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Group classification of semilinear Kohn–Laplace equations

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Abstract

We study the Lie point symmetries of semilinear Kohn–Laplace equations on the Heisenberg group H^1 and obtain a complete group classification of these equations.

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1. Introduction

The Heisenberg group H^n topologically is the real vector space $\mathbb{R}^{2n+1}$. Its Lie group structure is determined by the product

$$(x, y, t)(x^0, y^0, t^0) = \left(x + x^0, y + y^0, t + t^0 + 2 \sum_{i=1}^n (y_i x_i^0 - x_i y_i^0) \right),$$

where $(x, y, t), (x^0, y^0, t^0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = H^n$. It is easy to verify that the operators

$$T = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},$$

where $i = 1, 2, \dots, n$, form a basis of the left-invariant vector fields on H^n and satisfy the following commutation relations:

$$[X_i, Y_i] = -4\delta_{ij}T, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

These formulae present in an abstract form the commutation relations for the quantum-mechanical position and momentum operators in n -dimensional configuration space. This justifies the name Heisenberg group.

In the last few decades the Heisenberg group H^n has been intensively and extensively studied by a considerable number of authors using methods and approaches which come from algebraic and differential geometry, real and

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complex analysis, mathematical physics and applications. A big part of the corresponding work treats partial differential equations on H^n . In this regard various authors have obtained existence and nonexistence results for equations involving Kohn–Laplace operators. Recall that the Kohn–Laplace operator Δ_{H^n} is the natural subelliptic Laplacian on H^n defined by

$$\Delta_{H^n} = \sum_{i=1}^n (X_i^2 + Y_i^2).$$

Although there are similarities between Δ_{H^n} and the classical Laplacian they are essentially different. For example, the Kohn–Laplace operator is not a strongly elliptic operator. It is a typical representative of the hypoelliptic operators [19]. (Since the study of hypoellipticity properties is not the subject of this paper we shall not comment further on this point.)

In [18] Garofalo and Lanconelli established existence, regularity and nonexistence results for the Kohn–Laplace equation

$$\Delta_{H^n} u + f(u) = 0$$

in an open bounded or unbounded subset of H^n with homogeneous Dirichlet boundary conditions. One of the motivations for studying such semilinear equations is the fact that they may arise as Euler–Lagrange equations in some variational problems on Cauchy–Riemann (CR) manifolds as in the works of Jerison and Lee [25,26] on the CR Yamabe problem. The existence of weak solutions is proved in [18] provided the nonlinear term satisfies some growth conditions of the form $f(u) = o(|u|^{(Q+2)/(Q-2)})$ as $|u| \rightarrow \infty$, where $Q = 2n + 2$ is the so-called homogeneous dimension of H^n [15]. The exponent $(Q + 2)/(Q - 2)$ is the critical exponent for Stein’s Sobolev space [26]. The nonexistence results follow from remarkable Pohozaev identities established in [18] for the solutions of Kohn–Laplace equations on the Heisenberg group. The Dirichlet problem for the Kohn Laplacian on H^n was studied before by Jerison in [23,24]. See also [4] for the existence of classical nonnegative solutions of semilinear Kohn–Laplace equations. General nonexistence results for solutions of semilinear differential inequalities on the Heisenberg group were obtained by Pohozaev and Veron in [29]. Since there are a huge number of works dedicated to Heisenberg groups (see [1]) and the study of PDE on H^n , in order not to increase the volume of this paper, we shall not present here further details, directing the interested reader to the already cited works as well as to [2,3,5,6,14–17,20,21] and the references therein.

The purpose of the present paper is to illuminate the properties of the Kohn–Laplace equations from the point of view of the S. Lie Symmetry Theory, which to our knowledge has not been previously done. We shall obtain a complete group classification of semilinear partial differential equations on H^1 of the following form:

$$\Delta_{H^1} u + f(u) = 0, \tag{1}$$

where Δ_{H^1} is the Kohn–Laplace operator on H^1 and f is a generic function.

The importance of group classification of differential equations was first emphasized by Ovsiannikov in 1950s–1960s, when he and his school began a systematic research program of successfully applying modern group analysis methods to wide range of physically important problems. Following Olver ([27], p. 182), we recall that to perform a group classification on a differential equation involving a generic function f consists of finding the Lie point symmetries of the given equation with arbitrary f , and, then, to determine all possible particular forms of f for which the symmetry group can be enlarged. It is worth observing that for problems which arise from physics, quite often there is physical motivation for considering such specific cases.

The Heisenberg group H^1 itself possesses the rich properties of H^n (see [3]) and the calculations of the symmetry group of this model problem give insights for the general case $n > 1$. For this reason, and for the sake of simplicity and clarity we restrict ourselves to H^1 .

We write the Kohn–Laplace operator as

$$\Delta_{H^1} = X^2 + Y^2, \tag{2}$$

where

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t} \tag{3}$$

and

$$Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}. \quad (4)$$

Then Eq. (1) for $u = u(x, y, t)$ in more detail reads

$$u_{xx} + u_{yy} + 4(x^2 + y^2)u_{tt} + 4yu_{xt} - 4xu_{yt} + f(u) = 0. \quad (5)$$

We shall not present preliminaries concerning Lie point symmetries of differential equations, supposing that the reader is familiar with the basic notions and methods of contemporary group analysis [8,22,27,28].

The main result in this paper is the following

Theorem. *The widest Lie point symmetry group of the Kohn–Laplace equation (1) with an arbitrary $f(u)$ is determined by the operators*

$$T = \frac{\partial}{\partial t}, \quad R = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t}, \quad (6)$$

that is, by a translation in t , a rotation in the x – y plane and the generators of right multiplication in the Heisenberg group H^1 .

For some special choices of the right-hand side $f(u)$ it can be extended in the cases listed below. We shall write only the generators additional to (6).

(i) If $f(u) = 0$, then

$$V_1 = (xt - x^2y - y^3) \frac{\partial}{\partial x} + (yt + x^3 + xy^2) \frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2) \frac{\partial}{\partial t} - tu \frac{\partial}{\partial u} \quad (7)$$

$$V_2 = (t - 4xy) \frac{\partial}{\partial x} + (3x^2 - y^2) \frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2) \frac{\partial}{\partial t} + 2yu \frac{\partial}{\partial u} \quad (8)$$

$$V_3 = (x^2 - 3y^2) \frac{\partial}{\partial x} + (t + 4xy) \frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3) \frac{\partial}{\partial t} - 2xu \frac{\partial}{\partial u} \quad (9)$$

$$Z_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t}, \quad Z_2 = u \frac{\partial}{\partial u}, \quad W = \beta(x, y, t) \frac{\partial}{\partial u}, \quad (10)$$

where $\Delta_{H^1}\beta = 0$.

(ii) If $f(u) = c = \text{const}$, then this case is reduced to (i) by the change $u = v - cx^2/2$.

(iii) If $f(u) = k \cdot u$, k a constant, then

$$Z_2 = u \frac{\partial}{\partial u}, \quad W = \beta(x, y, t) \frac{\partial}{\partial u}, \quad (11)$$

where $\Delta_{H^1}\beta + k\beta = 0$.

(iv) If $f(u) = k \cdot u^p$, $p \neq 0$, $p \neq 1$, we have the generator of dilations

$$Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} + \frac{2}{1-p} u \frac{\partial}{\partial u}. \quad (12)$$

In the critical case $f(u) = k \cdot u^3$, there are three additional generators, namely V_1 , V_2 , V_3 given in (7), (8), (9) respectively.

(v) If $f(u) = k \cdot e^u$ then the operator

$$Z_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial u} \quad (13)$$

generates a subgroup of the Lie point symmetry group of (1).

This classification is similar to that for semilinear equations in $\mathbb{R}^n$ involving Laplace or polyharmonic operators [30]. We also observe that for power nonlinearity $f(u) = ku^p$ exactly in the critical case $p = 3 =$

$(Q + 2)/(Q - 2)$, $Q = 2 \cdot 1 + 2 = 4$ being the homogeneous dimension of H^1 , the symmetry group is expanded by three additional generators (see (iv) of the main theorem). This fact suggests that in the critical case maybe there are further properties as pointed out in [9,10] regarding other differential equations. This is our motivation to use the above group classification in three subsequent papers [11,13,12]. In [11] we study the variational properties of Kohn–Laplace equations and we find out which of the already found Lie point symmetries are variational/divergence symmetries. Further in [12] we establish the corresponding conservation laws via the Noether Theorem. In [13] we discuss the invariant solutions of various Kohn–Laplace equations on the Heisenberg group.

The group classification of Kohn–Laplace equations on the Heisenberg group H^n , $n > 1$, will be treated elsewhere.

This paper is organized as follows. In the next section we obtain the determining equations for the Lie point symmetries of Eq. (1). This process is essentially simplified by the use of two theorems of Bluman [7,8]. Then in Section 3 we obtain some formulae which are consequences of the determining equations. They are used in the proof of the main theorem, given in Sections 4–9.

2. The determining equations

In this section we obtain the determining equations for a Lie point symmetry of the Kohn–Laplace equation (1) with infinitesimal generator

$$S = \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \quad (14)$$

To begin with, we observe that the symmetry calculation is drastically simplified if we apply two theorems of Bluman [7,8]. Indeed, Theorem 4.2.3-1, [8], p. 174, implies that ξ , ϕ and τ do not depend on u . Then by Theorem 4.2.3-6, [8], p. 175, we conclude that η is a linear function of u . Therefore the infinitesimals are of the following form:

$$\begin{cases} \xi = \xi(x, y, t), \\ \phi = \phi(x, y, t), \\ \tau = \tau(x, y, t), \\ \eta = \alpha(x, y, t)u + \beta(x, y, t), \end{cases} \quad (15)$$

where $\alpha = \alpha(x, y, t)$ and $\beta = \beta(x, y, t)$ are functions to be determined.

We define

$$H := \Delta_{H^1} u + f(u).$$

Eq. (1) admits the symmetry (14) if and only if

$$\hat{S}H = 0$$

when $H = 0$ [8,27], where

$$\begin{aligned} \hat{S} = & \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \eta_x^{(1)} \frac{\partial}{\partial u_x} + \eta_y^{(1)} \frac{\partial}{\partial u_y} + \eta_t^{(1)} \frac{\partial}{\partial u_t} + \eta_{xx}^{(2)} \frac{\partial}{\partial u_{xx}} + \eta_{yy}^{(2)} \frac{\partial}{\partial u_{yy}} \\ & + \eta_{xt}^{(2)} \frac{\partial}{\partial u_{xt}} + \eta_{yt}^{(2)} \frac{\partial}{\partial u_{yt}} + \eta_{tt}^{(2)} \frac{\partial}{\partial u_{tt}} + \eta_{xy}^{(2)} \frac{\partial}{\partial u_{xy}} \end{aligned} \quad (16)$$

is the second-order extension of S [8,27]. Then the symmetry condition can be written as

$$(8x\xi + 8y\phi)u_{tt} + 4\phi u_{xt} - 4\xi u_{yt} + \eta f'(u) + \eta_{xx}^{(2)} + \eta_{yy}^{(2)} + 4(x^2 + y^2)\eta_{tt}^{(2)} + 4y\eta_{xt}^{(2)} - 4x\eta_{yt}^{(2)} = 0, \quad (17)$$

when $H = 0$. (The subscripts denote partial derivatives, e.g. $u_x = \frac{\partial u}{\partial x}$. Only in the extension coefficients like $\eta_{xt}^{(j)}$ are the subscripts indices. We also suppose that the functions considered are sufficiently smooth for the derivatives we write to exist.) Further, using the corresponding formulae for the extended infinitesimals [8,27] we calculate

$$\eta_x^{(1)} = \beta_x + \alpha_x u + (\alpha - \xi_x)u_x - \phi_x u_y - \tau_x u_t, \quad (18)$$

$$\eta_y^{(1)} = \beta_y + \alpha_y u - \xi_y u_x + (\alpha - \phi_y) u_y - \tau_y u_t, \quad (19)$$

$$\eta_t^{(1)} = \beta_t + \alpha_t u - \xi_t u_x - \phi_t u_y + (\alpha - \tau_t) u_t, \quad (20)$$

$$\eta_{xx}^{(2)} = \beta_{xx} + \alpha_{xx} u + (2\alpha_x - \xi_{xx}) u_x - \phi_{xx} u_y - \tau_{xx} u_t + (\alpha - 2\xi_x) u_{xx} - 2\phi_x u_{xy} - 2\tau_x u_{xt}, \quad (21)$$

$$\eta_{yy}^{(2)} = \beta_{yy} + \alpha_{yy} u - \xi_{yy} u_x + (2\alpha_y - \phi_{yy}) u_y - \tau_{yy} u_t - 2\xi_y u_{xy} + (\alpha - 2\phi_y) u_{yy} - 2\tau_y u_{yt}, \quad (22)$$

$$\eta_{tt}^{(2)} = \beta_{tt} + \alpha_{tt} u - \xi_{tt} u_x - \phi_{tt} u_y + (2\alpha_t - \tau_{tt}) u_t - 2\xi_t u_{xt} - 2\phi_t u_{yt} + (\alpha - 2\tau_t) u_{tt}, \quad (23)$$

$$\begin{aligned} \eta_{xt}^{(2)} = & \beta_{xt} + \alpha_{xt} u + (\alpha_t - \xi_{xt}) u_x - \phi_{xt} u_y + (\alpha_x - \tau_{xt}) u_t \\ & - \xi_t u_{xx} - \phi_t u_{xy} - \phi_x u_{yt} + (\alpha - \xi_x - \tau_t) u_{xt} - \tau_x u_{tt}, \end{aligned} \quad (24)$$

$$\begin{aligned} \eta_{yt}^{(2)} = & \beta_{yt} + \alpha_{yt} u - \xi_{yt} u_x + (\alpha_t - \phi_{yt}) u_y + (\alpha_y - \tau_{yt}) u_t \\ & - \xi_y u_{xt} - \xi_t u_{xy} - \phi_t u_{yy} + (\alpha - \phi_y - \tau_t) u_{yt} - \tau_y u_{tt}. \end{aligned} \quad (25)$$

Now substituting (18)–(25) into the symmetry condition (17), after some tedious work, we obtain

$$\begin{aligned} & (\alpha u + \beta) f'(u) + \Delta_{H^1} \beta + (\Delta_{H^1} \alpha) u + [2\alpha_x + 4y\alpha_t - \Delta_{H^1} \xi] u_x \\ & + [2\alpha_y - 4x\alpha_t - \Delta_{H^1} \phi] u_y + [8(x^2 + y^2)\alpha_t + 4y\alpha_x - 4x\alpha_y - \Delta_{H^1} \tau] u_t \\ & + [-2\phi_x - 2\xi_y - 4y\phi_t + 4x\xi_t] u_{xy} + [\alpha - 2\xi_x - 4y\xi_t] u_{xx} + [\alpha - 2\phi_y + 4x\phi_t] u_{yy} \\ & + [8x\xi + 8y\phi + 4(x^2 + y^2)(\alpha - 2\tau_t) - 4y\tau_x + 4x\tau_y] u_{tt} \\ & + [4\phi - 2\tau_x - 8(x^2 + y^2)\xi_t + 4y(\alpha - \xi_x - \tau_t) + 4x\xi_y] u_{xt} \\ & + [-4\xi - 2\tau_y - 8(x^2 + y^2)\phi_t - 4y\phi_x - 4x(\alpha - \phi_y - \tau_t)] u_{yt} = 0, \end{aligned} \quad (26)$$

when $H = 0$. Then, expressing u_{xx} using (5) and substituting in (26), we obtain an identity for all values of $(x, y, t, u, u_x, u_y, u_t, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt})$. Equating to zero the coefficients of the derivatives of u and the free term, we obtain the following nine determining equations:

$$\xi_x + 2y\xi_t - \phi_y + 2x\phi_t = 0, \quad (27)$$

$$\xi_y - 2x\xi_t + \phi_x + 2y\phi_t = 0, \quad (28)$$

$$\Delta_{H^1} \xi = 2X\alpha, \quad (29)$$

$$\Delta_{H^1} \phi = 2Y\alpha, \quad (30)$$

$$\Delta_{H^1} \tau = 4yX\alpha - 4xY\alpha, \quad (31)$$

$$\alpha u f'(u) + \beta f'(u) + (\Delta_{H^1} \alpha) u + \Delta_{H^1} \beta + [4y\xi_t + 2\xi_x - \alpha] f(u) = 0, \quad (32)$$

$$2y\xi_x + 2x\xi_y + 4(y^2 - x^2)\xi_t + 2\phi - \tau_x - 2y\tau_t = 0, \quad (33)$$

$$4x\xi_x + 8xy\xi_t + 2\xi + 2y\phi_x - 2x\phi_y + 4(x^2 + y^2)\phi_t + \tau_y - 2x\tau_t = 0, \quad (34)$$

$$2(x^2 + y^2)\xi_x + 4y(x^2 + y^2)\xi_t + 2x\xi + 2y\phi - y\tau_x + x\tau_y - 2(x^2 + y^2)\tau_t = 0, \quad (35)$$

where the operators X and Y are defined by (3) and (4). Multiplying correspondingly Eqs. (27), (28), (33) and (34) we obtain a relation which can be written symbolically as “(35) = $y \cdot (33) + x \cdot (34) - x \cdot (27) - y \cdot (28)$ ”. Hence, Eq. (35) is a consequence of (27), (28), (33) and (34). Another straightforward calculation shows that (31) also follows from these equations. Therefore there are seven independent determining equations which in terms of the operators X and Y can be written in the following simplified form:

$$X\xi - Y\phi = 0, \quad (36)$$

$$Y\xi + X\phi = 0, \quad (37)$$

$$\Delta_{H^1} \xi = 2X\alpha, \quad (38)$$

$$\Delta_{H^1} \phi = 2Y\alpha, \quad (39)$$

$$\alpha u f'(u) + \beta f'(u) + (\Delta_{H^1} \alpha) u + \Delta_{H^1} \beta + (2X\xi - \alpha) f(u) = 0, \quad (40)$$

$$X\tau = 2yX\xi + 2xY\xi + 2\phi, \quad (41)$$

$$Y\tau = -2xX\xi + 2yY\xi - 2\xi. \quad (42)$$

We conclude this section by noting that the system of two equations (36) and (37) may be considered as a Heisenberg group generalization of the Cauchy–Riemann equations.

3. Some consequences of the determining equations

Proposition 1. *If the infinitesimals ξ and ϕ satisfy (36) and (37) then*

$$\Delta_{H^1}\phi = 4\xi_t \quad (43)$$

and

$$\Delta_{H^1}\xi = -4\phi_t. \quad (44)$$

Proof. We apply X to Eq. (37), Y to Eq. (36) and subtract the resulting equations. In this way we obtain

$$(XY - YX)\xi + (X^2 + Y^2)\phi = 0,$$

which implies (43) since the commutator

$$[X, Y] = -4\partial_t \quad (45)$$

and $\Delta_{H^1} = X^2 + Y^2$. Eq. (44) can be derived in an analogous way. $\square$

Corollary. *If α , ξ , ϕ satisfy (36)–(39), then*

$$X\alpha = -2\phi_t, \quad (46)$$

$$Y\alpha = 2\xi_t. \quad (47)$$

Proposition 2. *If ξ , ϕ and τ satisfy (36), (37), (41) and (42), then*

$$\tau_t = 2y\xi_t - 2x\phi_t + 2X\xi. \quad (48)$$

Proof. We just sketch the proof. We apply the operator X to Eq. (42), the operator Y to Eq. (41) and subtract. The resulting equation, with the use of the commutator (45) and (36), (37), leads to (48). $\square$

Proposition 3.

$$\alpha_t = -(X\xi)_t. \quad (49)$$

Proof. We apply the operator X to (47), the operator Y to (46) and subtract. Then by

$$[X, \partial_t] = [Y, \partial_t] = 0, \quad (50)$$

(2) and (37) we obtain (49). $\square$

4. The Lie point symmetries for arbitrary $f(u)$

In this section we prove the main theorem for general right-hand side of the Kohn–Laplace equation (1). Since $f(u)$ is an arbitrary function, then $\alpha = \beta = 0$ by (40). Thus Eqs. (38) and (39) imply that

$$\Delta_{H^1}\xi = \Delta_{H^1}\phi = 0. \quad (51)$$

Then by (51), (43) and (44) it follows that

$$\xi_t = \phi_t = 0$$

and therefore ξ and ϕ are functions of x and y only. On the other hand, from Eq. (40),

$$2X\xi - \alpha = 0,$$

which implies that ξ depends only on y since $\alpha = 0$ and ξ does not depend on t . From Eq. (36) for $\xi(y)$ and $\phi(x, y)$ it follows that ϕ depends only on x . Further, from (51), we obtain that

$$\xi = a_1 y + a_2,$$

$$\phi = Ax + a_3,$$

where A, a_1, a_2, a_3 are arbitrary constants. Now from (37) we get that $A = -a_1$, that is

$$\phi = -a_1 x + a_3.$$

Substituting ξ and ϕ into (41) and (42) gives

$$X\tau - 2a_3 = 0,$$

$$Y\tau + 2a_2 = 0.$$

Applying Y and X to the latter two equations, subtracting and using the commutator $[X, Y] = -4\partial_t$ we obtain that τ does not depend on t . Therefore

$$\tau_x - 2a_3 = 0,$$

$$\tau_y + 2a_2 = 0,$$

from which we conclude easily that $\tau = 2a_3 x - 2a_2 y + a_4$ where a_4 is another arbitrary constant. In this way

$$\begin{cases} \xi = a_1 y + a_2, \\ \phi = -a_1 x + a_3, \\ \tau = 2a_3 x - 2a_2 y + a_4, \\ \eta = 0, \end{cases} \quad (52)$$

which proves the first statement of the main theorem.

5. The Lie point symmetries for $f(u) = ke^u$

In this section we prove item (v) of the main theorem.

We substitute $f(u) = ke^u$ into (40):

$$\alpha k u e^u + \beta k e^u + (\Delta_{H^1} \alpha) u + (\Delta_{H^1} \beta) + [2X\xi - \alpha] k e^u = 0.$$

Hence

$$\alpha = 0 \quad (53)$$

and

$$\beta + 2X\xi - \alpha = 0. \quad (54)$$

From (53) and (46), (47) it follows that $\xi = \xi(x, y)$ and $\phi = \phi(x, y)$. From (54),

$$\beta + 2\xi_x = 0 \quad (55)$$

and hence $\beta = \beta(x, y)$.

Further, the relation (48) implies that

$$\tau_t = 2\xi_x$$

since ξ and ϕ do not depend on t . Therefore there exists a function $h(x, y)$ such that

$$\tau = 2t\xi_x + h(x, y). \quad (56)$$

We observe that the right-hand side of Eq. (41) does not depend on t , while the left-hand side is

$$2t \xi_{xx} + h_x + 4y\xi_x.$$

Thus

$$\xi_{xx} = 0. \quad (57)$$

Analogously, from (42), we obtain that

$$\xi_{xy} = 0. \quad (58)$$

On the other hand $\Delta_{H^1}\xi = 0$ by (38) since $\alpha = 0$ (see (53)). But $\xi = \xi(x, y)$ and therefore ξ is harmonic:

$$\xi_{xx} + \xi_{yy} = 0. \quad (59)$$

Eqs. (57)–(59) imply that

$$\xi = a_5x + a_1y + a_2,$$

where a_1, a_2, a_5 are arbitrary constants. Then the Cauchy–Riemann equations (36) and (37) imply that

$$\phi = -a_1x + a_5y + a_3,$$

where a_3 is a constant. Now we substitute ξ, ϕ and τ into (41) and (42). The resulting equations, by a simple argument, imply

$$\tau = 2a_3x - 2a_2y + a_4 + 2a_5t.$$

Finally, $\beta = -2a_5$ from (55). Summarizing, the infinitesimals are given by

$$\begin{cases} \xi = a_1y + a_2 + a_5x, \\ \phi = -a_1x + a_3 + a_5y, \\ \tau = 2a_3x - 2a_2y + a_4 + 2a_5t, \\ \eta = -2a_5, \end{cases} \quad (60)$$

which concludes the proof of the main theorem in the case of exponential nonlinearity.

6. The Lie point symmetries for $f(u) = ku^p$

In this section we prove the main theorem in the case of nonlinearity of power type $f(u) = ku^p$. We suppose that $p \neq 0, p \neq 1, p \neq 2, p \neq 3$. We do not consider $p = 2$ since in this case, by a nonexistence result of Pohozaev and Veron [29], there is no solution of the corresponding Kohn–Laplace equation even in a very weak sense. The case $p = 3$ will be treated in the next section. The case $p = 1$ will be studied in Section 9. Finally, if $p = 0$ this is the item (ii), which is reduced to (i) as stated in the theorem.

By (40) we have

$$k(\alpha p + [2X\xi - \alpha])u^p + kp\beta u^{p-1} + (\Delta_{H^1}\alpha)u + (\Delta_{H^1}\beta) = 0.$$

Hence $\beta = 0, \Delta_{H^1}\alpha = 0$ and

$$\alpha = \frac{2}{1-p}X\xi. \quad (61)$$

By (49) and (61) it follows that

$$(p-3)\alpha_t = 0.$$

Thus $\alpha_t = 0$ since $p \neq 3$. Therefore α depends only on x and y , and Eqs. (46) and (47) read

$$\begin{aligned} \alpha_x &= -2\phi_t, \\ \alpha_y &= 2\xi_t. \end{aligned}$$

Hence there exist functions $B_1(x, y)$ and $B_2(x, y)$ such that

$$\begin{aligned}\xi &= \frac{1}{2}\alpha_y(x, y)t + B_1(x, y), \\ \phi &= -\frac{1}{2}\alpha_x(x, y)t + B_2(x, y).\end{aligned}$$

Substituting ξ and ϕ into (36) and (37), we obtain

$$\alpha_{xy} = 0 \quad (62)$$

and

$$\alpha_{xx} - \alpha_{yy} = 0. \quad (63)$$

Since $\Delta_{H^1}\alpha = 0$ and $\alpha = \alpha(x, y)$, it follows that α is harmonic:

$$\alpha_{xx} + \alpha_{yy} = 0. \quad (64)$$

From (62)–(64) we conclude that

$$\alpha = Ax + By + C,$$

where A, B, C are constants. Thus

$$\xi = \frac{B}{2}t + B_1(x, y), \quad (65)$$

$$\phi = -\frac{A}{2}t + B_2(x, y), \quad (66)$$

$$\alpha = Ax + By + C. \quad (67)$$

Further, we substitute ξ and ϕ into (48). In this way we see that τ_t is a function of x and y only. Hence, there are functions $M(x, y)$ and $N(x, y)$ such that

$$\tau = M(x, y)t + N(x, y). \quad (68)$$

We put (65), (66) and (68) into (41). We get that

$$M_x t + N_x + 2yM = g_1(x, y) - At,$$

where g_1 is a function of x and y only. Thus $M_x = -A$. Hence

$$M = -Ax + m(y)$$

for some function $m = m(y)$. Now we substitute (65) and (68) into (42). We have

$$M_y t + N_y - 2xM = g_2(x, y) - Bt,$$

where g_2 is a function of x and y only. Thus $M_y = -B$. Hence $m'(y) = -B$ and $m(y) = -By + D$, $D = \text{constant}$. Therefore

$$\tau = (-Ax - By + D)t + N(x, y). \quad (69)$$

By (61) and (67) we obtain

$$B_{1,x} = c_1x + c_2y + c_3 \quad (70)$$

where $c_1 = (1-p)A/2$, $c_2 = -(p+1)B/2$, $c_3 = (1-p)/2$. By integration

$$B_1 = \frac{c_1}{2}x^2 + c_2xy + c_3x + \varphi(y) \quad (71)$$

for some function φ of y only. From (38), (65), (67) and (71) we obtain

$$\varphi''(y) = (3+p)A/2 =: c_4.$$

Hence

$$\varphi(y) = c_4 y^2 + c_5 y + c_6,$$

where c_5, c_6 are constants. Then

$$B_1 = \frac{c_1}{2} x^2 + c_2 xy + c_3 x + c_4 y^2 + c_5 y + c_6. \quad (72)$$

By (36), (65), (66) and (72):

$$B_{2,y} = d_1 x + d_2 y + c_3,$$

where $d_1 = -(p+1)A/2$, $d_2 = (1-p)B/2$. Thus

$$B_2 = d_1 xy + \frac{d_2}{2} y^2 + c_3 y + \psi(x) \quad (73)$$

for some function $\psi(x)$. From (39), (66) and (73),

$$\psi''(x) = (3+p)B/2 =: c_7$$

and therefore

$$\psi(x) = \frac{c_7}{2} x^2 + c_8 x + c_9,$$

where c_8, c_9 are constants. Hence

$$B_2 = d_1 xy + \frac{d_2}{2} y^2 + c_3 y + \frac{c_7}{2} x^2 + c_8 x + c_9. \quad (74)$$

Substituting (65) with B_1 given in (72) and (66) with B_2 given in (74) into (37) we get that $c_8 = -c_5$.

It remains to determine the function $N(x, y)$ in (69). For this purpose we substitute τ from (69) into (41) and (42), taking into account the already found expressions for ξ, ϕ, α, B_1 and B_2 . In this way we obtain

$$\begin{aligned} N_x &= (2c_2 + 2c_7 - 2B)x^2 + (2c_2 + d_2 + 4B)y^2 + (2c_1 + 2d_1 + 2A + 4c_4)xy + (4c_3 - 2D)y + 2c_9, \\ N_y &= (-5c_1 + 2d_1)x^2 + (2A - 2c_4 - 2d_1)y^2 + (2d_2 - 6c_2 - 6B - 4c_7)xy + (2D - 4c_3)x - 2c_6. \end{aligned}$$

This system can be solved if and only if

$$D = 2c_3$$

and $A = B = 0$. Hence

$$c_1 = c_2 = c_4 = c_7 = d_1 = d_2 = 0$$

and the system is reduced to

$$\begin{aligned} N_x &= 2c_9, \\ N_y &= -2c_6, \end{aligned}$$

whose solution is $N = 2c_9 x - 2c_6 y + c_{10}$. After renaming the constants we obtain

$$\begin{cases} \xi = a_1 y + a_2 + a_5 x, \\ \phi = -a_1 x + a_3 + a_5 y, \\ \tau = 2a_3 x - 2a_2 y + a_4 + 2a_5 t, \\ \eta = \frac{2}{1-p} a_5 u. \end{cases} \quad (75)$$

Observe that the dilation Z comes from the constant a_5 , while the rest corresponds to the generators in (6).

7. The Lie point symmetries for $f(u) = ku^3$

In this section we prove the second part of item (iv) of the main theorem.

Let $f(u) = ku^3$. Then $\beta = 0$,

$$\alpha = -X\xi, \quad (76)$$

and

$$\Delta_{H^1}\alpha = 0 \quad (77)$$

from (40). Applying X to (46), Y to (47) and adding, we obtain

$$\Delta_{H^1}\alpha + 2X\phi_t - 2Y\xi_t = 0.$$

(Above, we used (2) and (50).) By (77)

$$X\phi_t - Y\xi_t = 0$$

which together with (37), differentiated with respect to t , implies

$$Y\xi_t = 0 \quad (78)$$

and

$$X\phi_t = 0. \quad (79)$$

Hence there exists a function $\varphi = \varphi(x, y)$ such that

$$Y\xi = \varphi \quad (80)$$

and, necessarily,

$$X\phi = -\varphi. \quad (81)$$

We also have, by (76),

$$X\xi = -\alpha, \quad (82)$$

$$Y\phi = -\alpha. \quad (83)$$

Then by (38), (82) and (83):

$$2X\alpha = \Delta_{H^1}\xi = X^2\xi + Y^2\xi = X(-\alpha) + Y\varphi = -X\alpha + \varphi_y,$$

that is

$$3X\alpha = \varphi_y.$$

Hence, and from (46), we obtain that there is a function $B_2(x, y)$ such that

$$\phi = -\frac{1}{6}\varphi_y t + B_2(x, y). \quad (84)$$

Analogously

$$\xi = -\frac{1}{6}\varphi_x t + B_1(x, y). \quad (85)$$

From (85) and (78) it follows that

$$\varphi_{xy} = 0. \quad (86)$$

From (36), (85) and (84) we have that

$$\varphi_{xx} - \varphi_{yy} = 0. \quad (87)$$

Clearly, from (86) and (87), the function φ is of the following form:

$$\varphi = \frac{k_1}{2}x^2 + \frac{k_1}{2}y^2 + k_2x + k_3y + k_4, \quad (88)$$

where k_1, k_2, k_3, k_4 are arbitrary constants. In this way

$$\xi = -\frac{1}{6}(k_1x + k_2)t + B_1(x, y), \quad (89)$$

$$\phi = -\frac{1}{6}(k_1y + k_3)t + B_2(x, y). \quad (90)$$

Substituting ξ from (89) and φ from (88) into (80), and integrating with respect to y , we obtain

$$B_1 = \frac{1}{6}k_1x^2y + \frac{1}{6}k_1y^3 + \frac{2}{3}k_2xy + \frac{k_3}{2}y^2 + k_4y + h_1(x), \quad (91)$$

where h_1 is a function of x only. Analogously, from (81) we find

$$B_2 = -\frac{1}{6}k_1x^3 - \frac{1}{6}k_1xy^2 - \frac{k_2}{2}x^2 - \frac{2}{3}k_3xy - k_4x + h_2(y), \quad (92)$$

where h_2 is a function of y only. After substitution of ξ and ϕ from (89) and (90) with B_1 and B_2 given by (91) and (92), into (36), we obtain

$$h_1'(x) + \frac{1}{3}k_3x = h_2'(y) - \frac{1}{3}k_3y.$$

Obviously, the last two equations can be easily integrated. In this way we find the functions h_1, h_2 , and hence the functions B_1 and B_2 . Summarizing, we have found

$$\begin{aligned} \xi &= -\frac{1}{6}(k_1x + k_2)t + \frac{1}{6}k_1x^2y + \frac{1}{6}k_1y^3 + \frac{2}{3}k_2xy - \frac{1}{6}k_3x^2 + \frac{k_2}{2}y^2 + k_4y + k_5x + k_6, \\ \phi &= -\frac{1}{6}(k_1y + k_3)t - \frac{1}{6}k_1x^3 - \frac{1}{6}k_1xy^2 - \frac{k_2}{2}x^2 - \frac{2}{3}k_3xy + \frac{1}{6}k_3y^2 - k_4x + k_5y + k_7, \\ \alpha &= \frac{1}{6}k_1t - \frac{1}{3}k_2y + \frac{1}{3}k_3x - k_5. \end{aligned} \quad (93)$$

It remains to find τ . In order to do this, we substitute (93) into (48) and obtain

$$\tau_t = -\frac{1}{3}k_1t + \frac{1}{3}k_2y - \frac{1}{3}k_3x + 2k_5.$$

Hence

$$\tau = -\frac{1}{6}k_1t^2 + \left(\frac{1}{3}k_2y - \frac{1}{3}k_3x + 2k_5\right)t + N(x, y) \quad (94)$$

and the problem is reduced to the problem of finding the function N in (94). Substituting (93) and (94) into Eqs. (41) and (42), after some work, we finally obtain

$$\begin{aligned} N_x &= \frac{2}{3}k_1x^3 + \frac{2}{3}k_1xy^2 + k_2x^2 + \frac{2}{3}k_3xy + \frac{1}{3}k_2y^2 + 2k_7, \\ N_y &= \frac{2}{3}k_1y^3 + \frac{2}{3}k_1x^2y + k_3y^2 + \frac{2}{3}k_2xy + \frac{1}{3}k_3x^2 - 2k_6. \end{aligned}$$

The latter system can be easily solved. After renaming the constants, we have

$$\begin{cases} \xi = a_1(xt - x^2y - y^3) + a_2(t - 4xy) + a_3(x^2 - 3y^2) + a_4x + a_5y + a_6, \\ \phi = a_1(yt + x^3 + xy^2) + a_2(3x^2 - y^2) + a_3(t + 4xy) + a_4y - a_5x + a_7, \\ \tau = a_1[t^2 - (x^2 + y^2)^2] + a_2(-2yt - 2x^3 - 2xy^2) + a_3(2xt - 2x^2y - 2y^3) \\ \quad + 2a_4t + 2a_7x - 2a_6y + a_8, \\ \eta = -a_1tu + 2a_2yu - 2a_3xu - a_4u. \end{cases} \quad (95)$$

This completes the proof of item (iv) of the main theorem.

We observe that the dilation Z is included in (95). Indeed, it corresponds to the constant a_4 .

8. The Lie point symmetries for $f(u) = 0$

The proof of item (i) of the main theorem is presented in this section. In order not to increase the volume of this paper, some of the calculations will be sketched, leaving the details to the interested reader.

From (40) with $f(u) = 0$ we obtain

$$\Delta_{H^1}\beta = 0$$

and

$$\Delta_{H^1}\alpha = 0.$$

From the latter equation we conclude, as in the beginning of Section 7, that there exists a function $\varphi = \varphi(x, y)$ such that

$$Y\xi = \varphi, \quad (96)$$

and thus

$$X\phi = -\varphi \quad (97)$$

by (37). On the other hand, from (49) it follows that there is a function $\psi = \psi(x, y)$ such that

$$\alpha = -X\xi + \psi, \quad (98)$$

$$X\xi = -\alpha + \psi. \quad (99)$$

Following the arguments in obtaining (84) and (85) in the preceding section, we conclude that there exist functions $A = A(x, y)$ and $B = B(x, y)$ such that

$$\xi = (-\varphi_x + \psi_y)t/6 + A(x, y), \quad (100)$$

$$\phi = -(\varphi_y + \psi_x)t/6 + B(x, y). \quad (101)$$

Substituting ξ and ϕ from (100) and (101) into (96), (97) and (36), we obtain, respectively, that

$$\psi_{yy} = \varphi_{xy}, \quad (102)$$

$$\psi_{xx} = -\varphi_{xy}, \quad (103)$$

$$\varphi_{yy} - \varphi_{xx} + 2\psi_{xy} = 0. \quad (104)$$

Integrating (102) and (103) we have

$$\psi_y = \varphi_x + h_1(x), \quad (105)$$

$$\psi_x = -\varphi_y + h_2(y) \quad (106)$$

for some functions $h_1 = h_1(x)$ and $h_2 = h_2(y)$. Then, differentiating (105) with respect to x and (106) with respect to y , adding and using (104), we get

$$h'_1(x) + h'_2(y) = 0.$$

Hence $h_1(x) = k_1x + k_2$ and $h_2(y) = -k_1y + k_3$ for some constants k_1, k_2, k_3 . After renaming the constants and using (105) and (106), we obtain

$$\xi = (a_1x + a_2)t + A(x, y), \quad (107)$$

$$\phi = (a_1x + a_3)t + B(x, y). \quad (108)$$

Further, from (49),

$$\alpha_t = -a_1.$$

Hence

$$\alpha = -a_1t + g(x, y), \quad (109)$$

where the function $g = g(x, y)$ does not depend on t . Substituting (107)–(109) into (46) and (47) we find

$$\alpha = -a_1t - 2a_3x + 2a_2y + a_9, \quad (110)$$

where a_9 is an arbitrary constant and a_1, a_2, a_3 are the same as those appearing in (107) and (108). Now, from (48), (107) and (108), we deduce, after integration with respect to t , that there is a function $N = N(x, y)$ such that

$$\tau = a_1t^2 + (2A_x + 4a_1xy + 6a_2y - 2a_3x)t + N(x, y). \quad (111)$$

We substitute ξ from (107), ϕ from (108) into the determining equation (41). In this way we obtain an identity which is linear in t . Equating to zero the corresponding coefficient of t , we obtain

$$A_{xx} = -2a_1y + 2a_3. \quad (112)$$

In an analogous way, using (42),

$$A_{xy} = -2a_1x - 4a_2. \quad (113)$$

We also have that (from Eqs. (38), (107) and (108))

$$A_{xx} + A_{yy} = -4a_3 - 8a_1y. \quad (114)$$

Then from (112)–(114) we find

$$A = -a_1y^3 - a_1x^2y + a_3x^2 - 3a_3y^2 - 4a_2xy + a_4x + a_5y + a_6, \quad (115)$$

where a_5, a_6 are constants.

By substituting ξ from (107) with A given in (115), and ϕ from (108) into Eqs. (36) and (37), and, then, integrating the resulting system for B , we find

$$B = a_1xy^2 + a_1x^3 + 4a_1xy + 3a_2x^2 - a_5x - a_2y^2 + a_4y + a_7, \quad (116)$$

where a_7 is a constant.

We have found ξ , ϕ and η . To find τ , it remains to determine the function $N(x, y)$ in (111). From (41), (42), (107), (108), (115) and (116), we obtain the system

$$\begin{aligned} N_x &= -4a_1x^3 - 4a_1xy^2 - 6a_2x^2 - 4a_3xy - 2a_2y^2 + 2a_7, \\ N_y &= -4a_1x^2y - 4a_1y^3 - 2a_3x^2 - 4a^2xy - 6a_3y^2 - 2a_6, \end{aligned}$$

which can be easily solved. Our calculations can be summarized as

$$\begin{cases} \xi = a_1(xt - x^2y - y^3) + a_2(t - 4xy) + a_3(x^2 - 3y^2) + a_4x + a_5y + a_6, \\ \phi = a_1(yt + x^3 + xy^2) + a_2(3x^2 - y^2) + a_3(t + 4xy) + a_4y - a_5x + a_7, \\ \tau = a_1[t^2 - (x^2 + y^2)^2] + a_2(-2yt - 2x^3 - 2xy^2) + a_3(2xt - 2x^2y - 2y^3) \\ \quad + 2a_4t + 2a_7x - 2a_6y + a_8, \\ \eta = -a_1tu + 2a_2yu - 2a_3xu + a_9u + \beta(x, y, t) \end{cases} \quad (117)$$

where

$$\Delta_{H^1}\beta = 0, \quad (118)$$

$a_1, \dots, a_9$ are arbitrary constants.

9. The Lie point symmetries for $f(u) = ku$

In this section we complete the proof of the main theorem.

Let $f(u) = ku$, $k \neq 0$. Then by (40)

$$\Delta_{H^1}\beta + k\beta = 0$$

and

$$\Delta_{H^1}\alpha = -2kX\xi. \quad (119)$$

Applying X to (46) and Y to (47), and adding, we obtain

$$\Delta_{H^1}\alpha = 2Y\xi_t - 2X\phi_t = 4Y\xi_t, \quad (120)$$

where we used (37) and (50). Then from (119) and (120),

$$Y\xi_t = -kX\xi/2. \quad (121)$$

Hence and from (37),

$$X\phi_t = kX\xi/2. \quad (122)$$

Differentiating (38) we have

$$2X\alpha_t = \Delta_{H^1}\xi_t = X(X\xi_t) + Y(Y\xi_t) = X(-\alpha_t) - kY(X\xi_t)/2 \quad (123)$$

by (49) and (121). On the other hand, by (99)

$$X\xi = -\alpha + \psi, \quad (124)$$

where $\psi = \psi(x, y)$. From (123) and (124) we have

$$2X\alpha_t = -X\alpha_t + kY\alpha/2 - k\psi_y/2$$

and hence

$$3X\alpha_t = -k\psi_y/2 + kY\alpha/2. \quad (125)$$

Similarly

$$3Y\alpha_t = k\psi_x/2 - kX\alpha/2. \quad (126)$$

We apply X to (125), Y to (126), and add:

$$3(X^2 + Y^2)\alpha_t = k(XY - YX)\alpha/2 = k[X, Y]\alpha/2 = -2k\alpha_t,$$

that is,

$$3\Delta_{H^1}\alpha_t = -2k\alpha_t. \quad (127)$$

Further, we differentiate (119) with respect to t and use (49) to obtain

$$\Delta_{H^1}\alpha_t = 2k\alpha_t. \quad (128)$$

Since $k \neq 0$, from (127) and (128) it follows that $\alpha_t = 0$ and hence $\alpha = \alpha(x, y)$. Thus, from (46) and (47), there exist functions $A = A(x, y)$ and $B = B(x, y)$ such that

$$\xi = \alpha_y t/2 + A(x, y), \quad (129)$$

$$\phi = -\alpha_x t/2 + B(x, y). \quad (130)$$

Since α does not depend on t , by (129) and (49) we have

$$\alpha_{xy} = 0. \quad (131)$$

From (37), (129) and (130)

$$\alpha_{yy} - \alpha_{xx} = 0. \quad (132)$$

Eqs. (131) and (132) can be easily solved. The solution is

$$\alpha = k_1 x^2 + k_1 y^2 + 2k_3 x + 2k_2 y + k_4, \quad (133)$$

where k_1, k_2, k_3, k_4 are arbitrary constants. Hence

$$\xi = (k_1 y + k_2)t + A(x, y), \quad (134)$$

$$\phi = -(k_1 x + k_3)t + B(x, y). \quad (135)$$

On the other hand, from (48), in which (134) and (135) are substituted, after an integration with respect to t , we obtain

$$\tau = [2y(k_1 y + k_2) + 2x(k_1 x + k_3) + 2(A_x + 2k_1 y^2 + k_2 y)]t + N(x, y), \quad (136)$$

where the function $N = N(x, y)$ is to be determined. Further we substitute (134)–(136) in (41) and (42). In this way we obtain two identities, linear in t . Equating the corresponding coefficients of t implies

$$A_{xx} = -2k_1 x - 2k_3, \quad (137)$$

$$A_{xy} = -3k_2. \quad (138)$$

We observe now that Eq. (119) reads

$$4k_1 = -2k A_x - 4k(k_1 y^2 + k_2 y). \quad (139)$$

Differentiating (139) with respect to x we obtain $0 = -2k A_{xx}$ and hence

$$A_{xx} = 0 \quad (140)$$

since $k \neq 0$. From (137) and (140) it follows that $k_1 = k_3 = 0$. Then from (139), since $k \neq 0$, we have $A_x = -2k_2 y$ which, together with (138), implies that $k_2 = 0$ and hence $A_x = 0$. That is, $A = A(y)$. Summarizing, we have obtained that

$$\xi = A(y), \quad \phi = B(x, y), \quad \tau = N(x, y), \quad \alpha = k_4.$$

Now the arguments in Section 4 imply that the infinitesimals are given by

$$\begin{cases} \xi = a_1 y + a_2, \\ \phi = -a_1 x + a_3, \\ \tau = 2a_3 x - 2a_2 y + a_4, \\ \eta = a_5 u + \beta(x, y, t), \end{cases} \quad (141)$$

with

$$\Delta_{H^1} \beta + k\beta = 0. \quad (142)$$

This completes the proof of the theorem.

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