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The Mathematical Structure of Newtonian Spacetime: Classical Dynamics and Gravitation

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We give a precise and modern mathematical characterization of the Newtonian spacetime structure ($\mathbb{N}$). Our formulation clarifies the concepts of absolute space, Newton's relative spaces, and absolute time. The concept of reference frames (which are "timelike" vector fields on $\mathbb{N}$) plays a fundamental role in our approach, and the classification of all possible reference frames on $\mathbb{N}$ is investigated in detail. We succeed in identifying a Lorentzian structure on $\mathbb{N}$ and we study the classical electrodynamics of Maxwell and Lorentz relative to this structure, obtaining the important result that there exists only one intrinsic generalization of the Lorentz force law which is compatible with Maxwell equations. This is at variance with other proposed intrinsic generalizations of the Lorentz force law appearing in the literature. We present also a formulation of Newtonian gravitational theory as a curve spacetime theory and discuss its meaning.

Contents

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1. INTRODUCTION

In the Principia Mathematica,[1] Newton introduces the concepts of absolute space and absolute time as follows:

> *Absolute space, in its own nature, without relation to anything external, remains always similar and immovable.*
> *Absolute, true, and mathematical time, of itself and from its own nature, flows equally without relation to anything external.*

In what follows we give a precise and modern mathematical characterization of these concepts, i.e., we are going to present the Newtonian spacetime and Newtonian kinematics and dynamics as a spacetime theory in the sense of Ref. 2. The study of the Newtonian theory as a spacetime theory has been done with varying degrees of mathematical rigor by several authors, as, e.g., in Refs. 3–7. In Ref. 7 we can find citations of the original works on the subject by, e.g., E. Cartan and K. Friedrichs.

In Sec. 2 we discuss the geometrical structure of Newtonian spacetime giving the main definitions and introducing the subsidiary elements of the structure.

Section 2.2 is dedicated to the formulation of the Newtonian dynamics. After the presentation of the concepts of Newtonian particles, momentum and co-momentum of a particle, kinetic energy, and force fields, we present the Newton's laws of motion.

In Sec. 3 we give an original discussion of the concept of reference frames and moving frames as well as the coordinate chart naturally adapted to a given reference frame.

Reference frames are classified according to various criteria, and the notion of a inertial reference frame appears with full rigor, in constrast to the usual elementary presentations. This is possible once we recognize that one of the important mathematical objects of the geometrical structure of Newtonian spacetime is a flat connection D (Sec. 3.2.1). A discussion gives a clear physical meaning of the components of D in a given reference frame. Also in Sec. 3.3 we give a rigorous description of the concept of relative rest spaces introduced by Newton in the "Principia."

We also discuss in Sec. 3.4 the meaning of Galileo's Principle of Relativity. The fundamental meaning of this (and other relativity principles) has been fully discussed in Ref. 2.

In Sec. 4 we identify a natural Lorentzian structure in Newtonian spacetime $\mathbb{N}$ by introducing a Lorentzian metric for $\mathbb{N}$. The identification of a Lorentzian metric in $\mathbb{N}$ has already been done, e.g., in Ref. 5, but there we do more. Indeed, we show that D, the connection of Newtonian spacetime, is the Levi-Civita connection of this Lorentzian metric. The classification of vectors, curves, and the concepts of Lorentzian reference frames in $\mathbb{N}$ is presented according to the Lorentzian structure of $\mathbb{N}$, and a Lorentzian dynamics on Newtonian spacetime (LDN) is formulated. We comment on the similarities and differences between LDN and the Einstein dynamics of special relativity after examining in Sec. 5 the classical electrodynamics of Maxwell and Lorentz in our formalism.

We succeed in proving that there is a unique coupling between the electromagnetic field with the current of a charged particle which is different from the postulate classical Lorentz force law when the latter is written in intrinsic form and which is a function of the four-velocity of the particle. This result is at variance with the one obtained, e.g., in Ref. 5 where the author identifies the intrinsic generalization of the Lorenz force law with Eq. (141) instead of Eq. (143), which is the correct one.

In Sec. 6 we present Newton's gravitational theory as a curved spacetime theory and we emphasize that this exercise suggests by itself to interpret Einstein's gravitational theory as a field theory in Minkowski spacetime. Such a theory formulated with the Clifford bundle formalism has been developed by two of us in Ref. 8.

Finally in Sec. 7 we present our conclusions. We observe that this paper is one of a series we are proposing about the mathematical structure of spacetime theories.

2. THE NEWTONIAN SPACETIME

2.1. Geometrical Structure

2.1.1. Definitions. The structure $\mathbb{N}$ describing the Newtonian spacetime is introduced through

Axiom 2.1. The *Newtonian spacetime* is characterized geometrically by a pentuple $\mathbb{N} = \langle N, D, \Omega, V, \hat{h} \rangle$, where

1. N is a paracompact, connected, oriented and noncompact four-dimensional smooth manifold.
2. D is a linear connection on N such that its tensors of torsion and curvature satisfy

$$T[D] = 0 \quad \text{and} \quad R[D] = 0 \tag{1}$$

3. $\Omega \in \sec(T^*N)$, $\Omega \neq 0$, is a differentiable 1-form field on N such that

$$D\Omega = 0 \tag{2}$$

4. $V \in \sec(TN)$ is a differentiable vector field on N such that

$$\Omega(V) = 1 \tag{3}$$

5. $\hat{h} \in T_0^2 N$ is a two-covariant, symmetric, and differentiable tensor field on N such that for every $p \in N$
   (a) $\hat{h}_p(u_p, v_p) = 0 \ \forall u_p \in T_p N \Leftrightarrow v_p = k V_p, k \in \mathbb{R}$
   (b) $\hat{h}_p(u_p, u_p) > 0 \ \forall u_p \in T_p N$
   (c) $D\hat{h}|_p = 0$

Each point of the manifold N is called an *event* of the Newtonian spacetime. The structure of the theory permits one to "stratify" the manifold N into a continuous succession of three-dimensional spaces, so that each event is characterized by the instant and the place of its occurrence. This stratification is employed as follows.

Definition 2.2. Any function $t: N \to \mathbb{R}$ for which $\Omega_p = a\,dt_p \neq 0$ for all $p \in N$, $a \in \mathbb{R}$, $a > 0$, is called an *(admissible) time function* for $\mathbb{N}$. If $a = 1$, the function t is also said to be *normalized*. For each admissible time function $t: N \to \mathbb{R}$, the number $t(p) \in \mathbb{R}$ is called *time* (relative to t) of the event $p \in N$ and given two events p, $q \in N$, the number $|t(q) - t(p)|$ is called *temporal interval* (relative to t) between p and q.

Two events p, $q \in N$ are said to be *simultaneous* if and only if $t(p) = t(q)$. For each $p \in N$, the set

$$S_p = \{q \in N, t(q) = t(p)\} \tag{4}$$

of all events simultaneous with the event p is called *absolute simultaneity space* at p. (Obviously, $S_q = S_p$ if $t(p) = t(q)$.)

Observation 2.3. The question relative to the existence of physical devices that can realize the Newtonian time will be discussed in another publication.

Proposition 2.4. For each $p \in N$ the set S_p is a flat three-dimensional submanifold of N.

Proof. This follows at once from the assumptions that $\Omega \neq 0$, $D\Omega = 0$, and $R[D] = 0$. □

In this way, the spacetime manifold N is split into a continuous succession of flat three-dimensional spaces, which serve as models for the "instantaneous" Newtonian space.

The stratification of the manifold N produced in this way depends uniquely on the field Ω and not on the admissible time function used. To see this, it is enough to observe that any admissible time functions t, t': $N \to \mathbb{R}$ are related by

$$t' = at + b \tag{5}$$

with a, $b \in \mathbb{R}$, $a > 0$. Then, if p, $q \in N$ are such that $t(q) = t(p)$, we have $t'(q) - t'(q) = a(t(q) - t(p)) = 0$, that is, events which are simultaneous relatively to t are also simultaneous relatively to any other admissible time function.

Definition 2.5. Simultaneity of events is an equivalence relation. Each absolute simultaneity space S_p, $p \in N$, is an equivalence class of this equivalence relation. The collection of all these equivalence classes, denoted by

$$T = \frac{N}{\Omega} \tag{6}$$

will be called *Newtonian absolute time manifold.*

Definition 2.6. A vector $u_p \in T_pN$, $p \in N$, is called *spacelike* if and only if

$$\Omega_p(u_p) = 0 \tag{7}$$

Otherwise, if $\Omega_p(u_p) \neq 0$, u_p is called *timelike.*

A timelike vector $u_p \in T_pN$, $p \in N$, is called *future-pointing* if and only if $\Omega_p(u_p) > 0$ and it is called *past-pointing* if and only if $\Omega_p(u_p) < 0$.

Observation 2.7. For each $p \in N$, the set

$$\Sigma_p = \{u_p \in T_pN, \Omega_p(u_p) = 0\} \tag{8}$$

of all spacelike vectors at p coincides with the set T_pS_p of all tangent vectors to the submanifold S_p.

Definition 2.6 above is naturally extended to vector fields and curves on N. Namely, a vector field $u \in T\mathscr{U}$, $\mathscr{U} \subseteq N$, is called *spacelike* [*timelike*] if and only if $u(p)$ is spacelike [timelike] for every $p \in U$. A curve $\varphi\colon J \to N$, $J \subseteq \mathbb{R}$, is called *spacelike* [*timelike*] if and only if, for each $s \in J$, the vector $\varphi_*(s)$, tangent to φ at $\varphi(s)$, is spacelike [timelike]. The definitions of future- and past-pointing timelike vectors and curves are obtained analogously.

Definition 2.8. The vector field V in $\mathbb{N} = \langle N, D, \Omega, V, \hat{h}\rangle$ is called *absolute reference frame* of $\mathbb{N}$. We say that two events $p, q \in N$ *occur at the same place* in the space if and only if they belong to the same integral line of V.

The property of two events occuring at the same place in the space is an equivalence relation. The quotient space of N by this equivalence relation will be called *Newtonian absolute space* and denoted by

$$S = \frac{N}{V} \tag{9}$$

Proposition 2.9. The absolute reference frame V satisfies

$$DV = 0 \tag{10}$$

Proof. Taking the covariant derivative of Eq. (3) in the direction of an arbitrary vector field $v \in \sec(TN)$, we get $(D_v\Omega)(V) + \Omega(D_v V) = 0$ or, taking into account Eq. (2)

$$\Omega(D_v V) = 0 \tag{11}$$

On the other hand, we know that $\hat{h}(u, V) = 0$ for all $u \in \sec(TN)$, so that $D_v(\hat{h}(u, V)) = \hat{h}(u, D_v V) = 0$, or

$$D_v V = kV \tag{12}$$

for some function $k\colon N \to \mathbb{R}$. But in view of Eq. (11), we have $\Omega(D_v V) = \Omega(kV) = k = 0$ for all $v \in \sec(TN)$, that is, $DV = 0$. □

Definition 2.10. The tensor field $\hat{h} \in \sec(T_0^2 N)$ in $\mathbb{N} = \langle N, D, \Omega, V, \hat{h} \rangle$ is called *metric tensor* of the Newtonian spacetime.

For each vector $u_p \in T_p N$ the scalar $\|u_p\| \in \mathbb{R}$, $\|u_p\| \geqslant 0$, given by

$$\|u_p\|^2 = \hat{h}_p(u_p, u_p) \tag{13}$$

is called *norm* of u_p. Any vector $u_p \in T_p N$ such that $\|u_p\| = 0$ is called *null vector*. For each $p \in N$, the set

$$\Gamma_p = \{u_p \in T_p N, \ \|u_p\| = 0\} \tag{14}$$

of all null vectors at $p \in N$ is a 1-dimensional subspace of $T_p N$. The vectors in Γ_p have the form kV_p, $k \in \mathbb{R}$.

Observation 2.11. The field $\hat{h}$ is singular in the direction of V, so it does not represent a "genuine" four-dimensional metric on N. However, for every $p \in N$, the restriction $\hat{h}_p|_{\Sigma_p}$ of $\hat{h}_p$ to the set of all spacelike vectors at p is a two-covariant, symmetric, nondegenerate, and positive-definite bilinear form, that is, it is an Euclidean metric.

2.1.2. Subsidiary Elements. In order to continue our study of the Newtonian spacetime structure and to formulate the concept of reference frame field, we need some subsidiary elements which will be introduced below.

Definition 2.12. We shall denote by $\hat{H}: p \mapsto \hat{H}_p$, $\hat{H}_p: T_p N \to T_p^* N$, $p \in N$, the map that to each $u_p \in T_p N$ associates a 1-form $\hat{H}_p u_p \in T_p^* N$, defined in such a way that

$$(\hat{H}_p u_p)(v_p) = \hat{h}_p(u_p, v_p) \tag{15}$$

for every $v_p \in T_p N$.

Observation 2.13. $\hat{H}$ is a differentiable mapping. For each $p \in N$, $\hat{H}_p$ is linear, but it is neither injective ($\ker \hat{H}_p = \Gamma_p$) nor surjective (there does not exist, e.g., any vector $X_p \in T_p N$ for which $\hat{H}_p X_p = \Omega_p$). Note also that

$$D_u(\hat{H} v) = \hat{H}(D_u v) \tag{16}$$

for every u, $v \in \sec(TN)$.

Definition 2.14. We shall denote by $\tilde{h} \in \sec(T_2^0 N)$ the two-contravariant, symmetric, and differentiable tensor field on N such that, for every $p \in N$,

1. $\tilde{h}_p(\hat{H}_p u_p, \hat{H}_p v_p) = \hat{h}_p(u_p, v_p)$, $\forall u_p, v_p \in T_p N$,
2. $\tilde{h}_p(\alpha_p, \beta_p) = 0$, $\forall \alpha_p \in T_p^* N \Leftrightarrow \beta_p = k\Omega_p$, $k \in \mathbb{R}$.

Observation 2.15. It can be proved that there exists a unique tensor field in the conditions of Definition 2.14 such that

$$D\tilde{h} = 0 \tag{17}$$

Definition 2.16. We shall denote by $\tilde{H}: p \mapsto \tilde{H}_p$, $\tilde{H}_p: T_p^* N \to T_p N$, $p \in N$, the mapping that to each 1-form $\alpha_p \in T_p^* N$ associates a vector $\tilde{H}_p \alpha_p \in T_p N$ defined in such a way that

$$\beta_p(\tilde{H}_p \alpha_p) = \tilde{h}_p(\alpha_p, \beta_p) \tag{18}$$

for all $\beta_p \in T_p^* N$.

Observation 2.17. The mapping $\tilde{H}$ is differentiable and (like $\hat{H}$) for each $p \in N$ the linear mapping $\tilde{H}_p$ is neither injective ($\ker \tilde{H}_p = \{\alpha_p \in T_p^* N, \alpha_p = k\Omega_p, k \in \mathbb{R}\}$) nor surjective (there does not exist, e.g., any 1-form $\alpha_p \in T_p^* N$ for which $\tilde{H}_p \alpha_p = V_p$). Furthermore, we have

$$D_u(\tilde{H}\alpha) = \tilde{H}(D_u \alpha) \tag{19}$$

for all $\alpha \in \sec(T^* N)$, $u \in \sec(TN)$.

Moreover, note that if we take $\beta_p = \Omega_p$ in Definition 2.16, then we get $\Omega_p(\tilde{H}_p \alpha_p) = \tilde{h}_p(\alpha_p, \Omega_p) = 0$. Therefore, $\tilde{H}_p \alpha_p$ is always a spacelike vector.

Definition 2.18. We shall denote by $H_p^*: T_p^* N \to T_p^* N$ and by $H_{p*}: T_p N \to T_p N$, $p \in N$, the mappings defined respectively by

$$H_p^* = \hat{H}_p \circ \tilde{H}_p \tag{20}$$

and

$$H_{p*} = \tilde{H}_p \circ \hat{H}_p \tag{21}$$

Observation 2.19. The mappings $H^*: p \mapsto H_p^*$ and $H_*: p \mapsto H_{p*}$, $p \in N$, are differentiable. For each $p \in N$, H_p^* and H_{p*} are linear and satisfy:

1. $\ker H_p^* = \{\alpha_p \in T_p^* N, \alpha_p = k\Omega_p, k \in \mathbb{R}\}$,
   $\ker H_{p*} = \{u_p \in T_p N, u_p = kV_p, k \in \mathbb{R}\}$,

2. $D_u(H^*\alpha)|_p = H_p^*(D_u\alpha|_p)$,
   $D_u(H_* v)|_p = H_{p*}(D_u v|_p)$,
3. $\alpha(H_* u) = (H^*\alpha)(u)$,
4. $H^* \circ H^* = H^*$,
   $H_* \circ H_* = H_*$,
5. $(H^*\alpha)(v) = 0$,
   $\Omega(H_* u) = 0$,

where u, $v \in \sec(TN)$ and $\alpha \in \sec(T^*N)$ are arbitrary fields.

2.2. Newtonian Dynamics

The Newtonian dynamics is primarily concerned with the study of the motion of the material *particles*. (Extended bodies are seen as collections of particles and have their motion established once we know the motion of their particles.)

Particles are sources of *force fields*, influencing and being influenced by the motion of other particles: given some distribution of particles in motion through the space, the configuration of the system at each instant produces a resulting force field which changes the motion of each particle and in turn the configuration of the system and the force field it generates.

In the intermix of this process lies the notion of *mass*, which, accordingly, comprises a double meaning: it is a measurement of the intensity of the field generated by the particle and of the resistance it opposes to the changes of its state of motion.

2.2.1. Newtonian Particles

Definition 2.20. We call *Newtonian particle* (or simply *particle*) a pair $\langle m, \varphi \rangle$, where $m \in \mathbb{R}^+$ is a real and positive constant, called its (*inertial*) *mass* and where $\varphi: J \to N$, $J \subseteq \mathbb{R}$, is a future-pointing timelike curve on N such that

$$\Omega_p(\varphi_{*p}) = 1 \tag{22}$$

for every $p \in \varphi(J)$, where $\varphi_{*p} \in T_pN$, $p \in \varphi(J)$ denotes the tangent vector of φ at p. The curve φ is called *trajectory* of $\langle m, \varphi \rangle$.

Observation 2.21. Recall that $\varphi_{*p} \in T_pN$, $p = \varphi(s)$, $s \in J$, is given by

$$\varphi_{*p} = \frac{d\varphi}{ds}(s)$$

and then we have

$$\Omega_p(\varphi_{*p}) = 1 \Leftrightarrow \frac{dt}{ds}(\varphi(s)) = 1$$

that is,

$$t(\varphi(s)) = s + b$$

where $b \in \mathbb{R}$ is a constant. This means that the trajectory of every Newtonian particle is parameterized in such a way that is inclusion parameter measures the absolute time.

Definition 2.22. The vector $\varphi_{*p} \in T_pN$, $p \in \varphi(J)$, is called *absolutely velocity* of the particle $\langle m, \varphi \rangle$ at p and the vector

$$A_{\varphi *}(p) = D_{\varphi *}\varphi_* |_p$$

is called *absolute acceleration* of $\langle m, \varphi \rangle$ at p.

Observation 2.23. Since $\Omega_p(\varphi_{*p}) = 1$ for all $p \in \varphi(J)$ and $D\Omega = 0$, we have that $\Omega_p(A_{\varphi * p}) = \Omega_p(D_{\varphi *}\varphi_* |_p) = D_{\varphi *}(\Omega(\varphi_*))|_p = 0$ for all $p \in \varphi(J)$ and therefore the absolute acceleration of a particle is always a spacelike vector.

We also define the *absolute co-velocity* and the *absolute co-acceleration* of $\langle m, \varphi \rangle$ by the 1-form given, respectively, by

$$\varphi_p^* = -\Omega + \hat{H}_p \varphi_{*p} \tag{23}$$

and

$$A_p^*(p) = D_{\varphi *}\varphi^* |_p \tag{24}$$

for each $p \in \varphi(\mathbb{R})$.

Definition 2.24. We call *momentum* of a particle $\langle m, \varphi \rangle$ the vector $P_{\varphi p} \in T_pN$, $p \in \varphi(\mathbb{R})$, given by

$$P_{\varphi p} = m\varphi_{*p} \tag{25}$$

Observation 2.25. In addition, the *co-momentum* of $\langle m, \varphi \rangle$ is defined by the 1-form $P_{\varphi p}^* \in T_p^* N$, $p \in \varphi(\mathbb{R})$, given by

$$P_{\varphi p}^* = m\varphi_p^* \tag{26}$$

Definition 2.26. The *kinetic energy* of a particle $\langle m, \varphi \rangle$ is the scalar function $T_{\varphi p}\colon \varphi(J) \to \mathbb{R}$ given, at each $p \in \varphi(J)$, by

$$T_{\varphi p} = \tfrac{1}{2} m \varphi_{*p}^2 \tag{27}$$

where $\varphi_{*p}^2 = \hat{h}_p(\varphi_{*p}, \varphi_{*p})$.

Observation 2.27. It is easily seen that

$$T_{\varphi p} = \frac{1}{2m} \hat{h}_p(P_{\varphi p}, P_{\varphi p}) = \frac{1}{2m} \tilde{h}_p(P^*_{\varphi p}, P^*_{\varphi p})$$

for each $p \in \varphi(J)$.

Obviously, the momentum and the kinetic energy of a free particle are constant along its trajectory, i.e.,

$$D_{\varphi *} p_\varphi = 0, \qquad D_{\varphi *} T_\varphi = 0$$

2.2.2. Force Fields

Definition 2.28. We call *force field* on $\mathbb{N}$ an 1-form field $F^* \in T^*\mathcal{U}$, $\mathcal{U} \subseteq N$, such that

$$H_p^* F_p^* = F_p^* \tag{28}$$

for every $p \in \mathcal{U}$.

Observation 2.29. The condition above will be satisfied if and only if F^* is purely spatial, that is, it does not have component in the direction of Ω. This, of course, does not mean that the field F^* is time-independent. Indeed we have

Definition 2.30. A force field $F^* \in \sec(T^*\mathcal{U})$ is called *time-independent* (or *stationary*) if and only

$$D_{\mathfrak{t}} F^* = 0 \tag{29}$$

at every point of $\mathcal{U}$.

For the rest of this section, let $F^* \in T^*\mathcal{U}$, $\mathcal{U} \subseteq N$, be a force field on $\mathbb{N}$, not necessarily time-independent.

Definition 2.31. The force field F^* is said to be

1. *locally holonomic* iff $F^* \wedge dF^* = 0$,
2. *locally conservative* iff $dF^* = 0$,
3. *holonomic* iff $F^* = -\alpha d\phi$, α, $\phi\colon \mathcal{U} \to \mathbb{R}$,
4. *conservative* iff $F^* = -d\phi$, $\phi\colon \mathcal{U} \to \mathbb{R}$.

Observation 2.32. We have introduced the concept of holonomic force field in analogy with the holonomic constraint forces of the analytical mechanics, which are essentially those force fields which satisfy Frobenius' integrability condition. Since we are not requiring F^* to be stationary, these definitions of holonomic and conservative force fields appear naturally in their most general form.

Definition 2.33. If the force field F^* is holonomic or conservative, any function $\phi: \mathscr{U} \to \mathbb{R}$ as in Definition 2.31 is called a *potential function* for F.

Definition 2.34. The *work* of the force field F over a particle $\langle m, \varphi \rangle (\varphi(J) \cap \mathscr{U} \neq \varnothing)$ between the points $p_1 = \varphi(a)$ and $p_2 = \varphi(b)$ $(a, b \in \mathbb{R},\ a < b,\ \varphi(s) \in \mathscr{U},\ s \in [a, b])$ is defined by

$$W = \int_a^b F^*_{\varphi(s)}(\varphi_*(s))\, ds \tag{30}$$

Observation 2.35. In particular, if F is conservative, $F = -d\phi$, we have

$$\begin{aligned} F^*_{\varphi(s)}(\varphi_*(s)) &= -d\phi(\varphi_*)|_{\varphi(s)} \\ &= -\frac{d\phi}{ds}(\varphi(s)) \end{aligned}$$

and we get

$$W = -\int_a^b \frac{d\phi}{ds}(\varphi(s))\, ds = -\phi(\varphi(s))|_a^b = \phi(p_2) - \phi(p_1) \tag{31}$$

2.2.3. Newton's Laws of Motion. The Newtonian dynamics is a theory of the motion of material systems. Its basic laws are introduced through

Axiom 2.36.

1. Any material particle $\langle m, \gamma \rangle$ subject to the action of a force $f \in \sec(TN)$ has its motion described by the equation

$$m D_{\gamma *} \gamma_* = f|_{\gamma^*} \tag{32}$$

2. The mutual forces between two particles $\langle m_a, \gamma_a \rangle$ and $\langle m_b, \gamma_b \rangle$ satisfy

$$f_{ab}(t) = -f_{ba}(t) \tag{33}$$

   where f_{ab} is the force that $\langle m_a, \gamma_a \rangle$ exercises over $\langle m_b, \gamma_b \rangle$.

Observation 2.37. When $D_{\gamma_*}\gamma_* = 0$, i.e., $f|_\gamma = 0$, the particle $\langle m, \gamma \rangle$ is said to be *free* and its motion is geodetic. The property of a curve γ to be a geodetic is an intrinsic property of the curve and does not depend, of course, on the reference frame and the coordinates relative to which the curve is referred.

The usual presentation of Newton's laws includes still, as is well known, the so-called *first law of motion*. This law is a statement concerning the existence of inertial frames. Since in our presentation the existence of inertial frames is already guaranteed, the first law of motion is unnecessary.

Since $f \in \sec(TN)$, $f|_\gamma$ must be a linear combination of the vectors at our disposal (in Newtonian theory) with coefficients that are scalar functions. For a given particle $\langle m, \gamma \rangle$, the vectors at our disposal are γ_* and V. The most general expression for $f|_\gamma$ such that $\Omega(f|_\gamma) = 0$ is

$$f|_\gamma = \alpha(\gamma_* - V|_\gamma) + \operatorname{div} \phi|_\gamma \tag{34}$$

where α: $(\gamma_*, \gamma_*) \mapsto f[h(\bar{\gamma}_*, \bar{\gamma}_*)]$, f: $h(\bar{\gamma}_*, \bar{\gamma}_*) \to \mathbb{R}$ and $\bar{\gamma}_* = \gamma_* - \Omega(\gamma_*) V$ and ϕ: $N \to \mathbb{R}$ is the potential function.

Let $\langle x^\mu \rangle$ be the coordinate functions associated to $(\mathscr{U}, \varphi)$, $\mathscr{U} \subset N$. The most general force acting on the particle $\langle m_a, \gamma_a \rangle$ due to the particle $\langle m_b, \gamma_b \rangle$ which does not depend on γ_{a*}, γ_{b*}, and V is

$$f^i_{ab} = -h^{ij}(z_a) \frac{\partial \phi}{\partial z^j_a} (l_{ab}) \tag{35}$$

where l_{ab}: $(x^i, z^i_b) \mapsto h_{ij} x^i z^j_b$, $z^i_a = x^i \circ \gamma_a$ and $z^j_b = x^j \circ \gamma_b$.

3. REFERENCE AND MOVING FRAMES

In what follows we present the general theory of reference frames on the Newtonian spacetime. This study can be extended, with few modifications, to the reference frames of relativistic spacetimes. The original reference for the relativistic case is Ref. 10 (see also Ref. 11). Indeed, the unique relevant difference beetween the reference frames of these theories is that while in the Newtonian theory the fields Ω and $\hat{h}$ are absolutes (independent of the reference frame), in the special relativity they depend on the choice of a reference frame for the spacetime. This will be discussed in detail in another publication.

Reference frames are important in the mathematical formulation of the spacetime theories because they are the objects which model the idea of an observer and that ultimately permit us to associated numerical quantities to the elements of the theory.

In essence, an *observer* (in the spacetime theories in general) is some apparatus which enables one to perform measurements of physical quantities. (Recall that in the purely mechanical systems all physical magnitudes are expressed in terms of measurements of length, time, and mass, so that observers should be equipped to perform [at least] measurements of these quantities.)

Observers are concerned with *local* measurements (i.e., those taking place in the immediate vicinity of their instantaneous position) as well as with *nonlocal* measurements (related to regions of the spacetime far away from them). The need to perform nonlocal measurements is what motivates the introduction of the notion of a reference frame: an infinite collection of observers sparsed by the whole of [a region of] the spacetime.

Observers are modeled as particles, that is, by unit "norm" timelike curves. Along their trajectories there are defined the fields Ω, V, and $\hat{h}$ which models their measurement instruments together with the notion of derivation introduced by the connection D.

3.1. Reference Frames

3.1.1. Acceleration, Rotation, Shear, and Expansion

Definition 3.1. A reference frame on $\mathbb{N}$ is characterized by a future-pointing timelike vector field $E \in \sec(T\mathcal{U})$, $\mathcal{U} \subseteq N$, such that

$$\Omega_p(E_p) = 1 \tag{36}$$

for all $p \in \mathcal{U}$. Each integral line of the field $E \in \sec(T\mathcal{U})$ is called an *observer* of E. For each $p \in \mathcal{U}$, the vector E_p is called *absolute velocity* of E at p.

Observation 3.2. The absolute reference frame V of $\mathbb{N}$ is a reference frame in the sense of definition 3.1. We call V the *absolute rest frame* of $\mathbb{N}$, and each observer associated with V is called an observer in *absolute rest.*

For the rest of this section, let $E \in \operatorname{scc}(T\mathcal{U})$, $\mathcal{U} \in N$, be a reference frame on $\mathbb{N}$. We can characterize E through the behavior of its absolute derivative DE. This is made as follows.

Definition 3.3. We say that E is *inertial* if and only if

$$DE = 0 \tag{37}$$

at all points of $\mathcal{U}$.

Observation 3.4. The existence of such a kind of vector fields is guaranteed by the assumption that the spacetime manifold is flat. In particular, the absolute rest frame V is an inertial reference frame on the whole of the spacetime manifold N.

Definition 3.5. We call *absolute acceleration* of E at $p \in \mathscr{U}$ the vector $A_E(p) \in T_p\mathscr{U}$ given, at each point of $\mathscr{U}$, by the vector field

$$A_E = D_E E \tag{38}$$

The reference frame E is said to be *geodetic* if and only if

$$A_E = D_E E = 0 \tag{39}$$

on all points of $\mathscr{U}$. The observers associated with a geodetic reference frame are called *free-falling observers.*

Observation 3.6. Since $\Omega_p(E_p) = 1$ for all $p \in \mathscr{U}$ and since $D\Omega|_p = 0$ for all $p \in N$, we have $\Omega_p(A_E|_p) = \Omega_p(D_E E|_p) = D_E(\Omega(E))|_p = 0$, for all $p \in \mathscr{U}$. Therefore, the absolute acceleration of a reference frame is always a spacelike vector.

In order to continue the characterization of the reference frame E, we shall need the following object:

Definition 3.7. We denote by $\hat{h}_E \in \sec(T_0^2\mathscr{U})$ the two-covariant tensor field on N defined in such a way that, for every $p \in \mathscr{U}$,

$$\hat{h}_E(u, v) = \hat{h}(P_E(u), P_E(v)) \tag{40}$$

for every $u, v \in \sec(T\mathscr{U})$, where

$$P_E(u) = u - \Omega(u)E \tag{41}$$

Observation 3.8. It is easily seen that we can write

$$\hat{h}_E = \|E\|^2\, \Omega \otimes \Omega - \Omega \otimes \hat{H}E - \hat{H}E \otimes \Omega + \hat{h} \tag{42}$$

In particular, $\hat{h}_V \equiv \hat{h}$ and $\hat{h}_E(u, v) = \hat{h}(u, v)$ for all spacelike vector fields $u, v \in \sec(T\mathscr{U})$.

We can still prove the following properties of the field $\hat{h}_E$:

Proposition 3.9. The tensor field $\hat{h}_E$ satisfy, for each $p \in \mathscr{U}$,

1. $\hat{h}_{E_p}(u_p, v_p) = \hat{h}_{E_p}(v_p, u_p)$, $\forall u_p, v_p \in T_p\mathscr{U}$,
2. $\hat{h}_{E_p}(u_p, v_p) = 0$, $\forall u_p \in T_p\mathscr{U}$ iff $v_p = kE_p$, $k \in \mathbb{R}$,
3. $\hat{h}_{E_p}(u_p, v_p) \geqslant 0$, $\forall u_p \in T_p\mathscr{U}$.

Proof. Properties (i) and (iii) are proved trivially. We shall prove property (ii).

We have that $\hat{h}_{E_p}(u_p, v_p) = 0$ for all $u_p \in T_p\mathscr{U}$ iff $\hat{h}_p(u_p - \Omega(u_p)\ E_p, P_E v|_p) = 0$ for all $u_p \in T_p\mathscr{U}$, which will be verified iff $P_E v|_p = kV_p$, $k \in \mathbb{R}$. However, since $\Omega_p(P_E v|_p) = 0$, we must necessarily set $k = 0$, and then we conclude that $\hat{h}_{E_p}(u_p, v_p) = 0$ for all $u_p \in T_p\mathscr{U}$ iff $P_E v|_p = v_p - \Omega(v_p)\ E_p = 0$, which will occur iff $v_p = kE_p$, $k \in \mathbb{R}$. □

The tensor field $\hat{h}_E$ can be used to characterize the inertiality and geodeticity of the reference frame E. In fact,

Proposition 3.10. The reference frame E is inertial if and only if

$$D\hat{h}_E = 0 \tag{43}$$

and it is geodetic if and only if

$$D_E \hat{h}_E = 0 \tag{44}$$

Proof. The absolute derivative of $\hat{h}_E$ in the direction of an arbitrary vector field $u \in \sec(T\mathscr{U})$ is written

$$\begin{aligned} D_u \hat{h}_E &= D_u(\hat{h}(E, E))\ \Omega \otimes \Omega - \Omega \otimes (D_u \hat{H} E) - (D_u \hat{H} E) \otimes \Omega \\ &= 2\hat{h}(D_u E, E)\ \Omega \otimes \Omega - \Omega \otimes (\hat{H} D_u E) - (\hat{H} D_u E) \otimes \Omega \end{aligned}$$

Then $D_u \hat{h}_E = 0$ iff $D_u E = 0$. Hence $D\hat{h}_E = 0$ iff $DE = 0$ and $D_E \hat{h}_E = 0$ iff $D_E E = 0$.

We can now continue te characterization of the reference frame E.

Definition 3.11. We call *velocity gradient* of E the tensor field $\nabla E \in \sec(T_0^2 \mathscr{U})$ defined by

$$\nabla E(u, v) = \hat{h}_E(D_u E, v) \tag{45}$$

for every u, $v \in \sec(T\mathscr{U})$. The vector field $\nabla_u E \in \sec(T\mathscr{U})$, $u \in \sec(T\mathscr{U})$, defined by

$$(\nabla_u E)(v) = \nabla E(u, v) \tag{46}$$

for every $v \in T\mathscr{U}$, is called velocity gradient of E *in the direction* of the vector field u.

The velocity gradient ∇E gives an alternative way to characterize the inertiality and geodeticity of E, namely,

Proposition 3.12. The reference frame E is inertial if and only if

$$\nabla E = 0 \tag{47}$$

and it is geodetic if and only if

$$\nabla_E E = 0 \tag{48}$$

Proof. For future reference, we shall prove this proposition expressing the velocity gradient of E in the direction of an arbitrary vector field $u \in \sec(T\mathscr{U})$ in terms of the absolute derivative of $\hat{h}_E$ in the direction of u. We have

$$D_u \hat{h}_E = 2\hat{h}(D_u E, E)\, \Omega \otimes \Omega - \Omega \otimes (\hat{H} D_u E) - (\hat{H} D_u E) \otimes \Omega \tag{49}$$

and we note now that

$$\begin{aligned}(\hat{H} D_u E)(v) &= \hat{h}(D_u E, v) \\ &= \hat{h}(D_u E, P_E v + \Omega(v) E) \\ &= \hat{h}(D_u E, P_E v) + \Omega(v)\, \hat{h}(D_u E, E) \\ &= \hat{h}_E(D_u E, v) + \hat{h}(D_u E, E)\, \Omega(v)\end{aligned}$$

that is,

$$\hat{H} D_u E = \nabla_u E + \hat{h}(D_u E, E)\, \Omega$$

and therefore,

$$D_u \hat{h}_E = -\Omega \otimes \nabla_u E - \nabla_u E \otimes \Omega \tag{50}$$

Then it is easy to conclude that $D_u \hat{h}_E = 0$ if and only if $\nabla_u E = 0$. Consequently, recalling Proposition 3.10, the reference frame E will be inertial if and only if $\nabla E = 0$ and it will be geodetic if and only if $\nabla_E E = 0$. □

Definition 3.13. We call *rotation tensor* of E the tensor field $\omega_E \in \sec(T_0^2 \mathscr{U})$ defined by

$$\omega_E(u, v) = \tfrac{1}{2}(\nabla E(P_E(u), P_E(v)) - \nabla E(P_E(v), P_E(u))) \tag{51}$$

for every u, $v \in \sec(T\mathscr{U})$, where $P_E(u)$ is given by Eq. (41).

We call *deformation tensor* of E the tensor field $\sigma_E \in \sec(T_0^2 \mathscr{U})$ defined by

$$\sigma_E(u, v) = \tfrac{1}{2}(\nabla E(P_E(u), P_E(v)) + \nabla E(P_E(v), P_E(u))) \tag{52}$$

for every u, $v \in \sec(T\mathscr{U})$.

Definition 3.14. The reference E will be said to be *irrotational* if and only if

$$\omega_E = 0 \tag{53}$$

and it will be said to be *Euclidean-rigid* if and only if

$$\sigma_E = 0 \tag{54}$$

Proposition 3.15. The velocity gradient of E can be written

$$\nabla E = \Omega \otimes \nabla_E E + \omega_E + \sigma_E \tag{55}$$

Proof. Since any vector field $u \in \sec(T\mathcal{U})$ can be written as $u = P_E(u) + \Omega(u) E$, we have that

$$\begin{aligned} \nabla E(u, v) &= \nabla E(P_E(u) + \Omega(u) E, v) \\ &= \Omega(u)\, \nabla E(E, v) + \nabla E(P_E(u), v) \end{aligned}$$

However, $\nabla E(P_E(u), v) = \nabla E(P_E(u), P_E(v))$ for all $v \in \sec(T\mathcal{U})$ and therefore

$$\nabla E(u, v) = \Omega(u)\, \nabla_E E(v) + \nabla E(P_E(u), P_E(v))$$

Thus, from $\nabla E(P_E(u), P_E(v)) = \omega_E(u, v) + \sigma_E(u, v)$, we conclude that

$$\nabla E(u, v) = \Omega(u)\, \nabla_E E(v) + \omega_E(u, v), \sigma_E(u, v)$$

for all $u, v \in \sec(T\mathcal{U})$. □

Corollary 3.16. The reference frame E is inertial if and only if it is geodetic, irrotational, and Euclidean-rigid.

Proof. E is inertial if and only if $\nabla E = 0$, which will be verified if and only if $\nabla_E E = 0$, $\omega_E = 0$ and $\sigma_E = 0$, i.e., if and only if E is (respectively) geodetic, irrotational, and Euclidean-rigid. □

Proposition 3.17. The deformation tensor of E satisfies

$$\sigma_E = \tfrac{1}{2} \mathcal{L}_E \hat{h}_E \tag{56}$$

where $\mathcal{L}_E$ denotes the Lie derivative in the direction of the vector field E.

Proof. Since the connection D is torsion-free, $T[D]=0$, we have $D_u v - D_v u = [u, v] = \mathscr{L}_u v$, for all $u, v \in \sec(TN)$. Then,

$$\begin{aligned}(\mathscr{L}_E \hat{h}_E)(u, v) &= \mathscr{L}_E(\hat{h}_E(u, v)) - \hat{h}_E(\mathscr{L}_E u, v) - \hat{h}_E(u, \mathscr{L}_E v)\\ &= D_E(\hat{h}_E(u, v)) - \hat{h}_E(D_E u, v) - \hat{h}_E(u, D_E v)\\ &\quad + \hat{h}_E(D_u E, v) + \hat{h}_E(u, D_v E)\\ &= (D_E \hat{h}_E)(u, v) + \hat{h}_E(D_u E, v) + \hat{h}_E(D_v E, u)\end{aligned}$$

for every $u, v \in \sec(T\mathscr{U})$. On the other hand,

$$\begin{aligned}\sigma_E(u, v) &= \tfrac{1}{2}(\hat{h}_E(D_u E - \Omega(u)\, D_E E, v) + \hat{h}_E(D_v E - \Omega(v)\, D_E E, u)\\ &= \tfrac{1}{2}(\hat{h}_E(D_u E, v) + \hat{h}_E(D_v E, u)) - \tfrac{1}{2}(\Omega(u)\, \nabla_E E(v) + \Omega(v)\, \nabla_E E(u))\\ &= \tfrac{1}{2}[(D_E \hat{h}_E)(u, v) + \hat{h}_E(D_u E, v) + \hat{h}_E(D_v E, u)]\end{aligned}$$

for every $u, v \in \sec(T\mathscr{U})$, where we have used Eq. (50). □

Definition 3.18. We call (four-dimensional) *divergence* of a vector field $X \in \sec(T\mathscr{U})$ the scalar field $\operatorname{Div} X: \mathscr{U} \to \mathbb{R}$ defined by

$$\operatorname{Div} X = \operatorname{Tr}(DX) \tag{57}$$

where $\operatorname{Tr}(DX)$ is the trace function of the absolute derivative $DX \in \sec(T^1_1\mathscr{U})$.

Definition 3.19. We call *expansion tensor* of the reference frame E the tensor field $\theta_E \in \sec(T^2_0\mathscr{U})$ defined by

$$\theta_E = \tfrac{1}{3}(\operatorname{Div} E)\, \hat{h}_E \tag{58}$$

and we call *shear tensor* of E the tensor field $\overset{\triangle}{\sigma}_E \in \sec(T^2_0\mathscr{U})$ defined by

$$\overset{\triangle}{\sigma}_E = \sigma_E - \theta_E \tag{59}$$

Definition 3.20. The reference frame E will be said to be *nonexpanding* (or *expansion-free*) if and only if

$$\theta_E = 0 \tag{60}$$

and it will be said to be *shear-free* if and only if

$$\overset{\triangle}{\sigma}_E = 0 \tag{61}$$

Observation 3.21. Of course, E is Euclidean-rigid if and only if it is expansion- and shear-free. Moreover, the velocity gradient of E can be written

$$\nabla E = \Omega \otimes \nabla_E E + \omega_E + \overset{\triangle}{\sigma}_E + \theta_E \tag{62}$$

so that E is inertial if and only if it is geodetic, irrotational, nonexpanding, and shear-free.

3.2. Moving Frames

Definition 3.22. A *(proper) moving frame* on $\mathbb{N}$, defined in $\mathscr{U} \subset N$, is a quadruple $\langle e_\mu \rangle = \langle e_0, e_1, e_2, e_3 \rangle$, $e_\mu \in \sec(T\mathscr{U})$, $\mu = 0, 1, 2, 3$, of linearly independent differentiable vector fields on $\mathscr{U} \subset N$ such that, at each $p \in \mathscr{U}$,

1. $\Omega_p(e_{0p}) = 1$,
2. $\Omega_p(e_{kp}) = 0$, $k = 1, 2, 3$.

Observation 3.23. The restriction that a moving frame should be constituted by one timelike and three spacelike vector fields is purely conventional. The Newtonian theory does not impose, by itself, any restriction on the kind of vectors one uses to construct a basis of the tangent space of the manifold at each point, except that such a basis should contain at least one timelike vector.

Definition 3.24. A moving frame $\langle e_\mu \rangle$ on $\mathscr{U} \subset N$ will be called *$\hat{h}$-orthonormal* if and only if, at each $p \in \mathscr{U}$,

$$\hat{h}_p(e_{kp}, e_{lp}) = \delta_{kl} \tag{63}$$

with $k, l = 1, 2, 3$.

Definition 3.25. A moving frame $\langle e_\mu \rangle$ on $\mathscr{U} \subset N$ will be called *coordinate* if and only if, at each $p \in \mathscr{U}$,

$$\mathscr{L}_{e_\mu} e_\nu |_p = 0$$

Observation 3.26. The condition $\mathscr{L}_{e_\mu} e_\nu = 0$ in an open set $\mathscr{U} \subset N$ is necessary and sufficient for the existence of coordinate functions x^μ: $\mathscr{U} \to \mathbb{R}$ for which

$$e_\mu = \frac{\partial}{\partial x^\mu} \tag{64}$$

at every point of $\mathscr{U}$.

Definition 3.27. We say that a set of coordinate functions under the conditions of Observation 3.26 (if it exists) forms a *chart naturally adapted* to the moving frame $\langle e_\mu \rangle$.

Definition 3.28. We call *dual moving frame* of a moving frame $\langle e_\mu \rangle$ on $\mathscr{U} \subset N$ the quadruple $\langle \theta^\rho \rangle = \langle \theta^0, \theta^1, \theta^2, \theta^3 \rangle$, $\theta^\rho \in \sec(T^*\mathscr{U})$, of differentiable fields of 1-forms on $\mathscr{U}$ such that, for all $p \in \mathscr{U}$,

$$\theta^\rho_p(e_{\mu p}) = \delta^\rho_\mu \tag{65}$$

Observation 3.29. If $\langle e_\mu \rangle$ is an orthonormal moving frame, then its dual moving frame $\langle \theta^\rho \rangle$ satisfies

$$\hat{h}_p(\theta^\rho_p, \theta^\sigma_p) = \delta^{\rho\sigma} \tag{66}$$

for every $p \in \mathscr{U}$. In addition, if $\langle e_\mu \rangle$ is coordinate and if $\langle x^\mu \rangle$ is a chart naturally adapted to it, then we have that

$$\theta^\rho = dx^\rho \tag{67}$$

at every point of $\mathscr{U}$.

Definition 3.30. Let $T \in \sec(T^r_s \mathscr{V})$ be an arbitrary field on $\mathscr{V} \subseteq N$. We call *components* of T with respect to a moving frame $\langle e_\mu \rangle$ on $\mathscr{U} \subset N$, $\mathscr{U} \cap \mathscr{V} \neq \varnothing$, the functions $T^{\rho_1 \cdots \rho_s}_{\mu_1 \cdots \mu_r}: \mathscr{U} \cap \mathscr{V} \to \mathbb{R}$ ($\rho_i, \mu_i = 0, 1, 2, 3$, $i = 1, \ldots, s$, $j = 1, \ldots, r$) given at each $p \in \mathscr{U} \cap \mathscr{V}$ by

$$T^{\rho_1 \cdots \rho_s}_{\mu_1 \cdots \mu_r}(p) = T(e_{\mu_1}, \ldots, e_{\mu_r}, e_{\mu_r}, \theta^{\rho_1}, \ldots, \theta^{\rho_s})|_p \tag{68}$$

Notation 3.31. The components with respect to a moving frame $\langle e_\mu \rangle$ on $\mathscr{U} \subset N$ of the objects which define the geometrical structure of the Newtonian spacetime will be denoted as follows:

1. $\Gamma^\rho_{\mu\nu} = \theta^\rho(D_{e_\mu} e_\nu)$,
2. $\Omega_\mu = e_\mu(\Omega)$,
3. $V^\rho = \theta^\rho(V)$,
4. $h_{\mu\nu} = \hat{h}(e_\mu, e_\nu)$, $h^{\rho\sigma} = \tilde{h}(\theta^\rho, \theta^\sigma)$, $h^\rho_\mu = \theta^\rho(H_* e_\mu) = (H^*\theta^\rho)(e_\mu)$, where $\Gamma^\rho_{\mu\nu}$, Ω_μ, V^ρ, $h_{\mu\nu}$, $h^{\rho\sigma}$, and h^ρ_μ are functions from $\mathscr{U}$ into $\mathbb{R}$.

Observation 3.32. For any proper moving frame $\langle e_\mu \rangle$ on $\mathbb{N}$ we have

1. $\Omega_0(p)=1$ and $\Omega_k(p)=0$ implies $\Omega_p=\theta^0_p$, $\forall p\in\mathcal{U}$,
2. $V^0(p)=\theta^0_p(V_p)=\Omega_p(V_p)=1$, $\forall p\in\mathcal{U}$,
3. $h_{00}=h_{kl}V^kV^l$, $h_{k0}=h_{0k}=-h_{kl}V^l$, $h^{00}=h^{j0}=h^{0j}=0$, $h^0_0=h^0_k=0$, $h^j_0=-V^j$.

In particular, if $\langle e_\mu \rangle$ is orthonormal, then we have also that $h_{kl}=\delta_{kl}$, $h^{ij}=\delta_{kl}$, $h^{ij}=\delta^{ij}$, and $h^i_k=\delta^i_k$.

Proposition 3.33. A moving frame $\langle e_\mu \rangle$ on $\mathbb{N}$, defined in $\mathcal{U}\subset N$, is coordinate if and only if

$$\Gamma^\rho_{\mu\nu}=\Gamma^\rho_{\nu\mu} \tag{69}$$

at every point of $\mathcal{U}$.

Proof. Since the torsion tensor $T[D]=0$, we have

$$c^\rho_{\mu\nu}=\Gamma^\rho_{\mu\nu}-\Gamma^\rho_{\nu\mu}$$

at every point of $\mathcal{U}$. Then, $\langle e_\mu \rangle$ will be coordinate iff $c^\rho_{\mu\nu}=0$ in $\mathcal{U}$, that is, iff

$$\Gamma^\rho_{\mu\nu}=\Gamma^\rho_{\nu\mu}$$

at every point of $\mathcal{U}$. □

Definition 3.34. Let $E\in\sec(T\mathcal{U})$, $\mathcal{U}\subseteq N$, be a reference frame field. A moving frame $\langle e_\mu \rangle$ on $\mathbb{N}$, defined in $\mathcal{U}\subset\mathcal{U}$, will be said to be *naturally adapted* to E in $\mathcal{U}$ if and only if

$$e_{0p}=E_p$$

for all $p\in\mathcal{U}$

Observation 3.35. It is always possible, at least locally, to find a moving frame naturally adapted to a given reference frame. Moreover, such a moving frame can always be chosen to be coordinate. Any chart naturally adapted to a coordinate moving frame which is naturally adapted to a reference frame E is called *naturally adapted co-ordinate system to E* ($\langle$nacs $|$ $E\rangle$).

Proposition 3.36. A moving frame $\langle e_\mu \rangle$ naturally adapted to an inertial reference frame $I \in \sec(T\mathcal{U})$, $\mathcal{U} \subseteq N$, $\mathcal{U} \subset \mathcal{U}$ is coordinate if and only if

$$\Gamma^\rho_{\mu\nu} = 0 \tag{70}$$

in all points of $\mathcal{U}$, where $\Gamma^\rho_{\mu\nu} = \theta^\rho(D_{e_\mu} e_\nu)$.

Proof. If $\Gamma^\rho_{\mu\nu} = 0$ in $\mathcal{U}$, then obviously $\langle e_\mu \rangle$ is coordinate, since in this case $c^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = 0$ in $\mathcal{U}$.

Conversely, if $\langle e_\mu \rangle$ is coordinate, we have $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ in $\mathcal{U}$ and therefore, taking into account that $e_0 = I \Rightarrow D_{e_\mu} e_0 = D_{e_\mu} I = 0$ in $\mathcal{U}$, we have

$$\Gamma^\rho_{\mu 0} = \Gamma^\rho_{0\mu} = 0$$

in $\mathcal{U}$. This concludes our proof, since the components Γ^0_{kl} and Γ^j_{kl} will be always null in $\mathcal{U}$, the first because $\Omega(e_l) = 0$ and the other because the spacelike submanifolds of N are flat.

3.2.1. The Physical Meaning of the Components of D in a Reference Frame. Throughout this section, let E be a reference frame on $\mathbb{N}$, defined in $\mathcal{U} \subseteq N$.

Proposition 3.37. The components of the acceleration, rotation, and deformation of E with respect to a moving frame $\langle e_\mu \rangle$ on $\mathcal{U} \subset \mathbb{N}$ such that $\Omega(e_0) = 1$ and $\Omega(e_k) = 0$ at each point of $\mathcal{U}$, are given respectively, by

1. $A^k = E^\rho D_\rho E^k$,
2. $\omega_{kl} = \frac{1}{2}(h_{ml} D_k E^m - h_{mk} D_l E^m)$,
3. $\sigma_{kl} = \frac{1}{2}(h_{ml} D_k E^m + h_{mk} D_l E^m)$,

where $A^k = \theta^k(A_E)$, $\omega_{kl} = \omega_E(e_k, e_l)$, $\sigma_{kl} = \sigma_E(e_k, e_l)$ and

$$D_\rho E^\sigma = e_\rho(E^\sigma) + \Gamma^\sigma_{\rho\lambda} E^\lambda \tag{71}$$

Proof.

(i) The covariant derivative of E with respect to itself is given, in the moving frame $\langle e_\mu \rangle$, by

$$\begin{aligned} D_E E &= E^\rho D_{e_\rho}(E^\sigma e_\sigma) \\ &= E^\rho((D_{e_\rho} E^\sigma)\, e_\sigma + E^\sigma D_{e_\rho} e_\sigma) \\ &= E^\rho(e_\rho(E^\sigma) + \Gamma^\sigma_{\rho\lambda} E^\lambda)\, e_\sigma \end{aligned}$$

Hence, taking into account that $\Omega(E)=E^0=1$, we have

$$A^k = E^\rho(e_\rho(E^k) + \Gamma^k_{\rho\lambda}E^\lambda)$$

(ii) The components of the rotation and deformation tensors are obtained in an analogous way. We shall derive the components of the rotation tensor. We have

$$\omega_E(u, v) = \tfrac{1}{2}(\hat{h}(D_u E, v) - \hat{h}(D_v E, u))$$

for every spacelike vector field $u, v \in \sec(T\mathscr{U})$. Hence,

$$\begin{aligned}\omega_{kl} = \omega_E(e_k, e_l) &= \tfrac{1}{2}(\hat{h}(D_{e_k}E, e_l) - \hat{h}(D_{e_l}E, e_k))\\ &= \tfrac{1}{2}(\hat{h}((D_k E^m)\, e_m, e_l) - \hat{h}((D_l E^m)\, e_m, e_k))\end{aligned}$$

that is, $\omega_{kl} = \frac{1}{2}(h_{ml}D_k E^m - h_{mk}D_l E^m)$. □

Observation 3.38. It follows from the expressions for the components of the acceleration, rotation, and deformation derived above that

$$A_l = h_{ml}A^m = h_{kl}D_0 E^k + \omega_{kl}E^k + \sigma_{kl}E^k \tag{72}$$

Corollary 3.39. The components of the acceleration, rotation, and deformation of E with respect to a moving frame $\langle e_{\bar{\mu}}\rangle$ naturally adapted to E itself are given by

1. $A^{\bar{k}} = \Gamma^{\bar{k}}_{\bar{0}\bar{0}}$
2. $\omega_{\bar{k}\bar{l}} = \frac{1}{2}(h_{\bar{m}\bar{l}}\Gamma^{\bar{m}}_{\bar{k}\bar{0}} - h_{\bar{m}\bar{k}}\Gamma^{\bar{m}}_{\bar{l}\bar{0}})$
3. $\sigma_{\bar{k}\bar{l}} = \frac{1}{2}(h_{\bar{m}\bar{l}}\Gamma^{\bar{m}}_{\bar{k}\bar{0}} + h_{\bar{m}\bar{k}}\Gamma^{\bar{m}}_{\bar{l}\bar{0}})$

Corollary 3.40. Let $\langle x^{\mu'}\rangle$ be a $\langle \text{nacs} \mid I\rangle$, where I is some inertial reference frame on $\mathscr{U}$. Suppose that $\Gamma^{\rho'}_{\mu'\nu'} = 0$ in $\langle x^{\mu'}\rangle$. The components of the acceleration, rotation, and deformation of E with respect to $\langle x^{\mu'}\rangle$ are

1. $A^{k'} = (\partial E^{k'}/\partial x^{0'}) + E^{j'}(\partial E^{k'}/\partial x^{j'})$
2. $\omega_{k'l'} = \frac{1}{2}(h_{m'l'}(\partial E^{m'}/\partial x^{k'}) - h_{m'k'}(\partial E^{m'}/\partial x^{l'}))$
3. $\sigma_{k'l'} = \frac{1}{2}(h_{m'l'}(\partial E^{m'}/\partial x^{k'}) + h_{m'k'}(\partial E^{m'}/\partial x^{l'}))$

Proposition 3.41. Let $E \in \sec(T\mathscr{U})$ be an irrotational and Euclidean-rigid reference frame on $\mathbb{N}$, and let $\langle e_\mu\rangle$ be a moving frame naturally adapted to an arbitrary reference frame $E' \in \sec(T\mathscr{U})$. Then we have

$$A_l = h_{ml}A^m = h_{ml}E'(E^m) + A'_l + \omega'_{kl}E^k + \sigma'_{kl}E^k - h_{ml}c^m_{k0}E^k \tag{73}$$

where $A'_l = h_{ml}A'^m$, $\omega'_{kl} = \omega_{E'}(e_k, e_l)$, σ'_{kl} and $c^m_{k0}e_m = [e_k, e_0]$.

Proof. Since by hypothesis E is irrotational and Euclidean-rigid, the components of the absolute acceleration of E in the moving frame $\langle e_\mu \rangle$ are

$$A^m = E'(E^m) + \Gamma^m_{0\lambda} E^\lambda$$

and therefore

$$A_l = h_{ml} A^m = h_{ml} E'(E^m) + h_{ml} \Gamma^m_{00} + h_{ml} \Gamma^m_{0k} E^k$$

Now, from corollary 3.40 we get $\Gamma^m_{00} = A'^m$ and also $h_{ml} \Gamma^m_{k0} = \omega'_{kl} + \sigma'_{kl}$. Then recalling that $\Gamma^m_{0k} = \Gamma^m_{k0} - c^m_{k0}$, we conclude that $A_l = h_{ml} E'(E^m) + A'_l + \omega'_{kl} E^k + \sigma'_{kl} E^k - h_{ml} c^m_{k0} E^k$. □

Observation 3.42. Proposition 3.41 is generalized trivially to the case in which the reference frame E is also arbitrary. For this, we must only take into account Eq. (72) and write

$$A_l = h_{ml} E'(E^m) + \omega_{kl} E^k + \sigma_{kl} E^k + A'_l + \omega'_{kl} E^k + \sigma'_{kl} E^k - h_{ml} c^m_{k0} E^k \tag{74}$$

The term $h_{ml} E'(E^m)$ in Eq. (74) can also be written as

$$\begin{aligned} h_{kl} E'(E^k) &= E'(h_{kl} E^k) - E^k E'(h_{kl}) \\ &= E'(E_l) - E^k E'(h_{kl}) \end{aligned}$$

and we have

$$\begin{aligned} E'(h_{kl}) &= D_{e_0}(\hat{h}(e_k, e_l)) \\ &= \hat{h}(D_{e_0} e_k, e_l) + \hat{h}(e_k, D_{e_0} e_l) \\ &= h_{ml} \Gamma^m_{0k} + h_{mk} \Gamma^m_{0l} \\ &= h_{ml} \Gamma^m_{k0} + h_{mk} \Gamma^m_{0l} - h_{ml} c^m_{k0} - h_{mk} c^m_{l0} \\ &= 2\sigma'_{kl} - h_{ml} c^m_{k0} - h_{mk} c^m_{l0} \end{aligned}$$

So, we can write also

$$A_l = E'(E_l) + \omega_{kl} E^k + \sigma_{kl} E^k + A'_l + \omega'_{kl} E^k - \sigma'_{kl} E^k + c^m_{l0} E_m \tag{75}$$

where $E_l = h_{ml} E^m$ and the other terms are as before.

3.2.2. The Jacobian from $\langle e_\mu \rangle$ to $\langle e_{\mu'} \rangle$

Definition 3.43. Let $\langle e_\mu \rangle$ and $\langle e_\mu \rangle$ be two arbitrary moving frames on $\mathbb{N}$, defined respectively in $\mathcal{U} \subset N$ and $\mathcal{U}' \subset N$, with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$. For each $p \in \mathcal{U} \cap \mathcal{U}'$, the matrix $a^\mu_{\mu'}(p)$ such that

$$e_{\mu' p} = a^\mu_{\mu'}(p)\, e_{\mu p} \tag{76}$$

is called *Jacobian* of the transformation from $\langle e_\mu \rangle$ to $\langle e_{\mu'} \rangle$.

Observation 3.44. In order for $\langle e_\mu \rangle$ and $\langle e_{\mu'} \rangle$ to effectively represent moving frames on $\mathbb{N}$, it is required that $\det(a^\mu_{\mu'}(p)) \neq 0$ for all $p \in \mathscr{U}$, e.g., at every $p \in \mathscr{U}$, the matrix $(a^\mu_{\mu'}(p))$ must be invertible.

Since e_μ and $e_{\mu'}$ are all differentiable fields, the matrix $(a^\mu_{\mu'}(p))$ (and consequently $\det(a^\mu_{\mu'}(p))$ must also be a differentiable function of p. Then, since $\det(a^\mu_{\mu'}(p)) \neq 0$ for all $p \in \mathscr{U}$, we conclude that this function never changes its sign (e.g., either $\det(a^\mu_{\mu'}(p)) > 0$ for all $p \in \mathscr{U}$ or $\det(a^\mu_{\mu'}(p)) < 0$ for all $p \in \mathscr{U}$).

If $\langle \theta^\sigma \rangle$ and $\langle \theta^{\sigma'} \rangle$ are the dual moving frames of $\langle e_\mu \rangle$ and $\langle e_{\mu'} \rangle$, respectively, then we have that

$$\theta^{\sigma'}_p = a^{\sigma'}_\sigma(p)\, \theta^\sigma_p \tag{77}$$

where $a^{\sigma'}_\sigma(p)$ denotes the inverse of the Jacobian $a^\mu_{\mu'}(p)$, e.g., $a^{\sigma'}_\sigma(p) = (a^{-1})^{\sigma'}_\sigma\,(p)$.

Definition 3.45. Let $T^{\sigma_1 \cdots \sigma_r}_{\mu_1 \cdots \mu_s}: \mathscr{U} \cap \mathscr{V} \to \mathbb{R}$ be the components of an arbitrary tensor field $T \in \sec(T^r_s \mathscr{V})$, $\mathscr{V} \subseteq N$, with respect to $\langle e_\mu \rangle$. The components of T with respect to $\langle e_{\mu'} \rangle$ will be given (in $\mathscr{U} \cap \mathscr{U}' \cap \mathscr{V}$) by

$$T^{\sigma'_1 \cdots \sigma'_r}_{\mu'_1 \cdots \mu'_s} = a^{\sigma'_1}_{\sigma_1} \cdots a^{\sigma'_r}_{\sigma_r} a^{\mu_1}_{\mu_1}, \ldots\, a^{\mu_s}_{\mu_{s'}} T^{\sigma_1 \cdots \sigma_r}_{\mu_1 \cdots \mu_s} \tag{78}$$

Proposition 3.46. Let $\langle e_\mu \rangle$ and $\langle e_{\mu'} \rangle$ be moving frames naturally adapted to two reference frames E and E, respectively. Then the Jacobian of the transformation from $\langle e_\mu \rangle$ and $\langle e_{\mu'} \rangle$ satisfies

$$a^0_{0'} = 1$$

$$a^0_{k'} = 0$$

$$a^k_{0'} = \bar{E}^k$$

$$\det(a^l_{k'}) = 1$$

where $\bar{E}^k = \theta^k(\bar{E})$.

Proof. This follows from the fact that $e_0 = E$, $e_{0'} = \bar{E}$, $\Omega(E) = \Omega(\bar{E}) = 1$, $\Omega(e_k) = 0$, and $\Omega(e'_k) = 0$.

Observation 3.47. In particular, if $E = \bar{E}$, i.e., if $\langle e_\mu \rangle$ and $\langle e_{\mu'} \rangle$ are both naturally adapted to the same reference frame, then the above conditions remain valid, but with $a^k_{0'} = \bar{E}^k = 0$.

3.3. Newtonian Space and Newtonian Time. Relative Rest Spaces

Throughout this section, let $E \in \sec(TN)$ be a reference frame defined globally on $\mathbb{N}$ which in addition will be supposed to be the generator of a one-parameter group of diffeomorphisms of $\mathbb{N}$, which will be denoted by $G_E = \{\sigma_\alpha, \alpha \in \mathbb{R}\}$.

Our aim here is to recover the Newtonian theory as a theory of space and time, starting from the Newtonian theory of spacetime. As will be seen, the theory of the reference frames plays a central role in this subject. The motivation of the objects we have introduced axiomatically in the previous section will be clarified, in particular, in order to characterize a given reference frame. Our starting point is the following:

Definition 3.48. We say that two events $p, q \in N$ are *co-local* in E if and only if they belong to the same integral line of E.

Observation 3.49. Co-locality of events (with respect to a global reference frame on $\mathbb{N}$) is an equivalence relation. The quotient space of $\mathbb{N}$ by this equivalence relation will be denoted

$$S_E = \frac{N}{E} \tag{79}$$

and called *relative rest space* of the reference frame E. In particular, the relative rest space of the absolute rest frame V is just the Newtonian absolute space we have introduced in observation 2.1.11.

Proposition 3.50. The relative rest space S_E of the reference frame E is an affine three-dimensional space.

Proof. Since E is a global vector field and $\Omega_p(E_p) = 1$ for all $p \in N$, any integral line of E intersects each absolute simultaneity space in one and only one point. Therefore, each absolute simultaneity space is a representative of the quotient space S_E. Then, since each absolute simultaneity space is a flat three-dimensional space, it follows that S_E is also a flat three-dimensional space. □

Observation 3.51. The statement of the proposition above is very strong, since we are not introducing any restriction on E other than that it be global and that it is the generator of a one-parameter group of diffeomorphisms of N.

A remarkable consequence of the proof given above is that

Corollary 3.52. The relative rest spaces S_E and $S_{E'}$ of any two global reference frames E and E' are isomorphic.

Proof. This follows from fact that any absolute simultaneity space is, at the same time, a representative for the quotient sets S_E and $S_{E'}$. □

Observation 3.53. More strictly, Corollary 3.52 states that the spaces S_E and $S_{E'}$ are affinely isomorphic, i.e., they are isomorphic from the point of view of their affine structures. Any other geometric structures eventually existing on those spaces are not necessarily preserved by such isomorphism.

Definition 3.54. We shall denote by $\phi_E: N \to S_E$ the canonical map from N into the quotient space $S_E = N/E$ and by $\phi_{E_p}: S_p \to S_E$, $p \in N$, the restriction of ϕ_E to the absolute simultaneity space through $p \in N$, i.e.,

$$\phi_{E_p} = \phi_E|_{S_p} \tag{80}$$

Observation 3.55. Any map $\phi_p: S_E \to S_{E'}$ such that

$$\phi_p = \phi_{E'_p} \circ \phi_{E_p}^{-1} \tag{81}$$

is an affine isomorphism between S_E and $S_{E'}$.

For every $\sigma_\alpha \in G_E$ ($\alpha \in \mathbb{R}$) we have

$$\phi_{E_{\sigma_\alpha(p)}} = \phi_{E_p} \circ \sigma_{-\alpha} \tag{82}$$

Definition 3.56. We shall denote by $h_E(\tau) \in \sec(T_0^2 N)$, $\tau = t(p)$, $p \in N$, the tensor field on S_E defined by

$$h_E(\tau) = (\phi_{E_p}^{-1})^* \hat{h}_E|_{S_p} \tag{83}$$

where $\hat{h}_E|_{S_p}$ denotes the restriction of $\hat{h}_E \in \sec(T_0^2 N)$ to the absolute simultaneity space S_p and $(\phi_{E_p}^{-1})^*$ denotes the pull-back of the map $\phi_{E_p}^{-1}: S_E \to S_p$.

Observation 3.57. If p, $p' \in N$ are two simultaneous events, then $\phi_{E_p} = \phi_{E_{p'}}$ and $\hat{h}_E|_{S_p} = \hat{h}_E|_{S_{p'}}$, which implies that $(\phi_{E_p}^{-1})^* \hat{h}_E|S_p = (\phi_{E_{p'}}^{-1})^* \hat{h}_E|_{S_{p'}}$. For this, the number $\tau = t(p)$ is sufficient to label the various lifts of $\hat{h}_E$ to S_E.

Note also that the field $h_E(\tau)$ depends differentiably on the parameter τ.

Proposition 3.58. For each $\tau \in \mathbb{R}$ the tensor field $h_E(\tau) \in \sec(T_0^2 N)$ is an Euclidean metric on S_E.

Proof. We have to prove that, for every $x \in S_E$,

1. $h_E^\tau|_x(u_x, v_x) = h_E^\tau|_x(v_x, u_v)$, $\forall u_x, v_x \in T_x S_E$,
2. $h_E^\tau|_x(u_x, v_x) = 0$, $\forall u_x \in T_x S_E$ iff $v_x = 0$,
3. $h_E^\tau|_x(u_x, u_x) = 0$, $\forall u_x \in T_x S_E$ and $h_E^\tau(u_x, u_x) = 0$ iff $u_x = 0$,

where we have written $h_E^\tau = h_E(\tau)$.

Let $p \in N$ be the (unique) point of N such that $\phi_{E_p}^{-1}(x) = p$ and $t(p) = \tau$. Then,

$$h_E^\tau|_x(\mu_x, v_x) = [(\phi_{E_p}^{-1})^* \hat{h}_E|_{S_p}](\mu_x, v_x)$$
$$= \hat{h}_{E_p}(\phi_{E_p *}^{-1} \mu_x, \phi_{E_p *}^{-1} v_x)$$

for every μ_x, $v_x \in T_x S_E$, where $\phi_{E_p *}^{-1}$ denotes the derivative map of $\phi_{E_p *}^{-1} : S_E \to S_p$.

From the above expression the validity of the statement (i) and of the first part of statement (iii) is obvious. Taking into account that $\phi_{E_p *}^{-1}$ is an isomorphism between $T_x S_E$ and $T_p S_p$, it is easily seen that the statement (ii) and the second part of statement (iii) are also valid. □

Observation 3.59. Then, at each instant of time (i.e., instantaneously) the relative rest space of a global reference frame on $\mathbb{N}$ is an Euclidean space. But since the metric tensor may be time-dependent, we cannot say, in the more general case, that space constitute a "genuine" Euclidean space. There is however, a particular case in which the relative rest space of a global reference frame is genuinely Euclidean. In fact,

Proposition 3.60. The relative rest space S_E of a global Euclidean-rigid reference frame E is an (affine three-dimensional) Euclidean space.

Proof. We have only to prove that $h_E(\tau)$ on S_E for a global Euclidean-rigid reference frame E is independent of τ, that is, $h_E(\tau) = h_E(\tau')$ for all $\tau, \tau' \in \mathbb{R}$.

Recall that E will be globally Euclidean-rigid if and only if

$$\mathscr{L}_E \hat{h}_E = 0 \tag{84}$$

on N, a condition which will be verified if and only if

$$\sigma_\alpha^* \hat{h}_E = \hat{h}_E$$

for all $\alpha \in \mathbb{R}$, where $\sigma_\alpha \in \sigma_E$.

Then, writing $\tau = t(p)$, $\tau' = t(\sigma_\alpha(p))$ with $p \in N$ and $\alpha \in \mathbb{R}$ arbitraries, we conclude that, in a globally Euclidean-rigid frame E,

$$\begin{aligned} h_E(\tau') &= (\phi^{-1}_{E_{\sigma_\alpha(p)}})^* \hat{h}_E|_{S_{\sigma_\alpha(p)}} \\ &= (\phi^{-1}_{E_{\sigma_\alpha(p)}})^* \sigma^*_{-\alpha} \hat{h}_E|_{S_p} \\ &= (\sigma_{-\alpha} \circ \phi^{-1}_{E_{\sigma_\alpha(p)}})^* \hat{h}_E|S_p \\ &= (\phi^{-1}_{E_p})^* \hat{h}_E|_{S_p} = h_E(\tau) \end{aligned}$$

for all τ, $\tau' \in \mathbb{R}$ (since $p \in N$ and $\alpha \in \mathbb{R}$ were arbitraries). So, if E is a global Euclidean-rigid reference frame, the label τ in $h_E(\tau)$ is immaterial and can be omitted, e.g., we write $h_E(\tau) = h_E$. □

Proposotion 3.61. The relative rest spaces S_E and $S_{E'}$ of any two global reference frames E and E' on $\mathbb{N}$ are isometric at each instant of time.

Proof. Consider the map $\phi_\tau = S_E \to S_{E'}$, $\tau \in \mathbb{R}$, given by

$$\phi_\tau = \phi_{E'_p} \circ \phi^{-1}_{E_p}$$

where $p \in N$ is any point such that $t(p) = \tau$. As we have already mentioned, this map is an affine isomorphism between S_E and $S_{E'}$. We shall prove that, in addition, it pulls back $h_{E'}(\tau)$ onto $h_E(\tau)$ for each $\tau \in \mathbb{R}$. In fact,

$$\begin{aligned} \phi^*_\tau h_{E'}(\tau) &= (\phi_{E'_p} \circ \phi^{-1}_{E'_p})^* h_{E'}(\tau) \\ &= (\phi^{-1}_{E_p})^* \phi^*_{E'_p} h_{E'}(\tau) \\ &= (\phi^{-1}_{E_p})^* (\phi^*_{E'_p}(\phi^{-1}_{E'_p})^* (h_{E'}|_{S_p}) \\ &= (\phi^{-1}_{E_p})^* h_{E'}|_{S_p} \end{aligned}$$

Moreover, since $\hat{h}_{E_p}(u_p, v_p) = \hat{h}(u_p, v_p) = \hat{h}_{E'_p}(u_p, v_p)$ for every spacelike vector u_p, $v_p \in T_pN$ and for every $p \in N$, we have that $(\phi^{-1}_{E_p})^* h_{E'}|_{S_p} = (\phi^{-1}_{E_p})^* h_E|_{S_p}$ and therefore

$$\phi^*_\tau h_{E'}(\tau) = (\phi^{-1}_{E_p})^* h_E|_{S_p} = h_E(\tau)$$

for each $\tau \in \mathbb{R}$. □

3.4. Galileo's Principle of Relativity

Galileo's principle of relativity is the statement that the laws governing the evolution of the mechanical systems do not permit us to distinguish, by means of mechanical experiments, between one or another of two inertial reference frames.

The validity of this statement follows as a consequence of the structure of the Newtonian theory if we make the additional hypothesis that there are no (mechanical) velocity-dependent forces. In Ref. 1 (Corollary V, "Axioms or Laws of Motion"), Newton gives a proof of Galileo's principle. However, the proof is true only if there are no velocity-dependent forces in Nature, as is clear from the discussion of Sec. 5 [see Eq. (141)].

Let us suppose for a moment that this is really the case. Then it is clear that the absolute reference frame V becomes immaterial in the theory, since there is no means to identify it. So, by a question of coherence, we should drop out any reference to V from the theory. This can be done introducing a new model for the Newtonian spacetime, which we call Galileo's spacetime.

Definition 3.62. We call *Galileo spacetime* the quadruple $\mathscr{G} = \langle N, D, \Omega, \tilde{h} \rangle$, where

$$R[D] = 0$$

$$T[D] = 0$$

$$D\Omega = 0$$

$$D\tilde{h} = 0$$

$$\hat{h}(\Omega, \cdot) = 0$$

and the symbols have the same meaning as in Sec. 2.

Observe that the elimination of the field V from the structure of the Newtonian spacetime requires the field $\hat{h}$ to be eliminated too, because its definition depends on the field V.

In order to maintain the metric notions of the theory, we are thus led to formulate their axioms in terms of the "dual" metric $\tilde{h}$, which is a (degenerate) metric over the cotangent bundle. In consequence, there does not exist anymore an absolute notion of "length" (norm) of vectors. The norm of a vector will now depend explicitly on the choice of a reference frame in the spacetime, according to the result of the following proposition.

Proposition 3.63. For each reference frame E over N, there exists a unique tensor field $\hat{h}_E \in \sec(T_0^2 N)$ such that:

1. $\hat{h}_E(\tilde{H}\alpha, \tilde{H}\beta) = \tilde{h}(\alpha, \beta)$,
2. $\hat{h}_E(u_p, v_p) = 0$ for all $v_p \in T_p N$ iff $u_p = kE_p$, $k \in \mathbb{R}$.

If E is an inertial reference frame, then the field $\hat{h}_E$ satisfies

$$D\hat{h}_E = 0 \tag{85}$$

and if E is a geodetic reference frame, then

$$D_E \hat{h}_E = 0 \tag{86}$$

By the way, it is important to observe that the absence of an absolute covariant metric tensor is not incoherent with the ideas of the Newtonian theory, because the norm of spacelike vectors is always the same for each one of the infinitely many tensor fields introduced by the proposition above. In fact, given $u, v \in T_pN$, there exists $\alpha, \beta \in T_p^*N$ such that $u = \tilde{H}\alpha$ and $v = \tilde{H}\beta$. Then, for any reference frames E and $\bar{E}$ on N, we get from (i) that $\hat{h}_E(u, v) = \hat{h}_{\bar{E}}(u, v) = \tilde{h}(\alpha, \beta)$.

4. LORENTZIAN STRUCTURE IN THE NEWTONIAN SPACETIME

4.1. Lorentzian Geometry in the Newtonian Spacetime

4.1.1. Lorentzian Metric

Definition 4.1. We call *Lorentzian metric* on Newtonian spacetime the tensor field $\hat{g} \in T_0^2 N$ given by (5)

$$\hat{g} = \Omega \otimes \Omega - \hat{h} \tag{87}$$

Observation 4.2. $\hat{g}$ is a symmetric and nondegenerate two-covariant tensor field of signature -2, that is, it is in fact a Lorentzian metric in the original sense of this concept.[9] It is also easy to see that $D\hat{g} = 0$ and since $T[D] = 0$, we conclude that D actually is the Levi-Civita connection of $\hat{g}$.

The fact that we have identified a Lorentzian metric in $\mathbb{N}$ is not surprising. Indeed the condition for a manifold N with properties given by Axiom 1, Sec. 1 to admit a Lorentz metric is the existence of a field of directions, which in N is V.

Definition 4.3. We denote by $\hat{G}: p \mapsto \hat{G}_p$, $\hat{G}_p: T_pN \to T_p^*N$, $p \in N$, the mapping that to each $u_p \in T_pN$ associates an 1-form $\hat{G}_p u_p \in T_p^*N$ defined by

$$(\hat{G}_p u_p)(v_p) = \hat{g}_p(u_p, v_p)$$

for every $v_p \in T_pN$.

Observation 4.4. The mapping $\hat{G}$ is differentiable and for each $p \in N$ the linear mapping $\hat{G}_p: T_pN \to T_p^*N$ is an isomorphism. We denote by

$\tilde{G}_p: T_p^* N \to T_p N$, $p \in N$, the inverse isomorphism of $\hat{G}_p$, that is, $\tilde{G}_p = \tilde{G}_p^{-1}$, and in addition we denote by $\tilde{G}: p \mapsto \tilde{G}_p$ the differentiable mapping that to each $p \in N$ associates the isomorphism $\tilde{G}_p$.

Definition 4.5. We denote by $\tilde{g} \in T_2^0 N$ the reciprocal of the tensor field $\hat{g}$, that is,

$$\tilde{g}_p(\alpha_p, \beta_p) = \hat{g}_p(\tilde{G}_p \alpha_p, \tilde{G}_p \beta_p) \tag{88}$$

for every $\alpha_p, \beta_p \in T_p^* N$ and for every $p \in N$.

Observation 4.6. It is easily seen that we have

$$\tilde{g} = V \otimes V - \tilde{h} \tag{89}$$

and consequently the tensor field $\tilde{g} \in T_2^0 N$ is a symmetric and non-degenerate two-contravariant tensor field of signature -2 and satisfies

$$D\tilde{g} = 0 \tag{90}$$

Observation 4.7. From the expressions for $\hat{g}$ and $\tilde{g}$, we see that

$$\hat{G}_p u_p = \Omega_p(u_p)\, \Omega_p - \hat{H}_p u_p \tag{91}$$

and

$$\tilde{G}_p \alpha_p = \alpha_p(V_p)\, V_p - \tilde{H}_p \alpha_p \tag{92}$$

for every $u_p \in T_p N$, $\alpha_p \in T_p^* N$, and $p \in N$. In addition, since $D\hat{g} = 0$ and $D\tilde{g} = 0$, we get

$$D_v(\hat{G}u)|_p = \hat{G}_p(D_v u|_p) \tag{93}$$

and

$$D_v(\hat{G}\alpha)|_p = \hat{G}_p(D_v \alpha|_p) \tag{94}$$

for every $u, v \in \sec(TN)$, $\alpha \in \sec(T^*N)$, and $p \in N$.

4.1.2. Classification of Vectors and Curves according to $\hat{g}$

Definition 4.8. A vector $u_p \in T_p N$, $p \in N$, will be called

1. *$\hat{g}$-spacelike* iff $\hat{g}_p(u_p, u_p) \leqslant 0$ and $\hat{g}_p(u_p, u_p) = 0$ iff $u_p = 0$,
2. *$\hat{g}$-lightlike* iff $\hat{g}_p(u_p, u_p) = 0$ and $u_p \neq 0$,
3. *$\hat{g}$-timelike* iff $\hat{g}_p(u_p, u_p) > 0$.

Observation 4.9. As usual, this definition can be extended to vector fields and curves on N.

If $u_p \in T_pN$ is a spacelike vector, that is, if $\Omega_p(u_p)=0$, then it is $\hat{g}$-spacelike since in this case $\hat{g}_p(u_p, u_p) = -\hat{h}_p(u_p, u_p) \leqslant 0$ and $\hat{h}_p(u_p, u_p)=0$ iff $u_p=0$.

If $u_p \in T_pN$ is a null vector, that is, $u_p = bV_p$, $b \in \mathbb{R}$, $b \neq 0$, then $\hat{g}_p(u_p, u_p) = (\Omega_p(u_p))^2 = b^2 > 0$ and therefore u_p is $\hat{g}$-timelike.

Finally, if $u_p \in T_pN$ is a timelike vector, that is, $\Omega_p(u_p)=a$, $a \in \mathbb{R}$, $a \neq 0$, then $\hat{g}_p(u_p, u_p) = a^2 - \hat{h}_p(u_p, u_p)$ and u_p will be $\hat{g}$-spacelike iff $\hat{h}_p(u_p, u_p) > a^2$, $\hat{g}$-lightlike iff $\hat{h}_p(u_p, u_p) = a^2$, and $\hat{g}$-timelike iff $\hat{h}_p(u_p, {}_p) < a^2$.

In view of this last observation we see that there are three kinds of reference frames on the Newtonian spacetime, namely

Definition 4.10. A reference frame $E \in \sec(T\mathcal{U})$, $\mathcal{U} \subseteq N$, will be called

1. *bradyonic* iff $E_p^2 = \hat{h}_p(E_p, E_p) < 1$, $\forall p \in \mathcal{U}$,
2. *luxonic* iff $E_p^2 = \hat{h}_p(E_p, E_p) = 1$, $\forall p \in \mathcal{U}$,
3. *tachyonic* iff $E_p^2 = \hat{h}_p(E_p, E_p) > 1$, $\forall_p \in \mathcal{U}$.

This definition is naturally extended to observers and particles on the Newtonian spacetime.

4.2. Lorentzian Reference Frames

From now on, we shall refer to the Ω-normalized reference frames (e.g., the vector fields $E \in \sec(TN)$ for which $\Omega(E) = 1$ as *Galilean reference frames* and we shall now introduce another kind of reference frames, the $\hat{g}$-normalized ones, as follows.

Definition 4.11. We call *Lorentzian reference frame* a future-pointing vector field $L \in \sec(T\mathcal{U})$, $\mathcal{U} \subset N$, such that

$$\hat{g}_p(L_p, L_p) = 1 \tag{95}$$

for every $p \in \mathcal{U}$.

Observation 4.12. Like for Galilean frames, for each $p \in \mathcal{U}$ the vector L_p is called *absolute velocity* of L at P and each integral line of the field L is called *observer* of the Lorentzian reference frame L.

Observation 4.13. Note that the absolute rest frame V is at the same time a Galilean and a Lorentzian reference frame. Moreover, it is the unique reference frame in these conditions.

Observation 4.14. Given Galilean reference frame $E \in \sec(T\mathscr{U})$, $\mathscr{U} \subseteq N$, we have

$$\hat{g}(E, E) = 1 - E^2 \tag{96}$$

with $E^2 = \hat{h}(E, E)$. When $\hat{g}(E, E) > 0$, that is, E is bradyonic, we can define a vector field $L(E) \in \sec(T\mathscr{U})$ by

$$L(E) = \frac{E}{\sqrt{1 - E^2}} \tag{97}$$

which is, obviously, a Lorentzian reference frame on $\mathscr{U}$, which will be called Lorentzian frame *associated* to the Galilean frame E.

Conversely, given a Lorentzian reference frame $L \in \sec(T\mathscr{U})$, $\mathscr{U} \subseteq N$, we have

$$\Omega(L) = \sqrt{1 + L^2} \tag{98}$$

$L^2 = \hat{h}(L, L)$, and we can define a vector field $E(L) \in \sec(T\mathscr{U})$ by

$$E(L) = \frac{L}{\sqrt{1 + L^2}} \tag{99}$$

which is, evidently, a bradyonic Galilean frame on $\mathscr{U}$, called Galilean frame *associated* to the Lorentzian frame L.

Note, moreover, that it is impossible to associate a Lorentzian frame to a luxonic or a tachyonic Galilean frame and reciprocally, the Galilean frame associated to a Lorentzian one is always bradyonic.

For the rest of this section, let $L \in \sec(T\mathscr{U})$, $\mathscr{U} \subseteq N$, be a Lorentzian reference frame on $\mathbb{N}$.

Definition 4.15. We denote by $\Omega_L \in \sec(\sec T^*\mathscr{U})$ the 1-form field on $\mathscr{U}$ defined by

$$\Omega_{L_p}(u_p) = \hat{g}_p(L_p, u_p) \tag{100}$$

for every $u_p \in T_pN$ and for every $p \in N$.

Observation 4.16. It is easy to see that Ω_L can be written

$$\Omega_L = \sqrt{1 + L^2}\, \Omega - \hat{H}L \tag{101}$$

and therefore if, in particular, $L = V$, then we shall have

$$\Omega_L = \Omega \tag{102}$$

Note also that if E is the Galilean frame associated to L, then we can write Ω_L as

$$\Omega_L = \frac{1}{\sqrt{1-E^2}}(\Omega - \hat{H}E) \tag{103}$$

Definition 4.17. The Lorentzian reference frame L will be called

1. *locally synchronizable* iff $\Omega_L \wedge d\Omega_L = 0$,
2. *locally proper time synchronizable* iff $d\Omega_L = 0$,
3. *synchronizable* iff $\Omega_L = f\,dx^0$, f, x^0: $\mathcal{U} \to \mathbb{R}$, $f > 0$,
4. *proper time synchronizable* iff $\Omega_L = dx^0$, x^0: $\mathcal{U} \to \mathbb{R}$.

Observation 4.18. If is clear that (iii)$\Rightarrow$(i), (iv)$\Rightarrow$(ii) and the reciprocals are valid only locally.

Definition 4.19. If L is a [proper time] synchronizable Lorentzian frame, any function x^0 as in Definition 4.17 is called a [*proper*] *time function* for L.

Observation 4.20. We denote by $\hat{h}_L \in \sec(T_0^2\mathcal{U})$ the tensor field on $\mathcal{U}$ defined by

$$\hat{h}_L = -\Omega_L \otimes \Omega_L + \hat{g} \tag{104}$$

Observation 4.21. Recalling the expressions of Ω_L and $\hat{g}$, we easily conclude that $\hat{h}_L$ can be written

$$\hat{h}_L = -L^2\Omega \otimes \Omega + \sqrt{1+L^2}(\Omega \otimes \hat{H}L + \hat{H}L \otimes \Omega) - \hat{H}L \otimes \hat{H}L - \hat{h} \tag{105}$$

Moreover, if in particular $L = V$, then it follows that

$$\hat{h}_L = \hat{h} \tag{106}$$

Observation 4.22. The tensor field $\hat{h}_L$ plays an analogous role to that of the tensor field $\hat{h}_E$ introduced in Definition 2.2.9 for a Galilean reference frame E. With the introduction of this tensor field we can prove the following

Proposition 4.23. The absolute derivative $D\Omega_L$ of the 1-form fields Ω_L can be uniquely written as

$$D\Omega_L = \Omega_L \otimes A_L + \omega_L + \sigma_L + \theta_L \tag{107}$$

where

$$\begin{aligned} A_L &= D_L \Omega_L \\ \omega_L(u, v) &= \tfrac{1}{2}(\nabla L(u_L, v_L) - \nabla L(v_L, u_L)) \\ \sigma_L(u, v) &= \tfrac{1}{2}(\nabla L(u_L, v_L) + \nabla L(v_L, u_L)) - \theta_L(u, v) \\ \theta_L &= \tfrac{1}{3}(\operatorname{Div} L)\, \hat{h}_L \\ \nabla L(u, v) &= \hat{h}_L(D_x L, v) \end{aligned} \tag{108}$$

with $u, v \in \sec(T\mathscr{U})$, $u_L = u - \Omega_L(u)L$, $v_L = v - \Omega_L(v)L$, and $\operatorname{Div} L = \operatorname{Tr}(DL)$.

Observation 4.24. Of course, the objects introduced through Proposition 4.23 have similar meaning and properties to those introduced in Sec. 3 for a Galilean frame. We call $A_L \in \sec(T^*\mathscr{U})$ the *acceleration 1-form* of L; $A_L = D_L L$ is its *absolute acceleration*; $\omega_L \in \sec(T^2_0\mathscr{U})$ is the *rotation tensor* of L; $\sigma_L \in \sec(T^2_{\underline{0}})$ is the *shear tensor* of L; $\theta_L \in \sec(T^2_0\mathscr{U})$ is the *expansion tensor* of L, and $\bar{\sigma}_L = \sigma_L + \theta_L$ is the *deformation tensor* of L. In addition, we define:

Definition 4.25. The Lorentzian reference frame L will be said to be

1. *inertial* iff $DL = 0$,
2. *geodetic* iff $D_L L = 0$,
3. *irrotational* iff $\omega_L = 0$,
4. *shear-free* iff $\sigma_L = 0$,
5. *expansion-free* iff $\theta_L = 0$,
6. *rigid* iff $\sigma_L = 0$ and $\theta_L = 0$.

Observation 4.26. All results we have stated in Sec. 2.2 relating the concepts of inertiality, geodeticity, rotationality, etc., remain valid for a Lorentzian frame and we shall not reformulate them here. We shall only state some new results that are specific for the Lorentzian frames.

Proposition 4.27. A Lorentzian reference frame L is inertial if and only if

$$D\Omega_L = 0 \tag{109}$$

and it is geodetic if and only if

$$D_L \Omega_L = 0 \tag{110}$$

Observation 4.28. The Lorentzian frame L is locally synchronizable iff it is irrotational, and it is locally proper time synchronizable iff it is geodetic and irrotational.

Proof. To prove this we calculate the differential of the 1-form field Ω_L. We have

$$
\begin{aligned}
2d\Omega_L(u, v) &= u(\Omega_L(v)) - v(\Omega_L(u)) - \Omega_L([u, v]) \\
&= D_x(\hat{g}(L, v)) - D_y(\hat{g}(L, v)) \\
&\quad - \hat{g}(L, D_x v - D_y u) \\
&= \hat{g}(D_x L, v) + \hat{g}(L, D_x v) - \hat{g}(D_y L, u) \\
&\quad - \hat{g}(L, D_y u) - \hat{g}(, D_x v - D_y u) \\
&= \hat{g}(D_x L, v) - \hat{g}(D_y L, u) \\
&= \hat{h}_L(D_x L, v) + \hat{h}_L(D_y L, u) \\
&= -\hat{h}_L(D_{x_L} L, v) - \Omega_L(u)\, \hat{h}_L(D_L L, v) \\
&\quad + \hat{h}_L(D_{v_L} L, u) + \Omega_L(v)\, \hat{h}_L(D_L(D_L L, u) \\
&= \nabla L(u_L, v_L) - \nabla L(v_L, u_L) + \Omega_L(u)\, A_L(v) \\
&\quad - \Omega_L(v)\, A_L(u) \\
&= 2\omega_L(u, v) + \Omega_L(u)\, A_L(v) - \Omega_L A_L(u)
\end{aligned}
$$

for every $u, v \in \sec(T\mathscr{U})$, that is,

$$d\Omega_L = \omega_L + \Omega_L \wedge A_L \tag{111}$$

with $\Omega_L \wedge A_L = \frac{1}{2}(\Omega_L \otimes A_L - A_L \otimes \Omega_L)$. From this expression it follows that

$$d\Omega_L \wedge \Omega_L = \omega_L \wedge \Omega_L \tag{112}$$

Therefore, $d\Omega_L = 0$ iff $\omega_L + \Omega_L \wedge A_L = 0$, which is impossible unless $\omega_L = 0$ and $\Omega_L \wedge A_L = 0$. But $\Omega_L \wedge A_L = 0$ iff $A_L = 0$, and we conclude that $d\Omega_L = 0$ (L is locally proper time synchronizable) iff $\omega_L = 0$ (L is irrotational) and $A_L = 0$ (L is geodetic).

Moreover, $d\Omega_L \wedge \Omega_L = 0$ iff $\omega_L \wedge \Omega_L = 0$, iff $\omega_L = 0$, that is, L is locally synchronizable iff it is irrotational. □

4.3. Lorentzian Dynamics on Newtonian Spacetime

4.3.1. Lorentzian Particles. From now on we shall refer to the Ω-normalized curves on N as *Galilean curves*. Now we define

Definition 4.29. A *Lorentzian curve* on N is a future-pointing curve φ: $\mathbb{R} \to N$ satisfying

$$\hat{g}_p(\varphi_{*p}, \varphi_{*p}) = 1 \tag{113}$$

for every $p \in \varphi(\mathbb{R})$.

Observation 4.30. The observers in a Lorentzian reference frame are Lorentzian curves.

Given a future-pointing bradyonic and Galilean curve on N, we can reparametrize it to get a Lorentzian curve. We proceed as follows.

Definition 4.31. Let $\bar{\varphi}$: $\mathbb{R} \to N$, $u \mapsto \bar{\varphi}(u)$, be a future-pointing bradyonic and Galilean curve on N. We call *proper time length* of $\bar{\varphi}$ between the points $p_1 = \bar{\varphi}(a)$ and $p_2 = \bar{\varphi}(b)$, a, $b \in \mathbb{R}$, $a < b$, to the number

$$\tau = \int_a^b [-\hat{g}_{\bar{\varphi}(u)}(\bar{\varphi}_*(u), \bar{\varphi}_*(u))]^{1/2} \, du \tag{114}$$

Observation 4.32. We can now introduce the function s: $[a, b] \to [0, \tau]$ by

$$s(k) = \int_a^k [-\hat{g}_{\varphi(u)}(\bar{\varphi}_*(u), \bar{\varphi}_*(u))]^{1/2} \, du \tag{115}$$

which gives us the proper time length of $\varphi|_{[a,b]}$ between the points $\bar{\varphi}(a)$ and $\bar{\varphi}(k)$, $k < b$. The mapping s is smooth and surjective. If it has a smooth inverse s^{-1} (which always holds when $\bar{\varphi}_*(u) \neq 0$ for all $u \in [a, b]$), then $\varphi|_{[0,\tau]} = \bar{\varphi}|_{[a,b]} \circ s^{-1}$: $[0, \tau] \to N$ is a smooth reparametrization of $\bar{\varphi}|_{[a,b]}$ with

$$(\text{proper time length of } \varphi \text{ between } \varphi(0) \text{ and } \varphi(k)) = k$$

and we say that the curve φ is parameterized by the proper time.

The relation between the tangent vectors $\bar{\varphi}_*(u)$ and $\varphi_*(s(u))$, $u \in [a, b]$, is easily obtained. Indeed we have

$$\bar{\varphi}_*(u) = \frac{d\bar{\varphi}}{du}(u) = \frac{d\varphi}{ds}(s(u)) \frac{ds}{du}(u) = \frac{ds}{du}(u) \, \varphi_*(s(u))$$

and

$$\frac{ds}{du}(u) = [-\hat{g}_{\bar{\varphi}(u)}(\bar{\varphi}_*(u), \bar{\varphi}_*(u))]^{1/2}$$

and therefore

$$\varphi_*(s(u)) = [-\hat{g}_{\bar{\varphi}(u)}(\bar{\varphi}(u), \bar{\varphi}(u), \bar{\varphi}_*(u))]^{-1/2}\,\bar{\varphi}_*(u)$$

that is,

$$\varphi_*(s(u)) = \frac{\bar{\varphi}_*(u)}{\sqrt{1-\bar{\varphi}_*^2(u)}} \tag{116}$$

for $a \leqslant u \leqslant b$, where $\bar{\varphi}_*^2(u) = \hat{h}_{\bar{\varphi}(u)}(\bar{\varphi}_*(u), \bar{\varphi}_*(u))$ and we have taken into account that $\bar{\varphi}$ is Galilean.

Definition 4.33. We call *Lorentzian particle* a pair $\langle m, \varphi \rangle$, where $m \in \mathbb{R}^+$ is a real and positive constant, called *rest mass* of the particle, and $\varphi\colon \mathbb{R} \to N$ is a Lorentzian curve on N, called *trajectory* of the particle.

Definition 4.34. The vector $\varphi_{*p} \in T_pN$, $p \in \varphi(\mathbb{R})$, tangent to the trajectory φ of a Lorentzian particle $\langle m, \varphi \rangle$ at the point p is called (*Lorentzian*) *absolute velocity* of $\langle m, \varphi \rangle$ at p, and the vector

$$A_{\varphi_*}(p) = D_{\varphi_*}\varphi_*|_p \tag{117}$$

$p \in \varphi(\mathbb{R})$, is called (*Lorentzian*) *absolute acceleration* of $\langle m, \varphi \rangle$ *at* p.

Observation 4.35. We also define the (Lorentzian absolute) co-velocity of $\langle m, \varphi \rangle$ and the (*Lorentzian absolute*) *co-acceleration* of $\langle m, \varphi \rangle$ as the 1-form given, respectively, by

$$\varphi_p^* = \hat{G}_p \varphi_{*p} \tag{118}$$

and

$$A_\varphi^*(p) = D_{\varphi_{*p}}\varphi^*|_p \tag{119}$$

for each $p \in \varphi(\mathbb{R})$.

Definition 4.36. We call (*Lorentzian*) *momentum* of a Lorentzian particle $\langle m, \varphi \rangle$ at $p \in \varphi(\mathbb{R})$ the vector

$$\Pi_{\varphi p} = m\varphi_{*p} \tag{120}$$

and we call (*Lorentzian*) *co-momentum* (*or the momentum 1-form*) of $\langle m, \varphi \rangle$ at $p \in \varphi(\mathbb{R})$ the 1-form

$$\Pi^*_{\varphi p} = m\varphi^*_p \tag{121}$$

Definition 4.37. Note that the momentum and the co-momentum of $\langle m, \varphi \rangle$ satisfy, for each $p \in \varphi(\mathbb{R})$,

$$\Pi^*_{\varphi p}(\Pi_{\varphi p}) = \hat{g}_p(\Pi_{\varphi p}, \Pi_{\varphi p}) = \tilde{g}(\Pi^*_{\varphi p}, \Pi^*_{\varphi p}) = m^2 \tag{122}$$

4.3.2. Lorentz's Laws of Motion. The simplest invariant generalization of Newton's second law of motion is

$$D_\varphi(\Pi_\varphi) = k \tag{123}$$

where k is a four-force. Since $D_\varphi(\Pi_\varphi) = mD_\varphi(\varphi_*)$ is a Lorentzian spacelike vector and since $\tilde{g}(D_\varphi \varphi_*, \varphi_*) = 0$, it follows that $\tilde{g}(k, \varphi_*) = 0$ and therefore $k^0 = \vec{k} \cdot \vec{v}$, $k^0 = \Omega(k)$ and $\vec{k} \cdot \vec{v} = \tilde{h}(k, v)$. This in turn implies that some or all components of k are velocity dependent. For example, a constant Newtonian force $\vec{F}$ corresponds to a "Minkowski" force $\vec{k}(v) = \vec{F}/(1 - v^2)^{1/2}$, which is increasing with velocity with $k^0 = \vec{F} \cdot \vec{v}/(1 - v^2)^{1/2}$. Even, if $\vec{k}$ were made constant, k^0 would be velocity dependent and vice-versa.

Equation (123) will be shown to describe the true physical facts in the case of Maxwell–Lorentz electrodynamics (Sec. 5).

5. CLASSICAL ELECTRODYNAMICS OF MAXWELL AND LORENTZ

5.1. Introduction

We present now a formulation of the classical electrodynamics of Maxwell and Lorentz as a spacetime theory, i.e., we are going to present Maxwell equations and the Lorentz force law as intrinsic equations for geometrical objects on N. Our intrinsic formulation is based on the identification of the Lorentzian structure in N discussed in Sec. 4. It shows

(i) that there is a unique coupling between the electromagnetic field with the current, which is different from the *postulate* of the classical Lorentz force law when the latter is written in intrinsic form, and which is a function of the velocity v;

(ii) there is a particularity inherent in the mode of propagation of electromagnetic signals that permits, in principle, the detection of V for experiments done inside an inertial frame $\bar{I}$ if we insist that $\langle x^\mu \rangle$, the Galilean inertial coordinates associated to $\bar{I}$, is such that $x^0 = t$, e.g., the time registered by a set of clocks at rest in $\bar{I}$ and synchronized by an internal synchronization procedure (ISP) is the absolute time (an ISP is a synchronization procedure done inside I, without looking for the "exterior"—this concept will be made more precise below).

5.2. Maxwell–Lorentz Equations

The Maxwell–Lorentz equations are usually presented by

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho, \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \vec{J} \\ \vec{\nabla}\cdot\vec{B} &= 0, \qquad \vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \end{aligned} \tag{124}$$

where $\vec{E}$ is the electric field vector, $\vec{B}$ is the magnetic (induction) vector, $\vec{J}$ is the current density, $\vec{E}$, $\vec{B}$, $\vec{J}$: $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, ρ: $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is the charge density, $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$, and c is a constant with dimension of velocity, called electromagnetic velocity of light (we are going to use units such that the numerical value of c is one, i.e., $c = 1$).

The fields $\vec{E}$ and $\vec{B}$ act on charged particles [which besides their masses and their trajectories are also characterized by their charges, a parameter $q \in (-\infty, \infty)$]. The resulting force is known as the Lorentz force law and is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{125}$$

where $\vec{v}$ is the 3-velocity of the particle, $\vec{v} = v^\mu \partial/\partial x^\mu$ ($x^1 = x$, $x^2 = y$, $x^3 = z$).

We shall show that this coupling of the electric and magnetic fields to the charge and current of a particle, which is usually introduced as a postulate, is *not* compatible with the structure of Maxwell equations. This fact in turn implies that we cannot assume the validity of Newton's law of motion for a charged particle, i.e.,

$$\frac{d}{dt}(m\vec{v}) = q(E + \vec{v} \times \vec{B}) \tag{126}$$

Before we prove the above claims, let us recall that Maxwell–Lorentz equations have explicit solutions when ρ, $\vec{J}=0$. In this case, any one of the Cartesian components of $\vec{E}$ and $\vec{B}$ satisfies the wave equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{127}$$

where ∇^2 denotes the Laplacian in $\mathbb{R}^3$. Such an equation has a solution of the form

$$\phi = \phi_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \tag{128}$$

where $\omega k = c$, $k = |\vec{k}|$. This solution (the real part, of course) represents an electromagnetic wave propagating with velocity c. The following question cannot be avoided:

> With relation to which reference frame does an electromagnetic wave (light, for short) propagate with velocity c?

For Maxwell and his contemporaries the answer was obvious: Light propagates with velocity c with respect to its carrier, the *ether*. From the point of view of Lorentz and Poincaré, the ether would be the physical substance which would give material support to V. We should then understand the coordinate functions $\langle x^\mu \rangle$ used to write down the Maxwell–Lorentz equations as constituting a $\langle \text{nacs} \mid V \rangle$.

It is quite obvious that if $\langle x'^\mu \rangle$ is a $\langle \text{nacs} \mid \bar{I} \rangle$, where $\bar{I}$ is a Galilean inertial frame moving with 3-velocity $v(\partial/\partial x^1)$ relative to V, i.e.,

$$\bar{I} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x^1} \tag{129}$$

then

$$\begin{aligned} x'^1 &= x^1 - vt \\ x'^2 &= x^2 \\ x'^3 &= x^3 \\ t' &= t \end{aligned} \tag{130}$$

and the velocity of the light in $\bar{I}$, in the coordinates of $\langle x'^\mu \rangle$, would depend on v.

Then it would appear that the measurement of the light velocity in two different directions inside $\bar{I}$ would enable one to determine the components of V in the $\langle x'^\mu \rangle$ coordinates. We would, obviously, get

$$V = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'^1} \tag{131}$$

As proved in Refs. 2, 12, an internal synchronization procedure (ISP) in I' which gives $x'^0 = t$ is possible only if there exists in Nature a physical system which does not have its Lorentz deformed version (LDV). The concept of LDV of a physical system together with an example showing explicitly how this permits one to obtain an ISP in $\bar{I}$ giving $x'^0 = t$ is discussed in Ref. 12. We do not have space to present this issue here.

Now, in order to present Maxwell–Lorentz electrodynamics in intrinsic form over Newtonian structure $\mathbb{N}$, we need

Definition 5.1. Let $\langle x^\mu \rangle$, $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be a $\langle \text{nacs} \mid V \rangle$. The current 1-form field $J \in \sec \bigwedge (T^*N)$ is

$$J = \rho\, dx^0 - j_x\, dx^1 - j_y\, dx^2 - j_z\, dx^3 = J_\mu\, dx^\mu \tag{132}$$

with

$$J_0 = \rho, \qquad J_1 = -j_x, \qquad J_2 = -j_y, \qquad J_3 = -j_z \tag{133}$$

Definition 5.2. The electromagnetic 2-form field $F \in \sec \bigwedge^2(T^*N)$ is

$$F = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu\, dx^{nu} \tag{134}$$

with

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & B_x \\ E_z & -B_y & -B_x & 0 \end{pmatrix} \tag{135}$$

With the above definitions, the Maxwell equations read

$$\begin{aligned} dF &= 0 \\ \delta F &= -J \end{aligned} \tag{136}$$

where d is the exterior differentiation and δ is the Hodge co-derivative operator

$$\delta \omega_p = (-1)^p *^{-1} d * \omega_p \tag{137}$$

where $\omega_p \in \sec \bigwedge^p (T^*)$ and $*$ is the Hodge-star operator, defined through $\hat{g}$.

Let us suppose that a charged particle is modeled by

Definition 5.3. A charged Newtonian particle is a triple (σ_N, m, q), where $\sigma_N: I \to N$, $I \subset \mathbb{R}$, $t \to \sigma_N(t)$ (t being the absolute time function) is a timelike curve, $m \in (0, \infty)$ is the mass, and $q \in \mathbb{R}$ is the charge of the particle.

In this case σ_{N_*} is the Newtonian four-velocity of the particle and we have

$$\vec{v} = \sigma_{N_*} - \Omega(\sigma_{N_*}) V \tag{138}$$

Thus, for a point charge modeled according to Definition 5.3, the current density $J = J_\mu \, dx^\mu$ is such that

$$\hat{J} = \hat{g}(J, \cdot) = J^\mu(x) \partial/\partial x^\mu$$

for

$$J^\mu(x) = q \int dt \, \dot{x}^\mu(\sigma_N(t)) \, \delta(x - x^\mu(\sigma_N(t))) \tag{139}$$

Then, the Newtonian four-force $\mathscr{F}_N$ corresponding to $\vec{F}$ [see Eq. (125)] can be written intrinsically by

$$\mathscr{F}_N = q[\tilde{g}(\hat{J} \lrcorner F, \cdot) - \Omega(\hat{g}(\hat{J} \lrcorner F, \cdot) V] \tag{140}$$

Newton's law of motion for a charged particle [Eq. (126)] can then be written in intrinsic form as

$$mD_{\sigma_{N_*}} \sigma_{N_*} = \tilde{g}(\hat{J} \lrcorner F, \cdot) - \Omega(\tilde{g}(\hat{J} \lrcorner F, \cdot)) V \tag{141}$$

This equation suggests that the "effect" of V on an appropriate experiment on charged particles, done inside an inertial frame, might be experimentally detected, and this has indeed been what Trouton and Noble[13] tried in their experiment, which, as is well known, could not detect V, showing that something is wrong with the postulates of the classical Maxwell–Lorentz theory. Thanks to Einstein we know now the true law of motion for charged particles. Indeed, if $\sigma_E: s \to N$ is a *Lorentzian* timelike curve, $\hat{g}(\sigma_{E_*}, \sigma_{E_*}) = 1$, and taking into account that we can write Eq. (139) as

$$J^\mu(x) = q \int ds \frac{dx^\mu}{ds}(\sigma_E(s)) \, \delta(x - x^\mu(\sigma_E(s))) \tag{142}$$

the true law of motion is

$$mD_{\sigma_{E_*}} \sigma_{E_*} = \tilde{g}(\hat{J} \lrcorner F, \cdot) \tag{143}$$

Equation (143) written in the coodinates $\langle x^{\mu} \rangle$ naturally adapted to V reads

$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2}} = q(E + \vec{v} \times \vec{B}) \tag{144}$$

Equation (144) has only the first member different from the classical law of motion given by Eq. (141), when expressed in $\langle \text{nacs} \mid V \rangle$ [Eq. (126)].

Since the true law of motion for charged particles is Eq. (143), the true coupling of the electromagnetic field F to the current must be

$$\tilde{g}(\hat{J} \lrcorner F, \cdot) \tag{145}$$

Also the true momentum of the particle must be [Eq. (144)]

$$\Pi_{\sigma_E} = m\sigma_{E_*} \tag{146}$$

Equation (146) can be called an equation empirically discovered, whose justification is the Kaufmann[14] experiment. We now prove that the right coupling of F and J is indeed to one given by Eq. (145).

In order to do that, we take advantage of the fact that the existence of $\tilde{g}$ and $\hat{g}$ permits us to give the cotangent bundle of N the structure of a local Clifford algebra, thereby permitting us to introduce the Clifford bundle of differential forms $\mathrm{Cl}(N, \hat{g})$. Then, using the Clifford bundle formalism,[8] one can show that the Dirac operator $\partial = \gamma^{\mu} D_{e_{\mu}}$ (where $\gamma^{\mu} \in \sec \bigwedge^1 (T^*N) \subset \mathrm{Cl}(N, \hat{g})$ is the dual frame of $e_{\mu} \in \sec TN$, $\tilde{g}(e_{\mu}, e_{\nu}) = \eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}$) can be written

$$\partial = d - \delta \tag{147}$$

Then, since we can suppose that $J \in \sec \bigwedge^1 (T^*N) \subset \mathrm{Cl}(N, \hat{g})$, $F \in \sec \bigwedge^2 (T^*N) \subset \mathrm{Cl}(N, \hat{g})$, we can write the Maxwell equations [Eq. (136)] as a single equation, namely,

$$\partial F = J \tag{148}$$

Applying to Eq. (148) the anti-automorphism called reversion, indicated by $\tilde{\ }$, we get[15,16]

$$\tilde{F}\tilde{\partial} = J \tag{149}$$

with

$$\tilde{F}\tilde{\partial} = -\partial_{\alpha}(F_{\mu\nu})\, \gamma^{\mu}\gamma^{\nu}\gamma^{\alpha} \tag{150}$$

Multiplying Eq. (148) by $\tilde{F}$ on the left and Eq. (149) by F on the right and summing the resulting equations we get

$$\tfrac{1}{2}(\tilde{F}\,\partial F - \tilde{F}\tilde{\partial} F) = \tfrac{1}{2}(JF - FJ) \tag{151}$$

Define

$$S^{\mu} = -\tfrac{1}{2}\tilde{F}\gamma^{\mu}F \tag{152}$$

Then Eq. (151) can be written

$$\partial_{\mu} S^{\mu} = F \cdot J \tag{153}$$

where "$\cdot$" is the interior product in the Clifford algebra.[3]

We can verify[15,16] from Eq. (152) that the S^{μ} are 1-form fields, i.e., $S^{\mu} \in \sec \bigwedge^1 (T^*N) \subset \mathcal{C}\ell(N, \hat{g})$ and will be called energy-momentum 1-forms. The reason for this is that $E^{\mu\nu} = S^{\mu} \cdot \gamma^{\nu}$ are the components of the symmetric energy-momentum of the electromagnetic field. Indeed, if $\langle\ \rangle$ is the 0-form part of the Clifford product, we have

$$E^{\mu\nu} = S^{\mu} \cdot \gamma^{\nu} = -\langle \tfrac{1}{2} F\gamma^{\mu} F\gamma^{\nu} \rangle = F^{\mu\alpha} F^{\lambda\nu} \eta_{\alpha\lambda} + \tfrac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{154}$$

We note that due to the symmetry $E^{\mu\nu} = E^{\nu\mu}$ we can write

$$\partial_{\mu} E^{\mu\nu} = \partial_{\mu} S^{\nu} \cdot \gamma^{\mu} = \partial \cdot S^{\nu}$$

∂ being the Dirac operator. Then Eq. (153) can be written

$$\partial \cdot S^{\nu} = (F \cdot J) \cdot \gamma^{\nu} \tag{155}$$

or

$$\partial_{\nu} E^{\mu\nu} = F^{\mu}_{\ \nu} J^{\nu} \tag{156}$$

Equation (156) expresses very clearly the fact that the energy-momentum of the field is not conserved, $\partial_{\mu} E^{\mu\nu} \neq 0$, when matter, described by J is present. But, on the other hand, it implies that the right coupling between J and F is $-F \cdot J = J \cdot F = \hat{J} \lrcorner F$. Actually, one expects that only the total energy-momentum of fields and currents are conserved. If we write the second member of Eq. (156) as $-\partial_{\nu} M^{\mu\nu}$, then

$$\partial_{\nu}(E^{\mu\nu} + M^{\mu\nu}) = 0 \tag{157}$$

where $M^{\mu\nu}$ plays the role of the symmetric energy-momentum tensor of matter.

Now, suppose that a real charged particle must be described by

Definition 5.4. A real charged particle is a triple (σ_E, m, q), where $\sigma_E: s \to N$ is a Lorentzian timelike curve, i.e., $\hat{g}(\sigma_{E_*}, \sigma_{E_*}) = 1$, $m \in \mathbb{R}^+$ is the mass, and $q \in \mathbb{R}$ is the charge of the particle.

Then, $\hat{J} = J^\mu(x)\partial/\partial dx^\mu$ with $J^\mu(x)$ given by Eq. (142), and the most general symmetric tensor that we can write, representing matter, and which does not depend on q, is

$$M = M^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \tag{158}$$

with

$$M^{\mu\nu}(x) = m \int ds\, \delta(x - \sigma_E(s)) \frac{dx^\mu}{ds}(\sigma_E(s)) \frac{dx^\mu}{ds}(\sigma_E(s)) \tag{159}$$

Using the fact that, by definition, $\partial_\nu M^{\mu\nu} = -F^\mu{}_\nu J^\nu$, we get

$$D_{\sigma_{E_*}}(\Pi_{\sigma_E}) = \tilde{g}(\hat{J} \lrcorner F, \cdot) \tag{160}$$

In conclusion, the Maxwell equations imply that the right coupling of J with F is given by Eq. (145). This result, coupled with the most general definition of M [Eq. (158)], leads to the relativistic Lorentz-force equation of motion. This is a nontrivial result never derived within the realm of the Newtonian theory.

The electromagnetic experiments done by man are, in the first approximation, done in an inertial frame I moving with respect to V and for charged particles with small velocity relative to I. This means that for the coordinates $\langle \bar{x}^\mu \rangle$ naturally adapted to I an equation like Eq. (124), that is, the Maxwell equations, and an equation like Eq. (144) hold true. This necessarily implies, as is well known, that the $\langle x^\mu \rangle$, the $\langle \text{nacs} \mid V \rangle$, and $\langle \bar{x}^\mu \rangle$, the $\langle \text{nacs} \mid I \rangle$, are related by a Poincaré transformation, which is the isometry group of $\hat{g}$. This observation does not imply that we cannot use the Galilean coordinates $\langle x^{\mu'} \rangle$ given by Eq. (130). $\langle x^{\mu'} \rangle$ cannot be used as a $\langle \text{nacs} \mid I \rangle$. Obviously the time $t' = t$ will not be the time as given by clocks synchronized a la Einstein in I and $x^{i'}$ will not have the meaning of physical distances measured along the corresponding axis. Even, more, if these coordinates are used in I, the form of the Maxwell equations will not be the one given by Eq. (124), as can be easily verified.

6. NEWTON'S THEORY OF UNIVERSAL GRAVITATION AND THE EXISTENCE OF INERTIAL REFERENCE FRAMES

6.1. Gravitational Theory on Galilean Spacetime[3,4,6,7]

We saw in Sec. 2.2.3 that the Newtonian laws of motion of a material particle $\langle m, \gamma \rangle$ postulate that

$$mD_{\gamma_*}\gamma_* = f|_\gamma \tag{161}$$

We now assume that the spacetime is $\mathscr{G} = \langle N, D, \Omega, \hat{h} \rangle$, the Galileo's spacetime. In a coordinate system $\langle x^\mu \rangle$ naturally adapted to an inertial reference frame I, i.e., $DI = 0$, Eq. (161) reads

$$m \frac{d^2}{dt^2}(x^i \circ \gamma) = f^i|_\gamma = F^i \tag{162}$$

In Newton's gravitational theory,

$$\vec{F} = m\vec{G} \tag{163}$$

where

$$\vec{G} = -\text{grad}\, \phi \tag{164}$$

is the gravitational potential generated by a distribution of mass (excluding m) represented by the mass density ρ. We have

$$\nabla^2 \phi = \kappa \rho \tag{165}$$

where κ is the universal gravitational constant and ∇^2 is the Laplacian.

To formulate Newton's gravitational theory as a spacetime theory in intrinsic form, we observe that once ϕ is a function ϕ: $N \to \mathbb{R}$, we can write (using the definition of H_* given in Sec. 2.1.2)

$$\text{grad}\, \phi = H_*\, d\phi \tag{166}$$

$$\nabla^2 \phi = \text{Div}(\text{grad}\, \phi) \tag{167}$$

Then, if ϕ is the gravitational potential (in N), the gravitational force is written

$$f|_\gamma = -m\, \text{grad}\, \phi|_\gamma \tag{168}$$

Newton's gravitational theory over Galileo's spacetime has then as model the heptuple $\tau_{NG} = \langle N, D, \hat{h}, \phi, \rho, \{m, \gamma\} \rangle$, where

$$\begin{aligned} R[D] &= 0 \\ D\Omega &= 0 \\ \hat{h}(\Omega, \omega) &= 0, \qquad \forall \omega \in T^*N \end{aligned} \tag{169}$$

$$\begin{aligned} \nabla^2 \phi &= 0 \\ mD_{\gamma_*}\gamma_* &= -m \operatorname{grad} \phi \end{aligned} \tag{170}$$

It is quite clear that the invariance group of τ_{NG} is Galileo's group.

6.2. The Problem of the Physical Existence of Inertial Reference Frames

According to the Galilean spacetime structure $\mathscr{G}$, the existence of inertial reference frames $I \in \sec(TU)$, $U \subset N$, satisfying $DI = 0$ is warranted. The question is: Can I be materialized by a system of physical particles, e.g., a rigid body. To answer this question we recall that since our physical universe is filled with matter and since the gravitational potential ϕ satisfies Poisson's equation, it follows that all pieces of matter are always in gravitational interaction. Then the world line of the center of mass of each piece of matter is always accelerating according to D [cf. Eq. (161)]. It follows that the inertial frames in general cannot be materialized by any piece of real matter. This conclusion, although relevant, does not imply that inertial frames are not important for Newton's gravitational theory. Indeed, according to our view, reference frames are theoretical instruments that do not need to be physically materialized. This is, of course, the view of an astronomer who applies the Newtonian's gravitational theory. Another important point is that using inertial reference frames, an easy formulation of the conservation laws follows.

Suppose now that in $U \subset N$, $\phi|_U = constant$ and we have a reference frame $Z \in \sec(TU)$, $U \subset N$, such that

$$D_Z Z = A \tag{171}$$

where A is a constant vector field on $U \subset N$. Suppose that Z is materialized by a rigid body which is nonrotating according to D. Let σ be an integral line of Z, parameterized by t, the absolute time function. Then Eq. (171) restricted to σ reads

$$D_{\sigma_*}\sigma_* = A|_\sigma \tag{172}$$

Suppose that γ is an integral line of a material reference frame $L \in \sec(TU)$, $U \subset N$, such that $L|_\gamma = \gamma_*$ and

$$mD_L L = -m \operatorname{grad} \phi \tag{173}$$

Suppose also that grad ϕ is approximately constant on U and that in first order we have grad $\phi|_U = A|_U$, where A is given by Eq. (171). Then, how can observer in U know that he is sitting on the integral line σ being accelerated or on the integral line γ in free fall? Obviously, an answer implies second-order effects, which do not exist in Z but exist in L (tidal forces). These are the arguments for the so-called equivalence principle which we do not discuss here. Instead, we now present a new formulation of Newton's gravitational theory supposing that real observers do not take care of separating inertial from gravitational accelerations.

6.3. Curved Spacetime Formulation of Newton's Gravitational Theory[3–7]

Let $\langle x^\mu \rangle$, $x^0 = t$, be a $\langle$nacs $| L\rangle$, where L satisfies Eq. (173). Then in the coordinates $\langle x^\mu \rangle$ Eq. (173) reads, for γ an integral curve of L, $L|_\gamma = \gamma_*$, parameterized by the absolute time t,

$$\frac{d^2}{dt^2}(x^\rho \circ \gamma) + \Gamma^\rho_{\mu\nu} \frac{d(x^\mu \circ \gamma)}{dt} \frac{d(x^\nu \circ \gamma}{dt} = h^{\rho\nu} \phi_{;\nu}|_\gamma \tag{174}$$

If we introduce on N a new connection ∇ such that

$$\nabla_{\partial/\partial x^\mu} \partial/\partial x^\nu = (\Gamma^\rho_{\mu\nu} + h^{\rho\sigma} \phi_{;\sigma} t_\mu t_\nu) \partial/\partial x^\rho \tag{175}$$

where $\Omega = t_\mu \, dx^\mu$, then Eq. (174) reads

$$\nabla_{\gamma_*} \gamma_* = 0 \tag{176}$$

Then the free fall is characterized by a geodetic equation relative to the connection ∇.

According to this connection we can define a local inertial frame $E \in \sec TU$, $U \subset N$, by

$$\nabla_E E = 0 \tag{177}$$

and such that E is nonrotating relative to ∇ (cf. Sec. 3).

Obviously ∇ is not a flat connection, i.e., $R(\nabla) \neq 0$. Indeed, the components of R, the Riemann tensor in $\langle x^\mu \rangle$, the (nacs $| L\rangle$, are

$$R^\mu_{\nu\rho\sigma} = 2 t_\nu h^{\mu\delta} \phi_{;\delta;[\rho} t_{\sigma]} \tag{178}$$

where the brackets in Eq. (178) mean antisymmetrization with respct to the indices ρ and σ. We can verify that $\nabla(\Omega)=0$, but $\nabla(\hat{h})\neq 0$. This last result is at variance with the one found in Ref. 5, but since we already found $D\hat{g}=0$ and $T[D]=0$ if $\nabla\hat{h}=0$, we could have $\nabla\hat{g}=0$ and $T[\nabla]=0$, which is obviously impossible. Equation (178) implies that

$$t_{[\alpha}R^{\mu}_{\mu]\rho\sigma}=0 \quad \text{and} \quad h^{\alpha\rho}R^{\mu}_{\nu\rho\sigma}=-h^{\mu\rho}R^{\alpha}_{\sigma\rho\nu} \tag{179}$$

Equations (179) imply the existence of a scalar field $\phi: N\to\mathbb{R}$ and a nonflat connection ∇ such that its coefficient in the basis $\partial/\partial x^{\mu}$ is given by Eq. (175).

If we suppose then that the true connection in a space generated by a gravitational field is ∇, then Eqs. (179) must be postulated as field equations if we want to produce a theory equivalent to the original one described in Sec. 6.1.

Recalling that the components of the Ricci tensor are

$$R_{\mu\nu}=R^{\rho}_{\mu\nu\rho}=-h^{\rho\alpha}\phi_{:\rho;\alpha}t_{\mu}t_{\nu} \tag{180}$$

a comparison with Poison's equation $\nabla^2\phi=k\rho$ yields

$$R_{\mu\nu}=-k\rho t_{\mu}t_{\nu} \tag{181}$$

We then arrive at a formulation of Newton's gravitational theory where the spacetime is curved due to the presence of matter. The equations of this theory can be written in intrinsic form as

$$(\Omega\otimes R)_{\text{Antis}}=0$$

$$H_{\mp}R(\omega,\alpha,X,Y)=H_{\mp}R(\alpha,\omega,X,Y),\ \alpha,\omega\in T^*N,\ X,Y\in TN \tag{182}$$

$$\nabla(\Omega)=0,\qquad \nabla(\hat{h})\neq 0,\qquad \text{Ricci}=-k\,dt\otimes dt \tag{183}$$

In Eq. (182) $(\ )_{\text{Antis}}$ means antisymmetrization with respect to the first and second indices of the tensor $\Omega\otimes R$ and

$$H_{\mp}: R(\omega,\alpha,X,Y)\to R(\omega,H_*\alpha,X,Y)$$

where H_* has been defined in Sec. 2.12.

Finally, the equation of motion of a particle (m,σ) subject only to the gravitational field is

$$\nabla_{\sigma_*}\sigma_*=0 \tag{184}$$

To conclude we can ask: What do we learn from the curved spacetime formulation of gravitational theory? The answer is that this approach suggests to us to formulate Einstein's gravitational theory as a field theory in the sense of Faraday formulated on a Minkowski spacetime. This has been done in Ref. 9 where the motivations for the enterprise are described in detail.

7. CONCLUSIONS

In this paper we presented a rigorous study of the structure of the Newtonian spacetime and its classical dynamics and gravitation. Sure, there are many studies similar to ours (e.g., Refs. 3–7), but ours is more rigorous and fresh in many aspects. In particular, we quote: (i) our study of the reference frames in the Newtonian spacetime; (ii) the identification of the Lorentzian structure of the Newtonian spacetime; (iii) the determination of the correct coupling of the electromagnetic field with the current of a charged particle compatible with the Maxwell equations (we found that the field V, the absolute reference frame of the Newtonian spacetime, does not appear in the invariant generalization of the right Lorentz force, a result that, as is well known, agrees with experience); (iv) we studied also the formulation of Newton's gravitational theory as a curved spacetime theory, and our results correct (as discussed in Sec. 6.2 and 6.3) some misconceptions found in the literature.

As we have said in the Introduction, this is the first of a series of papers we are proposing about the mathematical structure of spacetime theories. There remain many important points concerning the relation of Newtonian spacetime with other spacetime theories.

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