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Simple ARS on nonnilpotent, solvable three dimensional Lie groups

Simples ARS em grupos de Lie Solúveis não nilpotentes de dimensão três

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Resumo

O objetivo desta tese é estudar estruturas quase riemannianas simples sobre grupos de Lie solúveis não nilpotentes de dimensão três. Para isso, começamos descrevendo geometrica e algebricamente o locus singular de uma estrutura quase Riemanniana simples nos grupos em questão. E em seguida estabelecemos que tal locus singular é uma subvariedade. Além disso, analisamos como as curvas exponenciais cruzam o locus singular. Em seguida definimos ARS's de rank dois e analisamos isometrias de tais ARS's. O resultado principal mostra que, isometrias de rank dois são necessariamente automorfismos do grupo. Tal resultado permite separar as ARS dos grupos de dimensão três estudados em classes dependendo dos autovalores da matriz associada ao produto semi-direto.

Palavras-chave: Estrutura quase Riemanniana, locus singular, grupo de Lie, condição do posto de Lie, isometrias.

Abstract

The aim of this Thesis is to study the simple almost-Riemannian structure on nonnilpotent, solvable three dimensional Lie Groups. For doing that, we begin by describing geometrically and algebraically the singular locus of a simple almost Riemannian structure in the groups in question. Subsequently we establish that such a singular locus is a submanifold. Moreover, we analyze how exponential curves cross the singular locus. Then we define rank two ARS's and analyze isometries of such ARS's. The main result shows that rank two isometries are necessarily group of automorphisms. This result allows us separating the ARS of the groups of dimension three studied into classes depending on the eigenvalues of the matrix associated with the semi-direct product.

Keywords: Almost-Riemannian structure; singular locus; Lie group; rank condition; isometries.

Contents

Introduction

A simple almost-Riemannian structure, hencefort ARS, on a Lie group G is defined by a linear vector field and $\dim(G) - 1$ left-invariant vector fields. Those vectors fields form a local orthonormal frame and satisfy the Lie bracket generating condition, they can become collinear, and degenerate on some singular set. ARS's are generalized Riemannian structures and are a particular case of rank-varying Sub-Riemannian Structures, e.g [2, 9, 32]. The singular locus is defined as the set of points where the vector fields fail to be linearly independent. It is an analytic and in general is not a subgroup neither a submanifold as was pointed out in the example given in [6] on the nilpotent case. The singular Locus presents a particular behavior related with another important geometric quantities or elements like the metric, the Riemannian area, the curvature. For instance, through the study of the properties of the Laplace-Beltrami operator. The authors in [18] concluded that a quantum particle cannot cross the singular set of an associated ARS and heat cannot flow through the singularity.

We can also relate the singular locus with the singularities of metrics for example, in the Grushin case, when we approach the singular set all Riemannian quantities explode, but geodesics are still well defined and can cross the singular set without singularities [18].

In dimension two, ARS's have been studied in [4, 13, 17, 19]. 2-ARS were introduced in the context of hypoelliptic operator, [22, 23], and they are generalizations of the Grushin plane. Recently many authors are interested in the study of quantum confinement [10].

For 2-dimensional ARS, it was proved in [1] that generically the singular set $\mathcal{Z}$ is an embedded submanifold of dimension 1. For other references of ARS see [3, 14]. This theory also is related with the geometric control theory, [5, 30, 35, 36, 2].

In this work, we investigate the algebraic and geometric stucture of the Singular locus of ARS's on three-dimensional solvable nonnilpotent Lie groups. For these groups we show that the singular locus is a submanifold. In this way we investigate how the exponential flow crosses the the Singular locus. We exhibit a explicit form of the singular locus for each class.

The definition of linear vector field of rank two is given, with this, we define the ARS's of rank two. Moreover, Isometries of ARS's are studied. In this way, the main result of this thesis is stated, that is, we prove that isometries between rank two ARS's are automorphisms. We also extended the results to the connected case.

In this study, we use the classification of the solvable nonilpotent three dimensional Lie groups, for do that, we use the classification of three dimensional Lie algebras. This is because generically a Lie group can be recover from the Lie algebra. In such classification we have five classes of Lie algbras [34].

Chapter I

In this chapter we will explain some general concepts necessary for the understanding of this thesis. We start by defining Linear vector fields on Lie groups and stating their main properties. Next, we define general ARS's on manifolds and subsequently simple ARS on Lie groups. The definition of singular locus is given. We also present the definition of isometries together with the principal results provided in the literature. Finally, we present two main examples of ARS, the first one is a general example, the Grushin plane, and the second one is the Heisenberg Lie group, which is an example where we contrast the difference between the nilpotent and the solvable nonnilpotent Lie group with respect to the geometry of the singular locus.

Chapter II

In this chapter we present the classification of the solvable nonnilpotent three dimensional Lie groups. Then we study the automorphism and derivations of these groups and algebras. Afterwards, an equivalence form of the Lie algebra rank condition is discussed. Simple ARS isometry-related are studied.

Chapter III

In this chapter we study some algebraic and geometric properties of the singular locus. Here, we prove that the singular locus is a submanifold. Moreover, we analyze when the singular locus is a connected subset. We also investigate how the exponential curve crosses the singular locus.

Chapter IV

This chapter present the main result of this work. We show that Isometries between ARS of rank two are automorphisms. To prove such theorem, the fundamental lemma is proved, this lemma give us a sufficient condition for an isometry between ARS to be an automorphism. Next, we study the invariance of the nilradical. Finally, we give the proof of the theorem of isometries.

1 Preliminaries

In this chapter we give the basic definitions needed for the development of this thesis. For more details see [6, 8, 20, 25, 27, 31, 37, 38, 39, 40].
The layout of this chapter is as follows: In section 1.1 the definition of linear vector field is given. In Section 1.2 the general definition of almost Riemannian structure and the simple almost Riemannian structures are presented. Afterwards, the Lie Algebra rank condition is discussed. The definition of simple ARS in a connected Lie group is introduced in Section 1.3. Here we also give the definitions of singular locus and the singularities of a linear vector field. In Section 1.4 the definition of isometries of ARS is given and some theorems about the isometries of ARS's are also provided. Section 1.5 is concerned to the extension of the results of ARS provided in a simply connected Lie group to the connected one. Finally, in Section 1.6 we present two examples of ARS. The first example, called the Grushin plane, is a nontrivial two dimensional ARS. This example presents an interesting phenomenon with respect to the singular locus and the riemannian quantities associated to this group. The second one is on the Heisenberg group. In this group we show an example where the singular locus is not a submanifold. As we will see in Chapter 3 this is not the case in three nonnilpotent solvable Lie groups.

1.1 Linear vector fields

In this section the definition of linear vector fields and some of their properties are recalled. More details can found in [8, 27, 28].
Let G be a connected Lie group and $\mathfrak{g}$ its Lie Algebra, identified with the set of left-invariant vector fields. A vector field $\mathcal{X}$ on G is said to be *linear* if its flow $(\varphi_t)_{t\in\mathbb{R}}$ is a one-parameter group of automorphisms, i.e.,

$$\text{for all} \quad t \in \mathbb{R} \quad \varphi_t(gh) = \varphi_t(g)\varphi_t(h)$$

A linear vector field is consequently complete. Indeed, by the definition of one-parameter group of automorphism its integral curves are defined in the whole real line.
Associate to $\mathcal{X}$ there is a derivation $\mathcal{D}$ of $\mathfrak{g}$ defined by the formula

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathfrak{g},$$

which satisfies

$$(d\varphi_t)_e = e^{t\mathcal{D}}, \text{ for all } t \in \mathbb{R}. \tag{1.1}$$

1.2 Almost-Riemannian structures (ARS)

Here we present the definitions of ARS and simple ARS. For all that concern general sub-Riemannian geometry, including the almost-Riemannian case, the reader is refered to [3].

Definition 1.2.1. An almost-Riemannian structure (ARS) on a smooth n-dimensional manifold M is a triple $(E, f, \langle .,. \rangle)$ where:

1. E is a rank n vector bundle on M;
2. $f: \; E \longmapsto TM$ is a morphism of vector bundles;
3. $(E, \langle .,. \rangle)$ is an Euclidean bundle, that is $\langle .,. \rangle_q$ is an inner product on the fiber E_q of E, smoothly varying w.r.t. q;

assumed to satisfy the following properties:

(i) The set of points $q \in M$ such that the restriction of f to E_q is onto is a proper open and dense subset of M;

(ii) The modulus $\overline{\Xi}$ of vector fields of M, defined as the image by f of the modulus of smooth sections of E satisfies the Lie algebra rank condition (see definition below).

Definition 1.2.2. Let $\mathcal{F}$ be a family of smooth vector fields on a smooth manifold M of dimension n. We say that $\mathcal{F}$ satisfies the *Lie algebra rank condition* on a point $p_0 \in M$ if

$$T_{p_0}M = \mathrm{Span}\{X(p_0) : X \in \mathcal{LA}(\mathcal{F})\},$$

where $\mathcal{LA}(\mathcal{F})$ denotes the Lie algebra of the vector fields generated by $\mathcal{F}$. We say that the family $\mathcal{F}$ satisfies the Lie algebra rank condition (LARC) if it satisfies the previous of all $p_0 \in M$.

In other words, the LARC asks that the family of vector fields $\mathcal{F}$ of M spans in each point of the manifold M a Lie algebra of the same dimension as the tangent space in that point.

Definition 1.2.3. The *singular locus*, denoted by $\mathcal{Z}$, is the set of points of M where the rank of $f(E_q) = \Xi_q$ is less than n. If $\mathcal{Z}$ is empty the structure is Riemannian.

Remark 1.2.4. The structure is trivializable if $(E, \langle .,. \rangle)$ is isomorphic to the trivial Euclidean bundle $M \times \mathbb{R}^n$. In that case we can choose an orthonormal basis $(e_1, \ldots, e_n)$ on $\mathbb{R}^n$, and define n vector fields on M by $f_i(q) = f(q, e_i)$, $i = 1, \ldots, n$. The set $(f_1, \ldots, f_n)$ is an orthonormal frame on $M \backslash \mathcal{Z}$.

1.3 Simple ARS's

Let G a connected Lie group. A simple ARS on G defined by a set of n vector fields

$$\{\mathcal{X}, Y_1, \ldots, Y_{n-1}\},$$

where

(i) $\mathcal{X}$ is linear;

(ii) $Y_1, \ldots, Y_{n-1}$ are linearly independent left-invariant vector fields;

(iii) $n = \dim G$ and the rank of $\mathcal{X}, Y_1, \ldots, Y_{n-1}$ is full on a nonempty subset of G;

(iv) the set $\{\mathcal{X}, Y_1, \ldots, Y_{n-1}\}$ satisfies the LARC.

The metric is defined by declaring the frame $\{\mathcal{X}, Y_1, \ldots, Y_{n-1}\}$ to be ortonormal.

Remark 1.3.1. An equivalent definition of ARS on a Lie group is defined by $(n-1)-$dimensional left-invariant distribution $\Delta = \text{span}\{Y_1, Y_2 \ldots Y_{n-1}\}$, a left-invariant Euclidean metric on Δ and a linear vector field $\mathcal{X}$ which satisfies (iii) and (iv) of the above definition, In short, we write an ARS as $\Sigma = \{\mathcal{X}, \Delta\}$. The metric of the ARS is defined by declaring $\mathcal{X}$ unitary and orthogonal a Δ.

Now, Since we represent $\Delta := \Delta(e)$, where e is the identity element, then we write the next remark to establish our study in any element $g \in G$.

Remark 1.3.2. We denote by Δ^L the left-invariant distribution on G which is defined as the map $\Delta^L : G \to TG$ given by

$$\Delta^L(g) = \left(dL_g\right)_e \Delta,$$

where $\Delta \subset \mathfrak{g}$ is a $(n-1)-$ dimensional vector subspace. We can endow Δ^L with a left-invariant Euclidean metric by considering on Δ an inner product $\langle \cdot, \cdot \rangle$ and defining

$$\forall X, Y \in \Delta^L(g), \quad \langle X, Y \rangle_g := \left\langle \left(dL_{g^{-1}}\right)_g X, \left(dL_{g^{-1}}\right)_g Y \right\rangle.$$

Therefore, for this case the singular locus is given by

$$\mathcal{Z} = \{g \in G : \mathcal{X}(g) \in \Delta^L(g)\}. \tag{1.2}$$

We denote by $\mathcal{Z}_{\mathcal{X}}$ the set of the singular points of the linear vector field $\mathcal{X}$, that is,

$$\mathcal{Z}_{\mathcal{X}} = \{g \in G; \quad \mathcal{X}(g) = 0\}.$$

In particular $\mathcal{Z}_{\mathcal{X}} \subset \mathcal{Z}$.
In [6, proposition 2] was proven that $\mathcal{Z}_{\mathcal{X}}$ is a closed subgroup with Lie algebra given by $\ker D$. The singular locus is an analytic subset of G. According to (iii) it is not equal to G, and by analiticy its interior is empty. Since $\mathcal{X}(e) = 0$ the singular locus is not an empty set. Finally $G/\mathcal{Z}$ is an open, dense and proper subset of G. The points of $G/\mathcal{Z}$ will be called the Riemannian points.

To have a practical way to verify LARC. The next remark establish a necessary and sufficient condition to the LARC.

Remark 1.3.3. Notice that if $[\Delta, \Delta] \subset \Delta$ and $D(\Delta) \subset \Delta$, then the fact that the Lie algebra generated by $\mathcal{X}, Y_1, \ldots, Y_{n-1}$ is equal to $\mathbb{R}\mathcal{X} \oplus \Delta$ [6], together with $\mathcal{X}(e) = 0$ imply that Σ does not satisfies the LARC at the identity. Consequently the LARC implies that at least one of the following conditions holds

(i) $[\Delta, \Delta] \nsubseteq \Delta$,

(ii) $D(\Delta) \nsubseteq \Delta$,

Reciprocally, if (i) or (ii) is satisfied, then $\Sigma = \{\mathcal{X}, \Delta\}$ satisfies the LARC at identity element which by translations implies the LARC at any point.

1.4 Isometries of ARS

In this section we define isometries of simple ARS.

Definition 1.4.1. Let Σ be a simple ARS on G. The almost-Riemannian norm on G defined by Σ is

$$\text{For } X \in T_gG, \; \|X\|_{\Sigma,g} = \min\left\{\sqrt{v^2 + \sum_1^n u_i^2}; v\mathcal{X}_g + u_1Y_1(g) + \cdots + u_{n-1}Y_{n-1}(g) = X\right\}. \quad (1.3)$$

It is infinite if the point g belongs to the singular locus and X does not belong to Δ.

Definition 1.4.2. A diffeomorphism $\psi : G \to G$ between two ARS's Σ_1 and Σ_2 on a connected Lie group G is an isometry if

$$\forall g \in G, v \in T_gG \quad \|(d\psi)_g v\|_{\Sigma_2,\phi(g)} = \|v\|_{\Sigma_1,g}.$$

We denote by $\mathrm{Iso}_G(\Sigma_1; \Sigma_2)$ the group of isometries between the simple ARS's Σ_1 and Σ_2.
The following results can be found in [29].

Theorem 1.4.3. Any isometry $\psi \in \mathrm{Iso}_G(\Sigma_1; \Sigma_2)$ can be decomposed as $\psi = L_g \circ \psi_0$, where $g \in \mathcal{Z}_2$ and $\psi_0 \in \mathrm{Iso}_G(\Sigma_1; \Sigma_2)_0$. Where $\mathcal{Z}_2$ is the singular locus of Σ_2 and

$$\mathrm{Iso}_G(\Sigma_1; \Sigma_2)_0 = \{\psi_0 \in \mathrm{Iso}_G(\Sigma_1; \Sigma_2); \psi_0(e) = e\}.$$

Therefore, in order to understand the group $\mathrm{Iso}_G(\Sigma_1; \Sigma_2)$ it is enough to analyze the singular locus $\mathcal{Z}_2$ of Σ_2 and the subgroup $\mathrm{Iso}_G(\Sigma_1; \Sigma_2)_0$.

Theorem 1.4.4. For any $\psi \in \mathrm{Iso}_G(\Sigma_1; \Sigma_2)_0$ it holds that

1. $(d\psi)_g \Delta_1^L(g) = \Delta_2^L(\psi(g))$.
2. $\psi \circ \varphi_s^1 = \varphi_{\pm s}^2 \circ \psi$ where $\{\varphi_s^i\}_{s \in \mathbb{R}}$ is the flow associated with $\mathcal{X}_i$ for $i = 1, 2$.
3. $\psi(\mathcal{Z}_1) = \mathcal{Z}_2$.
4. $\psi(\mathcal{Z}_{\mathcal{X}_1}) = \mathcal{Z}_{\mathcal{X}_2}$.

Remark 1.4.5. Through this work and without loss of generality we use the the positive sign of the flow given by 2 in Theorem 1.4.4, that is, we employ the next formula

$$\psi \circ \varphi_s^1 = \varphi_s^2 \circ \psi. \tag{1.4}$$

Remark 1.4.6. Since the restriction

$$\psi : G \backslash \mathcal{Z}_1 \to G \backslash \mathcal{Z}_2,$$

is an isometry between Riemannian manifolds, Myers-Steenrod Theorem [33] implies that this restriction is of class $\mathcal{C}^\infty$ on the connected components of $G \backslash \mathcal{Z}_1$

1.5 Simply connected case to the connected one

In this section we show that the main results of this work can be proved only for connected simply connected groups.

Let $\Sigma = \{\mathcal{X}, \Delta\}$ be a simple ARS on a connected Lie group G. If $\widetilde{G}$ is the simply connected covering of G, we define the lift $\tilde{\Sigma}$ of Σ to be the simple ARS

$$\tilde{\Sigma} = \left\{\widetilde{\mathcal{X}}, \widetilde{\Delta}\right\}, \quad \forall g \in \widetilde{G},$$

where $\tilde{\mathcal{X}}$ and $\mathcal{X}$ are π -related, for the canonical projection $\pi : \widetilde{G} \to G$, that is, for

$$(d\pi)_g \widetilde{\mathcal{X}}(g) = \mathcal{X}(\pi(g)).$$

We also define the Left-invariant vector field in the the left-invariant distribution $\widetilde{\Delta}$ by consider $\widetilde{Y} \in \widetilde{\Delta}$ and $Y \in \Delta$ to be π conjugated.
The following proposition shows that any isometry $\psi \in \mathrm{Iso}_G(\Sigma;\Sigma)_0$ can be lifted to an isometry $\widetilde{\psi}$ of $\mathrm{Iso}_{\widetilde{G}}\left(\widetilde{\Sigma},\widetilde{\Sigma}\right)_0$ satisfying $\widetilde{\psi}\circ\pi = \pi\circ\psi$. Moreover, $\psi \in \mathrm{Aut}(G)$ as soon as $\widetilde{\psi} \in \mathrm{Aut}\left(\widetilde{G}\right)$

Proposition 1.5.1. Let G be a connected Lie group ans consider Σ_1 and Σ_2 to be simple ARS on G. For any isometry $\psi : G \to G$ there exist a unique isometry $\widetilde{\psi} : \widetilde{G} \to \widetilde{G}$ between the lifts $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ of Σ_1 and Σ_2, respectively, satisfying

$$\pi\circ\widetilde{\psi} = \psi\circ\pi.$$

In particular,

$$\widetilde{\psi} \in \mathrm{Aut}(\widetilde{G}) \Longrightarrow \psi \in \mathrm{Aut}(G).$$

Proof. Existence, uniqueness and the commutation $\pi\circ\widetilde{\psi} = \psi\circ\pi$ follows direct from the fact that $\widetilde{G}$ is simply connected and π a covering map [38]. Moreover, the fact that π is a local diffeomorphism and ψ is a difeomorphims implies that $\widetilde{\psi}$ is a diffeormorphism and we only have to show that $\widetilde{\psi}$ is in fact an isometry between the lifts $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ of Σ_1 and Σ_2. However,

$$\pi\circ\widetilde{\psi} = \psi\circ\pi \quad \Longrightarrow \quad \pi_*\circ\widetilde{\psi}_* = \psi_*\circ\pi_*,$$

where the subscript notation, for example, in π_* denote the differential. Therefore, we have

$$\begin{aligned}
\pi_*\left(\widetilde{\psi}_*\widetilde{\mathcal{X}}_1\right) =& \psi_*\pi_*\left(\widetilde{\mathcal{X}}_1\right)\\
=& \psi_*\mathcal{X}_1\pi\\
=& \mathcal{X}_2\psi\pi\\
=& \mathcal{X}_2\pi\widetilde{\psi}\\
=& \pi_*\left(\widetilde{\mathcal{X}}_2\widetilde{\psi}\right)\\
\Longrightarrow \quad & \widetilde{\psi}_*\widetilde{\mathcal{X}}_1 = \widetilde{\mathcal{X}}_2\widetilde{\psi},
\end{aligned}$$

since π_* is an isomorphism.
In order to show that $\widetilde{\psi}$ is an isometry, it is enough to show that $\left(d\widetilde{\psi}\right)_g\widetilde{\Delta}_1(g) = \widetilde{\Delta}_2\left(\widetilde{\psi}(g)\right)$ and that the restriction $\left(d\widetilde{\psi}\right)_g\Big|_{\widetilde{\Delta}_1(g)} : \widetilde{\Delta}_1(g) \to \widetilde{\Delta}_2\left(\widetilde{\psi}(g)\right)$ is an isometry.
For the first part, note that if $X \in \widetilde{\Delta}_1(g)$, then

$$\begin{aligned}
(d\pi)_{\widetilde{\psi}(g)}\left(d\widetilde{\psi}\right)_g X(g) =& d\left(\pi\circ\widetilde{\psi}\right)_g X(g)\\
=& d\left(\psi\circ\pi\right)_g X(g)\\
=& \left(d\psi\right)_{\pi(g)}\left(d\pi\right)_g X(g).
\end{aligned}$$

Since $(d\pi)_g X(g) \in \Delta_1(\pi(g))$ we get

$$\begin{aligned}(d\psi)_{\pi(g)} \Delta_1(\pi(g)) &= \Delta_2(\psi(\pi(g))) \\ &= \Delta_2\left(\pi\left(\widetilde{\psi}(g)\right)\right) \\ &= (d\pi)_{\widetilde{\psi}(g)} \Delta_2\left(\widetilde{\psi}(g)\right).\end{aligned}$$

The fact that $(d\pi)_g$ is isomorphism implies $\left(d\widetilde{\psi}\right)_g X(g) \in \Delta_2\left(\widetilde{\psi}(g)\right)$.
Now, we prove that

$$\left\|\left(d\widetilde{\psi}\right)_g X(g)\right\|_{\widetilde{\Sigma}_1,\widetilde{\psi}(g)} = \|X(g)\|_{\widetilde{\Sigma}_1,g} \quad \text{for all } g \in \widetilde{G},\ X \in \widetilde{\Delta}_1.$$

First notice that, for any $Y \in \widetilde{\Delta}_i,\ i = 1, 2$, we have that

$$\begin{aligned}\|(d\pi)_g Y(g)\|_{\Sigma_i,\pi(g)} &= \|(d\pi)_g (dL_g)_e Y\|_{\Sigma_i,\pi(g)} \\ &= \|d(\pi \circ L_g)_e Y\|_{\Sigma_i,\pi(g)} \\ &= \|d\left(L_{\pi(g)} \circ \pi\right)_e Y\|_{\Sigma_i,\pi(g)} \quad \text{since } \pi \text{ is homomorphism} \\ &= \|d\left(L_{\pi(g)}\right)_e (d\pi)_e Y\|_{\Sigma_i,\pi(g)} \\ &= \|(d\pi)_e Y\|_{\Sigma_i,\pi(e)} \quad \text{since the metric is invariant by left translations} \\ &= \|Y\|_{\Sigma_i,e} \quad \text{since } (d\pi)_e = \mathrm{id}_{\mathfrak{g}} \\ &= \|Y(g)\|_{\widetilde{\Sigma}_i,g}.\end{aligned}$$

Showing that $(d\pi)_g : \widetilde{\Delta}_i(g) \to \Delta_i(g)$ is an isometry of the left-invariant metrics. Therefore, for any $Y \in \widetilde{\Delta}_1$

$$\begin{aligned}\left\|\left(d\widetilde{\psi}\right)_g Y(g)\right\|_{\widetilde{\Sigma}_2,\widetilde{\psi}(g)} &= \left\|(d\pi)_{\widetilde{\psi}(g)} \left(d\widetilde{\psi}\right)_g Y(g)\right\|_{\Sigma_2,\pi(\widetilde{\psi}(g))} \\ &= \left\|d\left(\pi \circ \widetilde{\psi}\right)_g Y(g)\right\|_{\Sigma_2,\pi(\widetilde{\psi}(g))} \\ &= \left\|d(\psi \circ \pi)_g Y(g)\right\|_{\Sigma_2,\pi(\widetilde{\psi}(g))} \\ &= \left\|(d\psi)_{\pi(g)} (d\pi)_g Y(g)\right\|_{\Sigma_2,\psi(\pi(g))} \\ &= \left\|(d\pi)_g Y(g)\right\|_{\Sigma_1,\pi(g)} \quad \text{since } \psi \in \mathrm{Iso}_G(\Sigma_1, \Sigma_2)_0 \\ &= \|Y(g)\|_{\widetilde{\Sigma}_1,g}.\end{aligned}$$

Showing that $\widetilde{\psi} \in \mathrm{Iso}_{\widetilde{G}}\left(\widetilde{\Sigma}_1; \widetilde{\Sigma}_2\right)_0$ as stated.

Now, assume that $\widetilde{\psi} \in \mathrm{Aut}(\widetilde{G})$. Let $x_i \in G$ and $\widetilde{x}_i \in \widetilde{G}$ such that $\pi\left(\widetilde{x}_i\right) = x_i$, for $i = 1, 2$. Then,

$$\begin{aligned}
\psi\left(x_1 x_2\right) =& \psi\left(\pi\left(\widetilde{x}_1\right)\pi\left(\widetilde{x}_2\right)\right)\\
=& \psi\left(\pi\left(\widetilde{x}_1\widetilde{x}_2\right)\right)\\
=& \pi\left(\widetilde{\psi}\left(\widetilde{x}_1\widetilde{x}_2\right)\right)\\
=& \pi\left(\widetilde{\psi}\left(\widetilde{x}_1\right)\widetilde{\psi}\left(\widetilde{x}_2\right)\right)\\
=& \pi\left(\widetilde{\psi}\left(\widetilde{x}_1\right)\right)\pi\left(\widetilde{\psi}\left(\widetilde{x}_2\right)\right)\\
=& \psi\left(\pi\left(\widetilde{x}_1\right)\right)\psi\left(\pi\left(\widetilde{x}_2\right)\right)\\
=& \psi\left(x_1\right)\psi\left(x_2\right)
\end{aligned}$$

showing that $\psi \in \mathrm{Aut}(G)$ concluding the proof. □

Proposition 1.5.2. Let Σ be an ARS on G and $\widetilde{\Sigma}$ its lift to the simply connected covering $\widetilde{G}$ of G. If $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$ stands for the singular locus of Σ and $\widetilde{\Sigma}$, respectively, then

$$\pi\left(\widetilde{\mathcal{Z}}\right) = \mathcal{Z}, \quad \text{and} \quad \pi^{-1}\left(\mathcal{Z}\right) = \widetilde{\mathcal{Z}},$$

where $\pi : \widetilde{G} \to G$ stands for the canonical projection.

Proof. It follows from the equivalence

$$g \in \widetilde{\mathcal{Z}} \Leftrightarrow \widetilde{\mathcal{X}}(g) \in \widetilde{\Delta}(g) \Leftrightarrow (d\pi)_g \widetilde{\mathcal{X}}(g) \in (d\pi)_g \widetilde{\Delta}^L(g) \Leftrightarrow \mathcal{X}\left(\pi(g)\right) \in \Delta^L\left(\pi(g)\right),$$

where in the second equivalence we use that π is a local diffeomorphism. □

1.6 Examples of ARS

The purpose of this section is to give examples of ARS's. The first one is the classical example of the two dimensional ARS, namely the Grushin plane. The second one is on the three dimensional nilpotent Heisenberg Lie group and it is important to show that in general the singular locus does not need to be a subgroup or even a submanifold, note that such phenomenon which occurs in the nilpotent case, does not in the solvable nonilpotent one, which will be approached in this work.

Example 1.6.1. The Grushin Plane.
This example was named after Grushin, who studied in [22, 23, 24] the analytical properties of the operator $\partial_x^2 + x^2\partial_y^2$ and its generalizations. The models were introduced in the context of hypoelliptic operators, and appeared in problems of population transfer in quantum systems [15, 16, 21]. Moreover, it has applications to orbital transfer in space mechanics [11, 12].

In this example the manifold is $M = \mathbb{R}^2$, which is an abelian Lie group. A local orthonormal basis is given by

$$Y(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} := \partial_x, \quad \mathcal{X}(x,y) = \begin{pmatrix} 0 \\ x \end{pmatrix} := x\partial_y. \tag{1.5}$$

Since $\mathcal{X}$ is linear vector field and Y is left invariant one, the Grushin plane is a simple ARS. The singular locus is the line $\{x = 0\}$. Indeed, let a , b real numbers, by definition the singular locus is the set where the vector fields $\mathcal{X}$ and Y are linearly dependent. Therefore

$$a\mathcal{X}(x,y) + bY(x,y) = \begin{pmatrix} b \\ ax \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.6}$$

implies $b = 0$, and $ax = 0$. Notice that if $a = 0$ then the vector fields are linearly independent, thus necessarily $x = 0$.
The Riemannian metric, the Riemannian area and the Gaussian curvature are given respectively by:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}, \quad d\omega = \frac{1}{|x|}dx\,dy, \quad K = -\frac{2}{x^2}. \tag{1.7}$$

Observe that the Riemannian quantities namely, the metric, the Riemannian area, the curvature, explode while approaching $\mathcal{Z}$, [18].

Example 1.6.2. The Heisenberg group

The next example shows that $\mathcal{Z}$ does not need to be a subgroup or even a submanifold [6],[26]. Let G the simply connected Heisenberg Lie group of dimension three

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in \mathbb{R} \right\}.$$

Its Lie algebra $\mathfrak{g}$ is generated by X, Y, Z is such that $[X,Y] = Z$ and $[X,Z] = [Y,Z] = 0$. In natural coordinates the left invariant vector fields can be written as:

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

The derivations of $\mathfrak{g}$ are endomorphisms $\mathcal{D}$ whose matrix in the basis $\{X, Y, Z\}$ have the form:

$$\mathcal{D} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix},$$

and the associated linear vector field is:

$$\mathcal{X}(g) = (ax+by)\frac{\partial}{\partial x} + (cx+dy)\frac{\partial}{\partial y} + \left(ex+fy+(a+d)z+\frac{1}{2}cx^2+\frac{1}{2}by^2\right)\frac{\partial}{\partial z}.$$

Let us to consider the ARS, $\Sigma = \{\mathcal{X}, \Delta\}$, where $\Delta = \operatorname{span}\{X, Y\}$, then

$$\mathcal{Z} = \left\{ex+fy+(a+d)z-\frac{1}{2}cx^2+\frac{1}{2}by^2-dxy=0\right\}.$$

The singular locus defined by these quadratic forms need not be subgroups, not even submanifolds. For instance, by considering

$$\mathcal{D} = \begin{pmatrix} a & b & 0 \\ 0 & -a & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

we obtain

$$\mathcal{Z} = \left\{y+\frac{1}{2}by^2+axy=0\right\}. \tag{1.8}$$

Which is not a submanifold, since it is given by the intersection of two planes, (see figure(1)). In particular $\mathcal{Z}$ is also not a Lie subgroup.

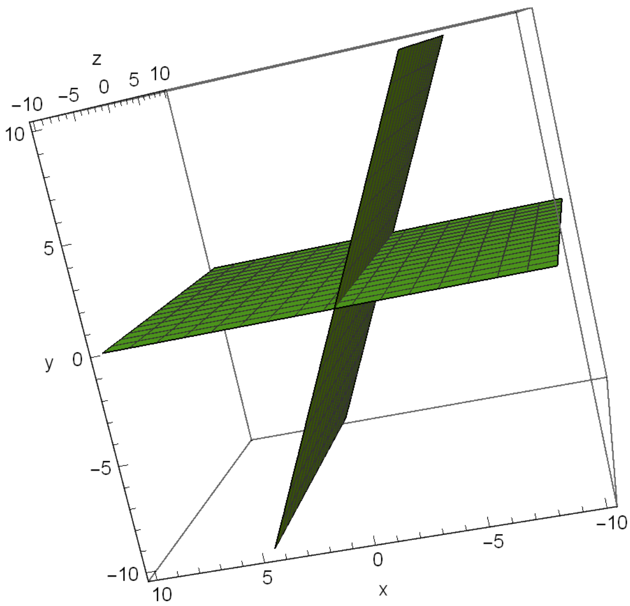

Figure 1 – singular locus (1.8), with $b = 2 \quad a = 1$.

2 Nonnilpotent, solvable three dimensional Lie groups

In this chapter we introduce the groups we are interested, namely the solvable nonnilpotent three dimensional Lie groups [7]. Also, several tools, which will be used in the development of this work, are presented here.

This chapter is arranged as follows: In section 2.1 the classification of the solvable nonnilpotent three dimensional Lie group is presented. Section 2.2 is about the automorphisms and derivations of these groups and algebras. In section 2.3 we introduce an operator, which plays an important role in the main results and state its main properties. By using such operator, the explicit form of the automorphisms is established and in particular the flow of a linear vector field is exhibit. Finally, in Section 2.4 the LARC and simple ARS isometry-related results are presented.

2.1 Classification of three-dimensional nonnilpotent solvable Lie groups

In this section we present the classification of the solvable nonnilpotent three dimensional Lie groups and algebras. By the results in Section 1.5, such classification will only consider connected, simply connected groups.

According to the Lie theory [25, 31, 38], for a given Lie algebra $\mathfrak{g}$ there exist, up to isomorphisms, a unique simply connected, connected Lie group $\widetilde{G}$ which Lie algebra is $\mathfrak{g}$.

For the classification of the nonnilpotent solvable three dimensional Lie groups, we begin with the classification of their respective Lie algebras [34].

We have three classes of Lie algebras, each of then can be written as the semi-direct product $\mathfrak{g}(\theta) = \mathbb{R} \times_\theta \mathbb{R}^2$, where θ is a two dimensional matrix with one of the following forms

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \text{ and } (\gamma \in \mathbb{R}, |\gamma| \leqslant 1) \quad \text{or} \quad \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix} \text{ and } \gamma \in \mathbb{R}. \tag{2.1}$$

The bracket in such algebras are given by

$$[(a_1, w_1), (a_2, w_2)] = (0, \theta(a_1 w_2 - a_2 w_1)), \tag{2.2}$$

and is therefore determined by the relation

$$[(a, 0), (0, w)] = (0, a\theta w), \quad a \in \mathbb{R} \quad w \in \mathbb{R}^2. \tag{2.3}$$

For each Lie algebra $\mathfrak{g}(\theta)$ the associated simply connected Lie groups, are given also by the semi-direct product $G = \mathbb{R} \times_\rho \mathbb{R}^2$ with $\rho_t = e^{t\theta}$, and the product of group given by

$$(t_1, v_1)(t_2, v_2) = (t_1 + t_2, v_1 + \rho_{t_1} v_2), \quad (t_1, v_1), (t_2, v_2) \in \mathbb{R} \times_\rho \mathbb{R}^2.$$

2.2 Automorphism and Derivations of three dimensional Lie algebras

In this section the automorphisms and the derivations of the solvable nonnilpotent three dimensional Lie group are presented.
We have that automorphisms and derivations of a Lie algebra $\mathfrak{g}$ are linear maps, that is $\phi \in \mathrm{Gl}(\mathfrak{g})$ and $\mathcal{D} \in \mathfrak{gl}(\mathfrak{g})$, respectively, which satisfy the following relationships

$$\forall X, Y \in \mathfrak{g}, \quad \phi[X,Y] = [\phi X, \phi Y] \quad \text{and} \quad \mathcal{D}[X,Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y].$$

Consider $M \in \mathfrak{gl}(\mathfrak{g}(\theta))$ and suppose that $M\mathbb{R}^2 \subset \mathbb{R}^2$,where here $\mathbb{R}^2 \cong \{0\} \times \mathbb{R}^2$. Then M can be written in the canonical basis, as

$$M = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix}, \eta \in \mathbb{R}^2, \varepsilon \in \mathbb{R} \text{ and } P \in \mathfrak{gl}(2, \mathbb{R}). \tag{2.4}$$

In fact, let $\{e_2, e_3\}$ a basis for $\mathbb{R}^2$ and complete a basis to a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}(\theta)$, writing $M = \{C^i_j\}$ then $M(e_j) = \sum_{i=1}^{j=3} C^i_j e_i$ for $j = 1, 2, 3$. Since that by assumption $M(\mathbb{R}^2) \subset \mathbb{R}^2$ necessarily we obtain $C^1_j = 0$ for $j = 2, 3$. Therefore the matrix M is given by equation (2.4).
Now the fact that $\mathbb{R}^2$ is the nilradical of $\mathfrak{g}(\theta)$, implies that it is invariant by automorphisms and derivations. Consequently, any $\phi \in \mathrm{Aut}\,(\mathfrak{g})$ can be written in the form (2.4).
Applying such property to the relation (2.3) gives us that

$$\phi[(a, 0), (0, w)] = [\phi(a, 0), \phi(0, w)], \quad \forall\, a \in \mathbb{R}, \; w \in \mathbb{R}^2 \tag{2.5}$$

and hence

$$\phi = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}(\theta)) \text{ if and only if } P\theta = \varepsilon\theta P. \tag{2.6}$$

In fact, since $[(a, 0), (0, w)] = (0, a\theta w)$ the left-hand side of (2.5) gives rise to

$$\begin{aligned} \phi[(a, 0), (0, w)] =& \phi(0, a\theta w) \\ =& \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix} \begin{pmatrix} 0 \\ a\theta w \end{pmatrix} \\ =& (0, aP\theta w) \end{aligned}$$

For the right-hand side of (2.5), we have that

$$\phi(a, 0) = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = (a\varepsilon, a\eta),$$

and

$$\phi(0, w) = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix} \begin{pmatrix} 0 \\ w \end{pmatrix} = (0, Pw).$$

Therefore

$$\begin{aligned}[\phi(a,0),\phi(0,w)] =&[(a\varepsilon,a\eta),(0,Pw)]\\ =&[(a\varepsilon,0)+(0,a\eta),(0,Pw)]\\ =&[(a\varepsilon,0),(0,Pw)]+[(0,a\eta),(0,Pw)]\\ =&(0,a\varepsilon\theta Pw)+(0,0)\\ =&(0,a\varepsilon\theta Pw).\end{aligned}$$

Now, by equaliting the left-hand side and the right-hand side of (2.5), we obtain for every $a\in\mathbb{R}$ $w\in\mathbb{R}^2$

$$(0,aP\theta w)=(0,a\varepsilon\theta Pw), \tag{2.7}$$

and then

$$P\theta=\varepsilon\theta P.$$

Similarly, if $\mathcal{D}\in \mathrm{Der}(\mathfrak{g}(\theta))$ the relation

$$\mathcal{D}[(a,0),(0,w)]=[\mathcal{D}(a,0),(0,w)]+[(a,0),\mathcal{D}(0,w)],\quad \forall\, a\in\mathbb{R}\ w\in\mathbb{R}^2 \tag{2.8}$$

gives us that

$$\mathcal{D}=\begin{pmatrix}\varepsilon & 0\\ \eta & A\end{pmatrix}\in \mathrm{Der}(\mathfrak{g}(\theta))\quad\Longleftrightarrow\quad A\theta-\theta A=\varepsilon\theta.$$

Indeed, since $[(a,0),(0,w)]=(0,a\theta w)$ the left-hand side of (2.8) gives rise to

$$\mathcal{D}[(a,0),(0,w)]=\mathcal{D}(0,a\theta w)=(0,aA\theta w).$$

On the other hand, $\mathcal{D}(a,0)=(a\varepsilon,a\eta)$ and $\mathcal{D}(0,w)=(0,Aw)$. Then, we get

$$\begin{aligned}[\mathcal{D}(a,0),(0,w)] =&[(a\varepsilon,a\eta),(0,w)]\\ =&[(a\varepsilon,0)+(0,a\eta),(0,w)]\\ =&[(a\varepsilon,0),(0,w)]+[(0,a\eta),(0,w)]\\ =&(0,a\varepsilon\theta w)+(0,0)\\ =&(0,a\varepsilon\theta w)\end{aligned}$$

and

$$\begin{aligned}[(a,0),\mathcal{D}(0,w)] =&[(a,0),(0,Aw)]\\ =&(0,a\theta Aw).\end{aligned}$$

Hence, the right-hand side of the equation (2.8) turns out to be

$$\begin{aligned}[\mathcal{D}(a,0),(0,w)]+[(a,0),\mathcal{D}(0,w)] =&(0,a\varepsilon\theta w)+(0,a\theta Aw)\\ =&(0,a\varepsilon\theta w+a\theta Aw).\end{aligned}$$

As a result, by equaliting the left-hand side and the right-hand side of (2.8), we get that for every $a \in \mathbb{R}, \;\; v \in \mathbb{R}^2$

$$(0, aA\theta w) = (0, a\varepsilon\theta w + a\theta A w), \tag{2.9}$$

and by choosing $a \neq 0$, we obtain $A\theta = \varepsilon\theta + \theta A$, or equivalently

$$A\theta - \theta A = \varepsilon\theta.$$

The previous calculations imply the following:

Proposition 2.2.1. For the Lie algebra $\mathfrak{g}(\theta)$ it holds that

$$\mathrm{Der}(\mathfrak{g}(\theta)) = \left\{ \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix}, \xi \in \mathbb{R}^2, A \in \mathfrak{gl}(2, \mathbb{R}), \text{ with } A\theta = \theta A \right\},$$

and

$$\mathrm{Aut}(\mathfrak{g}(\theta)) = \left\{ \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix}, \eta \in \mathbb{R}^2, P \in \mathrm{Gl}\left(\mathbb{R}^2\right), \text{ with } P\theta = \varepsilon\theta P \right\},$$

where $\varepsilon = 1$ when $\mathrm{tr}(\theta) \neq 0$ or $\varepsilon \in \{-1, 1\}$ if $\mathrm{tr}\,\theta = 0$.

Proof. Based on our previous calculations

$$\phi = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}(\theta)) \quad \text{ if and only if } \quad P\theta = \varepsilon\theta P.$$

Furthermore, since the map ϕ is invertible we get that $\varepsilon \det P = \det \phi \neq 0$ showing that P is also invertible. Consequently

$$P\theta = \varepsilon\theta P \quad \Longleftrightarrow \quad \varepsilon\theta = P\theta P^{-1}\ . \tag{2.10}$$

Assuming $\mathrm{tr}\,\theta \neq 0$ gives us that

$$\varepsilon\,\mathrm{tr}(\theta) = \mathrm{tr}\left(P\theta P^{-1}\right) = \mathrm{tr}(\theta) \quad \Longrightarrow \quad \varepsilon = 1 \quad \text{ and } \quad P\theta = \theta P.$$

On the other hand, $\mathrm{tr}(\theta) = 0$ implies necessarily that

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{ or } \quad \theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and in both cases $\det(\theta) \neq 0$. By applying the determinant function to the relation (2.10) gives us that

$$\varepsilon = 1 \quad \text{and} \quad P\theta = \theta P \quad \text{or} \quad \varepsilon = -1 \quad \text{and} \quad P\theta = -\theta P.$$

For a derivation $\mathcal{D} \in \mathrm{Der}(\mathfrak{g}(\theta))$ we have that

$$\mathcal{D} = \begin{pmatrix} \varepsilon & 0 \\ \xi & A \end{pmatrix} \in \mathrm{Der}(\mathfrak{g}(\theta)) \quad \text{if and only if} \quad A\theta - \theta A = \varepsilon\theta.$$

Therefore, $\operatorname{tr}\theta \neq 0$ gives us that

$$\varepsilon \operatorname{tr}\theta = \operatorname{tr}(A\theta - \theta A) = 0$$

implying that $\varepsilon = 0$.
On the other hand, if $\operatorname{tr}\theta = 0$ then necessarily $\det\theta \neq 0$ and

$$\begin{aligned} & A\theta - \theta A = \varepsilon\theta \\ \implies \quad & \varepsilon \mathrm{id}_{\mathbb{R}^2} = A - \theta A \theta^{-1} \\ \implies \quad & 2\varepsilon = \operatorname{tr}\left(A - \theta A\theta^{-1}\right) = 0 \end{aligned}$$

showing that, in any case, $\varepsilon = 0$ and $A\theta = \theta A$, concluding the proof. □

2.3 The operator Lambda and the automorphisms of the three dimensional nonnilpotent solvable Lie groups

In this section we define the operator Λ which will appear in the expression of a linear vector field. We also establish the explicit form of the automorphisms. In particular the flow associated to the linear vector field of a connected simply connected solvable nonnilpotent three dimensional Lie groups is obtained, allowing us to obtain an expression of a linear vector field.
Let A be a 2×2 matrix and define

$$\Lambda^A : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (t, w) \mapsto \Lambda_t^A w := \int_0^t \mathrm{e}^{sA} w ds. \tag{2.11}$$

The operator Λ^A is well defined and, for any $t, s \in \mathbb{R}$, it satisfies

1. $\Lambda_0^A = 0$,
2. $\dfrac{d}{dt}\Lambda_t^A = \mathrm{e}^{tA}$,
3. $\Lambda_{t+s}^A = \Lambda_t^A + \mathrm{e}^{tA}\Lambda_s^A$,
4. $\mathrm{e}^{tA} - A\Lambda_t^A = \mathrm{id}_{\mathbb{R}^2}$,
5. $\mathrm{e}^{sA}\Lambda_t^A = \Lambda_t^A \mathrm{e}^{sA}$,
6. $\Lambda_t^A = \left(\mathrm{e}^{tA} - \mathrm{id}_{\mathbb{R}^2}\right) A^{-1}$ if $\det A \neq 0$,
7. $A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \lambda \neq 0 \quad \implies \quad \Lambda_s^A = \begin{pmatrix} \frac{1}{\lambda}\left(e^{s\lambda} - 1\right) & 0 \\ 0 & s \end{pmatrix}$

The properties of the operator (2.11) can be found in [7].
We can now prove the following result concerning the automorphisms of the groups.

Proposition 2.3.1. For the three dimensional Lie group given by the semi-direct product $G = \mathbb{R} \times_\rho \mathbb{R}^2$, it holds that:

$$\mathrm{Aut}(G) = \left\{\phi(t, v) = \left(\varepsilon t, Pv + \varepsilon\Lambda^\theta_{\varepsilon t}\eta\right), \eta \in \mathbb{R}^2, P \in \mathrm{Gl}\left(\mathbb{R}^2\right), \text{ with } P\theta = \varepsilon\theta P\right\},$$

where $\varepsilon = 1$ if $\mathrm{tr}\,\theta \neq 0$ or $\varepsilon \in \{-1, 1\}$ if $\mathrm{tr}\,\theta = 0$.

Proof. Let ϕ be in the group of automorphism $\mathrm{Aut}(G)$. Then we have that $(d\phi)_{(0,0)} \in \mathfrak{g}(\theta)$. Therefore, we can write

$$(d\phi)_{(0,0)} = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix}, \quad \text{with} \quad P\theta = \varepsilon\theta P \quad \text{and} \quad \varepsilon^2 = 1.$$

The map $\psi(t, v) = \left(\varepsilon t, Pv + \varepsilon\Lambda^\theta_{\varepsilon t}\eta\right)$ satifies $(d\psi)_{(0,0)} = (d\phi)_{(0,0)}$ and, since G is connected, if we show that $\psi \in \mathrm{Aut}(G)$, the equality $\psi = \phi$ follows by general results in the Lie theory [38, Chapter 7].
For any $(t_1, v_1), (t_2, v_2) \in G$ it holds that

$$\begin{aligned}
\psi\left((t_1, v_1)(t_2, v_2)\right) =& \psi\left(t_1 + t_2, v_1 + \rho_{t_1}v_2\right) \\
=& \left(\varepsilon\left(t_1 + t_2\right), P\left(v_1 + \rho_{t_1}v_2\right) + \varepsilon\Lambda^\theta_{\varepsilon(t_1+t_2)}\eta\right) \\
=& \left(\varepsilon t_1 + \varepsilon t_2, Pv_1 + \rho_{\varepsilon t_1}Pv_2 + \varepsilon\Lambda^\theta_{\varepsilon t_1}\eta + \varepsilon\rho_{\varepsilon t_1}\Lambda^\theta_{\varepsilon t_2}\eta\right) \\
=& \left(\varepsilon t_1 + \varepsilon t_2, Pv_1 + \varepsilon\Lambda^\theta_{\varepsilon t_1}\eta + \rho_{\varepsilon t_1}\left(Pv_2 + \varepsilon\Lambda^\theta_{\varepsilon t_2}\eta\right)\right) \\
=& \left(\varepsilon t_1, Pv_1 + \varepsilon\Lambda^\theta_{\varepsilon t_1}\eta\right)\left(\varepsilon t_2, Pv_2 + \varepsilon\Lambda^\theta_{\varepsilon t_2}\eta\right) \\
=& \psi\left(t_1, v_1\right)\psi\left(t_2, v_2\right),
\end{aligned}$$

above we use that $P\theta = \varepsilon\theta P$, we also used the property 3 of the operator given by the equation (2.11). Therefore, $\psi \in \mathrm{Aut}(G)$ and consequently $\psi = \phi$ concluding the proof. □

By definition a linear vector field $\mathcal{X}$, is a complete vector field and its flow $\{\varphi_s\}_{s\in\mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{Aut}(G)$.
Since by differentiation, $\left\{(d\varphi_s)_{(0,0)}\right\}_{s\in\mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{Aut}(\mathfrak{g}(\theta))$, there exists a derivation $\mathcal{D} \in \mathrm{Der}(\mathfrak{g}(\theta))$ such that

$$\forall s \in \mathbb{R}. \quad (d\varphi_s)_{(0,0)} = \mathrm{e}^{s\mathcal{D}}.$$

But nevertheless

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix} \quad \text{implies} \quad \mathrm{e}^{s\mathcal{D}} = \begin{pmatrix} 1 & 0 \\ \Lambda^A_s\xi & \mathrm{e}^{sA} \end{pmatrix}.$$

Indeed, Consider the curves $\alpha_1(s) = e^{s\mathcal{D}}$ and $\alpha_2(s) = \begin{pmatrix} 1 & 0 \\ \Lambda_s^A \xi & \mathrm{e}^{sA} \end{pmatrix}$. Then

$$\alpha_1'(s)\mathcal{D}e^{s\mathcal{D}} = \mathcal{D}\alpha_1(s),$$

and

$$\begin{aligned}
\alpha_2'(s) &= \begin{pmatrix} 0 & 0 \\ \mathrm{e}^{sA}\xi & A\mathrm{e}^{sA} \end{pmatrix} && \text{by property 2 of the operator } \Lambda \\
&= \begin{pmatrix} 0 & 0 \\ \xi + A\Lambda_s^A \xi & A\mathrm{e}^{sA} \end{pmatrix} && \text{by property 4 of the operator } \Lambda \\
&= \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Lambda_s^A \xi & \mathrm{e}^{sA} \end{pmatrix} \\
&= \mathcal{D}\alpha_2(s).
\end{aligned}$$

That is the curves α_1 and α_2 satisfy the linear differential equation $\alpha' = \mathcal{D}\alpha$ and have the same initial condition $\alpha_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \alpha_2(0)$. Hence $\alpha_1(s) = \alpha_2(s)$ which shows that

$$\mathrm{e}^{s\mathcal{D}} = \begin{pmatrix} 1 & 0 \\ \Lambda_s^A \xi & \mathrm{e}^{sA} \end{pmatrix}.$$

Therefore, by Proposition 2.3.1 we get that

$$\varphi_s(t, v) = \left(t, \mathrm{e}^{sA} v + \Lambda_t^\theta \Lambda_s^A \xi\right), \quad \xi \in \mathbb{R}^2, A \in \mathfrak{gl}(2, \mathbb{R}), \text{ with } A\theta = \theta A. \tag{2.12}$$

Derivation at $s = 0$, gives us that the linear vector field $\mathcal{X}$ can be written as

$$\mathcal{X}(t, v) = \left(0, Av + \Lambda_t^\theta \xi\right), \quad \xi \in \mathbb{R}^2, A \in \mathfrak{gl}(2, \mathbb{R}), \text{ with } A\theta = \theta A.$$

Remark 2.3.2. By the fact that the vector $\xi \in \mathbb{R}^2$ and the matrix $A \in \mathfrak{gl}(2, \mathbb{R})$, together with the previous properties, determine $\mathcal{X}$, we will usually write $\mathcal{X} = (\xi, A)$ to denote the linear vector field $\mathcal{X}$.

Next, we give the explicitly expression for the exponential map of a three dimensional Lie group and the differential of the left translation. These expressions will be used throughout this work. For any $(a, w) \in \mathfrak{g}(\theta)$, the exponential map is given by

$$\exp(a, w) = \begin{cases} (0, w) & \text{if } a = 0, \\ \left(a, \dfrac{1}{a}\Lambda_a^\theta w\right), & \text{if } a \neq 0. \end{cases} \tag{2.13}$$

Let $(t_i, v_i) \in G, i = 1, 2$ and $(a, w) \in \mathfrak{g}(\theta)$. Then the left translations satisfies

$$\left(dL_{(t_1,v_1)}\right)_{(t_2,v_2)}(a, w) = (a, \rho_{t_1} w). \tag{2.14}$$

Therefore the left invariant vector field associated with $Y = (a, w) \in \mathfrak{g}(\theta)$ is given by

$$Y^L(t, v) = (a, \rho_t w)\,. \tag{2.15}$$

The formulas given in the equations (2.13), (2.14) and (2.15) were obtained in [7].

2.4 Simple ARS isometry-related

In this section we analyze some properties of the LARC and prove a proposition where we can construct simple ARS's isometry-related.

First, note that the family $\Sigma = \{\mathcal{X}, \Delta^L\}$, with $\Delta = \{0\} \times \mathbb{R}^2$, does not satisfies the LARC. In fact, in this case Δ is a subalgebra and

$$\mathcal{D}(0, w) = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix} \begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ Aw \end{pmatrix} = (0, Aw) \quad \Longrightarrow \quad \mathcal{D}\Delta \subset \Delta,$$

which contradicts Remark 1.3.3.

Therefore, if $\Sigma = \{\mathcal{X}, \Delta^L\}$ satisfies the LARC then the subspace Δ admits a basis

$$\{(\sigma_1, u_1), (\sigma_2, u_2)\}, \quad \text{with} \quad \sigma_1^2 + \sigma_2^2 \neq 0.$$

Notice that Δ is a two dimensional space of $(\{0\} \times \mathbb{R}^2)$ and the intersection $\Delta \cap (\{0\} \times \mathbb{R}^2)$ is two or one dimensional. However, by the previous discussion we conclude that the intersection $\Delta \cap (\{0\} \times \mathbb{R}^2)$ is one-dimensional. Let us denote by l_Δ the line in $\mathbb{R}^2$ satisfying

$$\{0\} \times l_\Delta = \Delta \cap (\{0\} \times \mathbb{R}^2)\,.$$

In the next proposition we establish an equivalent relation to Δ be a subalgebra and an important consequence of the LARC.

Proposition 2.4.1. For a family $\Sigma = \{\mathcal{X} = (\xi, A), \Delta^L\}$, it holds that:

1. Δ is a subalgebra if and only if l_Δ is an eigenspace of θ;
2. If $(1, 0) \in \Delta$, then Σ satisfies the LARC if and only if Δ is not a subalgebra or Δ is a subalgebra and $Al_\Delta \not\subset l_\Delta$ or $\xi \notin l_\Delta$

Proof. 1. Let $\{(\sigma_1, u_1), (\sigma_2, u_2)\}$ be a basis of Δ. Then,

$$\Delta \ni -\sigma_2 (\sigma_1, u_1) + \sigma_1 (\sigma_2, u_2) = (0, -\sigma_2 u_1 + \sigma_1 u_2) \in \{0\} \times \mathbb{R}^2,$$

is a nonzero vector, and the element $(0, -\sigma_2 u_1 + \sigma_1 u_2) \in \Delta \cap \{0\} \times \mathbb{R}^2$ implying that $l_\Delta = \mathbb{R}(\sigma_1 u_2 - \sigma_2 u_1)$.

On the other hand,

$$\Delta \text{ is a subalgebra} \iff [(\sigma_1, u_1), (\sigma_2, u_2)] \in \Delta\,.$$

However, by defnition

$$[(\sigma_1, u_1), (\sigma_2, u_2)] = (0, \theta(\sigma_1 u_2 - \sigma_2 u_1)),$$

implying that

$$\Delta \text{ is a subalgebra } \iff l_\Delta \text{ is an eigenspace of } \theta \, .$$

2. Let $\{(\sigma_1, u_1), (\sigma_2, u_2)\}$ be an orthonormal basis of Δ. If Δ is not a subalgebra, then necessarily $[(\sigma_1, u_1), (\sigma_2, u_2)] \notin \Delta$ and the LARC is satisfied. On the other hand, if Δ is a subalgebra, then by part 1 in Proposition 2.4.1, l_Δ is an eigenspace of θ. Since by hypothesis $(1,0)$ also belongs to Δ we have that $\Delta = \mathbb{R}(1,0) \oplus l_\Delta$. We have that if $\mathcal{D}$ is the derivation associated with $\mathcal{X}$, by Remark 1.3.3 the LARC is satisfied if and only

$$\mathcal{D}(\{0\} \times l_\Delta) \not\subset \Delta \quad \text{or} \quad \mathcal{D}(1,0) \notin \Delta.$$

However, the fact that

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix} \text{ gives us that } \mathcal{D}(\{0\} \times l_\Delta) = (\{0\} \times Al_\Delta) \text{ and } \mathcal{D}(1,0) = (0, \xi),$$

showing that $\mathcal{D}\Delta \not\subset \Delta$ if and only if $Al_\Delta \not\subset l_\Delta$ or $\xi \notin l_\Delta$, concluding the proof.

□

Remark 2.4.2. It is important to notice that if $\theta \neq \mathrm{id}_{\mathbb{R}^2}$, the fact that $A\theta = \theta A$ gives us that

$$\theta l_\Delta \subset l_\Delta \quad \implies \quad Al_\Delta \subset l_\Delta.$$

Consequently, in this case, if $(1,0) \in \Delta$ the LARC holds if and only if $\xi \notin l_\Delta$.

The next result shows that elements in $\mathrm{Aut}(G)$ can be seen as isometries between ARS's.

Proposition 2.4.3. Let $\Sigma = \{\mathcal{X} = (\xi, A), \Delta^L\}$ be an *ARS* on G and

$$\psi(t, v) = \left(\varepsilon t, Pv + \varepsilon \Lambda^\theta_{\varepsilon t} \eta\right) \text{ an automophism of } G \, .$$

The family

$$\Sigma_\psi = \left\{ \mathcal{X}_\psi = \left(P^{-1}(\varepsilon\xi + A\eta), P^{-1}AP\right), \Delta_\psi = (d\psi)^{-1}_{(0,0)}\Delta \right\}.$$

is an ARS and ψ is an isometry between Σ_ψ and Σ, where the left-invariant metric on Δ_ψ is the one that makes $(d\psi)_{(0,0)}\big|_{\Delta_\psi}$ an isometry.

Proof. By definition, we have that

$$
\begin{aligned}
(d\psi)_{(t,v)}\mathcal{X}_\psi(t,v) &= \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix} \begin{pmatrix} 0 \\ P^{-1}APv + \Lambda_t^\theta P^{-1}(\varepsilon\xi + A\eta) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ APv + P\Lambda_t^\theta P^{-1}(\varepsilon\xi + A\eta) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ APv + \varepsilon\Lambda_{\varepsilon t}^\theta(\varepsilon\xi + A\eta) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ APv + \varepsilon\Lambda_{\varepsilon t}^\theta A\eta + \Lambda_{\varepsilon t}^\theta \varepsilon^2\xi \end{pmatrix} \qquad \text{with} \quad \varepsilon^2 = 1 \\
&= \begin{pmatrix} 0 \\ A\left(Pv + \varepsilon\Lambda_{\varepsilon t}^\theta\eta\right) + \Lambda_{\varepsilon t}^\theta\xi \end{pmatrix} \\
&= \mathcal{X}\left(\varepsilon t, Pv + \varepsilon\Lambda_{\varepsilon t}^\theta\eta\right) \\
&= \mathcal{X}(\psi(t,v)),
\end{aligned}
$$

where in the third equality we used that

$$
P\theta = \varepsilon\theta P \quad \Longrightarrow \quad P\mathrm{e}^{t\theta} = \mathrm{e}^{\varepsilon t\theta}P \quad \text{and by integration} \quad P\Lambda_t^\theta P^{-1} = \varepsilon\Lambda_{\varepsilon t}^\theta\,,
$$

and in the fourth equality that $A\theta = \theta A$, showing that $\mathcal{X}_\psi$ and $\mathcal{X}$ are ψ -conjugated. Since automorphisms preserves left-invariant vector fields, it holds that Σ_ψ is in fact an ARS on G. Moreover, if we define the left invariant metric on Δ_ψ that makes $(d\psi)_{(0,0)}\big|_{\Delta_\psi}$ an isometry, we get that $(d\psi)_{(t,v)}$ carries orthonormal frames in $\Delta_\psi^L(t,v)$ onto orthonormal frames in $\Delta^L(\psi(t,v))$ implying that ψ is in fact an isometry between Σ_ψ and Σ. □

Remark 2.4.4. Let $\psi \in \mathrm{Iso}_G(\Sigma_1, \Sigma_2)_0$ and consider $\hat{\psi} \in \mathrm{Aut}(G)$. By the previous proposition there exist an ARS's $\Sigma_{\hat{\psi}}$ such that $\hat{\psi} \in \mathrm{Iso}_G\left(\Sigma_{\hat{\psi}}, \Sigma_1\right)$. As a consequence, the composition $\psi \circ \hat{\psi}$ is an isometry between $\Sigma_{\hat{\psi}}$ and Σ_{ψ_2}. In particular, the maps

$$
\psi_1(t,v) = \left(t, v - \Lambda_t^\theta\left(A_1^{-1}\xi_1\right)\right), \qquad \text{if } \det A_1 \neq 0,
$$

and

$$
\psi_2(t,v) = \left(t, v - \frac{1}{\sigma}\Lambda_t^\theta u\right), \qquad \text{if } (\sigma, u) \in \Delta_1 \text{ with } \sigma \neq 0,
$$

are automorphisms of G and their induced ARS's satisfies $\mathcal{X}_{\psi_1} = (0, A_1)$ and $(1,0) \in \Delta_{\psi_2}$. Therefore, up to automorphisms we can assume that $(1,0) \in \Delta_1$ or $\xi_1 = 0$ if $\det A_1 = 0$.

3 Singular Locus

The aim of this chapter is to study some geometric an algebric properties of the singular locus. The chapter is structured as follows: In Section 3.1 we show that the singular locus of a simple ARS on the three dimensional Lie groups under consideration is a submanifold. We also analyze when the singular locus is a connected subset. In Section 3.2 we investigate how the exponential curves crosses the singular locus. In Section 3.3 some examples of singular locus are presented.

3.1 The singular locus

In this section we analyze the singular locus simple ARS's on three dimensional solvable nonnilpotent Lie groups. By the results in Proposition 1.5.1 we can assume without loss of generality that of the ARS's on $G = \mathbb{R} \times_\rho \mathbb{R}^2$. As showed in [6, Theorem 1], the singular locus of a simple ARS $\Sigma = \left\{\mathcal{X}, \Delta^L\right\}$ such that Δ is a subalgebra, is a submanifold. In this section we show that for the class of groups in question the same holds independent of any condition on the distribution. Such property is not true, for instance, on the Heisenberg group as showed in the Example 1.6.2.

Let $\Sigma = \left\{\mathcal{X} = (\xi, A), \Delta^L\right\}$ be a simple ARS on G. We recall that by remark 1.3.2, the singular locus $\mathcal{Z}$ is defined as

$$\mathcal{Z} := \{(t, v) \in G; \mathcal{X}(t, v) \in \Delta^L(t, v)\}.$$

Using the expression for $\mathcal{X}$ gives us that

$$\mathcal{X}(t, v) = \left(0, Av + \Lambda_t^\theta \xi\right) \in \Delta^L(t, v) \iff (0, Av + \Lambda_t \xi) \in \Delta^L(t, v) \cap \left(\{0\} \times \mathbb{R}^2\right) = \{0\} \times \rho_t\left(l_\Delta\right).$$

Therefore, if $\mathbf{u}$ is a vector normal to l_Δ, it holds that

$$\mathcal{Z} = \left\{(t, v) \in G; \left\langle \rho_{-t}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} = 0\right\}.$$

Defining

$$F_\mathbf{u} : G \to \mathbb{R}, \quad F_\mathbf{u}(t, v) = \left\langle \rho_{-t}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2}, \tag{3.1}$$

gives us that $\mathcal{Z} = F^{-1}(0)$.

Proposition 3.1.1. For the function $F_\mathbf{u}$ it holds that

$$\partial_1 F_\mathbf{u}(t, v) = \langle \rho_{-t}(\xi - \theta Av), \mathbf{u}\rangle_{\mathbb{R}^2} \quad \text{and} \quad \partial_2 F_\mathbf{u}(t, v)w = \langle \rho_{-t} Aw, \mathbf{u}\rangle_{\mathbb{R}^2}. \tag{3.2}$$

Proof. Note that, for fixed t, the function

$$v \mapsto F_\mathbf{u}(t, v) = \langle \rho_{-t} Av, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \rho_{-t}\Lambda_t^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}$$

is the sum of the linear map with a constant. Therefore,

$$\partial_2 F_{\mathbf{u}}(t,v)w = \langle \rho_{-t}Aw, \mathbf{u}\rangle_{\mathbb{R}^2}.$$

For the first derivative, notice that by the properties of Λ

$$\begin{aligned}\left\langle \rho_{-t}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} &= \langle \rho_{-t}Av, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \rho_{-t}\Lambda_t^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ &= \langle \rho_{-t}Av, \mathbf{u}\rangle_{\mathbb{R}^2} - \left\langle \Lambda_{-t}^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}.\end{aligned}$$

Diferentiation in t gives us that

$$\begin{aligned}\partial_1 F_{\mathbf{u}}(t,v) =& \frac{d}{dt}\left(\langle \rho_{-t}Av, \mathbf{u}\rangle_{\mathbb{R}^2} - \left\langle \Lambda_{-t}^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}\right) \\ =& \langle \rho_{-t}(-\theta Av), \mathbf{u}\rangle_{\mathbb{R}^2} + \langle \rho_{-t}\xi, \mathbf{u}\rangle_{\mathbb{R}^2} \\ =& \langle \rho_{-t}(\xi - \theta Av+), \mathbf{u}\rangle_{\mathbb{R}^2}.\end{aligned}$$

as stated. □

Now we can show that the singular locus of a simple ARS on three dimensional Lie groups is a submanifold.

Theorem 3.1.2. The singular locus of any ARS on a $G = \mathbb{R} \times_\rho \mathbb{R}^2$ is an embedded submanifold of G.

Proof. Let us consider an ARS $\Sigma = \left\{\mathcal{X}, \Delta^L\right\}$. Since the image of a submanifold by a diffeomorphism is also a submanifold, Proposition 2.4.3 allows us to assume w.l.o.g. that $(1,0) \in \Delta$. Assume first that $A \equiv 0$. By definition, in this case, the singular locus is given by

$$X \times \mathbb{R}^2, \quad \text{where} \quad X = \left\{t \in \mathbb{R};\ \left\langle \rho_{-t}\left(\Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} = 0\right\}. \tag{3.3}$$

We have two cases

(i) $\det\theta = 0$. In this case

$$\Lambda_t^\theta = \begin{pmatrix} e^t - 1 & 0 \\ 0 & t \end{pmatrix} \quad \text{and} \quad \rho_t = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\rho_{-t}\Lambda_t^\theta = \begin{pmatrix} 1 - e^{-t} & 0 \\ 0 & t \end{pmatrix}.$$

Now, if $\xi = (\xi_1, \xi_2)$ and $\mathbf{u} = (u_1, u_2)$ then

$$\left\langle \rho_{-t}\left(\Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} = t\xi_2 u_2 - u_1\xi_1(e^{-t} - 1) \tag{3.4}$$

By considering the right-hand side of the equation (3.4) for ξ and $\mathbf{u}$ fixed, we associate a real-valued function and then we obtain that X has at most two elements, that is, X is a discrete set. Therefore, by the expression of the singular locus given in equation (3.3), we conclude that the singular locus is a submanifold.

(ii) $\det \theta \neq 0$. In this case

$$\Lambda_t^\theta = (\rho_t - 1)\theta^{-1}.$$

Thus, we get

$$\left\langle \rho_{-t}\left(\Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} = -\left\langle \rho_{-t}\theta^{-1}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} + \left\langle \theta^{-1}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}, \tag{3.5}$$

where in the equality of equation (3.5) we use the property 3 of the operator Λ.
Thus by using the Lemma A.1.1, we obtain that X is an enumerable subset of $\mathbb{R}$ with cardinality depending on the eigenvalues of θ. As a consequence, the singular locus of Σ is a submanifold.

Let us now consider the case where $A \not\equiv 0$. In order to show that the singular locus of Σ is a submanifold, it is enough to guarantee that $0 \in \mathbb{R}$ is a regular value of the map $F_{\mathbf{u}}$ defined in equation (3.1).
Notice that the map $F_{\mathbf{u}}$ is an application with values is $\mathbb{R}$, and this map is a submersion if the rank is 1, that is if its differential is onto. Consequently, we must have that at least one of its derivatives is nonzero.
By equation (3.2) $0 \in \mathbb{R}$ is not a regular value of F if there exists $(t, v) \in \mathcal{Z}$ such that

$$\partial_1 F_{\mathbf{u}}(t, v) = \langle \rho_{-t}(\xi - \theta A v), \mathbf{u}\rangle_{\mathbb{R}^2} = 0 \tag{3.6}$$

and

$$\forall w \in \mathbb{R}^2 \quad \partial_2 F_{\mathbf{u}}(t, v) w = \langle \rho_{-t} A w, \mathbf{u}\rangle_{\mathbb{R}^2} = 0. \tag{3.7}$$

From the equation (3.7) and the fact that $\rho_{-t}A = A\rho_{-t}$, we get that

$$0 = \partial_2 F_{\mathbf{u}}(t, v)\rho_t w = \langle \rho_{-t} A \rho_t w, \mathbf{u}\rangle_{\mathbb{R}^2} = \langle A w, \mathbf{u}\rangle_{\mathbb{R}^2}$$

implying that $Aw \in l_\Delta, \quad \forall w \in \mathbb{R}^2$ or equivalently, that $\operatorname{Im} A \subset l_\Delta$.
By using the previous in equation (3.6) allows us to obtain

$$\begin{aligned}
0 &= \langle \rho_{-t}(\xi - \theta A v), \mathbf{u}\rangle_{\mathbb{R}^2} \\
&= \langle \rho_{-t}\xi, \mathbf{u}\rangle_{\mathbb{R}^2} - \langle \rho_{-t} A\theta v, \mathbf{u}\rangle_{\mathbb{K}^2} \\
&= \langle \rho_{-t}\xi, \mathbf{u}\rangle_{\mathbb{R}^2} - \langle A\rho_{-t}\theta v, \mathbf{u}\rangle_{\mathbb{K}^2} \\
&= \langle \rho_{-t}\xi, \mathbf{u}\rangle_{\mathbb{R}^2} \quad \text{since} \quad A\rho_{-t}\theta v \in \operatorname{Im} A,
\end{aligned}$$

where we used that $A\theta = \theta A$. Since $A \not\equiv 0$ we have that $\operatorname{Im} A = l_\Delta$ implying that $Al_\Delta \subset l_\Delta$ and $\theta l_\Delta \subset l_\Delta$. Therefore Δ is a subalgebra and

$$\begin{aligned} & \langle \rho_{-t}\xi, \mathbf{u}\rangle_{\mathbb{R}^2} = 0 \\ \implies \quad & \rho_{-t}\xi \in l_\Delta \\ \implies \quad & \xi \in \rho_t\left(l_\Delta\right) = l_\Delta, \end{aligned}$$

By Proposition 2.4.1 Σ cannot satisfies the LARC. Therefore, 0 is a regular value of the map $F_{\mathbf{u}}$ showing that $\mathcal{Z}$ is in fact an embedded submanifold. □

The next result analyzes the connectedness of the singular locus.

Theorem 3.1.3. If $\Sigma = \{\mathcal{X} = (0, A), \Delta^L\}$ is a simple ARS and $A \neq 0$, then $\mathcal{Z}$ is connected. Moreover, $G\backslash\mathcal{Z}$ has two connected components given by

$$\mathcal{C}^- := F_{\mathbf{u}}^{-1}(-\infty, 0) \quad \text{and} \quad \mathcal{C}^+ := F_{\mathbf{u}}^{-1}(0, +\infty),$$

where $F_{\mathbf{u}}$ is the function defined in (3.1).

Proof. Assume first that $\det A \neq 0$ and consider the map

$$H : G \to G, \quad H(t, v) = \left(t, A^{-1}\left(\rho_t v - \Lambda_t^\theta \xi\right)\right).$$

The map H is continuous and has continuous inverse given by

$$H^{-1}(t, v) = (t, \rho_{-t}(Av + \Lambda_t^\theta \xi)).$$

Moreover,

$$\begin{aligned} (t, v) \in \mathcal{Z} \quad & \iff \quad \left\langle \rho_{-t}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} = 0 \\ & \iff \quad \exists s \in \mathbb{R}; \rho_{-t}\left(Av + \Lambda_t^\theta \xi\right) = su \\ & \iff \quad v = A^{-1}\left(s\rho_t u - \Lambda_t^\theta \xi\right) \\ & \iff \quad (t, v) = H(t, su), \end{aligned}$$

showing that $\mathcal{Z}$ is homeomorphic to the plane $\mathbb{R} \times l_\Delta \subset G$. As a consequency $\mathcal{Z}$ is connected and $G\backslash\mathcal{Z}$ has two connected components. Also,

$$\begin{aligned} F_{\mathbf{u}}(H(t, v)) = & F_{\mathbf{u}}\left(t, A^{-1}\left(\rho_t v - \Lambda_t^\theta \xi\right)\right) \\ = & \left\langle \rho_{-t}\left(A\left(A^{-1}\left(\rho_t v - \Lambda_t^\theta \xi\right)\right) + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ = & \langle v, \mathbf{u}\rangle_{\mathbb{R}^2}, \end{aligned}$$

implying that $F_{\mathbf{u}}(-\infty,0)$ and $F_{\mathbf{u}}(0,+\infty)$ are (pathwise) connected. Since

$$F_{\mathbf{u}}\left(\mathbb{R}\backslash\{0\}\right) = G\backslash\mathcal{Z},$$

we get that $\mathcal{C}^-$ and $\mathcal{C}^+$ are in fact the connected components of $G\backslash\mathcal{Z}$.
Let us now consider the case where $\dim \operatorname{Im} A = 1$ and assume w.l.o.g. that $(1,0) \in \Delta$. Since $\ker A$ has also dimension one and $A\theta = \theta A$, we can easily construct an orthonormal basis $\{w_1, w_2\}$ of $\mathbb{R}^2$ such that

$$Aw_1 \neq 0, \quad Aw_2 = 0 \quad \text{and} \quad \theta Aw_1 = \beta Aw_1.$$

Let $(t,v) \in \mathcal{Z}$ and write $v = \langle v, w_1\rangle_{\mathbb{R}^2} w_1 + \langle v, w_2\rangle_{\mathbb{R}^2} w_2$. Then

$$\begin{aligned}
0 &= \left\langle \rho_{-t}\left(Av + \Lambda_t^{\theta}\xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\
&= \left\langle \rho_{-t}\left(A\left[\langle v, w_1\rangle_{\mathbb{R}^2} w_1 + \langle v, w_2\rangle_{\mathbb{R}^2} w_2\right] + \Lambda_t^{\theta}\xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\
&= \langle v, w_1\rangle_{\mathbb{R}^2} \langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} + \langle v, w_2\rangle_{\mathbb{R}^2} \langle \rho_{-t}Aw_2, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \rho_{-t}\Lambda_t^{\theta}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} \\
&= \langle v, w_1\rangle_{\mathbb{R}^2} \langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} - \left\langle \Lambda_{-t}^{\theta}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}, \quad \text{since } \langle \rho_{-t}Aw_2, \mathbf{u}\rangle_{\mathbb{R}^2} = 0
\end{aligned}$$

implying that

$$\langle v, w_1\rangle_{\mathbb{R}^2} \langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} = \left\langle \Lambda_{-t}^{\theta}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}.$$

Since Aw_1 is a nonzero eigenvector of θ we have that

$$\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} = e^{-\beta t} \langle Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2},$$

where β is the associated eigenvalue. In particular, if $\langle Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} = 0$ we get by orthogonality that $Aw_1 \in l_\Delta$, implying that $l_\Delta = \operatorname{Im} A$ and hence $Al_\Delta \subset l_\Delta$. Also, in this case, $\forall t \in \mathbb{R}$,

$$\begin{aligned}
0 &= \langle v, w\rangle_{\mathbb{R}^2} e^{-\beta t} \langle Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} \\
&= \langle v, w\rangle_{\mathbb{R}^2} \langle \rho_t Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} \\
&= \left\langle \Lambda_{-t}^{\theta}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}.
\end{aligned}$$

Differentiation at $t = 0$ gives us that

$$\langle \xi, \mathbf{u}\rangle_{\mathbb{R}^2} = 0 \quad \Longleftrightarrow \quad \xi \in l_\Delta$$

Therefore,

$$\langle Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} = 0 \quad \Longrightarrow \quad Al_\Delta \subset l_\Delta \quad \text{and} \quad \xi \in l_\Delta,$$

which together with the assumption $(1,0) \in \Delta$ contradicts the LARC. Therefore, $\langle Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} \neq 0$ and we obtain that

$$\langle v, w_1\rangle_{\mathbb{R}^2} = \frac{\left\langle \Lambda_{-t}^{\theta}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}.$$

By the previous, the map

$$I: G \to G, (t,v) \in G \mapsto I(t,v) = \left(t, \left(\langle v, w_1\rangle_{\mathbb{R}^2} + \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right) w_1 + \langle v, w_2\rangle_{\mathbb{R}^2} w_2\right),$$

is well defined, continuous and a simple calculation shows that its inverse is the continuous map

$$I^{-1}(t,v) = \left(t, \left(\langle v, w_1\rangle_{\mathbb{R}^2} - \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right) w_1 + \langle v, w_2\rangle_{\mathbb{R}^2} w_2\right).$$

By the previous calculations we get that

$$(t,v) \in \mathcal{Z} \iff \langle v, w_1\rangle_{\mathbb{R}^2} = \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}} \iff I^{-1}(t,v) \in \mathbb{R} \times \mathbb{R}w_2,$$

showing that is the homemorphic image of $\mathbb{R} \times \mathbb{R}w_2$ by I. Also,

$$\begin{aligned}
F_{\mathbf{u}}(I(t,v)) =& F_{\mathbf{u}}\left(t, \left(\langle v, w_1\rangle_{\mathbb{R}^2} + \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right) w_1 + \langle v, w_2\rangle_{\mathrm{R}^2} w_2\right)\\
=& \left\langle \rho_{-t}\left[A\left(\left(\langle v, w_1\rangle_{\mathbb{R}^2} + \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right) w_1 + \langle v, w_2\rangle_{\mathbb{R}^2} w_2\right) + \Lambda^{\theta}_{t}\xi\right], \mathbf{u}\right\rangle_{\mathbb{R}^2}\\
=& \left\langle \rho_{-t}A\left(\left(\langle v, w_1\rangle_{\mathbb{R}^2} + \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right) w_1 + \langle v, w_2\rangle_{\mathbb{R}^2} w_2\right) + \rho_{-t}\Lambda^{\theta}_{t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}\\
=& \left(\langle v, w_1\rangle_{\mathbb{R}^2} + \frac{\left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right)\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \rho_{-t}\Lambda^{\theta}_{t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}\\
=& \left(\frac{\langle v, w_1\rangle_{\mathbb{R}^2}\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}}{\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2}}\right)\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \rho_{-t}\Lambda^{\theta}_{t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}\\
=& \langle v, w_1\rangle_{\mathbb{R}^2}\langle \rho_{-t}Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2} + \left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} - \left\langle \Lambda^{\theta}_{-t}\xi, \mathbf{u}\right\rangle_{\mathbb{R}^2}\\
=& \mathrm{e}^{-\beta t}\langle v, w_1\rangle_{\mathbb{R}^2}\langle Aw_1, \mathbf{u}\rangle_{\mathbb{R}^2},
\end{aligned}$$

which as previously implies that $\mathcal{C}^-$ and $\mathcal{C}^+$ are the connected components of $G \backslash \mathcal{Z}$, concluding the proof. □

3.2 Crossing the singular locus

In this section we analyze how the exponential curves crosses the singular locus. Such analysis will be necessary in the proof of the Fundamental Lemma ahead.
Let us consider as previously $\Sigma = \{\mathcal{X} = (\xi, A), \Delta^L\}$ be a simple ARS with $A \neq 0$, and define the function $F_{\mathbf{u}}$, where $\mathbf{u}$ is a normal vector to l_Δ fixed. Let $(t,v) \in G$ and consider the exponential curve $s \in \mathbb{R} \mapsto (t,v)\exp s(a,w)$. By Theorem 4.5.1 in order to see how such curve behaves with relation to the singular locus, it is enough to analyze the sign of the function

$$s \in \mathbb{R} \mapsto F_{\mathbf{u}}((t,v)\exp s(a,w)). \tag{3.8}$$

Using the formula for the exponential in equation (2.13) we have the following cases:

(i) $a = 0$: In this case, $\exp s(0,w) = (0, sw)$ and $(t,v)(0,sw) = (t, v + \rho_t sw)$. Therefore

$$\begin{aligned} F_{\mathbf{u}}((t,v)\exp s(0,w)) =& F_{\mathbf{u}}(t, v+\rho_t sw) \\ =& \left\langle \rho_{-t}\left(A(v+\rho_t sw) + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t}\left(Av+\rho_t Asw + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t}Av + sAw + \rho_{-t}\Lambda_t^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} + s\left\langle Aw, \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& F_{\mathbf{u}}(t,v) + s\left\langle Aw, \mathbf{u}\right\rangle_{\mathbb{R}^2}. \end{aligned}$$

(ii) $a \neq 0$: In this case, $\exp(s(a,w)) = \left(sa, \frac{1}{a}\Lambda_{sa}^\theta w\right)$ and $(t,v)\left(sa, \frac{1}{a}\Lambda_{sa}^\theta w\right) = \left(t+sa, v+\rho_t\left(\frac{1}{a}\Lambda_{sa}^\theta w\right)\right)$. Therefore,

$$\begin{aligned} F_{\mathbf{u}}((t,v)\exp s(a,w)) =& F_{\mathbf{u}}\left(t+sa, v+\rho_t\left(\frac{1}{a}\Lambda_{sa}^\theta w\right)\right) \\ =& \left\langle \rho_{-t-sa}\left[A\left(v+\rho_t\left(\frac{1}{a}\Lambda_{sa}^\theta w\right)\right) + \Lambda_{t+sa}^\theta \xi\right], \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t-sa}\left[A\left(v+\rho_t\left(\frac{1}{a}\Lambda_{sa}^\theta w\right)\right) + \Lambda_t^\theta \xi + \rho_t\Lambda_{sa}^\theta \xi\right], \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t-sa}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} + \left\langle \rho_{-t-sa}\left(\frac{1}{a}\rho_t\Lambda_{sa}^\theta Aw + \rho_t\Lambda_{sa}^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t-sa}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} + \left\langle \frac{1}{a}\rho_{-sa}\Lambda_{sa}^\theta Aw + \rho_{-sa}\Lambda_{sa}^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t-sa}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} + \left\langle -\frac{1}{a}\Lambda_{-sa}^\theta Aw - \Lambda_{-sa}^\theta \xi, \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \left\langle \rho_{-t-sa}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} - \frac{1}{a}\left\langle \Lambda_{-sa}^\theta (Aw + a\xi), \mathbf{u}\right\rangle_{\mathbb{R}^2}. \end{aligned}$$

Summarizing, we have that

$$F_{\mathbf{u}}((t,v)\exp s(a,w)) = \begin{cases} F_{\mathbf{u}}(t,v) + s\langle Aw, \mathbf{u}\rangle_{\mathbb{R}^2} & \text{if } a = 0 \\ \left\langle \rho_{-t-as}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} - \frac{1}{a}\left\langle \Lambda_{-as}^\theta (Aw + a\xi), \mathbf{u}\right\rangle_{\mathbb{R}^2} & \text{if } a \neq 0, \end{cases} \tag{3.9}$$

The following lemma states what happens with the function given in equation (3.8) when the starting point (t,v) belongs to the singular locus $\mathcal{Z}$ of Σ.

Lemma 3.2.1. Let $(t,v) \in \mathcal{Z}$ and consider the function

$$s \mapsto F_{\mathbf{u}}((t,v)\exp s(a,w)).$$

Then, $F_{\mathbf{u}}((t,v)\exp s(a,w)) \equiv 0$ or there exists $\delta > 0$ such that $F_{\mathbf{u}}((t,v)\exp s(a,w)) \neq 0$ for all $s \in (-\delta,\delta)\backslash\{0\}$.

Proof. Let us first consider the case $a = 0$. By equation (3.9), if $(t,v) \in \mathcal{Z}$, $F_{\mathbf{u}}((t,v) = 0$ and

$$\begin{aligned} F_{\mathbf{u}}((t,v)\exp s(a,w)) =& F_{\mathbf{u}}(t,v) + s\langle Aw, \mathbf{u}\rangle_{\mathbb{R}^2} \\ =& s\langle Aw, \mathbf{u}\rangle_{\mathbb{R}^2}. \end{aligned}$$

Therefore, if $Aw \in l_\Delta$ we get that $F_{\mathbf{u}}((t,v)\exp s(a,w)) = 0$ for all $s \in \mathbb{R}$ and if $Aw \notin l_\Delta$ we get that $F_{\mathbf{u}}((t,v)\exp s(a,w) \neq 0$ for all $s \in \mathbb{R}\backslash\{0\}$

Assume now that $a \neq 0$ and fix $0 \neq u \in l_\Delta$. By equation(3.9) we have that

$$F_{\mathbf{u}}((t,v)\exp s(a,w)) = \left\langle \rho_{-t-as}\left(Av + \Lambda_t^\theta \xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} - \frac{1}{a}\left\langle \Lambda_{-as}(Aw + a\xi), \mathbf{u}\right\rangle_{\mathbb{R}^2}.$$

Since $(t,v) \in \mathcal{Z}$, there exists $\mu = \mu(t,v) \in \mathbb{R}$ such that $\rho_{-t}\left(Av + \Lambda_t\xi\right) = \mu u$. In particular,

$$\begin{aligned} \left\langle \rho_{-t-as}\left(Av + \Lambda_t^\theta\xi\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} =& \left\langle \rho_{-as}\left(\rho_{-t}\left(Av + \Lambda_t\xi\right)\right), \mathbf{u}\right\rangle_{\mathbb{R}^2} \\ =& \mu\left\langle \rho_{-as}u, \mathbf{u}\right\rangle_{\mathbb{R}^2}, \end{aligned}$$

and so

$$F_{\mathbf{u}}((t,v)\exp s(a,w)) = \mu\left\langle \rho_{-as}u, \mathbf{u}\right\rangle_{\mathbb{R}^2} - \frac{1}{a}\left\langle \Lambda_{-as}(Aw + a\xi), \mathbf{u}\right\rangle_{\mathbb{R}^2}.$$

Derivation at s give us that

$$\frac{d}{ds}F_{\mathbf{u}}((t,v)\exp s(a,w)) = \left\langle \rho_{-as}(Aw + a\xi - a\mu\theta u), \mathbf{u}\right\rangle_{\mathbb{R}^2},$$

and we have the following cases:

1. $Aw + a\xi - a\mu\theta u \in l_\Delta$

In this case, there exists $\tau \in \mathbb{R}$ such that $Aw + a\xi - a\mu\theta u = \tau u$ and hence

$$\begin{aligned} \mu\rho_{-as}u - \frac{1}{a}\Lambda_{-as}^\theta(Aw + a\xi) =& \mu\rho_{-as}u - \frac{1}{a}\Lambda_{-as}^\theta(a\mu\theta u + \tau) \\ =& \mu\rho_{-as}u - \frac{1}{a}\Lambda_{-as}^\theta a\mu\theta u - \frac{1}{a}\Lambda_{-as}^\theta\tau u \\ =& \mu\left(\left(\rho_{-as} - \theta\Lambda_{-as}^\theta\right)u\right) - \frac{\tau}{a}\Lambda_{-as}^\theta u \\ =& \mu u - \frac{\tau}{a}\Lambda_{-as}^\theta u, \end{aligned}$$

where for the last equality we used property 4. of the operator Λ^θ. Hence,

$$\begin{aligned}F_{\mathbf{u}}((t,v)\exp s(a,w)) =&\mu\langle\rho_{-as}u,\mathbf{u}\rangle_{\mathbb{R}^2}-\frac{1}{a}\langle\Lambda_{-as}(Aw+a\xi),\mathbf{u}\rangle_{\mathbb{R}^2}\\ =&\mu\langle u,\mathbf{u}\rangle_{\mathbb{R}^2}-\frac{\tau}{a}\left\langle\Lambda^\theta_{-as}u,\mathbf{u}\right\rangle_{\mathbb{R}^2}\\ =&-\frac{\tau}{a}\left\langle\Lambda^\theta_{-as}u,\mathbf{u}\right\rangle_{\mathbb{R}^2}.\end{aligned}$$

Therefore, if $\tau=0$ or Δ is a subalgebra, $-\frac{\tau}{a}\left\langle\Lambda^\theta_{-as}u,\mathbf{u}\right\rangle_{\mathbb{R}^2}=0$, implying that $F_{\mathbf{u}}((t,v)\exp s(a,w))=0$ for all $s\in\mathbb{R}$. Now, if $\tau\neq 0$ and Δ is not a subalgebra, we have that

$$\frac{d}{ds}F_{\mathbf{u}}((t,v)\exp s(a,w))=\tau\langle\rho_{-as}u,\mathbf{u}\rangle_{\mathbb{R}^2}.$$

In particular, since Δ is not a subalgebra, u is not an eigenvalue of θ and hence, there exists $\delta>0$ such that

$$\frac{d}{ds}F_{\mathbf{u}}((t,v)\exp s(a,w))=\tau\langle\rho_{-as}u,\mathbf{u}\rangle_{\mathbb{R}^2}\neq 0,\quad s\in(-\delta,\delta)\backslash\{0\},$$

showing that $0\in\mathbb{R}$ is an isolated critical point. In particular, $F_{\mathbf{u}}((t,v)\exp s(a,w))\neq 0$ in $s\in(-\delta,\delta)\backslash\{0\}$ as desired.
2. $Aw+a\xi-a\mu\theta u\notin l_\Delta$ In this case,

$$\frac{d}{ds}F_{\mathbf{u}}((t,v)\exp s(a,w))=\frac{1}{a}\langle\rho_{-as}(Aw+a\xi-a\mu\theta u),\mathbf{u}\rangle_{\mathbb{R}^2}.$$

Since

$$\frac{d}{ds}_{|s=0}F_{\mathbf{u}}((t,v)\exp s(a,w))=\frac{1}{a}\langle Aw+a\xi-a\mu\theta u,\mathbf{u}\rangle_{\mathbb{R}^2}\neq 0,$$

we get by continuity that there exists $\delta>0$ such that

$$\frac{d}{ds}F_{\mathbf{u}}((t,v)\exp s(a,w))\neq 0,\quad\forall s\in(-\delta,\delta),$$

and hence $F_{\mathbf{u}}((t,v)\exp s(a,w))$ is strictly increasing or strictly decreasing the interval $(-\delta,\delta)$. In particular,

$$F_{\mathbf{u}}((t,v)\exp s(a,w))\neq 0,\qquad\text{for all } s\in(-\delta,\delta)\backslash\{0\},$$

proving the result. □

Using the previous lemma we have the following.

Theorem 3.2.2. Let $\Sigma=\{\mathcal{X}=(\xi,A),\Delta^L\}$ be a simple ARS with $A\neq 0$ and $(a,w)\in\mathfrak{g}(\theta)$. If $(t,v)\in G\backslash\mathcal{Z}$, the exponential curve $s\mapsto(t,v)\exp s(a,w)$ satisfies:

1. $s\mapsto(t,v)\exp s(a,w)$ remains in the same component that contains (t,v) or
2. $s\mapsto(t,v)\exp s(a,w)$ intersects $\mathcal{Z}$ discretely.

Proof. Let us assume that for $s_0 \in \mathbb{R}$ it holds that $(t, v) \exp s_0(a, w) \in \mathcal{Z}$. Since

$$(t, v) \exp (s_0 + s) (a, w) = \underbrace{((t, v) \exp s_0(a, w))}_{\in \mathcal{Z}} \exp s(a, w),$$

the previous lemma implies the following:

1. For all $s \in \mathbb{R}$ it holds that $F_{\mathbf{u}} ((t, v) \exp (s_0 + s) (a, w)) \equiv 0$. If this condition holds, we would have that $(t, v) \exp (s_0 + s) (a, w) \in \mathcal{Z}, \forall s \in \mathbb{R}$ and consequently, $(t, v) \in (G \backslash \mathcal{Z}) \cap \mathcal{Z} = \varnothing$ which is not possible. Therefore, $s \mapsto (t, v) \exp s(a, w)$ remains in the same component that contains (t, v).
2. There exists $\delta > 0$ such that $F_{\mathbf{u}} ((t, v) \exp (s_0 + s) (a, w)) \neq 0$ for all $s \in (-\delta, \delta) \backslash \{0\}$. In this case, $(t, v) \exp (s_0 + s) (a, w)$ intersects $\mathcal{Z}$ discretely at the point $(t, v) \exp s_0(a, w)$ and $(t, v) \exp(s_0 + s)(a, w)$ remains in the same component if $F_{\mathbf{u}} ((t, v) \exp (s_0 + s) (a, w))$ does not changes sign for all $s \in (-\delta, \delta) \backslash \{0\}$ or $\{(t, v) \exp (s_0 + s) (a, w), \ s \in (-\delta, 0)\}$ and $\{(t, v) \exp (s_0 + s) (a, w), s \in (0, \delta)\}$ belong to different connected components if the sign of $F_{\mathbf{u}} ((t, v) \exp (s_0 + s) (a, w))$ changes in $(-\delta, \delta) \backslash \{0\}$. □

For the flow of the linear vector field of Σ we have the following:

Proposition 3.2.3. Let $\Sigma = \{\mathcal{X}, \Delta^L\}$ be a simple ARS on G. For any $(t, v) \in G \backslash \mathcal{Z}_{\mathcal{X}}$, define the set,

$$J_{(t,v)} = \{s \in \mathbb{R}; \quad \varphi_s(t, v) \in \mathcal{Z}\}.$$

It holds:

1. l_Δ is an eigenspace of A and $J_{(t,v)} = \varnothing$ or $J_{(t,v)} = \mathbb{R}$.
2. l_Δ is not a eigenspace of A and $J_{(t,v)}$ is discrete.

Proof. Since

$$\begin{aligned} F_{\mathbf{u}} (\varphi_s(t, v)) &= \left\langle \rho_{-t} \left(A \left(e^{sA} v + \Lambda_s^A \Lambda_t^\theta \xi \right) + \Lambda_t^\theta \xi \right), \mathbf{u} \right\rangle_{\mathbb{R}^2} \\ &= \left\langle \rho_{-t} \left(A e^{sA} v + \left(A \Lambda_s^A + \mathrm{id}_{\mathbb{R}^2} \right) \Lambda_t^\theta \xi \right), \mathbf{u} \right\rangle_{\mathbb{R}^2} \\ &= \left\langle e^{sA} \left(\rho_{-t} \left(A v + \Lambda_t^\theta \xi \right) \right), \mathbf{u} \right\rangle_{\mathbb{R}^2} \quad \text{by using the property 4 of } \Lambda^A, \end{aligned}$$

the result follows from Lemma A.1.2. □

The previous result shows that if (t, v) is not a fixed point of $\mathcal{X}$ then the orbit of $\mathcal{X}$ starting at such point is contained in $\mathcal{Z}$ do not touch $\mathcal{Z}$ or crosses $\mathcal{Z}$ dicretely.

3.3 Examples of singular locus

In this section some examples of singular locus are presented.

Example 3.3.1. Let us consider the Lie algebra $\mathfrak{g}(\theta)$ with $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\Sigma_1 = \{\mathcal{X}, \Delta\}$ to be a simple ARS, where $\Delta = \operatorname{span}\{(1,0),(0,e_1)\}$, e_1 and e_2 stand for the canonical basis of $\mathbb{R}^2$. Notice that $[(1,0),(0,e_1)] = (0,e_2)$, that is, Δ is not a subalgebra consequently the LARC is satisfied.
The associated Lie group is called the Euclidean motion group.
The linear vector field is $\mathcal{X}(t,v) = (0, Av + \Lambda_t^\theta \xi)$, here $\xi = (a,b)$ and $v = (x,y)$ with $\xi, v \in \mathbb{R}^2$. Since $(dL_{(t,v)})_{(0,0)}(1,0) = (1,0)$ and $(dL_{(t,v)})_{(0,0)}(0,e_1) = (0,\rho_t e_1)$. The left-invariant distribution Δ^L is given by $\Delta^L(t,v) = \operatorname{span}\{(1,0),(0,\rho_t e_1)\}$. In this case $\rho_t = e^{t\theta} = \begin{pmatrix} \cos t & -\text{sin}t \\ \text{sin}t & \cos t \end{pmatrix}$, and $\theta^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so

$$\Lambda_t^\theta = (\rho_t - \mathrm{id}_{\mathbb{R}^2})\theta^{-1} = \begin{pmatrix} \text{sin}t & \cos t - 1 \\ 1 - \cos t & \text{sin}t \end{pmatrix}.$$

Notice that the equality $[A,\theta] = 0$ implies that A has the form $A = \begin{pmatrix} \lambda_1 & -\mu_1 \\ \mu_1 & \lambda_1 \end{pmatrix}$. Then

$$\mathcal{X}(t,v) = (0, Av + \Lambda_t^\theta \xi) = (0, \lambda_1 x - \mu_1 y + a\text{sin}t + b(\cos t - 1), \mu_1 x + \lambda_1 y + a(1-\cos t) + b\text{sin}t).$$

By definition of singular locus $(t,v) \in \mathcal{Z}$ if and only if $\mathcal{X}(t,v) \in \Delta^L(t,v)$ that is

$$Av + \Lambda_t^\theta = \rho_t e_1,$$

or equivalently

$$\langle Av + \Lambda_t^\theta \xi, \rho_t e_2 \rangle_{\mathbb{R}^2} = 0,$$

using that $\rho_t e_2 = (-\text{sin}t, \cos t)$ and developing the inner product in the previous equality, allows us to obtain that

$$(t,v) \in \mathcal{Z} \iff (\mu_1 y - \lambda_1 x)\text{sin}t + (\mu_1 x + \lambda_1 y)\cos t + b\text{sin}t + a\cos t - a = 0.$$

Consequently,

$$\mathcal{Z} = \{(t,v) : \ (\mu_1 y - \lambda_1 x)\text{sin}t + (\mu_1 x + \lambda_1 y)\cos t + b\text{sin}t + a\cos t - a = 0\}.$$

Considering $A = 0$. By the previous calculations, the singular locus is given by

$$\mathcal{Z} = \{(t,v) : b\,\text{sin}t + a\cos t = a\}. \tag{3.10}$$

This locus singular has infinites connected components. In fact, if γ is the angle between ξ and e_1 we have that $\cos\gamma = \dfrac{a}{a^2+b^2}$ and $\sin\gamma = \dfrac{b}{a^2+b^2}$. Therefore, if $(t,v) \in \mathcal{Z}$ we have that $\cos\gamma = \dfrac{a}{a^2+b^2}\cos t + \dfrac{b}{a^2+b^2}\sin t = \cos\gamma\cos t + \sin\gamma\sin t = \cos(t-\gamma)$. Therefore

$$(t,v) \in \mathcal{Z} \text{ if and only if } \cos(\gamma) - \cos(\gamma - t).$$

by using the identity

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),$$

we get that $x = 2\pi k - y$ or $x = 2\pi k + y$, for some $k \in \mathbb{Z}$. By replacing $x = \gamma$ and $y = \gamma - t$, we obtain that $t = 2(\gamma - \pi k)$ or $t = 2\pi k$, writing $\Gamma = \{2(\gamma - \pi k)\ k \in \mathbb{Z}\} \cup \{2\pi k\ k \in \mathbb{Z}\}$, we see that $\mathcal{Z} = \Gamma \times \mathbb{R}^2$ and hence $\mathcal{Z}$ is a submanifold with infinites connected components

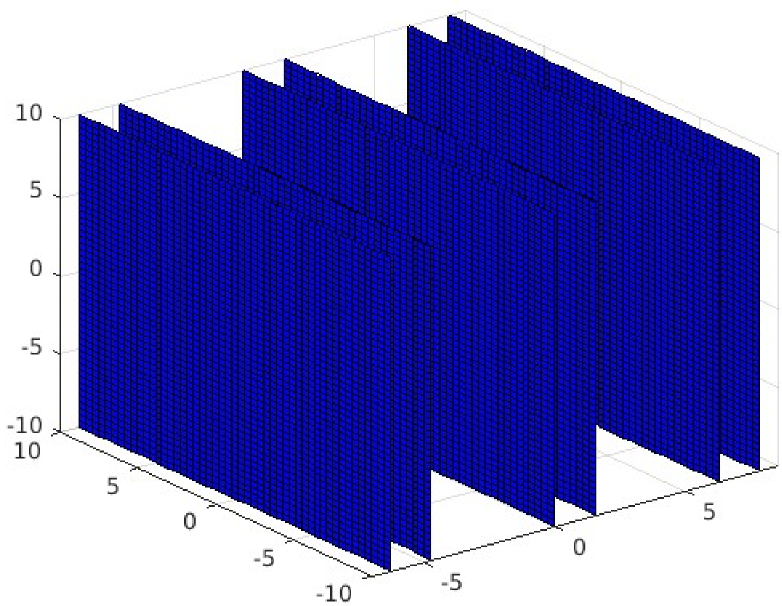

Figure 2 – Singular locus of (3.10) with , $a = b = 1$. $t, x, y \in [-10, 10]$.

Example 3.3.2. Consider the Lie algebra $\mathfrak{g}(\theta)$ with $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\Sigma_1 = \{\mathcal{X}, \Delta\}$ to be a simple ARS, where $\Delta = \operatorname{span}\{(1,0),(0,e_1)\}$. Since $[(1,0),(0,e_1)] = (0,e_1)$ we get that Δ is a subalgebra. However, if we assume that $\xi \notin \Delta$, Proposition 2.4.1 assures that LARC is satisfied. In this case

$$A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_1 \end{pmatrix}, \quad \rho_t = e^{t\theta} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \quad \text{and} \quad \theta^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

For $v = (x,y)$ and $\xi = (a,b)$, we have that

$$Av = (\lambda_1 x + \lambda_2 y, \lambda_1 y) \quad \text{and} \quad \Lambda_t^\theta = \begin{pmatrix} e^t - 1 & -e^t + 1 + te^t \\ 0 & e^t - 1 \end{pmatrix},$$

implying that

$$\mathcal{X}(t,v) = (0, (b-a) + e^t(bt - b + a) + \lambda_1 x + \lambda_2 y, b(e^t - 1) + \lambda_1 y).$$

By straightforward calculations, we get that

$$\mathcal{Z} = \{(t,v) : \lambda_1 y = (1 - e^t)b\}, \tag{3.11}$$

Where $b \neq 0$ by LARC.

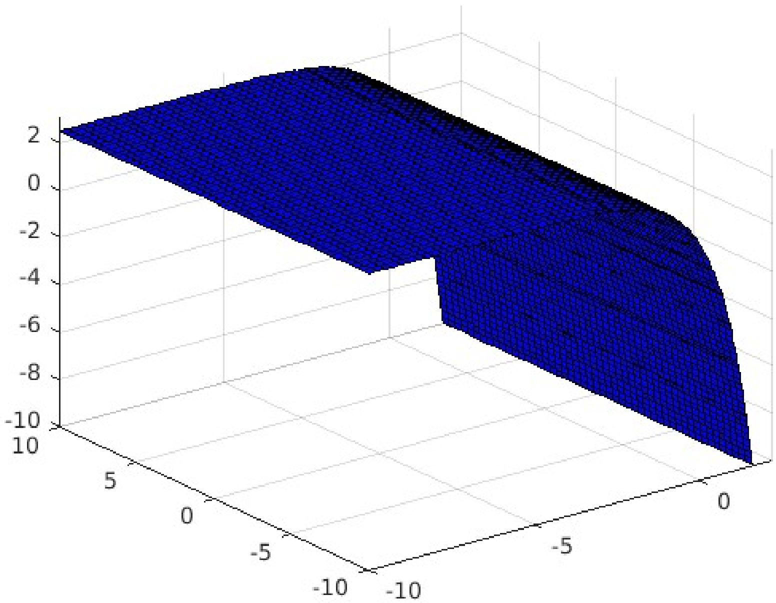

Figure 3 – Singular locus of (3.11) with $\lambda_1 = 2, b = 5$.

4 Isometries between rank two ARS's

In this chapter we define rank two ARS on the group $G = \mathbb{R} \times_\rho \mathbb{R}^2$. Our main goal of here is to prove that isometries between ARS's of rank two are automorphisms. We begin in Section 4.1 by introducing the definition of linear vector field of rank two and using that we define rank two ARS's. Subsequently in Section 4.2 the fundamental lemma is proved. This lemma provide a sufficient condition for an isometry between ARS to be a automorphism. In Section 4.3 we prove that isometries between rank two ARS's let the nilradical of $\mathfrak{g}(\theta)$ invariant. In Section 4.4 the main theorem is proved, that is, we establish that the only isometries between rank two of ARS are automorphisms of G. By using the previous result we are able to present a classification of rank two ARS's in Section 4.5.

4.1 Isometries of rank two ARS's

In this section we prove that the only isometries between rank two ARS's are automorphisms of the group. This result allow us to classify, up to automorphisms and reescalonation, that the only possible rank two ARS on connected three-dimensional solvable nonnilpotent Lie groups. We begin with the definition of linear vector field of rank two:

Definition 4.1.1. We say that a linear vector field is a rank two linear vector field if its associated derivation has rank two.

Definition 4.1.2. A simple ARS Σ is said to be a rank two ARS if the associated linear vector field has rank two.

By formula (1.4) we see that isometries preserves rank two ARS's.

Remark 4.1.3. Since the derivation associated with a linear field $\mathcal{X} = (\xi, A)$ is given by $\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix}$, we have that

$$\mathcal{X} \text{ has rank two } \iff \mathbb{R}^2 = \operatorname{Im} A + \mathbb{R}\xi \ .$$

Moreover, the expression of $\mathcal{D}$ shows that the rank of $\mathcal{D}$ is at most two and consequently, the set of rank two derivations is open and dense in $\operatorname{Der}(\mathfrak{g}(\theta))$. Also, the fact that $\operatorname{Der}(\mathfrak{g}(\theta))$ is isomorphic to the set of linear vector fields [8] implies that rank two linear vector fields, and consequently rank two ARS's, are (topologically) big.

Let $\psi : G \to G$ be an isometry between rank two ARS's Σ_1 and Σ_2 and consider $f : G \to \mathbb{R}$ and $g : G \to \mathbb{R}^2$ the coordinate functions of ψ. Write, on the canonical basis,

$$\psi_* = \begin{pmatrix} \partial_1 f & (\partial_2 f)^T \\ \partial_1 g & \partial_2 g \end{pmatrix},$$

where for $(t, v) \in G, \partial_2 f(t, v)$ is the gradient vector of the partial map $v \in \mathbb{R}^2 \mapsto f(t, v) \in \mathbb{R}$. If $\left\{\varphi_s^i\right\}_{t\in\mathbb{R}}$ is the flow associated with the linear vector fields of Σ_i, we have by formula (1.4) that Using the expression for the flows of $\mathcal{X}_i$ provided by equation (2.12) and equation (1.4) we get that

$$\begin{aligned}\left(f\left(t, e^{sA_1}v + \Lambda_t^\theta\Lambda_s^{A_1}\xi\right), g\left(t, e^{sA_1}v + \Lambda_t^\theta\Lambda_s^{A_1}\xi\right)\right) =& \psi\left(t, e^{sA_1}v + \Lambda_t^\theta\Lambda_s^{A_1}\xi\right)\\ =& \psi\left(\varphi_s^1(t, v)\right)\\ =& \varphi_s^2\left(\psi(t, v)\right)\\ =& \varphi_s^2\left(f(t, v), g(t, v)\right)\\ =& \left(f(t, v), e^{sA_2}g(t, v) + \Lambda_s^{A_2}\Lambda_{f(t,v)}^\theta\xi_2\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} f\left(t, \mathrm{e}^{sA_1}v + \Lambda_s^{A_1}\Lambda_t^\theta\xi_1\right) &= f(t, v),\\ g\left(t, \mathrm{e}^{sA_1}v + \Lambda_s^{A_1}\Lambda_t^\theta\xi_1\right) &= \mathrm{e}^{sA_2}g(t, v) + \Lambda_s^{A_2}\Lambda_{f(t,v)}^\theta\xi_2 \end{aligned} \tag{4.1}$$

Remark 4.1.4. The main result of this thesis: Theorem 4.4.1 shows that isometries between rank two ARS's on nonnilpotent, solvable three-dimensional Lie groups are automorphisms. For the proof of this theorem we employ the following steps.

- First we show that, if ψ preserves a left-invariant vector field of the associated distribution, then ψ is in fact an automorphism of G. This result simplifies our problem to look for vectors in the distribution which are preserved by the isometry.
- In the second part, we show that the differential of any rank two isometry let the nilradical $\{0\} \times \mathbb{R}^2$ of $\mathfrak{g}(\theta)$ invariant, or equivalently, if $\psi = (f, g)$ then $\partial_2 f \equiv 0$.
- In the third and last step, we show that if ψ_* let the nilradical invariant, then it preserves any left-invariant vector field in the intersection of the nilradical with the distribution of the ARS.

As a consequence, ψ is an automorphism by the first step.

4.2 The fundamental lemma

We use this section to prove a technical lemma which gives us a sufficient condition for an isometry between simple ARS's to be a group automorphism.

Lemma 4.2.1. (Fundamental Lemma) Let $\Sigma_1 = \{\mathcal{X}_1 = (\xi_1, A_1), \Delta_1^L\}$ and $\Sigma_2 = \{\mathcal{X}_2 = (\xi_2, A_2), \Delta_2^L\}$ be simple ARS's on the Lie group $G = \mathbb{R} \times_\rho \mathbb{R}^2$ with $A_1 \neq 0$, and consider $\psi \in \mathrm{Iso}_G(\Sigma_1; \Sigma_2)_0$. If there exists a nonzero vector $X \in \Delta_1$ such that

$$\forall (t,v) \in G; \quad (d\psi)_{(t,v)} X^L(t,v) = \left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} X, \tag{4.2}$$

then $\psi \in \mathrm{Aut}(G)$.

Proof. : Let us assume w.l.o.g. that $(1,0) \in \Delta_1$. We prove the lemma in four steps:
Step 1: For all $Z \in \Delta_1$ it holds that

$$\forall (t,v) \in G; (d\psi)_{(t,v)} Z^L(t,v) = \left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} Z.$$

Let us consider $Y \in \Delta_1$ such that

$$\{X, Y\} \text{ is an orthogonal basis of } \Delta_1 \text{ with } \|Y\|_{\Sigma_1,(0,0)} = \|X\|_{\Sigma_1,(0,0)}.$$

By linearity, it is enough to show that the relation (4.2) holds for Y. From the left-invariance of the metric in Δ_1^L we get that $\left\{X^L(t,v), Y^L(t,v)\right\}$ is an orthogonal basis of $\Delta_1^L(t,v)$ also satisfying

$$\begin{aligned}\|Y^L(t,v)\|_{\Sigma_1,(t,v)} =& \|Y\|_{\Sigma_1,(0,0)} \\ =& \|X\|_{\Sigma_1,(0,0)} \\ =& \|X^L(t,v)\|_{\Sigma_1,(t,v)}.\end{aligned}$$

Using that ψ is an isometry, it holds that $(d\psi)_{(t,v)} Y^L(t,v) \in \Delta_2^L(\psi(t,v))$ is orthogonal to $(d\psi)_{(t,v)} X^L(t,v)$ and

$$\begin{aligned}\left\|(d\psi)_{(t,v)} Y^L(t,v)\right\|_{\Sigma_2,\psi(t,v)} =& \left\|Y^L(t,v)\right\|_{\Sigma_1,(t,v)} \\ =& \|Y\|_{\Sigma_1,(0,0)}.\end{aligned}$$

On the other hand, the left-invariance of the metric in Δ_2^L implies that $\left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} Y \in \Delta_2(\psi(t,v))$ is orthogonal to $\left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} X$ and

$$\begin{aligned}\left\|\left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} Y\right\|_{\Sigma_2,\psi(t,v)} =& \left\|(d\psi)_{(0,0)} Y\right\|_{\Sigma_2,(0,0)} \\ =& \|Y\|_{\Sigma_1,(0,0)}.\end{aligned}$$

Since by hypothesis,

$$(d\psi)_{(t,v)}X^L(t,v) = \left(dL_{\psi(t,v)}\right)_{(0,0)}(d\psi)_{(0,0)}X,$$

the fact that $\dim\left(\Delta_2^L(\psi(t,v)\right) = 2$ forces that

$$(d\psi)_{(t,v)}Y^L(t,v) = \varepsilon(t,v)\left(dL_{\psi(t,v)}\right)_{(0,0)}(d\psi)_{(0,0)}Y, \quad \text{where} \quad \varepsilon(t,v) = \pm 1. \tag{4.3}$$

Moreover, by orthogonality, we obtain that

$$\varepsilon(t,v) = \frac{\left\langle (d\psi)_{(t,v)}Y^L(t,v), \left(dL_{\psi(t,v)}\right)_{(0,0)}(d\psi)_{(0,0)}Y\right\rangle_{\Sigma_2,\psi(t,v)}}{\|X\|_{\Sigma_1,(0,0)}},$$

showing that ε is a continuous function on $G\backslash\mathcal{Z}_1$ and hence, ε is constant on the connected components of $G\backslash\mathcal{Z}_1$.

Since $A_1 \neq 0$, Theorem 4.5.1 implies that $G\backslash\mathcal{Z}_1$ has two connected $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$. Let us consider $\varepsilon^+ := \varepsilon|_{\mathcal{C}_1^+}$ and $\varepsilon^- := \varepsilon|_{\mathcal{C}_1^-}$.

If for some $(t,v) \in G\backslash\mathcal{Z}$ the curve $s \mapsto (t,v)\exp sY$ intersects $\mathcal{Z}_1$, then by Theorem 4.5.1 such intersection is a discrete set. Assume w.l.o.g. that $(t,v) \in \mathcal{C}_1^+$ and consider the sets

$$I^+ := \left\{s \in \mathbb{R}; (t,v)\exp sY \in \mathcal{C}_1^+\right\} \quad \text{and} \quad I^- := \left\{s \in \mathbb{R}; (t,v)\exp sY \in \mathcal{C}_1^-\right\}.$$

These sets are open and their union is, by Theorem 4.5.1, dense in $\mathbb{R}$. Moreover, the curves

$$\gamma_\pm : I^\pm \to G, \quad \gamma_\pm(s) := \psi\left((t,v)\exp sY\right)$$

are differentiable and by equation (4.3) satisfies

$$\begin{aligned}\frac{d}{ds}\gamma_\pm(s) &= (d\psi)_{(t,v)\exp sY}Y^L((t,v))\exp sY\\ &= \left(\varepsilon^\pm Z\right)^L\left(\psi((t,v)\exp sY)\right)\\ &= \left(\varepsilon^\pm Z\right)^L\left(\gamma_\pm(s)\right),\end{aligned}$$

where for simplicity $Z := (d\psi)_{(0,0)}Y$. Therefore, $\gamma_\pm(s)$ coincides with the solution of the ODE defined by the vector field $\varepsilon^\pm Z^L$ on the open set $I^\pm$. By uniqueness we get that

$$\psi((t,v)\exp sY) = \psi(t,v)\exp s\varepsilon^\pm Z, \quad \text{for all} \quad s \in I^\pm.$$

Since $\mathbb{R}\backslash I^+ \cup I^-$ are the points where the curve $s \mapsto (t,v)\exp sY$ intersects $\mathcal{Z}_1$, we have by Theorem 3.2.2 that $\mathbb{R}\backslash I^+ \cup I^-$ is discrete. Let us assume that $I^- \neq \emptyset$. In this case, there exist $s_0 \in \mathbb{R}\backslash I^+ \cup I^-$ and $\delta > 0$ such that

$$(t,v)\exp sY \in \mathcal{C}_1^+ \;\; s \in (s_0, s_0+\delta) \quad \text{and} \quad (t,v)\exp sY \in \mathcal{C}_1^- \;\; s \in (s_0-\delta, s_0),$$

which by continuity implies

$$\begin{aligned}\psi(t,v)\exp s_0\varepsilon^+Z =& \psi\left((t,v)\exp s_0Y\right)\\ =& \psi(t,v)\exp s_0\varepsilon^-Z\end{aligned}$$

implying $\exp s_0\varepsilon^+Z = \exp s_0\varepsilon^-Z$. By the expression of the exponential in equation (2.13) and the fact that $s_0 \neq 0$ we conclude that $\varepsilon^+ = \varepsilon^-$.
On the other hand, if $I^- = \emptyset$, let us consider an open set $(t,v) \in U \subset \mathcal{C}_1^+$ and $s_0 \in \mathbb{R}\backslash I^+$. The open set $U\exp s_0Y$ intersects $\mathcal{Z}_1$ at the point $(t,v)\exp s_0Y$ and, since $\mathcal{Z}_1$ is a two-dimensional embedded manifold, we have that

$$U\exp s_0 \cap \mathcal{C}_1^- \neq \emptyset.$$

In particular, there exists $(\hat{t},\hat{v}) \in \mathcal{C}_1^+$ such that $(\hat{t},\hat{v})\exp s_0Y \in \mathcal{C}_1^-$. As a consequence, both of the sets

$$\hat{I}^+ := \left\{s\in\mathbb{R}; (\hat{t},\hat{v})\exp sY \in \mathcal{C}_1^+\right\} \quad \text{and} \quad \hat{I}^- := \left\{s\in\mathbb{R}; (\hat{t},\hat{v})\exp sY \in \mathcal{C}_1^-\right\},$$

are nonempty. By doing the previous analysis with $(\hat{t},\hat{v})$ instead of (t,v) allows us to conclude that $\varepsilon^+ = \varepsilon^-$.
Now, if for all $(t,v)\in G\backslash\mathcal{Z}_1$ and $s\in\mathbb{R}$ we have that $(t,v)\exp s(a,w) \in G\backslash\mathcal{Z}_1$, then as previously for all $(t,v)\in\mathcal{C}_1^{\pm}$ we get

$$\psi((t,v)\exp sY) = \psi(t,v)\exp s\varepsilon^{\pm}Z, \quad \text{for all} \quad s\in\mathbb{R}.$$

Again by the fact that $\mathcal{Z}_1$ is an embedded two-dimensional manifold, there exists $\gamma : (-\delta,\delta) \to G$ satisfying

$$\gamma(0)\in\mathcal{Z}, \quad \gamma(-\delta,0)\subset\mathcal{C}_1^- \quad \text{and} \quad \gamma(0,\delta)\subset\mathcal{C}^+,$$

and hence, for all $s\in\mathbb{R}$ it holds that

$$\psi(\gamma(\tau)\exp sY) = \psi(\gamma(\tau))\exp s\varepsilon^+Z, \tau\in(0,\delta) \text{ and } \psi(\gamma(\tau)\exp sY) = \psi(\gamma(\tau))\exp s\varepsilon^-Z, \tau\in(-\delta,0).$$

By considering the limit $\tau\to 0$ we get by continutity that

$$\psi(\gamma(0))\exp s\varepsilon^+Z = \psi(\gamma(0)\exp sY) = \psi(\gamma(0))\exp s\varepsilon^-Z, \quad \forall s\in\mathbb{R},$$

implying that $\varepsilon^+ = \varepsilon^-$ and concluding the proof of step 1.
Step 2: It holds that

$$f(t,v) = at \quad \text{and} \quad g(t,v) = g(0,v) + \Lambda_t^{a\theta}\partial_1 g(0,0).$$

Indeed. Since $\left(dL_{(t,v)}\right)_{(0,0)}(1,0) = (1,0)$ we have that

$$\begin{aligned}(\partial_1 f(t,v), \partial_1 g(t,v)) =& (d\psi)_{(t,v)}(1,0)\\ =& \left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)}(1,0)\\ =& \left(dL_{\psi(t,v)}\right)_{(0,0)} (\partial_1 f(0,0), \partial_1 g(0,0))\\ =& \left(\partial_1 f(0,0), \rho_{f(t,v)}\partial_1 g(0,0)\right)\end{aligned}$$

implying that

$$\partial_1 f(t,v) = \partial_1 f(0,0) \quad \text{and} \quad \partial_1 g(t,v) = \rho_{f(t,v)}\partial_1 g(0,0). \tag{4.4}$$

Analogously, for $(0,u) \in \Delta \cap \{0\} \times \mathbb{R}^2$

$$\begin{aligned}(\langle \partial_2 f(t,v), \rho_t u\rangle_{\mathbb{R}^2}, \partial_2 g(t,v)\rho_t u) =& (d\psi)_{(t,v)}(0,\rho_t u)\\ =& \left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)}(0,u)\\ =& \left(dL_{\psi(t,v)}\right)_{(0,0)} (\langle \partial_2 f(0,0), u\rangle_{\mathbb{R}^2}, \partial_2 g(0,0)u)\\ =& \left(\langle \partial_2 f(0,0), u\rangle_{\mathbb{R}^2}, \rho_{f(t,v)}\partial_2 g(0,0)u\right),\end{aligned}$$

implying that

$$\langle \partial_2 f(t,v), \rho_t u\rangle_{\mathbb{R}^2} = \langle \partial_2 f(0,0), u\rangle_{\mathbb{R}^2} \quad \text{and} \quad \partial_2 g(t,v)\rho_t u = \rho_{f(t,v)}\partial_2 g(0,0)u \tag{4.5}$$

By integration, equation (4.4) implies that

$$f(t,v) = at + h(v), \quad a = \partial_1 f(0,0) \quad \text{and} \quad h(v) = f(0,v).$$

Therefore, from equation (4.5) we get that for all $(t,v) \in G$,

$$\begin{aligned}& \langle \nabla h(v), \rho_t u\rangle_{\mathbb{R}^2} = \langle \nabla h(0), u\rangle_{\mathbb{R}^2}\\ \Longrightarrow\ & \langle \nabla h(v), \rho_t u\rangle_{\mathbb{R}^2} = \langle \nabla h(v), u\rangle_{\mathbb{R}^2}\\ \Longrightarrow\ & \langle \nabla h(v), (\rho_t u - u)\rangle_{\mathbb{R}^2} = 0\end{aligned}$$

On the other hand, $f \circ \varphi_s^1 = f$ implies that

$$h\left(\mathrm{e}^{sA_1}v + \Lambda_t^\theta \Lambda_s^{A_1}\xi_1\right) = h(v), \tag{4.6}$$

and so,

$$h\left(\mathrm{e}^{sA_1}v\right) = h(v) \quad \Longrightarrow \quad \nabla h\left(\mathrm{e}^{sA_1}v\right) = \mathrm{e}^{-sA_1^T}\nabla h(v).$$

Hence, $\forall v \in \mathbb{R}^2, \;\; s \in \mathbb{R}$,

$$\begin{aligned}\langle \nabla h(v), u\rangle_{\mathbb{R}^2} =& \left\langle \nabla h\left(e^{sA_1}v\right), u\right\rangle_{\mathbb{R}^2}\\ =& \left\langle \mathrm{e}^{-sA_1^T}\nabla h(v), u\right\rangle_{\mathbb{R}^2}\\ =& \left\langle \nabla h(v), \mathrm{e}^{-sA_1}u\right\rangle_{\mathbb{R}^2}\\ \Longrightarrow\ & \left\langle \nabla h(v), \left(e^{sA_1}u - u\right)\right\rangle_{\mathbb{R}^2} = 0.\end{aligned}$$

Now, derivation of (4.6) first at $t = 0$ and then at $s = 0$, gives us

$$\langle \nabla h(v), \xi_1 \rangle_{\mathbb{R}^2} = 0,$$

which together with the previous equations imply that $\left\{\xi_1, \mathrm{e}^{sA_1}u - u, \rho_t u - u,\ t, s \in \mathbb{R}\right\}$ belongs to the same line if $\nabla h(v) \neq 0$ for some $v \in \mathbb{R}^2$.
However, if $\left\{\xi_1, \mathrm{e}^{sA_1}u - u, \rho_t u - u,\ t, s \in \mathbb{R}\right\}$ belongs to the same line we necessarily have that u is a common eigenvalue of θ and A_1 which gives us that Δ_1 is a subalgebra that contains $(1, 0)$, $A_1 l_{\Delta_1} \subset l_{\Delta_1}$ and $\xi_1 \in l_{\Delta_1}$, which by Proposition 2.4.1 contradicts the LARC. Therefore

$$\nabla h(v) = 0 \quad \text{for all} \quad v \in \mathbb{R}^2 \quad \Rightarrow \quad h \quad \text{is constant.}$$

Since $f(0, 0) = 0$ we get that $h = 0$ and hence $f(t, v) = at$ as stated.
Now, since $f(t, v) = at$, equation (4.4) implies that

$$\partial_1 g(t, v) = \rho_{at} \partial_1 g(0, 0)$$

which by integration gives us

$$\begin{aligned} g(t, v) - g(0, v) &= \int_0^t \partial_1 g(s, v) ds \\ &= \int_0^t \rho_{as} \partial_1 g(0, 0) d \\ &= \Lambda_t^{a\theta} \partial_1 g(0, 0), \end{aligned}$$

then

$$g(t, v) = g(0, v) + \Lambda_t^{a\theta} \partial_1 g(0, 0),$$

showing the assertion for g and proving Step 2.
Step 3 : For all $(t, v) \in G$ and $s \in \mathbb{R}$, it holds that

$$\partial_2 g(0, v) \rho_t \xi_1 = \rho_{at} \partial_2 g(0, 0) \xi_1 \quad \text{and} \quad \partial_2 g(0, v) \rho_t \mathrm{e}^{-sA_1} u = \rho_{at} \partial_2 g(0, 0) \mathrm{e}^{-sA_1} u. \tag{4.7}$$

Since ψ commutes the flows of the linear vector fields of Σ_1 and Σ_2, it holds that

$$g\left(t, \mathrm{e}^{sA_1} v + \Lambda_s^{A_1} \Lambda_t^{\theta} \xi_1\right) = \mathrm{e}^{sA_2} g(t, v) + \Lambda_s^{A_2} \Lambda_{at}^{\theta} \xi_2,$$

and by Step 2,

$$g\left(0, \mathrm{e}^{sA_1} v + \Lambda_s^{A_1} \Lambda_t^{\theta} \xi_1\right) + \Lambda_{at}^{\theta} \eta = \mathrm{e}^{sA_2} g(0, v) + \mathrm{e}^{sA_2} \Lambda_{at}^{\theta} \eta + \Lambda_s^{A_2} \Lambda_{at}^{\theta} \xi_2,$$

where $\eta = \partial_1 g(0, 0)$. Therefore,

$$\begin{aligned} g\left(0, \mathrm{e}^{sA_1} v + \Lambda_s^{A_1} \Lambda_t^{\theta} \xi_1\right) - \mathrm{e}^{sA_2} g(0, v) &= \left(\mathrm{e}^{sA_2} - \mathrm{id}_{\mathbb{R}^2}\right) \Lambda_{at}^{\theta} \eta + \Lambda_s^{A_2} \Lambda_{at}^{\theta} \xi_2 \\ &= \Lambda_s^{A_2} \Lambda_{at}^{\theta} \left(\xi_2 + A_2 \eta\right). \end{aligned} \tag{4.8}$$

Derivation of equation (4.8) on t gives us, by the chain rule, that

$$\partial_2 g\left(0, \mathrm{e}^{sA_1}v + \Lambda_s^{A_1}\Lambda_t^{\theta}\xi_1\right)\rho_t\Lambda_s^{A_1}\xi_1 = \rho_{at}\Lambda_s^{A_2}\left(\xi_2 + A_2\eta\right).$$

Since the previous equations is true for any $s \in \mathbb{R}$ and any $(t, v) \in G$, we can substitute v by $e^{-sA_1}\left(v - \Lambda_s^{A_1}\Lambda_t^{\theta}\xi_1\right)$ in order to obtain

$$\partial_2 g(0, v)\rho_t\Lambda_s^{A_1}\xi_1 = \rho_{at}\Lambda_s^{A_2}\left(\xi_2 + A_2\eta\right).$$

Derivating now the last equation at $s = 0$ gives us

$$\begin{aligned}\partial_2 g(0, v)\rho_t\xi_1 =& \rho_{at}\left(\xi_2 + A_2\eta\right)\\ =&\rho_{at}\partial_2 g(0, 0)\xi_1,\end{aligned}$$

proving the first equality. For the second equality, let us notice that the right-hand side of equation (4.8) does not depends on v and hence

$$v \in \mathbb{R}^2 \mapsto g\left(0, \mathrm{e}^{sA_1}v + \Lambda_s^{A_1}\Lambda_t^{\theta}\xi_1\right) - \mathrm{e}^{sA_2}g(0, v),$$

has differential zero. By the chain rule, we obtain

$$\partial_2 g\left(0, \mathrm{e}^{sA_1}v + \Lambda_s^{A_1}\Lambda_t^{\theta}\xi_1\right)\mathrm{e}^{sA_1} = \mathrm{e}^{sA_2}\partial_2 g(0, v).$$

On the other hand, by Step 2.

$$\forall (t, v) \in G, \quad \partial_2 g(t, v) = \partial_2 g(0, v) \quad \text{and} \quad f(t, v) = at.$$

Therefore, for any $(t, v) \in G$ and and $s \in \mathbb{R}$ we get

$$\begin{aligned}\partial_2 g(0, v)\rho_t\mathrm{e}^{-sA_1} =& \partial_2 g(0, v)\mathrm{e}^{-sA_1}\rho_t u\\ =&\mathrm{e}^{-sA_2}\partial_2 g\left(0, \mathrm{e}^{sA_1}v + \Lambda_s^{A_1}\Lambda_t^{\theta}\xi_1\right)\rho_t u\\ =&\mathrm{e}^{-sA_2}\rho_{at}\partial_2 g(0, 0)u\\ =&\rho_{at}\mathrm{e}^{-sA_2}\partial_2 g(0, 0)u\\ =&\rho_{at}\partial_2 g(0, 0)\mathrm{e}^{-sA_1}u.\end{aligned}$$

showing the assertion.
Step 4: $\psi \in \mathrm{Aut}(G)$
We have to analyze the following ones:

1. Δ_1 is not a subalgebra: In this case, $u \neq 0$ is not an eigenvalue of θ. Since, for $t = s = 0$ the second equation in (4.7) implies that

$$\partial_2 g(0, v)\rho_t u = \partial_2 g(0, 0)u,$$

it holds that

$$\forall v \in \mathbb{R}^2, \quad \partial_2 g(0,v)\rho_t u = \rho_{at}\partial_2 g(0,0)u = \rho_{at}\partial_2 g(0,v)u.$$

Since $a\partial_2 g(0,v) = \det(d\psi)_{(t,v)} \neq 0$ we have necessarily that $\partial_2 g(0,v) \in \mathrm{Gl}(2,\mathbb{R})$. Applying Lemma A.1.1 to the linear maps $\partial_2 g(0,v)$ gives us that,

$$\forall v \in \mathbb{R}^2, \quad \partial_2 g(0,v) = \partial_2 g(0,0) \quad \text{and} \quad \partial_2 g(0,0)\circ\theta = a\theta\circ\partial_2 g(0,0),$$

where $a = 1$ if $\operatorname{tr}\theta \neq 0$ or $a \in \{-1,1\}$ if $\operatorname{tr}\theta = 0$.

2. Δ_1 is a subalgebra and $\theta \neq \mathrm{id}_{\mathbb{R}^2}$:
In this case, the fact that $A_1\theta = \theta A_1$ implies that u is also an eigenvector of A_1. Since we are assuming $(1,0) \in \Delta_1$ the LARC implies by Proposition 2.4.1 that $\{\xi_1, u\}$ is linearly independent. Considering $t = s = 0$ in equations (4.7) gives us that

$$\partial_2 g(0,v)u = \partial_2 g(0,0)u \quad \text{and} \quad \partial_2 g(0,v)\xi_1 = \partial_2 g(0,v)\xi_1,$$

which by linearity implies that,

$$\forall v \in \mathbb{R}^2 \quad \partial_2 g(0,v) = \partial_2 g(0,0) \quad \text{and} \quad \partial_2 g(0,v)\circ\rho_t = \rho_{at}\circ\partial_2 g(0,v).$$

By differentiation, we get that $\partial_2 g(0,0)\circ\theta = a\theta\circ\partial_2 g(0,0)$ implying that $a = 1$ if $\operatorname{tr}\theta \neq 0$ or $a \in \{-1,1\}$ if $\operatorname{tr}\theta = 0$.

3. Δ_1 is a subalgebra and $\theta = \mathrm{id}_{\mathrm{R}^2}$:
In this case,

$$\begin{aligned} \mathrm{e}^t\partial_2 g(0,v)u =& \mathrm{e}^t\partial_2 g(0,v)\rho_t u \\ =& \rho_{at}\partial_2 g(0,0)u \\ =& \mathrm{e}^{at}\partial_2 g(0,v)u, \end{aligned}$$

implying that $a = 1$. If $\{\xi, u\}$ is linearly independent, we can conclude as in the previous item that

$$\forall v \in \mathbb{R}^2, \quad \partial_2 g(0,v) = \partial_2 g(0,0).$$

On the other hand, if $\{\xi_1, u\}$ is linearly dependent, the LARC implies necessarily that u cannot be an eigenvector of A_1 (see Proposition 2.4.1). In particular, for some $s_0 \in \mathbb{R}$, $\left\{u, \mathrm{e}^{-s_0A_1}u\right\}$ is a basis of $\mathbb{R}^2$ and by Step 3. the linear maps $\partial_2 g(0,v)$ and $\partial_2 g(0,0)$ coincides on such basis, implying that $\partial_2 g(0,v) = \partial_2 g(0,0)$ for all $v \in \mathbb{R}^2$.

In any case, we get that

$$f(t,v) = \varepsilon t \quad \text{and} \quad g(t,v) = Pv + \varepsilon \Lambda_t^{\varepsilon\theta}\eta,$$

where $\eta = \partial_1 g(0,0), P = \partial_2 g(0,0)$ and $P\theta = \varepsilon\theta P$ with $\varepsilon = 1$ if $\operatorname{tr}\theta \neq 0$ or $\varepsilon \in \{-1,1\}$ if $\operatorname{tr}\theta = 0$. Moreover, by definition,

$$\begin{aligned}\Delta_t^{\varepsilon\theta}\eta = &\int_0^t \mathrm{e}^{\varepsilon s\theta}\eta ds \\ =&\varepsilon\int_0^{\varepsilon t} \mathrm{e}^{\mu\theta}\eta d\mu \\ =&\varepsilon\Lambda_{\varepsilon t}^{\theta}\eta\end{aligned}$$

implying that $\psi \in \operatorname{Aut}(G)$ and concluding the proof. □

4.3 Invariance of the nilradical

In this section we show that for an isometry $\psi \in \operatorname{Iso}_G(\Sigma_1, \Sigma_2)_0$ between rank two ARS's with $\psi = (f,g)$, the coordinate function f only depends on the first variable of G or equivalently, it satisfies $\partial_2 f \equiv 0$. Henceforth, we denote the constants $\alpha_i(t,v)$ that depend of (t,v) as α_i.
Let us consider $\{X_i, Y_i\} \subset \Delta_i$ be orthonormal basis. Since $\psi_*\Delta_1^L = \Delta_2^L(\psi)$, we can write uniquely

$$\psi_* X_1^L = \alpha_1 X_2^L(\psi) + \alpha_2 Y_2^L(\psi),$$

where

$$\begin{aligned}\alpha_1^2 + \alpha_2^2 &= \left\|\psi_* X_1^L\right\|_{\Sigma_2}^2 \\ &= \left\|X_1^L\right\|_{\Sigma_1}^2 \\ &= \|X_1\|_{\Sigma_1} \\ &=1\end{aligned}$$

Moreover, each α_i can be recovered from the orthonormality of the basis as

$$\alpha_1 = \left\langle \psi_* X_1^L, X_2^L(\psi)\right\rangle_{\Sigma_2} \quad \text{and} \quad \alpha_2 = \left\langle \psi_* X_1^L, Y_2^L(\psi)\right\rangle_{\Sigma_2},$$

showing that $\alpha_i : G\backslash\mathcal{Z}_1 \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. Moreover, the fact that $\left\{X_1^L, Y_1^L\right\}$ is an orthonormal basis of Δ_1^L implies that

$$\psi_* Y_1^L = \epsilon\left(-\alpha_2 X_2^L(\psi) + \alpha_1 Y_2^L(\psi)\right),$$

where $\epsilon = \pm 1$ is constant in each connected component $G\backslash\mathcal{Z}_1$.
Let us assume w.l.o.g. that $\epsilon = 1$ and consider a orthonormal basis of Δ_1 satisfying

$$X_i = (0, u_i) \text{ and } Y_i = (\sigma_i, w_i), \quad \text{with} \quad \sigma_i w_i \neq 0.$$

By writting $\psi = (f, g)$ and $\psi_* = \begin{pmatrix} \partial_1 f & (\partial_2 f)^T \\ \partial_1 g & \partial_2 g \end{pmatrix}$, the previous consideration, implies that

$$\psi_* (0, \rho_t u_1) = \alpha_1 (0, \rho_f u_2) + \alpha_2 (\sigma_2, \rho_f w_2) \quad \text{and} \quad \psi_* (\sigma_1, \rho_t w_1) = -\alpha_2 (0, \rho_f u_2) + \alpha_1 (\sigma_2, \rho_f w_2),$$

or equivalently

$$\begin{cases} \langle \partial_2 f, \rho_t u_1 \rangle_{\mathbb{R}^2} = \alpha_2 \sigma_2 \\ \sigma_1 \partial_1 f + \langle \partial_2 f, \rho_t w_1 \rangle_{\mathbb{R}^2} = \alpha_1 \sigma_2 \end{cases} \text{and} \begin{cases} \partial_2 g \rho_t u_1 = \rho_f (\alpha_1 u_2 + \alpha_2 w_2) \\ \sigma_1 \partial_1 g + \partial_2 g \rho_t w_1 = \rho_f (-\alpha_2 u_2 + \alpha_1 w_2). \end{cases} \tag{4.9}$$

Now,

$$\psi \circ \varphi_s^1 = \varphi_s^2 \circ \psi \Longrightarrow f \circ \varphi_s^1 = f,$$

that is,

$$f \left(t, e^{sA_1} v + \Lambda_t^\theta \Lambda_t^A \xi \right) = f(t, v).$$

By differentiation,

$$\begin{aligned} &\left\langle \partial_2 f, A_1 v + \Lambda_t^\theta \xi_1 \right\rangle_{\mathbb{R}^2} = 0 \\ &\partial_1 f = \partial_1 f \left(\varphi_s^1 \right) + \left\langle \partial_2 f \left(\varphi_s^1 \right), \rho_t \Lambda_s^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} \\ &\partial_2 f \left(\varphi_s^1 \right) = \mathrm{e}^{-sA_1^T} \partial_2 f. \end{aligned} \tag{4.10}$$

where the first equation in (4.10) is obtained by differentiating with relation to s and then making $s = 0$.

Now, using again the equations in (4.9) we get that

$$\begin{aligned} \sigma_2^2 =& \sigma_2^2 \left((\alpha_1^2 \left(\varphi_s^1 \right) + \alpha_2^2 \left(\varphi_s^1 \right) \right) \\ =& \sigma_2^2 \alpha_1^2 \left(\varphi_s^1 \right) + \sigma_2^2 \alpha_2^2 \left(\varphi_s^1 \right) \\ =& \left(\sigma_2 \alpha_2 \left(\varphi_s^1 \right) \right)^2 + \left(\sigma_2 \alpha_1 \left(\varphi_s^1 \right) \right)^2 \\ =& \left\langle \partial_2 f \left(\varphi_s^1 \right), \rho_t u_1 \right\rangle_{\mathbb{R}^2}^2 + \left(\sigma_1 \partial_1 f \left(\varphi_s^1 \right) + \left\langle \partial_2 f \left(\varphi_s^1 \right), \rho_t w_1 \right\rangle_{\mathbb{R}^2} \right)^2 \\ =& \left\langle \mathrm{e}^{-sA_1^T} \partial_2 f, \rho_t u_1 \right\rangle_{\mathbb{R}^2}^2 + \left(\sigma_1 \left(\partial_1 f - \left\langle \mathrm{e}^{-sA_1^T} \partial_2 f, \rho_t \Lambda_s^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} \right) + \left\langle \mathrm{e}^{-sA_1^T} \partial_2 f, \rho_t w_1 \right\rangle_{\mathbb{R}^2} \right)^2 \\ =& \left\langle \partial_2 f, \mathrm{e}^{-sA_1} \rho_t u_1 \right\rangle_{\mathbb{R}^2}^2 + \left(\sigma_1 \left(\partial_1 f + \left\langle \partial_2 f, \rho_t \Lambda_{-s}^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} \right) + \left\langle \partial_2 f, \mathrm{e}^{-sA_1} \rho_t w_1 \right\rangle_{\mathbb{R}^2} \right)^2. \end{aligned}$$

Where for the last equality we used that $\mathrm{e}^{-sA_1} \Lambda_s^{A_1} = -\Lambda_{-s}^{A_1}$
Derivation at $s = 0$ gives us

$$\begin{aligned} 0 =& 2 \langle \partial_2 f, \rho_t u_1 \rangle_{\mathbb{R}^2} \langle \partial_2 f, -A_1 \rho_t u_1 \rangle_{\mathbb{R}^2} \\ &+ 2 \left(\sigma_1 \partial_1 f + \langle \partial_2 f, \rho_t w_1 \rangle_{\mathbb{R}^2} \right) \cdot \left(-\sigma_1 \langle \partial_2 f, \rho_t \xi_1 \rangle_{\mathbb{R}^2} + \langle \partial_2 f, -A_1 \rho_t w_1 \rangle_{\mathbb{R}^2} \right) \\ =& 2 \alpha_2 \sigma_2 \langle \partial_2 f, -A_1 \rho_t u_1 \rangle_{\mathbb{R}^2} + 2 \alpha_1 \sigma_2 \cdot \left(-\sigma_1 \langle \partial_2 f, \rho_t \xi_1 \rangle_{\mathbb{R}^2} + \langle \partial_2 f, -A_1 \rho_t w_1 \rangle_{\mathbb{R}^2} \right) \\ =& -2 \langle \partial_2 f, \rho_t \left(\sigma_1 \sigma_2 \alpha_1 \xi_1 + \sigma_2 A_1 \left(\alpha_2 u_1 + \alpha_1 w_1 \right) \right) \rangle_{\mathbb{R}^2}, \end{aligned}$$

showing that

$$\partial_2 f \quad \text{is orthogonal to} \quad \rho_t\left(\sigma_1\sigma_2\alpha_1\xi_1 + \sigma_2 A_1\left(\alpha_2 u_1 + \alpha_1 w_1\right)\right) \quad \text{on} \quad G\backslash\mathcal{Z}_1\,.$$

Using the fist equation in (4.10), allow us to conclude that, on $G\backslash\mathcal{Z}_1$

$$\partial_2 f \neq 0 \Longrightarrow \left\{A_1 v + \Lambda_t^\theta \xi_1, \rho_t\left(\sigma_1\sigma_2\alpha_1\xi_1 + \sigma_2 A_1\left(\alpha_2 u_1 + \alpha_1 w_1\right)\right)\right\} \text{ is LD.} \tag{4.11}$$

We show that $\partial_2 f \equiv 0$ by analyzing the possibilities for the eigenvalues of A_1 in the next propositions.

Proposition 4.3.1. If A_1 has only eigenvalues with nonzero real parts, $\partial_2 f \equiv 0$

Proof. According to Proposition 2.4.3 (see also Remark 2.4.4) we can assume without loss of generality that $(1,0) \in \Delta_1$ Under this assumption, we have that $w_1 = cu_1$ and equations (4.9) gives us that

$$\left\langle \partial_2 f, \rho_t u_1\right\rangle_{\mathbb{R}^2} = \alpha_2\sigma_2 \quad \text{and} \quad \partial_1 f = \frac{\sigma_2}{\sigma_1}\left(\alpha_1 - c\alpha_2\right).$$

Where the second equation is obtained as follows. Since $\sigma_1\partial_1 f + \langle\partial_2 f, \rho_t w_1\rangle_{\mathbb{R}^2} = \alpha_1\sigma_2$, as $w_1 = cu_1$ we get $\sigma_1\partial_1 f + c\alpha_2\sigma_2 = \alpha_1\sigma_2$ consequently $\partial_1 f = \dfrac{\sigma_2}{\sigma_1}(\alpha_1 - c\alpha_2)$.
Using now equations (4.9) gives us that on $G\backslash\mathcal{Z}_1$,

$$\begin{aligned}\alpha_2\left(\varphi_s^1\right)\sigma_2 &= \left\langle \partial_2 f\left(\varphi_s^1\right), \rho_t u_1\right\rangle_{\mathbb{R}^2}\\ &= \left\langle \mathrm{e}^{-sA_1^T}\partial_2 f, \rho_t u_1\right\rangle_{\mathbb{R}^2}\end{aligned}$$

is bounded for $s \in \mathbb{R}$ outside a discrete subset (see Proposition 3.2.3). However, if A_1 has only eigenvalues with nonzero real parts, the fact that $u_1 \neq 0$ implies by Lemma A.1.2 that $\alpha_2 \equiv 0$ and $\alpha_1 = 1$ showing that $\partial_1 f$ is constant on $G\backslash\mathcal{Z}_1$. Using the continuity of f and the fact that $G\backslash\mathcal{Z}_1$ is an open and dense subset of G allow us to conclude that

$$\forall (t,v) \in G, \quad f(t,v) = at + h(v), \quad \text{where } a = \sigma_2/\sigma_1 \quad \text{and} \quad h(v) = f(0,v).$$

The same analysis as in Step 2. of Lemma 4.2.1 allows us to conclude that under the LARC and rank two assumptions it holds that $h \equiv 0$ showing that $\partial_2 f \equiv 0$ as stated. □

Proposition 4.3.2. If A_1 has a pair of pure imaginary eigenvalues, $\partial_2 f \equiv 0$

Proof. By Remark 2.4.4 we can assume that $\xi_1 = 0$. Consider the set

$$\mathcal{A} := \left\{(t,v) \in G; \quad \partial_2 f(t,v) \neq 0\right\}.$$

The first to notice is that

$$\partial_2 f\left(\varphi_s^1\right) = \mathrm{e}^{-sA^T}\partial_2 f \quad \Longrightarrow \quad \varphi_s^1(\mathcal{A}) \subset \mathcal{A}.$$

Also, the fact that A_1 has a pair of imaginary eigenvalues and that $A_1\theta = \theta A_1$ imply that, on the canonical basis, $A_1 = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}$, for some $\mu \neq 0$.
The assumption that $\xi_1 = 0$ implies

$$\mathcal{Z}_1 = \left\{\left(t, s\rho_t A_1^{-1} u_1\right), t, s \in \mathbb{R}\right\} \quad \text{and} \quad \mathcal{Z}_{\mathcal{X}_1} = \mathbb{R} \times \{0\}.$$

Consequently, for any $(t, v) \in \mathcal{Z}_1 \backslash \mathcal{Z}_{\mathcal{X}_1}$ there exists $s_0 \in \mathbb{R}$ such that $\varphi_{s_0}(t, v) \in G \backslash \mathcal{Z}_1$. In particular we get that

$$\mathcal{A} \backslash \mathcal{Z}_{\mathcal{X}_1} \neq \varnothing \quad \Longrightarrow \quad \mathcal{A} \cap (G \backslash \mathcal{Z}_1) \neq \varnothing.$$

Let us consider $(t_0, v_0) \in \mathcal{A} \cap (G \backslash \mathcal{Z}_1)$ and suppose that $\alpha_2(t_0, v_0)\rho_{t_0} u_1 + \alpha_1(t_0, v_0)\rho_{t_0} w_1 \in \mathbb{R}v_0$. Let's choose v_0^* such that, $\langle v_0, v_0^* \rangle_{\mathbb{R}^2} = 0$. Therefore, by using the relation (4.11) we get that

$$\begin{aligned} 0 =& \langle \alpha_2(t_0, v_0)\rho_{t_0} u_1 + \alpha_1(t_0, v_0)\rho_{t_0} w_1, v_0^* \rangle_{\mathbb{R}^2} \\ =& \alpha_2(t_0, v_0) \langle \rho_{t_0} u_1, v_0^* \rangle_{\mathbb{R}^2} + \alpha_1(t_0, v_0) \langle \rho_{t_0} w_1, v_0^* \rangle_{\mathbb{R}^2} \end{aligned}$$

and since $\alpha_1^2 + \alpha_2^2 = 1$ we obtain

$$\alpha_1(t_0, v_0)^2 = \frac{\langle \rho_{t_0} u_1, v_0^* \rangle_{\mathbb{R}^2}^2}{\langle \rho_{t_0} u_1, v_0^* \rangle_{\mathbb{R}^2}^2 + \langle \rho_{t_0} w_1, v_0^* \rangle_{\mathbb{R}^2}^2} \quad \text{and} \quad \alpha_2(t_0, v_0)^2 = \frac{\langle \rho_t w_1, v_0^* \rangle_{\mathbb{R}^2}}{\langle \rho_{t_0} u_1, v_0^* \rangle_{\mathbb{R}^2}^2 + \langle \rho_{t_0} w_1, v_0^* \rangle_{\mathbb{R}^2}^2}.$$

With the previous we will show that $\mathcal{A} \subset \mathcal{Z}_{\mathcal{X}_1}$, by considering two cases:

1. u_1 and w_1 are not orthogonal vectors;
   Let $(t, v) \in \mathcal{A} \backslash \mathcal{Z}_{\mathcal{X}_1}$. The fact that $e^{sA_1} v$ is a circumference around the origin gives us that

   $$\left\{e^{sA_1}, \ s \in \mathbb{R}\right\} \cap \{(t, s\rho_t v), s \in \mathbb{R}\} \neq \varnothing \quad \Longrightarrow \quad \exists s_0, s_1 \in \mathbb{R}^*; e^{s_0 A_1} = s_2 \rho_t v.$$

   In particular, if u_1 and w_1 are not orthogonal vectors, $A_1 w_1 \notin \mathbb{R}u_1$ implying that $(t, s_1\rho_t w_1) \in G \backslash \mathcal{Z}_1$ and hence

   $$\mathcal{A} \ni \varphi_{s_0}(t, v) = \left(t, e^{s_0 A_1} v\right) = (t, s_1 \rho_t w_1) \in G \backslash \mathcal{Z}_1.$$

   By the previous formula we get

   $$\begin{aligned} & \left\langle \rho_t w_1, (\rho_t w_1)^* \right\rangle_{\mathbb{R}^2} = 0 \\ \Longrightarrow \quad & \alpha_2(t, s_1 \rho_t w_1) = 0 \\ \overset{(4.9)}{\Longrightarrow} \quad & \langle \partial_2 f(t, s_1 \rho_t w_1), \rho_t u_1 \rangle_{\mathbb{R}^2} = 0. \end{aligned}$$

   On the other hand, by the first equation in (4.10) we get that

   $$\langle \partial_2 f(t, s_1 \rho_t w_1), A_1(s_1 \rho_t w_1) \rangle_{\mathbb{R}^2} = 0 \quad \Longrightarrow \quad \langle \partial_2 f(t, s_1 \rho_t w_1), \rho_t A_1 w_1 \rangle_{\mathbb{R}^2} = 0.$$

Since $A_1w_1 \notin \mathbb{R}u_1$, the set $\{\rho_t u_1, \rho_t A_1 w_1\}$ is a basis of $\mathbb{R}^2$, which by the previous equalities imply

$$0 = \partial_2 f(t, s_1\rho_t w_1) = \partial_2 f(\varphi_{s_0}(t,v)) = e^{-s_0 A_1^T}\partial_2 f(t,v) \implies \partial_2 f(t,v) = 0,$$

contradicting the fact that $(t,v) \in \mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1}$. Therefore, if u_1 and w_1 are not orthogonal vectors we must have that $\mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1} = \varnothing$.

2. u_1 and w_1 are orthogonal vectors;
Let $(t,v) \in \mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1}$. As previously,

$$\exists s_0, s_1 \in \mathbb{R}^*; \quad \mathcal{A} \ni \varphi^1_{s_0}(t,v) = (t, s_1\rho_t u_1) \in G\backslash\mathcal{Z}_1 \Longrightarrow \alpha_1\left(t, s_1\rho_t u_1\right) = 0.$$

Since u_1 and w_1 orthogonal are equivalent to $A_1u_1 \in \mathbb{R}w_1$, the first equation in (4.10) gives us that

$$\langle \partial_2 f\left(t, s_1\rho_t u_1\right), A_1\left(s_1\rho_t u_1\right)\rangle_{\mathbb{R}^2} = 0 \quad \Longrightarrow \quad \langle \partial_2 f\left(t, s_1\rho_t w_1\right), \rho_t w_1\rangle_{\mathbb{R}^2} = 0.$$

and hence,

$$\begin{aligned}\partial_1 f\left(t, s_1\rho_t u_1\right) =& \partial_1 f\left(t, s_1\rho_t u_1\right) + \underbrace{\langle \partial_2 f\left(t, s_1\rho_t u_1\right), \rho_t w_1\rangle_{\mathbb{R}^2}}_{=0} \\ =& \alpha_1\left(t, s_1\rho_t u_1\right)\sigma_2 = 0\end{aligned}$$

Since we are assuming $\xi_1 = 0$, the second equation in (4.10) implies $\partial_1 f\left(\varphi^1_s\right) = \partial_1 f$ for all $s \in \mathbb{R}$. In particular,

$$\begin{aligned}\partial_1 f(t,v) =& \partial_1 f\left(\varphi^1_{s_0}(t,v)\right) \\ =& \partial_1 f\left(t, s_1\rho_t u_1\right) \\ =& 0,\end{aligned}$$

which by the arbitrariness of $(t,v) \in \mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1}$ implies that

$$\mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1} \subset \left\{(t,v) \in G\backslash\mathcal{Z}_{\mathcal{X}_1}; \partial_1 f(t,v) = 0\right\}.$$

Now, if $(t_0, v_0) \in (\overline{\mathcal{A}}\backslash\mathcal{A}) \cap G\backslash\mathcal{Z}_1$, the fact that f restrict to $G\backslash\mathcal{Z}_1$ is $\mathcal{C}^\infty$ implies that

$$\partial_1 f\left(t_0, v_0\right) = 0 \quad \text{and} \quad \partial_2 f\left(t_0, v_0\right) = 0$$

which is a contradition to the fact that ψ is a diffeomorphism. Therefore, $\mathcal{A} \cap (G\backslash\mathcal{Z}_1)$ is open and closed in $G\backslash\mathcal{Z}_1$ and so, $\mathcal{A}$ contains any connected component of $G\backslash\mathcal{Z}_1$ that it intersects. However, by Theorem 4.5.1, $G\backslash\mathcal{Z}_1$ has two connected components and, since

A_1 has a pair of imaginary eigenvalues, any point $(t,v) \in G\backslash\mathcal{Z}_1$ crosses the singular locus by the action of the flow of $\mathcal{X}_1$. Consequently, the invariance $\varphi_s(\mathcal{A}) \subset \mathcal{A}$ implies that

$$\mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1} \neq \varnothing \quad \Longrightarrow \quad \mathcal{A} \cap (G\backslash\mathcal{Z}_1) \neq \varnothing \quad \Longrightarrow \quad G\backslash\mathcal{Z}_1 \subset \mathcal{A}.$$

From that, we conclude that $\partial_1 f|_{G\backslash\mathcal{Z}_1} \equiv 0$ and by the continuity of f we actually get

$$f(t,v) = f(0,v) \quad \Longrightarrow \quad \partial_1 f \equiv 0.$$

In particular, since $\det A_1 \neq 0$,

$$\partial_2 f\left(\varphi_s^1\right) = \mathrm{e}^{-sA^T}\partial_2 f \quad \Longrightarrow \quad \partial_2 f(0,0) = 0,$$

which together with $\partial_1 f \equiv 0$ is a contradiction with the fact that ψ is a diffeomorphism.

Therefore, in both cases, we conclude that $\mathcal{A}\backslash\mathcal{Z}_{\mathcal{X}_1} = \varnothing$ or equivalently $\mathcal{A} \subset \mathcal{Z}_{\mathcal{X}_1}$. By considering the complementary, we get that

$$\forall (t,v) \in G\backslash\mathcal{Z}_{\mathcal{X}_1}, \quad f(t,v) = at, \quad \text{where} \quad a = \sigma_2/\sigma_1 \in \mathbb{R}^*,$$

and by continuity $f(t,v) = at$ for all $(t,v) \in G$ implying that $\partial_2 f \equiv 0$ and concluding the proof.

□

Proposition 4.3.3. If $\mathbb{R}^2 = \operatorname{Im} A_1 \oplus \mathbb{R}\xi_1$ then $\partial_2 f \equiv 0$.

Proof. Suppose that $(1,0) \in \Delta_1$. In this case, $w_1 = cu_1$ and from equations (4.9) we get that

$$\partial_1 f = (\alpha_1 - c\alpha_2)\,\sigma_2 \Longrightarrow \partial_1 f \text{ is bounded on } G\backslash\mathcal{Z}_1 .$$

On the other hand, by our hypothesis $\dim \operatorname{Im} A_1 = 1$ and so we have the following possibilities:

1. A_1 has a pair of distinct eigenvalues: Since $A_1\theta = \theta A_1$ we have that, on the canonical basis,
$$A_1 = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$
Let us assume that the first case holds for A_1 since the analysis of the second case is analogous. By our hypothesis, we can write $\xi_1 = ae_1 + be_2$ with $b \neq 0$ and hence
$$\rho_t\Lambda_{-s}^{A_1}\xi_1 = a\mathrm{e}^t\frac{1}{\beta}\left(\mathrm{e}^{-s\beta} - 1\right)e_1 - bs\mathrm{e}^{t\lambda}e_2.$$
Also, from the second and third equations in (4.10) we have that
$$\begin{aligned}\partial_1 f =& \partial_1 f\left(\varphi_s^1\right) + \left\langle \partial_2 f\left(\varphi_s^1\right), \rho_t\Lambda_s^{A_1}\xi_1\right\rangle_{\mathbb{R}^2} \\ =& \partial_1 f\left(\varphi_s^1\right) - \left\langle \partial_2 f, \rho_t\Lambda_{-s}^{A_1}\xi_1\right\rangle_{\mathbb{R}^2},\end{aligned}$$

and hence $\left\langle \partial_2 f, \rho_t \Lambda_{-s}^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} = \partial_1 f\left(\varphi_s^1\right) - \partial_1 f$ is bounded for $s \in \mathbb{R}$ outside a discrete set. Hence

$$\left\langle \partial_2 f, \rho_t \Lambda_{-s}^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} = a\mathrm{e}^t \frac{1}{\beta}\left(\mathrm{e}^{-s\beta} - 1\right)\left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} - b\, s\mathrm{e}^{t\lambda}\left\langle \partial_2 f, e_2 \right\rangle_{\mathbb{R}^2}$$

is bounded and then

$$a\left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} = 0 \quad \text{and} \quad \left\langle \partial_2 f, e_2 \right\rangle_{\mathbb{R}^2} = 0.$$

If $a = 0$ we obtain that $\xi_1 \in \ker A_1 = \mathbb{R}e_2$. In particular, $\Lambda_t^\theta \xi \in \ker A_1$ and consequently

$$\left\langle \partial_2 f, e_2 \right\rangle_{\mathbb{R}^2} = 0 \quad \Longrightarrow \quad \left\langle \partial_2 f, \Lambda_t^\theta \xi \right\rangle_{\mathbb{R}^2} = 0.$$

Using the first equation in (4.10) we obtain that

$$0 = \left\langle \partial_2 f, A_1 v + \Lambda_t^\theta \xi \right\rangle_{\mathbb{R}^2} = \left\langle \partial_2 f, A_1 v \right\rangle_{\mathbb{R}^2}.$$

Thus, we get

$$\forall (t, v) \in G \backslash \left(\{0\} \times \ker A_1\right), \quad \left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} = 0$$

and in particular $\partial_2 f = 0$ on $G \backslash \mathcal{Z}_1$. Then $\partial_1 f = \dfrac{\sigma_2}{\sigma_1}$ on $G \backslash \mathcal{Z}_1$ implying, by the continuity of f that $f(t, v) = at$ on G as stated.

2. A_1 is nilpotent: Using that $A_1\theta = \theta A_1$ gives us that, on the canonical basis, it holds

$$A_1 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \quad \delta \in \{0, 1\}$$

or

$$A_1 = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By the rank two assumption we have that $\xi_1 = ae_1 + be_2$ with $b \neq 0$. Thus

$$\rho_t \Lambda_{-s}^{A_1} \xi_1 = \mathrm{e}^t\left(b\frac{s^2}{2} - (a + b\delta t)s\right)e_1 - \mathrm{e}^t bse_2$$

and being $\partial_1 f$ bounded on $G \backslash \mathcal{Z}_1$ it holds that

$$\forall s \in \mathbb{R}, \quad \left\langle \partial_2 f, \rho_t \Lambda_{-s}^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} \quad \text{is bounded.}$$

As a consequence,

$$\left\langle \partial_2 f, \rho_t \Lambda_{-s}^{A_1} \xi_1 \right\rangle_{\mathbb{R}^2} = \mathrm{e}^t\left(b\frac{s^2}{2} - (a + b\delta t)s\right)\left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} - \mathrm{e}^t bs\left\langle \partial_2 f, e_2 \right\rangle_{\mathbb{R}^2}$$

is bounded for $s \in \mathbb{R}$ outside a discrete set and hence

$$\frac{\mathrm{e}^t b}{2}\left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} = 0 \quad \text{and} \quad \mathrm{e}^t\left((a + b\delta t)\left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} + b\left\langle \partial_2 f, e_2 \right\rangle_{\mathbb{R}^2}\right) = 0. \tag{4.12}$$

As $b \neq 0$ the equation (4.12) is equivalenty to

$$\left\langle \partial_2 f, e_1 \right\rangle_{\mathbb{R}^2} = \left\langle \partial_2 f, e_2 \right\rangle_{\mathbb{R}^2} = 0,$$

which gives us that $\partial_2 f = 0$ on $G \backslash \mathcal{Z}_1$ and, as a result $f(t, v) = at$, concluding the proof.

□

4.4 The theorem of isometries

Theorem 4.4.1. The only isometries between rank two ARS's on $G = \mathbb{R} \times_\rho \mathbb{R}^2$ that fixated the identity are the automorphisms.

Proof. To prove this theorem we follow the steps given in the remark 4.1.4.
Let us consider $\psi = (f, g)$ an isometry between rank two ARS's Σ_1 and Σ_2, then by the invariance of the nilradical proved in Section 4.3, we have $\partial_2 f \equiv 0$. Therefore, if

$$X = (0, u) \in \Delta_1 \cap \left(\{0\} \times \mathbb{R}^2\right), \quad \text{with} \quad \|X\|_{\Sigma_1,(0,0)} = 1,$$

we obtain that

$$(d\psi)_{(t,v)} X^L(t,v) = \begin{pmatrix} \partial_1 f(t,v) & 0 \\ \partial_1 g(t,v) & \partial_2 g(t,v) \end{pmatrix} \begin{pmatrix} 0 \\ \rho_t u \end{pmatrix} = (0, \partial_2 g(t,v)\rho_t u)\,,$$

showing that

$$(d\psi)_{(t,v)} X^L(t,v) \in \Delta_2(\psi(t,v)) \cap \left(\{0\} \times \mathbb{R}^2\right).$$

On other hand, $(d\psi)_{(0,0)} X \in \Delta_2(0,0)$ implies that

$$\left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} X \in \Delta_2(\psi(t,v)) \cap \left(\{0\} \times \mathbb{R}^2\right).$$

Moreover, since ψ is an isometry and the metrics on Δ_1 and Δ_2 are left-invariant,

$$\begin{aligned} \left\|(d\psi)_{(t,v)} X^L(t,v)\right\|_{\Sigma_2,(\psi(t,v))} &= \left\|X^L(t,v)\right\|_{\Sigma_1,(t,v)} \\ &= \|X\|_{\Sigma_1,(0,0)} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \left\|\left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} X\right\|_{\Sigma_2,(\psi(t,v))} &= \left\|(d\psi)_{(0,0)} X\right\|_{\Sigma_2,(0,0)} \\ &= \left\|(d\psi)_{(0,0)} X\right\|_{\Sigma_2,(0,0)} \\ &= 1. \end{aligned}$$

Since $\dim\left(\Delta_2(\psi(t,v))\right) \cap \left(\{0\} \times \mathbb{R}^2\right) = 1$ it holds that

$$(d\psi)_{(t,v)} X^L(t,v) = \pm \left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} X,$$

where the sign is constant on any connected component of $G \backslash \mathcal{Z}_1$. Doing a similar analysis as done in Lemma 4.2.1, specifically in the Step 1, allows us to obtain that the sign is constant on G and then

$$(d\psi)_{(t,v)} X^L(t,v) = \left(dL_{\psi(t,v)}\right)_{(0,0)} (d\psi)_{(0,0)} X.$$

By using the Fundamental Lemma 4.2.1 we conclude that ψ is an automorphism. □

The next example shows that Theorem 4.4.1 is not true if the ARS has rank less than two.

Example 4.4.2. Let us consider $G = \mathbb{R} \times_\rho \mathbb{R}^2$ and the simple ARS given by $\Sigma = \{\mathcal{X}, \Delta^L\}$, where

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{X} = \left((0,0), \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \quad \text{and} \quad \alpha = \{(1,0), (0, e_2)\}$$

is an orthonormal basis of Δ.

By Proposition 2.4.1 we have that Σ satisfies the LARC and it is direct to see that it is not a rank two ARS. Note also that, for any $(t, v) \in G$, it holds

$$\left(dL_{(t,v)}\right)_{(0,0)}(1,0) = (1,0) \quad \text{and} \quad \left(dL_{(t,v)}\right)_{(0,0)}(0,e_2) = (0,e_2)\,.$$

Let us consider the diffeomorphism

$$\psi : G \to G, \quad (t, v) \mapsto (-t, v),$$

and note that, by Proposition 2.3.1 it holds that $\psi \notin \mathrm{Aut}(G(\theta))$. On the other hand,

$$\begin{aligned} \psi(\mathcal{X}(t,v)) =& \psi(0, Av) \\ =& (0, Av) \\ =& \mathcal{X}(-t, v) \\ =& \mathcal{X}(\psi(t,v)), \end{aligned}$$

$$\psi(1,0) = (-1,0) \quad \text{and} \quad \psi(0,e_2) = (0,e_2)\,,$$

and since $\psi_* = \psi$, we conclude that ψ carries the orthonormal frame $\{\mathcal{X}, (1,0), (0,e_2)\}$ onto the orthonormal frame $\{\mathcal{X}, -(1,0), (0,e_2)\}$ showing that $\psi \in \mathrm{Iso}_G(\Sigma;\Sigma)$.

4.5 On classification of simple rank two ARS

The results obtained in Chapter 4 allows obtaining the following classification theorem. In this section, we consider the notation $G(\theta) = \mathbb{R} \times_\rho \mathbb{R}^2$ for the connected, simply connected Lie group with Lie algebra $\mathfrak{g}(\theta)$. For any $\sigma \in \mathbb{R}^+$ we consider the subsets of $\mathfrak{g}(\theta)$ given by

$$\alpha_1 = \{(1,0), (\sigma, e_1)\}\,, \quad \alpha_2 = \{(1,0), (\sigma, e_2)\} \quad \text{and} \quad \alpha_3 = \{(1,0), (\sigma, e_1 + e_2)\}\,.$$

Define the simple ARS's of rank two $\Sigma^i_{\mathcal{X},\sigma} = \left\{\mathcal{X}, \Delta^L_{i,\sigma}\right\}$ on $G(\theta)$, where

1. $\mathcal{X}$ is a rank two linear vector field on $G(\theta)$;

2. α_i is an orthonormal basis of $\Delta_{i,\sigma}$.

Denote by $\mathcal{E}_\theta$ the set of all rank two simple ARS on $G(\theta)$ and consider the sets

$$\mathcal{E}_\theta^i = \left\{\Sigma \in \mathcal{E}_\theta; \mathrm{Iso}\left(\Sigma, \Sigma^i_{\mathcal{X},\sigma}\right)_0 \neq \varnothing \text{ for some } \Sigma^i_{\mathcal{X},\sigma}\right\}$$

that is, $\mathcal{E}_\theta^i$ is the set of rank two ARS's on $G(\theta)$ that are isometric to some of the ARS's $\Sigma^i_{\mathcal{X},\sigma}$

Theorem 4.5.1. Up to a rescaling, it holds that

(i) $\mathcal{E}_\theta = \mathcal{E}_\theta^1$ if $\theta \in \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, \lambda \in \mathbb{R}\right\}$;

(ii) $\mathcal{E}_\theta = \mathcal{E}_\theta^1 \dot\cup \mathcal{E}_\theta^3$ if $\theta \in \left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$;

(iii) $\mathcal{E}_\theta = \mathcal{E}_\theta^1 \dot\cup \mathcal{E}_\theta^2 \dot\cup \mathcal{E}_\theta^3$ if $\theta \in \left\{\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in (-1, 1)\right\}$.

Proof. Since, by Theorem 4.4.1 the only isometries between rank two ARS are automorphisms, we only have to show that any given ARS Σ is isometric to some ARS in $\mathcal{E}_\theta^i$ for $i = 1, 2, 3$ and that for $i \neq j$ we have that $\mathcal{E}_\theta^i \cap \mathcal{E}_\theta^j = \varnothing$ if θ is in the cases (ii) or (iii).
Let us consider $\Sigma = \left\{\mathcal{X}, \Delta^L\right\}$ be a rank two ARS. By Proposition 2.4.3 and Remark 2.4.4, the ARS Σ is isometric to an ARS whose distribution contains $(1, 0)$. Hence, we can assume without loss of generality that $(1, 0) \in \Delta$. By rescaling the norm on Δ if necessary we can assume that $\|(1, 0)\|_{\Sigma,(0,0)} = 1$. Choose $(\sigma, u) \in \Delta$ such that $\sigma > 0$ and $\{(1, 0), (\sigma, u)\}$ is an orthonormal basis of Δ. Note that, in this case

$$(0, u) = -\sigma(1, 0) + (\sigma, u) \quad \Longrightarrow \quad l_\Delta = \mathbb{R}u.$$

Write $u = (x, y)$ and by considering the following cases:

Case 1 : $\theta \in \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, \lambda \in \mathbb{R}\right\}$

By regarding $\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$ we obtain that

$$P\theta = \theta P, \quad \det P = x^2 + y^2 \neq 0 \quad \text{and} \quad Pe_1 = u.$$

The automorphism $\phi(t, v) = (t, Pv)$ is an isometry between $\Sigma^1_{\mathcal{X}_\phi,\sigma}$ and Σ.

Case 2 : $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

By regarding

$$P_1 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \quad \text{if} \quad y = 0 \quad \text{or} \quad P_3 = \begin{pmatrix} y & x-y \\ 0 & y \end{pmatrix} \text{ if } y \neq 0,$$

we obtain that $P_1\theta = \theta P_1$, $\det P_i \neq 0$ and

$$P_1 e_1 = u \text{ if } y = 0 \text{ and } P_3\,(e_1 + e_2) = u \text{ if } y \neq 0.$$

Therefore, if $y = 0$ the automorphism $\phi_1(t, v) = (t, P_1 v)$ is an isometry between $\Sigma^1_{\mathcal{X}_1,\sigma}$ and Σ, and if $y \neq 0$ the automorphism $\phi_3(t, v) = (t, P_3 v)$ is an isometry between $\Sigma^3_{\mathcal{X}_{\phi_3},\sigma}$ and Σ.

Case 3 : $\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

By regarding

$$P_1 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \quad \text{if} \quad y = 0, \quad P_2 = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \quad \text{if} \quad x = 0 \quad \text{or} \quad P_3 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \text{ if } xy \neq 0.$$

It holds that $P_i\theta = \theta P_i$ for $i = 1$ or 3 and $P_2\theta = -\theta P_2$, $\det P_i \neq 0$ and

$$P_1 e_1 = u \text{ if } y = 0, \quad P_2 e_1 = u \quad \text{ if } y = 0 \text{ and } P_3\,(e_1 + e_2) = u \quad \text{ if } xy \neq 0.$$

Thus, if $y = 0$ the automorphism $\phi_1(t, v) = (t, P_1 v)$ (respectively if $x = 0$) the automorphism $\phi_2(t, v) = (-t, P_2 v))$ is an isometry between $\Sigma^1_{\mathcal{X}_{\phi 1},\sigma}$ (resp. $\Sigma^1_{\mathcal{X}_{\phi_2},\sigma}\big)$ and Σ, and $\phi_3(t, v) = (t, P_3 v)$ is an isometry between $\Sigma^3_{\mathcal{X}_{\phi_3},\sigma}$ and Σ if $xy \neq 0$

Case 4 : $\theta = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \quad \lambda \in (-1, 1)$

By regarding

$$P_1 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \quad \text{if} \quad y = 0, \quad P_2 = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \quad \text{if} \quad x = 0 \quad \text{or} \quad P_3 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \text{ if } xy \neq 0.$$

Hence $P_i\theta = \theta P_i$, $\det P_i \neq 0$ and

$$P_1 e_1 = u \text{ if } y = 0, \quad P_2 e_2 = u \text{ if } x = 0 \text{ and } P_3\,(e_1 + e_2) = u \quad \text{ if } xy \neq 0.$$

and the automorphisms $\phi_i(t, v) = (t, P_i v)$ are isometries between $\Sigma_{\mathcal{X}_{\phi_1},\sigma}$ and Σ, for $i = 1, 2, 3$. Since the cases $1, 2, 3$ and case 4 cover all the possibilities, we have that $\mathcal{E}$ is in fact decomposed by the classes $\mathcal{E}^i_\theta$ as given in items (i), (ii) and (iii). The only thing that remains to show is that, in cases (ii) and (iii) we have that $\mathcal{E}^i_\theta \cap \mathcal{E}^j_\theta = \varnothing$ for $i \neq j$

Since both cases are analogous, let us show case (ii). In this case, if $\mathcal{E}_\theta^1 \cap \mathcal{E}_\theta^3 \neq \varnothing$, there exists rank two linear vector fields $\mathcal{X}_1, \mathcal{X}_3$ and positive real numbers σ_1, σ_2 such that $\Sigma^1_{\mathcal{X}_1,\sigma_1}$ and $\Sigma^3_{\mathcal{X}_3,\sigma_3}$ are isometrics. Nevertheless, since $\Sigma^1_{\mathcal{X}_1,\sigma_1}$ and $\Sigma^3_{\mathcal{X}_3,\sigma_3}$ are, by definition, rank two ARS's, Theorem 4.4.1 implies that

$$\mathrm{Iso}\ \left(\Sigma^1_{\mathcal{X}_1,\sigma_1};\Sigma^3_{\mathcal{X}_3,\sigma_3}\right)_0 \subset \mathrm{Aut}(G(\theta)),$$

Thus, any $\psi \in \mathrm{Iso}\left(\Sigma^1_{\mathcal{X}_1,\sigma_1};\Sigma^3_{\mathcal{X}_3,\sigma_3}\right)_0$ satisfies

$$(d\psi)_{(0,0)} = \begin{pmatrix} \varepsilon & 0 \\ \eta & P \end{pmatrix}, \qquad \text{with} \qquad P\theta = \varepsilon\theta P,$$

which implies,

$$(d\psi)_{(0,0)}\left(\Delta_{1,\sigma} \cap \{0\} \times \mathbb{R}^2\right) = \Delta_{3,\sigma} \cap \{0\} \times \mathbb{R}^2 \text{ and hence } Pe_1 \in \mathbb{R}\left(e_1 + e_2\right).$$

However, by the hypothesis on θ, the subspace $\mathbb{R}e_1$ is a one dimensional eigenspace of θ. $P\theta = \varepsilon\theta P$ implies that

$$Pe_1 \in \mathbb{R}e_1 \text{ if } \varepsilon = 1 \text{ and } Pe_1 \in \mathbb{R}e_2 \text{ if } \varepsilon = -1,$$

which contradicts $Pe_1 \in \mathbb{R}\left(e_1 + e_2\right)$. Consequently, $\mathcal{E}_\theta^1 \cap \mathcal{E}_\theta^3 = \varnothing$ as stated. □

Remark 4.5.2. In the notation of the Theorem 4.5.1, notice that

$$(0,u) = -\sigma(1,0) + (\sigma,u) \quad \Longrightarrow \quad \|(0,u)\|_{\Sigma,(0,0)} = 1 + \sigma^2.$$

In particular, $\|(0,u)\|_{\Sigma,(0,0)} = 1$ if and only if $\sigma = 0$. Therefore, the metric on Δ is Euclidean if and only if $\sigma = 0$.

Bibliography

[1] Agrachev, A., , Boscain, U., Sigalotti, M., and and. A gauss-bonnet-like formula on two-dimensional almost-riemannian manifolds. *Discrete & Continuous Dynamical Systems - A 20*, 4 (2008), 801–822.

[2] Agrachev, A., Barilari, D., and Boscain, U. *A comprehensive introduction to sub-Riemannian geometry*, vol. 181. Cambridge University Press, 2019.

[3] Agrachev, A. A., Barilari, D., and Boscain, U. Introduction to riemannian and sub-riemannian geometry.

[4] Agrachev, A. A., Boscain, U., Charlot, G., Ghezzi, R., and Sigalotti, M. Two-dimensional almost-riemannian structures with tangency points. In *Annales de l'IHP Analyse non linéaire* (2010), vol. 27, pp. 793–807.

[5] Agrachev, A. A., and Sachkov, Y. *Control theory from the geometric viewpoint*, vol. 87. Springer Science & Business Media, 2013.

[6] Ayala, V., and Jouan, P. Almost-Riemannian geometry on Lie groups. *SIAM Journal on Control and Optimization 54*, 5 (2016), 2919–2947.

[7] Ayala, V., and Silva, A. D. On the characterization of the controllability property for linear control systems on nonnilpotent, solvable three-dimensional lie groups. *Journal of Differential Equations 266*, 12 (June 2019), 8233–8257.

[8] Ayala, V., and Tirao, J. Linear control systems on Lie groups and controllability. In *Proceedings of symposia in pure mathematics* (1999), vol. 64, American Mathematical Society, pp. 47–64.

[9] Bellaïche, A. The tangent space in sub-riemannian geometry. In *Sub-Riemannian geometry*. Springer, 1996, pp. 1–78.

[10] Beschastnyi, I., Boscain, U., and Pozzoli, E. Quantum confinement for the curvature laplacian $-\frac{1}{2}\delta + ck$ on 2d-almost-riemannian manifolds. *Potential Analysis* (Aug. 2021).

[11] Bonnard, B., and Caillau, J. B. Singular metrics on the two-sphere in space mechanics. *preprint* (2008), 1–25.

[12] Bonnard, B., Caillau, J.-B., Sinclair, R., and Tanaka, M. Conjugate and cut loci of a two-sphere of revolution with application to optimal control. In *Annales de l'IHP Analyse non linéaire* (2009), vol. 26, pp. 1081–1098.

[13] Bonnard, B., Charlot, G., Ghezzi, R., and Janin, G. The sphere and the cut locus at a tangency point in two-dimensional almost-riemannian geometry. *Journal of dynamical and control systems 17*, 1 (2011), 141–161.

[14] Boscain, U., , Charlot, G., Gaye, M., Mason, P., , and and. Local properties of almost-riemannian structures in dimension 3. *Discrete & Continuous Dynamical Systems - A 35*, 9 (2015), 4115–4147.

[15] Boscain, U., and Charlot, G. Resonance of minimizers for n-level quantum systems with an arbitrary cost. *ESAIM: Control, Optimisation and Calculus of Variations 10*, 4 (2004), 593–614.

[16] Boscain, U., Charlot, G., Gauthier, J.-P., Guérin, S., and Jauslin, H.-R. Optimal control in laser-induced population transfer for two-and three-level quantum systems. *Journal of Mathematical Physics 43*, 5 (2002), 2107–2132.

[17] Boscain, U., Charlot, G., and Ghezzi, R. Normal forms and invariants for 2-dimensional almost-Riemannian structures. *Differential Geometry and its Applications 31*, 1 (2013), 41–62.

[18] Boscain, U., and Laurent, C. The laplace-beltrami operator in almost-riemannian geometry. In *Annales de l'institut Fourier* (2013), vol. 63, pp. 1739–1770.

[19] Boscain, U., and Sigalotti, M. High-order angles in almost-riemannian geometry. *Séminaire de théorie spectrale et géométrie 25* (2008), 41–54.

[20] Bourbaki, N. *Groupes et algèbres de Lie: Chapitre 1*. Springer Science & Business Media, 2007.

[21] Charlot, G., Boscain, U., and Chambrion, T. Nonisotropic 3-level quantum systems: complete solutions for minimum time and minimum energy. *Discrete and Continuous Dynamical Systems-Series B 4* (2006), 957–990.

[22] Franchi, B., and Lanconelli, E. Une métrique associéea une classe d'opérateurs elliptiques dégénérés. *Rend. Sem. Mat. Univ. Politec. Torino 105* (1983), 114.

[23] Grushin, V. Grushin. *A certain class of hypoelliptic operators*, 458–476.

[24] Grushin, V. On a class of elliptic pseudodifferential operators degenerate on a submanifold. *Mathematics of the USSR-Sbornik 13*, 2 (1971), 155.

[25] Helgason, S. *Differential geometry and symmetric spaces*, vol. 341. American Mathematical Soc., 2001.

[26] Hernández, D. A. G. Geometria quase-riemanniana sobre grupos de lie. In *Dissertação (Mestrado em Matemática)*. UNICAMP, 2018, p. 80.

[27] Jouan, P. Equivalence of control systems with linear systems on Lie groups and homogeneous spaces. *ESAIM: Control, Optimisation and Calculus of Variations 16*, 4 (2010), 956–973.

[28] Jouan, P. Controllability of linear systems on lie groups. *Journal of Dynamical and control systems 17*, 4 (2011), 591–616.

[29] Jouan, P., Zsigmond, G., and Ayala, V. Isometries of almost-riemannian structures on lie groups. *Differential Geometry and its Applications 61* (Dec. 2018), 59–81.

[30] Jurdjevic, V. *Geometric control theory*, vol. 52. Cambridge university press, 1997.

[31] Knapp, A. W. *Lie groups beyond an introduction*, vol. 140. Springer Science & Business Media, 2013.

[32] Montgomery, R. *A tour of subriemannian geometries, their geodesics and applications*. No. 91. American Mathematical Soc., 2002.

[33] Myers, S. B., and Steenrod, N. E. The group of isometries of a riemannian manifold. *Annals of Mathematics* (1939), 400–416.

[34] Onishchik, A., and Vinberg, E. *Lie groups and Lie algebras III-Structure of Lie groups and Lie algebras*, vol. 41 of *Encyclopaedia of Mathematical Sciences*. Springer Verlag, Berlin, 1994.

[35] Pontryagin, L. S. *Mathematical theory of optimal processes*. CRC Press, 1987.

[36] Sachkov, Y. L. Control theory on Lie groups. *Journal of Mathematical Sciences 156*, 3 (2009), 381.

[37] San Martin, L. A. B. *Algebras de Lie*. Editora Unicamp, 2010.

[38] San Martin, L. A. B. *Grupos de Lie*. Editora Unicamp, 2017.

[39] Spivak, M. A comprehensive introduction to differential geometry.

[40] Takasu, T. Generalized Riemannian Geometry. i.

APPENDIX A – Dynamics of 2x2 matrices

A.1 Dynamics of 2x2 matrices

Lemma A.1.1. Let $T \in \mathrm{Gl}(2,\mathbb{R})$ and $u \in \mathbb{R}^2 \backslash \{(0,0)\}$. If u is not an eigenvalue of θ and there exists $a \in \mathbb{R}^*$

$$1)\, T\rho_t u = \rho_{at} T u, \quad \forall t \in \mathbb{R},$$

then

$$2)\, T \circ \theta = a\theta \circ T,$$

and $a = 1$ if $\operatorname{tr}\theta \neq 0$ or $a \in \{-1, 1\}$ if $\operatorname{tr}\theta = 0$. Moreover, if $S \in \mathrm{Gl}(2,\mathbb{R})$ also satisfies 1. and 2 . and $Tu = Su$ then $T = S$.

Proof. If $\rho_t u \cdot u^* = 0$ for all $t \in \mathbb{R}$ then, by derivation, $\theta u \cdot u^* = 0$ implying that $\theta u \in \operatorname{span}\{u\}$ and hence u is an eigenvector of θ. Therefore, if u is not an eigenvalue of θ there exist $t_0, t_1 \in \mathbb{R}$ such that $\{u, \rho_{t_0} u\}$ is linearly independent. Hence, for any $w \in \mathbb{R}^2$ there exists $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $w = \gamma_1 \rho_{t_0} u + \gamma_2 \rho_{t_1} u$ and so

$$\begin{aligned}
T\rho_t w =& T\rho_t \left(\gamma_1 \rho_{t_0} u + \gamma_2 \rho_{t_1} u\right) \\
=& \gamma_1 T\rho_{t+t_0} u + \gamma_2 T\rho_{t+t_1} u \\
\overset{2}{=}& \gamma_1 \rho_{a(t+t_0)} T u + \gamma_2 \rho_{a(t+t_1)} T u \\
=& \rho_t \left(\gamma_1 \rho_{t_0} T u + \gamma_2 \rho_{t_1} T u\right) \\
\overset{2}{=}& \rho_t \left(\gamma_1 T\rho_{t_0} + \gamma_2 T\rho_{t_1} u\right) \\
=& \rho_t T \left(\gamma_1 \rho_{t_0} u + \gamma_2 \rho_{t_1} u\right) \\
=& \rho_{at} T w,
\end{aligned}$$

showing that $T \circ \rho_t = \rho_{at} \circ T$. Derivation at $t = 0$ gives us that

$$T \circ \theta = a\theta \circ T,$$

which implies the result. Now, if S also satisfies 1. and 2 . we have as previously that $S \circ \rho_t = \rho_{at} \circ S$. Therefore, if $Tu = Su$ we get that

$$\forall t \in \mathbb{R}, \quad T\rho_t u = \rho_{at} T u = \rho_{at} S u = S\rho_t u.$$

Since $\{u, \rho_{t_0} u\}$ is a basis of $\mathbb{R}^2$ for some $t_0 \in \mathbb{R}$, linearity of S and T implies $T = S$. Let $a, b, c, \theta, \lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, with $\lambda_1 \neq 0 \neq \lambda_2$ and consider the functions

$$\gamma_1(t) = a\mathrm{e}^{t\lambda_1} + b\mathrm{e}^{t\lambda_2} + c, \quad \gamma_2(t) = \mathrm{e}^{\lambda t}(at + b) + c \quad \text{and} \quad \chi(t) = \mathrm{e}^{t\lambda}\cos(t + \theta) + c. \tag{A.1}$$

It is straightforward to see that the following properties holds:

1. $a = b = 0$ and $\gamma_i \equiv c$;
2. $ab = 0$ with $a^2 + b^2 \neq 0$ then γ_i is unbounded and has at most one zero;
3. $ab \neq 0$ then γ is unbounded and has at most two zeros;
4. χ has a finite number of zeros if $|c| > 1$
5. The zeros of χ form an infinite discrete subset of $\mathbb{R}$ if $|c| \leqslant 1$.

□

Lemma A.1.2. Let $A \in \mathfrak{gl}(2, \mathbb{R}), \tau \in \mathbb{R}$ and $u, v \in \mathbb{R}^2$ nonzero vectors. For the function

$$\gamma : \mathbb{R} \to \mathbb{R}, \quad \gamma(t) = \mathrm{e}^{tA} u \cdot v + \tau,$$

it holds:

1. The eigenvalues of A are real and
   1.1. γ is an unbouded function with at most two zeros or
   1.2. $\gamma \equiv \tau, u \cdot v = 0$ and u is an eigenvalue of A.
2. The eigenvalues of A are complex and
   2.1. γ is unbounded with an enumerable discrete set of zeros if $\operatorname{tr} A \neq 0$ or
   2.2. γ is bounded with an enumerable discrete set of zeros if $\operatorname{tr} A = 0$.

Proof. By considering an appropriated orthonormal basis α of $\mathbb{R}^2$ and writing $[u]_\alpha = (a, b)$ and $[v]_\alpha = (c, d)$ it holds that

$$\gamma(t) = ace^{t\lambda_1} + bde^{t\lambda_2} + \tau, \quad \text{if} \quad [A]_\alpha = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

$$\gamma(t) = \mathrm{e}^{\lambda t}(u \cdot v + \varepsilon bct) + \tau \quad \text{if} \quad [A]_\alpha = \begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix},$$

or

$$\gamma(t) = \mathrm{e}^{\lambda t}\|u\|\|v\| \cos(\mu t + \theta_0) + \tau, \quad \text{if} \quad [A]_\alpha = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix},$$

where θ_0 is the angle between u and v. In particular, the assertions follows from the analysis of the maps γ_1, γ_2 and χ commented previously. □