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Numerical Modeling of Degenerate Equations in Porous Media Flow

Degenerate Multiphase Flow Equations in Porous Media

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Abstract In this paper is introduced a new numerical formulation for solving degenerate nonlinear coupled convection dominated parabolic systems in problems of flow and transport in porous media by means of a mixed finite element and an operator splitting technique, which, in turn, is capable of simulating the flow of a distinct number of fluid phases in different porous media regions. This situation naturally occurs in practical applications, such as those in petroleum reservoir engineering and groundwater transport. To illustrate the modelling problem at hand, we consider a nonlinear three-phase porous media flow model in one- and two-space dimensions, which may lead to the existence of a simultaneous one-, two- and three-phase flow regions and therefore to a degenerate convection dominated parabolic system. Our numerical formulation can also be extended for the case of three space dimensions. As a consequence of the standard mixed finite element approach for this flow problem the resulting linear algebraic system is singular. By using an operator splitting combined with mixed finite element, and a decomposition of the domain into different flow regions, compatibility conditions are obtained to bypass the degeneracy in order to the degenerate convection dominated parabolic system of equations be numerically tractable without any mathematical trick to remove the singularity, i.e., no use of a parabolic regularization. Thus, by using this procedure, we were able to write the full nonlinear system in an appropriate way in order to obtain a nonsingular system for its numerical solution. The

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689

robustness of the proposed method is verified through a large set of high-resolution numerical experiments of nonlinear transport flow problems with degenerating diffusion conditions and by means of a numerical convergence study.

Keywords Degenerate convection-diffusion · Operator splitting · Mixed finite elements · Finite volume central scheme · Porous media · Three-phase flow

1 Introduction

In this paper an operator splitting approach combined with mixed finite element methods are used to model and simulate numerically the flow of three immiscible and incompressible phases in porous media.

The nonlinear three-phase system is expressed in terms of a volumetric formulation for water, gas and oil saturations and fluid pressures for the immiscible phases. The pressure equation is a standard elliptic problem. The component mass balance equations are convection-dominated in the presence of capillary pressure diffusive effects in reservoir transport problems; they are purely hyperbolic in the absence of such diffusion. Instead of solving the governing system of differential equations in the form which results directly from the basic conservation laws (supplemented by constitutive relations), the three-phase system of equations is rewritten in such a way as to exhibit clearly its mathematical nature [14, 15, 21]. Thus, the governing set of partial differential equations are written as a nondegenerate pressure equation for Darcy's law that is strongly coupled with two saturation equations describing the mass conservation of fluid phases water, oil and gas [14, 15, 21]. Notice that our choice is due to the generality of the so-called phase formulation [15], in which we do not care about the form of the relative permeabilities [18, 26, 48, 54] and so it also allows the use of general capillary pressure models employed on practical applications [11, 46, 47]. The correct modelling of capillarity effects plays an important role in three-phase models [7, 41, 48]. Indeed, such phase-formulation is suitable for application of an operator splitting technique for the three-phase flow problem at hand [4–6]. This means that one can use appropriate numerical methods for distinct differential equations resulting from the procedure that leads to a subsystem of hyperbolic conservation laws, a subsystem of parabolic type, and an elliptic subsystem associated with the pressure-velocity problem. Moreover, this formulation might translate into a drastically reduced computational effort to produce numerical results within a given accuracy requirement [5, 6]. See also [42] and [16, 33, 43, 45] for recent developments in the analysis of time-splitting errors for onedimensional nonlinear convection-diffusion problems.

Distinct operator splitting techniques for transport problems in multiscale heterogeneous porous media flow are established upon an operator splitting formulation, where convectivediffusive forces and pressure velocity mechanisms are accounted for in separate substeps. Among Eulerian-Lagrangian procedures, we mention the MMOC [24], the MMOCAA [23], the LCELM [25] and the ELLAM [20]; see also [9, 17, 51]. For these methods, it is assumed a reasonable fluid injection rate and a capillary dissipation strength. Then, the flow is essentially along the characteristic curves of a modified transport differential operator that take into account both convective and diffusive mechanisms in the associated characteristic direction leading to long time steps. Although such methods have long been applied successfully for several scalar transport equations, the extension for systems is not well understood. More recently, the authors began to use the central schemes [49] (finite-volume based formulation) for two-phase and three-phase problems in porous media flows by means of an operator splitting formulation (see, e.g., [5, 6]). An important feature of the latter approach compared to the other methods cited above is the advantage of a simple straightforward generalization for systems and for multidimensional heterogeneous flow problems [5, 6, 10]. Many others successfully numerical methods based upon an operator splitting formulation have also been discussed recently in the literature [16, 42, 43, 45]. In those papers different techniques are used to handle the nonlinear parabolic equations.

The splitting approach proposed in [16] is based on the solution of a parabolic problem by means of a discretization of the exact kernel solution of a linear "heat equation", with constant coefficients. Here we have a coupled set of strongly nonlinear parabolic system of equations with a degenerate character and such that the parabolic operator related to the parabolic Green function (integral kernel of the heat equation) is not well defined as in the linear case; i.e., instead of having an equation in non-conservative form like $u_t = Du_{xx}$, u = u(x, t) with D being a diagonal matrix with constant entries, here we have a coupled nonlinear parabolic system of conservative form $u_t = (D(x, u)u_x)_x$, where D(x, u) is a 2×2 matrix with explicitly constrained entries upon x and u. It is well established in the literature that mixed finite elements (MFEM) are appropriate for the numerical approximation of parabolic and elliptic equations in conservative form (see, e.g., [23, 24, 28, 31]). In particular, MFEM are appropriate for accurate velocity field computation in the presence of highly variable rock permeability typical in petroleum geology (see e.g., [5, 6, 23, 24, 28]). Although MFEM are suitable for problems of this class, its standard variational formulation is not suited to treat parabolic-type degenerate problems. The issue of this fundamental problem has not been systematically addressed before, although distinct attempts have been made (see, e.g., [12, 13, 51]). In this work we propose a way to circumvent this difficulty. By means of an operator splitting and a decomposition of the domain into different flow regions, compatibility conditions were obtained to bypass the degeneracy. Thus, we were able to write the nonlinear system in an appropriate way in order to obtain a nonsingular system for its numerical solution. The resulting system is suitable for mixed finite element formulation as well as for multiscale-based finite element methods in complex flow domains, on non-uniform and unstructured grids [1, 3, 30, 39, 44].

The work in [42, 43] is based in a front tracking method for systems of conservation laws that relies heavily on a Riemann solver, which, in turn, is not entirely straightforward for multidimensional problems without exact or approximate Riemann solutions (see, e.g., [5, 7, 8, 19, 35, 48]). In [45] the authors showed that a semi-discrete central scheme may fail to converge to the unique entropy solution of a nonconvex conservation law. Thus, our hyperbolic solver for a pertinent system of hyperbolic conservation laws modelling the convective transport of the fluid phases is based on the former central scheme [5, 49]. Indeed, our work is in the same lines of papers [16, 17, 20, 23–25, 42, 51] and [1–3, 37, 38, 44] that are based on a operator splitting formulation, although with the use of a distinct finite element formulation approach with the ability of correctly resolving the nonlinear balance between the convective and diffusive terms, which is quite delicate for the three-phase flow problem, see, e.g., [5, 7, 8, 48]. We shall treat the case in which $\{\Omega_i\}$ is a partition of Ω (fluid flow domain region under consideration) into individual elements in a Cartesian grid, though an inspection of the procedure would indicate that larger subdomains are permissible (simplices, rectangles and prisms). We note that following ideas reported in papers [40, 53, 55], our formulation can be applied not only on structured Cartesian meshes but also on more quite general unstructured meshes; for more details, see papers [1-3, 37, 38,44].

It is worth pointing out that for the analysis of the complexities of multiphase flow displacements and transport processes in heterogeneous porous media (e.g., [5, 6, 29, 30,

34–36]) it is important to consider the study of three-phase flow generalizing the Buckley– Leverett's solution for immiscible two-phase flow [10, 32]. Moreover, three-phase flow models in porous media are important in a wide number of scientific and technological fields: water-alternating-gas injection and thermal flow techniques in oil recovery as well as injection of carbon dioxide directly into underground geological formations (e.g., declining oil fields, deep saline aquifers, and unminable deep coal seams) as an alternative tool for global warming contention; see [14, 15, 22, 34, 52]. For the sake of simplicity, in this paper we describe the fundamental ideas in one-space dimension. Examples are used to demonstrate the application of the model and sensitivity of the results to fluid properties in potential practical flow situations. The extension of the method and its application for multidimensional flow problems is straightforward.

The remaining of the paper is organized as follows. In Sect. 2, we discuss the mathematical model for the three-phase problem at hand and its computational modelling is discussed in Sect. 3. We briefly describe the operator splitting procedure in Sect. 4. The numerical solution of the convective transport system is indicated in Sect. 5. In Sect. 6 we present in detail our approach for addressing the degenerate diffusion problem. Numerical examples to demonstrate the potential of the proposed method are given in Sect. 7. Finally, our concluding remarks are presented in Sect. 8.

2 Governing Differential Equations

We consider the flow of three immiscible, incompressible fluid phases in a porous medium. Let us call the phases water, gas and oil, and indicate them by the subscripts w, g and o, respectively. We assume that pressure variations are small so that they do not affect the gas volume. We assume that there are no internal sources or sinks. We have neglected mass transfer, thermal and gravitational effects, but we consider capillary pressure (diffusive) effects. Next, we assume also that the three fluid phases saturate the pores,

$$S_w + S_g + S_o = 1.$$
 (1)

For brevity we will denote $\partial_{\ell}(\cdot) = \partial(\cdot)/\partial \ell$, where ℓ is the variable under consideration, and " ∇ ·" and " ∇ " stands for the classical differential operators. The equations expressing conservation of mass of water, gas and oil are

$$\partial_t (\phi S_i \rho_i) + \nabla \cdot (\rho_i \mathbf{v}_i) = 0, \quad i = w, g, o, \ \mathbf{x} \in \Omega, \ t \ge 0,$$
(2)

where $\phi(\mathbf{x})$ denotes the porosity of the medium. For phase *i*, *S_i* denotes its saturation, ρ_i its density, and \mathbf{v}_i its volumetric rate of flow, which is given by the multiphase extension of Darcy's law [14]:

$$\mathbf{v}_i = -K(\mathbf{x})\lambda_i \nabla p_i, \quad i = w, g, o, \ \mathbf{x} \in \Omega, \ t \ge 0,$$
(3)

where $K(\mathbf{x})$ denotes the absolute permeability tensor of the porous medium; λ_i and p_i are the mobility and pressure of phase *i*, respectively. The mobility of phase *i* is usually expressed as $\lambda_i = k_i/\mu_i \ge 0$, which is the ratio between the relative permeability k_i and the viscosity μ_i of the respective phase. Each relative permeability k_i depends on the saturation vector. Experimentally, k_i increases when S_i increases, and the relative permeabilities never vanish simultaneously. Furthermore, $k_i > 0$ for $S_i > S_{ir}$, and $k_i = 0$ for $S_i \le S_{ir}$, where S_{ir} is the residual saturation of phase *i*. Since thermal effects and compressibility are neglected, μ_i and ρ_i are constant, and (2) can be rewritten as:

$$\partial_t(\phi S_i) + \nabla \cdot \mathbf{v}_i = 0, \quad i = w, g, o, \text{ in } \Omega, t \ge 0.$$
 (4)

We denote the capillary pressure by $p_{ij} = p_i - p_j$, $i \neq j$, i, j = w, g, o. An independent pair of p_{ij} must be measured experimentally as functions of saturations. We also assume that p_{ij} and its derivative is bounded, which is a physically reasonable assumption [46]. Of course, boundary and initial conditions must be specified to complete the mathematical definition of the three-phase flow system. We will discuss further this issue in Sect. 4.

3 Computational Modeling of the Three-Phase Flow

We choose to describe our model using the *phase formulation* for three-phase flow, since it is general with regard to the choices of capillary pressures and relative permeability functions [14, 15, 21]. For completeness, we review these equations briefly. In the fractional flow formulation the governing system given by basic and constitutive equations (1)–(4) are written in terms of two saturation equations (expressing conservation of water, gas and oil) [5, 15]:

$$\partial_t(\phi S_w) + \nabla \cdot \left[\mathbf{v} f_w(S_w, S_g) \right] = \nabla \cdot \mathbf{w}_w, \quad \text{in } \Omega, \ t \ge 0, \\ \partial_t(\phi S_g) + \nabla \cdot \left[\mathbf{v} f_g(S_w, S_g) \right] = \nabla \cdot \mathbf{w}_g, \quad \text{in } \Omega, \ t \ge 0,$$
(5)

where $[\mathbf{w}_w, \mathbf{w}_g]^{\mathrm{T}} = K(\mathbf{x})B(S_w, S_g)[\nabla S_w, \nabla S_g]^{\mathrm{T}}$ with,

$$B(S_w, S_g) = \varepsilon Q(S_w, S_g) P'(S_w, S_g), \tag{6}$$

where Q and P' are given by,

$$Q = \begin{bmatrix} \lambda_w (1 - f_w) & -\lambda_w f_g \\ -\lambda_g f_w & \lambda_g (1 - f_g) \end{bmatrix}, \qquad P' = \begin{bmatrix} \partial_{S_w} p_{wo} & \partial_{S_g} p_{wo} \\ \partial_{S_w} p_{go} & \partial_{S_g} p_{go} \end{bmatrix}.$$
(7)

The parameter ε is an adimensional parameter that controls the amount of diffusion with respect to convection present in the problem; $\varepsilon = 1/Pe$, where *Pe* is the Péclet number. The nonlinear system (5)–(7) is coupled with,

$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} = -K(x)\lambda(S_w, S_g)\nabla p_o + \mathbf{v}_{wo} + \mathbf{v}_{go}, \tag{8}$$

the pressure-velocity (elliptic) system, where $\mathbf{v} = \sum_i \mathbf{v}_i$ is the total velocity (expressing the sum of Darcy's velocities), and $\mathbf{v}_{io} = -K(x)\lambda_i(S_w, S_g)\nabla p_{io}$, for i = w, g, and $\lambda = \sum_i \lambda_i$, i = w, g, o, is the total mobility and $f_i = \lambda_i/\lambda$ are the fractional flow functions. For the sake of simplicity, and without loss of generality, we develop the basic ideas to bypass the degeneracy in one space dimension since the decomposition of the flow domain (saturation space) into different flow regions to derive compatibility conditions in higher dimensions is quite straightforward. Notice that in one space dimension the pressure equation implies that total fluid velocity is independent of position, thus we take it to be constant in space and time. After nondimensionalizing time and space variables, fluid viscosities, and capillary pressures in a standard way, it reads:

$$\partial_t S_w + \partial_x f_w(S_w, S_g) = \partial_x w_w, \quad \text{in } \Omega, \ t \ge 0,$$

$$\partial_t S_g + \partial_x f_g(S_w, S_g) = \partial_x w_g, \quad \text{in } \Omega, \ t \ge 0,$$

(9)

$$w_w = B_{11}\partial_x S_w + B_{12}\partial_x S_g, \qquad w_g = B_{21}\partial_x S_w + B_{22}\partial_x S_g. \tag{10}$$

The diffusive term is represented by the right-hand side of the system (9)–(10), and it incorporates capillary pressure effects (6)–(7). In the diffusive fluxes w_w and w_g , the quantities B_{ij} , i, j = 1, 2 are the entries of the matrix $B(S_w, S_g)$ defined in (6). Boundary and initial conditions for these equations will be introduced in what follows.

4 The Operator Splitting Procedure

In our splitting scheme we take into account convection and diffusion effects separately and sequentially. Moreover, the operator splitting for the system of equations given by (9)– (10) allows the use of time steps for the diffusive computation to be longer than the steps used for the convection part of the saturation calculation; computational efficiency can be achieved for the three-phase flow system at hand by solving fewer diffusion steps (implicitly) than convection steps (explicitly), given an accuracy requirement with respect to the width numerical shock layer $O(\sqrt{\varepsilon \Delta t_c})$ because of the nonlinear self-sharpening mechanisms of system (11)–(16), see, e.g., [5, 6, 42, 43].

We introduce two time steps: Δt_c for the solution of the convection and Δt_d for the diffusion. We take $\Delta t_d = i_c \Delta t_c$, where i_c is a positive integer representing the number of evolution time steps of the convective problem before computing the diffusive effects. This means that $\Delta t_d \geq \Delta t_c$, so that, $t_n = n \Delta t_d$ and $t_{n,k} = t_n + k \Delta t_c$, $0 \leq k < i_c$, where the convection microsteps are determined dynamically by a necessary Courant–Friedrichs–Levy condition for stability. To simplify the description of the operator splitting, we assume that there is a single value for each of the time steps Δt_d and Δt_c . The integer n_s defines the total simulation time $T = n_s \Delta t_d$. Then, the algorithm is as follows:

```
Set S_w^0 and S_g^0 (initial saturations)
for n=0:(n_s - 1)
for k=0:(i_c - 1)
A) Solve, in [t_{n,k}, t_{n,k+1}], the convective (hyperbolic)
system:
```

$$\begin{aligned} \partial_t S_w &+ \partial_x f_w(S_w, S_g) = 0, & \text{in } \Omega \times [t_{n,k}, t_{n,k+1}], \\ \partial_t S_g &+ \partial_x f_g(S_w, S_g) = 0, & \text{in } \Omega \times [t_{n,k}, t_{n,k+1}], \end{aligned}$$
(11)

subject to initial conditions,

$$S_w(x, t_{n,k}) = S_w^0(x)$$
 and $S_g(x, t_{n,k}) = S_g^0(x)$, (12)

and to injection boundary conditions,

$$S_w(0,t) = S_{w,in}$$
 and $S_g(0,t) = S_{g,in}, t > 0.$ (13)

B) Set $S^0_w(x) = S_w(x,t_{n,k+1})$ and $S^0_g(x) = S_g(x,t_{n,k+1})$. end

C) Classify the mesh elements taking into account the saturations S_w^0 and S_g^0 from B) by means of the mobility (20), so that each element of the mesh belong to one of the regions (21a)-(21f).

D) Solve, in $[t_n, t_{n+1}]$, the diffusive system over the sets (21a) - (21f):

$$\partial_t S_w = \partial_x w_w, \quad w_w = B_{11} \partial_x S_w + B_{12} \partial_x S_g, \text{ in } \Omega \times [t_n, t_{n+1}], \\ \partial_t S_g = \partial_x w_g, \quad w_g = B_{21} \partial_x S_w + B_{22} \partial_x S_g, \text{ in } \Omega \times [t_n, t_{n+1}],$$

$$(14)$$

with boundary conditions,

$$w_w = w_g = 0, \quad \text{on } \partial \Omega, \tag{15}$$

and initial conditions,

$$S_w(x, t_n) = S_w^0(x)$$
 and $S_g(x, t_n) = S_g^0(x)$. (16)

E) Set
$$S_w^0(x) = S_w(x, t_{n+1})$$
 and $S_g^0(x) = S_g(x, t_{n+1})$.
end

Concerning step C) of the above algorithm, we point out that S_w^0 and S_g^0 are constant in each element.

We are setting up to calculate the above nonlinear initial boundary value problem (11)–(16), but more general domains and other boundary conditions (e.g., $w_w \neq 0$ and $w_g \neq 0$) can be treated by our techniques. Notice also that the nonlinear coefficients in the parabolic equations (14) are "frozen" with the latterly solution of the hyperbolic problem (11)–(13). Indeed, for the numerical solution of nonlinear system (14) we have performed some local iteration like in a fixed-point iteration (e.g., Piccard iteration) with no qualitative difference in the results. Remember, here we have a coupled nonlinear parabolic subproblem with nonconstant coefficients (6)–(7). Now, we proceed by describing the numerical methods used to solve systems (11)–(13) (convection) and (14)–(16) (diffusion).

5 Numerical Solution of the Convective Problem

The nonlinear convection (11)–(13) is solved using the explicit high-resolution central scheme developed by Nessyahu and Tadmor (NT) [49], which has been used by the current authors for numerical simulation of transport flow problems in porous media [4–6]. The NT scheme can be viewed as an extension of the Lax-Friedrichs scheme into a second order one. The main ingredients of the numerical method are: a nonoscillatory piecewise linear reconstruction of the solution using their averages and a central differencing based on the staggered evolution of reconstructed averages. The piecewise linear reconstruction reduces the excessive numerical diffusion of the Lax-Friedrichs scheme and, like upwind schemes, uses nonlinear limiters to guarantee the overall nonoscillatory nature of the approximate solution. In this scheme characteristic decompositions are unnecessary and, therefore, Riemann problems are avoided. The work in [42, 43] is based in a front tracking method for a triangular model for the fractional flow functions for three-phase flow equations, which in turn relies heavily on a Riemann solver, which in turn is not entirely straightforward for multidimensional problems [5, 7, 8, 19, 35, 48]. In [16, 45] the authors have been shown that the use of semidiscrete schemes may fail to converge to the unique entropy solution of nonconvex conservation laws. We note that following ideas reported in papers [40, 53, 55], our formulation can be applied not only on structured Cartesian meshes but also on more quite general unstructured meshes. Thus, our hyperbolic solver for solving the convection system is based on the former central scheme [5, 49] rather than a semidiscrete central scheme method [16, 45] nor based in a front tracking approach [42, 43].

Fig. 1 Flow regions in the state space with residual saturations: mobile oil (resp. water & gas) phase region Ω_o (resp. Ω_w & Ω_g) with residual or immobile gas & water (resp. gas & oil and water & oil) phases. Mobile gas & water region (resp. gas & oil and water & oil) is denoted by Ω_{wg} (resp. Ω_{go} and Ω_{wo}). Mobile three-phase flow region is denoted by $\tilde{\Omega}$



The NT scheme is locally conservative and it takes into account naturally the local balance of water, gas, and oil phases for simultaneous and distinct *one-phase* regions, *twophase* regions, and *three-phase* regions. This property is quite important since the hyperbolic differential equation is used to track the degenerate flow regions with respect to the parabolic problem. These flow regions are, respectively, represented by Ω_i , $i = w, o, g, \Omega_{ij}$, $i \neq j$, i, j = w, o, g, and $\tilde{\Omega}$. We illustrate in Fig. 1 the mapping of "physical" flow regions in the state space (saturation triangle).

Since our main concern is the numerical solution of the degenerate diffusive-parabolic system (14), we will not present details on the NT central scheme [49]; its formulation for the three-phase system can be found in [4-6].

6 Numerical Solution of the Degenerate Diffusive Problem

Now we wish to highlight the main ideas involved in solving the degenerate parabolic system (14)–(16) that arises in the mixed finite element formulation, after the application of the operator splitting approach for (9)–(10). In order to complete the time evolution for each time interval $t_n < t \le t_{n+1}$ for the full coupled convection-diffusion problem, we should incorporate the diffusive effects (9)–(10) subjected to the solution of the convective problem (11)–(13). Thus we need to solve the parabolic system (14) subject to the boundary condition (15) and to the initial conditions (16). The splitting procedure allows the use of the following approximation for (6)–(7), [5, 6],

$$B(S_w, S_g) \equiv B(S_w^0(x), S_g^0(x)).$$
(17)

We are solving the differential equations (6)–(7) by splitting diffusive and convective effects. As pointed out, the hyperbolic problem is first solved and then next we solve the parabolic system (14)–(16) in $[t_n, t_{n+1}]$; the nonlinear matrix *B* makes use of solution for convection equation (11) at t_{n+1} . The error in this approximation is expected to be smaller than the error of a zero order linearization at t_n (i.e., one Piccard iteration). Moreover, the introduction of this approximation makes the diffusive system "locally" linear for each time step Δt_d , but yet coupled and with the degenerate nature, such that coefficient matrix *B* becomes a function of space variable *x*:

$$B = B(S_w^0(x), S_g^0(x)) = \begin{bmatrix} B_{11}(x) & B_{12}(x) \\ B_{21}(x) & B_{22}(x) \end{bmatrix}.$$
 (18)

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We reiterate that the mixed finite element method has been shown to be very appropriate for solving elliptic and parabolic problems in conservative form (14)–(16), in particular in the presence of highly heterogeneous porous medium [5, 6, 27]. However, in order to use the finite element method to solve parabolic equation (14) it is essential that the coefficient matrix *B* be invertible. As we will see next, *B* is singular in regions where the mobility of a phase is zero (i.e., in the residual saturation regions). As a consequence, we need to provide compatibility conditions to solve the parabolic system by means of mixed finite elements. The matrix *B* is invertible in regions where det $Q \neq 0$ and det $P' \neq 0$, i.e.:

$$\det Q = \frac{\lambda_w \lambda_g \lambda_o}{\lambda} \neq 0, \tag{19a}$$

$$\partial_{S_w} p_{wo} \partial_{S_g} p_{go} - \partial_{S_g} p_{wo} \partial_{S_w} p_{go} \neq 0.$$
(19b)

As is usual in the modelling of the capillary pressures functions [46], we assume $p_{wo} = p_{wo}(S_w)$ and $p_{go} = p_{go}(S_g)$. Therefore the former condition (19b) in the capillary pressure becomes $\partial_{S_w} p_{wo} \partial_{S_g} p_{go} \neq 0$. Furthermore, the matrix P' becomes a diagonal one. Notice that the determinant of Q is nonvanishing only in regions of Ω where the mobility of all three phases are strictly positive. Thus, to solve the degenerate problem we decompose the domain Ω based on the mobility of the phases at time t_n from (11)–(13), as the initial condition for the parabolic problem (14)–(16). Thus, we denote:

$$\lambda_{j}^{0}(x) = \lambda_{j} \left(S_{w}^{0}(x), S_{g}^{0}(x) \right), \quad j = w, g, o,$$
(20)

and define the following non-overlapping decomposition of Ω :

$$\Omega = \widetilde{\Omega} \cup \Omega_{go} \cup \Omega_{wo} \cup \Omega_{wg} \cup \Omega_w \cup \Omega_g \cup \Omega_o, \quad \text{where,}$$
(21a)

$$\widetilde{\Omega} = \left\{ x \in \Omega; \, \lambda_j^0(x) > 0, \, j = w, g, o \right\},\tag{21b}$$

$$\Omega_{go} = \left\{ x \in \Omega; \, \lambda_w^0(x) = 0, \, \lambda_j^0(x) > 0, \, j = g, \, o \right\},\tag{21c}$$

$$\Omega_{wo} = \left\{ x \in \Omega; \lambda_g^0(x) = 0, \lambda_j^0(x) > 0, j = w, o \right\},\tag{21d}$$

$$\Omega_{wg} = \left\{ x \in \Omega; \, \lambda_o^0(x) = 0, \, \lambda_j^0(x) > 0, \, j = w, g \right\}, \quad \text{and}$$
(21e)

$$\Omega_j = \left\{ x \in \Omega; \, \lambda_k^0(x) = 0, \, k \in \{w, g, o\}, \, k \neq j \right\}, \quad j = w, g, o.$$
(21f)

We assume that each subdomain in the decomposition of Ω is an union of non-degenerate intervals or is an empty set. In short, as announced in Sect. 5, the key idea is to track the mobile and immobile saturation flow regions with the splitted-transport convective equation (11)–(13).

Lemma 1 If $x \in \Omega_{ij}$ and $i, j, k \in \{w, g, o\}$ are distinct indices, then:

$$S_k(x,t) = S_k(x,t_n) = S_k^0(x)$$
 in Ω_{ij} for $t \in [t_n, t_{n+1}]$. (22)

Furthermore, if $x \in \Omega_j$ then $S_k(x, t) = S_k(x, t_n) = S_k^0(x)$ in Ω_j for $t \in [t_n, t_{n+1}]$, for any $k \in \{w, g, o\}$.

Proof Let us consider $x \in \Omega_{go}$. In this physical region $\lambda_w(x, t_n) = 0$. Hence, from the definition of w_w we obtain that $w_w(x, t) = 0$ in $\Omega_{go} \times [t_n, t_{n+1}]$. Consequently, it follows from (14) that,

$$\partial_t S_w(x,t) = 0 \quad \text{in } \Omega_{go}. \tag{23}$$

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Therefore $S_w(x, t) = S_w(x, t_n) = S_w^0(x)$, for any $t \in [t_n, t_{n+1}]$, in Ω_{go} . The proof for Ω_{wg} , Ω_{wo} , and Ω_j (j = w, g, o) is analogous.

Remark 1 For a given phase $k \in \{w, g, o\}$, $\lambda_k^0 = 0$ is equivalent to $S_k^0 \leq S_{kr}$, where S_{kr} is the residual saturation of phase k. Now, since $S_k \geq S_{kr}$, then in the region Ω_{ij} (for $i, j \neq k$), $S_k = S_{kr}$, and therefore $\partial_x S_k = 0$.

In the region $\widetilde{\Omega}$ the matrix *B* is invertible. So, denoting the entries of matrix B^{-1} by B_{ij}^{-1} , *i*, *j* = 1, 2 we can write the system (14) as follows,

$$B_{11}^{-1}w_w + B_{12}^{-1}w_g - \partial_x S_w = 0, \qquad \partial_x w_w - \partial_t S_w = 0, \qquad \text{in } \widetilde{\Omega} \times [t_n, t_{n+1}], \\ B_{21}^{-1}w_w + B_{22}^{-1}w_g - \partial_x S_g = 0, \qquad \partial_x w_g - \partial_t S_g = 0, \qquad \text{in } \widetilde{\Omega} \times [t_n, t_{n+1}],$$
(24)

The system (24) is suitable for a mixed finite element formulation [5, 6, 23, 24, 28, 31] as well as for multiscale-based finite element methods [1–3, 30, 37–39, 44]. Since the matrix *B* is positive definite in $\tilde{\Omega}$ [7], so is matrix B^{-1} . Therefore system (24) has a unique solution in the region $\tilde{\Omega}$. Now, let us analyze the case when the matrix B^{-1} is not invertible.

6.1 Regions Ω_{go} and Ω_{wo}

The mobility of the water phase vanishes in the region Ω_{go} . Then $B_{11} = B_{12} = B_{21} = 0$, and therefore, $w_w = 0$. Thus, the system (14) is simplified to,

$$\partial_t S_g - \partial_x w_g = 0, \quad \text{in } \Omega_{go} \times [t_n, t_{n+1}], w_g = B_{22}(x) \partial_x S_g, \quad \text{in } \Omega_{go} \times [t_n, t_{n+1}],$$
(25)

subject to boundary condition $w_g(x, t) = 0$ for all $x \in \partial \Omega_{go} \cap \partial \Omega$ and to the initial condition $S_g(x, t_n) = S_g^0(x)$ in Ω_{go} ; remember (15)–(16) as introduced in Sect. 4. In order to obtain a compatibility condition, it is imposed naturally that the flux w_g should be continuous across the interface $\partial \Omega_{go} \setminus \partial \Omega$. Since $B_{22}(x) = \lambda_g f_o \partial_{S_g} p_{go}$, if $\partial_{S_g} p_{go} > 0$ then $B_{22}(x) > 0$. Therefore, equation (25) can be written in a computationally suitable form,

$$\frac{1}{B_{22}}w_g - \partial_x S_g = 0, \quad \text{in } \Omega_{go} \times [t_n, t_{n+1}], \\ \partial_x w_g - \partial_t S_g = 0, \quad \text{in } \Omega_{go} \times [t_n, t_{n+1}].$$
(26)

From Lemma 1, the water saturation reads to $S_w(x, t_{n+1}) = S_w^0(x)$ in Ω_{go} . In addition, as pointed out before, we impose continuity of w_g across the boundary $\partial \Omega_{go} \setminus \partial \Omega$. Since $w_w = 0$ in Ω_{go} and since w_w is continuous, then one should also impose $w_w = 0$ on $\partial \Omega_{go}$ in the neighboring subdomains in order to modify the diffusive system to be well posed in terms of boundary condition. On the other hand, in the region Ω_{wo} , it hold that $\partial_t S_g = 0$ and $w_g = 0$, and the system (14) reduces to,

$$\frac{1}{B_{11}}w_w - \partial_x S_w = 0, \quad \text{in } \Omega_{wo} \times [t_n, t_{n+1}], \\ \partial_x w_w - \partial_t S_w = 0, \quad \text{in } \Omega_{wo} \times [t_n, t_{n+1}],$$
(27)

subject to boundary condition $w_w = 0$ on $\partial \Omega_{wo} \cap \partial \Omega$. In this region, we need a similar compatibility condition for w_w by imposing its continuity across boundary $\partial \Omega_{wo} \setminus \partial \Omega$. Similarly to the previous case if $\partial_{S_w} p_{wo} > 0$ then $B_{11}(x) = \lambda_w f_o \partial_{S_w} p_{wo}$, is strictly positive in Ω_{wo} .

6.2 Region Ω_{wg}

In region Ω_{wg} mobility of oil phase vanishes. Thus, from Lemma 1 and Remark 1, $\partial_t S_o = 0$ and $\partial_x S_o^0 = 0$ in $\Omega_{wg} \times [t_n, t_{n+1}]$. Therefore, $S_o(x, t) = S_o^0(x)$ in $\Omega_{wg} \times [t_n, t_{n+1}]$. Since $S_o + S_w + S_g = 1$,

$$\partial_x S_g = -\partial_x S_w, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}], \quad \text{and}$$
 (28)

$$\partial_t S_g = -\partial_t S_w, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}].$$
 (29)

On the other hand, since $f_o = 0$ in this region, we have that $B_{11} = -B_{21}$ and $B_{12} = -B_{22}$. Therefore, it follows from (10) that,

$$w_w = -w_g. \tag{30}$$

The relations (28)–(30) also imply that the differential equations (14) for conservation of water and gas are equivalent. Indeed, we should choose one of the saturation equations in order to obtain a nonsingular system. Therefore, without loss of generality, we choose to use the water saturation, and the system in Ω_{wg} now reads,

$$w_w - B_{11}\partial_x S_w - B_{12}\partial_x S_g = 0, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}],$$

$$\partial_x w_w - \partial_t S_w = 0, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}].$$
(31)

By using relation (28), equation (31) is equivalent to,

$$w_w - [B_{11} - B_{12}]\partial_x S_w = 0, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}],$$

$$\partial_x w_w - \partial_t S_w = 0, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}].$$
(32)

Now, since $B_{12} = -B_{22}$ we get $B_{11} - B_{12} = tr(B)$. Thus, if tr(B) is nonvanishing in Ω_{wg} then we can write the system (32) in the following way:

$$\frac{1}{\operatorname{tr}(B)} w_w - \partial_x S_w = 0, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}], \\ \partial_x w_w - \partial_t S_w = 0, \quad \text{in } \Omega_{wg} \times [t_n, t_{n+1}].$$
(33)

Since $f_o = 0$ and $1 - f_w = f_g$, then:

$$\operatorname{tr}(B) = \lambda_w f_g[\partial_{S_w} p_{wo} + \partial_{S_g} p_{go}]. \tag{34}$$

This lead us to another condition in the capillary pressure functions,

$$\partial_{S_w} p_{wo} + \partial_{S_g} p_{go} \neq 0 \quad \text{in } \Omega_{wg}. \tag{35}$$

After solving (33), the gas saturation can be recovered from $S_w + S_g = 1 - S_o^0$. Notice that capillary pressure and relative permeability are important parameters in reservoir engineering [15, 21, 34]. Indeed, one can see that the Brooks–Corey capillary pressure model has a solid theoretical basis [11, 46]. Recently [47], a more general capillary pressure model was derived theoretically from fractal modelling of a porous medium and it was found that this model could be reduced to the frequently-used Brooks–Corey capillary pressure model in some applications. Thus, condition (35) is not so restrictive since it is quite general for porous media transport problems.

6.3 Regions Ω_i , j = w, g, o

Let us fix $j \in \{w, g, o\}$. Remember that in region Ω_j only one phase has positive mobility. Thus, it follows from Lemma 1 and from the saturation constraint (1) that:

$$\partial_t S_w = \partial_t S_g = \partial_t S_o = 0, \quad \text{in } \Omega_j \times [t_n, t_{n+1}].$$
 (36)

So, in Ω_i the saturations are given by,

$$S_i(x, t_{n+1}) = S_i(x, t_n) = S_i^0(x), \quad i = w, g, o.$$
(37)

Now, since the mobility of two phases vanishes, we obtain $w_w = w_g = 0$ in $\Omega_j \times [t_n, t_{n+1}]$. This implies a compatibility condition for fluxes w_w and w_g on the boundary interface with neighboring subdomains and now reads:

$$w_w = w_g = 0, \quad \text{on } \partial \Omega_j. \tag{38}$$

This completes the derivation of the compatibility conditions to bypass the degeneracy associated to the diffusion problem on the two-phase and three-phase flow regions.

6.4 Mixed Finite Element Formulation

As mentioned above, we discuss the key ideas in one-space dimension. Thus for the numerical approximation of the parabolic system (14)–(16), let \mathcal{T}_h be a uniform partition of domain $[a, b] \subset \mathbb{R}$ with characteristic size *h* (but non-uniform size is allowed). For this partition, we can define the ordered set of nodes and the set of elements, respectively:

$$\mathcal{N}_h = \{ a = x_1 < x_2 < \dots < x_{N+1} = b \}, \tag{39a}$$

$$\mathcal{E}_h = \{ T_i = [x_i, x_{i+1}], \ i = 1, \dots, N \},$$
(39b)

where the boundary of element T_i are nodes x_i and x_{i+1} . After the decomposition of the domain and desingularization of local diffusive equations, we are led to the system of differential equations (24), (26), (27), (33), which are subjected to compatibility conditions,

 $w_w = 0, \quad \text{on } \partial \Omega_{go} \cup \partial \Omega_k, \ (k = w, g, o),$ (40a)

$$w_g = 0, \quad \text{on } \partial \Omega_{wo} \cup \partial \Omega_k, \ (k = w, g, o),$$
 (40b)

$$w_w = -w_e, \quad \text{on } \partial \Omega_{we} \backslash \partial \Omega, \tag{40c}$$

and boundary and initial conditions, respectively:

$$w_w = w_g = 0, \quad \text{on } \partial \Omega, \tag{41a}$$

$$S_w(x, t_n) = S_w^0(x)$$
 and $S_g(x, t_n) = S_g^0(x)$. (41b)

We stress that one does not need to compute solutions of the associated diffusion subproblem in different subdomains in some specific order due to the third condition (40c). To this end, a Lagrange multiplier is introduced to impose the condition $w_w = -w_g$. In each region a distinct variational formulation is first implemented (see (26), (27), (33)) and then a fully assembled matrix is performed for its numerical solution. In the regions Ω_j , j = w, g, o as discussed in Sect. 6.3, we have

$$S_w(x, t_{n+1}) = S_w^0(x)$$
 and $S_g(x, t_{n+1}) = S_g^0(x)$. (42)

In each subdomain of the decomposition (21a), except in the regions Ω_j , we employ the zero order Raviart–Thomas finite element space to discretize (24), (26), (27), (33). Thus, we approximate the pair $(S_i, w_i) \in \mathcal{M}_h \times \mathcal{X}_h$, i = w, g, given by

$$\mathcal{M}_{h} = \left\{ \bar{s} \in L^{2}(\Omega); \ \bar{s} \text{ is constant in each element } T_{i} \in \mathcal{T}_{h} \right\},$$
(43a)

$$\mathcal{X}_h = \left\{ \bar{w} \in L^2(\Omega); \ \bar{w}|_{T_i} \text{ is linear, } i = 1, \dots, N, \text{ and } T_i \in \mathcal{T}_h \right\}.$$
(43b)

The initial saturation functions S_w^0 and S_g^0 are also in the space \mathcal{M}_h , which implies that the mobility functions λ_w^0 , λ_g^0 and λ_o^0 , defined in (20) are also in \mathcal{M}_h . Therefore, each one of the subdomain that define the decomposition of the domain Ω is an union of elements in \mathcal{T}_h or is the empty set. Now, note that the functions in \mathcal{X}_h can be discontinuous, which implies that this space does not satisfy the compatibility condition $\mathcal{X}_h \subset H(\operatorname{div}, \Omega)$. Then, we introduce a Lagrange multiplier in the discrete variational formulation which imposes the continuity of the flux functions in \mathcal{X}_h . Those Lagrange multipliers will also impose a regularity in the discrete system, since it would be indefinite [31]. The Lagrange multiplier space is the set of functions,

$$\Lambda_h = \{ \bar{l} : \mathcal{N}_h \to \mathbb{R} \},\tag{44}$$

which associates a real value to each node of the mesh [31]. The discrete variational formulation of the system (24) in region $\widetilde{\Omega}$ is the following: Find $(w_w, w_g, S_w, S_g, l_w, l_g) \in \mathcal{X}_h \times \mathcal{X}_h \times \mathcal{M}_h \times \mathcal{M}_h \times \Lambda_h \times \Lambda_h$ such that,

$$\begin{pmatrix} B_{11}^{-1}w_w, \bar{w}_w \end{pmatrix} + \begin{pmatrix} B_{12}^{-1}w_g, \bar{w}_w \end{pmatrix} + (S_w, \partial_x \bar{w}_w) - L(l_w, \bar{w}_w) = 0, \\ \begin{pmatrix} B_{21}^{-1}w_w, \bar{w}_g \end{pmatrix} + \begin{pmatrix} B_{22}^{-1}w_g, \bar{w}_g \end{pmatrix} + (S_g, \partial_x \bar{w}_g) - L(l_g, \bar{w}_g) = 0, \\ (\partial_x w_w, \bar{s}_w) - \frac{1}{\Delta t_d} (S_w, \bar{s}_w) = \frac{1}{\Delta t_d} \begin{pmatrix} S_w^0, \bar{s}_w \end{pmatrix}_{\widetilde{\Omega}}, \\ (\partial_x w_g, \bar{s}_g) - \frac{1}{\Delta t_d} (S_g, \bar{s}_g) = \frac{1}{\Delta t_d} \begin{pmatrix} S_g^0, \bar{s}_g \end{pmatrix}_{\widetilde{\Omega}}, \\ L(\bar{l}_w, w_w) = 0, \\ L(\bar{l}_g, w_g) = 0, \end{cases}$$
(45)

for any $\bar{w}_w, \bar{w}_g \in \mathcal{X}_h, \bar{s}_w, \bar{s}_g \in \mathcal{M}_h$ and $\bar{l}_w, \bar{l}_g \in \Lambda_h$, where for $(\bar{l}, v) \in \Lambda_h \times \mathcal{X}_h$,

$$L(\bar{l}, v) = \sum_{T_i \in \mathcal{E}_h} \{ \bar{l}(x_{i+1})v|_{T_i}(x_{i+1}) - \bar{l}(x_i)v|_{T_i}(x_i) \},$$
(46)

and (\cdot, \cdot) stands for integration over $\widetilde{\Omega}$. In order to impose transmission conditions in the coupling of the boundaries between a given pair of distinct porous media flow regions (21b)–(21f) we perform a computational classification of the elements in \mathcal{T}_h based on phase mobilities (20). Thus, in each element one can use a different formulation and impose the appropriate compatibility condition on the interface between elements. The discrete variational formulation for the equations (26), (27), (33) is analogous to the one presented above.

7 Numerical Experiments

In the first set of experiments, we consider a problem whose capillary pressure model is largely used on practical applications and is based on the features of the Brooks-Corey model [11] and on the J-function model [46],

$$p_{wo}(S_w) = -\varepsilon \left(S_w^*\right)^{-1/\eta} \quad \text{and} \quad p_{go}(S_g) = \varepsilon \left(S_g^*\right)^{-1/\eta},\tag{47}$$

where

$$S_w^* = \frac{S_w - S_{wr}}{1 - S_{wr} - S_{or}} \quad \text{and} \quad S_g^{\#} = \frac{1 - S_g - S_{wr} - S_{or}}{1 - S_{wr} - S_{or}}, \tag{48}$$

are normalized saturations and the coefficient ε measures the relative importance of diffusive and convective mechanisms. This capillary pressure model allow us to show the robustness of the proposed method. It follows from (47) that capillary pressure terms in the matrix P'are given by:

$$\partial_{S_w} p_{wo} = \frac{\varepsilon}{\eta} \frac{(S_w^*)^{-1/\eta - 1}}{(1 - S_{wr} - S_{or})} \quad \text{and} \quad \partial_{S_g} p_{go} = \frac{\varepsilon}{\eta} \frac{(S_g^\#)^{-1/\eta - 1}}{(1 - S_{wr} - S_{or})},\tag{49}$$

where $\partial_{S_w} p_{wo}$ and $\partial_{S_g} p_{go}$ are strictly positive functions, and therefore, condition (35) holds in Ω_{wg} . The similar conditions (24), (26), (27), (37) and (38) obtained for regions Ω_{go} , Ω_{wo} and Ω_j , j = w, g, o also hold by Lemma 1; see Sects. 6.1, 6.2 and 6.3. Note that $\partial_{S_g} p_{wo} = 0$ and $\partial_{S_w} p_{go} = 0$. In fact, the capillary pressure model implies that the determinant det P'is strictly positive in the interior of the saturation triangle (region $\tilde{\Omega}$). For all numerical experiments we consider $\eta = 2$ and $\varepsilon = 1.0 \times 10^{-1}$ unless explicitly specified otherwise. We take fluid viscosities to be $\mu_g = 0.3$, $\mu_w = 1.0$ and $\mu_o = 2.0$. Relative permeabilities are described by the Corey-Pope model [18, 26], in which each relative permeability is a function of its own phase saturation:

$$k_w = (S_w^*)^2, \qquad k_o = (S_o^*)^2 \text{ and } k_g = (S_g^*)^2,$$
 (50)

where the normalized saturations S_o^* and S_g^* are given by,

$$S_o^* = \frac{S_o - S_{or}}{1 - S_{wr} - S_{or}}$$
 and $S_g^* = \frac{S_g}{1 - S_{wr} - S_{or}}$. (51)

Furthermore, the residual saturations in the numerical simulations are $S_{wr} = 0.2$ and $S_{or} = 0.3$. We consider the domain $\Omega = [0, 4]$ and the results are taken at the computational dimensionless time t = 0.9. Indeed, it is a common practice circumvent the singularity in matrix B = QP' by introducing a regularity parameter ξ times the identity matrix, thus replacing the diffusion matrix B by a modified one:

$$B = \varepsilon \left(Q P' + \xi I_2 \right), \tag{52}$$

where I_2 is the identity matrix. It is important to note that such mathematical trick adds "extra" diffusion dictated by ξ . For comparisons purpose, we have considered in some numerical experiments the "numerical diffusion" value dictated by ξ . It is worth mentioning that our method does not require any kind of mathematical and numerical regularization. In the computations we consider $\xi = 1.0 \times 10^{-2}$, unless stated otherwise.

$(1/h)_{\text{fine}} \times (1/h)_{\text{coarse}}$	Sw (.)	Sg (.)	So (.)
60×30	7.22e-3 (-)	6.90e-3 (-)	1.22e-2 (-)
125×60	3.40e-3 (2.1)	3.49e-3 (2.0)	5.94e-3 (2.0)
250×125	1.08e-3 (3.1)	1.24e-3 (2.9)	2.03e-3 (2.9)
500×250	3.68e-4 (3.0)	4.49e-4 (2.7)	7.51e-4 (2.7)
1000×500	1.72e-4 (2.2)	1.92e-4 (2.4)	3.43e-4 (2.2)
2000×1000	8.40e-5 (2.0)	8.43e-5 (2.3)	1.58e-4 (2.1)

Table 1 Comparison of successive refined solutions of the New FEM using L^1 -norm

Table 2 Comparison of refined solutions of the *New FEM* using L^{∞} -norm

$(1/h)_{\text{fine}} \times (1/h)_{\text{coarse}}$	Sw (.)	Sg (.)	So (.)
60×30	5.25e-2 (-)	5.50e-2 (-)	5.50e-2 (-)
125×60	3.81e-2 (1.4)	6.49e-2 (0.8)	6.49e-2 (0.8)
250×125	1.89e-2 (2.0)	5.47e-2 (1.2)	5.47e-2 (1.2)
500×250	6.46e-3 (2.9)	4.38e-2 (1.2)	4.38e-2 (1.2)
1000×500	2.73e-3 (2.4)	2.92e-2 (1.5)	2.92e-2 (1.5)
2000×1000	1.22e-3 (2.3)	2.66e-2 (1.1)	2.66e-2 (1.1)

The finite element discretization considered in both methods for numerical solution of the nonlinear parabolic system, associated for the full problem (11)–(16), is the Raviart–Thomas space of order 1. The resulting linear problem is performed numerically by means of a LU factorization. The robustness of the proposed method is obtained through a large set of high-resolution numerical simulations of transport flow problems in regions with degenerated diffusion matrix occurring in the physical domain and by means of a numerical convergence study. Furthermore, numerical experiments are subsequently conducted to illustrate the applicability of the methodology and to carry out a sensitivity study regarding practical flow situations.

7.1 An Numerical Investigation of Convergence Rates

In this experiment we compare the converge rates of the New FEM (NFEM) method and the regularized FEM, with a regularization parameter $\xi = 1.0 \times 10^{-2}$. More specifically, we compare approximate solutions $||S_{fine} - S_{coarse}||$ for norms L^1 and L^{∞} ; S_{fine} stands for the solution on a fine mesh grid and S_{coarse} stands for the solution on a coarse mesh grid. In Tables 1 and 2 are shown the L^1 and L^{∞} errors, respectively, in a numerical refinement study. In parentheses presented in Tables 1 and 2 are shown the calculated numerical convergence rates showing some evidence of consistency of our approximate solutions.

In Tables 3 and 4 are shown the L^1 and L^{∞} errors (respectively) of a numerical mesh refinement study of the regularized FEM method. As it would be expected, since the finite element discretization are the same, the numerical convergence rates for both methods using NFEM and Regularized FEM are closely related. Of course, our NFEM methodology does not make use of any artificial numerical regularization.

$(1/h)_{\text{fine}} \times (1/h)_{\text{coarse}}$	S_w (.)	S_g (.)	$S_o(.)$
60×30	7.12e-3 (-)	6.83e-3 (-)	1.21e-2 (-)
125×60	3.30e-3 (2.1)	3.37e-3 (2.0)	5.74e-3 (2.1)
250 × 125	9.82e-4 (3.4)	1.14e-3 (3.1)	1.86e-3 (3.0)
500×250	3.27e-4 (2.9)	4.14e-4 (2.7)	6.81e-4 (2.8)
1000×500	1.49e-4 (2.2)	1.65e-4 (2.4)	2.92e-4 (2.3)

Table 3 Comparison of solutions of the regularized FEM using L^1 -norm

Table 4 Comparison of solutions of the regularized FEM using L^{∞} -norm

$(1/h)_{\text{fine}} \times (1/h)_{\text{coarse}}$	S_w (.)	$S_g(.)$	$S_o(.)$
60×30	5.04e-2 (-)	5.55e-2 (-)	5.55e-2 (-)
125×60	3.97e-2 (1.2)	6.55e-2 (0.9)	6.55e-2 (0.9)
250×125	1.67e-2 (2.4)	4.56e-2 (1.4)	4.56e-2 (1.4)
500×250	5.80e-3 (2.9)	3.27e-2 (1.4)	3.27e-2 (1.4)
1000×500	2.44e-3 (2.4)	2.42e-2 (1.4)	2.42e-2 (1.4)

Table 5 Comparison of the solution with New FEM and the reference solution (h = 1/2000) in the L^1 and L^{∞} norms for each individual phase. In the L^{∞} error, the result of oil saturation is qualitatively the same of gas saturation

1/h	L^1			L^{∞}	L^{∞}		
	$\overline{S_w}$ (.)	S_g (.)	$S_o(.)$	S_w (.)	S_g (.)		
30	1.17e-2 (-)	1.20e-2 (-)	2.06e-2 (-)	9.50e-2 (-)	1.14e-1 (-)		
60	4.70e-3 (2.6)	5.25e-3 (2.3)	8.60e-3 (2.4)	7.10e-2 (1.3)	1.18e-1 (0.9)		
125	1.38e-3 (3.6)	1.79e-3 (2.9)	2.79e-3 (3.1)	2.78e-2 (2.5)	7.72e-2 (1.5)		
250	4.51e-4 (3.1)	6.38e-4 (2.9)	1.01e-3 (2.8)	1.03e-2 (2.8)	6.73e-2 (1.2)		
500	1.89e-4 (2.4)	2.43e-4 (2.6)	4.09e-4 (2.4)	3.95e-3 (2.5)	5.58e-2 (1.2)		
1000	8.40e-5 (2.3)	8.43e-5 (2.9)	1.58e-4 (2.6)	1.22e-3 (3.3)	2.66e-2 (2.1)		

7.2 Convergence to a Reference Solution

Let us now consider a reference solution computed with the NFEM in a mesh h = 1/2000. Since the regularized FEM introduces numerical diffusion to the problem for removing the singularity of the parabolic operator, the solution of the NFEM is expected to exhibit better results.

In Table 5 we present the error in the L^1 and L^{∞} norms, respectively, of the solutions computed with the NFEM method with respect to the reference solution. The numerical results obtained computed by NFEM, without any artificial regularization, seems to be little better with respect to those obtained with the regularized FEM. In parenthesis we also present a convergence rate estimative. In Table 6 we present the error in the L^1 and L^{∞} norms, respectively, of the solutions computed with the regularized FEM method with respect to the reference solution. In parenthesis we also present a convergence rate estimative. 704

saturation								
1/h	L^1			L^{∞}				
	S_w (.)	S_g (.)	$S_o(.)$	S_w (.)	S_g (.)			
30	1.21e-2 (-)	1.23e-2 (-)	2.11e-2 (-)	9.63e-2 (-)	1.14e-1 (-)			
60	5.14e-3 (2.4)	5.62e-3 (2.1)	9.31e-3 (2.3)	7.45e-2 (1.3)	1.20e-1 (0.9)			
125	1.92e-3 (2.7)	2.25e-3 (2.5)	3.65e-3 (2.6)	3.49e-2 (2.1)	8.42e-2 (1.4)			
250	1.04e-3 (1.9)	1.16e-3 (1.8)	1.95e-3 (1.8)	1.94e-2 (1.7)	8.35e-2 (1.0)			
500	8.41e-4 (1.2)	8.21e-4 (1.5)	1.47e-3 (1.3)	1.37e-2 (1.4)	8.11e-2 (1.0)			
1000	7.94e-4 (1.1)	7.36e-4 (1.1)	1.35e-3 (1.2)	1.13e-2 (1.3)	7.46e-2 (1.1)			

Table 6 Comparison of the solution with Regularized FEM and the reference solution (h = 1/2000) in the L^1 and L^∞ norms for each individual phase. In the L^∞ error, the result of oil saturation is the same of gas saturation

In Fig. 2 we present a comparison of the computed solution with the regularized FEM in mesh $h = 1/30 \times h = 1/60$, $h = 1/60 \times h = 1/125$ and $h = 1/125 \times 1/250$. This is a grid refinement study for the numerical solution of the fully coupled convection-diffusion degenerate system (11)–(16) for Riemann data with initial discontinuity at x = 0, separating left (injection) $(S_w^L, S_g^L) = (0.67, 0.33)$ and right degenerate state $(S_w^R, S_g^R) = (S_{wr}, 0)$, that is, a reservoir saturated with pure oil. The numerical simulation is performed in a onedimensional discretized region by an uniform grid. Figure 2 show a study of numerical convergence under grid refinement, where we compare the solutions computed with a mesh size $h = 1/30 \times h = 1/60$ (top), $h = 1/60 \times h = 1/125$ (middle) and $h = 1/125 \times h = 1/250$ (bottom). The transport flow experiment described in Fig. 2 leads to a zero (degenerate) capillary pressure region, from interior of the triangle saturation Ω to the oil vertex Ω_{o} . Figure 2 (left column) display results for water and gas $(1 - S_{e})$ saturation profiles as a function of distance at the dimensionless time t = 0.9. At this time the solution has stabilized to a scale-invariant form [8]. The solutions shown in Fig. 2 comprise (from right to left) a "shock" and a composite "shock"-rarefaction wave. For complementeness, see Fig. 3 for the numerical solution of the pure hyperbolic case. We also show in Fig. 2 (right column) a mesh refinement comparison of the solutions obtained with the regularized method with $h = 1/30 \times h = 1/60, h = 1/60 \times h = 1/125$, and $h = 1/125 \times h = 1/250$, in which in turn exhibits the same solution structure. Moreover, it is clear from Fig. 2 (right) that as the grid is refined the layer region in the solutions becomes sharper and thus provides some evidence of numerical convergence of the procedure. In addition, we note that the comparison between numerical solutions in fine and coarse mesh studies shown in Table 6 corroborate our results.

The NFEM methodology resolves the traveling wave profile of the fast "shock" wave followed by a "shock"-rarefaction wave, as it is expected from the analysis of this model [8], yielding a validation of our computations. Furthermore, the solutions computed on a grid with h = 1/250 virtually coincide with the ones obtained on finer grid. Of course, this is not a prove of convergence, but rather a evidence of good consistency of our methodology. We also compare the solutions of both methods in each given mesh. In Table 7 it is shown the L^1 and L^{∞} norms of the error $||u_1 - u_2||$, where u_1 is the solution with the new FEM method and u_2 is the solution computed with the regularized FEM.

7.3 Convergence to the Pure Convective Problem

In the next experiment we performed a numerical qualitative analysis of the computed solution with a given diffusion ε for system (11)–(16) with respect to the pure convective solution as the diffusion parameter ε vanish to "zero".



Fig. 2 Numerical mesh refinement study considering two successive mesh grids with $h = 1/30 \times h = 1/60$ (*top*), $h = 1/60 \times h = 1/125$ (*middle*), and $h = 1/125 \times h = 1/250$ (*bottom*): New FEM (*left*) and Regularized FEM (*right*). Although the above results are qualitatively similar, we point out to the reader the results reported in Fig. 4, in which the regularized FEM clearly fails to converge to the correct solution



Fig. 3 Structure of the solution for the pure hyperbolic system in absence of diffusion: at this time the solution has stabilized to a scale-invariant form, and the solutions comprise (from *right* to *left*) a shock and a composite shock-rarefaction wave. See also the solutions in the saturation triangle (*bottom-left* picture) and in the reduced triangle of vertices O^* , G^* , W^* (*bottom-right* picture)

1/ <i>h</i>	S_w	S_g	So
30	3.68e-4 (1.59e-3)	2.96e-4 (1.09e-3)	5.95e-4 (1.25e-3)
60	4.84e-4 (3.56e-3)	4.02e-4 (2.78e-3)	7.86e-4 (2.89e-3)
125	6.09e-4 (7.13e-3)	5.25e-4 (7.06e-3)	9.97e-4 (7.06e-3)
250	6.95e-4 (9.49e-3)	6.17e-4 (1.61e-2)	1.15e-3 (1.61e-2)
500	7.41e-4 (1.00e-2)	6.69e-4 (3.51e-2)	1.24e-3 (3.51e-2)
1000	7.69e-4 (1.02e-2)	6.99e-4 (5.39e-2)	1.29e-3 (5.39e-2)

Table 7 Difference in L^1 norm (and L^{∞} in parenthesis) between the solutions of Regularized FEM and New FEM in a given mesh

ε	$S_w(.)$	S _g (.)	<i>S</i> ₀ (.)
0.8	3.98e-2 (1.34e-1)	1.46e-2 (1.22e-1)	5.08e-2 (1.22e-1)
0.4	2.28e-2 (1.38e-1)	9.76e-3 (1.26e-1)	3.01e-2 (1.26e-1)
0.2	1.28e-2 (1.42e-1)	6.05e-3 (1.28e-1)	1.73e-2 (1.28e-1)
0.1	7.12e-3 (1.45e-1)	3.74e-3 (1.26e-1)	9.95e-3 (1.27e-1)
0.01	2.38e-3 (1.42e-1)	1.92e-3 (1.19e-1)	3.76e-3 (1.19e-1)
0.001	2.05e-3 (1.38e-1)	1.82e-3 (1.19e-1)	3.34e-3 (1.19e-1)
0.0001	2.01e-3 (1.38e-1)	1.81e-3 (1.19e-1)	3.30e-3 (1.19e-1)
0.00001	2.01e-3 (1.38e-1)	1.81e-3 (1.19e-1)	3.30e-3 (1.19e-1)

Table 8 Numerical converge study of the approximate solution for system (11)–(16) given by for New FEM with respect to the pure convection solution in the L^1 -norm (and L^{∞} -norm in parenthesis) as the diffusion parameter ε tends to zero

Again, for complementeness, see Fig. 3 for the numerical approximate of the structure of solution for the pure hyperbolic case. A detailed description of the solution can be found in the Sect. 7.2. For all these experiments we have considered a mesh of h = 1/250. In Table 8 are shown the L^1 and L^{∞} errors, respectively, for the above mentioned study of the computed solution u_{ε} with the given ε parameter to the pure convection solution.

7.4 A Numerical Sensitivity Study on the Variability of Parameter ε

In Fig. 4 is shown a numerical sensitivity study of the NFEM and regularized FEM methods under the variability of parameter ε , as ε tends to zero. In Fig. 5 we show the results in the saturation triangle. A detailed description of the solution can be found in the Sect. 7.2. The solution for the pure hyperbolic system in absence of diffusion has stabilized to a scaleinvariant form, and the solutions comprise (from right to left) a shock and a composite shock-rarefaction wave. Here, our issue is to show the qualitative agreement among the reference solution in Fig. 3, and those obtained with NFEM (Fig. 4 top) and with the regularized FEM method (Fig. 4 bottom) under the variability of parameter ε . See the solutions in the saturation triangle (Fig. 5) for both NFEM and regularized FEM in the saturation triangle.

7.5 An Numerical Investigation of Time Splitting Errors

Here we want to assess that our operator splitting algorithm, indeed, exhibit the nice property of resolving with accuracy internal layers in the three-phase solutions with steep gradients, by exploiting the use of large time splitting relations, and at the same time, capturing the nonlinear balance between the convective and diffusive mechanisms. Using a fractional time step Δt_c the width of the numerical shock layer will be $O(\sqrt{\epsilon \Delta t_c})$, because the nonlinear self-sharpening mechanisms of the fractional flow functions " $f_i(S_w, S_g)$ ", i = w, g, in system (11)–(16), are thrown away by the unphysical entropy loss due to Oleinik's convexification [50] introduced in the convective step by the hyperbolic solver; see [5, 6, 42, 43]. This important error mechanism was introduced and carefully discussed in details in [42, 43]. Therefore, to not overestimate the shock layer the splitting time step should not be larger than ε . In this experiment we simulate and compare the qualitative behavior of the approximate solutions as the splitting time steps increases, with mesh h = 1/250 for both methods: the New FEM and the regularized FEM. In Table 9 is shown the numerical results



Fig. 4 A numerical sensitivity study is performed by varying decreasing values of parameter ε , with respect to the methods "NFEM" (*top*) and "regularized FEM" (*bottom*). All numerical experiments are performed in a fixed mesh grid with h = 1/250. Notice the *black dashed circles* surrounding the inflow boundary condition and the wave fronts in the numerical solutions, which exhibit different shapes depending on the size of the parameter ε . As it would be expected, the mathematical trick adds an "extra" artificial diffusion dictated by $\xi = 0.01$ (*bottom frame*). It is possible observe a deterioration of solution quality obtained by the "regularized FEM" method under decreasing values of parameter ε . These results clearly show that the "regularized FEM" method might fail to converge to the correct solution

of the regularized FEM method. In Table 10 is shown numerical results of the New FEM method.

In Table 9 is shown the discrepancy in L^1 -norm (left) L^{∞} -norm (right) of approximate solutions with the regularized FEM with h = 1/250 for several Time Splitting (TS) steps with respect to a reference solution with a single time splitting step Δt_c such that $O(\sqrt{\varepsilon \Delta t_c})$, as discussed above, and a mesh grid of width h = 1/2000. In Table 10 is shown numerical results in the same setting as described previously, for approximate solutions using our NFEM methodology with h = 1/250 and for several Time Splitting (TS) with respect to a reference solution with a single time splitting step and h = 1/2000. Here we also have discrepancy in L^1 -norm (left) L^{∞} -norm (right); see Table 9 and Table 10.



Fig. 5 Sensitive study of decreasing values of parameter ε for the New FEM method (*left frame*) and the Regularized FEM method (*right frame*). A mesh grid with h = 1/250 is fixed in all numerical experiments. The solutions for water, gas, and oil saturations are displayed in the saturation triangle (state space)

Table 9 Discrepancy in L^1 -norm (*left*) L^∞ -norm (*right*) of approximate solutions with the regularized FEM with h = 1/250 for several Time Splitting (TS) steps with respect to a reference solution with a single time splitting step and h = 1/2000

TS	$S_w(.)$	$S_g(.)$	$S_o(.)$	TS	S_w (.)	S_g (.)	$S_o\left(.\right)$
2	1.04e-3	1.16e-3	1.96e-3	2	1.92e-2	8.34e-2	8.34e-2
4	1.06e-3	1.18e-3	1.99e-3	4	1.95e-2	8.44e-2	8.44e-2
8	1.07e-3	1.18e-3	2.00e-3	8	1.91e-2	8.47e-2	8.47e-2
16	1.08e-3	1.20e-3	2.03e-3	16	1.90e-2	8.49e-2	8.49e-2
32	1.16e-3	1.29e-3	2.18e-3	32	2.05e-2	8.86e-2	8.86e-2
64	1.19e-3	1.41e-3	2.22e-3	64	2.56e-2	8.63e-2	8.63e-2
128	1.37e-3	1.54e-3	2.56e-3	128	2.83e-2	9.48e-2	9.48e-2
256	1.78e-3	2.09e-3	3.00e-3	256	4.10e-2	8.00e-2	8.00e-2

Table 10 Error of the solution of the New FEM as we increase the split step. Comparison in the L^1 -norm (*left*) in the L^{∞} -norm (*right*) of the New FEM to the reference solution with h = 1/2000 and split step equal 1

TS	S_w	S_g	So	TS	S_w (.)	S_g (.)	$S_o\left(. ight)$
2	4.49e-4	6.30e-4	1.00e-3	2	1.01e-2	6.63e-2	6.63e-2
4	4.59e-4	6.47e-4	1.02e-3	4	1.04e-2	6.77e-2	6.77e-2
8	4.64e-4	6.50e-4	1.03e-3	8	1.05e-2	6.72e-2	6.72e-2
16	4.99e-4	6.95e-4	1.06e-3	16	1.19e-2	6.63e-2	6.63e-2
32	5.42e-4	7.39e-4	1.18e-3	32	1.33e-2	7.05e-2	7.05e-2
64	8.11e-4	9.03e-4	1.30e-3	64	1.99e-2	6.49e-2	6.49e-2
128	8.72e-4	9.68e-4	1.58e-3	128	2.25e-2	7.23e-2	7.23e-2
256	1.56e-3	1.59e-3	2.28e-3	256	3.98e-2	5.23e-2	5.23e-2

7.6 A Triangular Model for Three-phase Flow

In what follows we consider a triangular model for the fractional flow functions associated to the system of three-phase flow equations (5)-(8), as is discussed in [43]. The numerical methodology described in [43] (see also [42]) relies heavily upon the method of Dafermos by means of polygonal approximations in which is based on solving local Riemann problems, see [19] for details. It is well known that such approach requires a priori construction of certain *residual flux functions* [42, 43]. Moreover, it is worth mentioning that such construction is not general for systems of conservation laws as is the "non-triangular" three-phase flow model with umbilic point we consider in this manuscript (see, e.g., [5, 6, 41, 48]). Indeed, even in the scalar case such residual flux function must be carefully constructed in order to ensure convergence to entropy solutions [42, 43]. Instead, we used the central differencing scheme [49] as a building block (see, e.g., [5, 6]) to bypass the need of Riemann solvers and/or approximate Riemann solvers as well as the use of a dimensional splitting approach. In addition, we mention that in [45] is shown that semidiscrete central-upwind schemes for convection-dominated flow regime may fail to converge to the unique entropy solution or the convergence may be so slow that achieving a proper resolution would require the use of impractically fine meshes. It has been shown in [5, 6] that the central differencing approach [49] can correctly capture the nonclassical solution in one spatial dimension as well as in multidimensional heterogeneous three-phase flows [5, 6]. We observe that in the triangular model discussed here (see also [43]) there is no residual saturations and the flux functions are given by,

$$f_w(S_w, S_g) = \frac{(1 - S_g)^2 + S_g^2/10}{10S_g^2 + (1 - S_g)^2} \cdot \frac{S_w^2}{S_w^2 + (1 - S_w^2)^2/10},$$
(53)

and

$$f_g(S_w, S_g) = \frac{S_g^2}{S_g^2 + (1 - S_g)^2 / 10}.$$
(54)

The regularized diffusive matrix is given by,

$$B = \varepsilon \begin{bmatrix} 4S_w(S_w - 1) + \xi & 0\\ 0 & 4S_g(S_g - 1) + \xi \end{bmatrix},$$
(55)

where the term ξ controls the amount of regularization introduced. For comparison purposes for our NFEM method we set $\xi = 0$ to get the correct "degenerate" diffusive matrix associated to the parabolic system (14)–(16). In the numerical experiments reported in this section the physical diffusion parameter ε is set to be 0.1. The initial condition of the problem at hand is,

$$(S_w, S_g)(x) = \begin{cases} (0.6, 0.4) & x \le 1.0, \\ (0.0, 0.0) & x > 1.0. \end{cases}$$
(56)

We consider the solution at time t = 1.0 for the results reported in the Table 11, whereby we computed the discrepancies L^1 -norm and L^{∞} -norm for the New FEM methodology. We use a CFL condition given by $\Delta t < 18.8 \ (\Delta x)$.

7.7 Modeling of Capillary Pressure Effects

When water and gas are injected alternately (WAG), the "hydrodynamic" flow sweep within the porous medium behaves as if an effective mixture had been injected [8, 52], then WAG

5.89e-4

2.92e-4

 512×256

 1024×512



Fig. 6 Water and gas saturation profiles are shown as a function of distance at times 0.5 and 1.0 computed in a mesh with 1024 elements. The numerical solutions presented here are in very good agreement with the semi-analytic results reported in [43], yielding a validation of our computations

mesh grids in the analysis of approximate solutions for the triangular model using the NFEM methodolog						
$h_f \times h_c$	$S_w(.)$	$S_g(.)$	$S_o(.)$	$S_w(.)$	S_g (.)	$S_o\left(. ight)$
64 × 32	5.55e-3	4.05e-3	9.46e-3	2.05e-2	4.31e-2	4.31e-2
128×64	2.51e-3	1.74e-3	4.22e-3	9.92e-3	2.62e-2	2.62e-2
256×128	1.20e-3	8.37e-4	2.03e-3	4.69e-3	1.51e-2	1.51e-2

2.21e-3

1.07e-3

9.91e-4

4.89e-4

4.02e-4

1.97e-4

Table 11 Discrepancies in L^1 -norm (*left*) and in L^∞ -norm (*right*) for the same fine (h_f) and coarse (h_c) mesh grids in the analysis of approximate solutions for the triangular model using the NFEM methodology

solutions tends towards a stabilized scale-invariant continuous injection shape. This longtime asymptotic behavior is in accordance with theoretical analysis in the literature [48]. For practical applications the transient regime is of importance, in particular when gravity and/or diffusion dominates the flow processes. Thus, for this early transient time capillary pressure must be taken into account. For the three-phase model considered throughout this paper, that generalize Buckley–Leverett's solution for immiscible two-phase, it has been shown recently that the injection of a water/gas mixture in the time-average proportions effectively behaves as the injection of a single phase [8]. Thus, we can regard the flow of this phase (water/gas mixture) and oil as another two-phase flow, similar to the Buckley– Leverett solution.

In this solution, a lead shock wave is followed by a continuous centered rarefaction wave, with the shock speed matching the characteristic speed of the state just behind it. In this three-phase flow experiment we consider a specific mixture of water and gas injected into a porous rock initially containing pure oil. Here, we focus on the lead shock wave (solid line in Fig. 7) that displaces the oil *in situ*. This numerical experiment is shown in Fig. 7, for the numerical solution of the degenerate system (11)–(16) with Riemann data $(S_w^L, S_g^L) = (0.8154, 0.1846)$ and $(S_w^R, S_g^R) = (S_{wr}, 0)$ (with initial discontinuity at x = 0), which leads to a classical Buckley–Leverett structure solution as described above. Notice that we use the

7.24e-3

3.22e-3

7.24e-3

3.22e-3



Fig. 7 On the *top frames* are shown water, oil and water saturation profiles as function of distance for a numerical simulation of the three-phase Buckley–Leverett problem. On the *bottom frames* are shown the corresponding numerical solutions in the saturation triangle, i.e., the state space. Notice that the *right-bottom frame* is a zoom of the region $O^*G^*W^*$ inside of the state space of the *left-bottom frame*. It is shown here a numerical sensitive study on the effects of parameter ε in the solution of the one-dimensional three-phase flow model (11)–(16) with the fractional flow functions (57). Our NFEM method was able to capture the qualitative behavior of solutions when the parameter ε vanishes to zero, with respect to the purely hyperbolic case [8]

fractional flow functions,

$$f_w(S_w, S_g) = \frac{\frac{S_w^*}{\mu_w}}{\frac{S_g^*}{\mu_g} + \frac{S_w^*}{\mu_w} + \frac{S_o^*}{\mu_o}} \quad \text{and} \quad f_g(S_w, S_g) = \frac{\frac{S_g^*}{\mu_g}}{\frac{S_g^*}{\mu_g} + \frac{S_w^*}{\mu_w} + \frac{S_o^*}{\mu_o}},$$
(57)

where S_g^* , S_w^* and S_o^* are defined in (48) and (51) with fluid viscosities $\mu_g = 0.3$, $\mu_w = 1.0$ and $\mu_o = 2.0$. Moreover, given the importance of accurate modelling of diffusive effects (capillary pressure) in the simulation of three-phase flow problems [8, 48] we have conducted a sensitivity study of solutions with respect to the strength of physical diffusion by varying parameter ε . The changes in the solutions produced by ε mimic the effects of physical-diffusion which in turn select distinct solutions. The analysis reported in [8] has shown that the injection of water/gas mixture in the time-average proportions effectively behaves similar to the classical, two-phase, Buckley–Leverett solution, from the edge G^*W^* to O^* (region saturated with pure oil). The Buckley–Leverett solution (without diffusion) is shown in Fig. 7. It comprises a composite wave (solid line) from right $(S_w^R, S_g^R) = (S_{wr}, 0)$ to left $(S_w^L, S_g^L) = (0.67, 0.33)$ states: a shock from $(S_{wr}, 0)$ to (S_w^*, S_g^*) and a rarefaction from (S_w^*, S_g^*) to (0.67, 0.33).

The deviations of the solutions through ε values from the oil-water edge to the Buckley– Leverett O^*B^* solution show how difficult it is to capture the nonlinear balance between effects of convection and diffusion. This illustrates the importance of an accurate numerical modelling of diffusive effects for the computation of physically correct solutions of threephase transport flow problems in porous media [7]. We note that this qualitative difference between numerical solutions with and without capillary pressure effects is better observed in the saturation triangle (bottom right picture in Fig. 7).

7.8 Two-Dimensional Numerical Experiments

As a model for multiscale rock heterogeneity, we consider scalar, log-normal permeability fields $K(\mathbf{x})$, so that log $K(\mathbf{x})$ is Gaussian and its distribution is determined by its mean and covariance function, We consider a distribution which is stationary, isotropic and fractal (self-similar); see [5, 6] for more details. The problem is defined on slab geometry reservoir with aspect ratio X/Y = 4, the spatial distributions of the permeability and porosity fields are defined on a 512 m × 128 m domain discretized into 1 m × 1 m cells. The two-dimensional three-phase flow model (5)–(8) are computed employing the same onedimensional discretization settings along with the fractional flow functions (57). Here we present numerical simulations for horizontal slab (gravity is not included).

A mixture (of water and gas) is injected at a uniform and constant rate of 0.2 pore volume per year along the left boundary $(x, y) \in \{0\} \times [0, Y], Y = 128$, with production of the three-phase fluid at the same uniform rate on the right boundary. No flow is allowed across the boundaries with y = 0 and y = Y. We used the fluid viscosities $\mu_w = 1.0$, $\mu_o = 2.0$ and $\mu_g = 0.3$, and the capillary pressure model (47). The injection saturation states are $S_w = 0.67$ and $S_g = 0.33$. The residual saturations are $S_{wr} = 0.2$ and $S_{or} = 0.3$. The initial condition for the problem correspond to the initial saturation state of the reservoir is $S_w = 0.2$ and $S_g = 0.0$ which means a reservoir full of oil (virgin reservoir) in which in turn exhibits degeneracy flow regions along the simulation time. As shown in Fig. 8, our NFEM method for solving degenerate parabolic systems is also capable of simulating the flow of a distinct number of fluid phases in distinct porous media regions in two-dimensional heterogeneous formations, a situation that naturally occurs in practical applications, such as petroleum reservoir engineering and groundwater transport. The proposed FEM method is able to capture the development of viscous fingering induced by the leading moving front waves associated to the nonlinear degenerate coupled two-dimensional three-phase flow problem (5)–(8).

8 Concluding Remarks

In this work we have introduced a new method for solving degenerate nonlinear coupled convection dominated parabolic systems in porous media with degenerate diffusion. By means of operator splitting, modified mixed finite element, and a simple strategy for decomposition of the domain into regions based on the mobility of phases, compatibility conditions at local interfaces were obtained in order to remove the singularity and thus allow us to reformulate



Fig. 8 From *top* to *bottom* are shown water, gas and oil saturation after one year of numerical simulation for a heterogeneous reservoir with variable porosity and permeability fields

the original nonlinear parabolic system in an appropriate form for their solution. Our NFEM method does not need to employ any artificial mathematical regularization.

A high-resolution numerical convergence study was performed and semi-analytical results have been computationally reproduced to test the robustness of the proposed method. The numerical experiments show that the solutions computed on a coarse grid virtually coincide with the ones obtained on finer grid, yielding evidence of numerical convergence. We note that the numerical solutions presented here for the pure hyperbolic case are in agreement with semi-analytic results reported in the literature [8], and thus corroborate our numerical computations. The robustness of the proposed method is obtained through a large set of numerical simulations of transport flow problems in regions with degenerate diffusion matrix. It is worth to mention that the implementation of the numerical technique in two-dimensional space (resp. three-dimensional space) does not present major difficulties because the method is based on the classification of elements and on the imposition of compatibility conditions on the edges (resp. on the faces) of the elements.

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