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Corrigendum and addendum to "Moduli spaces of framed sheaves and quiver varieties"

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ABSTRACT

This paper is an erratum to our paper *Moduli spaces of framed sheaves and quiver varieties* (J. Geom. Phys., 2016). As a byproduct, we prove a result (Prop. 2.5) providing a description of the fibre $T_V^\vee\,\mathrm{Gr}(a,r)^{\oplus n-1}$, for each $V \in \mathrm{Gr}(a,r)$, as the space of isomorphism classes of certain extensions of sheaves on Hirzebruch surfaces.

1. Introduction

The claim (ii) in Proposition 6.7 of [1] is incorrect, as next Example 2.3 will make clear. As a consequence, the claim (iii) in Proposition 6.7 and Corollary 6.9 are false as well, whilst Proposition 6.8 must be replaced by a slightly weaker statement (see Proposition 2.4). All results stated in Sections 2 to 5, in Subsections 6.1 to 6.3, and in Section 7 hold true, and their proofs remain unchanged; also the final part of Section 6.4, after the proof of Corollary 6.9, remains valid as it stands. In the Introduction, the sentence "the fibres of the direct sum of (copies of) the cotangent bundle classify the sheaves away from the line at infinity" [1, p. 2] has to be replaced by the sentence "each fibre of the direct sum of (copies of) the cotangent bundle can be identified with the vector space $\mathrm{Ext}^1_{\mathcal{O}_{\Sigma_n}}\left(\mathcal{O}_{\Sigma_n}^{\oplus r-a}, \mathcal{O}_{\Sigma_n}(E)^{\oplus a}\right)$".

If not otherwise stated, the notation is the same as in [1]. For the reader's convenience, we briefly recall which is the setting we are working in. We denote by Σ_n the n-th Hirzebruch surface, which can be defined as the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$; we assume the condition $n > 0$. The fibre of the natural ruling $\Sigma_n \to \mathbb{P}^1$ determines a class $F \in \mathrm{Pic}(\Sigma_n)$ and we denote by H and E the classes of sections squaring, respectively, to n and $-n$. As it is well-known, $\mathrm{Pic}(\Sigma_n)$ is freely generated on $\mathbb{Z}$ by H and F; we put $\mathcal{O}_{\Sigma_n}(p,q) = \mathcal{O}_{\Sigma_n}(pH+qF)$. We fix a "line at infinity", $\ell_\infty \simeq \mathbb{P}^1$, belonging to the class H and not intersecting E. A *framed sheaf* on Σ_n is a pair $(\mathcal{E},\theta)$, where $\mathcal{E}$ is a rank r torsion-free sheaf trivial along ℓ_∞ and $\theta : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism. Notice that the condition of being trivial at infinity implies $c_1(\mathcal{E}) \propto E$.

The moduli space $\mathcal{M}^n(r,a,c)$ parameterizing isomorphism classes of framed sheaves $(\mathcal{E},\theta)$ on Σ_n with Chern character $\mathrm{ch}(\mathcal{E}) = \gamma = (r, aE, -c - \frac{1}{2}na^2)$ has been extensively studied in [1–3]. It is a fine moduli space, which is nonempty if and only if

$$c \geq \frac{na(1-a)}{2}. \tag{1.1}$$

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When the lower bound of the inequality (1.1) is attained, the moduli space $\mathcal{M}^n(r, a, \frac{na(1-a)}{2})$ has a particularly simple and explicit form.

Theorem 1.1 (= [1, *Theorem 6.2]*). *There are isomorphisms*

$$\mathcal{M}^n\left(r, a, \frac{na(1-a)}{2}\right) \simeq \begin{cases} \mathrm{Gr}(a, r) & \text{if } n = 1; \\ T^\vee \mathrm{Gr}(a, r)^{\oplus n-1} & \text{if } n \geq 2, \end{cases}$$

where $\mathrm{Gr}(a, r)$ *is the Grassmannian of* a*-planes in* $\mathbb{C}^r$.

The main result we shall prove in the next section – sc. Proposition 2.5 – provides a description of the fibre $T_V^\vee \mathrm{Gr}(a, r)^{\oplus n-1}$, for each $V \in \mathrm{Gr}(a, r)$, as the space of isomorphism classes of extensions of the form

$$0 \longrightarrow V \otimes \mathcal{O}_{\Sigma_n}(E) \xrightarrow{i} \mathcal{E} \xrightarrow{p} \left(\mathbb{C}^r/V\right) \otimes \mathcal{O}_{\Sigma_n} \longrightarrow 0$$

2. A description of the fibre $T_V^\vee \mathrm{Gr}(a, r)^{\oplus n-1}$

Let us recall that, if a torsion-free sheaf $\mathcal{E}$ on Σ_n is trivial at infinity and satisfies the "minimality" condition (1.1), then it is locally free. These sheaves can be explicitly realized as extensions.

Proposition 2.1. *(=* [1, Proposition 6.7(i)]*) A torsion-free sheaf* $\mathcal{E}$ *is trivial at infinity and satisfies condition* (1.1) *if and only if it fits into an extension of the form*

$$0 \longrightarrow \mathcal{O}_{\Sigma_n}(E)^{\oplus a} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{O}_{\Sigma_n}^{\oplus r-a} \longrightarrow 0 \tag{2.1}$$

for some integers $r > 0$ *and* $0 \leq a < r$.

The following result replaces the erroneous claim [1, Proposition 6.7(ii)].

Proposition 2.2. *Two vector bundles* $\mathcal{E}$ *and* $\mathcal{E}'$ *which are trivial at infinity and satisfy condition* (1.1) *are isomorphic if and only if they fit into extensions of the form* (2.1) *which are isomorphic* as complexes.

Proof. The "if" part is trivial. To prove necessity, we have to distinguish the case $n = 1$ from the case $n \geq 2$. When $n = 1$, [2, Lemma 3.1] implies that

$$\mathrm{Ext}^1_{\mathcal{O}_{\Sigma_1}}\left(\mathcal{O}_{\Sigma_1}^{\oplus r-a}, \mathcal{O}_{\Sigma_1}(E)^{\oplus a}\right) = 0.$$

It follows that all extensions of the form (2.1) split, and this proves the claim in this case. Let us assume $n \geq 2$ and let $\mathcal{E}$ and $\mathcal{E}'$ be two isomorphic vector bundles which are trivial at infinity and satisfy condition (1.1). As shown in [1, § 6.1], $\mathcal{E}$ is the cohomology of a monad of the form

$$0 \longrightarrow \mathcal{O}_{\Sigma_n}(1, -1)^{\oplus na} \oplus \mathcal{O}_{\Sigma_n}^{\oplus r-a} \xrightarrow{\beta} \mathcal{O}_{\Sigma_n}(1, 0)^{\oplus (n-1)a} \longrightarrow 0, \tag{2.2}$$

where β is surjective (therefore, $\mathcal{E} \simeq \ker \beta$); analogously for $\mathcal{E}'$. By [2, Lemma 4.7], an isomorphism $\Lambda : \mathcal{E} \longrightarrow \mathcal{E}'$ lifts uniquely to an isomorphism of monads. We proved in [1, § 6.3] that such an isomorphism is uniquely determined by an invertible matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A \in \mathrm{GL}(a, \mathbb{C})$ and $C \in \mathrm{GL}(r-a, \mathbb{C})$. By using the diagram [1, eq. (6.18)] it can be shown that there is an induced diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\Sigma_n}(E)^{\oplus a} \arrow[r, "i"] \arrow[d, "A"] & \mathcal{E} \arrow[r, "p"] \arrow[d, "\Lambda"] & \mathcal{O}_{\Sigma_n}^{\oplus r-a} \arrow[r] \arrow[d, "C"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\Sigma_n}(E)^{\oplus a} \arrow[r, "i'"] & \mathcal{E}' \arrow[r, "p'"] & \mathcal{O}_{\Sigma_n}^{\oplus r-a} \arrow[r] & 0 \end{tikzcd}$$

which is commutative. □

Example 2.3. It should be pointed out that two isomorphic complexes of the form (2.1) may fail to be isomorphic *as extensions*. Indeed, if $\mathcal{E}$ fits into an extension of the form (2.1), of course it fits also into the extension

$$0 \longrightarrow \mathcal{O}_{\Sigma_n}(E)^{\oplus a} \xrightarrow{\lambda i} \mathcal{E} \xrightarrow{p} \mathcal{O}_{\Sigma_n}^{\oplus r-a} \longrightarrow 0\ , \tag{2.3}$$

for any $\lambda \in \mathbb{C}^*$. It is easy to see that the two sequences (2.1) and (2.3) are isomorphic as complexes, but, if $\lambda \neq 1$, not as extensions. □

Let $X_n = \Sigma_n \setminus \ell_\infty$. This open subset can be naturally regarded as the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$. The statement of [1, Lemma 6.8] needs to be replaced by the following result; the proof we provided in [1], however, remains unchanged.

Proposition 2.4. *Two extensions of the form (2.1) are isomorphic if and only if their restrictions to X_n are isomorphic as extensions of $\mathcal{O}_{X_n}$-modules.*

Finally, as a substitute for [1, Corollary 6.9], one has the following proposition.

Proposition 2.5. *For $n \geq 2$, for each point $V \in \mathrm{Gr}(a, r)$, thought of as an a-dimensional subspace of $\mathbb{C}^r$, there is a canonical isomorphism*

$$T_V^\vee \,\mathrm{Gr}(a,r)^{\oplus n-1} \simeq \mathrm{Ext}^1_{\mathcal{O}_{\Sigma_n}}\big((\mathbb{C}^r/V)\otimes\mathcal{O}_{\Sigma_n}, V\otimes\mathcal{O}_{\Sigma_n}(E)\big).$$

Proof. Let $V \subset \mathbb{C}^r$ be a point of $\mathrm{Gr}(a, r)$ and let $W = \mathbb{C}^r/V$. It is well-known (see e.g. [4, Lemma 10.7]) that there is a canonical isomorphism

$$T_V \,\mathrm{Gr}(a,r) \simeq \mathrm{Hom}_{\mathbb{C}}(V, W).$$

So, one has an induced (canonical) isomorphism

$$T_V^\vee \,\mathrm{Gr}(a,r) \simeq \mathrm{Hom}_{\mathbb{C}}(V, W)^* \simeq \mathrm{Hom}_{\mathbb{C}}(W, V).$$

Every morphism $b \in \mathrm{Hom}_{\mathbb{C}}(W, V)^{\oplus n-1}$ can be associated with an extension $e(b)$ of the form

$$0 \longrightarrow V\otimes\mathcal{O}_{\Sigma_n}(E) \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ p\ } W\otimes\mathcal{O}_{\Sigma_n} \longrightarrow 0 \tag{2.4}$$

in the following way. We set

$$\mathcal{V} = \left[V^{\oplus n}\otimes\mathcal{O}_{\Sigma_n}(1,-1)\right]\oplus(W\otimes\mathcal{O}_{\Sigma_n}), \qquad \mathcal{W} = V^{\oplus n-1}\otimes\mathcal{O}_{\Sigma_n}(1,0)$$

(notice that these sheaves are isomorphic to those in (2.2)) and define the morphism

$$\begin{aligned} &\beta_b : \mathcal{V} \longrightarrow \mathcal{W} \\ &\quad \beta_b = \chi \oplus b\, s_\infty, \end{aligned} \tag{2.5}$$

where

$$\chi = \mathrm{id}_V \otimes_{\mathbb{C}} \left[\left(\begin{array}{c|c} \mathbf{1}_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right) y_1 + \left(\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \mathbf{1}_{n-1} \end{array}\right) y_2\right],$$

s_∞ is a global section of $\mathcal{O}_{\Sigma_n}(1, 0)$ whose zero locus is ℓ_∞, and $\{y_1, y_2\}$ is a basis for $H^0(\mathcal{O}_{\Sigma_n}(0, 1))$ (cf. [1, eq. (6.7)]). The morphism β_b is surjective and fits into the commutative diagram $\mathbb{E}(b)$, which is analogous to that introduced in [1, eq. (6.18)]:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & V\otimes\mathcal{O}_{\Sigma_n}(E) \arrow[r, "i"] \arrow[d, "\kappa"] & \ker\beta_b \arrow[r, "p"] \arrow[d, "\tau"] & W\otimes\mathcal{O}_{\Sigma_n} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V^{\oplus n}\otimes\mathcal{O}_{\Sigma_n}(1,-1) \arrow[r, "j"] \arrow[d, "\chi"] & \mathcal{V} \arrow[r, "\pi"] \arrow[d, "\beta_b"] & W\otimes\mathcal{O}_{\Sigma_n} \arrow[r] & 0 \\ & \mathcal{W} \arrow[r, equal] \arrow[d] & \mathcal{W} \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd} \tag{2.6}$$

where τ is the canonical injection,

$$j = \begin{pmatrix} \mathrm{id}_{V^{\oplus n}} \\ 0 \end{pmatrix}, \qquad \pi = \begin{pmatrix} 0 & \mathrm{id}_W \end{pmatrix}, \qquad \kappa = \mathrm{id}_V \otimes {}^t\left(y_2^{n-1}, -y_1 y_2^{n-2}, \ldots, (-y_1)^{n-2} y_2, (-y_1)^{n-1}\right),$$

$p = \pi \circ \tau$, and i is induced by the other morphisms. The top row of the diagram (2.6) defines the extension $e(b)$ (2.4) we were looking for. Actually, by construction, the sheaf $\ker\beta_b$ fits into a monad of the form (2.2), so that it is locally free and trivial at infinity.

The previous procedure enables us to define a map

$$\begin{array}{rccl} \varphi_V: & \mathrm{Hom}_{\mathbb{C}}(W,V)^{\oplus n-1} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_{\Sigma_n}}\left(W\otimes\mathcal{O}_{\Sigma_n}, V\otimes\mathcal{O}_{\Sigma_n}(E)\right), \\ & b & \longmapsto & [e(b)] \end{array}$$

where we denote by $[e(b)]$ the isomorphism class of the extension $e(b)$. This map is canonical in the sense that it does not depend on the choice of a basis for $\mathrm{Hom}_{\mathbb{C}}(W,V)^{\oplus n-1}$. Our purpose is now to prove that φ_V is a vector space isomorphism. To this aim, we proceed in 5 steps.

Step 1. The diagram $\mathbb{E}(b)$ (2.6) can be regarded as a short exact sequence in the abelian category $\mathfrak{Coh}_b(\Sigma_n)$ of bounded complexes of coherent sheaves on Σ_n. Explicitly, $\mathbb{E}(b)$ can be written in the form

$$0 \longrightarrow M_{-1} \xrightarrow{G} M_0(b) \xrightarrow{F} M_1 \longrightarrow 0\ ,$$

where

$$M_{-1}:\quad 0 \longrightarrow V\otimes\mathcal{O}_{\Sigma_n}(E) \xrightarrow{\kappa} V^{\oplus n}\otimes\mathcal{O}_{\Sigma_n}(1,-1) \xrightarrow{\chi} \mathcal{W} \longrightarrow 0,$$

$$M_0(b):\quad 0 \longrightarrow \ker\beta_b \xrightarrow{\tau} \mathcal{V} \xrightarrow{\beta_b} \mathcal{W} \longrightarrow 0,$$

$$M_1:\quad \mathcal{W}\otimes\mathcal{O}_{\Sigma_n} = \mathcal{W}\otimes\mathcal{O}_{\Sigma_n} \longrightarrow 0,$$

$G=(i,j,\mathrm{id}_{\mathcal{W}})$,

$F=(p,\pi,0)$.

Step 2. It is easy to define a vector space homomorphism

$$\begin{array}{rccl} \xi: & \mathrm{Ext}^1_{\mathfrak{Coh}_b(\Sigma_n)}(M_1,M_{-1}) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_{\Sigma_n}}\left(W\otimes\mathcal{O}_{\Sigma_n}, V\otimes\mathcal{O}_{\Sigma_n}(E)\right), \\ & [\mathbb{E}] & \longmapsto & [e] \end{array}$$

where e is the top row of $\mathbb{E}$. (Notice that the additive structure of both spaces is provided by the Baer sum.) At the same time, one has a map $\widetilde{\varphi}_V : \mathrm{Hom}_{\mathbb{C}}(W,V)^{\oplus n-1} \to \mathrm{Ext}^1_{\mathfrak{Coh}_b(\Sigma_n)}(M_1,M_{-1})$ given by $b\mapsto[\mathbb{E}(b)]$, so that $\varphi_V=\xi\circ\widetilde{\varphi}_V$.

Step 3. Straightforward, though rather cumbersome computations show that the map $\widetilde{\varphi}_V$ is a vector space homomorphism. As a consequence, the same is true for the map φ_V.

Step 4. Next, we prove the injectivity of the homomorphism φ_V. Let $b\in\mathrm{Hom}(W,V)^{\oplus n-1}$ and suppose that $\varphi_V(b)=0$, i.e., that $[e(b)]$ is the split extension class. In particular, this entails the existence of an isomorphism

$$\Psi:\left(V\otimes\mathcal{O}_{\Sigma_n}(E)\right)\oplus(W\otimes\mathcal{O}_{\Sigma_n}) \xrightarrow{\sim} \ker\beta_b.$$

As a consequence, the morphism β_b fits into the short exact sequence

$$0 \longrightarrow \left(V\otimes\mathcal{O}_{\Sigma_n}(E)\right)\oplus(W\otimes\mathcal{O}_{\Sigma_n}) \xrightarrow{\tau\circ\Psi} \mathcal{V} \xrightarrow{\beta_b} \mathcal{W} \longrightarrow 0\ .$$

It is easy to check that the morphism $\tau\circ\Psi$ can be put into the block matrix form $\begin{pmatrix}\star & 0\\ 0 & A\end{pmatrix}$ for some $A\in\mathrm{Aut}_{\mathbb{C}}(W)$. Therefore, the condition

$$\beta_b\circ(\tau\circ\Psi)=0$$

implies $b=0$ (cf. Eq. (2.5)), and φ_V is injective.

Step 5. To conclude the proof, it is enough to show that the vector spaces $T^\vee_V\,\mathrm{Gr}(a,r)^{\oplus n-1}$ and $\mathrm{Ext}^1_{\mathcal{O}_{\Sigma_n}}\left(W\otimes\mathcal{O}_{\Sigma_n}, V\otimes\mathcal{O}_{\Sigma_n}(E)\right)$ have the same dimension. The former has dimension $a(r-a)(n-1)$. As for the latter, one has the canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{\Sigma_n}}\left(W\otimes\mathcal{O}_{\Sigma_n}, V\otimes\mathcal{O}_{\Sigma_n}(E)\right)\simeq\mathrm{Hom}(W,V)\otimes H^1(\mathcal{O}_{\Sigma_n}(E)).$$

Since $h^0(\mathcal{O}_{\Sigma_n}(E))=1$ and $h^2(\mathcal{O}_{\Sigma_n}(E))=0$ (cf. [2, Lemma 3.1]), the Riemann–Roch formula implies $h^1(\mathcal{O}_{\Sigma_n}(E))=n-1$. □

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