**DOI: 10.18178/ijscer.7.1.15-21**

# The Use of Natural Period of Vibration as a Simplified Indicator of Second-Order Effects for RC Frames

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***Abstract*—In this study, the derivation of a simplified parameter ( $\chi_t$ ), based on the modal analysis of reinforced concrete frames, is presented. This simplified parameter can be used as an indicator of the susceptibility to global second-order effects as well as an amplifier to multiply first-order results in order to satisfactorily obtain the results of second-order analysis. The simplified amplification factor formulation is based on the Rayleigh's method, virtual work principles and the use of generalized coordinates to represent the behavior of flexible structures. The main advantage of a formulation based on a simple modal analysis is that natural period of vibration can be easily obtained using a finite element software. The simplified factor $\chi_t$ is developed for an equivalent cantilever beam-column and the results compared with the values of the overturning moment amplification ratio obtained from first- and second-order analyzes performed using a finite element software. The results obtained demonstrate a satisfactory agreement between the simplified amplification factor and the amplification overturning moment amplification ratio.**

***Index Terms*—Second-Order Effects, Modal Analysis, Structural Stability**

## I. Introduction

Given an increasingly need for slender structures, considering global second-order effects have become essential during a structural analysis; the effect of vertical loads on the deformed shape of the structure is no longer negligible, requiring a much more complex structural analysis to determine the final design internal loadings.

In order to determine the global second-order effects on a structure, different methods can be used. Among these methods, the most precise are the iterative methods, in which structural stiffness matrix is updated within each increment of load or displacement, as presented by Crisfield [1] and McGuire et al. [2]. In addition to iterative methods, there are direct methods as presented by Ruttenberg [3], Wilson and Habilullah [4] and White et al. [5], in which simplified amplification factors, or reduction factors applied to the structural stiffness matrix.

Recently, Statler et al. [6] demonstrated that the natural period of vibration could be used as an indicator of the global second-order effect for steel frames since both of these properties depend essentially on the mass and the stiffness of the structure. Further developments have demonstrated that this concept can be extended to reinforced concrete frames [7]. Reis [8] presented a study in which a simplified equation is derived using the principle of generalized coordinates and Rayleigh's method, in which the generalized displacement function has been chosen in order to respect the essential boundary conditions of the problem.

The work presented herein is a continuation of previous research efforts to derive a simplified equation based on the natural period of vibration (T), which can be used for reinforced concrete frame both as an indicator of the susceptibility to global second-order effects, as well as, a multiplier that amplifies first-order internal loadings and lead to results similar than the ones obtained through a second-order analysis. The main contribution presented herein is based on the fact that throughout the derivations the generalized displacement function respects the not only the essential, but also the natural boundary conditions.

The derivation of $\chi_t$ is presented and compared to a single bar example and a square symmetrical frame example; both example where modeled in SAP2000 [9] using the P-Delta function [10]. The total height of both examples are varied in order to compare the equation to a larger gamma of structure slenderness. The results demonstrate satisfactory agreement between the ratio of second-to-first order moments and the simplified factor $\chi_t$ proposed herein

Manuscript received May 11, 2017; revised December 18, 2017.


doi: 10.18178/ijscer.7.1.15-21

## II. Derivation of Amplification Factor $\chi_t$

The parameter $\chi_t$ is derived using Rayleigh's method [11] for flexible conservative systems, where is possible to determine the natural period of vibration of a generalized equivalent system. The system adopted in the derivation is a simplified equivalent cantilever beam with mass and stiffness distributed uniformly along its height; in addition, it is considered a continuous mass equally spaced along the height of the system representing the mass of each floor.

According to Paultre [12], the natural period of vibration of a flexible system can be obtained using a generalized coordinate system. The deformed shape of a flexible bar is taken as the product of two functions; one function describing the displacement of a convenient reference point of the system, which is known as the generalized coordinate, and another shape function describing the deformed position along its height, as presented in (1). The cantilever beam-column free-body diagram considered herein is depicted in Fig. 1.

$$u(x,t) = \psi(x) z(t) \tag{1}$$

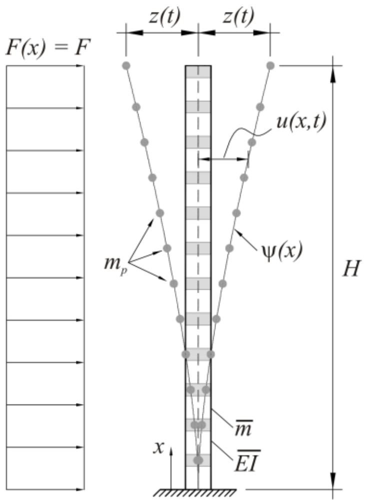


Figure 1. Equivalent cantilever beam-column representing the flexible system.

Using the virtual works principle, the following equation can be written

$$mz(t) + k_E z(t) + k_G z(t) = p(t). \tag{2}$$

Where $m$ is a generalized mass, $k_E$ is the generalized elastic stiffness and $k_G$ is the generalized geometric stiffness of the system. Thus:

$$m = \int_0^H \overline{m}(x)[\psi(x)]^2 \partial x + \sum m_i \psi_i^2 \tag{3}$$

$$k_E = \int_0^H EI(x)[\psi''(x)]^2 \partial x \tag{4}$$

$$k_G = -\int_0^H N(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^2 \partial x \tag{5}$$

$$p(t) = F \tag{6}$$

Applying Rayleigh's method [11] to this problem, the angular frequency is given by:

$$\omega^2 = \frac{\int_0^H EI(x)[\psi''(x)]^2 \partial x - \int_0^H N(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^2 \partial x}{\int_0^H \overline{m}(x)[\psi(x)]^2 \partial x + \sum m_i \psi_i^2} \tag{7}$$

The shape function chosen herein for the cantilever beam-column with axial and shear forces uniformly distributed respects the essential and natural boundary conditions of the problem, equation (8).

$$\psi(x) = \frac{1}{3}\left(\frac{x}{H}\right)^4 - \frac{4}{3}\left(\frac{x}{H}\right)^3 + 2\left(\frac{x}{H}\right)^2 \tag{8}$$

Substituting (8) into (7), neglecting the geometric stiffness variation, and considering the elastic modulus (E) and the inertia moment (I) constant along the height of the cantilever beam element, the angular frequency is given by:

$$\omega^2 = \frac{\overline{EI}\,\frac{16}{5H^3}}{\overline{m}\,\frac{104H}{405} + \sum_{i=1}^{n} m_i \psi_i^2} \tag{9}$$

Where, $m_i$ is the mass of each floor, which is also equal to $m_p$ . On the other hand, $\psi_i = \psi(x_i)$ and $x_i = i\frac{H}{n}$. By substituting the terms described above in the summation found in the denominator of equation (9), we have:

$$\sum m_i \psi_i^2 = m_p\left(\frac{208n^8 + 405n^7 + 180n^6 + 20n^2 - 3}{810n^7}\right) \tag{10}$$

Therefore, the angular frequency of the system is given by:

$$\omega^2 = \frac{\overline{EI}\,\frac{16}{5H^3}}{\overline{m}\,\frac{104H}{405} + m_p\left(\frac{208n^8 + 405n^7 + 180n^6 + 20n^2 - 3}{810n^7}\right)} \tag{11}$$

Given the angular frequency, equation (11), it is then possible to calculate the natural vibration period of vibration of the flexible system as:

$$T^2 = 4\pi^2 \frac{\bar{m}\frac{104H}{405} + m_p\left(\frac{208n^8 + 405n^7 + 180n^6 + 20n^2 - 3}{810n^7}\right)}{\overline{EI}\frac{16}{5H^3}} \quad (12)$$

The distributed mass of the columns ($\bar{m}$) and also the concentrated mass of the floors ($m_p$) can be given as a function of the total weight of the structure ($P$) and weight of each floor ($P_{sto}$):

$$k_{pav} = \frac{P_{sto}}{P} = 1 - \frac{P_{col}}{P} \quad (13)$$

$$\bar{m} = \frac{P_{col}}{gH} = \frac{P(1-k_{sto})}{gH} \quad (14)$$

$$m_p = \frac{k_{pav}P}{gn} \quad (15)$$

Substituting (13) into (15) and the product into (12), we have:

$$T^2 = \frac{P}{g}\frac{5\pi^2}{4}\frac{H^3}{\overline{EI}}\frac{208n^8 + k_{pav}\left(405n^7 + 180n^6 + 20n^2 - 3\right)}{810n^8} \quad (16)$$

According to Franco and Vasconcelos [13], the ratio between second order and first order bending moments ($M_2/M_1$), as defined in the Brazilian Standard [14] for reinforced concrete design, can be given by:

$$\frac{M_2}{M_1} = \frac{1}{1 - \frac{\Delta M_2}{M_1}} \quad (17)$$

The parcel $\Delta M_2$ is the bending moment caused by the vertical loads acting on the deformed shape of the structure and $M_1$ is the overturning moment due to lateral loads. Thus, equation (14) can be rewritten as:

$$\frac{M_2}{M_1} = \frac{1}{1 - \frac{\sum P_i u_i}{\sum F_i h_i}} \quad (18)$$

Where $u_i$ is the horizontal distance between the deformed and undeformed shape, where a vertical force $P_i$ is applied. The displaced shape is given by a linear analysis, where a horizontal force, $F_i$, is applied at the respective height, $h_i$. In the case of a cantilever beam-column with an uniformly distributed lateral load $F$, the deformed shape of the structure from a linear analysis can be given by:

$$u(x) = \frac{FH^4}{8EI}\left[\frac{1}{3}\left(\frac{x}{H}\right)^4 - \frac{4}{3}\left(\frac{x}{H}\right)^3 + 2\left(\frac{x}{H}\right)^2\right] \quad (19)$$

In addition, it shall also be mentioned that:

$$P_i = \frac{P}{n} \quad (20)$$

$$F_i = \frac{FH}{n} \quad (21)$$

$$h_i = i\frac{H}{n} = x_i \quad (22)$$

$$u_i = u(x_i) \quad (23)$$

Thus:

$$M_1 = \sum_{i=i}^{n} F_i \frac{h_i^2}{n^2} = FH^2\frac{n+1}{2n} \quad (24)$$

$$\Delta M_2 = \frac{PFH^4}{24EI}(n+1)\frac{36n^3 + 9n^2 + n - 1}{30n^4} \quad (25)$$

Substituting (24) and (25) in (17), the ratio between the second and first order bending moments is given by:

$$\frac{M_2}{M_1} = \frac{1}{1 - \frac{PH^2}{EI}\frac{36n^3 + 9n^2 + n - 1}{360n^3}} \quad (26)$$

As expected, the ratio between $M_2$ and $M_1$ is mainly a function of the mass and the stiffness of the structure. For any frame, the only parameter that needs to be determined in (26) is the global stiffness ($EI$). In order to overcome the difficulty of determining the global stiffness ($EI$), an equivalent stiffness ($\overline{EI}$) is obtained from the using (16), thus:

$$\overline{EI} = \frac{P}{g}\frac{5\pi^2}{4}\frac{H^3}{T^2}\frac{208n^8 + k_{pav}\left(405n^7 + 180n^6 + 20n^2 - 3\right)}{810n^8} \quad (27)$$

Substituting in (26) the equivalent stiffness obtained in (27), the amplification parameter $\chi_t$ is given by:

$$\chi_t = \frac{1}{1 - \frac{4T^2 g}{5\pi^2 H}\frac{810n^8}{208n^8 + k_{pav}\left(405n^7 + 180n^6 + 20n^2 - 3\right)}\frac{36n^3 + 9n^2 + n - 1}{360n^3}} \quad (28)$$

This equation can be rewritten as:

$$\chi_t = \frac{1}{1 - \frac{T^2 g}{\pi^2 H}\mu_n} \quad (29)$$

Where:

$$\mu_n = \frac{324n^8 + 81n^7 + 9n^6 - 9n^5}{1040n^8 + k_{pav}\left(2025n^7 + 900n^6 + 100n^2 - 15\right)} \quad (30)$$

In order to simplify the equation of the parameter $\chi_t$, a verification of the importance of each term in the numerator and in the denominator of the parameter $\mu_n$ was carried out. Fig. 2 and 3 depicts the representativeness of each term in the numerator and in the denominator of the expression of $\mu_n$. It is important to note that $k_{pav}$ is equal to 1, which is the value that leads to greater representativeness of the terms with smaller order.

Note in Fig. 2 that the first term in the numerator represents more than 90% of the total value when the number of stories is greater than or equal to three. The parcels in the denominator are depicted in Fig. 3; note that the first two terms of the polynomial equation represent more than 90% of the total value when the number of stories is greater than or equal to two. Thus, the parameter $\mu_n$ can be simplified as follows:

$$\mu_n = \frac{324n}{1040n + 2025k_{pav}} \quad (31)$$

Substituting (31) into (29) the following expression for the simplified factor $\chi_t$ can be obtained, where $k_{pav}$ can vary from 0.5 to 1.0; however, the authors consider 0.8 a value that represents well common practice.

$$\chi_t = \frac{1}{1 - \frac{T^2 g}{\pi^2 H}\frac{324n}{1040n + 2025k_{pav}}} \quad (32)$$

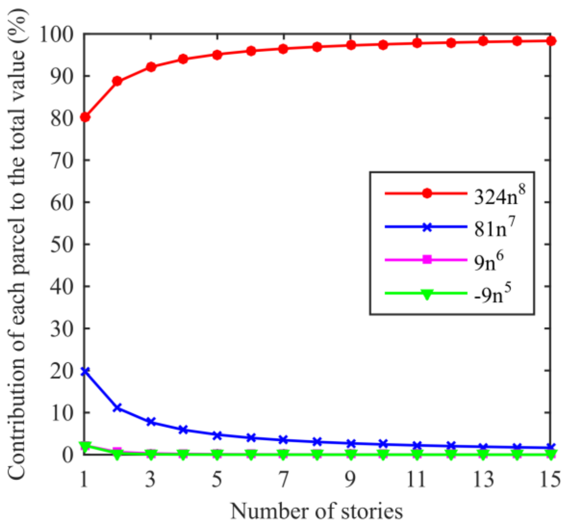


Figure 2. Sensibility analysis for the terms in the numerator of $\mu_n$.

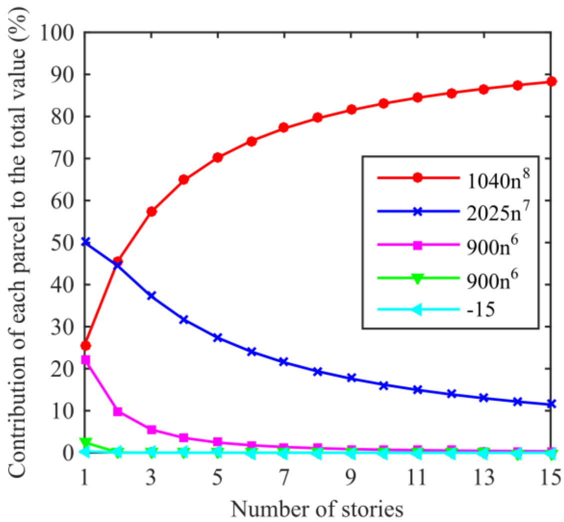


Figure 3. Sensibility analysis for the terms in the denominator of $\mu_n$.

## III. Case Study

In order to validate the accuracy of using the proposed simplified parameter, two case studies were carried out. The first model studied consists in a simple equivalent bar with concentrated masses representing the stories, and the second model consists in a simple frame with square floor plan. In both cases, the analyses are performed varying the number of stories from 1 to 15, resulting in 30 analyzes, for one orthogonal direction.

The structural analysis are performed in the finite element software SAP2000 [9]. The structural analyses are subdivided in: (i) modal analysis, (ii) first-order analysis, and (iii) second-order analysis. The modal analysis is necessary to obtain the natural period of vibration used to determine $\chi_t$, while first and second-order analysis are necessary to validate the method proposed herein. During the analysis, second-order effects are determined based on the method proposed by Wilson and Habibullah [10] using the P-Delta analysis from SAP2000.

The first case studied herein is a reinforced concrete cantilever beam-column with a square cross-section with 1.85 m sides. As one can see, the side size of the cross-section is unrealistic; however, the great size side is an artificial method to lead the model to a great bending stiffness and, thus, a small, but realistic, horizontal displacement. Along the height of the model, a concentrated load is applied every 3 m, to simulate the mass of each story, additionally to a uniformly distributed load representing the self-weight of the beam.

The second case studied herein is a fictitious tridimensional single bay reinforced concrete frame with a square floor plan as depicted in Fig. 4 and Fig. 5. The same dimensions for the structural members are adopted along the height of the frame (see Fig. 4) and no inference about cracking is considered. A value of 7.0 kN/m$^2$ is uniformly applied on the slabs to represent the total vertical gravitational load and a lateral uniformly distributed load of 1.39 kN/m$^2$ is applied along the height of the frame in one of the orthogonal directions. The natural period of vibration, as well as, first and second-order overturning moments are obtained for all the columns in the frame; the ratio $M_2/M_1$ is compared to the parameter $\chi_t$ calculated for different number of stories, but same floor plan.

It is worth mentioning that we consider that the most loaded column, leeward column, is most representative of the model. The ratio between first and second-order overturning moments is determined at the ends of the column being analyzed.

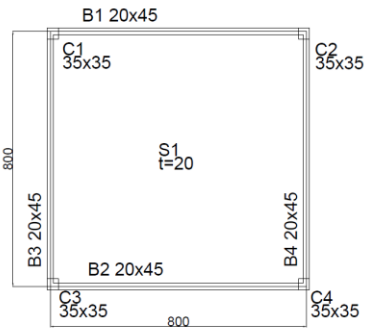


Figure 4. Floor plan for the square reinforced concrete frame analyzed (units in cm).

## IV. Results

In order to determine the accuracy of the method proposed herein, the natural period of vibration values obtained using equation (12) are compared to those obtained by Finite Element (FE) analysis; Fig. 6 depicts the comparison for the single equivalent cantilever beam-column case. As one can see, FE and simplification equation lead to very similar results, which demonstrates the robustness of the method proposed herein.

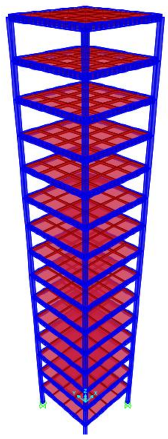

Figure 5. Isometric view of the square reinforced concrete frame.

In Fig. 7, the values of the ratio between $M_2/M_1$ obtained from the results of first-order and second-order FE analysis are compared with the values of the simplified amplification factor $\chi_t$ in the case of the equivalent cantilever beam-column. As depicted in Fig. 7, the simplified amplification factor are in satisfactory agreement with the ratio of second to first-order overturning moments obtained in the FE models.

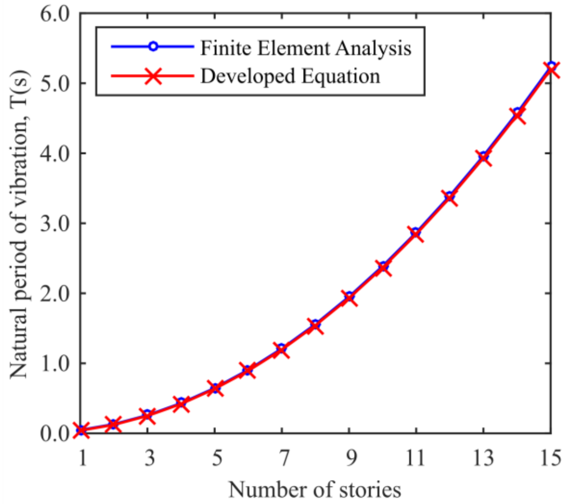


Figure 6. Natural period of vibration in function of the number of stories for the equivalent cantilever beam-column.

Fig. 8 depicts the comparison between overturning moment amplification ratio ( $M_2/M_1$ ) from the FE models of a concrete frame with square floor plan and the values found using the simplified amplification factor $\chi_t$ . As one can see, the simplified parameter leads to results in close agreement to those obtained for the most loaded columns in the FE models; the most loaded columns are also the columns of most interest to the structural engineer, since it will be determinant in the frame's structural design. The results obtained in this study are in accordance with those obtained by Statler et al. [6].

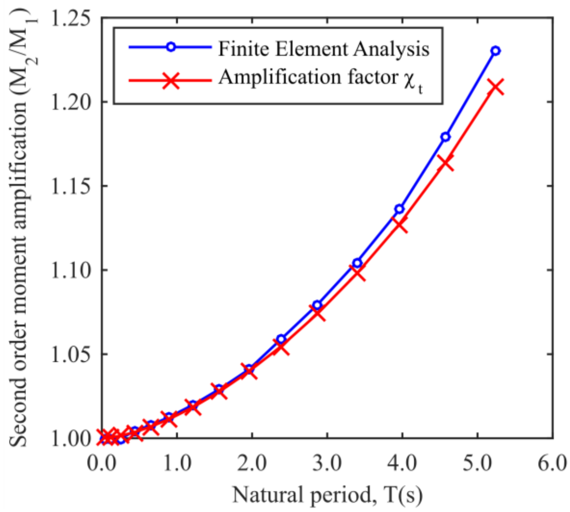


Figure 7. Plot of overturning moment amplification ($M_2/M_1$) against natural period of vibration ($T$) for the equivalent cantilever beam-column.

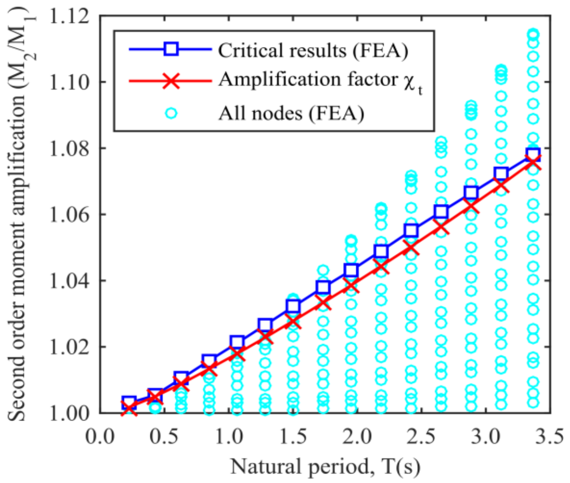


Figure 8. Plot of overturning moment amplification ($M_2/M_1$) against natural period of vibration ($T$) to the square concrete frame.

## V. Conclusions

The simplified parameter $\chi_t$ is mostly based on the natural period of vibration and geometric information; it can be used as an indicator of the susceptibility to global second order effects as well as an amplifier that transform first-order overturning moments into the respective overturning moment if a second-order analysis is carried out.

The derivation was carried out by means of the virtual works principle and Rayleigh's method applied to the solution of an equivalent cantilever beam-column. Taking in account the expression of bending moment amplification proposed by Franco and Vasconcelos [13], the simplified parameter is shown that natural period of vibration and second-order effects are essentially dependent on the mass and bending stiffness of the structure.

Two case studies were carried out: an equivalent cantilever beam-column and a fictitious regular reinforced concrete square floor plan frame. In both cases, the structure's height was varied from 3 to 45 m in intervals of 3 m, totalizing 15 different frame's height.

Although the method proposed herein can be easily calculated, it leads to satisfactory results and, in a few cases, a conservative approximation. It is important to note that only a simple beam-column and a regular frame were studied and a parametric study shall be carried out in order to verify the method's accuracy in a not regular floor plan. Additionally, a more realistic lateral load shall be applied that best represents wind loading on a frame facade; we have considered a uniform lateral load, but a triangular lateral loading is recommended to be implemented in the future.

## Acknowledgements

The authors would like to acknowledge the financial support of the São Paulo Research Foundation (FAPESP) under Grant No. 2015/18450 - 8, FAEPEX - UNICAMP under PAPDIC program No. 1274/2015, and S4 Sistemas Estruturais.

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