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The Noether Approach to Pokhozhaev's Identities

Yuri Bozhkov and Enzo Mitidieri

Abstract. We propose a general method to generate Pokhozhaev identities and apply this approach to various nonlinear differential equations and systems.

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1. Introduction

The celebrated Pokhozhaev's identity [14],[15] is an important tool in the theory of differential equations. Among a variety of its applications, it is particularly useful in establishing nonexistence results. Commonly its specific form for each concrete problem is obtained by using ad hoc procedures, e.g. multiplying the considered equations by some appropriate functions, integrating by parts and then summing up the results.

In this paper we look at the Pokhozhaev's identity from a more general point of view. Our main purpose is to propose a unified method to generate identities of this type. The suggested algorithm consists of three steps. The first step is to obtain an identity for arbitrary sufficiently smooth functions, without using the fact that some of them satisfy the given differential equations or systems. The second step is to integrate the identity in question taking into account the corresponding equations and boundary conditions. The third step is to apply the divergence theorem.

Obviously the essential point is how to obtain a 'generic' starting identity to be used in the first step. This can be done in the following way.

To begin with, let $u^\alpha(x)$, $\alpha = 1, 2, \dots, m$, be a set of $C^k(\Omega)$ functions, where $k \geq 1$ and $x \in \Omega \subseteq \mathbb{R}^n$, $n \geq 1$. We denote by A_k the space of all locally analytic

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functions of x, u^α and the partial derivatives of u^α up to order k. The elements $f(x, u^\alpha, u^\alpha_{(1)}, \ldots, u^\alpha_{(k)})$ of A_k are called differential functions [1],[13].

Given n differential functions $\xi^i \in A_k$ and m differential functions $\eta^\alpha \in A_k$, we associate to them a partial differential operator X in the following way:

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}.$$

Above and throughout this paper we use the Einstein summation convention, that is, we assume summation from 1 to n over a repeated Latin index and summation from 1 to m over repeated Greek indices.

The cornerstone of the proposed approach to Pokhozhaev identity is the following

Theorem 1.1. *Let $k \geq 1$, $u^\alpha(x) \in C^k(\Omega)$ and $\xi^i, \eta^\alpha \in A_k$. Let*

$$L = L(x, u^\alpha, u^\alpha_{(1)}, \ldots, u^\alpha_{(k)}) \in A_k$$

be an arbitrary differential function. Then the following identity holds

$$X^{(k)}L + LD_i\xi^i = E_\alpha(L)(\eta^\alpha - u^\alpha_j\xi^j) + D_i[L\xi^i + W_i[u, \eta - u_j\xi^j]], \tag{1.1}$$

where $u = (u^1, \ldots, u^m)$, $\eta = (\eta^1, \ldots, \eta^m)$, $u_i = (\frac{\partial u^1}{\partial x_i}, \ldots, \frac{\partial u^m}{\partial x_i})$ and the k-th order extension $X^{(k)}$ of X, the Euler operator E_α and the operator W_i are defined in Section 2.

This identity is called the *Noether identity.* We emphasize that the identity (1.1) is obtained here without any use of variational structure.

It is now clear from (1.1) why the usual 'ad hoc' procedure for obtaining Pokhozhaev identities, described briefly in the beginning, works - simply for each specific problem one reproves, in practice, the Noether identity, which is valid in a general context. In particular, the 'appropriate' multipliers of the Euler-Lagrange equations $E_\alpha(L) = 0$ are nothing but the Lie characteristic functions $v^\alpha = \eta^\alpha - u^\alpha_j\xi^j$ [1],[13] of the conservation laws obtained via the Noether's theorem [12], whose conclusion, in fact, follows easily from (1.1).

Our further comments on the fundamental identity (1.1) will be facilitated if we denote

$$A = X^{(k)}L + LD_i\xi^i, \quad E = E_\alpha(L)(\eta^\alpha - u^\alpha_j\xi^j),$$
$$N_i = L\xi^i + W_i[u, \eta - u_j\xi^j], \quad N = (N_1, \ldots, N_n),$$
$$C = E_\alpha(L)(\eta^\alpha - u^\alpha_j\xi^j) + D_i[L\xi^i + W_i[u, \eta - u_j\xi^j]].$$

Obviously in this notation (1.1) reads:

$$A = C = E + \mathrm{Div}(N). \tag{1.2}$$

Now we make the following observations:

(i) If ξ^i, η^α were the components of a variational generalized (Lie-Bäcklund) symmetry [1],[13] of the Euler-Lagrange equation $E = 0$, then $A = 0$ and hence $\mathrm{Div}(N) = 0$ which is the conclusion of the Noether's theorem [12].

(ii) If X were a divergence generalized (Lie-Bäcklund) symmetry [1],[13] of $E = 0$, there exists B such that $\mathrm{Div}(B) = A$. Then by (1.2)

$$\mathrm{Div}(B) = E + \mathrm{Div}(N) = \mathrm{Div}(N)$$

which implies $\mathrm{Div}[N - B] = 0$. Again we obtain the conservation law guaranteed by the Noether's theorem [12].

(iii) Usually one first gets $A = 0$ as a necessary condition for the 'absolute invariance' of the Euler functional [10]. Then $0 = E + \mathrm{Div}(N)$ is a consequence of the absolute invariance, which, together with $E = 0$, imply $\mathrm{Div}(N) = 0$. That is, $A = 0$ and $C = 0$ are commonly obtained by some variational arguments, while the equality $A = C$ holds always for solutions of $E = 0$ and thus $A = 0$ immediately implies $C = 0$.

(iv) In fact, one writes the invariance condition for the Euler functional and obtains a relation between the expressions for the Lagrange Function in two coordinate systems. This relation is given in the terms of the corresponding Jacobian. Then expanding the relation in question in ε (the group parameter) and taking only the $O(\varepsilon)$ terms one gets $A = 0$. See, for instance, [1, pages 274-275], or [13, page 278].

(v) The identity (1.1) is obtained without any reference and use of invariance, symmetries, etc. It holds for any $u, \xi_i(x, u, u_i, \dots), \eta(x, u, u_i, \dots)$ and $L(x, u, u_i, \dots)$ - sufficiently smooth functions of their arguments.

(vi) Substituting $\xi^i = h_i(x)$ and $\eta^\alpha = -a(x)u^\alpha$ into (1.1) one obtains the identity which leads, by integration, to the variant of the Pokhozhaev's identity discussed in [16]. See page 683 of [16] for a comment on Noether approach to variational identities.

So far, the first step in obtaining a Pokhozhaev type identity for a given problem is reduced to choosing the operator X which appears in the Noether identity. Different choices of X will lead to different Pokhozhaev identities. A choice closest to the original Pokhozhaev's idea is suggested by the following observation of Schoen and Yau: 'One can use conformal vector fields to derive certain identities for some special differential equations. Such a fact was first discovered by S. I. Pokhozhaev [14], who made use of $X = r\frac{\partial}{\partial r}$ on R^n.'([20, page 196],). This X is the infinitesimal generator of a dilation of the independent variable x, namely $x_j^* = \lambda x_j$, where λ is a parameter. In this way, for each problem we shall commonly use a dilation X, extended to the dependent variable(s). The parameters of such scaling transformations will be chosen to assume critical values. The latter term is related to the notion of critical exponents which are found as critical values for embedding theorems. The critical exponents can be also viewed as numbers which divide the existence and nonexistence cases for various differential equations and systems. The notion of criticality of differential equations, its relations to scaling transformations and to the present approach is discussed in another work [2].

The practical aspects of the proposed method to generate Pokhozhaev Identities can be summarized as follows:

1. Take as X a dilation (in x and u) whose coefficients assume critical values.

2. Calculate the k-th order extension $X^{(k)}$ of X using the formulae in Section 2. ($2k$ is the order of the considered variational problem.)
3. Find the corresponding Function of Lagrange $L \in A_k$.
4. Calculate and simplify the expression $A = X^{(k)}L + LD_i\xi^i$.
5. Integrate $A = \mathrm{Div}(N)$. (See (1.2) and recall that the Euler-Lagrange equation reads $E = 0$.)
6. Apply the divergence theorem.

We point out that the approach permits to use other conformal vector fields X (e.g. inversions) and general ξ^i and η^α which may depend on the derivatives of u^α. Such applications will be treated elsewhere.

In this paper we apply the devised method to a number of nonlinear differential equations and systems. In order to convince the reader that it works properly, we first obtain some well-known Pokhozhaev identities established by Pokhozhaev [14] for the Poisson equation, by Pucci and Serrin [16] for potential systems, by Mitidieri [11] for elliptic Hamiltonian systems, by Giga and Kohn [8] for a semilinear equation with power nonlinearity and by Clément and van der Vorst [5] for unbounded Hamiltonian systems. Then we establish new Pokhozhaev identities corresponding to: potential systems, mixed Hamiltonian-potential systems, unbounded Hamiltonian systems and hyperbolic Hamiltonian systems both involving polyharmonic operators. The obtained Pokhozhaev identities are the starting point of the next step in this research, namely the corresponding nonexistence results, which will also be treated elsewhere. For other aspects and applications of the Pokhozhaev's identity the interested reader is directed to [6],[7],[17],[18],[19],[22], [23].

This paper is organized as follows. In Section 2 we introduce notation and define the basic notions and operators. In Section 3 we prove the fundamental identity (1.1). In Section 4 we present the applications described above.

2. Preliminaries

In this section we introduce notations and present some formulae which we shall use in the next sections.

We shall suppose that all considered functions, vector fields, tensors, functionals, etc. are sufficiently smooth in order that the derivatives we write to exist. When we say that a function is an arbitrary function we mean any sufficiently smooth function of its arguments.

The independent variable $x \in \Omega \subseteq \mathbb{R}^n$ – a bounded or unbounded domain. The partial derivatives of a smooth function $v = v(x)$ are denoted by subscripts:

$$v_i := \frac{\partial v}{\partial x_i} \quad v_{ij} := \frac{\partial^2 v}{\partial x_i \partial x_j},$$

etc. We shall also assume summation over a repeated index. The Latin indices vary from 1 to n, while the Greek ones – from 1 to m. The latter will denote collections of functions, e.g. $v^\alpha(x)$.

We introduce the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots + u_{ii_1i_2\ldots i_l}^\alpha \frac{\partial}{\partial u_{i_1i_2\ldots i_l}^\alpha} + \ldots$$

where $u^\alpha(x)$ are given functions. (See [1],[13].) If $v \in A_l$, where the space A_l was defined in the introduction, then

$$D_i v = \frac{\partial v}{\partial x_i} + u_i^\alpha \frac{\partial v}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial v}{\partial u_j^\alpha} + \cdots + u_{ii_1i_2\ldots i_l}^\alpha \frac{\partial v}{\partial u_{i_1i_2\ldots i_l}^\alpha}.$$

The Euler-Lagrange equations, corresponding to a functional

$$J[u] = \int_\Omega L(x, u^\alpha, u_{(k)}^\alpha)\mathrm{d}x,$$

where $L = L(x, u^\alpha, u_{(k)}^\alpha) \in A_k$, are given by

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i \frac{\partial L}{\partial u_i^\alpha} + D_i D_j \frac{\partial L}{\partial u_{ij}^\alpha} + \cdots + (-1)^k D_{i_1} D_{i_2} \ldots D_{i_k} \frac{\partial L}{\partial u_{i_1i_2\ldots i_k}^\alpha} = 0,$$

where the operator

$$E_\alpha = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha} + D_i D_j \frac{\partial}{\partial u_{ij}^\alpha} + \cdots + (-1)^k D_{i_1} D_{i_2} \ldots D_{i_k} \frac{\partial}{\partial u_{i_1i_2\ldots i_k}^\alpha} + \ldots \quad (2.1)$$

is the Euler operator. See [1],[13].

Further, let $\xi^i, \eta^\alpha \in A_k$ and consider the differential operator

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}.$$

The functions ξ^i and η^α are called infinitesimals of the Lie point transformation gene-rated by X, that is the transformation

$$x_j^* = x_j^*(x, u, \varepsilon), \quad u^{*\alpha} = u^{*\alpha}(x, u, \varepsilon),$$

where ε is a parameter and

$$\xi^i = \left.\frac{\partial x_i^*}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta^\alpha = \left.\frac{\partial \eta^{*\alpha}}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

We associate to X its k-th order extension $X^{(k)}$ given by

$$X^{(k)} = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha} + \cdots + \eta_{i_1i_2\ldots i_k}^{(k)\alpha} \frac{\partial}{\partial u_{i_1i_2\ldots i_k}^\alpha}, \quad (2.2)$$

where

$$\eta_i^{(1)\alpha} = D_i \eta^\alpha - (D_i \xi^j) u_j^\alpha, \ \ i = 1, 2, \ldots, n;$$

$$\eta_{i_1i_2\ldots i_l}^{(l)\alpha} = D_{i_l} \eta_{i_1i_2\ldots i_{l-1}}^{(l-1)\alpha} - (D_{i_l} \xi^j) u_{i_1i_2\ldots i_{l-1}j}^\alpha,$$

with $i_l = 1, 2, \dots, n$ for $l = 2, 3, \dots, k$, $k = 2, 3, \dots$. See [1],[13] for further details. The functions $\eta^{(l)\alpha}_{i_1 i_2 \dots i_l}$ are called extended infinitesimals. They can also be determined by the general prolongation formula

$$\eta^{(l)\alpha}_{i_1 i_2 \dots i_l} = D_{i_1} D_{i_2} \dots D_{i_l}(\eta^\alpha - \xi^i u^\alpha_i) + \xi^i u^\alpha_{i_1 i_2 \dots i_k i}. \tag{2.3}$$

([13, page 113].)

For any smooth u^α, v^α and $L = L(x, u^\alpha, u^\alpha_{(k)}) \in A_k$ we define:

$$\begin{aligned} W_j[u,v] = {} & v^\alpha \left[\frac{\partial L}{\partial u^\alpha_j} + \dots (-1)^{k-1} D_{i_1} D_{i_2} \dots D_{i_{k-1}} \frac{\partial L}{\partial u^\alpha_{j i_1 i_2 \dots i_{k-1}}} \right] \\ & + (D_{i_1} v^\alpha) \left[\frac{\partial L}{\partial u^\alpha_{i_1 j}} + \dots + (-1)^{k-2} D_{i_2} D_{i_3} \dots D_{i_{k-1}} \frac{\partial L}{\partial u^\alpha_{i_1 j i_2 \dots i_{k-1}}} \right] \\ & + \dots + (D_{i_1} D_{i_2} \dots D_{i_{k-1}} v^\alpha) \frac{\partial L}{\partial u^\alpha_{i_1 i_2 \dots i_{k-1} j}}, \end{aligned} \tag{2.4}$$

(see [1, pages 254–255]). In particular, if $m = k = 1$, then, by (2.4), $W_i[u,v] = v \frac{\partial L}{\partial u_i}$.

The introduced notions play an important role in the calculus of variations, in particular in the Noether's theorem [12] on conservation laws. We point out, however, that these objects are defined for arbitrary smooth functions ξ^i, η^α and L which, in general, need not be concerned with any variational setting. This fact is manifested by the Noether identity which makes the proof of Noether's theorem 'purely algebraic'.

3. The Noether Identity

In this section we prove the Noether identity (1.1). We shall use induction on k – the number of the derivatives of u^α which appear in the function $L \in A_k$.

(i) Let $k = 1$, $m = 1$, $u^1 = u$ and $L = L(x, u, \nabla u)$ - arbitrary function of x, u and the first derivatives of u.

Consider the differential operator

$$X = \xi^i(x,u) \frac{\partial}{\partial x_i} + \eta(x,u) \frac{\partial}{\partial u},$$

where ξ^i and η are *arbitrary* functions of x and u. The first order extension of X is given by

$$X^{(1)} = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + (D_i \eta - u_j D_i \xi^j) \frac{\partial}{\partial u_i}.$$

See Section 2. Then

$$X^{(1)} L + L D_i \xi^i = \xi^i \frac{\partial L}{\partial x_i} + \eta \frac{\partial L}{\partial u} + D_i \eta \frac{\partial L}{\partial u_i} - u_j D_i \xi^j \frac{\partial L}{\partial u_i} + L D_i \xi^i. \tag{3.1}$$

On the other hand, let

$$E(L) = \frac{\partial L}{\partial u} - D_l \frac{\partial L}{\partial u_l}$$

be the Euler operator. We calculate

$$
\begin{aligned}
&E(L)(\eta - u_j\xi^j) + D_i[L\xi^i + \frac{\partial L}{\partial u_i}(\eta - u_j\xi^j)] \\
&\quad = \eta\frac{\partial L}{\partial u} - \eta D_l\frac{\partial L}{\partial u_l} - u_j\xi_j\frac{\partial L}{\partial u} + u_j\xi^j D_l\frac{\partial L}{\partial u_l} \\
&\qquad + LD_i\xi^i + \xi^i D_i L + (D_i\frac{\partial L}{\partial u_i})(\eta - u_j\xi^j) + \frac{\partial L}{\partial u_i}D_i\eta - \frac{\partial L}{\partial u_i}D_i(u_j\xi^j) \\
&\quad = \eta\frac{\partial L}{\partial u} - \eta D_l\frac{\partial L}{\partial u_l} - u_j\xi_j\frac{\partial L}{\partial u} + u_j\xi^j D_l\frac{\partial L}{\partial u_l} + LD_i\xi^i \\
&\qquad + \xi^i\frac{\partial L}{\partial x_i} + \xi^i u_i\frac{\partial L}{\partial u} + \xi^i u_{il}\frac{\partial L}{\partial u_l} + \eta D_i\frac{\partial L}{\partial u_i} \\
&\qquad - u_j\xi^j D_i\frac{\partial L}{\partial u_i} + \frac{\partial L}{\partial u_i}D_i\eta - \frac{\partial L}{\partial u_i}\xi^j u_{ij} - \frac{\partial L}{\partial u_i}u_j D_i\xi^j,
\end{aligned}
$$

that is,

$$
\begin{aligned}
E(L)(\eta - u_j\xi^j) + D_i[L\xi^i + \frac{\partial L}{\partial u_i}(\eta - u_j\xi_j)] = \eta\frac{\partial L}{\partial u} + LD_i\xi^i \\
+ \xi^i\frac{\partial L}{\partial x_i} + \frac{\partial L}{\partial u_i}D_i\eta - \frac{\partial L}{\partial u_i}u_j D_i\xi^j.
\end{aligned}
\tag{3.2}
$$

Then, (3.1) and (3.2) imply:

$$
X^{(1)}L + LD_i\xi^i = E(L)(\eta - u_j\xi^j) + D_i[L\xi^i + \frac{\partial L}{\partial u_i}(\eta - u_j\xi_j)]. \tag{3.3}
$$

The identity (3.3) can be easily generalized for an arbitrary number m of functions $u^\alpha, \alpha = 1, 2, \ldots, m$, and arbitrary $\xi^i(x, u^\alpha), i = 1, 2, \ldots, n$, $\eta^\alpha(x, u^\alpha)$ and L depending on x, u^α and the first derivatives of u^α - just repeat the preceding calculation with u^α in the place of u. In this way the identity (3.3) assumes the form:

$$
X^{(1)}L + LD_i\xi^i = E_\alpha(L)(\eta^\alpha - u_j^\alpha\xi^j) + D_i[L\xi^i + \frac{\partial L}{\partial u_i^\alpha}(\eta^\alpha - u_j^\alpha\xi^j)] \tag{3.4}
$$

which is (1.1) for $k = 1$.

(ii) Now we suppose that (1.1) holds for any $L = L^{(k)} \in A_k$:

$$
X^{(k)}L^{(k)} + L^{(k)}D_i\xi^i = E_\alpha^{(k)}(L^{(k)})(\eta^\alpha - u_j^\alpha\xi^j) + D_i[L^{(k)}\xi^i + W_i^{(k)}[u, \eta - u_j\xi^j]]. \tag{3.5}
$$

We have to prove that (1.1) holds for $L = L^{(k+1)} \in A_{k+1}$.

By Definitions (2.1), (2.2) and (2.4) one can obtain the following recurent relations:

$$
X^{(k+1)} = X^{(k)} + \eta_{i_1 i_2 \ldots i_{k+1}}^{(k+1)\alpha}\frac{\partial}{\partial u_{i_1 i_2 \ldots i_{k+1}}^\alpha}, \tag{3.6}
$$

$$\begin{aligned} W_i^{(k+1)}[u,v] &= W_i^{(k)}[u,v] + (-1)^k v^\alpha D_{i_1} \dots D_{i_k} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{ii_1\dots i_k}} \\ &\quad + (-1)^{(k-1)} (D_{i_1} v^\alpha) D_{i_2} \dots D_{i_k} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 i i_2 \dots i_k}} + \dots \\ &\quad - (D_{i_1} \dots D_{i_{k-1}} v^\alpha) D_{i_k} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_{k-1} i i_k}} \\ &\quad + (D_{i_1} \dots D_{i_k} v^\alpha) \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_k i}} \end{aligned} \tag{3.7}$$

and

$$E_\alpha^{(k+1)}(L^{(k+1)}) = E_\alpha^{(k)}(L^{(k)}) + (-1)^{k+1} D_{i_1} \dots D_{i_k} D_{i_{k+1}} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_k i_{k+1}}}. \tag{3.8}$$

In (3.5) and (3.7) the notation $W^{(l)}$ means that in the formula (2.4) for W_l the function $L = L^{(l)} \in A_l$. Similarly, in the left-hand side of (3.8) the Euler operator is applied to $L = L^{(k+1)} \in A_{k+1}$, while in the right-hand side of (3.8) the Euler operator is applied to the same $L \in A_{k+1}$ but viewed as a function $L^{(k)} \in A_k$, the $(k+1)$-th derivatives of u^α being considered parameters.

Then by (2.3), (3.5) and (3.6) we have

$$\begin{aligned} X^{(k+1)} L^{(k+1)} + L^{(k+1)} D_i \xi^i &= X^{(k)} L^{(k)} + L^{(k)} D_i \xi^i + \eta^{(k+1)\alpha}_{i_1 i_2 \dots i_{k+1}} \tfrac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 i_2 \dots i_{k+1}}} \\ &= E_\alpha^{(k)}(L^{(k)}) v^\alpha + D_i [L^{(k)} \xi^i + W_i^{(k)}[u,v]] \\ &\quad + (D_{i_1} \dots D_{i_k} D_{i_{k+1}} v^\alpha) \tfrac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_k i_{k+1}}} + \xi^i u^\alpha_{i i_1 \dots i_k i_{k+1}} \tfrac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_k i_{k+1}}} \end{aligned} \tag{3.9}$$

where $v^\alpha = \eta^\alpha - u^\alpha_j \xi^j$.

On the other hand

$$\begin{aligned} E_\alpha^{(k+1)}(L^{(k+1)}) v^\alpha + D_i [L^{(k+1)} \xi^i + W_i^{(k+1)}[u,v]] &= E_\alpha^{(k)}(L^{(k)}) v^\alpha \\ &\quad + (-1)^{k+1} v^\alpha D_{i_1} \dots D_{i_k} D_{i_{k+1}} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_k i_{k+1}}} + D_i [L^{(k+1)} \xi^i] \\ &\quad + D_i W_i^{(k)}[u,v] + D_i \Big\{ (-1)^k v^\alpha D_{i_1} \dots D_{i_k} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{ii_1 \dots i_k}} \\ &\quad + (-1)^{(k-1)} (D_{i_1} v^\alpha) D_{i_2} \dots D_{i_k} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 i i_2 \dots i_k}} + \dots \\ &\quad - (D_{i_1} \dots D_{i_{k-1}} v^\alpha) D_{i_k} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_{k-1} i i_k}} + (D_{i_1} \dots D_{i_k} v^\alpha) \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1 \dots i_k i}} \Big\} \end{aligned} \tag{3.10}$$

by (3.7) and (3.8). Performing the differentiation of the expression in braces in (3.10), most of the resulting terms cancel. Then from (3.10) and using

$$D_i[L^{(k+1)}\xi^i] = D_i[L^{(k)}\xi^i] + \xi^i u^\alpha_{ii_1\dots i_k i_{k+1}} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1\dots i_k i_{k+1}}}$$

we obtain

$$E^{(k+1)}_\alpha(L^{(k+1)})v^\alpha + D_i[L^{(k+1)}\xi^i + W^{(k+1)}_i[u,v]] = E^{(k)}_\alpha(L^{(k)})v^\alpha + D_i[L^{(k)}\xi^i + W^{(k)}_i[u,v]] + D_{i_1}\dots D_{i_k} D_{i_{k+1}} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1\dots i_k i_{k+1}}} + \xi^i u^\alpha_{ii_1\dots i_k i_{k+1}} \frac{\partial L^{(k+1)}}{\partial u^\alpha_{i_1\dots i_k i_{k+1}}}. \tag{3.11}$$

From (3.9) and (3.11) it follows that

$$X^{(k+1)}L^{(k+1)} + L^{(k+1)}D_i\xi^i = E^{(k+1)}_\alpha(L^{(k+1)})v^\alpha + D_i[L^{(k+1)}\xi^i + W^{(k+1)}_i[u,v]],$$

which completes the proof.

4. Applications

In this section we present several examples of nonlinear differential equations and systems for which we establish the corresponding Pokhozhaev identity. The calculations of the extended infinitesimals determining the prolongations of the considered operators X are omitted since this is a straightforward substitution of the coefficients of X into the formulae stated in Section 2. Furthermore, for the same reason, only the final simplified form of $X^{(k)}L + LD_i\xi^i$ is presented. We note that the group parameter λ which we shall use in this section and the parameter ε in Section 2 are related by $\lambda = \exp[\varepsilon]$. Hence $\varepsilon = 0$ corresponds to $\lambda = 1$.

4.1. Poisson equations

For the sake of completeness we shall obtain here the 1965 Pokhozhaev's identity [14] for the equation

$$\Delta u + f(u) = 0 \tag{4.1}$$

in a bounded domain $\Omega \subset R^n$, $n \geq 3$, with homogeneous Dirichlet condition

$$u = 0 \tag{4.2}$$

on $\partial\Omega$. The corresponding Function of Lagrange is

$$L = \frac{1}{2}u_j^2 - F(u), \quad F(u) = \int_0^u f(z)\mathrm{d}z.$$

We consider the dilation

$$X = x_i\frac{\partial}{\partial x_i} + \frac{2-n}{2}u\frac{\partial}{\partial u}.$$

Then

$$\xi^i = x_i, \quad \eta = \frac{2-n}{2}u.$$

The first order extension of X is given by

$$X^{(1)} = x_i \frac{\partial}{\partial x_i} + \frac{2-n}{2} u \frac{\partial}{\partial u} - \frac{n}{2} u_i \frac{\partial}{\partial u_i}.$$

By a straightforward calculation

$$X^{(1)} L + L D_i \xi^i = \frac{n-2}{2} u f(u) - nF(u). \tag{4.3}$$

Then by (3.3), (4.2), (4.3) and the divergence theorem, we obtain easily

$$\int_\Omega [\frac{n-2}{2} u f(u) - nF(u)] \mathrm{d}x = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x, \nu) \mathrm{d}s,$$

where ν is the outward unit normal to $\partial\Omega$.

Before concluding this subsection we would like to comment on the choice of the operator X.

We note that the one parameter Lie group of point transformations generated by

$$Y = x_i \frac{\partial}{\partial x_i} + \frac{2}{1-p} u \frac{\partial}{\partial u}$$

is a symmetry of the equation

$$\Delta u + u^p = 0,$$

as it can be easily verified. Moreover, by a direct calculation one can show that it is a variational symmetry of this equation, that is

$$Y^{(1)} L + L D_i \xi_i = 0,$$

if and only if

$$p = \frac{n+2}{n-2},$$

the critical Sobolev exponent. With this choice of p, the operator Y is exactly the conformal vector field X used above to obtain the 1965 Pokhozhaev's identity. In the next subsections we shall choose critical values of the coefficients of the operator X using similar heuristic arguments.

4.2. Elliptic potential systems

The next example is the following potential system of m equations

$$\left\{ \begin{array}{rcl} -\Delta u^1 & = & c^1 F_{u^1}(u^1, u^2, \ldots, u^m), \\ -\Delta u^2 & = & c^2 F_{u^2}(u^1, u^2, \ldots, u^m), \\ \vdots & & \vdots \\ -\Delta u^m & = & c^m F_{u^m}(u^1, u^2, \ldots, u^m), \end{array} \right. \tag{4.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with the Dirichlet boundary condition

$$u^1 = u^2 = \cdots = u^m = 0 \tag{4.5}$$

on $\partial\Omega$. We shall suppose that the real constants $c^\alpha \neq 0$ for all $\alpha = 1, \ldots, m$ and that $F(0, \ldots, 0) = 0$. The case $c^1 = \cdots = c^m = 1$ has been treated in [16] by Pucci and Serrin.

This system has a variational structure and its Function of Lagrange is given by

$$L = \frac{1}{2c^\alpha}(u_j^\alpha)^2 - F(u^1, u^2, \ldots, u^m).$$

The dilation

$$\begin{cases} x_j^* &= \lambda x_j, \\ u^{*\alpha} &= \lambda^{(2-n)/2} u^\alpha, \end{cases}$$

has infinitesimals

$$\begin{cases} \xi^j &= x_j, \\ \eta^\alpha &= \frac{(2-n)}{2} u^\alpha, \end{cases}$$

and its first order generator is given by

$$X^{(1)} = x_i \frac{\partial}{\partial x_i} + \frac{2-n}{2} u^\alpha \frac{\partial}{\partial u^\alpha} - \frac{n}{2} u_i^\alpha \frac{\partial}{\partial u_i^\alpha}.$$

Then

$$X^{(1)} L + L D_i \xi^i = \frac{n-2}{2} u^\alpha F_{u^\alpha} - nF(u^1, u^2, \ldots, u^m). \tag{4.6}$$

Finally, by (3.4), (4.5), (4.6) and the divergence theorem, we obtain the following identity:

$$\int_\Omega [\frac{n-2}{2} u^\alpha F_{u^\alpha} - nF(u^1, \ldots, u^m)] \mathrm{d}x = -\frac{1}{2c^\alpha} \int_{\partial\Omega} |\nabla u^\alpha|^2 (x, \nu) \mathrm{d}s.$$

If $c^1 = \cdots = c^m = 1$ the above identity is well known.

We emphasize that some of the constants c^α might be negative. E.g in the special case $m = 2$, $u^1 = u$, $u^2 = v$ for the system

$$\begin{cases} -\Delta u &= F_u(u, v), \\ \Delta v &= F_v(u, v), \end{cases}$$

whose Lagrangian is given by

$$L = \frac{1}{2}|\nabla u|^2 - \frac{1}{2}|\nabla v|^2 - F,$$

the variational identity reads

$$\int_\Omega [\frac{n-2}{2}(uF_u + vF_v) - nF(u, v)] \mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} (|\nabla v|^2 - |\nabla u|^2)(x, \nu) \mathrm{d}s.$$

4.3. Elliptic Hamiltonian systems

Our third example is the semilinear Hamiltonian system of $2m$ equations

$$\left\{\begin{array}{lcl} -\Delta u^1 & = & H_{v^1}(u^1,\dots,u^m,v^1,\dots,v^m),\\ -\Delta v^1 & = & H_{u^1}(u^1,\dots,u^m,v^1,\dots,v^m),\\ \vdots & & \vdots \\ -\Delta u^m & = & H_{v^m}(u^1,\dots,u^m,v^1,\dots,v^m),\\ -\Delta v^m & = & H_{u^m}(u^1,\dots,u^m,v^1,\dots,v^m), \end{array}\right. \tag{4.7}$$

in Ω with the homogeneous Dirichlet boundary conditions on $\partial\Omega$. Clearly, its variational structure is determined by the following Function of Lagrange:

$$L = u_j^\alpha v_j^\alpha - H(u^1,\dots,u^m,v^1,\dots,v^m).$$

Further, the dilation

$$\left\{\begin{array}{lcl} x_j^* & = & \lambda x_j,\\ u^{*\alpha} & = & \lambda^{a^\alpha(2-n)/2}u^\alpha,\\ v^{*\alpha} & = & \lambda^{b^\alpha(2-n)/2}v^\alpha, \end{array}\right.$$

where a^α and b^α are real numbers such that

$$a^\alpha + b^\alpha = 2,$$

has infinitesimals

$$\left\{\begin{array}{lcl} \xi^j & = & x_j,\\ \eta^\alpha & = & \frac{a^\alpha(2-n)}{2}u^\alpha,\\ \phi^\alpha & = & \frac{b^\alpha(2-n)}{2}v^\alpha. \end{array}\right.$$

(Note that in the expressions $a^\alpha u^\alpha$ and $b^\alpha v^\alpha$ above there is no summation!)

Its first order generator is:

$$X^{(1)} = x_i\frac{\partial}{\partial x_i} + \frac{a^\alpha(2-n)}{2}u^\alpha\frac{\partial}{\partial u^\alpha} + \frac{b^\alpha(2-n)}{2}v^\alpha\frac{\partial}{\partial v^\alpha} - \frac{A^\alpha n}{2}u_i^\alpha\frac{\partial}{\partial u_i^\alpha} - \frac{B^\alpha n}{2}v_i^\alpha\frac{\partial}{\partial v_i^\alpha},$$

where $A^\alpha = (2-n)a^\alpha - 2$, $B^\alpha = (2-n)b^\alpha - 2$ and the summation over α is assumed again. We have

$$X^{(1)}L + LD_i\xi^i = \frac{n-2}{2}(a^\alpha u^\alpha H_{u^\alpha} + b^\alpha v^\alpha H_{v^\alpha}) - nH. \tag{4.8}$$

Then by (3.4), (4.8), the boundary conditions and the divergence theorem, we obtain

$$\int_\Omega [\frac{n-2}{2}(a^\alpha u^\alpha H_{u^\alpha} + b^\alpha u^\alpha H_{v^\alpha}) - nH]\mathrm{d}x = -\int_{\partial\Omega}\frac{\partial u^\alpha}{\partial\nu}\frac{\partial v^\alpha}{\partial\nu}(x,\nu)\mathrm{d}s,$$

where $a^\alpha + b^\alpha = 2$, $\alpha = 1,\dots,m$.

In particular if $m = 1$, $u^1 = u$ and $v^1 = v$, the resulting identity for the system

$$\left\{\begin{array}{lcl} -\Delta u & = & H_v(u,v),\\ -\Delta v & = & H_u(u,v), \end{array}\right.$$

in Ω with the homogeneous Dirichlet boundary conditions on $\partial\Omega$ is as follows:

$$\int_{\Omega}[\frac{n-2}{2}(auH_u+bvH_v)-nH(u,v)]\mathrm{d}x=-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu}(x,\nu)\mathrm{d}s.$$

Up to notation, the latter identity is exactly the formula (3.5) on [11, page 137]. Thus we can recover immediately the corresponding nonexistence result of Mitidieri [11, page 136].

We observe that the approach applies to more general Hamiltonian systems of type:

$$\begin{cases} Lu &= H_v(u,v), \\ L^*v &= H_u(u,v), \end{cases}$$

where L is a linear higher order elliptic operator in divergence form and L^* is its formally adjoint operator.

4.4. Mixed elliptic systems

In this subsection we shall obtain the Pokhozhaev's identity for the following mixed Hamiltonian-potential system consisting of $2m+r$ equations

$$\begin{cases} -\Delta u^1 &= H_{v^1}, \\ -\Delta v^1 &= H_{u^1} \\ \vdots & \vdots \\ -\Delta u^m &= H_{v^m}, \\ -\Delta v^m &= H_{u^m}, \\ -\Delta w^1 &= c^1 H_{w^1}, \\ \vdots & \vdots \\ -\Delta w^r &= c^r H_{w^r}, \end{cases} \tag{4.9}$$

in Ω with homogeneous Dirichlet boundary conditions:

$$u^\alpha = v^\alpha = w^\beta = 0 \tag{4.10}$$

on $\partial\Omega$. Here $H=H(u^1,\dots,u^m,v^1,\dots,v^m,w^1,\dots,w^r)$, $H(0,\dots,0)=0$ and the constants $c^\beta\neq 0$ for all $\beta=1,\dots,r$.

The Function of Lagrange of (4.9) is given by

$$L=u_j^\alpha v_j^\alpha+\frac{1}{2c^\beta}(w_j^\beta)^2-H(u^1,\dots,u^m,v^1,\dots,v^m,w^1,\dots,w^r).$$

The dilation

$$\begin{cases} x_j^* &= \lambda x_j, \\ u^{*\alpha} &= \lambda^{a^\alpha(2-n)/2}u^\alpha, \\ v^{*\alpha} &= \lambda^{b^\alpha(2-n)/2}v^\alpha, \\ w^{*\beta} &= \lambda^{(2-n)/2}w^\beta, \end{cases}$$

where the real numbers a and b satisfy

$$a^\alpha+b^\alpha=2,$$

has infinitesimals

$$\begin{cases} \xi^j &= x_j, \\ \eta^\alpha &= \frac{a^\alpha(2-n)}{2}u^\alpha, \\ \phi^\alpha &= \frac{b^\alpha(2-n)}{2}v^\alpha, \\ \psi^\beta &= \frac{2-n}{2}w^\beta. \end{cases}$$

(Note that in the expressions $a^\alpha u^\alpha$ and $b^\alpha v^\alpha$ above there is no summation!)

Its first order generator is

$$\begin{aligned} X^{(1)} = x_i\frac{\partial}{\partial x_i} + \frac{a^\alpha(2-n)}{2}u^\alpha\frac{\partial}{\partial u^\alpha} + \frac{b^\alpha(2-n)}{2}v^\alpha\frac{\partial}{\partial v^\alpha} + \frac{2-n}{2}w^\beta\frac{\partial}{\partial w^\beta} \\ -\frac{A^\alpha n}{2}u_i^\alpha\frac{\partial}{\partial u_i^\alpha} - \frac{B^\alpha n}{2}v_i^\alpha\frac{\partial}{\partial v_i^\alpha} - \frac{n}{2}w_i^\beta\frac{\partial}{\partial w_i^\beta}, \end{aligned} \tag{4.11}$$

where $A^\alpha = (2-n)a^\alpha - 2$, $B^\alpha = (2-n)b^\alpha - 2$ and the summation over α is assumed again. We have

$$X^{(1)}L + LD_i\xi^i = \frac{n-2}{2}(a^\alpha u^\alpha H_{u^\alpha} + b^\alpha v^\alpha H_{v^\alpha} + w^\beta H_{w^\beta}) - nH. \tag{4.12}$$

Then by (3.4), (4.10), (4.12) and the divergence theorem, we obtain

$$\int_\Omega[\frac{n-2}{2}(a^\alpha u^\alpha H_{u^\alpha} + b^\alpha v^\alpha H_{v^\alpha} + w^\beta H_{w^\beta}) - nH]\mathrm{d}x$$

$$= -\int_{\partial\Omega}[\frac{\partial u^\alpha}{\partial\nu}\frac{\partial v^\alpha}{\partial\nu} + \frac{1}{2c^\beta}|\nabla w^\beta|^2](x,\nu)\mathrm{d}s,$$

where $a^\alpha + b^\alpha = 2$, $\alpha = 1,\dots,m$, $\beta = 1,\dots r$, $H = H(u^1,\dots,u^m,v^1,\dots,v^m,$ $w^1,\dots,w^r)$.

In particular if $m = 1$, $r = 1$, $u^1 = u$, $v^1 = v$ and $w^1 = w$, the resulting identity for the system

$$\begin{cases} -\Delta u &= H_v(u,v,w), \\ -\Delta v &= H_u(u,v,w), \\ -\Delta w &= cH_w(u,v,w), \end{cases}$$

in Ω with the homogeneous Dirichlet boundary conditions reads:

$$\int_\Omega[\frac{n-2}{2}(auH_u + bvH_v + wH_w) - nH(u,v,w)]\mathrm{d}x$$

$$= -\int_{\partial\Omega}[\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu} + \frac{1}{2c}|\nabla w|^2](x,\nu)\mathrm{d}s,$$

where $a + b = 2$ and the constant $c \neq 0$.

4.5. Unbounded Hamiltonian systems

In [4] Clément, Felmer and Mitidieri obtained some results concerning the existence of positive periodic and of homoclinic solutions to Hamiltonian systems of the type

$$\begin{cases} u_t - \Delta u &= H_v(u,v), \\ -v_t - \Delta v &= H_u(u,v), \end{cases} \tag{4.13}$$

in $\Omega \times \mathbb{R}$ and $u = v = 0$ on $\partial\Omega \times \mathbb{R}$. (Here Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ and $H(0,0) = H_u(0,0) = H_v(0,0) = 0$.) Further a Pokhozhaev type identity was established in [5].

Our purpose is to obtain the corresponding Pokhozhaev's identity using the Noether approach.

To begin with, we observe that (4.13) has a variational structure determined by the following Function of Lagrange:

$$L = \frac{1}{2} v u_t - \frac{1}{2} u v_t + u_j v_j - H(u,v).$$

Indeed, the Euler operator

$$E_1 = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_i \frac{\partial}{\partial u_i}$$

applied to L gives

$$E_1(L) = -\frac{1}{2} v_t - H_u - D_t(\frac{1}{2} v) - D_i v_i = -v_t - \Delta v - H_u = 0,$$

which is the second equation of (4.13). Similarly

$$\begin{aligned} E_2(L) &= \left(\frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_i \frac{\partial}{\partial v_i} \right) L \\ &= \frac{1}{2} u_t - H_v - D_t(-\frac{1}{2} u) - D_i u_i = u_t - \Delta u - H_v = 0, \end{aligned}$$

which is the first equation of (4.13). We note that the total derivative operators in this case read

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + u_{tj} \frac{\partial}{\partial u_j} + v_{tt} \frac{\partial}{\partial v_t} + v_{tj} \frac{\partial}{\partial v_j} \cdots$$

and

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + v_i \frac{\partial}{\partial v} + u_{it} \frac{\partial}{\partial u_t} + u_{ij} \frac{\partial}{\partial u_j} + v_{it} \frac{\partial}{\partial v_t} + v_{ij} \frac{\partial}{\partial v_j} \cdots .$$

We now consider a dilation

$$\begin{cases} x_j^* &= \lambda x_j, \\ t^* &= \lambda^2 t, \\ u^* &= \lambda^A u, \\ v^* &= \lambda^B v, \end{cases}$$

where the constants A and B will be specified later. Clearly its generator X has infinitesimals

$$\begin{cases} \xi^j &= x_j, \\ \phi &= 2t, \\ \eta^1 &= Au, \\ \eta^2 &= Bv. \end{cases}$$

By (2.2) the first order extension of X is given by

$$\begin{aligned} X^{(1)} = &\; x_i\frac{\partial}{\partial x_i} + 2t\frac{\partial}{\partial t} + Au\frac{\partial}{\partial u} + Bv\frac{\partial}{\partial v} \\ &+(A-2)u_t\frac{\partial}{\partial u_t} + (A-1)u_i\frac{\partial}{\partial u_i} + (B-2)v_t\frac{\partial}{\partial v_t} + (B-1)v_i\frac{\partial}{\partial v_i}. \end{aligned} \tag{4.14}$$

Then, after some work, we calculate

$$\begin{aligned} X^{(1)}L + L(D_t\phi + D_i\xi^i) = &\; \frac{1}{2}(A+B+n)[vu_t - uv_t + u_iv_i] \\ &-AuH_u - BvH_v - (n+2)H(u,v). \end{aligned} \tag{4.15}$$

Choosing $A = -an$, $B = -bn$, where $a + b = 1$, in the above equality, we obtain

$$X^{(1)}L + L(D_t\phi + D_i\xi^i) = n(auH_u + bvH_v) - (n+2)H(u,v).$$

Hence, from the boundary conditions and the divergence theorem, we finally obtain

$$\begin{aligned} &n\int_\Omega (auH_u + bvH_v)dx - (n+2)\int_\Omega H(u,v)\mathrm{d}x \\ &= \frac{d}{dt}\int_\Omega [2t(u_iv_i - H(u,v)) + \frac{n}{2}(b-a)uv + \frac{1}{2}uv_ix_i - \frac{1}{2}vu_ix_i]\mathrm{d}x \\ &- \int_{\partial\Omega}[\frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu}(x,\nu) + 2tu_t\frac{\partial v}{\partial \nu} + 2tv_t\frac{\partial u}{\partial \nu}]\mathrm{d}s, \end{aligned} \tag{4.16}$$

where $a + b = 1$ and we have taken into account the vanishing of u and v on $\partial\Omega$.

4.6. Unbounded Hamiltonian systems involving polyharmonic operators

In this subsection we state the Pokhozhaev identity for the more general problem of higher order:

$$\begin{cases} u_t + (-1)^k\Delta^k u &= H_v(u,v), \\ -v_t + (-1)^k\Delta^k v &= H_u(u,v), \end{cases}$$

with Navier boundary conditions

$$u = \Delta u = \cdots = \Delta^{k-1}u = v = \Delta v = \cdots = \Delta^{k-1}v = 0$$

on $\partial\Omega$.

In order not to increase the volume of this paper we shall not present the corresponding details merely pointing out the following main points:

- we suppose that $k \geq 2$ is an even number (the case k odd can be treated in a similar way);
- we follow the approach in [3] and use similar formulae to those obtained there;

– the Function of Lagrange for the considered problem is given by

$$L = \frac{1}{2} v u_t - \frac{1}{2} u v_t + (\Delta^{k/2} u)(\Delta^{k/2} v) - H(u, v);$$

– the used dilation has infinitesimals

$$\begin{cases} \xi^j &= x_j, \\ \phi &= 2k\, t, \\ \eta^1 &= -an\, u, \\ \eta^2 &= -bn\, v, \end{cases}$$

where the constants a and b are such that $a + b = 1$. The corresponding k-th order extension is given by

$$\begin{aligned} X^{(k)} = \quad & x_i \frac{\partial}{\partial x_i} + 2kt \frac{\partial}{\partial t} - an\, u \frac{\partial}{\partial u} - bn\, v \frac{\partial}{\partial v} \\ & -(an + 2k) u_t \frac{\partial}{\partial u_t} - (an + 1) u_i \frac{\partial}{\partial u_i} - (bn + 2k) v_t \frac{\partial}{\partial v_t} - (bn + 1) v_i \frac{\partial}{\partial v_i} \\ & -(an + 2) u_{ij} \frac{\partial}{\partial u_{ij}} - (bn + 2) v_{ij} \frac{\partial}{\partial v_{ij}} - \dots \\ & -(an + k) u_{i_1 \dots i_k} \frac{\partial}{\partial u_{i_1 \dots i_k}} - (bn + k) v_{i_1 \dots i_k} \frac{\partial}{\partial v_{i_1 \dots i_k}} - \dots \end{aligned} \tag{4.17}$$

Then after some tedious work we obtain the following Pokhozhaev identity:

$$\begin{aligned} & \int_\Omega (auH_u + bvH_v) dx - (n + 2k) \int_\Omega H(u, v) \mathrm{d}x \\ & = \frac{d}{dt} \int_\Omega [2kt\, ((\Delta^{k/2} u)(\Delta^{k/2} v) - H(u, v)) + \frac{n}{2}(b - a) uv + \frac{1}{2} u v_i x_i - \frac{1}{2} v u_i x_i] \mathrm{d}x \\ & + \int_{\partial\Omega} [(x, \nu) \sum_{l=0}^{k-1} \frac{\partial}{\partial \nu} (\Delta^l u). \frac{\partial}{\partial \nu} (\Delta^{k-1-l} v) \\ & + 2kt \sum_{l=0}^{k/2-1} (\Delta^l u_t). \frac{\partial}{\partial \nu} (\Delta^{k-1-l} v) + 2kt \sum_{l=0}^{k/2-1} (\Delta^l v_t). \frac{\partial}{\partial \nu} (\Delta^{k-1-l} u)] \mathrm{d}s \end{aligned}$$

with $a + b = 1$ and Navier boundary conditions. In the case $k = 2$ this identity reads

$$\begin{aligned} & n \int_\Omega (auH_u + bvH_v) \mathrm{d}x - (n + 4) \int_\Omega H(u, v) \mathrm{d}x \\ & = \frac{d}{dt} \int_\Omega [4t(\Delta u \Delta v - H(u, v)) + \frac{n}{2}(b - a) uv + \frac{1}{2} u v_i x_i - \frac{1}{2} v u_i x_i] \mathrm{d}x \\ & \quad + \int_{\partial\Omega} [\frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu} (\Delta v) + \frac{\partial v}{\partial \nu} \frac{\partial}{\partial \nu} (\Delta u) + 4t u_t \frac{\partial}{\partial \nu} (\Delta v) + 4t v_t \frac{\partial}{\partial \nu} (\Delta u)] (x, \nu) \mathrm{d}s, \end{aligned}$$

where $a + b = 1$ and $u = v = \Delta u = \Delta v = 0$ on $\partial\Omega$.

4.7. Hyperbolic Hamiltonian systems

In this subsection we consider the nonlinear hyperbolic system of Hamiltonian type

$$\left\{\begin{array}{lcl} u_{tt} - \Delta u + H_v(u,v) & = & 0, \\ v_{tt} - \Delta v + H_u(u,v) & = & 0 \end{array}\right. \tag{4.18}$$

in $\Omega \times \mathbb{R}$ with the homogeneous Dirichlet boundary conditions on $\partial\Omega \times \mathbb{R}$. Its Function of Lagrange is defined as follows:

$$L = u_t v_t - u_j v_j - H(u,v).$$

In order to obtain the Pokhozhaev's identity for (4.18) we introduce the operator X, which is the infinitesimal generator of the dilation

$$\left\{\begin{array}{lcl} x_j^* & = & \lambda x_j, \\ t^* & = & \lambda t, \\ u^* & = & \lambda^{a(1-n)/2} u, \\ v^* & = & \lambda^{b(1-n)/2} v, \end{array}\right.$$

where the real numbers a and b are such that

$$a + b = 2.$$

Clearly X has infinitesimals

$$\left\{\begin{array}{lcl} \xi^j & = & x_j, \\ \phi & = & t, \\ \eta^1 & = & \frac{a(1-n)}{2} u, \\ \eta^2 & = & \frac{b(1-n)}{2} v. \end{array}\right.$$

The first order extension of X is given by

$$\begin{aligned} X^{(1)} = \quad & x_i \frac{\partial}{\partial x_i} + t \frac{\partial}{\partial t} + \frac{a(1-n)}{2} u \frac{\partial}{\partial u} + \frac{b(1-n)}{2} v \frac{\partial}{\partial v} \\ & + \frac{A}{2} u_t \frac{\partial}{\partial u_t} + \frac{A}{2} u_i \frac{\partial}{\partial u_i} + \frac{B}{2} v_t \frac{\partial}{\partial v_t} + \frac{B}{2} v_i \frac{\partial}{\partial v_i}, \end{aligned}$$

where $A = (1-n)a - 2$ and $B = (1-n)b - 2$. Then

$$X^{(1)} L + L(D_t\phi + D_i \xi^i) = \frac{n-1}{2}(auH_u + bvH_v) - (n+1)H(u,v). \tag{4.19}$$

Then by (3.4), (4.19), the boundary conditions and the divergence theorem, we obtain

$$\begin{aligned} \frac{d}{dt} \int_\Omega & [te(u,v) + (x_j u_j v_t + x_j v_j u_t) + \frac{n-1}{2}(auv_t + bvu_t)] \mathrm{d}x \\ = & \int_\Omega [(n+1)H(u,v) - \frac{n-1}{2}(auH_u + bvH_v)] \mathrm{d}x \\ & + \int_{\partial\Omega} [(u_t v_t + \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu})(x,\nu) + tu_t \frac{\partial v}{\partial \nu} + tv_t \frac{\partial u}{\partial \nu}] \mathrm{d}s, \end{aligned}$$

where $e(u,v)$ is the energy (density)

$$e(u,v) = u_t v_t + u_i v_i + H(u,v).$$

Let Ω be a ball of radius R and centre at the origin of $\mathbb{R}^n$. If we assume for (u,v) appropriate vanishing conditions as $R \to \infty$ then we obtain the following conformal identity for the nonlinear hyperbolic system (4.18):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int [te(u,v) + (x_j u_j v_t + x_j v_j u_t) + \frac{n-1}{2}(auv_t + bvu_t)]\mathrm{d}x$$
$$= \int [(n+1)H(u,v) - \frac{n-1}{2}(auH_u + bvH_v)]\mathrm{d}x,$$

where $a+b=2$ and the integration is performed over the whole $\mathbb{R}^n$.

The established identities generalize to systems the Morawetz's dilational identity for nonlinear wave equations presented and discussed by Strauss in [21].

4.8. Hyperbolic Hamiltonian Euler-Bernoulli systems involving polyharmonic operators

In this subsection we apply the Noether approach to the following problem

$$\begin{cases} u_{tt} + (-1)^k \Delta^k u + H_v(u,v) &= 0, \\ v_{tt} + (-1)^k \Delta^k v + H_v(u,v) &= 0 \end{cases}$$

with Navier boundary conditions on $\partial\Omega$:

$$u = \Delta u = \cdots = \Delta^{k-1} u = v = \Delta v = \cdots = \Delta^{k-1} v = 0.$$

The latter problem is related to that in the preceding section. If $k=2$ then this is a generalization to Hamiltonian systems of the Euler-Bernoulli equation

$$u_{tt} + \Delta^2 u = 0$$

describing the transverse oscillations of plates. This equation has been extensively studied, among others, by Lasiecka and Triggiani (see [9] and the refernces therein).

Using the Function of Lagrange (for k even)

$$u_t v_t - (\Delta^{k/2} u)(\Delta^{k/2} v) - H(u,v),$$

the dilation

$$\begin{cases} x_j^* &= \lambda x_j, \\ t^* &= \lambda^k t, \\ u^* &= \lambda^{a(k-n)} u, \\ v^* &= \lambda^{b(k-n)} v, \end{cases}$$

where the real numbers a and b are such that $a+b=1$ and

$$\begin{aligned} X^{(k)} = {} & x_i \frac{\partial}{\partial x_i} + kt\frac{\partial}{\partial t} + a(k-n)u\frac{\partial}{\partial u} + b(k-n)v\frac{\partial}{\partial v} \\ & + (a(k-n)-k)u_t\frac{\partial}{\partial u_t} + (a(k-n)-1)u_i\frac{\partial}{\partial u_i} \\ & + (b(k-n)-k)v_t\frac{\partial}{\partial v_t} + (b(k-n)-1)v_i\frac{\partial}{\partial v_i} \\ & + \cdots + (a(k-n)-k)u_{i_1\dots i_k}\frac{\partial}{\partial u_{i_1\dots i_k}} + (b(k-n)-k)v_{i_1\dots i_k}\frac{\partial}{\partial v_{i_1\dots i_k}} - \dots, \end{aligned}$$

and the same assumptions and the approach as in Subsection 4.6, we obtain the following Pokhozhaev identity:

$$
\begin{aligned}
&(n-k)\int_{\Omega}(auH_u+bvH_v)\mathrm{d}x-(n+k)\int_{\Omega}H(u,v)\mathrm{d}x\\
&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}[kt\,e(u,v)+(n-k)(auv_t+bvu_t)+u_tv_ix_i+v_tu_ix_i]\mathrm{d}x\\
&\quad+\int_{\partial\Omega}[(u_tv_t-\sum_{l=0}^{k-1}\frac{\partial}{\partial\nu}(\Delta^l u).\frac{\partial}{\partial\nu}(\Delta^{k-1-l}v))(x,\nu)\\
&\quad-kt\sum_{l=0}^{k/2-1}(\Delta^l u_t).\frac{\partial}{\partial\nu}(\Delta^{k-1-l}v)-kt\sum_{l=0}^{k/2-1}(\Delta^l v_t).\frac{\partial}{\partial\nu}(\Delta^{k-1-l}u)]\mathrm{d}s
\end{aligned}
$$

with Navier boundary conditions, $a+b=1$ and

$$e(u,v)=u_tv_t+(\Delta^{k/2}u)(\Delta^{k/2}v)+H(u,v)$$

is the higher order energy density.

If Ω is a ball of radius R and centre at the origin of $\mathbb{R}^n$ as in the preceding section we obtain the following conformal identity for the considered nonlinear hyperbolic system involving the polyharmonic operator:

$$
\begin{aligned}
&\frac{d}{dt}\int[kt\,e(u,v)+(n-k)(auv_t+bvu_t)+u_tv_ix_i+v_tu_ix_i]\mathrm{d}x\\
&\qquad=\int[(n+k)H(u,v)+(k-n)(auH_u+bvH_v)]\mathrm{d}x,
\end{aligned}
$$

where $a+b=1$ and the integration is performed over the whole $\mathbb{R}^n$.

4.9. Giga-Kohn equations

The semilinear partial differential equation

$$-\Delta w+\frac{1}{2}y_iw_i+\beta w=w^p \tag{4.20}$$

was studied by Giga and Kohn in [8]. Here $w=w(y)$ is a positive function of $y\in\mathbb{R}^n$, $p>1$ is a real number and $\beta=1/(p-1)$. This equation can be written in a divergence form as follows:

$$-\operatorname{Div}(\rho\nabla w)+\beta\rho w=\rho w^p,$$

where $\rho=\exp(-|y|^2/4)$, $|y|^2=y_i^2=\Sigma_{i=1}^n y_i^2$. Hence the corresponding Function of Lagrange is given by

$$L=\frac{1}{2}\rho w_j^2+\frac{\beta}{2}\rho w^2-\frac{1}{p+1}\rho w^{p+1}.$$

Indeed, by (2.1) the Euler-Lagrange equation for L is

$$E(L)=\frac{\partial L}{\partial w}-D_i\frac{\partial L}{\partial w_i}=\beta\rho w-\rho w^p-D_i(e^{-|y|^2/4}w_i)=0$$

which is (4.20) multiplied by $\rho>0$.

Now we consider the operator

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial w}$$

with infinitesimals

$$\xi^i = -y_i, \quad \eta = \left(\frac{n}{p+1} - \frac{1}{2(p+1)} |y|^2 \right) w.$$

The first order extension $X^{(1)}$ of X is given by

$$\begin{aligned} X^{(1)} = \quad & -y_i \frac{\partial}{\partial x_i} + \left(\frac{n}{p+1} - \frac{1}{2(p+1)} |y|^2 \right) w \frac{\partial}{\partial w} \\ & + \left[\left(\frac{n}{p+1} + 1 - \frac{1}{2(p+1)} |y|^2 \right) w_i - \frac{1}{p+1} y_i w \right] \frac{\partial}{\partial w_i}. \end{aligned}$$

Then, after a not very tedious calculation, we obtain

$$\begin{aligned} \int [X^{(1)} L + L D_i \xi^i] \mathrm{d}y \quad = \quad & \left(\frac{n}{p+1} + \frac{2-n}{2} \right) \int \rho |\nabla w|^2 \mathrm{d}y \\ & + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{p+1} \right) \int \rho |y|^2 |\nabla w|^2 \mathrm{d}y, \end{aligned} \tag{4.21}$$

where the integration on $\mathbb{R}^n$ is justified as in [8] due to the presence of the factor ρ.

The identity (4.21) is exactly the identity established by Giga and Kohn in [8]. In fact, the *exact* form of the corresponding Pokhozhaev identity is that obtained by choosing $\xi^i = -y_i$ and $\eta = (n/(p+1))w$, which is an identity also obtained and used in [8] to get (4.21).

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Yuri Bozhkov
Departamento de Matemática Aplicada – DMA
Instituto de Matemática, Estatistica e Computação Científica – IMECC
Universidade Estadual de Campinas – UNICAMP
C.P. 6065
13083-970 – Campinas – SP
Brasil
e-mail: bozhkov@ime.unicamp.br

Enzo Mitidieri
Dipartimento di Matematica e Informatica
Università degli Studi di Trieste
Via Valerio 12/1
34127 Trieste
Italy
e-mail: mitidier@units.it

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