the exactness and involution property of the derived dual. That is, exact sequences are kept exact after applying the derived dual $(-)^\vee : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ and that $(-)^{\vee\vee} = \mathrm{Id}_{\mathrm{D^b}(X)}$. The fact that the sheaves E_i are locally free and $E_i^\vee = E_{n-i+1}$ establishes that the dual linear complex is still in $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$. $\square$

Proposition 4.3.3. *Suppose $(\beta, \alpha) \in R_1 \cap \{(0, \alpha) \in \mathbb{H} | \alpha^2 \geqslant \frac{1}{3}\}$ then any object E with* $\dim(E) = [0, c, 2c + k, c]$ *and λ-semistable at both $\tilde{\Upsilon}_1$ and $\tilde{\Upsilon}_3$ is $\lambda_{\beta,\alpha}$-stable, unless there exists an object $F \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ such that $\lambda_{\beta,\alpha}(F) = \lambda_{\beta,\alpha}(E)$ for every $(\beta, \alpha) \in R_1$.*

Proof. Let E be an object with $\dim(E) = [0, c, 2c+k, c]$ and F a destabilizer λ-semistable subobject for E at $\alpha^2 > 1/3$. For any point $(0, \alpha) \in R_1$, since $E, F \in \mathcal{A}^{0,\alpha}$ with $\lambda_{0,\alpha,1/3}(F) \geqslant \lambda_{0,\alpha,1/3}(E)$ for every α in then open interval $(1/\sqrt{3}, 1/\sqrt{3} + \epsilon)$ for some small $\epsilon > 0$, then $F \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ and $\dim(F) = [0, f, g, h]$ for some $f, g, h \in \mathbb{Z}_{\geqslant 0}$. We can determine the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T(F) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & \mathcal{O}_X(-1)^{\oplus f} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T(E) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathcal{O}_X(-1)^{\oplus c} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T(Q) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathcal{O}_X(-1)^{\oplus c-f} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.13}$$

where Q is the quotient of $F \hookrightarrow E$ in $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ and $T(F), T(E), T(Q)$ is the stupid truncation functor $\tau_{\geqslant -1}$ applied to the respective objects of the exact sequence. Since the λ-wall $\tilde{\Upsilon}_1$ is actual, we can see that $T(E)$ is $\lambda_{\beta,\alpha}$-semistable and by Lemma 4.2.1 we know that $T(E)$ satisfies the determinant condition. Now, applying said condition to the exact sequence $T(F) \hookrightarrow T(E)$ we obtain the inequality $(c+k)h \geqslant c(g-h)$. Similarly, we can do the same argument for the stupid truncation $\tau_{\leqslant -1}$ applied to the exact sequence $F \hookrightarrow E \twoheadrightarrow Q$ to obtain the inequality $(c+k)f \leqslant c(g-f)$.

Considering Lemma 4.2.3, we see that if both of these inequalities were an equality then $f_{E,F} \equiv 0$. If that is not the case then one of those inequalities is a strict inequality

Now, analyze the λ-slope of F and E at the line $\beta = 0$ using that $\lambda_{0,\alpha,1/3}(E) = \tau_{0,\alpha,1/3}(E) = 0$ for all $\alpha > 0$ so that if F destabilizes E for $\tilde{\alpha} \in (1/\sqrt{3}, 1/\sqrt{3}+\epsilon)$ for some small $\epsilon > 0$ then $\tau_{0,\tilde{\alpha},1/3}(F) > 0$. Applying the linearity of the operator $\tau_{0,\tilde{\alpha},1/3}$ over the Chern characters we arrive at

$$\begin{aligned} \tau_{0,\tilde{\alpha},1/3}(F) &= f\tau_{0,\tilde{\alpha},1/3}(\mathcal{O}_X(-1)) - g\tau_{0,\tilde{\alpha},1/3}(\mathcal{O}_X) + h\tau_{0,\tilde{\alpha},1/3}(\mathcal{O}_X(1)) \\ &= -f(\frac{1}{6} - \frac{\alpha^2}{2}) + h(\frac{1}{6} - \frac{\alpha^2}{2}) = (h-f)(\frac{1}{6} - \frac{\alpha^2}{2}) \end{aligned} \tag{4.14}$$

which is non-negative above $\alpha^2 = 1/3$ if and only if $f \geqslant h$.

Combining these inequalities it is clear that we arrived at a contradiction. □

Theorem 4.3.4. *Let $E \in \langle \mathrm{S}(\tilde{\mathcal{E}}) \rangle$ be an object with* $\dim(E) = [0, c, 2c+k, c]$ *and suppose it has an actual λ-wall over any two of the three canonical walls $\tilde{\Upsilon}_i$, $i = 1, 2, 3$. Then these two walls are the only actual walls for the object E in R_1. Moreover, E is $\lambda_{\beta,\alpha}$-semistable outside the respective actual walls, with strict semistability if and only if there exists an subobject with dimension vector a multiple of* $[0, c, 2c+k, c]$.

We will divide the proof in two situations: Either the walls to be considered are $\tilde{\Upsilon}_1$ and $\tilde{\Upsilon}_3$ or one of them is $\tilde{\Upsilon}_2$. The former case is a combination of the latter ones and Proposition 4.3.3. The proof of the case $(\tilde{\Upsilon}_1, \tilde{\Upsilon}_2)$ is the same argument as the case $(\tilde{\Upsilon}_2, \tilde{\Upsilon}_3)$, so that we will only show one.

Proof. Case $(\tilde{\Upsilon}_1, \tilde{\Upsilon}_2)$: Assume that F is an object with $\dim(F) = [0, f, g, h]$ that determines an actual wall for E. Because E is $\lambda_{\beta,\alpha}$-semistable at $\tilde{\Upsilon}_2$, we know that $h \geqslant f$ by (4.14) . By the functoriality of the truncation functor we know that $\tau_{\geqslant -1}(F)$ is a subobject of $\tau_{\geqslant -1}(E)$, and by Lemma 4.2.1 we have with $\det(A) := h(2c+k) - cg \geqslant 0$. Now we can apply Lemma 4.2.3 to determine the equation for the λ-wall determined by F as

$$f_{E,F} = \begin{vmatrix} f_{\mathcal{O}_X(1),\mathcal{O}_X} & f_{\mathcal{O}_X(1),\mathcal{O}_X(-1)} & f_{\mathcal{O}_X,\mathcal{O}_X(-1)} \\ c & 2c+k & c \\ f & g & h \end{vmatrix}. \tag{4.15}$$

The fact that its given by a determinant is useful to reduce the wall equation to

$$f_{E,F} = \begin{vmatrix} f_{\mathcal{O}_X(1),\mathcal{O}_X} & f_{\mathcal{O}_X(1),\mathcal{O}_X(-1)} & f_{\mathcal{O}_X,\mathcal{O}_X(-1)} \\ c & 2c+k & c \\ f-h & \dfrac{-\det(A)}{c} & 0 \end{vmatrix}. \tag{4.16}$$

So that we can describe $f_{E,F}$ as

$$f_{E,F} = \frac{\det(A)}{c}\frac{1}{6}(-\beta^3) + (f-h)f_{E,\mathcal{O}_X(-1)[2]}. \tag{4.17}$$

Moreover, if a point in $\tilde{R}_-$ is outside the numerical λ-wall defined by $\mathcal{O}_X(-1)[2]$ then $f_{E,\mathcal{O}_X(-1)[2]}$ is strictly negative, making $f_{E,F} = 0$ if and only if F defines either the λ-wall Υ_2, Υ_3 or $\dim(F)$ is proportional to $\dim(E)$. To conclude we just have to prove that it does not have any λ-wall in $\tilde{R}_+$, but that is just a consequence of it having the wall $\tilde{\Upsilon}_2$ as actual λ-wall and from observing that the outside of this wall is exactly $\beta < 0$, making it that any object defining the wall $\tilde{\Upsilon}_2$ should destabilize E for $\beta > 0$.

Case ($\tilde{\Upsilon}_1, \tilde{\Upsilon}_3$): For that, we just need to apply Proposition 4.3.3 because in this case we have that E will $\lambda_{\beta,\alpha}$-semistable at $\tilde{\Upsilon}_2$ and therefore, from cases $i = 1, 2$ and $i = 2, 3$, we can conclude that E does not have any other wall beyond $\tilde{\Upsilon}_1$ and $\tilde{\Upsilon}_3$ □

Remark 4.3.5. The statement in Theorem 4.3.4 can be regarded in another, less geometric, manner, where instead of demanding that the canonical walls are actual we assumed the respective truncation functor applied to E satisfies the determinant condition. For that we would just need to adjust the notion of the determinant condition to the case of $\tilde{\Upsilon}_2$ to which we would say E satisfies the middle determinant condition if every for every subobject F with $\dim(F) = [0, f, g, h]$ it satisfied $f > h$.

Remark 4.3.6. Another way to generalize this result is by considering other exceptional collections. It is clear that in the cases $i = 1$ and $i = 2$ or $i = 2$ and $i = 3$ the same result applies to other varieties. The case $i = 1$ and $i = 3$ would require an analogous result to Proposition 4.3.3, which would require the curve determined by $f_{E_3,E_1} - f_{E_1,E_2}$ to be the same as a component of the curve $\Gamma_{E,s}$.

As a direct consequence of Theorem 4.3.4, locally free instanton sheaves shifted by 1 are $\lambda_{\beta,\alpha}$-stable at every point of R_1, outside the walls $\tilde{\Upsilon}_1$ and $\tilde{\Upsilon}_3$.

Corollary 4.3.7. *For any locally free instanton I with charge c, $I[1]$ is $\lambda_{\beta,\alpha}$-stable for every $(\beta,\alpha) \in R_1$ outside of both λ-walls $\tilde{\Upsilon}_1$ and $\tilde{\Upsilon}_3$.*

The description of any wall by equation (4.17) gives us a corollary about the intersection of the numerical λ-walls with $\tilde{\Upsilon}_i$.

Corollary 4.3.8. *No other numerical λ-wall for an object E with* $\dim(E) = [0, c, 2c+k, c]$ *destabilized by $F \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ and* $\dim(F) = [0, a, b, c]$ *can cross $\tilde{\Upsilon}_i$ at a point different then* $(0, 1/\sqrt{3})$*, unless this numerical λ-wall determines the same λ-wall as the respective $\tilde{\Upsilon}_i$.*

We will prove this result only for $i = 1$, as the other ones are analogous.

Proof. First, observe that $\tilde{\Upsilon}_1$ is only an actual λ-wall for the points (β, α) where $\beta \leqslant 0$. The point $\beta = 0$ in $\tilde{\Upsilon}_1$ has $\alpha^2 = 1/3$. Now, we assume there exists a point P at the intersection $\tilde{\Upsilon}_1 \cap \Upsilon_{E,F}$ so that $f_{E,F}(P) = 0$, but we know from $P \in \tilde{\Upsilon}_1$ that $f_{E,\mathcal{O}_X(-1)[2]}(P) = 0$ and $\tau_{\geqslant -1}(E)$ satisfies the determinant condition. Moreover, using equation (4.17) and its notation, we also can conclude that $\det(A) = 0$. This implies that either $\tilde{\Upsilon}_1 = \Upsilon_{E,F}$, without taking in to account the orientation, or $\dim(F)$ is proportional to $\dim(E)$ and in that case E has the same λ-slope as F at every point. □

Next, we analyze the stability of the rank 0 instanton sheaves. This will be important in determining the uniqueness of the walls for the non-locally free instantons sheaf.

Lemma 4.3.9. *Rank 0 instanton sheaves are Bridgeland semistable for every point in the intersection* $R_1 \cap \{(0, \alpha) \in \mathbb{H} | \alpha^2 \geqslant \frac{1}{3}\}$.

Proof. From Lemma 4.2.7 we conclude that $Q \in \mathcal{A}^{0,\frac{1}{\sqrt{3}}}$ and that Q is an extension of $\mathcal{O}_X(-1)^{\oplus d}[2]$, $\mathcal{O}_X^{\oplus 2d}[1]$ and $\mathcal{O}_X(1)^{\oplus d}$ in $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ but all of these objects have the same λ-slope in $(0, 1/\sqrt{3})$. Therefore Q is λ-semistable at that point.

Suppose that F is a destabilizing object for Q at $\alpha \in (1/\sqrt{3}, 1/\sqrt{3}+\epsilon)$ for some $\epsilon > 0$. Then F is λ-semistable and $F \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ with $\dim(F) = [0, f, g, h]$. As in equation (4.14), we know that $\lambda_{0,\alpha}(Q) = 0$ for every α and if F destabilizes Q then $f > h$.

But now we can study the following exact sequence in $\mathcal{A}^{0,1/\sqrt{3}}$

$$0 \to F \to Q \to G \to 0. \tag{4.18}$$

Applying the cohomology functor $\mathcal{H}^0_{\mathrm{Coh}(X)}$ and $\mathcal{H}^0_\beta$ for $\beta = 0$ to this sequence we obtain that $\mathcal{H}^{-2}(F) = 0$, $\mathcal{H}^{-2}(G) = \mathcal{H}^{-1}(F)$, $\mathcal{H}^{-1}_0(F) = 0$. But $\mathrm{ch}^0_1(F) = \mathrm{ch}^0_1(\mathcal{H}^0_0(F)) = h - f$ should be positive because $F \in \mathcal{B}^0$, which is impossible due to F destabilizing Q. □

Corollary 4.3.10. *At the intersection* $R_1 \cap \{(0, \alpha) \in \mathbb{H} | \alpha^2 \geqslant \frac{1}{3}\}$ *the instanton sheaves are Bridgeland semistable.*

Proof. Notice that the double dual of an instanton sheaf is a locally free instanton sheaf, see (JARDIM; MAICAN; TIKHOMIROV, 2017), and the existence of the shifted double dual sequence

$$0 \to Q \to I[1] \to I^{**}[1] \to 0$$

in $\mathcal{A}^{0,\alpha}$. □

Remark 4.3.11. The above results could be generalized for higher rank instanton sheaves, that is, sheaves obtained by monads of dimension $[0, c, 2c + k, c]$ if an analogous result as (ANCONA; OTTAVIANI, 1994, Proposition 2.8) is found to be true.

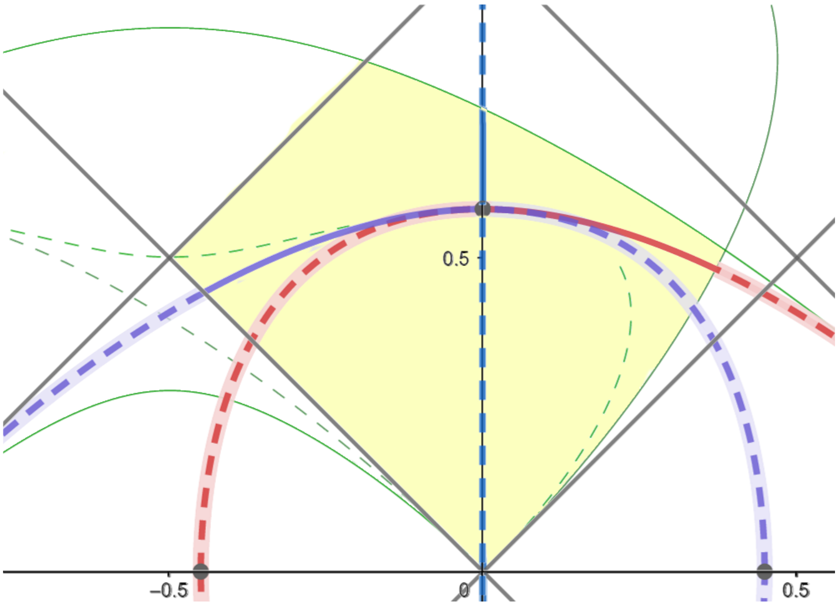

Figure 4.3.12.1 – For $X = \mathbb{P}^3$: the yellow region represents $\tilde{R}$, the purple curve is $\tilde{\Upsilon}_1$, the blue curve is $\tilde{\Upsilon}_2$, the red curve is $\tilde{\Upsilon}_3$ and the doted curves represent the non-necessarily actual parts of these walls. The green curves, either doted or not, represent the curves determining the quiver region.

Example 4.3.12. Image 4.3.12.1 represents the actual λ-walls discussed in this section for the instantons with charge $c = 2$ in $\mathbb{P}^3$. For different c the picture does not change much, at least in this restricted region.

4.4 Moduli spaces

To close the thesis, we relate the results of the previous sections to a description of the space of Bridgeland stable objects close to the β-axis in $\mathbb{H}$ with Chern character $(-2, 0, c, 0)$. The first result is a theorem in (ARCARA et al., 2013) proving that Bridgeland stability over these quiver regions behave exactly like quivers and therefore the moduli problem is given by a GIT problem.

Theorem 4.4.1. *(ARCARA et al., 2013, Theorem 8.1) Let $\mathcal{E}$ be a full Ext-exceptional collection of a smooth projective threefold X, (β, α) a point in the quiver region $R_{\mathcal{E}}$ in $\mathbb{H}$. Then the moduli space $\bar{\mathcal{M}}[a, b, c, d]$ of Bridgeland semistable objects with dimension vector $[a, b, c, d]$ is a projective variety. Furthermore, if we consider only the stable objects in $\bar{\mathcal{M}}[a, b, c, d]$ then this space is a quasi-projective variety.*

Proposition 4.4.2. *Let*

$$\mathcal{I}(c) = \{I \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle |\ \dim(I) = [0, c, 2c+2, c],\ \mathcal{H}^{-2}(I) = 0 \text{ and } \mathcal{H}^0(I) = 0\}.$$

Then any object in $\mathcal{I}(c)$ is $\lambda_{\beta,\alpha}$-stable, for every $(\beta,\alpha) \in \tilde{R}_-$ outside of $\tilde{\Upsilon}_1$.

Proof. Let $c \geqslant 1$ be an integer and $(\beta,\alpha) \in \tilde{R}_-$ outside of $\tilde{\Upsilon}_1$. Firstly, observe that the objects in $\mathcal{I}(c)$ are cohomology of the linear monad (4.5) and so are a shift by [1] of the instanton sheaves. Let $I[1]$ be one of these objects. By a combination of Lemmas 4.3.10, 4.3.1 and Theorem 4.3.4 we can conclude that $I[1]$ is $\lambda_{\beta,\alpha}$-semistable at the walls $\tilde{\Upsilon}_1, \tilde{\Upsilon}_2$ and it does not have any other wall in $\tilde{R}_-$, unless it is semistable at every point of $\tilde{R}_-$.

To see how the later is impossible, assume that there exists a $F \hookrightarrow I[1]$ in $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ such that $\lambda_{\beta,\alpha}(F) = \lambda_{\beta,\alpha}(I[1])$ for every point in $\tilde{R}_-$ and outside of $\tilde{\Upsilon}_1$. We know from Lemma 4.2.3 that an object F has the same $\lambda_{\beta,\alpha}$-slope as $I[1]$ for a 2-dimensional region if and only if its dimension vector is a multiple of $[0, c, 2c+2, c]$. But this impossible because $\tau_{\geqslant -1}(I[1])$ would have a subobject with dimension vector a multiple of its dimension, which is impossible because $\tau_{\geqslant -1}(I[1])$ is $\lambda_{\beta,\alpha}$-stable by Proposition 4.2.5.

□

For c odd we are able to provide a stratification of the moduli of $\lambda_{\beta,\alpha}$-stable objects with dimension vector $[0, c, 2c+2, c]$ in terms of the moduli of $\lambda_{\beta,\alpha}$-stable 2-step complexes with dimension vector $[0, 0, 2c+2, c]$. For $c = 2k$, the same description only deals with a subset of the moduli space of $\lambda_{\beta,\alpha}$-stable objects because it is possible for a 2-step complex with dimension vector $[0, 0, 4k+2, 2k]$ to be strictly $\lambda_{\beta,\alpha}$-semistable.

Main Theorem 3. *For any $(\beta,\alpha) \in \tilde{R}_-$ outside and sufficiently close to the λ-wall $\tilde{\Upsilon}_1$, inside the Bridgeland moduli space*

$$\mathcal{M}_{\beta,\alpha}(c) = \{E \in \mathrm{D^b}(\mathbb{P}^3) | E \text{ is } \lambda_{\beta,\alpha,1/3}\text{-stable with } \mathrm{ch}(E) = (-2, 0, c, 0)\}$$

we have the set

$$\mathcal{N}(c) := \bigcup_{T \in \mathcal{K}} \{F \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c}[2], T) |\ \text{with } \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1)[2], F) = 0\}$$

where $\mathcal{K} = \{T \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle | \dim(T) = [0, 0, 2c+2, c]$ and T is $\lambda_{\beta,\alpha,1/3}$-stable $\}$.

If the charge c is odd then $\mathcal{N}(c)$ is equal to $\mathcal{M}_{\beta,\alpha}(c)$ and $\mathcal{M}_{\beta,\alpha}(c)$ is a projective space. Otherwise, if c is even, $\mathcal{M}_{\beta,\alpha}(c)$ is a quasi-projective variety. In both cases $\mathcal{I}(c)$ is a subset of $\mathcal{M}_{\beta,\alpha}(c)$.

Proof. Applying Corollary 4.3.8 it is clear that there exists a region in $\tilde{R}_-$ outside $\tilde{\Upsilon}_1$ where no other numerical λ-wall go through. Let W be this open neighborhood. Since the wall

$\tilde{\Upsilon}_1$ is a vanishing wall for the dimension vector $[0, c, 2c+2, c]$ then every $\lambda_{\beta,\alpha}$-stable object for $(\beta, \alpha) \in W$ is of the form $F \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c}[2], T)$ with T satisfying the determinant condition. Now we just have to prove that the necessary and sufficient condition for F to be $\lambda_{\beta,\alpha}$-stable.

Suppose that F is unstable after crossing $\tilde{\Upsilon}_1$ to its outside and assume that L is a subobject destabilizing F in $(\beta, \alpha) \in W$. Then by Corollary 4.3.8 L determines the same wall as $\tilde{\Upsilon}_1$ and by equation (4.17) we have $\det(A) = 0$. If T were in $\mathcal{K}$ we would have either $\tau_{\geqslant -1}(L)$ with dimension $[0, 0, 2c+2, c]$ or equal to zero, in the former case L would never destabilize F due to Equation (4.17) and in the later case $L = \mathcal{O}_X(-1)[2]^k$ for some $k > 0$.

Now, if c is odd then every two step complex with dimension $[0, 0, 2c+2, c]$ satisfies the determinant condition if and only if it is -stable after crossing its λ-wall. Even more, an object F is $\lambda_{\beta,\alpha}$-semistable for $(\beta, \alpha) \in W$ if and only if it is $\lambda_{\beta,\alpha}$-stable. □

Remark 4.4.3. The condition $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1)[2], F) = 0$ for an object F to be $\lambda_{\beta,\alpha}$-stable is a necessary condition for all (β, α) outside of $\tilde{\Upsilon}_1$. This condition is equivalent to the vector space of global sections of $\mathcal{H}^{-2}(F)(1)$ being zero.

The analogous occurs to the right-hand side of $\tilde{\Upsilon}_2$, but now we have the locally free and *perverse instanton sheaves* instead of the instanton sheaves. For that we will need the following definition present in (HENNI; JARDIM; MARTINS, 2015, Definition 5.6).

Definition 4.4.4. An object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is a perverse instanton sheaf if it is isomorphic to a linear complex of the form

$$\mathcal{O}_X(-1)^{\oplus c} \xrightarrow{f} \mathcal{O}_X^{\oplus a} \xrightarrow{g} \mathcal{O}_X(1)^{\oplus c}$$

such that the left derived dual of the restriction to a line, $Lj^*(E)$, is a sheaf object, where $j : l \hookrightarrow X$ is a line inside of X.

As noted in (JARDIM; SILVA, 2020), every derived dual of an instanton sheaf is a perverse instanton sheaf but the converse is not true.

Example 4.4.5. We can take, for example, $E \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$ to be the linear complex

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\begin{pmatrix} -y \\ x \\ 0 \\ w \end{pmatrix}} \mathcal{O}_{\mathbb{P}^3}{}^{\oplus 4} \xrightarrow{\begin{pmatrix} x & y & z & 0 \end{pmatrix}} \mathcal{O}_{\mathbb{P}^3}$$

which has its last-step cohomology equal to $\mathcal{O}_p$, for the point $p \in \mathbb{P}^3$ satisfying the equation $x = y = z = 0$. Therefore, we can choose a line that does not go through the point p in order to conclude that $Lj^*(E)$ is a sheaf element, making E into a perverse instanton sheaf.

Moreover, it is not the derived dual of any instanton sheaf as its last-step cohomology would nescessarily need to be pure of dimension 1, (JARDIM; MAICAN; TIKHOMIROV, 2017). Even more, we can calculate $E^\vee$ to determine that its last-step cohomology would be $\mathcal{O}_q$, for the point q determined by $x = y = w = 0$, and in that way it cannot be the derived dual of an instanton sheaf because the derived dual is an involution.

Proposition 4.4.6. *Let $\mathcal{L}(c) \subset \mathcal{I}(c)$ be the subset of shifted locally free instanton sheaves of charge c and $\mathcal{P}(c) = \{E \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle | \ E^\vee \in \mathcal{I}(c)\}$ the subset of perverse instanton sheaves that are dual of instanton sheaves of charge c. Then, for every $(\beta, \alpha) \in \tilde{R}_+$ outside of $\tilde{\Upsilon}_3$, every object in $\mathcal{L}(c)$ and $\mathcal{P}(c)$ is $\lambda_{\beta,\alpha}$-stable.*

Proof. This statement is a direct consequence of Lemmas 4.4.2,4.3.2 and 4.3.1. □

Main Theorem 4. *For any $(\beta, \alpha) \in \tilde{R}_+$ outside and sufficiently close to the λ-wall $\tilde{\Upsilon}_3$, inside the moduli space*

$$\mathcal{M}_{\beta,\alpha}(c) = \{E \in \mathrm{D^b}(\mathbb{P}^3) | E \textit{ is } \lambda_{\beta,\alpha,1/3}\textit{-stable with } \mathrm{ch}(E) = (-2, 0, c, 0)\}$$

we have the set

$$\tilde{\mathcal{N}}(c) := \bigcup_{T \in \tilde{\mathcal{K}}} \{F \in \mathrm{Ext}^1(T, \mathcal{O}_X(1)^{\oplus c}) | \textit{ with } \mathrm{Hom}(E, \mathcal{O}_X(1)) = 0\}$$

where $\tilde{\mathcal{K}} = \{T \in \langle \mathrm{S}(\tilde{\mathcal{E}})\rangle | \dim(T) = [0, c, 2c+2, 0]$ and T is $\lambda_{\beta,\alpha,1/3}$-stable $\}$.

If the charge c is odd then $\tilde{\mathcal{N}}(c)$ is equal to $\mathcal{M}_{\beta,\alpha}(c)$ and $\mathcal{M}_{\beta,\alpha}(c)$ is a projective space. Otherwise, if c is even, $\mathcal{M}_{\beta,\alpha}(c)$ is a quasi-projective variety. In both cases $\mathcal{L}(c)$ and $\mathcal{P}(c)$ are subsets of $\mathcal{M}_{\beta,\alpha}(c)$.

Remark 4.4.7. In the case of $X = Q_3$ we can only obtain that the objects in $\mathcal{L}(c)$ are inside the moduli space of stable objects $\mathcal{M}_{\beta,\alpha}(c)$, due to the fact that we do not have a well-stablished notion of non-locally free instanton sheaves in Q_3 to obtain both our version of Main Theorem 3 and, consequently, the stability of the objects in $\mathcal{P}(c)$.

The above construction can be made more explicit in the case of $c = 1$. The quiver approach to the study of the moduli space of objects with dimension vector $[0, 1, 4, 1]$ was done by Jardim–daSilva, see (JARDIM; SILVA, 2020). In the next example, we will show how to obtain the same description of the walls by using this different method.

Example 4.4.8. Let $c = 1$ and E an object in $\langle \mathrm{S}(\tilde{\mathcal{E}})\rangle$ with $\dim(E) = [0, 1, 4, 1]$. Firstly, observe that $\tau_{\geqslant -1}(E)$ or $\tau_{\leqslant -1}(E)$ satisfies the determinant condition if and only if E is $\lambda_{\beta,\alpha}$-semistable at $\tilde{\Upsilon}_2$. This is a consequence of Lemma 4.2.3, by observing that an object destabilizes E at $\tilde{\Upsilon}_2$ if and only if it λ-destabilizes both $\tau_{\geqslant -1}(E)$ and $\tau_{\leqslant -1}(E)$.

Applying Theorem 4.3.4 we note that the only possible actual λ-walls for an object with dimension vector $[0,1,4,1]$ are $\tilde{\Upsilon}_i$. Hence, we can apply the Theorem 4.3.4 to prove that instanton sheaves are stable at $\tilde{R}_-$, outside of $\tilde{\Upsilon}_1$, while locally free and perverse instanton sheaves which are duals of instanton sheaves are stable at $\tilde{R}_+$, outside of $\tilde{\Upsilon}_3$.

To prove that these are the only stable objects in their respective spaces, we just have to show that if an object E is stable at $\tilde{R}_-$ outside $\tilde{\Upsilon}_1$ then $\mathcal{H}^{-2}(E) = 0$ and $\mathcal{H}^0(E) = 0$.

$\mathcal{H}^{-2}(E) = 0$: If $\mathcal{H}^{-2}(E) = \mathcal{O}_{\mathbb{P}^3}(-1)$ then $\mathcal{O}_{\mathbb{P}^3}(-1)$ is a direct summand of E and therefore E is nowhere stable, outside the wall $\tilde{\Upsilon}_1$. Otherwise, suppose $0 \neq \mathcal{H}^{-2}(E) \neq \mathcal{O}_{\mathbb{P}^3}(-1)$ then we would have $Q = \mathcal{O}_{\mathbb{P}^3}(-1)/\mathcal{H}^{-2}(E)$ as a subsheaf of ${\mathcal{O}_{\mathbb{P}^3}}^{\oplus 4}$, which is absurd because Q is a torsion sheaf.

$\mathcal{H}^0(E) = 0$: Suppose that $\mathcal{H}^0(E) \neq 0$. Then $\mathcal{H}^0(E) = \mathcal{O}_S(1)$ with $S \hookrightarrow \mathbb{P}^3$ a subvariety due to $\mathcal{H}^0(E)$ being a quotient of $\mathcal{O}(1)$. Its twisted ideal $J(1)$ is the image of the map $f : \mathcal{O}^{\oplus 4} \to \mathcal{O}(1)$ defining $\tau_{\geqslant -1}(E)$ and we can see that its space of global sections has dimension $h^0(J(1)) \leqslant 3$, making it so that $h^0(\ker(f)) \neq 0$.

Moreover, this implies that $h^0(\mathcal{H}^{-1}(E)) \neq 0$ because it is a quotient of $\ker(f)$ by $\mathcal{O}(-1)$, and therefore there exists a non zero map $\mathcal{O}[1] \to \mathcal{H}^{-1}(E)[1] \to E$. Hence, there exists a destabilizer for E in $\tilde{R}_-$ by Lemma 4.2.3, an absurd.

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APPENDIX A – Computational Observations

As seen in the paper, the construction of examples of Ext-exceptional collections is very important to establishing exceptional regions. One way of constructing these collections is by shifting the objects in a strong exceptional collection by the appropriate degree. For $\mathbb{P}^2$, it is a known result by Rudakov (RUDAKOV, 1988) that the only strong exceptional collections we can create are mutations of the canonical exceptional collection , proved to be strong by Beilinson (BEILINSON, 1978).

Question A.0.1. *Is there an Ext-exceptional collection in $\mathbb{P}^3$ satisfying the upper-half plane for any point in $\mathbb{H}$ with $\alpha > 2$ and $s = \dfrac{1}{3}$?*

In this appendix we describe a computational approach that lead to Question 1. There is no known list of all full exceptional collections in $\mathbb{P}^3$ but there exists a conjecture about the transitivity of the action of the group $B_n \rtimes \mathbb{Z}^n$ over the space of strong exceptional collections, where B_n is the n-braid group acting by mutations and $\mathbb{Z}^n$ is the action by shifts, see (BONDAL; POLISHCHUK, 1994). Our idea was to use mutations to inductively define strong exceptional collections, with easy to compute Chern characters due to their additive property, and test for the upper-half plane condition using a parametrization of lines going through the origin.

We start by defining the notion of mutation of an exceptional collection and their computational properties.

Definition A.0.2. Let $\{E_0, E_1\}$ be a exceptional pair in $\mathrm{D^b}(X)$, a left (resp. right) mutation of the exceptional pair is the object $\mathcal{L}_{E_0}E_1$ (resp. $\mathcal{R}_{E_1}E_0$) defined by the following distinguished triangles

$$\mathcal{L}_{E_0}E_1 \to \mathrm{Hom}^\bullet(E_0, E_1) \otimes E_0 \to E_1$$

$$(E_0 \to \mathrm{Hom}^\bullet(E_0, E_1)^* \otimes E_1 \to \mathcal{R}_{E_1}E_0).$$

Definition A.0.3. For an exceptional collection $\mathcal{E} = \{E_0, ..., E_k\}$ in $\mathrm{D^b}(X)$ we can define the Left and Right mutations of $\mathcal{E}$ by

$$\mathcal{L}_i\mathcal{E} = \{E_0, ..., E_{i-1}, \mathcal{L}_{E_i}E_{i+1}, E_i, E_{i+2}, ..., E_k\}$$

$$\mathcal{R}_i\mathcal{E} = \{E_0, ..., E_{i+1}, \mathcal{R}_{E_{i+1}}E_i, E_{i+2}, ..., E_k\}$$

It is known that a mutation of an exceptional collection is also an exceptional collection, they also preserve fullness. That is not true for strong exceptional collections, in general, but in our case this is actually true.

The following theorem describe what happens when mutating a strong exceptional collection and the nature of the objects in a strong exceptional collection.

Theorem A.0.4. *(BONDAL, 1989, Section 9) Suppose that $\mathcal{E} = \langle E_0, ..., E_n \rangle$ is a full exceptional collection of sheaves in $\mathrm{D^b}(X)$ for a n-dimensional manifold X. Then $\mathcal{E}$ is a strong exceptional collection and any mutation of $\mathcal{E}$ is a full strong exceptional collection.*

Theorem A.0.5. *(POSITSELSKI, 2013) Let X be a smooth projective variety for which $n = \dim \mathrm{K}_0(X) - 1 = \dim(X)$. Then for any full strong exceptional collection $E_0, ..., E_n$ in $\mathrm{D^b}(X)$, the objects E_i are shifts of locally free sheaves by the same number $a \in \mathbb{Z}$.*

Remark A.0.6. For a smooth n dimensional projective variety X, the Theorems A.0.4 and A.0.5 can be applied in conjunction so that $\mathrm{Hom}^\bullet(E_i, E_{i+1}) = \mathrm{Hom}^0(E_i, E_{i+1}) \neq 0$ if $\mathcal{E} = \{E_i\}_i$ is a strong exceptional collection, because otherwise $\mathcal{L}_{E_i} E_{i+1} = E_{i+1}[1]$ and that would be a contradiction due to every strong exceptional collection, in this case $\mathcal{L}_i(\mathcal{E})$, consisting of sheaves shifted by the same number in $\mathbb{Z}$. Another consequence of this is that if the E_i are all μ-stable then $\mu(E_i) < \mu(E_{i+1})$.

With this in hand we can describe a algorithm to produce candidates of full Ext-exceptional collections capable of generating a heart of a bounded t-structure satisfying the upper-half plane condition. As a consequence, we were not able to find a point (β, α) for $s = 1/3$ in the upper-half plane of stability conditions $\mathbb{H}$ with $\alpha > 2$ where a single example of full Ext-exceptional collection was capable of satisfying the conditions imposed by the algorithm(a numerical version of the upper-half plane condition).

Algorithm A.0.7. *We start with a set pre-determined of Chern characters of a given strong exceptional collection of locally free sheaves $\mathcal{E} = \{E_0, ..., E_3\}$ that is the strings*

$$c[i] = (\mathrm{ch}_0(E_i), \mathrm{ch}_1(E_i), \mathrm{ch}_2(E_i), \mathrm{ch}_3(E_i)) = (\mathrm{ch}_j(E_i))_{0 \leqslant j \leqslant n}.$$

With this string at hand we can calculate the Chern character of the mutations $\mathcal{L}_i\mathcal{E}$ and $\mathcal{R}_j\mathcal{E}$ by realizing that in a strong exceptional collection of locally free sheaves we have $\mathrm{Hom}^\bullet(E_i, E_j) = \mathrm{Hom}(E_i, E_j)$ *and*

$$\dim(\mathrm{Hom}(E_i, E_j)) = \chi(E_i, E_j) = \chi(E_i \otimes E_j^\vee, \mathcal{O}_{\mathbb{P}^3}),$$

where $\chi(E_i \otimes E_j^\vee, \mathcal{O}_{\mathbb{P}^3})$ can be calculated by the Hirzebruch-Riemann-Roch, as in (BAYER; MACRÌ, 2011, Theorem 9.3), and $\mathrm{ch}(E_i \otimes E_j^\vee) = \mathrm{ch}(E_i) \cdot \mathrm{ch}(E_j)^\vee$ with $(\mathrm{ch}(E_j)^\vee)_i = (-1)^i \mathrm{ch}_i(E_j)$.

Once we know $\tilde{c}[j] = (\mathrm{ch}_j(\mathcal{L}_k\mathcal{E}))_{0\leqslant j\leqslant 3}$ *or* $\bar{c}[j] = (\mathrm{ch}_j(\mathcal{R}_k\mathcal{E}))_{0\leqslant j\leqslant 3}$*, for all* $k = 0, 1, 2$*, we can calculate if this string satisfy the numerical conditions to which any object* Q_j *with* $\mathrm{ch}(Q_j) = \tilde{c}[j]$ *or* $\mathrm{ch}(Q_j) = \bar{c}[j]$ *and* $Q_j \in <\mathcal{A}, \mathcal{A}[1]>$ *is subject to, for all* j*, such as*

(i) $\nu_{\beta,\alpha}(Q_0) > 0$ *and* $\mu(Q_0) > \beta$ *(if* $Q_0[3] \in \mathcal{A}[1]$*)*

(ii) Either:

$\nu_{\beta,\alpha}(Q_1) > 0$ *and* $\mu(Q_1) > \beta$ *(if* $Q_1[2] \in \mathcal{A}$*),*

$\nu_{\beta,\alpha}(Q_1) \leqslant 0$ *and* $\mu(Q_1) > \beta$ *(if* $Q_1[2] \in \mathcal{A}[1]$*) or*

$\nu_{\beta,\alpha}(Q_1) > 0$ *and* $\mu(Q_1) \leqslant \beta$ *(if* $Q_1[2] \in \mathcal{A}[1]$*)*

(iii) Either:

$\nu_{\beta,\alpha}(Q_2) < 0$ *and* $\mu(Q_2) > \beta$ *(if* $Q_2[1] \in \mathcal{A}$*),*

$\nu_{\beta,\alpha}(Q_2) \leqslant 0$ *and* $\mu(Q_2) \leqslant \beta$ *(if* $Q_2[1] \in \mathcal{A}$*) or*

$\nu_{\beta,\alpha}(Q_2) > 0$ *and* $\mu(Q_2) \leqslant \beta$ *(if* $Q_2[1] \in \mathcal{A}[1]$*)*

(iv) $\nu_{\beta,\alpha}(Q_3) > 0$ *and* $\mu(Q_3) > \beta$ *(if* $Q_3 \in \mathcal{A}$*)*

If any of the conditions $(i), (ii), (iii), (iv)$ *is not satisfid the algorithm returns* 0 *with respect to this mutation* $\mathcal{L}_k\mathcal{E}$ *or* $\mathcal{R}_k\mathcal{E}$*, otherwise we continue the algorithm to see if* $\mathcal{L}_i\mathcal{E}$ *or* $\mathcal{R}_i\mathcal{E}$ *satisfy the upper-half plane condition. This is done by testing if all the images of* $Z_{\beta,\alpha,1/3}(\tilde{c}[j])$ *or* $Z_{\beta,\alpha,1/3}(\bar{c}[j])$ *are above a given line with slope* ϕ*, with* ϕ *varying from* 0 *to* π *with increments of* 0.01π*. In the end, if the collection mutated satisfies* $(i), (ii), (iii), (iv)$ *and the upper-half plane condition then the algorithm returns* 1*. This can be iterated further for any of the mutated collections.*

One other computational approach to the problem of finding real λ-walls in an exceptional region R for a fixed dimension vector $[a, b, c, d]$ is to determine its numerical walls, these will be a finite set due to Lemma 4.1.4, and apply the numerical conditions

(i) (Positivity) $\rho_{\beta,\alpha}(v) \geqslant 0$,

(ii) (Generalized Bogomolov inequality) $Q_{\beta,\alpha}(v) \geqslant 0$

to any possible destabilizing dimension vector $v = [a', b', c', d']$. These necessary conditions restrict the numerical walls to a finer set which can be studied using Proposition A.0.10.

Definition A.0.8. We call a region C of the Upper-half plane $\mathbb{H}$ to be λ-bounded with respect to $E \in \mathrm{D^b}(X)$ if there exists $a, b \in \mathbb{R}$ such that, for every $(\beta, \alpha) \in \mathbb{H}$ and a fixed $s \in \mathbb{R}_{>0}$, the $\mathbb{C}$-slope of E is in $[a, b] \subset (0, 1)$.

Example A.0.9. For any object $E \in \mathrm{D}^{\mathrm{b}}(X)$, one clear example of a C is a compact subset of the upper-half plane not intersecting Θ_E,

Proposition A.0.10. *Let $E \in \mathrm{D}^{\mathrm{b}}(X)$ and P, Q be points in a λ-bounded region C such that exists $\gamma : [0,1] \to C$ a continuous curve with endpoints in P and Q respectively. Also, suppose that $E \notin \mathcal{A}^P$ and $E \in \mathcal{P}(\lambda_{Q,s}(E)) \subset \mathcal{A}^Q$, there exists $w \in (0,1]$ such that for every $t \in [w,1]$, $E \in \mathcal{A}^{\gamma(t)}$ and E is $\lambda_{\gamma(w),s}$-unstable.*

Proof. We start by noting that $\phi^+_{\beta,\alpha}(E)$ and $\phi^-_{\beta,\alpha}(E)$ are continuous functions on $\mathbb{H}$ and therefore $f(t) : [0,1] \to \mathbb{R}$

$$f(t) = \phi^+_{\gamma(t)}(E) - \phi^-_{\gamma(t)}(E)$$

is a non-negative continuous function. By the condition in P we know that f is not a trivial function with image $\{0\}$ and since it is a continuous image of a connected set, we know that $f([0,1]) = [0,c]$ for some $c \in \mathbb{R}_{>0}$.

Now let $\epsilon > 0$ be a real number which makes $[a-\epsilon, b+\epsilon] \subset (0,1)$. It is clear that for every $t \in f^{-1}([0,\epsilon])$ we have $E \in \mathcal{A}^{\gamma(t)}$, we just have to choose the connected component W of $f^{-1}([0,\epsilon])$ containing 1 and this set will be $W = [w,1]$ for some $w \in R_{>0}$.

□

The consequence of this theorem is that for an object to be in a real λ-wall, that is for it to became semistable, it has to first be unstable in $\mathcal{A}^{\gamma(w)}$. This allows for an inductive approach in this exceptional regions, where we use the fact that twisting by $\mathcal{O}(-1)$ the exceptional region R_1 intersect the original region R_1 at the line $\{-1\} \times (0, 0.7)$, and if we can prove that an object is semistable at region R_1 then we can draw curves through the twisted region that don't go inside the regions "numerically possible" to prove that none of this numerical λ-walls are real.

ANNEX A – Code

```c
#include <stdio.h>
#include <stdlib.h>
#include <math.h>

struct Object{    //Fixed information about object

    int ch[4];  // Chern Character
    double Z[2];   // Stability Function
};

struct ExcpSet{ //Exceptional Collection

    struct Object E[4];

};

int max_of_four(int a, int b, int c, int d);
int CondA(struct ExcpSet C, double beta, double alpha);
int ConUpp(struct ExcpSet C);
void StaZ(struct Object * E,double beta, double alpha, double s);
int EulerChar(int a[4],int b[4]);
struct Object LMut(struct Object E, struct Object F);
struct Object RMut(struct Object E, struct Object F);

int main()
{
    struct ExcpSet C[2][7][7]; // Exceptional Sets
    int i=0,j=0,k=0,p,l=0,f=0,g=0, u=0;  // Counter
    double alpha,beta,s; // Parameters
    int L[2][7][7][2]; // Number of Mutations to the Left and to the Right, respectively
    FILE * fpointer = fopen("Results.txt", "w");

    /* O(-3),O(-2),O(-1),O */
```

```

    C[0][0][0].E[0].ch[0]=1; // Chern characters of Base Exceptional Object 0
    C[0][0][0].E[0].ch[1]=-3;
    C[0][0][0].E[0].ch[2]=9;
    C[0][0][0].E[0].ch[3]=-27;

    C[0][0][0].E[1].ch[0]=1; // Chern Character of Base Exceptional Object 1
    C[0][0][0].E[1].ch[1]=-2;
    C[0][0][0].E[1].ch[2]=4;
    C[0][0][0].E[1].ch[3]=-8;


    C[0][0][0].E[2].ch[0]=1; // Chern character of Base Exceptional Object 2
    C[0][0][0].E[2].ch[1]=-1;
    C[0][0][0].E[2].ch[2]=1;
    C[0][0][0].E[2].ch[3]=-1;

    C[0][0][0].E[3].ch[0]=1; // Chern Character of Base Exceptional Object 3
    C[0][0][0].E[3].ch[1]=0;
    C[0][0][0].E[3].ch[2]=0;
    C[0][0][0].E[3].ch[3]=0;

    /* O(-2),O(-1),O,O(1) */

    C[1][0][0].E[0].ch[0]=1;
    C[1][0][0].E[0].ch[1]=-2;
    C[1][0][0].E[0].ch[2]=4;
    C[1][0][0].E[0].ch[3]=-8;

    C[1][0][0].E[1].ch[0]=1;
    C[1][0][0].E[1].ch[1]=-1;
    C[1][0][0].E[1].ch[2]=1;
    C[1][0][0].E[1].ch[3]=-1;

    C[1][0][0].E[2].ch[0]=1;
```

```c
    C[1][0][0].E[2].ch[1]=0;
    C[1][0][0].E[2].ch[2]=0;
    C[1][0][0].E[2].ch[3]=0;

    C[1][0][0].E[3].ch[0]=1;
    C[1][0][0].E[3].ch[1]=1;
    C[1][0][0].E[3].ch[2]=1;
    C[1][0][0].E[3].ch[3]=1;




        printf("Insert beta: \n" );
        scanf("%lf", &beta);
        printf("Insert alpha: \n");
        scanf("%lf", &alpha);
        printf("insert s: \n");
        scanf("%lf", &s);
        fprintf(fpointer, "Test for (beta,alpha,s)=( %lf, %lf, %lf) \n", beta, alpha,s);

        for(u=0 ; u<=1 ; u++){

        L[u][0][0][0]=CondA(C[u][0][0],beta,alpha);

        for(i=0; i<=3 ; i++){ // Calculating their stability vectors
                StaZ(&C[u][0][0].E[i],beta,alpha,s);
        }

        L[u][0][0][1]=ConUpp(C[u][0][0]);



            for(f=0; f<=6 ; f++){

                for(g=0 ; g<=3 ; g++){

                    C[u][f][0].E[g]=C[u][0][f].E[g];  //
```

```
                      Fixing the Exceptional class from
                      where the mutations are coming |
                      Second iteration.
                }

                 L[u][f][0][0]=CondA(C[u][f][0],beta,alpha);

                for(i=0; i<=3 ; i++){ // Calculating their stability vectors
                    StaZ(&C[u][f][0].E[i],beta,alpha,s);
                }

                L[u][f][0][1]=ConUpp(C[u][f][0]);


                    /* Mutation to the Left */

                for(j=0; j<=2 ; j++){

                    p=j+1;

                    for(k=0; k<=3; k++){ // Constructing the Exceptional Sequence
                        if(k!=j && k!=j+1){
                                    C[u][f][p].E[k]= C[u][f][0].E[k];
                        }else if(k==j){
                                    C[u][f][p].E[k]= LMut(C[u][f][0].E[k],C[u][f][0].E[k+1]);
                        }else{
                                    C[u][f][p].E[k]= C[u][f][0].E[k-1];
                        }
                    }

                    L[u][f][p][0]=CondA(C[u][f][p],beta,alpha);

```

```
            for(i=0; i<=3 ; i++){ // Calculating their stability vectors
                    StaZ(&C[u][f][p].E[i],beta,alpha,s);
            }

            L[u][f][p][1]=ConUpp(C[u][f][p]);

        }

        /* Mutation to the Right */
        j=0;
        k=0;

        for(j=0; j<=2 ; j++){

            p=j+4;

            for(k=0; k<=3; k++){ // Constructing the Exceptional Sequence
                if(k!=j && k!=j+1){
                            C[u][f][p].E[k]=C[u][f][0].E[k];

                }else if(k==j+1){

                            C[u][f][p].E[j+1]= RMut(C[u][f][0].E[j+1],C[u][f][0].E[j]);
                    }
                else{
                            C[u][f][p].E[k]=C[u][f][0].E[j+1];
                }
            }

            L[u][f][p][0]=CondA(C[u][f][p],beta,alpha);

```

```c

                    for(i=0; i<=3 ; i++){ // Calculating their stability vectors
                        StaZ(&C[u][f][p].E[i],beta,alpha,s);
                    }

                    L[u][f][p][1]=ConUpp(C[u][f][p]);
                }

                /* Printing the Results */
                j=1;
                k=0;
                    for(j=0; j<=6; j++){
                        fprintf(fpointer, "\n Set %d | Mutation %d , %d  \n", u, j,f);

                        for(k=0; k<=3; k++){
                            fprintf(fpointer,"(%d,%d,%d,%d), ", C[u][f][j].E[k].ch[0],C[u][f][j].E[k].ch[1],C[u][f][j].E[k].ch[2],C[u][f][j].E[k].ch[3]);
                        }

                        fprintf(fpointer, " %d, %d" , L[u][f][j][0],L[u][f][j][1]);


                    }

                }

                fprintf(fpointer, "\n \n \n \n");

        }
         fclose(fpointer);


```

```
    return 0;
};


int CondA(struct ExcpSet C, double beta, double alpha){ // Verifying if the objects satisfy the condition for <C> to be contained in <A,A[1]>
    int i = 0;
    double mu;   // mu-slope
    double nu;   // nu-slope


    for(i=0; i<=3 ; i++ ){

        mu = (double) C.E[i].ch[1]/C.E[i].ch[0];
        if(mu==beta){
            nu=50;
        }else{
        nu = (double)(C.E[i].ch[2] - 2*C.E[i].ch[1]*beta + (C.E[i].ch[0]*beta*beta)-(C.E[i].ch[0]*alpha*alpha))/2*(C.E[i].ch[1]-beta*C.E[i].ch[0]);
        }

        switch(i){

    case 0:                         // E_0[3] \in A
        if(nu>0 || mu>beta){
            return 1;
        };
        break;

    case 1:                        // E_1[2] \in A
        if((nu>0 || mu>beta) && (nu>0 || mu<=beta) && (nu<=0 || mu>beta)){
            return 2;
        };
        break;

    case 2:                        // E_2[1] \in A
```

```
        if((nu>0 || mu<=beta) && (nu<0 || mu>beta) && (nu<=0
           || mu<=beta)){
            return 3;
        };
        break;

    case 3:                           // E_3    \in A
        if(nu<=0 || mu<=beta){
            return 4;
            };
        break;
        };

    };

           return 0;




};

int max_of_four(int a, int b, int c, int d){
 int max;
 if((a>b) && (a>c) && (a>d))
 max = a;
 if((b>a) && (b>c) && (b>d))
 max = b;
 if((c>a) && (c>b) && (c>d))
 max = c;
 if((d>a) && (d>b) && (d>c))
 max = d;
 return max;
}

int ConUpp(struct ExcpSet C){ // Verifying the Upper-half
   plane condition for the objects in a giver Exceptional
   Collection

```

```
    double k; // Slope of the line
    double s; // Increment on the slope of the line
    int r[4]={0,0,0,0}; // For a fixed line, if r[i]=r[j]
       then Z(E_i) and Z(E_j) are on the same side of the
       Upper-Half plane
    int i; //Counter

    k=-1;
    s=0.001;

        /* One side of the lines */

    while(k<=1 && (r[0]==0 || r[1]==0 || r[2]==0 || r[3]==0))
       { // Lines bounded on the sides by x=y and x=-y
        for(i=0; i<=3; i++){
            if(i==0 || i==2){
                if(-C.E[i].Z[0]>= -k*C.E[i].Z[1]){
                r[i]=1;

                }else{r[i]=0;}
            }else{
                if(C.E[i].Z[0]>= k*C.E[i].Z[1]){
                r[i]=1;

                }else{r[i]=0;}

            }
        }
        if(r[0]==0 && r[1]==0 && r[2]==0 && r[3]==0){
            return 0;
        }
    k=k+s;
    }
    k=-1;
    if((r[0]==0 || r[1]==0 || r[2]==0 || r[3]==0)){ // To see
        if it is finished
        if(r[0]==0 && r[1]==0 && r[2]==0 && r[3]==0){
            return 0;
        }
```

```
        while(k<=1 && (r[0]==0 || r[1]==0 || r[2]==0 || r
            [3]==0)){ // Lines bounded above by x=y and below
            by x=-y
            for(i=0 ; i<=3; i++){
                if(i==0 || i==2){
                    if(-k*C.E[i].Z[0]<= -C.E[i].Z[1]){
                    r[i]=1;

                    }else{r[i]=0;}
                }else{
                    if(k*C.E[i].Z[0]<= C.E[i].Z[1]){
                    r[i]=1;

                    }else{r[i]=0;}
                }
            }
             if(r[0]==0 && r[1]==0 && r[2]==0 && r[3]==0){
            return 0;
            }
        k=k+s;
        }

    }else{return 0;}


    if(r[0]==1 && r[1]==1 && r[2]==1 && r[3]==1){
        return 0;
    }else{return 3;};



};

void StaZ(struct Object * E,double beta, double alpha, double
    s){ // Calculating the stability function for an object E

    E->Z[0]= (double)(-E->ch[3]+(E->ch[2]*beta*3)-(E->ch[1]*
       beta*beta*3)+(E->ch[0]*beta*beta*beta)+(6*s+1)*(E->ch
       [1]-E->ch[0]*beta)*(alpha*alpha))/6;
```

```

    E->Z[1]= (double)(E->ch[2] -((double)2*E->ch[1]*beta)+((
       double)E->ch[0]*beta*beta)-((double)E->ch[0]*alpha*
       alpha))/2;

    return;


}

int EulerChar(int a[4],int b[4]){ // Calculation of the Euler
    Characteristic of two Vector bundles with ch(E_1)=a[4]
   and ch(E_2)=b[4]

    float Chi;  // Euler Characteristic of (E_1,E_2)
    float C[4]; // Chern Character of E_1* X E_2

    C[0]=a[0]*b[0];
    C[1]=b[1]*a[0]-a[1]*b[0];
    C[2]=b[2]*a[0]-2*a[1]*b[1]+a[2]*b[0];
    C[3]=b[3]*a[0]-3*a[1]*b[2]+3*a[2]*b[1]-a[3]*b[0];

    Chi=(C[3]+6*C[2]+11*C[1]+6*C[0])/6; // Theorem 9.3
       Schmidt and Macri Lectures on Bridgeland Stability

    return Chi;
}

struct Object LMut(struct Object E, struct Object F){ // Left
    Mutation Object  L_E(F)

    struct Object L;
    int Chi;
    int j=0;

    Chi=EulerChar(E.ch,F.ch);

    for(j=0; j<=3; j++){
        L.ch[j]=Chi*E.ch[j]- F.ch[j];
```

```
    }

    return L;

};

struct Object RMut(struct Object E, struct Object F){ //
   Right Mutation Object R_E(F)

    struct Object R;
    int Chi;
    int j=0;

    Chi=EulerChar(F.ch,E.ch);

    for(j=0 ; j<=3; j++){
        R.ch[j]=Chi*E.ch[j] - F.ch[j];
    }

    return R;

};
```