DOI: 10.1093/jigpal/jzx006

Cut-elimination and deductive polarization in complementary classical logic

WALTER A. CARNIELLI* *Centre for Logic, Epistemology and History of Science, and Department of Philosophy, State University of Campinas.*

GABRIELE PULCINI** *Departamento de Matemática, Universidade Nova de Lisboa.*

Abstract

In this article, we consider $\overline{\mathsf{LK}}$, a cut-free sequent calculus able to faithfully characterize classical (propositional) non-theorems, in the sense that a formula φ is provable in $\overline{\mathsf{LK}}$ if, and only if, φ is *not* provable in LK, i.e., φ is *not* a classical tautology. The $\overline{\mathsf{LK}}$ calculus is here enriched with two admissible (unary) cut rules, which allow for a simple and efficient cut-elimination algorithm. We observe two facts: (i) complementary cut-elimination always returns the simplest proof for a given provable sequent, and (ii) provable complementary sequents turn out to be deductively polarized by the empty sequent.

Keywords: Complementary classical logic, refutation calculi, cut-elimination theorem.

1 Introduction

Two deductive systems $\mathscr{S}$ and $\overline{\mathscr{S}}$, sharing a same language, are said to be *complementary*[1] when:

$$\vdash_{\overline{\mathscr{S}}} \varphi \text{ if, and only if, } \nvdash_{\mathscr{S}} \varphi.$$

In other words, a system $\overline{\mathscr{S}}$ turns out to be complementary with respect to another system $\mathscr{S}$ if it proves exactly the non-theorems of $\mathscr{S}$ [20, 21]. The conceptual idea underlying the study of complementarity is that of the characterization of a decidable system $\mathscr{S}$ by taking, so to speak, its picture in the negative. The term 'characterization' has here a precise meaning in the sense that theorems of the positive part $\mathscr{S}$ can be ascertained by excluding the possibility of their provability in the complementary system $\overline{\mathscr{S}}$.

As a matter of fact, logical complementarity should be thought of as a way to sharpen our proof-theoretical understanding of decidable calculi in two main respects. First, it allows us to widen the space of proofs so as to include complementary derivations. Second, resorting to complementary characterizations has the effect of making semantics almost dispensable. This latter aspect can be better appreciated when considering a typical case in which one needs to prove that a formula φ is a

*E-mail: walter.carnielli@cle.unicamp.br

**E-mail: g.pulcini@fct.unl.pt

[1] Formal calculi proving the non-theorems of a given logic are better known in the literature as 'refutation systems' [9, 14, 15]. Actually, this kind of terminology turns out to be slightly inaccurate as, in line with the intuitionistic tradition, the act of *refuting* a formula φ is usually taken as the act of providing a proof for the implication $\varphi \to \bot$ defining $\neg\varphi$. Now, when φ is a contingent formula (neither a tautology nor a contradiction), provability in $\overline{\mathsf{LK}}$ does not imply refutability in LK since $\neg\varphi$ is not a LK-theorem either. For this reason, we prefer to follow Varzi and call these systems 'complementary' [20, 21].

Vol. 25 No. 3,
doi:10.1093/jigpal/jzx006 Advance Access published 10 April 2017

theorem of $\mathscr{S}$ if, and only if, some mathematical fact $\mathcal{F}$ occurs. Having at disposal the complementary formalism $\overline{\mathscr{S}}$, the biconditional at issue can be proved just by supplying two soundness-style inductive proofs for the following two claims: (i) if $\vdash_{\mathscr{S}} \varphi$, then it is the case that $\mathcal{F}$, and (ii) if $\vdash_{\overline{\mathscr{S}}} \varphi$, then it is *not* the case that $\mathcal{F}$.

Needless to say, the most interesting complementary systems are those complementing well-known logical calculi, classical logic *in primis*. In this case, whereas the semantical characterization proves straightforward (just consider all the formulas for which there is at least one falsifying valuation), the proof-theoretic characterization can be a more challenging task. Łukasiewicz's calculus of refutations can be seen as the first proof system complementing classical logic [11, 14, 15]. More than twenty years later, Caicedo provided the first Hilbert calculus for complementary classical logic in [3]. Another, much simpler, Hilbert calculus was proposed by Varzi at the beginning of the 1990s; we owe the term 'complementarity' to him [20, 21]. Almost in the same years, Tiomkin issued the first sequent system for complementary classical logic with rules for negation and disjunction [18]. This system was independently extended by Bonatti and Goranko so as to include rules for the whole spectrum of classical connectives [2, 9]. Finally, to complete the proof-theoretical picture, Tamminga designed two natural deduction systems, one for classical non-theorems and the other for classical contradictions [17]. However, the problem of giving a well-behaved natural deduction system for complementary classical logic still remains open in many respects due to the fact that these specific formalisms cannot enjoy the deduction theorem.

The aforementioned sequent calculi, however, do not consider cut rules and so the possibility of implementing a cut-elimination algorithm is *ipso facto* excluded. Indeed, in complementary classical logic, the cut rule in its standard multiplicative formulation is not admissible:

$$\frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma' \vdash \Delta', \varphi}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ cut.}$$

The following example shows how a (classically) valid sequent can be obtained by cutting two (classically) invalid sequents:

$$\frac{p \nvdash p \to q \qquad p \to q \nvdash p}{p \vdash p} \text{ cut.}$$

In [18], Tiomkin considers a couple of 'hybrid' rules that he calls 'cuts for the unprovability' since they are obtained by 'reversing' the (additive) standard cut rule:

$$\frac{\Gamma \nvdash \Delta \qquad \Gamma \vdash \Delta, \varphi}{\Gamma, \varphi \nvdash \Delta} \qquad \frac{\Gamma \nvdash \Delta \qquad \Gamma, \varphi \vdash \Delta}{\Gamma \nvdash \Delta, \varphi}.$$

However, such a denomination turns out to be proof-theoretically inaccurate since both of these rules do not display any cut formulas.

In this article, we consider the complementary sequent calculus as it appear in [2, 9] and we enrich it with two unary cut rules which prove admissible in $\overline{\mathsf{LK}}$. The enriched complementary system is here indicated by $\overline{\mathsf{LK}}^{+}$. The completeness proof reported, for instance, in [9] is developed without resorting to any complementary version of the cut rule, therefore it clearly expresses a semantical proof of cut-eliminability. Anyway, indirect proofs of this kind are usually not very informative insofar as they do not indicate how to effectively *transform* proofs with cuts into proofs without cuts. As a natural consequence, semantic proofs of cut-elimination usually fail to provide new proof-theoretical insights. This is the reason why we provide here an efficient and simple normalization procedure.

The specific proof-theoretical insights offered by the cut-elimination algorithm devised here highlight the difference with respect to the implementation of the analogous process in the positive part LK. These differences can be summarized as follows.

(1) It is well-known how cut-elimination in LK might make the size of proofs explode, so that cut-free proofs are analytical — i.e., they do enjoy the subformula property — but analyticity does not necessarily imply simplicity, at least when the term 'simplicity' is taken in its intuitive meaning. On the contrary, cut-elimination in $\overline{\mathsf{LK}}^{+}$ can be thought of as a procedure which returns the *shortest* $\overline{\mathsf{LK}}$ proof for a complementary sequent, given any of its proofs in input. This means that, unlike what happens in LK, analyticity in the complementary part can actually be taken as synonymous with simplicity.

(2) The set of (classically) invalid sequents, that is those sequents provable in $\overline{\mathsf{LK}}$, is deductively polarized by the empty sequent in the sense that any $\overline{\mathsf{LK}}^{+}$ proof can be seen as a subproof of a longer proof ending with the empty sequent (but not *vice versa*). Consequently, as per the previous point, $\overline{\mathsf{LK}}^{+}$ proofs turn out to be polarized, via cut-elimination, by the complementary axiom introducing the empty sequent.

2 The sequent calculus $\overline{\mathsf{LK}}$

Table 1 displays the $\overline{\mathsf{LK}}$ sequent calculus as it has been introduced in [2, 9]. According to the standard notation, capital Greek letters $\Gamma, \Delta, \ldots$ stand for finite sequences of formulas. We indicate with $[\Gamma]$ the *multiset* of all the formulas occurring in Γ. Differently from the notation adopted in [2, 9, 18], we denote the complementary turnstile with '$\vdash\!\!\sim$' so that complementary sequents will come with the form $\Gamma \vdash\!\!\sim \Delta$.

Remark 1
The empty sequent $\vdash\!\!\sim$ is provable in $\overline{\mathsf{LK}}$ as a limit case of the axiom when $\{\Gamma\} = \{\Delta\} = \varnothing$.

Example 2.1
We prove that the following formulas are both theorems of $\overline{\mathsf{LK}}$.

- $(p \to q) \to (\neg p \to q)$
- $(p \wedge \neg p) \wedge \neg(p \wedge \neg p)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\!\!\sim q, p, p}\,ax.}{\neg p \vdash\!\!\sim q, p}\,\neg\vdash\!\!\sim}{p \to q, \neg p \vdash\!\!\sim q}\,\to\vdash\!\!\sim(1)}{p \to q \vdash\!\!\sim \neg p \to q}\,\vdash\!\!\sim\to}{\vdash\!\!\sim (p \to q) \to (\neg p \to q)}\,\vdash\!\!\sim\to \qquad\qquad \dfrac{\dfrac{\dfrac{}{\vdash\!\!\sim p}\,ax.}{\vdash\!\!\sim p \wedge \neg p}\,\vdash\!\!\sim\wedge(1)}{\vdash\!\!\sim (p \wedge \neg p) \wedge \neg(p \wedge \neg p)}\,\vdash\!\!\sim\wedge(1)$$

It is worth observing here how the $\overline{\mathsf{LK}}$ logical rules can be heuristically generated by a bottom-up reading of the tableau rules for classical logic once: (i) formulas labelled as true are stored on the left-hand side of the sequent symbol, (ii) formulas labelled as false are listed on the right-hand side,

TABLE 1. The $\overline{\mathsf{LK}}$ sequent calculus

Axiom:

$$\frac{}{\Gamma \mid\!\sim \Delta}\ ax. \qquad [\Gamma], [\Delta] \text{ disjoint multisets of atoms}$$

Structural rules:

$$\frac{\Gamma, \alpha, \beta \mid\!\sim \Delta}{\Gamma, \beta, \alpha \mid\!\sim \Delta}\ exch.\mid\!\sim \qquad \frac{\Gamma \mid\!\sim \Delta, \alpha, \beta}{\Gamma \mid\!\sim \Delta, \beta, \alpha}\ \mid\!\sim exch.$$

Logical rules:

$$\frac{\Gamma, \alpha, \beta \mid\!\sim \Delta}{\Gamma, \alpha \wedge \beta \mid\!\sim \Delta}\ \wedge\mid\!\sim \qquad \frac{\Gamma \mid\!\sim \Delta, \alpha}{\Gamma \mid\!\sim \Delta, \alpha \wedge \beta}\ \mid\!\sim\wedge(1) \qquad \frac{\Gamma \mid\!\sim \Delta, \beta}{\Gamma \mid\!\sim \Delta, \alpha \wedge \beta}\ \mid\!\sim\wedge(2)$$

$$\frac{\Gamma, \alpha \mid\!\sim \Delta}{\Gamma, \alpha \vee \beta \mid\!\sim \Delta}\ \vee\mid\!\sim(1) \qquad \frac{\Gamma, \beta \mid\!\sim \Delta}{\Gamma, \alpha \vee \beta \mid\!\sim \Delta}\ \vee\mid\!\sim(2) \qquad \frac{\Gamma \mid\!\sim \alpha, \beta, \Delta}{\Gamma \mid\!\sim \alpha \vee \beta, \Delta}\ \mid\!\sim\vee$$

$$\frac{\Gamma \mid\!\sim \alpha, \Delta}{\Gamma, \alpha \to \beta \mid\!\sim \Delta}\ \to\mid\!\sim(1) \qquad \frac{\Gamma, \beta \mid\!\sim \Delta}{\Gamma, \alpha \to \beta \mid\!\sim \Delta}\ \to\mid\!\sim(2) \qquad \frac{\Gamma, \alpha \mid\!\sim \beta, \Delta}{\Gamma \mid\!\sim \alpha \to \beta, \Delta}\ \mid\!\sim\to$$

$$\frac{\Gamma \mid\!\sim \alpha, \Delta}{\Gamma, \neg\alpha \mid\!\sim \Delta}\ \neg\mid\!\sim \qquad \frac{\Gamma, \alpha \mid\!\sim \Delta}{\Gamma \mid\!\sim \neg\alpha, \Delta}\ \mid\!\sim\neg$$

and (iii) a context variable is put on both sides of the sequent. The leading idea is that branching tableau rules generate two distinct $\overline{\mathsf{LK}}$ logical rules, whereas non-branching rules generate a single $\overline{\mathsf{LK}}$ logical rule. As an illustrative example, consider the tableau rules for classical implication.

$$\begin{array}{c} \mathtt{T}{:}\alpha \to \beta \\ \diagup \quad \diagdown \\ \mathtt{F}{:}\alpha \qquad \mathtt{T}{:}\beta \end{array} \qquad\qquad \begin{array}{c} \mathtt{F}{:}\alpha \to \beta \\ | \\ \mathtt{T}{:}\alpha \\ | \\ \mathtt{F}{:}\beta \end{array}$$

If we take the first branch in which α comes out as false, the rule for $\mathtt{T}{:}\alpha \to \beta$ induces the $\overline{\mathsf{LK}}$ rule $\to\mid\!\sim(1)$. Just write the true formula on the left of the sequent symbol and the false one on the right: $\frac{\alpha \to \beta \mid\!\sim}{\mid\!\sim \alpha}$. Then reverse the rule so as to get $\frac{\mid\!\sim \alpha}{\alpha \to \beta \mid\!\sim}$, and finally add context variables as follows: $\frac{\Gamma \mid\!\sim \alpha, \Delta}{\Gamma, \alpha \to \beta \mid\!\sim \Delta}$. The rule $\to\mid\!\sim(2)$ can be produced similarly by taking the branch in which β is labelled as true. Let us now consider the tableau rule for $\mathtt{F}{:}\alpha \to \beta$. Here we have only one branch and so only one rule is expected to be produced. Following the same pattern, put the true formula on the left and the false ones on the right: $\frac{\mid\!\sim \alpha \to \beta}{\alpha \mid\!\sim \beta}$. Then reverse the rule and add

TABLE 2. Some rules admissible in $\overline{\mathsf{LK}}$

$\dfrac{\Gamma,\alpha \mathrel{\vert\!\!\sim} \Delta}{\Gamma \mathrel{\vert\!\!\sim} \Delta}\ \mathit{strength.}\!\mathrel{\vert\!\!\sim}$	$\dfrac{\Gamma \mathrel{\vert\!\!\sim} \Delta,\alpha}{\Gamma \mathrel{\vert\!\!\sim} \Delta}\ \mathrel{\vert\!\!\sim}\!\mathit{strength.}$
$\dfrac{\Gamma,\alpha \mathrel{\vert\!\!\sim} \Delta}{\Gamma,\alpha,\alpha \mathrel{\vert\!\!\sim} \Delta}\ \mathit{copy}\!\mathrel{\vert\!\!\sim}$	$\dfrac{\Gamma \mathrel{\vert\!\!\sim} \Delta,\alpha}{\Gamma \mathrel{\vert\!\!\sim} \Delta,\alpha,\alpha}\ \mathrel{\vert\!\!\sim}\!\mathit{copy}$
$\dfrac{\Gamma,\neg\alpha \mathrel{\vert\!\!\sim} \Delta}{\Gamma \mathrel{\vert\!\!\sim} \Delta,\alpha}\ \mathit{inverse}\ \neg\!\mathrel{\vert\!\!\sim}$	$\dfrac{\Gamma \mathrel{\vert\!\!\sim} \Delta,\neg\alpha}{\Gamma,\alpha \mathrel{\vert\!\!\sim} \Delta}\ \mathit{inverse}\ \mathrel{\vert\!\!\sim}\!\neg$
$\dfrac{\Gamma,\alpha\wedge\beta \mathrel{\vert\!\!\sim} \Delta}{\Gamma,\alpha,\beta \mathrel{\vert\!\!\sim} \Delta}\ \mathit{inverse}\ \wedge\!\mathrel{\vert\!\!\sim}$	$\dfrac{\Gamma \mathrel{\vert\!\!\sim} \Delta,\alpha\vee\beta}{\Gamma \mathrel{\vert\!\!\sim} \Delta,\alpha,\beta}\ \mathit{inverse}\ \mathrel{\vert\!\!\sim}\!\vee$
$\dfrac{\Gamma \mathrel{\vert\!\!\sim} \alpha\to\beta,\Delta}{\Gamma,\alpha \mathrel{\vert\!\!\sim} \beta,\Delta}\ \mathit{inverse}\ \vdash\to$	

contexts so as to obtain: $\dfrac{\Gamma,\alpha \mathrel{|\!\!\sim} \beta,\Delta}{\Gamma \mathrel{|\!\!\sim} \alpha\to\beta,\Delta}$. It is easy to check that each one the other $\overline{\mathsf{LK}}$ rules can be produced in a similar fashion.

COROLLARY 2.1
The rules displayed in Table 2 are admissible in $\overline{\mathsf{LK}}$, i.e., their inclusion does not extend the set of $\overline{\mathsf{LK}}$ theorems.

PROOF. Easy, by observing that, for each one of the nine rules $\dfrac{\Gamma \mathrel{|\!\!\sim} \Delta}{\Gamma' \mathrel{|\!\!\sim} \Delta'}$ reported in Table 2, the very same valuation that falsifies the premise formula $\bigwedge\Gamma\to\bigvee\Delta$ also falsifies the conclusion $\bigwedge\Gamma'\to\bigvee\Delta'$. ∎

DEFINITION 1 (contingencies)
Formulas that are neither tautologies nor contradictions are said to be *contingent* [19].

PROPOSITION 2.2
(i) $\overline{\mathsf{LK}}$ proves $\mathrel{|\!\!\sim}\alpha$ if, and only if, α is *not* a tautology;
(ii) $\overline{\mathsf{LK}}$ proves $\alpha\mathrel{|\!\!\sim}$ if, and only if, α is *not* a contradiction;
(iii) if $\overline{\mathsf{LK}}$ proves $\Gamma \mathrel{|\!\!\sim} \Delta,\alpha$ then α is *not* a tautology;
(iv) if $\overline{\mathsf{LK}}$ proves $\Gamma,\alpha \mathrel{|\!\!\sim} \Delta$ then α is *not* a contradiction;
(v) $\overline{\mathsf{LK}}$ proves both $\mathrel{|\!\!\sim}\alpha$ and $\alpha\mathrel{|\!\!\sim}$ if, and only if, α is contingent.

PROOF. (*i*) and (*ii*). Straightforward, by the fact that $\overline{\mathsf{LK}}$ is sound and complete with respect to the set of classical non-tautologies [9].

(*iii*) Apply a cluster of left and/or right *strengthening* rules so as to get $\mathrel{|\!\!\sim}\alpha$ from $\Gamma \mathrel{|\!\!\sim} \Delta,\alpha$. Then apply point (*i*).

(*iv*) Similar to point (*iii*).

(*v*) By points (*i*) and (*ii*), α is neither a tautology nor a contradiction, so it is contingent. ∎

Remark 2
Due to the fact that the $\overline{\mathsf{LK}}$ sequent calculus displays only axioms and logical rules, and no structural rules, the length of any $\overline{\mathsf{LK}}$ proof π of a sequent $\Gamma \mid\!\sim \Delta$ always equals the number of occurrences of logical connectives in $[\Gamma] \uplus [\Delta]$ plus one, the axiom rule.

3 Cut-elimination and deductive polarization

In this section, the *strengthening* rules reported in Table 2 are regained as unary cut rules:

$$\frac{\Gamma,\alpha \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta}\,\mathit{cut}\mid\!\sim \qquad\qquad \frac{\Gamma \mid\!\sim \Delta,\alpha}{\Gamma \mid\!\sim \Delta}\mid\!\sim\mathit{cut}.$$

Tiomkin and Goranko consider these rules as inverse weakenings [9, 18]. Indeed, since they display a cut formula, it seems proof-theoretically more appropriate to consider them as cuts at all intents and purposes and then pose the question of their procedural eliminability.

We indicate with $\overline{\mathsf{LK}}^{+}$ the calculus obtained by enriching $\overline{\mathsf{LK}}$ with the aforementioned unary cut rules. $\overline{\mathsf{LK}}$ also admits the following binary cuts:

$$\frac{\Gamma \mid\!\sim \Delta,\alpha \qquad \alpha \mid\!\sim}{\Gamma \mid\!\sim \Delta} \qquad\qquad \frac{\Gamma,\alpha \mid\!\sim \Delta \qquad \mid\!\sim \alpha}{\Gamma \mid\!\sim \Delta} \qquad\qquad \frac{\Gamma,\alpha \mid\!\sim \Delta \qquad \Gamma \mid\!\sim \Delta,\alpha}{\Gamma \mid\!\sim \Delta}.$$

Nonetheless, we can ignore them since they are easily derivable from $\mid\!\sim\mathit{cut}$ and $\mathit{cut}\mid\!\sim$.

Theorem 3.1 (cut-elimination)
Any sequent provable in $\overline{\mathsf{LK}}^{+}$ is also provable in $\overline{\mathsf{LK}}$.

Proof. As for the analogous process in the positive side LK, we always reduce the uppermost cut.[2] Following the standard pattern, reduction steps are partitioned into commutations and logical reductions. We report below a few cases, the others can be detailed similarly.

- Commutation $\mathit{cut}/\mathit{cut}$:

$$\frac{\dfrac{\Gamma,\beta \mid\!\sim \Delta,\alpha}{\Gamma,\beta \mid\!\sim \Delta}\mid\!\sim\mathit{cut}}{\Gamma \mid\!\sim \Delta}\,\mathit{cut}\mid\!\sim \quad\longrightarrow\quad \frac{\dfrac{\Gamma,\beta \mid\!\sim \Delta,\alpha}{\Gamma \mid\!\sim \Delta,\alpha}\,\mathit{cut}\mid\!\sim}{\Gamma \mid\!\sim \Delta}\mid\!\sim\mathit{cut}$$

- (axiom) Here we use the fact that the set of axiomatic complementary sequents are closed under cut applications.

[2]Actually, $\overline{\mathsf{LK}}$ does not enjoy the *strong* normalization property, i.e., there are reduction strategies which form a loop configuration, and so never achieve a normal form. Consider for instance the reduction strategy illustrated below, which consists in persistently reducing the lowest cut.

$$\frac{\dfrac{\dfrac{}{\mid\!\sim p,q}\,ax.}{\mid\!\sim p}\mid\!\sim\mathit{cut}}{\mid\!\sim}\mid\!\sim\mathit{cut} \;\longrightarrow\; \frac{\dfrac{\dfrac{}{\mid\!\sim p,q}\,ax.}{\mid\!\sim q}\mid\!\sim\mathit{cut}}{\mid\!\sim}\mid\!\sim\mathit{cut} \;\longrightarrow\; \frac{\dfrac{\dfrac{}{\mid\!\sim p,q}\,ax.}{\mid\!\sim p}\mid\!\sim\mathit{cut}}{\mid\!\sim}\mid\!\sim\mathit{cut}$$

Clearly, the process never terminates since the second reduction returns the very first proof.

$$\cfrac{\cfrac{}{\Gamma, p \mathrel{\vert\!\sim} \Delta}\ ax.}{\Gamma \mathrel{\vert\!\sim} \Delta}\ cut \mathrel{\vert\!\sim} \quad \longrightarrow \quad \cfrac{}{\Gamma \mathrel{\vert\!\sim} \Delta}\ ax.$$

- $(\wedge \mathrel{\vert\!\sim})$

$$\cfrac{\cfrac{\Gamma, \alpha, \beta \mathrel{\vert\!\sim} \Delta}{\Gamma, \alpha \wedge \beta \mathrel{\vert\!\sim} \Delta}\ \wedge \mathrel{\vert\!\sim}}{\Gamma \mathrel{\vert\!\sim} \Delta}\ cut \mathrel{\vert\!\sim} \quad \longrightarrow \quad \cfrac{\cfrac{\Gamma, \alpha, \beta \mathrel{\vert\!\sim} \Delta}{\Gamma, \beta \mathrel{\vert\!\sim} \Delta}\ cut \mathrel{\vert\!\sim}}{\Gamma \mathrel{\vert\!\sim} \Delta}\ cut \mathrel{\vert\!\sim}$$

- $(\mathrel{\vert\!\sim} \wedge)$

$$\cfrac{\cfrac{\Gamma \mathrel{\vert\!\sim} \Delta, \alpha}{\Gamma \mathrel{\vert\!\sim} \Delta, \alpha \wedge \beta}\ \mathrel{\vert\!\sim} \wedge}{\Gamma \mathrel{\vert\!\sim} \Delta}\ \mathrel{\vert\!\sim} cut \quad \longrightarrow \quad \cfrac{\Gamma \mathrel{\vert\!\sim} \Delta, \alpha}{\Gamma \mathrel{\vert\!\sim} \Delta}\ \mathrel{\vert\!\sim} cut$$

■

EXAMPLE 3.1
We propose here a concrete example of normalization.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{p \mathrel{\vert\!\sim} q, r}\ ax.}{p, q \to t \mathrel{\vert\!\sim} r}\ \to\mathrel{\vert\!\sim}(1)}{p \mathrel{\vert\!\sim} r}\ cut \mathrel{\vert\!\sim}}{\mathrel{\vert\!\sim} r}\ cut \mathrel{\vert\!\sim} \quad \longrightarrow \quad \cfrac{\cfrac{\cfrac{}{p \mathrel{\vert\!\sim} q, r}\ ax.}{p \mathrel{\vert\!\sim} r}\ \mathrel{\vert\!\sim} cut}{\mathrel{\vert\!\sim} r}\ cut \mathrel{\vert\!\sim} \quad \longrightarrow \quad \cfrac{\cfrac{}{p \mathrel{\vert\!\sim} r}\ ax.}{\mathrel{\vert\!\sim} r}\ cut \mathrel{\vert\!\sim} \quad \longrightarrow \quad \cfrac{}{\mathrel{\vert\!\sim} r}\ ax.$$

REMARK 3
Unlike standard cut-elimination algorithms for LK, 'complementary' cut-elimination always induces a remarkable simplification of the size of proofs. In case of $\overline{\mathsf{LK}}^+$, we can say that cut-elimination has the effect of returning the *shortest* proof for any given provable sequent (cf. Remark 2). This peculiar phenomenon is essentially due to the fact that complementary classical logic allows for a sequent formulation $\overline{\mathsf{LK}}$ which does not need to resort to the structural rules weakening and contraction, neither explicitly nor implicitly. This fact is clearly relative to the specific sequent system under consideration. Indeed, it is worth observing here that an alternative sequent calculus proving all and only classically invalid sequents can be given by considering Kleene's system G4 [10]. The complementary system $\overline{\mathsf{G4}}$ can be obtained just by: (i) enriching G4 with the complementary axiom schema as it appear in $\overline{\mathsf{LK}}$, and (ii) requiring that any $\overline{\mathsf{G4}}$ proof displayed at least one complementary axiom. Now, being based on G4, the system $\overline{\mathsf{G4}}$ comes with implicit structural rules (weakening is implicit in generalized axioms and contraction is implicit in the additive formulation of the logical rules) thus in some cases the cut-elimination procedure may have the effect of increasing the size of $\overline{\mathsf{G4}}$ proofs as may be the case for G4 proofs.

REMARK 4 (uniqueness of the normal form)
It is easy to check that cut-elimination in $\overline{\mathsf{LK}}^+$ is confluent and so any $\overline{\mathsf{LK}}^+$ proof has only one normal form. However, there is a deeper sense in which this property holds true in $\overline{\mathsf{LK}}^+$. As already observed in Remark 2, any cut-free $\overline{\mathsf{LK}}^+$ proof (i.e., any $\overline{\mathsf{LK}}$ proof) displays exactly one axiom and a sequence of logical rules introducing one by one each specific occurrence of the logical connectives in the final sequent. This means that, if we consider complementary proofs modulo permutations of (permutable) logical rules, there is exactly one cut-free proof for any provable sequent.

REMARK 4 (deductive polarization)
We say that provable sequents of a Gentzen system $\mathcal{S}$ are deductively polarized by a particular $\mathcal{S}$-sequent s if: (i) for any $\mathcal{S}$-proof π there is a $\mathcal{S}$-proof δ of s such that π is a subproof of δ, and (ii) s is the unique sequent which enjoys (i). Roughly speaking, in a deductively polarized system any deduction points towards the polarizing sequent. Given this preliminary definition, it is easy to see that $\overline{\mathsf{LK}}^{+}$ provable sequents turn out to be deductively polarized by the empty sequent. In fact, any $\overline{\mathsf{LK}}^{+}$ proof π of a sequent $\Gamma \mid\!\sim \Delta$ can always be lengthened by means of a sequence of cut rules so as to finally become a proof δ of $\mid\!\sim$. Moreover, no $\overline{\mathsf{LK}}^{+}$-proof can display a proof of the empty sequent as a proper subproof, so condition (ii) is satisfied as well. Since provable sequents are deductively polarized by the empty sequent, proofs are polarized, via cut-elimination, by the axiom introducing the empty sequent. To sum up, given any proof π of a sequent $\Gamma \mid\!\sim \Delta$:

$$\begin{array}{c}\pi\\ \vdots\\ \Gamma \mid\!\sim \Delta\end{array} \quad \text{can be turned into } \delta\text{:} \quad \begin{array}{c}\pi\\ \vdots\\ \Gamma \mid\!\sim \Delta\\ \hline\hline \Gamma \mid\!\sim \quad {\mid\!\sim}\, cut\\ \hline\hline \mid\!\sim \quad cut\, {\mid\!\sim}\end{array} \quad \text{which normalizes to } \delta'\text{:} \quad \dfrac{}{\mid\!\sim}\, ax. \; .$$

This fact seems to be of a certain logical and philosophical relevance. In fact, from a proof-theoretical point of view, whereas (classically) valid sequents form a wide and multifarious galaxy (actually, any sequent provable in LK enjoys condition (i), therefore condition (ii) cannot be satisfied in any way), the complementary set of (classically) invalid sequents is, so to speak, organized like a gravitationally bound system in which provable sequents all 'orbit' the empty sequent.

4 Future Work

Many-valued logics represent the most immediate generalization of classical logic and have attracted a great deal of attention among philosophers and computer scientists, besides logicians, due to their philosophical interest from several viewpoints, and to their potentialities of application in knowledge-based systems, fuzzy reasoning, software verification, etc. Although several sequent systems and analytic calculi for many-valued logics are known in the literature, the results we address here seem to be naturally generalizable to practically any propositional many-valued logic by means of all-purpose proof systems such as the ones in [4]. The question of complementarity for many-valued logics and for at least certain paraconsistent logics (as in [1]), even if combinatorially more complex and deferred for further work, seems to follow the same pattern of the relationship between abstract normalization, weak normalization and confluence that is found in the classical two-valued case.

As we have already seen, a byproduct of proof-theoretical approaches to complementarity is that of the widening of the space of proofs. Once the space of proofs is complemented, a natural problem to pose is that of restricting this actual space in such a way as to characterize some philosophically or computationally relevant intermediate logics [12, 13]. This idea of approximating falsities is in some way specular to that of approximating truths proposed in [6, 7].

Other possible directions of research may concern the so-called SAT problem. It is easy to check that the complexity of testing the satisfiability of a given formula by implementing a proof-search algorithm in $\overline{\mathsf{LK}}$ is the same as that of accomplishing the same task by resorting to the tableaux method (most likely exponential). An interesting result would be to enhance the effectiveness of the proof-search algorithm by resorting to a complementary calculus dealing with analytic cuts [5, 16].

Acknowledgements

The authors would like to thank the anonymous reviewers for their many insightful comments and suggestions. Moreover, G.P. acknowledges the support from FCT grant PTDC/MHC-FIL/2583/2014.

References

[1] A. Avron, B. Konikowska and A. Zamansky. Modular construction of cut-free sequent calculi for paraconsistent logics. In *Proceedings of Logic in Computer Science (LICS 2012)*, M. Baaz et al., eds, pp. 85–94. IEEE, 2012.
[2] P. Bonatti and Olivetti N. Sequent calculi for propositional nonmonotonic logics. *ACM Transactions on Computational Logic*, **3**, 226–278, 2002.
[3] X. Caicedo. A formal system for the non-theorems of the propositional calculus. *Notre Dame Journal of Formal Logic*, **19**, 147–151, 1978.
[4] W. A. Carnielli. On sequents and tableaux for many-valued logics. *Journal of Non-Classical Logic*, **8**, 59–76, 1991.
[5] M. D'Agostino and M. Mondadori. The taming of the cut. classical refutations with analytic cut. *Journal of Logic and Computation*, **4**, 285–319, 1994.
[6] M. D'Agostino, M. Finger and D. M. Gabbay. Semantics and proof-theory of depth-bounded Boolean logics. *Theoretical Computer Science*, **480**, 43–68, 2013.
[7] M. D'Agostino. An informational view of classical logic. *Theoretical Computer Science*, **606**, 79–97, 2015.
[8] J.-Y. Girard, Y. Lafont and P. Taylor. *Proofs and types*. Cambridge University Press, (Cambridge Tracts in Theoretical Computer Science, 7), 1989.
[9] V. Goranko. Refutation systems in modal logic. *Studia Logica*, **53**, 299–324, 1994.
[10] S. Kleene. *Mathematical Logic*. John Wiley and Sons, 1967.
[11] J. Łukasiewicz. *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*. Clarendon Press, Oxford, 1951.
[12] M. Piazza and G. Pulcini. Unifying logics via context-sensitiveness. *Journal of Logic and Computation*, **27**, 21–40, 2017.
[13] M. Piazza and G. Pulcini. Uniqueness of axiomatic extensions of cut-free classical propositional logic. *Logic Journal of the IGPL*, 5, 708–718.
[14] T. Skura. A refutation theory. *Logica Universalis*, **3**, 293–302, 2009.
[15] T. Skura. Refutation systems in propositional logic. In *Handbook of Philosophical Logic*, D. M. Gabbay and F. Guenthner, eds, 2nd edn, Vol. 16, pp. 115–157. Kluwer, 2011.
[16] R. Smullyan. Analytic cut. *Journal of Symbolic Logic*, **33**, 560–564, 1968.
[17] A. Tamminga. Logics of rejection: two systems of natural deduction. *Logique & Analyse*, **146**, 169–208, 1994.
[18] M. L. Tiomkin. Proving unprovability. *Proceedings of LICS-88*, pp. 22–26. Edinburgh, 1988.
[19] M. L. Tiomkin. A sequent calculus for a logic of contingencies. *Journal of Applied Logic*, **11**, 530–535, 2013.

[20] Varzi, A. C. Complementary sentential logics. *Bulletin of the Section of Logic*, **19**, 112–116, 1990.

[21] A. C. Varzi. Complementary logics for classical propositional languages. *Kriterion*, **4**, 20–24, 1992.

Received 31 July 2016