DOI: 10.1134/S0012266111080143

ISSN 0012-2661, Differential Equations, 2011, Vol. 47, No. 8, pp. 1210–1214. © Pleiades Publishing, Ltd., 2011.
Original Russian Text © Yu.D. Bozhkov, I.L. Freire, 2011, published in Differentsial'nye Uravneniya, 2011, Vol. 47, No. 8, pp. 1196–1200.

SHORT COMMUNICATIONS

Divergence Symmetries of Critical Kohn–Laplace Equations on Heisenberg Groups

Yu. D. Bozhkov and I. L. Freire

Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Brazil
Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Brazil

Received October 23, 2009

Abstract—We show that every Lie point symmetry of semilinear Kohn–Laplace equations with a power-law nonlinearity on the Heisenberg group H^1 is a divergence symmetry if and only if the corresponding exponent takes a critical value.

DOI: 10.1134/S0012266111080143

1. INTRODUCTION

Recently, there has been a renewed significant and steadily increasing interest in Heisenberg groups in both analysis and geometry. In the last decades, a number of authors studied partial differential equations (PDE) on such groups by various methods. We note only some of the numerous results. Results on the existence, regularity, and absence of solutions for PDE containing Kohn–Laplace operators were obtained in [1]. General results on the absence of solutions of differential inequalities on Heisenberg groups were obtained in [2]. The work [3] represented a review of a number of results dealing with critical semilinear equations on the Heisenberg group.

In the present paper, we use the symmetry theory of Lie differential equations [4, 5] for the investigation of a sample Kohn–Laplace equation on a Heisenberg group. Namely, we consider variational and divergent symmetries of the following differential equation on the Heisenberg group H^1:

$$\Delta_{H^1} u + u^p = 0, \tag{1}$$

where $\Delta_{H^1} = X^2 + Y^2$ is a Kohn–Laplace operator,

$$X = \partial/\partial x + 2y\,\partial/\partial t, \qquad Y = \partial/\partial y - 2x\,\partial/\partial t$$

generate a *left* multiplication in H^1. More precisely, Eq. (1) with $u = u(x, y, t) : \mathbb{R}^3 \to \mathbb{R}$ has the form

$$u_{xx} + u_{yy} + 4(x^2 + y^2)u_{tt} + 4yu_{xt} - 4xu_{yt} + u^p = 0. \tag{2}$$

In [6] we constructed a complete group classification of Kohn–Laplace semilinear equations on H^1. In the case of a power-law nonlinearity, the result implies that the group of symmetries (1) with $p \neq 0$ and $p \neq 1$ consists of displacements with respect to t, rotations in the plane $x - y$, right multiplications in the Heisenberg group H^1, and an elongation induced by

$$T = \frac{\partial}{\partial t}, \qquad R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad \tilde{X} = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial t}, \qquad \tilde{Y} = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t} \tag{3}$$

and

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}. \tag{4}$$

Moreover, if $p = 3$, then the group of symmetry can be complemented by the following generating elements:

$$V_1 = (xt - x^2y - y^3)\frac{\partial}{\partial x} + (yt + x^3 + xy^2)\frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u},$$
$$V_2 = (t - 4xy)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2)\frac{\partial}{\partial t} + 2yu\frac{\partial}{\partial u},$$
$$V_3 = (x^2 - 3y^2)\frac{\partial}{\partial x} + (t + 4xy)\frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3)\frac{\partial}{\partial t} - 2xu\frac{\partial}{\partial u}.$$

The aim of the present paper is to clarify what of the above-listed symmetries are variational or divergent.

By G we denote the five-parameter Lie group of pointwise transformations generated by T, R, $\tilde{X}$, $\tilde{Y}$, and Z. Then our first result can be formulated as the following.

Theorem 1. *The Lie group G of pointwise symmetries of the Kohn–Laplace equation* (1) *is a variational group of symmetries if and only if $p = 3$.*

Recall that the homogeneous dimension of the Heisenberg group H^n is equal to $Q = 2n + 2$, and the Sobolev critical exponent is equal to $(Q + 2)/(Q - 2)$. Therefore, Theorem 1 implies that the elongation Z is a variational symmetry if and only if p is equal to the critical exponent. Virtually, this property is valid for H^n, $n > 1$, and we return to this situation later.

Next we show that the additional symmetries V_1, V_2, and V_3 are divergent in the critical case. To this end, we find a close form of vector-valued potentials defining V_1, V_2, and V_3 as divergent symmetries.

The following assertion is the main result of the present paper.

Theorem 2. *Any Lie pointwise symmetry of the Kohn–Laplace equation*

$$\Delta_{H^1} u + u^3 = 0 \tag{5}$$

is divergent.

It is well known that divergent symmetries generate conservation laws by the Noether theorem [4]. Therefore, the following investigation stages imply to find conservation laws [7] corresponding to considered variational and divergent symmetries and to analyze invariant solutions of the Kohn–Laplace equations.

Note that, by Theorem 2, all Lie pointwise symmetries of the Kohn–Laplace equation (5) are divergent. This justifies the general property proved in [8], that property implies that Lie pointwise symmetries of critical quasilinear differential equations with power-law nonlinearities are divergent. The conjecture on the validity of this property for differential equations on Heisenberg groups was suggested by E. Mitidieri in June, 2003.

In Sections 2 and 3, we prove Theorems 1 and 2, respectively, and in Section 3, we discuss the generalization of obtained results to the Heisenberg group H^n, $n > 1$.

2. VARIATIONAL SYMMETRIES

Firstly, we note that the Kohn–Laplace equation $\Delta_{H^1} u + f(u) = 0$ is an Euler–Lagrange equation for the functional

$$\int L(x, y, t, u, u_x, u_y, u_t)\, dx\, dy\, dt,$$

where the integration is performed over $\mathbb{R}^3$, the Lagrange function is given by the formula

$$L = \frac{1}{2}(Xu)^2 + \frac{1}{2}(Yu)^2 - \int\limits_0^u f(s)\, ds$$

or, which is equivalent,

$$L = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \int\limits_0^u f(s)\,ds, \tag{6}$$

and the function u is assumed to satisfy proper decreasing conditions as $d = (t^2+(x^2+y^2))^{1/4} \to \infty$.

Proof of Theorem 1. By the general symmetry theory of differential equations [4], it suffices to show that generating elements G define variational symmetries.

Indeed, by the infinitesimal invariance criterion [4, p. 257], G is a group of variational symmetry if and only if

$$W^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = 0 \tag{7}$$

for all $(x, y, t, u, u_x, u_y, u_t)$ and for any infinitesimal generator

$$W = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}.$$

(Recall that $W^{(1)}$ is a first-order extension of W, see [4].)

To prove relation (7) for T, R, $\tilde{X}$, $\tilde{Y}$, and Z, we firstly evaluate the corresponding first-order extensions with the use of formulas for extended infinitesimal operators [4]:

$$\begin{gathered} T^{(1)} = T, \quad R^{(1)} = R + u_y\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial u_y}, \quad \tilde{X}^{(1)} = \tilde{X} + 2u_t\frac{\partial}{\partial u_y}, \quad \tilde{Y}^{(1)} = \tilde{Y} - 2u_t\frac{\partial}{\partial u_x}, \\ Z^{(1)} = Z + \frac{1+p}{1-p}u_x\frac{\partial}{\partial u_x} + \frac{1+p}{1-p}u_y\frac{\partial}{\partial u_y} + \frac{2p}{1-p}u_t\frac{\partial}{\partial u_t}. \end{gathered} \tag{8}$$

Then from (3), (6), and (8), one can readily find that T, R, $\tilde{X}$, and $\tilde{Y}$ satisfy (7). Therefore, they define variational symmetries for an arbitrary $f(u)$.

Next, let $\xi = x$, $\phi = y$, $\tau = 2t$, and $\eta = 2u/(1-p)$ be infinitesimal generators of the elongation Z. Then the left-hand side of relation (7) with $W = Z$ acquires the form

$$\begin{aligned} &Z^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) \\ &\quad = \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u} + \frac{1+p}{1-p}u_x\frac{\partial}{\partial u_x} + \frac{1+p}{1-p}u_y\frac{\partial}{\partial u_y} + \frac{2p}{1-p}u_t\frac{\partial}{\partial u_t}\right]L + 4L, \end{aligned}$$

where

$$L = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{1}{p+1}u^{p+1}.$$

After the differentiation and simplifications, we obtain

$$\begin{aligned} &Z^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) \\ &\quad = \frac{3-p}{1-p}(u_x^2 + u_y^2 + 4(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t) + \frac{2(3-p)}{p^2-1}u^{p+1}. \end{aligned}$$

Therefore, Z is a variational symmetry if and only if $p = 3$, which completes the proof of the theorem.

3. DIVERGENT SYMMETRIES

Let us prove Theorem 2. Recall that a pointwise transformation with the infinitesimal generator

$$W = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}$$

is a divergent symmetry for $\int L$ if and only if there exists a vector function $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ of u and its derivatives up to some finite order such that

$$W^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = \operatorname{Div}\varphi. \tag{9}$$

Since variational symmetries are divergent symmetries with $\varphi = 0$, it follows from Theorem 1 that it suffices to show that V_1, V_2, and V_3 are divergent symmetries. To this end, we find the corresponding potentials φ.

For the symmetry V_1, we have $\xi = xt - x^2y - y^3$, $\phi = yt + x^3 + xy^2$, $\tau = t^2 - (x^2 + y^2)^2$, and $\eta = -tu$. Let us evaluate the first-order extension of V_1 :

$$V_1^{(1)} = V_1 + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_y^{(1)}\frac{\partial}{\partial u_y} + \eta_t^{(1)}\frac{\partial}{\partial u_t},$$

where extensions of infinitesimal operators are given by the formulas

$$\eta_x^{(1)} = 2(xy - t)u_x - (3x^2 + y^2)u_y + 4x(x^2 + y^2)u_t,$$
$$\eta_y^{(1)} = (x^2 + 3y^2)u_x - 2(t + xy)u_y + 4y(x^2 + y^2)u_t, \qquad \eta_t^{(1)} = -u - xu_x - yu_y - 3tu_t.$$

Next, after some manipulations, we obtain

$$V_1^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = 2xuu_y - 2yuu_x - 4(x^2 + y^2)u_t. \tag{10}$$

Therefore, V_1 is not a variational symmetry. Let

$$A_1 = -yu^2, \qquad A_2 = xu^2, \qquad A_3 = -2(x^2 + y^2)u^2. \tag{11}$$

Then, by virtue of relations (10) and (11), we have

$$V_1^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = \operatorname{Div}(A),$$

where $A = (A_1, A_2, A_3)$. Therefore, V_1 is a divergent symmetry. Similarly, for the symmetries V_2 and V_3, we obtain

$$V_2^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = 2uu_y - 4xuu_t, \tag{12}$$
$$V_3^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = -2uu_x - 4yuu_t. \tag{13}$$

Now we set

$$B = (0, u^2, -2xu^2), \qquad C = (-u^2, 0, -2yu^2). \tag{14}$$

Then it follows from relations (12)–(14) that V_2 and V_3 satisfy relation (9) with φ replaced by B and C, respectively. Therefore, V_2 and V_3 are divergent symmetries.

4. ON THE GENERALIZATION TO H^n, $n > 1$

Here we outline possibilities of the generalization of the above-suggested approach to the Heisenberg group H^n, $n > 1$.

Note firstly that the Kohn–Laplace equation

$$\Delta_{H^n}u + f(u) = 0 \tag{15}$$

or, which is equivalent,

$$u_{x_ix_i} + u_{y_iy_i} + 4(x_i^2 + y_i^2)u_{tt} + 4y_iu_{x_it} - 4x_iu_{y_it} + f(u) = 0,$$

is an Euler–Lagrange equation for the functional $J[u] = \int L$ with

$$L = \frac{1}{2}(X_i u)^2 + \frac{1}{2}(Y_i u)^2 - \int\limits_0^u f(s)\,ds$$
$$= \frac{1}{2}u_{x_i}^2 + \frac{1}{2}u_{y_i}^2 + 2(x_i^2 + y_i^2)u_t^2 + 2y_i u_{x_i} u_t - 2x_i u_{y_i} u_t - \int\limits_0^u f(s)\,ds,$$

where $X_i = \dfrac{\partial}{\partial x_i} + 2y_i \dfrac{\partial}{\partial t}$, $Y_i = \dfrac{\partial}{\partial y_i} - 2x_i \dfrac{\partial}{\partial t}$, and the summation is performed over $i = 1, 2, \ldots, n$.

By using the definition of a Lie pointwise symmetry of a differential equation, one can show that the scaling transformation

$$x_j^* = \lambda x_j, \qquad y_j^* = \lambda y_j, \qquad t^* = \lambda^2 t, \qquad u^* = \lambda^{2/(1-p)} u$$

preserves the equation

$$\Delta_{H^n} u + u^p = 0. \tag{16}$$

Then, by performing this substitution in the functional J, one can readily find that the elongation

$$Z = x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} + 2t \frac{\partial}{\partial t} + \frac{2}{1-p} u \frac{\partial}{\partial u}$$

is a variational symmetry if and only if

$$p = \frac{n+2}{n} = \frac{Q+2}{Q-2}.$$

Therefore, Eq. (16) admits a group of variational symmetries containing Z if and only if p takes the critical value.

In conclusion, we note that the complete classification of groups of Kohn–Laplace equations (15) has been constructed only for $n = 1$ (see [6]).

ACKNOWLEDGMENTS

The authors are grateful to E. Mitidieri for the useful discussion of results. The first author was financially supported by FAPESP and CNPq, Brazil, and the second author was supported by CAPES, Brazil.

REFERENCES

1. Garofalo, N. and Lanconelli, E., Existence and Nonexistence Results for Semilinear Equations on the Heisenberg Group, *Indiana Univ. Math. J.*, 1992, vol. 41, pp. 71–98.
2. Pohozaev, S.I. and Veron, L., Nonexistence Results of Solutions of Semilinear Differential Inequalities on the Heisenberg Group, *Manuscripta Math.*, 2000, vol. 102, pp. 85–99.
3. Lanconelli, E., Critical Semilinear Equations on the Heisenberg Group, Workshop on Partial Differential Equations (Ferrara, 1999), *Ann. Univ. Ferrara Sez. VII (N.S.)*, 1999, vol. 45, Suppl, pp. 187–195.
4. Olver, P.J., *Application of Lie Groups to Differential Equations*, GTM 107, New York, 1986.
5. Ovsiannikov, L., *Group Analysis of Differential Equations*, London, 1982.
6. Bozhkov, Y. and Freire, I.L., Group Classification of Semilinear Kohn–Laplace Equations, *Nonlinear Anal.*, 2008, vol. 68, pp. 2552–2568.
7. Bozhkov, Y. and Freire, I.L., Conservations Laws for Critical Kohn–Laplace Equations on the Heisenberg Group, *J. Nonlinear Math. Phys.*, 2008, vol. 15, pp. 35–47.
8. Bozhkov, Y., Noether Symmetries and Critical Exponents, *Sigma Symmetry Integrability Geom. Methods Appl.*, 2005, vol. 1, paper 022.