UNIVERSIDADE ESTADUAL DE
CAMPINAS

Instituto de Matemática, Estatística e
Computação Científica

NATHALIA LIMA CHAVES

Generalized Augmented Mixed Birnbaum-Saunders regression models

Modelos de regressão Birnbaum-Saunders Mistos Aumentados Generalizados

Campinas

2019

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em Estatística.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Statistics.

Supervisor: Caio Lucidius Naberezny Azevedo

Este exemplar corresponde à versão final da Tese defendida pela aluna Nathalia Lima Chaves e orientada pelo Prof. Dr. Caio Lucidius Naberezny Azevedo.

Campinas

2019

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

C398g Chaves, Nathalia Lima, 1989-
Generalized augmented mixed Birnbaum-Saunders regression models / Nathalia Lima Chaves. – Campinas, SP : [s.n.], 2019.

Orientador: Caio Lucidius Naberezny Azevedo.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Birnbaum-Saunders, Distribuição de. 2. Misturas de escala (Estatística). 3. Distribuição normal assimétrica. 4. Modelos lineares generalizados. 5. Inferência bayesiana. I. Azevedo, Caio Lucidius Naberezny, 1979-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Modelos de regressão Birnbaum-Saunders mistos aumentados generalizados
Palavras-chave em inglês:
Birnbaum-Saunders distribution
Scale mixtures (Statistics)
Skew-normal distributions
Generalized linear models
Bayesian inference
Área de concentração: Estatística
Titulação: Doutora em Estatística
Banca examinadora:
Caio Lucidius Naberezny Azevedo [Orientador]
Filidor Edilfonso Vilca Labra
Gilberto Alvarenga Paula
Juvêncio Santos Nobre
Mário de Castro Andrade Filho
Data de defesa: 26-04-2019
Programa de Pós-Graduação: Estatística

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0001-8932-8755
- Currículo Lattes do autor: http://lattes.cnpq.br/7379678423570217

Tese de Doutorado defendida em 26 de abril de 2019 e aprovada

pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Acknowledgements

First of all, I would like to thank God for this opportunity and for giving me health, strength, courage, and serenity to complete this work. Without Him, I would not be able to overcome the challenges of this journey.

I would like to thank my mother Ana, my father Afrânio (*In Memoriam*), and my grandmother Teresinha (*In Memoriam*) for their love, education, examples of honesty and constant encouragement. I would like to thank my sister Mariana and all my family for their love and for the good moments lived.

I am also very grateful to Roberta for her love, care and dedication during all these years. Thank you for sharing your knowledge with me and for being by my side at all times of this journey.

I would like to express my gratitude to my advisor, Professor Caio Lucidius Naberezny Azevedo, for his guidance, friendship, patience, teachings and suggestions.

I thanks also to the Department of Statistics and Applied Mathematics of the Federal University of Ceará and the Department of Statistics of the State University of Campinas.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

"Tranquiliza-te, nenhuma tarefa é
maior do que o teu espírito".

"Sonhar é verbo: é seguir, é pensar,
inspirar e fazer força, insistir, é lutar,
transpirar. São mil verbos que
vem antes do verbo realizar".
(Bráulio Bessa)

Resumo

Dados positivos (não-negativos), transversais ou longitudinais, com ou sem a presença de zeros, apresentando assimetria e/ou caudas pesadas, são frequentes em diversas áreas do conhecimento como: Biologia, Química, Física, Medicina, Psicometria, entre outras. Nesse sentido, modelos de regressão baseados na distribuição Birnbaum-Saunders (BS) e na correspondente distribuição log-BS, tem tido um papel bastante importante. No entanto, para alguns desses modelos, a resposta original deve ser transformada através da transformação logarítmica, o que pode levar a dificuldades de interpretação de resultados e problemas inferenciais. Com o intuito de contornar esse problema, foram desenvolvidos modelos de regressão baseados em uma reparametrização da distribuição BS, que permitem a análise dos dados em sua escala original e possibilitam que tanto a média quanto o parâmetro de dispersão sejam modelados por preditores apropriados através de funções de ligação adequadas. Neste trabalho, com base nessa parametrização, desenvolvemos uma ampla família de modelos de regressão BS mistos aumentados (ou não) no zero, para dados positivamente ou negativamente assimétricos, que apresentam ou não caudas pesadas. Inicialmente, propusemos uma classe de distribuições de probabilidade BS aumentadas e não-aumentadas, considerando a família de distribuições de mistura de escala normal assimétrica centrada. Várias de suas propriedades foram desenvolvidas. Com base nessas famílias, foram propostas classes de modelos de regressão BS de efeitos fixos e mistos, aumentadas e não-aumentadas. Sob o ponto de vista Bayesiano, desenvolvemos estimação paramétrica, análise de resíduos, estatísticas de comparação de modelos e checagem preditiva a posteriori, baseadas nos algoritmos MCMC. Realizamos estudos de simulação considerando diferentes cenários de interesse prático a fim de avaliar o desempenho das metodologias propostas, incluindo as classes de modelos, os métodos de estimação e as medidas de diagnóstico e comparação de modelos. Além disso, ilustramos as ferramentas desenvolvidas através da análise de conjuntos de dados reais, os quais serviram como motivação para este trabalho.
Palavras-chave: Distribuição Birnbaum-Saunders. Distribuições de mistura de escala normal assimétrica centrada. Modelos aumentados em zero. Modelos lineares generalizados mistos. Inferência Bayesiana. Algoritmos MCMC. Verificação da qualidade do ajuste. Comparação de modelos.

Abstract

Positive (non-negative), cross-sectional and longitudinal data, with or without the presence of zeros, presenting asymmetry and/or heavy tails, are frequently observed in several fields of knowledge such as: Biology, Chemistry, Physics, Medicine, Psychometrics, among others. In this sense, regression models based on the Birnbaum-Saunders (BS) and the correspondent log-BS distribution have been playing an important. However, for some of these models, the original response must be transformed to a logarithmic scale, which could lead to some difficulties of the interpretation of results and inferential problems. To overcome this problem, regression models based on a reparameterizated BS distribution were proposed. This parameterization allows to analyze data in their original scale and allows for modeling both the mean and the dispersion parameter through suitable predictors using appropriate link functions. In this work, based on this reparameterizated BS distribution, we developed a general family of mixed BS regression models, augmented (or not) by zero, for positively or negatively skewed data, presenting or not heavy-tails. Initially, we propose families of non-augmented and zero-augmented BS distributions, considering the family of scale mixture of the centred skew-normal distributions. Several of their properties are developed. Based on these families, fixed and random effects BS regression models were proposed. We developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking under the Bayesian paradigm based on MCMC algorithms. We conducted simulation studies considering different scenarios of practical interest, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation method, the diagnostic measures and the statistics of model comparison. Furthermore, we illustrate the developed tools through the analysis of real data sets, which motivated the developments for this work.
Keywords: Birnbaum-Saunders distribution. Skew scale-mixture of normal distributions. Zero-augmented models. Generalized linear mixed models. Bayesian inference. MCMC algorithms. Model fit assessment. Model comparison.

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List of abbreviations and acronyms

BS	Birnbaum-Saunders
CSN	Centred skew-normal
CSSBS	Centred skew scale-mixture Birnbaum-Saunders
CSSMN	Centred skew scale-mixture of normal
SCN	Centred skew contaminated normal
SCNBS	Centred skew contaminated normal Birnbaum-Saunders
SGt	Centred skew generalized Student-t
SGtBS	Centred skew generalized Student-t Birnbaum-Saunders
SN	Skew-normal
SNBS	Centred skew-normal Birnbaum-Saunders
SSBS	Skew scale-mixture Birnbaum-Saunders
SSL	Centred skew slash
SSLBS	Centred skew slash Birnbaum-Saunders
SSMN	Skew scale-mixture of normal
St	Centred skew Student-t
StBS	Centred skew Student-t Birnbaum-Saunders
ZABS	Zero-adjusted Birnbaum-Saunders
ZA-SCNBS	Zero-augmented centred skew contaminated normal Birnbaum-Saunders
ZA-SGtBS	Zero-augmented centred skew generalized Student-t Birnbaum-Saunders
ZA-SNBS	Zero-augmented centred skew normal Birnbaum-Saunders
ZA-SSBS	Zero-augmented centred skew scale-mixture Birnbaum-Saunders
ZA-SSLBS	Zero-augmented centred skew slash Birnbaum-Saunders
ZA-StBS	Zero-augmented centred skew Student-t Birnbaum-Saunders

Contents

Introduction

The class of Birnbaum-Saunders (BS) models was developed from problems that arose in the field of material reliability. These problems generally are related to the study of material fatigue. However, this class of models has been applied in areas outside that context, such as in health sciences, environmental, forestry, demographic, actuarial, financial, among others, due to its great versatility and its attractive properties. The BS distribution is unimodal, positively skewed and has two parameters, which correspond to the shape and scale of the distribution.

Since the pioneering work of Birnbaum and Saunders (1969b) was published, several BS-type distributions have been developed in the last years. Furthermore, regression models based on the BS and the correspondent log-BS distribution have been playing an important. Rieck and Nedelman (1991) were pioneers in this line. They proposed log-linear regression models based on the log-BS distribution and applied them to fatigue data, whereas Galea et al. (2004) and Xie and Wei (2007) developed several diagnostic tools for this model. Barros et al. (2008) assumed that the cumulative damage follows a Student-t distribution, developed the BS-t distribution and then introduced the BS-t log-linear regression models and their diagnostics, and applied them to the survival data of patients with lung cancer. Furthermore, Paula et al. (2012) applied the BS-t log-linear models to insurance data. Extensions of the BS distribution based on the skew-elliptical distributions can be found in Vilca and Leiva (2006), and in Vilca et al. (2011) and Chaves et al. (2019b) that proposed BS distributions based on the usual (Azzalini, 1985) and the centred (Azzalini, 2013) versions of the skew-normal (SN) distribution, respectively, and developed the correspondent log-BS regression models. Recently, for positive data, presenting asymmetry and heavy tails, Balakrishnan et al. (2017) proposed a new family of skew scale-mixture Birnbaum-Saunders (SSBS) distributions, considering the usual BS distribution and a skewed version of the scale-mixture of normals (SSMN) model (da Silva Ferreira et al., 2011). Recently, Sánchez (2018) developed the modeling of extreme percentiles through the family of SSBS distributions based on the frequentist approach. In the context of longitudinal data, Villegas et al. (2011) proposed a random-effects log-linear model based on the BS distribution, and Desmond et al. (2012) applied them to the failure times of a particular kind carbon fiber.

For all of these probability and regression models, the original response must be transformed to a logarithmic scale, which could lead to some difficulties of the interpretation of the results and inferential problems (see Huang and Qu (2006)). In this context, Santos-Neto et al. (2012) developed a new parameterization for the BS distribution, which allow us to analyze data in their original scale. Based on this reparameterizated BS distribution, Leiva et al. (2014) developed a BS regression model for modeling the mean through suitable predictors using appropriate link functions. Recently, Santos-Neto et al. (2016) extended the work by Leiva et al. (2014) and proposed a BS regression model with precision varying.

Despite the wide use of the BS distribution, it is well defined only for positive values. However, zero-augmented positive data are frequently observed in several fields of knowledge such as: Biology, Chemistry, Physics, Medicine, Psychometrics, among others. In this context, Leiva et al. (2016) adapted the reparametrizated BS distribution (Santos-Neto et al., 2012) that considers positive probability (for this event) giving rise to the zero-adjusted BS (ZABS) model. Recently, Tomazella et al. (2018), proposed the ZABS regression model with fixed-effects, and applied them to the fumonosin production by *Fusarium verticilliodes* in corn grains. In the work of Batista (2018), the ZABS regression model with random-effects was developed under the Bayesian approach.

In this work, based on the reparameterizated BS distribution (Santos-Neto et al., 2012), we developed a general family of mixed BS regression models, augmented (or not) by zero, for positively or negatively skewed data, presenting or not heavy-tails. Initially, we propose families of non-augmented and zero-augmented BS distributions, considering the family of scale mixture of the centred skew-normal distributions. The centred parameterization of the SN distribution circumvents some inferential problems, which were inherited from the usual SN distribution, and facilitates the calculations of the moments for the proposed models. Our family allows to analyze data in their original scale, and allows for modeling the mean, the dispersion parameter and the probability of a point mass at zero through suitable predictors using appropriate link functions. Based on the proposed probability models, fixed and random-effects BS regression models were proposed. We developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking under the Bayesian paradigm based on MCMC algorithms. We conducted simulation studies considering different scenarios of practical interest, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the diagnostic measures and the statistics for model comparison. Furthermore, we illustrate the developed tools through the analysis of real data sets, which motivated the developments for this work.

Motivation

The motivating data set for this work comes from a bilirubin concentration study in newborns conducted in the Escola Paulista de Medicina (UNIFESP) in 2005, see Draque (2005). In this study it was measured the bilirubin concentration (μ mol/L) in 89 healthy full-term newborns on breastfeeding for 1, 2, 3, 4, 5, 6, 8, 10 and 12 days after birth. The main objective of the researchers was to explain the bilirubin concentration as a function of age. In Table 1, Figure 1 and Figure 2, we present some descriptive analysis.

Neonatal jaundice, or jaundice, is the discoloration of the skin and sclera color to yellowish in a newborn by bilirubin excess. Is a common clinical problem encountered during the neonatal period, especially in the first week of life. Nearly 8% to 11% of neonates develop hyperbilirubinemia. In neonates, the dermal icterus is first noted in the face and when the bilirubin level rises, it proceeds to the body and then to the extremities.

This study is irregular, balanced and complete (89 observations for each evaluation condition and 9 per subject). Longitudinal studies are classified as regular, when the interval between two consecutive measurements is constant; otherwise, they are called irregular. In addition, they can be classified as balanced, if the measurements are obtained at the same evaluation moments in all the experimental units; otherwise, they are said to be unbalanced. Also, these studies are complete if there are no missing observations (missing data) and incomplete observations, if any.

The response variable, bilirubin concentration, is non negative, presenting a mixed structure of a continuous (positive) part and observations equals to zero, see Table 1. Also, there is a negative/positive asymmetry in the empirical distribution of the bilirubin concentration (see Figure 1), which is confirmed by the sample skewness, see Table 1. Figure 2 present individual and mean longitudinal profiles for 89 healthy full-term newborns. In general, the bilirubin concentration decreases over time for most patients but with substantial between subject variability.

More specific details concerning of the bilirubin concentration study in newborns are presented in Chapter 6.

Table 1 – Descriptive statistics for the bilirubin concentration.

	Days of birth								
	1	2	3	4	5	6	8	10	12
% of zeros	3	3	8	7	8	10	10	9	10
Mean	4.436	5.758	5.874	5.585	5.020	4.600	3.991	3.587	3.118
Median	4.500	6.100	5.800	4.600	4.100	3.300	2.800	2.700	2.100
SD	1.994	2.916	3.863	4.120	4.025	3.905	3.701	3.391	2.974
CV	44.952	50.643	65.767	73.756	80.170	84.890	92.732	94.558	95.391
CS	-.099	-.266	.108	.269	.470	.523	.839	1.040	.994
CK	2.557	2.257	2.025	1.838	2.043	1.951	2.584	3.357	3.285
n	89	89	89	89	89	89	89	89	89

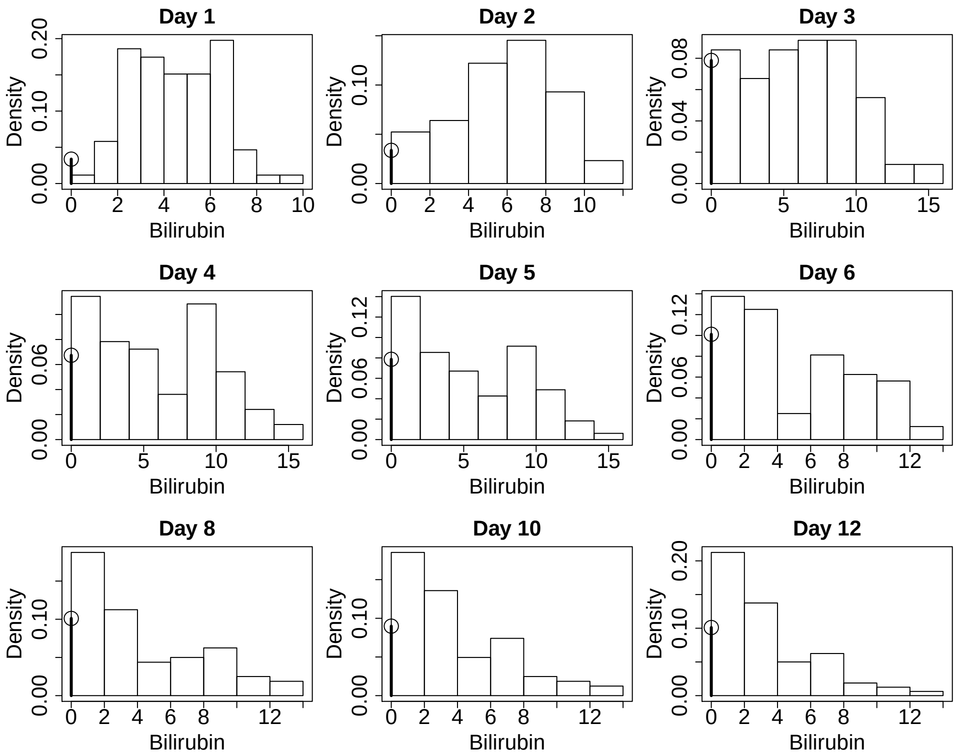

Figure 1 – Distribution of the bilirubin concentration.

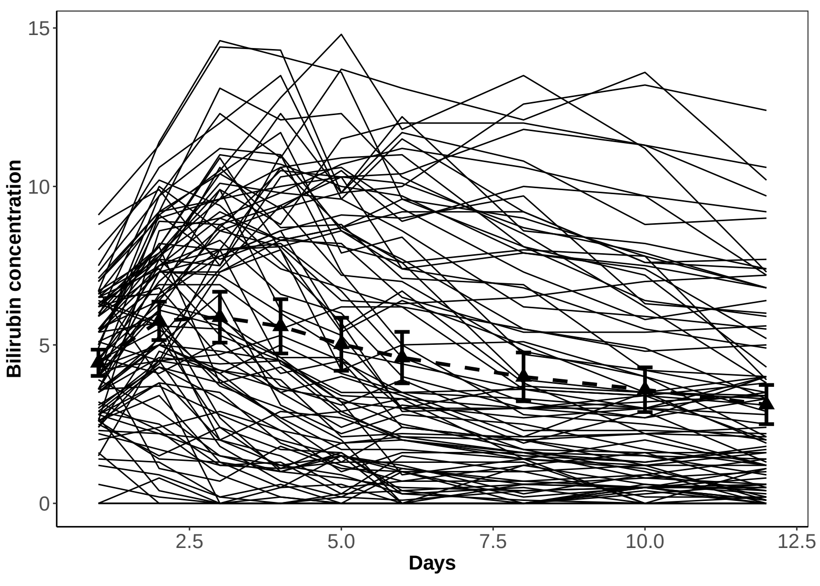

Figure 2 – Individual and mean longitudinal profiles for 89 healthy full-term newborns.

1 A new class of generalized Birnbaum-Saunders distributions

1.1 Introduction

The Birnbaum-Saunders (BS) distribution is a positively skewed model that has received considerable attention in the last two decades. This is as result of its theoretical arguments associated with cumulative damage processes, its close relationship to the normal distribution, and its attractive properties, such as (i) it has two parameters, modifying its shape and scale, (ii) it has positive skewness, but due to its flexibility, symmetric data can also be modeled, (iii) its scale parameter is also its median, among others. These aspects of the BS model render it as an alternative to the distributions for data with positive support and positive skewness, such as the gamma, inverse gamma, lognormal and Weibull distributions. The BS distribution is related to the normal distribution through the following stochastic representation

$$T = \eta\left[\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right]^2, \tag{1.1}$$

where $Z \sim \mathrm{N}(0,1)$. The notation $T \sim \mathrm{BS}(\alpha,\eta)$ is used in this case, where $\alpha > 0$, $\eta > 0$ are shape and scale parameters, respectively. Since the pioneering work of Birnbaum and Saunders (1969b) was published, several extensions of the BS distribution have been proposed in the literature.

From a frequentist view point, Birnbaum and Saunders (1969a) presented a discussion on the maximum likelihood estimation of the parameters of this model. Mann et al. (1974) showed that the BS distribution is unimodal. Engelhardt et al. (1981) developed confidence intervals and hypothesis tests for each one of the two parameters. Desmond (1985) developed a BS-type distribution based on a biological model. Desmond (1986) investigated the relationship between the BS distribution and the inverse Gaussian distribution. Lu and Chang (1997) used bootstrap methods to construct prediction intervals for future observations. On the other hand, from a Bayesian perspective there are few works on the BS distribution. The first one is due to Achcar (1993) who developed Bayesian estimation using numerical approximations for the marginal posterior distributions of interest based on the Laplace approximation. Also, Xu and Tang (2011) presented a Bayesian study with partial information while Wang et al. (2016) assumed that the parameters follow inverse gamma distributions. All these results were studied considering the definition presented in Equation (1.1).

Most of the generalizations of the BS distribution are based on the elliptical and skew-elliptical distributions, obtaining more robust and flexible models. Diaz-Garcia and Leiva (2005), for example, generalized the BS distribution using the elliptical distributions that includes the Cauchy, Laplace, Logistic, Normal and Student-t distributions as particular cases. Other works are: the generalized BS distribution Leiva et al. (2008), the Student-t BS distribution Barros et al. (2008), and the scale-mixture of normal BS distributions Balakrishnan et al. (2009), which correspond to a flexible heavy-tailed family of distributions. More information can be found in Barros et al. (2009) and Leiva (2016), which present a review of the BS distribution. Other generalizations have been obtained, as Owen and Padgett (1999) who developed a three-parameter BS distribution and the β-BS distribution presented in Cordeiro and Lemonte (2011). Extensions of the BS distribution based on the skew-elliptical distributions can be found in Vilca and Leiva (2006), and in Vilca et al. (2011) and Chaves et al. (2019b) that proposed BS distributions based on the usual (Azzalini, 1985) and the centred (Azzalini, 2013) versions of the skew-normal (SN) distribution, respectively. Recently, Poursadeghfard et al. (2018) developed an extended BS based on the skew-t-normal distribution.

Usually, in many practical situations, besides the presence of heavy tails, data such as lifetimes, family incomes, and pollutant concentrations also present skewness. Therefore, these two characteristics should be properly modeled. In this context, Balakrishnan et al. (2017) proposed a new family of the skew scale-mixture Birnbaum-Saunders (SSBS) distributions, considering the usual BS distribution and the skew scale-mixture of normal (SSMN) models (da Silva Ferreira et al., 2011). Recently, Sánchez (2018) developed the modeling of extreme percentiles through the family of SSBS distributions based on the frequentist approach. The SSBS distributions are based on the usual SN distribution which, despite of has been applied in many situations, it presents problems of singularity of the Fisher information matrix, when the asymmetric parameter is equals to zero. To overcome this problem, Arellano-Valle and Azzalini (2008) and Azzalini (2013) explored the SN distribution under a convenient parameterization, named *the centred parameterization*, known as centred SN (CSN) distribution, which leads the Fisher information matrix be non-singular. Moreover, the relative profile log-likelihood function (RPLL) for Pearson's index of skewness (γ) exhibits a more regular behavior, much closer to quadratic functions, and without a stationary point at $\gamma = 0$. Recently, Chaves et al. (2019b) showed that all these desired properties are transferred to the BS distribution based on the CSN distribution.

In this chapter, we developed a general family of BS distributions, named centred skew scale-mixture Birnbaum-Saunders (CSSBS) distributions. Besides to consider the reparameterization of the BS distribution proposed by Santos-Neto et al. (2012), which allows to write the respective mean on the related density and which can be very useful for regression models on the original scale of the response variable, we considered the centred skew scale-mixture of normal (CSSMN) distributions (Maioli, 2018), which facilitates the calculations of the moments of our distributions. Our family accommodate properly both positively or negatively skewed data, presenting or not heavy tails, and generalizes the scale-mixture of normal BS distributions Balakrishnan et al. (2009). Several of its properties are developed and we provided empirical evidences that the CSSBS distributions have advantages in inferential terms, over the SSBS distributions (Balakrishnan et al., 2017), similarly to the advantages of the CSN distribution (Azzalini, 2013) compared with usual SN distribution (Azzalini, 1985). Furthermore, we developed parameter estimation, statistics of model comparison, and posterior predictive checking through Bayesian inference based on MCMC algorithms. The impact of some factors of interest (sample size, asymmetry levels, and different degrees of freedom) on the estimates, are measured through of a simulation study on parameter recovery. Finally, we have presented an application to a real data set related to the breaking stress of carbon fibres (in Gba), showing the usefulness of the inferential methods developed here. Also, the results indicate that the heavy-tailed models outperforms the centred skew-normal BS in terms of model fit.

1.2 Skew scale-mixture of normal distributions

1.2.1 The centred skew-normal distribution

A r.v Z is said to have a centred skew-normal (CSN) distribution (Azzalini, 2013), denoted by $Z \sim \mathrm{CSN}(\varepsilon, \varpi^2, \gamma)$, where $\varepsilon \in \mathbb{R}$, $\varpi^2 \in (0, \infty)$ and $\gamma \in (-.99527, .99527)$ are the mean, the variance and Pearson's skewness coefficient, respectively, if its density is given by:

$$f(z|\varepsilon, \varpi^2, \gamma) = 2\frac{\sigma_z}{\varpi}\phi\Big[\mu_z + \frac{\sigma_z}{\varpi}(z-\varepsilon)\Big]\Phi\left\{\lambda\Big[\mu_z + \frac{\sigma_z}{\varpi}(z-\varepsilon)\Big]\right\}\mathbb{1}\left\{z \in \mathbb{R}\right\}, \tag{1.2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the density and the cumulative distribution function (cdf) of the N(0,1) distribution, respectively. Also, $\mu_z = r\delta$, $\sigma_z^2 = 1-\mu_z^2$, $\lambda = \gamma^{1/3}s/\sqrt{r^2 + s^2\gamma^{2/3}(r^2-1)}$, $r = \sqrt{2/\pi}$, $s = [2/(4-\pi)]^{1/3}$, $\gamma = r\delta^3(4/\pi - 1)(\sigma_z^2)^{-3/2}$ and $\delta = \dfrac{\lambda}{\sqrt{1+\lambda^2}}$. For $\varepsilon = 0$ and $\varpi^2 = 1$, we have the standard CSN distribution, denoted by $Z \sim \mathrm{CSN}(0, 1, \gamma)$, and its density is given by $f(z|\gamma) = 2\sigma_z\phi\Big(\mu_z + \sigma_z\, z\Big)\Phi\Big[\lambda\Big(\mu_z + \sigma_z\, z\Big)\Big]\mathbb{1}\left\{z \in \mathbb{R}\right\}$. The CSN is a particular case of the class distributions presented below.

1.2.2 Centred skew scale-mixture of normal distributions

The family of scale-mixtures of normal distributions was first introduced by Branco and Dey (2001) and it is often used to model symmetrical data. However, very often, we observe data set skewed and/or heavy-tailed behavior. Some examples are the data on family income (Azzalini and Capitanio, 2003) or substance concentration (Galea-Rojas et al. (2003) and Bolfarine and Lachos (2007)). In this context, da Silva Ferreira et al. (2011) proposed the family of the SSMN distributions. In order to circumvent some inferential problems, which were inherited from the usual SN distribution, Maioli (2018) developed a new family of the CSSMN distributions.

Definition 1. *A r.v Y is said to have a CSSMN distribution (Maioli, 2018), if it can be represented as*

$$Y = \mu_0 + \sigma_0 k(U)^{1/2} Z, \tag{1.3}$$

where $\mu_0 \in \mathbb{R}$, $\sigma_0 > 0$ are mean and scale parameters, respectively, Z follows the CSN distribution (Azzalini, 2013), denoted by $Z \sim CSN(0, 1, \gamma)$, with zero mean, variance one, and Pearson's skewness coefficient γ. Furthermore, U is a positive r.v, independent of Z, with cdf $G(\cdot; \boldsymbol{\nu})$ known as mixing scale distribution, indexed by a (possibly multivariate) parameter $\boldsymbol{\nu}$. We use the notation $Y \sim CSSMN(\mu_0, \sigma_0^2, \gamma, \boldsymbol{\nu})$.

When $\mu_0 = 0$, $\sigma_0 = 1$, and $k(U) = 1$, we recovery the standard CSN distribution. Furthermore, when $\gamma = 0$, we recovery the scale-mixture of normal distributions.

From Definition 1 it follows that the mean, variance, the Pearson's skewness (γ_Y) and kurtosis (kurt(Y)) coefficients of Y are given, by

$$\mathbb{E}(Y) = \mu_0, \quad \mathbb{V}(Y) = \sigma_0^2 \mathbb{E}[k(U)],$$

$$\gamma_Y = r k_3 \delta^3 (4/\pi - 1)[k_2 \sigma_z^2]^{-3/2} \text{ and } \operatorname{kurt}(Y) = k_2^{-2} k_4 \left[2(\pi - 3) \frac{4}{\pi^2} \delta^4 \left(1 - \frac{2}{\pi} \delta^2 \right)^{-2} + 3 \right],$$

where $k_m = \mathbb{E}[k(U)^{m/2}], m = 2, 3, 4$ and the other quantities were defined in Equation (1.2).

Although we can consider some functions $k(\cdot)$, in this work we restrict our attention to $k(U) = U^{-1}$. Thus, the density of Y is given by

$$\begin{aligned} \phi_{\text{SSMN}}(y|\mu_0, \sigma_0^2, \gamma, \boldsymbol{\nu}) \quad &= \quad 2 \int_0^\infty \phi \left[y \middle| \mu_0 - \frac{\sigma_0 \mu_z}{\sqrt{u} \sigma_z}; \frac{\sigma_0^2}{u \sigma_z^2} \right] \Phi \left\{ \lambda \left[\mu_z + \sigma_z \sqrt{u} \left(\frac{y - \mu_0}{\sigma_0} \right) \right] \right\} \\ &\quad \times dG(u|\boldsymbol{\nu}) \mathbb{1} \{ y \in \mathbb{R} \}, \end{aligned} \tag{1.4}$$

where $\phi(\cdot|\varrho, \tau^2)$ denotes the density of the $\mathrm{N}(\varrho, \tau^2)$ distribution and all other quantities were defined in Equation (1.2).

1.2.3 Examples of CSSMN distributions

In this section, we present some particular cases of the class of CSSMN distributions, which are detailed explored in this work.

- *The centred skew generalized Student-t (SGt) distribution*, denoted by $Y \sim \text{SGt}(\mu_0, \sigma_0^2, \gamma, \nu_1, \nu_2)$. Considering $U \sim \text{gamma}(\nu_1/2, \nu_2/2); \nu_1 > 2, \nu_2 > 0$ in Equation (1.4), the density of Y is given by

$$\begin{aligned} f(y|\mu_0, \sigma_0^2, \gamma, \nu_1, \nu_2) &= 2(\nu_2/2)^{\nu_1/2} \left[\Gamma(\nu_1/2)\right]^{-1} \int_0^\infty \phi\left[y\middle|\mu_0 - \frac{\sigma_0\mu_z}{\sqrt{u}\sigma_z}; \frac{\sigma_0^2}{u\sigma_z^2}\right] \\ &\times \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{u}\left(\frac{y-\mu_0}{\sigma_0}\right)\right]\right\} u^{\nu_1/2-1} \exp\left\{\frac{-\nu_2}{2}u\right\} du \mathbb{1}\left\{y \in \mathbb{R}\right\}. \end{aligned}$$

Using the moments of U, $\mathbb{E}(U^{-m}) = (\nu_2/2)^m\Gamma(\nu_1/2 - m)/\Gamma(\nu_1/2), m < \nu_1/2$, the mean and variance of Y are given, respectively, by

$$\mathbb{E}(Y) = \mu_0 \text{ and } \mathbb{V}(Y) = \sigma_0^2\left(\frac{\nu_2}{\nu_1 - 2}\right), \nu_1 > 2, \nu_2 > 0. \tag{1.5}$$

Note that when the parameters ν_1, $\nu_2 \to \infty$ the SGt distribution reduces to the CSN distribution.

When $\nu_1 = \nu_2 = \nu$, Y follows the skew Student-t (St) distribution, denoted by $Y \sim \text{St}(\mu_0, \sigma_0^2, \gamma, \nu)$.

- *The centred skew slash (SSL) distribution, denoted by $Y \sim SSL(\mu_0, \sigma_0^2, \gamma, \nu)$.* Considering $U \sim \text{beta}(\nu, 1); \nu > 1$ in Equation (1.4) the density of Y is given by

$$\begin{aligned} f(y|\mu_0, \sigma_0^2, \gamma, \nu) &= 2\nu \int_0^1 \phi\left[y\middle|\mu_0 - \frac{\sigma_0\mu_z}{\sqrt{u}\sigma_z}; \frac{\sigma_0^2}{u\sigma_z^2}\right] \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{u}\left(\frac{y-\mu_0}{\sigma_0}\right)\right]\right\} \\ &\times u^{\nu-1}\mathbb{1}\left\{y \in \mathbb{R}\right\}. \end{aligned}$$

Furthermore, using the moments of U, $\mathbb{E}(U^{-m}) = \nu/(\nu - m), \nu > m$, the mean and variance of Y are given, respectively, by

$$\mathbb{E}(Y) = \mu_0 \text{ and } \mathbb{V}(Y) = \sigma_0^2\left(\frac{\nu}{\nu - 1}\right), \nu > 1. \tag{1.6}$$

Note that the CSN distribution is a special case of the SSL distribution when $\nu \to \infty$.

- *The centred skew contaminated normal (SCN) distribution, denoted by $Y \sim SCN(\mu_0, \sigma_0^2, \gamma, \nu_1, \nu_2)$.* Taking $g(u|\nu_1, \nu_2) = \nu_1\mathbb{1}_{\{\nu_2\}}(u) + (1 - \nu_1)\mathbb{1}_{\{1\}}(u); \nu_1, \nu_2 \in (0, 1)$, in Equation (1.4), the density of Y is given by

$$\begin{aligned} f(y|\mu_0, \sigma_0^2, \gamma, \nu_1, \nu_2) &= 2\left\{\nu_1\phi\left[y\middle|\mu_0 - \frac{\sigma_0\mu_z}{\sqrt{\nu_2}\sigma_z}; \frac{\sigma_0^2}{\nu_2\sigma_z^2}\right] \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{\nu_2}\left(\frac{y-\mu_0}{\sigma_0}\right)\right]\right\}\right. \\ &+ \left.(1-\nu_1)\phi\left[y\middle|\mu_0 - \frac{\sigma_0\mu_z}{\sigma_z}; \frac{\sigma_0^2}{\sigma_z^2}\right] \Phi\left\{\lambda\left[\mu_z + \sigma_z\left(\frac{y-\mu_0}{\sigma_0}\right)\right]\right\}\right\} \\ &\times \mathbb{1}\left\{y \in \mathbb{R}\right\}. \end{aligned}$$

Furthermore, using the moments of U, $\mathbb{E}(U^{-m}) = \nu_1/\nu_2^m + 1 - \nu_1$, the mean and variance of Y are given, respectively, by

$$\mathbb{E}(Y) = \mu_0 \text{ and } \mathbb{V}(Y) = \sigma_0^2\left(\frac{\nu_1}{\nu_2} + 1 - \nu_1\right). \tag{1.7}$$

Note that when $\nu_2 \to 1$ and/or $\nu_1 \to 0$, the SCN distribution reduces to the CSN distribution.

Figures 3-5 present the densities of the SGt, SSL and SCN distributions, respectively, for different values of γ and $\boldsymbol{\nu}$. For all distributions, we can notice that the negative asymmetry is observed when γ assumes negative values, whereas the positive asymmetry is observed when γ assumes positive values. The SGt, SSL and SCN distributions have tails (much) havier that than the CSN distribution, when $\nu_1 \to 0$, $\nu \to 0$ and $\nu_2 \to 0$ and $\nu_1 \to 1$, respectively.

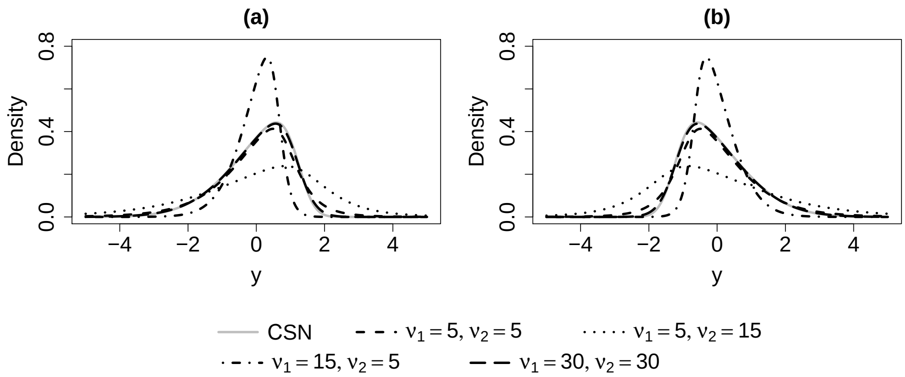

Figure 3 – Densities of the SGt distribution for different values of γ, ν_1 and ν_2, with $\mu_0 = 0$ and $\sigma_0^2 = 1$. (a) $\gamma = -.8$ and (b) $\gamma = .8$.

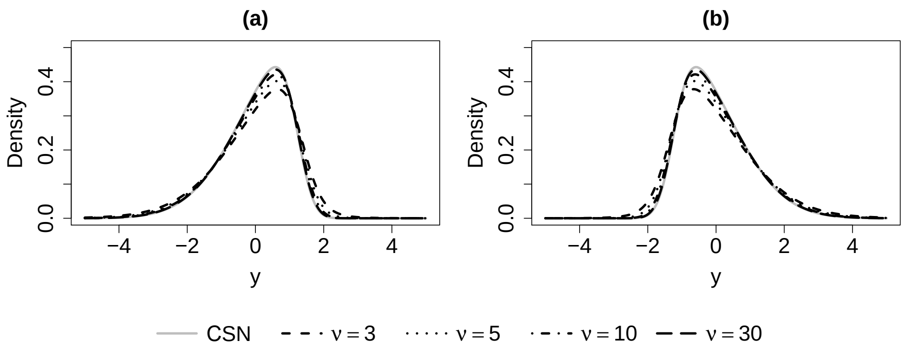

Figure 4 – Densities of the SSL distribution for different values of γ and ν, with $\mu_0 = 0$ and $\sigma_0^2 = 1$. (a) $\gamma = -.8$ and (b) $\gamma = .8$.

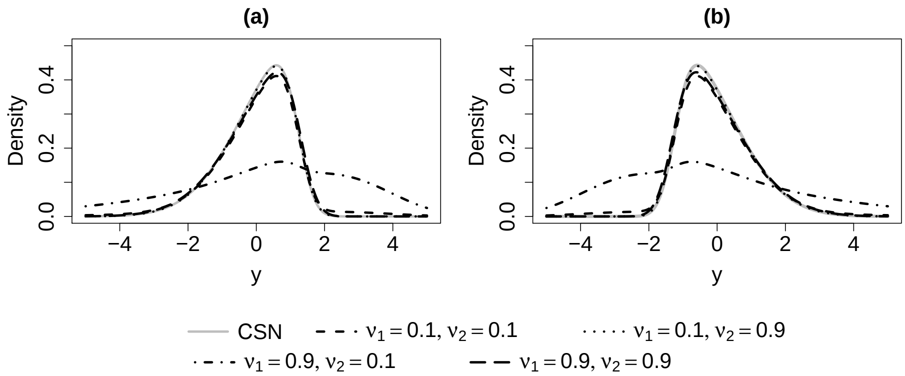

Figure 5 – Densities of the SCN distribution for different values of γ, ν_1 and ν_2, with $\mu_0 = 0$ and $\sigma_0^2 = 1$. (a) $\gamma = -.8$ and (b) $\gamma = .8$.

1.3 Centred skew scale-mixture Birnbaum-Saunders distributions

In this section, we developed a new family of BS distributions, named centred skew scale-mixture Birnbaum-Saunders (CSSBS) distributions. Besides to consider the reparameterization of the BS distribution proposed by Santos-Neto et al. (2012), we considered the centred skewed version of the scale-mixture of normals distribution. Several of its properties are developed and we provided empirical evidences that the CSSBS distributions have advantages in inferential terms, over the SSBS distributions (Balakrishnan et al., 2017). Also, we developed a set of tools of statistical analysis through Bayesian inference based on MCMC algorithms, such as: parameter estimation, statistics of model comparison, and posterior predictive checking. The methodology proposed is illustrated with data sets from both simulation studies and real data sets.

Proposition 1. *If a r.v Y follows the standard CSSMN distribution, denoted by $Y \sim$ CSSMN$(0, 1, \gamma, \boldsymbol{\nu})$, then a r.v T is said to have a CSSBS distribution, denoted by $T|\boldsymbol{\theta} \sim$ CSSBS$(\mu, \phi, \gamma, \boldsymbol{\nu})$, where $\boldsymbol{\theta} = (\mu, \phi, \gamma, \boldsymbol{\nu}^\top)^\top$, if it admits the following stochastic representation*

$$T = \frac{\mu}{[1 + \phi\mathbb{E}(Y^2)]}\left[\frac{\sqrt{2\phi}Y}{2} + \sqrt{\left(\frac{\sqrt{2\phi}Y}{2}\right)^2 + 1}\right]^2, \tag{1.8}$$

where $\mu > 0$ is a scale parameter and the mean of the distribution, $\phi > 0$ is a shape and dispersion parameter, $\gamma \in (-.99527, .99527)$ is the asymmetry parameter, and $\mathbb{E}(Y^2)$ varies according to the particular cases of the CSSMN distribution (see Section 1.2.3). Its density is given by

$$\begin{aligned} f(t|\boldsymbol{\theta}) &= \phi_{\gamma,\boldsymbol{\nu}}\left[a_t(\mu, \phi)\right] A_t(\mu, \phi) \\ &= 2A_t(\mu, \phi)\int_0^\infty \phi\left[a_t(\mu, \phi)\middle| -\frac{\mu_z}{\sqrt{u}\sigma_z}; \frac{1}{u\sigma_z^2}\right]\Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{u}a_t(\mu, \phi)\right]\right\} \\ &\quad \times dG(u|\boldsymbol{\nu})\mathbb{1}\left\{z \in \mathbb{R}^+\right\}, \end{aligned} \tag{1.9}$$

and its cdf is given by

$$F_T(t|\boldsymbol{\theta}) \quad = \quad \Phi_{\gamma,\boldsymbol{\nu}}\left[a_t(\mu,\phi)\right], \tag{1.10}$$

where $\phi_{\gamma,\boldsymbol{\nu}}(\cdot) \equiv \phi_{SSMN}(\cdot|0,1,\gamma,\boldsymbol{\nu})$*,* $a_t(\mu,\phi) = \dfrac{\sqrt{t[1+\phi\mathbb{E}(Y^2)]/\mu} - \sqrt{\mu/t[1+\phi\mathbb{E}(Y^2)]}}{\sqrt{2\phi}}$*,* $A_t(\mu,\phi) = \dfrac{t^{-3/2}\left\{t+\mu/\left[1+\phi\mathbb{E}(Y^2)\right]\right\}}{2\sqrt{2\phi}\sqrt{\mu/\left[1+\phi\mathbb{E}(Y^2)\right]}}$*. Furthermore,* $\Phi_{\gamma,\boldsymbol{\nu}}(\cdot)$ *represents the cdf of the r.v* Y*,* μ_z*,* σ_z *and* λ *were defined in Equation (1.2).*

The proof of the density presented in Equation (1.9) is in Section A.1 of Appendix A.

The moments of T (see Section A.1 of Appendix A for more details) are given by

$$\mathbb{E}(T^r|\boldsymbol{\theta}) = \frac{\mu^r}{[1+\phi m_2]^r}\sum_{j=0}^{r}\binom{2r}{2j}\sum_{i=0}^{j}\mathbb{E}\left[Y^{2(r-j+i)}\right]\left(\sqrt{2\phi}/2\right)^{2(r-j+i)}. \tag{1.11}$$

Particularly, the mean, variance and Pearson's skewness coefficient (γ_T) of T are given by

$$\begin{aligned}
\mathbb{E}(T|\boldsymbol{\theta}) &= \mu, \\
\mathbb{V}(T|\boldsymbol{\theta}) &= \frac{2\phi\mu^2}{[1+\phi m_2]^2}\left\{m_2 + \frac{\phi}{2}\Big[2m_4 - m_2^2\Big]\right\}, \\
\gamma_T &= \frac{\sqrt{\phi}[\phi(4m_6 - 6m_2m_4 + 2m_2^3) + 6(m_4 - m_2^2)]}{\left\{2\Big[m_2 + \frac{\phi}{2}\Big(2m_4 - m_2^2\Big)\Big]\right\}^{3/2}},
\end{aligned} \tag{1.12}$$

where $m_k = \mathbb{E}(Y^k), k = 2,4,6$ represents the kth moment of $Y \sim \text{SSMN}(0,1,\gamma,\boldsymbol{\nu})$. Note that γ_T does not depend on μ.

1.3.1 Examples of CSSBS distributions

In this section, we present some particular cases of CSSBS distributions, which are detailed studied in this work.

- *The centred skew-normal Birnbaum-Saunders (SNBS) distribution, denoted by* $T \sim$ *SNBS*(μ,ϕ,γ). Considering $U = 1$ in Equation (1.9), the respective density is given by

$$f(t|\mu,\phi,\gamma) = 2\sigma_z\phi\left[\mu_z + \sigma_z a_t(\mu,\phi)\right]\Phi\left\{\lambda\left[\mu_z + \sigma_z a_t(\mu,\phi)\right]\right\}\mathbb{1}\left\{t \in \mathbb{R}^+\right\}. \tag{1.13}$$

Note that the density above is a reparameterization of that proposal by Chaves et al. (2019b).

- *The centred skew generalized Student-t Birnbaum-Saunders (SGtBS) distribution, denoted by* $T \sim$ *SGtBS*$(\mu, \phi, \gamma, \nu_1, \nu_2)$. Considering $U \sim \text{gamma}(\nu_1/2, \nu_2/2); \nu_1 > 4, \nu_2 > 0$ in Equation (1.9), the respective density is given by

$$\begin{aligned} f(t|\mu,\phi,\gamma,\nu_1,\nu_2) &= 2A_t(\mu,\phi)(\nu_2/2)^{\nu_1/2}[\Gamma(\nu_1/2)]^{-1}\int_0^\infty \phi\left[a_t(\mu,\phi)\Big| -\frac{\mu_z}{\sqrt{u}\sigma_z};\frac{1}{u\sigma_z^2}\right] \\ &\quad\times \Phi\left\{\lambda\left[\mu_z+\sigma_z\sqrt{u}a_t(\mu,\phi)\right]\right\}u^{\nu_1/2-1} \\ &\quad\times \exp\left\{-\frac{\nu_2}{2}u\right\}du\mathbb{1}\left\{t\in\mathbb{R}^+\right\}. \end{aligned} \tag{1.14}$$

  Using the moments of Y defined in Equation (1.5), the variance of T is given by

$$\mathbb{V}(T) = \frac{2\phi\mu^2}{\{1+[\phi\nu_2/(\nu_1-2)]\}^2}\left\{\left(\frac{\nu_2}{\nu_1-2}\right)+\frac{\phi}{2}\left[2m_4-\left(\frac{\nu_2}{\nu_1-2}\right)^2\right]\right\}, \nu_1>4, \nu_2>0,$$

  where $m_4 = \left[\nu_2^2/(\nu_1-2)(\nu_1-4)\right]\left\{2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2}+3\right\}$.

  Note that the SNBS distribution is a special case of the SGtBS distribution when $\nu_1, \nu_2 \to \infty$.

  When $\nu_1 = \nu_2 = \nu$, T has a skew Student-t Birnbaum-Saunders (StBS) distribution, denoted by $T \sim \text{StBS}(\mu, \phi, \gamma, \nu)$, which will be also explored in this work.

- *The centred skew slash Birnbaum-Saunders (SSLBS) distribution, denoted by* $T \sim$ *SSLBS*(μ, ϕ, γ, ν). Considering $U \sim \text{beta}(\nu, 1); \nu > 2$ in Equation (1.9), the respective density is given by

$$\begin{aligned} f(t|\mu,\phi,\gamma,\nu) &= 2\nu A_t(\mu,\phi)\int_0^1 \phi\left[a_t(\mu,\phi)\Big| -\frac{\mu_z}{\sqrt{u}\sigma_z};\frac{1}{u\sigma_z^2}\right]\Phi\left\{\lambda\left[\mu_z+\sigma_z\sqrt{u}a_t(\mu,\phi)\right]\right\} \\ &\quad\times u^{\nu-1}du\mathbb{1}\left\{t\in\mathbb{R}^+\right\}. \end{aligned} \tag{1.15}$$

  Using the moments of Y defined in Equation (1.6), the variance of T is given by

$$\mathbb{V}(T) = \frac{2\phi\mu^2}{\{1+[\phi\nu/(\nu-1)]\}^2}\left\{\left(\frac{\nu}{\nu-1}\right)+\frac{\phi}{2}\left[2m_4-\left(\frac{\nu}{\nu-1}\right)^2\right]\right\}, \nu>2,$$

  where $m_4 = \left[\nu/(\nu-2)\right]\left\{2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2}+3\right\}$.

  Note that when $\nu \to \infty$ the SSLBS distribution reduces to the SNBS distribution.

- *The centred skew contaminated normal Birnbaum-Saunders (SCNBS) distribution, denoted by* $T \sim$ *SCNBS*$(\mu, \phi, \gamma, \nu_1, \nu_2)$. Taking $g(u|\nu_1, \nu_2) = \nu_1\mathbb{1}_{\{\nu_2\}}(u) + (1-\nu_1)\mathbb{1}_{\{1\}}(u); \nu_1, \nu_2 \in (0, 1)$ in Equation (1.9), the respective density is given by

$$\begin{aligned} f(t|\mu,\phi,\gamma,\nu_1,\nu_2) &= 2A_t(\mu,\phi)\Bigg\{\nu_1\,\phi\left[a_t(\mu,\phi)\Big| -\frac{\mu_z}{\sqrt{\nu_2}\sigma_z};\frac{1}{\nu_2\sigma_z^2}\right] \\ &\quad\times\Phi\left\{\lambda\left[\mu_z+\sigma_z\sqrt{\nu_2}a_t(\mu,\phi)\right]\right\}+(1-\nu_1)\phi\left[a_t(\mu,\phi)\Big| -\frac{\mu_z}{\sigma_z};\frac{1}{\sigma_z^2}\right] \\ &\quad\times\Phi\left\{\lambda\left[\mu_z+\sigma_z a_t(\mu,\phi)\right]\right\}\Bigg\}\mathbb{1}\left\{t\in\mathbb{R}^+\right\}. \end{aligned} \tag{1.16}$$

Using the moments of Y defined in Equation (1.7), the variance of T is given by

$$\mathbb{V}(T) = \frac{2\phi\mu^2}{\{1+[\phi(\nu_1/2+1-\nu_1)]\}^2}\left\{\left(\frac{\nu_1}{\nu_2}+1-\nu_1\right)+\frac{\phi}{2}\left[2m_4-\left(\frac{\nu_1}{\nu_2}+1-\nu_1\right)^2\right]\right\},$$

where $m_4 = \left[(\nu_1/\nu_2^2)+1-\nu_1\right]\left\{2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2}+3\right\}$.

Note that the SNBS distribution is a special case of the SCNBS distribution when $\nu_2 \to 1$ and/or $\nu_1 \to 0$.

Figures 6-8 present the densities of the SGtBS, SSLBS and SCNBS distributions, respectively, for different values of μ, ϕ and γ, considering fixed values of $\boldsymbol{\nu}$. We can notice that μ affects both the scale and position of the distributions and, the higher/smaller it is, the higher/smaller the variance is. It is also possible to notice that ϕ and γ control the skewness and kurtosis, respectively. More specifically, as ϕ increases and γ assumes positive values, the densities become more dispersed and positively skewed. In addition, Figure 9 presents the densities of the SGtBS, SSLBS and SCNBS distributions, respectively, for different values of $\boldsymbol{\nu}$. The SGtBS, SSLBS and SCNBS distributions have tails much heavier than the centred SNBS distribution when $\nu_1 \to 0$, $\nu \to 0$ and $\nu_2 \to 0$ and $\nu_1 \to 1$, respectively.

In short, the distributions tend to be symmetric around μ, for $\gamma = 0$ and for small values of ϕ. Positive asymmetry is observed as ϕ increases and/or γ assumes positive values. On the other hand, negative asymmetry is observed as ϕ decreases and/or γ assumes negative values. Also, $\boldsymbol{\nu}$ controls the weight of the tails. Thus, the proposed family provides flexible skewed heavy-tailed distributions, allowing also, negative asymmetry, which is a uncommon feature for positive random variables.

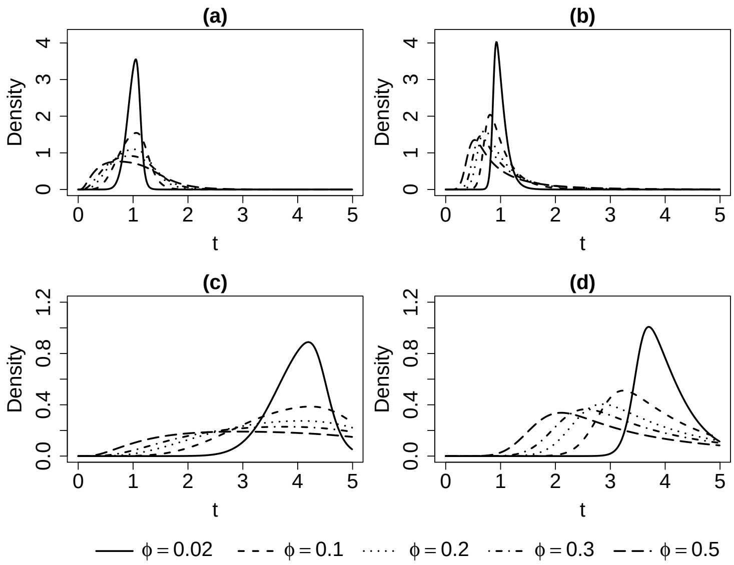

Figure 6 – Densities of the SGtBS distribution for different values of μ, ϕ and γ, with $\nu_1 = \nu_2 = 5$: (a) $\mu = 1, \gamma = -.8$; (b) $\mu = 1, \gamma = .8$; (c) $\mu = 4, \gamma = -.8$ and (d) $\mu = 4, \gamma = .8$.

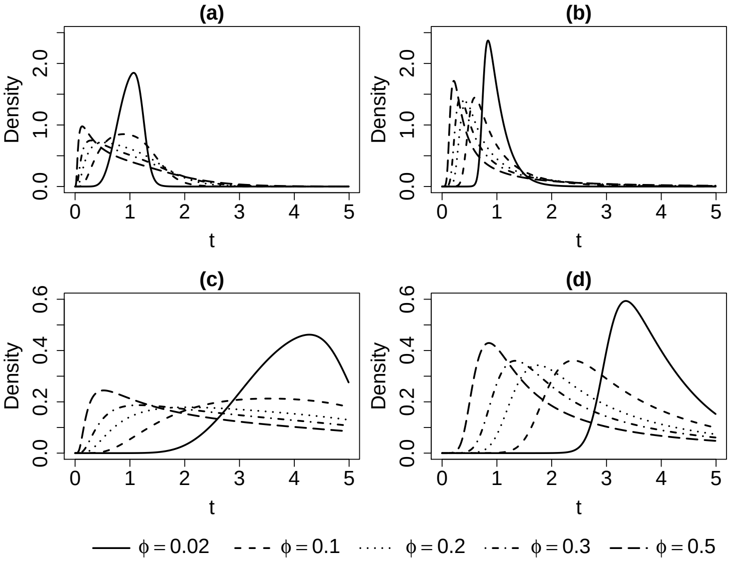

Figure 7 – Densities of the SSLBS distribution for different values of μ, ϕ and γ, with $\nu = 5$: (a) $\mu = 1, \gamma = -.8$; (b) $\mu = 1, \gamma = .8$; (c) $\mu = 4, \gamma = -.8$ and (d) $\mu = 4, \gamma = .8$.

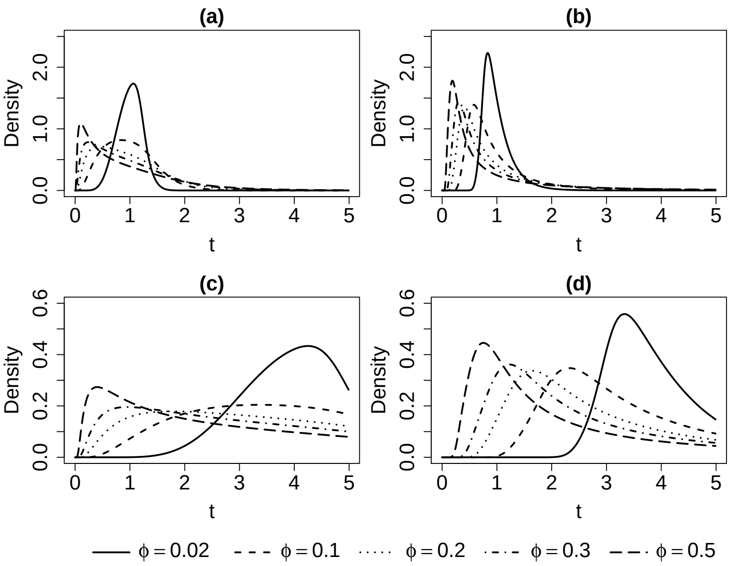

Figure 8 – Densities of the SCNBS distribution for different values of μ, ϕ and γ, with $\nu_1 = \nu_2 = .5$: (a) $\mu = 1, \gamma = -.8$; (b) $\mu = 1, \gamma = .8$; (c) $\mu = 4, \gamma = -.8$ and (d) $\mu = 4, \gamma = .8$.

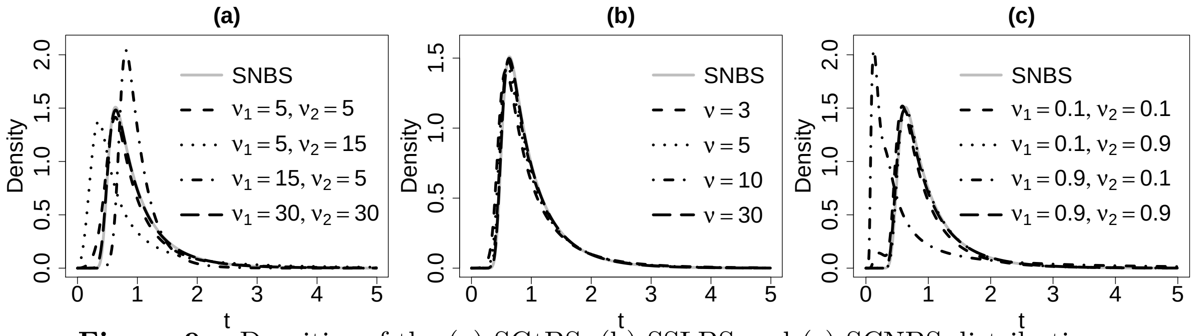

Figure 9 – Densities of the (a) SGtBS, (b) SSLBS and (c) SCNBS distributions.

1.3.2 Some advantages of the proposed model

In this section, we present some advantages of the proposed family.

- It is well know that it is not an easy task to estimate the parameters of the usual SN distribution, by maximum likelihood, when the asymmetry parameter (λ) is close to zero (Arellano-Valle and Azzalini, 2008). The skew scale-mixture of normal (SSMN) distributions (da Silva Ferreira et al., 2011) and the SSBS distributions (Balakrishnan et al., 2017), inherit such problems. Even under the Bayesian approach, this issue remains, unless a strongly informative prior is considered, as pointed out by Arellano-Valle and Azzalini (2008). On the other hand, as we will show below, the family of CSSBS circumvents these problems.

- The centred parameterization of the SN distribution removes the singularity of the expected Fisher information, which occurs when the asymmetry parameter is equals to zero. Moreover, it circumvents the problem concerning the existence of an inflection point in the relative profiled log-likelihood (RPLL) (Arellano-Valle and Azzalini, 2008) of this parameter. The RPLL, corresponds to $\ell(\widehat{\mu}(\gamma), \widehat{\phi}(\gamma), \widehat{\boldsymbol{\nu}}(\gamma), \gamma) - \ell(\widehat{\mu}(\gamma), \widehat{\phi}(\gamma), \widehat{\boldsymbol{\nu}}(\gamma), \widehat{\gamma})$, where ℓ represents the log-likelihood. Figures 10 - 13 present the plots of twice the RPLL for λ (left panels) for the SSBS distributions and the RPLL for the γ (right panels) for the CSSBS distributions. The corresponding graphs were constructed considering random samples of size 200 of the distributions SSBS and CSSBS, under suitable values of μ, ϕ, $\boldsymbol{\nu}$, and the respective asymmetry parameters. We can notice a non-quadratic form of the RPLL related to SSBS distributions, making it difficult the parameters estimation process. However, the RPLL related to the CSSBS distributions is well-behaved and it presents a concave shape.

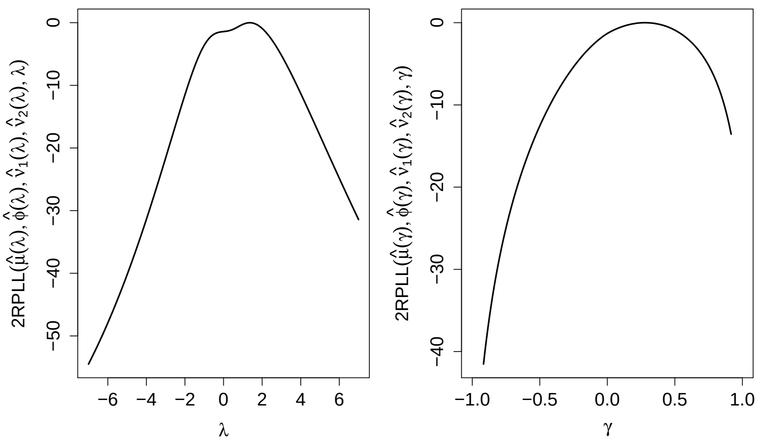

Figure 10 – Twice the RPLL for the asymmetry parameters λ, (left panel) and γ, (right panel) for the SGtBS distribution.

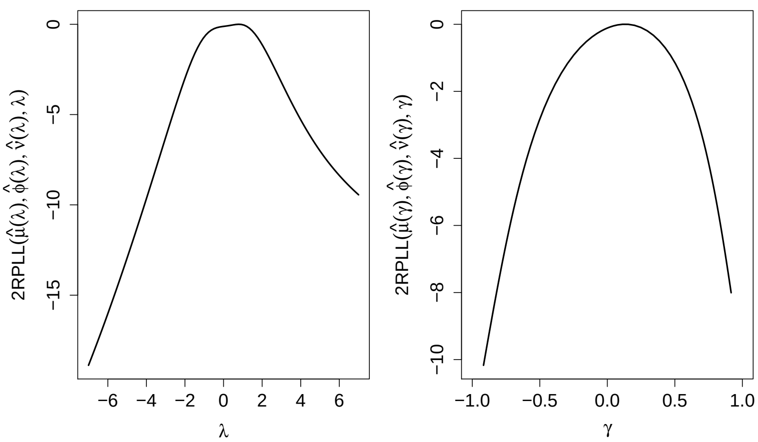

Figure 11 – Twice the RPLL for the asymmetry parameters λ, (left panel) and γ, (right panel) for the StBS distribution.

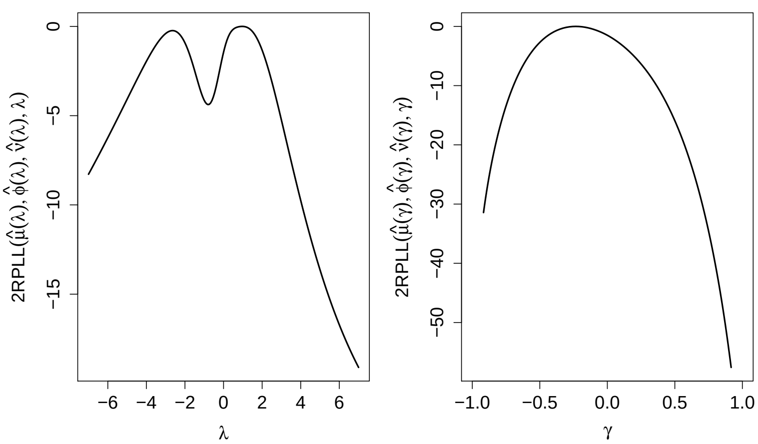

Figure 12 – Twice the RPLL for the asymmetry parameters λ, (left panel) and γ, (right panel) for the SSLBS distribution.

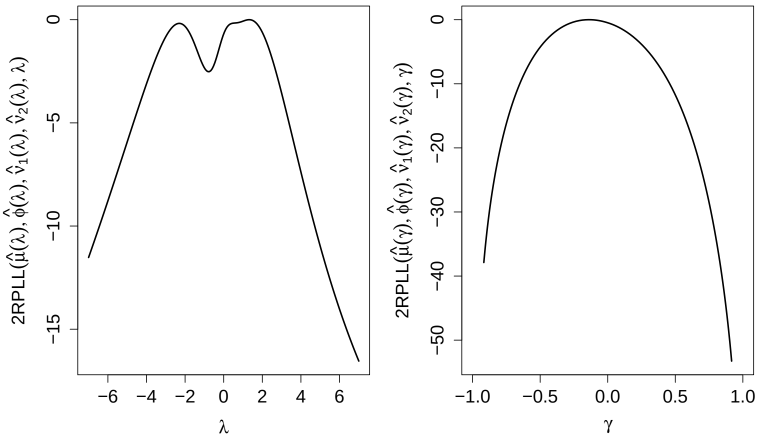

Figure 13 – Twice the RPLL for the asymmetry parameters λ, (left panel) and γ, (right panel) for the SCNBS distribution.

1.4 Bayesian inference

In this section, we present Bayesian inference for our family of distributions. The adopted approach, since the marginal posterior distributions of interest can not be analytically obtained, relies on the MCMC algorithms, to obtain numerical approximations for those distributions. Bayesian hierarchical modelling is very attractive due to its flexibility. It allows for full parameter uncertainty and Bayesian inference does not depend on asymptotic results, see Gelman et al. (2013). Interval estimates for the parameters or functions of them can be easily obtained directly from the MCMC output.

1.4.1 Likelihoods

Let $T_i|\boldsymbol{\theta} \stackrel{\text{iid}}{\sim} \text{SSBS}(\mu, \phi, \gamma, \boldsymbol{\nu}), i = 1, \ldots, n$, where $\boldsymbol{\theta} = (\mu, \phi, \gamma, \boldsymbol{\nu}^\top)^\top$. The likelihood for $\boldsymbol{\theta}$, given the observed sample $\mathbf{t} = (t_1, \ldots, t_n)^\top$, takes on the form

$$L(\boldsymbol{\theta}|\mathbf{t}) = \prod_{i=1}^{n} \phi_{\gamma,\boldsymbol{\nu}}\left[a_{t_i}(\mu, \phi)\right] A_{t_i}(\mu, \phi), \tag{1.17}$$

where $\phi_{\gamma,\boldsymbol{\nu}}(\cdot)$ was defined in Equation (1.9), $a_{t_i}(\mu, \phi) = \left\{\sqrt{t_i[1 + \phi\mathbb{E}(Y^2)]/\mu} - \sqrt{\mu/t_i[1 + \phi\mathbb{E}(Y^2)]}\right\}/\sqrt{2\phi}$ and $A_{t_i}(\mu, \phi) = \dfrac{t_i^{-3/2}\left\{t_i + \mu/\left[1 + \phi\mathbb{E}(Y^2)\right]\right\}}{2\sqrt{2\phi}\sqrt{\mu/\left[1 + \phi\mathbb{E}(Y^2)\right]}}$.

It is possible to consider a hierarchical representation of the CSSBS distribution (see Section A.1 of Appendix A for more details), which is useful for stochastic simulation and parameter estimation (both frequentist and Bayesian), which is given by

$$\begin{aligned} T|\left(H = h, U = u\right) &\sim \text{EBS}(\phi_\delta, \mu/[1 + \phi\mathbb{E}(Y^2)], \kappa = 2, \vartheta_h) \\ H &\sim \text{HN}(0, 1) \\ U|\boldsymbol{\nu} &\sim G(u|\boldsymbol{\nu}), \end{aligned} \tag{1.18}$$

where $\phi_\delta = \sqrt{2\phi}\Big(\dfrac{u^{-1/2}\sqrt{1-\delta^2}}{\sigma_z}\Big)$ and $\vartheta_h = \dfrac{\mu_z - \delta h}{\sqrt{1-\delta^2}}$. The acronym EBS represents the extended Birnbaum-Saunders (EBS) distribution, which is properly discussed in Vilca et al. (2010). Thus, defining $\mathbf{t}_c = (\mathbf{t}^\top, \mathbf{h}^\top, \mathbf{u}^\top)$, where $\mathbf{t} = (t_1, \ldots, t_n)^\top$, $\mathbf{h} = (h_1, \ldots, h_n)^\top$ and $\mathbf{u} = (u_1, \ldots, u_n)^\top$, we have that the so-called complete likelihood is given by

$$L(\boldsymbol{\theta}|\mathbf{t}_c) \propto \prod_{i=1}^{n} \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi) \exp\left\{-h_i^2/2\right\} g(u_i|\boldsymbol{\nu}), \tag{1.19}$$

where ϑ_{h_i} was defined in Equation (1.18), $a_{t_i,\tau_i}(\mu,\phi) = \tau_i^{-1} a_{t_i}(\mu,\phi)$, $A_{t_i,\tau_i}(\mu,\phi) = \tau_i^{-1} A_{t_i}(\mu,\phi)$, where $a_{t_i}(\mu,\phi)$ and $A_{t_i}(\mu,\phi)$ were defined in Equation (1.17).

1.4.2 Prior, posterior and full conditional distributions

To complete the Bayesian specification we need to consider convenient prior distribution. Assuming independence among the elements of $\boldsymbol{\theta}$, that is

$$\pi(\boldsymbol{\theta}) = \pi(\mu)\pi(\phi)\pi(\gamma)\pi(\boldsymbol{\nu}). \tag{1.20}$$

Based on the work of Chaves et al. (2019b) and the results of some results of a priori sensitivity study (not presented for the sake of simplicity), we chose the following prior distributions:

$$\mu \sim \text{gamma}(a_\mu, b_\mu), \quad \phi \sim \text{gamma}(a_\phi, b_\phi) \text{ and } \gamma \sim \text{U}(c,d), \tag{1.21}$$

where gamma(a,b) stands for the gamma distribution with mean a/b and variance a/b^2 and $U(c,d)$ stands for a continuous uniform distribution over the interval (c,d). The prior distribution of $\boldsymbol{\nu}$ depends on the particular cases of the CSSBS distribution (more details are provided ahead). Combining the complete likelihood presented in Equation (1.19) and the prior distributions presented in Equation (1.20), the joint posterior distribution is given by

$$\pi(\boldsymbol{\theta},\mathbf{h},\mathbf{u}|\mathbf{t}) \propto \left\{\prod_{i=1}^{n} \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi) \exp\left\{-h_i^2/2\right\} g(u_i|\boldsymbol{\nu})\right\} \pi(\boldsymbol{\theta}).$$

and the full conditional distributions, are given by

$$\begin{aligned} \pi(h_i|\boldsymbol{\theta}, t_i, u_i) &\propto \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] \exp\left\{-h_i^2/2\right\} \\ \pi(u_i|\boldsymbol{\theta}, t_i, h_i) &\propto \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\left[g(u_i|\boldsymbol{\nu})\right] \end{aligned} \tag{1.22}$$

$$
\begin{aligned}
\pi(\mu|\phi,\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\mu)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\right\}\\
\pi(\phi|\mu,\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\phi)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\right\}\\
\pi(\gamma|\mu,\phi,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\gamma)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\right\}\\
\pi(\boldsymbol{\nu}|\mu,\phi,\gamma,\mathbf{t}_c) &\propto \pi(\boldsymbol{\nu})\left\{\prod_{i=1}^{n}g(u_i|\boldsymbol{\nu})\right\}. \qquad (1.23)
\end{aligned}
$$

The shape of distributions presented in Equations (1.22) and (1.23) vary according to the particular cases of the CSSBS distribution and the adopted prior for $\boldsymbol{\nu}$. The full conditional distributions of u_i and $\boldsymbol{\nu}$, and $\pi(\boldsymbol{\nu})$ for each CSSBS distribution are presented bellow. We made all implementations considering the `OpenBUGS` software (Spiegelhalter et al., 2014), through the `R` program (R Core Team, 2014), using the package `R2OpenBUGS` (Sturtz et al., 2010). That is, the auxiliary algorithms used to simulate from the full conditional distributions follow the predefined hierarchy implemented in `OpenBUGS`.

1.4.3 Prior distribution of $\boldsymbol{\nu}$ and related full conditional distributions

1. *The centred skew generalized Student-t Birnbaum-Saunders (SGtBS) distribution.* Here, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$. We adopt a trucated exponential prior for ν_j, $j = 1, 2$, that is $\nu_j \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}}(\nu_j)$, such that $\Lambda \sim \mathrm{U}(\Lambda_0, \Lambda_1)$ (Cabral et al., 2012), which leads to the density

$$\pi(\nu_j) = \frac{1}{\nu_j^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_j+1)\exp\{-\Lambda_0\nu_j\}-(\Lambda_1\nu_j+1)\exp\{-\Lambda_1\nu_j\}\right].$$

   The full conditional distributions of u_i and ν_j take the form

$$
\begin{aligned}
\pi(u_i|\boldsymbol{\theta},t_i,h_i) &\propto \phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\left[u_i^{\nu_j/2-1}\right]\exp\left\{-\frac{\nu_j}{2}u_i\right\},\\
\pi(\nu_j|\mu,\phi,\gamma,\mathbf{t}_c) &\propto \frac{1}{\nu_j^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_j+1)\exp\{-\Lambda_0\nu_j\}-(\Lambda_1\nu_j+1)\exp\{-\Lambda_1\nu_j\}\right]\times\\
&\quad\times\left\{\prod_{i=1}^{n}(\nu_j/2)^{\nu_j/2}\left[\Gamma(\nu_j/2)\right]^{-1}u_i^{\nu_j/2-1}\exp\left\{-\frac{\nu_j}{2}u_i\right\}\right\}.
\end{aligned}
$$

2. *The centred skew slash Birnbaum-Saunders (SSLBS) distribution.*
   We adopt a truncated gamma distribution for $\boldsymbol{\nu} = \nu$, that is, $\nu \sim \mathrm{gamma}(a, b)\mathbb{1}_{\mathbb{A}}(\nu)$, with small positive values of a and b ($b << a$), see Vilca et al. (2016). The full conditional distributions of u_i and ν become

$$
\begin{aligned}
\pi(u_i|\boldsymbol{\theta},t_i,h_i) &\propto \phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)u_i^{\nu-1}\\
\pi(\nu|\mu,\phi,\gamma,\mathbf{t}_c) &\propto \nu^{a-1}\exp\{-b\nu\}\prod_{i=1}^{n}\nu\, u_i^{\nu-1}.
\end{aligned}
$$

3. *The centred skew contaminated normal Birnbaum-Saunders (SCNBS) distribution.* The possible states of the "weights" u_i are ν_2 or 1, with $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$, and its density can be expressed as

$$g(u_i|\boldsymbol{\nu}) = \nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}.$$

The full conditional distribution of each u_i can be written as:

$$\pi(u_i|\boldsymbol{\theta}, t_i, h_i) \propto \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}.$$

Thus, the distribution is proportional to

$$\begin{cases} \nu_1\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi), \text{if}\, u_i = \nu_2 \\ (1-\nu_1)\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi), \text{if}\, u_i = 1 \end{cases}$$

In this case, we consider $\nu_j \sim \text{beta}(a_j, b_j)$ (Lachos et al., 2017). The full conditional distribution of ν_j, $j = 1, 2$, is given by

$$\pi(\nu_j|\mu,\phi,\gamma,\mathbf{t}_c) \propto \nu_j^{a_j+a_{n,\nu_2}-1}(1-\nu_j)^{b_j+b_{n,\nu_2}-1},$$

where $a_{n,\nu_2} = \left(n - \sum_{i=1}^n u_i\right)/\left(1-\nu_2\right)$ and $b_{n,\nu_2} = \left(\sum_{i=1}^n u_i - n\,\nu_2\right)/\left(1-\nu_2\right)$, which is proportional to the $\text{beta}(a_j + a_{n,\nu_2}, b_j + b_{n,\nu_2})$ distribution.

1.5 Model fit assessment and model comparison

1.5.1 Statistics for model comparison

When MCMC algorithms are used to obtain the posterior distributions, some statistics for model comparison can be easily calculated, see Spiegelhalter et al. (2002). To introduce these statistics we first define $\mathrm{D}(\boldsymbol{\theta}) = -2\log\left[L(\boldsymbol{\theta}|\mathbf{t})\right]$, where $\boldsymbol{\theta} = (\mu, \phi, \gamma, \boldsymbol{\nu}^\top)^\top$ and $L(\boldsymbol{\theta}|\mathbf{t})$ is the (incomplete) likelihood presented in Equation (1.17). Also, let $\boldsymbol{\theta}^{(m)}$, $m = 1, ..., M$, be the mth value of the valid simulated MCMC sample, that is, the MCMC sample obtained after discarding the burn-in and a proper thinning (lag) between the values. Finally, let $\overline{\boldsymbol{\theta}}$ be the vector with the posterior expectation of all parameters, based on the valid MCMC sample, and $\overline{\mathrm{D}(\boldsymbol{\theta})} = \frac{1}{M}\sum_{m=1}^M D(\boldsymbol{\theta}^{(m)})$. Denote also the deviance by $\mathrm{D}(\overline{\boldsymbol{\theta}}) = -2\ell(\overline{\boldsymbol{\theta}}|\mathbf{t})$, and the deviance information criterion (DIC) by $\mathrm{DIC} = \mathrm{D}(\overline{\boldsymbol{\theta}}) + 2p_D$, where $p_D = \overline{\mathrm{D}(\boldsymbol{\theta})} - \mathrm{D}(\overline{\boldsymbol{\theta}})$. The EAIC (posterior expectation of AIC) and EBIC (posterior expectation of BIC) are given, respectively, by $\mathrm{EAIC} = D(\bar{\boldsymbol{\theta}}) + 2k$ and $\mathrm{EBIC} = \mathrm{D}(\bar{\boldsymbol{\theta}}) + k\log(n)$, where k is the number of parameters and n is the number of observations. Finally, the LPML (logarithm of the pseudo-marginal likelihood) is calculated as $\mathrm{LPML} = \sum_{i=1}^n \ln(\widehat{\mathrm{CPO}}_i)$, where $\widehat{\mathrm{CPO}}_i = \left\{\frac{1}{M}\sum_{m=1}^M \left[1/L\left(\boldsymbol{\theta}^{(m)}|t_i\right)\right]\right\}^{-1}$ represents the conditional predictive ordinate, see Ibrahim et al. (2004) and Gelfand et al. (1992). The smaller the values of DIC, EAIC, EBIC and deviance, the better the model fit, occurring the opposite with the LPML.

The EAIC and EBIC tend to select the model with the smallest number of parameters (k) since it gives more penalties to models with more parameters. On the other hand, the DIC tends to select the most complex (or the most general) model, that is, it tends to select the overfitted model, see Ando (2007). Finally, the LPML statistic tends to select the model that presents the largest likelihood. This corresponds to the most general model when the competing models are nested.

1.5.2 Posterior predictive checking

Under a Bayesian perspective, one way to check the goodness of model fit, is to compare the response predictive distribution with the observed distribution of the data. Let $\boldsymbol{t}^{\text{obs}}$ be the observed response and $\boldsymbol{t}^{\text{rep}}$ the replicated response generated from its posterior predictive distribution, which is given by

$$p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{t}^{\text{obs}}) = \int p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}}) d\boldsymbol{\theta}, \tag{1.24}$$

where $\boldsymbol{\theta} = (\mu, \phi, \gamma, \boldsymbol{\nu}^\top)^\top$. A suitable discrepancy measure $D(\boldsymbol{t}, \boldsymbol{\theta})$ are defined by Gelman et al. (1996) and the posterior distribution of $D(\boldsymbol{t}^{\text{obs}}, \boldsymbol{\theta})$ is compared to the posterior predictive distribution of $D(\boldsymbol{t}^{\text{rep}}, \boldsymbol{\theta})$ and substantial differences between them indicating model misfit. Gelman et al. (2013) suggest several graphs to compare the replicated and the observed data, under the given measure of divergence.

Another measure used to quantify the goodness of fit, it is the Bayesian p-value, which for an adopted discrepancy measure is defined as

$$\begin{aligned} \mathbb{P}[D(\boldsymbol{t}^{\text{rep}}, \boldsymbol{\theta})] &\geqslant D(\boldsymbol{t}^{\text{obs}}, \boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}}) \\ &= \int_{D(\boldsymbol{t}^{\text{rep}}, \boldsymbol{\theta}) \geqslant D(\boldsymbol{t}^{\text{obs}}, \boldsymbol{\theta})} p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}}) d\boldsymbol{t}^{\text{rep}} d\boldsymbol{\theta}. \end{aligned} \tag{1.25}$$

Due to the difficulty in dealing with Equations (1.24) and (1.25) analytically, Rubin (1984) suggests simulating replicated data sets from the posterior predictive distribution. One draws M simulations $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_M$ from the posterior distribution $p(\boldsymbol{\theta}|\boldsymbol{t})$ of $\boldsymbol{\theta}$ and then draws $\boldsymbol{t}^{\text{rep},n}$ from the distribution $p(\mathbf{t}|\boldsymbol{\theta}^n)$ for $n = 1, \ldots, M$. The proportion of the M replications for which $D(\boldsymbol{t}^{\text{rep},n}, \boldsymbol{\theta}^n)$ exceeds $D(\boldsymbol{t}, \boldsymbol{\theta}^n)$ provides an estimate of the p-value Bayesian. Extreme values of the Bayesian p-value (less than .05 or greater than .95, depending on the nature of the discrepancy measure) indicate model misfit, see Sinharay et al. (2006). Based on Gelman et al. (1996), the measure of discrepancy here adopted is $D(\mathbf{t}|\boldsymbol{\theta}) = \sum_{i=1}^n \left\{[t_i - \mathbb{E}(T_i|\boldsymbol{\theta})]^2\right\}/\mathbb{V}(T_i|\boldsymbol{\theta})$, where $\mathbb{E}(T_i|\boldsymbol{\theta})$ and $\mathbb{V}(T_i|\boldsymbol{\theta})$ are given by Equation (1.12).

1.6 Simulation study

In this section, we presented a parameter recovery study in order to evaluate the performance of the methodology. We considered different scenarios of interest, which correspond to the combination of the levels of some factors. They (with the respective levels within parenthesis) are: sample size (n) (100, 500), that is, small and large sample sizes, asymmetry parameter γ (-.8, 0, .8), that is high negative skewness, symmetry and high positive skewness, and different values of $\boldsymbol{\nu}$, which induce either a normal shape, or a heavy tails behavior. Specifically, we considered $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ for the SGtBS model, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the SCNBS model, and $\nu \in \{5, 30\}$ for the StBS and SSLBS models. Also, we fix $\mu = 1$ for all models and $\phi = .5$, which induce a strong positively skewed behavior of the proposed distributions. Furthermore, to overcome the identifiability problem in the SGtBS model, we fix $\phi = 1$.

Based on the work of Chaves et al. (2019b) and the results of some studies related to the sensitivity to the prior choice that we previously conducted, we chose the following prior distributions: $\mu \sim \text{gamma}(.001, .001)$, $\phi \sim \text{gamma}(1, .5)$ and $\gamma \sim \text{U}(-.99527, .99527)$. The first prior is quite flat and the second is reasonably concentrated in the interval (0, 4.5) (90% of the mass), and was based on works available in the literature which indicates that, in general, the estimates usually lie in this interval. The third prior, suggested by Azevedo et al. (2011), is non-informative. For the SGtBS model we set $\nu_j \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_j}(\nu_j)$; $j = 1, 2$, with $\Lambda \sim U(.02, .5)$ (Cabral et al., 2012). In order to guarantee the existence of the first two moments we consider $\mathbb{A}_1 = (4, \infty)$ and $\mathbb{A}_2 = (2, \infty)$. Similarly, for the StBS model we set $\nu \sim \exp(\Lambda)\mathbb{1}_{(4,\infty)}(\nu)$, such that $\Lambda \sim U(.02, .5)$. For the SCNBS model we chose $\nu_1 \sim \text{beta}(2, 2)$ and $\nu_2 \sim \text{beta}(1, 1)$ (Lachos et al., 2017).

For the SSLBS model, we investigated the sensitivity to the prior choice for ν, by using different suggestions found in the literature, such as: $\nu \sim \exp(\Lambda)\mathbb{1}_{(2,\infty)}(\nu)$, where $\Lambda \sim U(.02, .5)$, and $\nu \sim \text{gamma}(.001, .001)\mathbb{1}_{(2,\infty)}(\nu)$, both suggested by Cabral et al. (2012), $\nu \sim \text{gamma}(1, .1)\mathbb{1}_{(2,\infty)}(\nu)$ (Vilca et al., 2016) and $\nu \sim \text{gamma}(.01, .001)\mathbb{1}_{(2,\infty)}(\nu)$ (Bandyopadhyay et al., 2010). The results showed, for all distributions, estimates concentrated in the interval $(2, 7)$, independently of the respective true value. Therefore, we decided to consider two situations: when the true ν value is small, for example $\nu = 5$, we chose $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$. When the true ν value is large, for example $\nu = 30$, we chose $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$. We will refer to the SSLBS model as SSLBS1 and as SSLBS2, when we consider $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$ and $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$, respectively. In a real data analysis, we can fit the two models and consider QQ plots, the statistics for model comparison, and Bayesian p-values, to choose the most properly one.

From the results related to a convergence study (not presented for the sake of simplicity) we observed that to set a burn-in of 60,000, with a spacing of 40, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the SGtBS model. On the other hand, when $\nu_1 = \nu_2$, that is, for the StBS model, we considered a burn-in of 30,000, with spacing of 50, generating a total of 80,000 values. For the SCNBS and SSLBS1, we set a burn-in of 60,000 and a total of 100,000 values were simulated, and samples were collected at a spacing of 40 iterations. Finally, we observed that to set a burn-in of 50,000, with a spacing of 30, generating a total of 80,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the SSLBS2 model. In general, for all models, the Geweke statistic did not indicate lack of convergence for any parameter. Furthermore, the values for the Gelman-Rubin statistic and the inspection of the traceplots and autocorrelation plots indicated that the MCMC algorithms converged and the autocorrelations were almost negligible.

We considered R=10 replicas (simulated responses from the models) and calculated the usual statistics to measure the accuracy of the bias, standard deviation (SD), square root of the mean squared error (RMSE), absolute value of the relative bias (AVRB), coverage probability (CP) of the 95% equi-tailed credibility interval and average length (LCI) of the 95% equi-tailed credibility interval. Let θ be the parameter of interest and let $\widehat{\theta}_r$ be some estimate (posterior mean, median or mode) related to the replica r, and $\overline{\widehat{\theta}} = (1/R)\sum_{r=1}^{R}\widehat{\theta}_r$. The aforementioned statistics are: bias $= \overline{\widehat{\theta}} - \theta$; SD $= \sqrt{(1/R)\sum_{r=1}^{R}(\widehat{\theta}_r - \overline{\widehat{\theta}})^2}$, RMSE $= \sqrt{(1/R)\sum_{r=1}^{R}(\theta - \widehat{\theta}_r)^2}$ and AVRB $= |\overline{\widehat{\theta}} - \theta|/|\theta|$, CP $= (1/R)\sum_{r=1}^{R} \mathrm{I}(\theta \in [\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}])$ and LCI $= (1/R)\sum_{r=1}^{R}[\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}]$, where $\widehat{\theta}_{r,LCL}$ and $\widehat{\theta}_{r,UCL}$ are the estimated lower and upper 95% limits of the CI's, respectively. We considered $(< .001)$ to represent positive values (statistics and/or estimates) and $(> .001)$ to denote negative values, when they are close to zero. All results of the parameter recovery study can be found in Section A.2 of Appendix A.

In Tables 18-23, the results of simulation studies for the SGtBS distribution are presented. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$, as the sample size increases, the estimates for all parameters tend to the correspondent true values. When $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$, which corresponds to the StBS distribution with $\nu = 30$, we can notice that μ and γ were well recovered for both sample sizes. In this scenario, although ν_1 and ν_2 are underestimated, it is clear that the estimates lead to an equivalence between the SGtBS and SNBS models. Therefore, we have indications that ν_1 and ν_2 are reasonably estimated in scenarios that induce a normal shape behavior.

Tables 24-29 present the results for the StBS distribution and Tables 30-35 present the results for the SSLBS. For both models, under $\nu = 5$, we can notice that the estimates, for all parameters, tend to the correspondent true values in all scenarios. Also, as the sample size increases, the width of the credibility intervals decreases. Under $\nu = 30$, the estimates of ν tend to the correspondent true value. However, the width of credibility intervals are too large. Concerning μ, ϕ and γ, the estimates are close to the respective true values in all scenarios.

In Tables 36-41, the results for the SCNBS distribution are presented. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$, we can notice that the estimates of μ and γ tend to the correspondent true values. However, the estimates of ϕ were not so good for both sample sizes. In this case, larger sample size is required so that the estimates of ϕ to be close to the respective true value. As mentioned earlier, when the estimates of the hyperparameters ν_1 and ν_2 of SCNBS distribution are such that $\nu_2 \to 0$ and $\nu_1 \to 1$, we have that this distribution has much heavier tails than the SNBS one. Based only on the posterior mode, $\widehat{\nu}_1 = .691$ and $\widehat{\nu}_2 = .461$ (see Table 38), for example, it is not clear that the SCNBS distribution has heavy tails. However, when we also consider the estimates of μ, ϕ, and γ, we can notice that the SCNBS distribution has a behavior compatible with that of the heavy-tailed distribution. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, we can notice that the estimates of all parameters tend to the correspondent true values in all scenarios. Also, as the sample size increases, the bias, RMSE and AVRB decrease.

In general, we can notice that μ, ϕ and γ are well recovered in all models. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed distributions, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed distributions are equivalent to SNBS distribution, the $\boldsymbol{\nu}$ estimates are biased and the width of the credibility interval are large. However, as sample size increases, the estimates become more accurate.

1.7 Real data analysis

In this section, we illustrate the proposed methodology by analyzing a real data set with our methodology. The data set considered here refers to the breaking stress of carbon fibres (in Gba), see Nichols and Padgett (2006) and Cordeiro et al. (2013). Consider a process that is producing carbon fibers to be used in constructing fibrous composite materials. A total of 100 carbon fibers of 50 mm in length were sampled from the process, tested, and their tensile strength, observed. Some descriptive statistics, including location measures, standard deviation (SD), coefficient of skewness (CS), and kurtosis (CK), are provided in Table 2. From these statistics and Figure 14 (a) we notice that the CSSBS models can be reasonably assumed for modeling these data, mainly due to their asymmetric nature and level of kurtosis. We fitted the CSSBS models using the Bayesian approach. The prior distributions were the same presented in Section 1.4.2.

Similarly to what was done in Vilca et al. (2011) and Chaves et al. (2019b), we replaced the Bayesian estimates of μ and ϕ in $d(\mu, \phi) = (1/2\phi)\left\{T[1 + \phi\mathbb{E}(Y^2)]/\mu - \mu/T[1 + \phi\mathbb{E}(Y^2)] - 2\right\}$. If $T \sim SSBS(\mu, \phi, \gamma, \boldsymbol{\nu})$, thus $d(\mu, \phi) \sim SSMN(0, 1, \gamma, \boldsymbol{\nu})$. Since the observations $d(\widehat{\alpha}, \widehat{\beta})$ are expected to follow a SSMN distribution, under the well fit the model, the envelopes are simulated from SSMN distribution. These plots are presented in Figure 17 (lines represent the 5th percentile, the mean, and the 95th percentile of 100 simulated points of each observations). In general, we can notice that most observations are inside of the envelope, without show any systematic behavior. Thus, we can say that the models present a similar and a good fitting.

Table 3 presents the posterior expectations (PE), posterior standard deviations (PSD) and equi-tailed 95% credibility intervals (CI). We can see that the asymmetry parameter was statistically significant, since the zero does not belong to the credibility interval. Based only on the posterior expectations of the degrees of freedom, it is not clear that the SGtBS, StBS and SCNBS are preferable to the SNBS model, since the credibility intervals of the degrees of freedom are large. On the other hand, analyzing the posterior densities of $\boldsymbol{\nu}$ (see Figure 16), we can notice that for the SGtBS, StBS and SCNBS distributions, the densities are concentrated around small values. As discussed earlier, this behavior is compatible with that of the heavy-tailed distributions. The estimated densities are shown in Figure 14 (b). We can notice that the heavy-tailed SSBS models present a slight advantage over the SNBS distributions. Figure 15 presents the observed and predicted responses (indicated by gray) under the proposed models. We can notice that the SCNBS model predicts better the observations present on the right tail.

Table 4 shows the comparison among the different models by using the EAIC, EBIC, DIC and LPML (see Section 1.5.1) and Bayesian p-value (the bold values indicate the chosen model by each statistic). Notice that two (DIC and LPML) of the four criteria selected the SCNBS model as the most appropriate one. Also, when comparing the Bayesian p-values, we can say that the SCNBS model presents a better fit compared to the other models, since its Bayesian p-value is closer to .5. Also, it is possible to note that the SNBS model is the least appropriate for this data set. In conclusion, we can say that the SCNBS distribution offers an excellent fit to the carbon fibers data.

Table 2 – Descriptive statistics for the tensile strength of carbon fibers.

n	Mean	Median	Minimum	Maximum	SD	CS	CK
100	2.621	2.700	0.390	5.560	1.014	0.368	3.105

Table 3 – Posterior expectations (PE), posterior standard deviations (PSD) and equi-tailed 95% credibility intervals.

	SGtBS			SCNBS		
Parameter	PE	PSD	$CI_{95\%}$	PE	PSD	$CI_{95\%}$
μ	2.656	.107	[2.466; 2.891]	2.637	.102	[2.442; 2.838]
ϕ	-	-	-	.065	.023	[.027 ; .113]
γ	-.739	.171	[-.980; -.327]	-.789	.148	[-.973; -.428]
ν_1	11.373	6.441	[4.347; 27.622]	.469	.193	[.186; .832]
ν_2	.9789	.670	[.253; 2.767]	.431	.203	[.189 ; .913]
	StBS			SSLBS1		
Parameter	PE	PSD	$CI_{95\%}$	PE	PSD	$CI_{95\%}$
μ	2.663	.115	[2.452; 2.890]	2.650	.108	[2.442 ; 2.862]
ϕ	.082	.017	[.053; .122]	.069	.013	[.048; .097]
γ	-.738	.154	[-.956; -.364]	-.666	.154	[-.880; -0.306]
ν	11.281	10.988	[4.236; 44.789]	3.247	.814	[2.227 ; 5.144]
	SSLBS2			SNBS		
Parameter	PE	PSD	$CI_{95\%}$	PE	PSD	$CI_{95\%}$
μ	2.634	.098	[2.448; 2.823]	2.608	.102	[2.416; 2.798]
ϕ	.100	.016	[.072 ; .136]	.106	.017	[.078; .142]
γ	-.654	.159	[-.863; -.191]	-.647	.134	[-.862 ; -.356]
ν	31.318	22.390	[5.628 ; 84.551]	-	-	-

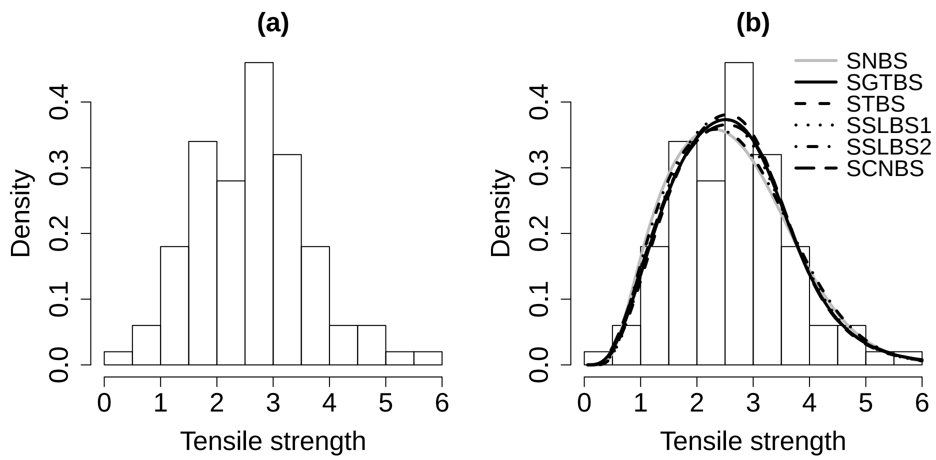

Figure 14 – (a) Histogram of the tensile strength of carbon fibers (b) Histogram of the tensile strength of carbon fibers and estimated densities.

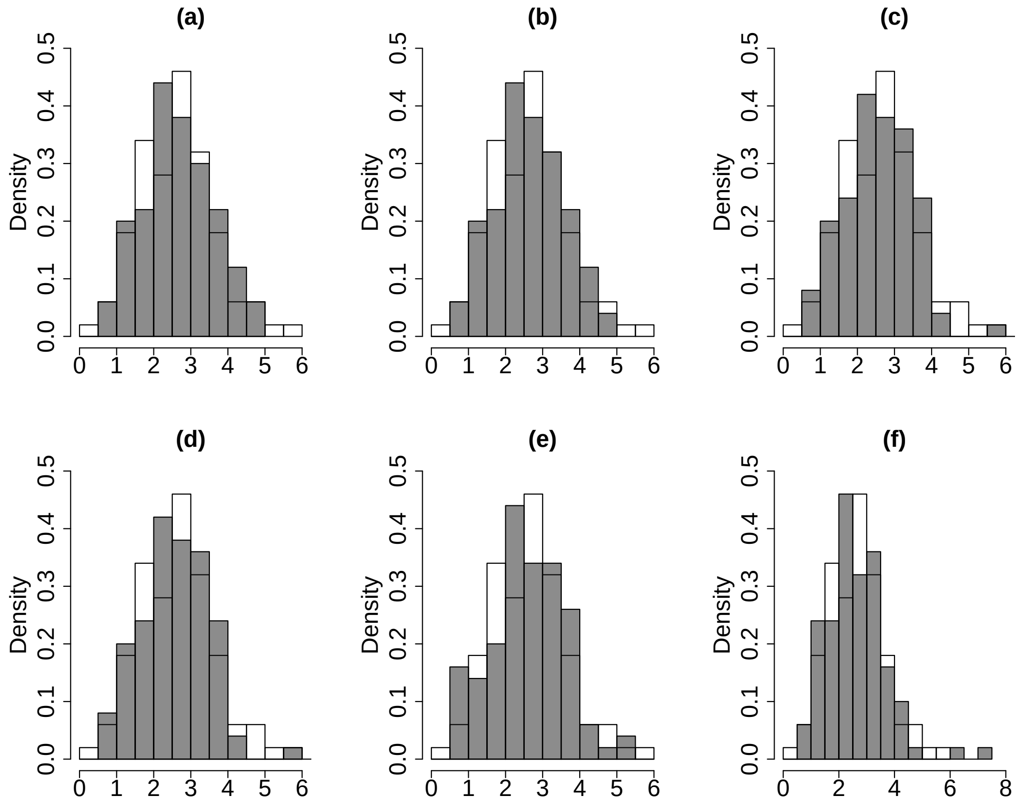

Figure 15 – Histogram of the predicted distributions for the models: (a) SGtBS, (b) StBS, (c) SSLBS1, (d) SSLBS2, (e) SCNBS, and (f) SNBS.

Table 4 – Model selection criteria and Bayesian p-value.

	Models					
Criteria	SGtBS	StBS	SSLBS1	SSLBS2	SCNBS	SNBS
EAIC	293.449	**293.312**	294.330	296.005	294.022	294.066
EBIC	303.870	303.732	304.751	306.426	307.048	**301.881**
DIC	853.622	853.204	856.321	861.163	**850.026**	861.406
LPML	-144.243	-144.045	-144.599	-145.905	**-143.965**	-145.859
p-value	.649	.674	.614	.682	**.554**	.806

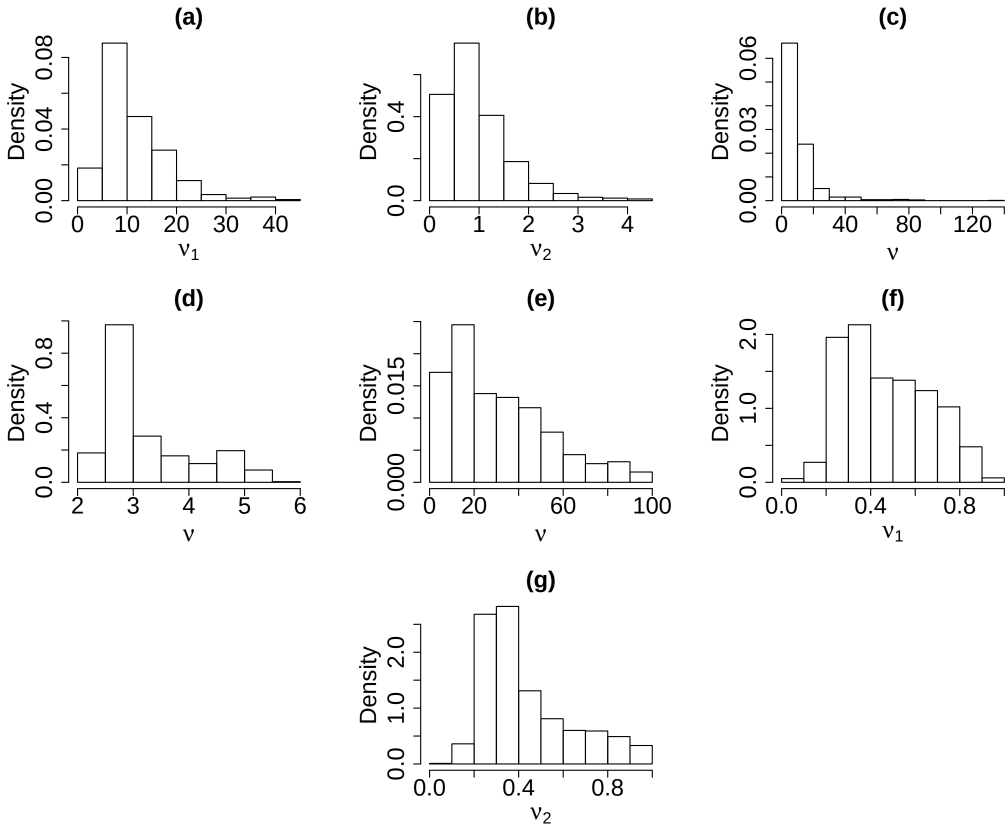

Figure 16 – (a) Posterior density of the parameter: (a) ν_1 of the SGtBS, (b) ν_2 of the SGtBS, (c) ν of the StBS, (d) ν of the SSLBS1, (e) ν of the SSLBS2, (f) ν_1 of the SCNBS and (g) ν_2 of the SCNBS distribution.

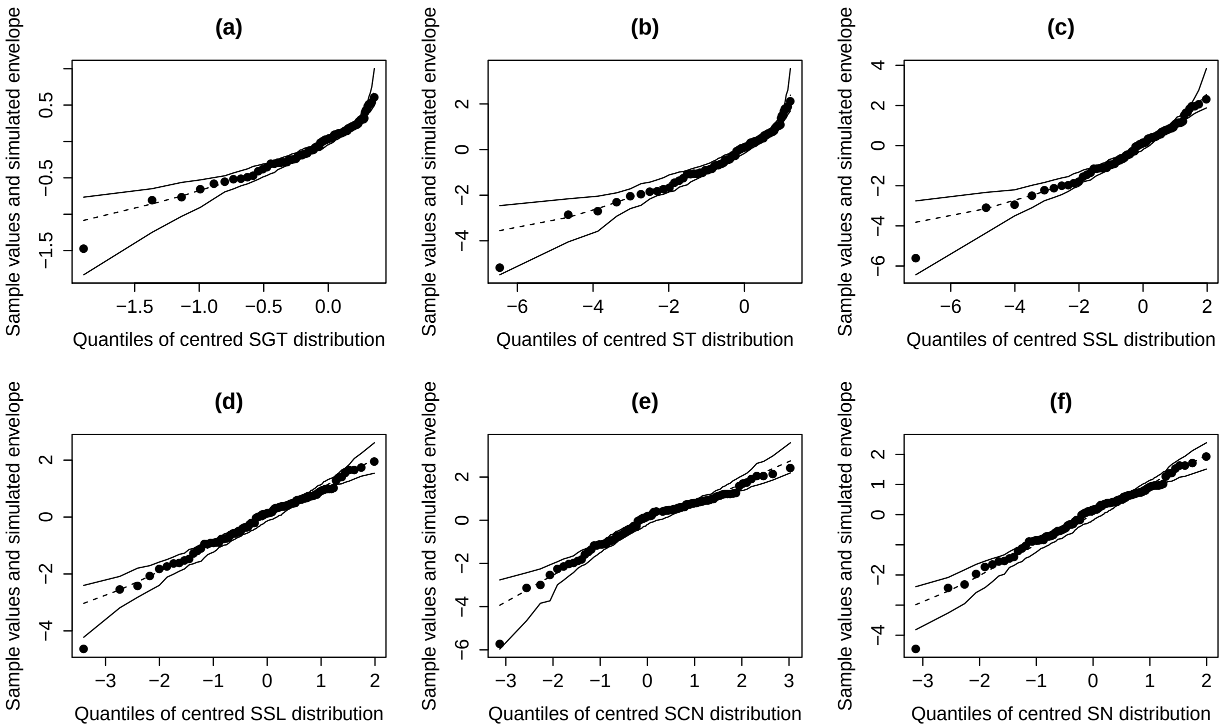

Figure 17 – QQ plot with envelopes for (a) SGtBS, (b) StBS, (c) SSLBS1, (d) SSLBS2, (e) SCNBS and (f) SNBS for the tensile strength of carbon fibers.

1.8 Concluding Remarks

In this chapter, we developed a new class of probability models, named centred skew scale-mixture Birnbaum-Saunders distributions, considering the reparameterization of the BS distribution proposed by Santos-Neto et al. (2012), which allows to write the respective mean on the related density and which can be very useful for regression models on the original scale of the response variable. Also, we consider the family of the skew scale-mixture of normals distributions under the centred parameterization, which, as we have shown, circumvent inferential problems, related to the usual skew scale-mixture of normals distributions used by Balakrishnan et al. (2017). Our family accommodates properly both positively or negatively skewed data, presenting or not heavy tails, and generalizes the scale-mixture of normal BS distributions Balakrishnan et al. (2009). Under Bayesian approach, we developed parameter estimation, statistics for model comparison, and posterior predictive checking based on MCMC algorithms. We conducted parameter recovery studies considering different scenarios of practical interest. In general, the results indicated, for all models, that μ, ϕ and γ were well recovered in all scenarios. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed distributions, the estimates of $\boldsymbol{\nu}$ were close to true values. On the other hand, in scenarios where the proposed distributions were equivalent to the SNBS distribution, the estimates of $\boldsymbol{\nu}$ are biased and the width of the credibility interval are large. However, as the sample size increases, the estimates of $\boldsymbol{\nu}$

get more accurate. Finally, we have presented applications to a real data set related to the breaking stress of carbon fibres (in Gba), showing that the our approach can be much more useful than the traditional ones. The results indicated that the SCNBS distribution offers an excellent fit to the carbon fibers data and that the SNBS model is the least appropriate for this data set.

2 A new class of generalized zero-augmented Birnbaum-Saunders distributions

2.1 Introduction

Statistical modeling of zero-augmented positive data has been received much attention in the last few years. In this context, there are various examples of zero-augmented distributions, for example those presented by: Iwasaki and Daidoji (2009), Ospina and Ferrari (2012), Pereira et al. (2012), Tu (2014), Galvis et al. (2014), among others. Recently, Leiva et al. (2016) developed a new BS distribution to model data of this nature. It is well known that the BS distribution presents many attractive features and properties. Despite the wide use of the BS distribution, it is well defined only for positive values. Therefore, such data sets can not be properly analyzed through this model. Thus, Leiva et al. (2016) adapted the reparametrizated BS distribution (Santos-Neto et al., 2012) that considers positive probability for this event giving rise to the zero-adjusted BS (ZABS) model. The ZABS distribution was also recently explored by Batista (2018) and Tomazella et al. (2018), who proposed ZABS regression models with and without random effects, respectively.

Some positive variables, such as bilirubin concentration in newborns, are typically characterized by the presence of zeros, heavy-tails and skewness. In this chapter, in order to adequately model such characteristics, we proposed a general family of zero-augmented BS distributions, named zero-augmented centred skew scale-mixture Birnbaum-Saunders (ZA-SSBS) distributions. In summary, we used a mixture distribution of two components: the CSSBS models proposed in Chapter 1 (continuous component) and a degenerate distribution at the zero (discrete component). It is well known that the mixture models are powerful and popular tools to generate flexible distributions with good properties (Geoffrey McLachlan (2000) and Kotz et al. (2010)). Our family accommodates properly both positively or negatively skewed data, presenting or not heavy tails, and generalizes the zero-augmented BS distribution (Leiva et al., 2016). In addition, the ZA-SSBS distributions inherit the advantages in inferential terms of the CSSBS distributions. Several of its properties are developed. Furthermore, we developed parameter estimation, statistics for model comparison, and posterior predictive checking through Bayesian inference based on MCMC algorithms. The impact of some factors of interest (sample size, asymmetry levels, and different degrees of freedom) on the estimates, are measured through of a simulation study on parameter recovery. Finally, we have presented applications to a real data set related to the bilirubin concentration, showing the usefulness of the inferential methods developed here. Also, the results indicate that our models outperforms the ZABS in terms of model fit.

2.2 Zero-augmented centred skew scale-mixture Birnbaum-Saunders distributions

Definition 2. *An r.v T has a ZA-SSBS, denoted by $T \sim$ ZA-SSBS$(p, \mu, \phi, \gamma, \boldsymbol{\nu})$, where $p \in (0, 1)$ is the mixture parameter, $\mu > 0$ is a scale parameter, $\phi > 0$ is a shape and dispersion parameter, $\gamma \in (-.99527, .99527)$ is the asymmetry parameter and $\boldsymbol{\nu}$ are degrees of freedom, if its density is given by*

$$h(t|\boldsymbol{\theta}) = p\,\mathbb{1}_{\{0\}}(t) + (1-p)f(t|\mu, \phi, \gamma, \boldsymbol{\nu})\mathbb{1}_{(0,\infty)}(t), t \geqslant 0, \tag{2.1}$$

where $\boldsymbol{\theta} = (p, \mu, \phi, \gamma, \boldsymbol{\nu}^\top)^\top$ and $f(t|\mu, \phi, \gamma, \boldsymbol{\nu})$ was defined in Equation (1.9). Another way of writing Equation (2.1) is

$$h(t|\theta) = p^{\mathbb{1}\{t=0\}}\left[(1-p)f(t|\mu, \phi, \gamma, \boldsymbol{\nu})\right]^{1-\mathbb{1}\{t=0\}}\mathbb{1}_{(0,\infty)}(t), t \geqslant 0. \tag{2.2}$$

The moments of T (see Section B.1 of Appendix B for more details) are given by $\mathbb{E}(T^r|\boldsymbol{\theta}) = (1-p)\mu_r$, where μ_r is the rth moment of the CSSBS distributions and can be calculated using Equation (1.11). Particularly, the mean and the variance of T are given by

$$\begin{aligned} \mathbb{E}(T|\boldsymbol{\theta}) &= (1-p)\mu \\ \mathbb{V}(T|\boldsymbol{\theta}) &= (1-p)\left\{\frac{2\phi\mu^2}{[1+\phi m_2]^2}\left[m_2 + \frac{\phi}{2}\left\{2m_4 - m_2^2\right\}\right]\right\} + p(1-p)\mu^2, \end{aligned} \tag{2.3}$$

where $m_k = \mathbb{E}[Y^k], k = 2, 4$ represents the k moment of $Y \sim$ CSSMN$(0, 1, \gamma, \boldsymbol{\nu})$.

2.2.1 Examples of ZA-SSBS distributions

In this section, we present some particular cases of the ZA-SSBS distributions, which will be considered more detailed in this work.

- *The zero-augmented centred skew normal Birnbaum-Saunders (ZA-SNBS) distribution, denoted by $T \sim$ ZA-SNBS(p, μ, ϕ, γ).* Considering Equation (1.13) in Equation (2.1), the respective density is given by

$$\begin{aligned} h(t|, p, \mu, \phi, \gamma) &= p^{\mathbb{1}\{t=0\}} + \Big[(1-p)\Big\{2\sigma_z\phi\left[\mu_z + \sigma_z a_t(\mu, \phi)\right]\Phi\left\{\lambda\left[\mu_z + \sigma_z a_t(\mu, \phi)\right]\right\} \\ &\quad \times A_t(\mu, \phi)\Big\}\Big]^{1-\mathbb{1}\{t=0\}}\mathbb{1}_{(0,\infty)}(t), \end{aligned}$$

  and the variance, given by

$$\mathbb{V}(T) = (1-p)\left\{\frac{2\phi\mu^2}{[1+\phi]^2}\left[1 + \frac{\phi}{2}\left\{2m_4 - 1\right\}\right]\right\} + p(1-p)\mu^2,$$

  where $m_4 = 2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2} + 3$.

- *The zero-augmented centred skew generalized Student-t Birnbaum-Saunders (ZA-SGtBS) distribution, denoted by $T \sim$ ZA-SGtBS$(p, \mu, \phi, \gamma, \nu_1, \nu_2); \nu_1 > 4, \nu_2 > 0$.* Considering Equation (1.14) in Equation (2.1), the respective density is given by

$$\begin{aligned} h(t|, p, \mu, \phi, \gamma, \nu_1, \nu_2) &= p^{\mathbb{1}\{t=0\}} + \Big[(1-p)\Big\{2A_t(\mu, \phi)(\nu_2/2)^{\nu_1/2}[\Gamma(\nu_1/2)]^{-1} \\ &\times \int_0^\infty \phi\left[a_t(\mu,\phi)\Big| - \frac{\mu_z}{\sqrt{u}\sigma_z}; \frac{1}{u\sigma_z^2}\right] \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{u}a_t(\mu,\phi)\right]\right\} \\ &\times u^{\nu_1/2-1}\exp\left\{-\frac{\nu_2}{2}u\right\}du\Big\}\Big]^{1-\mathbb{1}\{t=0\}} \mathbb{1}_{(0,\infty)}(t), \end{aligned}$$

and the variance, given by

$$\begin{aligned} \mathbb{V}(T) &= (1-p)\left\{\frac{2\phi\mu^2}{\{1+[\phi\nu_2/(\nu_1-2)]\}^2}\left[\left(\frac{\nu_2}{\nu_1-2}\right) + \frac{\phi}{2}\left\{2m_4 - \left(\frac{\nu_2}{\nu_1-2}\right)^2\right\}\right]\right\} \\ &+ p(1-p)\mu^2, \nu_1 > 4, \nu_2 > 0, \end{aligned}$$

where $m_4 = \left[\nu_2^2/(\nu_1-2)(\nu_1-4)\right]\left\{2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2}+3\right\}$. Note that the ZA-SNBS distribution is a special case of the ZA-SGtBS distribution when $\nu_1, \nu_2 \to \infty$. When $\nu_1 = \nu_2 = \nu$, T has a zero-augmented centred skew Student-t Birnbaum-Saunders (ZA-StBS) distribution, denoted by $T \sim$ ZA-StBS(μ, ϕ, γ, ν), which will be also explored in this work.

- *The zero-augmented centred skew slash Birnbaum-Saunders (ZA-SSLBS) distribution, denoted by $T \sim$ ZA-SSLBS$(p, \mu, \phi, \gamma, \nu); \nu > 2$.* Considering Equation (1.15) in Equation (2.1), the respective density is given by

$$\begin{aligned} h(t|, p, \mu, \phi, \gamma, \nu) &= p^{\mathbb{1}\{t=0\}} + \left[(1-p)\left\{2\nu A_t(\mu,\phi)\int_0^1 \phi\left[a_t(\mu,\phi)\Big| - \frac{\mu_z}{\sqrt{u}\sigma_z}; \frac{1}{u\sigma_z^2}\right]\right.\right. \\ &\left.\left.\times \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{u}a_t(\mu,\phi)\right]\right\}u^{\nu-1}du\right\}\right]^{1-\mathbb{1}\{t=0\}} \mathbb{1}_{(0,\infty)}(t), \end{aligned}$$

and the variance, given by

$$\begin{aligned} \mathbb{V}(T) &= (1-p)\left[\frac{2\phi\mu^2}{\{1+[\phi\nu/(\nu-1)]\}^2}\left\{\left(\frac{\nu}{\nu-1}\right) + \frac{\phi}{2}\left[2m_4 - \left(\frac{\nu}{\nu-1}\right)^2\right]\right\}\right] \\ &+ p(1-p)\mu^2, \nu > 2, \end{aligned}$$

where, in this case, $m_4 = \left[\nu/(\nu-2)\right]\left\{2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2}+3\right\}$. Note that when $\nu \to \infty$ the ZA-SSLBS distribution reduces to the ZA-SNBS distribution.

- *The zero-augmented centred skew contaminated normal Birnbaum-Saunders (ZA-SCNBS) distribution, denoted by $T \sim$ ZA-SCNBS$(p, \mu, \phi, \gamma, \nu_1, \nu_2)$; $\nu_1, \nu_2 \in (0, 1)$.* Considering Equation (1.16) in Equation (2.1), the density of T is given by

$$\begin{aligned} h(t|p, \mu, \phi, \gamma, \nu_1, \nu_2) &= p^{\mathbb{1}\{t=0\}} + \left[(1-p)\left\{2A_t(\mu,\phi)\left(\nu_1\phi\left[a_t(\mu,\phi)| - \frac{\mu_z}{\sqrt{\nu_2}\sigma_z}; \frac{1}{\nu_2\sigma_z^2}\right]\right.\right.\right. \\ &\times \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{\nu_2}a_t(\mu,\phi)\right]\right\} + (1-\nu_1)\phi\left[a_t(\mu,\phi)| - \frac{\mu_z}{\sigma_z}; \frac{1}{\sigma_z^2}\right] \\ &\left.\left.\left.\times\ \Phi\left\{\lambda\left[\mu_z + \sigma_z a_t(\mu,\phi)\right]\right\}\right)\right\}\right]^{1-\mathbb{1}\{t=0\}} \mathbb{1}_{(0,\infty)}(t), \end{aligned}$$

and the variance, given by

$$\mathbb{V}(T) = (1-p)\left[\frac{2\phi\mu^2}{\{1+[\phi(\nu_1/2+1-\nu_1)]\}^2}\left\{\left(\frac{\nu_1}{\nu_2}+1-\nu_1\right)+\frac{\phi}{2}\left[2m_4 - \left(\frac{\nu_1}{\nu_2}+1-\nu_1\right)^2\right]\right\}\right]+p(1-p)\mu^2,$$

where, in this case, $m_4 = \left[(\nu_1/\nu_2^2)+1-\nu_1\right]\left\{2(\pi-3)(4/\pi^2)\delta^4[1-(2\delta^2/\pi)]^{-2}+3\right\}$. Note that the ZA-SCNBS distribution is a special case of the SCNBS distribution when $\nu_2 \to 1$ and/or $\nu_1 \to 0$.

Figures 18-20 present the densities of the ZA-SGtBS, ZA-SSLBS and ZA-SCNBS distributions, respectively, for different values of μ, ϕ and γ, and fixed values of p and $\boldsymbol{\nu}$. Also, Figure 21 presents the density of the ZA-SGtBS, ZA-SSLBS and ZA-SCNBS distributions for different values of p, and fixed values of μ, ϕ, γ, and $\boldsymbol{\nu}$. We observed that p affects only the scale of distribution. On the other hand, we can notice that μ affects both the scale and position of the distributions and, the higher/smaller it is, the higher/smaller the variance is. It is also possible to note that ϕ and γ control the skewness and kurtosis, respectively. More specifically, as ϕ increases and γ assumes positive values, the densities become more dispersed and positively skewed. Also, $\boldsymbol{\nu}$ controls the weight of the tails.

In short, the distributions tend to be symmetric around μ, for $\gamma = 0$ and for small values of ϕ. Also, p only changes the scale of the distributions. Positive asymmetry is observed as ϕ increases and/or γ assumes positive values. On the other hand, negative asymmetry is observed as ϕ decreases and/or γ assumes negative values. The ZA-SGtBS, ZA-SSLBS and ZA-SCNBS distributions have tails much heavier than the ZA-SNBS distribution when $\nu_1 \to 0$, $\nu \to 0$ and $\nu_2 \to 0$ and $\nu_1 \to 1$, respectively. Thus, the proposed family provides flexible skewed heavy-tailed distributions.

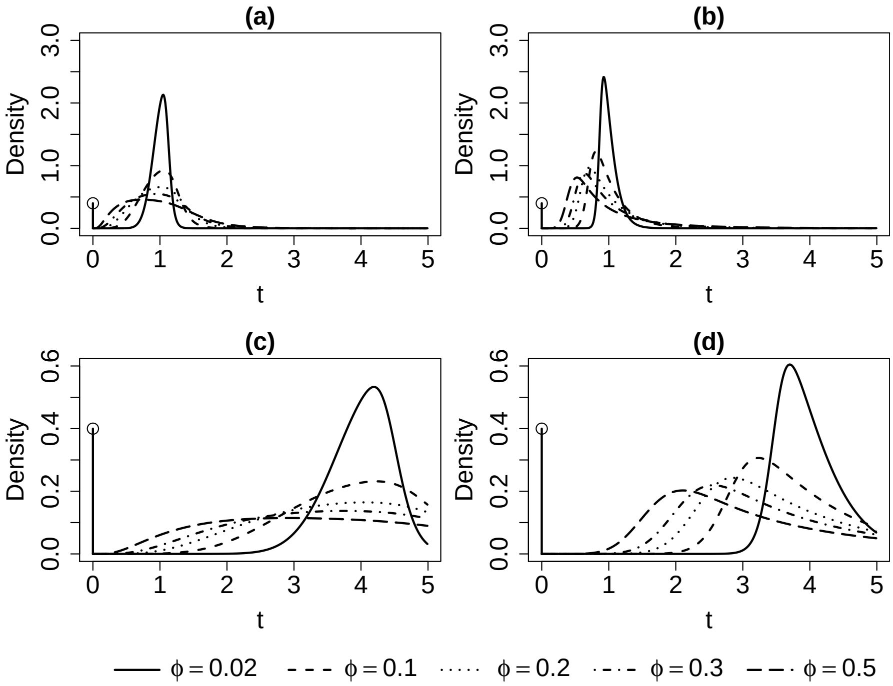

Figure 18 – Densities of the ZA-SGtBS distribution for different values of μ, ϕ and γ, with $\nu_1 = \nu_2 = 5$: (a) $\mu = 1, \gamma = -.8$; (b) $\mu = 1, \gamma = .8$; (c) $\mu = 4, \gamma = -.8$ and (d) $\mu = 4, \gamma = .8$.

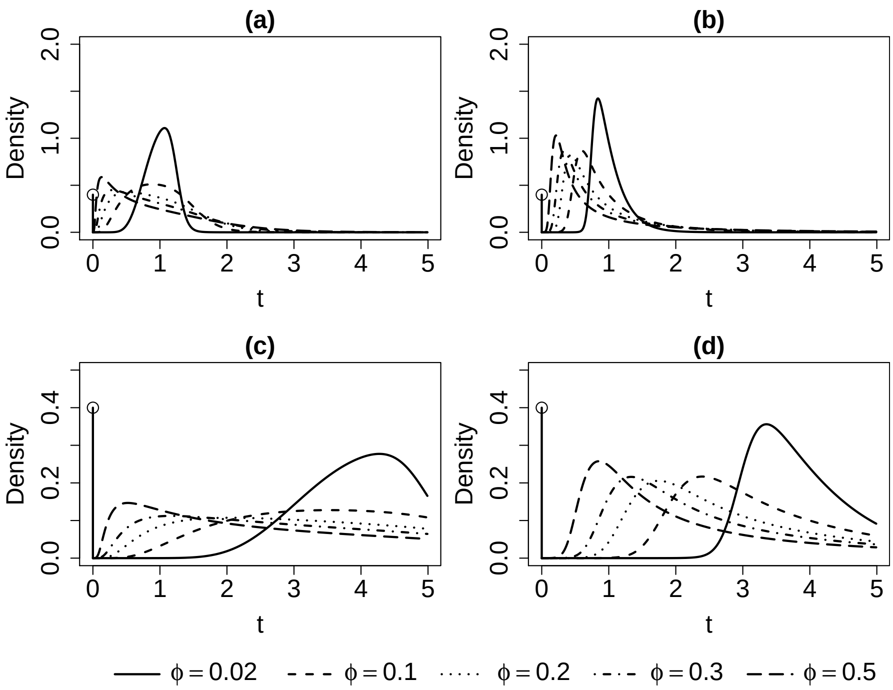

Figure 19 – Densities of the ZA-SSLBS distribution for different values of μ, ϕ and γ, with $\nu = 5$: (a) $\mu = 1, \gamma = -.8$; (b) $\mu = 1, \gamma = .8$; (c) $\mu = 4, \gamma = -.8$ and (d) $\mu = 4, \gamma = .8$.

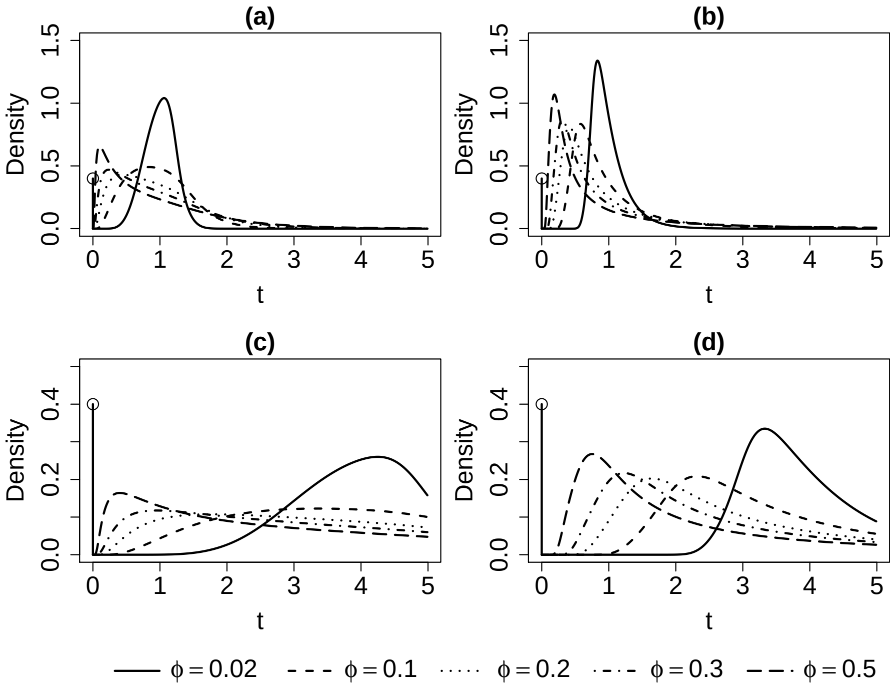

Figure 20 – Densities of the ZA-SCNBS distribution for different values of μ, ϕ and γ, with $\nu_1 = \nu_2 = .5$: (a) $\mu = 1, \gamma = -.8$; (b) $\mu = 1, \gamma = .8$; (c) $\mu = 4, \gamma = -.8$ and (d) $\mu = 4, \gamma = .8$.

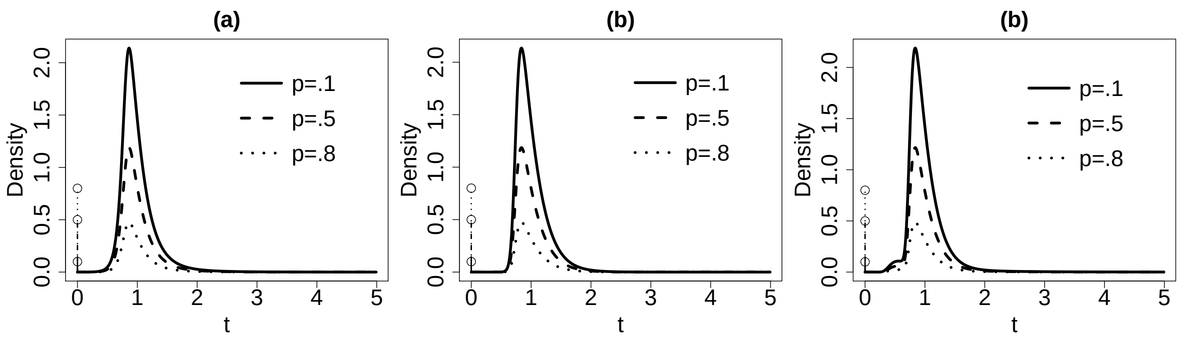

Figure 21 – Densities of the (a) ZA-SGtBS, (b) ZA-SSLBS and (c) ZA-SCNBS distributions for different values of p.

2.3 Bayesian inference

In this section, we present the Bayesian inference for ZA-SSBS models. The adopted approach, since the marginal posterior distributions on interest can not be analytically obtained, relies on the MCMC algorithms to obtain numerical approximations for those distributions.

2.3.1 Likelihoods

Let $T_i|\boldsymbol{\theta} \overset{\text{iid}}{\sim} \text{ZA-SSBS}(p,\mu,\phi,\gamma,\boldsymbol{\nu}), i = 1,\ldots,n,$ where $\boldsymbol{\theta} = (p,\mu,\phi,\gamma,\boldsymbol{\nu}^\top)^\top$. The likelihood for $\boldsymbol{\theta}$, given the observed sample $\mathbf{t} = (t_1,\ldots,t_n)^\top$, takes on the form

$$L(\boldsymbol{\theta}|\mathbf{t}) = \left[\prod_{i=1}^{n} p^{\mathbb{1}\{t_i=0\}}(1-p)^{1-\mathbb{1}\{t_i=0\}}\right]\left[\prod_{i=1}^{n} f(t_i|\mu,\phi,\gamma,\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}. \tag{2.4}$$

where $f(t|\mu,\phi,\gamma,\boldsymbol{\nu})$ was defined in Equation (1.9). Furthermore, considering the hierarchical representation presented in Equation (1.18), which is useful for stochastic simulation and parameter estimation (both frequentist and Bayesian), we have that the so-called complete likelihood is given by

$$\begin{aligned} L(\boldsymbol{\theta}|\mathbf{t}_c) \;\propto\; & \left[\prod_{i=1}^{n} p^{\mathbb{1}\{t_i=0\}}(1-p)^{1-\mathbb{1}\{t_i=0\}}\right]\left[\prod_{i=1}^{n} \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\exp\left\{-h_i^2/2\right\}\right. \\ & \left.\times\; g(u_i|\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}, \end{aligned} \tag{2.5}$$

where $\mathbf{t}_c = (\mathbf{t}^\top,\mathbf{h}^\top,\mathbf{u}^\top)$, with $\mathbf{t} = (t_1,\ldots,t_n)^\top$, $\mathbf{h} = (h_1,\ldots,h_n)^\top$ and $\mathbf{u} = (u_1,\ldots,u_n)^\top$. Also, ϑ_{h_i} was defined in Equation (1.18), $a_{t_i,\tau_i}(\mu,\phi)$ and $A_{t_i,\tau_i}(\mu,\phi)$ were defined in Equation (1.19).

2.3.2 Prior, posterior and full conditional distributions

To complete the Bayesian specification we need to consider convenient prior distribution. Assuming that the parameters are independent a priori, we consider that the respective joint prior distribution is given by

$$\pi(\boldsymbol{\theta}) = \pi(p)\pi(\mu)\pi(\phi)\pi(\gamma)\pi(\boldsymbol{\nu}). \tag{2.6}$$

We chose for μ, ϕ and γ the prior distributions presented in Equation (1.21). Additionally, we consider $p \sim \text{beta}(c,d)$, where $\text{beta}(c,d)$ stands for the beta distribution with mean $c/(c+d)$ and variance $cd/[(c+d)^2(c+d+1)]$. The prior distribution of $\boldsymbol{\nu}$ depends on the particular cases of the ZA-SSBS distribution (more details are provided ahead). Combining the complete likelihood presented in Equation (2.5) and the prior distribution presented in Equation (2.6), the joint posterior distribution is given by

$$\begin{aligned} \pi(\boldsymbol{\theta},\mathbf{h},\mathbf{u}|\mathbf{t}) \;\propto\; & \left\{\left[\prod_{i=1}^{n} p^{\mathbb{1}\{t_i=0\}}(1-p)^{1-\mathbb{1}\{t_i=0\}}\right]\left[\prod_{i=1}^{n} \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\right.\right. \\ & \left.\left.\times\; \exp\left\{-h_i^2/2\right\} g(u_i|\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}\right\}\pi(\boldsymbol{\theta}). \end{aligned}$$

and the full conditional distributions, are given by

$$\begin{aligned} \pi(h_i|\boldsymbol{\theta},t_i,u_i) \;&\propto\; \left\{\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right]\exp\left\{-h_i^2/2\right\}\right\}^{1-\mathbb{1}\{t_i=0\}} \\ \pi(u_i|\boldsymbol{\theta},t_i,h_i) \;&\propto\; \left\{\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi) g(u_i|\boldsymbol{\nu})\right\}^{1-\mathbb{1}\{t_i=0\}} \end{aligned} \tag{2.7}$$

$$\begin{aligned}
\pi(p|\mu,\phi,\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(p)\left\{\prod_{i=1}^{n} p^{\mathbb{1}\{t_i=0\}}(1-p)^{1-\mathbb{1}\{t_i=0\}}\right\}\\
\pi(\mu|p,\phi,\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\mu)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\right\}^{1-\mathbb{1}\{t_i=0\}}\\
\pi(\phi|p,\mu,\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\phi)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\right\}^{1-\mathbb{1}\{t_i=0\}}\\
\pi(\gamma|p,\mu,\phi,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\gamma)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\right\}^{1-\mathbb{1}\{t_i=0\}}\\
\pi(\boldsymbol{\nu}|p,\mu,\phi,\gamma,\mathbf{t}_c) &\propto \pi(\boldsymbol{\nu})\left\{\prod_{i=1}^{n} g(u_i|\boldsymbol{\nu})\right\}^{1-\mathbb{1}\{t_i=0\}}.
\end{aligned} \tag{2.8}$$

The shape of distributions presented in Equations (2.7) and (2.8) vary according to the particular cases of the ZA-SSBS distribution and the adopted prior for $\boldsymbol{\nu}$. The full conditional distributions of u_i and $\boldsymbol{\nu}$, and $\pi(\boldsymbol{\nu})$ for each ZA-SSBS distribution are presented bellow. We made all implementations considering the `OpenBUGS` software (Spiegelhalter et al., 2014), through the `R` program (R Core Team, 2014), using the package `R2OpenBUGS` (Sturtz et al., 2010). That is, the auxiliary algorithms used to simulate from the full conditional distributions follow the predefined hierarchy implemented in `OpenBUGS`.

2.3.3 Prior distribution of $\boldsymbol{\nu}$ and full conditional distributions

1. *The zero-augmented centred skew generalized Student-t Birnbaum-Saunders (ZA-SGtBS) distribution.* Here, $\boldsymbol{\nu}=(\nu_1,\nu_2)^\top$. We adopt a truncated exponential prior for ν_j, that is $\nu_j\sim\exp(\Lambda)\mathbb{1}_{\mathbb{A}}(\nu_j)$, such that $\Lambda\sim\mathrm{U}(\Lambda_0,\Lambda_1)$ (Cabral et al., 2012), which leads to the density

$$\pi(\nu_j)=\frac{1}{\nu_j^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu+1)\exp\{-\Lambda_0\nu_j\}-(\Lambda_1\nu_j+1)\exp\{-\Lambda_1\nu_j\}\right].$$

The full conditional distributions of u_i and ν_j take the form

$$\begin{aligned}
\pi(u_i|\boldsymbol{\theta},t_i,h_i) &\propto \left\{\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu,\phi)\right]A_{t_i,\tau_i}(\mu,\phi)\,u_i^{\nu_j/2-1}\exp\left\{-\frac{\nu_j}{2}u_i\right\}\right\}^{1-\mathbb{1}\{t_i=0\}}\\
\pi(\nu_j|p,\mu,\phi,\gamma,\mathbf{t}_c) &\propto \frac{1}{\nu_j^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_j+1)\exp\{-\Lambda_0\nu_j\}-(\Lambda_1\nu_j+1)\exp\{-\Lambda_1\nu_j\}\right]\\
&\quad\times\left\{\prod_{i=1}^{n}(\nu_j/2)^{\nu_j/2}\left[\Gamma(\nu_j/2)\right]^{-1}u_i^{\nu_j/2-1}\exp\left\{-\frac{\nu_j}{2}u_i\right\}\right\}^{1-\mathbb{1}\{t_i=0\}}.
\end{aligned}$$

2. *The zero-augmented centred skew slash Birnbaum-Saunders (ZA-SSLBS) distribution.* We adopt a truncated gamma distribution for $\boldsymbol{\nu}=\nu$, that is, $\nu\sim\mathrm{gamma}(a,b)\mathbb{1}_{\mathbb{A}}(\nu)$,

with small positive values of a and b ($b << a$), see Vilca et al. (2016). The full conditional distributions of u_i and ν become

$$\begin{aligned}\pi(u_i|\boldsymbol{\theta}, t_i, h_i) &\propto \left\{\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi) u_i^{\nu-1}\right\}^{1-\mathbb{1}\{t_i=0\}}\\ \pi(\nu|p,\mu,\phi,\gamma,\mathbf{t}_c) &\propto \nu^{a-1}\exp\{-b\nu\}\left\{\prod_{i=1}^{n}\nu\, u_i^{\nu-1}\right\}^{1-\mathbb{1}\{t_i=0\}}.\end{aligned}$$

3. *The zero-augmented centred skew contaminated normal Birnbaum-Saunders (ZA-SCNBS) distribution.* The possible states of the "weights" u_i are ν_2 or 1, with $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$, and its density can be expressed as

$$g(u_i|\boldsymbol{\nu}) = \nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}.$$

The full conditional distribution of each u_i can be written as:

$$\pi(u_i|\boldsymbol{\theta}, t_i, h_i) \propto \left\{\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}\right\}^{1-\mathbb{1}\{t_i=0\}}.$$

Thus, the distribution is proportional to

$$\begin{cases}\left[\nu_1\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\right]^{1-\mathbb{1}\{t_i=0\}}, \text{if } u_i = \nu_2\\ \left[(1-\nu_1)\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu,\phi)\right] A_{t_i,\tau_i}(\mu,\phi)\right]^{1-\mathbb{1}\{t_i=0\}}, \text{if } u_i = 1\end{cases}$$

In this case, we consider $\nu_j \sim \text{beta}(a_j, b_j)$ (Lachos et al., 2017). The full conditional distribution of $\nu_j, j = 1, 2$ is given by

$$\pi(\nu_j|p,\mu,\phi,\gamma,\mathbf{t}_c) \propto \nu_j^{a_j+a_{n,\nu_2}-1}(1-\nu_j)^{b_j+b_{n,\nu_2}-1},$$

where $a_{n,\nu_2} = [(1-\mathbb{1}\{t_i=0\})(n-\sum_{i=1}^n u_i)]/(1-\nu_2)$ and $b_{n,\nu_2} = [(1-\mathbb{1}\{t_i=0\})\times$ $\times(\sum_{i=1}^{n} u_i - n\nu_2)]/(1-\nu_2)$, which is proportional to the beta$(a_j + a_{n,\nu_2}, b_j + b_{n,\nu_2})$ distribution.

2.4 Model fit assessment and model comparison

2.4.1 Statistics for model comparison

When MCMC algorithms are used to obtain the posterior distributions, some statistics for model comparison can be easily calculated, see Spiegelhalter et al. (2002). To introduce these statistics, we first define $\mathrm{D}(\boldsymbol{\theta}) = -2\log\left[L(\boldsymbol{\theta}|\mathbf{t})\right]$, where $\boldsymbol{\theta} = (p,\mu,\phi,\gamma,\boldsymbol{\nu}^\top)^\top$ and $L(\boldsymbol{\theta}|\mathbf{t})$ is the (incomplete) likelihood presented in Equation (2.4). Also, let $\boldsymbol{\theta}^{(m)}$, $m = 1, ..., M$, be the mth value of the valid simulated MCMC sample, that is, the MCMC sample obtained after discarding the burn-in and a proper thinning (lag) between the values. Finally, let $\overline{\boldsymbol{\theta}}$ be the vector with the posterior expectation of all parameters, based on the valid MCMC sample, and $\overline{\mathrm{D}(\boldsymbol{\theta})} = \frac{1}{M}\sum_{m=1}^{M} D(\boldsymbol{\theta}^{(m)})$. Denote also the deviance by $\mathrm{D}(\overline{\boldsymbol{\theta}}) = -2\ell(\overline{\boldsymbol{\theta}}|\mathbf{t})$, and the deviance information criterion (DIC) by $\mathrm{DIC} = \mathrm{D}(\overline{\boldsymbol{\theta}}) + 2p_D$, where $p_D = \overline{\mathrm{D}(\boldsymbol{\theta})} - \mathrm{D}(\overline{\boldsymbol{\theta}})$. The EAIC (posterior expectation of AIC) and EBIC (posterior expectation of BIC) are given, respectively, by $\mathrm{EAIC} = D(\bar{\boldsymbol{\theta}}) + 2k$ and $\mathrm{EBIC} = \mathrm{D}(\bar{\boldsymbol{\theta}}) + k\log(n)$, where k is the total number of parameters of the model and n is the number of observations. Finally, the LPML (logarithm of the pseudo-marginal likelihood) is calculated as $\mathrm{LPML} = \sum_{i=1}^{n}\ln(\widehat{\mathrm{CPO}}_i)$, where $\widehat{\mathrm{CPO}}_i = \left\{\frac{1}{M}\sum_{m=1}^{M}\left[1/L\left(\boldsymbol{\theta}^{(m)}|t_i\right)\right]\right\}^{-1}$. The smaller the values of DIC, EAIC, EBIC and deviance, the better the model fit, occurring the opposite with the LPML.

2.4.2 Posterior predictive checking

Under a Bayesian perspective, one way to check the goodness of model fit, is to compare the response predictive distribution with the observed distribution of the data. Let $\boldsymbol{t}^{\mathrm{obs}}$ be the observed response and $\boldsymbol{t}^{\mathrm{rep}}$ the replicated response generated from its posterior predictive distribution, which is given by

$$p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{t}^{\mathrm{obs}}) = \int p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\mathrm{obs}}) d\boldsymbol{\theta}, \tag{2.9}$$

where $\boldsymbol{\theta} = (p, \mu, \phi, \gamma, \boldsymbol{\nu}^{\top})^{\top}$. A suitable discrepancy measure $D(\boldsymbol{t}, \boldsymbol{\theta})$ are defined by Gelman et al. (1996) and the posterior distribution of $D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\theta})$ is compared to the posterior predictive distribution of $D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\theta})$, and substantial differences between them indicating model misfit. Gelman et al. (2013) suggest several graphs to compare the replicated and the observed data, under the given measure of divergence.

Another measure used to quantify the goodness of fit, it is the Bayesian p-value, which for an adopted discrepancy measure is defined as

$$\begin{aligned}\mathbb{P}[D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\theta})] &\geqslant D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\theta}|\boldsymbol{t}^{\mathrm{obs}}) \\ &= \int_{D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\theta}) \geqslant D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\theta})} p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\mathrm{obs}}) d\boldsymbol{t}^{\mathrm{rep}} d\boldsymbol{\theta}.\end{aligned} \tag{2.10}$$

Due to the difficulty in dealing with Equations (2.9) and (2.10) analytically, Rubin (1984) suggests simulating replicated data sets from the posterior predictive distribution. One draws M simulations $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_M$ from the posterior distribution $p(\boldsymbol{\theta}|\boldsymbol{t})$ of $\boldsymbol{\theta}$ and then draws $\boldsymbol{t}^{\mathrm{rep},n}$ from the distribution $p(\boldsymbol{t}|\boldsymbol{\theta}^n)$ for $n = 1, \ldots, M$. The proportion of the M replications for which $D(\boldsymbol{t}^{\mathrm{rep},n}, \boldsymbol{\theta}^n)$ exceeds $D(\boldsymbol{t}, \boldsymbol{\theta}^n)$ provides an estimate of the p-value Bayesian. Extreme values of the Bayesian p-value (less than .05 or greater than .95, depending on the nature of the discrepancy measure) indicate model misfit, see Sinharay et al. (2006). Based on Gelman et al. (1996), the measure of discrepancy here adopted is $D(\mathbf{t}|\boldsymbol{\theta}) = \sum_{i=1}^{n}\{[t_i - \mathbb{E}(T_i|\boldsymbol{\theta})]^2\}/\mathbb{V}(T_i|\boldsymbol{\theta})$, where $\mathbb{E}(T_i|\boldsymbol{\theta})$ and $\mathbb{V}(T_i|\boldsymbol{\theta})$ are given by Equation (2.3).

2.5 Simulation study

In this section, we presented a parameter recovery study in order to evaluate the performance of the methodology. We considered different scenarios of interest, which correspond to the combination of the levels of some factors. They (with the respective levels within parenthesis) are: sample size (n) (100, 500), that is, small and large sample sizes, values of the parameter γ (-.8, 0, .8), that is, small and large sample sizes, asymmetry parameter γ (-.8, 0, .8), that is high negative skewness, symmetry and high positive skewness, and different values of $\boldsymbol{\nu}$, which induce either a normal shape, or a heavy tails behavior. Specifically, we considered $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ for the ZA-SGtBS model, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the ZA-SCNBS model, and $\nu \in \{5, 30\}$ for the ZA-StBS and ZA-SSLBS models. Also, we fixed $p = .1$, which reflects the proportion of zeros in real data. In addition, we fix $\mu = 1$ for all models and $\phi = .5$ for the ZA-StBS, ZA-SSLBS and ZA-SCNBS models, which induce a strong positively skewed behavior of the proposed distributions. To overcome the identifiability problem in the ZA-SGtBS model, we fix $\phi = 1$.

Based on the results obtained in the previous chapter, we assume for μ, ϕ, γ the same prior distributions considered in Section 1.6, that is, $\mu \sim \text{gamma}(.001, .001)$, $\phi \sim \text{gamma}(1, .5)$ and $\gamma \sim \text{U}(-.99527, .99527)$. The first prior is quite flat and the second is reasonably concentrated in the interval (0, 4.5) (90% of the mass), and was based on works available in the literature which indicate that, in general, the estimates usually lie in this interval. The third prior, suggested by Azevedo et al. (2011), is non-informative. Also, for p, we considered $p \sim \text{beta}(1, 1)$. For the ZA-SGtBS model we set $\nu_j \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_j}(\nu_j)$; $j = 1, 2$, such that $\Lambda \sim U(.02, .5)$ (Cabral et al., 2012). In order to guarantee the existence of the first two moments we consider $\mathbb{A}_1 = (4, \infty)$ and $\mathbb{A}_2 = (2, \infty)$. Similarly, for the ZA-StBS model we set $\nu \sim \exp(\Lambda)\mathbb{1}_{(4,\infty)}(\nu)$, such that $\Lambda \sim U(.02, .5)$. For the ZA-SCNBS model we chose $\nu_1 \sim \text{beta}(2, 2)$ and $\nu_2 \sim \text{beta}(1, 1)$ (Lachos et al., 2017).

Specifically for the ZA-SSLBS distribution, we investigated the sensitivity to the prior choice for ν. , by using different suggestions found in the literature, such as: $\nu \sim \exp(\Lambda)\mathbb{1}_{(2,\infty)}(\nu)$, where $\Lambda \sim U(.02, .5)$, and $\nu \sim \text{gamma}(.001, .001)\mathbb{1}_{(2,\infty)}(\nu)$, both suggested by Cabral et al. (2012), $\nu \sim \text{gamma}(1, .1)\mathbb{1}_{(2,\infty)}(\nu)$ (Vilca et al., 2016) and $\nu \sim \text{gamma}(.01, .001)\mathbb{1}_{(2,\infty)}(\nu)$ (Bandyopadhyay et al., 2010). The results showed, for all distributions, estimates concentrated in the interval $(2, 7)$, independently of the respective true value. Therefore, we decided to consider two situations: when the true ν value is small, for example $\nu = 5$, we chose $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$. When the true ν value is large, for example $\nu = 30$, we chose $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$. We will refer to the ZA-SSLBS model as ZA-SSLBS1 and as ZA-SSLBS2, when we consider $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$ and $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$, respectively. In a real data analysis, we can fit the two models and consider QQ plots, the statistics for model comparison, and Bayesian p-values, to choose the most properly one.

From the results related to a convergence study (not presented for the sake of simplicity) we observed that to set a burn-in of 50,000 iterations, with a spacing of 50 iterations, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the ZA-SGtBS model. On the other hand, when $\nu_1 = \nu_2$, that is, for the ZA-StBS model, we considered a burn-in of 50,000 iterations, with spacing of 30 iterations, generating a total of 80,000 values. For ZA-SSLBS1 model, we set a burn-in of 60,000 iterations and a total of 100,000 values were simulated, and samples were collected at a spacing of 40 iterations. For ZA-SSLBS2 model, we observed that to set a burn-in of 80,000, with a spacing of 40 iterations, generating a total of 120,000 values was enough to have valid MCMC samples of 1,000 values for each parameter. Finally, for ZA-SCNBS model, we set a burn-in of 40,000 iterations, with spacing of 40, generating a total of 80,000 values. In general, for all models, the Geweke statistic did not indicate lack of convergence for any parameter. Furthermore, the values for the Gelman-Rubin statistic and the inspection of the traceplots and autocorrelation plots indicated that the MCMC algorithms converged and the autocorrelations were almost negligible.

We considered R=10 replicas (simulated responses from the models) and calculated the usual statistics to measure the accuracy of the bias, standard deviation (SD), square root of the mean squared error (RMSE), absolute value of the relative bias (AVRB), coverage probability (CP) of the 95% equi-tailed credibility interval and average length (LCI) of the 95% equi-tailed credibility interval. Let θ be the parameter of interest and let $\widehat{\theta}_r$ be some estimate (posterior mean, median or mode) related to the replica r, and $\overline{\widehat{\theta}} = (1/R)\sum_{r=1}^{R}\widehat{\theta}_r$. The aforementioned statistics are: bias $= \overline{\widehat{\theta}} - \theta$; SD $= \sqrt{(1/R)\sum_{r=1}^{R}(\widehat{\theta}_r - \overline{\widehat{\theta}})^2}$, RMSE $= \sqrt{(1/R)\sum_{r=1}^{R}(\theta - \widehat{\theta}_r)^2}$ and AVRB $= |\overline{\widehat{\theta}} - \theta|/|\theta|$, CP $= (1/R)\sum_{r=1}^{R} \mathrm{I}(\theta \in [\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}])$ and LCI $= (1/R)\sum_{r=1}^{R}[\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}]$, where $\widehat{\theta}_{r,LCL}$ and $\widehat{\theta}_{r,UCL}$ are the estimated lower and upper 95% limits of the CI's, respectively. We considered ($< .001$) to represent positive values (statistics and/or estimates) and ($> .001$) to denote negative values, when they are close to zero. All results of the parameter recovery study can be found in Section B.2 in Appendix B.

In Tables 42 and 47, the results for the ZA-SGtBS distribution are presented. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$, as the sample size increases, the estimates for all parameters tend to the correspondent true values. When $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$, which corresponds to the ZA-StBS distribution with $\nu = 30$, we can notice that p, μ and γ were well recovered for both sample sizes. In this scenario, although ν_1 and ν_2 are underestimated, the estimates lead to an equivalence between the ZA-SGtBS and ZA-SNBS models. Therefore, we have indications that ν_1 and ν_2 are reasonably estimated in scenarios that induce a normal shape behavior.

Tables 48-53 present the results for the ZA-StBS distribution and Tables 54-59 present the results for the ZA-SSLBS. For both models, under $\nu = 5$, we can notice that the estimates, for all parameters, tend to the correspondent true values in all scenarios. Also, as the sample size increases, the width of the credibility intervals, bias, RMSE, and AVRB decrease. Under $\nu = 30$, the estimates of ν tend to the correspondent true value. However, the width of credibility intervals are too large. Concerning p, μ, ϕ, and γ, the estimates are close to the respective true values in all scenarios.

In Tables 60-65, the results for the ZA-SCNBS distribution are presented. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$, we can notice that the estimates of p, μ and γ tend to the correspondent true values. However, the estimates of ϕ were not so good for both sample sizes. In this case, larger sample size is required so that the estimates of ϕ to be close to the respective true value. As mentioned earlier, when the estimates of the hyperparameters ν_1 and ν_2 of ZA-SCNBS distribution are such that $\nu_2 \to 0$ and $\nu_1 \to 1$, we have that this distribution has much heavier tails than the ZA-SNBS one. Based only on the posterior mode, $\nu_1 = .638$ and $\nu_2 = .431$ (see Table 62), for example, it is not clear that the ZA-SCNBS distribution has heavy tails. However, when we also consider the estimates of μ, ϕ, and γ, we can notice that the ZA-SCNBS distribution has a behavior compatible with that of the heavy-tailed distribution. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, we can notice that the estimates of all parameters tend to the correspondent true values in all scenarios. Also, as the sample size increases, the bias, RMSE and AVRB decrease.

In general, we can notice that p, μ, ϕ, and γ are well recovered in all models. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed distributions, we can notice that the estimates are close to respective true values. On the other hand, when the proposed distributions are equivalent to ZA-SNBS distribution, the $\boldsymbol{\nu}$ estimates are biased and the width of the credibility interval are large. However, as sample size increases, the estimates become more accurate.

2.6 Real data analysis

In this section, we illustrate the proposed methodology by analyzing a real data set with our methodology. The data set considered here refers to the bilirubin concentration study in newborns conducted in the Escola Paulista de Medicina (UNIFESP) in 2005, see Draque (2005). The concentration of bilirubin (μ mol/L) was measured in 89 healthy full-term newborns on breastfeeding for 1, 2, 3, 4, 5, 6, 8, 10 and 12 days after birth, resulting in a total of 801 observations, being 61 equal to zero. Some descriptive statistics, including location measures, standard deviation (SD), coefficient of skewness (CS), and kurtosis (CK), are provided in Table 5. From these statistics and Figure 22 (a), we notice that the ZA-SSBS models can be reasonably assumed for modeling these data mainly due to the presence of observations equal to zero, and their asymmetric nature. Thus, we fitted the proposed distributions and the ZABS distribution (Leiva et al., 2016) using the Bayesian approach. The prior distributions were the same presented in Section 2.3.2.

We constructed the QQ plots with simulated envelopes. Here, we used an adaptation of the randomized quantile residual (Dunn and Smyth, 1996). In summary, we replaced the Bayesian estimates of μ and ϕ in

$$R_i^q = \begin{cases} \Phi^{-1}[F_{T_i|\boldsymbol{\theta}}(t_i)], \text{if } t_i > 0, \\ \Phi^{-1}(u_i), \text{if } t_i = 0, \end{cases}$$

where $F_{T_i|\boldsymbol{\theta}}(t_i)$ was defined in Equation (1.10) and u_i is the observed value of $U_i \sim (0, \widehat{p})$, where $\widehat{p}$ is the Bayesian estimate of p. The methodology used for the construction of simulated envelopes follows the usual one. That is, since the observed are expected to follow a standard normal distribution, under the well fit of the model, the envelopes are simulated from a standard normal distributions, as described in Atkinson (1985). These plots are presented in Figure 23 (lines represent the 5th percentile, the mean, and the 95th percentile of 100 simulated points of each observations). From Figure 23 (a)-(f), we can notice that the proposed models present a similar and a good fitting. On the other hand, from Figure 23 (g), we can see a systematic behavior, i.e., the observations appear to form a downward-facing. This behavior is compatible with a negatively skewed distribution. Thus, we have indications that the ZABS distribution is not appropriate for this data.

Table 6 presents the posterior expectations (PE), posterior standard deviations (PSD) and equi-tailed 95% credibility intervals (CI). We can see that all parameters were statistically significant, since the zero does not belong to the credibility intervals. Also, we can notice that the estimates of p, μ, ϕ, and γ are quite similar to each other. The estimates of $\boldsymbol{\nu}$ and the respective credibility intervals, which include values > 30, suggest that the ZA-SGtBS, ZA-StBS, ZA-SSLBS, and ZA-SCNBS distributions can be equivalent to the ZA-SNBS one. Figure 24 presents the posterior densities of $\boldsymbol{\nu}$ (see Figure 16). We can notice that for the ZA-StBS, ZA-SSLBS2 and ZA-SCNBS, the densities are concentrated around large values. As discussed earlier, this behavior is compatible with that of the normal shape distributions. The estimated densities are shown in Figure 22 (b). We can notice that the proposed distributions present a similar fitting and the ZABS is the least appropriate for this data set. Figure 25 presents the observed and predicted responses (indicated by gray) under the proposed and ZABS models. We can notice that the proposed distributions present a large advantage over the ZABS distribution. Also, the ZA-SNBS distribution predicts better the observations than the other models.

Table 7 shows the criteria for model selection and the Bayesian p-values (the bold values indicate the chosen model by each statistic). Notice that three (EAIC, EBIC and LPML) of the four criteria selected the ZA-SNBS model as the most appropriate one. On the other hand, when comparing the Bayesian p-values, we can say that the proposed models present a similar fitting. A possible explanation for this would be the fact that the posterior predictive checking methods are conservative (indicating that the model is well fitted when it is not). In conclusion, we can say that the ZA-SNBS distribution is the most indicate and offers an excellent fit to the bilirubin concentration data.

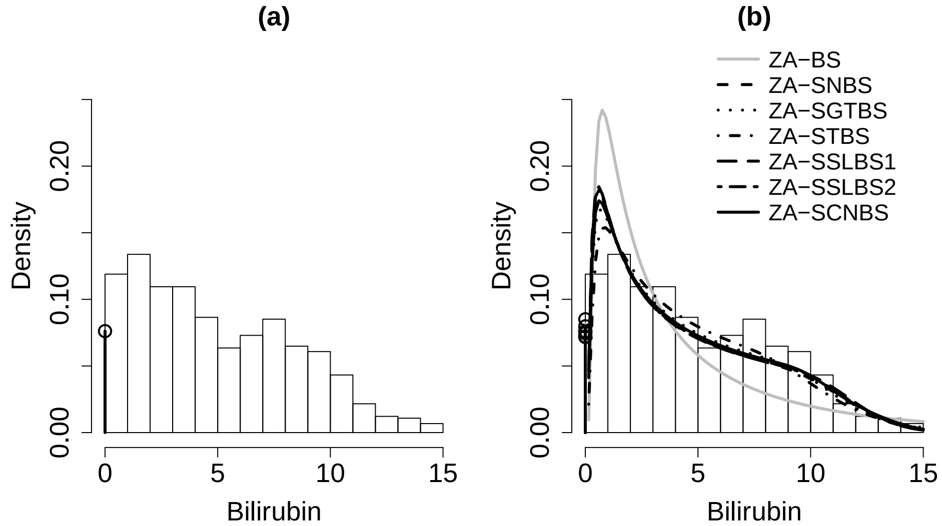

Figure 22 – (a) Histogram of the concentration of bilirubin (b) Histogram of the concentration of bilirubin and estimated densities.

Table 5 – Descriptive statistics for the concentration of bilirubin (μ mol/L).

n	Mean	Median	Minimum	Maximum	SD	CS	CK
801	4.663	4.000	.000	14.800	.606	.531	2.317

Table 7 – Model selection criteria and Bayesian p-value.

Model	EAIC	EBIC	DIC	LPML	p-value
ZA-SGtBS	4,181.145	4,204.574	12,510.620	-2,087.020	.149
ZA-StBS	4,197.010	4,220.439	12,559.340	-2,094.560	.239
ZA-SSLBS1	4,184.291	4,207.720	12,519.870	-2,088.483	.062
ZA-SSLBS2	4,184.434	4,207.863	12,520.090	-2,088.936	.046
ZA-SCNBS	4,181.665	4,209.780	12,506.730	-2,086.759	.177
ZA-SNBS	**1,022.595**	**1,035.789**	3,043.321	**-507.100**	.057
ZABS	4,586.469	4,600.526	**1,3738.110**	-2,293.024	.996

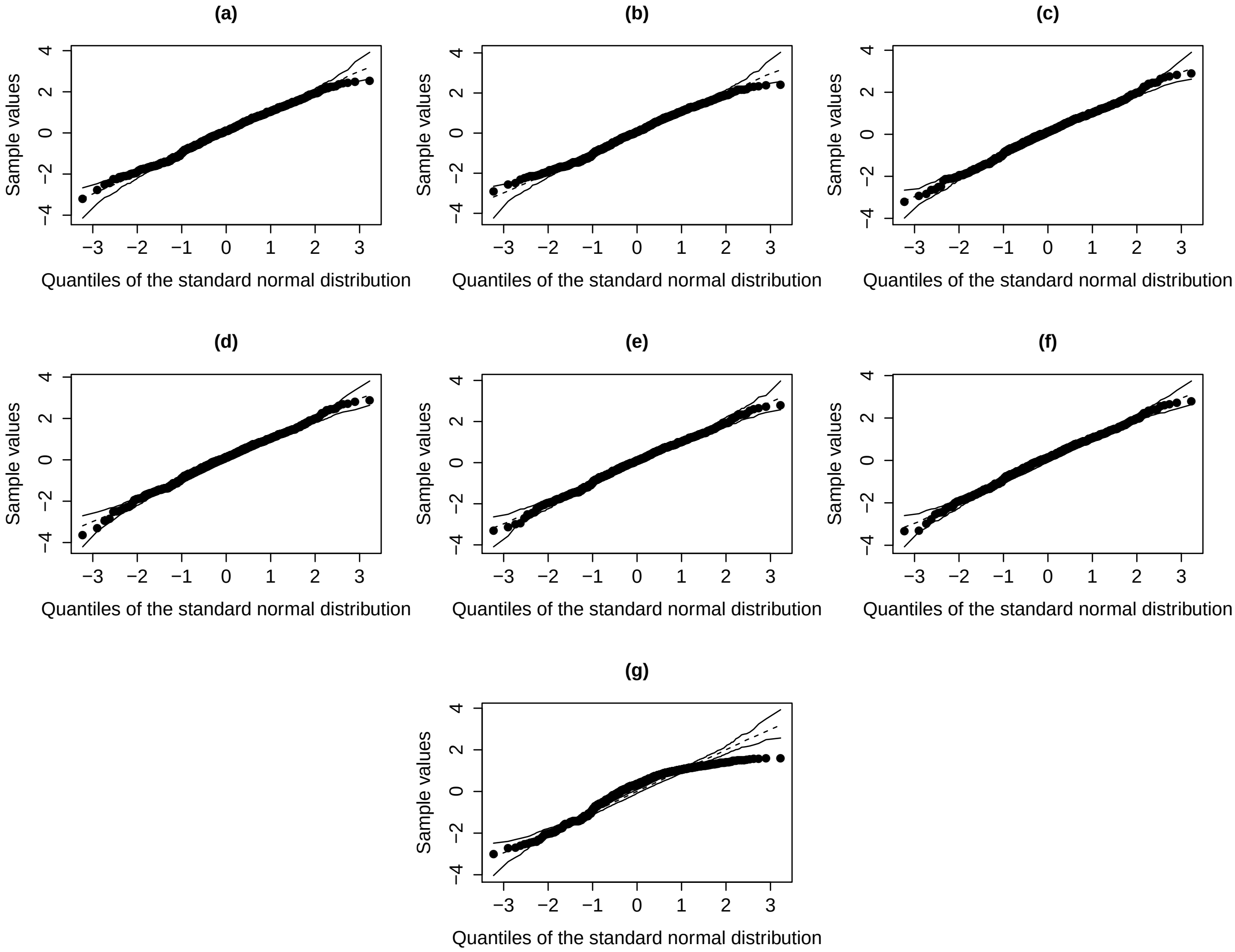

Figure 23 – QQ plot with envelopes for (a) ZA-SGtBS, (b) ZA-StBS, (c) ZA-SSLBS1, (d) ZA-SSLBS2, (e) ZA-SCNBS, (f) ZA-SNBS, and (g) ZABS for the concentration of bilirubin.

Table 6 – Posterior expectations (PE), posterior standard deviations (PSD) and equi-tailed 95% credibility intervals.

	ZA-SGtBS			ZA-SCNBS		
Parameter	PE	PSD	$CI_{95\%}$	PE	PSD	$CI_{95\%}$
p	.075	.009	[.056; .092]	.072	.008	[.063; .091]
μ	5.125	.064	[5.016; 5.236]	5.129	.120	[4.874; 5.300]
ϕ	-	-	-	.491	.049	[.411; .615]
γ	-.984	.005	[-.990; -.971]	-.976	.017	[-.993; -.938]
ν_1	28.506	7.042	[17.509; 45.901]	.562	.143	[.397; .871]
ν_2	15.857	4.381	[9.147; 26.580]	.696	.100	[.589; .974]
	ZA-StBS			ZA-SSLBS1		
Parameter	PE	PSD	$CI_{95\%}$	PE	PSD	$CI_{95\%}$
p	.072	.003	[.067; .076]	.076	.009	[.059; .096]
μ	5.134	.082	[4.981; 5.273]	5.072	.093	[4.884; 5.248]
ϕ	.490	.016	[.461; .526]	.626	.020	[.587; .662]
γ	-.940	.017	[-.950; -.873]	-.965	.011	[-.981; -.941]
ν	31.177	17.737	[14.310; 85.428]	45.229	.925	[43.560; 46.951]
	ZA-SSLBS2			ZA-SNBS		
Parameter	PE	PSD	$CI_{95\%}$	PE	PSD	$CI_{95\%}$
p	.080	.011	[.060; .104]	.085	.009	[.067; .098]
μ	5.049	.091	[4.868; 5.234]	5.025	.085	[4.806; 5.145]
ϕ	.606	.030	[.553; .675]	.631	.053	[.569; .765]
γ	-.960	.010	[-.972; -.937]	-.952	.007	[-.963; -.934]
ν	28.825	19.387	[11.063; 82.129]	-	-	-
	ZA-BS					
Parameter	PE	PSD	$CI_{95\%}$			
p	.077	.010	[.059; .099]			
μ	4.844	.202	[4.468; 5.280]			
ϕ	.689	.036	[.619; .766]			

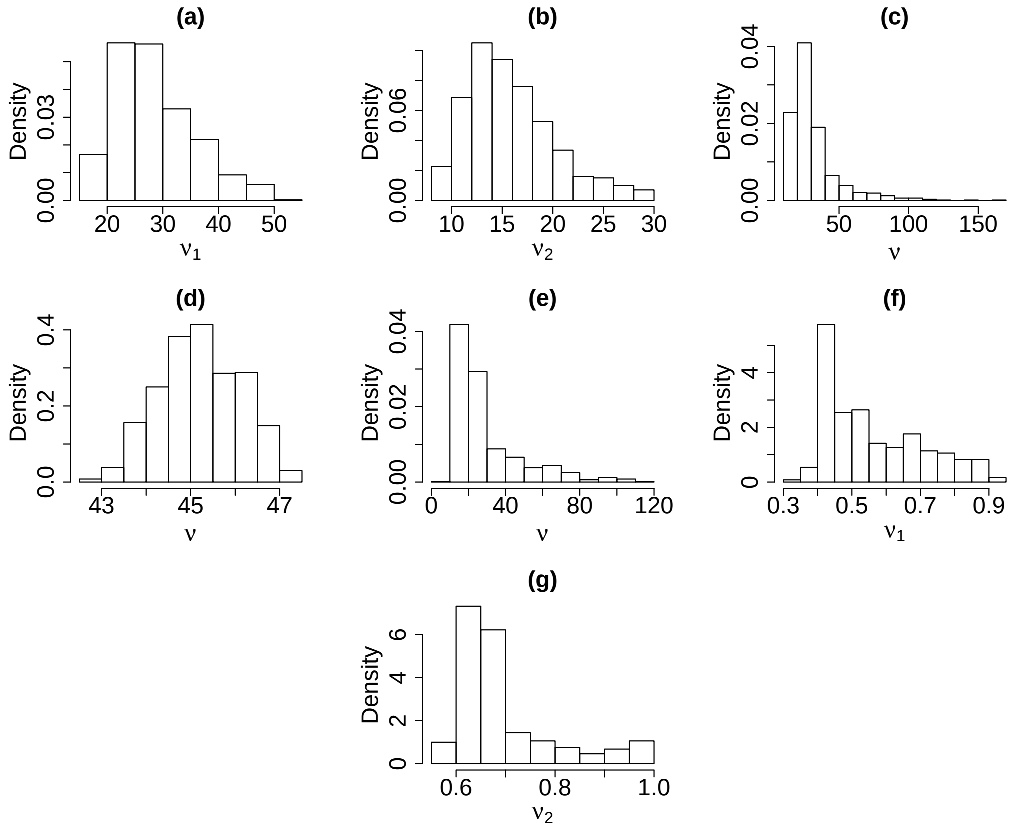

Figure 24 – (a) Posterior density of the parameter: (a) ν_1 of the ZA-SGtBS, (b) ν_2 of the ZA-SGtBS, (c) ν of the ZA-StBS, (d) ν of the ZA-SSLBS1, (e) ν of the ZA-SSLBS2, (f) ν_1 of the ZA-SCNBS and (g) ν_2 of the ZA-SCNBS distribution.

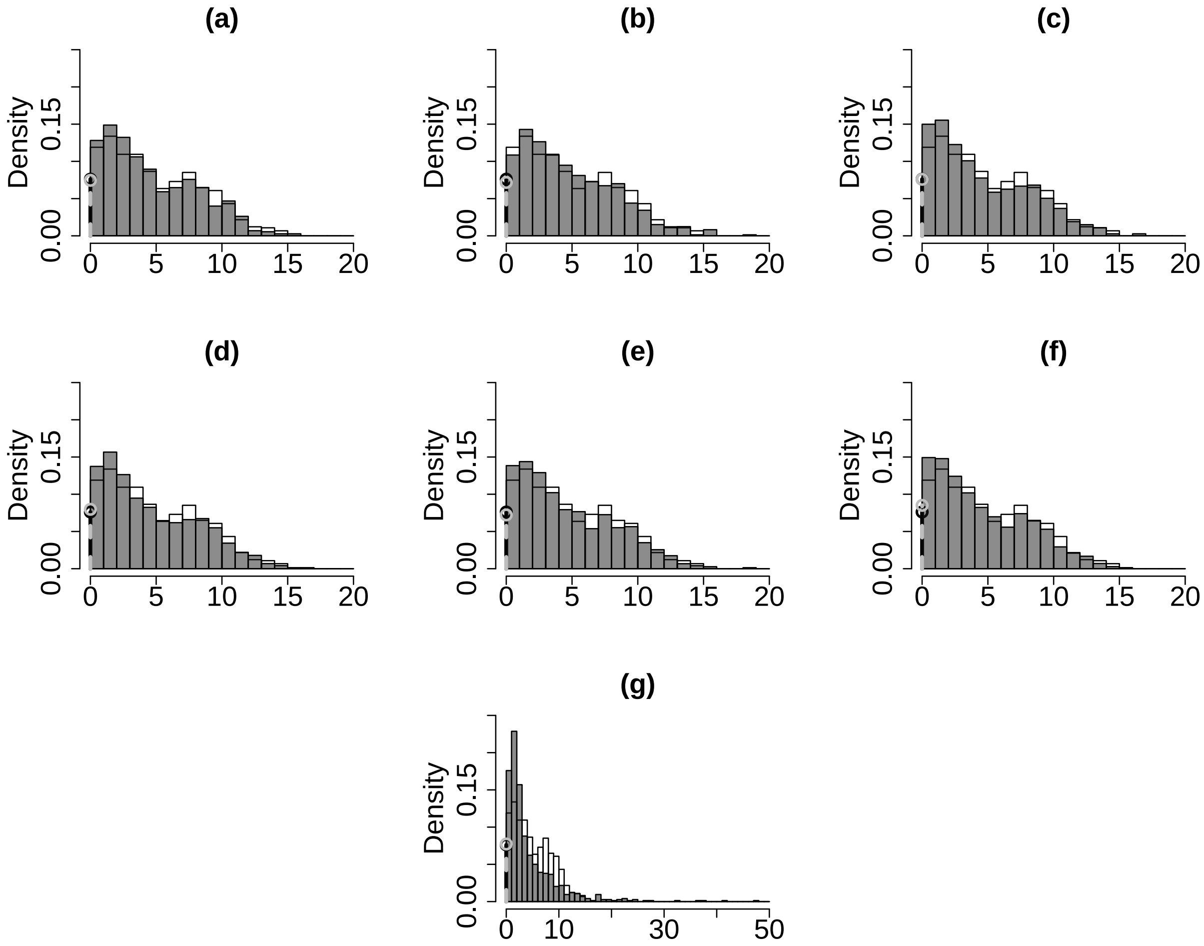

Figure 25 – Histogram of the predicted distributions for the models: (a) ZA-SGtBS, (b) ZA-StBS, (c) ZA-SSLBS1, (d) ZA-SSLBS2, (e) ZA-SCNBS, (f) ZA-SNBS, and (g) ZABS.

2.7 Concluding Remarks

In this chapter, we developed a new class of zero-augmented probability models, named zero-augmented centred skew scale-mixture Birnbaum-Saunders distributions. The proposed family accommodates properly both positively or negatively skewed data, presenting or not heavy tails, and generalize the zero-augmented BS distribution (Leiva et al., 2016). Under Bayesian approach, we developed parameter estimation, statistics for model comparison, and posterior predictive checking based on MCMC algorithms. We conducted parameter recovery studies considering different scenarios of practical interest. In general, the results indicated, for all models, that p, μ, ϕ and γ were well recovered in all scenarios. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed distributions, the estimates of $\boldsymbol{\nu}$ were close to true values. On the other hand, in scenarios where the proposed distributions were equivalent to the ZA-SNBS distribution, the estimates of $\boldsymbol{\nu}$ are biased and the width of the credibility interval are large. However, as the sample size increases, the estimates of $\boldsymbol{\nu}$ get more accurate. Finally, we have presented applications to a real data set related to the bilirubin concentration, showing that the our approach can be much more useful than the traditional ones. The results indicated that the ZA-SNBS distribution offers an excellent fit to the bilirubin concentration data and that the ZABS model (Leiva et al., 2016) is the least appropriate for this data set.

3 Generalized Birnbaum-Saunders regression models

3.1 Introduction

Regression models based on the BS and the correspondent log-BS distributions have been widely studied and applied in the few last years. Rieck and Nedelman (1991) were pioneers in this line. They defined that if $Y \sim BS(\alpha, \eta)$, then $V = \log(Y)$ follows a log-BS distribution with shape and location parameters α and $\rho = \log(\eta) \in \mathbb{R}$, respectively, denoted by $V \sim$ log-BS(α, η). They proposed log-linear regression models based on the log-BS distribution to model fatigue data, whereas Galea et al. (2004) and Xie and Wei (2007) developed several diagnostic tools for this model. Leiva et al. (2007) formulated BS log-linear regression models and their diagnostics, and applied them to the survival data of patients with blood cell cancer. Barros et al. (2008) assumed that the cumulative damage follows a Student-t distributions, then introducing BS-t log-linear regression models and related diagnostics tools. They consider an application them to survival data of patients with lung cancer. Paula et al. (2012) applied the BS-t log-linear models to insurance data. Lemonte and Cordeiro (2009) proposed the BS non-linear regression models, generalizing that proposed by Rieck and Nedelman (1991). Lemonte and Patriota (2011) and Vanegas et al. (2012) performed diagnostic procedures for these nonlinear models.

Some authors developed log-BS regression models based on the skew-elliptical distributions, in order to obtain more robust and flexible models. Santana et al. (2011) and Chaves et al. (2019a) developed the log-BS models based on usual (Azzalini, 1985) and centred (Azzalini, 2013) versions of the skew-normal distribution, respectively. Recently, Sánchez (2018) developed a family of log-BS models based on the skew scale-mixture of normal distributions (da Silva Ferreira et al., 2011).

For all of these regression models, the original response must be transformed to a logarithmic scale, which could lead to some difficulties of interpretation and inferential problems (see Huang and Qu (2006)). To overcome this problem, using the reparameterizated BS distribution (Santos-Neto et al., 2012), Leiva et al. (2014) developed an approach based on the BS regression models similar to the generalized linear models. Recently, Santos-Neto et al. (2016) extended the work of Leiva et al. (2014) and proposing a BS regression model with varying precision.

In this chapter, we developed a general family of BS regression models, named CSSBS regression models, which generalizes the regression model proposed by Santos-Neto et al. (2016). Our family inherits the properties of the CSSBS distribution. Furthermore, it allows to analyze data in their original scale, consider the modeling of both mean and the dispersion parameter, through suitable predictors using appropriate link functions. Also, the proposed models accommodate properly both positively or negatively skewed data, presenting or not heavy tails. Under the Bayesian paradigm, we developed parameter estimation, residual analysis, statistics for model comparison and posterior predictive checking, based on MCMC algorithms. We conducted simulation studies, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the model fit assessment tools and the statistics for model comparison. Finally, we consider an application to a real data set related to the lean body mass of 202 elite athletes, showing the usefulness of the inferential methods developed here. The results indicate that the SSLBS regression model outperforms others, in terms of goodness of model fit.

3.2 Centred skew scale-mixture Birnbaum-Saunders regression models

Let $T_i|\boldsymbol{\theta} \overset{\text{ind}}{\sim} \text{CSSBS}(\mu_i, \phi_i, \gamma, \boldsymbol{\nu})$, where $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, whose density is given by Equation (1.9). Suppose that the mean and dispersion parameter of T_i satisfy the following functional relations: $g_1(\mu_i) = \eta_i = f_1(\boldsymbol{x}_i; \boldsymbol{\beta})$ and $g_2(\phi_i) = \varsigma_i = f_2(\boldsymbol{w}_i; \boldsymbol{\psi})$, for $i = 1, \ldots, n$, where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$, $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_q)^\top$ are vectors of regression parameters, $p + q < n$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)^\top$ and $\boldsymbol{\varsigma} = (\varsigma_1, \ldots, \varsigma_n)^\top$ are predictors vectors, and $g_r(\cdot;\cdot), r = 1, 2$ are linear or nonlinear twice continuously differentiable functions, in the second argument. Furthermore, $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})^\top$ and $\boldsymbol{w}_i = (w_{i1}, \ldots, w_{iq})^\top$ are vectors with p and q explanatory variables, respectively. Here, the link functions $g_r : \mathbb{R}^+ \longrightarrow \mathbb{R}, r = 1, 2$ are strictly monotone, positive, and at least twice differentiable. In this work, we connect μ_i and ϕ_i to covariates through the log-linear function as follows

$$\mu_i = \exp\left\{\boldsymbol{x}_i^\top \boldsymbol{\beta}\right\} \text{ and } \phi_i = \exp\left\{\boldsymbol{w}_i^\top \boldsymbol{\psi}\right\}. \tag{3.1}$$

Eventually, for data sets in which the observations are divided into, say, k groups, we can allow γ and $\boldsymbol{\nu}$ to vary according the groups, that is, $\gamma = \gamma_j$ and $\boldsymbol{\nu} = \boldsymbol{\nu}_j$, $j = 1, \ldots, k$.

3.3 Bayesian inference

In this section, we present the Bayesian inference for SSBS models. The adopted approach, since the marginal posterior distributions on interest can not be analytically obtained, relies on the MCMC algorithms to obtain numerical approximations for those distributions.

3.3.1 Likelihoods

The likelihood for $\boldsymbol{\theta}$, given the observed sample $\mathbf{t} = (t_1, \ldots, t_n)^\top$, takes on the form

$$L(\boldsymbol{\theta}|\mathbf{t}) = \prod_{i=1}^{n} \phi_{\gamma,\boldsymbol{\nu}} \left[a_{t_i}(\mu_i, \phi_i)\right] A_{t_i}(\mu_i, \phi_i), \tag{3.2}$$

where $\phi_{\gamma,\boldsymbol{\nu}}(\cdot)$ was defined in Equation (1.9), $a_{t_i}(\mu_i, \phi_i) = \left\{\sqrt{t_i[1+\phi_i\mathbb{E}(Y^2)]/\mu_i} - \sqrt{\mu_i/t[1+\phi_i\mathbb{E}(Y^2)]}\right\}/\sqrt{2\phi_i}$ and $A_{t_i}(\mu_i, \phi_i) = \dfrac{t_i^{-3/2}\left\{t_i + \mu_i/\left[1+\phi_i\mathbb{E}(Y^2)\right]\right\}}{2\sqrt{2\phi_i}\sqrt{\mu_i/\left[1+\phi_i\mathbb{E}(Y^2)\right]}}$. Furthermore, considering the hierarchical representation presented in Equation (1.18), which is useful for stochastic simulation and parameter estimation (both frequentist and Bayesian), we have that the so-called complete likelihood is given by

$$L(\boldsymbol{\theta}|\mathbf{t}_c) \propto \prod_{i=1}^{n} \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i) \exp\left\{-h_i^2/2\right\} g(u_i|\boldsymbol{\nu}), \tag{3.3}$$

where $\mathbf{t}_c = (\mathbf{t}^\top, \mathbf{h}^\top, \mathbf{u}^\top)$, with $\mathbf{t} = (t_1, \ldots, t_n)^\top$, $\mathbf{h} = (h_1, \ldots, h_n)^\top$ and $\mathbf{u} = (u_1, \ldots, u_n)^\top$. Also, ϑ_{h_i} was defined in Equation (1.18).

3.3.2 Prior, posterior and full conditional distributions

To complete the Bayesian specification we need to consider convenient prior distribution. Assuming that the parameters are independent a priori, we consider that the respective joint prior distribution is given by

$$\pi(\boldsymbol{\theta}) = \pi(\boldsymbol{\beta})\pi(\boldsymbol{\psi})\pi(\gamma)\pi(\boldsymbol{\nu}), \tag{3.4}$$

where $\boldsymbol{\beta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\beta)$, $\boldsymbol{\psi} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\psi)$, and $\gamma \sim U(c, d)$. The prior distribution for $\boldsymbol{\nu}$ depends on the particular distribution adopted for the CSSBS regression model (more details are provided ahead). Combining the likelihood presented in Equation (3.3) and the prior distribution presented in Equation (3.4), the joint posterior distribution is given by

$$\pi(\boldsymbol{\theta}, \mathbf{h}, \mathbf{u}|\mathbf{t}) \propto \left\{\prod_{i=1}^{n} \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i) \exp\left\{-h_i^2/2\right\} g(u_i|\boldsymbol{\nu})\right\} \pi(\boldsymbol{\theta}).$$

and the full conditional distributions, are given by

$$\begin{aligned}
\pi(h_i|\boldsymbol{\theta}, t_i, u_i) &\propto \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] \exp\left\{-h_i^2/2\right\}, \\
\pi(u_i|\boldsymbol{\theta}, t_i, h_i) &\propto \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i) \left[g(u_i|\boldsymbol{\nu})\right],
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
\pi(\boldsymbol{\beta}|\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\boldsymbol{\beta})\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\right\},\\
\pi(\boldsymbol{\psi}|\boldsymbol{\beta},\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\boldsymbol{\psi})\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\right\},\\
\pi(\gamma|\boldsymbol{\beta},\boldsymbol{\psi},\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\gamma)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\right\},\\
\pi(\boldsymbol{\nu}|\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\mathbf{t}_c) &\propto \pi(\boldsymbol{\nu})\left\{\prod_{i=1}^{n}g(u_i|\boldsymbol{\nu})\right\}.
\end{aligned} \tag{3.6}
$$

The shape of distributions presented in Equations (3.5) and (3.6) depend on the particular distribution adopted for CSSBS regression model and the adopted prior for $\boldsymbol{\nu}$. The full conditional distributions of u_i and $\boldsymbol{\nu}$, and $\pi(\boldsymbol{\nu})$ for each CSSBS model are presented bellow. We made all implementations considering the `OpenBUGS` software (Spiegelhalter et al., 2014), through the `R` program (R Core Team, 2014), using the package `R2OpenBUGS` (Sturtz et al., 2010). That is, the auxiliary algorithms used to simulate from the full conditional distributions follow the predefined hierarchy implemented in `OpenBUGS`.

3.3.3 Prior distribution of $\boldsymbol{\nu}$ and full conditional distributions

1. *The centred skew generalized Student-t Birnbaum-Saunders (SGtBS) regression model.* Here, $\boldsymbol{\nu}=(\nu_1,\nu_2)^\top$. We adopt a truncated exponential prior for ν_s, $s=1,2$, that is $\nu_s\sim\exp(\Lambda)\mathbb{1}_{\mathbb{A}}(\nu_s)$, such that $\Lambda\sim\mathrm{U}(\Lambda_0,\Lambda_1)$ (Cabral et al., 2012), which leads to the density

$$
\pi(\nu_s)=\frac{1}{\nu_s^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_s+1)\exp\{-\Lambda_0\nu_s\}-(\Lambda_1\nu_s+1)\exp\{-\Lambda_1\nu_s\}\right].
$$

The full conditional distributions of u_i and ν_s take the form

$$
\begin{aligned}
\pi(u_i|\boldsymbol{\theta},t_i,h_i) &\propto \phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\left[u_i^{\nu_s/2-1}\right]\exp\left\{-\frac{\nu_s}{2}u_i\right\},\\
\pi(\nu_s|\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\mathbf{t}_c) &\propto \frac{1}{\nu_s^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_s+1)\exp\{-\Lambda_0\nu_s\}-(\Lambda_1\nu_s+1)\exp\{-\Lambda_1\nu_s\}\right]\\
&\quad\times\left\{\prod_{i=1}^{n}(\nu_s/2)^{\nu_s/2}\left[\Gamma(\nu_s/2)\right]^{-1}u_i^{\nu_s/2-1}\exp\left\{-\frac{\nu_s}{2}u_i\right\}\right\}.
\end{aligned}
$$

2. *The centred skew slash Birnbaum-Saunders (SSLBS) regression model.* We adopt a truncated gamma distribution for $\boldsymbol{\nu}=\nu$, that is, $\nu\sim\mathrm{gamma}(a,b)\mathbb{1}_{\mathbb{A}}(\nu)$, with small positive values of a and b ($b<<a$), see Vilca et al. (2016). The full conditional distributions of u_i and ν become

$$
\begin{aligned}
\pi(u_i|\boldsymbol{\theta},t_i,h_i) &\propto \phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)u_i^{\nu-1}\\
\pi(\nu|\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\mathbf{t}_c) &\propto \nu^{a-1}\exp\{-b\nu\}\prod_{i=1}^{n}\nu\,u_i^{\nu-1}.
\end{aligned}
$$

3. *The centred skew contaminated normal Birnbaum-Saunders (SCNBS) regression model.*
   The possible states of the "weights" u_i are ν_2 or 1, with $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$, and its density can be expressed as

$$g(u_i|\boldsymbol{\nu}) = \nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}.$$

   The full conditional distribution of each u_i can be written as:

$$\pi(u_i|\boldsymbol{\theta}, t_i, h_i) \propto \phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i)\nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}.$$

   Thus, the distribution is proportional to

$$\begin{cases} \nu_1\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i), \text{if } u_i = \nu_2 \\ (1-\nu_1)\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i), \text{if } u_i = 1 \end{cases}$$

   In this case, we consider $\nu_s \sim \text{beta}(a_s, b_s)$ (Lachos et al., 2017). The full conditional distribution of ν_s, $s = 1, 2$, is given by

$$\pi(\nu_s|\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c) \propto \nu_s^{a_s + a_{n,\nu_2} - 1}(1-\nu_s)^{b_s + b_{n,\nu_2} - 1},$$

   where $a_{n,\nu_2} = \left(n - \sum_{i=1}^{n} u_i\right)/\left(1-\nu_2\right)$ and $b_{n,\nu_2} = \left(\sum_{i=1}^{n} u_i - n\,\nu_2\right)/\left(1-\nu_2\right)$, which is proportional to the $\text{beta}(a_s + a_{n,\nu_2}, b_s + b_{n,\nu_2})$ distribution.

3.4 Model fit assessment and model comparison

3.4.1 Residual analysis

The residual analysis is an important tool for model fit assessment. It is possible, through the residual analysis, checking the presence of outliers, as well as the departing from (specific) model assumptions. Following the methodology proposed by Dunn and Smyth (1996), we consider the quantile residual, since its reference distribution, in our case, is known, which facilitates the detection of the model misfit. On the other hand, once we expect that the Bayesian estimates are consistent (in the frequentist sense) the residual can be viewed in a similar way when the maximum likelihood approach is employed. However, it is also possible to study the posterior distribution of the residual of each observation, in order to identify possible outliers and person level model misfit (Fox, 2005).

Let $T_i|\boldsymbol{\theta} \overset{\text{ind}}{\sim} \text{CSSBS}(\mu_i, \phi_i, \gamma, \boldsymbol{\nu})$, where $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, be a r.v with a conditional cdf, $F_{T_i|\boldsymbol{\theta}}(t_i)$, presented in Equation (1.10). Therefore we can define the quantile residual as

$$R_i^q = \Phi^{-1}\Big[F_{T_i|\boldsymbol{\theta}}(t_i)\Big] = \Phi^{-1}\left\{\Phi_{\widehat{\gamma},\widehat{\boldsymbol{\nu}}}[a_{t_i}(\widehat{\mu}_i, \widehat{\phi}_i)]\right\}, \tag{3.7}$$

where $\Phi_{\widehat{\gamma},\widehat{\boldsymbol{\nu}}}(\cdot)$ and $a_{t_i}(\mu_i, \phi_i)$ are given in Equations (1.9) and (3.2), respectively and $\widehat{\mu}_i = \exp\left\{\boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}\right\}$ and $\widehat{\phi}_i = \exp\left\{\boldsymbol{w}_i^\top \widehat{\boldsymbol{\psi}}\right\}$. Furthermore, $\widehat{(\cdot)}$ is the respective Bayesian estimator (i.e., posterior expectation, posterior median or posterior mode). Therefore, with $\widehat{\boldsymbol{\beta}}$, $\widehat{\boldsymbol{\psi}}$, $\widehat{\gamma}$, and $\widehat{\boldsymbol{\nu}}$ being consistent estimators (in the frequentist sense) of $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, γ, and $\boldsymbol{\nu}$, respectively, we have that R_i^q converges in distribution to the standard normal distribution. The methodology used for the construction of simulated envelopes follows the usual one. That is, since the observed residuals are expected to follow a standard normal distribution, under the well fit of the model, the envelopes are simulated from a standard normal distribution, as described in Atkinson (1985) (see also Vilca et al. (2016)).

3.4.2 Statistics for model comparison

When MCMC algorithms are used to obtain the posterior distributions, some statistics for model comparison can be easily calculated, see Spiegelhalter et al. (2002). To introduce these statistics we first define $\mathrm{D}(\boldsymbol{\theta}) = -2\log\left[L(\boldsymbol{\theta}|\mathbf{t})\right]$, where $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$ and $L(\boldsymbol{\theta}|\mathbf{t})$ is the (incomplete) likelihood presented in Equation (3.2). Also, let $\boldsymbol{\theta}^{(m)}$, $m = 1, ..., M$, be the mth value of the valid simulated MCMC sample, that is, the MCMC sample obtained after discarding the burn-in and a proper thinning (lag) between the values. Finally, let $\overline{\boldsymbol{\theta}}$ be the vector with the posterior expectation of all parameters, based on the valid MCMC sample, and $\overline{\mathrm{D}(\boldsymbol{\theta})} = \frac{1}{M}\sum_{m=1}^{M} D(\boldsymbol{\theta}^{(m)})$. Denote also the deviance by $\mathrm{D}(\overline{\boldsymbol{\theta}}) = -2\ell(\overline{\boldsymbol{\theta}}|\mathbf{t})$, and the deviance information criterion (DIC) by $\mathrm{DIC} = \mathrm{D}(\overline{\boldsymbol{\theta}}) + 2p_D$, where $p_D = \overline{\mathrm{D}(\boldsymbol{\theta})} - \mathrm{D}(\overline{\boldsymbol{\theta}})$. The EAIC (posterior expectation of AIC) and EBIC (posterior expectation of BIC) are given, respectively, by $\mathrm{EAIC} = D(\bar{\boldsymbol{\theta}}) + 2k$ and $\mathrm{EBIC} = \mathrm{D}(\bar{\boldsymbol{\theta}}) + k\log(n)$, where k is the total number of parameters of the model and n is the number of observations. The LPML (logarithm of the pseudo-marginal likelihood) is calculated as $\mathrm{LPML} = \sum_{i=1}^{n}\ln(\widehat{\mathrm{CPO}}_i)$, where $\widehat{\mathrm{CPO}}_i = \left\{\frac{1}{M}\sum_{m=1}^{M}\left[1/L\left(\boldsymbol{\theta}^{(m)}|t_i\right)\right]\right\}^{-1}$. The smaller the values of DIC, EAIC, EBIC and deviance, the better the model fit, occurring the opposite with the LPML.

3.4.3 Posterior predictive checking

Under a Bayesian perspective, one way to check the goodness of the model fit, is to compare the predictive distribution with the observed distribution of the data. Let $\boldsymbol{t}^{\mathrm{obs}}$ be the observed response and $\boldsymbol{t}^{\mathrm{rep}}$ the replicated response generated from its posterior predictive distribution, which is given by

$$p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{t}^{\mathrm{obs}}) = \int p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\mathrm{obs}}) d\boldsymbol{\theta}, \tag{3.8}$$

where $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$. A suitable discrepancy measure $D(\boldsymbol{t}, \boldsymbol{\theta})$ are defined by Gelman et al. (1996) and the posterior distribution of $D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\theta})$ is compared to the posterior predictive distribution of $D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\theta})$, and substantial differences between them indicating model misfit. Gelman et al. (2013) suggest several graphs to compare the replicated and the observed data, under the given measure of divergence.

Another measure used to quantify the goodness of fit, it is the Bayesian p-value, which for an adopted discrepancy measure is defined as

$$\begin{aligned}\mathbb{P}[D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\theta})] &\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}})\\ &= \int_{D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\theta})\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\theta})} p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}}) d\boldsymbol{t}^{\text{rep}} d\boldsymbol{\theta}. \end{aligned} \quad (3.9)$$

Due to the difficulty in dealing with Equations (3.8) and (3.9) analytically, Rubin (1984) suggests simulating replicated data sets from the posterior predictive distribution. One draws M simulations $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_M$ from the posterior distribution $p(\boldsymbol{\theta}|\boldsymbol{t})$ of $\boldsymbol{\theta}$ and then draws $\boldsymbol{t}^{\text{rep},n}$ from the distribution $p(\boldsymbol{t}|\boldsymbol{\theta}^n)$ for $n = 1, \ldots, M$. The proportion of the M replications for which $D(\boldsymbol{t}^{\text{rep},n}, \boldsymbol{\theta}^n)$ exceeds $D(\boldsymbol{t}, \boldsymbol{\theta}^n)$ provides an estimate of the p-value Bayesian. Extreme values of the Bayesian p-value (less than .05 or greater than .95, depending on the nature of the discrepancy measure) indicate model misfit, see Sinharay et al. (2006). Based on Gelman et al. (1996), the measure of discrepancy here adopted is $D(\mathbf{t}|\boldsymbol{\theta}) = \sum_{i=1}^{n} \{[t_i - \mathbb{E}(T_i|\boldsymbol{\theta})]^2\} / \mathbb{V}(T_i|\boldsymbol{\theta})$, where $\mathbb{E}(T_i|\boldsymbol{\theta})$ and $\mathbb{V}(T_i|\boldsymbol{\theta})$ are given by Equation (1.12), considering $\mu_i = \exp\left\{\boldsymbol{x}_i^\top \boldsymbol{\beta}\right\}$ and $\phi_i = \exp\left\{\boldsymbol{w}_i^\top \boldsymbol{\psi}\right\}$.

3.4.4 Bayesian case influence diagnostics

Since regression models are sensitive to the underlying assumptions, it is important to perform sensitivity analysis. Here, we consider the measure of divergence within the Bayesian context proposed by Cho et al. (2009). They developed case deletion influence diagnostics for both joint and marginal posterior distributions based on the Kullback-Leibler (K-L) divergence, and presented a simple way of calculating such influence measure by using MCMC outputs. Let $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ be the K-L divergence between P and $\mathrm{P}_{(-i)}$, where P stands for the posterior distribution of $\boldsymbol{\theta}$, where $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$, for the full data and $\mathrm{P}_{(-i)}$ stands for the posterior distribution of $\boldsymbol{\theta}$ without the ith observation. Then, we have

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = \int \pi(\boldsymbol{\theta}|\mathbf{t}) \ln \left\{ \frac{\pi(\boldsymbol{\theta}|\boldsymbol{t})}{\pi(\boldsymbol{\theta}|\boldsymbol{t}_{(-i)})} \right\} d\boldsymbol{\theta}, \quad (3.10)$$

where $\mathbf{t}_{(-i)}$ corresponds to the vector $\mathbf{t}$ without the ith observation. Also, using the notation introduced in Section 3.4.2, the MCMC estimate of $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ is $\widehat{\mathrm{K}}(\mathrm{P}, \mathrm{P}_{(-i)}) = -\ln(\widehat{\mathrm{CPO}}_i) + \frac{1}{M} \times \sum_{m=1}^{M} \ln[L(\boldsymbol{\theta}^{(m)}|\mathbf{t}_i)]$, where $\widehat{\mathrm{CPO}}_i = \left\{ \frac{1}{M} \sum_{m=1}^{M} \left[1/L\left(\boldsymbol{\theta}^{(m)}|t_i\right)\right]\right\}^{-1}$. As usual, we need to establish a cut-off point, in order to determine whether an observation is influential or not. As pointed by Cho et al. (2009), the calibration of K-L divergence can be done by solving for p_i the equation

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K\left[Ber(1/2), Ber(p_i)\right] = 0.5 \log\left[4p_i(1-p_i)\right], \quad (3.11)$$

where $Ber(p_i)$ is the Bernoulli distribution with success probability p_i. From the equality $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K[Ber(1/2), Ber(p_i)]$ we have that describing outcomes using $\pi(\boldsymbol{\theta}|\mathbf{t})$ instead of $\pi(\boldsymbol{\theta}|\mathbf{t}_{(-i)})$ is compatible with describing an unobserved event as having probability p_i when the correct probability is .5 (Cho et al., 2009). Solving Equation (3.11), the calibration of the K-L divergence is

$$p_i = 0.5\left[1 + \sqrt{1 - \exp\left\{-2\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})\right\}}\right].$$

This equation implies that $.5 \leqslant p_i \leqslant 1$. For p_i much greater than .5 implies that the ith observation is influential. In this work, we considered an observation to be influential $p_i \geqslant .8$, as used by Garay et al. (2011) and Chaves et al. (2019a). So, for K-L divergence measure greater than $K\left[Ber(1/2), Ber(.8)\right] \approx .223$, the observation is considered influential (Maioli, 2018).

3.5 Simulation study

In this section, we presented five simulation studies, namely: parameter recovery (PR), behavior of the residuals (R), behavior of the K-L divergence measure (D), performance of the statistics for model comparison (SMC), and study of the posterior predictive checking (PPC) tools.

We considered different scenarios of interest, which correspond to the combination of the levels of some factors. They (with the respective levels within parenthesis) are: sample size (n) (100, 500), that is, small and large sample sizes, asymmetry parameter γ (-.8, 0, .8), that is high negative skewness, symmetry and high positive skewness, and different values of $\boldsymbol{\nu}$, which induce either a normal shape, or a heavy tails behavior.

The general structure of the model considered is

$$\begin{aligned}
\log(\mu_i) &= \beta_0 + \beta_1 x_i, i = 1, \ldots, n \\
\log(\phi_i) &= \psi_0 + \psi_1 w_i,
\end{aligned}$$

where x_i and w_i, the explanatory variables, are generated as independent draws from a continuous uniform distribution over the interval $(0, 1)$. Also, we fix $\boldsymbol{\beta} = (-.5, 1)^\top$ and $\boldsymbol{\psi} = (-1, .5)^\top$ for the StBS, SSLBS and SCNBS regression models and, to overcome the identifiability issue in the SGtBS model, we fitted two different structures: in the first model, named SGtBS1, we fixed $\nu_2 = 1$ and in the second model, named SGtBS2, we fixed $\phi = 1$. Furthermore, we considered $\nu \in \{5, 30\}$ for the StBS and SSLBS regression models, $\nu_1 \in \{5, 30\}$ for the SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ for the SGtBS2 regression model, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the SCNBS model.

The prior distributions (which were used in all studies) were: $\beta_k \sim N(0, 10^4)$, $\psi_k \sim N(0, 10^4)$, $k = 0, 1$, and $\gamma \sim$ U$(-.99527, .99527)$. The first and second priors are quite flats, and the third (Azevedo et al., 2011), is non-informative. For the SGtBS1 regression model we set $\nu_1 \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_1}$, for SGtBS2 we consider $\nu_i \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_i}(\nu_i)$; $i = 1, 2$, such that $\Lambda \sim U(.02, .5)$ (Cabral et al., 2012). In order to guarantee the existence of the first two moments we consider $\mathbb{A}_1 = (4, \infty)$ and $\mathbb{A}_2 = (2, \infty)$. Similarly, for the StBS regression model we set $\nu \sim \exp(\Lambda)\mathbb{1}_{(4,\infty)}(\nu)$, such that $\Lambda \sim U(.02, .5)$. For the SCNBS regression model we chose $\nu_1 \sim$ beta$(2, 2)$ and $\nu_2 \sim$ beta$(1, 1)$ (Lachos et al., 2017).

For the SSLBS regression model, we investigated the sensitivity to the prior choice for ν, by using different suggestions found in the literature, such as: $\nu \sim \exp(\Lambda)\mathbb{1}_{(2,\infty)}(\nu)$, where $\Lambda \sim U(.02, .5)$, and $\nu \sim$ gamma$(.001, .001)\mathbb{1}_{(2,\infty)}(\nu)$, both suggested by Cabral et al. (2012), $\nu \sim$ gamma$(1, .1)\mathbb{1}_{(2,\infty)}(\nu)$ (Vilca et al., 2016) and $\nu \sim$ gamma$(.01, .001)\mathbb{1}_{(2,\infty)}(\nu)$ (Bandyopadhyay et al., 2010). The results showed, for all distributions, estimates concentrated in the interval $(2, 7)$, independently of the respective true value. Therefore, we decided to consider two situations: when the true ν value is small, for example $\nu = 5$, we chose $\nu \sim$ gamma$(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$. When the true ν value is large, for example $\nu = 30$, we chose $\nu \sim$ gamma$(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$. We will refer to the SSLBS model as SSLBS1 and as SSLBS2, when we consider $\nu \sim$ gamma$(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$ and $\nu \sim$ gamma$(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$, respectively. In a real data analysis, we can fit the two models and consider QQ plots, the statistics for model comparison, and Bayesian p-values, to choose the most properly one.

From the results related to a convergence study (not presented for the sake of simplicity) we observed that to set a burn-in of 60,000, with a spacing of 40, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the SGtBS1, SGtBS2, SSLBS1, and SSLBS2 regression models. For the StBS and SCNBS regression models, we set a burn-in of 40,000, a total of 80,000 values were simulated, and samples were collected at a spacing of 40 iterations. In general, for all models, the Geweke statistic did not indicate lack of convergence for any parameter. Furthermore, the values for the Gelman-Rubin statistic and the inspection of the traceplots and autocorrelation plots indicated that the MCMC algorithms converged and the autocorrelations were almost negligible.

For the PR and SMC studies, R = 5 and R=10 replicas (simulated responses from the model) were considered, respectively. For the three others, one replica and only one scenario were used. All results of the simulation studies can be found in Sections C.1-C.5 of Appendix C. Some specific details concerning each study are presented in the following sections.

3.5.1 Parameter recovery

We calculated the usual statistics to measure the accuracy of the bias, standard deviation (SD), square root of the mean squared error (RMSE), absolute value of the relative bias (AVRB), coverage probability (CP) of the 95% equi-tailed credibility interval and average length (LCI) of the 95% equi-tailed credibility interval. Let θ be the parameter of interest and let $\widehat{\theta}_r$ be some estimate (posterior mean, median or mode) related to the replica r, and $\overline{\widehat{\theta}} = (1/R)\sum_{r=1}^{R}\widehat{\theta}_r$. The aforementioned statistics are: bias $= \overline{\widehat{\theta}} - \theta$; SD $= \sqrt{(1/R)\sum_{r=1}^{R}(\widehat{\theta}_r - \overline{\widehat{\theta}})^2}$, RMSE $= \sqrt{(1/R)\sum_{r=1}^{R}(\theta - \widehat{\theta}_r)^2}$ and AVRB $= |\overline{\widehat{\theta}} - \theta|/|\theta|$, CP $= (1/R)\sum_{r=1}^{R}\mathrm{I}(\theta \in [\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}])$ and LCI $= (1/R)\sum_{r=1}^{R}[\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}]$, where $\widehat{\theta}_{r,LCL}$ and $\widehat{\theta}_{r,UCL}$ are the estimated lower and upper 95% limits of the CI's, respectively. We considered $(< .001)$ to represent positive values (statistics and/or estimates) and $(> .001)$ to denote negative values, when they are close to zero. All results of the PR study can be found in Section C.1 the Appendix C.

Tables 66-71 present the results for the SGtBS1 regression model and Tables 72-77 present the results for the SGtBS2 model. For both models, as the sample size increases, we can notice that the estimates of β_0, β_1, ψ_0, ψ_1, and γ tend to the correspondent true values and the bias, RMSE and AVRB, decrease. Specifically, when $\nu_1 = 30$ in the SGtBS1 model and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ in the SGtBS2 model, although ν_1 and ν_2 are underestimated, it is clear that the estimates (see Table 69 and 75, for example) lead to an equivalence between the proposed models and the correspondent SNBS models. Therefore, we have indications that ν_1 and ν_2 are also reasonably estimated in scenarios that induce a normal shape behavior.

In Tables 78-83 and Tables 84-89, the results for the StBS and SSLBS regression models are presented, respectively. For both models, under $\nu = 5$, we can notice that the estimates, for all parameters, tend to the correspondent true values in all scenarios. Also, as the sample size increases, the width of the credibility intervals, bias, RMSE, and AVRB, decrease. Under $\nu = 30$, the estimates of β_0, β_1, ψ_0, ψ_1, and γ are close to the respective true values. Concerning ν, although the estimates tend to true value, we can notice that the width of credibility intervals are too large.

Tables 90-95 present the results for the SCNBS regression model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$, we can notice that the estimates of β_0, β_1, ψ_1, and γ tend to the correspondent true values, and the width of the credibility intervals, bias, RMSE, and AVRB decrease. However, the estimates of ψ_0 were not so good for both sample sizes. In this case, a larger sample size is required so that the estimates of ψ_0 tend to be close to the respective true value. Based only on the posterior mean, $\widehat{\nu}_1 = .558$ and $\widehat{\nu}_2 = .634$ (see Table 90), for example, it is not clear that the SCNBS model is suitable to model response variable with heavy-tails. However, when we also consider the estimates of $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, and γ, we can notice that the SCNBS model has a behavior compatible with that of the heavy-tailed model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, we can notice that the estimates of all parameters tend to the correspondent true values, in all scenarios. Also, as the sample size increases, the bias, RMSE and AVRB, decrease.

In general, we can notice that β_0, β_1, ψ_0, ψ_1, and γ are well recovered in all models. Concerning $\boldsymbol{\nu}$, specifically in the scenarios of heavy-tailed response variable, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed models are equivalent to SNBS model, the estimates of $\boldsymbol{\nu}$ are biased and the width of the respective credibility intervals are large. However, as the sample size increases, the estimates become more accurate.

3.5.2 Behavior of the residuals

We considered the scenario where $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, and $\gamma = .8$, which induce a strong positively skewed behavior on the conditional distribution of the response variable. We simulated only one set of observations from each model, considering $\nu_1 = 5$ for the SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$ for the SGtBS2, $\nu = 5$ for the StBS, $\nu = 3$ for the SSLBS, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the SCNBS regression model. For each simulated data set we fitted the proposed models. Four plots were built for each situation, including an envelope for the residuals, and they are presented in Figures 86-120 (see Section C.2 of Appendix C).

In general, when the underlying model is the SGtBS1, SGtBS2, StBS, SSLBS or SCNBS, and we fit the SSLBS2 or SNBS models, we can notice some residuals with large absolute values (close to 3), i.e., possible outliers, with some points falling outside the confidence bands. Also, we can see that the residuals present a behavior compatible with a heavy-tailed distribution. However, the SGtBS1, SGtBS2, StBS, SSLBS1 and SCNBS models accommodate well the observations present in the tails, regardless the model used to generate the data. When we fit the true model to the simulated data, we can notice that the residuals present a symmetric behavior, resembling a standard normal distribution, with the most points well within the interval (-2, 2), with no systematic behavior. In conclusion, we can say that the proposed residuals are appropriate to detect model misfit, concerning the generating distribution.

3.5.3 Behavior of the K-L divergence measure

The scenario considered here are exactly those presented in Section 3.5.2. That is, we fitted the proposed models to each one the five data sets, generated according to the SGtBS1, SGtBS2, StBS, SSLBS1 and SCNBS regression models. All results of the study D can be found in Section C.3 of Appendix C.

In general, we can notice a number of large values for the K-L divergence, when we fit the SSLBS2, SNBS models to the data sets generated from the SGtBS1, SGtBS2, StBS, SSLBS or SCNBS models, indicating that these models do not accommodate, properly, all observations. Furthermore, when the SGtBS1, SGtBS2, StBS and SSLBS1 models are used to generated the data, at least two observations are considered potentially influential by the SCNBS regression model. This does not happen when the data are simulated from the SCNBS model. This indicates that the SCNBS model does not accommodate so well the extreme observations, compared with other models.

3.5.4 Statistics for model comparison

In order to asses the performance of the statistics for model comparison, we conducted a simulation study considering five different scenarios. In the first, we simulated R=10 replicas of the StBS regression model with $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, $\gamma = .8$, and $\nu = 5$, considering two samples sizes ($n = 100$ and $n = 500$) and we fit all models. The other four scenarios are equivalent to the first, but the replicas were simulated from the SGtBS1, SGtBS2, SSLBS and SCNBS models, considering $\nu_1 = 5$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$, $\nu = 3$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, respectively. The results of the study SMC can be found in Section C.4 of Appendix C. Table 96 presents the average criteria for the five scenarios and Table 98 presents the percentage of times that the correct model was selected. The number of times (in percentage) that the SSLBS model was selected is given by the sum of times that the SSLBS1 or SSLBS2 models were chosen by the criteria.

In Table 96, we can notice that the true underlying model was selected in all almost the replicas by the criteria, when the underlying model is the StBS, SGtBS1 or SGtBS2. Also, notice that when the underlying model is the SCNBS and $n = 500$, all criteria have chosen the correct model. On the other hand, when the SSLBS model is used to generated the data, the EAIC and EBIC criteria chose the SNBS model, regardless sample size. This probably occurred since the estimates of the degrees of freedom of the competing models were not so accurate. In general, we can see that the percentage of times that the correct model is selected increases as the sample size increases (see Table 98).

3.5.5 Posterior predictive checking

The scenario considered here are exactly those presented in Section 3.5.4. That is, we fitted the proposed models to the each one the five data sets, generated according to the SGtBS1, SGtBS2, StBS, SSLBS and SCNBS regression models. The results of the study PPD can be found in Section C.5 of Appendix C.

In Table 98, we can notice that when the underlying model is the SGtBS1, SGtBS2, StBS, SSLBS1 or SCNBS, the Bayesian p-values indicate that the SSLBS2 and

SNBS are misfit, as expected. In general, we can say that the Bayesian p-values, together with the other proposed tools, help to choose the best model.

3.6 Real data analysis

The data set considered here is a subset of the data collected by the Australian Institute of Sports (AIS data set). The AIS data set is available in R software and can be accessed from the *sn* package through the command *data(ais)*. They refer to the lean body mass (LBM), height (in cm) and weight (in kg) of 202 elite athletes, being 102 men and 100 women. The objective is to predict the lean body mass based on the height and weight, and to study the difference between the LBM of men and women. Figure 26 presents the boxplots of the LBM for male and female. We can notice that the variability of the LBM is higher for males when compared to females. Figure 27 displays the scatterplot between response and independent variables. It can be seen, for both sexes, that the LBM increases as the height and weight increase. However, this tendency seems to be more pronounced for men.

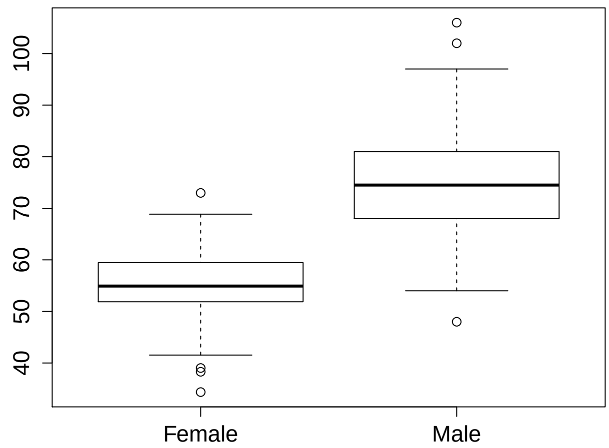

Figure 26 – Boxplots of the LBM for male and female.

We assume that the response follows a CSSBS distribution, that is $T_{ij} \overset{\text{ind}}{\sim}$ CSSBS$(\mu_{ij}, \phi_{ij}, \gamma, \nu)$. Based on the descriptive analysis, the systematic components of the regression models are expressed as

$$\log(\mu_{ij}) = \beta_{0j} + \beta_{1j}x_{1ij} + \beta_{2j}x_{2ij} \quad \text{and} \quad \log(\phi_{ij}) = \psi_{0j} + \psi_{1j}z_{1ij} + \psi_{2j}z_{2ij}, \tag{3.12}$$

where j=1 (female), 2(male), $i = 1, \ldots, n_j$, $\boldsymbol{\beta} = (\beta_{01}, \beta_{02}, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})^\top$ and $\boldsymbol{\psi} = (\psi_{01}, \psi_{02}, \psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})^\top$ are the regression coefficients, $x_{1ij} \equiv z_{1ij}$ is the height of the ith patient of sex j, and $x_{2ij} \equiv z_{2ij}$ is the weight of the ith patient of sex j. Furthermore,

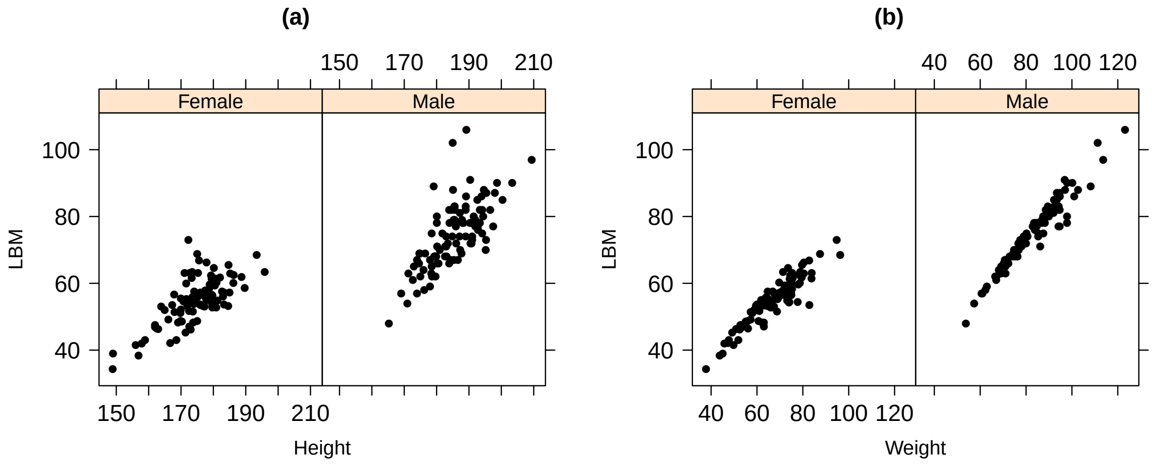

Figure 27 – Scatterplot between LBM and height (a), and between LBM and weight (b).

$e^{\beta_{0j}}$ represents the mean of the LBM for athletes of the sex j, when the height and weight are equal to their respective means. Also, $e^{\beta_{1j}}$ ($e^{\beta_{2j}}$) represents the rate of the change in the mean of the LBM for athletes of the sex j, when the height (weight) increases by one unit and the weight (height) remains constant. Moreover, $e^{\phi_{0j}}$ represents the dispersion of the LBM for athletes of the sex j, when the height and weight are equal to their respective means. Finally, $e^{\phi_{1j}}$ ($e^{\phi_{2j}}$) represents the rate of the change in the dispersion of the LBM for athletes of the sex j, when the height (weight) increases by one unit and the weight (height) remains constant. For numerical stability of the OpenBUGS program, the height and weight were standardized, that is, they were subtracted from their respective means and divided by the respective standard deviations.

We fitted all models according to Equation (3.12). Due to numerical instability of the OpenBUGS program, it was not possible to fit the SGtBS2 regression model. Figures 28-33 display the residual analysis for the other six models. For the StBS, SCNBS and SNBS models, the residuals present systematic behaviors, compatible with that of heavy-tailed and/or skewed distributions, with many points falling outside the confidence bands. On the other hand, we can notice that the SGtBS1, SSLBS1 and SSLBS2 models fit the data very well, since the residuals behave as expected. Figure 34 presents the K-L divergence measure for the six models. We can notice a considerable number of large values for the K-L divergence under the SSLBS2 and SCNBS models. From the results presented in Table 8 (where the bold values indicate the chosen model by each criteria), we can see that the SSLBS1 regression model was selected by EAIC, EBIC, and DIC. Also, from the Bayesian p-value, we can say that the SSLBS1 model presents a slight advantage over the others. In conclusion, we have evidences that the SSLBS1 is more appropriate than the other models, fitting the data quite well.

Table 9 presents Bayesian estimates, that is, the posterior expectations (PE), posterior standard deviations (PSD) and the 95% equi-tailed credibility intervals for the chosen model. The results of the other models, presented in Table 99, can be found in Section C.6 of Appendix C. In general, we can notice that the estimates of $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, and γ_j are quite similar among the models. Specifically, from Table 9, we have indications that all parameters are different from zero, since the respective credibility intervals do not contain this value. The chosen model indicates that the mean of the LBM is 59.383 [58.557; 60.280] for women and 68.033 [67.627; 68.443] for men, when the height and weight are equal to their respective means. We can notice that, for both sexes, the mean of the LBM increases as height and weight increase. Also, the impact of the height on the mean of the LBM is larger for women than for men. However, the impact of the weight on the mean of the LBM is larger for men than for women. Concerning to dispersion of the LBM, we can notice that it is equals to 9.829×10^{-4} [5.487×10^{-4}; 1.843×10^{-3}] for women and 2.503×10^{-4} [$1.512 \times 10^{-4}; 3.980 \times 10^{-4}$] for men, when the height and weight are equal to their respective means. We can notice that, for both sexes, the variability of the LBM decreases as height increases, and increases as weight increases. Also, the impact of the height and weight on the variability of the LBM is larger for women than for men. In general, the weight impacts more than the height in both the mean and the variability of the LBM.

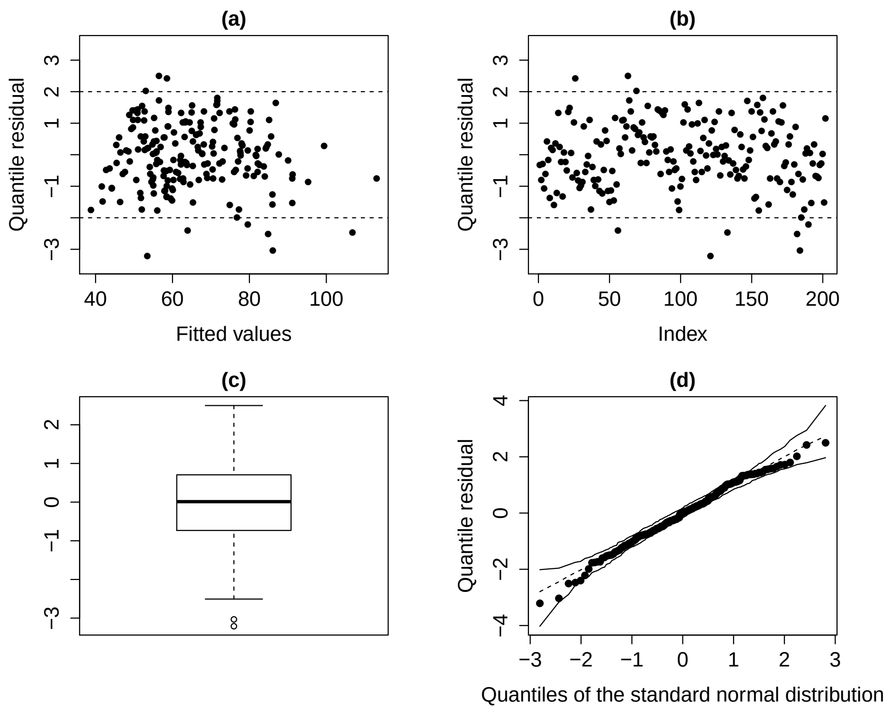

Figure 28 – Residual plots for the SGtBS1 regression model.

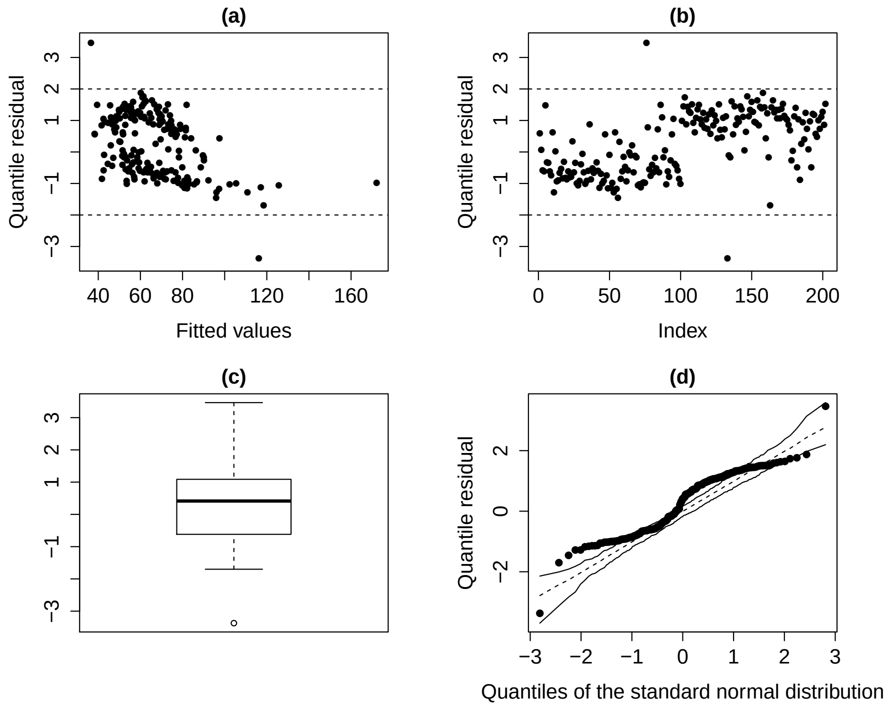

Figure 29 – Residual plots for the StBS regression model.

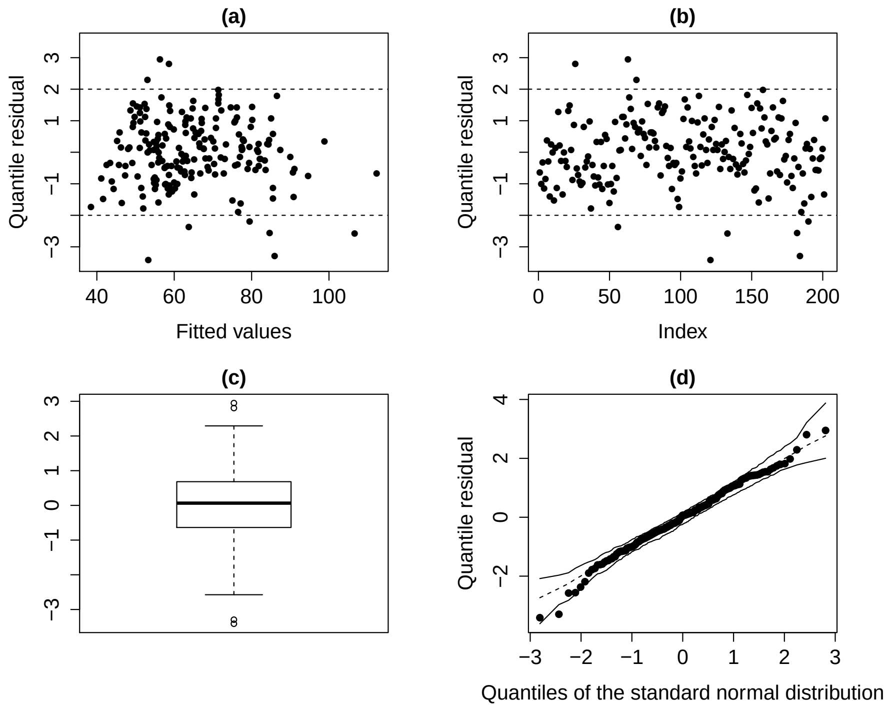

Figure 30 – Residual plots for the SSLBS1 regression model.

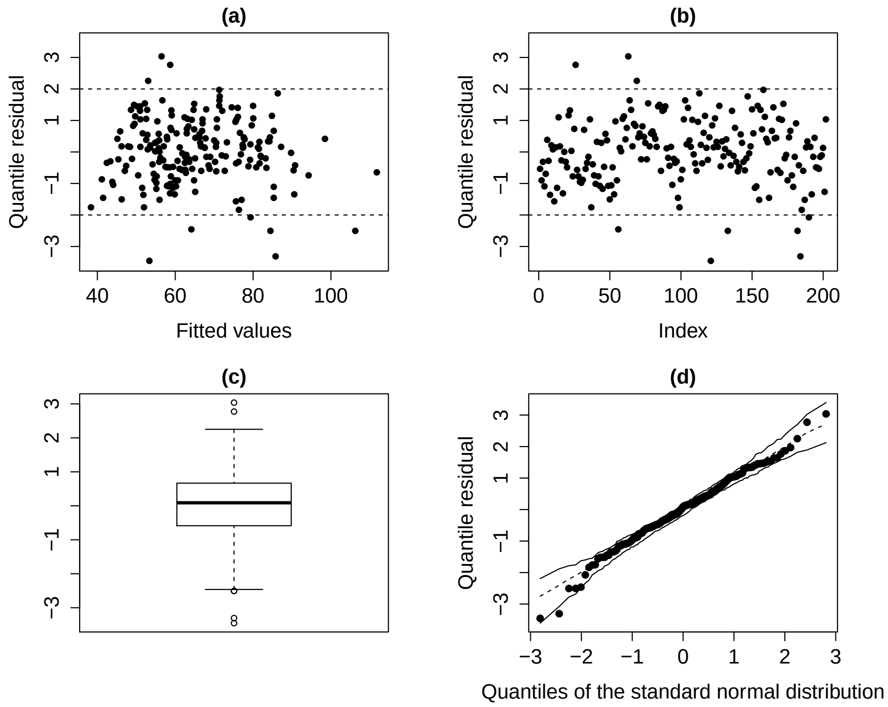

Figure 31 – Residual plots for the SSLBS2 regression model.

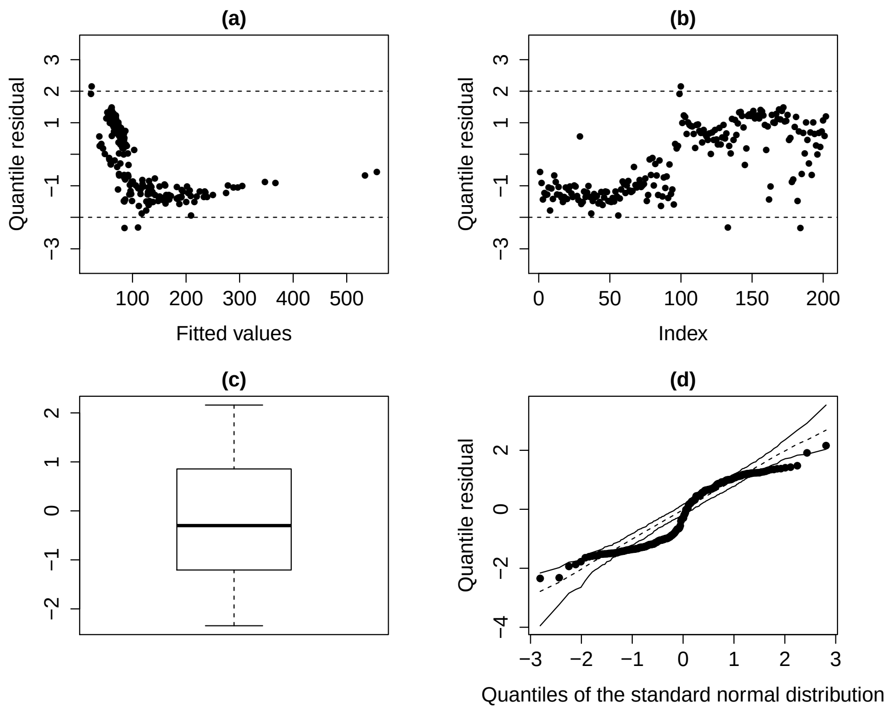

Figure 32 – Residual plots for the SCNBS regression model.

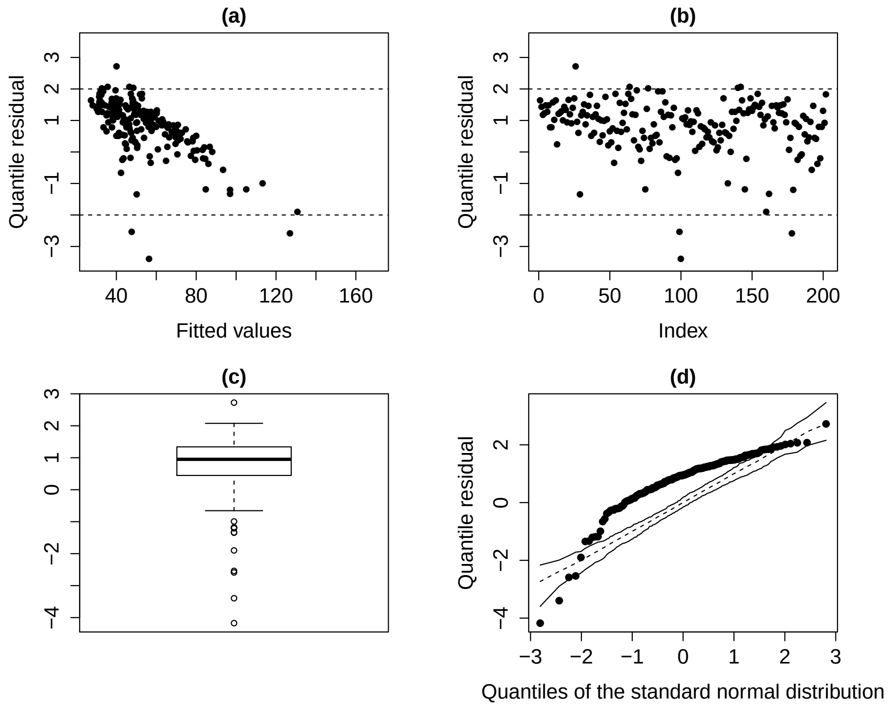

Figure 33 – Residual plots for the SNBS regression model.

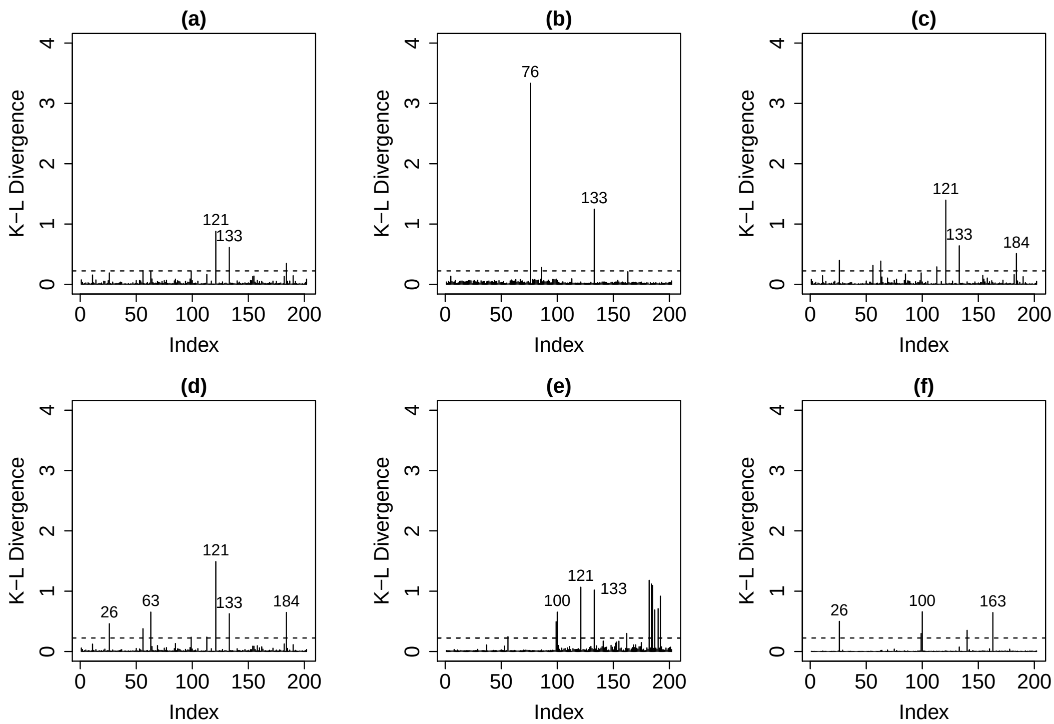

Figure 34 – K-L divergence measure for the models: (a) SGtBS1, (b) StBS, (c) SSLBS1, (d) SSLBS2, (e) SCNBS, and (f) SNBS.

Table 8 – Model selection criteria and Bayesian p-value.

Model	EAIC	EBIC	DIC	LPML	p-value
SGtBS1	937.799	984.114	2,717.331	**-462.832**	.328
StBS	1,473.950	1,520.266	4,338.210	-735.238	.393
SSLBS1	**937.368**	**983.683**	**2,716.470**	-463.457	**.544**
SSLBS2	938.875	985.191	2,720.317	-464.083	.328
SCNBS	1,525.465	1,575.089	4,497.765	-763.574	.997
SNBS	1,571.376	1,614.384	4,632.814	-776.141	$< .001$

Table 9 – Bayesian estimates for the SSLBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_{01}	4.084	.007	[4.070; 4.099]
β_{02}	4.220	.003	[4.214; 4.226]
β_{11}	.029	.009	[.009; .046]
β_{12}	.021	.004	[.013; .028]
β_{21}	.127	.011	[.108; .147]
β_{22}	.139	.005	[.130; .149]
ψ_{01}	-6.925	.303	[-7.508; -6.296]
ψ_{02}	-8.293	.259	[-8.797; -7.829]
ψ_{11}	-.568	.303	[-1.168; .040]
ψ_{12}	-.810	.247	[-1.235; -.300]
ψ_{21}	.590	.321	[-.037; 1.209]
ψ_{22}	.979	.273	[.414; 1.449]
γ	-.795	.122	[-.982; -.524]
ν	4.852	3.899	[2.060; 16.192]

3.7 Concluding Remarks

In this chapter, we developed a new family of BS regression models, named CSSBS regression models. Our family, which generalizes the regression model proposed by Santos-Neto et al. (2016), allows to analyze data in their original scale, the modeling of both mean and the dispersion parameter, through suitable predictors, using appropriate link functions. Furthermore, the proposed methodology accommodates properly both postively or negatively skewed data, presenting or not heavy tails. We developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking through Bayesian inference, based on MCMC algorithms. We conducted simulation studies, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the diagnostic measures and the statistics for model comparison. Specifically, we conducted a parameter recovery study considering different scenarios of practical interest. In general, the results indicated, for all models, that $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and γ are well recovered, in all scenarios. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed data, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed models are equivalent to SNBS model, the related estimates are biased and the width of the credibility interval are large. However, as the sample size increases, the estimates become more accurate. Furthermore, the results indicated that the proposed tools of model fit assessment and model comparison are suitable to choose the best model. Finally, we have presented and application to a real data set related to the lean body mass of 202 elite athletes, showing that the our approach can be much more useful than the traditional ones. The results indicated that the SSLBS regression model offers an excellent fit to the LBM data.

4 Generalized zero-augmented Birnbaum-Saunders regression models

4.1 Introduction

In several areas, there are many examples of zero-augmented positive data: in car insurance studies, the total claim amount reported to a given contract is often equal to zero, if no claims have been filed against the insurer, but may also be strictly positive if one or diverse accidents occurred; in microbiology, such positive data could happen from assays, virus titers, or metabolomic and proteomic data (Taylor and Pollard, 2009). Finally, in economic studies, the amount an individual or household spends on a determined category during the study period is positive (see Tu and Zhou (1999) and Xiao-Hua and Tu (1999)). In this context, Tomazella et al. (2018) developed an approach, named zero-adjusted Birnbaum-Saunders (ZABS) regression model, which considers a positive probability at zero and a continuous component based on the reparameterized BS distribution (Santos-Neto et al., 2012).

In this chapter, we developed a new family of BS regression models for zero-augmented positive data, named zero-augmented centred skew scale-mixture Birnbaum-Saunders (ZA-SSBS) regression models. Our family allows to analyze data in their original scale, modeling the mean, the dispersion parameter, and the probability mass at zero through suitable predictors using appropriate link functions. Also, the ZA-SSBS models accommodate properly both positively or negatively skewed data, presenting or not heavy tails. Our family generalizes the ZABS regression model (Tomazella et al., 2018).

Under the Bayesian paradigm, we developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking based on MCMC algorithms. We conducted simulation studies, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the diagnostic measures and the statistics for model comparison. Finally, we have presented applications to a real data set related to the bilirubin concentration, showing the usefulness of the inferential methods developed here. Also, the results indicate that our models outperforms the ZABS model in terms of model fit.

4.2 Zero-augmented centred skew scale-mixture Birnbaum-Saunders regression model

Let $T_i|\boldsymbol{\theta} \overset{\text{ind}}{\sim}$ ZA-SSBS$(p_i, \mu_i, \phi_i, \gamma, \boldsymbol{\nu})$, where $\boldsymbol{\theta} = (\boldsymbol{\zeta}^\top, \boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, whose density is given by Equation (2.2). Suppose that the mixture parameter, mean, and dispersion parameter of T_i satisfy the following functional relations: $g_1(\mu_i) = \eta_i = f_1(\boldsymbol{x}_i; \boldsymbol{\beta})$, $g_2(\phi_i) = \varsigma_i = f_2(\boldsymbol{w}_i; \boldsymbol{\psi})$ and $g_3(p_i) = \tau_i = f_3(\boldsymbol{v}_i; \boldsymbol{\zeta})$, for $i = 1, \dots, n$, where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$, $\boldsymbol{\psi} = (\psi_1, \dots, \psi_q)^\top$, and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_r)^\top$ are vectors of regression parameters, $p+q+r < n$, $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)^\top$, $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_n)^\top$, and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_r)^\top$ are predictors vectors, and $g_j(\cdot; \cdot), j = 1, 2, 3$ are linear or nonlinear twice continuously differentiable functions, in the second argument. Furthermore, $\boldsymbol{x}_i = (x_{i1}, \dots, x_{ip})^\top$, $\boldsymbol{w}_i = (w_{i1}, \dots, w_{iq})^\top$, and $\boldsymbol{v}_i = (v_{i1}, \dots, v_{ir})^\top$ are vectors with p, q and r explanatory variables, respectively. Here, the link functions $g_j : \mathbb{R}^+ \longrightarrow \mathbb{R}, j = 1, 2$ are strictly monotone, positive, and at least twice differentiable and $g_3 : (0, 1) \longrightarrow \mathbb{R}$ is strictly monotone and and twice differentiable. In this work, we connect μ_i and ϕ_i to covariates through the log-linear function as defined in Equation (3.1). Also, we connect p_i to covariates through the logit function, that is

$$p_i = \frac{\exp\{\boldsymbol{v}_i^\top \boldsymbol{\zeta}\}}{\left(1 + \exp\{\boldsymbol{v}_i^\top \boldsymbol{\zeta}\}\right)}. \tag{4.1}$$

Eventually, for data sets in which the observations are divided into, say, k groups, we can allow γ and $\boldsymbol{\nu}$ to vary according the groups, that is, $\gamma = \gamma_j$ and $\boldsymbol{\nu} = \boldsymbol{\nu}_j$, $j = 1, \dots, k$.

4.3 Bayesian inference

In this section, we present the Bayesian inference for the ZA-SSBS models. The adopted approach, since the marginal posterior distributions on interest can not be analytically obtained, relies on the MCMC algorithms to obtain numerical approximations for those distributions.

4.3.1 Likelihoods

The likelihood for $\boldsymbol{\theta}$, given the observed sample $\mathbf{t} = (t_1, \dots, t_n)^\top$, takes on the form

$$L(\boldsymbol{\theta}|\mathbf{t}) = \left[\prod_{i=1}^{n} p_i^{\mathbb{1}\{t_i=0\}} (1 - p_i)^{1-\mathbb{1}\{t_i=0\}}\right] \left[\prod_{i=1}^{n} f(t_i|\mu_i, \phi_i, \gamma, \boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}, \tag{4.2}$$

where p_i is defined in Equation (4.1) and $f(t_i|\mu_i, \phi_i, \gamma, \boldsymbol{\nu})$ is given by Equation (1.9). Furthermore, considering the hierarchical representation presented in Equation (1.18), which is useful for stochastic simulation and parameter estimation (both frequentist and

Bayesian), we have that the so-called complete likelihood is given by

$$
\begin{aligned}
L(\boldsymbol{\theta}|\mathbf{t}_c) \propto & \left[\prod_{i=1}^{n} p_i^{\mathbb{1}\{t_i=0\}}(1-p_i)^{1-\mathbb{1}\{t_i=0\}}\right]\left[\prod_{i=1}^{n} \phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right] A_{t_i,\tau_i}(\mu_i,\phi_i)\exp\left\{-h_i^2/2\right\}\right. \\
& \left.\times\; g(u_i|\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}, \qquad (4.3)
\end{aligned}
$$

where $\mathbf{t}_c = (\mathbf{t}^\top, \mathbf{h}^\top, \mathbf{u}^\top)$, with $\mathbf{t} = (t_1, \ldots, t_n)^\top$, $\mathbf{h} = (h_1, \ldots, h_n)^\top$ and $\mathbf{u} = (u_1, \ldots, u_n)^\top$. Also, ϑ_{h_i} was defined in Equation (1.18).

4.3.1.1 Prior, posterior and full conditional distributions

To complete the Bayesian specification we need to consider convenient prior distribution. Assuming that the parameters are independent a priori, we consider that the respective joint prior distribution is given by

$$
\pi(\boldsymbol{\theta}) = \pi(\boldsymbol{\zeta})\pi(\boldsymbol{\beta})\pi(\boldsymbol{\psi})\pi(\gamma)\pi(\boldsymbol{\nu}), \qquad (4.4)
$$

where $\boldsymbol{\zeta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\zeta)$, $\boldsymbol{\beta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\beta)$, $\boldsymbol{\psi} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\psi)$, and $\gamma \sim U(c, d)$. The prior distribution of $\boldsymbol{\nu}$ depends on the particular distribution adopted for ZA-SSBS regression model (more details are provided ahead). Combining the likelihood presented in Equation (4.3) and prior distribution presented in Equation (4.4), the joint posterior distribution is given by

$$
\begin{aligned}
\pi(\boldsymbol{\theta}, \mathbf{h}, \mathbf{u}|\mathbf{t}) \propto & \left\{\left[\prod_{i=1}^{n} p_i^{\mathbb{1}\{t_i=0\}}(1-p_i)^{1-\mathbb{1}\{t_i=0\}}\right]\left[\prod_{i=1}^{n} \phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right] A_{t_i,\tau_i}(\mu_i,\phi_i)\right.\right. \\
& \left.\left.\times\; \exp\left\{-h_i^2/2\right\} g(u_i|\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}\right\}\pi(\boldsymbol{\theta}).
\end{aligned}
$$

and the full conditional distributions, are given by

$$
\begin{aligned}
\pi(h_i|\boldsymbol{\theta}, t_i, u_i) &\propto \left\{\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]\exp\left\{-h_i^2/2\right\}\right\}^{1-\mathbb{1}\{t_i=0\}}, \\
\pi(u_i|\boldsymbol{\theta}, t_i, h_i) &\propto \left\{\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right] A_{t_i,\tau_i}(\mu_i,\phi_i) g(u_i|\boldsymbol{\nu})\right\}^{1-\mathbb{1}\{t_i=0\}}, \qquad (4.5) \\
\pi(\boldsymbol{\zeta}|\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}, \mathbf{t}_c) &\propto \pi(\boldsymbol{\zeta})\left\{\prod_{i=1}^{n} p_i^{\mathbb{1}\{t_i=0\}}(1-p_i)^{1-\mathbb{1}\{t_i=0\}}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\pi(\boldsymbol{\beta}|\boldsymbol{\zeta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\boldsymbol{\beta})\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\right\}^{1-\mathbb{1}\{t_i=0\}},\\
\pi(\boldsymbol{\psi}|\boldsymbol{\zeta},\boldsymbol{\beta},\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\boldsymbol{\psi})\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\right\}^{1-\mathbb{1}\{t_i=0\}},\\
\pi(\gamma|\boldsymbol{\zeta},\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\gamma)\left\{\prod_{i=1}^{n}\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\right\}^{1-\mathbb{1}\{t_i=0\}},\\
\pi(\boldsymbol{\nu}|\boldsymbol{\zeta},\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c) &\propto \pi(\boldsymbol{\nu})\left\{\prod_{i=1}^{n}g(u_i|\boldsymbol{\nu})\right\}^{1-\mathbb{1}\{t_i=0\}}. \qquad (4.6)
\end{aligned}
$$

The shape of distributions presented in Equations (4.5) and (4.6) depend on the particular distribution adopted for ZA-SSBS model and the adopted prior for $\boldsymbol{\nu}$. The full conditional distributions of u_i and $\boldsymbol{\nu}$, and $\pi(\boldsymbol{\nu})$ for each ZA-SSBS model are presented bellow. We made all implementations considering the `OpenBUGS` software (Spiegelhalter et al., 2014), through the `R` program (R Core Team, 2014), using the package `R2OpenBUGS` (Sturtz et al., 2010). That is, the auxiliary algorithms used to simulate from the full conditional distributions follow the predefined hierarchy implemented in `OpenBUGS`.

4.3.2 Prior distribution of $\boldsymbol{\nu}$ and full conditional distributions

1. *The zero-augmented centred skew generalized Student-t Birnbaum-Saunders (ZA-SGtBS) regression model.*
   Here, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$. We adopt a truncated exponential prior for ν_s, $s = 1, 2$, this is $\nu_s \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}}(\nu_s)$, such that $\Lambda \sim \mathrm{U}(\Lambda_0, \Lambda_1)$ Cabral et al. (2012), which leads to the density

$$
\pi(\nu_s) = \frac{1}{\nu_s^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_s+1)\exp\{-\Lambda_0\nu_s\}-(\Lambda_1\nu_s+1)\exp\{-\Lambda_1\nu_s\}\right].
$$

   The full conditional distributions of u_i and ν_s take the form

$$
\begin{aligned}
\pi(u_i|\boldsymbol{\theta},t_i,h_i) &\propto \left\{\phi\left[\vartheta_{h_i}+a_{t_i,\tau_i}(\mu_i,\phi_i)\right]A_{t_i,\tau_i}(\mu_i,\phi_i)\,u_i^{\nu_s/2-1}\exp\left\{-\frac{\nu_s}{2}u_i\right\}\right\}^{1-\mathbb{1}\{t_i=0\}},\\
\pi(\nu_s|\boldsymbol{\zeta},\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\mathbf{t}_c) &\propto \frac{1}{\nu_s^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_s+1)\exp\{-\Lambda_0\nu_s\}-(\Lambda_1\nu_s+1)\exp\{-\Lambda_1\nu_s\}\right]\\
&\quad\times\left\{\prod_{i=1}^{n}(\nu_s/2)^{\nu_s/2}\left[\Gamma(\nu_s/2)\right]^{-1}u_i^{\nu_s/2-1}\exp\left\{-\frac{\nu_s}{2}u_i\right\}\right\}^{1-\mathbb{1}\{t_i=0\}}.
\end{aligned}
$$

2. *The zero-augmented centred skew slash Birnbaum-Saunders (ZA-SSLBS) regression model.*
   We adopt a truncated gamma distribution for $\boldsymbol{\nu} = \nu$, that is, $\nu \sim \mathrm{gamma}(a, b)\mathbb{1}_{\mathbb{A}}(\nu)$, with small positive values of a and b ($b << a$), see Vilca et al. (2016). The full

conditional distributions of u_i and ν in (4.5) and (4.6) become

$$\begin{aligned}\pi(u_i|\boldsymbol{\theta}, t_i, h_i) &\propto \left\{\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i) u_i^{\nu-1}\right\}^{1-\mathbb{1}\{t_i=0\}} \\ \pi(\nu|\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c) &\propto \nu^{a-1} \exp\{-b\nu\} \left\{\prod_{i=1}^{n} \nu\, u_i^{\nu-1}\right\}^{1-\mathbb{1}\{t_i=0\}}.\end{aligned}$$

3. *The zero-augmented centred skew contaminated normal Birnbaum-Saunders (ZA-SCNBS) regression model.*
The possible states of the "weights" u_i are ν_2 or 1, with $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$, and its density can be expressed as

$$g(u_i|\boldsymbol{\nu}) = \nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}.$$

The full conditional distribution of each u_i can be written as:

$$\pi(u_i|\boldsymbol{\theta}, t_i, h_i) \propto \left\{\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i)\nu_1^{[(1-u_i)/(1-\nu_2)]}(1-\nu_1)^{[(u_i-\nu_2)/(1-\nu_2)]}\right\}^{1-\mathbb{1}\{t_i=0\}}.$$

Thus, the distribution is proportional to

$$\begin{cases} \left[\nu_1\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i)\right]^{1-\mathbb{1}\{t_i=0\}}, \text{if } u_i = \nu_2 \\ \left[(1-\nu_1)\phi\left[\vartheta_{h_i} + a_{t_i,\tau_i}(\mu_i, \phi_i)\right] A_{t_i,\tau_i}(\mu_i, \phi_i)\right]^{1-\mathbb{1}\{t_i=0\}}, \text{if } u_i = 1 \end{cases}$$

In this case, we consider $\nu_s \sim \text{beta}(a_s, b_s)$ Lachos et al. (2017). The full conditional distribution of $\nu_s, s = 1, 2$, is given by

$$\pi(\nu_s|\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c) \propto \nu_s^{a_s + a_{n,\nu_2}-1}(1-\nu_i)^{b_s+b_{n,\nu_2}-1},$$

where $a_{n,\nu_2} = [(1-\mathbb{1}\{t_i=0\})(n - \sum_{i=1}^n u_i)]/(1-\nu_2)$ and $b_{n,\nu_2} = [(1-\mathbb{1}\{t_i=0\})\times \times(\sum_{i=1}^{n} u_i - n\nu_2)]/(1-\nu_2)$, which is proportional to $\text{beta}(a_s + a_{n,\nu_2}, b_s + b_{n,\nu_2})$ density.

4.4 Model fit assessment and model comparison

4.4.1 Residual analysis

To assess goodness of fit and departure from the assumptions of the ZA-SSBS regression models, we adapted the randomized quantile residual (Dunn and Smyth, 1996) for our models, which is randomized version of Cox and Snell (1968) residual, and it is given by

$$R_i^q = \begin{cases} \Phi^{-1}[F_{T_i|\boldsymbol{\theta}}(t_i)], \text{if } t_i > 0, \\ \Phi^{-1}(u_i), \text{if } t_i = 0, \end{cases}$$

where $F_{T_i|\boldsymbol{\theta}}(t_i)$ was defined in Equation (1.10) and u_i is the observed value of $U_i \sim (0, \widehat{p}_i)$, where $\widehat{p}_i$ is the Bayesian estimate of p. Furthermore, $\widehat{(\cdot)}$ is the respective Bayesian estimator (i.e., posterior expectation, posterior median or posterior mode). According to Tomazella et al. (2018), if the model is correctly specified, then R_i^q is approximately normally distributed. The methodology used for the construction of simulated envelopes follows the usual one. That is, since the observed residuals are expected to follow a standard normal distribution, under the well fit of the model, the envelopes are simulated from a standard normal distributions, as described in Atkinson (1985) (see also Vilca et al. (2016)).

4.4.2 Statistics for model comparison

When MCMC algorithms are used to obtain the posterior distributions, some statistics for model comparison can be easily calculated, see Spiegelhalter et al. (2002). To introduce these statistics we first define $\mathrm{D}(\boldsymbol{\theta}) = -2\log\left[L(\boldsymbol{\theta}|\mathbf{t})\right]$, where $\boldsymbol{\theta} = (\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$ and $L(\boldsymbol{\theta}|\mathbf{t})$ is the (incomplete) likelihood presented in Equation (4.2). Also, let $\boldsymbol{\theta}^{(m)}$, $m = 1, ..., M$, be the mth value of the valid simulated MCMC sample, that is, the MCMC sample obtained after discarding the burn-in and a proper thinning (lag) between the values. Finally, let $\overline{\boldsymbol{\theta}}$ be the vector with the posterior expectation of all parameters, based on the valid MCMC sample, and $\overline{\mathrm{D}(\boldsymbol{\theta})} = \frac{1}{M}\sum_{m=1}^{M} D(\boldsymbol{\theta}^{(m)})$. Denote also the deviance by $\mathrm{D}(\overline{\boldsymbol{\theta}}) = -2\ell(\overline{\boldsymbol{\theta}}|\mathbf{t})$, and the deviance information criterion (DIC) by $\mathrm{DIC} = \mathrm{D}(\overline{\boldsymbol{\theta}}) + 2p_D$, where $p_D = \overline{\mathrm{D}(\boldsymbol{\theta})} - \mathrm{D}(\overline{\boldsymbol{\theta}})$. The EAIC (posterior expectation of AIC) and EBIC (posterior expectation of BIC) are given, respectively, by $\mathrm{EAIC} = D(\bar{\boldsymbol{\theta}}) + 2k$ and $\mathrm{EBIC} = \mathrm{D}(\bar{\boldsymbol{\theta}}) + k\log(n)$, where k is the total number of parameters of the model and n is the number of observations. he LPML (logarithm of the pseudo-marginal likelihood) is calculated as $\mathrm{LPML} = \sum_{i=1}^{n}\ln(\widehat{\mathrm{CPO}_i})$, where $\widehat{\mathrm{CPO}}_i = \left\{\frac{1}{M}\sum_{m=1}^{M}\left[1/L\left(\boldsymbol{\theta}^{(m)}|t_i\right)\right]\right\}^{-1}$. The smaller the values of DIC, EAIC, EBIC and deviance, the better the model fit, occurring the opposite with the LPML.

4.4.3 Posterior predictive checking

Under Bayesian perspective, a way to check the goodness of the model fit, is to compare the predictive distribution with the distribution of the observed data. Let $\boldsymbol{t}^{\mathrm{obs}}$ be the observed response and $\boldsymbol{t}^{\mathrm{rep}}$ the replicated response generated from its posterior predictive distribution, which is given by

$$p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{t}^{\mathrm{obs}}) = \int p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\mathrm{obs}})d\boldsymbol{\theta}. \tag{4.7}$$

where $\boldsymbol{\theta} = (\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$. Discrepancy measures $D(\boldsymbol{t}, \boldsymbol{\theta})$ are defined by Gelman et al. (1996) and the posterior distribution of $D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\theta})$ is compared to the posterior predictive distribution of $D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\theta})$, an substantial differences between them indicating model misfit. Gelman et al. (2013) suggest several graphs to compare the replicated and the observed data, under the given measure of divergence.

Another measure used to quantify the goodness of fit, is the Bayesian p-value, which for an adopted discrepancy measure, and is defined as

$$\begin{aligned} \mathbb{P}[D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\theta})] &\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}}) \\ &= \int_{D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\theta})\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\theta})} p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\boldsymbol{t}^{\text{obs}}) d\boldsymbol{t}^{\text{rep}} d\boldsymbol{\theta}. \end{aligned} \tag{4.8}$$

Due to the difficulty in dealing with Equations (4.7) and (4.8) analytically, Rubin (1984) suggests simulating replicated data sets from the posterior predictive distribution. One draws M simulations $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_M$ from the posterior distribution $p(\boldsymbol{\theta}|\boldsymbol{t})$ of $\boldsymbol{\theta}$ and then draws $\boldsymbol{t}^{\text{rep},n}$ from the distribution $p(\boldsymbol{t}|\boldsymbol{\theta}^n)$ for $n = 1, \ldots, M$. The proportion of the M replications for which $D(\boldsymbol{t}^{\text{rep},n}, \boldsymbol{\theta}^n)$ exceeds $D(\boldsymbol{t}, \boldsymbol{\theta}^n)$ provides an estimate of the p-value Bayesian. Extreme values of the Bayesian p-value (less than .05 or greater than .95, depending on the nature of the discrepancy measure) indicate model misfit, see Sinharay et al. (2006). Based on Gelman et al. (1996), the measure of discrepancy used was $D(\mathbf{t}|\boldsymbol{\theta}) = \sum_{i=1}^{n} \{[t_i - \mathbb{E}(T_i|\boldsymbol{\theta})]^2\} / \mathbb{V}(T_i|\boldsymbol{\theta})$, where $\mathbb{E}(T_i|\boldsymbol{\theta})$ and $\mathbb{V}(T_i|\boldsymbol{\theta})$ are given by Equation (2.3), considering $\mu_i = \exp\left\{\boldsymbol{x}_i^\top \boldsymbol{\beta}\right\}$ and $\phi_i = \exp\left\{\boldsymbol{w}_i^\top \boldsymbol{\psi}\right\}$.

4.4.4 Bayesian case influence diagnostics

Since regression models are sensitive to the underlying model assumptions, it is important to perform sensitivity analysis. Here, we consider the measure of divergence within the Bayesian context proposed by Cho et al. (2009). They developed case deletion influence diagnostics for both joint and marginal posterior distributions based on the Kullback-Leibler (K-L) divergence, and presented a simple way of calculating such influence measure by using MCMC outputs. Let $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ be the K-L divergence between P and $\mathrm{P}_{(-i)}$, where P stands for the posterior distribution of $\boldsymbol{\theta}$, where $\boldsymbol{\theta} = (\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$, for the full data and $\mathrm{P}_{(-i)}$ stands for the posterior distribution of $\boldsymbol{\theta}$ without the ith observation. Then, we have

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = \int \pi(\boldsymbol{\theta}|\mathbf{t}) \ln \left\{ \frac{\pi(\boldsymbol{\theta}|\boldsymbol{t})}{\pi(\boldsymbol{\theta}|\boldsymbol{t}_{(-i)})} \right\} d\boldsymbol{\theta},$$

where $\mathbf{t}_{(-i)}$ corresponds to the vector $\mathbf{t}$ without the ith observation. Also, using the notation introduced in Section 4.4.2, the MCMC estimate of $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ is $\widehat{\mathrm{K}}(\mathrm{P}, \mathrm{P}_{(-i)}) = -\ln(\widehat{\mathrm{CPO}}_i) + \frac{1}{M} \times \sum_{m=1}^{M} \ln[L(\boldsymbol{\theta}^{(m)}|\mathbf{t}_i)]$, where $\widehat{\mathrm{CPO}}_i = \left\{ \frac{1}{M} \sum_{m=1}^{M} \left[1/L\left(\boldsymbol{\theta}^{(m)}|t_i\right)\right]\right\}^{-1}$. As usual, we need to establish a cut-off point, in order to determine whether an observation is influential or not. As pointed by Cho et al. (2009), the calibration of K-L divergence can be done by solving for p_i the equation

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K\left[Ber(1/2), Ber(p_i)\right] = 0.5 \log\left[4p_i(1-p_i)\right], \tag{4.9}$$

where $Ber(p_i)$ is the Bernoulli distribution with sucess probability p_i. From the equality $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K[Ber(1/2), Ber(p_i)]$ we have that describing outcomes using $\pi(\boldsymbol{\theta}|\mathbf{t})$ instead of $\pi(\boldsymbol{\theta}|\mathbf{t}_{(-i)})$ is compatible with describing an unobserved event as having probability p_i when correct probability is .5 (Cho et al., 2009). Solving Equation (4.9), the calibration of the K-L divergence is

$$p_i = 0.5\left[1 + \sqrt{1 - \exp\left\{-2\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})\right\}}\right].$$

This equation implies that $.5 \leqslant p_i \leqslant 1$. For p_i much greater than .5 implies that the ith observation is influential. In this work, we considered an observation to be influential $p_i \geqslant .8$, as used by Garay et al. (2011) and Chaves et al. (2019a). So, for K-L divergence measure greater than $K\left[Ber(1/2), Ber(.8)\right] \approx .223$, the observation is considered influential (Maioli, 2018).

4.5 Simulation study

In this section, we present five simulation studies, namely: parameter recovery of the MCMC algorithm (PR), the behavior of the proposed residuals (R), the behavior of the K-L divergence measure (D), the performance of the statistics for model comparison (SMC), and the study of the posterior predictive checking techniques (PPC).

We considered different relevant scenarios, which correspond to the combination of the levels of some factors of interest. The factors (with the respective levels within parenthesis) are: sample size (n) (100, 500), that is, small and large sample sizes, values of the parameter γ (-.8, 0, .8), that is high negative skewness, symmetry and high positive skewness, and different values of $\boldsymbol{\nu}$, which induce either a normal shape, or a heavy tails behavior.

The general structure of the model considered is

$$\begin{aligned}
\log(\mu_i) &= \beta_0 + \beta_1 x_i, i = 1, \ldots, n \\
\log(\phi_i) &= \psi_0 + \psi_1 w_i \\
\mathrm{logit}(p_i) &= \zeta_0 + \zeta_1 v_i,
\end{aligned}$$

where x_i, w_i and v_i, the explanatory variables, are generated as independent draws from a continuous uniform distribution over the interval $(0, 1)$. Also, we fix $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, and $\boldsymbol{\zeta} = (-2.5, .8)^\top$ for the ZA-StBS, ZA-SSLBS and ZA-SCNBS regression models and, to overcome the identifiability issue in the ZA-SGtBS model, we fitted two different structures: in the first model, named ZA-SGtBS1, we fixed $\nu_2 = 1$ and in the second model, named ZA-SGtBS2, we fixed $\phi = 1$. Furthermore, we considered $\nu \in \{5, 30\}$ for the ZA-StBS and ZA-SSLBS regression models, $\nu_1 \in \{5, 30\}$ for the ZA-SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ for the ZA-SGtBS2 regression model, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the ZA-SCNBS model.

The prior distributions used in all studies were: $\beta_k \sim N(0, 10^4)$, $\psi_k \sim N(0, 10^4)$, $\zeta_k \sim N(0, 10^4)$, $k = 0, 1$, and $\gamma \sim \text{U}(-.99527, .99527)$. The first three priors are quite flats, and the fourth prior (Azevedo et al., 2011), is non-informative. For ZA-SGtBS1 regression model we set $\nu_1 \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_1}$, for ZA-SGtBS2 we consider $\nu_s \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_i}(\nu_s)$; $s = 1, 2$, such that $\Lambda \sim U(.02, .5)$ (Cabral et al., 2012). In order to guarantee the existence of the first two moments, we consider $\mathbb{A}_1 = (4, \infty)$ and $\mathbb{A}_2 = (2, \infty)$. Similarly, for the ZA-StBS regression model, we set $\nu \sim \exp(\Lambda)\mathbb{1}_{(4,\infty)}(\nu)$, such that $\Lambda \sim U(.02, .5)$. For the ZA-SCNBS regression model, we chose $\nu_1 \sim \text{beta}(2, 2)$ and $\nu_2 \sim \text{beta}(1, 1)$ (Lachos et al., 2017).

For the ZA-SSLBS regression model, we investigated the sensitivity to the prior choice for ν, by using different suggestions found in the literature, such as: $\nu \sim \exp(\Lambda)\mathbb{1}_{(2,\infty)}(\nu)$, where $\Lambda \sim U(.02, .5)$, and $\nu \sim \text{gamma}(.001, .001)\mathbb{1}_{(2,\infty)}(\nu)$, both suggested by Cabral et al. (2012), $\nu \sim \text{gamma}(1, .1)\mathbb{1}_{(2,\infty)}(\nu)$ (Vilca et al., 2016) and $\nu \sim \text{gamma}(.01, .001)\mathbb{1}_{(2,\infty)}(\nu)$ (Bandyopadhyay et al., 2010). The results showed, for all distributions, estimates concentrated in the interval $(2, 7)$, independently of the respective true value. Therefore, we decided to consider two situations: when the true ν value is small, for example $\nu = 5$, we chose $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$. When the true ν value is large, for example $\nu = 30$, we chose $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$. We will refer to the ZA-SSLBS model as ZA-SSLBS1 and as ZA-SSLBS2, when we consider $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$ and $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$, respectively. In a real data analysis, we can fit the two models and consider QQ plots, the statistics for model comparison, and Bayesian p-values, to choose the most properly one.

From the results related to a convergence study (not presented for the sake of simplicity) we observed that to set a burn-in of 60,000, with a spacing of 40, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the ZA-SGtBS1, ZA-SSLBS1, and ZA-SCNBS regression models. For the ZA-SGtBS2 model, we set a burn-in of 40,000 and a total of 80,000 values were simulated, and samples were collected at a spacing of 40 iterations. For the ZA-StBS model, we set a burn-in of 50,000 and a total of 80,000 values were simulated, and samples were collected at a spacing of 30 iterations. For the ZA-SSLBS2, model we set a burn-in of 80,000 and a total of 120,000 values were simulated, and samples were collected at a spacing of 40 iterations. Finally, for the ZA-SNBS model, we set a burn-in of 20,000 and a total of 60,000 values were simulated, and samples were collected at a spacing of 40 iterations. In general, for all models, the Geweke statistic did not indicate lack of convergence for all any parameter. Furthermore, the values for the Gelman-Rubin statistic and the inspection of the traceplots and autocorrelation plots indicated that the MCMC algorithms converged and the autocorrelations were almost negligible.

For the PR and SMC studies, R = 5 and R=10 replicas (simulated responses from the model) were considered, respectively. For the three others, only one replica and only one scenario were used. All the results of the simulation studies can be found in Sections D.1-D.5 of Appendix D. Some specific details concerning each study are presented in the following sections.

4.5.1 Parameter recovery

We calculated the usual statistics to measure the accuracy of the estimates, that is: bias, standard deviation (SD), square root of the mean squared error (RMSE), absolute value of the relative bias (AVRB), coverage probability (CP) of the 95% equi-tailed credibility interval and average length (LCI) of the 95% equi-tailed credibility interval. Let θ be the parameter of interest and let $\widehat{\theta}_r$ be some estimate (posterior mean, median or mode) related to the replica r, and $\overline{\widehat{\theta}} = (1/R)\sum_{r=1}^{R}\widehat{\theta}_r$. The aforementioned statistics are: bias $= \overline{\widehat{\theta}} - \theta$; SD $= \sqrt{(1/R)\sum_{r=1}^{R}(\widehat{\theta}_r - \overline{\widehat{\theta}})^2}$, RMSE $= \sqrt{(1/R)\sum_{r=1}^{R}(\theta - \widehat{\theta}_r)^2}$ and AVRB $= |\overline{\widehat{\theta}} - \theta|/|\theta|$, CP $= (1/R)\sum_{r=1}^{R}\mathrm{I}(\theta \in [\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}])$ and LCI $= (1/R)\sum_{r=1}^{R}[\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}]$, where $\widehat{\theta}_{r,LCL}$ and $\widehat{\theta}_{r,UCL}$ are the estimated lower and upper 95% limits of the CI's, respectively. We considered ($< .001$) to represent positive values (statistics and/or estimates) and ($> .001$) to denote negative values, when they are close to zero. All results of the PR study can be found in Section D.1 of Appendix D.

Tables 100-105 present the results for the ZA-SGtBS1 regression model and Tables 106-111 present the results for the ZA-SGtBS2 model. For both models, as the sample size increases, we can notice that ζ_0, ζ_1, β_0, β_1, ψ_0, ψ_1, and γ tend to the correspondent true values and bias, RMSE and AVRB decrease. Specifically, when $\nu_1 = 30$ in the ZA-SGtBS1 model and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ in the ZA-SGtBS2 model, although ν_1 and ν_2 are underestimated, it is clear that their estimates lead to an equivalence between the proposed models and the correspondent ZA-SNBS models. Therefore, we have indications that ν_1 and ν_2 are reasonably estimated in scenarios that induce a normal shape behavior.

In Tables 112-117 and Tables 118-123, the results of the ZA-StBS and ZA-SSLBS regression models are presented, respectively. For both models, under $\nu = 5$, we can notice that the estimates, for all parameters tend to the correspondent true values in all scenarios. Also, as the sample size increases, the width of the credibility intervals, bias, RMSE, and AVRB decrease. Under $\nu = 30$, the estimates for ζ_0, ζ_1, β_0, β_1, ψ_0, ψ_1, and γ are close to the correspondent true values. Concerning ν, although the estimates tend to the true value, we can notice that the width of credibility intervals are too large.

Tables 124-129 present the results for the ZA-SCNBS regression model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$, we can notice that the estimates of ζ_0, ζ_1, β_0, β_1, ψ_1, and γ tend to the correspondent true values, and the width of the credibility intervals, bias, RMSE, and AVRB decrease. However, the estimates of ψ_0 were not so good for both sample sizes. In this case, larger sample size is required so that the estimates of ψ_0 to be close to the respective true value. Based only on the posterior mean, $\widehat{\nu}_1 = .687$ and $\widehat{\nu}_2 = .463$ (see Table 124), for example, it is not clear that the ZA-SCNBS model is suitable to model heavy-tailed data sets. However, when we also consider the estimates of $\boldsymbol{\zeta}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, and γ, we can notice that the ZA-SCNBS distribution has a behavior compatible with that of the heavy-tailed model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, we can notice that the estimates obtained for all parameters tend to the correspondent true values in all scenarios. Also, as the sample size increases, the bias, RMSE and AVRB decrease.

In general, we can notice that ζ_0, ζ_1, β_0, β_1, ψ_1, and γ are well recovered in all models. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed models, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed models are equivalent to SNBS model, the $\boldsymbol{\nu}$ estimates are biased and the width of the credibility interval are large. However, as sample size increases, the estimates become more accurate.

4.5.2 Behavior of the residuals

We considered the scenario where $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, $\boldsymbol{\zeta} = (-2.5, .8)^\top$, and $\gamma = .8$, which induce a strong positively skewed behavior on the conditional distribution of the response variable. We simulated only one set of observations from each model, considering for the ZA-SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$ for the ZA-SGtBS2, $\nu = 5$ for the ZA-StBS, $\nu = 3$ for the ZA-SSLBS, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the ZA-SCNBS regression model. For each simulated data set we fitted the proposed models. Four plots were built for each situation, including an envelope for the residuals, and they are presented in Figures 126-160 (see Section D.2 of Appendix D).

In general, when the underlying model is the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS or ZA-SCNBS, and we fit the ZA-SNBS model, we can notice some residuals with large absolute values (close to 3), i.e., possible outliers, with some points falling outside the confidence bands. Also, we can see that the residuals present a behavior compatible with a heavy-tailed distribution. However, the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZASSLBS1 and ZA-SCNBS models accommodate well the observations present in the tails, regardless the model used to generate the data. When we fit the true model to the simulated data, we can notice that the residuals present a symmetric behavior, resembling a standard normal distribution, with the most points well within the interval (-2, 2), with no systematic behavior. In conclusion, we can say that the proposed residuals are appropriate to detect model misfit, concerning the generating distribution.

4.5.3 Behavior of the K-L divergence

The scenario considered here are exactly those presented in Section 4.5.2. That is, we fitted the proposed models to the each one the five data sets, generated according to the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS1 and ZA-SCNBS regression models. All results of the study D can be found in Section D.3 of Appendix D.

In general, we can notice a number of large values for the K-L divergence, when we fit the ZA-SSLBS2, ZA-SNBS models to the data sets generated from the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS or ZA-SCNBS models, indicating that these models do not accommodate, properly, all observations. Furthermore, when the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS and ZA-SSLBS1 models are used to generated the data, at least two observations are considered potentially influential by the ZA-SCNBS regression model. This does not happen when the data are simulated from the ZA-SCNBS model. This indicates that the ZA-SCNBS model does not accommodate so well the extreme observations, compared with other models.

4.5.4 Statistics for model comparison

In order to asses the performance of the statistics for model comparison, we conducted a simulation study considering five different scenarios. In the fist, we simulated R=10 replicas of the ZA-StBS regression model with $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, $\boldsymbol{\zeta} = (-2.5, .8)$, $\gamma = .8$, and $\nu = 5$, considering $n = 200$ and we fit all models, the ZA-StBS, ZA-SGtBS1, ZA-SGtBS2, ZA-SSLBS1, ZA-SSLBS2, ZA-SCNBS and ZA-SNBS. The other four scenarios are equivalent to the first, but the replicas were simulated from the ZA-SGtBS1, ZA-SGtBS2, ZA-SSLBS and ZA-SCNBS, considering $\nu_1 = 5$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$, $\nu = 3$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, respectively. The results of the study SMC can be found in Section D.4 of Appendix D. Table 130 presents the average criteria for the five scenarios and Table 131 presents the percentage of times that the correct model was selected. The number of times (in percentage) that the ZA-SSLBS model was selected is given by the sum of times that the ZA-SSLBS1 or ZA-SSLBS2 models were chosen by the criteria.

In Table 130, we can notice that the true underlying model was selected in all almost the replicas by the criteria, when the underlying model is the ZA-StBS, ZA-SGtBS1 or ZA-SGtBS2. Also, notice that when the underlying model is the ZA-SCNBS, three of the four criteria chose the correct model. On the other hand, when the ZA-SSLBS model is used to generated the data, none of the criteria chose the correct model. This probably occurred because the estimate of the degrees of freedom was not so accurate. From Table 131, we can notice that when the underlying model is the ZA-SGtBS1, ZA-StBS and SSLBS, the percentage of times the correct model is selected is low. However, we observed we observe that this percentage increases as the sample size increases.

4.5.5 Posterior predictive checking

The scenario considered here are exactly those presented in Section 4.5.4. That is, we fitted the proposed models to the each one the five data sets, generated according to the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS and ZA-SCNBS regression models. The results of the study PPD can be found in Section D.5 of Appendix D.

In Table 132, we can notice that when the underlying model is the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS1 or ZA-SCNBS, the Bayesian p-values indicate that the ZA-SNBS are misfit, as expected. In general, we can say that the Bayesian p-values, together with the other proposed tools, help to choose the best model.

4.6 Real data analysis

The data set considered here refers to the bilirubin concentration study in newborns conducted in the Escola Paulista de Medicina (UNIFESP) in 2005, see Draque (2005). The concentration of bilirubin (μ mol/L) was measured in 89 healthy full-term newborns on breastfeeding for 1, 2, 3, 4, 5, 6, 8, 10 and 12 days after birth. This study is irregular, balanced and complete (9 observations per subject and 89 for each evaluation condition). The main objective is to explain the variation of bilirubin concentration as a function of age. Some descriptive statistics of this data, including location measures, standard deviation (SD), coefficient of variation (CV), skewness (CS), and kurtosis (CK), are provided in Table 1. We can note that the number of observations equal to zero is lower in the first two days of life. Also, we can notice that the average of the bilirubin concentration increases from the first to second day of life, remains essentially constant on the third day and decreases from the fourth day of life. In Figure 35, it is possible to see that the variability of the bilirubin concentration increases from the third day of life and decreases after the sixth day. From Figure 1 and the skewness coefficients (see Table 1), we can notice that the empirical distribution of bilirubin concentration is symmetric for the first day, negatively skewed for the second, and positively skewed for the other days. Finally, Figures 2 and 36 present individual and longitudinal mean profiles for the 89 healthy full-term newborns. In general, the bilirubin concentration decreases over time for most patients but with substantial between subject variability. In conclusion, the descriptive analysis indicates that the bilirubin concentration depends on the individual characteristics of the patients, but the linear trend over time is similar among the newborns.

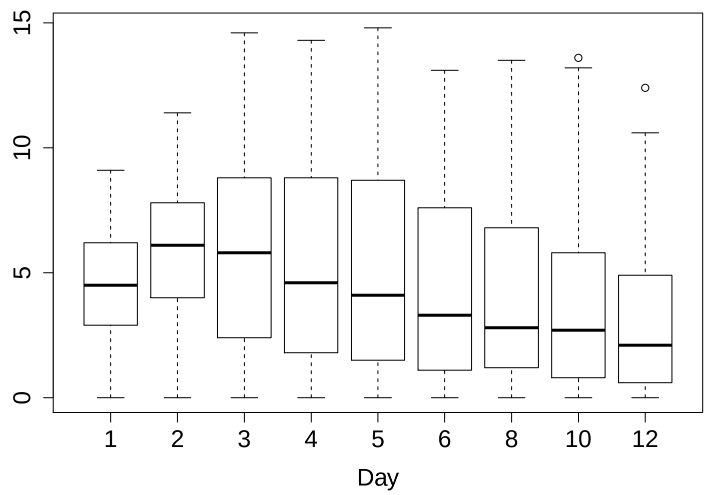

Figure 35 – Boxplots of the bilirubin concentration.

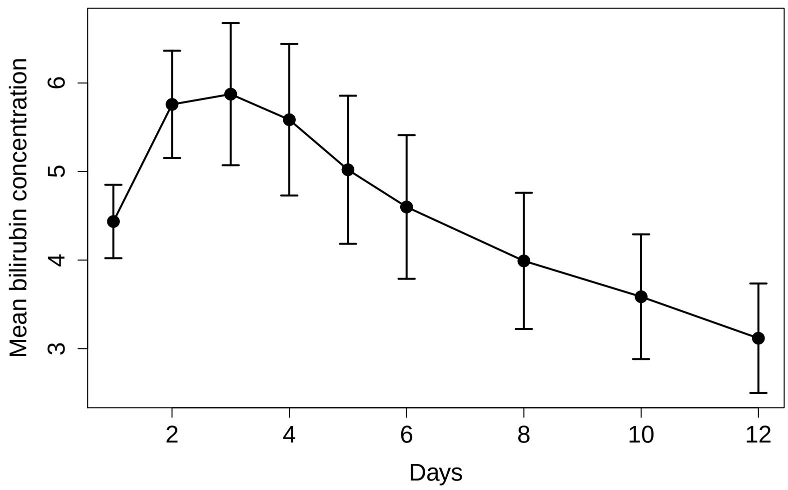

Figure 36 – Mean longitudinal profile of the bilirubin concentration.

We assumed the response follows a ZA-SSBS distribution, that is $T_{ij} \overset{\text{ind}}{\sim}$ ZA-SSBS$(p_{ij}, \mu_{ij}, \phi_{ij}, \gamma_j, \boldsymbol{\nu}_j)$. Based on the descriptive analysis, the systematic components of the regression models are expressed as

$$\begin{aligned} \log(\mu_{ij}) &= \beta_0 + \beta_1(x_{ij} - 1)\mathbb{1}(j \in \{1, 2\}) + \beta_2(x_{ij} - 1)\mathbb{1}(j \in \{3, 4, 5, 6, 8, 10, 12\}) \\ \log(\phi_{ij}) &= \psi_0 + \psi_1(z_{ij} - 1) \\ \text{logit}(p_{ij}) &= \zeta_0 + \zeta_1(v_{ij} - 1), \end{aligned} \tag{4.10}$$

where $i = 1, \ldots, 89$, $j = 1, \ldots, 9$, $\boldsymbol{\beta} = (\beta_0, \beta_1)^\top$, $\boldsymbol{\psi} = (\psi_0, \psi_1)^\top$ and $\boldsymbol{\zeta} = (\zeta_0, \zeta_1)^\top$ are the regression coefficients and $x_{ij} \equiv z_{ij} \equiv v_{ij}$ is the day after birth on which the concentration of bilirubin, corresponding to the jth instant, was measured in the ith newborn. Furthermore,

e^{β_0} represents the mean of the bilirubin concentration for the first day and e^{β_1} represents the mean of the bilirubin concentration for the second. Also, e^{β_2} represents the rate of the change in the mean of the bilirubin concentration at one day interval. Moreover, e^{ψ_0} represents the dispersion of the bilirubin concentration for the first day and e^{ψ_1} represents the dispersion of the bilirubin concentration for the second. Also, e^{ψ_2} represents the rate of the change in the dispersion of the bilirubin concentration at one day interval. Finally, ζ_0 e ζ_1 represent the effects of the age in $\text{logit}(p_{ij})$.

We fitted the proposed models and the ZABS regression model (Tomazella et al., 2018) according to Equation (4.10). Due to numerical instability in the OpenBUGS program, it was not possible to adjust the ZA-SGtBS2 model. It is important to emphasize that we are disregarding a possible dependence between the observations of the same individual. Figures 37-43 display the residual analysis for all models. When we fitted the ZA-SGtBS1, ZA-StBS, ZA-SSLBS1, ZA-SCNBS and ZABS, we can notice that the residual present a behavior compatible with that of heavy-tailed and/or skewed distributions, with some points falling outside the bands. On the other hand, from Figures 40 and 42, the behavior of the residuals reveal that the ZA-SSLBS2 and ZA-SNBS regression models fit the data very well, with show any tendency. Also, we notice that the observations are inside of simulated envelope. On the other hand, from the Figure 44, we can observe that at least ten observations appear as potentially influential under the ZA-SNBS model, whereas the ZA-SSLBS2 model highlights a maximum of three observations. Therefore, we can conclude that the ZA-SSLBS2 model presents an advantage, under this criterion. From the results presented in Table 10 (where the bold values indicate the chosen model by each statistic), we can see that the ZA-SSLBS2 model was selected by EAIC, EBIC and LPML. In conclusion, we can say that the ZA-SSLBS model is more appropriate than the other models.

Table 11 presents the Bayesian estimates, that is, the posterior expectations (PE), posterior standard deviations (PSD) and the 95% equi-tailed credibility intervals for the ZA-SSLBS2 model. The results for the other models, presented in Tables 133-138, can be found in Section D.6 of Appendix D. In general, we can notice that the estimates of $\boldsymbol{\zeta}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, and γ_j are quite similar among the models. Specifically, from Table 11, we have indications that only ζ_1 is equal to zero, once zero belongs the credibility interval. Furthermore, we can notice that its posterior distribution is practically symmetric around zero (see Figure 45). Thus, we excluded the non-significant covariate. The final selected regression structure is:

$$\begin{aligned} \log(\mu_{ij}) &= \beta_0 + \beta_1(x_{ij} - 1)\mathbb{1}(j \in \{1, 2\}) + \beta_2(x_{ij} - 1)\mathbb{1}(j \in \{3, 4, 5, 6, 8, 10, 12\}) \\ \log(\phi_{ij}) &= \psi_0 + \psi_1(z_{ij} - 1), \end{aligned}$$

where $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, x_{ij}, and z_{ij} were previously defined. Considering this structure, we fitted the ZA-SSLBS2 regression model. Figure 46 presents the residual analysis for the final model. We can notice that the observations are inside of simulated envelope, with show any tendency. Thus, we can say that the ZA-SSLBS2 offers a good fit to the bilirubin concentration data. Table 12 presents the Bayesian estimates, that is, the posterior expectations (PE), posterior standard deviations (PSD) and the 95% equi-tailed credibility intervals for the final model. We can notice that all parameters were statistically significant. Figure 47 presents the observed means and the predicted means by the ZA-SSLBS2 model (indicated by gray color). We can notice that the observed means do not belong to the predicted credibility intervals, except in the first two instants. The results presented in Table 12 indicate that the mean bilirubin concentration is equal to 6.567 [6.068; 7.142] on the first day after birth. Also, the variability is equal to .270 [.227; .319] on the first day. Finally, the percentage of zeros is constant, that is, it does not depend on the number of days after birth, and is approximately equal to 10%.

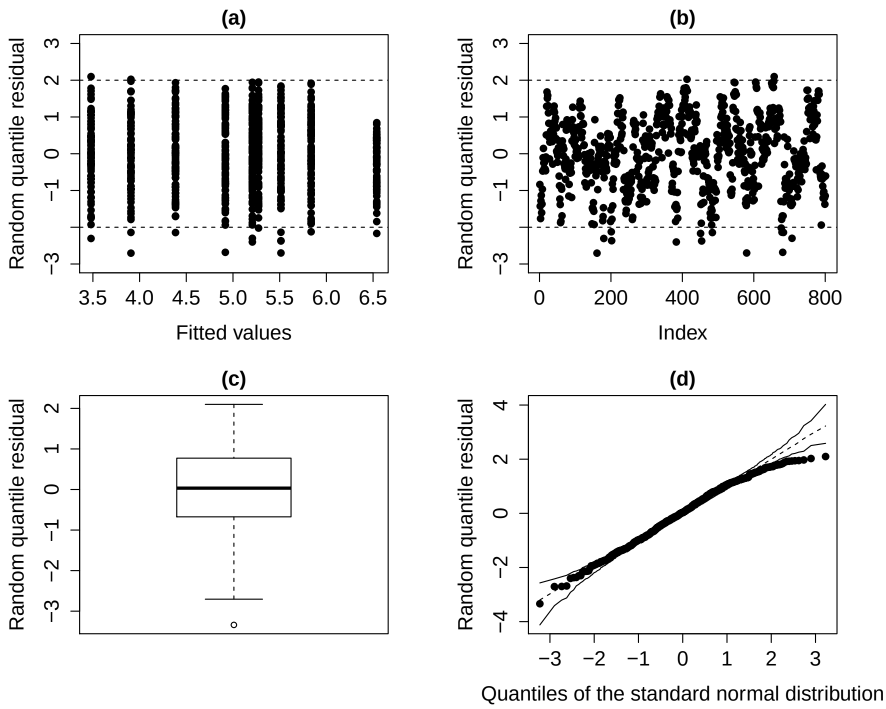

Figure 37 – Residual plots for the ZA-SGtBS1 regression model.

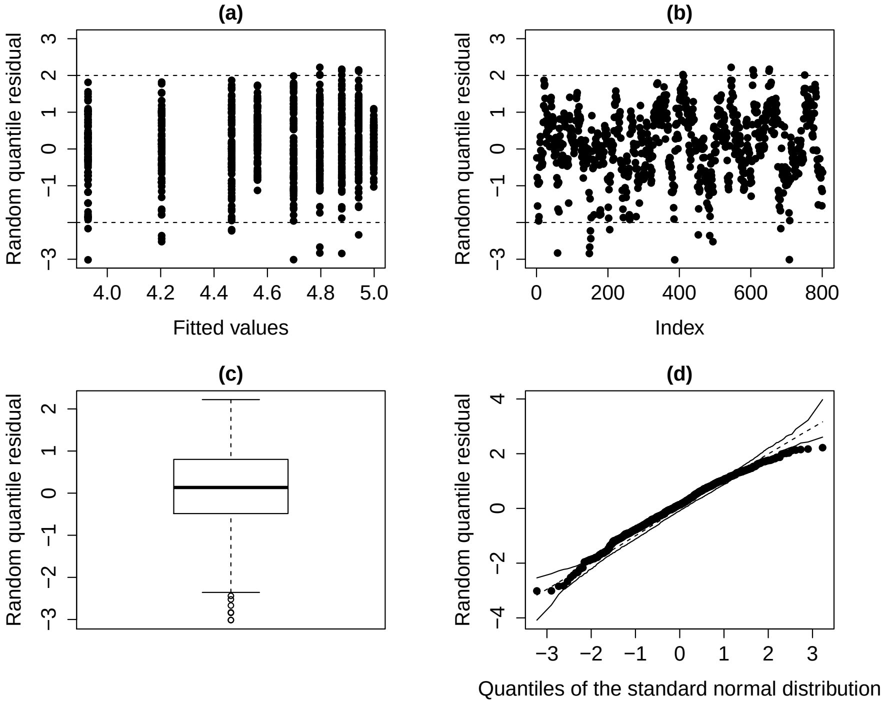

Figure 38 – Residual plots for the ZA-StBS regression model.

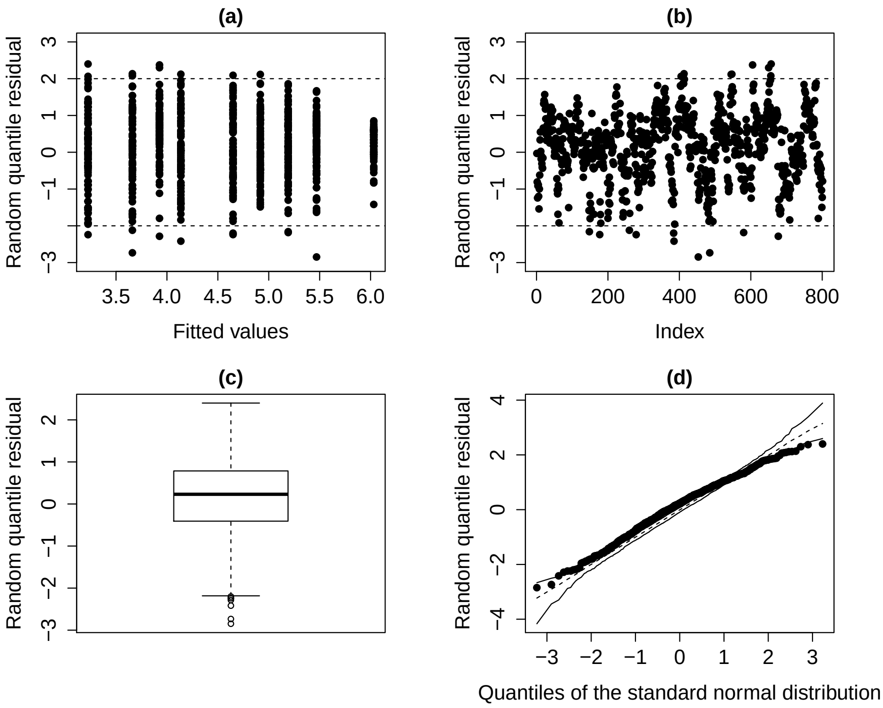

Figure 39 – Residual plots for the ZA-SSLBS1 regression model.

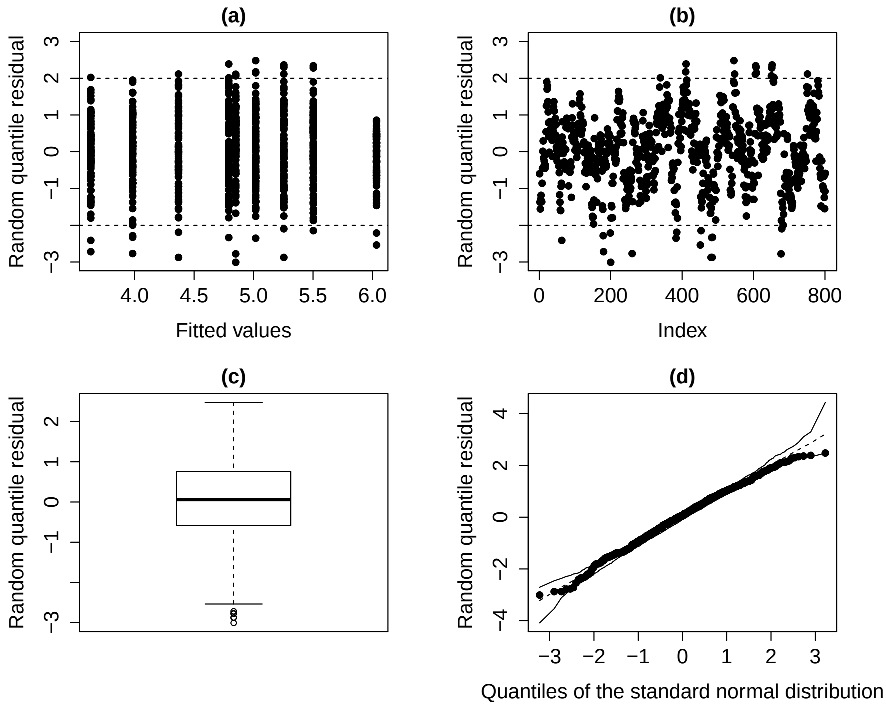

Figure 40 – Residual plots for the ZA-SSLBS2 regression model.

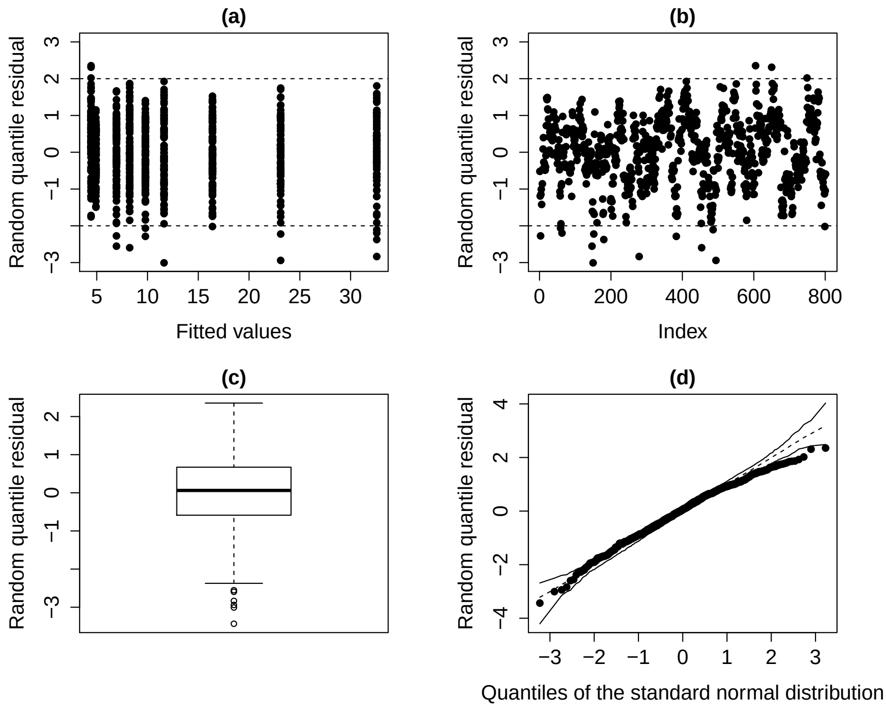

Figure 41 – Residual plots for the ZA-SCNBS regression model.

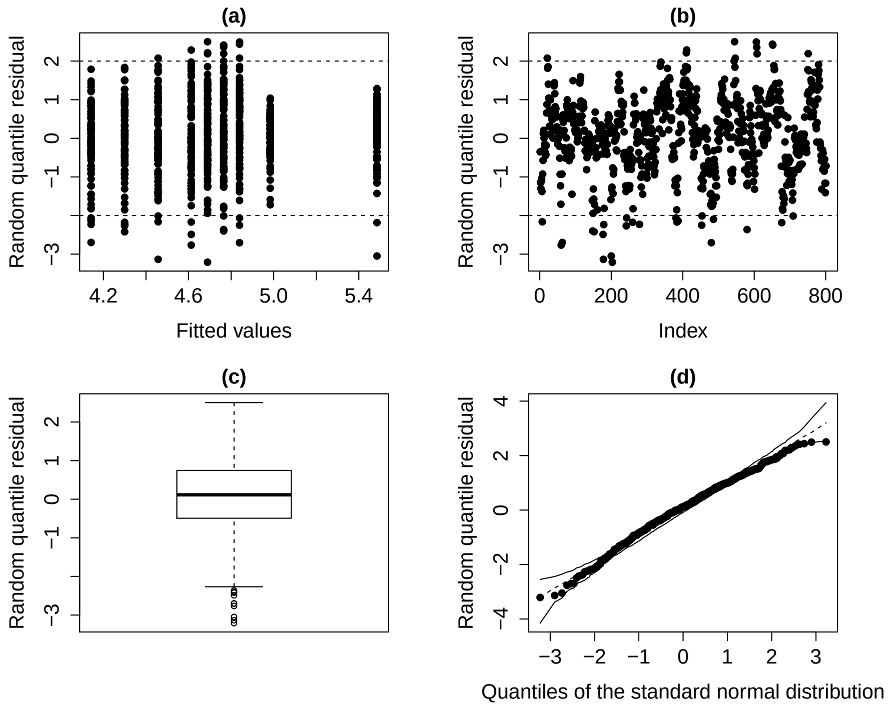

Figure 42 – Residual plots for the ZA-SNBS regression model.

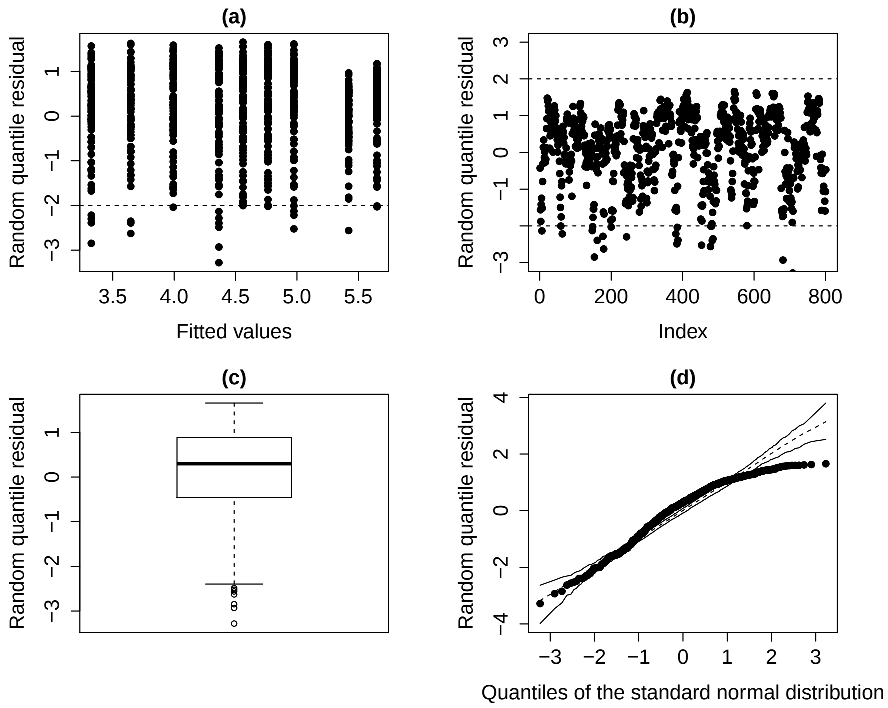

Figure 43 – Residual plots for the ZABS regression model.

Table 10 – Model selection criteria.

Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	4,655.347	4,772.493	13,308.880	-2,312.378
ZA-StBS	4,193.928	4,306.389	**8,751.755**	-2077.505
ZA-SSLBS1	4,276.635	4,393.782	12,662.710	-2,124.003
ZA-SSLBS2	**4,162.319**	**4,279.466**	12,329.300	**-2,062.958**
ZA-SCNBS	4,208.461	4,367.781	12,406.690	-2,079.665
ZA-SNBS	4,211.147	4,286.120	12,497.760	-2,121.040
ZABS	4,488.076	4,516.191	13,419.840	-2,243.666

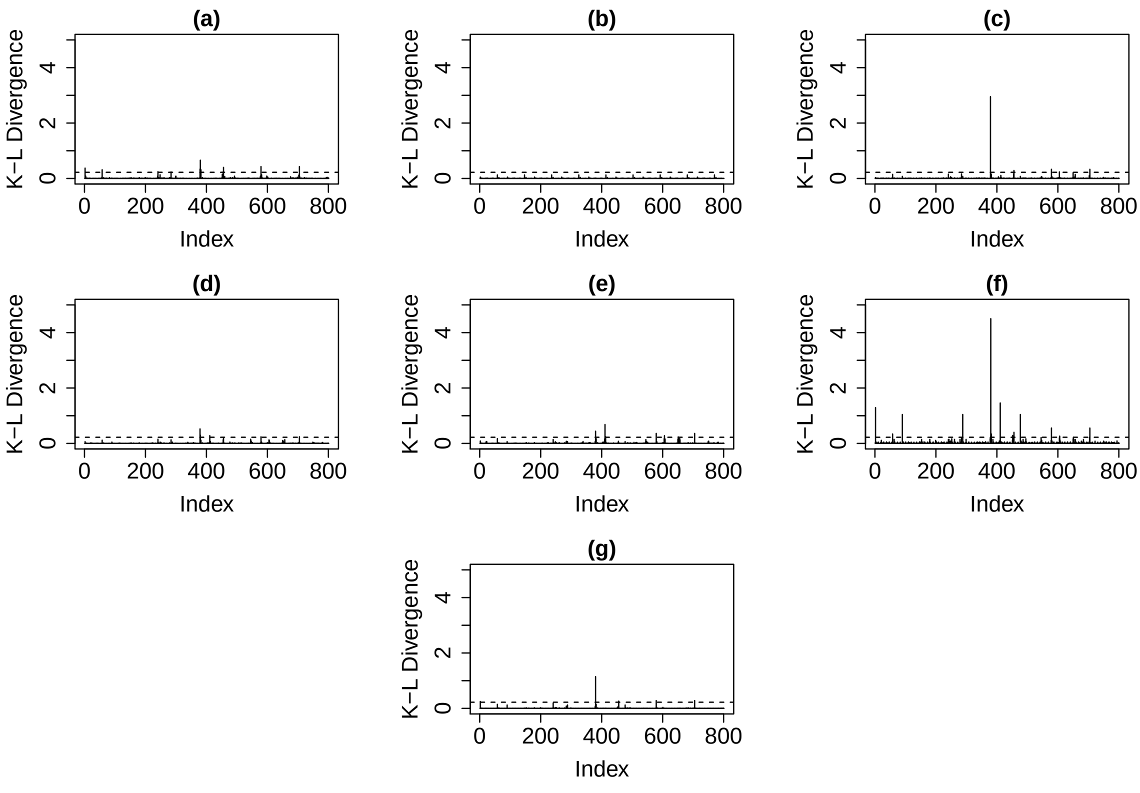

Figure 44 – K-L divergence measure for the models: (a) ZA-SGtBS1, (b) ZA-StBS, (c) ZA-SSLBS1, (d) ZA-SSLBS2, (e) ZA-SCNBS, (f) ZA-SNBS, and (g) ZABS.

Table 11 – Bayesian estimates for the ZA-SSLBS2 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.882	.041	[1.803; 1.966]
β_1	-.217	.061	[-.351; -.107]
β_2	-.045	.008	[-.062; -.031]
ψ_0	-1.307	.089	[-1.482; -1.144]
ψ_1	.129	.017	[.100; .165]
ζ_0	-2.425	.121	[-2.654; -2.180]
ζ_1	.015	.025	[-.033; .062]
γ_1	.274	.115	[.071; .483]
γ_2	-.953	.036	[-.990; -.857]
γ_3	-.927	.055	[-.982; -.815]
γ_4	-.937	.041	[-.984; -.836]
γ_5	-.927	.049	[-.980; -.797]
γ_6	-.973	.029	[-.994; -.895]
γ_7	-.888	.095	[-.984; -.651]
γ_8	-.764	.185	[-.964; -.213]
γ_9	-.873	.097	[-.976; -.616]
ν_1	8.694	1.804	[5.468; 12.341]
ν_2	38.263	24.141	[7.051; 94.333]
ν_3	23.928	20.911	[4.241; 76.247]
ν_4	25.154	18.203	[6.087; 76.442]
ν_5	24.062	21.398	[3.415; 83.778]
ν_6	22.401	13.640	[4.963; 63.819]
ν_7	31.792	23.506	[5.062; 97.096]
ν_8	38.173	25.279	[7.008; 97.473]
ν_9	34.383	22.350	[7.001; 87.826]

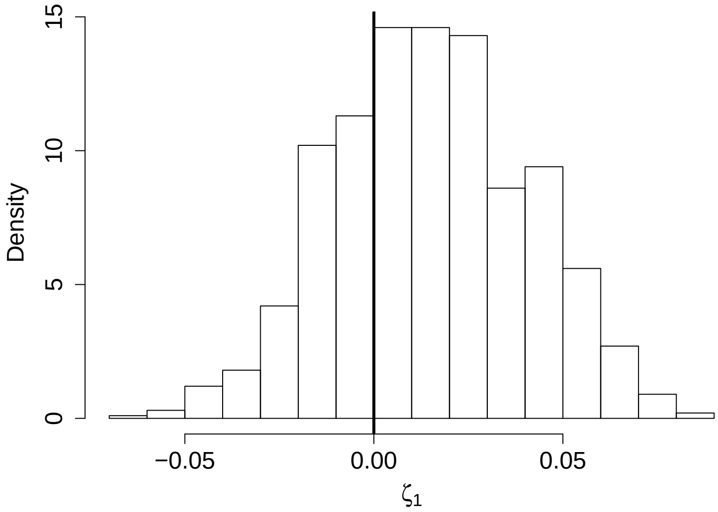

Figure 45 – Posterior distribution of ζ_1 of ZA-SSLBS2 regression model.

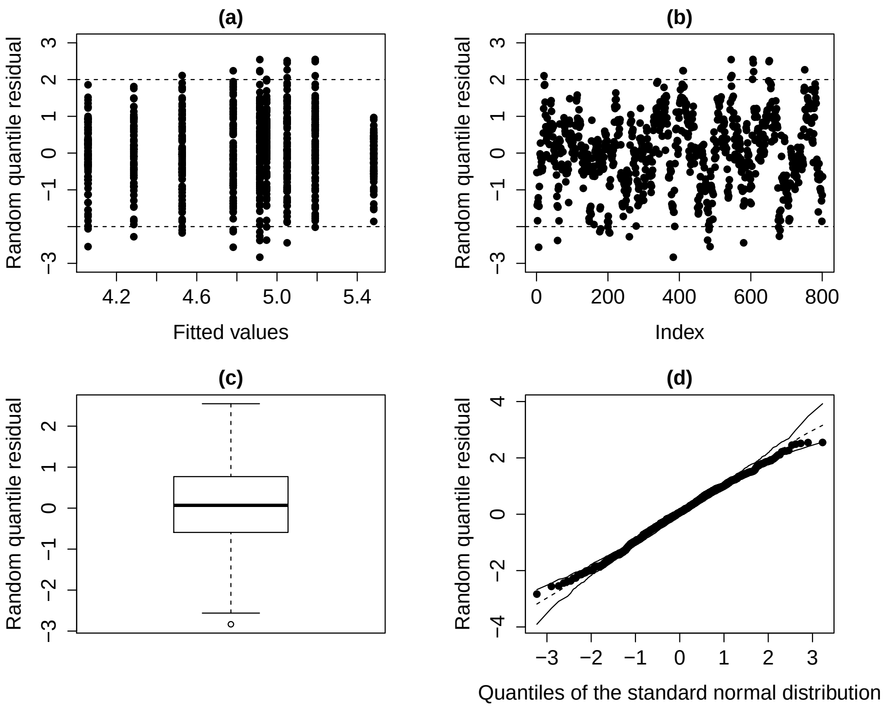

Figure 46 – Residual plots for the final regression model.

Table 12 – Bayesian estimates for the final regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.780	.037	[1.714; 1.857]
β_1	-.102	.061	[-.214; .017]
β_2	-.027	.008	[-.043; -.011]
ψ_0	-1.317	.145	[-1.552; -1.061]
ψ_1	.136	.025	[.099; .188]
p	.075	.008	[.062; .094]
γ_1	-.691	.119	[-.954; -.498]
γ_2	-.945	.032	[-.980; -.870]
γ_3	-.915	.050	[-.981; -.782]
γ_4	-.932	.040	[-.980; -.829]
γ_5	-.922	.053	[-.982; -.779]
γ_6	-.953	.039	[-.990; -.841]
γ_7	-.914	.102	[-.991; -.612]
γ_8	-.761	.225	[-.970; -.139]
γ_9	-.908	.099	[-.995; -.626]
ν_1	5.463	.969	[3.742; 7.412]
ν_2	38.166	25.427	[7.054; 113.325]
ν_3	22.661	20.454	[3.743; 72.233]
ν_4	20.920	21.812	[3.719; 80.839]
ν_5	21.913	20.816	[3.495; 80.155]
ν_6	28.722	18.759	[5.918; 70.751]
ν_7	30.824	25.160	[7.867; 99.603]
ν_8	38.222	24.631	[10.190; 100.935]
ν_9	33.790	21.719	[7.812; 87.496]

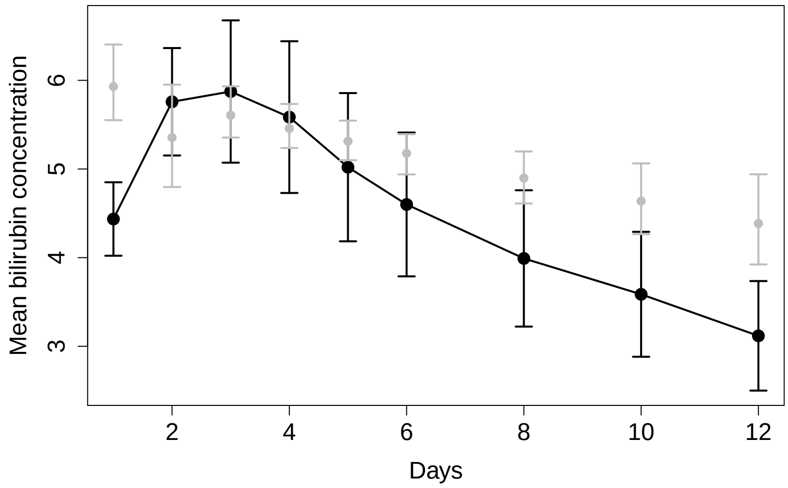

Figure 47 – Observed and predicted means.

4.7 Concluding Remarks

In this chapter, we developed a new family of BS regression models for zero-augmented positive data, named zero-augmented centred skew scale-mixture Birnbaum-Saunders (ZA-SSBS) regression models. Our family, which generalizes the ZABS regression model (Tomazella et al., 2018), allows to analyze data in their original scale and it allows for modeling the mean, the dispersion parameter, and the probability of a point mass at zero through suitable predictors using appropriate link functions. Also, the ZA-SSBS models accommodate properly both positively or negatively skewed data, presenting or not heavy tails. Under the Bayesian paradigm, we developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking based on MCMC algorithms. We conducted simulation studies, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the diagnostic measures and the statistics for model comparison. Specifically, we conducted parameter recovery studies considering different scenarios of practical interest. In general, the results indicated, for all models, that $\boldsymbol{\zeta}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and γ are well recovered in all scenarios. Concerning $\boldsymbol{\nu}$, specifically in the scenarios that lead to heavy-tailed models, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed models are equivalent to ZA-SNBS model, the $\boldsymbol{\nu}$ estimates are biased and the width of the credibility interval are large. However, as sample size increases, the estimates become more accurate. Furthermore, the results indicated that the proposed tools are suitable to choose the best model. Finally, we have presented applications to a real data set related to the bilirubin concentration, showing that the our approach can be much more useful than the traditional ones. The results indicate that our models outperforms the ZABS regression model in terms of model fit.

5 Generalized mixed Birnbaum-Saunders regression models

5.1 Introduction

Mixed-effect models have become a popular approach for the analysis of repeated-measures (see, e.g., McCulloch and Neuhaus (2005), Song and Song (2007), and Verbeke and Molenberghs (2009)), once they provide a common baseline for all the individuals, and enables the practitioner not only to describe the trend over time within each individual, but also to describe the variation among different individuals. However, regression models with random-effects based on the BS distribution has not been widely considered, and only two works have been developed. From a frequentist view point, Villegas et al. (2011) and Desmond et al. (2012) proposed and explored, respectively, a mixed-effect model based on the log-BS distribution for censored reliability data analysis. For this model, the original response must be transformed to a logarithmic scale, which could provoke difficulties of interpretation and inferential problems.

In this chapter, our purpose is to extend the fixed-effects CSSBS regression models proposed in Chapter 3 by including random-effects, which make it possible to: (i) study the correlation between observations of the same experimental unit, and (ii) consider the heterogeneity among different individuals. The family of mixed BS regression models inherits the properties and advantages in inferential terms of the fixed-effects CSSBS regression models. Furthermore, it accommodates properly both positively or negatively skewed data, presenting or not heavy tails, it allows to analyze data in their original scale, and it allows for modeling both the mean and the dispersion parameter through suitable predictors using appropriate link functions. Under the Bayesian paradigm, we developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking. We conducted simulation studies, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the diagnostic measures and the statistics for model comparison. Finally, we have presented applications to a real data set related to the male and female cholesterol levels, showing the usefulness of the inferential methods developed here.

5.2 Mixed centred skew scale-mixture Birnbaum-Saunders regression models

5.2.1 General model

Let $\mathbf{T}_1, \ldots \mathbf{T}_n$ be n independent continuous random vectors with $\boldsymbol{t}_i = (t_{i1}, \ldots, t_{ik_i})^\top$ being the response vector for ith sample unit with element $t_{ij} \in \mathbb{R}$, $j = 1, \ldots, k_i$. Let $\boldsymbol{\mu}_i = (\mu_{i1}, \ldots, \mu_{ik_i})^\top$ and $\boldsymbol{\phi}_i = (\phi_{i1}, \ldots, \phi_{ik_i})^\top$, where μ_{ij} and ϕ_{ij} is the mean and dispersion parameter of $\mathbf{T}_i$, respectively. Suppose that μ_{ij} and ψ_{ij} satisfy the following functional relations:

$$g_1(\mu_{ij}) = \eta_{ij} = f_1(\boldsymbol{x}_{ij}; \boldsymbol{\beta}) \text{ and } g_2(\phi_{ij}) = \varsigma_{ij} = f_2(\boldsymbol{w}_{ij}; \boldsymbol{\psi}),$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$, $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_q)^\top$ are $p \times 1$ and $q \times 1$ vectors of regression parameters, $\boldsymbol{\eta}_i = (\eta_{i1}, \ldots, \eta_{ik_i})^\top$ and $\boldsymbol{\varsigma}_i = (\varsigma_{i1}, \ldots, \varsigma_{ik_i})^\top$ are predictors vectors, and $g_s(\cdot;\cdot), s = 1, 2$ are linear or nonlinear twice continuously differentiable functions in the second argument. Furthermore, $\boldsymbol{x}_{ij} = (x_{ij1}, \ldots, x_{ijp})^\top$ and $\boldsymbol{w}_{ij} = (w_{ij1}, \ldots, w_{ijq})^\top$ are vectors with p and q explanatory variables, respectively. Her, the link functions $g_s : \mathbb{R}^+ \longrightarrow \mathbb{R}, s = 1, 2$ are strictly monotone, positive, and at least twice differentiable. In this work, we connect μ_{ij} and ϕ_{ij} to covariates through the log-linear function as follows

$$\mu_{ij} = \exp\left\{\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top \boldsymbol{b}_i\right\} \text{ and } \phi_{ij} = \exp\left\{\boldsymbol{w}_{ij}^\top \boldsymbol{\psi}\right\}, \tag{5.1}$$

where $\boldsymbol{b}_i = (b_{i1}, \ldots, b_{ir})^\top$ is a random-effects vector of the ith sample unit, which may be, for instance, random intercepts and/or random coefficients, $\boldsymbol{z}_{ij} = (z_{ij1}, \ldots, z_{ijr})^\top$ is a vector with r explanatory variable associated with $\boldsymbol{b}_i$. Also, $\boldsymbol{b}_i | \boldsymbol{\Sigma}_b \sim N(\mathbf{0}, \boldsymbol{\Sigma}_b)$, where $\boldsymbol{\Sigma}_b \in \mathbb{R}^{r \times r}$ is a matrix that contains the variance components of the model and the intraclass (within experimental unit) covariances.

Given the random-effects, $T_{ij} | \boldsymbol{b}_i, \boldsymbol{\Omega} \overset{\text{ind}}{\sim} \text{CSSBS}(\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}), i = 1, \ldots, n, j = 1, \ldots, k_i$, where $\boldsymbol{\Omega} = (\boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, μ_{ij} and ϕ_{ij} are defined in Equation (5.1). The hierarchical structure of the mixed SSBS regression models is given by

$$\begin{aligned} T_{ij} | \boldsymbol{b}_i, \boldsymbol{\Omega} &\sim \text{CSSBS}(\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}) \\ \boldsymbol{b}_i | \boldsymbol{\Sigma}_b &\sim N(\mathbf{0}, \boldsymbol{\Sigma}_b). \end{aligned} \tag{5.2}$$

Thus,

$$\begin{aligned} \mathbb{E}(T_{ij} | \boldsymbol{b}_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) &= \exp\left\{\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top \boldsymbol{b}_i\right\} \\ \mathbb{V}(T_{ij} | \boldsymbol{b}_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) &= c \exp\left\{2\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + 2\boldsymbol{z}_{ij}^\top \boldsymbol{b}_i\right\}, \end{aligned} \tag{5.3}$$

where $c = \dfrac{2\phi_{ij}}{[1 + \phi_{ij} m_2]^2} \left\{m_2 + \dfrac{\phi_{ij}}{2}\left[2m_4 - m_2^2\right]\right\}$ does not depend on $\boldsymbol{b}_i$, ϕ_{ij} is defined in Equation (5.1), and $m_k = \mathbb{E}(Y^k), k = 2, 4$ represents the kth moment of $Y \sim$

CSSMN$(0, 1, \gamma, \boldsymbol{\nu})$. Also, by using results from conditional distributions, we have that

$$\begin{aligned} \mathbb{E}(T_{ij}) &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\} M_b(\boldsymbol{z}_{ij}) \\ \mathbb{V}(T_{ij}) &= \exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\} \left\{(1+c)M_b(2\boldsymbol{z}_{ij}) - [M_b(\boldsymbol{z}_{ij})]^2\right\} \\ \text{Cov}(T_{ij}, T_{ij'}) &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\} [M_b(\boldsymbol{z}_{ij} + \boldsymbol{z}_{ij'}) - M_b(\boldsymbol{z}_{ij})M_b(\boldsymbol{z}_{ij'})]. \end{aligned} \tag{5.4}$$

where $M_b(\mathbf{s}) = \exp\left\{\frac{1}{2}\mathbf{s}^\top\Sigma_b\mathbf{s}\right\}$ is the moment generating function of a normally distributed random vector. The proof of these results can be found in Section E.1 of Appendix E.

5.2.2 Random intercepts model

In this work, we assume that the random intercepts are sufficient to capture heterogeneity between individuals. Thus, suppose that $\boldsymbol{b} \sim N(0, \sigma^2\mathbf{I})$ and $\boldsymbol{z}_{ij}$ has a single entry equal to 1. We may simplify the hierarchical structure presented in Equation (5.2) to

$$\begin{aligned} T_{ij}|b_i, \boldsymbol{\Omega} &\overset{\text{ind}}{\sim} \text{CSSBS}(\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}) \\ b_i|\sigma^2 &\overset{\text{iid}}{\sim} N(0, \sigma^2), \end{aligned} \tag{5.5}$$

where

$$\mu_{ij} = \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + b_i\right\} \text{ and } \phi_{ij} = \exp\left\{\boldsymbol{w}_{ij}^\top\boldsymbol{\psi}\right\}, \tag{5.6}$$

and $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$. Thus, we have that

$$\mathbb{E}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) = \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + b_i\right\} \tag{5.7}$$

$$\mathbb{V}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) = c\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + 2b_i\right\}. \tag{5.8}$$

We may simplify the expressions presented in Equation (5.3) to

$$\begin{aligned} \mathbb{E}(T_{ij}) &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\exp\left\{\sigma^2/2\right\} \\ \mathbb{V}(T_{ij}) &= \exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\exp\left\{\sigma^2\right\}\left[(1+c)\exp\left\{\sigma^2\right\} - 1\right] \\ \text{Cov}(T_{ij}, T_{ij'}) &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\}\left[\exp\left\{\sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right)\right]. \end{aligned}$$

5.3 Bayesian inference

In this section, we present the Bayesian inference for the mixed SSBS models. The adopted approach, since the marginal posterior distributions on interest can not be analytically obtained, relies on the MCMC algorithms to obtain numerical approximations for those distributions.

5.3.1 Likelihoods

Given the random-effects, we assume that $T_{i1}, \ldots, T_{ik_i}$ are independent. Let $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_n)^\top$, and $\boldsymbol{b} = (b_1, \ldots, b_n)^\top$. The joint likelihood (without integrating out the random-effects b_i) takes on the form

$$L(\boldsymbol{\Omega}|\mathbf{t}, \boldsymbol{b}) = \prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi_{\gamma,\boldsymbol{\nu}}\left[a_{t_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij}}(\mu_{ij}, \phi_{ij}), \tag{5.9}$$

where $\boldsymbol{\Omega} = (\boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, $\phi_{\gamma,\boldsymbol{\nu}}(\cdot)$ was defined in Equation (1.9), $a_{t_{ij}}(\mu_{ij}, \phi_{ij}) = \left\{\sqrt{t_{ij}[1+\phi_{ij}\mathbb{E}(Y^2)]/\mu_{ij}} - \sqrt{\mu_{ij}/t_{ij}[1+\phi_{ij}\mathbb{E}(Y^2)]}\right\}/\sqrt{2\phi_{ij}}$, and $A_{t_{ij}}(\mu_{ij}, \phi_{ij}) = \dfrac{t_{ij}^{-3/2}\left\{t_{ij} + \mu_{ij}/\left[1+\phi_{ij}\mathbb{E}(Y^2)\right]\right\}}{2\sqrt{2\phi_{ij}}\sqrt{\mu_{ij}/\left[1+\phi_{ij}\mathbb{E}(Y^2)\right]}}$. Furthermore, considering the hierarchical representation presented in Equation (1.18), which is useful for stochastic simulation and parameter estimation (both frequentist and Bayesian), we have that the so-called complete likelihood is given by

$$L(\boldsymbol{\Omega}|\mathbf{t}_c, \boldsymbol{b}) \propto \prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}) \exp\left\{-h_{ij}^2/2\right\} g(u_{ij}|\boldsymbol{\nu}) \tag{5.10}$$

where $\mathbf{t}_c = (\mathbf{t}^\top, \mathbf{h}^\top, \mathbf{u}^\top)$, where $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_n)^\top$, $\mathbf{h} = (\mathbf{h}_1, \ldots, \mathbf{h}_n)^\top$ and $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)^\top$. Also, $\vartheta_{h_{ij}}$ was defined in Equation (1.18).

5.3.2 Prior, posterior and full conditional distributions

To complete the Bayesian specification we need to consider convenient prior distribution. Assuming that the parameters are independent a priori, we consider that the respective joint prior distribution is given by

$$\pi(\boldsymbol{\theta}, \boldsymbol{b}) = \pi(\boldsymbol{\beta})\pi(\boldsymbol{\psi})\pi(\gamma)\pi(\boldsymbol{\nu})\pi(\boldsymbol{b}|\sigma^2)\pi(\sigma^2), \tag{5.11}$$

where $\boldsymbol{\theta} = (\boldsymbol{\Omega}, \sigma^2)^\top$. We specify weakly informative prior distributions on the fixed-effects regression parameters and random-effects $\boldsymbol{b}$. Specifically, we chose $\boldsymbol{\beta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\beta)$, $\boldsymbol{\psi} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\psi)$, and $\gamma \sim U(a, b)$. Also, we consider $\boldsymbol{b} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, where $\sigma^2 \sim \text{gamma}(c, d)$. The prior distribution of $\boldsymbol{\nu}$ depends on the particular distributions adopted for the mixed CSSBS model (more details will be presented below). Combining the likelihood presented in Equation (5.10) and prior distribution presented in Equation (5.11), the joint posterior distribution is given by

$$\begin{aligned}\pi(\boldsymbol{\theta}, \mathbf{h}, \mathbf{u}, \boldsymbol{b}|\mathbf{t}) \quad \propto \quad & \left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}) \exp\left\{-h_{ij}^2/2\right\}\right. \\ & \left. \times\, g(u_{ij}|\boldsymbol{\nu})\right\} \times \pi(\boldsymbol{\theta}, \boldsymbol{b}), \end{aligned} \tag{5.12}$$

and the full conditional distributions, are given by

$$\begin{aligned}
\pi(h_{ij}|\boldsymbol{\theta}, t_{ij}, u_{ij}, b_i) &\propto \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] \exp\left\{-h_{ij}^2/2\right\} \\
\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) &\propto \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}) \left[g(u_{ij}|\boldsymbol{\nu})\right] \qquad (5.13)
\end{aligned}$$

$$\begin{aligned}
\pi(\boldsymbol{\beta}|\boldsymbol{\psi}, \gamma, \boldsymbol{\nu}, \mathbf{t}_c, \boldsymbol{b}) &\propto \pi(\boldsymbol{\beta})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right\} \\
\pi(\boldsymbol{\psi}|\boldsymbol{\beta}, \gamma, \boldsymbol{\nu}, \mathbf{t}_c, \boldsymbol{b}) &\propto \pi(\boldsymbol{\psi})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right\} \\
\pi(\gamma|\boldsymbol{\beta}, \boldsymbol{\psi}, \boldsymbol{\nu}, \mathbf{t}_c, \boldsymbol{b}) &\propto \pi(\gamma)\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right\} \\
\pi(\boldsymbol{\nu}|\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c, \boldsymbol{b}) &\propto \pi(\boldsymbol{\nu})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} g(u_{ij})|\boldsymbol{\nu}\right\} \\
\pi(\boldsymbol{b}|\boldsymbol{\theta}, \mathbf{t}_c) &\propto \pi(\boldsymbol{b}|\sigma^2)\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right\} \\
\pi(\sigma^2|\boldsymbol{\theta}, \mathbf{t}_c, \boldsymbol{b}) &\propto \pi(\sigma^2)\pi(\boldsymbol{b}|\sigma^2), \qquad (5.14)
\end{aligned}$$

The shape of distributions presented in Equations (5.13) and (5.14) depend on the particular distribution adopted for the mixed CSSBS regression models and the adopted prior distribution of $\boldsymbol{\nu}$. The full conditional distributions of u_i and $\boldsymbol{\nu}$, and $\pi(\boldsymbol{\nu})$ for each mixed CSSBS model are presented bellow. We made all implementations considering the `OpenBUGS` software (Spiegelhalter et al., 2014), through the `R` program (R Core Team, 2014), using the package `R2OpenBUGS` (Sturtz et al., 2010). That is, the auxiliary algorithms used to simulate from the full conditional distributions follow the predefined hierarchy implemented in `OpenBUGS`.

5.3.2.1 Prior distribution of $\boldsymbol{\nu}$ and full conditional distributions

1. *The mixed SGtBS regression model.* Here, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$. We adopt a truncated exponential prior for ν_s, $s = 1, 2$, that is $\nu_s \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}}(\nu_s)$, such that $\Lambda \sim \mathrm{U}(\Lambda_0, \Lambda_1)$ (Cabral et al., 2012), which leads to the density
$$\pi(\nu_s) = \frac{1}{\nu_s^2(\Lambda_1 - \Lambda_0)}\left[(\Lambda_0\nu_s + 1)\exp\{-\Lambda_0\nu_s\} - (\Lambda_1\nu_s + 1)\exp\{-\Lambda_1\nu_s\}\right].$$
The full conditional distributions of u_{ij} and ν_s takes the form
$$\begin{aligned}
\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) &\propto \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}) \left[u_{ij}^{\nu_s/2-1}\right] \exp\left\{-\frac{\nu_s}{2}u_{ij}\right\} \\
\pi(\nu_s|\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c, \boldsymbol{b}, \sigma^2) &\propto \frac{1}{\nu_s^2(\Lambda_1 - \Lambda_0)}\left[(\Lambda_0\nu_s + 1)\exp\{-\Lambda_0\nu_s\} - (\Lambda_1\nu_s + 1)\exp\{-\Lambda_1\nu_s\}\right] \times \\
&\quad \times \left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} (\nu_s/2)^{\nu_s/2}\left[\Gamma(\nu_s/2)\right]^{-1} u_{ij}^{\nu_s/2-1} \exp\left\{-\frac{\nu_s}{2}u_{ij}\right\}\right\}.
\end{aligned}$$

2. *The mixed SSLBS regression model.* We adopt a truncated gamma distribution for $\boldsymbol{\nu} = \nu$, that is, $\nu \sim \text{gamma}(a, b)\mathbb{1}_{\mathbb{A}}(\nu)$, with small positive values of a and b ($b << a$), see Vilca et al. (2016). The full conditional distributions of u_{ij} and ν become

$$\begin{aligned}\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) &\propto \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}) u_i j^{\nu-1}\\ \pi(\nu|\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c, \boldsymbol{b}, \sigma^2) &\propto \nu^{a-1}\exp\{-b\nu\}\prod_{i=1}^{n}\prod_{j=1}^{k_i}\nu\, u_{ij}^{\nu-1}.\end{aligned}$$

3. *The mixed SCNBS regression model.* The possible states of the "weights" u_{ij} are ν_2 or 1, with $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$, and its density can be expressed as

$$g(u_{ij}|\boldsymbol{\nu}) = \nu_1^{[(1-u_{ij})/(1-\nu_2)]}(1-\nu_1)^{[(u_{ij}-\nu_2)/(1-\nu_2)]}.$$

The full conditional distribution of each u_{ij} can be written as:

$$\begin{aligned}\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) &\propto \phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\nu_1^{[(1-u_{ij})/(1-\nu_2)]}\\ &\quad \times (1-\nu_1)^{[(u_{ij}-\nu_2)/(1-\nu_2)]}.\end{aligned}$$

Thus, the distribution is proportional to

$$\begin{cases}\nu_1\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}), \text{if}\, u_i = \nu_2\\ (1-\nu_1)\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_i}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}), \text{if}\, u_i = 1\end{cases}$$

In this case, we consider $\nu_s \sim \text{beta}(a_s, b_s)$ (Lachos et al., 2017). The full conditional distribution of ν_s, $s = 1, 2$, is given by

$$\pi(\nu_s|\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c, \boldsymbol{b}, \sigma^2) \propto \nu_s^{a_s + a_{n,\nu_2}-1}(1-\nu_s)^{b_s+b_{n,\nu_2}-1},$$

where $a_{n,\nu_2} = \left(n - \sum_{i=1}^{n}\sum_{j=1}^{k_i} u_{ij}\right)/\left(1-\nu_2\right)$ and $b_{n,\nu_2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{k_i} u_{ij} - n\,\nu_2\right)/\left(1-\nu_2\right)$, which is proportional to $\text{beta}(a_s + a_{n,\nu_2}, b_s + b_{n,\nu_2})$ density.

5.4 Model fit assessment and model comparison

5.4.1 Residual analysis

Let $T_{ij}|b_i, \boldsymbol{\Omega} \sim \text{CSSBS}(\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu})$, be a r.v with a conditional cdf, $F_{T_{ij}|b_i,\boldsymbol{\Omega}}(t_{ij})$, defined in Equation (1.10). Therefore we can define the quantile residual as

$$R^q_{ij} = \Phi^{-1}\Big[F_{T_{ij}|b_i,\boldsymbol{\Omega}}(t_{ij})\Big] = \Phi^{-1}\left\{\Phi_{\widehat{\gamma},\widehat{\boldsymbol{\nu}}}[a_{t_{ij}}(\widehat{\mu}_{ij}, \widehat{\phi}_{ij})]\right\}, \quad (5.15)$$

where $\Phi_{\widehat{\gamma},\widehat{\boldsymbol{\nu}}}(\cdot)$ was defined in Equation (1.9), and $a_{t_{ij}}(\mu_{ij}, \phi_{ij})$ and $A_{t_i}(\mu_i, \phi_i)$ were defined in Equation (5.9), $\widehat{\mu}_{ij} = \exp\left\{\boldsymbol{x}_{ij}^\top\widehat{\boldsymbol{\beta}} + \tilde{b}_i\right\}$ and $\widehat{\phi}_{ij} = \exp\left\{\boldsymbol{w}_{ij}^\top\widehat{\boldsymbol{\psi}}\right\}$. Furthermore, $\widehat{(\cdot)}$ is the respective Bayesian estimator (i.e., posterior expectation, posterior median or posterior mode). Therefore, with $\widehat{\boldsymbol{\beta}}$, $\widehat{\boldsymbol{\psi}}$, $\widehat{\gamma}$, and $\widehat{\boldsymbol{\nu}}$ being consistent estimators (in the frequentist sense) of $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, γ, and $\boldsymbol{\nu}$, respectively, we have that R_i^q converges in distribution to the standard normal distribution. The methodology used for the construction of simulated envelopes follows the usual one. That is, since the observed residuals are expected to follow a standard normal distribution, under the well fit of the model, the envelopes are simulated from a standard normal distributions, as described in Atkinson (1985).

5.4.2 Statistics for model comparison

When MCMC algorithms are used to obtain the posterior distributions, some statistics for model comparison can be easily calculated, see Spiegelhalter et al. (2002). To introduce these statistics, we first define $\mathrm{D}(\boldsymbol{\Omega}) = -2\log\left[L(\boldsymbol{\Omega}|\mathbf{t})\right]$, where $\boldsymbol{\Omega} = (\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$ and $L(\boldsymbol{\Omega}|\mathbf{t})$ is the (incomplete) likelihood presented in Equation (5.9). Also, let $\boldsymbol{\Omega}^{(m)}$, $m = 1, ..., M$, be the mth value of the valid simulated MCMC sample, that is, the MCMC sample obtained after discarding the burn-in and a proper thinning (lag) between the values. Finally, let $\overline{\boldsymbol{\Omega}}$ be the vector with the posterior expectation of all parameters, based on the valid MCMC sample, and $\overline{\mathrm{D}(\boldsymbol{\Omega})} = \frac{1}{M}\sum_{m=1}^{M} D(\boldsymbol{\Omega}^{(m)})$. Denote also the deviance by $\mathrm{D}(\overline{\boldsymbol{\Omega}}) = -2\ell(\overline{\boldsymbol{\Omega}}|\mathbf{t})$, and the deviance information criterion (DIC) by $\mathrm{DIC} = \mathrm{D}(\overline{\boldsymbol{\Omega}}) + 2p_D$, where $p_D = \overline{\mathrm{D}(\boldsymbol{\Omega})} - \mathrm{D}(\overline{\boldsymbol{\Omega}})$. The EAIC (posterior expectation of AIC) and EBIC (posterior expectation of BIC) are given, respectively, by $\mathrm{EAIC} = D(\bar{\boldsymbol{\Omega}}) + 2k$ and $\mathrm{EBIC} = \mathrm{D}(\bar{\boldsymbol{\Omega}}) + k\log(n)$, where k is the total number of parameters of the model and n is the number of observations. Finally, the LPML (logarithm of the pseudo-marginal likelihood) is calculated as $\mathrm{LPML} = \sum_{i=1}^{n}\ln(\widehat{\mathrm{CPO}_i})$, where $\widehat{\mathrm{CPO}}_i = \left\{\frac{1}{M}\sum_{m=1}^{M}\left[1/L\left(\boldsymbol{\Omega}^{(m)}|t_i\right)\right]\right\}^{-1}$. The smaller the values of DIC, EAIC, EBIC and deviance, the better the model fit, occurring the opposite with the LPML.

5.4.3 Posterior predictive checking

Under Bayesian perspective, one way to check the goodness of the model fit, is to compare the predictive distribution with the distribution of the observed data. Let $\boldsymbol{t}^{\mathrm{obs}}$ be the observed response and $\boldsymbol{t}^{\mathrm{rep}}$ the replicated response generated from its posterior predictive distribution, which is given by

$$p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{t}^{\mathrm{obs}}) = \int p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{\Omega})\, p(\boldsymbol{\Omega}|\boldsymbol{t}^{\mathrm{obs}})d\boldsymbol{\Omega}. \tag{5.16}$$

where $\boldsymbol{\Omega} = (\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$. A suitable discrepancy measures $D(\boldsymbol{t}, \boldsymbol{\Omega})$ are defined by Gelman et al. (1996) and the posterior distribution of $D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\Omega})$ is compared to the posterior predictive distribution of $D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\Omega})$, an substantial differences between them indicating model misfit. Gelman et al. (2013) suggest several graphs to compare the replicated and the observed data, under the given measure of divergence.

Another measure used to quantify the goodness of fit, is the Bayesian p-value, which for an adopted discrepancy measure, and is defined as

$$\begin{aligned}\mathbb{P}[D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\Omega})] &\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\Omega}|\boldsymbol{t}^{\text{obs}}) \\ &= \int_{D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\Omega})\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\Omega})} p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{\Omega})\, p(\boldsymbol{\Omega}|\boldsymbol{t}^{\text{obs}}) d\boldsymbol{t}^{\text{rep}} d\boldsymbol{\Omega}. \end{aligned} \tag{5.17}$$

Due to the difficulty in dealing with Equations (5.16) and (5.17) analytically, Rubin (1984) suggests simulating replicated data sets from the posterior predictive distribution. One draws M simulations $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_M$ from the posterior distribution $p(\boldsymbol{\Omega}|\boldsymbol{t})$ of $\boldsymbol{\Omega}$ and then draws $\boldsymbol{t}^{\text{rep},n}$ from the distribution $p(\boldsymbol{t}|\boldsymbol{\Omega}^n)$ for $n = 1, \ldots, M$. The proportion of the M replications for which $D(\boldsymbol{t}^{\text{rep},n}, \boldsymbol{\Omega}^n)$ exceeds $D(\boldsymbol{t}, \boldsymbol{\Omega}^n)$ provides an estimate of the p-value Bayesian. Extreme values of the Bayesian p-value (less than .05 or greater than .95, depending on the nature of the discrepancy measure) indicate model misfit, see Sinharay et al. (2006). Based on Gelman et al. (1996), the measure of discrepancy here adopted is $D(\mathbf{t}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \{[t_{ij} - \mathbb{E}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)]^2\} / \mathbb{V}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)$, where $\mathbb{E}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)$ and $\mathbb{V}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)$ are given by Equation (5.7).

5.4.4 Bayesian case influence diagnostics

Since regression models are sensitive to the underlying model assumptions, it is important to perform sensitivity analysis. Here, we consider the measure of divergence within the Bayesian context proposed by Cho et al. (2009). They developed case deletion influence diagnostics for both joint and marginal posterior distributions based on the Kullback-Leibler (K-L) divergence, and presented a simple way of calculating such influence measure by using MCMC outputs. Let $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ be the K-L divergence between P and $\mathrm{P}_{(-i)}$, where P stands for the posterior distribution of $\boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = (\boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$, for the full data and $\mathrm{P}_{(-i)}$ stands for the posterior distribution of $\boldsymbol{\Omega}$ without the ith observation. Then, we have

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = \int \pi(\boldsymbol{\Omega}|\mathbf{t}) \ln \left\{ \frac{\pi(\boldsymbol{\Omega}|\boldsymbol{t})}{\pi(\boldsymbol{\Omega}|\boldsymbol{t}_{(-i)})} \right\} d\boldsymbol{\Omega}, \tag{5.18}$$

where $\mathbf{t}_{(-i)}$ corresponds to $\mathbf{t}$ without the ith observation. Also, using the notation introduced earlier in Section 5.4.2, the MCMC estimate of $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ is $\widehat{\mathrm{K}}(\mathrm{P}, \mathrm{P}_{(-i)}) = -\ln(\widehat{\mathrm{CPO}}_i) + \frac{1}{M} \times \sum_{m=1}^{M} \ln[L(\boldsymbol{\Omega}^{(m)}|\mathbf{t}_i)]$, where $\widehat{\mathrm{CPO}}_i = \left\{ \frac{1}{M} \sum_{m=1}^{M} \left[1/L\left(\boldsymbol{\Omega}^{(m)}|t_i\right)\right]\right\}^{-1}$. As usual, we need to establish a cut-off point, in order to determine whether an observation is influential or not. As pointed by Cho et al. (2009), the calibration of K-L divergence can be done by solving for p_i the equation

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K\left[Ber(1/2), Ber(p_i)\right] = 0.5 \log\left[4p_i(1-p_i)\right], \tag{5.19}$$

where $Ber(p_i)$ is the Bernoulli distribution with success probability p_i. From the equality $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K[Ber(1/2), Ber(p_i)]$ we have that describing outcomes using $\pi(\boldsymbol{\Omega}|\mathbf{t})$ instead of $\pi(\boldsymbol{\Omega}|\mathbf{t}_{(-i)})$ is compatible with describing an unobserved event as having probability p_i when correct probability is .5 (Cho et al., 2009). Solving Equation (5.19), the calibration of the K-L divergence is

$$p_i = 0.5\left[1+\sqrt{1-\exp\left\{-2\mathrm{K}(\mathrm{P},\mathrm{P}_{(-i)})\right\}}\right].$$

This equation implies that $.5 \leqslant p_i \leqslant 1$. For p_i much greater than .5 implies that the ith observation is influential. In this work, we considered an observation to be influential $p_i \geqslant .8$, as used by Garay et al. (2011) and Chaves et al. (2019a). So, for K-L divergence measure greater than $K\left[Ber(1/2), Ber(.8)\right] \approx .223$, the observation is considered influential (Maioli, 2018).

5.5 Simulation study

In this section, we presented five simulation studies, namely: parameter recovery (PR), behavior of the residuals (R), behavior of the K-L divergence measure (D), performance of the statistics for model comparison (SMC), and study of the posterior predictive checking (PPC) tools.

We considered different scenarios of interest, which correspond to the combination of the levels of some factors. They (with the respective levels within parenthesis) are: sample size (n) (50, 100), that is, small and large sample sizes, asymmetry parameter γ (-.8, 0, .8), that is high negative skewness, symmetry and high positive skewness, and different values of $\boldsymbol{\nu}$, which induce either a normal shape, or a heavy tails behavior.

The general structure of the model considered is

$$\begin{aligned}\log(\mu_{ij}) &= \beta_0+\beta_1 x_{ij}+b_i, i=1,\ldots,n, j=1,\ldots,k_i\\ \log(\phi_{ij}) &= \psi_0+\psi_1 w_{ij},\end{aligned}$$

where $b_i \sim \mathrm{N}(0,\sigma^2)$. Also, x_{ij} and w_{ij}, the explanatory variables, are generated as independent draws from a continuous uniform distribution over the interval $(0,1)$. We fix $\sigma^2 = 4$, $\boldsymbol{\beta} = (-.5, 1)^\top$ and $\boldsymbol{\psi} = (-1, .5)^\top$ for the mixed StBS, SSLBS and SCNBS regression models and, to overcome the identifiability issue in the mixed SGtBS model, we fitted two different structures: in the first model, named mixed SGtBS1, we fixed $\nu_2 = 1$ and in the second model, named mixed SGtBS2, we fixed $\phi = 1$. Furthermore, we considered $\nu \in \{5, 30\}$ for the mixed StBS and mixed SSLBS regression models, $\nu_1 \in \{5, 30\}$ for the mixed SGtBS1, $\boldsymbol{\nu} = (\nu_1,\nu_2)^\top = (15,5)^\top$ and $\boldsymbol{\nu} = (\nu_1,\nu_2)^\top = (30,30)^\top$ for the mixed SGtBS2 regression model, and $\boldsymbol{\nu} = (\nu_1,\nu_2)^\top = (.9,.1)^\top$ and $\boldsymbol{\nu} = (\nu_1,\nu_2)^\top = (.1,.1)^\top$ for the mixed SCNBS model.

The prior distributions (which were used in all studies) were: $\beta_k \sim N(0,10^4)$, $\psi_k \sim N(0,10^4)$, $k = 0,1$, $b_i \sim N(0,\sigma^2)$, where $\sigma^2 \sim \mathrm{gamma}(.01,.01)$, and $\gamma \sim \mathrm{U}\,(-.99527,$

.99527). The first, second and third priors are quite flats, and the fourth prior (Azevedo et al., 2011), is non-informative. For the mixed SGtBS1 regression model we set $\nu_1 \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_1}$, for mixed SGtBS2 model we consider $\nu_i \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_i}(\nu_i)$; $i = 1, 2$, such that $\Lambda \sim U(.02, .5)$ (Cabral et al., 2012). In order to guarantee the existence of the first two moments we consider $\mathbb{A}_1 = (4, \infty)$ and $\mathbb{A}_2 = (2, \infty)$. Similarly, for the mixed StBS regression model we set $\nu \sim \exp(\Lambda)\mathbb{1}_{(4,\infty)}(\nu)$, such that $\Lambda \sim U(.02, .5)$. For the mixed SCNBS regression model we chose $\nu_1 \sim \text{beta}(2, 2)$ and $\nu_2 \sim \text{beta}(1, 1)$ (Lachos et al., 2017).

For the mixed SSLBS regression model, we investigated the sensitivity to the prior choice for ν, by using different suggestions found in the literature, such as: $\nu \sim \exp(\Lambda)\mathbb{1}_{(2,\infty)}(\nu)$, where $\Lambda \sim U(.02, .5)$, and $\nu \sim \text{gamma}(.001, .001)\mathbb{1}_{(2,\infty)}(\nu)$, both suggested by Cabral et al. (2012), $\nu \sim \text{gamma}(1, .1)\mathbb{1}_{(2,\infty)}(\nu)$ (Vilca et al., 2016) and $\nu \sim \text{gamma}(.01, .001)\mathbb{1}_{(2,\infty)}(\nu)$ (Bandyopadhyay et al., 2010). The results showed, for all distributions, estimates concentrated in the interval $(2, 7)$, independently of the respective true value. Therefore, we decided to consider two situations: when the true ν value is small, for example $\nu = 5$, we chose $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$. When the true ν value is large, for example $\nu = 30$, we chose $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$. We will refer to the mixed SSLBS model as mixed SSLBS1 and as mixed SSLBS2, when we consider $\nu \sim \text{gamma}(1, .2)\mathbb{1}_{(2,\infty)}(\nu)$ and $\nu \sim \text{gamma}(1.5, .05)\mathbb{1}_{(2,\infty)}(\nu)$, respectively. In a real data analysis, we can fit the two models and consider QQ plots, the statistics for model comparison, and Bayesian p-values, to choose the most properly one.

From the results related to a convergence study (not presented for the sake of simplicity) we observed that to set a burn-in of 100,000, with a spacing of 20, generating a total of 120,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the mixed SGtBS1 regression model. For the mixed SGtBS2 model, we set a burn-in of 50,000 and a total of 80,000 values were simulated, and samples were collected at a spacing of 30 iterations. Also, we observed that to set a burn-in of 50,000, with a spacing of 50, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the mixed StBS model. We observed that to set a burn-in of 60,000, with a spacing of 40, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the mixed SSLBS1, SSLBS2 and SCNBS models. Finally, for the mixed SNBS regression model, we set a burn-in of 40,000 and a total of 80,000 values were simulated, and samples were collected at a spacing of 40 iterations.

For the PR and SMC studies, R = 5 and R=10 replicas (simulated responses from the model) were considered, respectively. For the three others, one replica and only one scenario were used. All the results of the simulation studies can be found in the Sections E.2-E.6 of Appendix E. More specific details concerning each study are presented in the following sections.

5.5.1 Parameter recovery

We calculated the usual statistics to measure the accuracy of the bias, standard deviation (SD), square root of the mean squared error (RMSE), absolute value of the relative bias (AVRB), coverage probability (CP) of the 95% equi-tailed credibility interval and average length (LCI) of the 95% equi-tailed credibility interval. Let θ be the parameter of interest and let $\widehat{\theta}_r$ be some estimate (posterior mean, median or mode) related to the replica r, and $\overline{\widehat{\theta}} = (1/R)\sum_{r=1}^{R}\widehat{\theta}_r$. The aforementioned statistics are: bias $= \overline{\widehat{\theta}} - \theta$; SD $= \sqrt{(1/R)\sum_{r=1}^{R}(\widehat{\theta}_r - \overline{\widehat{\theta}})^2}$, RMSE $= \sqrt{(1/R)\sum_{r=1}^{R}(\theta - \widehat{\theta}_r)^2}$ and AVRB $= |\overline{\widehat{\theta}} - \theta|/|\theta|$, CP $= (1/R)\sum_{r=1}^{R}\mathrm{I}(\theta \in [\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}])$ and LCI $= (1/R)\sum_{r=1}^{R}[\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}]$, where $\widehat{\theta}_{r,LCL}$ and $\widehat{\theta}_{r,UCL}$ are the estimated lower and upper 95% limits of the CI's, respectively. We considered $(< .001)$ to represent positive values (statistics and/or estimates) and $(> .001)$ to denote negative values, when they are close to zero. All results of the PR study can be found in Section E.2 the Appendix E.

Tables 139-144 present the results for the mixed SGtBS1 regression model and Tables 145-150 present the results for the mixed SGtBS2 model. For both models, as the sample size increases, we can notice that the estimates of β_0, β_1, ψ_0, ψ_1, γ and σ^2 tend to the correspondent true values and the bias, RMSE and AVRB, decrease. Specifically, when $\nu_1 = 30$ in the mixed SGtBS1 model and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ in the mixed SGtBS2 model, although ν_1 and ν_2 are underestimated, it is clear that the estimates (see Table 3.3.2 and 3.3.2, for example) lead to an equivalence between the proposed models and the correspondent mixed SNBS models. Therefore, we have indications that ν_1 and ν_2 are also reasonably estimated in scenarios that induce a normal shape behavior.

In Tables 151-156 and Tables 157-162, the results for the mixed StBS and SSLBS regression models are presented, respectively. For both models, under $\nu = 5$, we can notice that the estimates, for all parameters, tend to the correspondent true values in all scenarios. Also, as the sample size increases, the width of the credibility intervals, bias, RMSE, and AVRB, decrease. Under $\nu = 30$, the estimates of β_0, β_1, ψ_0, ψ_1, γ and σ^2 are close to the respective true values. Concerning ν, although the estimates tend to true value, we can notice that the width of credibility intervals are too large.

Tables 163-168 present the results for the mixed SCNBS regression model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$, we can notice that the estimates of β_0, β_1, ψ_1, γ and σ^2 tend to the correspondent true values, and the width of the credibility intervals, bias, RMSE, and AVRB decrease. However, the estimates of ψ_0 were not so good for both sample sizes. In this case, a larger sample size is required so that the estimates of ψ_0 tend to be close to the respective true value. Based only on the posterior mean, $\widehat{\nu}_1 = .518$ and

$\widehat{\nu}_2 = .585$ (see Table 163), for example, it is not clear that the mixed SCNBS model is suitable to model response variable with heavy-tails. However, when we also consider the estimates of $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, and γ, we can notice that the mixed SCNBS model has a behavior compatible with that of the heavy-tailed model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, we can notice that the estimates of all parameters tend to the correspondent true values, in all scenarios. Also, as the sample size increases, the bias, RMSE and AVRB, decrease.

In general, we can notice that β_0, β_1, ψ_0, ψ_1, γ and σ^2 are well recovered in all models. Concerning $\boldsymbol{\nu}$, specifically in the scenarios of heavy-tailed response variable, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed models are equivalent to mixed SNBS model, the estimates of $\boldsymbol{\nu}$ are biased and the width of the respective credibility intervals are large. However, as the sample size increases, the estimates become more accurate.

5.5.2 Behavior of the residuals

We considered the scenario where $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, and $\gamma = .8$, which induce a strong positively skewed behavior on the conditional distribution of the response variable. We simulated only one set of observations from each model, considering $\nu_1 = 5$ for the mixed SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$ for the mixed SGtBS2, $\nu = 5$ for the mixed StBS, $\nu = 3$ for the mixed SSLBS, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the mixed SCNBS regression model. For each simulated data set we fitted the proposed models. Four plots were built for each situation, including an envelope for the residuals, and they are presented in Figures 166-200 (see Section E.3 of Appendix E).

In general, when the underlying mixed model is the SGtBS1, SGtBS2, StBS, SSLBS or SCNBS, and we fit the mixed SSLBS2 or SNBS models, we can notice some residuals with large absolute values (close to 3), i.e., possible outliers, with some points falling outside the confidence bands. Also, we can see that the residuals present a behavior compatible with a heavy-tailed distribution. However, the mixed SGtBS1, SGtBS2, StBS, SSLBS1 and SCNBS models accommodate well the observations present in the tails, regardless the model used to generate the data. When we fit the true model to the simulated data, we can notice that the residuals present a symmetric behavior, resembling a standard normal distribution, with the most points well within the interval (-2, 2), with no systematic behavior. In conclusion, we can say that the proposed residuals are appropriate to detect model misfit, concerning the generating distribution.

5.5.3 Behavior of the K-L divergence

The scenario considered here are exactly those presented in Section 5.5.2. That is, we fitted the proposed models to each one the five data sets, generated according to the mixed SGtBS1, SGtBS2, StBS, SSLBS1 and SCNBS regression models. All results of the D study can be found in Section E.4 of Appendix E.

In general, we can notice a number of large values for the K-L divergence, when we fit the mixed SSLBS2, SNBS models to the data sets generated from the mixed SGtBS1, SGtBS2, StBS, SSLBS or SCNBS models, indicating that these models do not accommodate, properly, all observations. Furthermore, when the mixed SGtBS1, SGtBS2, StBS and SSLBS1 models are used to generated the data, at least two observations are considered potentially influential by the mixed SCNBS regression model. This does not happen when the data are simulated from the mixed SCNBS model. This indicates that the mixed SCNBS model does not accommodate so well the extreme observations, compared with other models.

5.5.4 Statistics for model comparison

In order to verify the performance of the statistics for model comparison, we conducted a simulation study considering five different scenarios. In the fist, we simulated R=10 replicas of the mixed StBS regression model with $\boldsymbol{\beta} = (-.5, 1)^{\top}$, $\boldsymbol{\psi} = (-1, .5)^{\top}$, $\gamma = .8$, and $\nu = 5$, considering $n = 50$ and we fit all models. The other four scenarios are equivalent to the first, but the replicas were simulated from the mixed SGtBS1, SGtBS2, SSLBS and SCNBS models, considering $\nu_1 = 5$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^{\top} = (5, 15)^{\top}$, $\nu = 3$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^{\top} = (.1, .1)^{\top}$, respectively. The results of the study SMC can be found in Section E.5 of Appendix E. Table 169 presents the average criteria for the five scenarios and Table 170 presents the percentage of times that the correct model was selected. The number of times (in percentage) that the mixed SSLBS model was selected is given by the sum of times that the mixed SSLBS1 or SSLBS2 models were chosen by the criteria.

In Table 169, we can notice that the true underlying model was selected in all almost the replicas by the criteria, when the underlying model is the mixed StBS, SGtBS1 or SGtBS2. Also, notice that when the underlying model is the mixed SCNBS and $n = 500$, all criteria have chosen the correct model. On the other hand, when the mixed SSLBS model is used to generated the data, the EAIC and EBIC criteria chose the mixed SNBS model, regardless sample size. This probably occurred since the estimates of the degrees of freedom of the competing models were not so accurate. In general, we can see that the percentage of times that the correct model is selected increases as the sample size increases (see Table 170).

5.5.5 Posterior predictive checking

The scenario considered here are exactly those presented in Section 5.5.4. That is, we fitted the proposed models to the each one the five data sets, generated according to the mixed SGtBS1, SGtBS2, StBS, SSLBS and SCNBS regression models. The results of the study PPD can be found in Section E.6 of Appendix E.

In Table 171, we can notice that when the underlying model is the mixed SGtBS1, SGtBS2, StBS, SSLBS1 or SCNBS, the Bayesian p-values indicate that the mixed SSLBS2 and SNBS are misfit, as expected. In general, we can say that the Bayesian p-values, together with the other proposed tools, help to choose the best model.

5.6 Real data analysis

The data set considered here is the Framingham cholesterol data. The Framingham study is perhaps one of the most well-known long-term studies to identify the relationship between various risk factors and diseases and to characterize the natural history of chronic circulatory disease processes. The data on various aspects have been, and continue to be, collected every two years on a cohort of individuals. It began in 1948 in Framingham, located 21 miles west of Boston with limited goals of investigating the serum cholesterol, smoking and elevated blood pressure as the risk factors of coronary heart disease. Over the years its goal has been greatly expanded to aid in understanding the numerous etiological factors of various diseases. The data set used in this work is the same used by Arellano-Valle et al. (2007). It consists of cholesterol levels of 133 patients measured at the beginning of the study and then every 2 years for 10 years, age at baseline, and gender. This study is regular, complete, balanced with respect to time, and unbalanced with respect to the groups (gender). Its main objective is to characterize the change in cholesterol levels over time, considering the age and gender of the patients.

Some descriptive statistics of this data, including central tendency statistics, standard deviation (SD), coefficient of variation (CV), skewness (CS), and kurtosis (CK), are provided in Table 13. At the beginning of the study, we can notice that the mean cholesterol is similar between men and women. Over time, the mean cholesterol levels increase for both sexes. After 10 years, the mean cholesterol is higher for male than for female. Also, we can see that the same occurs with median cholesterol levels. In general, from Figures 49 and 50 we can notice that there is a positive linear relationship between the cholesterol levels and time regardless gender. However, this tendency seems to be more pronounced for men. From Figure 48, it is possible to see the presence of some discrepant observations. In addition, it is noted that the variability of cholesterol levels of men and women are similar, except in the last year. From Figures 51 and 52, and the skewness coefficients (see Table 13), we can notice that the empirical distributions of the cholesterol levels are positively skewed. Figure 53 presents the individual and mean longitudinal profiles for 73 women (left panel) and 60 men (right panel). It suggests that cholesterol increases over time for most patients but with substantial inter-subject variation.

In conclusion, the descriptive analysis indicates that the cholesterol levels depends on the individual characteristics of the patients, but the linear trend over time is similar among the newborns.

Table 13 – Descriptive statistics for the cholesterol levels.

Gender	Year	Mean	SD	CV	Min.	Median	Max.	CS	CK	n
Female	0	221.140	43.480	19.660	134	215.000	340	.760	3.250	73
	2	228.330	42.830	18.760	159	223.000	360	.760	3.370	73
	4	231.450	42.630	18.420	154	220.000	380	1.000	4.350	73
	6	237.080	39.020	16.460	164	230.000	373	.880	4.200	73
	8	236.070	46.670	19.770	165	231.000	430	1.360	5.940	73
	10	244.160	34.040	13.940	188	236.000	343	.650	2.950	73
Male	0	219.020	42.150	19.240	133	213.000	317	.370	2.720	60
	2	221.180	40.400	18.260	150	216.000	334	.760	3.180	60
	4	229.770	45.180	19.660	145	229.000	339	.370	2.600	60
	6	241.830	48.050	19.870	144	234.500	403	.600	3.830	60
	8	243.270	47.330	19.460	166	242.500	356	.420	2.590	60
	10	254.800	48.510	19.040	153	253.000	378	.070	2.610	60

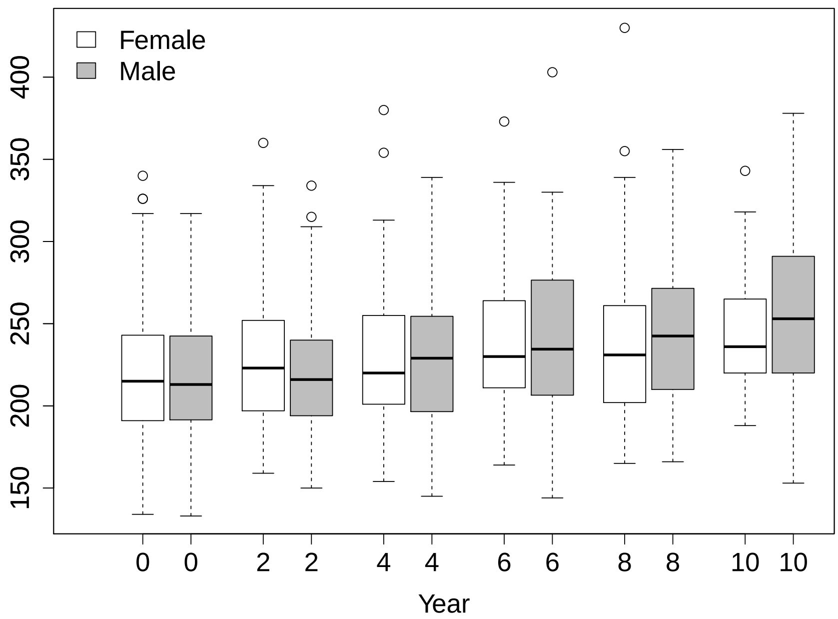

Figure 48 – Boxplot of the cholesterol levels.

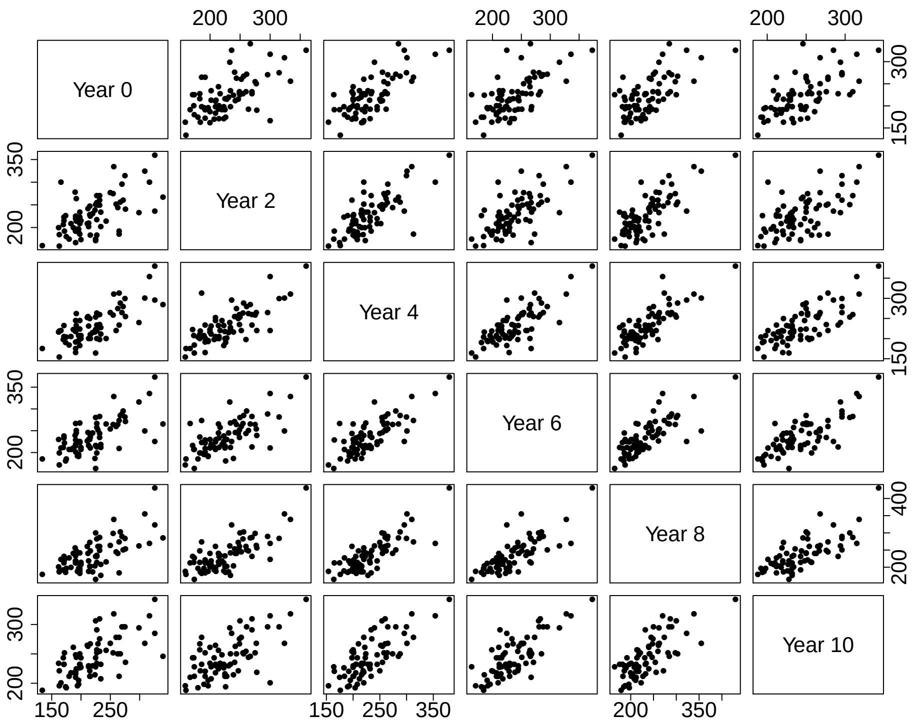

Figure 49 – Scatter plot between the female cholesterol and time.

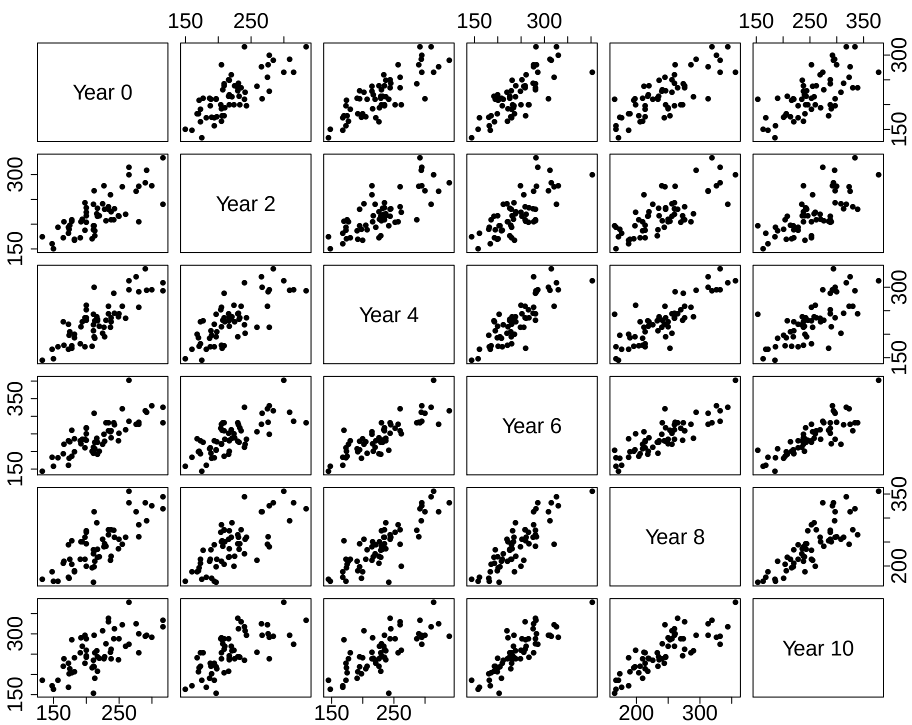

Figure 50 – Scatter plot between the male cholesterol and time.

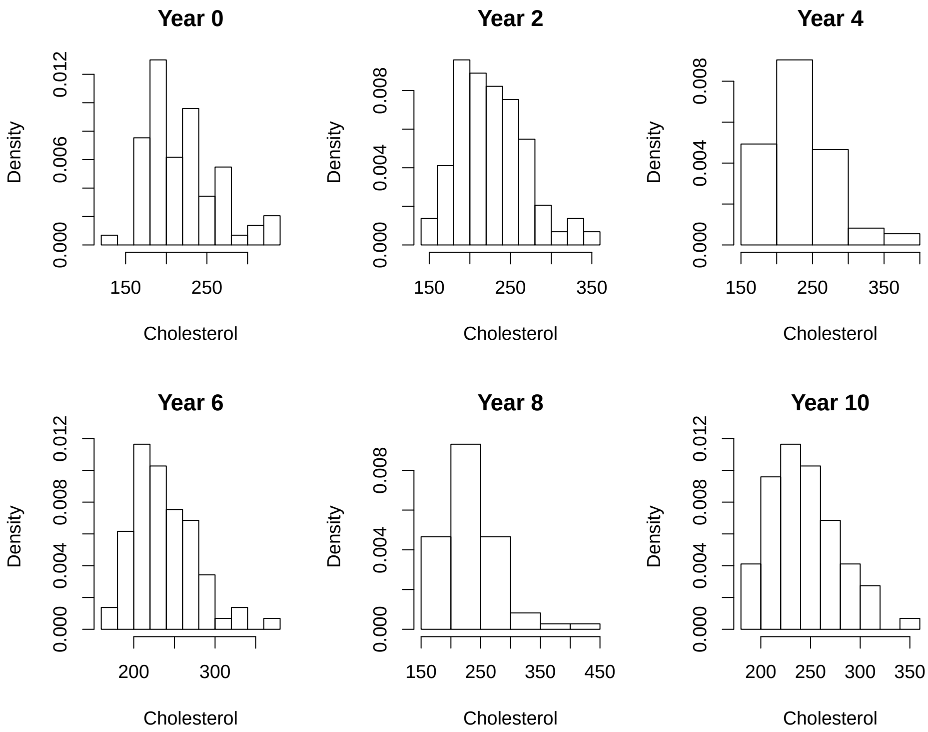

Figure 51 – Distribution of the cholesterol levels for female.

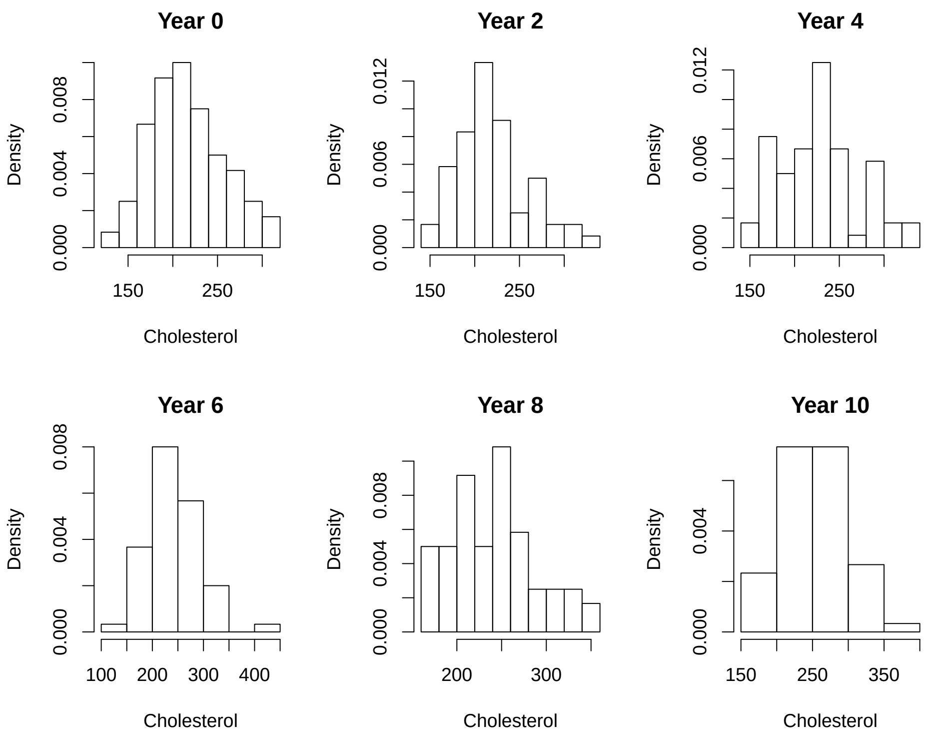

Figure 52 – Distribution of the cholesterol levels for male.

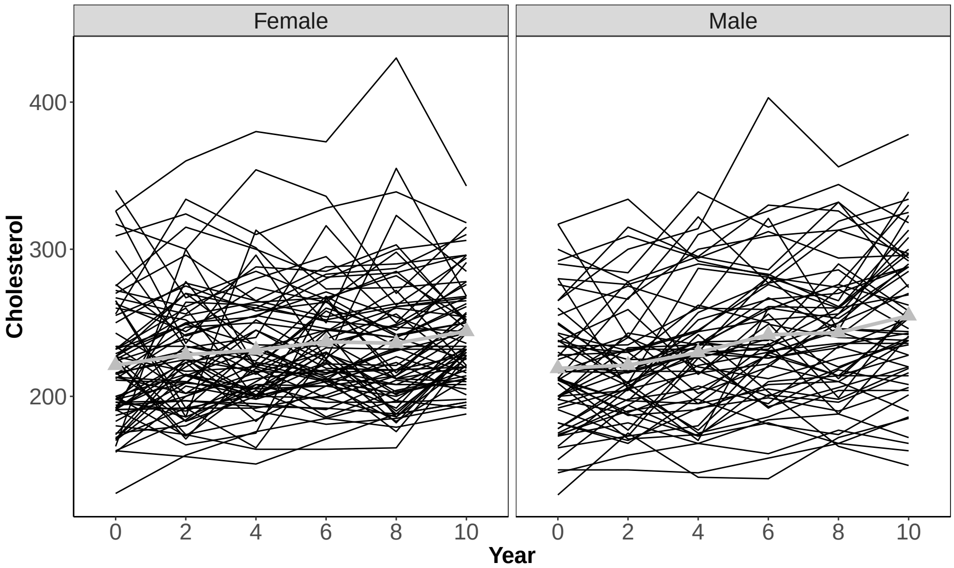

Figure 53 – Individual and mean longitudinal profiles for 73 women (left panel) and 60 men (right panel).

We assumed that $T_{ijk}|\boldsymbol{b}_i \stackrel{\text{ind}}{\sim} \text{CSSBS}(\mu_{ijk}, \phi_{ijk}, \gamma, \boldsymbol{\nu})$, where T_{ijk} is the cholesterol level (divided by 100) of the ith patient, which belongs to the kth group, measured at the jth instant. Based on the descriptive analysis systematic components of the regression models are expressed as

$$\begin{aligned} \log(\mu_{ijk}) &= \beta_0 + \beta_1(x_{1ik} - \bar{x}_{1ik}) + \beta_{2k}\, x_{2ijk} + b_{ik} \\ \log(\phi_{ijk}) &= \psi_0 + \psi_1(z_{1ik} - \bar{z}_{1ik}) + \psi_{2k}\, z_{2ijk}, \end{aligned} \tag{5.20}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_{21}, \beta_{22})^\top$, $\boldsymbol{\psi} = (\psi_0, \psi_1, \psi_{21}, \psi_{22})^\top$ are the regression coefficients, $x_{1ik} \equiv z_{1ik}$ is age at baseline of the ith patient, which belongs to the kth group, $x_{2ijk} \equiv z_{2ijk}$, $x_{2ijk} \equiv z_{2ijk}$ is $(\text{time} - 5)/10$, where time measured in years from baseline.

We fitted all models according to (5.20). Due to numerical instability in the OpenBUGS program, it was not possible to adjust the mixed SGtBS2 model. Figures 60-65 display the residuals analysis for all models. When we fit all models except the mixed SNBS regression model, we can notice that the residual present a behavior compatible with that of heavy-tailed and/or skewed distributions, with many points falling outside the bands. On the other hand, from Figure 65 (d), we can notice that the observations are inside of simulated envelope. The behavior of the residuals reveal that the mixed SNBS regression models fit the data very well, with show any tendency. Figures 60 - 65 present the posterior distributions of the random-effects for the proposed models. In general, we can notice that for the mixed SNBS regression model, the distributions are closer to zero, which indicates a certain advantage over the other models. From Figure 66, we can notice that the mixed SGtBS, StBS, and SNBS, and SCNBS model are equivalent, with respect to the number of observations that appear as potentially influential, with a slight advantage to the mixed SNBS model. From Table 14, we can notice that the mixed SNBS model was chosen by all criteria, indicating that the model provides a good fit to the data.

Table 15 presents the Bayesian estimates, that is, the posterior expectations (PE), posterior standard deviations (PSD) and the 95% equi-tailed credibility intervals for the mixed SSLBS1 regression model. The results of the other models, presented in Tables 172 - 176, can be found in Section E.6 of Appendix E. We can notice that $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_{21}, \beta_{22})^\top$, ψ_0, ψ_1, ψ_{22}, γ, and σ^2 are different from zero, once zero does not belong the correspondent credibility intervals. Furthermore, in Figure 67, we can notice that the posterior distribution of ψ_{21} is concentrated below $-.5$. In conclusion, we can say that the mean and the dispersion of the cholesterol increases for both sexes. However, this increase is larger for men than for women.

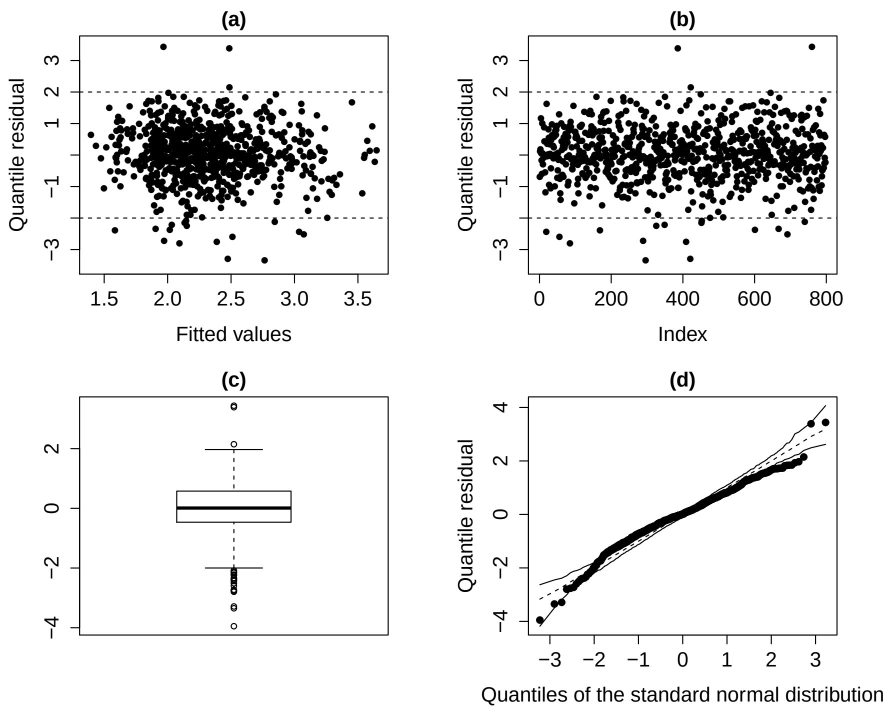

Figure 54 – Residual plots for the mixed SGtBS1 regression model.

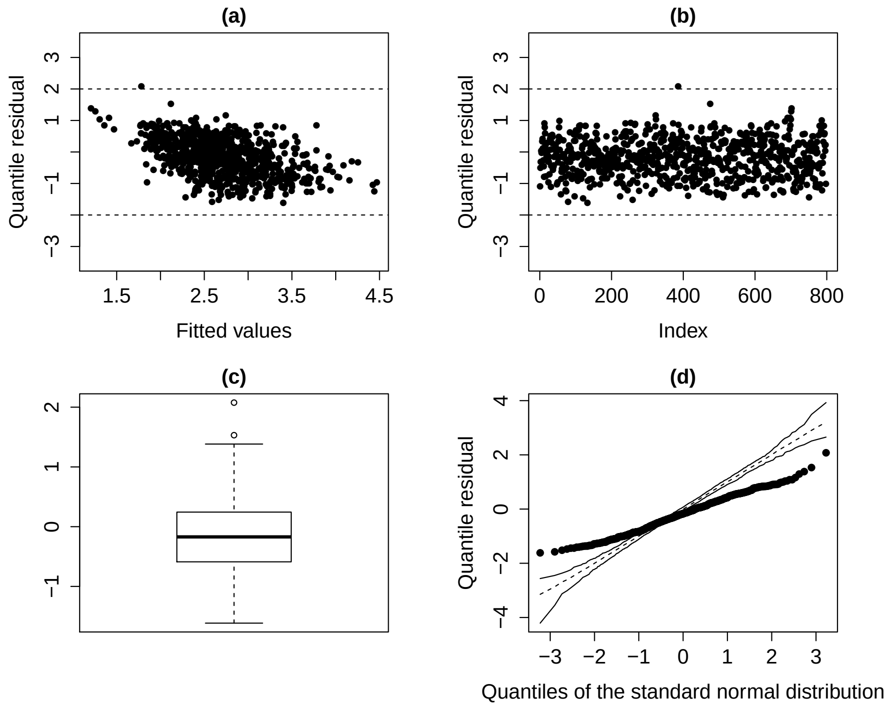

Figure 55 – Residual plots for the mixed StBS regression model.

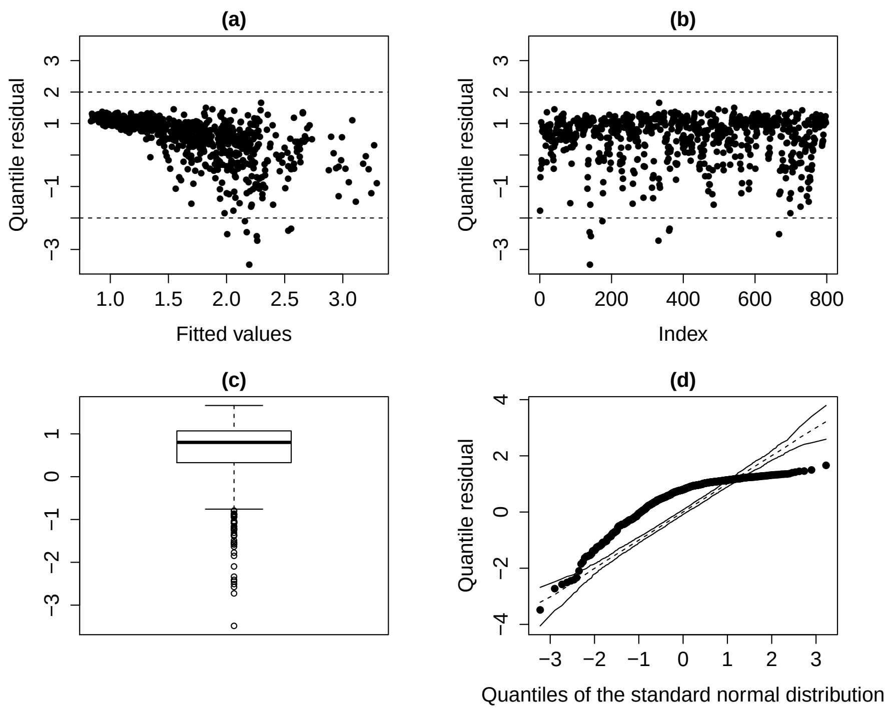

Figure 56 – Residual plots for the mixed SSLBS1 regression model.

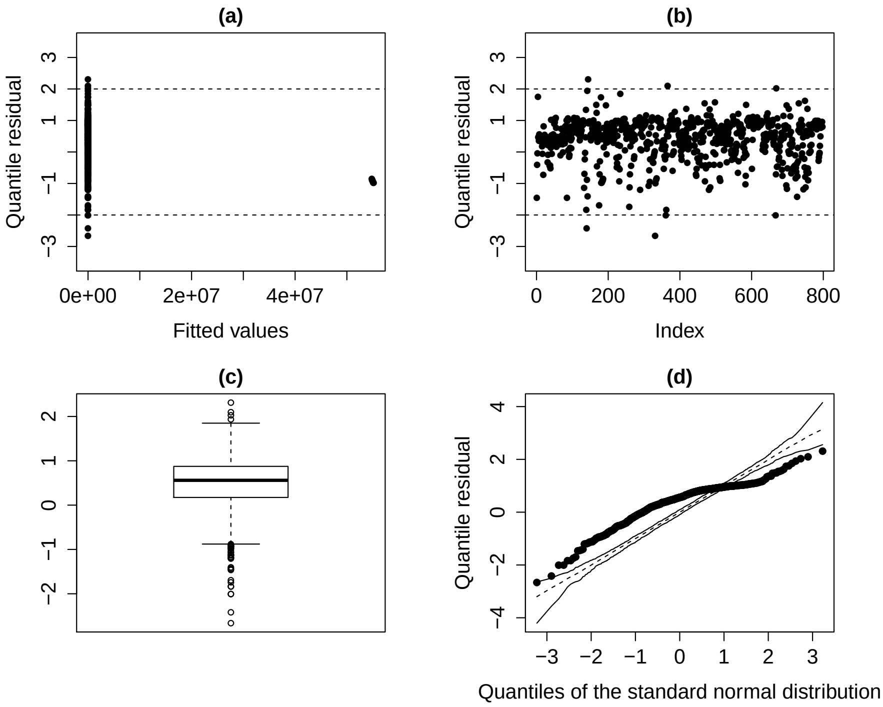

Figure 57 – Residual plots for the mixed SSLBS2 regression model.

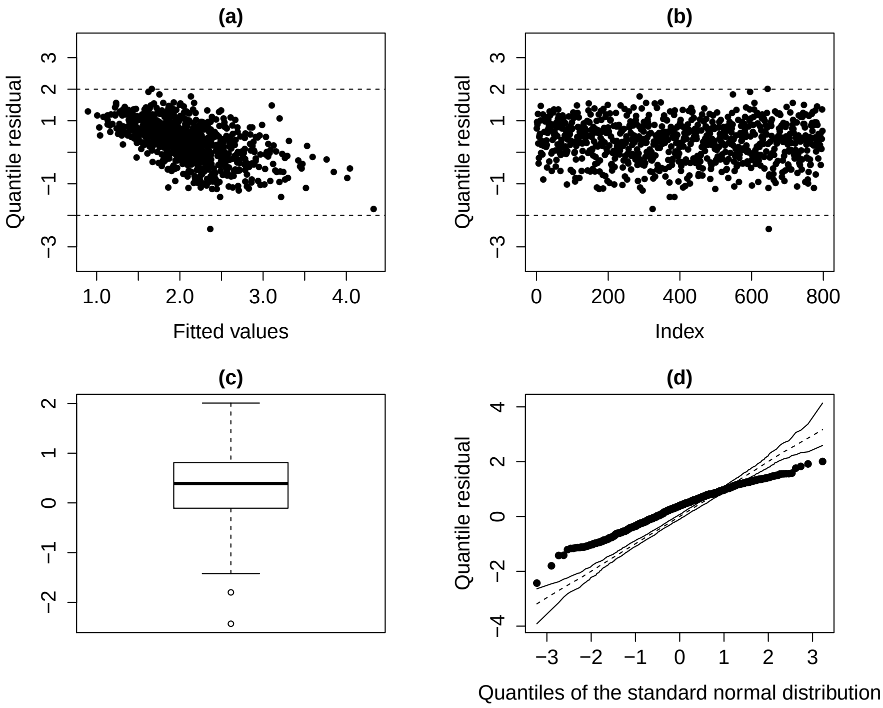

Figure 58 – Residual plots for the mixed SCNBS regression model.

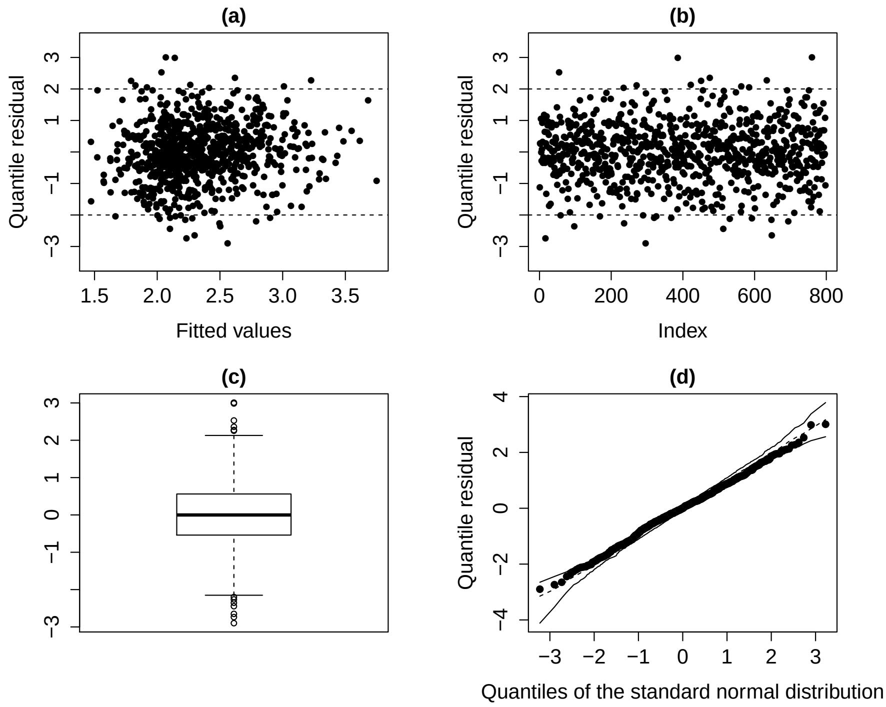

Figure 59 – Residual plots for the mixed SNBS regression model.

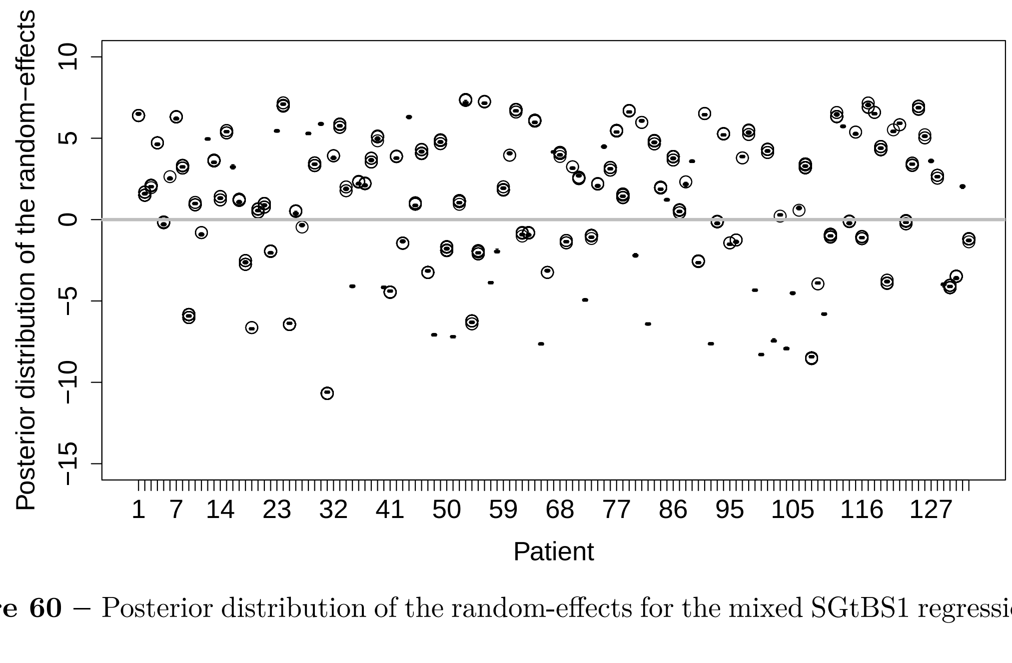

Figure 60 – Posterior distribution of the random-effects for the mixed SGtBS1 regression model.

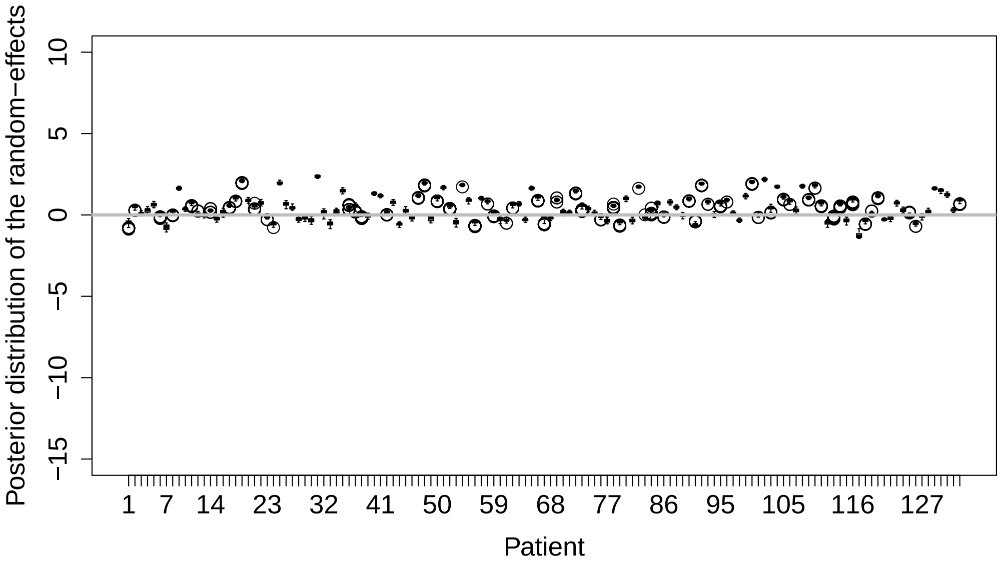

Figure 61 − Posterior distribution of the random-effects for the mixed StBS regression model.

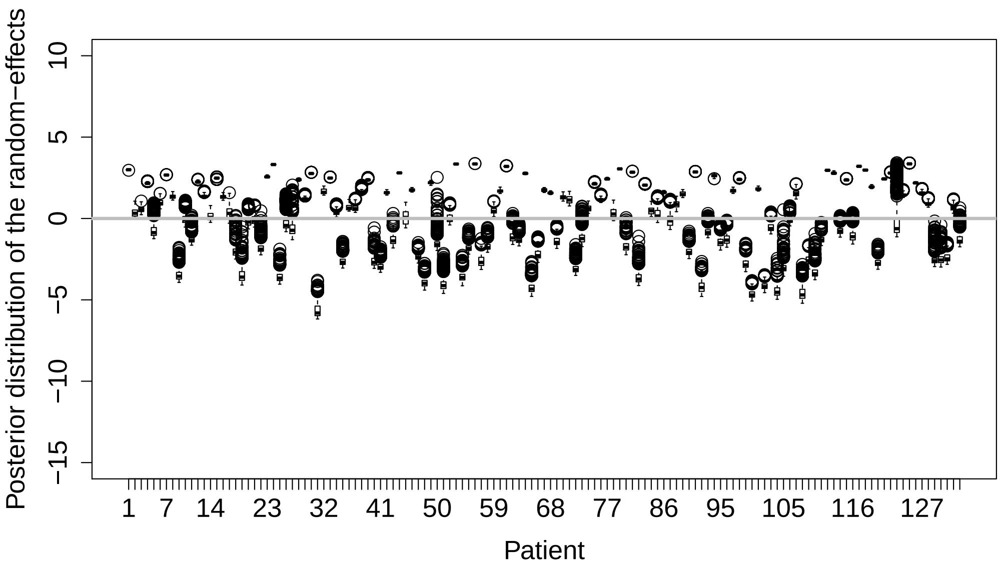

Figure 62 − Posterior distribution of the random-effects for the mixed SSLBS1 regression model.

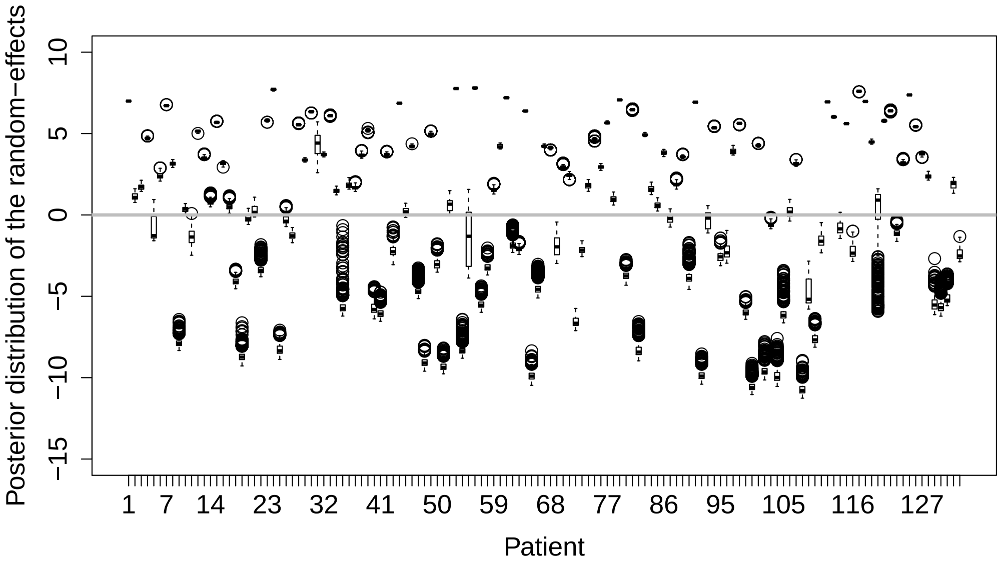

Figure 63 – Posterior distribution of the random-effects for the mixed SSLBS2 regression model.

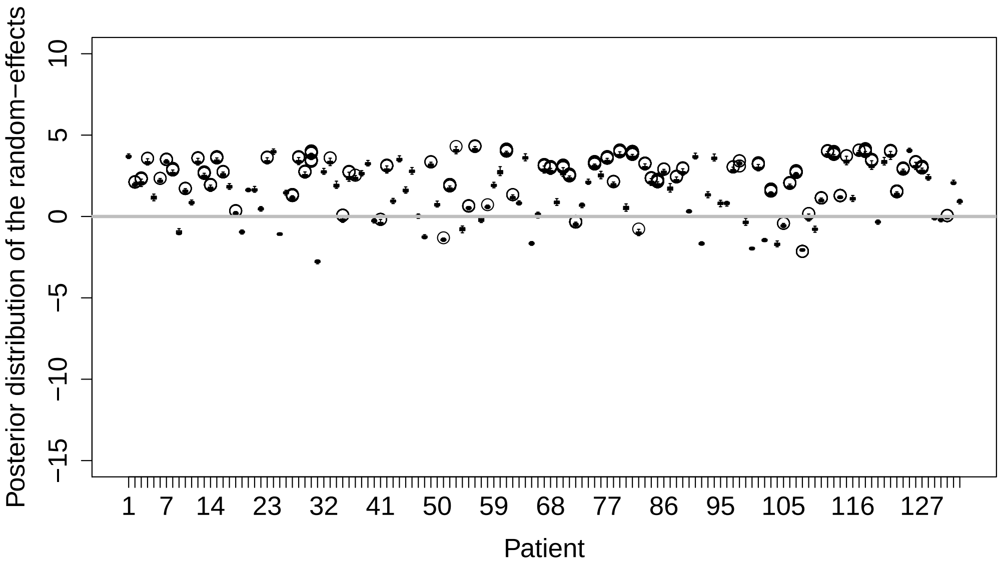

Figure 64 – Posterior distribution of the random-effects for the mixed SCNBS regression model.

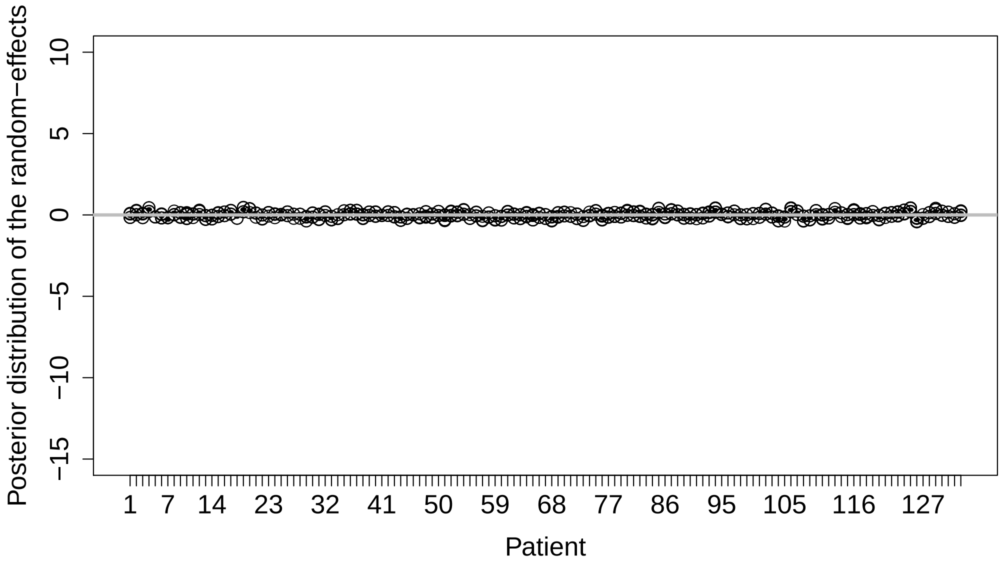

Figure 65 – Posterior distribution of the random-effects for the mixed SNBS regression model.

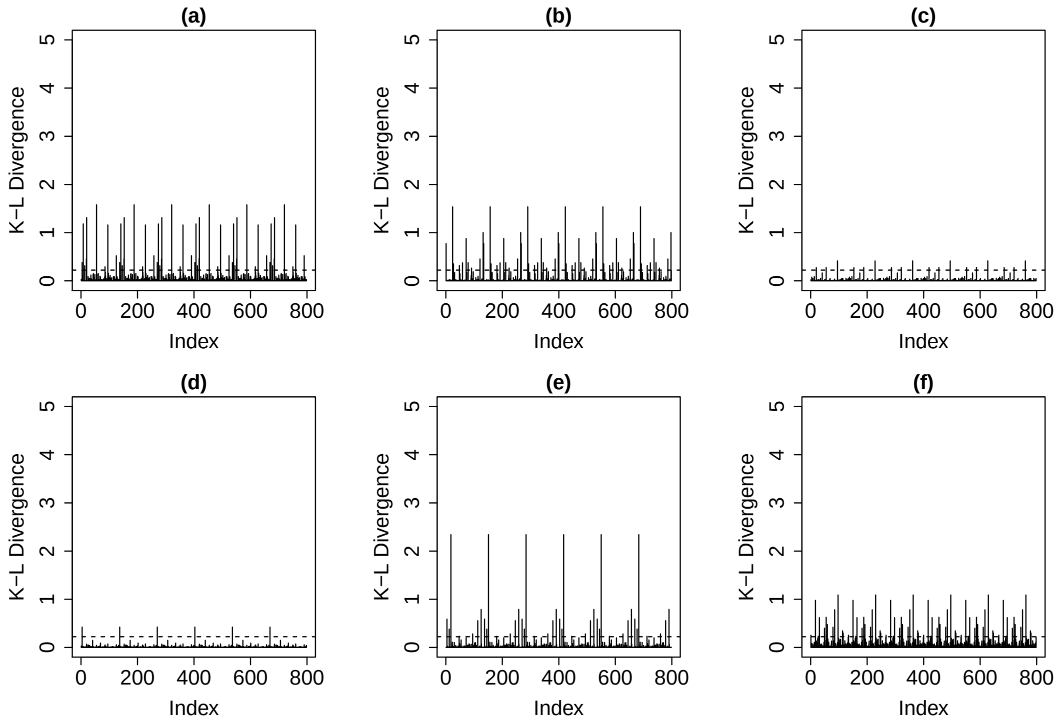

Figure 66 – K-L divergence measure for the mixed models: (a) SGtBS1, (b) StBS, (c) SSLBS1, (d) SSLBS2, (e) SCNBS, (f) SNBS

Table 14 – Model selection criteria.

Model	EAIC	EBIC	DIC	LPML
SGtBS1	-87.260	-40.439	-253.028	-13.520
StBS	1,145.962	1,192.783	2,426.170	-615.209
SSLBS1	2,077.538	2,124.359	4,450.825	-1,045.949
SSLBS2	1,905.215	1,952.037	4,075.670	-956.787
SCNBS	956.2216	1,007.725	2,010.016	-512.428
SNBS	**-255.198**	**-213.059**	**-614.205**	**59.902**

Table 15 – Bayesian estimates for the mixed SNBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.839	.013	[.813; .865]
β_1	.008	.001	[.005; .011]
β_{21}	.092	.013	[.068; .118]
β_{22}	.167	.016	[.137; .200]
ψ_0	-5.422	.072	[-5.543; -5.258]
ψ_1	.018	.008	[.004; .034]
ψ_{21}	-.569	.550	[-1.146; 1.533]
ψ_{22}	1.246	.045	[1.102; 1.303]
γ	-.033	.029	[-.107; -.001]
σ^2	.021	.003	[.016; .027]

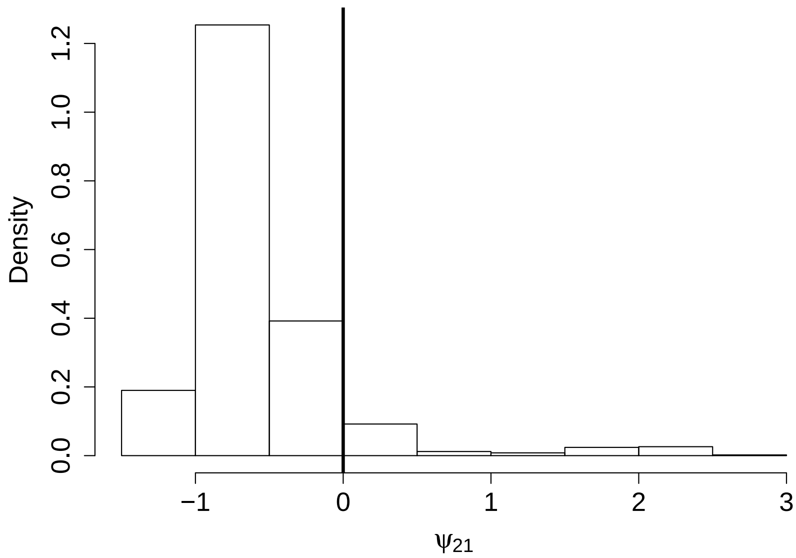

Figure 67 – Posterior distribution of ψ_{21}.

5.7 Concluding Remarks

In this chapter, we extend the fixed-effects CSSBS regression models by including random-effects. Several properties were developed. Our family inherits the properties and advantages in inferential terms of the fixed-effects CSSBS regression models. Furthermore, it accommodates properly both positively or negatively skewed data, presenting or not heavy tails, it allows to analyze data in their original scale, and it allows for modeling both the mean and the dispersion parameter through suitable predictors using appropriate link functions. Under the Bayesian paradigm, we developed parameter estimation, diagnostic measures, and statistics for model comparison based on MCMC algorithms. We conducted simulation studies, in order to evaluate the performance of the proposed methodologies. Finally, we have presented applications to a real data set related to the male and female cholesterol levels, showing the usefulness of the inferential methods developed here.

6 Generalized zero augmented mixed Birnbaum-Saunders regression models

Positive (non-negative) longitudinal data, with presence of zeros are frequently observed in several fields of knowledge such as: Biology, Chemistry, Physics, Medicine, Psychometrics, among others. In this context, considering the reparametrizated BS distribution (Santos-Neto et al., 2012), Batista (2018) developed the zero-adjusted BS (ZABS) regression model with random-effects, and applied them to a dietary assessment study.

In this chapter, our purpose is to extend the fixed-effects ZA-SSBS regression models proposed in Chapter 4 by including random-effects. We developed a flexible family of mixed regression models for modeling zero-augmented positive data, named mixed ZA-SSBS regression models, which generalizes the model proposed by Batista (2018). One of the main advantages of our family is the possibility of modeling data in the original scale, this thus discarding the need of either using transformation of the data or being forced to use an inappropriate model. Also, the proposed models allow for modeling the mean, the dispersion parameter, and the probability of a point mass at zero through suitable predictors using appropriate link functions. Furthermore, the mixed ZA-SSBS models accommodate properly both positively or negatively skewed data, presenting or not heavy tails. Under the Bayesian paradigm, we developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking based on MCMC algorithms. We conducted simulation studies, whose results will be inserted, in order to evaluate the performance of the proposed methodologies, including the classes of models, the estimation methods, the diagnostic measures and the statistics for model comparison. Finally, we have presented applications to a real data set related to the bilirubin concentration, showing the usefulness of the inferential methods developed here, and the advantages of the proposed model under the fixed-effects regression models.

6.1 Introduction

6.1.1 General model

Let $\mathbf{T}_1, \ldots \mathbf{T}_n$ be n independent continuous random vectors with $\boldsymbol{t}_i = (t_{i1}, \ldots, t_{ik_i})^\top$ being the response vector for ith sample unit with element $t_{ij} \in \mathbb{R}$, $j = 1, \ldots, k_i$. Let $\boldsymbol{p}_i = (p_{i1}, \ldots, p_{ik_i})^\top$, $\boldsymbol{\mu}_i = (\mu_{i1}, \ldots, \mu_{ik_i})^\top$ and $\boldsymbol{\phi}_i = (\phi_{i1}, \ldots, \phi_{ik_i})^\top$, where p_{ij} is the probability that T_{ij} is equal to zero, μ_{ij} and ϕ_{ij} is the mean and dispersion parameter of T_{ij}, respectively. Suppose that p_{ij}, μ_{ij} and ψ_{ij} satisfy the following functional relations:

$$g_1(\mu_{ij}) = \eta_{ij} = f_1(\boldsymbol{x}_{ij};\boldsymbol{\beta})\,,\;\; g_2(\phi_{ij}) = \varsigma_{ij} = f_2(\boldsymbol{w}_{ij};\boldsymbol{\psi}) \;\text{ and }\; g_3(p_{ij}) = \tau_{ij} = f_3(\boldsymbol{v}_{ij};\boldsymbol{\zeta}),$$

where $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_p)^\top$, $\boldsymbol{\psi} = (\psi_1,\ldots,\psi_q)^\top$, and $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_r)^\top$ are $p\times 1$, $q\times 1$, and $r\times 1$ vectors, respectively, of unknown regression parameters of fixed-effects to be estimated, $\boldsymbol{\eta}_i = (\eta_{i1},\ldots,\eta_{ik_i})^\top$, $\boldsymbol{\varsigma}_i = (\varsigma_{i1},\ldots,\varsigma_{ik_i})^\top$ and $\boldsymbol{\tau}_i = (\tau_{i1},\ldots,\tau_{ik_i})^\top$ are predictors vectors, and $g_s(\cdot;\cdot)$, $s = 1,2,3$ are linear or nonlinear twice continuously differentiable functions in the second argument. Furthermore, $\boldsymbol{x}_{ij} = (x_{ij1},\ldots,x_{ijp})^\top$, $\boldsymbol{w}_{ij} = (w_{ij1},\ldots,w_{ijq})^\top$, and $\boldsymbol{v}_{ij} = (v_{ij1},\ldots,v_{ijr})^\top$ are vectors that contain the values of p, q, and r explanatory variables, respectively. In this model, the link functions $g_j : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $j = 1,2$ are strictly monotone, positive, and at least twice differentiable and $g_3 : (0,1) \longrightarrow \mathbb{R}$ is strictly monotone and and twice differentiable. In this work, we connect p_{ij}, μ_{ij} and ϕ_{ij} to covariates through the linear function as follows

$$\mu_{ij} = \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top\boldsymbol{b}_i\right\}\,,\;\; \phi_{ij} = \exp\left\{\boldsymbol{w}_{ij}^\top\boldsymbol{\psi}\right\} \;\text{ and }\; p_{ij} = \frac{\exp\left\{\boldsymbol{v}_{ij}^\top\boldsymbol{\zeta}\right\}}{\left(1+\exp\left\{\boldsymbol{v}_{ij}^\top\boldsymbol{\zeta}\right\}\right)}, \tag{6.1}$$

where $\boldsymbol{b}_i = (b_{i1},\ldots,b_{is})^\top$ is a random-effects vector of the ith sample unit, which may be, for instance, random intercepts and/or random coefficients, $\boldsymbol{z}_{ij} = (z_{ij1},\ldots,z_{ijs})^\top$ is a vector that contains values of the covariates associated with $\boldsymbol{b}_i$. Also, $\boldsymbol{b}_i|\boldsymbol{\Sigma}_b \sim N(\boldsymbol{0},\boldsymbol{\Sigma}_b)$, where $\boldsymbol{\Sigma}_b \in \mathbb{R}^{s\times s}$ is a matrix that contains the variance components of the model and the intraclass (within experimental unit) covariances.

Given the random-effects, $T_{ij}|\boldsymbol{b}_i,\boldsymbol{\Omega} \overset{\text{ind}}{\sim} \text{ZA-SSBS}(p_{ij},\mu_{ij},\phi_{ij},\gamma,\boldsymbol{\nu})$, $i = 1,\ldots,n$, $j = 1,\ldots,k_i$, where $\boldsymbol{\Omega} = (\boldsymbol{\zeta}^\top,\boldsymbol{\beta}^\top,\boldsymbol{\psi}^\top,\gamma,\boldsymbol{\nu}^\top)^\top$, p_{ij}, μ_{ij} and ϕ_{ij} are defined in Equation (6.1). The hierarchical structure of the mixed ZA-SSBS regression models is given by

$$\begin{aligned} T_{ij}|\boldsymbol{b}_i,\boldsymbol{\Omega} &\sim \text{ZA-SSBS}(p_{ij},\mu_{ij},\phi_{ij},\gamma,\boldsymbol{\nu}) \\ \boldsymbol{b}_i|\boldsymbol{\Sigma}_b &\sim N(\boldsymbol{0},\boldsymbol{\Sigma}_b). \end{aligned} \tag{6.2}$$

Thus,

$$\begin{aligned} \mathbb{E}(T_{ij}|\boldsymbol{b}_i,\boldsymbol{\Omega},\boldsymbol{\Sigma}_b) &= (1-p_{ij})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top\boldsymbol{b}_i\right\} \\ \mathbb{V}(T_{ij}|\boldsymbol{b}_i,\boldsymbol{\Omega},\boldsymbol{\Sigma}_b) &= (1-p_{ij})(c+p_{ij})\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + 2\boldsymbol{z}_{ij}^\top\boldsymbol{b}_i\right\} \end{aligned} \tag{6.3}$$

where $c = \dfrac{2\phi_{ij}}{[1+\phi_{ij}m_2]^2}\left\{m_2 + \dfrac{\phi_{ij}}{2}\left[2m_4 - m_2^2\right]\right\}$ does not depend on $\boldsymbol{b}_i$, ϕ_{ij}, and $m_k = \mathbb{E}(Y^k)$, $k = 2,4$ represents the kth moment of $Y \sim \text{CSSMN}(0,1,\gamma,\boldsymbol{\nu})$. Also, by using results from conditional distributions, we have that

$$\begin{aligned} \mathbb{E}(T_{ij}) &= (1-p_{ij})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}M_b(\boldsymbol{z}_{ij}) \\ \mathbb{V}(T_{ij}) &= (1-p_{ij})\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\left\{(1-p_{ij})\left[M_b(2\boldsymbol{z}_{ij}) - \{M_b(\boldsymbol{z}_{ij})\}^2\right] + (p+c)M_b(2\boldsymbol{z}_{ij})\right\} \\ \text{Cov}(T_{ij},T_{ij'}) &= (1-p_{ij})(1-p_{ij'})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\}\left[M_b(\boldsymbol{z}_{ij}+\boldsymbol{z}_{ij'}) - M_b(\boldsymbol{z}_{ij})M_b(\boldsymbol{z}_{ij'})\right] \end{aligned} \tag{6.4}$$

where $M_b(\mathbf{s}) = \exp\left\{\dfrac{1}{2}\mathbf{s}^\top\Sigma_b\mathbf{s}\right\}$ is the moment generating function of a normally distributed random vector. The proof of these results can be found in Section F.1 of Appendix F.

6.1.2 Random intercepts model

In this work, we assume that the random intercepts are sufficient to capture heterogeneity between individuals. Thus, suppose that $\boldsymbol{b} \sim N(0, \sigma^2 \mathbf{I})$ and $\boldsymbol{z}_{ij}$ has a single entry equal to 1. We may simplify the hierarchical structure presented in Equation (6.2) to

$$\begin{aligned} T_{ij}|b_i, \boldsymbol{\Omega} &\overset{\text{ind}}{\sim} \text{ZA-SSBS}(p_{ij}, \mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}) \\ b_i|\sigma^2 &\overset{\text{iid}}{\sim} N(0, \sigma^2), \end{aligned} \tag{6.5}$$

where

$$\mu_{ij} = \exp\left\{\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + b_i\right\}, \ \phi_{ij} = \exp\left\{\boldsymbol{w}_{ij}^\top \boldsymbol{\psi}\right\} \text{ and } p_{ij} = \frac{\exp\left\{\boldsymbol{v}_{ij}^\top \boldsymbol{\zeta}\right\}}{\left(1 + \exp\left\{\boldsymbol{v}_{ij}^\top \boldsymbol{\zeta}\right\}\right)}, \tag{6.6}$$

and $i = 1, \dots, n$ and $j = 1, \dots, k_i$. Thus, we have that

$$\begin{aligned} \mathbb{E}(T_{ij}|\boldsymbol{b}_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) &= (1 - p_{ij}) \exp\left\{\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + b_i\right\} \\ \mathbb{V}(T_{ij}|\boldsymbol{b}_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) &= c \exp\left\{2\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + 2b_i\right\}, \end{aligned} \tag{6.7}$$

We may simplify the expressions presented in Equation (6.3) to

$$\begin{aligned} \mathbb{E}(T_{ij}) &= (1 - p_{ij}) \exp\left\{\boldsymbol{x}_{ij}^\top \boldsymbol{\beta}\right\} \exp\left\{\sigma^2/2\right\} \\ \mathbb{V}(T_{ij}) &= (1 - p_{ij}) \exp\left\{2\boldsymbol{x}_{ij}^\top \boldsymbol{\beta}\right\} \left\{(1 - p_{ij}) \exp\left\{\sigma^2\right\} \left[(1 + c + p_{ij}) \left\{\sigma^2\right\} - 1\right]\right\} \\ \text{Cov}(T_{ij}, T_{ij'}) &= (1 - p_{ij})(1 - p_{ij'}) \exp\left\{\boldsymbol{x}_{ij}^\top \boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top \boldsymbol{\beta}\right\} \left[\exp\left\{\sigma^2\right\} \left(\exp\left\{\sigma^2\right\} - 1\right)\right]. \end{aligned}$$

6.2 Bayesian inference

In this section, we present the Bayesian inference for the mixed ZA-SSBS models. The adopted approach, since the marginal posterior distributions on interest can not be analytically obtained, relies on the MCMC algorithms to obtain numerical approximations for those distributions.

6.2.1 Likelihoods

Given the random-effects, we assume that $T_{i1}, \dots, T_{ik_i}$ are independent. Let $\mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_n)^\top$, and $\boldsymbol{b} = (b_1, \dots, b_n)^\top$. The joint likelihood (without integrating out the random-effects b_i) takes on the form

$$L(\boldsymbol{\Omega}|\mathbf{t}, \boldsymbol{b}) = \left[\prod_{i=1}^{n} \prod_{j=1}^{k_i} p_{ij}^{\mathbb{1}\{t_{ij}=0\}} (1 - p_{ij})^{1-\mathbb{1}\{t_{ij}=0\}}\right] \left[\prod_{i=1}^{n} \prod_{j=1}^{k_i} f(t_{ij}|\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_{ij}=0\}}, \tag{6.8}$$

where $\boldsymbol{\Omega} = (\boldsymbol{\zeta}^\top, \boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, p_{ij}, μ_{ij}, and ϕ_{ij} were defined in Equation (6.6) and $f(t_{ij}|\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu})$ is given by Equation (1.9). Furthermore, considering the hierarchical representation presented in Equation (1.18), which is useful for stochastic simulation and parameter estimation (both frequentist and Bayesian), we have that the so-called complete likelihood is given by

$$
\begin{aligned}
L(\boldsymbol{\Omega}|\mathbf{t}_c, \boldsymbol{b}) \;\propto\; & \left[\prod_{i=1}^{n}\prod_{j=1}^{k_i} p_{ij}^{\mathbb{1}\{t_{ij}=0\}}(1-p_{ij})^{1-\mathbb{1}\{t_{ij}=0\}}\right]\left[\prod_{i=1}^{n}\prod_{j=1}^{k_i}\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]\right. \\
& \left.\times\; A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_i)\exp\left\{-h_{ij}^2/2\right\} g(u_{ij}|\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_{ij}=0\}}, \qquad (6.9)
\end{aligned}
$$

where $\mathbf{t}_c = (\mathbf{t}^\top, \mathbf{h}^\top, \mathbf{u}^\top)$, where $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_n)^\top$, $\mathbf{h} = (\mathbf{h}_1, \ldots, \mathbf{h}_n)^\top$ and $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)^\top$. Also, $a_{t_{ij}}$ and $A_{t_{ij}}$ were defined in Equation (5.9), and $\vartheta_{h_{ij}}$ was defined in Equation (1.18).

6.2.2 Prior, posterior and full conditional distributions

To complete the Bayesian specification we need to consider convenient prior distribution. Assuming that the parameters are independent a priori, we consider that the respective joint prior distribution is given by

$$
\pi(\boldsymbol{\theta}, \boldsymbol{b}) = \pi(\boldsymbol{\zeta})\pi(\boldsymbol{\beta})\pi(\boldsymbol{\psi})\pi(\gamma)\pi(\boldsymbol{\nu})\pi(\boldsymbol{b}|\sigma^2)\pi(\sigma^2), \qquad (6.10)
$$

where $\boldsymbol{\theta} = (\boldsymbol{\Omega}, \sigma^2)^\top$. We specify weakly informative prior distributions on the fixed-effects regression parameters and random-effects $\boldsymbol{b}$. Specifically, we chose $\boldsymbol{\zeta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\zeta)$, $\boldsymbol{\beta} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\beta)$, $\boldsymbol{\psi} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\psi)$, and $\gamma \sim U(a, b)$. Also, we consider $\boldsymbol{b} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, where $\sigma^2 \sim \text{gamma}(c, d)$. The prior distribution of $\boldsymbol{\nu}$ depends on the particular distributions adopted for the mixed ZA-SSBS model (more details will be presented below). Combining the likelihood presented in Equation (6.9) and prior distribution presented in Equation (6.10), the joint posterior distribution is given by

$$
\begin{aligned}
\pi(\boldsymbol{\theta}, \mathbf{h}, \mathbf{u}|\mathbf{t}) \;\propto\; & \left\{\left[\prod_{i=1}^{n}\prod_{j=1}^{k_i} p_{ij}^{\mathbb{1}\{t_{ij}=0\}}(1-p_{ij})^{1-\mathbb{1}\{t_{ij}=0\}}\right]\left[\prod_{i=1}^{n}\prod_{j=1}^{k_i}\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]\right.\right. \\
& \left.\left.\times\; A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\exp\left\{-h_{ij}^2/2\right\} g(u_{ij}|\boldsymbol{\nu})\right]^{1-\mathbb{1}\{t_i=0\}}\right\}\pi(\boldsymbol{\theta}),
\end{aligned}
$$

and the full conditional distributions, are given by

$$
\begin{aligned}
\pi(h_{ij}|\boldsymbol{\theta}, t_{ij}, u_{ij}, b_i) \;&\propto\; \left\{\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]\exp\left\{-h_{ij}^2/2\right\}\right\}^{1-\mathbb{1}\{t_{ij}=0\}}, \\
\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) \;&\propto\; \left\{\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij}) g(u_{ij}|\boldsymbol{\nu})\right\}^{1-\mathbb{1}\{t_{ij}=0\}} \qquad (6.11)
\end{aligned}
$$

$$
\begin{aligned}
\pi(\boldsymbol{\zeta}|\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c,\boldsymbol{b}) &\propto \pi(\boldsymbol{\zeta})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} p_{ij}^{\mathbb{1}\{t_{ij}=0\}}(1-p_{ij})^{1-\mathbb{1}\{t_{ij}=0\}}\right\},\\
\pi(\boldsymbol{\beta}|\boldsymbol{\zeta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c,\boldsymbol{b}) &\propto \pi(\boldsymbol{\beta})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i}\phi\left[\vartheta_{h_{ij}}+a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right\}^{1-\mathbb{1}\{t_{ij}=0\}},\\
\pi(\boldsymbol{\psi}|\boldsymbol{\zeta},\boldsymbol{\beta},\gamma,\boldsymbol{\nu},\mathbf{t}_c,\boldsymbol{b}) &\propto \pi(\boldsymbol{\psi})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i}\phi\left[\vartheta_{h_{ij}}+a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right\}^{1-\mathbb{1}\{t_{ij}=0\}},\\
\pi(\gamma|\boldsymbol{\zeta},\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c,\boldsymbol{b}) &\propto \pi(\gamma)\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i}\phi\left[\vartheta_{h_{ij}}+a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right\}^{1-\mathbb{1}\{t_{ij}=0\}},\\
\pi(\boldsymbol{\nu}|\boldsymbol{\zeta},\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\boldsymbol{\nu},\mathbf{t}_c,\boldsymbol{b}) &\propto \pi(\boldsymbol{\nu})\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i} g(u_{ij}|\boldsymbol{\nu})\right\}^{1-\mathbb{1}\{t_{ij}=0\}},\\
\pi(\boldsymbol{b}|\boldsymbol{\theta},\mathbf{t}_c) &\propto \pi(\boldsymbol{b}|\sigma^2)\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i}\phi\left[\vartheta_{h_{ij}}+a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right\}^{1-\mathbb{1}\{t_{ij}=0\}},\\
\pi(\sigma^2|\boldsymbol{\theta},\mathbf{t}_c,\boldsymbol{b}) &\propto \pi(\sigma^2)\pi(\boldsymbol{b}|\sigma^2), \qquad (6.12)
\end{aligned}
$$

The shape of distributions presented in Equations (6.11) and (6.12) depend on the particular distribution adopted for the mixed ZA-SSBS regression models and the adopted prior distribution of $\boldsymbol{\nu}$. The full conditional distributions of u_i and $\boldsymbol{\nu}$, and $\pi(\boldsymbol{\nu})$ for each mixed ZA-SSBS model are presented bellow. We made all implementations considering the `OpenBUGS` software (Spiegelhalter et al., 2014), through the `R` program (R Core Team, 2014), using the package `R2OpenBUGS` (Sturtz et al., 2010). That is, the auxiliary algorithms used to simulate from the full conditional distributions follow the predefined hierarchy implemented in `OpenBUGS`.

6.2.2.1 Prior distribution of $\boldsymbol{\nu}$ and full conditional distributions

1. *The mixed ZA-SGtBS regression model* Here, $\boldsymbol{\nu} = (\nu_1,\nu_2)^\top$. We adopt a trucated exponential prior for ν_m, $m = 1, 2$, this is $\nu_m \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}}(\nu_m)$, such that $\Lambda \sim \mathrm{U}(\Lambda_0,\Lambda_1)$ (Cabral et al., 2012), which leads to the density

   $$\pi(\nu_m) = \frac{1}{\nu_m^2(\Lambda_1-\Lambda_0)}\left[(\Lambda_0\nu_m+1)\exp\{-\Lambda_0\nu_m\}-(\Lambda_1\nu_m+1)\exp\{-\Lambda_1\nu_m\}\right].$$

   The full conditional distributions of u_{ij} and ν_m takes the form

   $$
   \begin{aligned}
   \pi(u_{ij}|\boldsymbol{\theta},t_{ij},h_{ij},b_i) &\propto \Bigg\{\phi\left[\vartheta_{h_{ij}}+a_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\right]A_{t_{ij},\tau_{ij}}(\mu_{ij},\phi_{ij})\,u_{ij}^{\nu_m/2-1}\\
   &\quad\times\exp\left\{-\frac{\nu_m}{2}u_{ij}\right\}\Bigg\}^{1-\mathbb{1}\{t_{ij}=0\}},\\
   \pi(\nu_m|\boldsymbol{\zeta},\boldsymbol{\beta},\boldsymbol{\psi},\gamma,\mathbf{t}_c,\boldsymbol{b},\sigma^2) &\propto \frac{1}{\nu_s^2(\Lambda_1-\Lambda_0)}\Big[(\Lambda_0\nu_m+1)\exp\{-\Lambda_0\nu_m\}-(\Lambda_1\nu_m+1)\\
   &\quad\times\exp\{-\Lambda_1\nu_m\}\Big]\Bigg\{\prod_{i=1}^{n}\prod_{j=1}^{k_i}(\nu_m/2)^{\nu_m/2}\left[\Gamma(\nu_m/2)\right]^{-1}u_{ij}^{\nu_m/2-1}\\
   &\quad\times\exp\left\{-\frac{\nu_m}{2}u_{ij}\right\}\Bigg\}^{1-\mathbb{1}\{t_{ij}=0\}}.
   \end{aligned}
   $$

2. *The mixed ZA-SSLBS regression model* Here, $\boldsymbol{\nu}$ is a scalar parameter. We adopt a truncated gamma distribution for ν, $\nu \sim \text{gamma}(a, b)\mathbb{1}_{\mathbb{A}}(\nu)$, with small positive values of a and b ($b << a$), see Vilca et al. (2016). The full conditional distributions of u_{ij} and ν in (6.11) and (6.12) become

$$\begin{aligned}\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) &\propto \left\{\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij}) u_{ij}^{\nu-1}\right\}^{1-\mathbb{1}\{t_{ij}=0\}}\\ \pi(\nu|\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c, \sigma^2) &\propto \nu^{a-1}\exp\{-b\nu\}\left\{\prod_{i=1}^{n}\prod_{j=1}^{k_i}\nu\, u_{ij}^{\nu-1}\right\}^{1-\mathbb{1}\{t_{ij}=0\}}.\end{aligned}$$

3. *The mixed ZA-SCNBS regression model* The possible states of the "weights" u_{ij} are ν_2 or 1, with $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$, and its density can be expressed as

$$g(u_{ij}|\boldsymbol{\nu}) = \nu_1^{[(1-u_{ij})/(1-\nu_2)]}(1-\nu_1)^{[(u_{ij}-\nu_2)/(1-\nu_2)]}.$$

The full conditional distribution of each u_{ij} can be written as:

$$\begin{aligned}\pi(u_{ij}|\boldsymbol{\theta}, t_{ij}, h_{ij}, b_i) &\propto \Big\{\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\nu_1^{[(1-u_{ij})/(1-\nu_2)]}\\ &\quad\times(1-\nu_1)^{[(u_{ij}-\nu_2)/(1-\nu_2)]}\Big\}^{1-\mathbb{1}\{t_{ij}=0\}}.\end{aligned}$$

Thus, the distribution is proportional to

$$\begin{cases}\left[\nu_1\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right]^{1-\mathbb{1}\{t_{ij}=0\}}, \text{if } u_{ij} = \nu_2\\ \left[(1-\nu_1)\phi\left[\vartheta_{h_{ij}} + a_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right] A_{t_{ij},\tau_{ij}}(\mu_{ij}, \phi_{ij})\right]^{1-\mathbb{1}\{t_{ij}=0\}}, \text{if } u_{ij} = 1\end{cases}$$

In this case, we consider $\nu_m \sim \text{beta}(a_m, b_m)$ (Lachos et al., 2017). The full conditional distribution of ν_m, $m = 1, 2$, is given by

$$\pi(\nu_m|\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \mathbf{t}_c, \boldsymbol{b}, \sigma^2) \propto \nu_m^{a_m + a_{n,\nu_2}-1}(1-\nu_m)^{b_m+b_{n,\nu_2}-1},$$

where $a_{n,\nu_2} = \Big(n - \sum_{i=1}^{n}\sum_{j=1}^{k_i} u_{ij}\Big)/\Big(1-\nu_2\Big)$ and $b_{n,\nu_2} = \Big(\sum_{i=1}^{n}\sum_{j=1}^{k_i} u_{ij} - n\,\nu_2\Big)/\Big(1-\nu_2\Big)$, which is proportional to $\text{beta}(a_m + a_{n,\nu_2}, b_m + b_{n,\nu_2})$ density.

6.3 Model fit assessment and model comparison

6.3.1 Residual analysis

To assess goodness of fit and departure from the assumptions of the mixed ZA-SSBS regression models, we adapted the randomized quantile residual (Dunn and Smyth, 1996) for our models, which is randomized version of Cox and Snell (1968) residual. Let $T_{ij}|b_i, \boldsymbol{\Omega} \sim \text{ZA-SSBS}(p_{ij}, \mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu})$. The proposed residual are given by

$$R_{ij}^q = \begin{cases}\Phi^{-1}[F_{T_{ij}|\boldsymbol{\theta}}(t_{ij})], \text{if } t_{ij} > 0,\\ \Phi^{-1}(u_{ij}), \text{if } t_{ij} = 0,\end{cases}$$

where $F_{T_{ij}|\boldsymbol{\theta}}(t_{ij})$ was defined in Equation (1.10) and u_{ij} is the observed value of $U_{ij} \sim (0, \widehat{p}_{ij})$, where $\widehat{p}_{ij}$ is the Bayesian estimate of p. Furthermore, $\widehat{(\cdot)}$ is the respective Bayesian estimator (i.e., posterior expectation, posterior median or posterior mode). If the model is correctly specified, then R^q_{ij} is approximately normally distributed. The methodology used for the construction of simulated envelopes follows the usual one. That is, since the observed residuals are expected to follow a standard normal distribution, under the well fit of the model, the envelopes are simulated from a standard normal distributions, as described in Atkinson (1985) (see also Vilca et al. (2016)).

6.3.2 Statistics for model comparison

When MCMC algorithms are used to obtain the posterior distributions, some statistics for model comparison can be easily calculated, see Spiegelhalter et al. (2002). To introduce these statistics, we first define $\mathrm{D}(\boldsymbol{\Omega}) = -2\log\left[L(\boldsymbol{\Omega}|\mathbf{t})\right]$, where $\boldsymbol{\Omega} = (\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$ and $L(\boldsymbol{\Omega}|\mathbf{t})$ is the (incomplete) likelihood presented in Equation (6.8). Also, let $\boldsymbol{\Omega}^{(m)}$, $m = 1, ..., M$, be the mth value of the valid simulated MCMC sample, that is, the MCMC sample obtained after discarding the burn-in and a proper thinning (lag) between the values. Finally, let $\overline{\boldsymbol{\Omega}}$ be the vector with the posterior expectation of all parameters, based on the valid MCMC sample, and $\overline{\mathrm{D}(\boldsymbol{\Omega})} = \frac{1}{M}\sum_{m=1}^{M} D(\boldsymbol{\Omega}^{(m)})$. Denote also the deviance by $\mathrm{D}(\overline{\boldsymbol{\Omega}}) = -2\ell(\overline{\boldsymbol{\Omega}}|\mathbf{t})$, and the deviance information criterion (DIC) by $\mathrm{DIC} = \mathrm{D}(\overline{\boldsymbol{\Omega}}) + 2p_D$, where $p_D = \overline{\mathrm{D}(\boldsymbol{\Omega})} - \mathrm{D}(\overline{\boldsymbol{\Omega}})$. The EAIC (posterior expectation of AIC) and EBIC (posterior expectation of BIC) are given, respectively, by $\mathrm{EAIC} = D(\bar{\boldsymbol{\Omega}}) + 2k$ and $\mathrm{EBIC} = \mathrm{D}(\bar{\boldsymbol{\Omega}}) + k\log(n)$, where k is the total number of parameters of the model and n is the number of observations. Finally, the LPML (logarithm of the pseudo-marginal likelihood) is calculated as $\mathrm{LPML} = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\ln(\widehat{\mathrm{CPO}}_{ij})$, where $\widehat{\mathrm{CPO}}_{ij} = \left\{\frac{1}{M}\sum_{m=1}^{M}\left[1/L\left(\boldsymbol{\Omega}^{(m)}|t_{ij}\right)\right]\right\}^{-1}$. The smaller the values of DIC, EAIC, EBIC and deviance, the better the model fit, occurring the opposite with the LPML.

6.3.3 Posterior predictive checking

Under Bayesian perspective, a way to check the goodness of the model fit, is to compare the predictive distribution with the distribution of the observed data. Let $\boldsymbol{t}^{\mathrm{obs}}$ be the observed response and $\boldsymbol{t}^{\mathrm{rep}}$ the replicated response generated from its posterior predictive distribution, which is given by

$$p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{t}^{\mathrm{obs}}) = \int p(\boldsymbol{t}^{\mathrm{rep}}|\boldsymbol{\Omega})\, p(\boldsymbol{\Omega}|\boldsymbol{t}^{\mathrm{obs}})d\boldsymbol{\Omega}. \tag{6.13}$$

where $\boldsymbol{\Omega} = (\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$. Discrepancy measures $D(\boldsymbol{t}, \boldsymbol{\Omega})$ are defined by Gelman et al. (1996) and the posterior distribution of $D(\boldsymbol{t}^{\mathrm{obs}}, \boldsymbol{\Omega})$ is compared to the posterior predictive distribution of $D(\boldsymbol{t}^{\mathrm{rep}}, \boldsymbol{\Omega})$, an substantial differences between them indicating model misfit.

Gelman et al. (2013) suggest several graphs to compare the replicated and the observed data, under the given measure of divergence.

Another measure used to quantify the goodness of fit, is the Bayesian p-value, which for an adopted discrepancy measure, and is defined as

$$\begin{aligned}\mathbb{P}[D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\Omega})] &\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\Omega}|\boldsymbol{t}^{\text{obs}}) \\ &= \int_{D(\boldsymbol{t}^{\text{rep}},\boldsymbol{\Omega})\geqslant D(\boldsymbol{t}^{\text{obs}},\boldsymbol{\Omega})} p(\boldsymbol{t}^{\text{rep}}|\boldsymbol{\Omega})\, p(\boldsymbol{\Omega}|\boldsymbol{t}^{\text{obs}}) d\boldsymbol{t}^{\text{rep}} d\boldsymbol{\Omega}. \end{aligned} \tag{6.14}$$

Due to the difficulty in dealing with Equations (6.13) and (6.14) analytically, Rubin (1984) suggests simulating replicated data sets from the posterior predictive distribution. One draws M simulations $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots, \boldsymbol{\Omega}_M$ from the posterior distribution $p(\boldsymbol{\Omega}|\boldsymbol{t})$ of $\boldsymbol{\Omega}$ and then draws $\boldsymbol{t}^{\text{rep},n}$ from the distribution $p(\boldsymbol{t}|\boldsymbol{\Omega}^n)$ for $n = 1, \ldots, M$. The proportion of the M replications for which $D(\boldsymbol{t}^{\text{rep},n}, \boldsymbol{\Omega}^n)$ exceeds $D(\boldsymbol{t}, \boldsymbol{\Omega}^n)$ provides an estimate of the p-value Bayesian. Extreme values of the Bayesian p-value (less than .05 or greater than .95, depending on the nature of the discrepancy measure) indicate model misfit, see Sinharay et al. (2006). Based on Gelman et al. (1996), the measure of discrepancy used was $D(\mathbf{t}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \{[t_{ij} - \mathbb{E}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)]^2\} / \mathbb{V}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)$, where $\mathbb{E}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)$ and $\mathbb{V}(T_{ij}|b_i, \boldsymbol{\Omega}, \boldsymbol{\Sigma}_b)$ are given by Equation (6.7).

6.3.4 Bayesian case influence diagnostics

Since regression models are sensitive to the underlying model assumptions, it is important to perform sensitivity analysis. Here, we consider the measure of divergence within the Bayesian context proposed by Cho et al. (2009). They developed case deletion influence diagnostics for both joint and marginal posterior distributions based on the Kullback-Leibler (K-L) divergence, and presented a simple way of calculating such influence measure by using MCMC outputs. Let $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ be the K-L divergence between P and $\mathrm{P}_{(-i)}$, where P stands for the posterior distribution of $\boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = (\boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\psi}, \gamma, \boldsymbol{\nu}^\top)^\top$, for the full data and $\mathrm{P}_{(-i)}$ stands for the posterior distribution of $\boldsymbol{\Omega}$ without the ith observation. Then, we have

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = \int \pi(\boldsymbol{\Omega}|\mathbf{t}) \ln\left\{\frac{\pi(\boldsymbol{\Omega}|\boldsymbol{t})}{\pi(\boldsymbol{\Omega}|\boldsymbol{t}_{(-i)})}\right\} d\boldsymbol{\Omega}, \tag{6.15}$$

where $\mathbf{t}_{(-i)}$ corresponds to $\mathbf{t}$ without the ith observation. Also, using the notation introduced earlier in Section 6.3.2, the MCMC estimate of $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})$ is $\widehat{\mathrm{K}}(\mathrm{P}, \mathrm{P}_{(-i)}) = -\ln(\widehat{\mathrm{CPO}}_i) + \frac{1}{M} \times \sum_{m=1}^{M} \ln[L(\boldsymbol{\Omega}^{(m)}|\mathbf{t}_i)]$, where $\widehat{\mathrm{CPO}}_i = \left\{\frac{1}{M}\sum_{m=1}^{M}\left[1/L\left(\boldsymbol{\Omega}^{(m)}|t_i\right)\right]\right\}^{-1}$. As usual, we need to establish a cut-off point, in order to determine whether an observation is influential or not. As pointed by Cho et al. (2009), the calibration of K-L divergence can be done by solving for p_i the equation

$$\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K\left[Ber(1/2), Ber(p_i)\right] = 0.5 \log\left[4p_i(1-p_i)\right], \tag{6.16}$$

where $Ber(p_i)$ is the Bernoulli distribution with success probability p_i. The equality $\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)}) = K[Ber(1/2), Ber(p_i)]$ we have that describing outcomes using $\pi(\boldsymbol{\Omega}|\mathbf{t})$ instead of $\pi(\boldsymbol{\Omega}|\mathbf{t}_{(-i)})$ is compatible with describing an unobserved event as having probability p_i when correct probability is .5 (Cho et al., 2009). Solving Equation (6.16), the calibration of the K-L divergence is

$$p_i = 0.5\left[1 + \sqrt{1 - \exp\left\{-2\mathrm{K}(\mathrm{P}, \mathrm{P}_{(-i)})\right\}}\right].$$

This equation implies that $.5 \leqslant p_i \leqslant 1$. For p_i much greater than .5 implies that the ith observation is influential. In this work, we considered an observation to be influential $p_i \geqslant .8$, as used by Garay et al. (2011) and Chaves et al. (2019a). So, for K-L divergence measure greater than $K\left[Ber(1/2), Ber(.8)\right] \approx .223$, the observation is considered influential (Maioli, 2018).

6.4 Simulation studies

In this section, we presented five simulation studies, namely: parameter recovery (PR), behavior of the residuals (R), behavior of the K-L divergence measure (D), performance of the statistics for model comparison (SMC), and study of the posterior predictive checking (PPC) tools.

We considered different scenarios of interest, which correspond to the combination of the levels of some factors. They (with the respective levels within parenthesis) are: sample size (n) (50, 100), that is, small and large sample sizes, asymmetry parameter γ (-.8, 0, .8), that is high negative skewness, symmetry and high positive skewness, and different values of $\boldsymbol{\nu}$, which induce either a normal shape, or a heavy tails behavior.

The general structure of the model considered is

$$\begin{aligned}
\log(\mu_{ij}) &= \beta_0 + \beta_1 x_{ij} + b_i, i = 1, \ldots, n, j = 1, \ldots, k_i \\
\log(\phi_{ij}) &= \psi_0 + \psi_1 w_{ij} \\
\mathrm{logit}(p_{ij}) &= \zeta_0 + \zeta_1 v_{ij},
\end{aligned}$$

where $b_i \sim \mathrm{N}(0, \sigma^2)$. Also, x_{ij}, w_{ij} and v_{ij}, the explanatory variables, are generated as independent draws from a continuous uniform distribution over the interval $(0, 1)$. We fix $\sigma^2 = 4$, $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, and $\boldsymbol{\zeta} = (-2.5, .8)^\top$ for the mixed ZA-StBS, ZA-SSLBS and ZA-SCNBS regression models and, to overcome the identifiability issue in the mixed ZA-SGtBS model, we fitted two different structures: in the first model, named mixed ZA-SGtBS1, we fixed $\nu_2 = 1$ and in the second model, named mixed ZA-SGtBS2, we fixed $\phi = 1$. Furthermore, we considered $\nu \in \{5, 30\}$ for the mixed ZA-StBS and ZA-SSLBS regression models, $\nu_1 \in \{5, 30\}$ for the mixed ZA-SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (15, 5)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (30, 30)^\top$ for the mixed ZA-SGtBS2 regression model, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the mixed ZA-SCNBS model.

The prior distributions used in all studies were: $\beta_k \sim N(0, 10^4)$, $\psi_k \sim N(0, 10^4)$, $\zeta_k \sim N(0, 10^4)$, $k = 0, 1$, $b_i \sim N(0, \sigma^2)$, where $\sigma^2 \sim \text{gamma}(.01, .01)$, and $\gamma \sim \text{U}(-.99527\,, .99527)$. The first three priors are quite flats, and the fourth prior (Azevedo et al., 2011), is non-informative. For mixed ZA-SGtBS1 regression model we set $\nu_1 \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_1}$, for mixed ZA-SGtBS2 we consider $\nu_s \sim \exp(\Lambda)\mathbb{1}_{\mathbb{A}_i}(\nu_s)$; $s = 1, 2$, such that $\Lambda \sim U(.02, .5)$ (Cabral et al., 2012). In order to guarantee the existence of the first two moments, we consider $\mathbb{A}_1 = (4, \infty)$ and $\mathbb{A}_2 = (2, \infty)$. Similarly, for the mixed ZA-StBS regression model, we set $\nu \sim \exp(\Lambda)\mathbb{1}_{(4,\infty)}(\nu)$, such that $\Lambda \sim U(.02, .5)$. For the mixed ZA-SCNBS regression model, we chose $\nu_1 \sim \text{beta}(2, 2)$ and $\nu_2 \sim \text{beta}(1, 1)$ (Lachos et al., 2017).

From the results related to a convergence study (not presented for the sake of simplicity) we observed that to set a burn-in of 60,000, with a spacing of 40, generating a total of 100,000 values was enough to have valid MCMC samples of 1,000 values for each parameter of the mixed ZA-SGtBS1, ZA-SSLBS1, and ZA-SCNBS regression models. For the mixed ZA-SGtBS2 model, we set a burn-in of 40,000 and a total of 80,000 values were simulated, and samples were collected at a spacing of 40 iterations. For the mixed ZA-StBS model, we set a burn-in of 50,000 and a total of 80,000 values were simulated, and samples were collected at a spacing of 30 iterations. For the mixed ZA-SSLBS2, model we set a burn-in of 80,000 and a total of 120,000 values were simulated, and samples were collected at a spacing of 40 iterations. Finally, for the mixed ZA-SNBS model, we set a burn-in of 20,000 and a total of 60,000 values were simulated, and samples were collected at a spacing of 40 iterations. In general, for all models, the Geweke statistic did not indicate lack of convergence for all any parameter. Furthermore, the values for the Gelman-Rubin statistic and the inspection of the traceplots and autocorrelation plots indicated that the MCMC algorithms converged and the autocorrelations were almost negligible.

For the PR and SMC studies, R = 5 and R=10 replicas (simulated responses from the model) were considered, respectively. For the three others, one replica and only one scenario were used. All the results of the simulation studies can be found in the Sections F.2-F.6 of Appendix F. More specific details concerning each study are presented in the following sections.

6.4.1 Parameter recovery

We calculated the usual statistics to measure the accuracy of the bias, standard deviation (SD), square root of the mean squared error (RMSE), absolute value of the relative bias (AVRB), coverage probability (CP) of the 95% equi-tailed credibility interval and average length (LCI) of the 95% equi-tailed credibility interval. Let θ be the parameter of interest and let $\widehat{\theta}_r$ be some estimate (posterior mean, median or mode) related to the replica r, and $\overline{\overline{\theta}} = (1/R)\sum_{r=1}^{R}\widehat{\theta}_r$. The aforementioned statistics are: bias $= \overline{\overline{\theta}} - \theta$; SD $= \sqrt{(1/R)\sum_{r=1}^{R}(\widehat{\theta}_r - \overline{\overline{\theta}})^2}$, RMSE $= \sqrt{(1/R)\sum_{r=1}^{R}(\theta - \widehat{\theta}_r)^2}$ and AVRB $= |\overline{\overline{\theta}} - \theta|/|\theta|$, CP $= (1/R)\sum_{r=1}^{R}\mathrm{I}(\theta \in [\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}])$ and LCI $= (1/R)\sum_{r=1}^{R}[\widehat{\theta}_{r,LCL}, \widehat{\theta}_{r,UCL}]$, where $\widehat{\theta}_{r,LCL}$ and $\widehat{\theta}_{r,UCL}$ are the estimated lower and upper 95% limits of the CI's, respectively. We considered $(< .001)$ to represent positive values (statistics and/or estimates) and $(> .001)$ to denote negative values, when they are close to zero. All results of the PR study can be found in Section F.2 the Appendix F.

Tables 177-182 present the results for the mixed ZA-SGtBS1 regression model. We can notice that as the sample size increases, we can notice that ζ_0, ζ_1, β_0, β_1, ψ_0, ψ_1, γ, and σ^2 tend to the correspondent true values and bias, RMSE and AVRB decrease. Specifically, when $\nu_1 = 30$ in the mixed ZA-SGtBS1 model, although ν_1 and ν_2 are underestimated, it is clear that their estimates lead to an equivalence between the proposed models and the correspondent mixed ZA-SNBS models. Therefore, we have indications that ν_1 and ν_2 are reasonably estimated in scenarios that induce a normal shape behavior.

In Tables 183-188 and Tables 189-194, the results of the mixed ZA-StBS and ZA-SSLBS regression models are presented, respectively. For both models, under $\nu = 5$, we can notice that the estimates, for all parameters tend to the correspondent true values in all scenarios. Also, as the sample size increases, the width of the credibility intervals, bias, RMSE, and AVRB decrease. Under $\nu = 30$, the estimates for ζ_0, ζ_1, β_0, β_1, ψ_0, ψ_1, γ, and σ^2 are close to the correspondent true values. Concerning ν, although the estimates tend to the true value, we can notice that the width of credibility intervals are too large.

Tables 195-200 present the results for the mixed ZA-SCNBS regression model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.9, .1)^\top$, we can notice that the estimates of ζ_0, ζ_1, β_0, β_1, ψ_1, γ, and σ_2 tend to the correspondent true values, and the width of the credibility intervals, bias, RMSE, and AVRB decrease. However, the estimates of ψ_0 were not so good for both sample sizes. In this case, larger sample size is required so that the estimates of ψ_0 to be close to the respective true value. Based only on the posterior mean, $\widehat{\nu}_1 = .542$ and $\widehat{\nu}_2 = .358$ (see Table 195), for example, it is not clear that the mixed ZA-SCNBS model is suitable to model heavy-tailed data sets. However, when we also consider the estimates of $\boldsymbol{\zeta}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, and γ, we can notice that the mixed ZA-SCNBS distribution has a behavior compatible with that of the heavy-tailed model. Under $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, we can notice that the estimates obtained for all parameters tend to the correspondent true values in all scenarios. Also, as the sample size increases, the bias, RMSE and AVRB decrease.

In general, we can notice that ζ_0, ζ_1, β_0, β_1, ψ_0, ψ_1, γ and σ^2 are well recovered in all models. Concerning $\boldsymbol{\nu}$, specifically in the scenarios of heavy-tailed response variable, we can notice that the estimates are close to the respective true values. On the other hand, when the proposed models are equivalent to mixed ZA-SNBS model, the estimates of $\boldsymbol{\nu}$ are biased and the width of the respective credibility intervals are large. However, as the sample size increases, the estimates become more accurate.

6.4.2 Behavior of the residuals

We considered the scenario where $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, $\boldsymbol{\zeta} = (-2.5, .8)^\top$, and $\gamma = .8$, which induce a strong positively skewed behavior on the conditional distribution of the response variable. We simulated only one set of observations from each model, considering for the mixed ZA-SGtBS1, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$ for the mixed ZA-SGtBS2, $\nu = 5$ for the mixed ZA-StBS, $\nu = 3$ for the mixed ZA-SSLBS, and $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$ for the mixed ZA-SCNBS regression model. For each simulated data set we fitted the proposed models. Four plots were built for each situation, including an envelope for the residuals, and they are presented in Figures 206-229 (see Section F.3 of Appendix F).

In general, when the underlying mixed model is the ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS or ZA-SCNBS, and we fit the mixed ZA-SNBS model, we can notice some residuals with large absolute values (close to 3), i.e., possible outliers, with some points falling outside the confidence bands. Also, we can see that the residuals present a behavior compatible with a heavy-tailed distribution. However, the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZASSLBS1 and ZA-SCNBS models accommodate well the observations present in the tails, regardless the model used to generate the data. When we fit the true model to the simulated data, we can notice that the residuals present a symmetric behavior, resembling a standard normal distribution, with the most points well within the interval (-2, 2), with no systematic behavior. In conclusion, we can say that the proposed residuals are appropriate to detect model misfit, concerning the generating distribution.

6.4.3 Behavior of the K-L divergence

The scenario considered here are exactly those presented in Section 6.4.2. That is, we fitted the proposed models to the each one the five data sets, generated according to the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS1 and ZA-SCNBS regression models. All results of the study D can be found in Section F.4 of Appendix F.

In general, we can notice a number of large values for the K-L divergence, when we fit the mixed ZA-SSLBS2, ZA-SNBS models to the data sets generated from the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS or ZA-SCNBS models, indicating that these models do not accommodate, properly, all observations. Furthermore, when the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-StBS and ZA-SSLBS1 models are used to generated the data, at least two observations are considered potentially influential by the mixed ZA-SCNBS regression model. This does not happen when the data are simulated from the mixed ZA-SCNBS model. This indicates that the mixed ZA-SCNBS model does not accommodate so well the extreme observations, compared with other models.

6.4.4 Statistics for model comparison

In order to asses the performance of the statistics for model comparison, we conducted a simulation study considering five different scenarios. In the fist, we simulated R=10 replicas of the mixed ZA-StBS regression model with $\boldsymbol{\beta} = (-.5, 1)^\top$, $\boldsymbol{\psi} = (-1, .5)^\top$, $\boldsymbol{\zeta} = (-2.5, .8)$, $\gamma = .8$, and $\nu = 5$, considering $n = 50$ and we fit all models, the mixed ZA-StBS, ZA-SGtBS1, ZA-SGtBS2, ZA-SSLBS1, ZA-SSLBS2, ZA-SCNBS and ZA-SNBS. The other four scenarios are equivalent to the first, but the replicas were simulated from the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-SSLBS and ZA-SCNBS, considering $\nu_1 = 5$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (5, 15)^\top$, $\nu = 3$, $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = (.1, .1)^\top$, respectively. The results of the study SMC can be found in Section F.5 of Appendix F. Table 201 presents the average criteria for the five scenarios and Table 202 presents the percentage of times that the correct model was selected. The number of times (in percentage) that the mixed ZA-SSLBS model was selected is given by the sum of times that the mixed ZA-SSLBS1 or ZA-SSLBS2 models were chosen by the criteria.

In Table 201, we can notice that the true underlying model was selected in all almost the replicas by the criteria, when the underlying model is the mixed ZA-StBS, ZA-SGtBS1 or ZA-SGtBS2. Also, notice that when the underlying model is the mixed ZA-SCNBS, three of the four criteria chose the correct model. On the other hand, when the mixed ZA-SSLBS model is used to generated the data, none of the criteria chose the correct model. This probably occurred because the estimate of the degrees of freedom was not so accurate. From Table 202, we can notice that when the underlying model is the mixed ZA-SGtBS1, ZA-StBS and SSLBS, the percentage of times the correct model is selected is low. However, we observed we observe that this percentage increases as the sample size increases.

6.4.5 Posterior predictive checking

The scenario considered here are exactly those presented in Section 6.4.4. That is, we fitted the proposed models to the each one the five data sets, generated according to the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS and ZA-SCNBS regression models. The results of the study PPD can be found in Section F.6 of Appendix F.

In Table 203, we can notice that when the underlying model is the mixed ZA-SGtBS1, ZA-SGtBS2, ZA-StBS, ZA-SSLBS1 or ZA-SCNBS, the Bayesian p-values indicate that the mixed ZA-SNBS are misfit, as expected. In general, we can say that the Bayesian p-values, together with the other proposed tools, help to choose the best model.

6.5 Real data analysis

In this section, we illustrate the proposed methodology by applying the mixed ZA-SSBS regression models to the bilirubin concentration data set, which was presented in details in Section 4.6. We assumed that $T_{ij}|\boldsymbol{b}_i \sim$ ZA-SSBS$(p_{ij}, \mu_{ij}, \phi_{ij}, \gamma_j, \boldsymbol{\nu}_j)$, where T_{ij} is the concentration of bilirubin of the ith newborn measured at the jth instant. Based on the descriptive analysis presented in Section 4.6, the systematic components of the regression models are expressed as

$$\begin{aligned} \log(\mu_{ij}) &= \beta_0 + \beta_1(x_{ij} - 1)\mathbb{1}(j \in \{1,2\}) + \beta_2(x_{ij} - 1)\mathbb{1}(j \in \{3,4,5,6,8,10,12\}) + b_i \\ \log(\phi_{ij}) &= \psi_0 + \psi_1(z_{ij} - 1) \\ \text{logit}(p_{ij}) &= \zeta_0 + \zeta_1(v_{ij} - 1), \end{aligned} \tag{6.17}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1)^\top$, $\boldsymbol{\psi} = (\psi_0, \psi_1)^\top$ and $\boldsymbol{\zeta} = (\zeta_0, \zeta_1)^\top$ are the regression parameters of fixed-effects, b_i is the random-effects related to the ith newborn, $b_i \overset{\text{ind}}{\sim} N(0, \sigma^2)$.

We fitted the mixed ZA-SGtBS1, ZA-StBS, ZA-SSLBS1, ZA-SSLBS2, ZA-SCNBS and ZA-SNBS regression models according to Equation (6.17). Due to numerical instability in the OpenBUGS program, it was not possible to adjust the mixed ZA-SGtBS2 model. Figures 68-73 display the residual analysis for the six models. When we fitted the mixed ZA-SGtBS1 regression model, we can notice that the residual present a behavior compatible with that of a heavy-tailed distribution, with some observations falling outside the bands. Also, we fitted the mixed ZA-SSLBS2 and ZA-SNBS models, we can notice that the residuals present a behavior compatible with that of a heavy-tailed and skewed distribution. On the other hand, from Figures 69 (d), 70(d), and 72(d), the behavior of the residuals reveal that the mixed ZA-StBS, ZA-SSLBS1 and ZA-SCNBS regression models fit the data very well, with show any tendency. Also, we notice that the observations are inside of simulated envelope. Figure 74 present the K-L divergence measure for the six models. Note that the mixed ZA-StBS and ZA-SSLBS1 models are similar, with respect to the number of observations that appear as potentially influential, with a slight advantage to the mixed ZA-StBS model. From the results presented in Table 16 (where the bold values indicate the chosen model by each statistic), we can see that the mixed ZA-SSLBS1 model was selected by EAIC, DIC and LPML, whereas that the mixed ZA-SNBS model was select by EBIC. Figures 75 - 80 present the posterior distribution of the random-effects for the six mixed models. In general, we can notice that for the mixed ZA-SCNBS regression model, the distributions are closer to zero, which indicates a certain advantage over the other models. Figures 81-83 present the observed means and the predicted means (indicated by gray color) by the mixed ZA-StBS, ZA-SSLBS1, and ZA-SCNBS regression models. Specifically, from Figure 83, we can notice that the observed means belong to the predicted credibility intervals in all instants. Thus, we will continue the analysis with the mixed ZA-SCNBS model.

Table 17 presents the Bayesian estimates, that is, the posterior expectations (PE), posterior standard deviations (PSD) and the 95% equi-tailed credibility intervals for the mixed ZA-SCNBS regression model. The results of the other models, presented in Tables 204 - 208, can be found in Section F.2 of Appendix F. In general, we can notice that all parameters were statistically significant. Specifically, from Figure 84 and 85, we can notice that the posterior distributions of ψ_1, ζ_1, and γ_4 are concentrated below zero, and the posterior distribution of γ_2 is concentrated above zero.Thus, we have indications that these parameters are different from zero. In general, we can notice that the logarithm of mean bilirubin concentration increases on the second day. After the third day of life, the logarithm of mean bilirubin concentration decreases. Furthermore, the variability and the percentage of zeros decrease within one day. Finally, from Figures 47 and 83, we noticed that the mixed ZA-SCNBS predict much better the mean bilirubin concentration in all instants than the fixed-effects model.

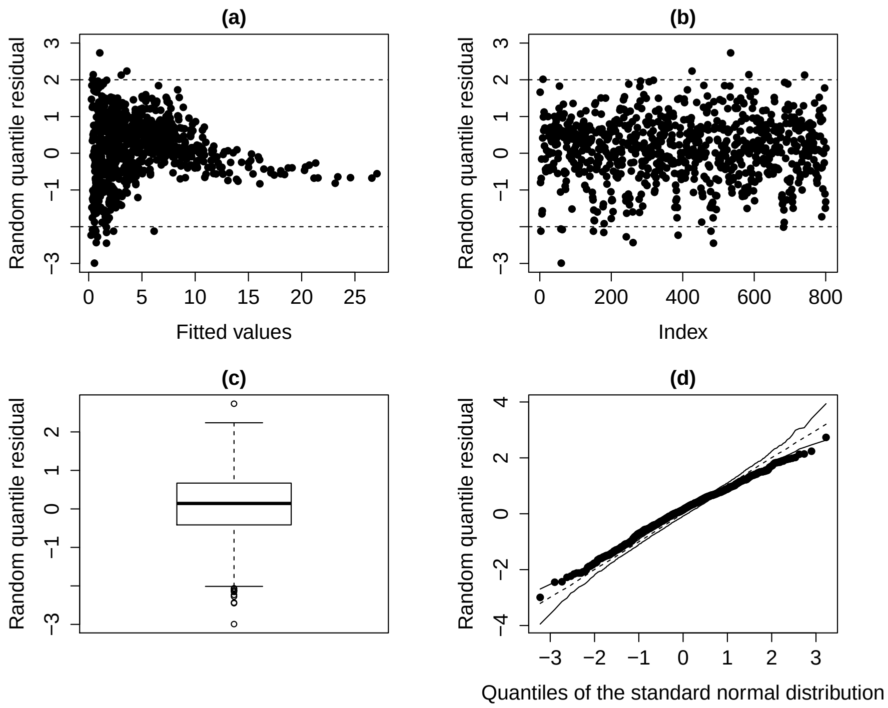

Figure 68 – Residual plots for the mixed ZA-SGtBS1 regression model.

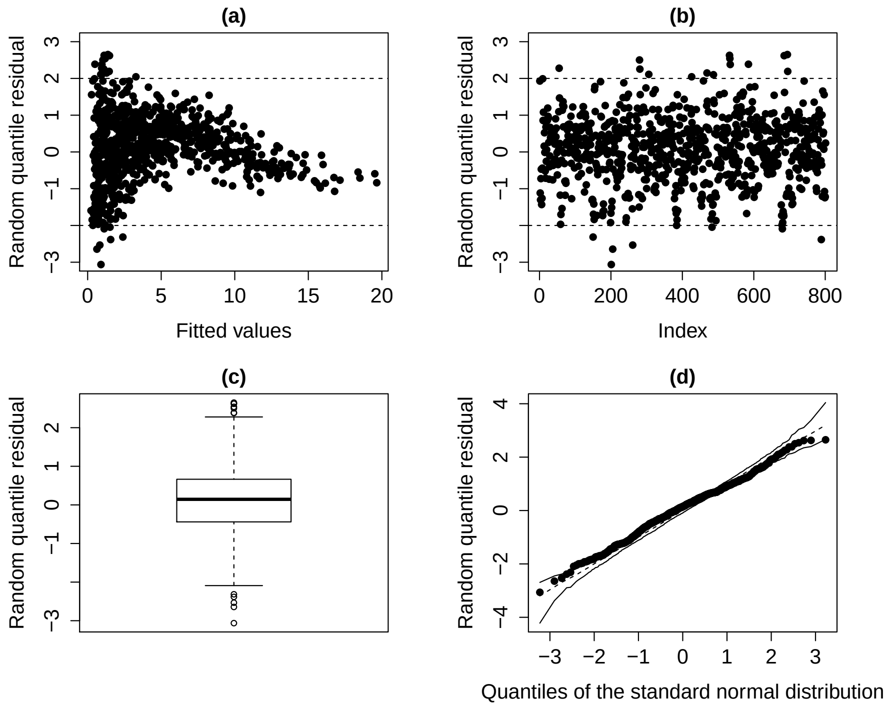

Figure 69 – Residual plots for the mixed ZA-StBS regression model.

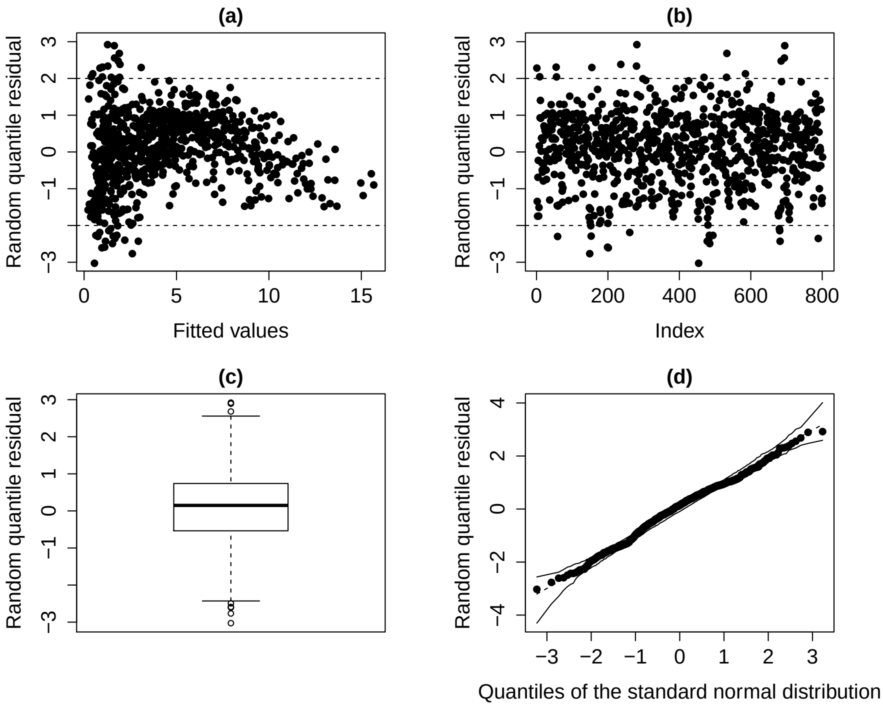

Figure 70 – Residual plots for the mixed ZA-SSLBS1 regression model.

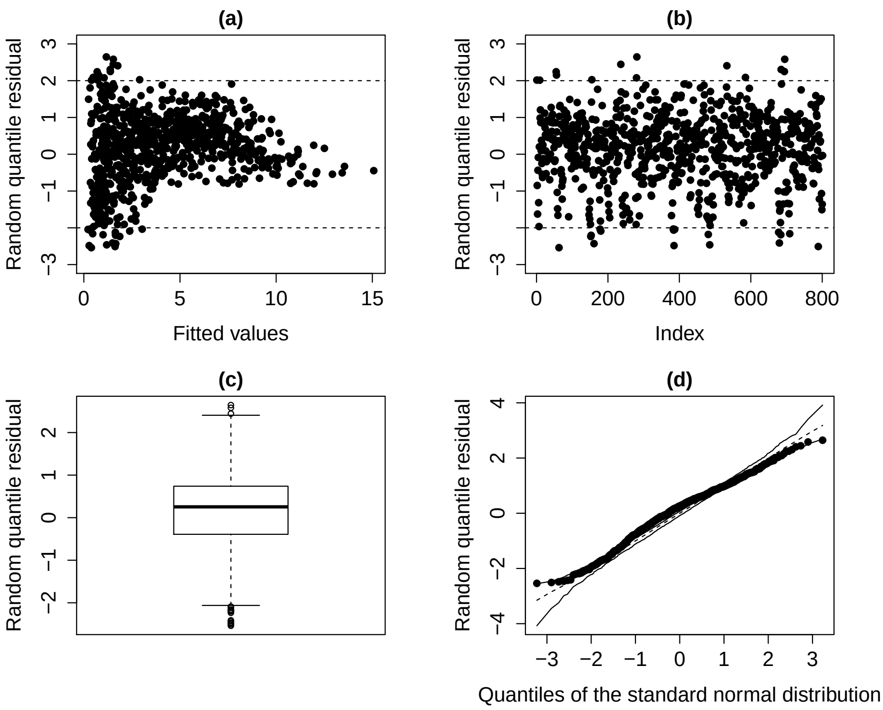

Figure 71 – Residual plots for the mixed ZA-SSLBS2 regression model.

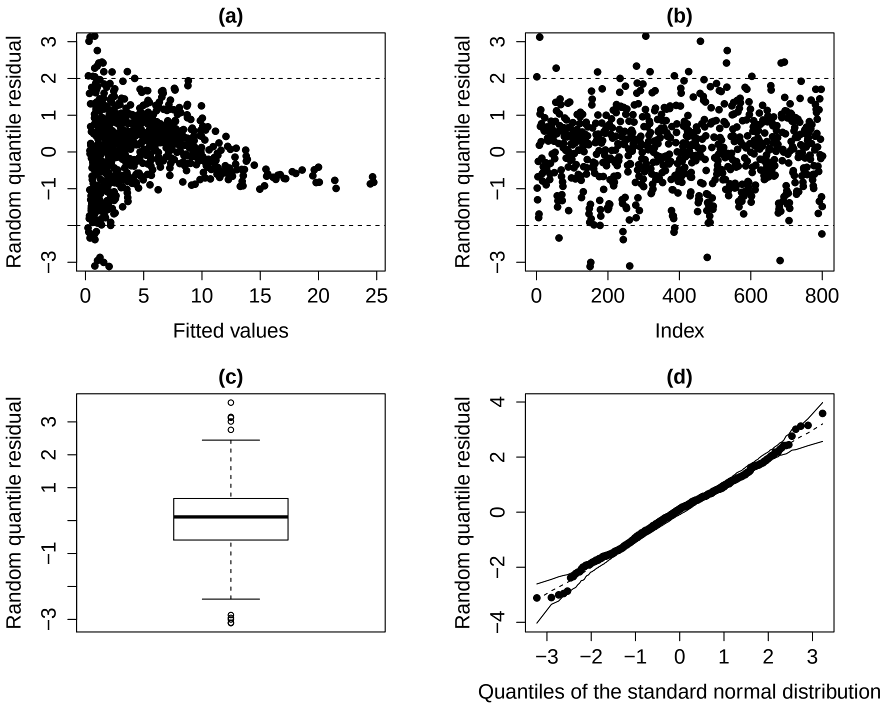

Figure 72 – Residual plots for the mixed ZA-SCNBS regression model.

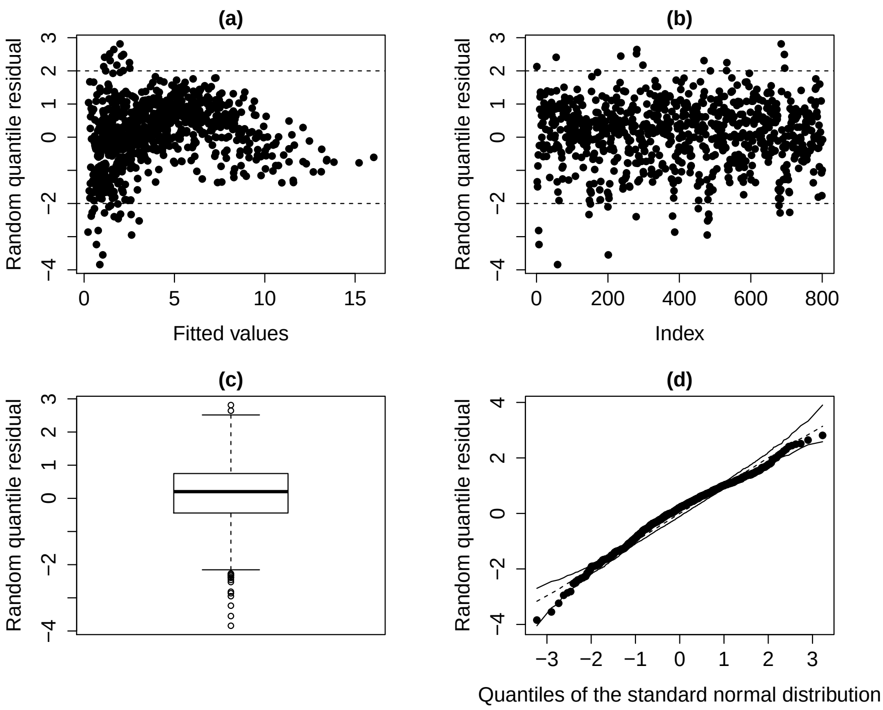

Figure 73 – Residual plots for the mixed ZA-SNBS regression model.

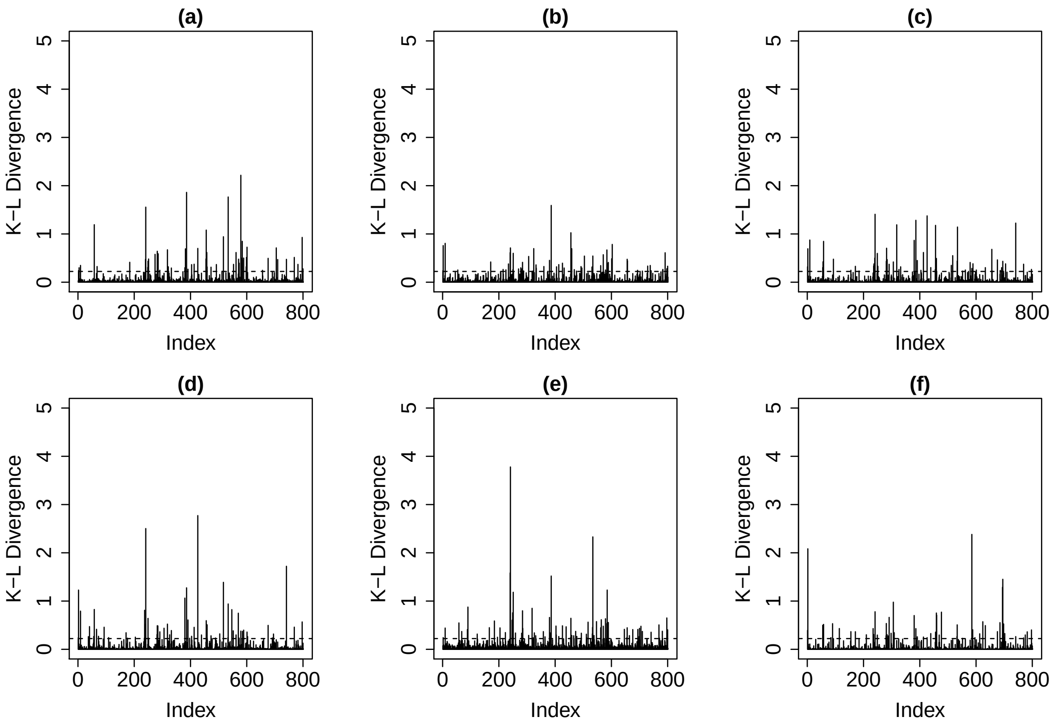

Figure 74 – K-L divergence measure for the mixed models: (a) ZA-SGtBS1, (b) ZA-StBS, (c) ZA-SSLBS1, (d) ZA-SSLBS2, (e) ZA-SCNBS, (f) ZA-SNBS.

Table 16 – Model selection criteria.

Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	3,135.834	3,257.666	9,157.093	-1,601.240
ZA-StBS	3,210.374	3,327.520	9,385.168	-1,638.943
ZA-SSLBS1	**3,118.870**	3,236.016	**9,117.897**	**-1,589.374**
ZA-SSLBS2	3,175.597	3,292.744	9,292.532	-1,622.568
ZA-SCNBS	3,123.196	3,282.516	9,118.264	-1,614.231
ZA-SNBS	3,145.045	**3,220.019**	9,270.133	-1,604.371

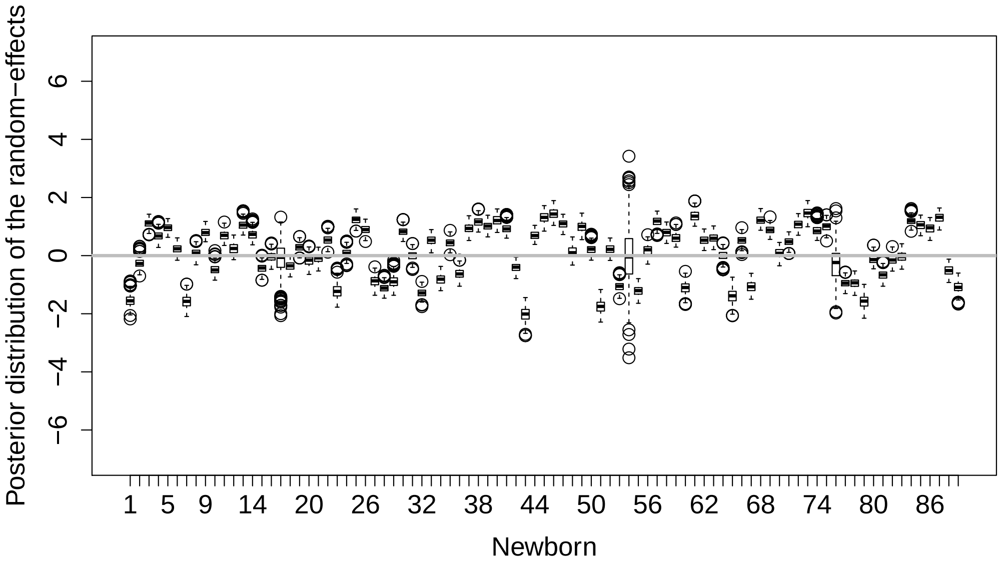

Figure 75 – Posterior distribution of the random-effects for the mixed ZA-SGtBS1 regression model.

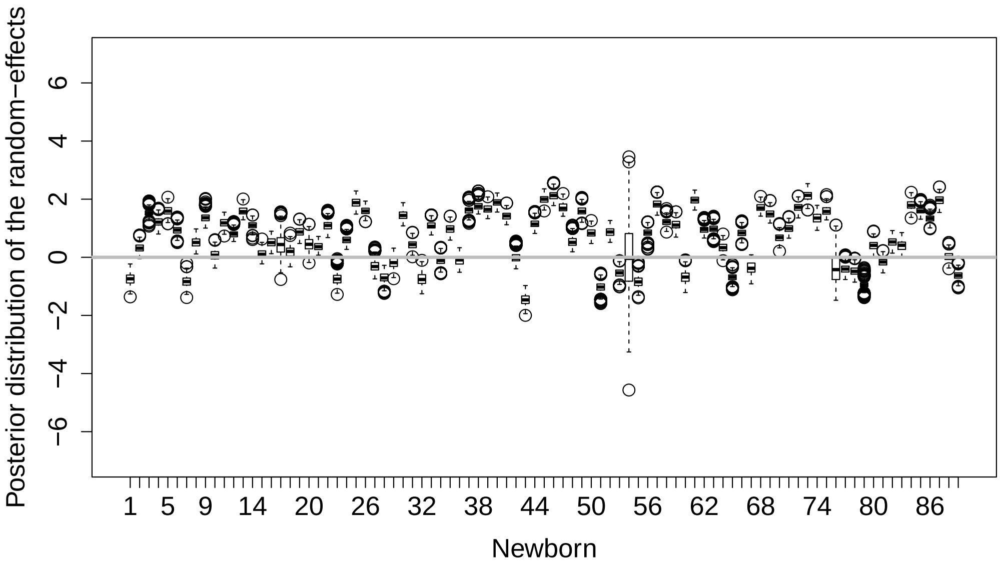

Figure 76 – Posterior distribution of the random-effects for the mixed ZA-StBS regression model.

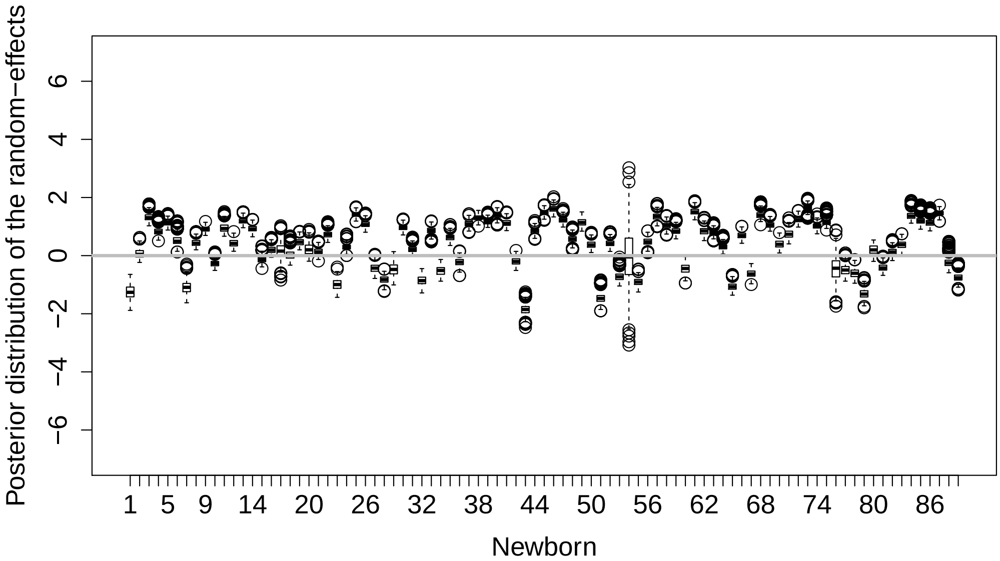

Figure 77 – Posterior distribution of the random-effects for the mixed ZA-SSLBS1 regression model.

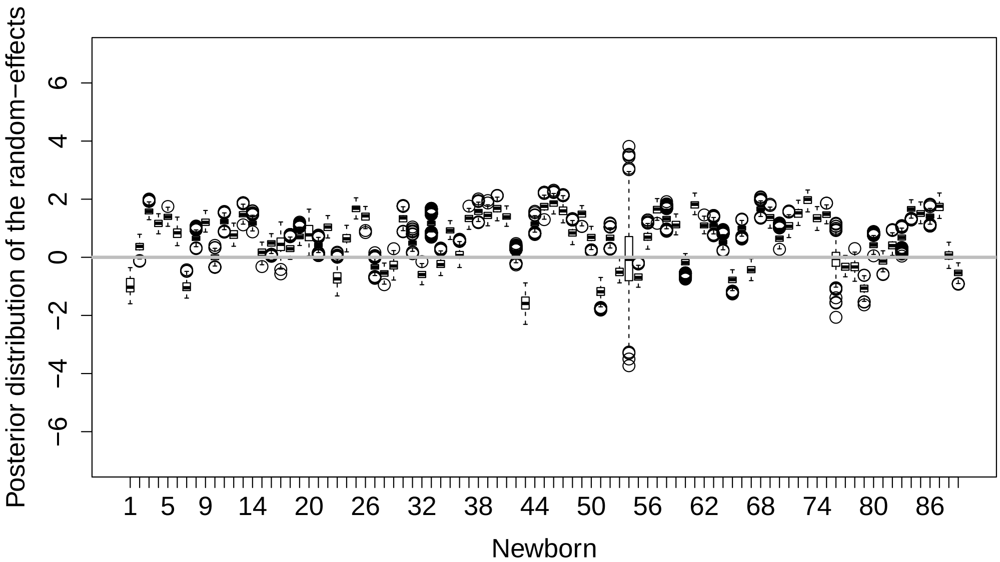

Figure 78 – Posterior distribution of the random-effects for the mixed ZA-SSLBS2 regression model.

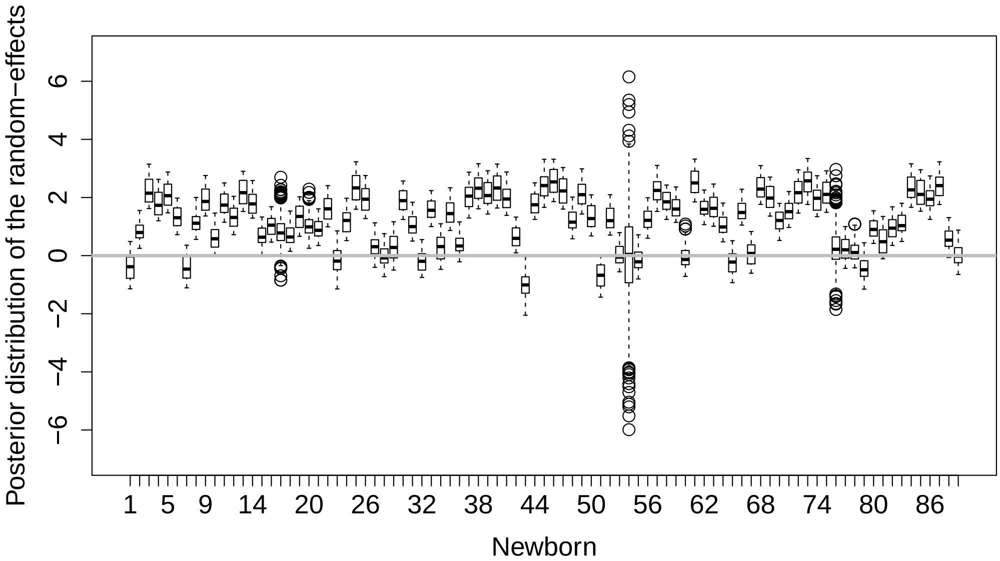

Figure 79 – Posterior distribution of the random-effects for the mixed ZA-SCNBS regression model.

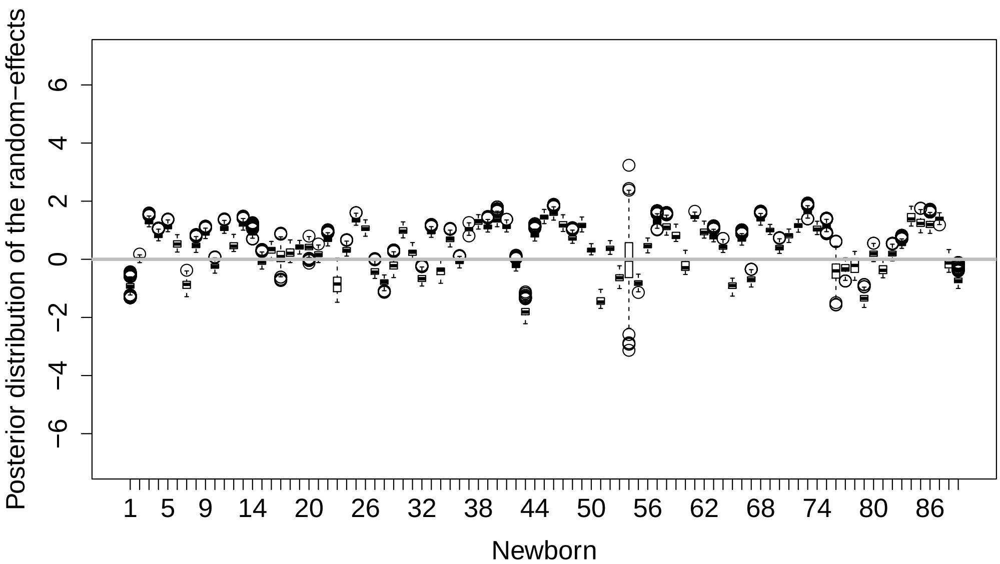

Figure 80 – Posterior distribution of the random-effects for the mixed ZA-SNBS regression model.

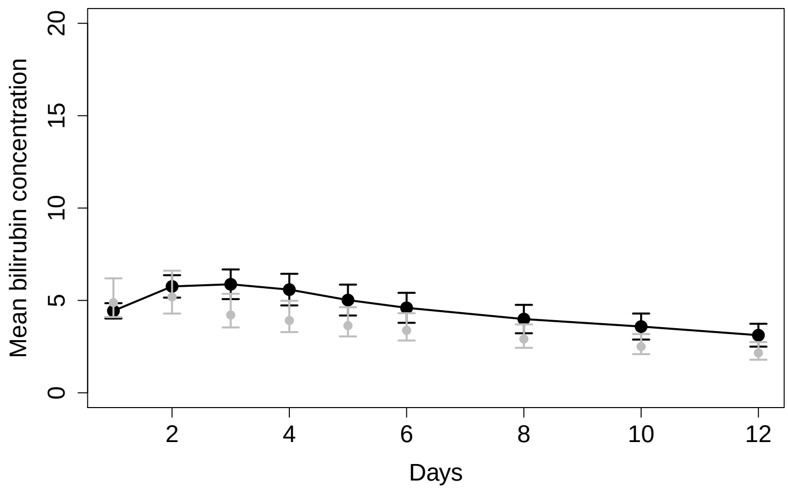

Figure 81 – Observed means and predicted means by the mixed ZA-StBS regression model.

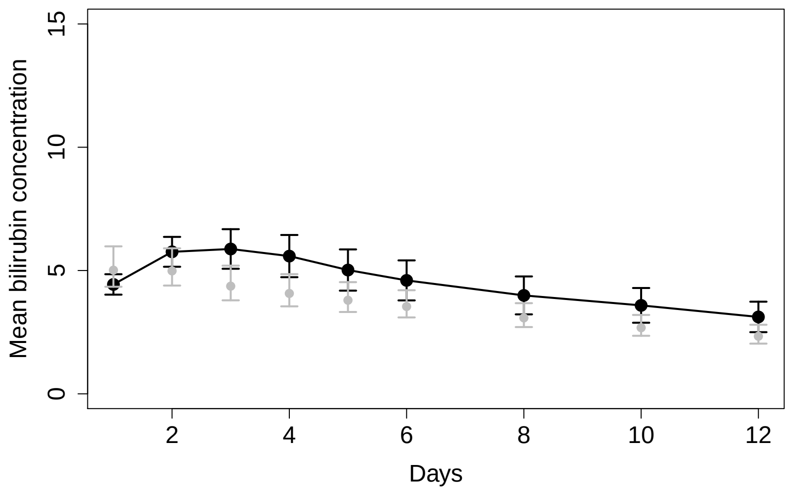

Figure 82 – Observed means and predicted means by the mixed ZA-SSLBS1 regression model.

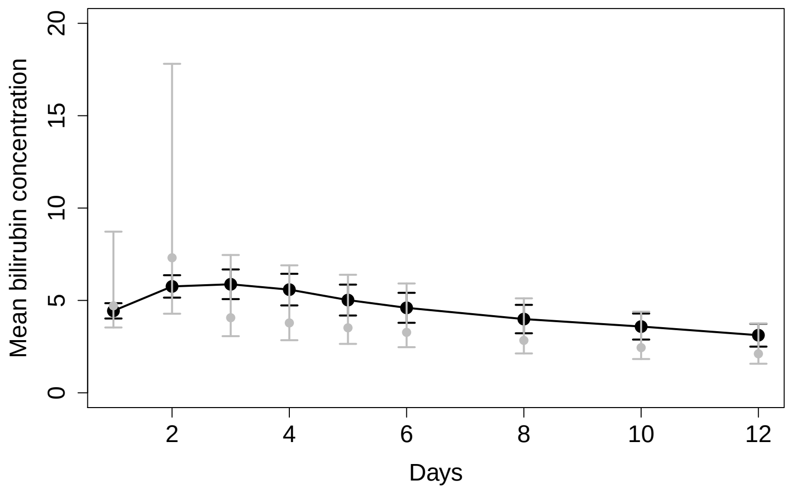

Figure 83 – Observed means and predicted means by the mixed ZA-SCNBS regression model.

Table 17 – Bayesian estimates for the mixed ZA-SCNBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.352	.298	[-.142; .716]
β_1	.450	.229	[.096; .806]
β_2	-.073	.007	[-.088; -.061]
ψ_0	-2.173	.702	[-3.073; -1.078]
ψ_1	-.243	.228	[-.575; .031]
ζ_0	-1.738	.459	[-2.360; -1.065]
ζ_1	-.100	.080	[-.233; .017]
γ_1	-.373	.340	[-.881; .086]
γ_2	.697	.270	[-.074; .949]
γ_3	-.652	.187	[-.964; -.256]
γ_4	-.768	.234	[-.979; .049]
γ_5	-.819	.158	[-.984; -.394]
γ_6	-.791	.200	[-.981; -.231]
γ_7	-.703	.244	[-.959; -.124]
γ_8	-.679	.243	[-.987; -.169]
γ_9	-.876	.074	[-.977; -.691]
ν_{11}	.350	.121	[.177; .513]
ν_{12}	.696	.160	[.369; .957]
ν_{13}	.418	.218	[.099; .862]
ν_{14}	.358	.213	[.051; .808]
ν_{15}	.327	0.175	[.055; .764]
ν_{16}	.288	.188	[.062; .766]

Table 17 (continued).

ν_{17}	.198	.167	[.031; .620]
ν_{18}	.573	.230	[.149; .937]
ν_{19}	.821	.170	[.412; .989]
ν_{21}	.247	.113	[.099; .442]
ν_{22}	.283	.069	[.166; .438]
ν_{23}	.544	.209	[.228; .967]
ν_{24}	.772	.185	[.322; .996]
ν_{25}	.469	.247	[.121; .955]
ν_{26}	.312	.275	[.044; .922]
ν_{27}	.077	.074	[.015; .291]
ν_{28}	.329	.280	[.027; .930]
ν_{29}	.105	.103	[.003; .314]
σ^2	2.485	.948	[1.247; 4.417]

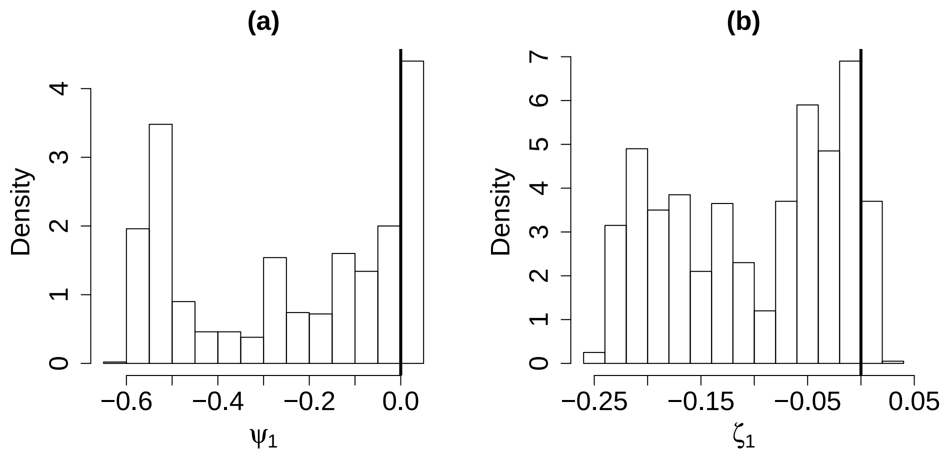

Figure 84 – (a) Posterior distribution of ψ_1 (b) Posterior distribution of ζ_1.

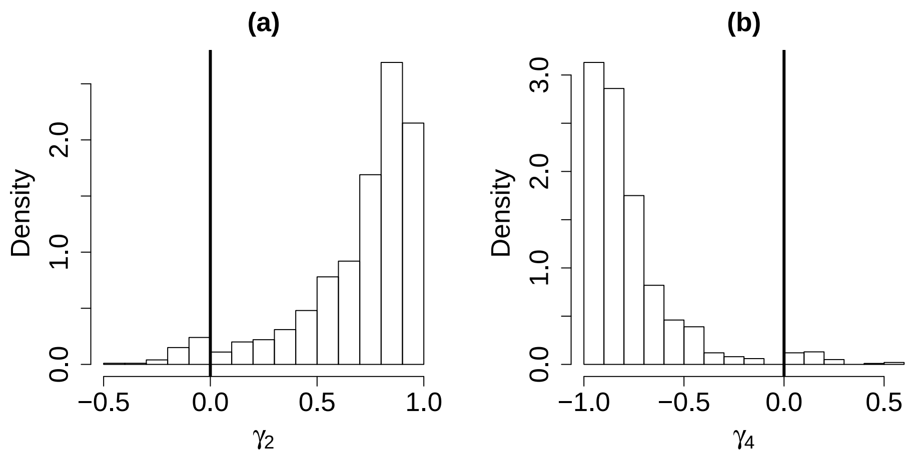

Figure 85 – (a) Posterior distribution of γ_2 (b) Posterior distribution of ψ_4.

6.6 Concluding Remarks

In this chapter, we extend the fixed-effects ZA-SSBS regression models by including random-effects. Several properties were developed. Our family inherits the properties and advantages in inferential terms of the fixed-effects ZA-SSBS regression models. Furthermore, it accommodates properly both positively or negatively skewed data, presenting or not heavy tails, it allows to analyze data in their original scale, and it allows for modeling the mean, the dispersion parameter, and the probability of a point mass at zero. Under the Bayesian paradigm, we developed parameter estimation, diagnostic measures, and statistics for model comparison based on MCMC algorithms. We conducted simulation studies, whose results will be inserted, in order to evaluate the performance of the proposed methodologies. Finally, we have presented applications to a real data set related to the bilirubin concentration, showing the usefulness of the inferential methods developed here. Also, we discuss that the proposed random-effects models are preferable to fixed-effects models.

Conclusions

In this work, we developed a general family of mixed BS regression models, augmented (or not) by zero, for positively or negatively skewed data, presenting or not heavy-tails. Initially, considering the reparameterization of the BS distribution proposed by Santos-Neto et al. (2012), we proposed families of non-augmented and zero-augmented BS distributions. Unlike the usual BS distribution, the reparameterized BS distribution allows us to analyze the data in its original scale, thus avoiding difficulties of interpretation of results and inferential problems. Besides to consider this parameterization, which allows us to write the respective mean on the related density and which can be very useful for regression models, we considered the centred skewed version of the scale-mixture of normals distribution, which facilitates the calculations of the moments of our distributions. Also, we provided empirical evidences that the proposed models have advantages in inferential terms over the models proposed by Balakrishnan et al. (2017). Under Bayesian paradigm, we developed parameter estimation, statistics for model comparison, and posterior predictive checking based on MCMC algorithms. We performed parameter recovery studies considering different scenarios, in order to evaluate the performance of estimation method. In general, the results indicate that the estimates of $\boldsymbol{\nu}$ were close to true values in the scenarios that lead to heavy-tailed distributions. On the other hand, in scenarios that lead to normal shape distributions, the estimates of $\boldsymbol{\nu}$ are biased and the width of credibility intervals are large. However, as sample size increases, the estimates become more accurate. We have presented applications to a real data set, showing that our non-augmented and zero-augmented BS distributions can be much more useful than those found in the literature.

Based on our class of probability models, we developed fixed and random-effects BS regression models. Several of their properties are developed. The proposed regression models inherit the properties of our BS-type distributions. We developed parameter estimation, residual analysis, statistics for model comparison, and posterior predictive checking under Bayesian approach. We conducted simulation studies considering different scenarios of interest, in order to evaluate the performance of estimates and diagnostic measures. In general, the results indicated that the proposed methodologies perform very well. We illustrate the proposed methodology by analyzing a real data sets with our methodology. The results indicate that the proposed heavy-tailed models fit well to the data sets. Specifically, for the data set that motivates this work, the bilirubin concentration data, we discuss that the proposed random-effects models are preferable to fixed-effects models.

Future research

Several works can be developed from the results of this work, such as

- To improve the estimates of $\boldsymbol{\nu}$, considering, for example, Jeffreys' prior.
- To extend the proposed models for the censored data.
- To model the random-effects b_i using skewed distributions.
- To develop the parameter estimation, the diagnostic measures and the statistics of model comparison for the proposed models under the frequentist approach.
- To consider nonlinear regression structures.

Bibliography

Achcar, J. A. (1993). Inferences for the birnbaum-saunders fatigue life model using bayesian methods. *Computational statistics & data analysis*, 15(4):367–380. Citado na página 36.

Ando, T. (2007). Bayesian predictive information criterion for the evaluation of hierarchical bayesian and empirical bayes models. *Biometrika*, 94(2):443–458. Citado na página 53.

Arellano-Valle, R., Bolfarine, H., and Lachos, V. (2007). Bayesian inference for skew-normal linear mixed models. *Journal of Applied Statistics*, 34(6):663–682. Citado na página 140.

Arellano-Valle, R. B. and Azzalini, A. (2008). The centred parametrization for the multivariate skew-normal distribution. *Journal of Multivariate Analysis*, 99(7):1362–1382. Citado 2 vezes nas páginas 37 and 47.

Atkinson, A. C. (1985). *Plots, Transformations and Regression. An Introduction to Graphical Methods of Diagnostic Regresion Analysis.* Oxford University Press. Citado 5 vezes nas páginas 76, 88, 108, 133, and 160.

Azevedo, C. L., Bolfarine, H., and Andrade, D. F. (2011). Bayesian inference for a skew-normal irt model under the centred parameterization. *Computational Statistics & Data Analysis*, 55(1):353–365. Citado 6 vezes nas páginas 54, 73, 91, 111, 136, and 163.

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian journal of statistics*, 12(2):171–178. Citado 4 vezes nas páginas 32, 37, 38, and 83.

Azzalini, A. (2013). *The skew-normal and related families.* Cambridge University Press. Citado 5 vezes nas páginas 32, 37, 38, 39, and 83.

Azzalini, A. and Capitanio, A. (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65(2):367–389. Citado na página 38.

Balakrishnan, N., Leiva, V., Sanhueza, A., Vilca, F., et al. (2009). Estimation in the birnbaum-saunders distribution based on scale-mixture of normals and the em-algorithm. *Statistics and Operations Research Transactions*, 33(2):171–192. Citado 3 vezes nas páginas 37, 38, and 61.

Balakrishnan, N., Saulo, H., and Leão, J. (2017). On a new class of skewed birnbaum–saunders models. *Journal of Statistical Theory and Practice*, 11(4):573–593. Citado 7 vezes nas páginas 32, 37, 38, 42, 47, 61, and 180.

Bandyopadhyay, D., Lachos, V. H., Abanto-Valle, C. A., and Ghosh, P. (2010). Linear mixed models for skew-normal/independent bivariate responses with an application to periodontal disease. *Statistics in medicine*, 29(25):2643–2655. Citado 5 vezes nas páginas 54, 73, 91, 111, and 136.

Barros, M., Paula, G. A., and Leiva, V. (2008). A new class of survival regression models with heavy-tailed errors: robustness and diagnostics. *Lifetime Data Analysis*, 14(3):316–332. Citado 3 vezes nas páginas 32, 37, and 83.

Barros, M., Paula, G. A., and Leiva, V. (2009). An r implementation for generalized birnbaum-saunders distributions. *Computational Statistics & Data Analysis*, 53(4):1511–1528. Citado na página 37.

Batista, E. M. (2018). Modelos birnbaum-saunders mistos aumentados em zero. Master's thesis, UNICAMP, IMECC. Citado 3 vezes nas páginas 33, 63, and 154.

Birnbaum, Z. W. and Saunders, S. C. (1969a). Estimation for a family of life distributions with applications to fatigue. *Journal of Applied Probability*, 6(2):328–347. Citado na página 36.

Birnbaum, Z. W. and Saunders, S. C. (1969b). A new family of life distributions. *Journal of applied probability*, 6(2):319–327. Citado 2 vezes nas páginas 32 and 36.

Bolfarine, H. and Lachos, V. (2007). Skew probit error-in-variables models. *Statistical Methodology*, 4(1):1–12. Citado na página 38.

Branco, M. D. and Dey, D. K. (2001). A general class of multivariate skew-elliptical distributions. *Journal of Multivariate Analysis*, 79(1):99–113. Citado na página 38.

Cabral, C. R. B., Lachos, V. H., and Madruga, M. R. (2012). Bayesian analysis of skew-normal independent linear mixed models with heterogeneity in the random-effects population. *Journal of Statistical Planning and Inference*, 142(1):181–200. Citado 12 vezes nas páginas 51, 54, 70, 73, 86, 91, 106, 111, 131, 136, 158, and 163.

Chaves, N. L., Azevedo, C. L. N., Vilca, F., and Nobre, J. S. (2019a). Bayesian inference for a birnbaum-saunders regression model based on the centered skew normal distribution. Submitted to REVSTAT - Statistical Journal. Citado 5 vezes nas páginas 83, 90, 110, 135, and 162.

Chaves, N. L., Azevedo, C. L. N., Vilca, F., and Nobre, J. S. (2019b). A new birnbaum-saunders type distribution based on the skew-normal model under a centered parameterization. *Chilean Journal of Statistics*, 10(1):55–76. Citado 6 vezes nas páginas 32, 37, 43, 50, 54, and 56.

Cho, H., Ibrahim, J. G., Sinha, D., and Zhu, H. (2009). Bayesian case influence diagnostics for survival models. *Biometrics*, 65(1):116–124. Citado 8 vezes nas páginas 89, 90, 109, 110, 134, 135, 161, and 162.

Cordeiro, G. M. and Lemonte, A. J. (2011). The beta-birnbaum-saunders distribution: an improved distribution for fatigue life modeling. *Computational Statistics & Data Analysis*, 55(3):1445–1461. Citado na página 37.

Cordeiro, G. M., Lemonte, A. J., and Ortega, E. M. (2013). An extended fatigue life distribution. *Statistics*, 47(3):626–653. Citado na página 56.

Cox, D. R. and Snell, E. J. (1968). A general definition of residuals. *Journal of the Royal Statistical Society: Series B (Methodological)*, 30(2):248–265. Citado 2 vezes nas páginas 107 and 159.

da Silva Ferreira, C., Bolfarine, H., and Lachos, V. H. (2011). Skew scale mixtures of normal distributions: properties and estimation. *Statistical Methodology*, 8(2):154–171. Citado 5 vezes nas páginas 32, 37, 38, 47, and 83.

Desmond, A. (1985). Stochastic models of failure in random environments. *Canadian Journal of Statistics*, 13(3):171–183. Citado na página 36.

Desmond, A. (1986). On the relationship between two fatigue-life models. *IEEE Transactions on Reliability*, 35(2):167–169. Citado na página 36.

Desmond, A. F., González, C. L. C., Singh, R., and Lu, X. (2012). A mixed effects log-linear model based on the birnbaum–saunders distribution. *Computational Statistics & Data Analysis*, 56(2):399–407. Citado 2 vezes nas páginas 32 and 127.

Diaz-Garcia, J. A. and Leiva, V. (2005). A new family of life distributions based on the elliptically contoured distributions. *Journal of Statistical Planning and Inference*, 128(2):445–457. Citado na página 37.

Draque, C. M. (2005). *Curva de bilirrubinemia total em récem-nascidos de termo em aleitamento materno nos primeiros doze dias de vida*. PhD thesis, Universidade Federal de São Paulo. Escola Paulista de Medicina. Curso de Pediatria e Ciências Aplicada à Pediatria. Citado 3 vezes nas páginas 33, 75, and 115.

Dunn, P. K. and Smyth, G. K. (1996). Randomized quantile residuals. *Journal of Computational and Graphical Statistics*, 5(3):236–244. Citado 4 vezes nas páginas 76, 87, 107, and 159.

Engelhardt, M., Bain, L. J., and Wright, F. (1981). Inferences on the parameters of the birnbaum-saunders fatigue life distribution based on maximum likelihood estimation. *Technometrics*, 23(3):251–256. Citado na página 36.

Fox, J.-P. (2005). Multilevel irt model assessment. *New developments in categorical data analysis for the social and behavioral sciences*, pages 227–252. Citado na página 87.

Galea, M., Leiva, V., and Paula, G. (2004). Influence diagnostics in log-birnbaum-saunders regression models. *Journal of Applied Statistics*, 31(9):1049–1064. Citado 2 vezes nas páginas 32 and 83.

Galea-Rojas, M., de Castilho, M. V., Bolfarine, H., and de Castro, M. (2003). Detection of analytical bias. *Analyst*, 128(8):1073–1081. Citado na página 38.

Galvis, D. M., Bandyopadhyay, D., and Lachos, V. H. (2014). Augmented mixed beta regression models for periodontal proportion data. *Statistics in medicine*, 33(21):3759–3771. Citado na página 63.

Garay, A. M., Lachos, V. H., and Abanto-Valle, C. A. (2011). Nonlinear regression models based on scale mixtures of skew-normal distributions. *Journal of the Korean Statistical Society*, 40(1):115–124. Citado 4 vezes nas páginas 90, 110, 135, and 162.

Gelfand, A. E., Dey, D. K., and Chang, H. (1992). Model determination using predictive distributions with implementation via sampling-based methods. Technical Report 462, Stanford University. Citado na página 52.

Gelman, A., Meng, X.-L., and Stern, H. (1996). Posterior predictive assessment of model fitness via realized discrepancies. *Statistica sinica*, 6:733–760. Citado 10 vezes nas páginas 53, 72, 88, 89, 108, 109, 133, 134, 160, and 161.

Gelman, A., Stern, H. S., Carlin, J. B., Dunson, D. B., Vehtari, A., and Rubin, D. B. (2013). *Bayesian data analysis*. Chapman and Hall/CRC, 3 edition. Citado 7 vezes nas páginas 49, 53, 72, 89, 108, 133, and 161.

Geoffrey McLachlan, D. (2000). Finite mixture models hoboken. Citado na página 63.

Huang, S. and Qu, Y. (2006). The loss in power when the test of differential expression is performed under a wrong scale. *Journal of Computational Biology*, 13(3):786–797. Citado 2 vezes nas páginas 32 and 83.

Ibrahim, J. G., Chen, M.-H., and Sinha, D. (2004). *Bayesian survival analysis*. Wiley Online Library. Citado na página 52.

Iwasaki, M. and Daidoji, K. (2009). Zero-inflated probability models and their applications to the analysis of test scores. *Japanese Journal of Behaviormetrics*, 36(1):25–34. Citado na página 63.

Kotz, S., Leiva, V., and Sanhueza, A. (2010). Two new mixture models related to the inverse gaussian distribution. *Methodology and Computing in Applied Probability*, 12(1):199–212. Citado na página 63.

Lachos, V. H., Dey, D. K., Cancho, V. G., and Louzada, F. (2017). Scale mixtures log-birnbaum-saunders regression models with censored data: a bayesian approach. *Journal of Statistical Computation and Simulation*, 87(10):2002–2022. Citado 12 vezes nas páginas 52, 54, 71, 73, 87, 91, 107, 111, 132, 136, 159, and 163.

Leiva, V. (2016). *The Birnbaum-Saunders Distribution*. Academic Press. Citado na página 37.

Leiva, V., Barros, M., Paula, G. A., and Galea, M. (2007). Influence diagnostics in log-birnbaum–saunders regression models with censored data. *Computational Statistics & Data Analysis*, 51(12):5694–5707. Citado na página 83.

Leiva, V., Barros, M., Paula, G. A., and Sanhueza, A. (2008). Generalized birnbaum-saunders distributions applied to air pollutant concentration. *Environmetrics: The official journal of the International Environmetrics Society*, 19(3):235–249. Citado na página 37.

Leiva, V., Santos-Neto, M., Cysneiros, F. J. A., and Barros, M. (2014). Birnbaum-saunders statistical modelling: a new approach. *Statistical Modelling*, 14(1):21–48. Citado 2 vezes nas páginas 33 and 83.

Leiva, V., Santos-Neto, M., Cysneiros, F. J. A., and Barros, M. (2016). A methodology for stochastic inventory models based on a zero-adjusted birnbaum-saunders distribution. *Applied Stochastic Models in Business and Industry*, 32(1):74–89. Citado 5 vezes nas páginas 33, 63, 76, 81, and 82.

Lemonte, A. J. and Cordeiro, G. M. (2009). Birnbaum–saunders nonlinear regression models. *Computational Statistics & Data Analysis*, 53(12):4441–4452. Citado na página 83.

Lemonte, A. J. and Patriota, A. G. (2011). Influence diagnostics in birnbaum–saunders nonlinear regression models. *Journal of Applied Statistics*, 38(5):871–884. Citado na página 83.

Lu, M.-C. and Chang, D. S. (1997). Bootstrap prediction intervals for the birnbaum-saunders distribution. *Microelectronics Reliability*, 37(8):1213–1216. Citado na página 36.

Maioli, M. C. (2018). Univariate and bivariate regression models based on centered skew scale mixture of normal distributions: Modelos de regressão univariados e bivariados baseados nas distribuições de mistura de escala normal assimétrica sob a parametrização centrada. Master's thesis, UNICAMP, IMECC. Citado 6 vezes nas páginas 38, 39, 90, 110, 135, and 162.

Mann, N. R., Singpurwalla, N. D., and Schafer, R. E. (1974). *Methods for statistical analysis of reliability and life data*. Wiley, 1 edition. Citado na página 36.

McCulloch, C. E. and Neuhaus, J. M. (2005). Generalized linear mixed models. *Encyclopedia of biostatistics*, 4. Citado na página 127.

Nichols, M. D. and Padgett, W. (2006). A bootstrap control chart for weibull percentiles. *Quality and reliability engineering international*, 22(2):141–151. Citado na página 56.

Ospina, R. and Ferrari, S. L. (2012). A general class of zero-or-one inflated beta regression models. *Computational Statistics & Data Analysis*, 56(6):1609–1623. Citado na página 63.

Ospina Martinez, R. (2008). *Inflated beta regression model*. PhD thesis, Universidade de São Paulo, IME. Citado na página 219.

Owen, W. J. and Padgett, W. J. (1999). Accelerated test models for system strength based on birnbaum-saunders distributions. *Lifetime Data Analysis*, 5(2):133–147. Citado na página 37.

Paula, G. A., Leiva, V., Barros, M., and Liu, S. (2012). Robust statistical modeling using the birnbaum-saunders-t distribution applied to insurance. *Applied Stochastic Models in Business and Industry*, 28(1):16–34. Citado 2 vezes nas páginas 32 and 83.

Pereira, G. H., Botter, D. A., and Sandoval, M. C. (2012). The truncated inflated beta distribution. *Communications in Statistics-Theory and Methods*, 41(5):907–919. Citado na página 63.

Poursadeghfard, T., Jamalizadeh, A., and Nematollahi, A. (2018). On the extended birnbaum-saunders distribution based on the skew-t-normal distribution. *Iranian Journal of Science and Technology, Transactions A: Science*, pages 1–15. Citado na página 37.

R Core Team (2014). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. Citado 6 vezes nas páginas 51, 70, 86, 106, 131, and 158.

Rieck, J. R. and Nedelman, J. R. (1991). A log-linear model for the birnbaum-saunders distribution. *Technometrics*, 33(1):51–60. Citado 2 vezes nas páginas 32 and 83.

Rubin, D. B. (1984). Bayesianly justifiable and relevant frequency calculations for the applies statistician. *The Annals of Statistics*, 12(4):1151–1172. Citado 6 vezes nas páginas 53, 72, 89, 109, 134, and 161.

Sánchez, R. P. M. (2018). *An extension of Birnbaum-Saunders distributions based on scale mixtures of skew-normal distributions with applications to regression models*. PhD thesis, Universidade de São Paulo. Citado 3 vezes nas páginas 32, 37, and 83.

Santana, L., Vilca, F., and Leiva, V. (2011). Influence analysis in skew-birnbaum-saunders regression models and applications. *Journal of Applied Statistics*, 38(8):1633–1649. Citado na página 83.

Santos-Neto, M., Cysneiros, F. J. A., Leiva, V., and Ahmed, S. E. (2012). On new parameterizations of the birnbaum-saunders distribution. *Pakistan Journal of Statistics*, 28(1):1–26. Citado 10 vezes nas páginas 32, 33, 37, 42, 61, 63, 83, 103, 154, and 180.

Santos-Neto, M., Cysneiros, F. J. A., Leiva, V., Barros, M., et al. (2016). Reparameterized birnbaum-saunders regression models with varying precision. *Electronic Journal of Statistics*, 10(2):2825–2855. Citado 3 vezes nas páginas 33, 83, and 102.

Sinharay, S., Johnson, M. S., and Stern, H. S. (2006). Posterior predictive assessment of item response theory models. *Applied psychological measurement*, 30(4):298–321. Citado 6 vezes nas páginas 53, 72, 89, 109, 134, and 161.

Song, X.-K. and Song, P. X.-K. (2007). *Correlated data analysis: modeling, analytics, and applications*. Springer Science & Business Media, 1 edition. Citado na página 127.

Spiegelhalter, D., Thomas, A., Best, N., and Lunn, D. (2014). Openbugs user manual, version 3.2.3. *URL http://www.openbugs.net/Manuals/Manual.html*. Citado 6 vezes nas páginas 51, 70, 86, 106, 131, and 158.

Spiegelhalter, D. J., Best, N. G., Carlin, B. P., and Van Der Linde, A. (2002). Bayesian measures of model complexity and fit. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 64(4):583–639. Citado 6 vezes nas páginas 52, 71, 88, 108, 133, and 160.

Sturtz, S., Ligges, U., and Gelman, A. (2010). R2openbugs: a package for running openbugs from r. *URL http://cran. rproject. org/web/packages/R2OpenBUGS/vignettes/R2OpenBUGS. pdf*. Citado 6 vezes nas páginas 51, 70, 86, 106, 131, and 158.

Taylor, S. and Pollard, K. (2009). Hypothesis tests for point-mass mixture data with application toomics data with many zero values. *Statistical Applications in Genetics and Molecular Biology*, 8(1):1–43. Citado na página 103.

Tomazella, V., Pereira, G. H. A., Nobre, J. S., and Santos-Neto, M. (2018). Zero-adjusted birnbaum-saunders regression model. *Statistics & Probability letters*, 149:142–145. Citado 6 vezes nas páginas 33, 63, 103, 108, 117, and 126.

Tu, W. (2014). Zero-inflated data. *Wiley StatsRef: Statistics Reference Online*. Citado na página 63.

Tu, W. and Zhou, X.-H. (1999). A wald test comparing medical costs based on log-normal distributions with zero valued costs. *Statistics in Medicine*, 18(20):2749–2761. Citado na página 103.

Vanegas, L. H., Rondón, L. M., and Cysneiros, F. J. A. (2012). Diagnostic procedures in birnbaum–saunders nonlinear regression models. *Computational Statistics & Data Analysis*, 56(6):1662–1680. Citado na página 83.

Verbeke, G. and Molenberghs, G. (2009). *Linear mixed models for longitudinal data*. Springer Science & Business Media. Citado na página 127.

Vilca, F., Azevedo, C. L., and Balakrishnan, N. (2016). Bayesian inference for sinh-normal/independent nonlinear regression models. *Journal of Applied Statistics*, 44(11):2052–2074. Citado 14 vezes nas páginas 51, 54, 71, 73, 86, 88, 91, 106, 108, 111, 132, 136, 159, and 160.

Vilca, F. and Leiva, V. (2006). A new fatigue life model based on the family of skew-elliptical distributions. *Communications in Statistics-Theory and Methods*, 35(2):229–244. Citado 2 vezes nas páginas 32 and 37.

Vilca, F., Sanhueza, A., Leiva, V., and Christakos, G. (2010). An extended birnbaum-saunders model and its application in the study of environmental quality in santiago, chile. *Stochastic Environmental Research and Risk Assessment*, 24(5):771–782. Citado na página 50.

Vilca, F., Santana, L., Leiva, V., and Balakrishnan, N. (2011). Estimation of extreme percentiles in birnbaum-saunders distributions. *Computational Statistics & Data Analysis*, 55(4):1665–1678. Citado 3 vezes nas páginas 32, 37, and 56.

Villegas, C., Paula, G. A., and Leiva, V. (2011). Birnbaum-saunders mixed models for censored reliability data analysis. *IEEE Transactions on Reliability*, 60(4):748–758. Citado 2 vezes nas páginas 32 and 127.

Wang, M., Sun, X., and Park, C. (2016). Bayesian analysis of birnbaum-saunders distribution via the generalized ratio-of-uniforms method. *Computational Statistics*, 31(1):207–225. Citado na página 36.

Xiao-Hua, Z. and Tu, W. (1999). Comparison of several independent population means when their samples contain log-normal and possibly zero observations. *Biometrics*, 55(2):645–651. Citado na página 103.

Xie, F.-C. and Wei, B.-C. (2007). Diagnostics analysis for log-birnbaum–saunders regression models. *Computational Statistics & Data Analysis*, 51(9):4692–4706. Citado 2 vezes nas páginas 32 and 83.

Xu, A. and Tang, Y. (2011). Bayesian analysis of birnbaum-saunders distribution with partial information. *Computational Statistics & Data Analysis*, 55(7):2324–2333. Citado na página 36.

APPENDIX A – Results of Chapter 1

In this section, we present in detail the density of the CSSBS distribution and its properties. Also, we present all results of the parameter recovery study.

A.1 The density of the CSSBS distribution and its properties

- **Density**

  If $Y \sim \text{CSSMN}(0, 1, \gamma, \boldsymbol{\nu})$ in Equation (1.8), then we have a r.v T follows the CSSBS distribution, whose cdf is given by

$$\begin{aligned} F(t|\boldsymbol{\theta}) &= P(T \leqslant t) \\ &= P\left\{ \frac{\mu}{[1+\phi\mathbb{E}(Y^2)]} \left[\frac{\sqrt{2\phi}Y}{2} + \sqrt{\left(\frac{\sqrt{2\phi}Y}{2}\right)^2 + 1} \right]^2 \leqslant t \right\} \\ &= P\left[Y \leqslant \underbrace{\frac{1}{\sqrt{2\phi}} \left\{ \sqrt{t[1+\phi\mathbb{E}(Y^2)]/\mu} - \sqrt{\mu/t[1+\phi\mathbb{E}(Y^2)]} \right\}}_{=a_t(\mu,\phi)} \right] \\ &= P\left[Y \leqslant a_t(\mu,\phi)\right] = \Phi_{\gamma,\boldsymbol{\nu}}\left[a_t(\mu,\phi)\right], \end{aligned}$$

  where $\boldsymbol{\theta} = (\mu, \phi, \gamma, \boldsymbol{\nu}^\top)^\top$ and $\Phi_{\gamma,\boldsymbol{\nu}}(\cdot)$ represents the cdf of Y. Therefore, the density of T is given by

$$\begin{aligned} f(t|\boldsymbol{\theta}) &= \frac{\partial F(t|\boldsymbol{\theta})}{\partial t} = \frac{\partial \Phi_{\gamma,\boldsymbol{\nu}}\left[a_t(\mu,\phi)\right]}{\partial t} = \Phi_{\gamma,\boldsymbol{\nu}}\left[a_t(\mu,\phi)\right] \underbrace{\frac{\partial a_t(\mu,\phi)}{\partial t}}_{=A_t(\mu,\phi)} \\ &= \Phi_{\gamma,\boldsymbol{\nu}}\left[a_t(\mu,\phi)\right] A_t(\mu,\phi) \\ &= 2A_t(\mu,\phi) \int_0^\infty \phi\left[a_t(\mu,\phi) \Big| -\frac{\mu_z}{\sqrt{u}\sigma_z}; \frac{1}{u\sigma_z^2} \right] \Phi\left\{\lambda\left[\mu_z + \sigma_z\sqrt{u}a_t(\mu,\phi)\right]\right\} dG(u|\boldsymbol{\nu}). \end{aligned}$$

- **Moments**

  Let $T \sim \text{CSSBS}(\mu, \phi, \gamma, \boldsymbol{\nu})$ and $Y \sim \text{CSSMN}(0, 1, \gamma, \boldsymbol{\nu})$. If $\mathbb{E}\left[Y^{2(r-j+i)}\right] < \infty$, then $\mathbb{E}(T^r)$ exist and are given by,

$$\mathbb{E}(T^r) = \frac{\mu^r}{[1+\phi m_2]^r} \sum_{j=0}^{r} \binom{2r}{2j} \sum_{i=0}^{j} \mathbb{E}\left[Y^{2(r-j+i)}\right] \left(\sqrt{2\phi}/2\right)^{2(r-j+i)}.$$

Proof. Using Equation (1.8), we have that

$$\mathbb{E}\left\{\left(\frac{T[1+\phi m_2]}{\mu}\right)^r\right\} = \mathbb{E}\left\{\left[\left(\frac{\sqrt{2\phi}Y}{2}+\sqrt{\left(\frac{\sqrt{2\phi}Y}{2}\right)^2+1}\right)^2\right]^r\right\}.$$

From the Binomial Theorem, that is $(a+b)^m = \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k$, we have that

$$\mathbb{E}\left\{\left(\frac{T[1+\phi m_2]}{\mu}\right)^r\right\} = \sum_{k=0}^{2r}\binom{2r}{k}\mathbb{E}\left\{\left[\left(\frac{\sqrt{2\phi}Y}{2}\right)^2+1\right]^{k/2}\left(\frac{\sqrt{2\phi}Y}{2}\right)^{2r-k}\right\}.$$

Considering $k=2j$, that is, $j=k/2$, it comes that

$$\mathbb{E}\left\{\left(\frac{T[1+\phi m_2]}{\mu}\right)^r\right\} = \sum_{j=0}^{r}\binom{2r}{2j}\mathbb{E}\left\{\left[\left(\frac{\sqrt{2\phi}Y}{2}\right)^2+1\right]^{j}\left(\frac{\sqrt{2\phi}Y}{2}\right)^{2(r-j)}\right\}.$$

From the Binomial Theorem again, it comes that

$$\begin{aligned}\mathbb{E}\left\{\left(\frac{T[1+\phi m_2]}{\mu}\right)^r\right\} &= \sum_{j=0}^{r}\binom{2r}{2j}\mathbb{E}\left\{\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\sqrt{2\phi}Y}{2}\right)^{2i}\left(\frac{\sqrt{2\phi}Y}{2}\right)^{2(r-j)}\right\}\\ &= \sum_{j=0}^{r}\binom{2r}{2j}\sum_{i=0}^{j}\binom{j}{i}\mathbb{E}\left[\left(\frac{\sqrt{2\phi}Y}{2}\right)^{2(r-j+i)}\right]\\ &= \sum_{j=0}^{r}\binom{2r}{2j}\sum_{i=0}^{j}\binom{j}{i}\mathbb{E}\left[Y^{2(r-j+i)}\right]\left(\frac{\sqrt{2\phi}}{2}\right)^{2(r-j+i)}.\end{aligned}$$

Therefore,

$$\mathbb{E}(T^r) = \frac{\mu^r}{[1+\phi m_2]^r}\sum_{j=0}^{r}\binom{2r}{2j}\sum_{i=0}^{j}\binom{j}{i}\mathbb{E}\left[Y^{2(r-j+i)}\right]\left(\sqrt{2\phi}/2\right)^{2(r-j+i)}.$$

From Equation (1.11), we obtain

$$\mathbb{E}(T) = \frac{\mu}{[1+\phi m_2]}\sum_{j=0}^{1}\binom{2}{2j}\sum_{i=0}^{j}\binom{j}{i}\mathbb{E}\left[Y^{2(1-j+i)}\right]\left(\sqrt{2\phi}/2\right)^{2(1-j+i)}. \qquad \text{(A.1)}$$

For $j=0$, the first term of the sum in Equation (A.1) is equals to $\frac{\mu}{[1+\phi m_2]}\mathbb{E}(Y^2)\times$ $\times(\sqrt{2\phi}/2)^2$. For $j=1$, the second term of the sum in Equation (A.1) is equals to $\frac{\mu}{[1+\phi m_2]}\times\left[1+\mathbb{E}(Y^2)(\sqrt{2\phi}/2)^2\right]$. Therefore, by adding these two terms, we have

$$\mathbb{E}(T) = \mu.$$

Furthermore, from Equation (1.11), we have

$$\mathbb{E}(T^2) = \left\{\frac{\mu}{[1+\phi m_2]}\right\}^2\sum_{j=0}^{2}\binom{4}{2j}\sum_{i=0}^{j}\binom{j}{i}\mathbb{E}\left[Y^{2(2-j+i)}\right]\left(\sqrt{2\phi}/2\right)^{2(2-j+i)}.$$

Developing the above sum, we obtain

$$\mathbb{E}(T^2) = \left\{\frac{\mu}{[1+\phi m_2]}\right\}^2 \left[1 + \frac{\left(\sqrt{2\phi}\right)^4}{2} m_4 + 2\left(\sqrt{2\phi}\right)^2 m_2\right].$$

Thus,

$$\begin{aligned}\mathbb{V}(T) &= \mathbb{E}(T^2) - [\mathbb{E}(T)]^2 \\ &= \frac{2\phi\mu^2}{[1+\phi m_2]^2}\left\{m_2 + \frac{\phi}{2}\left[2m_4 - m_2^2\right]\right\}.\end{aligned}$$

Finally, Pearson's skewness coefficient of T is given by

$$\gamma_T = \frac{\mathbb{E}([T-\mu)^3]}{[\mathbb{V}(T)]^{3/2}}.$$

From Equation (1.11), we have

$$\mathbb{E}(T^3) = \frac{\mu^3}{[1+\phi m_2]^3} \sum_{j=0}^{3} \binom{6}{2j} \sum_{i=0}^{j} \mathrm{E}\left[Y^{2(3-j+i)}\right] \left(\sqrt{2\phi}/2\right)^{2(3-j+i)}.$$

Developing the above sum, we obtain

$$\mathbb{E}(T^3) = \frac{\mu^3}{(1+\phi m_2)^3}\Big(1 + 9\phi m_2 + 12\phi^2 m_4 + 4\phi^3 m_6\Big).$$

After some algebra, we have

$$\gamma_T = \frac{\sqrt{\phi}[\phi(4m_6 - 6m_2m_4 + 2m_2^3) + 6(m_4 - m_2^2)]}{\left\{2\Big[m_2 + \frac{\phi}{2}\Big(2m_4 - m_2^2\Big)\Big]\right\}^{3/2}},$$

where $m_k = \mathbb{E}(Y^k), k = 2, 4, 6.$ □

- **Hierarchical representation**

If $T \sim \mathrm{CSSBS}(\mu, \phi, \gamma, \boldsymbol{\nu})$, then its hierarchical representation is given by

$$\begin{aligned} T|\,(H=h, U=u) &\sim \mathrm{EBS}(\phi_\delta, \mu/[1+\phi\mathbb{E}(Y^2)], \kappa=2, \vartheta_h) \\ H &\sim \mathrm{HN}(0,1) \\ U|\boldsymbol{\nu} &\sim G(u|\boldsymbol{\nu}),\end{aligned}$$

where $\phi_\delta = \sqrt{2\phi}\left(\frac{u^{-1/2}\sqrt{1-\delta^2}}{\sigma_z}\right)$, $\vartheta_h = \frac{\mu_z - \delta h}{\sqrt{1-\delta^2}}$.

Proof. If $Y \sim \mathrm{CSSMN}(0,1,\gamma,\boldsymbol{\nu})$, then its stochastic representation is given by $Y = \mu_0 + \sigma_0 U^{-1/2}\left[\frac{\mu_z}{\sigma_z} + \frac{1}{\sigma_z}\left(\delta H + \sqrt{1-\delta^2}X_0\right)\right]$. Therefore, $Y|(H=h, U=u) \sim \mathrm{N}(\varrho, \tau^2)$, where $\varrho = u^{-1/2}[(\delta h - \mu_z)/\sigma_z]$ and $\tau^2 = (1-\delta^2)/u\sigma_z^2$. Then

$$V = -\frac{\varrho}{\tau} + \frac{1}{\sqrt{2\phi}\tau}\left\{\sqrt{t[1+\phi\mathbb{E}(Y^2)]/\mu} - \sqrt{\mu/t[1+\phi\mathbb{E}(Y^2)]}\right\}\Big|(H=h, U=u) \sim \mathrm{N}(0,1).$$

Therefore, we can write

$$T = \frac{\mu}{[1 + \phi\mathbb{E}(Y^2)]}\left[\frac{\sqrt{2\phi}}{2}\,(\varrho + \tau V) + \sqrt{\left[\frac{\sqrt{2\phi}}{2}\,(\varrho + \tau V)\right]^2 + 1}\,\right].$$

From the above result, the proof is concluded. □

A.2 Results of the recovery parameter study

SGtBS distribution

Table 18 – Results of recovery parameter study for SGtBS distribution ($\nu_1 = 15, \nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mean	1.031	1.030	1.039	1.006	1.006	1.001
	SD	.024	.028	.043	.015	.012	.016
	LCI	.324	.410	.484	.129	.167	.192
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.031	.030	.039	.006	.006	.001
	RMSE	.040	.041	.058	.016	.013	.016
	AVRB	.031	.030	.039	.006	.006	.001
γ	Mean	-.664	.027	.786	-.807	-.012	.779
	SD	.140	.253	.092	.048	.157	.076
	LCI	.788	1.168	.592	.322	.417	.319
	CP	1.000	1.000	1.000	1.000	.200	1.000
	Bias	.136	.027	-.014	-.007	-.012	-.021
	RMSE	.195	.254	.093	.049	.158	.079
	AVRB	.170	-	.017	.009	-	.026
ν_1	Mean	11.904	12.504	14.962	18.125	16.592	15.299
	SD	2.945	3.168	6.692	11.284	8.954	5.108
	LCI	30.825	31.028	44.096	35.058	31.041	26.574
	CP	1.000	1.000	1.000	.900	.800	1.000
	Bias	-3.096	-2.496	-.038	3.125	1.592	.299
	RMSE	4.273	4.033	6.692	11.709	9.094	5.117
	AVRB	.206	.166	.003	.208	.106	.020
ν_2	Mean	3.994	4.256	5.052	6.237	5.651	5.119
	SD	1.210	1.256	2.335	4.558	3.426	2.002
	LCI	12.693	12.088	16.688	13.360	12.067	10.001
	CP	1.000	1.000	1.000	.900	.800	1.000
	Bias	-1.006	-.744	.052	1.237	.651	.119
	RMSE	1.573	1.460	2.335	4.722	3.487	2.005
	AVRB	.201	.149	.010	.247	.130	.024

Table 19 – Results of simulation study for SGtBS distribution ($\nu_1 = 15, \nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Median	1.025	1.020	1.022	1.006	1.004	.998
	SD	.025	.025	.037	.014	.012	.015
	LCI	.324	.410	.484	.129	.167	.192
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.025	.020	.022	.006	.004	-.002
	RMSE	.035	.032	.043	.016	.013	.016
	AVRB	.025	.020	.022	.006	.004	.002
γ	Median	-.700	.012	.824	-.810	-.014	.784
	SD	.146	.261	.090	.056	.146	.078
	LCI	.788	1.168	.592	.322	.417	.319
	CP	1.000	1.000	1.000	1.000	.200	1.000
	Bias	.100	.012	.024	-.010	-.014	-.016
	RMSE	.177	.261	.093	.056	.147	.080
	AVRB	.125	-	.030	.012	-	.020
ν_1	Median	9.527	9.880	11.259	14.651	14.441	13.556
	SD	2.203	2.319	3.088	6.475	7.035	3.730
	LCI	30.825	31.028	44.096	35.058	31.041	26.574
	CP	1.000	1.000	1.000	.900	.800	1.000
	Bias	-5.473	-5.120	-3.741	-.349	-.559	-1.444
	RMSE	5.900	5.621	4.850	6.485	7.058	4.000
	AVRB	.365	.341	.249	.023	.037	.096
ν_2	Median	3.070	3.234	3.665	4.845	4.817	4.465
	SD	.937	.923	1.155	2.468	2.702	1.480
	LCI	12.693	12.088	16.688	13.360	12.067	10.001
	CP	1.000	1.000	1.000	.900	.800	1.000
	Bias	-1.930	-1.766	-1.335	-.155	-.183	-.535
	RMSE	2.146	1.992	1.765	2.473	2.708	1.574
	AVRB	.386	.353	.267	.031	.037	.107

Table 20 – Results of simulation study for SGtBS distribution ($\nu_1 = 15, \nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mode	1.028	1.022	1.022	1.006	1.006	1.001
	SD	.024	.026	.034	.015	.012	.016
	LCI	.324	.410	.484	.129	.167	.192
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.028	.022	.022	.006	.006	.001
	RMSE	.037	.034	.041	.016	.013	.016
	AVRB	.028	.022	.022	.006	.006	.001
γ	Mode	-.716	-.005	.822	-.809	-.014	.782
	SD	.134	.284	.078	.049	.152	.077
	LCI	.788	1.168	.592	.322	.417	.319
	CP	1.000	1.000	1.000	1.000	.200	1.000
	Bias	.084	-.005	.022	-.009	-.014	-.018
	RMSE	.158	.284	.081	.050	.153	.079
	AVRB	.105	-	.028	.011	-	.023
ν_1	Mode	7.234	7.405	7.193	10.520	11.474	11.566
	SD	2.408	1.884	2.132	3.147	6.123	3.759
	LCI	30.825	31.028	44.096	35.058	31.041	26.574
	CP	1.000	1.000	1.000	.900	.800	1.000
	Bias	-7.766	-7.595	-7.807	-4.480	-3.526	-3.434
	RMSE	8.131	7.825	8.093	5.475	7.066	5.091
	AVRB	.518	.506	.520	.299	.235	.229
ν_2	Mode	2.092	2.104	2.195	3.334	4.063	3.379
	SD	.711	.587	.648	1.294	2.715	1.011
	LCI	12.693	12.088	16.688	13.360	12.067	10.001
	CP	1.000	1.000	1.000	.900	.800	1.000
	Bias	-2.908	-2.896	-2.805	-1.666	-.937	-1.621
	RMSE	2.993	2.955	2.879	2.110	2.872	1.911
	AVRB	.582	.579	.561	.333	.187	.324

Table 21 – Results of simulation study for SGtBS distribution ($\nu_1 = 30, \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mean	1.056	1.063	1.067	1.020	1.025	1.013
	SD	.045	.031	.049	.021	.015	.030
	LCI	.453	.689	.796	.187	.273	.299
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.056	.063	.067	.020	.025	.013
	RMSE	.072	.071	.083	.029	.029	.032
	AVRB	.056	.063	.067	.020	.025	.013
γ	Mean	-.759	.092	.787	-.846	.043	.817
	SD	.129	.248	.106	.063	.178	.052
	LCI	.634	1.179	.606	.259	.427	.284
	CP	1.000	.900	1.000	.900	.200	1.000
	Bias	.041	.092	-.013	-.046	.043	.017
	RMSE	.135	.265	.107	.078	.183	.055
	AVRB	.052	-	.016	.057	-	.022
ν_1	Mean	11.942	11.883	10.785	20.092	17.297	23.216
	SD	4.083	4.364	2.563	4.995	5.185	18.733
	LCI	28.260	28.668	26.084	37.410	34.375	47.395
	CP	.600	.300	.600	.800	.800	.600
	Bias	-18.058	-18.117	-19.215	-9.908	-12.703	-6.784
	RMSE	18.514	18.635	19.385	11.096	13.721	19.924
	AVRB	.602	.604	.640	.330	.423	.226
ν_2	Mean	11.019	11.237	9.848	19.782	16.850	22.808
	SD	4.196	4.870	2.916	5.610	5.440	18.670
	LCI	30.778	30.645	28.266	39.540	36.659	50.451
	CP	.600	.300	.600	.800	.800	.700
	Bias	-18.981	-18.763	-20.152	-10.218	-13.150	-7.192
	RMSE	19.439	19.385	20.362	11.656	14.231	20.008
	AVRB	.633	.625	.672	.341	.438	.240

Table 22 – Results of simulation study for SGtBS distribution ($\nu_1 = 30, \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Median	1.051	1.044	1.040	1.018	1.021	1.010
	SD	.050	.031	.047	.021	.015	.031
	LCI	.453	.689	.796	.187	.273	.299
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.051	.044	.040	.018	.021	.010
	RMSE	.072	.054	.062	.028	.026	.032
	AVRB	.051	.044	.040	.018	.021	.010
γ	Median	-.797	.102	.826	-.856	.038	.825
	SD	.135	.256	.106	.071	.170	.055
	LCI	.634	1.179	.606	.259	.427	.284
	CP	1.000	.900	1.000	.900	.200	1.000
	Bias	.003	.102	.026	-.056	.038	.025
	RMSE	.135	.276	.109	.090	.175	.060
	AVRB	.004	-	.032	.070	-	.031
ν_1	Median	9.626	9.415	8.731	17.692	15.100	18.578
	SD	2.682	2.791	1.923	4.742	4.843	12.457
	LCI	28.260	28.668	26.084	37.410	34.375	47.395
	CP	.600	.300	.600	.800	.800	.600
	Bias	-20.374	-20.585	-21.269	-12.308	-14.900	-11.422
	RMSE	20.550	20.773	21.355	13.189	15.667	16.901
	AVRB	.679	.686	.709	.410	.497	.381
ν_2	Median	8.609	8.608	7.610	17.213	14.461	17.809
	SD	2.849	3.250	2.188	5.291	5.137	12.021
	LCI	30.778	30.645	28.266	39.540	36.659	50.451
	CP	.600	.300	.600	.800	.800	.700
	Bias	-21.391	-21.392	-22.390	-12.787	-15.539	-12.191
	RMSE	21.580	21.637	22.497	13.838	16.366	17.120
	AVRB	.713	.713	.746	.426	.518	.406

Table 23 – Results of simulation study for SGtBS distribution ($\nu_1 = 30, \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mode	1.050	1.039	1.029	1.019	1.024	1.012
	SD	.046	.031	.042	.021	.015	.029
	LCI	.453	.689	.796	.187	.273	.299
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.050	.039	.029	.019	.024	.012
	RMSE	.068	.050	.051	.029	.028	.032
	AVRB	.050	.039	.029	.019	.024	.012
γ	Mode	-.798	.108	.826	-.849	.041	.820
	SD	.128	.266	.087	.063	.171	.051
	LCI	.634	1.179	.606	.259	.427	.284
	CP	1.000	.900	1.000	.900	.200	1.000
	Bias	.002	.108	.026	-.049	.041	.020
	RMSE	.128	.287	.091	.080	.176	.055
	AVRB	.003	-	.032	.061	-	.025
ν_1	Mode	7.305	6.147	6.114	14.231	12.546	14.670
	SD	2.388	1.404	1.455	3.115	3.289	7.106
	LCI	28.260	28.668	26.084	37.410	34.375	47.395
	CP	.600	.300	.600	.800	.800	.600
	Bias	-22.695	-23.853	-23.886	-15.769	-17.454	-15.330
	RMSE	22.820	23.894	23.931	16.074	17.761	16.897
	AVRB	.757	.795	.796	.526	.582	.511
ν_2	Mode	6.160	5.780	4.933	13.461	12.083	13.691
	SD	1.971	1.770	1.534	3.764	4.090	6.230
	LCI	30.778	30.645	28.266	39.540	36.659	50.451
	CP	.600	.300	.600	.800	.800	.700
	Bias	-23.840	-24.220	-25.067	-16.539	-17.917	-16.309
	RMSE	23.921	24.284	25.114	16.962	18.378	17.458
	AVRB	.795	.807	.836	.551	.597	.544

StBS distribution

Table 24 – Results of simulation study for StBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	1.175	1.135	1.051	1.008	1.014	.998
	SD	.402	.122	.099	.043	.062	.071
	LCI	1.352	1.129	.776	.259	.293	.321
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.175	.135	.051	.008	.014	-.002
	RMSE	.439	.182	.111	.044	.064	.071
	AVRB	.175	.135	.051	.008	.014	.002
	Mean	.573	.558	.596	.490	.543	.502
	SD	.058	.091	.054	.033	.054	.031
	LCI	.505	.500	.517	.203	.226	.218
ϕ	CP	1.000	1.000	1.000	1.000	.900	1.000
	Bias	.073	.058	.096	-.010	.043	.002
	RMSE	.094	.108	.110	.035	.069	.031
	AVRB	.146	.115	.191	.019	.085	.005
	Mean	-.594	-.004	.645	-.715	.147	.734
	SD	.154	.147	.088	.104	.175	.093
	LCI	.948	1.301	.826	.445	.531	.440
γ	CP	1.000	1.000	1.000	1.000	.300	1.000
	Bias	.206	-.004	-.155	.085	.147	-.066
	RMSE	.257	.147	.178	.134	.228	.115
	AVRB	.258	-	.193	.106	-	.083
	Mean	8.621	8.293	10.670	5.413	7.041	5.930
	SD	3.508	3.922	4.423	1.173	2.229	1.376
	LCI	25.777	27.761	36.761	5.211	9.075	6.441
ν	CP	.900	1.000	1.000	1.000	.800	1.000
	Bias	3.621	3.293	5.670	.413	2.041	.930
	RMSE	5.041	5.121	7.191	1.244	3.022	1.661
	AVRB	.724	.659	1.134	.083	.408	.186

Table 25 – Results of simulation study for StBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	1.076	1.073	1.016	1.002	1.007	.990
	SD	.171	.109	.084	.038	.058	.065
	LCI	1.352	1.129	.776	.259	.293	.321
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.076	.073	.016	.002	.007	-.01
	RMSE	.187	.131	.085	.038	.059	.066
	AVRB	.076	.073	.016	.002	.007	.010
	Median	.563	.546	.58	.488	.54	.501
	SD	.061	.094	.056	.034	.054	.032
	LCI	.505	.5	.517	.203	.226	.218
ϕ	CP	1.000	1.000	1.000	1.000	.900	1.000
	Bias	.063	.046	.08	-.012	.04	.001
	RMSE	.088	.104	.098	.036	.068	.032
	AVRB	.125	.091	.161	.025	.081	.002
	Median	-.637	-.004	.683	-.724	.146	.744
	SD	.142	.156	.084	.108	.17	.099
	LCI	.948	1.301	.826	.445	.531	.44
γ	CP	1.000	1.000	1.000	1.000	.300	1.000
	Bias	.163	-.004	-.117	.076	.146	-.056
	RMSE	.216	.156	.144	.132	.224	.113
	AVRB	.204	-	.147	.095	-	.069
	Median	6.383	5.799	7.47	5.13	6.504	5.58
	SD	2.063	2.091	2.313	.998	1.842	1.154
	LCI	25.777	27.761	36.761	5.211	9.075	6.441
ν	CP	.900	1.000	1.000	1.000	.800	1.000
	Bias	1.383	.799	2.470	.130	1.504	.580
	RMSE	2.483	2.238	3.384	1.007	2.378	1.291
	AVRB	.277	.160	.494	.026	.301	.116

Table 26 – Results of simulation study for StBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mode	1.050	1.053	1.006	1.007	1.010	.993
	SD	.099	.087	.073	.041	.057	.062
	LCI	1.352	1.129	.776	.259	.293	.321
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.050	.053	.006	.007	.010	-.007
	RMSE	.111	.102	.073	.041	.058	.063
	AVRB	.050	.053	.006	.007	.010	.007
ϕ	Mode	.564	.548	.584	.489	.541	.502
	SD	.058	.092	.054	.032	.054	.032
	LCI	.505	.500	.517	.203	.226	.218
	CP	1.000	1.000	1.000	1.000	.900	1.000
	Bias	.064	.048	.084	-.011	.041	.002
	RMSE	.087	.104	.100	.034	.068	.032
	AVRB	.128	.096	.168	.021	.083	.004
γ	Mode	-.673	-.005	.707	-.722	.145	.741
	SD	.114	.165	.069	.105	.169	.096
	LCI	.948	1.301	.826	.445	.531	.440
	CP	1.000	1.000	1.000	1.000	.300	1.000
	Bias	.127	-.005	-.093	.078	.145	-.059
	RMSE	.171	.165	.116	.131	.223	.112
	AVRB	.159	-	.116	.098	-	.073
ν	Mode	4.877	3.947	4.980	4.788	5.910	4.982
	SD	1.371	1.090	1.189	.996	1.462	.748
	LCI	25.777	27.761	36.761	5.211	9.075	6.441
	CP	.900	1.000	1.000	1.000	.800	1.000
	Bias	-.123	-1.053	-.020	-.212	.910	-.018
	RMSE	1.377	1.515	1.189	1.019	1.722	.748
	AVRB	.025	.211	.004	.042	.182	.004

Table 27 – Results of simulation study for StBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mean	1.05	1.048	1.048	1.007	1.009	1.015
	SD	.03	.043	.013	.011	.007	.022
	LCI	.342	.532	.575	.139	.201	.224
	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	.05	.048	.048	.007	.009	.015
	RMSE	.058	.065	.05	.013	.011	.027
	AVRB	.05	.048	.048	.007	.009	.015
ϕ	Mean	.495	.464	.464	.489	.481	.488
	SD	.038	.027	.037	.022	.023	.021
	LCI	.307	.354	.36	.161	.164	.157
	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	-.005	-.036	-.036	-.011	-.019	-.012
	RMSE	.038	.045	.051	.025	.03	.025
	AVRB	.01	.072	.072	.022	.038	.025
γ	Mean	-.653	.098	.639	-.825	.033	.862
	SD	.147	.321	.185	.07	.169	.051
	LCI	.763	1.08	.835	.277	.414	.253
	CP	.900	.800	1.000	1.000	.400	.900
	Bias	.147	.098	-.161	-.025	.033	.062
	RMSE	.207	.336	.246	.075	.172	.081
	AVRB	.183	-	.202	.032	-	.078
ν	Mean	78.866	17.484	17.969	27.739	28.862	29.194
	SD	170.282	4.796	5.699	5.584	12.484	10.862
	LCI	77.937	63.713	63.656	79.922	83.746	78.996
	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	48.866	-12.516	-12.031	-2.261	-1.138	-.806
	RMSE	177.155	13.404	13.312	6.025	12.536	10.892
	AVRB	1.629	.417	.401	.075	.038	.027

Table 28 – Results of simulation study for StBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	1.043	1.032	1.03	1.007	1.007	1.013
	SD	.032	.033	.013	.011	.007	.025
	LCI	.342	.532	.575	.139	.201	.224
μ	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	.043	.032	.030	.007	.007	.013
	RMSE	.054	.046	.033	.013	.010	.028
	AVRB	.043	.032	.030	.007	.007	.013
	Median	.489	.457	.456	.488	.48	.487
	SD	.037	.027	.037	.023	.022	.022
	LCI	.307	.354	.36	.161	.164	.157
ϕ	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	-.011	-.043	-.044	-.012	-.02	-.013
	RMSE	.039	.05	.058	.026	.03	.025
	AVRB	.023	.085	.088	.024	.039	.027
	Median	-.697	.112	.673	-.832	.033	.873
	SD	.161	.337	.196	.071	.162	.059
	LCI	.763	1.08	.835	.277	.414	.253
γ	CP	.900	.800	1.000	1.000	.400	.900
	Bias	.103	.112	-.127	-.032	.033	.073
	RMSE	.191	.355	.233	.078	.166	.093
	AVRB	.129	-	.159	.04	-	.091
	Median	71.691	11.431	12.207	20.698	22.079	22.441
	SD	172.777	3.171	3.87	4.016	9.88	9.206
	LCI	77.937	63.713	63.656	79.922	83.746	78.996
ν	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	41.691	-18.569	-17.793	-9.302	-7.921	-7.559
	RMSE	177.736	18.838	18.209	10.132	12.663	11.912
	AVRB	1.390	.619	.593	.31	.264	.252

Table 29 – Results of simulation study for StBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	1.046	1.033	1.030	1.008	1.008	1.015
	SD	.031	.030	.014	.011	.006	.023
	LCI	.342	.532	.575	.139	.201	.224
μ	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	.046	.033	.030	.008	.008	.015
	RMSE	.056	.045	.033	.014	.010	.027
	AVRB	.046	.033	.030	.008	.008	.015
	Mode	.491	.461	.460	.488	.481	.487
	SD	.036	.028	.037	.023	.023	.021
	LCI	.307	.354	.360	.161	.164	.157
ϕ	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	-.009	-.039	-.040	-.012	-.019	-.013
	RMSE	.038	.048	.054	.026	.029	.025
	AVRB	.018	.078	.079	.024	.037	.026
	Mode	-.719	.127	.682	-.828	.033	.865
	SD	.147	.359	.201	.070	.166	.051
	LCI	.763	1.080	.835	.277	.414	.253
γ	CP	.900	.800	1.000	1.000	.400	.900
	Bias	.081	.127	-.118	-.028	.033	.065
	RMSE	.168	.381	.233	.075	.170	.083
	AVRB	.102	-	.148	.035	-	.081
	Mode	65.770	6.553	7.410	13.889	14.304	14.741
	SD	174.845	1.686	2.077	2.784	5.565	4.689
	LCI	77.937	63.713	63.656	79.922	83.746	78.996
ν	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	35.770	-23.447	-22.590	-16.111	-15.696	-15.259
	RMSE	178.467	23.507	22.685	16.350	16.653	15.963
	AVRB	1.192	.782	.753	.537	.523	.509

SSLBS distribution

Table 30 – Results of simulation study for SSLBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	.999	1.045	1.058	1.003	1.01	1.007
	SD	.03	.029	.045	.01	.007	.02
	LCI	.34	.515	.589	.149	.201	.235
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.001	.045	.058	.003	.01	.007
	RMSE	.03	.054	.073	.011	.012	.021
	AVRB	.001	.045	.058	.003	.01	.007
	Mean	.543	.529	.536	.528	.541	.528
	SD	.027	.041	.032	.027	.016	.024
	LCI	.409	.409	.386	.215	.219	.202
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.043	.029	.036	.028	.041	.028
	RMSE	.051	.051	.048	.039	.044	.037
	AVRB	.086	.059	.072	.055	.082	.056
	Mean	-.696	.113	.659	-.759	-.02	.77
	SD	.138	.230	.161	.065	.123	.062
	LCI	.589	.991	.640	.267	.406	.248
γ	CP	.900	.900	.900	.900	.600	1.000
	Bias	.104	.113	-.141	.041	-.020	-.030
	RMSE	.173	.257	.214	.076	.125	.069
	AVRB	.131	-	.176	.051	-	.038
	Mean	7.85	6.788	7.718	8.371	9.097	8.024
	SD	1.19	1.696	1.289	1.882	1.244	1.713
	LCI	19.375	16.877	18.029	18.332	19.443	15.55
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	2.85	1.788	2.718	3.371	4.097	3.024
	RMSE	3.088	2.465	3.008	3.861	4.282	3.475
	AVRB	.570	.358	.544	.674	.819	.605

Table 31 – Results of simulation study for SSLBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	.996	1.033	1.042	1.003	1.007	1.005
	SD	.029	.028	.044	.010	.008	.020
	LCI	.340	.515	.589	.149	.201	.235
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.004	.033	.042	.003	.007	.005
	RMSE	.030	.043	.061	.010	.011	.020
	AVRB	.004	.033	.042	.003	.007	.005
	Median	.536	.522	.528	.527	.544	.528
	SD	.026	.045	.034	.028	.015	.024
	LCI	.409	.409	.386	.215	.219	.202
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.036	.022	.028	.027	.044	.028
	RMSE	.045	.050	.044	.039	.047	.037
	AVRB	.072	.043	.056	.055	.088	.057
	Median	-.719	.118	.685	-.766	-.017	.776
	SD	.142	.237	.160	.065	.117	.063
	LCI	.589	.991	.640	.267	.406	.248
γ	CP	.900	.900	.900	.900	.600	1.000
	Bias	.081	.118	-.115	.034	-.017	-.024
	RMSE	.164	.265	.197	.073	.118	.067
	AVRB	.101	-	.143	.043	-	.030
	Median	6.351	5.417	6.426	7.062	7.846	6.909
	SD	1.086	1.421	1.275	1.696	1.024	1.589
	LCI	19.375	16.877	18.029	18.332	19.443	15.55
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.351	.417	1.426	2.062	2.846	1.909
	RMSE	1.733	1.481	1.913	2.67	3.025	2.484
	AVRB	.270	.083	.285	.412	.569	.382

Table 32 – Results of simulation study for SSLBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	.997	1.034	1.042	1.003	1.01	1.006
	SD	.03	.029	.043	.010	.007	.020
	LCI	.340	.515	.589	.149	.201	.235
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.003	.034	.042	.003	.010	.006
	RMSE	.03	.044	.06	.010	.012	.021
	AVRB	.003	.034	.042	.003	.010	.006
	Mode	.538	.524	.532	.528	.541	.528
	SD	.025	.043	.034	.028	.015	.023
	LCI	.409	.409	.386	.215	.219	.202
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.038	.024	.032	.028	.041	.028
	RMSE	.046	.049	.046	.039	.044	.036
	AVRB	.076	.047	.063	.056	.082	.056
	Mode	-.719	.121	.689	-.761	-.020	.772
	SD	.132	.245	.146	.065	.121	.062
	LCI	.589	.991	.640	.267	.406	.248
γ	CP	.900	.900	.900	.900	.600	1.000
	Bias	.081	.121	-.111	.039	-.020	-.028
	RMSE	.155	.274	.183	.076	.122	.068
	AVRB	.101	-	.138	.049	-	.035
	Mode	3.748	3.247	4.286	5.294	6.152	5.38
	SD	.946	.857	1.394	1.467	1.346	1.181
	LCI	19.375	16.877	18.029	18.332	19.443	15.55
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-1.252	-1.753	-.714	.294	1.152	.38
	RMSE	1.569	1.951	1.566	1.496	1.771	1.241
	AVRB	.250	.351	.143	.059	.230	.076

Table 33 – Results of simulation study for SSLBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	.993	1.023	1.041	.999	1.000	1.006
	SD	.044	.011	.031	.016	.008	.010
	LCI	.301	.429	.556	.129	.184	.219
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.007	.023	.041	-.001	$<$ .001	.006
	RMSE	.045	.026	.052	.016	.008	.011
	AVRB	.007	.023	.041	.001	.000	.006
	Mean	.494	.508	.497	.485	.480	.487
	SD	.024	.014	.024	.028	.032	.016
	LCI	.338	.336	.325	.156	.181	.151
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.006	.008	-.003	-.015	-.020	-.013
	RMSE	.025	.017	.024	.031	.038	.021
	AVRB	.012	.016	.005	.030	.040	.027
	Mean	-.714	-.032	.778	-.781	-.008	.795
	SD	.157	.209	.091	.050	.163	.057
	LCI	.560	.885	.482	.216	.340	.211
γ	CP	1.000	.900	1.000	1.000	.200	.900
	Bias	.086	-.032	-.022	.019	-.008	-.005
	RMSE	.179	.211	.094	.053	.163	.058
	AVRB	.108	-	.027	.024	-	.006
	Mean	30.370	29.319	28.952	28.437	30.442	27.798
	SD	3.090	4.338	5.541	6.557	13.615	5.713
	LCI	82.194	85.022	77.390	70.396	77.893	70.605
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.370	-.681	-1.048	-1.563	.442	-2.202
	RMSE	3.112	4.392	5.639	6.741	13.622	6.123
	AVRB	.012	.023	.035	.052	.015	.073

Table 34 – Results of simulation study for SSLBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Median	.990	1.014	1.022	.999	1.000	1.000
	SD	.044	.008	.024	.017	.008	.011
	LCI	.301	.429	.556	.129	.184	.219
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.010	.014	.022	-.001	$>$ .001	$<$ .001
	RMSE	.045	.016	.032	.017	.008	.011
	AVRB	.010	.014	.022	.001	$<$ .001	$<$ .001
ϕ	Median	.489	.502	.492	.485	.484	.484
	SD	.019	.013	.022	.027	.029	.015
	LCI	.338	.336	.325	.156	.181	.151
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.011	.002	-.008	-.015	-.016	-.016
	RMSE	.022	.013	.023	.031	.033	.022
	AVRB	.022	.004	.017	.030	.032	.032
γ	Median	-.737	-.032	.802	-.785	-.013	.801
	SD	.164	.215	.097	.051	.158	.056
	LCI	.560	.885	.482	.216	.340	.211
	CP	1.000	.900	1.000	1.000	.200	.900
	Bias	.063	-.032	.002	.015	-.013	.001
	RMSE	.176	.217	.097	.053	.159	.056
	AVRB	.079	-	.003	.018	-	.001
ν	Median	24.813	23.585	23.198	22.531	25.226	22.476
	SD	3.204	4.180	6.461	9.044	13.441	5.348
	LCI	82.194	85.022	77.390	70.396	77.893	70.605
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-5.187	-6.415	-6.802	-7.469	-4.774	-7.524
	RMSE	6.097	7.657	9.381	11.730	14.263	9.231
	AVRB	.173	.214	.227	.249	.159	.251

Table 35 – Results of simulation study for SSLBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	.992	1.017	1.021	.999	1.000	1.005
	SD	.044	.009	.023	.017	.008	.011
	LCI	.301	.429	.556	.129	.184	.219
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.008	.017	.021	-.001	$>$ .001	.005
	RMSE	.045	.020	.031	.017	.008	.012
	AVRB	.008	.017	.021	.001	$<$ .001	.005
	Mode	.493	.504	.494	.486	.480	.486
	SD	.022	.014	.024	.029	.032	.016
	LCI	.338	.336	.325	.156	.181	.151
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.007	.004	-.006	-.014	-.020	-.014
	RMSE	.023	.015	.024	.032	.038	.021
	AVRB	.015	.008	.012	.029	.040	.028
	Mode	-.733	-.038	.796	-.781	-.010	.797
	SD	.163	.224	.090	.050	.162	.057
	LCI	.560	.885	.482	.216	.340	.211
γ	CP	1.000	.900	1.000	1.000	.200	.900
	Bias	.067	-.038	-.004	.019	-.010	-.003
	RMSE	.176	.227	.090	.053	.162	.057
	AVRB	.084	-	.005	.023	-	.004
	Mode	12.971	11.650	12.132	15.765	9.339	13.067
	SD	7.127	7.051	7.036	14.528	7.876	8.119
	LCI	82.194	85.022	77.390	70.396	77.893	70.605
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-17.029	-18.350	-17.868	-14.235	-20.661	-16.933
	RMSE	18.460	19.658	19.203	20.340	22.111	18.779
	AVRB	.568	.612	.596	.475	.689	.564

SCNBS distribution

Table 36 – Results of simulation study for SCNBS distribution ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mean	1.011	1.085	1.091	.998	1.002	1.016
	SD	.084	.098	.143	.044	.035	.054
	LCI	.633	1.032	1.053	.281	.379	.396
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.011	.085	.091	-.002	.002	.016
	RMSE	.085	.129	.170	.044	.035	.056
	AVRB	.011	.085	.091	.002	.002	.016
ϕ	Mean	3.305	2.969	3.124	2.554	2.376	2.629
	SD	.193	.461	.573	.964	.640	.905
	LCI	4.124	4.020	4.091	3.579	3.327	3.245
	CP	$< .001$	.100	$< .001$	.500	.200	.300
	Bias	2.805	2.469	2.624	2.054	1.876	2.129
	RMSE	2.811	2.512	2.686	2.269	1.982	2.313
	AVRB	5.610	4.938	5.249	4.108	3.752	4.258
γ	Mean	-.640	-.007	.637	-.741	-.080	.741
	SD	.215	.246	.144	.110	.150	.087
	LCI	.749	1.123	.829	.342	.443	.335
	CP	.900	.900	1.000	.800	.300	1.000
	Bias	.160	-.007	-.163	.059	-.080	-.059
	RMSE	.268	.246	.218	.125	.170	.106
	AVRB	.200	-	.204	.074	-	.074
ν_1	Mean	.562	.527	.541	.645	.665	.643
	SD	.058	.065	.071	.120	.062	.125
	LCI	.946	.904	.932	.811	.747	.779
	CP	1.000	.800	1.000	1.000	.900	1.000
	Bias	-.338	-.373	-.359	-.255	-.235	-.257
	RMSE	.343	.379	.366	.282	.243	.286
	AVRB	.376	.415	.399	.283	.261	.285
ν_2	Mean	.573	.471	.509	.457	.427	.481
	SD	.040	.106	.076	.164	.137	.153
	LCI	.696	.717	.705	.680	.606	.637
	CP	$< .001$	.200	$< .001$	.500	.200	.300
	Bias	.473	.371	.409	.357	.327	.381
	RMSE	.475	.386	.417	.392	.355	.411
	AVRB	4.734	3.713	4.095	3.568	3.271	3.815

Table 37 – Results of simulation study for SCNBS distribution ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Median	1.002	1.044	1.058	.998	.997	1.008
	SD	.085	.074	.137	.045	.035	.055
	LCI	.633	1.032	1.053	.281	.379	.396
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.002	.044	.058	-.002	-.003	.008
	RMSE	.085	.086	.149	.045	.035	.055
	AVRB	.002	.044	.058	.002	.003	.008
ϕ	Median	3.343	2.965	3.160	2.369	2.347	2.577
	SD	.215	.549	.665	1.261	.768	1.065
	LCI	4.124	4.020	4.091	3.579	3.327	3.245
	CP	$< .001$	.100	$< .001$	.500	.200	.300
	Bias	2.843	2.465	2.660	1.869	1.847	2.077
	RMSE	2.851	2.525	2.742	2.255	2.000	2.334
	AVRB	5.685	4.929	5.320	3.738	3.694	4.153
γ	Median	-.674	-.005	.680	-.751	-.080	.746
	SD	.223	.252	.153	.110	.141	.093
	LCI	.749	1.123	.829	.342	.443	.335
	CP	.900	.900	1.000	.800	.300	1.000
	Bias	.126	-.005	-.120	.049	-.080	-.054
	RMSE	.256	.252	.195	.121	.162	.107
	AVRB	.158	-	.150	.061	-	.067
ν_1	Median	.597	.551	.547	.703	.699	.671
	SD	.079	.097	.124	.137	.059	.153
	LCI	.946	.904	.932	.811	.747	.779
	CP	1.000	.800	1.000	1.000	.900	1.000
	Bias	-.303	-.349	-.353	-.197	-.201	-.229
	RMSE	.313	.363	.375	.239	.210	.276
	AVRB	.336	.388	.393	.218	.224	.255
ν_2	Median	.574	.458	.495	.416	.410	.462
	SD	.046	.124	.095	.211	.152	.179
	LCI	.696	.717	.705	.680	.606	.637
	CP	$< .001$	.200	$< .001$	.500	.200	.300
	Bias	.474	.358	.395	.316	.310	.362
	RMSE	.477	.379	.407	.380	.346	.404
	AVRB	4.744	3.576	3.953	3.159	3.104	3.623

Table 38 – Results of simulation study for SCNBS distribution ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mode	.999	1.016	1.055	.999	1.000	1.012
	SD	.083	.056	.182	.045	.035	.054
	LCI	.633	1.032	1.053	.281	.379	.396
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.001	.016	.055	-.001	>-.001	.012
	RMSE	.083	.058	.190	.045	.035	.055
	AVRB	.001	.016	.055	.001	$<$ .001	.012
ϕ	Mode	3.493	2.941	3.307	2.231	2.396	2.264
	SD	.378	.670	.980	1.600	1.006	1.458
	LCI	4.124	4.020	4.091	3.579	3.327	3.245
	CP	$<$ .001	.100	$<$ .001	.500	.200	.300
	Bias	2.993	2.441	2.807	1.731	1.896	1.764
	RMSE	3.017	2.532	2.973	2.357	2.147	2.289
	AVRB	5.986	4.883	5.613	3.462	3.792	3.528
γ	Mode	-.677	-.006	.708	-.744	-.081	.742
	SD	.212	.262	.131	.109	.145	.088
	LCI	.749	1.123	.829	.342	.443	.335
	CP	.900	.900	1.000	.800	.300	1.000
	Bias	.123	-.006	-.092	.056	-.081	-.058
	RMSE	.245	.262	.160	.122	.166	.105
	AVRB	.154	-	.115	.070	-	.072
ν_1	Mode	.666	.598	.530	.738	.726	.691
	SD	.155	.147	.209	.180	.045	.164
	LCI	.946	.904	.932	.811	.747	.779
	CP	1.000	.800	1.000	1.000	.900	1.000
	Bias	-.234	-.302	-.370	-.162	-.174	-.209
	RMSE	.281	.336	.425	.242	.180	.266
	AVRB	.260	.336	.411	.180	.193	.232
ν_2	Mode	.577	.453	.495	.412	.412	.461
	SD	.051	.127	.099	.215	.150	.190
	LCI	.696	.717	.705	.680	.606	.637
	CP	$<$ .001	.200	$<$ .001	.500	.200	.300
	Bias	.477	.353	.395	.312	.312	.361
	RMSE	.480	.375	.407	.379	.347	.408
	AVRB	4.768	3.530	3.949	3.117	3.125	3.613

Table 39 – Results of simulation study for SCNBS distribution ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
μ	Mean	.993	1.003	.984	.980	.988	.986
	SD	.130	.122	.117	.083	.087	.040
	LCI	.611	.651	.640	.245	.284	.311
	CP	.800	1.000	1.000	.800	.900	1.000
	Bias	-.007	.003	-.016	-.020	-.012	-.014
	RMSE	.130	.122	.118	.086	.088	.042
	AVRB	.007	.003	.016	.020	.012	.014
ϕ	Mean	.523	.514	.495	.507	.494	.479
	SD	.098	.063	.042	.022	.032	.045
	LCI	.527	.551	.510	.188	.216	.191
	CP	1.000	1.000	1.000	1.000	1.000	.900
	Bias	.023	.014	-.005	.007	-.006	-.021
	RMSE	.101	.064	.042	.023	.032	.049
	AVRB	.047	.028	.011	.014	.013	.042
γ	Mean	-.608	>-.001	.512	-.758	-.008	.740
	SD	.192	.326	.274	.075	.210	.087
	LCI	.726	1.005	.824	.270	.518	.316
	CP	.900	.900	.800	.900	.500	.900
	Bias	.192	>-.001	-.288	.042	-.008	-.060
	RMSE	.272	.326	.398	.086	.210	.106
	AVRB	.240	-	.360	.052	-	.075
ν_1	Mean	.246	.285	.294	.105	.143	.130
	SD	.155	.137	.140	.050	.051	.033
	LCI	.504	.607	.583	.115	.199	.121
	CP	1.000	.900	.900	.800	.900	.800
	Bias	.146	.185	.194	.005	.043	.030
	RMSE	.213	.230	.239	.050	.067	.044
	AVRB	1.457	1.851	1.936	.048	.433	.299
ν_2	Mean	.253	.262	.278	.114	.150	.118
	SD	.203	.146	.170	.037	.052	.015
	LCI	.383	.474	.434	.093	.152	.092
	CP	.600	.700	.700	.900	.700	1.000
	Bias	.153	.162	.178	.014	.050	.018
	RMSE	.254	.217	.246	.040	.072	.023
	AVRB	1.527	1.616	1.776	.139	.503	.182

Table 40 – Results of simulation study for SCNBS distribution ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	.970	.979	.960	.976	.981	.979
	SD	.123	.115	.109	.081	.087	.039
	LCI	.611	.651	.640	.245	.284	.311
μ	CP	.800	1.000	1.000	.800	.900	1.000
	Bias	-.030	-.021	-.040	-.024	-.019	-.021
	RMSE	.126	.117	.116	.085	.089	.044
	AVRB	.030	.021	.040	.024	.019	.021
	Median	.515	.507	.484	.506	.492	.476
	SD	.102	.065	.041	.022	.033	.045
	LCI	.527	.551	.510	.188	.216	.191
ϕ	CP	1.000	1.000	1.000	1.000	1.000	.900
	Bias	.015	.007	-.016	.006	-.008	-.024
	RMSE	.103	.066	.044	.023	.033	.050
	AVRB	.030	.014	.033	.011	.016	.047
	Median	-.632	.006	.533	-.766	-.013	.750
	SD	.194	.342	.290	.075	.208	.086
	LCI	.726	1.005	.824	.270	.518	.316
γ	CP	.900	.900	.800	.900	.500	.900
	Bias	.168	.006	-.267	.034	-.013	-.050
	RMSE	.257	.342	.395	.082	.209	.099
	AVRB	.210	-	.334	.043	-	.063
	Median	.230	.261	.278	.102	.136	.127
	SD	.157	.141	.159	.049	.050	.033
	LCI	.504	.607	.583	.115	.199	.121
ν_1	CP	1.000	.900	.900	.800	.900	.800
	Bias	.130	.161	.178	.002	.036	.027
	RMSE	.204	.214	.239	.049	.062	.042
	AVRB	1.303	1.614	1.778	.019	.362	.272
	Median	.237	.240	.260	.112	.148	.117
	SD	.196	.138	.176	.037	.052	.014
	LCI	.383	.474	.434	.093	.152	.092
ν_2	CP	.600	.700	.700	.900	.700	1.000
	Bias	.137	.140	.160	.012	.048	.017
	RMSE	.239	.197	.238	.039	.070	.022
	AVRB	1.369	1.399	1.602	.120	.476	.167

Table 41 – Results of simulation study for SCNBS distribution ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	.964	.975	.954	.978	.985	.983
	SD	.118	.110	.101	.082	.087	.039
	LCI	.611	.651	.640	.245	.284	.311
μ	CP	.800	1.000	1.000	.800	.900	1.000
	Bias	-.036	-.025	-.046	-.022	-.015	-.017
	RMSE	.123	.112	.111	.085	.088	.043
	AVRB	.036	.025	.046	.022	.015	.017
	Mode	.515	.507	.485	.506	.494	.479
	SD	.099	.063	.041	.022	.031	.045
	LCI	.527	.551	.510	.188	.216	.191
ϕ	CP	1.000	1.000	1.000	1.000	1.000	.900
	Bias	.015	.007	-.015	.006	-.006	-.021
	RMSE	.100	.063	.044	.023	.031	.049
	AVRB	.029	.014	.030	.013	.013	.042
	Mode	-.644	.016	.544	-.761	-.010	.744
	SD	.180	.360	.292	.073	.206	.086
	LCI	.726	1.005	.824	.270	.518	.316
γ	CP	.900	.900	.800	.900	.500	.900
	Bias	.156	.016	-.256	.039	-.010	-.056
	RMSE	.239	.361	.389	.082	.206	.102
	AVRB	.196	-	.320	.048	-	.070
	Mode	.226	.253	.291	.104	.142	.129
	SD	.144	.133	.180	.049	.051	.033
	LCI	.504	.607	.583	.115	.199	.121
ν_1	CP	1.000	.900	.900	.800	.900	.800
	Bias	.126	.153	.191	.004	.042	.029
	RMSE	.192	.202	.263	.049	.066	.044
	AVRB	1.264	1.528	1.910	.040	.417	.295
	Mode	.241	.241	.264	.114	.149	.118
	SD	.192	.131	.172	.038	.052	.014
	LCI	.383	.474	.434	.093	.152	.092
ν_2	CP	.600	.700	.700	.900	.700	1.000
	Bias	.141	.141	.164	.014	.049	.018
	RMSE	.238	.192	.237	.040	.072	.023
	AVRB	1.406	1.410	1.637	.138	.495	.179

APPENDIX B – Results of Chapter 2

In this section, we present in detail the moments of the ZA-SSBS distributions. Also, we present all results of the parameter recovery study.

B.1 The moments of the ZA-SSBS distribution

Le $T \sim$ ZA-SSBS$(p, \mu, \phi, \gamma, \boldsymbol{\nu})$ and $Y \sim$ SSMN$(0, 1, \gamma, \boldsymbol{\nu})$. If $\mathbb{E}\left[Y^{2(r-j+i)}\right] < \infty$, then $\mathbb{E}(T^r)$ exist and are given by,

$$\mathbb{E}(T^r) = (1-p)\mu_r.$$

Proof. To get $\mathbb{E}(T^r)$ and $\mathbb{V}(T)$ we use the equality: $\mathbb{E}(T^r) = \mathbb{E}\left[\mathbb{E}(T^r|\mathbb{1}\{t=0)\}\right]$ and $\mathbb{V}(T) = \mathbb{E}\left[\mathbb{V}(T|\mathbb{1}\{t=0\})\right] + \mathbb{V}[\mathbb{E}(T|\mathbb{1}\{t=0\})]$ (Ospina Martinez, 2008), where

$$\mathbb{E}(T^r|\mathbb{1}\{t=0\}) = \begin{cases} 0, \text{with probability}\, p \\ \mu_r, \text{with probability}\, (1-p) \end{cases}$$

$$\mathbb{V}(T|\mathbb{1}\{t=0\}) = \begin{cases} 0, \text{with probability}\, p \\ \dfrac{2\phi\mu^2}{[1+\phi m_2]^2}\left[m_2 + \dfrac{\phi}{2}\left\{2m_4 - m_2^2\right\}\right], \text{with probability}\, (1-p). \end{cases}$$

Therefore, we have that $\mathbb{E}(T^r) = (1-p)\mu_r$. Consequently, it comes that $\mathbb{E}(T) = (1-p)\mu$ and

$$\begin{aligned} \mathbb{V}\left[\mathbb{E}(T|\mathbb{1}\{t=0\})\right] &= \mathbb{E}\left[\mathbb{V}(T|\mathbb{1}\{t=0\})\right] + \mathbb{V}[\mathbb{E}(T|\mathbb{1}\{t=0\})] \\ &= (1-p)\left\{\frac{2\phi\mu^2}{[1+\phi m_2]^2}\left[m_2 + \frac{\phi}{2}\left\{2m_4 - m_2^2\right\}\right]\right\} \\ &\quad + (1-p)\mu^2 - (1-p)^2\mu^2 \\ &= (1-p)\left\{\frac{2\phi\mu^2}{[1+\phi m_2]^2}\left[m_2 + \frac{\phi}{2}\left\{2m_4 - m_2^2\right\}\right]\right\} \\ &\quad + p(1-p)\mu^2, \end{aligned}$$

where $m_k = \mathbb{E}[Y^k], k = 2, 4.$ □

B.2 Results of the parameter recovery study

ZA-SGtBS distribution

Table 42 – Results of simulation study for ZA-SGtBS distribution ($\nu_1 = 15, \nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mean	.096	.097	.098	.104	.104	.104
	SD	.031	.031	.032	.015	.015	.015
	LCI	.109	.112	.114	.053	.052	.052
	CP	.900	.900	1.000	.900	.900	.900
	Bias	-.004	-.003	-.002	.004	.004	.004
	RMSE	.031	.031	.032	.016	.016	.016
	AVRB	.036	.028	.020	.042	.039	.042
μ	Mean	.996	1.043	1.039	1.003	1.008	.999
	SD	.043	.034	.055	.008	.014	.020
	LCI	.304	.431	.507	.135	.169	.204
	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	-.004	.043	.039	.003	.008	-.001
	RMSE	.043	.055	.067	.009	.016	.020
	AVRB	.004	.043	.039	.003	.008	.001
γ	Mean	-.706	-.060	.698	-.769	-.043	.742
	SD	.162	.310	.099	.078	.124	.068
	LCI	.700	1.105	.697	.344	.447	.354
	CP	1.000	.9000	1.000	1.000	.500	1.000
	Bias	.094	-.060	-.102	.031	-.043	-.058
	RMSE	.187	.316	.142	.084	.131	.089
	AVRB	.118	-	.128	.039	-	.072
ν_1	Mean	16.510	14.819	15.385	19.114	18.016	19.643
	SD	4.038	3.260	2.850	8.675	4.376	6.574
	LCI	40.691	33.860	36.735	33.622	32.956	35.497
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.51	-.181	.385	4.114	3.016	4.643
	RMSE	4.311	3.265	2.876	9.601	5.315	8.048
	AVRB	.101	.012	.026	.274	.201	.310
ν_2	Mean	5.528	5.262	5.188	6.483	6.144	6.717
	SD	1.330	1.256	1.157	3.126	1.566	2.355
	LCI	15.185	13.085	13.752	12.332	12.387	13.165
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.528	.262	.188	1.483	1.144	1.717
	RMSE	1.431	1.283	1.173	3.46	1.94	2.914
	AVRB	.106	.052	.038	.297	.229	.343

Table 43 – Results of simulation study for ZA-SGtBS distribution ($\nu_1 = 15, \nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Median	.094	.095	.095	.104	.103	.104
	SD	.032	.031	.032	.015	.016	.015
	LCI	.109	.112	.114	.053	.052	.052
	CP	.900	.900	1.000	.900	.900	.900
	Bias	-.006	-.005	-.005	.004	.003	.004
	RMSE	.032	.032	.032	.016	.016	.016
	AVRB	.060	.055	.050	.038	.034	.037
μ	Median	.992	1.034	1.019	1.003	1.006	.996
	SD	.041	.033	.056	.008	.014	.020
	LCI	.304	.431	.507	.135	.169	.204
	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	-.008	.034	.019	.003	.006	-.004
	RMSE	.042	.047	.059	.009	.016	.020
	AVRB	.008	.034	.019	.003	.006	.004
γ	Median	-.744	-.065	.734	-.777	-.039	.748
	SD	.161	.332	.102	.081	.115	.070
	LCI	.700	1.105	.697	.344	.447	.354
	CP	1.000	.900	1.000	1.000	.500	1.000
	Bias	.056	-.065	-.066	.023	-.039	-.052
	RMSE	.171	.339	.122	.084	.121	.087
	AVRB	.070	-	.083	.029	-	.065
ν_1	Median	13.132	11.909	12.228	17.218	15.627	17.330
	SD	2.563	1.798	1.721	8.092	3.735	5.729
	LCI	40.691	33.86	36.735	33.622	32.956	35.497
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-1.868	-3.091	-2.772	2.218	.627	2.330
	RMSE	3.172	3.575	3.263	8.391	3.787	6.185
	AVRB	.125	.206	.185	.148	.042	.155
ν_2	Median	4.196	4.070	3.943	5.782	5.249	5.855
	SD	.843	.680	.651	2.924	1.323	2.072
	LCI	15.185	13.085	13.752	12.332	12.387	13.165
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.804	-.930	-1.057	.782	.249	.855
	RMSE	1.165	1.153	1.242	3.027	1.346	2.241
	AVRB	.161	.186	.211	.156	.050	.171

Table 44 – Results of simulation study for ZA-SGtBS distribution ($\nu_1 = 15, \nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.097	.097	.098	.104	.104	.104
	SD	.031	.031	.032	.015	.016	.016
	LCI	.109	.112	.114	.053	.052	.052
	CP	.900	.900	1.000	.900	.900	.900
	Bias	-.003	-.003	-.002	.004	.004	.004
	RMSE	.031	.031	.032	.016	.016	.016
	AVRB	.03	.028	.016	.044	.037	.043
μ	Mode	.994	1.035	1.024	1.003	1.007	.998
	SD	.042	.032	.053	.008	.015	.020
	LCI	.304	.431	.507	.135	.169	.204
	CP	.900	1.000	1.000	1.000	1.000	1.000
	Bias	-.006	.035	.024	.003	.007	-.002
	RMSE	.042	.048	.059	.009	.016	.020
	AVRB	.006	.035	.024	.003	.007	.002
γ	Mode	-.753	-.079	.742	-.774	-.039	.745
	SD	.144	.366	.092	.077	.122	.068
	LCI	.700	1.105	.697	.344	.447	.354
	CP	1.000	.900	1.000	1.000	.500	1.000
	Bias	.047	-.079	-.058	.026	-.039	-.055
	RMSE	.152	.375	.109	.081	.128	.087
	AVRB	.059	-	.073	.033	-	.068
ν_1	Mode	9.583	8.641	9.101	14.344	11.852	13.591
	SD	1.683	.837	.646	7.378	2.830	4.375
	LCI	40.691	33.86	36.735	33.622	32.956	35.497
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-5.417	-6.359	-5.899	-.656	-3.148	-1.409
	RMSE	5.672	6.413	5.934	7.407	4.233	4.596
	AVRB	.361	.424	.393	.044	.210	.094
ν_2	Mode	2.657	2.483	2.628	4.713	3.862	4.241
	SD	.527	.200	.492	2.601	1.025	1.765
	LCI	15.185	13.085	13.752	12.332	12.387	13.165
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-2.343	-2.517	-2.372	-.287	-1.138	-.759
	RMSE	2.401	2.525	2.422	2.616	1.532	1.922
	AVRB	.469	.503	.474	.057	.228	.152

Table 45 – Results of simulation study for ZA-SGtBS distribution ($\nu_1 = 30, \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mean	.108	.108	.108	.106	.106	.106
	SD	.031	.030	.032	.013	.013	.013
	LCI	.116	.118	.119	.053	.053	.054
	CP	.900	1.000	.900	1.000	1.000	1.000
	Bias	.008	.008	.008	.006	.006	.006
	RMSE	.032	.031	.033	.014	.014	.015
	AVRB	.083	.076	.079	.058	.057	.062
μ	Mean	1.065	1.073	1.039	1.034	1.006	.994
	SD	.054	.057	.109	.016	.030	.032
	LCI	.547	.69	.765	.205	.284	.315
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.065	.073	.039	.034	.006	-.006
	RMSE	.084	.093	.116	.037	.031	.033
	AVRB	.065	.073	.039	.034	.006	.006
γ	Mean	-.765	-.148	.732	-.850	.032	.856
	SD	.116	.262	.212	.042	.088	.054
	LCI	.646	1.227	.675	.289	.481	.276
	CP	1.000	.900	1.000	1.000	.800	.900
	Bias	.035	-.148	-.068	-.050	.032	.056
	RMSE	.121	.301	.223	.065	.094	.078
	AVRB	.044	-	.085	.062	-	.070
ν_1	Mean	11.131	9.640	11.771	17.906	18.952	18.645
	SD	2.044	2.063	2.501	4.849	4.999	6.365
	LCI	26.123	21.518	29.132	42.573	39.602	36.604
	CP	.600	.100	.700	.800	.800	.900
	Bias	-18.869	-20.360	-18.229	-12.094	-11.048	-11.355
	RMSE	18.979	20.465	18.400	13.03	12.126	13.017
	AVRB	.629	.679	.608	.403	.368	.378
ν_2	Mean	10.197	8.837	10.942	17.288	18.339	18.292
	SD	2.685	2.855	3.120	5.434	5.408	7.729
	LCI	28.091	23.677	32.183	44.948	42.186	38.802
	CP	.600	.200	.600	.800	.800	.900
	Bias	-19.803	-21.163	-19.058	-12.712	-11.661	-11.708
	RMSE	19.984	21.355	19.312	13.825	12.854	14.029
	AVRB	.660	.705	.635	.424	.389	.390

Table 46 – Results of simulation study for ZA-SGtBS distribution ($\nu_1 = 30, \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Median	.106	.105	.106	.105	.105	.106
	SD	.032	.031	.032	.012	.013	.013
	LCI	.116	.118	.119	.053	.053	.054
	CP	.900	1.000	.900	1.000	1.000	1.000
	Bias	.006	.005	.006	.005	.005	.006
	RMSE	.033	.031	.033	.014	.014	.014
	AVRB	.062	.050	.058	.055	.053	.056
μ	Median	1.050	1.052	1.021	1.032	1.002	.989
	SD	.054	.057	.111	.018	.029	.035
	LCI	.547	.690	.765	.205	.284	.315
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.050	.052	.021	.032	.002	-.011
	RMSE	.074	.077	.113	.037	.029	.036
	AVRB	.050	.052	.021	.032	.002	.011
γ	Median	-.805	-.146	.769	-.86	.029	.864
	SD	.118	.280	.218	.045	.083	.054
	LCI	.646	1.227	.675	.289	.481	.276
	CP	1.000	.900	1.000	1.000	.800	.900
	Bias	-.005	-.146	-.031	-.060	.029	.064
	RMSE	.118	.316	.220	.075	.088	.084
	AVRB	.006	-	.039	.075	-	.080
ν_1	Median	9.132	7.930	9.395	14.197	15.632	16.046
	SD	1.419	1.415	1.743	2.578	4.049	6.025
	LCI	26.123	21.518	29.132	42.573	39.602	36.604
	CP	.600	.100	.700	.800	.800	.900
	Bias	-20.868	-22.070	-20.605	-15.803	-14.368	-13.954
	RMSE	20.916	22.115	20.679	16.012	14.928	15.199
	AVRB	.696	.736	.687	.527	.479	.465
ν_2	Median	7.973	6.903	8.369	13.362	14.760	15.525
	SD	1.912	2.044	2.179	3.163	4.359	7.273
	LCI	28.091	23.677	32.183	44.948	42.186	38.802
	CP	.600	.200	.600	.800	.800	.900
	Bias	-22.027	-23.097	-21.631	-16.638	-15.240	-14.475
	RMSE	22.110	23.187	21.740	16.936	15.851	16.200
	AVRB	.734	.770	.721	.555	.508	.483

Table 47 – Results of simulation study for ZA-SGtBS distribution ($\nu_1 = 30, \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.108	.107	.108	.106	.106	.106
	SD	.031	.030	.032	.013	.013	.014
	LCI	.116	.118	.119	.053	.053	.054
	CP	.900	1.000	.900	1.000	1.000	1.000
	Bias	.008	.007	.008	.006	.006	.006
	RMSE	.032	.031	.033	.014	.014	.015
	AVRB	.077	.074	.083	.059	.056	.059
μ	Mode	1.050	1.048	1.006	1.033	1.005	.993
	SD	.053	.053	.103	.016	.030	.033
	LCI	.547	.69	.765	.205	.284	.315
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.050	.048	.006	.033	.005	-.007
	RMSE	.073	.071	.103	.037	.030	.034
	AVRB	.050	.048	.006	.033	.005	.007
γ	Mode	-.812	-.148	.773	-.854	.03	.86
	SD	.094	.303	.21	.041	.089	.053
	LCI	.646	1.227	.675	.289	.481	.276
	CP	1.000	.900	1.000	1.000	.800	.900
	Bias	-.012	-.148	-.027	-.054	.03	.06
	RMSE	.095	.337	.211	.068	.094	.08
	AVRB	.014	-	.034	.068	-	.075
ν_1	Mode	7.177	5.407	6.456	10.897	12.353	12.977
	SD	1.257	1.091	.738	1.562	2.105	3.706
	LCI	26.123	21.518	29.132	42.573	39.602	36.604
	CP	.600	.100	.700	.800	.800	.900
	Bias	-22.823	-24.593	-23.544	-19.103	-17.647	-17.023
	RMSE	22.858	24.617	23.556	19.167	17.773	17.421
	AVRB	.761	.82	.785	.637	.588	.567
ν_2	Mode	5.684	4.680	5.411	9.774	10.904	11.245
	SD	1.622	1.069	1.105	1.522	2.560	3.346
	LCI	28.091	23.677	32.183	44.948	42.186	38.802
	CP	.600	.200	.600	.800	.800	.900
	Bias	-24.316	-25.32	-24.589	-20.226	-19.096	-18.755
	RMSE	24.370	25.342	24.614	20.283	19.267	19.051
	AVRB	.811	.844	.820	.674	.637	.625

ZA-StBS distribution

Table 48 – Results of simulation study for ZA-StBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	.116	.116	.115	.099	.100	.100
	SD	.035	.037	.035	.015	.015	.015
	LCI	.114	.121	.118	.052	.052	.053
p	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.016	.016	.015	-.001	$< .001$	$< .001$
	RMSE	.038	.04	.038	.015	.015	.015
	AVRB	.157	.157	.153	.006	$< .001$	.005
	Mean	1.007	1.103	1.032	1.023	1.021	1.034
	SD	.074	.214	.126	.059	.057	.069
	LCI	.628	1.125	.772	.286	.374	.347
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.007	.103	.032	.023	.021	.034
	RMSE	.074	.237	.130	.063	.061	.077
	AVRB	.007	.103	.032	.023	.021	.034
	Mean	.611	.580	.555	.492	.502	.474
	SD	.131	.089	.075	.036	.043	.037
	LCI	.483	.505	.492	.211	.223	.206
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.111	.080	.055	-.008	.002	-.026
	RMSE	.172	.12	.093	.036	.043	.046
	AVRB	.223	.159	.109	.016	.003	.052
	Mean	-.586	.061	.572	-.730	-.068	.771
	SD	.160	.238	.137	.120	.181	.072
	LCI	.875	1.234	.931	.453	.538	.424
γ	CP	1.000	1.000	1.000	1.000	.300	1.000
	Bias	.214	.061	-.228	.070	-.068	-.029
	RMSE	.267	.246	.266	.139	.193	.078
	AVRB	.267	-	.286	.088	-	.036
	Mean	14.879	11.945	11.667	5.121	5.278	4.715
	SD	6.318	7.310	7.126	.992	1.459	1.158
	LCI	51.612	40.990	38.070	4.553	5.321	3.877
ν	CP	1.000	.900	.800	1.000	.800	1.000
	Bias	9.879	6.945	6.667	.121	.278	-.285
	RMSE	11.726	10.083	9.758	1.000	1.485	1.193
	AVRB	1.976	1.389	1.333	.024	.056	.057

Table 49 – Results of simulation study for ZA-StBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Median	.114	.113	.113	.099	.100	.099
	SD	.036	.037	.035	.015	.015	.015
	LCI	.114	.121	.118	.052	.052	.053
	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.014	.013	.013	-.001	$< .001$	-.001
	RMSE	.038	.039	.038	.015	.015	.015
	AVRB	.140	.132	.127	.009	.005	.010
μ	Median	.980	1.021	.998	1.015	1.008	1.029
	SD	.055	.101	.112	.055	.051	.070
	LCI	.628	1.125	.772	.286	.374	.347
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.020	.021	-.002	.015	.008	.029
	RMSE	.058	.103	.112	.058	.052	.075
	AVRB	.020	.021	.002	.015	.008	.029
ϕ	Median	.608	.568	.541	.491	.498	.471
	SD	.145	.089	.078	.036	.043	.037
	LCI	.483	.505	.492	.211	.223	.206
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.108	.068	.041	-.009	-.002	-.029
	RMSE	.180	.112	.089	.037	.043	.047
	AVRB	.216	.135	.082	.018	.003	.058
γ	Median	-.621	.055	.606	-.742	-.054	.786
	SD	.174	.257	.147	.126	.167	.073
	LCI	.875	1.234	.931	.453	.538	.424
	CP	1.000	1.000	1.000	1.000	.300	1.000
	Bias	.179	.055	-.194	.058	-.054	-.014
	RMSE	.25	.263	.244	.139	.175	.075
	AVRB	.223	-	.243	.073	-	.018
ν	Median	10.225	8.101	8.212	4.908	5.003	4.551
	SD	3.863	4.602	4.633	.874	1.230	1.093
	LCI	51.612	40.990	38.070	4.553	5.321	3.877
	CP	1.000	.900	.800	1.000	.800	1.000
	Bias	5.225	3.101	3.212	-.092	.003	-.449
	RMSE	6.498	5.549	5.638	.878	1.230	1.181
	AVRB	1.045	.620	.642	.018	.001	.090

Table 50 – Results of simulation study for ZA-StBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.117	.116	.116	.100	.100	.100
	SD	.036	.036	.035	.015	.014	.014
	LCI	.114	.121	.118	.052	.052	.053
	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.017	.016	.016	$< .001$	$< .001$	$< .001$
	RMSE	.039	.039	.039	.015	.014	.014
	AVRB	.166	.160	.155	.001	.001	.002
μ	Mode	.978	1.001	.987	1.020	1.012	1.031
	SD	.051	.082	.100	.056	.051	.068
	LCI	.628	1.125	.772	.286	.374	.347
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.022	.001	-.013	.020	.012	.031
	RMSE	.055	.082	.101	.059	.052	.074
	AVRB	.022	.001	.013	.020	.012	.031
ϕ	Mode	.607	.57	.544	.492	.501	.473
	SD	.143	.087	.074	.036	.043	.036
	LCI	.483	.505	.492	.211	.223	.206
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.107	.070	.044	-.008	.001	-.027
	RMSE	.179	.112	.087	.037	.043	.045
	AVRB	.214	.139	.088	.017	.003	.053
γ	Mode	-.636	.046	.637	-.738	-.058	.781
	SD	.171	.285	.148	.123	.169	.071
	LCI	.875	1.234	.931	.453	.538	.424
	CP	1.000	1.000	1.000	1.000	.300	1.000
	Bias	.164	.046	-.163	.062	-.058	-.019
	RMSE	.237	.289	.220	.138	.178	.074
	AVRB	.205	-	.204	.078	-	.024
ν	Mode	6.550	4.869	5.201	4.626	4.676	4.207
	SD	1.978	1.955	2.196	.683	1.165	.842
	LCI	51.612	40.990	38.070	4.553	5.321	3.877
	CP	1.000	.900	.800	1.000	.800	1.000
	Bias	1.550	-.131	.201	-.374	-.324	-.793
	RMSE	2.513	1.959	2.205	.779	1.210	1.156
	AVRB	.310	.026	.040	.075	.065	.159

Table 51 – Results of simulation study for ZA-StBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mean	.106	.108	.109	.100	.100	.100
	SD	.026	.025	.026	.010	.010	.011
	LCI	.117	.118	.117	.051	.052	.051
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.006	.008	.009	$< .001$	$< .001$	$< .001$
	RMSE	.027	.026	.028	.010	.010	.011
	AVRB	.064	.075	.086	.002	.002	$< .001$
μ	Mean	1.008	1.058	1.072	1.008	1.011	1.006
	SD	.035	.064	.063	.015	.019	.036
	LCI	.407	.615	.670	.160	.208	.243
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.008	.058	.072	.008	.011	.006
	RMSE	.036	.086	.096	.017	.022	.037
	AVRB	.008	.058	.072	.008	.011	.006
ϕ	Mean	.470	.490	.459	.485	.496	.489
	SD	.054	.051	.055	.031	.032	.038
	LCI	.380	.374	.380	.166	.171	.165
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.030	-.010	-.041	-.015	-.004	-.011
	RMSE	.062	.052	.069	.035	.033	.039
	AVRB	.061	.020	.082	.031	.009	.022
γ	Mean	-.757	-.005	.672	-.783	.052	.801
	SD	.078	.295	.101	.075	.114	.085
	LCI	.670	1.201	.839	.315	.514	.300
	CP	1.000	.900	1.000	1.000	.800	.800
	Bias	.043	-.005	-.128	.017	.052	.001
	RMSE	.089	.295	.163	.077	.126	.085
	AVRB	.053	-	.160	.021	-	.002
ν	Mean	20.086	18.069	19.122	28.620	32.301	30.809
	SD	7.181	6.790	7.077	8.776	8.943	10.567
	LCI	68.673	67.358	69.490	78.414	90.066	82.876
	CP	1.000	1.000	.900	1.000	1.000	.900
	Bias	-9.914	-11.931	-10.878	-1.380	2.301	.809
	RMSE	12.242	13.728	12.977	8.884	9.234	10.598
	AVRB	.330	.398	.363	.046	.077	.027

Table 52 – Results of simulation study for ZA-StBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	.104	.105	.106	.100	.099	.099
	SD	.027	.025	.027	.010	.010	.011
	LCI	.117	.118	.117	.051	.052	.051
p	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.004	.005	.006	$< .001$	-.001	-.001
	RMSE	.027	.026	.028	.010	.010	.011
	AVRB	.038	.051	.065	.004	.005	.009
	Median	.999	1.035	1.046	1.006	1.008	1.003
	SD	.034	.059	.049	.015	.019	.038
	LCI	.407	.615	.670	.160	.208	.243
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.001	.035	.046	.006	.008	.003
	RMSE	.034	.068	.067	.016	.020	.038
	AVRB	.001	.035	.046	.006	.008	.003
	Median	.458	.481	.448	.483	.494	.489
	SD	.051	.053	.056	.031	.032	.038
	LCI	.380	.374	.380	.166	.171	.165
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.042	-.019	-.052	-.017	-.006	-.011
	RMSE	.066	.056	.076	.035	.032	.040
	AVRB	.083	.039	.104	.035	.011	.023
	Median	-.798	.003	.721	-.789	.050	.809
	SD	.075	.306	.094	.075	.115	.087
	LCI	.670	1.201	.839	.315	.514	.300
γ	CP	1.000	.900	1.000	1.000	.800	.800
	Bias	.002	.003	-.079	.011	.050	.009
	RMSE	.075	.306	.123	.076	.125	.087
	AVRB	.002	-	.099	.014	-	.011
	Median	13.841	12.028	12.899	21.729	24.763	24.050
	SD	4.924	4.367	4.965	6.644	7.176	8.519
	LCI	68.673	67.358	69.490	78.414	90.066	82.876
ν	CP	1.000	1.000	.900	1.000	1.000	.900
	Bias	-16.159	-17.971	-17.101	-8.271	-5.237	-5.950
	RMSE	16.892	18.495	17.807	10.609	8.883	10.391
	AVRB	.539	.599	.570	.276	.175	.198

Table 53 – Results of simulation study for ZA-StBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	.106	.108	.108	.100	.099	.101
	SD	.026	.025	.026	.010	.010	.010
	LCI	.117	.118	.117	.051	.052	.051
p	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.006	.008	.008	$<$ .001	-.001	.001
	RMSE	.027	.027	.027	.010	.010	.010
	AVRB	.061	.075	.079	.001	.009	.006
	Mode	1.001	1.034	1.042	1.007	1.011	1.005
	SD	.034	.056	.044	.015	.019	.037
	LCI	.407	.615	.670	.160	.208	.243
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.001	.034	.042	.007	.011	.005
	RMSE	.034	.065	.061	.017	.022	.037
	AVRB	.001	.034	.042	.007	.011	.005
	Mode	.464	.485	.453	.484	.495	.489
	SD	.051	.052	.055	.031	.032	.038
	LCI	.380	.374	.380	.166	.171	.165
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.036	-.015	-.047	-.016	-.005	-.011
	RMSE	.063	.054	.073	.035	.033	.040
	AVRB	.073	.029	.095	.033	.009	.023
	Mode	-.804	.015	.744	-.786	.049	.805
	SD	.063	.312	.075	.075	.115	.085
	LCI	.670	1.201	.839	.315	.514	.300
γ	CP	1.000	.900	1.000	1.000	.800	.800
	Bias	-.004	.015	-.056	.014	.049	.005
	RMSE	.063	.313	.093	.077	.125	.085
	AVRB	.005	-	.07	.017	-	.006
	Mode	8.288	6.739	8.236	16.788	16.985	16.015
	SD	2.563	2.110	4.168	5.315	5.875	6.625
	LCI	68.673	67.358	69.490	78.414	90.066	82.876
ν	CP	1.000	1.000	.900	1.000	1.000	.900
	Bias	-21.712	-23.261	-21.764	-13.212	-13.015	-13.985
	RMSE	21.863	23.356	22.159	14.241	14.280	15.475
	AVRB	.724	.775	.725	.440	.434	.466

ZA-SSLBS distribution

Table 54 – Results of simulation study for ZA-SSLBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	.108	.108	.106	.107	.108	.108
	SD	.025	.025	.025	.008	.007	.008
	LCI	.118	.116	.114	.054	.053	.053
p	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.008	.008	.006	.007	.008	.008
	RMSE	.027	.027	.025	.011	.01	.011
	AVRB	.082	.079	.058	.075	.078	.082
	Mean	.992	1.036	1.025	1.004	1.003	1.010
	SD	.047	.043	.039	.024	.020	.028
	LCI	.340	.507	.595	.163	.226	.247
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.008	.036	.025	.004	.003	.010
	RMSE	.047	.056	.046	.025	.021	.030
	AVRB	.008	.036	.025	.004	.003	.010
	Mean	.519	.531	.520	.521	.522	.514
	SD	.020	.026	.026	.036	.034	.036
	LCI	.399	.397	.396	.217	.237	.217
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.019	.031	.020	.021	.022	.014
	RMSE	.027	.041	.033	.042	.04	.039
	AVRB	.038	.061	.039	.042	.044	.028
	Mean	-.747	.007	.648	-.742	.082	.771
	SD	.108	.243	.182	.065	.131	.038
	LCI	.594	1.014	.697	.291	.447	.298
γ	CP	1.000	1.000	.900	1.000	.500	1.000
	Bias	.053	.007	-.152	.058	.082	-.029
	RMSE	.121	.243	.237	.087	.155	.048
	AVRB	.067	-	.190	.073	-	.037
	Mean	7.99	7.655	7.519	8.533	8.167	7.385
	SD	.974	.946	.674	2.235	2.183	2.678
	LCI	19.005	19.365	17.193	18.376	17.278	14.966
ν	CP	1.000	1.000	1.000	.900	1.000	1.000
	Bias	2.99	2.655	2.519	3.533	3.167	2.385
	RMSE	3.145	2.819	2.608	4.181	3.847	3.586
	AVRB	.598	.531	.504	.707	.633	.477

Table 55 – Results of simulation study for ZA-SSLBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	.106	.105	.104	.107	.107	.107
	SD	.026	.026	.024	.008	.007	.007
	LCI	.118	.116	.114	.054	.053	.053
p	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.006	.005	.004	.007	.007	.007
	RMSE	.026	.026	.025	.010	.010	.010
	AVRB	.064	.054	.035	.068	.073	.074
	Median	.990	1.026	1.004	1.004	1.001	1.008
	SD	.045	.043	.035	.026	.018	.030
	LCI	.340	.507	.595	.163	.226	.247
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.010	.026	.004	.004	.001	.008
	RMSE	.047	.050	.035	.026	.018	.031
	AVRB	.010	.026	.004	.004	.001	.008
	Median	.511	.524	.512	.521	.524	.512
	SD	.022	.026	.026	.039	.034	.038
	LCI	.399	.397	.396	.217	.237	.217
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.011	.024	.012	.021	.024	.012
	RMSE	.025	.035	.028	.044	.042	.040
	AVRB	.023	.048	.024	.041	.049	.024
	Median	-.776	.001	.679	-.746	.079	.775
	SD	.107	.253	.184	.065	.129	.039
	LCI	.594	1.014	.697	.291	.447	.298
γ	CP	1.000	1.000	.900	1.000	.500	1.000
	Bias	.024	.001	-.121	.054	.079	-.025
	RMSE	.110	.253	.220	.084	.151	.046
	AVRB	.030	-	.151	.067	-	.031
	Median	6.427	6.119	6.246	7.113	6.935	6.359
	SD	.975	.988	.777	1.685	1.928	2.326
	LCI	19.005	19.365	17.193	18.376	17.278	14.966
ν	CP	1.000	1.000	1.000	.900	1.000	1.000
	Bias	1.427	1.119	1.246	2.113	1.935	1.359
	RMSE	1.729	1.493	1.469	2.703	2.731	2.694
	AVRB	.285	.224	.249	.423	.387	.272

Table 56 – Results of simulation study for ZA-SSLBS distribution ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.107	.108	.106	.107	.107	.108
	SD	.025	.025	.024	.008	.007	.008
	LCI	.118	.116	.114	.054	.053	.053
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.007	.008	.006	.007	.007	.008
	RMSE	.026	.026	.025	.011	.010	.011
	AVRB	.070	.084	.058	.075	.074	.081
μ	Mode	.989	1.026	1.003	1.004	1.003	1.009
	SD	.046	.042	.035	.025	.020	.029
	LCI	.340	.507	.595	.163	.226	.247
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.011	.026	.003	.004	.003	.009
	RMSE	.047	.049	.035	.025	.020	.030
	AVRB	.011	.026	.003	.004	.003	.009
ϕ	Mode	.515	.527	.516	.522	.522	.514
	SD	.020	.027	.026	.037	.033	.037
	LCI	.399	.397	.396	.217	.237	.217
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.015	.027	.016	.022	.022	.014
	RMSE	.025	.038	.030	.042	.040	.040
	AVRB	.030	.054	.031	.043	.044	.028
γ	Mode	-.777	.004	.690	-.743	.081	.773
	SD	.099	.261	.169	.065	.130	.038
	LCI	.594	1.014	.697	.291	.447	.298
	CP	1.000	1.000	.900	1.000	.500	1.000
	Bias	.023	.004	-.110	.057	.081	-.027
	RMSE	.102	.261	.201	.086	.153	.047
	AVRB	.029	-	.137	.071	-	.033
ν	Mode	4.183	3.674	3.951	4.741	5.47	4.859
	SD	1.068	.934	1.027	1.092	2.122	1.803
	LCI	19.005	19.365	17.193	18.376	17.278	14.966
	CP	1.000	1.000	1.000	.900	1.000	1.000
	Bias	-.817	-1.326	-1.049	-.259	.470	-.141
	RMSE	1.345	1.622	1.468	1.122	2.173	1.808
	AVRB	.163	.265	.210	.052	.094	.028

Table 57 – Results of simulation study for ZA-SSLBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mean	.100	.098	.100	.103	.103	.103
	SD	.029	.027	.027	.016	.017	.017
	LCI	.109	.113	.114	.050	.053	.053
	CP	.900	1.000	1.000	.800	.900	.800
	Bias	< .001	-.002	< .001	.003	.003	.003
	RMSE	.029	.027	.027	.017	.017	.017
	AVRB	.005	.02	.002	.025	.031	.029
μ	Mean	1.002	1.018	.984	1.000	1.011	1.012
	SD	.031	.020	.066	.010	.014	.011
	LCI	.315	.430	.470	.139	.194	.236
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.002	.018	-.016	< .001	.011	.012
	RMSE	.031	.027	.068	.010	.018	.016
	AVRB	.002	.018	.016	< .001	.011	.012
ϕ	Mean	.471	.491	.471	.505	.503	.499
	SD	.043	.034	.038	.018	.009	.016
	LCI	.319	.325	.316	.165	.162	.162
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.029	-.009	-.029	.005	.003	-.001
	RMSE	.052	.036	.048	.019	.010	.016
	AVRB	.059	.019	.058	.010	.006	.002
γ	Mean	-.685	-.095	.571	-.796	.062	.798
	SD	.159	.263	.279	.034	.127	.054
	LCI	.601	.973	.651	.242	.394	.238
	CP	1.000	.900	.800	1.000	.500	1.000
	Bias	.115	-.095	-.229	.004	.062	-.002
	RMSE	.196	.279	.361	.034	.141	.054
	AVRB	.144	-	.286	.005	-	.003
ν	Mean	32.456	32.477	32.643	33.832	32.437	34.718
	SD	4.714	2.758	4.889	3.209	4.036	8.992
	LCI	91.647	93.658	88.640	85.532	79.048	85.749
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	2.456	2.477	2.643	3.832	2.437	4.718
	RMSE	5.316	3.707	5.558	4.998	4.715	10.154
	AVRB	.082	.083	.088	.128	.081	.157

Table 58 – Results of simulation study for ZA-SSLBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	.098	.095	.097	.102	.103	.102
	SD	.029	.028	.027	.016	.017	.017
	LCI	.109	.113	.114	.050	.053	.053
p	CP	.900	1.000	1.000	.800	.900	.800
	Bias	-.002	-.005	-.003	.002	.003	.002
	RMSE	.029	.028	.027	.017	.017	.017
	AVRB	.024	.049	.027	.021	.027	.022
	Median	.999	1.009	.970	.998	1.010	1.009
	SD	.031	.021	.064	.010	.014	.011
	LCI	.315	.430	.470	.139	.194	.236
μ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.001	.009	-.030	-.002	.010	.009
	RMSE	.031	.023	.070	.010	.017	.014
	AVRB	.001	.009	.030	.002	.010	.009
	Median	.464	.483	.464	.503	.504	.497
	SD	.043	.033	.039	.018	.009	.016
	LCI	.319	.325	.316	.165	.162	.162
ϕ	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.036	-.017	-.036	.003	.004	-.003
	RMSE	.056	.038	.053	.018	.010	.017
	AVRB	.072	.035	.073	.005	.007	.006
	Median	-.712	-.094	.593	-.804	.066	.806
	SD	.159	.281	.287	.033	.118	.054
	LCI	.601	.973	.651	.242	.394	.238
γ	CP	1.000	.900	.800	1.000	.500	1.000
	Bias	.088	-.094	-.207	-.004	.066	.006
	RMSE	.181	.296	.353	.033	.135	.054
	AVRB	.110	-	.258	.005	-	.007
	Median	25.517	25.998	27.335	27.249	27.439	28.831
	SD	4.210	2.737	4.631	3.963	3.394	8.065
	LCI	91.647	93.658	88.640	85.532	79.048	85.749
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-4.483	-4.002	-2.665	-2.751	-2.561	-1.169
	RMSE	6.150	4.848	5.343	4.824	4.252	8.149
	AVRB	.149	.133	.089	.092	.085	.039

Table 59 – Results of simulation study for ZA-SSLBS distribution ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.100	.098	.100	.102	.103	.103
	SD	.029	.027	.027	.016	.017	.016
	LCI	.109	.113	.114	.050	.053	.053
	CP	.9	1.000	1.000	.800	.900	.800
	Bias	0	-.002	$< .001$	.002	.003	.003
	RMSE	.029	.027	.027	.017	.017	.017
	AVRB	.001	.016	.001	.024	.028	.032
μ	Mode	1.001	1.011	.973	.999	1.010	1.011
	SD	.031	.021	.063	.010	.013	.012
	LCI	.315	.430	.470	.139	.194	.236
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.001	.011	-.027	-.001	.010	.011
	RMSE	.031	.023	.069	.010	.016	.016
	AVRB	.001	.011	.027	.001	.010	.011
ϕ	Mode	.468	.486	.468	.505	.504	.499
	SD	.042	.034	.039	.019	.009	.016
	LCI	.319	.325	.316	.165	.162	.162
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.032	-.014	-.032	.005	.004	-.001
	RMSE	.053	.036	.05	.019	.010	.016
	AVRB	.065	.027	.063	.010	.008	.001
γ	Mode	-.714	-.098	.595	-.798	.063	.8
	SD	.144	.297	.285	.032	.122	.053
	LCI	.601	.973	.651	.242	.394	.238
	CP	1.000	.900	.800	1.000	.500	1.000
	Bias	.086	-.098	-.205	.002	.063	$< .001$
	RMSE	.168	.313	.351	.032	.137	.053
	AVRB	.108	-	.256	.003	-	$< .001$
ν	Mode	11.004	12.030	11.235	15.880	12.672	14.681
	SD	5.386	6.997	6.467	5.789	5.805	6.498
	LCI	91.647	93.658	88.64	85.532	79.048	85.749
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-18.996	-17.970	-18.765	-14.120	-17.328	-15.319
	RMSE	19.745	19.284	19.848	15.260	18.274	16.641
	AVRB	.633	.599	.625	.471	.578	.511

ZA-SCNBS distribution

Table 60 – Results of simulation study for ZA-SCNBS distribution ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mean	.108	.109	.109	.104	.104	.104
	SD	.024	.024	.024	.016	.015	.016
	LCI	.119	.116	.115	.051	.053	.052
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	.008	.009	.009	.004	.004	.004
	RMSE	.025	.026	.025	.016	.016	.017
	AVRB	.083	.088	.089	.043	.041	.04
μ	Mean	.971	1.056	1.068	.996	1.021	1.014
	SD	.081	.128	.061	.058	.080	.081
	LCI	.718	1.006	1.067	.305	.405	.449
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.029	.056	.068	-.004	.021	.014
	RMSE	.086	.140	.091	.058	.082	.083
	AVRB	.029	.056	.068	.004	.021	.014
ϕ	Mean	3.321	2.663	3.315	3.211	2.603	2.914
	SD	.276	1.055	.393	.671	.746	.969
	LCI	4.255	3.722	4.085	3.282	3.685	3.389
	CP	$< .001$	.300	$< .001$	.200	.300	.300
	Bias	2.821	2.163	2.815	2.711	2.103	2.414
	RMSE	2.834	2.406	2.842	2.793	2.231	2.601
	AVRB	5.642	4.325	5.630	5.423	4.206	4.827
γ	Mean	-.634	.107	.547	-.756	.027	.751
	SD	.152	.243	.234	.084	.101	.074
	LCI	.821	1.247	.897	.335	.452	.352
	CP	1.000	.900	.800	1.000	.600	1.000
	Bias	.166	.107	-.253	.044	.027	-.049
	RMSE	.225	.266	.345	.095	.104	.088
	AVRB	.207	-	.316	.055	-	.061
ν_1	Mean	.538	.526	.529	.579	.601	.625
	SD	.046	.06	.046	.100	.085	.108
	LCI	.802	.717	.822	.767	.717	.702
	CP	.700	.400	.900	.700	.600	.700
	Bias	-.362	-.374	-.371	-.321	-.299	-.275
	RMSE	.365	.379	.374	.336	.311	.296
	AVRB	.402	.416	.413	.356	.332	.306
ν_2	Mean	.608	.448	.602	.594	.464	.553
	SD	.057	.215	.068	.126	.149	.181
	LCI	.793	.690	.780	.701	.749	.708
	CP	$< .001$	.400	$< .001$	.200	.300	.300
	Bias	.508	.348	.502	.494	.364	.453
	RMSE	.511	.409	.506	.510	.393	.488
	AVRB	5.08	3.477	5.018	4.944	3.640	4.533

Table 61 – Results of simulation study for ZA-SCNBS distribution ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Median	.106	.107	.107	.104	.104	.103
	SD	.024	.025	.024	.015	.015	.016
	LCI	.119	.116	.115	.051	.053	.052
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	.006	.007	.007	.004	.004	.003
	RMSE	.025	.026	.025	.016	.016	.017
	AVRB	.060	.066	.066	.039	.036	.034
μ	Median	.956	1.024	1.029	.994	1.016	1.005
	SD	.076	.131	.060	.059	.078	.082
	LCI	.718	1.006	1.067	.305	.405	.449
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.044	.024	.029	-.006	.016	.005
	RMSE	.088	.134	.067	.059	.08	.082
	AVRB	.044	.024	.029	.006	.016	.005
ϕ	Median	3.403	2.662	3.398	3.357	2.614	2.839
	SD	.298	1.182	.406	.709	.961	1.223
	LCI	4.255	3.722	4.085	3.282	3.685	3.389
	CP	$< .001$	.300	$< .001$	.200	.300	.300
	Bias	2.903	2.162	2.898	2.857	2.114	2.339
	RMSE	2.918	2.464	2.927	2.944	2.322	2.64
	AVRB	5.806	4.323	5.797	5.714	4.227	4.678
γ	Median	-.673	.110	.578	-.763	.028	.762
	SD	.16	.252	.246	.084	.090	.076
	LCI	.821	1.247	.897	.335	.452	.352
	CP	1.000	.900	.800	1.000	.600	1.000
	Bias	.127	.110	-.222	.037	.028	-.038
	RMSE	.205	.275	.331	.092	.094	.085
	AVRB	.159	-	.278	.046	-	.048
ν_1	Median	.550	.536	.546	.605	.626	.661
	SD	.055	.069	.056	.133	.094	.125
	LCI	.802	.717	.822	.767	.717	.702
	CP	.700	.400	.900	.700	.600	.700
	Bias	-.350	-.364	-.354	-.295	-.274	-.239
	RMSE	.355	.371	.359	.324	.290	.269
	AVRB	.389	.405	.393	.328	.304	.265
ν_2	Median	.615	.429	.604	.598	.438	.524
	SD	.076	.232	.089	.142	.175	.225
	LCI	.793	.690	.780	.701	.749	.708
	CP	$< .001$	.400	$< .001$	.200	.300	.300
	Bias	.515	.329	.504	.498	.338	.424
	RMSE	.520	.402	.512	.518	.380	.480
	AVRB	5.146	3.286	5.039	4.978	3.378	4.245

Table 62 – Results of simulation study for ZA-SCNBS distribution ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.108	.109	.109	.105	.105	.104
	SD	.023	.023	.023	.017	.016	.016
	LCI	.119	.116	.115	.051	.053	.052
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	.008	.009	.009	.005	.005	.004
	RMSE	.025	.025	.025	.017	.016	.017
	AVRB	.085	.093	.092	.045	.047	.038
μ	Mode	.949	1.001	.998	.996	1.018	1.008
	SD	.075	.132	.059	.059	.079	.080
	LCI	.718	1.006	1.067	.305	.405	.449
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.051	.001	-.002	-.004	.018	.008
	RMSE	.090	.132	.059	.059	.081	.081
	AVRB	.051	.001	.002	.004	.018	.008
ϕ	Mode	3.540	2.702	3.503	3.387	2.539	2.868
	SD	.473	1.361	.571	1.205	1.321	1.618
	LCI	4.255	3.722	4.085	3.282	3.685	3.389
	CP	$< .001$	.300	$< .001$	.200	.300	.300
	Bias	3.040	2.202	3.003	2.887	2.039	2.368
	RMSE	3.077	2.589	3.057	3.128	2.430	2.868
	AVRB	6.080	4.405	6.006	5.774	4.079	4.736
γ	Mode	-.694	.124	.593	-.760	.027	.756
	SD	.141	.280	.239	.083	.096	.073
	LCI	.821	1.247	.897	.335	.452	.352
	CP	1.000	.900	.800	1.000	.600	1.000
	Bias	.106	.124	-.207	.040	.027	-.044
	RMSE	.176	.306	.316	.092	.100	.085
	AVRB	.132	-	.258	.050	-	.055
ν_1	Mode	.562	.543	.559	.608	.638	.668
	SD	.067	.076	.066	.153	.092	.125
	LCI	.802	.717	.822	.767	.717	.702
	CP	.700	.400	.900	.700	.600	.700
	Bias	-.338	-.357	-.341	-.292	-.262	-.232
	RMSE	.344	.365	.347	.330	.278	.263
	AVRB	.375	.396	.379	.325	.291	.258
ν_2	Mode	.627	.432	.607	.607	.431	.510
	SD	.096	.231	.106	.152	.188	.241
	LCI	.793	.690	.780	.701	.749	.708
	CP	$< .001$	.400	$< .001$	.200	.300	.300
	Bias	.527	.332	.507	.507	.331	.410
	RMSE	.536	.405	.518	.529	.381	.476
	AVRB	5.269	3.324	5.070	5.067	3.310	4.104

Table 63 – Results of simulation study for ZA-SCNBS distribution ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mean	.108	.109	.110	.104	.104	.104
	SD	.024	.023	.023	.016	.016	.017
	LCI	.117	.117	.114	.052	.054	.052
	CP	.900	1.000	1.000	1.000	.900	.900
	Bias	.008	.009	.010	.004	.004	.004
	RMSE	.026	.025	.025	.017	.017	.017
	AVRB	.084	.094	.099	.040	.044	.045
μ	Mean	.994	1.080	1.080	.999	.989	.992
	SD	.168	.177	.164	.065	.067	.061
	LCI	.616	.926	.834	.261	.296	.332
	CP	.900	.900	1.000	1.000	1.000	1.000
	Bias	-.006	.080	.080	-.001	-.011	-.008
	RMSE	.169	.194	.182	.065	.068	.061
	AVRB	.006	.08	.080	.001	.011	.008
ϕ	Mean	.537	.471	.549	.492	.483	.491
	SD	.096	.102	.111	.057	.038	.052
	LCI	.561	.554	.612	.193	.215	.195
	CP	1.000	1.000	1.000	.900	1.000	.900
	Bias	.037	-.029	.049	-.008	-.017	-.009
	RMSE	.103	.106	.121	.057	.042	.053
	AVRB	.074	.058	.098	.017	.035	.017
γ	Mean	-.670	.060	.597	-.764	-.031	.763
	SD	.154	.402	.286	.094	.159	.070
	LCI	.750	1.088	.770	.321	.532	.335
	CP	1.000	.800	.800	.900	.600	1.000
	Bias	.130	.060	-.203	.036	-.031	-.037
	RMSE	.202	.406	.351	.101	.162	.079
	AVRB	.162	-	.254	.046	-	.047
ν_1	Mean	.298	.311	.283	.124	.149	.130
	SD	.147	.130	.134	.033	.028	.034
	LCI	.537	.532	.559	.129	.218	.134
	CP	.900	.600	.800	.800	.900	.800
	Bias	.198	.211	.183	.024	.049	.030
	RMSE	.247	.248	.226	.040	.057	.045
	AVRB	1.983	2.111	1.826	.237	.495	.296
ν_2	Mean	.327	.234	.309	.120	.143	.126
	SD	.242	.167	.209	.021	.040	.025
	LCI	.485	.458	.546	.102	.150	.102
	CP	.600	.800	.700	1.000	.900	.800
	Bias	.227	.134	.209	.020	.043	.026
	RMSE	.332	.214	.296	.029	.059	.036
	AVRB	2.275	1.339	2.092	.201	.431	.257

Table 64 – Results of simulation study for ZA-SCNBS distribution ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Median	.105	.107	.108	.104	.104	.104
	SD	.025	.023	.024	.016	.017	.017
	LCI	.117	.117	.114	.052	.054	.052
	CP	.900	1.000	1.000	1.000	.900	.900
	Bias	.005	.007	.008	.004	.004	.004
	RMSE	.025	.024	.025	.017	.017	.017
	AVRB	.051	.065	.075	.035	.037	.038
μ	Median	.971	1.038	1.045	.995	.983	.986
	SD	.148	.145	.151	.064	.064	.060
	LCI	.616	.926	.834	.261	.296	.332
	CP	.900	.900	1.000	1.000	1.000	1.000
	Bias	-.029	.038	.045	-.005	-.017	-.014
	RMSE	.151	.150	.158	.064	.066	.062
	AVRB	.029	.038	.045	.005	.017	.014
ϕ	Median	.528	.458	.539	.491	.482	.490
	SD	.101	.103	.117	.057	.039	.053
	LCI	.561	.554	.612	.193	.215	.195
	CP	1.000	1.000	1.000	.900	1.000	.900
	Bias	.028	-.042	.039	-.009	-.018	-.010
	RMSE	.104	.111	.123	.057	.043	.054
	AVRB	.056	.084	.078	.019	.035	.019
γ	Median	-.710	.066	.624	-.775	-.029	.774
	SD	.162	.422	.294	.094	.153	.071
	LCI	.750	1.088	.770	.321	.532	.335
	CP	1.000	.800	.800	.900	.600	1.000
	Bias	.090	.066	-.176	.025	-.029	-.026
	RMSE	.186	.428	.342	.097	.156	.076
	AVRB	.112	-	.219	.031	-	.032
ν_1	Median	.287	.291	.257	.119	.140	.126
	SD	.153	.131	.144	.033	.031	.033
	LCI	.537	.532	.559	.129	.218	.134
	CP	.900	.600	.800	.800	.900	.800
	Bias	.187	.191	.157	.019	.040	.026
	RMSE	.241	.232	.213	.038	.051	.042
	AVRB	1.869	1.909	1.573	.194	.403	.260
ν_2	Median	.314	.204	.288	.118	.140	.124
	SD	.247	.153	.220	.020	.039	.025
	LCI	.485	.458	.546	.102	.150	.102
	CP	.600	.800	.700	1.000	.900	.800
	Bias	.214	.104	.188	.018	.040	.024
	RMSE	.327	.185	.289	.026	.056	.034
	AVRB	2.138	1.043	1.882	.176	.404	.237

Table 65 – Results of simulation study for ZA-SCNBS distribution ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
p	Mode	.108	.110	.109	.104	.104	.105
	SD	.024	.023	.023	.017	.017	.017
	LCI	.117	.117	.114	.052	.054	.052
	CP	.900	1.000	1.000	1.000	.900	.900
	Bias	.008	.010	.009	.004	.004	.005
	RMSE	.025	.026	.025	.017	.017	.017
	AVRB	.080	.099	.093	.037	.04	.051
μ	Mode	.963	1.017	1.031	.999	.987	.989
	SD	.132	.124	.139	.064	.065	.060
	LCI	.616	.926	.834	.261	.296	.332
	CP	.900	.900	1.000	1.000	1.000	1.000
	Bias	-.037	.017	.031	-.001	-.013	-.011
	RMSE	.137	.125	.143	.064	.066	.061
	AVRB	.037	.017	.031	.001	.013	.011
ϕ	Mode	.530	.459	.541	.492	.482	.491
	SD	.100	.103	.116	.057	.038	.053
	LCI	.561	.554	.612	.193	.215	.195
	CP	1.000	1.000	1.000	.900	1.000	.900
	Bias	.030	-.041	.041	-.008	-.018	-.009
	RMSE	.105	.111	.123	.058	.042	.054
	AVRB	.060	.082	.082	.016	.035	.018
γ	Mode	-.727	.083	.634	-.769	-.033	.769
	SD	.149	.429	.281	.092	.155	.070
	LCI	.750	1.088	.770	.321	.532	.335
	CP	1.000	.800	.800	.900	.600	1.000
	Bias	.073	.083	-.166	.031	-.033	-.031
	RMSE	.166	.437	.326	.097	.158	.077
	AVRB	.091	-	.208	.038	-	.039
ν_1	Mode	.287	.291	.261	.124	.147	.130
	SD	.146	.123	.138	.032	.029	.034
	LCI	.537	.532	.559	.129	.218	.134
	CP	.900	.600	.800	.800	.900	.800
	Bias	.187	.191	.161	.024	.047	.030
	RMSE	.237	.227	.212	.040	.055	.045
	AVRB	1.871	1.911	1.608	.243	.470	.297
ν_2	Mode	.318	.201	.296	.119	.143	.125
	SD	.249	.136	.229	.021	.039	.026
	LCI	.485	.458	.546	.102	.150	.102
	CP	.600	.800	.700	1.000	.900	.800
	Bias	.218	.101	.196	.019	.043	.025
	RMSE	.331	.170	.301	.028	.058	.036
	AVRB	2.175	1.011	1.961	.191	.430	.254

APPENDIX C – Results of Chapter 3

In this section, we present the results of the simulation studies for the CSSBS regression models. Furthermore, we present the results of the statistical analysis of the AIS data set.

C.1 Results of the parameter recovery study

SGtBS1 regression model

Table 66 – Results of simulation study for SGtBS1 regression model ($\nu_1 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.593	-.556	-.419	-.523	-.500	-.478
	SD	.046	.098	.063	.019	.033	.025
	LCI	.356	.407	.428	.163	.176	.181
β_0	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.093	-.056	.081	-.023	$< .001$	.022
	RMSE	.103	.113	.102	.030	.033	.034
	AVRB	.186	.113	.161	.045	$< .001$	.045
	Mean	1.175	1.125	.833	1.051	.993	.947
	SD	.125	.209	.153	.037	.103	.036
	LCI	.599	.675	.654	.270	.298	.272
β_1	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.175	.125	-.167	.051	-.007	-.053
	RMSE	.215	.243	.227	.063	.104	.064
	AVRB	.175	.125	.167	.051	.007	.053
	Mean	-.134	-.507	-.124	-.766	-.911	-.817
	SD	.435	.266	.331	.179	.090	.112
	LCI	2.330	2.178	2.508	1.067	1.165	1.020
ψ_0	CP	.600	1.000	.800	1.000	1.000	1.000
	Bias	.866	.493	.876	.234	.089	.183
	RMSE	.970	.560	.936	.295	.127	.214
	AVRB	.866	.493	.876	.234	.089	.183
	Mean	.469	.512	.177	.537	.526	.546
	SD	.760	.435	.528	.401	.371	.124
	LCI	2.350	2.109	1.834	.897	.910	.757

Table 66 (continued).

ψ_1	CP	.800	1.000	.800	1.000	1.000	1.000
	Bias	-.031	.012	-.323	.037	.026	.046
	RMSE	.760	.435	.619	.403	.372	.133
	AVRB	.062	.024	.646	.074	.052	.092
	Mean	-.660	.030	.538	-.697	.001	.756
	SD	.099	.462	.118	.148	.213	.109
	LCI	.711	1.084	1.037	.441	.494	.409
γ	CP	1.000	.800	1.000	.800	.200	1.000
	Bias	.140	.030	-.262	.103	.001	-.044
	RMSE	.171	.463	.287	.180	.213	.118
	AVRB	.175	-	.328	.129	-	.055
	Mean	11.775	7.870	9.980	6.204	5.619	5.932
	SD	5.432	2.030	1.807	1.016	.604	.542
	LCI	21.527	11.757	19.898	5.087	4.121	4.245
ν_1	CP	.600	1.000	1.000	1.000	1.000	1.000
	Bias	6.775	2.870	4.980	1.204	.619	.932
	RMSE	8.684	3.516	5.298	1.576	.865	1.078
	AVRB	1.355	.574	.996	.241	.124	.186

Table 67 – Results of simulation study for SGtBS1 regression model ($\nu_1 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.595	-.558	-.425	-.523	-.499	-.478
	SD	.042	.095	.057	.019	.032	.025
	LCI	.356	.407	.428	.163	.176	.181
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.095	-.058	.075	-.023	.001	.022
	RMSE	.104	.112	.095	.029	.032	.033
	AVRB	.191	.116	.150	.046	.002	.045
β_1	Median	1.180	1.125	.832	1.051	.993	.951
	SD	.124	.207	.155	.035	.104	.043
	LCI	.599	.675	.654	.270	.298	.272
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.180	.125	-.168	.051	-.007	-.049
	RMSE	.218	.242	.228	.062	.104	.065
	AVRB	.180	.125	.168	.051	.007	.049
ψ_0	Median	-.169	-.529	-.171	-.768	-.924	-.838
	SD	.448	.297	.373	.171	.091	.100
	LCI	2.330	2.178	2.508	1.067	1.165	1.020
	CP	.600	1.000	.800	1.000	1.000	1.000
	Bias	.831	.471	.829	.233	.076	.162
	RMSE	.944	.556	.910	.288	.119	.191
	AVRB	.831	.471	.829	.233	.076	.162
ψ_1	Median	.465	.522	.162	.542	.520	.546
	SD	.763	.418	.513	.401	.368	.132
	LCI	2.350	2.109	1.834	.897	.910	.757
	CP	.800	1.000	.800	1.000	1.000	1.000
	Bias	-.035	.022	-.338	.042	.020	.046
	RMSE	.764	.419	.614	.403	.369	.140
	AVRB	.070	.044	.676	.083	.039	.091
γ	Median	-.687	.038	.580	-.708	.011	.769
	SD	.102	.495	.122	.148	.202	.118
	LCI	.711	1.084	1.037	.441	.494	.409
	CP	1.000	.800	1.000	.800	.200	1.000
	Bias	.113	.038	-.220	.092	.011	-.031
	RMSE	.152	.497	.252	.174	.203	.122
	AVRB	.141	-	.275	.115	-	.039
ν_1	Median	1.466	6.996	8.242	5.981	5.420	5.690
	SD	5.195	1.868	1.771	.842	.630	.530
	LCI	21.527	11.757	19.898	5.087	4.121	4.245
	CP	.600	1.000	1.000	1.000	1.000	1.000
	Bias	5.466	1.996	3.242	.981	.420	.690
	RMSE	7.541	2.734	3.694	1.293	.757	.870
	AVRB	1.093	.399	.648	.196	.084	.138

Table 68 – Results of simulation study for SGtBS1 regression model ($\nu_1 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.595	-.557	-.423	-.524	-.499	-.477
	SD	.046	.099	.059	.020	.033	.025
	LCI	.356	.407	.428	.163	.176	.181
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.095	-.057	.077	-.024	.001	.023
	RMSE	.106	.114	.096	.031	.033	.034
	AVRB	.190	.114	.153	.047	.001	.046
β_1	Mode	1.179	1.124	.832	1.050	.993	.948
	SD	.126	.208	.151	.037	.103	.037
	LCI	.599	.675	.654	.270	.298	.272
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.179	.124	-.168	.050	-.007	-.052
	RMSE	.219	.242	.226	.062	.104	.064
	AVRB	.179	.124	.168	.050	.007	.052
ψ_0	Mode	-.192	-.595	-.291	-.788	-.937	-.862
	SD	.506	.359	.476	.182	.101	.095
	LCI	2.330	2.178	2.508	1.067	1.165	1.020
	CP	.600	1.000	.800	1.000	1.000	1.000
	Bias	.808	.405	.709	.212	.063	.138
	RMSE	.953	.541	.854	.279	.119	.167
	AVRB	.808	.405	.709	.212	.063	.138
ψ_1	Mode	.414	.543	.124	.543	.520	.550
	SD	.817	.389	.463	.402	.367	.132
	LCI	2.350	2.109	1.834	.897	.910	.757
	CP	.800	1.000	.800	1.000	1.000	1.000
	Bias	-.086	.043	-.376	.043	.020	.050
	RMSE	.822	.391	.596	.404	.368	.141
	AVRB	.172	.085	.753	.087	.041	.100
γ	Mode	-.695	.024	.613	-.703	.013	.764
	SD	.090	.531	.114	.145	.203	.111
	LCI	.711	1.084	1.037	.441	.494	.409
	CP	1.000	.800	1.000	.800	.200	1.000
	Bias	.105	.024	-.187	.097	.013	-.036
	RMSE	.138	.531	.219	.174	.203	.117
	AVRB	.131	-	.234	.121	-	.045
ν_1	Mode	7.620	5.431	5.309	5.631	4.988	5.311
	SD	3.155	1.489	1.017	.789	.708	.706
	LCI	21.527	11.757	19.898	5.087	4.121	4.245
	CP	.600	1.000	1.000	1.000	1.000	1.000
	Bias	2.620	.431	.309	.631	-.012	.311
	RMSE	4.101	1.550	1.063	1.010	.709	.771
	AVRB	.524	.086	.062	.126	.002	.062

Table 69 – Results of simulation study for SGtBS1 regression model ($\nu_1 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.498	-.510	-.501	-.508	-.505	-.495
	SD	.047	.060	.048	.019	.007	.011
	LCI	.123	.143	.137	.051	.063	.055
	CP	.800	.800	1.000	.800	1.000	1.000
	Bias	.002	-.01	-.001	-.008	-.005	.005
	RMSE	.047	.061	.048	.020	.009	.012
	AVRB	.003	.019	.001	.017	.010	.010
β_1	Mean	1.007	1.024	.999	1.016	1.011	.988
	SD	.087	.109	.094	.033	.015	.017
	LCI	.219	.240	.217	.091	.110	.092
	CP	.600	.800	.800	.600	1.000	1.000
	Bias	.007	.024	-.001	.016	.011	-.012
	RMSE	.087	.112	.094	.037	.018	.021
	AVRB	.007	.024	.001	.016	.011	.012
ψ_0	Mean	-1.992	-2.133	-1.968	-.910	-1.185	-1.450
	SD	.576	.292	.399	.754	.444	.550
	LCI	1.883	2.522	2.303	.845	1.436	.476
	CP	.600	.800	.800	.600	1.000	$< .001$
	Bias	-.992	-1.133	-.968	.090	-.185	-.450
	RMSE	1.147	1.170	1.047	.760	.481	.711
	AVRB	.992	1.133	.968	.090	.185	.450
ψ_1	Mean	.861	.765	.658	.229	.532	.358
	SD	.230	.416	.264	.196	.305	.170
	LCI	1.859	2.100	1.726	.826	1.016	.423
	CP	1.000	1.000	1.000	.800	.800	.600
	Bias	.361	.265	.158	-.271	.032	-.142
	RMSE	.428	.494	.308	.335	.307	.222
	AVRB	.723	.531	.315	.542	.065	.284
γ	Mean	-.652	-.038	.666	-.789	-.047	.879
	SD	.300	.354	.058	.060	.161	.046
	LCI	.674	1.135	.758	.249	.406	.207
	CP	1.000	1.000	1.000	1.000	.200	.800
	Bias	.148	-.038	-.134	.011	-.047	.079
	RMSE	.334	.357	.146	.061	.168	.092
	AVRB	.185	-	.168	.014	-	.099
ν_1	Mean	16.833	13.919	15.593	36.504	28.979	20.783
	SD	7.248	1.954	5.183	24.167	14.683	11.643
	LCI	24.098	3.438	3.135	34.000	31.252	8.239
	CP	.600	.600	.800	.600	.600	$< .001$
	Bias	-13.167	-16.081	-14.407	6.504	-1.021	-9.217
	RMSE	15.030	16.199	15.311	25.027	14.719	14.850
	AVRB	.439	.536	.480	.217	.034	.307

Table 70 – Results of simulation study for SGtBS1 regression model ($\nu_1 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.499	-.510	-.501	-.509	-.505	-.495
	SD	.047	.060	.049	.018	.007	.011
	LCI	.123	.143	.137	.051	.063	.055
	CP	.800	.800	1.000	.800	1.000	1.000
	Bias	.001	-.010	-.001	-.009	-.005	.005
	RMSE	.047	.061	.049	.020	.009	.012
	AVRB	.003	.020	.002	.017	.011	.010
β_1	Median	1.007	1.024	.997	1.017	1.011	.988
	SD	.086	.109	.093	.033	.015	.018
	LCI	.219	.240	.217	.091	.110	.092
	CP	.600	.800	.800	.600	1.000	1.000
	Bias	.007	.024	-.003	.017	.011	-.012
	RMSE	.087	.111	.093	.037	.019	.021
	AVRB	.007	.024	.003	.017	.011	.012
ψ_0	Median	-1.976	-2.144	-1.959	-.877	-1.168	-1.445
	SD	.616	.299	.395	.801	.444	.549
	LCI	1.883	2.522	2.303	.845	1.436	.476
	CP	.600	.800	.800	.600	1.000	$< .001$
	Bias	-.976	-1.144	-.959	.123	-.168	-.445
	RMSE	1.154	1.183	1.037	.810	.475	.707
	AVRB	.976	1.144	.959	.123	.168	.445
ψ_1	Median	.909	.773	.647	.217	.535	.366
	SD	.229	.416	.283	.194	.308	.168
	LCI	1.859	2.100	1.726	.826	1.016	.423
	CP	1.000	1.000	1.000	.800	.800	.600
	Bias	.409	.273	.147	-.283	.035	-.134
	RMSE	.469	.497	.319	.344	.310	.215
	AVRB	.818	.545	.293	.567	.071	.267
γ	Median	-.666	-.048	.699	-.794	-.055	.889
	SD	.335	.371	.062	.062	.146	.045
	LCI	.674	1.135	.758	.249	.406	.207
	CP	1.000	1.000	1.000	1.000	.200	.800
	Bias	.134	-.048	-.101	.006	-.055	.089
	RMSE	.361	.374	.119	.062	.156	.099
	AVRB	.168	-	.126	.007	-	.111
ν_1	Median	15.890	11.853	13.542	37.265	27.638	2.817
	SD	6.849	1.211	4.059	27.957	14.285	11.684
	LCI	24.098	3.438	3.135	34.000	31.252	8.239
	CP	.600	.600	.800	.600	.600	$< .001$
	Bias	-14.110	-18.147	-16.458	7.265	-2.362	-9.183
	RMSE	15.684	18.187	16.951	28.886	14.479	14.861
	AVRB	.470	.605	.549	.242	.079	.306

Table 71 – Results of simulation study for SGtBS1 regression model ($\nu_1 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.499	-.510	-.500	-.509	-.505	-.496
	SD	.047	.059	.047	.019	.008	.010
	LCI	.123	.143	.137	.051	.063	.055
β_0	CP	.800	.800	1.000	.800	1.000	1.000
	Bias	.001	-.010	$<$ .001	-.009	-.005	.004
	RMSE	.047	.060	.047	.021	.010	.011
	AVRB	.002	.020	.001	.018	.011	.009
	Mode	1.007	1.025	.998	1.016	1.012	.989
	SD	.088	.110	.094	.033	.014	.017
	LCI	.219	.240	.217	.091	.110	.092
β_1	CP	.600	.800	.800	.600	1.000	1.000
	Bias	.007	.025	-.002	.016	.012	-.011
	RMSE	.088	.113	.094	.036	.018	.021
	AVRB	.007	.025	.002	.016	.012	.011
	Mode	-1.993	-2.237	-1.933	-.892	-1.136	-1.448
	SD	.694	.321	.422	.841	.450	.547
	LCI	1.883	2.522	2.303	.845	1.436	.476
ψ_0	CP	.600	.800	.800	.600	1.000	$<$ .001
	Bias	-.993	-1.237	-.933	.108	-.136	-.448
	RMSE	1.211	1.278	1.023	.848	.470	.707
	AVRB	.993	1.237	.933	.108	.136	.448
	Mode	1.027	.761	.618	.209	.536	.363
	SD	.297	.381	.364	.196	.306	.163
	LCI	1.859	2.100	1.726	.826	1.016	.423
ψ_1	CP	1.000	1.000	1.000	.800	.800	.600
	Bias	.527	.261	.118	-.291	.036	-.137
	RMSE	.605	.462	.383	.351	.308	.213
	AVRB	1.055	.522	.237	.583	.072	.273
	Mode	-.638	-.061	.714	-.792	-.051	.880
	SD	.386	.392	.058	.059	.155	.044
	LCI	.674	1.135	.758	.249	.406	.207
γ	CP	1.000	1.000	1.000	1.000	.200	.800
	Bias	.162	-.061	-.086	.008	-.051	.080
	RMSE	.418	.397	.104	.060	.163	.091
	AVRB	.202	-	.107	.011	-	.100
	Mode	12.103	6.997	10.200	37.951	25.797	19.799
	SD	6.304	1.371	3.756	35.054	14.404	13.327
	LCI	24.098	30.438	30.135	34.000	31.252	8.239
ν_1	CP	.600	.600	.800	.600	.600	$<$ .001
	Bias	-17.897	-23.003	-19.800	7.951	-4.203	-10.201
	RMSE	18.974	23.044	20.153	35.945	15.004	16.783
	AVRB	.597	.767	.660	.265	.140	.340

SGtBS2 regression model

Table 72 – Results of simulation study for SGtBS2 ($\nu_1 = 15$, $\nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.438	-.365	-.561	-.495	-.504	-.496
	SD	.236	.111	.204	.056	.058	.070
	LCI	.495	.688	.583	.247	.298	.282
	CP	.600	1.000	1.000	1.000	1.000	1.000
	Bias	.062	.135	-.061	.005	-.004	.004
	RMSE	.243	.175	.213	.056	.058	.07
	AVRB	.123	.269	.122	.010	.008	.009
β_1	Mean	.835	.802	1.213	1.017	1.034	.997
	SD	.439	.227	.404	.091	.149	.107
	LCI	.933	1.137	.933	.414	.498	.424
	CP	.600	1.000	.600	1.000	1.000	1.000
	Bias	-.165	-.198	.213	.017	.034	-.003
	RMSE	.469	.302	.457	.092	.153	.107
	AVRB	.165	.198	.213	.017	.034	.003
γ	Mean	-.757	.239	.713	-.732	-.029	.757
	SD	.230	.104	.227	.086	.182	.030
	LCI	.544	1.081	.574	.392	.431	.337
	CP	1.000	1.000	1.000	1.000	.400	1.000
	Bias	.043	.239	-.087	.068	-.029	-.043
	RMSE	.234	.260	.243	.110	.184	.053
	AVRB	.054	-	.109	.085	-	.054
ν_1	Mean	19.641	14.336	16.344	14.037	14.467	22.313
	SD	5.702	2.547	5.450	3.633	3.273	15.916
	LCI	46.124	32.818	35.673	22.968	21.425	5.897
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	4.641	-.664	1.344	-.963	-.533	7.313
	RMSE	7.352	2.632	5.613	3.759	3.316	17.516
	AVRB	.309	.044	.090	.064	.036	.488
ν_2	Mean	6.530	5.091	5.601	4.762	4.889	7.724
	SD	1.718	1.006	1.980	1.466	1.217	5.722
	LCI	16.627	12.733	13.413	8.733	7.975	18.515
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.530	.091	.601	-.238	-.111	2.724
	RMSE	2.301	1.010	2.069	1.485	1.222	6.337
	AVRB	.306	.018	.120	.048	.022	.545

Table 73 – Results of simulation study for SGtBS2 ($\nu_1 = 15$, $\nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.438	-.370	-.560	-.494	-.508	-.498
	SD	.236	.110	.212	.056	.057	.071
	LCI	.495	.688	.583	.247	.298	.282
	CP	.600	1.000	1.000	1.000	1.000	1.000
	Bias	.062	.130	-.060	.006	-.008	.002
	RMSE	.244	.170	.220	.056	.058	.071
	AVRB	.124	.260	.121	.013	.016	.005
β_1	Median	.826	.800	1.23	1.014	1.035	.996
	SD	.446	.228	.418	.091	.148	.107
	LCI	.933	1.137	.933	.414	.498	.424
	CP	.600	1.000	.600	1.000	1.000	1.000
	Bias	-.174	-.200	.230	.014	.035	-.004
	RMSE	.479	.303	.477	.093	.152	.107
	AVRB	.174	.200	.230	.014	.035	.004
γ	Median	-.783	.244	.733	-.732	-.024	.760
	SD	.226	.114	.246	.082	.175	.026
	LCI	.544	1.081	.574	.392	.431	.337
	CP	1.000	1.000	1.000	1.000	.400	1.000
	Bias	.017	.244	-.067	.068	-.024	-.040
	RMSE	.227	.269	.255	.106	.176	.048
	AVRB	.021	-	.084	.085	-	.050
ν_1	Median	15.631	11.558	13.494	12.201	13.233	16.645
	SD	3.896	1.670	4.509	3.539	3.026	8.317
	LCI	46.124	32.818	35.673	22.968	21.425	5.897
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.631	-3.442	-1.506	-2.799	-1.767	1.645
	RMSE	3.947	3.826	4.754	4.512	3.504	8.478
	AVRB	.042	.229	.100	.187	.118	.110
ν_2	Median	5.040	3.964	4.508	4.038	4.385	5.574
	SD	1.081	.631	1.627	1.429	1.154	2.981
	LCI	16.627	12.733	13.413	8.733	7.975	18.515
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.040	-1.036	-.492	-.962	-.615	.574
	RMSE	1.082	1.213	1.699	1.723	1.308	3.036
	AVRB	.008	.207	.098	.192	.123	.115

Table 74 – Results of simulation study for SGtBS2 ($\nu_1 = 15$, $\nu_2 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.440	-.373	-.565	-.494	-.506	-.496
	SD	.238	.105	.211	.057	.060	.071
	LCI	.495	.688	.583	.247	.298	.282
	CP	.600	1.000	1.000	1.000	1.000	1.000
	Bias	.060	.127	-.065	.006	-.006	.004
	RMSE	.246	.165	.220	.057	.061	.071
	AVRB	.121	.254	.129	.012	.012	.008
β_1	Mode	.822	.797	1.228	1.015	1.035	.997
	SD	.441	.231	.407	.091	.148	.106
	LCI	.933	1.137	.933	.414	.498	.424
	CP	.600	1.000	.600	1.000	1.000	1.000
	Bias	-.178	-.203	.228	.015	.035	-.003
	RMSE	.475	.307	.467	.092	.152	.106
	AVRB	.178	.203	.228	.015	.035	.003
γ	Mode	-.785	.244	.724	-.735	-.028	.758
	SD	.199	.125	.250	.087	.177	.028
	LCI	.544	1.081	.574	.392	.431	.337
	CP	1.000	1.000	1.000	1.000	.400	1.000
	Bias	.015	.244	-.076	.065	-.028	-.042
	RMSE	.200	.275	.261	.108	.179	.051
	AVRB	.019	-	.095	.081	-	.053
ν_1	Mode	1.328	8.036	1.219	1.587	12.134	11.813
	SD	.879	.717	3.189	3.759	3.637	4.788
	LCI	46.124	32.818	35.673	22.968	21.425	5.897
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-4.672	-6.964	-4.781	-4.413	-2.866	-3.187
	RMSE	4.754	7.001	5.747	5.797	4.631	5.752
	AVRB	.311	.464	.319	.294	.191	.212
ν_2	Mode	3.067	2.463	2.673	3.364	3.444	3.541
	SD	.528	.195	.308	1.213	1.092	1.330
	LCI	16.627	12.733	13.413	8.733	7.975	18.515
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-1.933	-2.537	-2.327	-1.636	-1.556	-1.459
	RMSE	2.004	2.544	2.348	2.036	1.901	1.975
	AVRB	.387	.507	.465	.327	.311	.292

Table 75 – Results of simulation study for SGtBS2 ($\nu_1 = \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.556	-.403	-.418	-.500	-.448	-.462
	SD	.121	.168	.129	.110	.052	.037
	LCI	.772	1.031	.953	.357	.440	.449
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.056	.097	.082	$<$ .001	.052	.038
	RMSE	.133	.193	.153	.110	.074	.053
	AVRB	.112	.194	.165	$<$ .001	.104	.077
β_1	Mean	1.132	.925	.857	1.035	.940	.922
	SD	.228	.340	.334	.186	.118	.107
	LCI	1.290	1.737	1.411	.612	.708	.587
	CP	.800	1.000	1.000	1.000	1.000	11.000
	Bias	.132	-.075	-.143	.035	-.060	-.078
	RMSE	.263	.348	.364	.190	.133	.133
	AVRB	.132	.075	.143	.035	.060	.078
γ	Mean	-.836	.064	.747	-.807	.007	.840
	SD	.079	.170	.153	.077	.208	.066
	LCI	.471	1.232	.624	.321	.484	.253
	CP	1.000	1.000	1.000	1.000	.600	1.000
	Bias	-.036	.064	-.053	-.007	.007	.040
	RMSE	.087	.182	.162	.077	.208	.077
	AVRB	.045	-	.066	.009	-	.049
ν_1	Mean	13.948	1.187	12.152	19.332	23.077	22.630
	SD	3.427	1.876	2.678	8.945	11.086	6.626
	LCI	31.879	25.777	25.722	39.605	41.969	37.225
	CP	.400	.400	.400	.800	.800	1.000
	Bias	-16.052	-19.813	-17.848	-1.668	-6.923	-7.370
	RMSE	16.414	19.902	18.048	13.922	13.070	9.911
	AVRB	.535	.660	.595	.356	.231	.246
ν_2	Mean	12.531	9.356	1.929	18.787	23.052	22.418
	SD	3.005	2.048	2.894	9.887	12.070	7.970
	LCI	31.665	28.359	26.426	4.864	45.368	4.066
	CP	.400	.600	.400	.800	.800	1.000
	Bias	-17.469	-2.644	-19.071	-11.213	-6.948	-7.582
	RMSE	17.726	2.745	19.289	14.949	13.927	11.000
	AVRB	.582	.688	.636	.374	.232	.253

Table 76 – Results of simulation study for SGtBS2 ($\nu_1 = \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.552	-.409	-.427	-.498	-.449	-.460
	SD	.124	.171	.129	.110	.051	.039
	LCI	.772	1.031	.953	.357	.440	.449
β_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.052	.091	.073	.002	.051	.040
	RMSE	.134	.194	.148	.110	.072	.056
	AVRB	.105	.181	.146	.004	.102	.081
	Median	1.125	.926	.869	1.038	.94	.916
	SD	.212	.338	.333	.185	.117	.105
	LCI	1.29	1.737	1.411	.612	.708	.587
β_1	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.125	-.074	-.131	.038	-.060	-.084
	RMSE	.246	.346	.358	.189	.131	.134
	AVRB	.125	.074	.131	.038	.060	.084
	Median	-.870	.063	.780	-.817	.010	.845
	SD	.082	.189	.153	.080	.202	.066
	LCI	.471	1.232	.624	.321	.484	.253
γ	CP	1.000	1.000	1.000	1.000	.600	1.000
	Bias	-.070	.063	-.020	-.017	.010	.045
	RMSE	.108	.199	.154	.082	.203	.080
	AVRB	.088	-	.026	.021	-	.056
	Median	11.787	7.935	1.421	15.538	2.224	19.926
	SD	2.773	1.447	2.446	4.808	9.449	5.729
	LCI	31.879	25.777	25.722	39.605	41.969	37.225
ν_1	CP	.400	.400	.400	.800	.800	1.000
	Bias	-18.212	-22.065	-19.579	-14.462	-9.776	-1.074
	RMSE	18.422	22.113	19.732	15.240	13.596	11.589
	AVRB	.607	.736	.653	.482	.326	.336
	Median	1.088	6.940	9.006	14.848	19.946	19.616
	SD	2.253	1.598	2.443	5.464	1.289	6.825
	LCI	31.665	28.359	26.426	4.864	45.368	4.066
ν_2	CP	.400	.600	.400	.800	.800	1.000
	Bias	-19.912	-23.060	-2.994	-15.152	-1.054	-1.384
	RMSE	2.039	23.116	21.136	16.107	14.385	12.426
	AVRB	.664	.769	.700	.505	.335	.346

Table 77 – Results of simulation study for SGtBS2 ($\nu_1 = \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.558	-.417	-.432	-.499	-.450	-.463
	SD	.122	.169	.137	.110	.052	.037
	LCI	.772	1.031	.953	.357	.440	.449
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.058	.083	.068	.001	.050	.037
	RMSE	.135	.188	.153	.110	.072	.053
	AVRB	.116	.167	.135	.001	.099	.074
β_1	Mode	1.112	.934	.877	1.037	.942	.920
	SD	.210	.340	.330	.186	.118	.104
	LCI	1.290	1.737	1.411	.612	.708	.587
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.112	-.066	-.123	.037	-.058	-.080
	RMSE	.238	.346	.352	.190	.132	.131
	AVRB	.112	.066	.123	.037	.058	.080
γ	Mode	-.858	.055	.782	-.813	.009	.840
	SD	.071	.207	.134	.075	.203	.064
	LCI	.471	1.232	.624	.321	.484	.253
	CP	1.000	1.000	1.000	1.000	.600	1.000
	Bias	-.058	.055	-.018	-.013	.009	.040
	RMSE	.092	.214	.135	.077	.203	.075
	AVRB	.073	-	.023	.016	-	.050
ν_1	Mode	7.911	5.234	6.759	12.050	12.037	15.159
	SD	2.179	.830	1.142	2.287	2.642	2.334
	LCI	31.879	25.777	25.722	39.605	41.969	37.225
	CP	.400	.400	.400	.800	.800	1.000
	Bias	-22.089	-24.766	-23.241	-17.950	-17.963	-14.841
	RMSE	22.197	24.780	23.269	18.095	18.156	15.024
	AVRB	.736	.826	.775	.598	.599	.495
ν_2	Mode	6.183	4.855	5.107	11.303	12.397	14.107
	SD	1.984	1.074	.749	3.589	5.869	2.998
	LCI	31.665	28.359	26.426	4.864	45.368	4.066
	CP	.400	.600	.400	.800	.800	1.000
	Bias	-23.817	-25.145	-24.893	-18.697	-17.603	-15.893
	RMSE	23.900	25.168	24.905	19.039	18.555	16.173
	AVRB	.794	.838	.830	.623	.587	.530

StBS regression model

Table 78 – Results of simulation study for StBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.685	-.737	-.439	-.496	-.494	-.514
	SD	.105	.025	.129	.043	.151	.072
	LCI	.709	.886	.827	.352	.396	.397
β_0	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.185	-.237	.061	.004	.006	-.014
	RMSE	.213	.238	.142	.043	.151	.073
	AVRB	.371	.474	.121	.009	.012	.028
	Mean	1.251	1.538	.785	.984	.944	.970
	SD	.361	.062	.327	.126	.294	.136
	LCI	1.171	1.428	1.245	.603	.625	.560
β_1	CP	.800	.800	1.000	1.000	.600	1.000
	Bias	.251	.538	-.215	-.016	-.056	-.030
	RMSE	.439	.541	.391	.127	.300	.139
	AVRB	.251	.538	.215	.016	.056	.030
	Mean	-.931	-.877	-.882	-.945	-1.029	-1.022
	SD	.392	.157	.386	.157	.091	.089
	LCI	1.308	1.21	1.105	.634	.610	.509
ψ_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.069	.123	.118	.055	-.029	-.022
	RMSE	.398	.199	.403	.167	.095	.091
	AVRB	.069	.123	.118	.055	.029	.022
	Mean	.683	.528	.442	.419	.532	.556
	SD	.813	.371	.719	.220	.168	.148
	LCI	2.273	1.847	1.552	.982	.903	.652
ψ_1	CP	.800	1.000	.600	1.000	1.000	1.000
	Bias	.183	.028	-.058	-.081	.032	.056
	RMSE	.833	.372	.721	.235	.171	.158
	AVRB	.365	.056	.115	.162	.065	.112
	Mean	-.732	.261	.671	-.686	-.080	.749
	SD	.066	.381	.162	.098	.05	.049
	LCI	.633	.93	.731	.473	.584	.451
γ	CP	1.000	.600	1.000	1.000	.800	1.000
	Bias	.068	.261	-.129	.114	-.08	-.051
	RMSE	.095	.462	.207	.151	.094	.070

Table 78 (continued).

	AVRB	.086	-	.161	.143	-	.064
	Mean	18.120	9.364	13.598	5.738	5.643	5.758
	SD	6.364	1.990	6.343	.824	.663	.872
	LCI	61.840	27.732	46.417	4.760	4.615	4.754
ν	CP	.600	1.000	.800	1.000	1.000	1.000
	Bias	13.120	4.364	8.598	.738	.643	.758
	RMSE	14.582	4.796	10.685	1.106	.924	1.155
	AVRB	2.624	.873	1.720	.148	.129	.152

Table 79 – Results of simulation study for StBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.693	-.745	-.445	-.496	-.496	-.513
	SD	.099	.023	.129	.042	.152	.075
	LCI	.709	.886	.827	.352	.396	.397
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.193	-.245	.055	.004	.004	-.013
	RMSE	.216	.246	.141	.042	.152	.076
	AVRB	.385	.49	.111	.008	.009	.026
β_1	Median	1.246	1.537	.785	.982	.944	.970
	SD	.351	.06	.345	.123	.295	.138
	LCI	1.171	1.428	1.245	.603	.625	.560
	CP	.800	.800	1.000	1.000	.600	1.000
	Bias	.246	.537	-.215	-.018	-.056	-.030
	RMSE	.429	.54	.407	.125	.300	.141
	AVRB	.246	.537	.215	.018	.056	.030
ψ_0	Median	-.925	-.877	-.889	-.946	-1.027	-1.020
	SD	.389	.157	.376	.159	.092	.087
	LCI	1.308	1.210	1.105	.634	.610	.509
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.075	.123	.111	.054	-.027	-.020
	RMSE	.397	.199	.392	.168	.096	.090
	AVRB	.075	.123	.111	.054	.027	.020
ψ_1	Median	.666	.533	.463	.419	.536	.550
	SD	.792	.374	.707	.220	.170	.145
	LCI	2.273	1.847	1.552	.982	.903	.652
	CP	.800	1.000	.600	1.000	1.000	1.000
	Bias	.166	.033	-.037	-.081	.036	.050
	RMSE	.809	.376	.708	.234	.174	.154
	AVRB	.331	.066	.074	.161	.072	.101
γ	Median	-.760	.267	.702	-.694	-.072	.761
	SD	.060	.398	.153	.100	.064	.048
	LCI	.633	.930	.731	.473	.584	.451
	CP	1.000	.600	1.000	1.000	.800	1.000
	Bias	.040	.267	-.098	.106	-.072	-.039
	RMSE	.072	.479	.182	.146	.096	.062
	AVRB	.050	-	.123	.133	-	.049
ν	Median	12.379	6.695	9.331	5.453	5.367	5.481
	SD	4.337	.897	3.959	.673	.621	.803
	LCI	61.840	27.732	46.417	4.760	4.615	4.754
	CP	.600	1.000	.800	1.000	1.000	1.000
	Bias	7.379	1.695	4.331	.453	.367	.481
	RMSE	8.559	1.918	5.868	.811	.721	.936
	AVRB	1.476	.339	.866	.091	.073	.096

Table 80 – Results of simulation study for StBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.692	-.749	-.446	-.496	-.496	-.516
	SD	.098	.028	.134	.042	.150	.074
	LCI	.709	.886	.827	.352	.396	.397
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.192	-.249	.054	.004	.004	-.016
	RMSE	.216	.25	.145	.042	.15	.076
	AVRB	.385	.497	.108	.007	.007	.031
β_1	Mode	1.247	1.541	.781	.985	.945	.971
	SD	.340	.062	.353	.126	.296	.135
	LCI	1.171	1.428	1.245	.603	.625	.560
	CP	.800	.800	1.000	1.000	.600	1.000
	Bias	.247	.541	-.219	-.015	-.055	-.029
	RMSE	.420	.545	.416	.127	.301	.138
	AVRB	.247	.541	.219	.015	.055	.029
ψ_0	Mode	-.918	-.880	-.888	-.944	-1.027	-1.019
	SD	.382	.167	.378	.160	.090	.088
	LCI	1.308	1.210	1.105	.634	.610	.509
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.082	.120	.112	.056	-.027	-.019
	RMSE	.391	.206	.394	.169	.094	.090
	AVRB	.082	.120	.112	.056	.027	.019
ψ_1	Mode	.634	.542	.474	.417	.533	.554
	SD	.732	.372	.703	.222	.171	.147
	LCI	2.273	1.847	1.552	.982	.903	.652
	CP	.800	1.000	.600	1.000	1.000	1.000
	Bias	.134	.042	-.026	-.083	.033	.054
	RMSE	.744	.374	.704	.237	.174	.156
	AVRB	.268	.084	.052	.167	.066	.108
γ	Mode	-.765	.269	.719	-.692	-.078	.758
	SD	.052	.412	.132	.100	.063	.046
	LCI	.633	.93	.731	.473	.584	.451
	CP	1.000	.600	1.000	1.000	.800	1.000
	Bias	.035	.269	-.081	.108	-.078	-.042
	RMSE	.062	.492	.154	.147	.100	.062
	AVRB	.044	-	.101	.135	-	.052
ν	Mode	7.432	4.716	6.458	5.027	4.981	5.167
	SD	3.106	.305	2.469	.494	.725	.771
	LCI	61.840	27.732	46.417	4.76	4.615	4.754
	CP	.600	1.000	.800	1.000	1.000	1.000
	Bias	2.432	-.284	1.458	.027	-.019	.167
	RMSE	3.945	.416	2.868	.495	.725	.789
	AVRB	.486	.057	.292	.005	.004	.033

Table 81 – Results of simulation study for StBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.529	-.439	-.449	-.499	-.495	-.514
	SD	.105	.175	.135	.051	.051	.035
	LCI	.631	.769	.768	.280	.331	.298
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.029	.061	.051	.001	.005	-.014
	RMSE	.109	.185	.145	.051	.052	.038
	AVRB	.057	.122	.102	.001	.010	.029
β_1	Mean	1.086	.921	.946	1.016	1.018	1.035
	SD	.221	.344	.241	.087	.095	.068
	LCI	1.073	1.230	1.130	.476	.552	.417
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.086	-.079	-.054	.016	.018	.035
	RMSE	.237	.353	.247	.088	.097	.077
	AVRB	.086	.079	.054	.016	.018	.035
ψ_0	Mean	-1.155	-1.101	-1.109	-1.028	-.952	-1.075
	SD	.428	.234	.155	.126	.101	.095
	LCI	1.337	1.192	1.131	.578	.522	.420
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.155	-.101	-.109	-.028	.048	-.075
	RMSE	.455	.254	.189	.129	.112	.121
	AVRB	.155	.101	.109	.028	.048	.075
ψ_1	Mean	.640	.533	.561	.461	.363	.560
	SD	.814	.437	.235	.168	.140	.164
	LCI	2.179	1.887	1.522	.969	.814	.528
	CP	.800	1.000	1.000	1.000	1.000	.800
	Bias	.140	.033	.061	-.039	-.137	.060
	RMSE	.826	.438	.243	.172	.196	.175
	AVRB	.280	.065	.123	.078	.275	.121
γ	Mean	-.782	.095	.682	-.754	.038	.852
	SD	.104	.258	.147	.09	.116	.072
	LCI	.587	1.124	.775	.337	.414	.255
	CP	1.000	1.000	1.000	1.000	.600	.800
	Bias	.018	.095	-.118	.046	.038	.052
	RMSE	.105	.275	.189	.100	.122	.089
	AVRB	.023	-	.148	.057	-	.065
ν	Mean	24.787	19.231	19.061	24.805	27.655	30.970
	SD	4.520	5.567	2.163	12.516	7.706	18.264
	LCI	86.677	68.026	72.314	68.773	79.925	79.993
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-5.213	-10.769	-10.939	-5.195	-2.345	.970
	RMSE	6.900	12.123	11.151	13.551	8.055	18.290
	AVRB	.174	.359	.365	.173	.078	.032

Table 82 – Results of simulation study for StBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.535	-.441	-.449	-.501	-.496	-.519
	SD	.111	.172	.136	.051	.052	.031
	LCI	.631	.769	.768	.280	.331	.298
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.035	.059	.051	-.001	.004	-.019
	RMSE	.116	.182	.145	.051	.052	.036
	AVRB	.069	.118	.101	.002	.008	.039
β_1	Median	1.084	.920	.946	1.016	1.02	1.041
	SD	.211	.349	.248	.087	.099	.066
	LCI	1.073	1.230	1.130	.476	.552	.417
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.084	-.080	-.054	.016	.02	.041
	RMSE	.227	.358	.254	.088	.101	.078
	AVRB	.084	.080	.054	.016	.020	.041
ψ_0	Median	-1.153	-1.098	-1.102	-1.029	-.952	-1.079
	SD	.424	.233	.153	.124	.100	.097
	LCI	1.337	1.192	1.131	.578	.522	.420
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.153	-.098	-.102	-.029	.048	-.079
	RMSE	.451	.253	.184	.127	.111	.125
	AVRB	.153	.098	.102	.029	.048	.079
ψ_1	Median	.629	.528	.567	.468	.361	.561
	SD	.792	.444	.236	.159	.138	.160
	LCI	2.179	1.887	1.522	.969	.814	.528
	CP	.800	1.000	1.000	1.000	1.000	.800
	Bias	.129	.028	.067	-.032	-.139	.061
	RMSE	.802	.445	.246	.163	.196	.172
	AVRB	.257	.056	.135	.064	.277	.123
γ	Median	-.820	.084	.719	-.759	.031	.859
	SD	.104	.261	.147	.091	.108	.078
	LCI	.587	1.124	.775	.337	.414	.255
	CP	1.000	1.000	1.000	1.000	.600	.800
	Bias	-.020	.084	-.081	.041	.031	.059
	RMSE	.106	.275	.168	.100	.113	.097
	AVRB	.025	-	.102	.051	-	.073
ν	Median	16.717	12.787	12.788	18.762	20.928	24.24
	SD	3.647	3.899	1.936	8.723	5.877	14.137
	LCI	86.677	68.026	72.314	68.773	79.925	79.993
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-13.283	-17.213	-17.212	-11.238	-9.072	-5.76
	RMSE	13.775	17.649	17.320	14.226	10.809	15.265
	AVRB	.443	.574	.574	.375	.302	.192

Table 83 – Results of simulation study for StBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.534	-.444	-.451	-.499	-.496	-.515
	SD	.113	.175	.135	.050	.051	.033
	LCI	.631	.769	.768	.280	.331	.298
β_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.034	.056	.049	.001	.004	-.015
	RMSE	.118	.184	.143	.050	.052	.036
	AVRB	.069	.112	.098	.003	.008	.030
	Mode	1.078	.923	.940	1.017	1.017	1.037
	SD	.194	.354	.243	.085	.097	.068
	LCI	1.073	1.230	1.130	.476	.552	.417
β_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.078	-.077	-.060	.017	.017	.037
	RMSE	.210	.362	.251	.087	.098	.078
	AVRB	.078	.077	.060	.017	.017	.037
	Mode	-1.146	-1.099	-1.098	-1.030	-.953	-1.075
	SD	.408	.231	.154	.125	.099	.096
	LCI	1.337	1.192	1.131	.578	.522	.420
ψ_0	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.146	-.099	-.098	-.030	.047	-.075
	RMSE	.434	.252	.183	.129	.110	.122
	AVRB	.146	.099	.098	.030	.047	.075
	Mode	.606	.513	.575	.472	.362	.559
	SD	.738	.469	.245	.157	.138	.163
	LCI	2.179	1.887	1.522	.969	.814	.528
ψ_1	CP	.800	1.000	1.000	1.000	1.000	.800
	Bias	.106	.013	.075	-.028	-.138	.059
	RMSE	.746	.469	.256	.16	.195	.173
	AVRB	.212	.027	.149	.056	.277	.118
	Mode	-.818	.072	.734	-.756	.034	.854
	SD	.096	.277	.120	.091	.110	.072
	LCI	.587	1.124	.775	.337	.414	.255
γ	CP	1.000	1.000	1.000	1.000	.600	.800
	Bias	-.018	.072	-.066	.044	.034	.054
	RMSE	.097	.287	.137	.101	.115	.090
	AVRB	.023	-	.082	.056	-	.068
	Mode	8.881	7.126	6.911	13.335	14.636	14.958
	SD	3.435	1.689	.850	5.582	3.795	7.898
	LCI	86.677	68.026	72.314	68.773	79.925	79.993
ν	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-21.119	-22.874	-23.089	-16.665	-15.364	-15.042
	RMSE	21.396	22.936	23.105	17.575	15.826	16.989
	AVRB	.704	.762	.770	.556	.512	.501

SSLBS regression model

Table 84 – Results of simulation study for SSLBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.523	-.386	-.428	-.511	-.476	-.504
	SD	.104	.137	.132	.088	.093	.076
	LCI	.595	.762	.723	.281	.335	.325
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.023	.114	.072	-.011	.024	-.004
	RMSE	.107	.178	.151	.088	.096	.076
	AVRB	.045	.228	.144	.023	.049	.008
β_1	Mean	1.085	.753	.838	1.025	.937	.989
	SD	.186	.284	.247	.194	.190	.142
	LCI	1.030	1.315	1.124	.485	.593	.447
	CP	1.000	.800	1.000	.800	.800	1.000
	Bias	.085	-.247	-.162	.025	-.063	-.011
	RMSE	.204	.377	.295	.196	.200	.143
	AVRB	.085	.247	.162	.025	.063	.011
ψ_0	Mean	-1.109	-1.014	-1.004	-1.043	-1.065	-.980
	SD	.321	.196	.161	.050	.084	.060
	LCI	1.151	1.173	.998	.650	.556	.498
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.109	-.014	-.004	-.043	-.065	.020
	RMSE	.339	.196	.161	.066	.106	.063
	AVRB	.109	.014	.004	.043	.065	.020
ψ_1	Mean	.726	.611	.468	.631	.625	.571
	SD	.686	.352	.302	.102	.215	.081
	LCI	2.036	1.902	1.364	.877	.760	.529
	CP	.800	1.000	1.000	1.000	.800	1.000
	Bias	.226	.111	-.032	.131	.125	.071
	RMSE	.722	.369	.304	.166	.249	.107
	AVRB	.451	.223	.065	.261	.251	.142
γ	Mean	-.754	.006	.667	-.746	.004	.801
	SD	.135	.177	.143	.039	.161	.023
	LCI	.621	1.033	.676	.279	.432	.237
	CP	1.000	1.000	1.000	1.000	.400	1.000
	Bias	.046	.006	-.133	.054	.004	.001
	RMSE	.142	.178	.195	.067	.161	.023

Table 84 (continued).

	AVRB	.058	-	.166	.068	-	.001
	Mean	8.034	7.616	6.410	7.624	7.406	10.222
	SD	1.012	.907	1.348	1.334	2.783	2.188
	LCI	22.284	16.871	13.727	15.166	13.875	20.761
ν	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	3.034	2.616	1.410	2.624	2.406	5.222
	RMSE	3.199	2.769	1.950	2.944	3.679	5.662
	AVRB	.607	.523	.282	.525	.481	1.044

Table 85 – Results of simulation study for SSLBS regression model ($\nu = 5$).

		$n=100$			$n=500$		
		$\gamma=-.8$	$\gamma=0$	$\gamma=.8$	$\gamma=-.8$	$\gamma=0$	$\gamma=.8$
β_0	Median	-.523	-.394	-.426	-.511	-.477	-.505
	SD	.104	.136	.134	.089	.092	.077
	LCI	.595	.762	.723	.281	.335	.325
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.023	.106	.074	-.011	.023	-.005
	RMSE	.106	.173	.153	.090	.094	.077
	AVRB	.045	.213	.149	.021	.046	.009
β_1	Median	1.092	.759	.830	1.024	.936	.990
	SD	.193	.282	.251	.192	.192	.139
	LCI	1.030	1.315	1.124	.485	.593	.447
	CP	1.000	.800	1.000	.800	.800	1.000
	Bias	.092	-.241	-.170	.024	-.064	-.010
	RMSE	.214	.371	.303	.193	.203	.140
	AVRB	.092	.241	.170	.024	.064	.010
ψ_0	Median	-1.117	-1.022	-1.002	-1.041	-1.067	-.976
	SD	.329	.196	.149	.051	.083	.060
	LCI	1.151	1.173	.998	.650	.556	.498
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.117	-.022	-.002	-.041	-.067	.024
	RMSE	.349	.197	.149	.065	.107	.065
	AVRB	.117	.022	.002	.041	.067	.024
ψ_1	Median	.741	.611	.460	.642	.626	.571
	SD	.692	.343	.297	.102	.217	.082
	LCI	2.036	1.902	1.364	.877	.760	.529
	CP	.800	1.000	1.000	1.000	.800	1.000
	Bias	.241	.111	-.040	.142	.126	.071
	RMSE	.732	.360	.300	.175	.250	.108
	AVRB	.482	.223	.081	.284	.252	.142
γ	Median	-.788	-.002	.695	-.753	-.003	.806
	SD	.129	.195	.152	.038	.154	.024
	LCI	.621	1.033	.676	.279	.432	.237
	CP	1.000	1.000	1.000	1.000	.400	1.000
	Bias	.012	-.002	-.105	.047	-.003	.006
	RMSE	.129	.195	.185	.060	.154	.025
	AVRB	.015	-	.132	.058	-	.008
ν	Median	6.351	6.372	5.361	6.455	6.446	8.439
	SD	1.145	.953	1.068	1.261	2.332	1.545
	LCI	22.284	16.871	13.727	15.166	13.875	20.761
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	1.351	1.372	.361	1.455	1.446	3.439
	RMSE	1.771	1.670	1.127	1.925	2.743	3.770
	AVRB	.270	.274	.072	.291	.289	.688

Table 86 – Results of simulation study for SSLBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.522	-.395	-.428	-.512	-.475	-.505
	SD	.106	.134	.131	.086	.092	.078
	LCI	.595	.762	.723	.281	.335	.325
β_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.022	.105	.072	-.012	.025	-.005
	RMSE	.109	.17	.149	.087	.095	.078
	AVRB	.044	.209	.143	.023	.049	.01
	Mode	1.102	.766	.822	1.026	.936	.989
	SD	.198	.279	.261	.194	.192	.142
	LCI	1.03	1.315	1.124	.485	.593	.447
β_1	CP	1.000	.800	1.000	.800	.800	1.000
	Bias	.102	-.234	-.178	.026	-.064	-.011
	RMSE	.223	.364	.316	.196	.202	.143
	AVRB	.102	.234	.178	.026	.064	.011
	Mode	-1.133	-1.025	-.996	-1.042	-1.064	-.976
	SD	.336	.194	.146	.048	.084	.057
	LCI	1.151	1.173	.998	.65	.556	.498
ψ_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.133	-.025	.004	-.042	-.064	.024
	RMSE	.362	.196	.146	.064	.106	.062
	AVRB	.133	.025	.004	.042	.064	.024
	Mode	.737	.607	.448	.645	.622	.571
	SD	.700	.321	.292	.102	.218	.081
	LCI	2.036	1.902	1.364	.877	.760	.529
ψ_1	CP	.800	1.000	1.000	1.000	.800	1.000
	Bias	.237	.107	-.052	.145	.122	.071
	RMSE	.739	.338	.297	.178	.250	.108
	AVRB	.473	.214	.105	.291	.244	.142
	Mode	-.803	$<.001$	.701	-.746	.003	.802
	SD	.084	.212	.14	.039	.157	.023
	LCI	.621	1.033	.676	.279	.432	.237
γ	CP	1.000	1.000	1.000	1.000	.400	1.000
	Bias	-.003	$<.001$	-.099	.054	.003	.002
	RMSE	.084	.212	.172	.066	.157	.023
	AVRB	.004	-	.124	.067	-	.003
	Mode	4.180	4.060	3.682	4.446	4.547	6.602
	SD	1.727	.444	.786	1.544	1.474	2.297
	LCI	22.284	16.871	13.727	15.166	13.875	20.761
ν	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.820	-.940	-1.318	-.554	-.453	1.602
	RMSE	1.912	1.040	1.534	1.640	1.543	2.800
	AVRB	.164	.188	.264	.111	.091	.320

Table 87 – Results of simulation study for SSLBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.456	-.521	-.531	-.494	-.496	-.526
	SD	.136	.178	.101	.048	.071	.055
	LCI	.551	.719	.660	.265	.329	.302
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.044	-.021	-.031	.006	.004	-.026
	RMSE	.143	.179	.106	.048	.072	.060
	AVRB	.087	.041	.061	.012	.009	.052
β_1	Mean	.928	1.042	1.047	.982	.998	1.054
	SD	.286	.361	.228	.086	.146	.102
	LCI	1.013	1.284	1.076	.479	.550	.406
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.072	.042	.047	-.018	-.002	.054
	RMSE	.295	.364	.233	.088	.146	.115
	AVRB	.072	.042	.047	.018	.002	.054
ψ_0	Mean	-1.103	-1.036	-.922	-1.039	-1.086	-1.041
	SD	.272	.221	.184	.083	.210	.100
	LCI	1.155	1.035	.932	.550	.535	.450
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.103	-.036	.078	-.039	-.086	-.041
	RMSE	.291	.224	.200	.091	.227	.108
	AVRB	.103	.036	.078	.039	.086	.041
ψ_1	Mean	.642	.536	.263	.519	.488	.513
	SD	.524	.505	.334	.132	.166	.128
	LCI	2.03	1.592	1.210	.892	.799	.529
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.142	.036	-.237	.019	-.012	.013
	RMSE	.543	.506	.410	.133	.167	.129
	AVRB	.284	.072	.474	.038	.024	.026
γ	Mean	-.737	.138	.643	-.748	-.075	.804
	SD	.151	.261	.203	.029	.163	.037
	LCI	.607	.934	.694	.265	.39	.211
	CP	1.000	1.000	.800	1.000	.200	1.000
	Bias	.063	.138	-.157	.052	-.075	.004
	RMSE	.164	.295	.257	.059	.180	.037
	AVRB	.079	-	.196	.065	-	.005
ν	Mean	31.886	31.592	29.580	30.704	25.290	25.115
	SD	2.418	3.537	3.211	7.756	12.924	8.437
	LCI	82.164	86.702	80.447	77.048	59.015	56.518
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	1.886	1.592	-.420	.704	-4.710	-4.885
	RMSE	3.067	3.879	3.239	7.788	13.756	9.750
	AVRB	.063	.053	.014	.023	.157	.163

Table 88 – Results of simulation study for SSLBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.453	-.523	-.540	-.495	-.496	-.529
	SD	.138	.176	.103	.048	.072	.054
	LCI	.551	.719	.660	.265	.329	.302
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.047	-.023	-.040	.005	.004	-.029
	RMSE	.145	.178	.111	.049	.072	.061
	AVRB	.093	.047	.080	.010	.009	.058
β_1	Median	.919	1.034	1.059	.985	.999	1.055
	SD	.291	.362	.22	.084	.146	.101
	LCI	1.013	1.284	1.076	.479	.55	.406
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.081	.034	.059	-.015	-.001	.055
	RMSE	.302	.364	.228	.085	.146	.115
	AVRB	.081	.034	.059	.015	.001	.055
ψ_0	Median	-1.116	-1.034	-.923	-1.038	-1.088	-1.035
	SD	.277	.218	.180	.086	.212	.104
	LCI	1.155	1.035	.932	.550	.535	.450
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.116	-.034	.077	-.038	-.088	-.035
	RMSE	.300	.220	.196	.094	.230	.110
	AVRB	.116	.034	.077	.038	.088	.035
ψ_1	Median	.639	.534	.265	.511	.488	.509
	SD	.526	.512	.332	.137	.169	.134
	LCI	2.030	1.592	1.210	.892	.799	.529
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.139	.034	-.235	.011	-.012	.009
	RMSE	.544	.513	.406	.138	.169	.134
	AVRB	.277	.067	.47	.021	.024	.017
γ	Median	-.763	.133	.673	-.754	-.071	.811
	SD	.141	.261	.207	.028	.152	.037
	LCI	.607	.934	.694	.265	.390	.211
	CP	1.000	1.000	.800	1.000	.200	1.000
	Bias	.037	.133	-.127	.046	-.071	.011
	RMSE	.146	.293	.243	.054	.168	.038
	AVRB	.046	-	.159	.057	-	.013
ν	Median	26.953	25.947	24.463	25.853	21.526	21.929
	SD	1.964	3.085	2.017	6.617	10.939	7.245
	LCI	82.164	86.702	80.447	77.048	59.015	56.518
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-3.047	-4.053	-5.537	-4.147	-8.474	-8.071
	RMSE	3.625	5.094	5.893	7.809	13.837	10.846
	AVRB	.102	.135	.185	.138	.282	.269

Table 89 – Results of simulation study for SSLBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.455	-.524	-.542	-.494	-.497	-.526
	SD	.136	.175	.102	.048	.073	.054
	LCI	.551	.719	.660	.265	.329	.302
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.045	-.024	-.042	.006	.003	-.026
	RMSE	.143	.177	.110	.049	.073	.060
	AVRB	.090	.048	.084	.012	.007	.053
β_1	Mode	.918	1.019	1.066	.984	1.000	1.054
	SD	.299	.360	.210	.084	.147	.102
	LCI	1.013	1.284	1.076	.479	.550	.406
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.082	.019	.066	-.016	$< .001$	.054
	RMSE	.310	.361	.221	.086	.147	.116
	AVRB	.082	.019	.066	.016	$< .001$	.054
ψ_0	Mode	-1.121	-1.043	-.922	-1.039	-1.087	-1.037
	SD	.277	.217	.177	.085	.213	.100
	LCI	1.155	1.035	.932	.550	.535	.450
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.121	-.043	.078	-.039	-.087	-.037
	RMSE	.302	.221	.194	.094	.231	.107
	AVRB	.121	.043	.078	.039	.087	.037
ψ_1	Mode	.647	.538	.268	.502	.488	.510
	SD	.533	.516	.325	.143	.171	.131
	LCI	2.030	1.592	1.210	.892	.799	.529
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.147	.038	-.232	.002	-.012	.010
	RMSE	.553	.518	.399	.143	.172	.131
	AVRB	.294	.077	.464	.004	.023	.019
γ	Mode	-.779	.115	.68	-.749	-.072	.805
	SD	.106	.261	.191	.03	.158	.036
	LCI	.607	.934	.694	.265	.39	.211
	CP	1.000	1.000	.800	1.000	.200	1.000
	Bias	.021	.115	-.120	.051	-.072	.005
	RMSE	.108	.285	.225	.059	.173	.036
	AVRB	.026	-	.15	.063	-	.006
ν	Mode	13.982	14.143	10.901	12.423	10.392	11.003
	SD	8.816	4.218	6.925	9.029	8.416	4.122
	LCI	82.164	86.702	80.447	77.048	59.015	56.518
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-16.018	-15.857	-19.099	-17.577	-19.608	-18.997
	RMSE	18.284	16.408	20.315	19.760	21.338	19.438
	AVRB	.534	.529	.637	.586	.654	.633

SCNBS regression model

Table 90 – Results of simulation study for SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.688	-.595	-.428	-.448	-.449	-.574
	SD	.176	.255	.260	.049	.096	.085
	LCI	1.160	1.299	1.258	.479	.590	.543
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.188	-.095	.072	.052	.051	-.074
	RMSE	.257	.272	.27	.071	.109	.113
	AVRB	.375	.19	.144	.104	.101	.149
β_1	Mean	1.277	1.314	.888	.888	.958	1.162
	SD	.485	.475	.415	.136	.177	.208
	LCI	1.936	2.085	1.868	.738	.883	.750
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.277	.314	-.112	-.112	-.042	.162
	RMSE	.558	.569	.43	.176	.182	.263
	AVRB	.277	.314	.112	.112	.042	.162
ψ_0	Mean	.872	.823	.874	.716	.731	.523
	SD	.231	.203	.497	.425	.347	1.044
	LCI	1.662	1.833	1.595	1.416	1.855	.972
	CP	$< .001$	.200	$< .001$	.200	.600	.200
	Bias	1.872	1.823	1.874	1.716	1.731	1.523
	RMSE	1.886	1.834	1.939	1.768	1.765	1.846
	AVRB	1.872	1.823	1.874	1.716	1.731	1.523
ψ_1	Mean	.578	.597	.449	.495	.407	.501
	SD	.390	.224	.633	.260	.214	.117
	LCI	2.041	1.687	1.129	.799	.719	.442
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.078	.097	-.051	-.005	-.093	.001
	RMSE	.398	.244	.635	.26	.233	.117
	AVRB	.157	.195	.103	.009	.186	.003
γ	Mean	-.611	.013	.632	-.741	-.053	.773
	SD	.143	.090	.137	.041	.064	.081
	LCI	.799	1.174	.746	.359	.452	.312
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.189	.013	-.168	.059	-.053	-.027
	RMSE	.237	.091	.217	.072	.083	.085

Table 90 (continued).

	AVRB	.236	-	.210	.074	-	.034
	Mean	.509	.512	.516	.610	.562	.625
	SD	.020	.033	.063	.128	.077	.132
	LCI	.797	.788	.829	.716	.783	.661
ν_1	CP	.800	.400	.800	.800	.600	1.000
	Bias	-.391	-.388	-.384	-.290	-.338	-.275
	RMSE	.392	.389	.389	.317	.346	.305
	AVRB	.434	.431	.426	.322	.375	.306
	Mean	.641	.602	.628	.618	.566	.599
	SD	.021	.068	.078	.156	.166	.309
	LCI	.723	.779	.771	.717	.723	.48
ν_2	CP	$< .001$	.200	$< .001$	.200	.600	.200
	Bias	.541	.502	.528	.518	.466	.499
	RMSE	.541	.507	.533	.541	.495	.587
	AVRB	5.406	5.024	5.277	5.181	4.66	4.992

Table 91 – Results of simulation study for SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.693	-.597	-.439	-.445	-.448	-.572
	SD	.182	.252	.265	.047	.101	.087
	LCI	1.160	1.299	1.258	.479	.590	.543
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.193	-.097	.061	.055	.052	-.072
	RMSE	.265	.27	.272	.073	.113	.113
	AVRB	.386	.195	.123	.111	.104	.144
β_1	Median	1.290	1.312	.879	.896	.945	1.160
	SD	.490	.486	.422	.136	.181	.203
	LCI	1.936	2.085	1.868	.738	.883	.750
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.290	.312	-.121	-.104	-.055	.160
	RMSE	.569	.578	.439	.172	.190	.259
	AVRB	.290	.312	.121	.104	.055	.160
ψ_0	Median	.898	.881	.945	.689	.942	.570
	SD	.247	.170	.469	.651	.204	1.083
	LCI	1.662	1.833	1.595	1.416	1.855	.972
	CP	$< .001$	.200	$< .001$	.200	.600	.200
	Bias	1.898	1.881	1.945	1.689	1.942	1.570
	RMSE	1.914	1.889	2.001	1.810	1.953	1.907
	AVRB	1.898	1.881	1.945	1.689	1.942	1.570
ψ_1	Median	.563	.598	.44	.492	.406	.504
	SD	.374	.213	.623	.26	.213	.117
	LCI	2.041	1.687	1.129	.799	.719	.442
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.063	.098	-.06	-.008	-.094	.004
	RMSE	.379	.235	.626	.26	.233	.117
	AVRB	.127	.197	.119	.016	.188	.008
γ	Median	-.638	0	.66	-.75	-.06	.78
	SD	.161	.093	.146	.048	.045	.082
	LCI	.799	1.174	.746	.359	.452	.312
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.162	0	-.140	.050	-.060	-.020
	RMSE	.229	.093	.203	.07	.075	.085
	AVRB	.202	-	.175	.063	-	.025

Table 91 (continued).

	Median	.510	.515	.524	.638	.582	.640
	SD	.026	.049	.082	.140	.100	.141
	LCI	.797	.788	.829	.716	.783	.661
ν_1	CP	.800	.400	.800	.800	.600	1.000
	Bias	-.390	-.385	-.376	-.262	-.318	-.260
	RMSE	.391	.388	.385	.297	.334	.295
	AVRB	.433	.427	.418	.291	.354	.289
	Median	.644	.602	.636	.598	.58	.617
	SD	.017	.084	.092	.256	.174	.321
	LCI	.723	.779	.771	.717	.723	.480
ν_2	CP	$< .001$	.200	$< .001$	.200	.600	.200
	Bias	.544	.502	.536	.498	.480	.517
	RMSE	.545	.509	.544	.560	.511	.608
	AVRB	5.443	5.016	5.361	4.980	4.801	5.165

Table 92 – Results of simulation study for SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.699	-.602	-.450	-.445	-.448	-.575
	SD	.193	.249	.263	.049	.102	.089
	LCI	1.160	1.299	1.258	.479	.590	.543
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.199	-.102	.050	.055	.052	-.075
	RMSE	.277	.269	.267	.074	.114	.116
	AVRB	.399	.205	.101	.110	.104	.150
β_1	Mode	1.320	1.315	.854	.893	.940	1.158
	SD	.500	.525	.445	.136	.181	.203
	LCI	1.936	2.085	1.868	.738	.883	.750
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.320	.315	-.146	-.107	-.060	.158
	RMSE	.594	.612	.468	.173	.191	.257
	AVRB	.320	.315	.146	.107	.06	.158
ψ_0	Mode	.926	.951	1.018	.681	1.064	.595
	SD	.260	.141	.436	.785	.123	1.110
	LCI	1.662	1.833	1.595	1.416	1.855	.972
	CP	$< .001$	.200	$< .001$	.200	.600	.200
	Bias	1.926	1.951	2.018	1.681	2.064	1.595
	RMSE	1.944	1.956	2.065	1.855	2.068	1.943
	AVRB	1.926	1.951	2.018	1.681	2.064	1.595
ψ_1	Mode	.515	.605	.438	.489	.407	.504
	SD	.307	.200	.619	.263	.213	.119
	LCI	2.041	1.687	1.129	.799	.719	.442
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.015	.105	-.062	-.011	-.093	.004
	RMSE	.307	.226	.623	.264	.232	.119
	AVRB	.029	.209	.124	.021	.186	.008
γ	Mode	-.645	-.008	.669	-.745	-.060	.778
	SD	.160	.109	.136	.043	.053	.080
	LCI	.799	1.174	.746	.359	.452	.312
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.155	-.008	-.131	.055	-.060	-.022
	RMSE	.223	.109	.189	.070	.081	.083
	AVRB	.194	-	.164	.069	-	.028

Table 92 (continued).

	Mode	.515	.519	.528	.641	.628	.646
	SD	.032	.061	.098	.152	.103	.148
	LCI	.797	.788	.829	.716	.783	.661
ν_1	CP	.800	.400	.800	.800	.600	1.000
	Bias	-.385	-.381	-.372	-.259	-.272	-.254
	RMSE	.387	.386	.384	.300	.290	.294
	AVRB	.428	.423	.413	.288	.302	.282
	Mode	.658	.583	.639	.605	.603	.624
	SD	.015	.127	.109	.265	.179	.317
	LCI	.723	.779	.771	.717	.723	.480
ν_2	CP	$< .001$	.200	$< .001$	.200	.600	.200
	Bias	.558	.483	.539	.505	.503	.524
	RMSE	.558	.500	.550	.570	.534	.612
	AVRB	5.582	4.832	5.388	5.051	5.030	5.237

Table 93 – Results of simulation study for SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.471	-.646	-.559	-.487	-.431	-.546
	SD	.133	.114	.183	.109	.143	.054
	LCI	.804	.864	.906	.363	.433	.398
β_0	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.029	-.146	-.059	.013	.069	-.046
	RMSE	.136	.186	.193	.11	.158	.071
	AVRB	.058	.293	.119	.027	.138	.093
	Mean	.959	1.181	1.006	.953	.872	1.045
	SD	.263	.127	.250	.103	.187	.123
	LCI	1.278	1.455	1.316	.501	.623	.490
β_1	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.041	.181	.006	-.047	-.128	.045
	RMSE	.266	.221	.250	.114	.227	.131
	AVRB	.041	.181	.006	.047	.128	.045
	Mean	-1.444	-1.249	-1.346	-.961	-.982	-.964
	SD	.324	.401	.464	.261	.154	.091
	LCI	1.823	1.826	1.675	.608	.586	.457
ψ_0	CP	1.000	.800	.800	.600	1.000	1.000
	Bias	-.444	-.249	-.346	.039	.018	.036
	RMSE	.550	.472	.579	.264	.155	.098
	AVRB	.444	.249	.346	.039	.018	.036
	Mean	.880	.654	.842	.517	.454	.532
	SD	.567	.432	.643	.423	.198	.109
	LCI	2.426	2.052	1.853	.985	.819	.533
ψ_0	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.380	.154	.342	.017	-.046	.032
	RMSE	.682	.459	.729	.423	.203	.114
	AVRB	.760	.307	.683	.033	.093	.065
	Mean	-.522	-.132	.440	-.774	.018	.793
	SD	.177	.358	.238	.039	.192	.054
	LCI	.961	1.024	.967	.313	.482	.276
γ	CP	.800	.800	.800	1.000	.400	1.000
	Bias	.278	-.132	-.360	.026	.018	-.007
	RMSE	.329	.381	.432	.047	.193	.055
	AVRB	.347	-	.451	.032	-	.009

Table 93 (continued).

	Mean	.345	.448	.341	.090	.124	.093
	SD	.107	.032	.123	.038	.074	.033
	LCI	.573	.703	.610	.100	.155	.104
ν_1	CP	.400	< .001	1.000	1.000	.800	1.000
	Bias	.245	.348	.241	-.010	.024	-.007
	RMSE	.268	.349	.270	.039	.078	.034
	AVRB	2.452	3.478	2.407	.096	.241	.075
	Mean	.246	.348	.299	.108	.120	.115
	SD	.163	.138	.211	.021	.040	.015
	LCI	.471	.624	.500	.106	.132	.101
ν_2	CP	.800	.400	.600	1.000	.800	1.000
	Bias	.146	.248	.199	.008	.020	.015
	RMSE	.219	.283	.290	.023	.045	.021
	AVRB	1.462	2.475	1.987	.081	.203	.146

Table 94 – Results of simulation study for SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.476	-.650	-.559	-.489	-.433	-.548
	SD	.130	.120	.175	.107	.141	.055
	LCI	.804	.864	.906	.363	.433	.398
β_0	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.024	-.150	-.059	.011	.067	-.048
	RMSE	.132	.192	.185	.108	.156	.073
	AVRB	.048	.300	.118	.023	.134	.096
	Median	.960	1.181	1.014	.950	.875	1.042
	SD	.260	.129	.247	.100	.187	.124
	LCI	1.278	1.455	1.316	.501	.623	.490
β_1	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.040	.181	.014	-.050	-.125	.042
	RMSE	.263	.222	.247	.112	.225	.131
	AVRB	.040	.181	.014	.050	.125	.042
	Median	-1.427	-1.222	-1.312	-.958	-.981	-.963
	SD	.332	.399	.443	.266	.155	.089
	LCI	1.823	1.826	1.675	.608	.586	.457
ψ_0	CP	1.000	.800	.800	.600	1.000	1.000
	Bias	-.427	-.222	-.312	.042	.019	.037
	RMSE	.541	.456	.541	.269	.156	.097
	AVRB	.427	.222	.312	.042	.019	.037
	Median	.864	.648	.846	.518	.457	.528
	SD	.557	.445	.635	.421	.202	.107
	LCI	2.426	2.052	1.853	.985	.819	.533
ψ_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.364	.148	.346	.018	-.043	.028
	RMSE	.665	.469	.723	.422	.207	.111
	AVRB	.728	.296	.692	.035	.087	.056
	Median	-.556	-.144	.456	-.787	.014	.803
	SD	.206	.370	.252	.037	.185	.057
	LCI	.961	1.024	.967	.313	.482	.276
γ	CP	.800	.800	.800	1.000	.400	1.000
	Bias	.244	-.144	-.344	.013	.014	.003
	RMSE	.320	.398	.426	.039	.186	.057
	AVRB	.306	-	.430	.016	-	.003

Table 94 (continued).

ν_1	Median	.332	.435	.324	.088	.119	.089
	SD	.125	.041	.125	.037	.075	.032
	LCI	.573	.703	.610	.100	.155	.104
	CP	.400	$<$ .001	1.000	1.000	.800	1.000
	Bias	.232	.335	.224	-.012	.019	-.011
	RMSE	.263	.337	.256	.039	.077	.033
	AVRB	2.319	3.345	2.235	.120	.192	.109
ν_2	Median	.223	.307	.276	.106	.118	.113
	SD	.158	.126	.211	.021	.039	.014
	LCI	.471	.624	.500	.106	.132	.101
	CP	.800	.400	.600	1.000	.800	1.000
	Bias	.123	.207	.176	.006	.018	.013
	RMSE	.200	.242	.275	.022	.043	.019
	AVRB	1.230	2.070	1.761	.058	.176	.128

Table 95 – Results of simulation study for SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.479	-.655	-.564	-.486	-.433	-.547
	SD	.133	.118	.170	.109	.142	.054
	LCI	.804	.864	.906	.363	.433	.398
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.021	-.155	-.064	.014	.067	-.047
	RMSE	.135	.195	.181	.110	.157	.072
	AVRB	.042	.310	.127	.028	.135	.093
β_1	Mode	.959	1.184	1.019	.949	.873	1.045
	SD	.257	.142	.251	.102	.187	.124
	LCI	1.278	1.455	1.316	.501	.623	.490
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.041	.184	.019	-.051	-.127	.045
	RMSE	.26	.232	.252	.114	.226	.132
	AVRB	.041	.184	.019	.051	.127	.045
ψ_0	Mode	-1.390	-1.181	-1.265	-.959	-.980	-.962
	SD	.327	.384	.420	.267	.156	.092
	LCI	1.823	1.826	1.675	.608	.586	.457
	CP	1.000	.800	.800	.600	1.000	1.000
	Bias	-.390	-.181	-.265	.041	.020	.038
	RMSE	.509	.425	.497	.270	.157	.099
	AVRB	.390	.181	.265	.041	.020	.038
ψ_1	Mode	.855	.629	.849	.524	.457	.528
	SD	.556	.470	.624	.421	.201	.108
	LCI	2.426	2.052	1.853	.985	.819	.533
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.355	.129	.349	.024	-.043	.028
	RMSE	.659	.487	.715	.421	.206	.112
	AVRB	.710	.257	.698	.048	.086	.055
γ	Mode	-.576	-.159	.460	-.779	.019	.795
	SD	.219	.386	.247	.038	.188	.055
	LCI	.961	1.024	.967	.313	.482	.276
	CP	.800	.800	.800	1.000	.400	1.000
	Bias	.224	-.159	-.340	.021	.019	-.005
	RMSE	.313	.417	.420	.043	.189	.055
	AVRB	.280	-	.425	.026	-	.006

Table 95 (continued).

	Mode	.336	.430	.321	.091	.123	.092
	SD	.126	.045	.119	.038	.073	.034
	LCI	.573	.703	.610	.100	.155	.104
ν_1	CP	.400	$<$ .001	1.000	1.000	.800	1.000
	Bias	.236	.33	.221	-.009	.023	-.008
	RMSE	.268	.333	.252	.039	.076	.034
	AVRB	2.362	3.295	2.214	.093	.226	.081
	Mode	.223	.298	.273	.107	.121	.114
	SD	.134	.102	.204	.022	.041	.016
	LCI	.471	.624	.500	.106	.132	.101
ν_2	CP	.800	.400	.600	1.000	.800	1.000
	Bias	.123	.198	.173	.007	.021	.014
	RMSE	.182	.222	.268	.023	.046	.022
	AVRB	1.229	1.976	1.734	.066	.206	.145

C.2 Behavior of the residuals

Simulated observations from SGtBS1 regression model

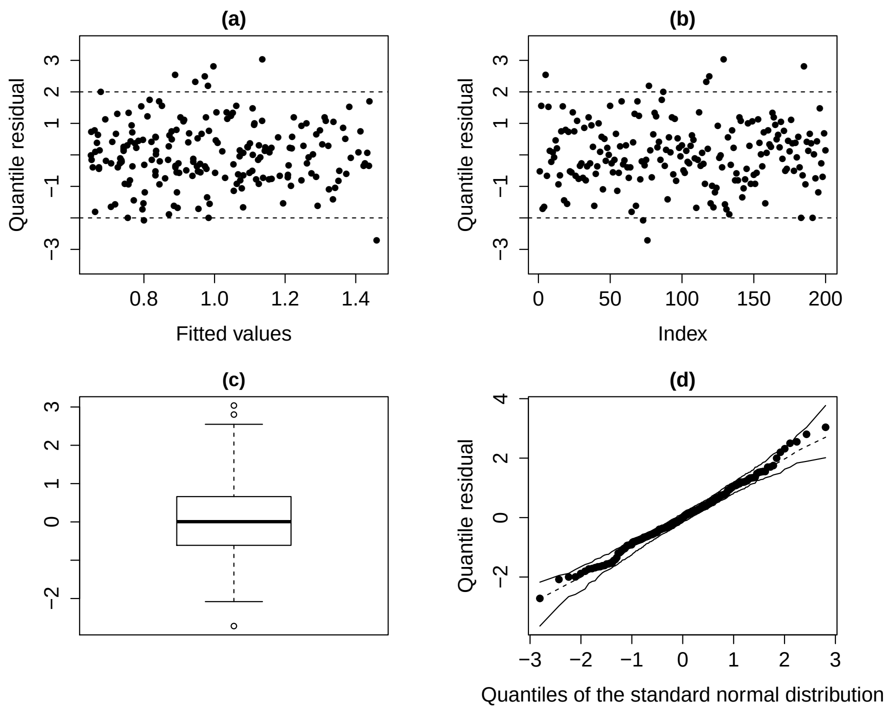

Figure 86 – Residual plots for the SGtBS1 regression model.

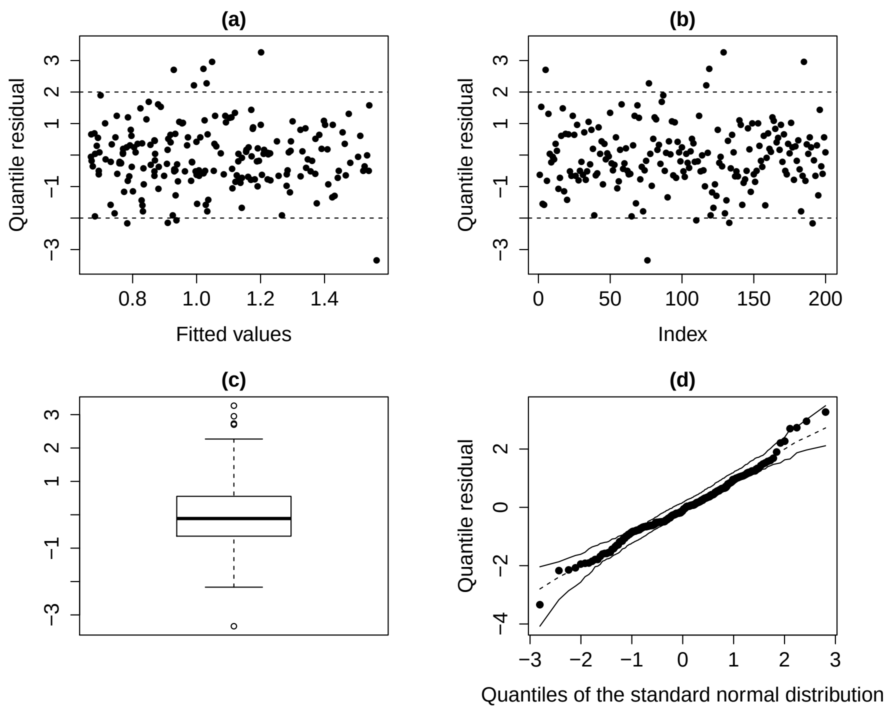

Figure 87 – Residual plots for the SGtBS2 regression model.

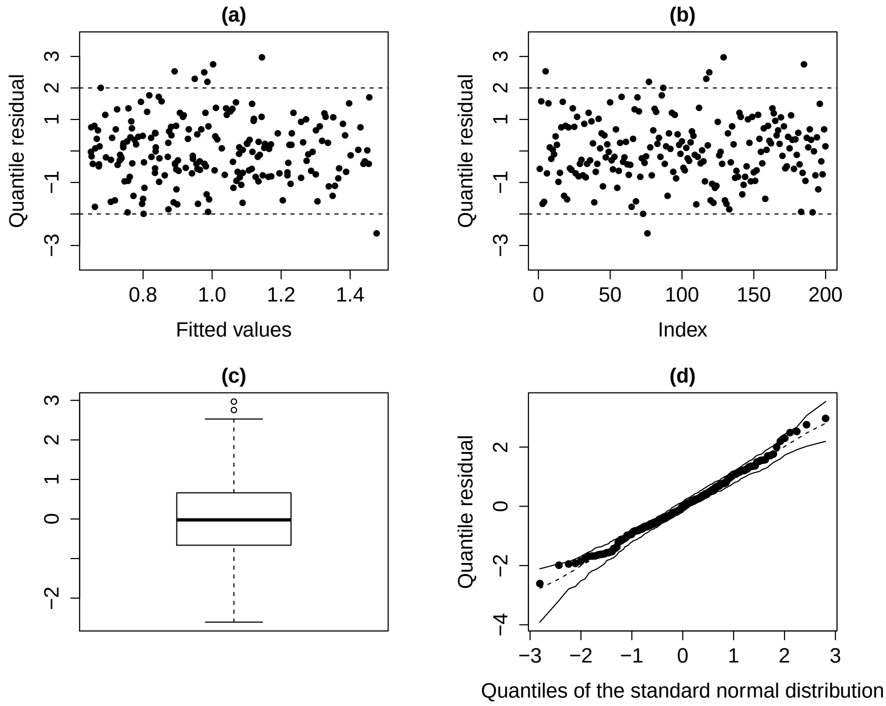

Figure 88 – Residual plots for the StBS regression model.

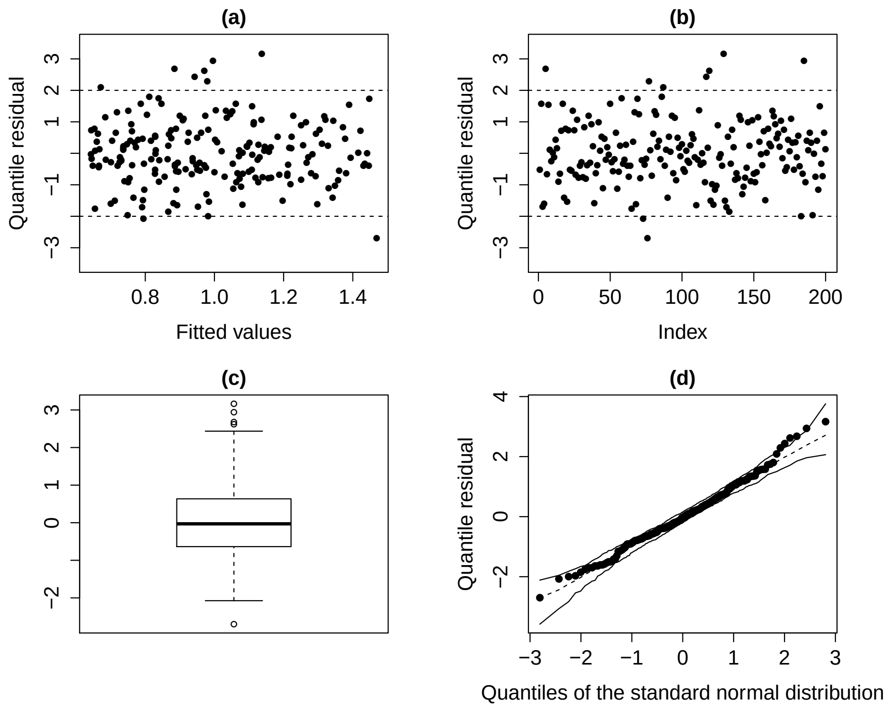

Figure 89 – Residual plots for the SSLBS1 regression model.

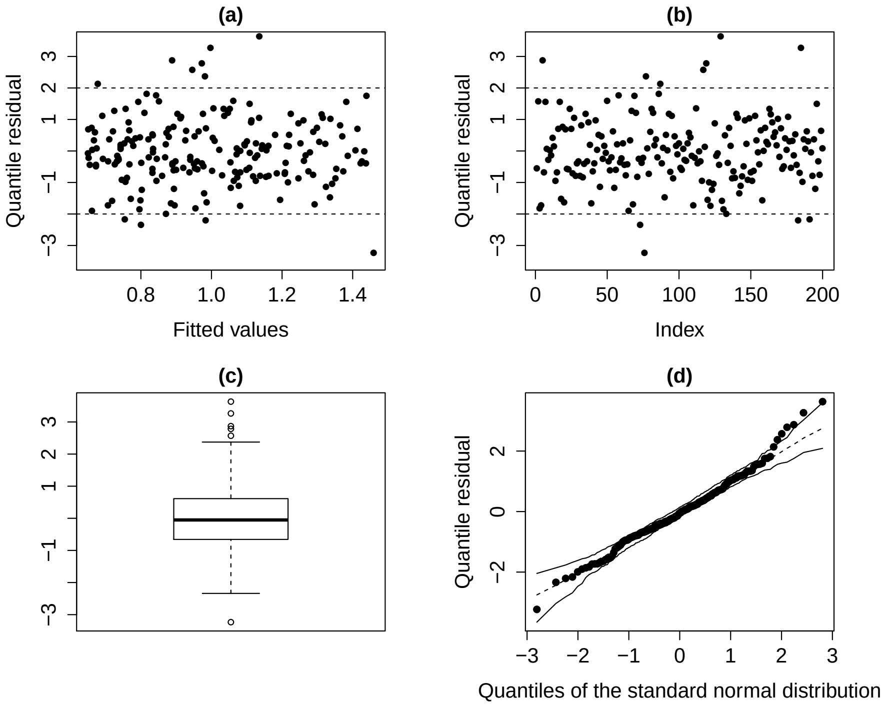

Figure 90 – Residual plots for the SSLBS2 regression model.

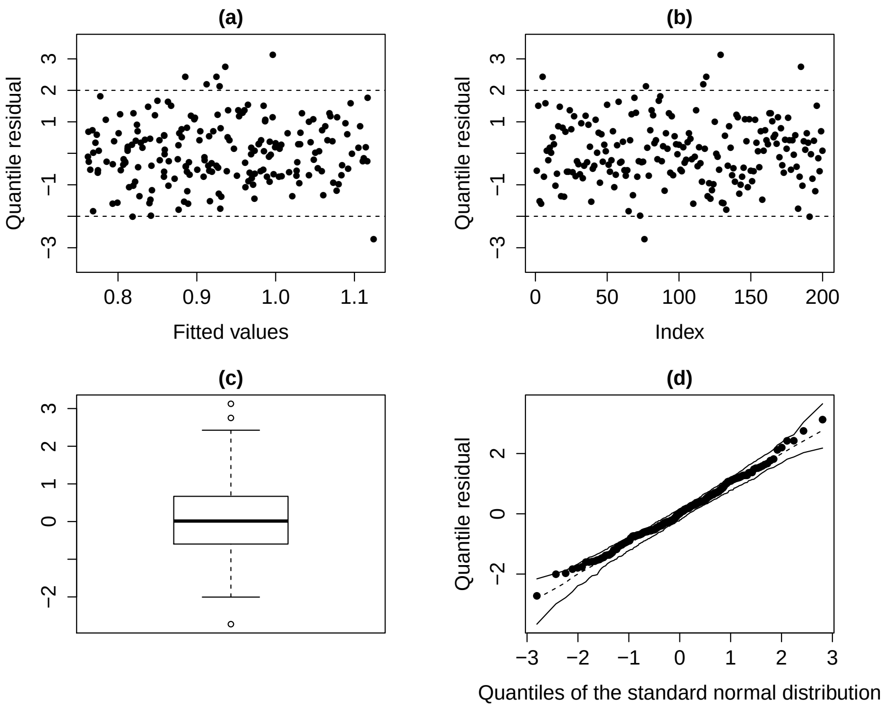

Figure 91 – Residual plots for the SCNBS regression model.

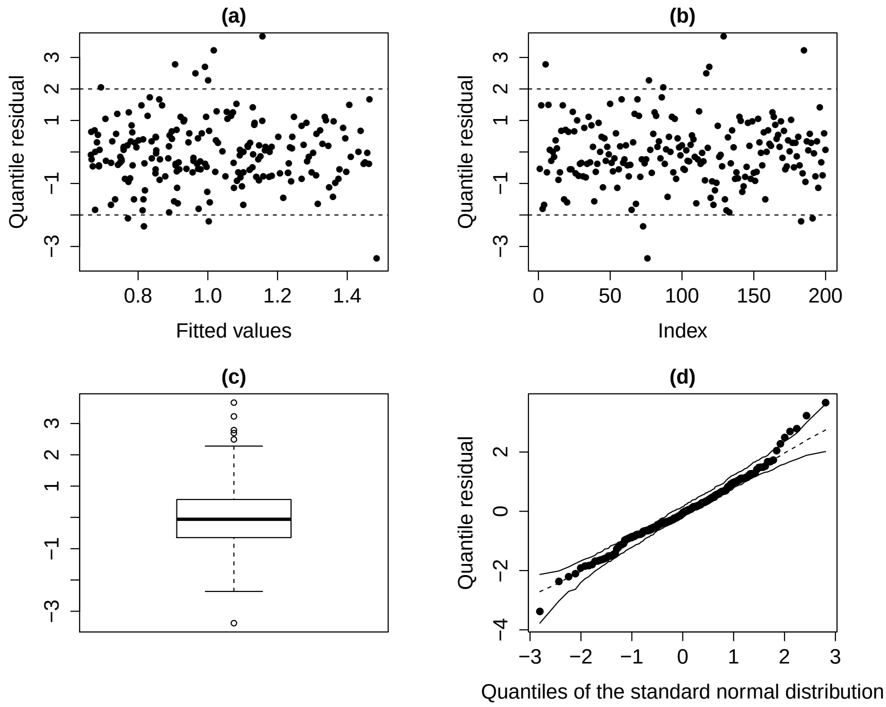

Figure 92 – Residual plots for the SNBS regression model.

Simulated observations from SGtBS2 regression model

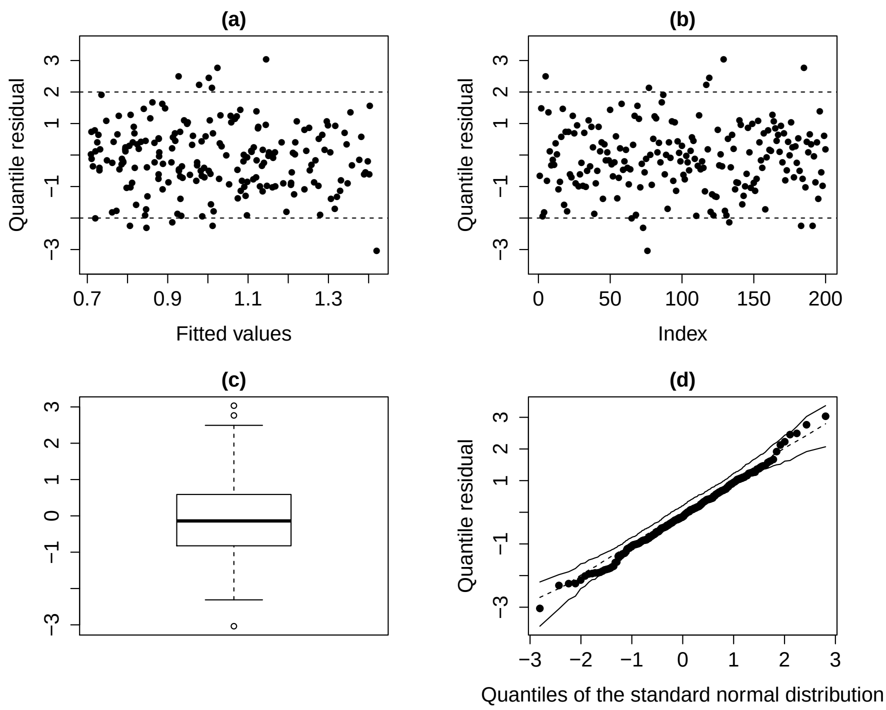

Figure 93 – Residual plots for the SGtBS2 regression model.

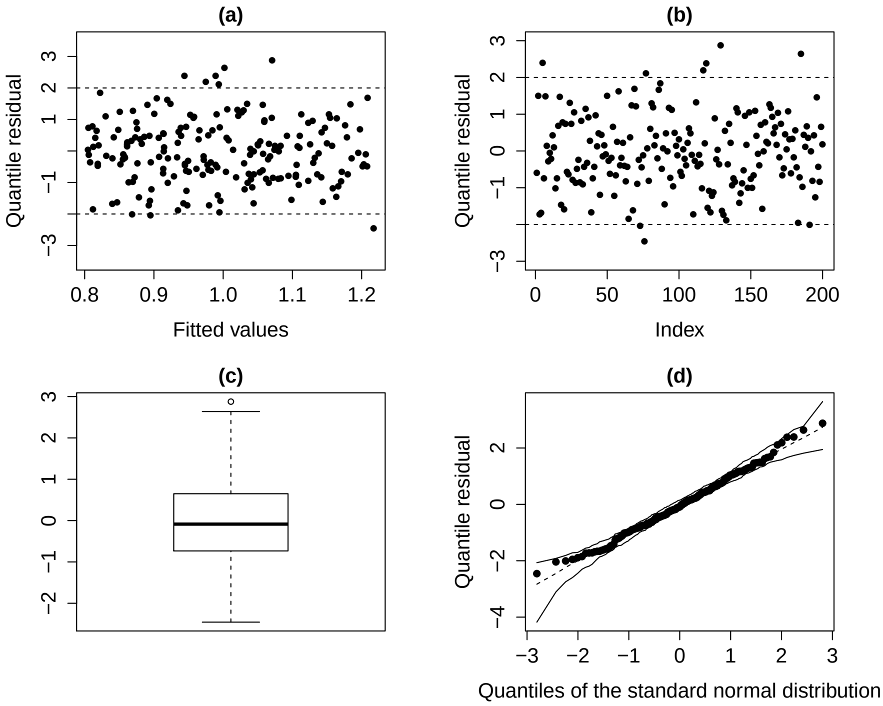

Figure 94 – Residual plots for the SGtBS1 regression model.

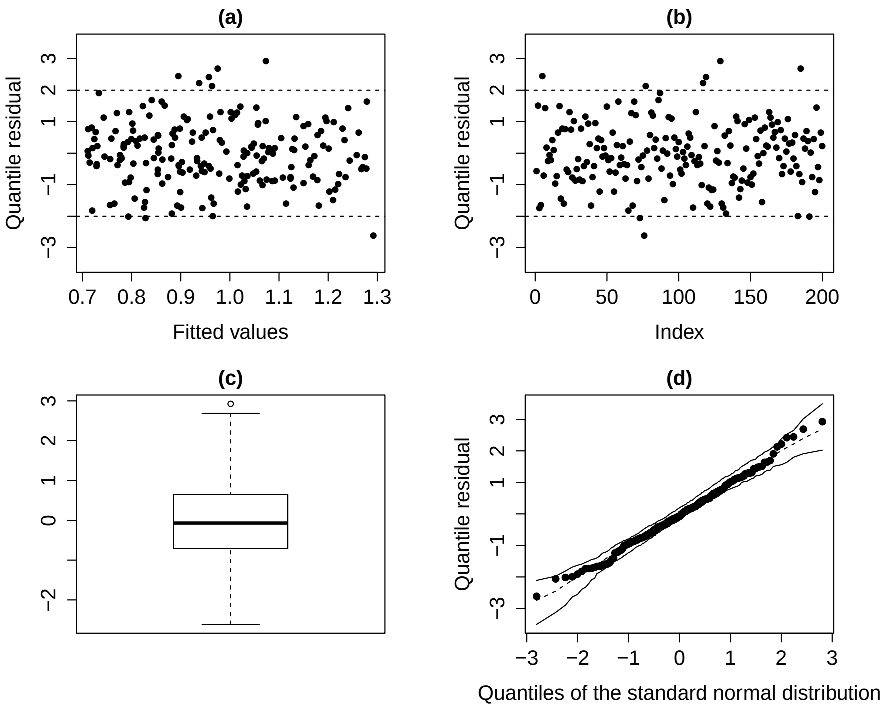

Figure 95 – Residual plots for the StBS regression model.

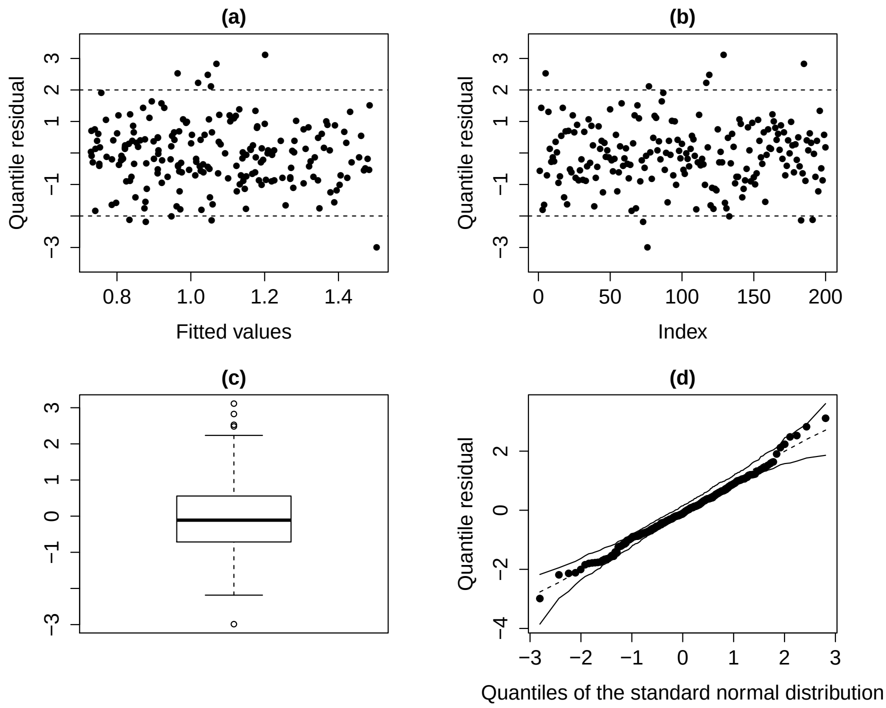

Figure 96 – Residual plots for the SSLBS1 regression model.

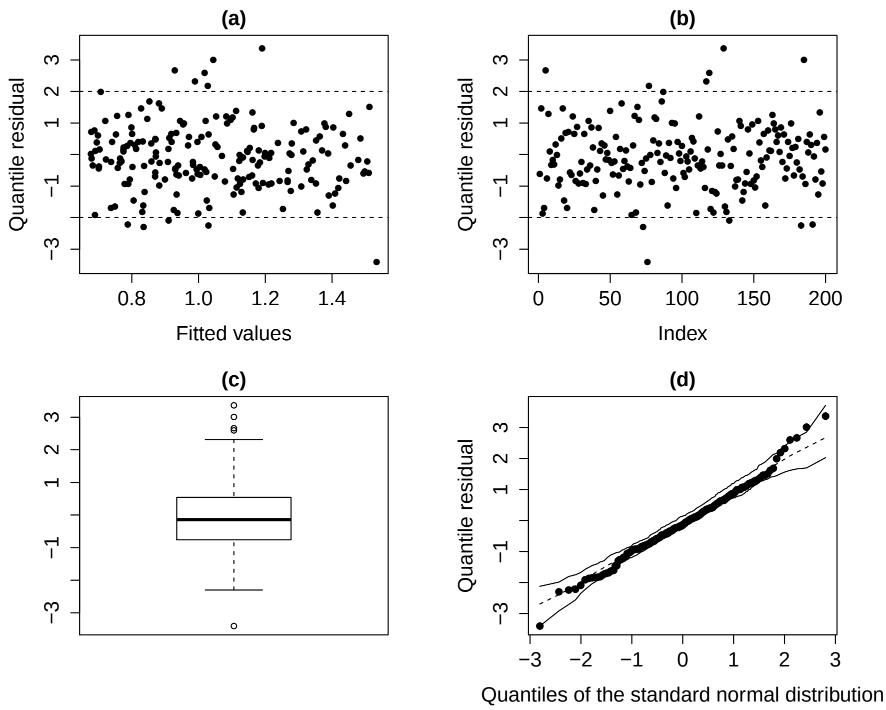

Figure 97 – Residual plots for the SSLBS2 regression model.

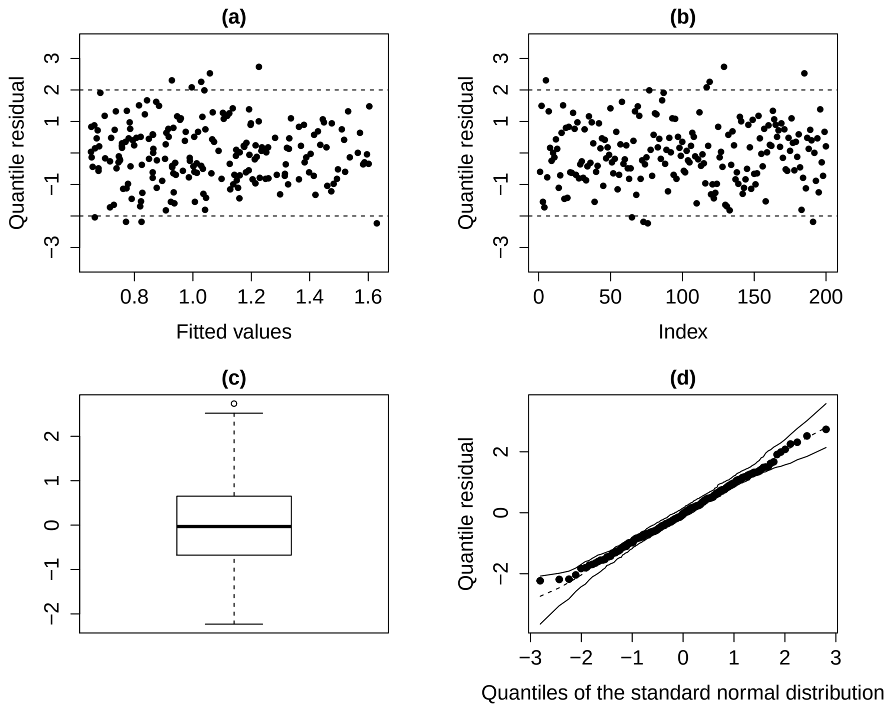

Figure 98 – Residual plots for the SCNBS regression model.

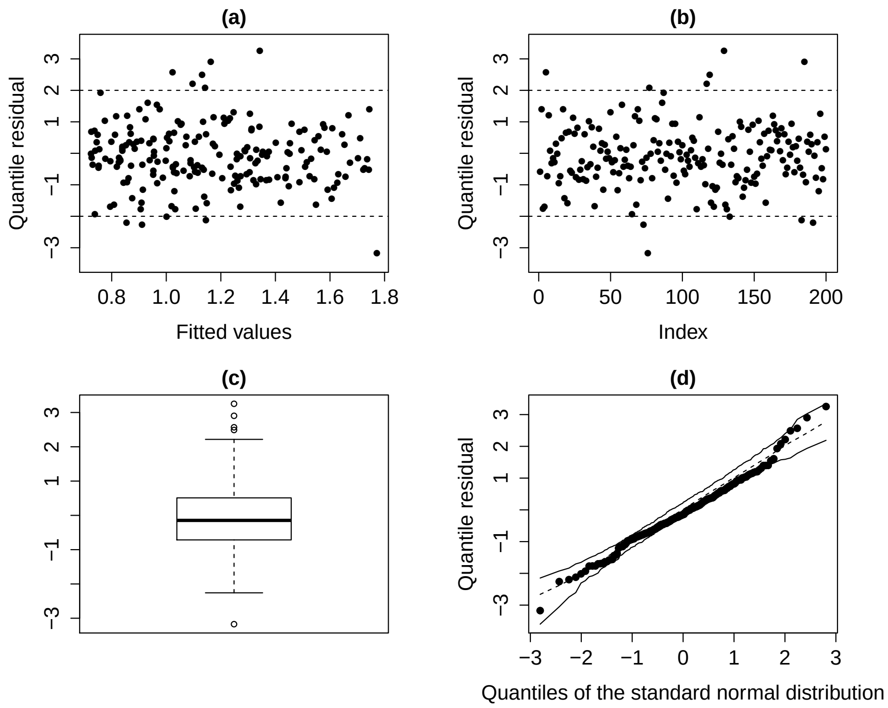

Figure 99 – Residual plots for the SNBS regression model.

Simulated observations from StBS regression model

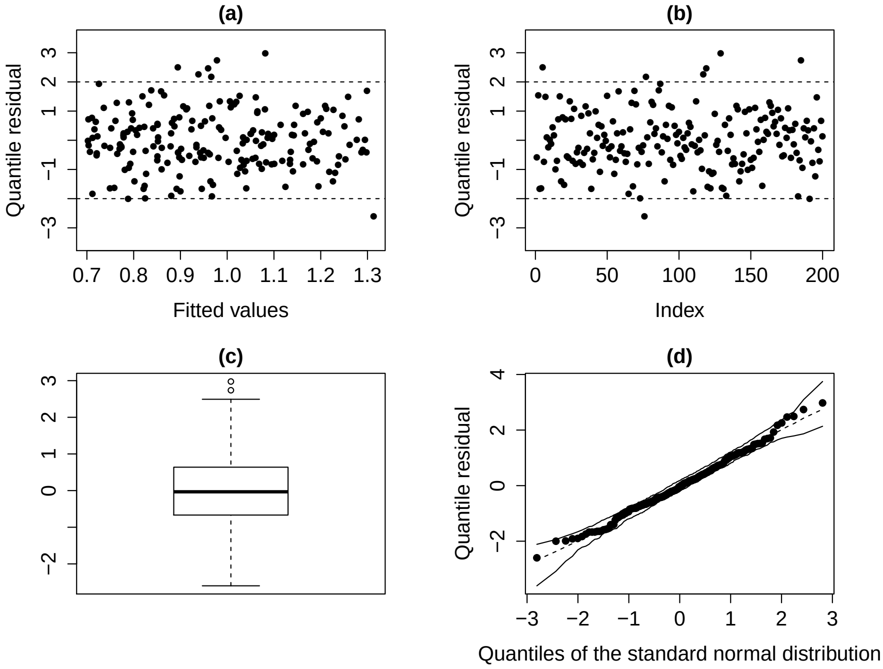

Figure 100 – Residual plots for the StBS regression model.

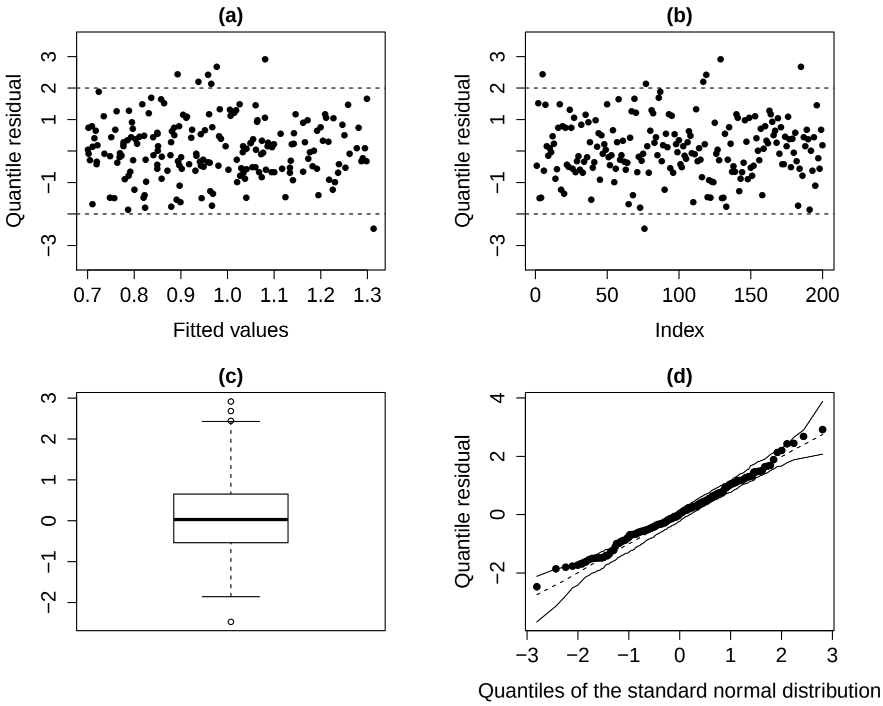

Figure 101 – Residual plots for the SGtBS1 regression model.

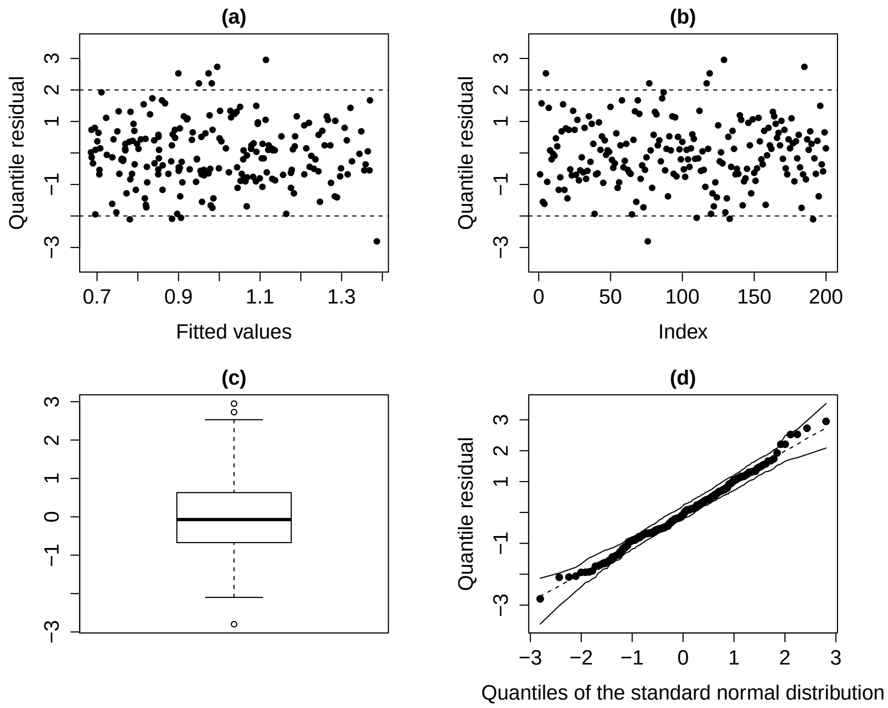

Figure 102 – Residual plots for the SGtBS2 regression model.

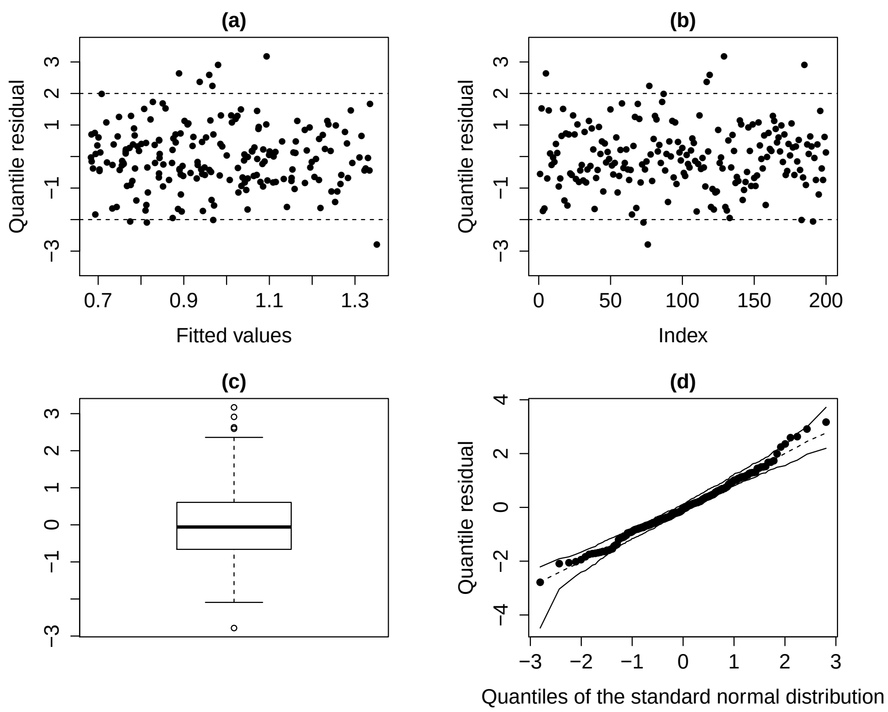

Figure 103 – Residual plots for the SSLBS1 regression model.

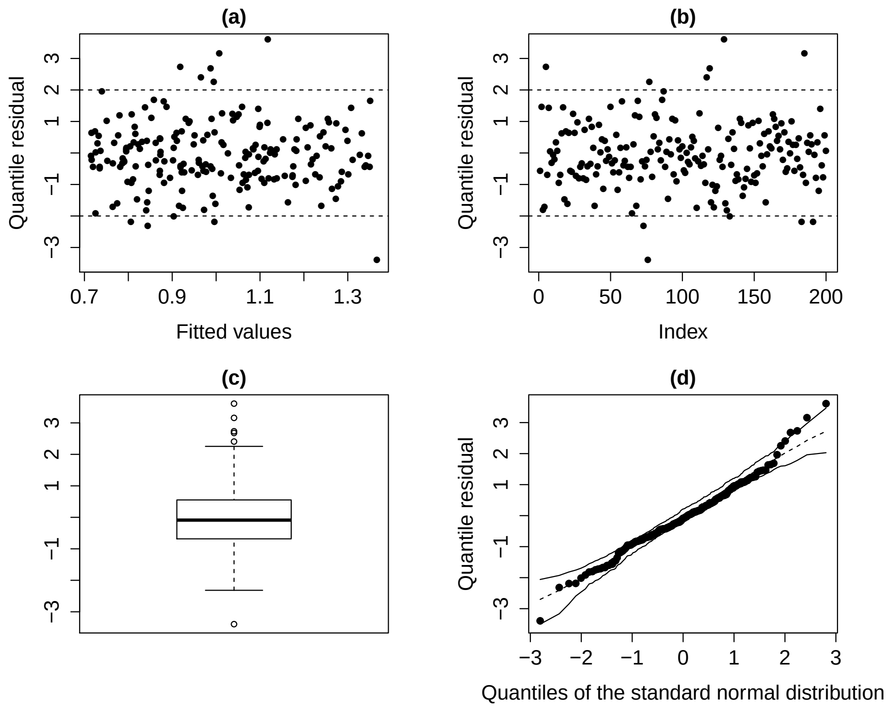

Figure 104 – Residual plots for the SSLBS2 regression model.

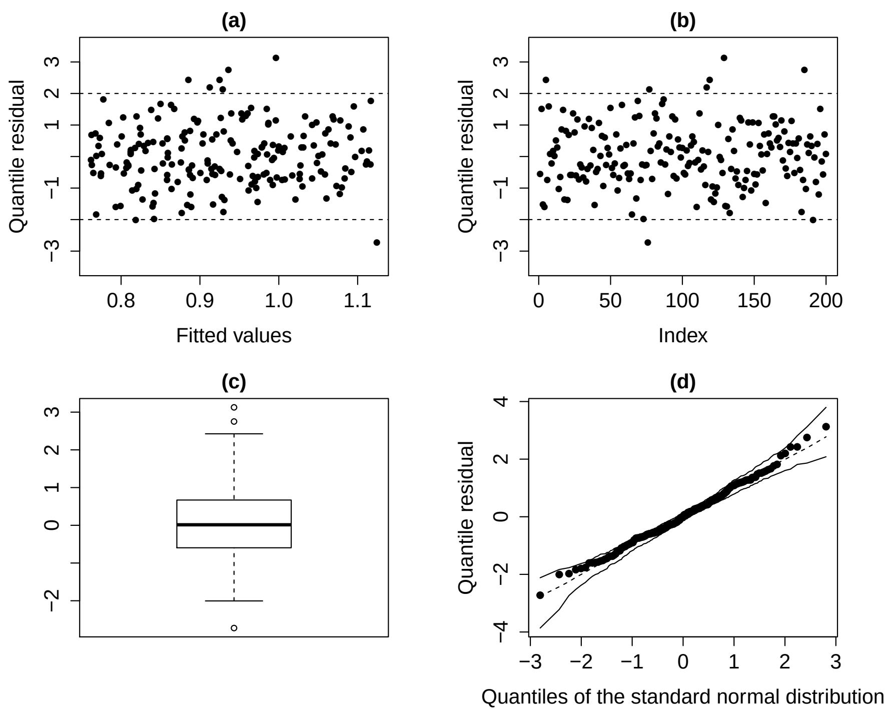

Figure 105 – Residual plots for the SCNBS regression model.

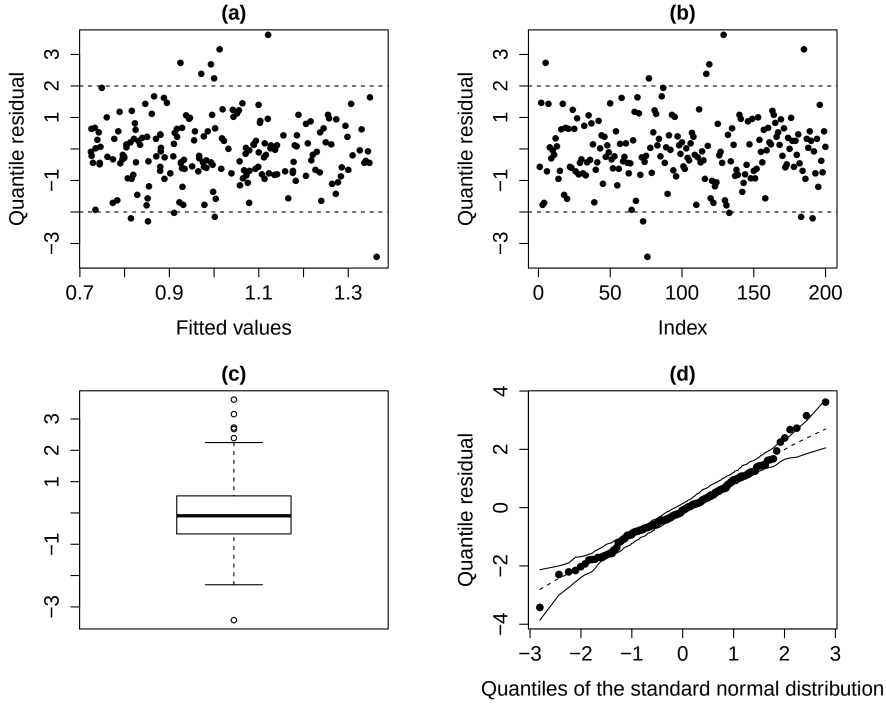

Figure 106 – Residual plots for the SNBS regression model.

Simulated observations from SSLBS regression model

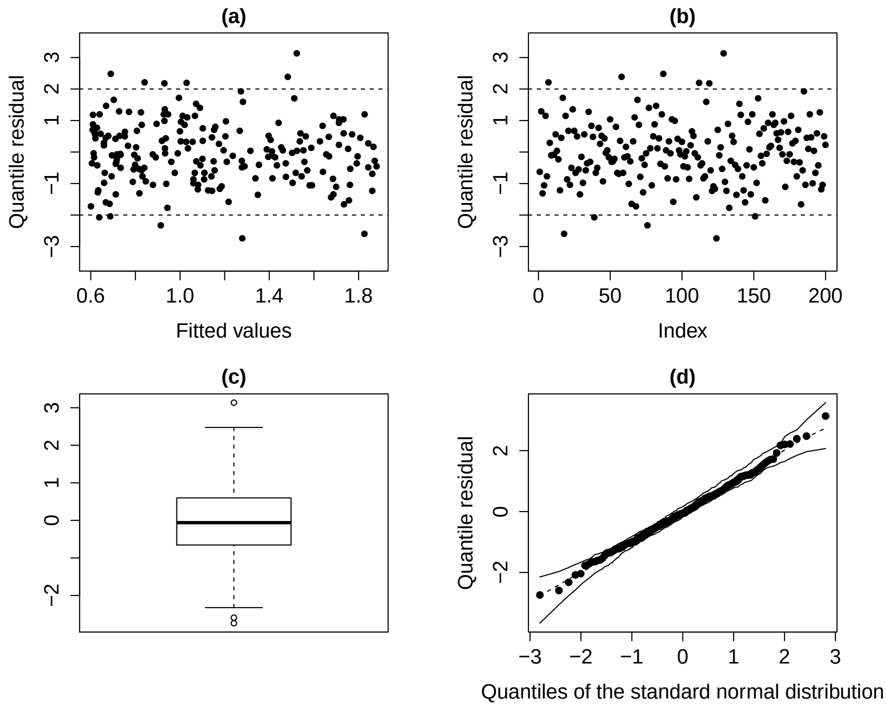

Figure 107 – Residual plots for the SSLBS1 regression model.

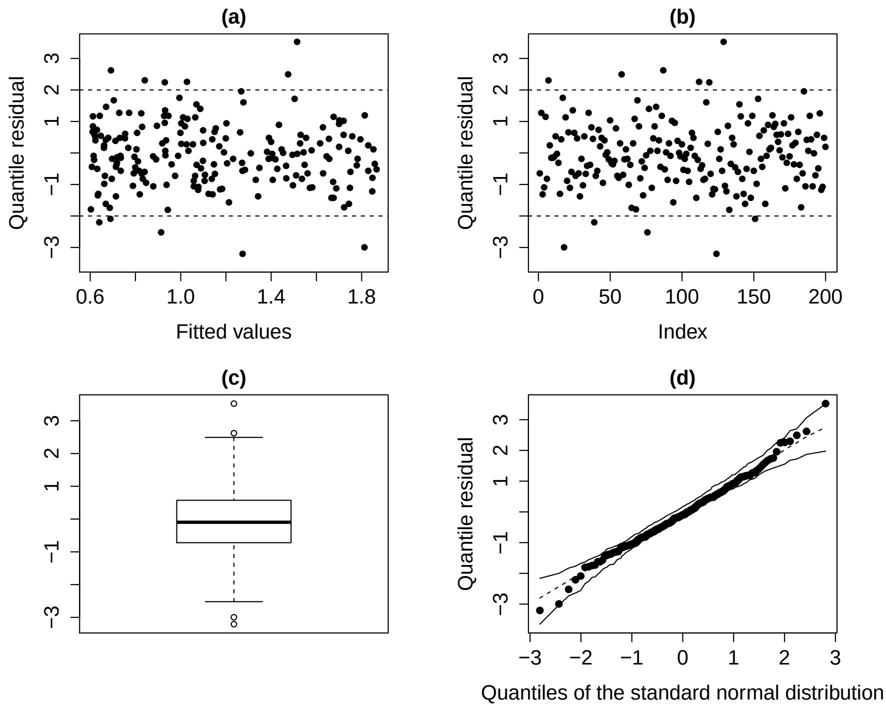

Figure 108 – Residual plots for the SSLBS2 regression model.

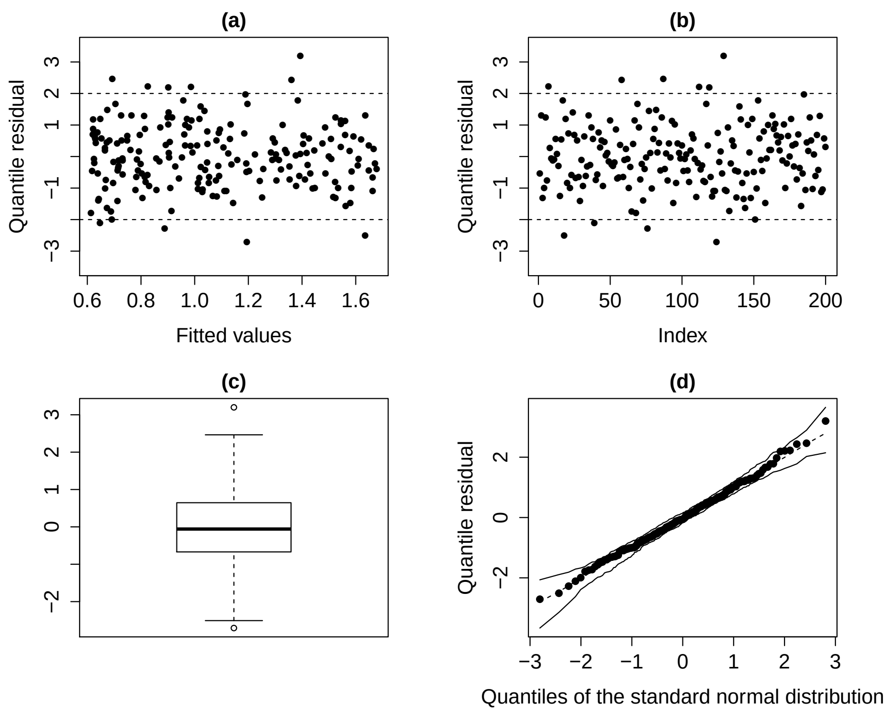

Figure 109 – Residual plots for the SGtBS1 regression model.

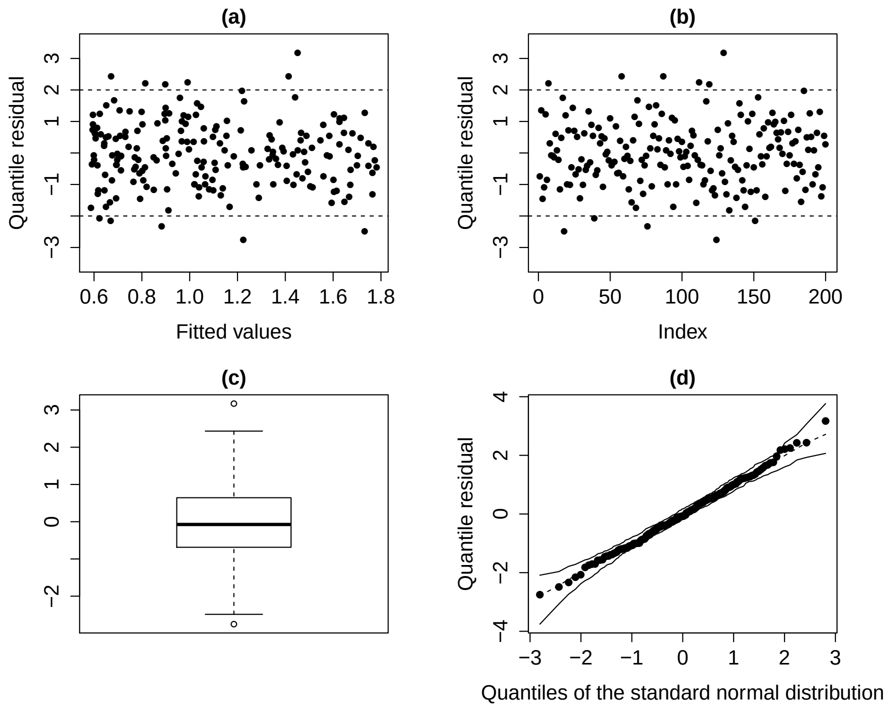

Figure 110 – Residual plots for the SGtBS2 regression model.

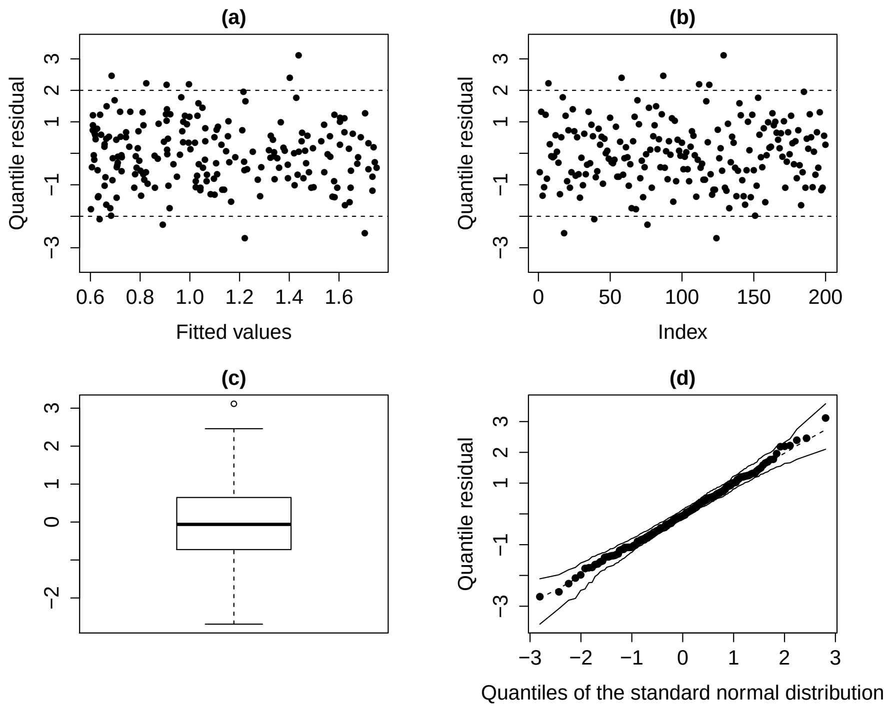

Figure 111 – Residual plots for the StBS regression model.

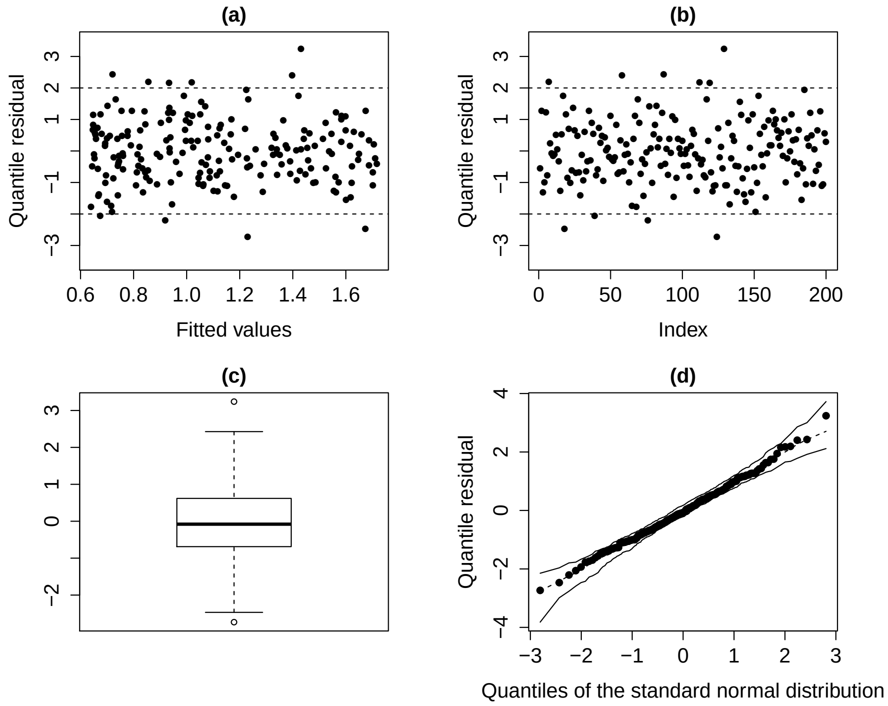

Figure 112 – Residual plots for the SCNBS regression model.

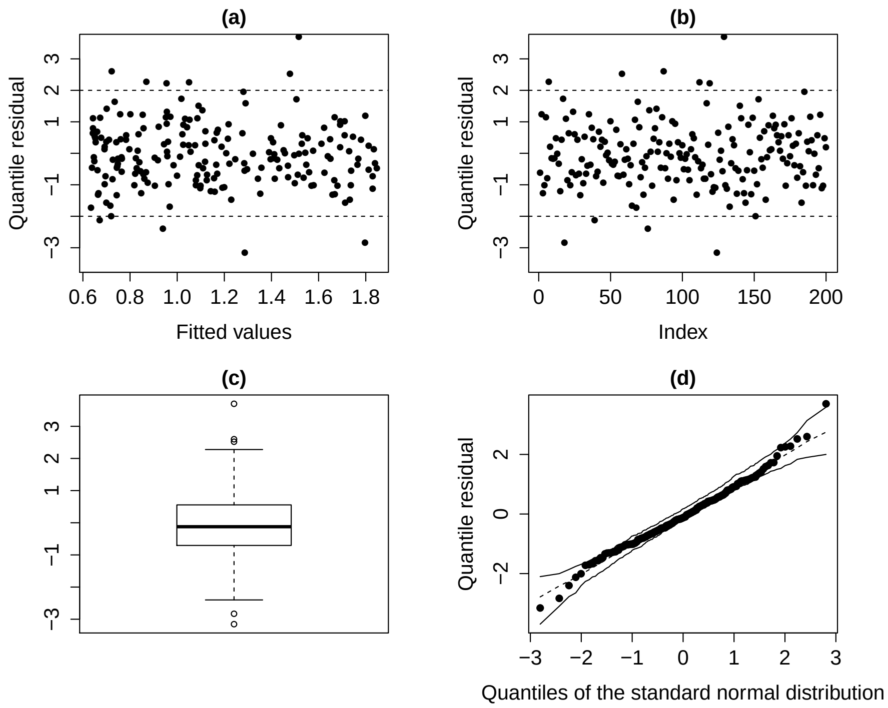

Figure 113 – Residual plots for the SNBS regression model.

Simulated observations from SCNBS regression model

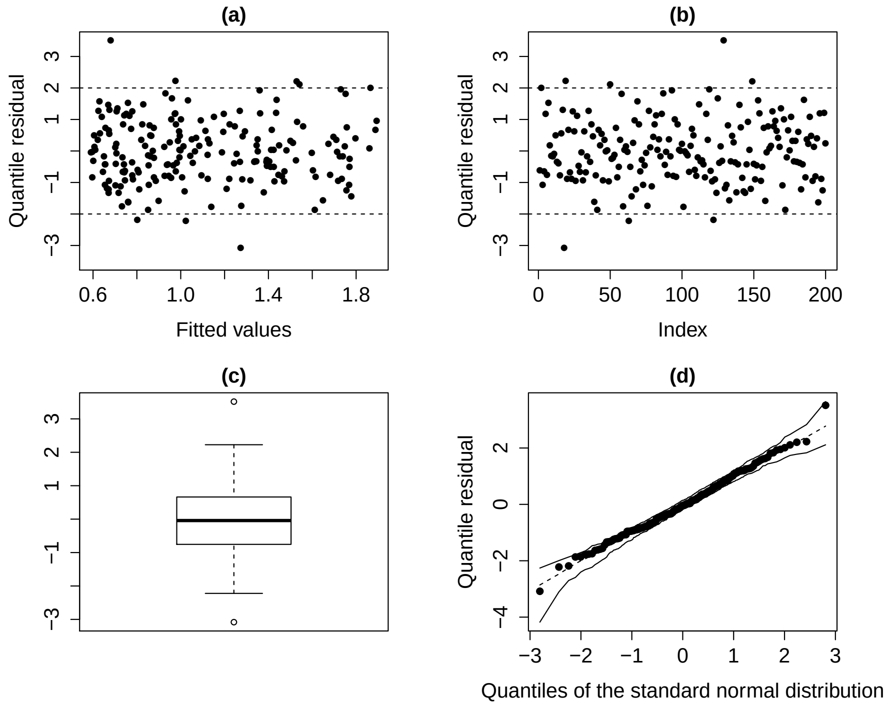

Figure 114 – Residual plots for the SCNBS regression model.

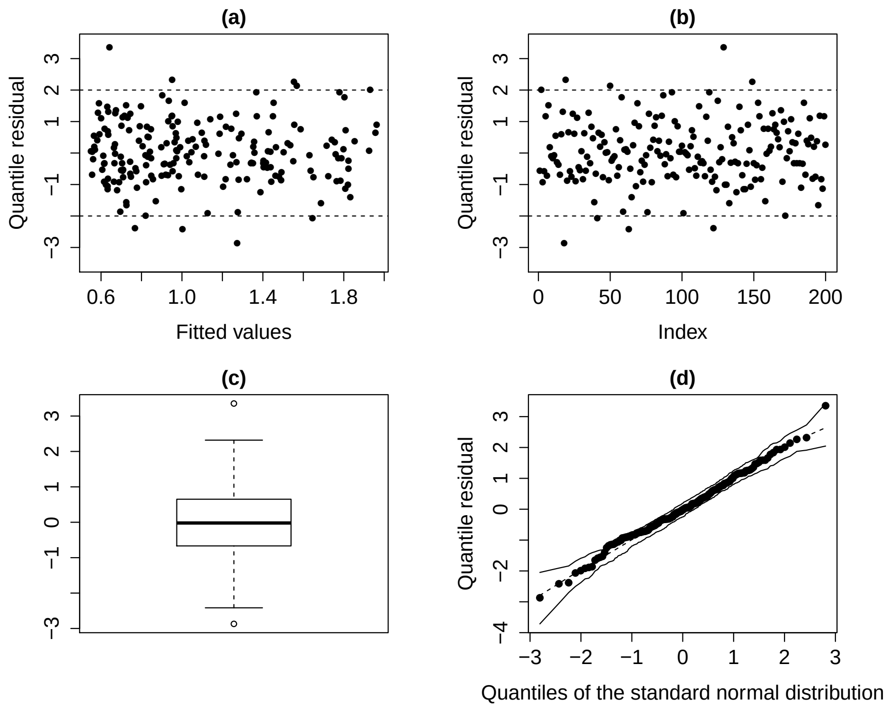

Figure 115 – Residual plots for the SGtBS1 regression model.

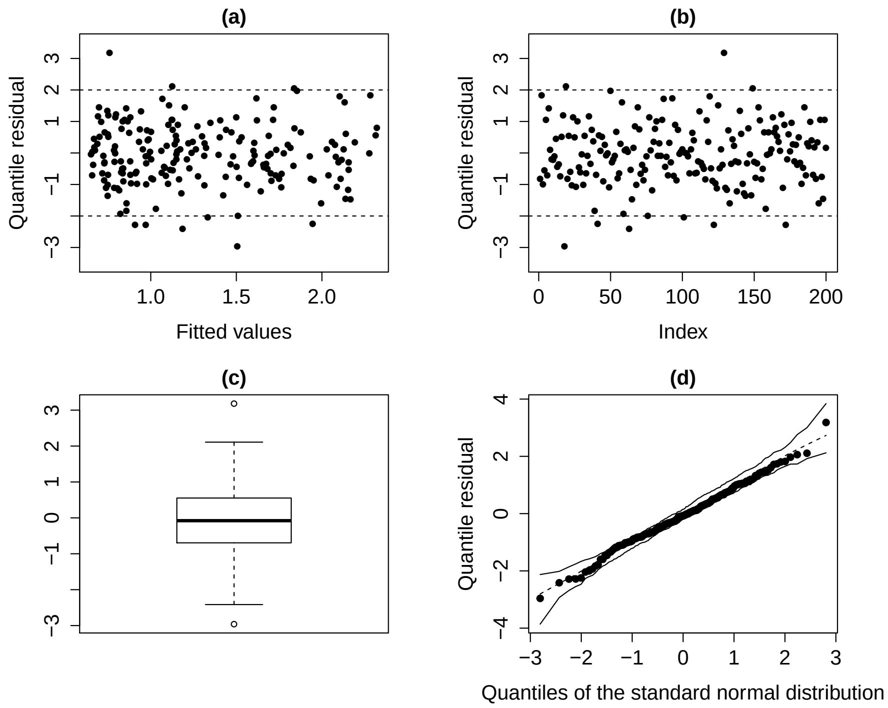

Figure 116 – Residual plots for the SGtBS2 regression model.

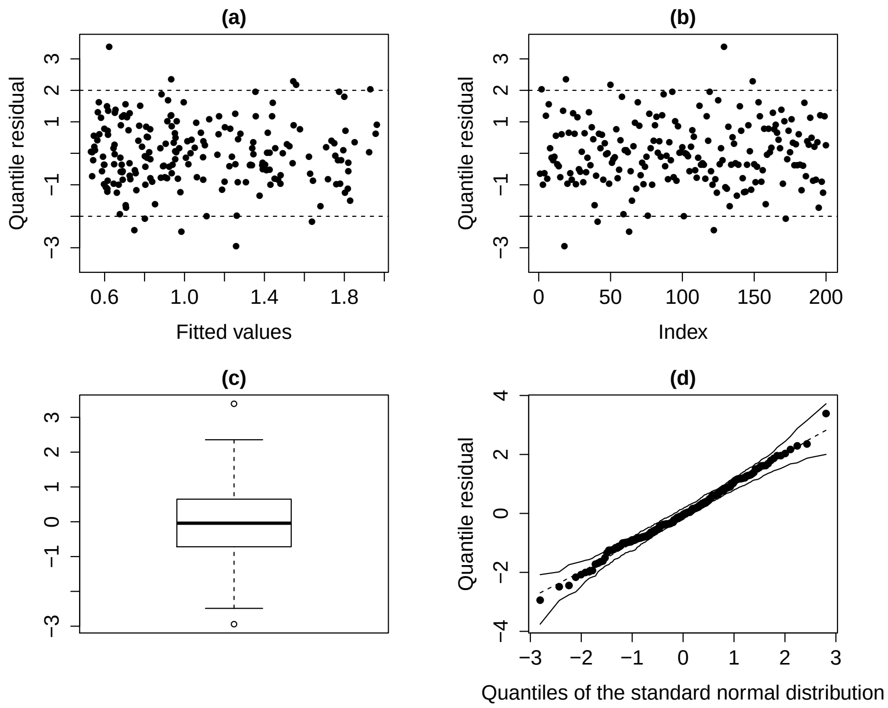

Figure 117 – Residual plots for the StBS regression model.

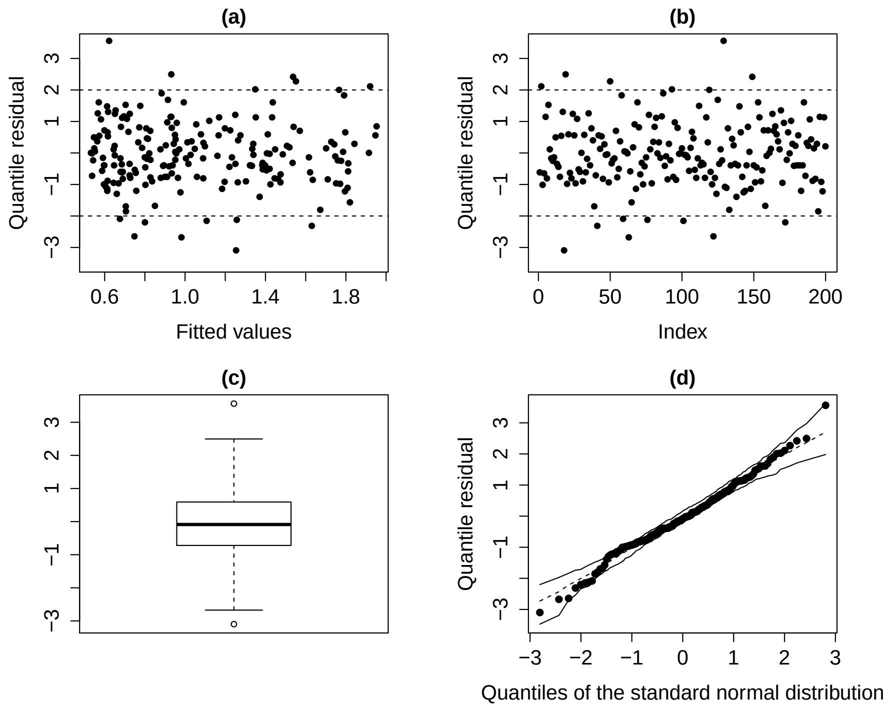

Figure 118 – Residual plots for the SSLBS1 regression model.

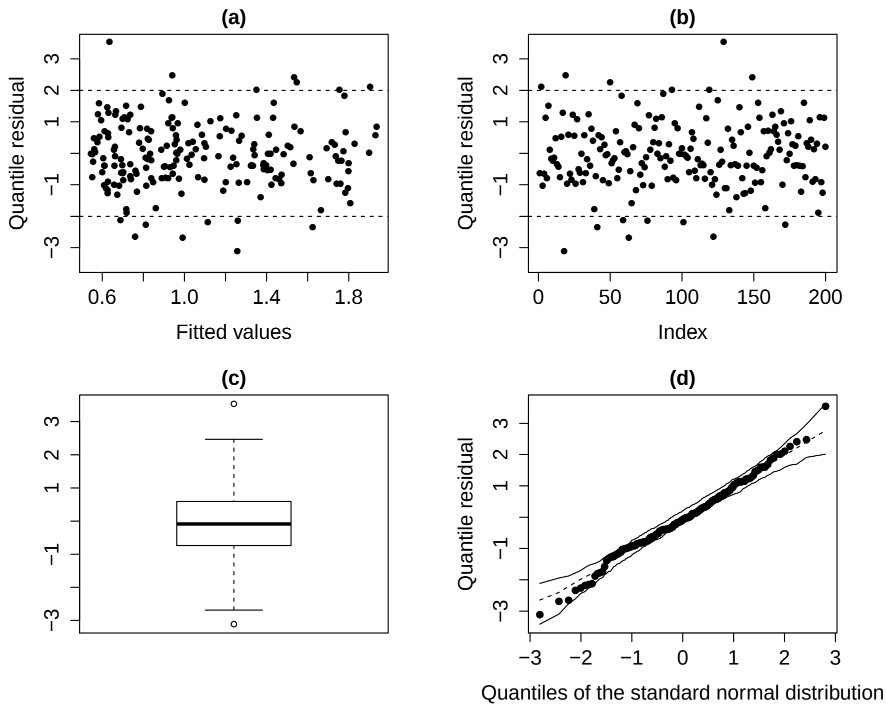

Figure 119 – Residual plots for the SSLBS2 regression model.

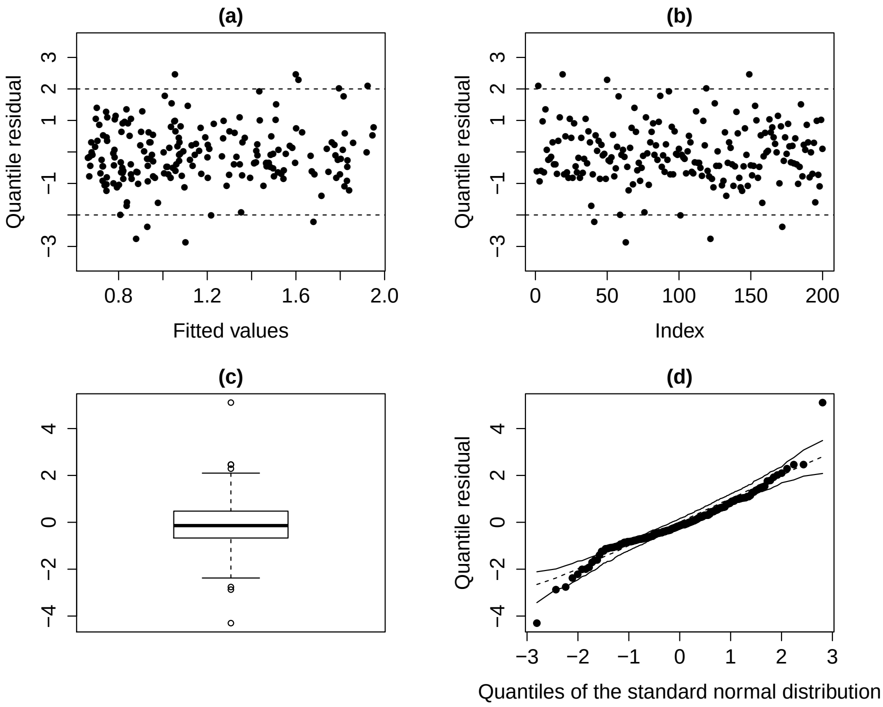

Figure 120 – Residual plots for the SNBS regression model.

C.3 Behavior of the K-L divergence

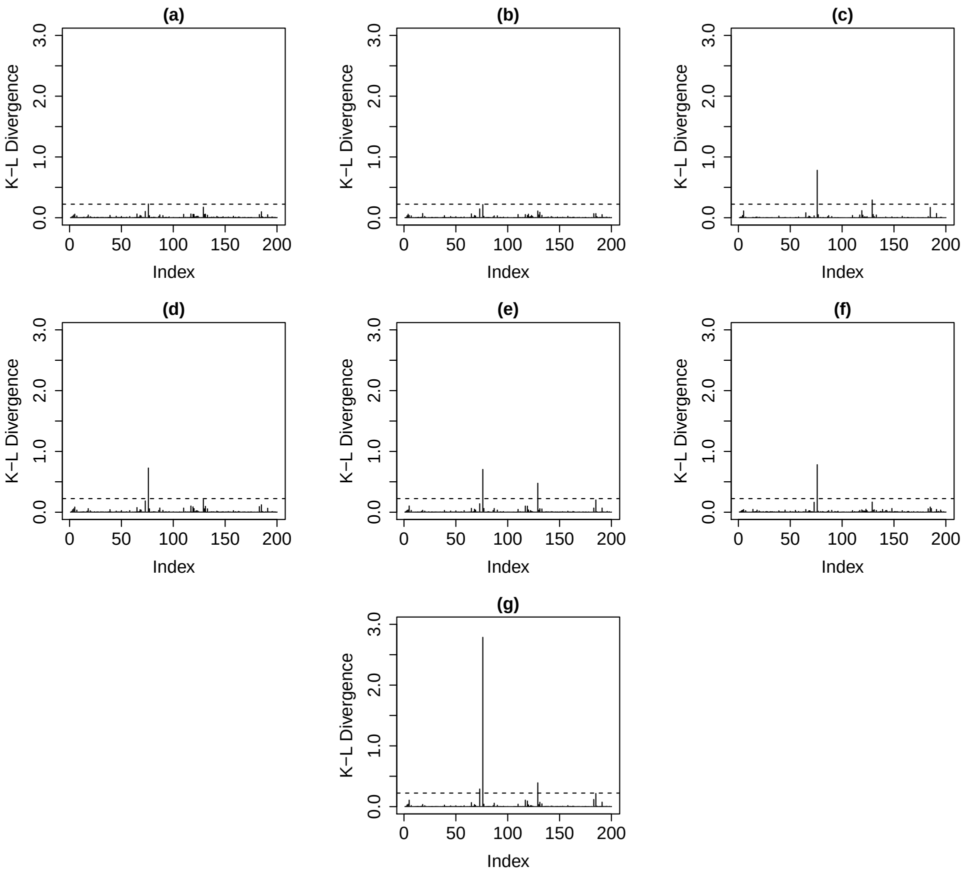

Figure 121 – K-L divergence when we generated the data set from SGtBS1 and fitted the following models: (a) StBS, (b) SGtBS1, (c)SGtBS2, (d) SSLBS1, (e) SSLBS2, (f) SCNBS, and (g) SNBS.

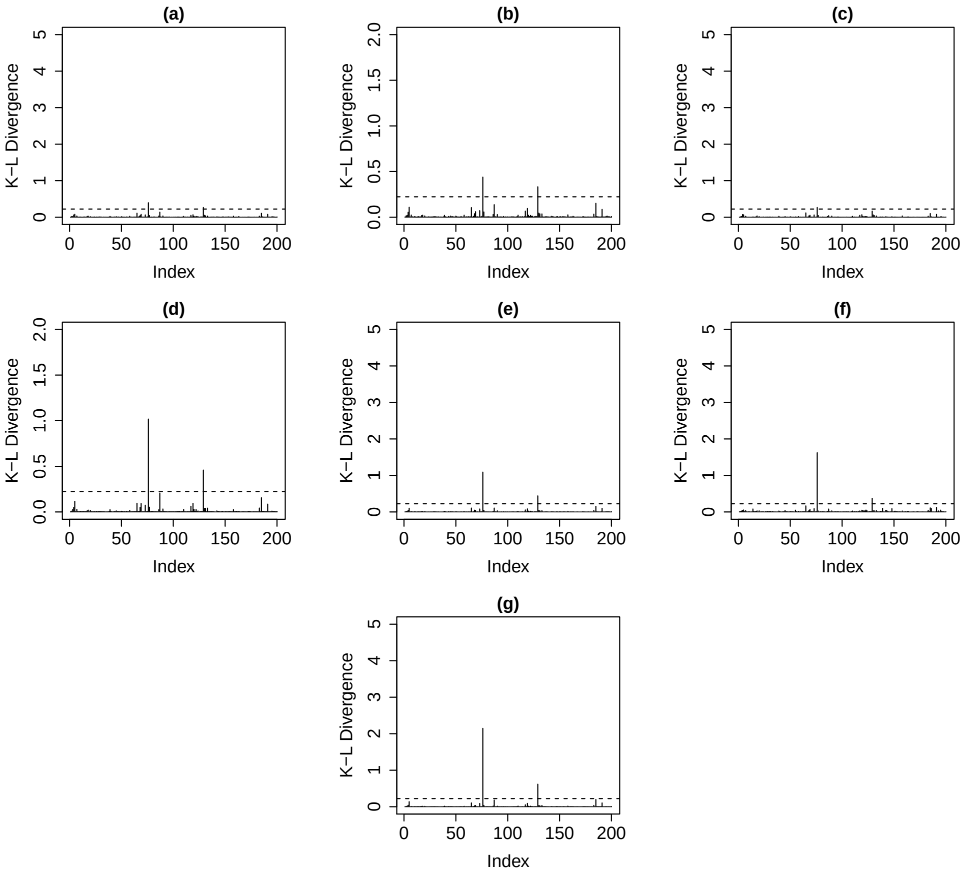

Figure 122 – K-L divergence when we generated the data set from SGtBS2 and fitted the following models: (a) StBS, (b) SGtBS1, (c)SGtBS2, (d) SSLBS1, (e) SSLBS2, (f) SCNBS, and (g) SNBS.

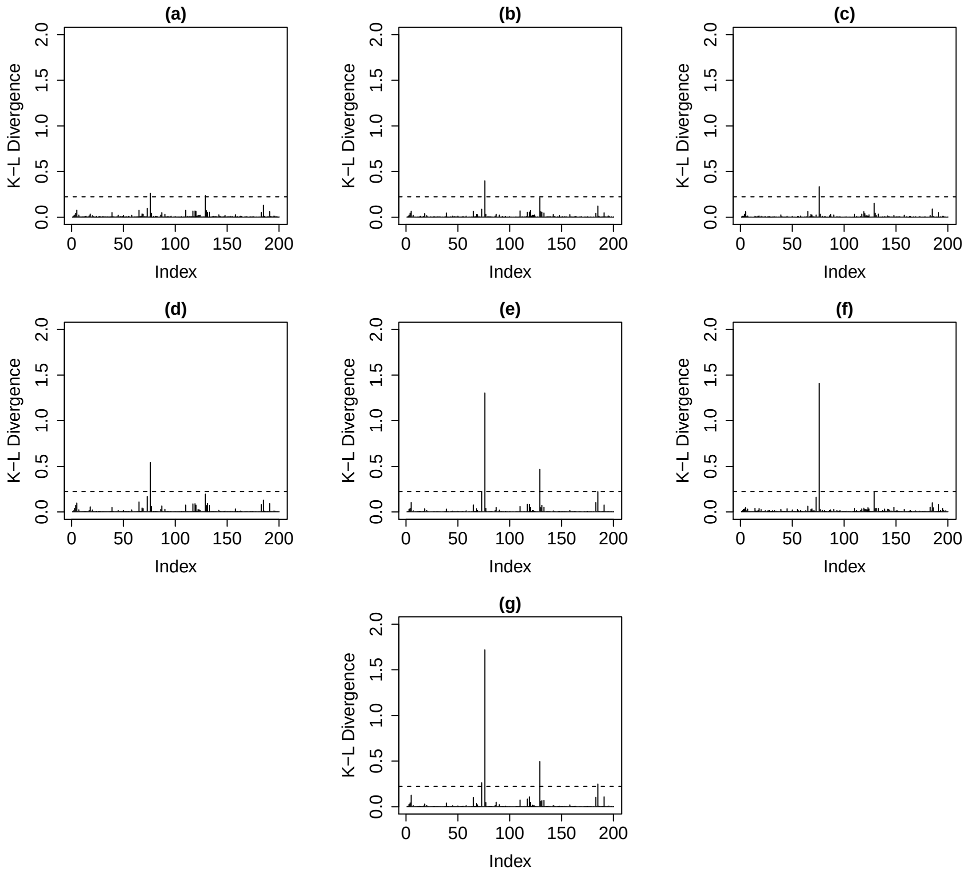

Figure 123 – K-L divergence when we generated the data set from StBS and fitted the following models: (a) StBS, (b) SGtBS1, (c)SGtBS2, (d) SSLBS1, (e) SSLBS2, (f) SCNBS, and (g) SNBS.

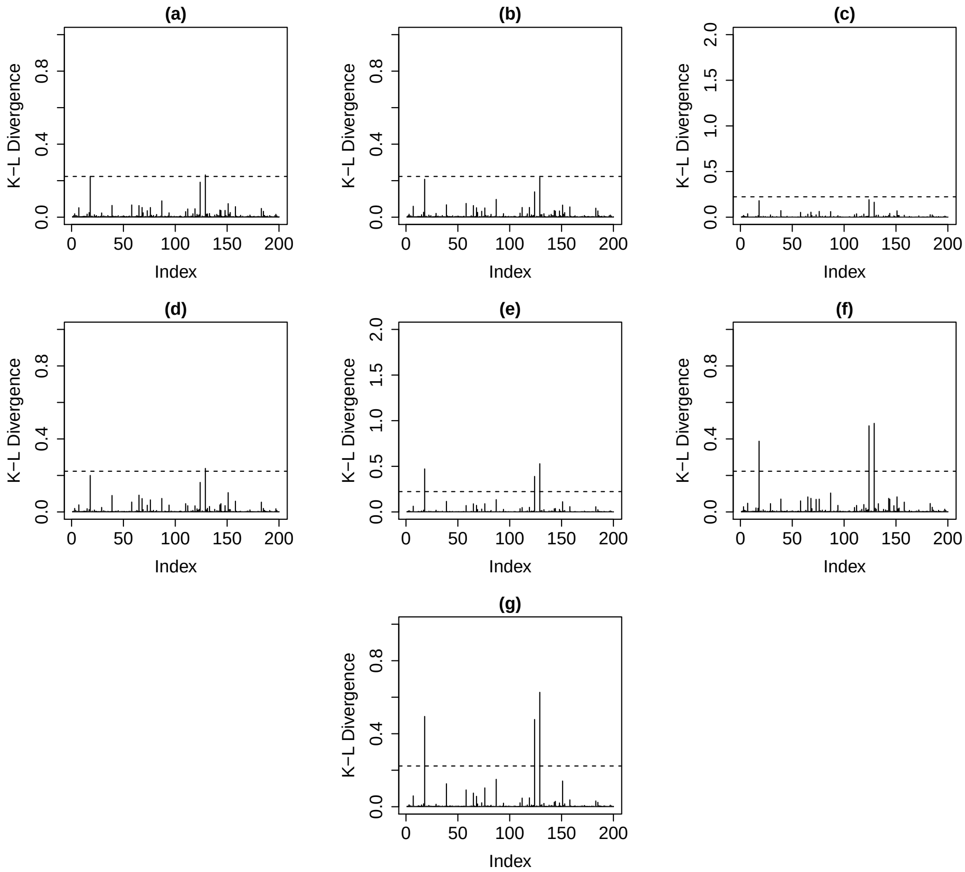

Figure 124 – K-L divergence when we generated the data set from SSLBS and fitted the following models: (a) StBS, (b) SGtBS1, (c)SGtBS2, (d) SSLBS1, (e) SSLBS2, (f) SCNBS, and (g) SNBS.

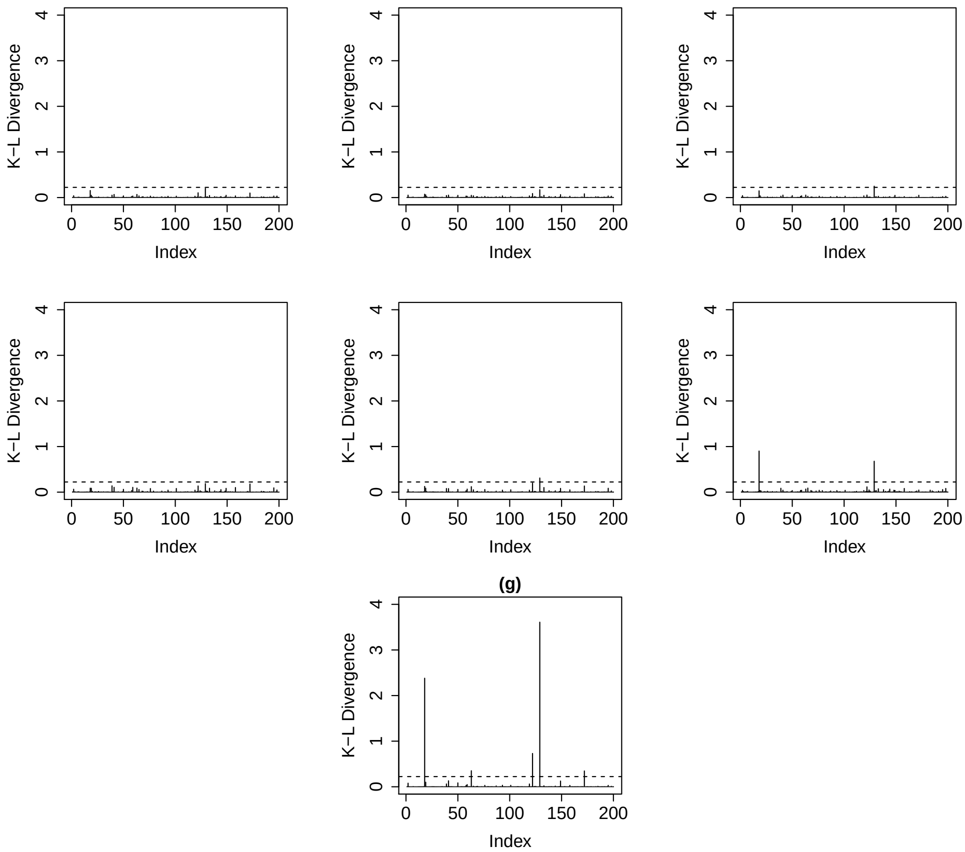

Figure 125 – K-L divergence when we generated the data set from SCNBS and fitted the following models: (a) StBS, (b) SGtBS1, (c)SGtBS2, (d) SSLBS1, (e) SSLBS2, (f) SCNBS, and (g) SNBS.

C.4 Statistics of model comparison

Table 96 – Averaged criteria for the simulation study.

True underlying model: StBS					
Model	n	EAIC	EBIC	DIC	LPML
StBS	100	180.942	196.573	**502.157**	-87.305
	500	855.913	881.200	2,526.629	-424.723
SGtBS1	100	181.243	196.874	504.260	-87.595
	500	**855.590**	880.878	**2,526.339**	**-424.308**
SGtBS2	100	**179.319**	**192.345**	504.399	**-86.768**
	500	859.539	**880.612**	2,544.710	-426.828
SSLBS1	100	181.922	197.553	505.541	-88.067
	500	862.181	887.468	2,545.521	-428.007
SSLBS2	100	182.564	198.195	508.665	-88.694
	500	862.241	887.528	2,545.780	-427.951
SCNBS	100	183.027	201.263	502.167	-88.080
	500	859.265	888.767	2,529.590	-426.628
SNBS	100	183.536	196.562	515.920	-90.516
	500	898.798	919.871	2,661.475	-449.342
True underlying model: SGtBS1					
Model	n	EAIC	EBIC	DIC	LPML
StBS	100	**129.956**	145.587	**349.262**	**-61.804**
	500	**595.819**	**621.106**	**1746.911**	**-294.346**
SGtBS1	100	130.281	145.912	350.759	-61.934
	500	596.287	621.574	1748.119	-294.888
SGtBS2	100	132.187	**145.213**	362.960	-63.824
	500	617.038	638.111	1817.262	-305.981
SSLBS1	100	131.052	146.683	352.950	-62.600
	500	602.776	628.064	1767.659	-298.01
SSLBS2	100	132.150	147.781	356.475	-63.483
	500	603.240	628.528	1772.010	-298.537
SCNBS	100	134.053	152.289	355.069	-63.663
	500	599.294	628.796	1749.920	-296.619
SNBS	100	133.176	146.201	364.786	-65.251
	500	640.655	661.728	1887.049	-320.045
True underlying model: SGtBS2					
Model	n	EAIC	EBIC	DIC	LPML
StBS	100	56.416	72.047	128.566	-25.072

Table 96 (continued).

	500	195.992	221.280	**547.031**	-94.854
SGtBS1	100	56.724	72.355	130.495	-25.225
	500	199.781	225.069	559.543	-96.438
SGtBS2	100	**53.997**	**67.023**	128.067	**-24.168**
	500	**193.921**	**214.994**	547.385	**-94.371**
SSLBS1	100	57.770	73.401	132.996	-26.162
	500	197.244	222.531	551.095	-95.360
SSLBS2	100	58.403	74.034	136.185	-26.770
	500	197.938	223.225	554.767	-96.147
SCNBS	100	56.828	75.064	**123.798**	-24.880
	500	199.660	229.162	571.816	-97.021
SNBS	100	60.173	73.199	145.769	-29.199
	500	232.112	253.185	661.340	-117.552
True underlying model: SSLBS					
Model	n	EAIC	EBIC	DIC	LPML
StBS	100	181.844	197.475	504.477	-87.342
	500	868.913	894.201	2,565.424	-431.193
SGtBS1	100	181.750	197.381	506.411	-87.152
	500	869.084	894.371	2,566.709	-431.069
SGtBS2	100	183.716	196.741	517.150	-88.765
	500	879.336	900.409	2,603.173	-436.807
SSLBS1	100	181.016	196.647	502.570	-87.069
	500	868.691	893.979	2,566.755	-431.082
SSLBS2	100	180.560	196.191	501.328	**-86.871**
	500	869.007	894.295	2,567.536	-431.322
SCNBS	100	182.522	200.758	501.333	-86.908
	500	869.830	899.332	**2,562.886**	**-430.748**
SNBS	100	**178.551**	**191.576**	**501.278**	-87.061
	500	**867.938**	**889.011**	2,568.903	-431.776
True underlying model: SCNBS					
Model	n	EAIC	EBIC	DIC	LPML
StBS	100	166.501	182.132	459.226	**-80.165**
	500	797.722	823.009	2,352.801	-395.254
SGtBS1	100	166.751	182.382	460.351	-80.234
	500	797.745	823.033	2,352.944	-395.257
SGtBS2	100	**166.078**	**179.104**	464.771	-80.298

Table 96 (continued).

	500	808.224	829.297	2,391.161	-401.148
SSLBS1	100	168.626	184.257	466.279	-81.704
	500	807.471	832.759	2,381.659	-400.598
SSLBS2	100	170.253	185.884	472.870	-83.022
	500	807.248	832.536	2,381.082	-400.450
SCNBS	100	167.299	185.535	**455.329**	-80.327
	500	**782.767**	**812.270**	**2,299.796**	**-387.802**
SNBS	100	172.983	186.009	484.199	-86.306
	500	896.059	917.132	2,653.258	-449.966

Table 97 – Percentage of times that the correct model was selected.

Model	n	EAIC	EBIC	DIC	LPML
StBS	100	10%	0%	10%	10%
	500	20%	10%	30%	30%
SGtBS1	100	20%	0%	30%	30%
	500	10%	20%	10%	10%
SGtBS2	100	70%	70%	40%	70%
	500	70%	100%	40%	50%
SSLBS	100	0%	0%	50%	60%
	500	20%	10%	30%	30%
SCNBS	100	20%	0%	50%	20%
	500	90%	90%	100%	90%

C.5 Posterior predictive checking

Table 98 – Posterior predictive checking for the CSSBS regression model.

True underlying model: SGtBS1							
	SGTBS1	SGTBS2	StBS	SSLBS1	SSLBS2	SCNBS	SNBS
p-value	.199	.275	.169	.156	.099	.142	.004
True underlying model: SGtBS2							
	SGtBS2	SGtBS1	StBS	SSLBS1	SSLBS2	SCNBS	SNBS
p-value	.366	.314	.288	.289	.143	.560	.094
True underlying model: StBS							
	StBS	SGtBS1	SGtBS2	SSLBS1	SSLBS2	SCNBS	SNBS
p-value	.214	.150	.202	.213	.044	.083	.012
True underlying model: SSLBS							
	SSLBS1	SSLBS2	SGtBS1	SGtBS2	StBS	SCNBS	SNBS
p-value	.422	.188	.263	.303	.330	.380	.037
True underlying model: SCNBS							
	SCNBS	SGtBS1	SGtBS2	StBS	SSLBS1	SSLBS2	SNBS
p-value	.158	.228	.133	.228	.015	.004	$< .001$

C.6 Results of the statistical analysis of the AIS data

Table 99 – Bayesian estimates for the CSSBS regression model.

Parameter	SGtBS1		
	PE	PSD	$CI_{95\%}$
β_{01}	4.083	.007	[4.071; 4.096]
β_{02}	4.223	.003	[4.217; 4.228]
β_{11}	.021	.007	[.005; .034]
β_{12}	.019	.004	[.010; .025]
β_{21}	.133	.010	[.111; .153]
β_{22}	.141	.005	[.132; .152]
ψ_{01}	-4.874	.509	[-5.743; -3.963]
ψ_{02}	-6.337	.552	[-7.229; -5.425]
ψ_{11}	-.428	.353	[-1.171; .258]
ψ_{12}	-.821	.264	[-1.374; -.283]
ψ_{21}	.376	.347	[-.337; 1.026]
ψ_{22}	1.020	.275	[.453; 1.518]
γ	-.810	.152	[-.991; -.454]
ν_1	8.032	3.142	[4.230; 14.881]

Table 99 (continued).

Parameter	StBS		
	PE	PSD	$CI_{95\%}$
β_{01}	4.285	.082	[4.181; 4.415]
β_{02}	4.033	.044	[3.969; 4.103]
β_{11}	-.227	.064	[-.325; -.139]
β_{12}	.056	.008	[.046; .070]
β_{21}	.480	.074	[.357; .592]
β_{22}	.199	.021	[.174; .231]
ψ_{01}	-3.558	.505	[-4.291; -2.790]
ψ_{02}	-3.982	.490	[-4.941;-3.314]
ψ_{11}	-.766	.343	[-1.258; -.141]
ψ_{12}	-.535	.093	[-.713; -0.359]
ψ_{21}	1.019	.406	[.430; 1.689]
ψ_{22}	-.590	.133	[-.774; -0.275]
γ	-0.986	.010	[-.995; -.957]
ν	71.073	47.326	[20.548; 193.225]
Parameter	SSLBS1		
	PE	PSD	$CI_{95\%}$
β_{01}	4.084	.007	[4.070; 4.099]
β_{02}	4.220	.003	[4.214; 4.226]
β_{11}	.029	.009	[.009; .046]
β_{12}	.021	.004	[.013; .028]
β_{21}	.127	.011	[.108; .147]
β_{22}	.139	.005	[.130; .149]
ψ_{01}	-6.925	.303	[-7.508; -6.296]
ψ_{02}	-8.293	.259	[-8.797; -7.829]
ψ_{11}	-.568	.303	[-1.168; .040]
ψ_{12}	-.810	.247	[-1.235; -.300]
ψ_{21}	.590	.321	[-.037; 1.209]
ψ_{22}	.979	.273	[.414; 1.449]
γ	-.795	.122	[-.982; -.524]
ν	4.852	3.899	[2.060; 16.192]
Parameter	SSLBS2		
	PE	PSD	$CI_{95\%}$
β_{01}	4.086	.007	[4.073; 4.100]
β_{02}	4.219	.003	[4.214; 4.226]
β_{11}	.025	.009	[.008; .043]

Table 99 (continued).

β_{12}	.021	.004	[.013; .028]
β_{21}	.134	.010	[.113; .153]
β_{22}	.138	.005	[.130; .148]
ψ_{01}	-6.684	.247	[-7.237; -6.236]
ψ_{02}	-8.017	.234	[-8.649; -7.638]
ψ_{11}	-.477	.292	[-1.032; .118]
ψ_{12}	-.761	.303	[-1.359; -.217]
ψ_{21}	.530	.306	[-.105; 1.082]
ψ_{22}	.907	.325	[.363; 1.595]
γ	-.763	.103	[-.931; -.517]
ν	26.805	25.754	[2.406; 94.255]

Parameter	SCNBS		
	PE	PSD	$CI_{95\%}$
β_{01}	5.311	.158	[4.864; 5.559]
β_{02}	4.171	.018	[4.142; 4.218]
β_{11}	.607	.039	[.481; .672]
β_{12}	.014	.007	[.003; .027]
β_{21}	.092	.083	[-.112; .180]
β_{22}	.176	.013	[.149; .192]
ψ_{01}	-1.086	.484	[-3.328; -.725]
ψ_{02}	-6.478	.534	[-8.653; -5.646]
ψ_{11}	.927	.127	[.730; 1.284]
ψ_{12}	-.160	.278	[-1.303; .118]
ψ_{21}	.018	.170	[-.511; .170]
ψ_{22}	.482	.181	[.200; 1.091]
γ	.974	.08	[.755; .995]
ν_1	.345	.234	[.037; .871]
ν_2	.806	.269	[.004; .997]

Parameter	SNBS		
	PE	PSD	$CI_{95\%}$
β_{01}	3.727	.022	[3.691; 3.755]
β_{02}	3.996	.011	[3.977; 4.011]
β_{11}	-.314	.011	[-.341; -.295]
β_{12}	-.081	.008	[-.093; -.065]
β_{21}	.325	.011	[.311; .349]
β_{22}	.352	.018	[.329; .386]
ψ_{01}	-2.514	.136	[-2.714; -2.270]

Table 99 (continued).

ψ_{02}	-3.356	.185	[-3.612; -3.092]
ψ_{11}	1.602	.059	[1.491; 1.712]
ψ_{12}	.301	.137	[.016; .547]
ψ_{21}	-.874	.086	[-1.010; -.713]
ψ_{22}	-.761	.047	[-.830; -.679]
γ	-.993	.002	[-.995; -.986]

APPENDIX D – Results of Chapter 4

In this section, we present the results of the simulation studies for the ZA-SSBS regression models. Furthermore, we present the results of the statistical analysis of the bilirubin concentration data set.

D.1 Results of the parameter recovery study

ZA-SGtBS1 regression model

Table 100 – Results of simulation study for ZA-SGtBS1 regression model ($\nu_1 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-2.204	-2.305	-2.176	-2.439	-2.411	-2.44
	SD	.672	.666	.736	.164	.118	.161
	LCI	2.636	2.553	2.555	1.053	1.079	.989
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.296	.195	.324	.061	.089	.060
	RMSE	.734	.694	.804	.175	.148	.172
	AVRB	.118	.078	.130	.024	.035	.024
ζ_1	Mean	.670	.799	.599	.759	.721	.790
	SD	1.011	.979	1.105	.417	.366	.451
	LCI	4.102	4.160	4.045	1.700	1.800	1.499
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	-.130	-.001	-.201	-.041	-.079	-.010
	RMSE	1.019	.979	1.123	.419	.374	.451
	AVRB	.162	.002	.251	.051	.098	.013
β_0	Mean	-.551	-.481	-.474	-.502	-.500	-.501
	SD	.085	.101	.135	.016	.043	.046
	LCI	.337	.561	.443	.156	.18	.211
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.051	.019	.026	-.002	$< .001$	-.001
	RMSE	.099	.103	.137	.016	.043	.046
	AVRB	.102	.039	.052	.003	$< .001$	.003
	Mean	1.098	1.017	.873	1.001	.981	.988
	SD	.172	.129	.272	.031	.118	.098
	LCI	.652	.947	.821	.292	.327	.383

Table 100 (continued).

β_1	CP	.800	1.000	.800	1.000	.800	1.000
	Bias	.098	.017	-.127	.001	-.019	-.012
	RMSE	.198	.130	.301	.031	.119	.099
	AVRB	.098	.017	.127	.001	.019	.012
	Mean	-.194	-.136	-.330	-.804	-.904	-.782
	SD	.501	.379	.463	.314	.176	.342
	LCI	2.506	2.599	2.258	1.109	1.079	1.155
ψ_0	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.806	.864	.670	.196	.096	.218
	RMSE	.949	.944	.814	.370	.201	.405
	AVRB	.806	.864	.670	.196	.096	.218
	Mean	.434	.126	.434	.504	.402	.528
	SD	.692	.482	.902	.270	.348	.288
	LCI	2.574	2.648	2.314	1.069	1.121	1.108
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.066	-.374	-.066	.004	-.098	.028
	RMSE	.695	.610	.904	.270	.362	.289
	AVRB	.132	.748	.131	.009	.196	.056
	Mean	-.624	.371	.557	-.718	-.056	.782
	SD	.115	.344	.133	.106	.235	.079
	LCI	.816	.997	1.010	.478	.509	.424
γ	CP	1.000	.800	1.000	1.000	.200	1.000
	Bias	.176	.371	-.243	.082	-.056	-.018
	RMSE	.210	.506	.277	.134	.241	.081
	AVRB	.220	-	.304	.102	-	.023
	Mean	11.639	10.333	10.404	6.132	5.525	6.388
	SD	3.185	3.050	2.365	1.320	.925	1.918
	LCI	20.912	22.054	18.672	4.759	3.852	5.213
ν_1	CP	.600	.800	.800	1.000	1.000	.800
	Bias	6.639	5.333	5.404	1.132	.525	1.388
	RMSE	7.363	6.144	5.899	1.739	1.064	2.368
	AVRB	1.328	1.067	1.081	.226	.105	.278

Table 101 – Results of simulation study for ZA-SGtBS1 regression model ($\nu_1 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.151	-2.292	-2.155	-2.433	-2.416	-2.439
	SD	.648	.632	.709	.158	.119	.155
	LCI	2.636	2.553	2.555	1.053	1.079	.989
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.349	.208	.345	.067	.084	.061
	RMSE	.736	.665	.789	.172	.146	.167
	AVRB	.140	.083	.138	.027	.034	.024
ζ_1	Median	.627	.803	.610	.776	.727	.794
	SD	.999	.947	1.033	.381	.364	.432
	LCI	4.102	4.160	4.045	1.700	1.800	1.499
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	-.173	.003	-.190	-.024	-.073	-.006
	RMSE	1.014	.947	1.050	.382	.371	.432
	AVRB	.216	.004	.237	.030	.091	.008
β_0	Median	-.555	-.493	-.477	-.502	-.501	-.503
	SD	.085	.087	.137	.017	.042	.044
	LCI	.337	.561	.443	.156	.180	.211
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.055	.007	.023	-.002	-.001	-.003
	RMSE	.101	.087	.139	.017	.042	.045
	AVRB	.109	.015	.045	.003	.001	.006
β_1	Median	1.097	1.030	.873	1.003	.982	.989
	SD	.168	.107	.275	.035	.116	.097
	LCI	.652	.947	.821	.292	.327	.383
	CP	.800	1.000	.800	1.000	.800	1.000
	Bias	.097	.030	-.127	.003	-.018	-.011
	RMSE	.194	.111	.303	.036	.117	.098
	AVRB	.097	.030	.127	.003	.018	.011
ψ_0	Median	-.182	-.180	-.339	-.816	-.902	-.793
	SD	.515	.357	.491	.336	.185	.341
	LCI	2.506	2.599	2.258	1.109	1.079	1.155
	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.818	.820	.661	.184	.098	.207
	RMSE	.967	.895	.823	.383	.210	.399
	AVRB	.818	.82	.661	.184	.098	.207

Table 101 (continued).

	Median	.451	.138	.450	.511	.401	.524
	SD	.677	.482	.905	.271	.349	.291
	LCI	2.574	2.648	2.314	1.069	1.121	1.108
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.049	-.362	-.050	.011	-.099	.024
	RMSE	.678	.602	.906	.271	.363	.292
	AVRB	.097	.723	.100	.022	.197	.048
	Median	-.652	.381	.598	-.728	-.043	.794
	SD	.137	.351	.123	.111	.225	.080
	LCI	.816	.997	1.010	.478	.509	.424
γ	CP	1.000	.800	1.000	1.000	.200	1.000
	Bias	.148	.381	-.202	.072	-.043	-.006
	RMSE	.202	.519	.237	.132	.229	.080
	AVRB	.185	——	.253	.090	——	.008
	Median	10.197	8.493	9.197	5.964	5.331	6.165
	SD	2.907	2.479	2.404	1.337	.824	1.750
	LCI	20.912	22.054	18.672	4.759	3.852	5.213
ν_1	CP	.600	.800	.800	1.000	1.000	.800
	Bias	5.197	3.493	4.197	.964	.331	1.165
	RMSE	5.954	4.283	4.837	1.648	.888	2.103
	AVRB	1.039	.699	.839	.193	.066	.233

Table 102 – Results of simulation study for ZA-SGtBS1 regression model ($\nu_1 = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mode	-2.016	-2.307	-2.032	-2.432	-2.422	-2.426
	SD	.748	.586	.508	.160	.124	.149
	LCI	2.636	2.553	2.555	1.053	1.079	.989
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.484	.193	.468	.068	.078	.074
	RMSE	.891	.617	.691	.174	.147	.166
	AVRB	.194	.077	.187	.027	.031	.029
ζ_1	Mode	.412	.754	.656	.796	.731	.774
	SD	1.131	.958	.979	.356	.356	.392
	LCI	4.102	4.160	4.045	1.700	1.800	1.499
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	-.388	-.046	-.144	-.004	-.069	-.026
	RMSE	1.196	.959	.989	.356	.363	.393
	AVRB	.484	.058	.179	.005	.086	.033
β_0	Mode	-.553	-.496	-.475	-.501	-.501	-.503
	SD	.086	.084	.132	.017	.043	.047
	LCI	.337	.561	.443	.156	.180	.211
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.053	.004	.025	-.001	-.001	-.003
	RMSE	.101	.084	.134	.017	.043	.047
	AVRB	.106	.007	.050	.003	.001	.006
β_1	Mode	1.098	1.036	.865	1.002	.980	.991
	SD	.173	.097	.270	.032	.117	.097
	LCI	.652	.947	.821	.292	.327	.383
	CP	.800	1.000	.800	1.000	.800	1.000
	Bias	.098	.036	-.135	.002	-.020	-.009
	RMSE	.199	.103	.302	.032	.119	.097
	AVRB	.098	.036	.135	.002	.020	.009
ψ_0	Mode	-.274	-.257	-.314	-.835	-.904	-.793
	SD	.625	.336	.622	.356	.192	.358
	LCI	2.506	2.599	2.258	1.109	1.079	1.155
	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.726	.743	.686	.165	.096	.207
	RMSE	.958	.816	.926	.393	.215	.414
	AVRB	.726	.743	.686	.165	.096	.207

Table 102 (continued).

	Mode	.492	.202	.495	.514	.396	.522
	SD	.661	.550	.937	.278	.356	.291
	LCI	2.574	2.648	2.314	1.069	1.121	1.108
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.008	-.298	-.005	.014	-.104	.022
	RMSE	.661	.626	.937	.279	.371	.292
	AVRB	.016	.597	.009	.028	.208	.043
	Mode	-.667	.386	.633	-.726	-.043	.789
	SD	.150	.354	.104	.108	.226	.075
	LCI	.816	.997	1.010	.478	.509	.424
γ	CP	1.000	.800	1.000	1.000	.200	1.000
	Bias	.133	.386	-.167	.074	-.043	-.011
	RMSE	.200	.524	.197	.131	.230	.076
	AVRB	.166	——	.209	.093	——	.013
	Mode	9.534	5.916	6.867	5.606	4.952	5.609
	SD	3.398	2.844	2.470	1.376	.674	1.603
	LCI	20.912	22.054	18.672	4.759	3.852	5.213
ν_1	CP	.600	.800	.800	1.000	1.000	.800
	Bias	4.534	.916	1.867	.606	-.048	.609
	RMSE	5.666	2.988	3.096	1.504	.676	1.714
	AVRB	.907	.183	.373	.121	.010	.122

Table 103 – Results of simulation study for ZA-SGtBS1 regression model ($\nu_1 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-2.446	-2.825	-2.959	-2.649	-2.358	-2.323
	SD	.970	.863	.967	.563	.205	.794
	LCI	2.638	3.129	2.856	.684	1.183	1.113
	CP	.800	1.000	1.000	.400	1.000	.400
	Bias	.054	-.325	-.459	-.149	.142	.177
	RMSE	.971	.922	1.070	.583	.249	.813
	AVRB	.021	.130	.183	.060	.057	.071
ζ_1	Mean	.457	1.155	1.346	1.064	.502	.417
	SD	1.285	.954	1.088	.778	.364	1.032
	LCI	4.224	4.648	4.33	.914	1.944	1.773
	CP	.800	1.000	1.000	.400	1.000	.600
	Bias	-.343	.355	.546	.264	-.298	-.383
	RMSE	1.330	1.018	1.217	.821	.470	1.101
	AVRB	.429	.444	.682	.330	.373	.479
β_0	Mean	-.482	-.500	-.530	-.520	-.496	-.454
	SD	.032	.015	.036	.036	.008	.063
	LCI	.138	.150	.144	.069	.062	.099
	CP	.800	1.000	1.000	.600	1.000	.200
	Bias	.018	$< .001$	-.030	-.020	.004	.046
	RMSE	.037	.015	.047	.041	.009	.078
	AVRB	.036	$< .001$	.059	.040	.008	.091
β_1	Mean	.967	1.013	1.065	1.029	.997	.931
	SD	.074	.032	.080	.067	.020	.106
	LCI	.259	.290	.275	.113	.114	.152
	CP	.800	1.000	.800	.600	1.000	.200
	Bias	-.033	.013	.065	.029	-.003	-.069
	RMSE	.081	.035	.103	.073	.020	.127
	AVRB	.033	.013	.065	.029	.003	.069
ψ_0	Mean	-1.640	-1.985	-1.989	-1.312	-1.369	-1.024
	SD	.386	.374	.322	.398	.347	.561
	LCI	2.414	2.666	2.39	.617	1.053	.997
	CP	1.000	.600	.600	.200	.800	.600
	Bias	-.640	-.985	-.989	-.312	-.369	-.024
	RMSE	.747	1.053	1.040	.506	.507	.562
	AVRB	.640	.985	.989	.312	.369	.024

Table 103 (continued).

	Mean	.325	.910	.638	.032	.430	.004
	SD	.848	.567	.842	1.308	.411	1.029
	LCI	2.506	2.375	2.274	.667	.891	1.004
ψ_1	CP	1.000	1.000	.800	.200	.800	$< .001$
	Bias	-.175	.410	.138	-.468	-.070	-.496
	RMSE	.866	.700	.854	1.389	.417	1.143
	AVRB	.350	.820	.276	.936	.140	.993
	Mean	-.703	.066	.716	-.853	.029	.866
	SD	.123	.157	.104	.047	.151	.053
	LCI	.737	1.193	.680	.243	.451	.214
γ	CP	1.000	1.000	1.000	.800	.600	.800
	Bias	.097	.066	-.084	-.053	.029	.066
	RMSE	.156	.170	.134	.071	.154	.084
	AVRB	.122	——	.105	.067	——	.082
	Mean	17.394	16.681	14.943	17.717	20.832	26.441
	SD	5.518	4.054	1.793	6.103	4.260	19.597
	LCI	28.826	35.192	25.784	8.962	18.916	17.199
ν_1	CP	.600	.600	.600	.200	.800	.200
	Bias	-12.606	-13.319	-15.057	-12.283	-9.168	-3.559
	RMSE	13.761	13.922	15.163	13.715	10.109	19.917
	AVRB	.420	.444	.502	.409	.306	.119

Table 104 – Results of simulation study for ZA-SGtBS1 regression model ($\nu_1 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-2.384	-2.754	-2.909	-2.640	-2.355	-2.328
	SD	.957	.831	.955	.564	.205	.788
	LCI	2.638	3.129	2.856	.684	1.183	1.113
ζ_0	CP	.800	1.000	1.000	.400	1.000	.400
	Bias	.116	-.254	-.409	-.140	.145	.172
	RMSE	.964	.869	1.039	.582	.251	.806
	AVRB	.046	.102	.163	.056	.058	.069
	Median	.372	1.079	1.295	1.064	.502	.410
	SD	1.316	.943	1.057	.772	.369	1.034
	LCI	4.224	4.648	4.330	.914	1.944	1.773
ζ_1	CP	.800	1.000	1.000	.400	1.000	.600
	Bias	-.428	.279	.495	.264	-.298	-.390
	RMSE	1.384	.983	1.167	.816	.474	1.105
	AVRB	.535	.348	.619	.330	.373	.488
	Median	-.481	-.502	-.532	-.519	-.496	-.464
	SD	.032	.013	.037	.035	.008	.056
	LCI	.138	.150	.144	.069	.062	.099
β_0	CP	.800	1.000	1.000	.600	1.000	.200
	Bias	.019	-.002	-.032	-.019	.004	.036
	RMSE	.037	.014	.049	.040	.009	.067
	AVRB	.037	.003	.064	.037	.008	.072
	Median	.968	1.012	1.067	1.028	.998	.941
	SD	.073	.032	.081	.067	.020	.099
	LCI	.259	.29	.275	.113	.114	.152
β_1	CP	.800	1.000	.800	.600	1.000	.200
	Bias	-.032	.012	.067	.028	-.002	-.059
	RMSE	.080	.034	.105	.072	.020	.115
	AVRB	.032	.012	.067	.028	.002	.059
	Median	-1.581	-1.964	-1.967	-1.317	-1.369	-.993
	SD	.429	.348	.314	.396	.356	.57
	LCI	2.414	2.666	2.39	.617	1.053	.997
ψ_0	CP	1.000	.600	.600	.200	.800	.600
	Bias	-.581	-.964	-.967	-.317	-.369	.007
	RMSE	.722	1.025	1.017	.507	.513	.570
	AVRB	.581	.964	.967	.317	.369	.007

Table 104 (continued).

ψ_1	Median	.342	.891	.639	.021	.432	.061
	SD	.857	.571	.86	1.313	.41	1.000
	LCI	2.506	2.375	2.274	.667	.891	1.004
	CP	1.000	1.000	.800	.200	.800	$<$ .001
	Bias	-.158	.391	.139	-.479	-.068	-.439
	RMSE	.872	.693	.871	1.397	.416	1.092
	AVRB	.316	.783	.278	.958	.135	.878
γ	Median	-.756	.069	.744	-.862	.028	.870
	SD	.116	.164	.117	.048	.143	.055
	LCI	.737	1.193	.68	.243	.451	.214
	CP	1.000	1.000	1.000	.800	.600	.800
	Bias	.044	.069	-.056	-.062	.028	.070
	RMSE	.124	.178	.130	.078	.145	.089
	AVRB	.056	——	.070	.077	——	.088
ν_1	Median	16.618	14.000	13.456	17.649	20.400	26.239
	SD	5.463	2.443	1.520	6.117	4.327	19.626
	LCI	28.826	35.192	25.784	8.962	18.916	17.199
	CP	.600	.600	.600	.200	.800	.200
	Bias	-13.382	-16.000	-16.544	-12.351	-9.600	-3.761
	RMSE	14.454	16.185	16.614	13.783	10.530	19.984
	AVRB	.446	.533	.551	.412	.320	.125

Table 105 – Results of simulation study for ZA-SGtBS1 regression model ($\nu_1 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-2.327	-2.561	-2.642	-2.637	-2.348	-2.243
	SD	.973	.677	.728	.565	.21	.678
	LCI	2.638	3.129	2.856	.684	1.183	1.113
ζ_0	CP	.800	1.000	1.000	.400	1.000	.400
	Bias	.173	-.061	-.142	-.137	.152	.257
	RMSE	.988	.680	.742	.582	.259	.725
	AVRB	.069	.025	.057	.055	.061	.103
	Mode	-.032	.866	1.065	1.068	.501	.543
	SD	1.560	.845	1.050	.753	.382	1.268
	LCI	4.224	4.648	4.330	.914	1.944	1.773
ζ_1	CP	.800	1.000	1.000	.400	1.000	.600
	Bias	-.832	.066	.265	.268	-.299	-.257
	RMSE	1.768	.847	1.083	.800	.485	1.294
	AVRB	1.040	.083	.331	.335	.374	.322
	Mode	-.484	-.500	-.531	-.520	-.497	-.457
	SD	.031	.015	.037	.036	.008	.062
	LCI	.138	.150	.144	.069	.062	.099
β_0	CP	.800	1.000	1.000	.600	1.000	.200
	Bias	.016	$< .001$	-.031	-.020	.003	.043
	RMSE	.035	.015	.048	.041	.009	.075
	AVRB	.033	.001	.062	.040	.007	.087
	Mode	.967	1.012	1.063	1.028	.998	.933
	SD	.075	.033	.082	.067	.020	.103
	LCI	.259	.290	.275	.113	.114	.152
β_1	CP	.800	1.000	.800	.600	1.000	.200
	Bias	-.033	.012	.063	.028	-.002	-.067
	RMSE	.082	.035	.103	.072	.020	.123
	AVRB	.033	.012	.063	.028	.002	.067
	Mode	-1.436	-1.845	-1.918	-1.314	-1.369	-.847
	SD	.591	.600	.383	.395	.359	.794
	LCI	2.414	2.666	2.390	.617	1.053	.997
ψ_0	CP	1.000	.600	.600	.200	.800	.600
	Bias	-.436	-.845	-.918	-.314	-.369	.153
	RMSE	.734	1.037	.994	.504	.514	.809
	AVRB	.436	.845	.918	.314	.369	.153

Table 105 (continued).

	Mode	.554	.806	.604	-.004	.435	.102
	SD	.894	.528	.895	1.287	.409	.992
	LCI	2.506	2.375	2.274	.667	.891	1.004
ψ_1	CP	1.000	1.000	.800	.200	.800	$< .001$
	Bias	.054	.306	.104	-.504	-.065	-.398
	RMSE	.895	.610	.901	1.382	.414	1.069
	AVRB	.109	.612	.209	1.009	.130	.796
	Mode	-.772	.066	.748	-.856	.031	.866
	SD	.098	.181	.110	.046	.147	.052
	LCI	.737	1.193	.680	.243	.451	.214
γ	CP	1.000	1.000	1.000	.800	.600	.800
	Bias	.028	.066	-.052	-.056	.031	.066
	RMSE	.102	.193	.122	.073	.150	.084
	AVRB	.035	——	.064	.070	——	.083
	Mode	15.896	9.822	9.420	17.142	21.113	25.182
	SD	3.958	1.546	2.910	6.009	5.419	21.201
	LCI	28.826	35.192	25.784	8.962	18.916	17.199
ν_1	CP	.600	.600	.600	.200	.800	.200
	Bias	-14.104	-20.178	-20.580	-12.858	-8.887	-4.818
	RMSE	14.649	20.237	20.784	14.193	10.409	21.741
	AVRB	.470	.673	.686	.429	.296	.161

ZA-SGtBS2 regression model

Table 106 – Results of simulation study for ZA-SGtBS2 regression model ($\nu_1 = 5, \nu_2 = 15$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-2.239	-2.245	-2.286	-2.415	-2.421	-2.405
	SD	.605	.668	.669	.106	.126	.127
	LCI	2.648	2.556	2.53	1.103	1.137	1.177
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.261	.255	.214	.085	.079	.095
	RMSE	.659	.715	.703	.136	.149	.159
	AVRB	.105	.102	.086	.034	.032	.038
ζ_1	Mean	.699	.702	.788	.75	.742	.727
	SD	.893	.983	.994	.381	.401	.392
	LCI	4.331	4.084	4.181	1.856	1.887	1.828
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.101	-.098	-.012	-.05	-.058	-.073
	RMSE	.899	.988	.994	.384	.405	.399
	AVRB	.126	.122	.015	.062	.073	.091
β_0	Mean	-.799	-.651	-.382	-.556	-.636	-.444
	SD	.415	.443	.214	.121	.203	.112
	LCI	1.408	1.657	1.573	.608	.728	.715
	CP	.800	.800	1.000	1.000	1.000	1.000
	Bias	-.299	-.151	.118	-.056	-.136	.056
	RMSE	.512	.468	.245	.133	.244	.125
	AVRB	.598	.302	.237	.112	.272	.112
β_1	Mean	1.461	1.464	.471	1.114	1.212	.881
	SD	.819	.646	.452	.185	.549	.050
	LCI	2.301	2.733	2.221	1.010	1.193	.991
	CP	.800	.800	1.000	1.000	.800	1.000
	Bias	.461	.464	-.529	.114	.212	-.119
	RMSE	.94	.795	.696	.217	.589	.129
	AVRB	.461	.464	.529	.114	.212	.119
γ	Mean	-.685	.175	.643	-.76	-.057	.796
	SD	.052	.311	.186	.107	.126	.115
	LCI	.873	1.265	.936	.46	.598	.413
	CP	1.000	1.000	1.000	1.000	.600	1.000
	Bias	.115	.175	-.157	.040	-.057	-.004
	RMSE	.126	.356	.244	.114	.138	.115

Table 106 (continued).

	AVRB	.143	-	.196	.050	-	.005
ν_1	Mean	7.65	6.436	6.726	5.761	5.201	5.663
	SD	1.431	.322	1.557	.800	.333	1.092
	LCI	14.446	10.539	10.565	5.318	3.642	4.551
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	2.650	1.436	1.726	.761	.201	.663
	RMSE	3.011	1.472	2.325	1.104	.389	1.278
	AVRB	.530	.287	.345	.152	.040	.133
ν_2	Mean	24.024	20.796	20.485	18.53	15.517	18.084
	SD	6.248	2.735	7.670	3.682	.997	4.489
	LCI	59.36	49.376	41.892	23.843	15.499	20.864
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	9.024	5.796	5.485	3.53	.517	3.084
	RMSE	10.976	6.409	9.429	5.101	1.123	5.447
	AVRB	.602	.386	.366	.235	.034	.206

Table 107 – Results of simulation study for ZA-SGtBS2 regression model ($\nu_1 = 5, \nu_2 = 15$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.201	-2.205	-2.227	-2.406	-2.414	-2.394
	SD	.595	.651	.635	.100	.121	.140
	LCI	2.648	2.556	2.530	1.103	1.137	1.177
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.299	.295	.273	.094	.086	.106
	RMSE	.666	.715	.691	.137	.149	.176
	AVRB	.119	.118	.109	.038	.034	.042
ζ_1	Median	.654	.673	.757	.749	.739	.731
	SD	.889	.993	.954	.383	.393	.401
	LCI	4.331	4.084	4.181	1.856	1.887	1.828
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.146	-.127	-.043	-.051	-.061	-.069
	RMSE	.901	1.001	.955	.386	.398	.406
	AVRB	.182	.158	.053	.063	.076	.087
β_0	Median	-.812	-.660	-.393	-.553	-.641	-.448
	SD	.400	.442	.229	.119	.205	.115
	LCI	1.408	1.657	1.573	.608	.728	.715
	CP	.800	.800	1.000	1.000	1.000	1.000
	Bias	-.312	-.160	.107	-.053	-.141	.052
	RMSE	.508	.470	.253	.131	.249	.126
	AVRB	.625	.321	.214	.106	.282	.104
β_1	Median	1.459	1.477	.472	1.116	1.22	.88
	SD	.811	.657	.469	.188	.550	.047
	LCI	2.301	2.733	2.221	1.010	1.193	.991
	CP	.800	.800	1.000	1.000	.800	1.000
	Bias	.459	.477	-.528	.116	.22	-.120
	RMSE	.932	.812	.706	.221	.592	.128
	AVRB	.459	.477	.528	.116	.220	.120
γ	Median	-.742	.186	.700	-.771	-.044	.817
	SD	.056	.34	.187	.115	.121	.120
	LCI	.873	1.265	.936	.46	.598	.413
	CP	1.000	1.000	1.000	1.000	.600	1.000
	Bias	.058	.186	-.100	.029	-.044	.017
	RMSE	.080	.387	.212	.119	.129	.121
	AVRB	.073	-	.125	.036	-	.021

Table 107 (continued).

	Median	6.457	5.495	5.896	5.447	4.984	5.371
	SD	.904	.124	1.103	.767	.314	.982
	LCI	14.446	10.539	10.565	5.318	3.642	4.551
ν_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.457	.495	.896	.447	-.016	.371
	RMSE	1.714	.51	1.422	.887	.314	1.05
	AVRB	.291	.099	.179	.089	.003	.074
	Median	19.39	16.749	17.128	17.111	14.679	16.732
	SD	4.501	2.644	5.746	3.207	.806	3.877
	LCI	59.360	49.376	41.892	23.843	15.499	20.864
ν_2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	4.390	1.749	2.128	2.111	-.321	1.732
	RMSE	6.287	3.17	6.128	3.84	.867	4.247
	AVRB	.293	.117	.142	.141	.021	.115

Table 108 – Results of simulation study for ZA-SGtBS2 regression model ($\nu_1 = 5, \nu_2 = 15$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mode	-2.147	-2.127	-2.145	-2.400	-2.407	-2.382
	SD	.540	.676	.626	.092	.120	.159
	LCI	2.648	2.556	2.53	1.103	1.137	1.177
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.353	.373	.355	.100	.093	.118
	RMSE	.645	.772	.720	.136	.152	.198
	AVRB	.141	.149	.142	.040	.037	.047
ζ_1	Mode	.743	.589	.744	.724	.713	.758
	SD	.753	1.086	.949	.399	.399	.419
	LCI	4.331	4.084	4.181	1.856	1.887	1.828
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.057	-.211	-.056	-.076	-.087	-.042
	RMSE	.755	1.106	.950	.406	.408	.421
	AVRB	.071	.264	.069	.095	.109	.053
β_0	Mode	-.835	-.680	-.407	-.554	-.641	-.450
	SD	.371	.424	.245	.121	.201	.116
	LCI	1.408	1.657	1.573	.608	.728	.715
	CP	.800	.800	1.000	1.000	1.000	1.000
	Bias	-.335	-.180	.093	-.054	-.141	.050
	RMSE	.500	.460	.263	.132	.246	.126
	AVRB	.670	.361	.186	.108	.282	.100
β_1	Mode	1.45	1.515	.496	1.111	1.22	.885
	SD	.790	.699	.540	.191	.551	.053
	LCI	2.301	2.733	2.221	1.010	1.193	.991
	CP	.800	.800	1.000	1.000	.800	1.000
	Bias	.450	.515	-.504	.111	.220	-.115
	RMSE	.909	.868	.739	.221	.594	.126
	AVRB	.450	.515	.504	.111	.220	.115
γ	Mode	-.772	.222	.724	-.770	-.049	.807
	SD	.051	.388	.168	.108	.123	.112
	LCI	.873	1.265	.936	.460	.598	.413
	CP	1.000	1.000	1.000	1.000	.600	1.000
	Bias	.028	.222	-.076	.030	-.049	.007
	RMSE	.058	.447	.185	.112	.132	.112
	AVRB	.035	-	.095	.037	-	.009

Table 108 (continued).

	Mode	4.697	4.388	4.537	4.864	4.55	4.829
	SD	.441	.037	.324	.408	.269	.602
	LCI	14.446	10.539	10.565	5.318	3.642	4.551
ν_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.303	-.612	-.463	-.136	-.45	-.171
	RMSE	.535	.613	.565	.430	.524	.626
	AVRB	.061	.122	.093	.027	.090	.034
	Mode	13.623	13.383	13.289	14.681	12.746	15.37
	SD	2.971	2.647	3.227	1.347	.485	2.656
	LCI	59.360	49.376	41.892	23.843	15.499	20.864
ν_2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-1.377	-1.617	-1.711	-.319	-2.254	.370
	RMSE	3.275	3.102	3.652	1.384	2.306	2.682
	AVRB	.092	.108	.114	.021	.150	.025

Table 109 – Results of simulation study for ZA-SGtBS2 regression model ($\nu_1 = \nu_2 = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-2.700	-2.693	-2.676	-2.367	-2.379	-2.378
	SD	.789	.806	.784	.194	.214	.190
	LCI	2.941	3.121	2.868	1.146	1.184	1.138
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	-.200	-.193	-.176	.133	.121	.122
	RMSE	.814	.829	.803	.235	.246	.226
	AVRB	.080	.077	.071	.053	.048	.049
ζ_1	Mean	1.000	1.006	.985	.529	.551	.551
	SD	.791	.795	.785	.275	.262	.285
	LCI	4.542	4.877	4.531	1.927	1.959	1.870
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.200	.206	.185	-.271	-.249	-.249
	RMSE	.816	.821	.806	.386	.362	.379
	AVRB	.250	.257	.232	.338	.311	.312
β_0	Mean	-.376	-.428	-.562	-.524	-.495	-.455
	SD	.268	.199	.226	.078	.025	.064
	LCI	.994	1.135	1.138	.369	.453	.468
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.124	.072	-.062	-.024	.005	.045
	RMSE	.296	.212	.235	.082	.026	.078
	AVRB	.248	.143	.125	.047	.011	.090
β_1	Mean	.828	1.000	1.201	1.071	1.069	.919
	SD	.519	.330	.472	.142	.068	.129
	LCI	1.661	1.812	1.732	.621	.766	.599
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.172	$< .001$	.201	.071	.069	-.081
	RMSE	.546	.330	.513	.159	.097	.152
	AVRB	.172	$< .001$	.201	.071	.069	.081
γ	Mean	-.648	.147	.536	-.842	-.023	.861
	SD	.233	.332	.235	.045	.174	.052
	LCI	.816	1.16	.993	.310	.485	.251
	CP	1.000	1.000	.800	1.000	.400	1.000
	Bias	.152	.147	-.264	-.042	-.023	.061
	RMSE	.278	.363	.353	.062	.175	.080
	AVRB	.190	-	.330	.053	-	.076

Table 109 (continued).

	Mean	11.258	11.143	9.752	16.949	17.538	22.681
	SD	2.542	1.536	1.129	4.366	3.643	4.325
	LCI	25.882	26.625	23.426	38.159	37.93	47.64
ν_1	CP	.400	.600	.400	.800	.800	1.000
	Bias	-18.742	-18.857	-20.248	-13.051	-12.462	-7.319
	RMSE	18.914	18.920	20.280	13.762	12.983	8.501
	AVRB	.625	.629	.675	.435	.415	.244
	Mean	10.273	10.588	8.734	16.381	17.506	22.454
	SD	2.911	1.458	1.334	4.949	4.528	5.081
	LCI	27.788	29.759	25.098	41.376	41.114	52.29
ν_2	CP	.400	.800	.400	.800	.800	1.000
	Bias	-19.727	-19.412	-21.266	-13.619	-12.494	-7.546
	RMSE	19.941	19.467	21.308	14.491	13.29	9.097
	AVRB	.658	.647	.709	.454	.416	.252

Table 110 – Results of simulation study for ZA-SGtBS2 regression model ($\nu_1 = 5, \nu_2 = 15$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.650	-2.644	-2.632	-2.362	-2.368	-2.364
	SD	.774	.786	.773	.199	.207	.194
	LCI	2.941	3.121	2.868	1.146	1.184	1.138
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	-.150	-.143	-.132	.138	.132	.136
	RMSE	.788	.799	.784	.243	.245	.237
	AVRB	.060	.057	.053	.055	.053	.055
ζ_1	Median	.997	.983	.957	.536	.558	.548
	SD	.804	.774	.808	.268	.259	.301
	LCI	4.542	4.877	4.531	1.927	1.959	1.870
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.197	.183	.157	-.264	-.242	-.252
	RMSE	.828	.795	.823	.376	.354	.393
	AVRB	.246	.229	.197	.33	.302	.315
β_0	Median	-.387	-.438	-.56	-.524	-.497	-.457
	SD	.269	.201	.228	.079	.026	.064
	LCI	.994	1.135	1.138	.369	.453	.468
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.113	.062	-.060	-.024	.003	.043
	RMSE	.292	.211	.236	.083	.026	.077
	AVRB	.225	.125	.120	.047	.007	.086
β_1	Median	.839	1.004	1.192	1.071	1.072	.917
	SD	.525	.327	.460	.138	.071	.127
	LCI	1.661	1.812	1.732	.621	.766	.599
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.161	.004	.192	.071	.072	-.083
	RMSE	.549	.327	.498	.155	.101	.152
	AVRB	.161	.004	.192	.071	.072	.083
γ	Median	-.680	.157	.560	-.850	-.023	.868
	SD	.251	.345	.240	.047	.162	.052
	LCI	.816	1.160	.993	.310	.485	.251
	CP	1.000	1.000	.800	1.000	.400	1.000
	Bias	.120	.157	-.240	-.050	-.023	.068
	RMSE	.278	.379	.339	.069	.164	.086
	AVRB	.150	-	.300	.063	-	.085

Table 110 (continued).

	Median	9.281	8.876	7.818	13.837	14.848	19.804
	SD	1.828	1.251	.918	2.854	2.462	3.547
	LCI	25.882	26.625	23.426	38.159	37.93	47.64
ν_1	CP	.400	.600	.400	.800	.800	1.000
	Bias	-20.719	-21.124	-22.182	-16.163	-15.152	-10.196
	RMSE	20.799	21.161	22.201	16.413	15.351	10.795
	AVRB	.691	.704	.739	.539	.505	.340
	Median	8.088	8.031	6.754	12.998	14.567	19.344
	SD	1.858	1.301	1.199	3.295	3.229	4.317
	LCI	27.788	29.759	25.098	41.376	41.114	52.29
ν_2	CP	.400	.800	.400	.800	.800	1.000
	Bias	-21.912	-21.969	-23.246	-17.002	-15.433	-10.656
	RMSE	21.99	22.007	23.277	17.318	15.767	11.497
	AVRB	.73	.732	.775	.567	.514	.355

Table 111 – Results of simulation study for ZA-SGtBS2 regression model ($\nu_1 = 5, \nu_2 = 15$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mode	-2.636	-2.569	-2.560	-2.359	-2.363	-2.349
	SD	.868	.788	.745	.204	.199	.188
	LCI	2.941	3.121	2.868	1.146	1.184	1.138
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	-.136	-.069	-.060	.141	.137	.151
	RMSE	.879	.791	.748	.248	.242	.242
	AVRB	.055	.028	.024	.056	.055	.060
ζ_1	Mode	1.129	.983	.971	.574	.573	.551
	SD	.770	.808	.953	.249	.231	.318
	LCI	4.542	4.877	4.531	1.927	1.959	1.870
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.329	.183	.171	-.226	-.227	-.249
	RMSE	.837	.829	.968	.337	.324	.404
	AVRB	.412	.229	.214	.283	.284	.311
β_0	Mode	-.401	-.448	-.557	-.524	-.495	-.456
	SD	.265	.203	.228	.079	.025	.064
	LCI	.994	1.135	1.138	.369	.453	.468
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.099	.052	-.057	-.024	.005	.044
	RMSE	.283	.210	.235	.082	.026	.078
	AVRB	.199	.104	.113	.048	.009	.087
β_1	Mode	.865	1.012	1.165	1.068	1.072	.918
	SD	.537	.317	.448	.141	.070	.128
	LCI	1.661	1.812	1.732	.621	.766	.599
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.135	.012	.165	.068	.072	-.082
	RMSE	.553	.317	.477	.156	.100	.152
	AVRB	.135	.012	.165	.068	.072	.082
γ	Mode	-.678	.172	.577	-.846	-.023	.864
	SD	.258	.355	.237	.043	.163	.051
	LCI	.816	1.16	.993	.310	.485	.251
	CP	1.000	1.000	.800	1.000	.400	1.000
	Bias	.122	.172	-.223	-.046	-.023	.064
	RMSE	.285	.394	.325	.063	.165	.082
	AVRB	.152	-	.279	.058	-	.080

Table 111 (continued).

	Mode	6.959	6.633	5.533	10.264	11.236	16.168
	SD	.634	1.331	.733	2.371	1.616	3.782
	LCI	25.882	26.625	23.426	38.159	37.930	47.640
ν_1	CP	.400	.600	.400	.800	.800	1.000
	Bias	-23.041	-23.367	-24.467	-19.736	-18.764	-13.832
	RMSE	23.050	23.405	24.478	19.878	18.834	14.340
	AVRB	.768	.779	.816	.658	.625	.461
	Mode	5.410	5.623	4.378	9.073	10.556	16.220
	SD	1.061	1.251	1.240	2.783	2.760	5.608
	LCI	27.788	29.759	25.098	41.376	41.114	52.290
ν_2	CP	.400	.800	.400	.800	.800	1.000
	Bias	-24.590	-24.377	-25.622	-20.927	-19.444	-13.780
	RMSE	24.613	24.409	25.652	21.111	19.639	14.878
	AVRB	.820	.813	.854	.698	.648	.459

ZA-StBS regression model

Table 112 – Results of simulation study for ZA-StBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-2.110	-2.161	-2.184	-2.364	-2.372	-2.359
	SD	.293	.201	.250	.306	.312	.284
	LCI	2.531	2.581	2.499	1.169	1.159	1.096
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.390	.339	.316	.136	.128	.141
	RMSE	.488	.394	.403	.335	.337	.317
	AVRB	.156	.136	.127	.054	.051	.057
ζ_1	Mean	.387	.441	.517	.621	.632	.622
	SD	.322	.413	.368	.334	.337	.310
	LCI	4.152	4.314	4.132	1.928	1.877	1.829
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.413	-.359	-.283	-.179	-.168	-.178
	RMSE	.524	.547	.465	.379	.376	.357
	AVRB	.516	.448	.354	.224	.21	.222
β_0	Mean	-.589	-.363	-.503	-.478	-.494	-.497
	SD	.238	.140	.298	.076	.111	.162
	LCI	.962	1.224	1.209	.373	.471	.534
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	-.089	.137	-.003	.022	.006	.003
	RMSE	.254	.196	.298	.079	.111	.162
	AVRB	.177	.273	.005	.043	.013	.007
β_1	Mean	1.211	.803	.949	.969	.951	.986
	SD	.466	.345	.674	.167	.265	.338
	LCI	1.785	2.043	2.087	.679	.85	.973
	CP	1.000	1.000	.800	1.000	.800	.800
	Bias	.211	-.197	-.051	-.031	-.049	-.014
	RMSE	.511	.397	.676	.170	.270	.338
	AVRB	.211	.197	.051	.031	.049	.014
ψ_0	Mean	-1.010	-.744	-.863	-.966	-.925	-.994
	SD	.271	.151	.344	.185	.231	.156
	LCI	1.643	1.592	1.679	.723	.683	.670
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.010	.256	.137	.034	.075	.006
	RMSE	.271	.298	.370	.188	.243	.156

Table 112 (continued).

	AVRB	.010	.256	.137	.034	.075	.006
ψ_1	Mean	.731	.366	.454	.456	.427	.508
	SD	.668	.471	.834	.276	.462	.239
	LCI	2.750	2.578	2.628	1.128	1.117	1.090
	CP	1.000	1.000	.800	1.000	.800	1.000
	Bias	.231	-.134	-.046	-.044	-.073	.008
	RMSE	.707	.490	.836	.280	.468	.240
	AVRB	.462	.267	.091	.087	.145	.017
γ	Mean	-.358	-.016	.278	-.681	.035	.751
	SD	.322	.189	.245	.117	.148	.084
	LCI	1.059	1.412	1.167	.522	.510	.457
	CP	.800	1.000	.800	1.000	.400	1.000
	Bias	.442	-.016	-.522	.119	.035	-.049
	RMSE	.546	.189	.576	.167	.152	.098
	AVRB	.552	-	.652	.148	-	.062
ν	Mean	11.608	10.931	11.473	5.624	5.875	5.627
	SD	5.097	4.482	5.553	1.463	.637	1.533
	LCI	38.435	36.389	35.529	4.650	5.328	4.493
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	6.608	5.931	6.473	.624	.875	.627
	RMSE	8.346	7.434	8.529	1.59	1.082	1.656
	AVRB	1.322	1.186	1.295	.125	.175	.125

Table 113 – Results of simulation study for ZA-StBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.079	-2.124	-2.155	-2.356	-2.365	-2.358
	SD	.278	.202	.244	.303	.315	.282
	LCI	2.531	2.581	2.499	1.169	1.159	1.096
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.421	.376	.345	.144	.135	.142
	RMSE	.504	.427	.423	.335	.343	.316
	AVRB	.168	.151	.138	.058	.054	.057
ζ_1	Median	.392	.417	.504	.615	.637	.631
	SD	.327	.443	.370	.330	.339	.308
	LCI	4.152	4.314	4.132	1.928	1.877	1.829
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.408	-.383	-.296	-.185	-.163	-.169
	RMSE	.523	.585	.474	.378	.376	.351
	AVRB	.510	.479	.370	.231	.204	.211
β_0	Median	-.601	-.389	-.528	-.478	-.495	-.502
	SD	.230	.140	.295	.077	.112	.159
	LCI	.962	1.224	1.209	.373	.471	.534
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	-.101	.111	-.028	.022	.005	-.002
	RMSE	.251	.178	.296	.080	.112	.159
	AVRB	.201	.222	.055	.044	.010	.005
β_1	Median	1.222	.805	.957	.968	.953	.988
	SD	.474	.349	.674	.168	.268	.338
	LCI	1.785	2.043	2.087	.679	.850	.973
	CP	1.000	1.000	.800	1.000	.800	.800
	Bias	.222	-.195	-.043	-.032	-.047	-.012
	RMSE	.523	.400	.675	.171	.272	.338
	AVRB	.222	.195	.043	.032	.047	.012
ψ_0	Median	-1.006	-.754	-.861	-.966	-.924	-.996
	SD	.268	.139	.343	.187	.233	.161
	LCI	1.643	1.592	1.679	.723	.683	.670
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.006	.246	.139	.034	.076	.004
	RMSE	.268	.282	.370	.190	.245	.161
	AVRB	.006	.246	.139	.034	.076	.004

Table 113 (continued).

	Median	.725	.367	.453	.456	.421	.517
	SD	.670	.458	.834	.274	.471	.244
	LCI	2.750	2.578	2.628	1.128	1.117	1.090
ψ_1	CP	1.000	1.000	.800	1.000	.800	1.000
	Bias	.225	-.133	-.047	-.044	-.079	.017
	RMSE	.706	.477	.835	.278	.478	.244
	AVRB	.450	.266	.094	.088	.158	.034
	Median	-.373	-.015	.271	-.694	.036	.757
	SD	.353	.202	.277	.122	.133	.092
	LCI	1.059	1.412	1.167	.522	.510	.457
γ	CP	.800	1.000	.800	1.000	.400	1.000
	Bias	.427	-.015	-.529	.106	.036	-.043
	RMSE	.554	.203	.597	.161	.138	.102
	AVRB	.533	-	.661	.133	-	.054
	Median	7.915	7.486	8.017	5.330	5.575	5.391
	SD	2.915	2.121	3.256	1.263	.551	1.424
	LCI	38.435	36.389	35.529	4.650	5.328	4.493
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	2.915	2.486	3.017	.330	.575	.391
	RMSE	4.123	3.268	4.439	1.305	.796	1.476
	AVRB	.583	.497	.603	.066	.115	.078

Table 114 – Results of simulation study for ZA-StBS regression model ($\nu = 5$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-2.038	-2.058	-2.115	-2.348	-2.345	-2.356
	SD	.269	.183	.228	.295	.315	.277
	LCI	2.531	2.581	2.499	1.169	1.159	1.096
ζ_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.462	.442	.385	.152	.155	.144
	RMSE	.535	.479	.448	.332	.351	.312
	AVRB	.185	.177	.154	.061	.062	.058
	Mode	.427	.370	.426	.627	.634	.625
	SD	.363	.654	.339	.347	.339	.302
	LCI	4.152	4.314	4.132	1.928	1.877	1.829
ζ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.373	-.430	-.374	-.173	-.166	-.175
	RMSE	.521	.782	.505	.387	.378	.349
	AVRB	.467	.537	.468	.216	.207	.218
	Mode	-.611	-.410	-.557	-.479	-.496	-.502
	SD	.230	.139	.287	.076	.112	.159
	LCI	.962	1.224	1.209	.373	.471	.534
β_0	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	-.111	.090	-.057	.021	.004	-.002
	RMSE	.255	.165	.292	.079	.112	.159
	AVRB	.222	.179	.113	.041	.009	.003
	Mode	1.232	.788	.956	.962	.949	.991
	SD	.493	.344	.689	.168	.265	.340
	LCI	1.785	2.043	2.087	.679	.850	.973
β_1	CP	1.000	1.000	.800	1.000	.800	.800
	Bias	.232	-.212	-.044	-.038	-.051	-.009
	RMSE	.545	.404	.691	.173	.270	.340
	AVRB	.232	.212	.044	.038	.051	.009
	Mode	-1.001	-.763	-.861	-.970	-.925	-.995
	SD	.268	.126	.349	.190	.232	.161
	LCI	1.643	1.592	1.679	.723	.683	.670
ψ_0	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.001	.237	.139	.030	.075	.005
	RMSE	.268	.268	.375	.192	.244	.161
	AVRB	.001	.237	.139	.030	.075	.005

Table 114 (continued).

ψ_1	Mode	.732	.353	.445	.456	.420	.523
	SD	.697	.447	.828	.274	.477	.244
	LCI	2.750	2.578	2.628	1.128	1.117	1.090
	CP	1.000	1.000	.800	1.000	.800	1.000
	Bias	.232	-.147	-.055	-.044	-.080	.023
	RMSE	.734	.470	.830	.278	.483	.245
	AVRB	.463	.293	.110	.088	.160	.047
γ	Mode	-.388	-.005	.262	-.693	.038	.757
	SD	.382	.215	.306	.119	.139	.087
	LCI	1.059	1.412	1.167	.522	.510	.457
	CP	.800	1.000	.800	1.000	.400	1.000
	Bias	.412	-.005	-.538	.107	.038	-.043
	RMSE	.562	.215	.619	.160	.144	.097
	AVRB	.515	-	.672	.134	-	.054
ν	Mode	5.477	4.920	5.105	4.842	4.887	5.032
	SD	1.570	.818	1.078	.912	.358	1.510
	LCI	38.435	36.389	35.529	4.650	5.328	4.493
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.477	-.080	.105	-.158	-.113	.032
	RMSE	1.641	.822	1.083	.925	.376	1.510
	AVRB	.095	.016	.021	.032	.023	.006

Table 115 – Results of simulation study for ZA-StBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-2.746	-2.707	-2.713	-2.268	-2.312	-2.272
	SD	1.139	1.081	1.132	.272	.277	.245
	LCI	3.258	3.189	3.237	1.125	1.169	1.110
ζ_0	CP	.800	1.000	.800	.800	.800	.800
	Bias	-.246	-.207	-.213	.232	.188	.228
	RMSE	1.165	1.101	1.152	.358	.335	.335
	AVRB	.098	.083	.085	.093	.075	.091
	Mean	.666	.588	.592	.385	.477	.406
	SD	1.170	1.162	1.274	.618	.628	.573
	LCI	4.972	5.099	4.923	1.898	1.919	1.806
ζ_1	CP	1.000	1.000	1.000	.800	.800	.800
	Bias	-.134	-.212	-.208	-.415	-.323	-.394
	RMSE	1.177	1.181	1.291	.744	.706	.695
	AVRB	.167	.265	.260	.519	.404	.493
	Mean	-.481	-.418	-.432	-.490	-.432	-.502
	SD	.080	.243	.227	.027	.071	.105
	LCI	.667	.885	1.106	.265	.367	.486
β_0	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	.019	.082	.068	.010	.068	-.002
	RMSE	.082	.256	.237	.029	.099	.105
	AVRB	.038	.165	.136	.021	.136	.003
	Mean	.974	.931	.925	.976	.901	1.057
	SD	.153	.534	.533	.053	.179	.214
	LCI	1.110	1.637	1.927	.493	.703	.851
β_1	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	-.026	-.069	-.075	-.024	-.099	.057
	RMSE	.156	.538	.538	.058	.204	.222
	AVRB	.026	.069	.075	.024	.099	.057
	Mean	-1.065	-1.094	-.968	-1.041	-1.048	-1.045
	SD	.334	.270	.267	.118	.118	.169
	LCI	1.541	1.513	1.453	.614	.661	.653
ψ_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.065	-.094	.032	-.041	-.048	-.045
	RMSE	.340	.286	.269	.125	.127	.174
	AVRB	.065	.094	.032	.041	.048	.045

Table 115 (continued).

ψ_1	Mean	.412	.472	.238	.518	.497	.488
	SD	.729	.667	.509	.286	.252	.286
	LCI	2.533	2.492	2.441	1.017	1.077	1.107
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.088	-.028	-.262	.018	-.003	-.012
	RMSE	.734	.668	.572	.287	.252	.286
	AVRB	.177	.057	.524	.037	.005	.023
γ	Mean	-.738	-.040	.699	-.830	.008	.842
	SD	.151	.248	.148	.066	.099	.041
	LCI	.658	1.230	.714	.278	.482	.303
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.062	-.040	-.101	-.030	.008	.042
	RMSE	.163	.251	.179	.072	.099	.059
	AVRB	.077	-	.127	.037	-	.053
ν	Mean	18.834	14.693	18.565	27.547	22.544	27.032
	SD	4.479	3.480	4.443	8.859	10.089	17.025
	LCI	67.463	52.609	64.153	81.845	64.764	68.442
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-11.166	-15.307	-11.435	-2.453	-7.456	-2.968
	RMSE	12.031	15.698	12.268	9.192	12.545	17.281
	AVRB	.372	.51	.381	.082	.249	.099

Table 116 – Results of simulation study for ZA-StBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.668	-2.634	-2.669	-2.252	-2.298	-2.265
	SD	1.078	1.035	1.086	.268	.273	.248
	LCI	3.258	3.189	3.237	1.125	1.169	1.110
	CP	.800	1.000	.800	.800	.800	.800
	Bias	-.167	-.134	-.169	.248	.202	.235
	RMSE	1.091	1.043	1.100	.365	.340	.342
	AVRB	.067	.053	.068	.099	.081	.094
ζ_1	Median	.599	.551	.544	.358	.464	.394
	SD	1.106	1.128	1.230	.623	.621	.544
	LCI	4.972	5.099	4.923	1.898	1.919	1.806
	CP	1.000	1.000	1.000	.800	.800	.800
	Bias	-.201	-.249	-.256	-.442	-.336	-.406
	RMSE	1.124	1.155	1.256	.764	.706	.679
	AVRB	.251	.311	.320	.553	.420	.507
β_0	Median	-.489	-.425	-.460	-.490	-.433	-.508
	SD	.089	.241	.226	.026	.072	.101
	LCI	.667	.885	1.106	.265	.367	.486
	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	.011	.075	.040	.010	.067	-.008
	RMSE	.090	.253	.229	.028	.098	.101
	AVRB	.022	.149	.080	.020	.133	.016
β_1	Median	.984	.929	.936	.973	.898	1.057
	SD	.164	.538	.536	.053	.178	.213
	LCI	1.110	1.637	1.927	.493	.703	.851
	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	-.016	-.071	-.064	-.027	-.102	.057
	RMSE	.165	.542	.540	.060	.205	.220
	AVRB	.016	.071	.064	.027	.102	.057
ψ_0	Median	-1.056	-1.108	-.967	-1.041	-1.047	-1.050
	SD	.337	.279	.283	.118	.118	.163
	LCI	1.541	1.513	1.453	.614	.661	.653
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.056	-.108	.033	-.041	-.047	-.050
	RMSE	.341	.299	.284	.125	.127	.170
	AVRB	.056	.108	.033	.041	.047	.050

Table 116 (continued).

	Median	.418	.484	.243	.520	.492	.492
	SD	.749	.666	.515	.285	.250	.274
	LCI	2.533	2.492	2.441	1.017	1.077	1.107
ψ_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.082	-.016	-.257	.020	-.008	-.008
	RMSE	.753	.666	.575	.286	.250	.274
	AVRB	.163	.032	.515	.040	.015	.016
	Median	-.775	-.026	.733	-.836	.004	.854
	SD	.155	.247	.149	.068	.094	.051
	LCI	.658	1.23	.714	.278	.482	.303
γ	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.025	-.026	-.067	-.036	.004	.054
	RMSE	.157	.249	.163	.077	.095	.074
	AVRB	.031	I-	.084	.045	-	.068
	Median	12.874	9.579	12.815	21.05	16.985	21.495
	SD	3.011	1.990	3.246	6.385	7.258	13.159
	LCI	67.463	52.609	64.153	81.845	64.764	68.442
ν	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-17.126	-20.421	-17.185	-8.950	-13.015	-8.505
	RMSE	17.389	20.517	17.489	10.994	14.902	15.668
	AVRB	.571	.681	.573	.298	.434	.283

Table 117 – Results of simulation study for ZA-StBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-2.630	-2.517	-2.514	-2.241	-2.284	-2.264
	SD	1.164	.942	.926	.269	.270	.249
	LCI	3.258	3.189	3.237	1.125	1.169	1.110
ζ_0	CP	.800	1.000	.800	.800	.800	.800
	Bias	-.130	-.017	-.014	.259	.216	.236
	RMSE	1.171	.943	.926	.373	.346	.343
	AVRB	.052	.007	.006	.103	.086	.094
	Mode	.405	.369	.573	.320	.432	.410
	SD	1.163	1.067	1.134	.662	.603	.401
	LCI	4.972	5.099	4.923	1.898	1.919	1.806
ζ_1	CP	1.000	1.000	1.000	.800	.800	.800
	Bias	-.395	-.431	-.227	-.480	-.368	-.390
	RMSE	1.228	1.151	1.157	.817	.707	.559
	AVRB	.494	.538	.284	.599	.460	.488
	Mode	-.493	-.436	-.479	-.490	-.433	-.506
	SD	.087	.241	.221	.027	.071	.103
	LCI	.667	.885	1.106	.265	.367	.486
β_0	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	.007	.064	.021	.010	.067	-.006
	RMSE	.087	.250	.222	.029	.098	.103
	AVRB	.015	.129	.041	.020	.135	.013
	Mode	.991	.932	.956	.975	.901	1.062
	SD	.180	.546	.551	.052	.177	.214
	LCI	1.110	1.637	1.927	.493	.703	.851
β_1	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	-.009	-.068	-.044	-.025	-.099	.062
	RMSE	.180	.550	.553	.058	.203	.222
	AVRB	.009	.068	.044	.025	.099	.062
	Mode	-1.045	-1.128	-.978	-1.040	-1.044	-1.051
	SD	.329	.297	.308	.118	.118	.164
	LCI	1.541	1.513	1.453	.614	.661	.653
ψ_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.045	-.128	.022	-.040	-.044	-.051
	RMSE	.332	.324	.309	.125	.126	.172
	AVRB	.045	.128	.022	.040	.044	.051

Table 117 (continued).

ψ_1	Mode	.470	.494	.244	.523	.489	.490
	SD	.819	.642	.491	.285	.247	.265
	LCI	2.533	2.492	2.441	1.017	1.077	1.107
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.030	-.006	-.256	.023	-.011	-.010
	RMSE	.819	.642	.554	.285	.247	.266
	AVRB	.061	.012	.511	.047	.021	.019
γ	Mode	-.774	-.007	.746	-.832	.002	.845
	SD	.142	.248	.125	.065	.101	.043
	LCI	.658	1.230	.714	.278	.482	.303
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.026	-.007	-.054	-.032	.002	.045
	RMSE	.145	.248	.136	.073	.101	.063
	AVRB	.033	-	.067	.040	-	.057
ν	Mode	7.152	6.335	6.590	13.983	11.758	13.886
	SD	1.016	2.221	.933	1.751	4.022	7.355
	LCI	67.463	52.609	64.153	81.845	64.764	68.442
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-22.848	-23.665	-23.410	-16.017	-18.242	-16.114
	RMSE	22.870	23.769	23.429	16.113	18.680	17.713
	AVRB	.762	.789	.780	.534	.608	.537

ZA-SSLBS regression model

Table 118 – Results of simulation study for ZA-SSLBS regression model ($\nu = 3$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-2.703	-2.763	-2.805	-2.279	-2.261	-2.313
	SD	.972	.947	.95	.375	.365	.374
	LCI	2.601	2.694	2.820	1.132	1.134	1.204
	CP	.800	1.000	1.000	.800	.800	.800
	Bias	-.203	-.263	-.305	.221	.239	.187
	RMSE	.993	.983	.998	.435	.436	.419
	AVRB	.081	.105	.122	.088	.096	.075
ζ_1	Mean	1.498	1.628	1.636	.419	.417	.497
	SD	1.433	1.386	1.343	.706	.678	.656
	LCI	4.132	4.28	4.445	1.902	1.917	1.969
	CP	.800	.800	.800	.800	.800	.800
	Bias	.698	.828	.836	-.381	-.383	-.303
	RMSE	1.594	1.614	1.582	.802	.779	.723
	AVRB	.873	1.034	1.045	.476	.479	.379
β_0	Mean	-.499	-.488	-.507	-.483	-.495	-.491
	SD	.149	.212	.12	.051	.136	.102
	LCI	.696	.912	1.023	.318	.443	.516
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.001	.012	-.007	.017	.005	.009
	RMSE	.149	.212	.120	.054	.136	.103
	AVRB	.001	.024	.014	.034	.010	.017
β_1	Mean	1.057	1.025	1.000	.945	.97	.981
	SD	.315	.424	.162	.101	.27	.179
	LCI	1.368	1.847	1.883	.57	.803	.973
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.057	.025	$< .001$	-.055	-.030	-.019
	RMSE	.320	.424	.162	.115	.272	.180
	AVRB	.057	.025	$< .001$	.055	.030	.019
ψ_0	Mean	-.957	-.875	-.889	-.923	-.909	-.924
	SD	.363	.200	.274	.089	.158	.209
	LCI	1.394	1.375	1.367	.721	.675	.687
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.043	.125	.111	.077	.091	.076
	RMSE	.366	.236	.296	.117	.183	.222

Table 118 (continued).

	AVRB	.043	.125	.111	.077	.091	.076
	Mean	.680	.595	.461	.395	.433	.445
	SD	.496	.321	.436	.171	.171	.200
	LCI	2.559	2.283	2.258	.987	.987	1.012
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.180	.095	-.039	-.105	-.067	-.055
	RMSE	.528	.335	.437	.201	.184	.208
	AVRB	.361	.19	.079	.21	.134	.110
	Mean	-.657	.004	.659	-.731	-.056	.741
	SD	.260	.174	.187	.067	.202	.053
	LCI	.705	1.105	.819	.342	.485	.316
γ	CP	1.000	1.000	1.000	.800	.600	.800
	Bias	.143	.004	-.141	.069	-.056	-.059
	RMSE	.296	.174	.234	.097	.21	.079
	AVRB	.178	-	.176	.087	-	.073
	Mean	6.282	6.996	5.59	3.894	5.025	4.384
	SD	1.281	1.039	1.872	.801	2.625	1.703
	LCI	13.909	17.997	15.237	6.356	8.868	7.699
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	3.282	3.996	2.590	.894	2.025	1.384
	RMSE	3.523	4.129	3.196	1.200	3.315	2.194
	AVRB	1.094	1.332	.863	.298	.675	.461

Table 119 – Results of simulation study for ZA-SSLBS regression model ($\nu = 3$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.656	-2.721	-2.740	-2.266	-2.262	-2.309
	SD	.969	.925	.939	.373	.367	.370
	LCI	2.601	2.694	2.820	1.132	1.134	1.204
	CP	.800	1.000	1.000	.800	.800	.800
	Bias	-.156	-.221	-.240	.234	.238	.191
	RMSE	.981	.951	.970	.441	.438	.417
	AVRB	.062	.088	.096	.094	.095	.077
ζ_1	Median	1.475	1.608	1.586	.413	.418	.503
	SD	1.448	1.371	1.352	.689	.675	.650
	LCI	4.132	4.280	4.445	1.902	1.917	1.969
	CP	.800	.800	.800	.800	.800	.800
	Bias	.675	.808	.786	-.387	-.382	-.297
	RMSE	1.598	1.592	1.564	.790	.775	.714
	AVRB	.844	1.010	.983	.483	.477	.371
β_0	Median	-.500	-.497	-.527	-.484	-.497	-.493
	SD	.151	.210	.118	.051	.140	.105
	LCI	.696	.912	1.023	.318	.443	.516
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	0	.003	-.027	.016	.003	.007
	RMSE	.151	.210	.121	.054	.140	.105
	AVRB	.001	.005	.053	.031	.005	.014
β_1	Median	1.053	1.016	.999	.942	.970	.977
	SD	.323	.406	.147	.102	.270	.182
	LCI	1.368	1.847	1.883	.57	.803	.973
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.053	.016	-.001	-.058	-.030	-.023
	RMSE	.327	.406	.147	.117	.272	.184
	AVRB	.053	.016	.001	.058	.03	.023
ψ_0	Median	-.956	-.871	-.895	-.929	-.909	-.930
	SD	.343	.211	.264	.098	.163	.211
	LCI	1.394	1.375	1.367	.721	.675	.687
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.044	.129	.105	.071	.091	.070
	RMSE	.345	.247	.284	.121	.187	.222
	AVRB	.044	.129	.105	.071	.091	.070

Table 119 (continued).

ψ_1	Median	.685	.601	.459	.393	.435	.449
	SD	.489	.318	.454	.163	.174	.200
	LCI	2.559	2.283	2.258	.987	.987	1.012
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.185	.101	-.041	-.107	-.065	-.051
	RMSE	.522	.333	.456	.195	.185	.206
	AVRB	.370	.201	.082	.215	.131	.101
γ	Median	-.684	.002	.710	-.738	-.059	.747
	SD	.275	.196	.190	.070	.197	.053
	LCI	.705	1.105	.819	.342	.485	.316
	CP	1.000	1.000	1.000	.800	.600	.800
	Bias	.116	.002	-.090	.062	-.059	-.053
	RMSE	.298	.196	.211	.094	.205	.075
	AVRB	.145	-	.112	.078	-	.066
ν	Median	4.974	5.475	4.286	3.396	4.256	3.777
	SD	1.128	1.102	1.426	.677	1.898	1.129
	LCI	13.909	17.997	15.237	6.356	8.868	7.699
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.974	2.475	1.286	.396	1.256	.777
	RMSE	2.273	2.709	1.920	.784	2.276	1.371
	AVRB	.658	.825	.429	.132	.419	.259

Table 120 – Results of simulation study for ZA-SSLBS regression model ($\nu = 3$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mode	-2.567	-2.676	-2.654	-2.252	-2.266	-2.301
	SD	.948	.898	.973	.369	.37	.365
	LCI	2.601	2.694	2.820	1.132	1.134	1.204
	CP	.800	1.000	1.000	.800	.800	.800
	Bias	-.067	-.176	-.154	.248	.234	.199
	RMSE	.951	.915	.985	.444	.438	.416
	AVRB	.027	.07	.062	.099	.094	.08
ζ_1	Mode	1.467	1.670	1.580	.409	.421	.509
	SD	1.612	1.331	1.611	.649	.663	.648
	LCI	4.132	4.28	4.445	1.902	1.917	1.969
	CP	.800	.800	.800	.800	.800	.800
	Bias	.667	.870	.780	-.391	-.379	-.291
	RMSE	1.745	1.590	1.790	.758	.763	.710
	AVRB	.833	1.087	.975	.489	.474	.364
β_0	Mode	-.502	-.503	-.541	-.483	-.497	-.492
	SD	.153	.205	.116	.050	.137	.102
	LCI	.696	.912	1.023	.318	.443	.516
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.002	-.003	-.041	.017	.003	.008
	RMSE	.153	.205	.123	.053	.137	.102
	AVRB	.004	.007	.081	.034	.007	.015
β_1	Mode	1.032	.971	.988	.944	.970	.975
	SD	.344	.353	.173	.102	.269	.182
	LCI	1.368	1.847	1.883	.570	.803	.973
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.032	-.029	-.012	-.056	-.03	-.025
	RMSE	.345	.354	.174	.117	.270	.184
	AVRB	.032	.029	.012	.056	.030	.025
ψ_0	Mode	-.963	-.867	-.901	-.932	-.910	-.93
	SD	.324	.232	.257	.096	.167	.208
	LCI	1.394	1.375	1.367	.721	.675	.687
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.037	.133	.099	.068	.090	.070
	RMSE	.326	.267	.276	.118	.190	.219
	AVRB	.037	.133	.099	.068	.090	.070

Table 120 (continued).

ψ_1	Mode	.743	.635	.472	.388	.432	.456
	SD	.483	.344	.495	.164	.175	.199
	LCI	2.559	2.283	2.258	.987	.987	1.012
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.243	.135	-.028	-.112	-.068	-.044
	RMSE	.541	.369	.496	.199	.188	.204
	AVRB	.487	.269	.056	.225	.135	.088
γ	Mode	-.682	.006	.723	-.733	-.057	.743
	SD	.277	.228	.168	.067	.198	.053
	LCI	.705	1.105	.819	.342	.485	.316
	CP	1.000	1.000	1.000	.800	.600	.800
	Bias	.118	.006	-.077	.067	-.057	-.057
	RMSE	.301	.228	.185	.094	.206	.078
	AVRB	.147	-	.097	.083	-	.071
ν	Mode	3.113	2.938	2.717	2.809	3.066	2.901
	SD	1.056	.564	.317	.512	.669	.455
	LCI	13.909	17.997	15.237	6.356	8.868	7.699
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.113	-.062	-.283	-.191	.066	-.099
	RMSE	1.062	.567	.425	.547	.672	.466
	AVRB	.038	.021	.094	.064	.022	.033

Table 121 – Results of simulation study for ZA-SSLBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-2.579	-2.606	-2.667	-2.372	-2.426	-2.452
	SD	.388	.482	.465	.367	.283	.322
	LCI	2.646	2.667	2.72	1.100	1.254	1.219
ζ_0	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.079	-.106	-.167	.128	.074	.048
	RMSE	.396	.494	.494	.389	.292	.326
	AVRB	.032	.042	.067	.051	.030	.019
	Mean	1.023	1.02	1.139	.561	.662	.729
	SD	.533	.678	.754	.511	.328	.384
	LCI	4.343	4.369	4.536	1.757	1.949	2.005
ζ_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.223	.220	.339	-.239	-.138	-.071
	RMSE	.577	.713	.826	.564	.356	.391
	AVRB	.278	.276	.424	.299	.173	.089
	Mean	-.414	-.654	-.596	-.491	-.521	-.535
	SD	.095	.198	.114	.058	.062	.093
	LCI	.52	.741	.908	.264	.327	.404
β_0	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	.086	-.154	-.096	.009	-.021	-.035
	RMSE	.129	.251	.149	.059	.065	.099
	AVRB	.173	.309	.192	.019	.041	.069
	Mean	.886	1.392	1.198	.969	1.066	1.112
	SD	.227	.382	.117	.163	.109	.236
	LCI	1.074	1.623	1.789	.460	.617	.744
β_1	CP	1.000	.800	1.000	.800	1.000	.800
	Bias	-.114	.392	.198	-.031	.066	.112
	RMSE	.254	.547	.23	.166	.127	.261
	AVRB	.114	.392	.198	.031	.066	.112
	Mean	-1.134	-1.209	-1.124	-1.082	-1.140	-1.115
	SD	.327	.146	.330	.127	.256	.165
	LCI	1.106	1.215	1.221	.590	.548	.522
ψ_0	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-.134	-.209	-.124	-.082	-.140	-.115
	RMSE	.354	.255	.352	.152	.292	.201
	AVRB	.134	.209	.124	.082	.140	.115

Table 121 (continued).

ψ_1	Mean	.706	.904	.667	.577	.687	.704
	SD	.510	.163	.561	.186	.301	.309
	LCI	2.099	2.189	2.176	.940	.923	.860
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	.206	.404	.167	.077	.187	.204
	RMSE	.550	.435	.586	.201	.354	.370
	AVRB	.413	.808	.334	.154	.374	.408
γ	Mean	-.739	.047	.661	-.742	-.020	.798
	SD	.156	.257	.216	.072	.118	.053
	LCI	.598	1.004	.681	.270	.427	.233
	CP	1.000	1.000	.800	1.000	.600	1.000
	Bias	.061	.047	-.139	.058	-.020	-.002
	RMSE	.168	.261	.257	.092	.120	.053
	AVRB	.076	-	.174	.073	-	.003
ν	Mean	28.004	30.448	29.875	22.653	25.55	27.682
	SD	10.89	2.511	4.580	10.012	12.570	9.954
	LCI	71.164	88.582	84.243	46.028	63.167	56.973
	CP	.800	1.000	1.000	.800	.800	1.000
	Bias	-1.996	.448	-.125	-7.347	-4.450	-2.318
	RMSE	11.071	2.550	4.582	12.418	13.334	10.220
	AVRB	.067	.015	.004	.245	.148	.077

Table 122 – Results of simulation study for ZA-SSLBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.548	-2.579	-2.616	-2.373	-2.415	-2.450
	SD	.400	.475	.450	.361	.273	.333
	LCI	2.646	2.667	2.72	1.100	1.254	1.219
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.048	-.079	-.116	.127	.085	.050
	RMSE	.403	.482	.465	.383	.286	.337
	AVRB	.019	.032	.047	.051	.034	.020
ζ_1	Median	.982	.988	1.115	.561	.66	.734
	SD	.507	.684	.743	.502	.315	.407
	LCI	4.343	4.369	4.536	1.757	1.949	2.005
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.182	.188	.315	-.239	-.140	-.066
	RMSE	.539	.709	.807	.556	.345	.413
	AVRB	.228	.236	.394	.299	.175	.083
β_0	Median	-.420	-.658	-.610	-.492	-.523	-.541
	SD	.091	.194	.107	.060	.062	.091
	LCI	.520	.741	.908	.264	.327	.404
	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	.080	-.158	-.110	.008	-.023	-.041
	RMSE	.122	.250	.153	.060	.066	.100
	AVRB	.160	.316	.220	.016	.045	.083
β_1	Median	.887	1.377	1.198	.971	1.064	1.112
	SD	.230	.376	.104	.162	.107	.234
	LCI	1.074	1.623	1.789	.460	.617	.744
	CP	1.000	.800	1.000	.800	1.000	.800
	Bias	-.113	.377	.198	-.029	.064	.112
	RMSE	.256	.533	.224	.165	.124	.26
	AVRB	.113	.377	.198	.029	.064	.112
ψ_0	Median	-1.136	-1.209	-1.129	-1.095	-1.139	-1.121
	SD	.331	.141	.322	.140	.256	.164
	LCI	1.106	1.215	1.221	.59	.548	.522
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-.136	-.209	-.129	-.095	-.139	-.121
	RMSE	.358	.252	.347	.169	.291	.204
	AVRB	.136	.209	.129	.095	.139	.121

Table 122 (continued).

ψ_1	Median	.708	.894	.660	.579	.682	.703
	SD	.505	.159	.563	.185	.302	.307
	LCI	2.099	2.189	2.176	.940	.923	.860
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	.208	.394	.160	.079	.182	.203
	RMSE	.546	.425	.585	.201	.352	.368
	AVRB	.417	.788	.319	.157	.364	.406
γ	Median	-.774	.045	.689	-.746	-.030	.805
	SD	.153	.274	.221	.072	.104	.053
	LCI	.598	1.004	.681	.27	.427	.233
	CP	1.000	1.000	.800	1.000	.600	1.000
	Bias	.026	.045	-.111	.054	-.030	.005
	RMSE	.155	.278	.247	.090	.108	.053
	AVRB	.032	-	.139	.067	-	.006
ν	Median	23.506	23.730	24.405	20.142	21.342	24.031
	SD	8.593	3.138	3.872	9.247	10.701	6.623
	LCI	71.164	88.582	84.243	46.028	63.167	56.973
	CP	.800	1.000	1.000	.800	.800	1.000
	Bias	-6.494	-6.27	-5.595	-9.858	-8.658	-5.969
	RMSE	10.771	7.011	6.804	13.516	13.765	8.916
	AVRB	.216	.209	.186	.329	.289	.199

Table 123 – Results of simulation study for ZA-SSLBS regression model ($\nu = 30$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-2.571	-2.575	-2.502	-2.374	-2.394	-2.446
	SD	.553	.482	.428	.366	.264	.339
	LCI	2.646	2.667	2.720	1.100	1.254	1.219
ζ_0	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.071	-.075	-.002	.126	.106	.054
	RMSE	.558	.487	.428	.387	.285	.343
	AVRB	.028	.030	.001	.050	.042	.022
	Mode	1.039	.868	1.045	.543	.639	.764
	SD	.530	.701	.775	.500	.292	.450
	LCI	4.343	4.369	4.536	1.757	1.949	2.005
ζ_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.239	.068	.245	-.257	-.161	-.036
	RMSE	.581	.704	.813	.562	.333	.451
	AVRB	.299	.085	.307	.321	.201	.045
	Mode	-.418	-.660	-.623	-.490	-.522	-.538
	SD	.091	.195	.106	.058	.061	.092
	LCI	.52	.741	.908	.264	.327	.404
β_0	CP	1.000	.800	1.000	1.000	1.000	1.000
	Bias	.082	-.160	-.123	.010	-.022	-.038
	RMSE	.122	.252	.163	.059	.065	.100
	AVRB	.164	.319	.247	.019	.044	.077
	Mode	.880	1.349	1.211	.971	1.067	1.113
	SD	.226	.376	.083	.162	.105	.231
	LCI	1.074	1.623	1.789	.460	.617	.744
β_1	CP	1.000	.800	1.000	.800	1.000	.800
	Bias	-.120	.349	.211	-.029	.067	.113
	RMSE	.256	.513	.227	.165	.125	.257
	AVRB	.12	.349	.211	.029	.067	.113
	Mode	-1.147	-1.213	-1.136	-1.095	-1.140	-1.118
	SD	.342	.136	.319	.143	.254	.165
	LCI	1.106	1.215	1.221	.59	.548	.522
ψ_0	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	-.147	-.213	-.136	-.095	-.140	-.118
	RMSE	.372	.253	.347	.171	.290	.203
	AVRB	.147	.213	.136	.095	.14	.118

Table 123 (continued).

	Mode	.714	.864	.65	.577	.681	.700
	SD	.498	.153	.553	.186	.300	.310
	LCI	2.099	2.189	2.176	.940	.923	.860
ψ_1	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	.214	.364	.150	.077	.181	.200
	RMSE	.542	.395	.573	.202	.351	.369
	AVRB	.428	.727	.301	.154	.362	.401
	Mode	-.776	.050	.692	-.744	-.023	.800
	SD	.128	.294	.207	.073	.112	.052
	LCI	.598	1.004	.681	.27	.427	.233
γ	CP	1.000	1.000	.800	1.000	.600	1.000
	Bias	.024	.050	-.108	.056	-.023	$<$ .001
	RMSE	.131	.298	.234	.092	.114	.052
	AVRB	.030	-	.135	.070	-	.001
	Mode	13.887	9.567	9.767	16.901	13.905	17.840
	SD	5.902	3.430	5.076	11.833	10.373	7.053
	LCI	71.164	88.582	84.243	46.028	63.167	56.973
ν	CP	.800	1.000	1.000	.800	.800	1.000
	Bias	-16.113	-20.433	-20.233	-13.099	-16.095	-12.160
	RMSE	17.159	20.719	20.860	17.652	19.148	14.057
	AVRB	.537	.681	.674	.437	.536	.405

ZA-SCNBS regression model

Table 124 – Results of simulation study for ZA-SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-3.019	-3.125	-3.052	-2.452	-2.493	-2.403
	SD	.653	.686	.733	.347	.212	.312
	LCI	3.237	3.409	3.548	1.206	1.163	1.002
ζ_0	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.519	-.625	-.552	.048	.007	.097
	RMSE	.834	.928	.918	.351	.213	.327
	AVRB	.207	.25	.221	.019	.003	.039
	Mean	.463	.696	.496	.740	.866	.686
	SD	2.024	1.957	2.221	.436	.259	.313
	LCI	6.204	5.958	6.143	1.893	1.806	1.638
ζ_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.337	-.104	-.304	-.060	.066	-.114
	RMSE	2.052	1.96	2.242	.440	.267	.333
	AVRB	.421	.130	.381	.075	.083	.142
	Mean	-.566	-.659	-.307	-.421	-.304	-.475
	SD	.2470	.434	.424	.076	.179	.297
	LCI	1.256	1.773	1.999	.493	.761	.823
β_0	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.066	-.159	.193	.079	.196	.025
	RMSE	.255	.462	.466	.110	.265	.298
	AVRB	.133	.319	.386	.158	.392	.049
	Mean	1.001	1.559	.934	.918	.688	.955
	SD	.4	.845	1.017	.108	.285	.516
	LCI	2.272	3.298	3.693	.902	1.264	1.417
β_1	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	.001	.559	-.066	-.082	-.312	-.045
	RMSE	.4	1.013	1.019	.135	.422	.518
	AVRB	.001	.559	.066	.082	.312	.045
	Mean	.932	.670	.837	.867	.419	.047
	SD	.169	.476	.29	.374	.7	1.016
	LCI	2.199	2.225	2.102	1.346	2.313	1.258
ψ_0	CP	$< .001$	.200	.200	.200	.600	.600
	Bias	1.932	1.670	1.837	1.867	1.419	1.047
	RMSE	1.939	1.737	1.859	1.905	1.582	1.459

Table 124 (continued).

	AVRB	1.932	1.67	1.837	1.867	1.419	1.047
ψ_1	Mean	.399	.580	.503	.370	.377	.376
	SD	.483	.294	.399	.196	.215	.369
	LCI	2.417	2.454	2.425	.908	.984	.935
	CP	1.000	1.000	1.000	1.000	.800	.600
	Bias	-.101	.080	.003	-.130	-.123	-.124
	RMSE	.493	.305	.399	.235	.248	.389
	AVRB	.201	.16	.006	.260	.247	.248
γ	Mean	-.575	.035	.487	-.756	-.038	.808
	SD	.182	.193	.192	.087	.106	.080
	LCI	.840	1.161	.989	.359	.468	.285
	CP	1.000	1.000	.800	1.000	.800	1.000
	Bias	.225	.035	-.313	.044	-.038	.008
	RMSE	.289	.196	.367	.097	.113	.080
	AVRB	.282	-	.391	.055	-	.009
ν_1	Mean	.497	.552	.535	.558	.687	.744
	SD	.040	.056	.043	.127	.100	.211
	LCI	.805	.800	.790	.689	.587	.401
	CP	.400	.800	.600	.400	1.000	.800
	Bias	-.403	-.348	-.365	-.342	-.213	-.156
	RMSE	.405	.352	.367	.365	.236	.262
	AVRB	.447	.387	.405	.381	.237	.173
ν_2	Mean	.606	.525	.602	.634	.463	.359
	SD	.045	.155	.088	.194	.205	.291
	LCI	.815	.799	.768	.631	.720	.389
	CP	$< .001$	.200	.200	.200	.600	.600
	Bias	.506	.425	.502	.534	.363	.259
	RMSE	.508	.452	.509	.568	.417	.390
	AVRB	5.059	4.248	5.017	5.342	3.629	2.592

Table 125 – Results of simulation study for ZA-SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-2.965	-3.071	-2.970	-2.448	-2.486	-2.407
	SD	.640	.651	.741	.344	.199	.315
	LCI	3.237	3.409	3.548	1.206	1.163	1.002
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.465	-.571	-.470	.052	.014	.093
	RMSE	.790	.866	.878	.348	.199	.329
	AVRB	.186	.228	.188	.021	.006	.037
ζ_1	Median	.491	.715	.476	.736	.862	.697
	SD	1.929	1.858	2.177	.446	.259	.306
	LCI	6.204	5.958	6.143	1.893	1.806	1.638
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.309	-.085	-.324	-.064	.062	-.103
	RMSE	1.954	1.86	2.201	.450	.267	.323
	AVRB	.387	.106	.405	.080	.077	.128
β_0	Median	-.567	-.682	-.34	-.419	-.309	-.478
	SD	.25	.412	.424	.077	.183	.295
	LCI	1.256	1.773	1.999	.493	.761	.823
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.067	-.182	.160	.081	.191	.022
	RMSE	.259	.451	.453	.112	.264	.296
	AVRB	.134	.365	.32	.162	.381	.044
β_1	Median	1.000	1.563	.904	.922	.695	.958
	SD	.409	.816	.972	.108	.285	.523
	LCI	2.272	3.298	3.693	.902	1.264	1.417
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	$< .001$	.563	-.096	-.078	-.305	-.042
	RMSE	.409	.992	.977	.133	.417	.525
	AVRB	$< .001$	.563	.096	.078	.305	.042
ψ_0	Median	.995	.721	.906	.932	.517	.062
	SD	.184	.479	.240	.367	.807	1.045
	LCI	2.199	2.225	2.102	1.346	2.313	1.258
	CP	$< .001$	.200	.200	.200	.600	.600
	Bias	1.995	1.721	1.906	1.932	1.517	1.062
	RMSE	2.004	1.787	1.921	1.966	1.718	1.49
	AVRB	1.995	1.721	1.906	1.932	1.517	1.062

Table 125 (continued).

ψ_1	Median	.399	.583	.498	.371	.372	.380
	SD	.468	.282	.387	.203	.211	.372
	LCI	2.417	2.454	2.425	.908	.984	.935
	CP	1.000	1.000	1.000	1.000	.800	.6
	Bias	-.101	.083	-.002	-.129	-.128	-.120
	RMSE	.479	.293	.387	.240	.247	.390
	AVRB	.203	.165	.004	.258	.257	.239
γ	Median	-.602	.038	.528	-.764	-.039	.815
	SD	.202	.188	.192	.096	.094	.082
	LCI	.84	1.161	.989	.359	.468	.285
	CP	1.000	1.000	.800	1.000	.800	1.000
	Bias	.198	.038	-.272	.036	-.039	.015
	RMSE	.283	.191	.333	.102	.101	.083
	AVRB	.247	-	.34	.045	-	.019
ν_1	Median	.499	.574	.548	.581	.702	.757
	SD	.055	.062	.061	.133	.104	.203
	LCI	.805	.800	.79	.689	.587	.401
	CP	.400	.800	.600	.400	1.000	.800
	Bias	-.401	-.326	-.352	-.319	-.198	-.143
	RMSE	.405	.332	.357	.346	.223	.248
	AVRB	.446	.362	.391	.355	.22	.159
ν_2	Median	.616	.508	.604	.6420	.455	.366
	SD	.057	.192	.103	.219	.238	.309
	LCI	.815	.799	.768	.631	.72	.389
	CP	$< .001$	.200	.200	.200	.600	.600
	Bias	.516	.408	.504	.542	.355	.266
	RMSE	.519	.451	.514	.585	.427	.408
	AVRB	5.159	4.081	5.039	5.423	3.547	2.658

Table 126 – Results of simulation study for ZA-SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mode	-2.884	-3.020	-2.851	-2.448	-2.469	-2.413
	SD	.720	.624	.795	.340	.183	.318
	LCI	3.237	3.409	3.548	1.206	1.163	1.002
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.384	-.520	-.351	.052	.031	.087
	RMSE	.816	.813	.869	.344	.185	.329
	AVRB	.154	.208	.140	.021	.012	.035
ζ_1	Mode	.352	.930	.498	.730	.848	.723
	SD	1.955	1.361	2.385	.458	.253	.306
	LCI	6.204	5.958	6.143	1.893	1.806	1.638
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.448	.130	-.302	-.070	.048	-.077
	RMSE	2.005	1.368	2.404	.464	.258	.316
	AVRB	.560	.162	.378	.088	.060	.096
β_0	Mode	-.567	-.721	-.384	-.420	-.315	-.484
	SD	.253	.378	.435	.078	.180	.302
	LCI	1.256	1.773	1.999	.493	.761	.823
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.067	-.221	.116	.080	.185	.016
	RMSE	.262	.438	.450	.112	.258	.302
	AVRB	.134	.443	.232	.160	.370	.033
β_1	Mode	1.014	1.680	.766	.922	.700	.969
	SD	.418	.659	.863	.107	.290	.545
	LCI	2.272	3.298	3.693	.902	1.264	1.417
	CP	1.000	1.000	1.000	1.000	.800	.800
	Bias	.014	.680	-.234	-.078	-.300	-.031
	RMSE	.418	.947	.895	.132	.417	.546
	AVRB	.014	.680	.234	.078	.300	.031
ψ_0	Mode	1.047	.892	.994	1.105	.563	.099
	SD	.201	.282	.188	.078	.961	1.129
	LCI	2.199	2.225	2.102	1.346	2.313	1.258
	CP	$< .001$	.200	.200	.200	.600	.600
	Bias	2.047	1.892	1.994	2.105	1.563	1.099
	RMSE	2.057	1.913	2.002	2.106	1.835	1.576
	AVRB	2.047	1.892	1.994	2.105	1.563	1.099

Table 126 (continued).

	Mode	.461	.594	.488	.371	.368	.383
	SD	.563	.281	.39	.215	.205	.375
	LCI	2.417	2.454	2.425	.908	.984	.935
ψ_1	CP	1.000	1.000	1.000	1.000	.800	.600
	Bias	-.039	.094	-.012	-.129	-.132	-.117
	RMSE	.564	.297	.390	.251	.243	.393
	AVRB	.077	.188	.024	.258	.264	.234
	Mode	-.608	.061	.551	-.76	-.041	.810
	SD	.209	.147	.181	.090	.101	.078
	LCI	.84	1.161	.989	.359	.468	.285
γ	CP	1.000	1.000	.800	1.000	.800	1.000
	Bias	.192	.061	-.249	.040	-.041	.010
	RMSE	.284	.159	.308	.099	.109	.079
	AVRB	.24	-	.311	.05	-	.013
	Mode	.498	.587	.560	.591	.707	.752
	SD	.070	.058	.070	.131	.095	.207
	LCI	.805	.800	.790	.689	.587	.401
ν_1	CP	.400	.800	.600	.400	1.000	.800
	Bias	-.402	-.313	-.340	-.309	-.193	-.148
	RMSE	.408	.318	.347	.336	.215	.255
	AVRB	.446	.348	.378	.343	.215	.164
	Mode	.636	.501	.563	.65	.467	.331
	SD	.075	.206	.184	.224	.242	.284
	LCI	.815	.799	.768	.631	.72	.389
ν_2	CP	$< .001$	.200	.200	.200	.600	.600
	Bias	.536	.401	.463	.550	.367	.231
	RMSE	.541	.451	.499	.594	.440	.366
	AVRB	5.357	4.008	4.633	5.499	3.673	2.311

Table 127 – Results of simulation study for ZA-SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mean	-3.119	-3.085	-3.047	-2.434	-2.52	-2.518
	SD	.652	.665	.661	.383	.241	.252
	LCI	3.612	3.366	3.465	1.152	1.227	1.173
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.619	-.585	-.547	.066	-.020	-.018
	RMSE	.899	.886	.858	.389	.242	.253
	AVRB	.248	.234	.219	.027	.008	.007
ζ_1	Mean	.693	.676	.563	.729	.883	.860
	SD	1.802	1.712	1.877	.429	.307	.300
	LCI	6.181	5.586	5.902	1.787	1.951	1.863
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.107	-.124	-.237	-.071	.083	.060
	RMSE	1.805	1.716	1.892	.434	.318	.306
	AVRB	.134	.155	.296	.088	.104	.075
β_0	Mean	-.547	-.772	-.698	-.424	-.421	-.393
	SD	.176	.233	.185	.118	.173	.074
	LCI	.863	.934	1.088	.443	.572	.635
	CP	1.000	.800	1.000	1.000	.800	1.000
	Bias	-.047	-.272	-.198	.076	.079	.107
	RMSE	.182	.358	.271	.140	.190	.130
	AVRB	.095	.543	.396	.152	.159	.214
β_1	Mean	.982	1.506	1.300	.899	.887	.777
	SD	.239	.587	.441	.151	.291	.175
	LCI	1.633	1.857	2.183	.655	.919	1.018
	CP	1.000	.8 00	1.000	1.000	.800	1.000
	Bias	-.018	.506	.300	-.101	-.113	-.223
	RMSE	.240	.774	.533	.182	.312	.283
	AVRB	.018	.506	.3	.101	.113	.223
ψ_0	Mean	-1.219	-1.332	-1.411	-.714	-.965	-.820
	SD	.266	.379	.272	.220	.180	.105
	LCI	1.806	2.123	2.159	.661	.754	.658
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.219	-.332	-.411	.286	.035	.180
	RMSE	.345	.504	.493	.361	.183	.208
	AVRB	.219	.332	.411	.286	.035	.180

Table 127 (continued).

ψ_1	Mean	.755	.896	.926	-.066	.391	.155
	SD	.718	.771	.78	.41	.261	.194
	LCI	2.477	2.447	2.712	1.181	1.163	1.073
	CP	1.000	.800	1.000	.400	1.000	.800
	Bias	.255	.396	.426	-.566	-.109	-.345
	RMSE	.762	.867	.889	.699	.283	.396
	AVRB	.509	.792	.853	1.131	.219	.690
γ	Mean	-.382	-.022	.284	-.758	.041	.75
	SD	.347	.329	.367	.022	.101	.094
	LCI	.971	1.054	.964	.373	.634	.326
	CP	.600	1.000	.400	1.000	1.000	.800
	Bias	.418	-.022	-.516	.042	.041	-.050
	RMSE	.544	.330	.634	.047	.109	.106
	AVRB	.523	-	.645	.052	-	.063
ν_1	Mean	.386	.460	.407	.115	.129	.109
	SD	.080	.083	.038	.035	.062	.040
	LCI	.722	.738	.698	.121	.173	.121
	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.286	.36	.307	.015	.029	.009
	RMSE	.297	.369	.309	.038	.069	.040
	AVRB	2.856	3.595	3.065	.155	.293	.086
ν_2	Mean	.372	.429	.350	.110	.110	.105
	SD	.190	.186	.159	.020	.022	.021
	LCI	.635	.725	.608	.101	.122	.098
	CP	.400	.400	.400	1.000	1.000	1.000
	Bias	.272	.329	.250	.010	.010	.005
	RMSE	.332	.378	.296	.022	.025	.022
	AVRB	2.719	3.292	2.497	.098	.100	.046

Table 128 – Results of simulation study for ZA-SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Median	-3.041	-3.029	-2.970	-2.418	-2.509	-2.507
	SD	.640	.648	.663	.384	.235	.244
	LCI	3.612	3.366	3.465	1.152	1.227	1.173
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.541	-.529	-.47	.082	-.009	-.007
	RMSE	.837	.837	.812	.393	.235	.244
	AVRB	.216	.212	.188	.033	.004	.003
ζ_1	Median	.692	.658	.535	.719	.877	.862
	SD	1.740	1.685	1.821	.429	.293	.293
	LCI	6.181	5.586	5.902	1.787	1.951	1.863
	CP	1.000	1	1.000	1.000	1.000	1.000
	Bias	-.108	-.142	-.265	-.081	.077	.062
	RMSE	1.744	1.690	1.840	.437	.303	.299
	AVRB	.135	.177	.331	.101	.097	.078
β_0	Median	-.562	-.782	-.713	-.421	-.430	-.402
	SD	.163	.232	.192	.126	.172	.074
	LCI	.863	.934	1.088	.443	.572	.635
	CP	1.000	.800	1.000	1.000	.800	1.000
	Bias	-.062	-.282	-.213	.079	.070	.098
	RMSE	.174	.365	.287	.149	.186	.123
	AVRB	.125	.563	.425	.158	.140	.196
β_1	Median	.993	1.499	1.290	.894	.889	.786
	SD	.226	.581	.469	.162	.281	.189
	LCI	1.633	1.857	2.183	.655	.919	1.018
	CP	1.000	.800	1.000	1.000	.800	1.000
	Bias	-.007	.499	.290	-.106	-.111	-.214
	RMSE	.226	.766	.551	.194	.302	.286
	AVRB	.007	.499	.290	.106	.111	.214
ψ_0	Median	-1.195	-1.290	-1.380	-.703	-.963	-.820
	SD	.268	.375	.278	.239	.181	.111
	LCI	1.806	2.123	2.159	.661	.754	.658
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.195	-.290	-.380	.297	.037	.180
	RMSE	.331	.474	.470	.381	.184	.212
	AVRB	.195	.290	.380	.297	.037	.180

Table 128 (continued).

ψ_1	Median	.747	.886	.917	-.089	.394	.158
	SD	.706	.757	.792	.459	.257	.204
	LCI	2.477	2.447	2.712	1.181	1.163	1.073
	CP	1.000	.800	1.000	.4	1.000	.800
	Bias	.247	.386	.417	-.589	-.106	-.342
	RMSE	.748	.850	.895	.747	.277	.398
	AVRB	.495	.771	.834	1.179	.211	.683
γ	Median	-.428	-.023	.292	-.772	.042	.760
	SD	.375	.341	.377	.021	.094	.096
	LCI	.971	1.054	.964	.373	.634	.326
	CP	.600	1.000	.400	1.000	1.000	.800
	Bias	.372	-.023	-.508	.028	.042	-.040
	RMSE	.528	.341	.633	.035	.102	.103
	AVRB	.465	-	.635	.034	-	.049
ν_1	Median	.360	.460	.394	.112	.122	.105
	SD	.106	.094	.035	.034	.058	.039
	LCI	.722	.738	.698	.121	.173	.121
	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.260	.360	.294	.012	.022	.005
	RMSE	.280	.372	.296	.036	.062	.039
	AVRB	2.598	3.596	2.939	.122	.220	.046
ν_2	Median	.342	.403	.322	.108	.107	.102
	SD	.199	.207	.161	.019	.023	.021
	LCI	.635	.725	.608	.101	.122	.098
	CP	.400	.400	.400	1.000	1.000	1.000
	Bias	.242	.303	.222	.008	.007	.002
	RMSE	.313	.367	.274	.021	.024	.021
	AVRB	2.419	3.031	2.223	.081	.074	.023

Table 129 – Results of simulation study for ZA-SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 100$			$n = 500$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
ζ_0	Mode	-2.913	-3.004	-2.896	-2.413	-2.498	-2.492
	SD	.669	.733	.630	.374	.230	.236
	LCI	3.612	3.366	3.465	1.152	1.227	1.173
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.413	-.504	-.396	.087	.002	.008
	RMSE	.786	.889	.744	.384	.23	.236
	AVRB	.165	.201	.158	.035	.001	.003
ζ_1	Mode	.595	.551	.439	.724	.877	.863
	SD	1.656	1.569	1.881	.428	.248	.281
	LCI	6.181	5.586	5.902	1.787	1.951	1.863
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.205	-.249	-.361	-.076	.077	.063
	RMSE	1.668	1.589	1.915	.435	.26	.288
	AVRB	.256	.311	.451	.095	.097	.079
β_0	Mode	-.576	-.790	-.725	-.423	-.431	-.402
	SD	.153	.231	.202	.123	.169	.076
	LCI	.863	.934	1.088	.443	.572	.635
	CP	1.000	.800	1.000	1.000	.800	1.000
	Bias	-.076	-.290	-.225	.077	.069	.098
	RMSE	.171	.370	.302	.145	.183	.124
	AVRB	.151	.579	.451	.153	.139	.196
β_1	Mode	1.012	1.506	1.314	.888	.898	.792
	SD	.207	.572	.543	.168	.267	.196
	LCI	1.633	1.857	2.183	.655	.919	1.018
	CP	1.000	.800	1.000	1.000	.800	1.000
	Bias	.012	.506	.314	-.112	-.102	-.208
	RMSE	.208	.764	.627	.201	.286	.286
	AVRB	.012	.506	.314	.112	.102	.208
ψ_0	Mode	-1.173	-1.235	-1.341	-.697	-.963	-.819
	SD	.276	.392	.312	.245	.181	.113
	LCI	1.806	2.123	2.159	.661	.754	.658
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.173	-.235	-.341	.303	.037	.181
	RMSE	.325	.457	.462	.390	.185	.213
	AVRB	.173	.235	.341	.303	.037	.181

Table 129 (continued).

	Mode	.675	.892	.902	-.099	.409	.169
	SD	.644	.735	.817	.484	.243	.209
	LCI	2.477	2.447	2.712	1.181	1.163	1.073
ψ_1	CP	1.000	.800	1.000	.400	1.000	.800
	Bias	.175	.392	.402	-.599	-.091	-.331
	RMSE	.667	.833	.911	.770	.259	.391
	AVRB	.35	.784	.804	1.198	.182	.662
	Mode	-.462	-.022	.296	-.768	.043	.756
	SD	.398	.352	.375	.020	.083	.094
	LCI	.971	1.054	.964	.373	.634	.326
γ	CP	.600	1.000	.400	1.000	1.000	.800
	Bias	.338	-.022	-.504	.032	.043	-.044
	RMSE	.522	.353	.628	.038	.094	.104
	AVRB	.423	-	.629	.04	-	.056
	Mode	.353	.466	.382	.115	.128	.108
	SD	.104	.102	.027	.035	.059	.038
	LCI	.722	.738	.698	.121	.173	.121
ν_1	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.253	.366	.282	.015	.028	.008
	RMSE	.273	.380	.283	.038	.065	.039
	AVRB	2.527	3.655	2.821	.151	.283	.084
	Mode	.335	.401	.316	.110	.110	.104
	SD	.182	.216	.145	.021	.022	.022
	LCI	.635	.725	.608	.101	.122	.098
ν_2	CP	.400	.400	.400	1.000	1.000	1.000
	Bias	.235	.301	.216	.010	.010	.004
	RMSE	.297	.371	.26	.023	.024	.023
	AVRB	2.350	3.013	2.155	.100	.095	.036

D.2 Behavior of the residuals

Simulated observations from ZA-SGtBS1 regression model

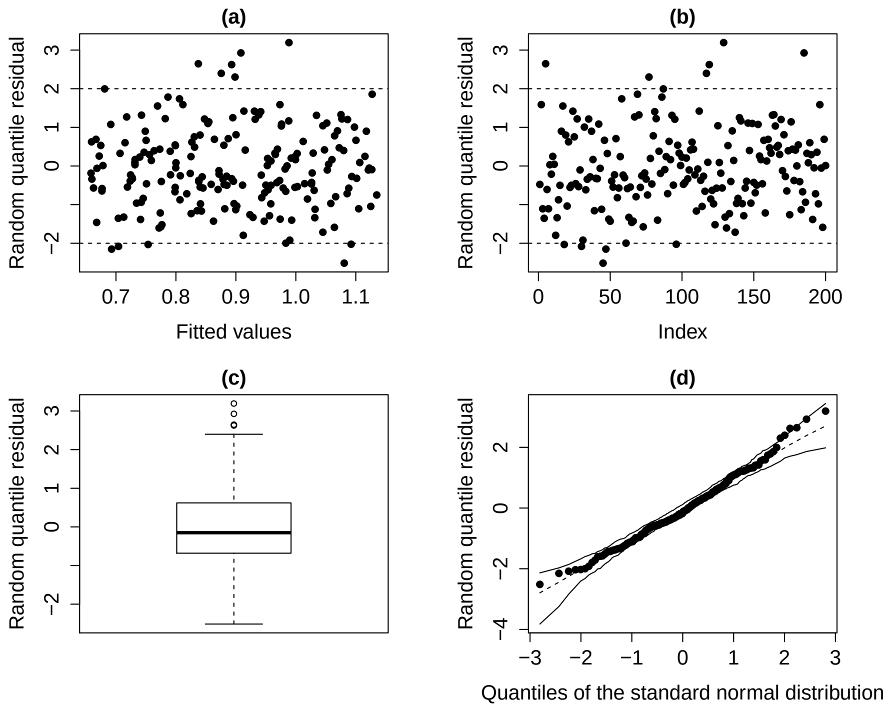

Figure 126 – Residual plots for the ZA-SGtBS1 regression model.

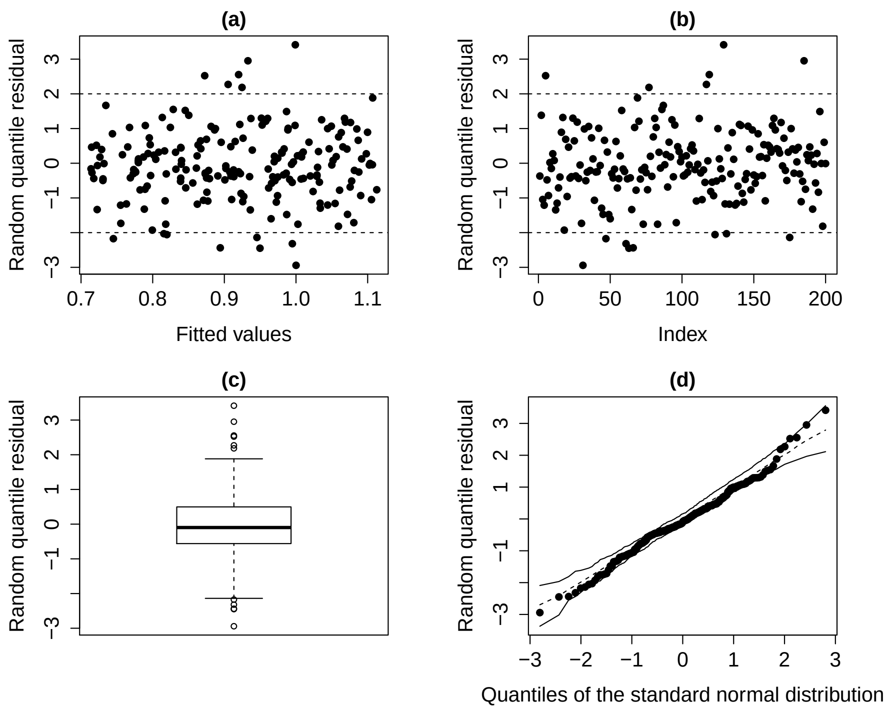

Figure 127 – Residual plots for the ZA-SGtBS2 regression model.

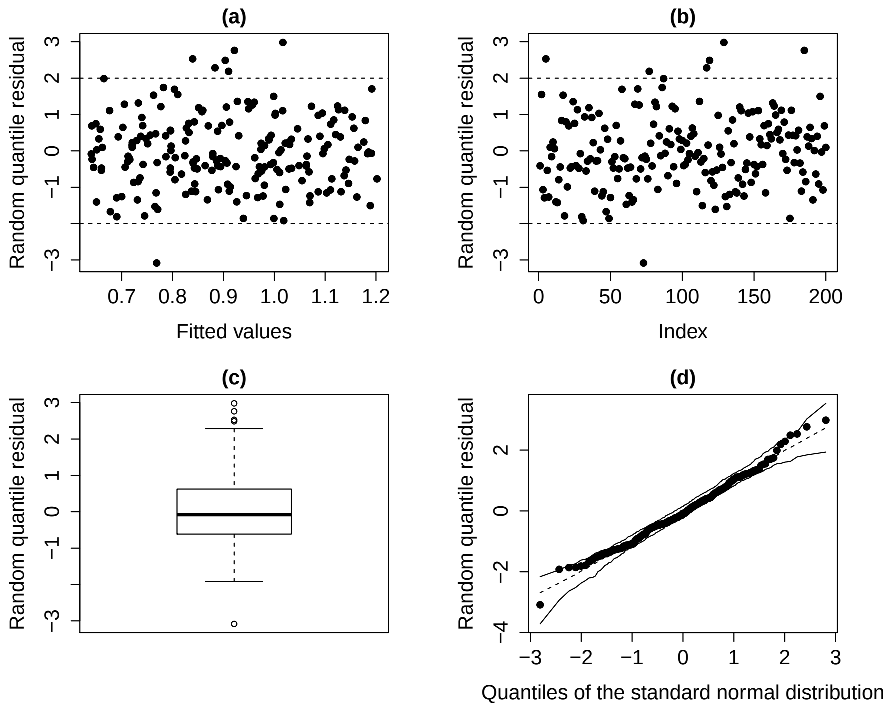

Figure 128 – Residual plots for the ZA-StBS regression model.

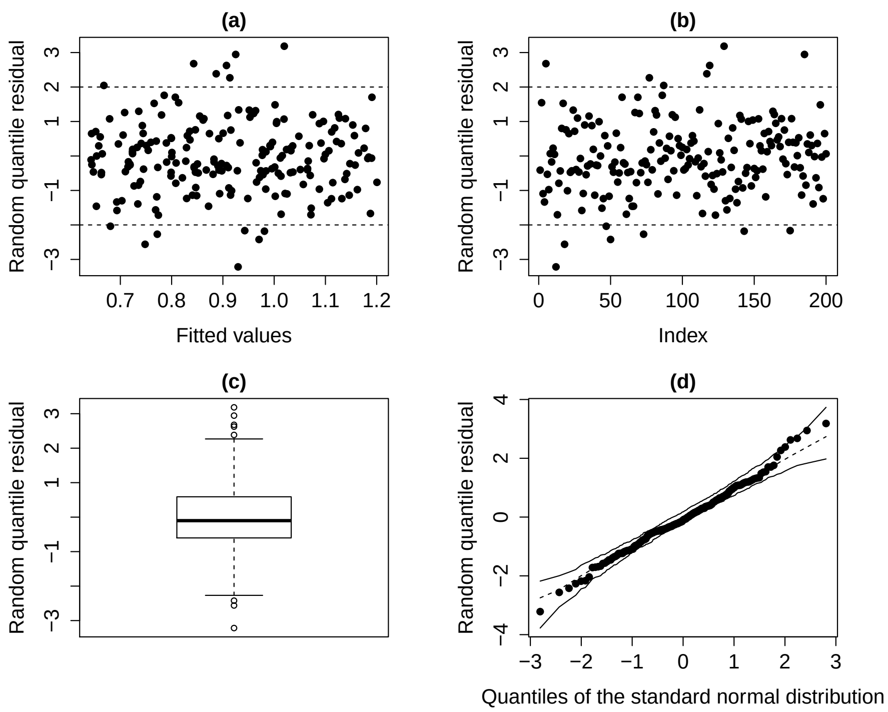

Figure 129 – Residual plots for the ZA-SSLBS1 regression model.

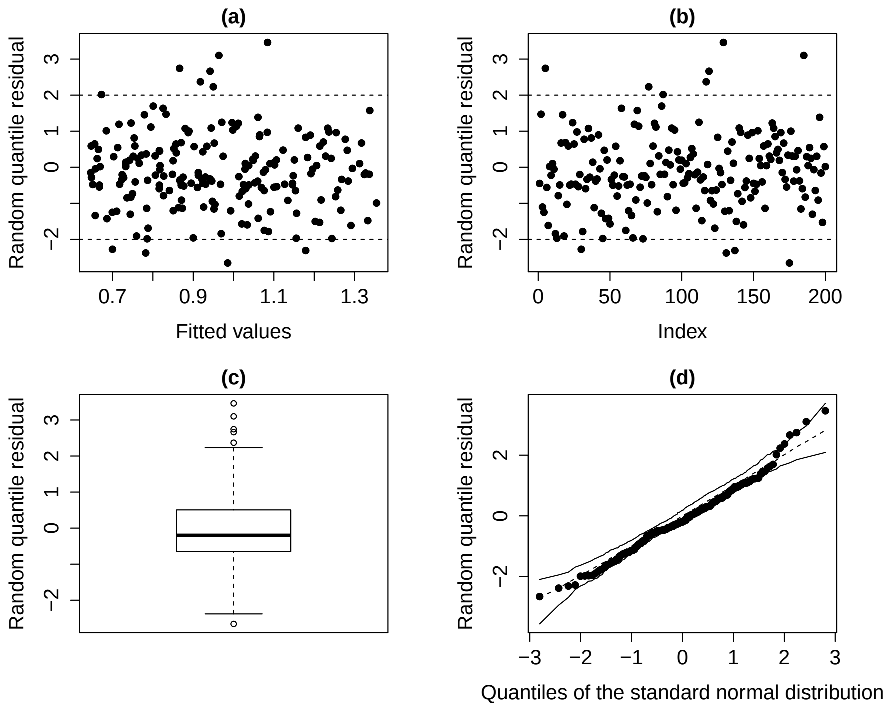

Figure 130 – Residual plots for the ZA-SSLBS2 regression model.

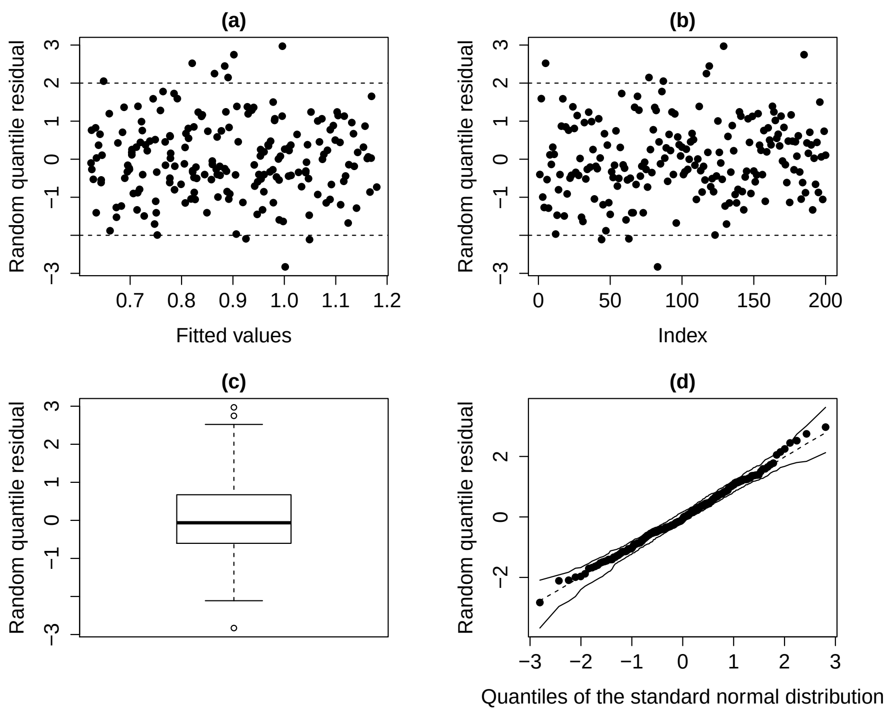

Figure 131 – Residual plots for the ZA-SCNBS regression model.

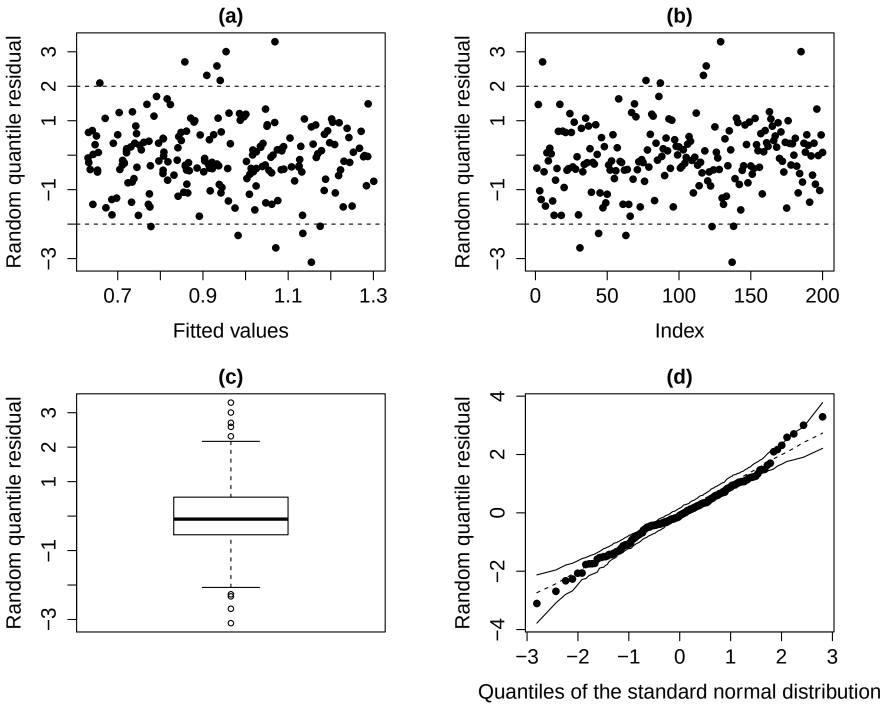

Figure 132 – Residual plots for the ZA-SNBS regression model.

Simulated observations from ZA-SGtBS2 regression model

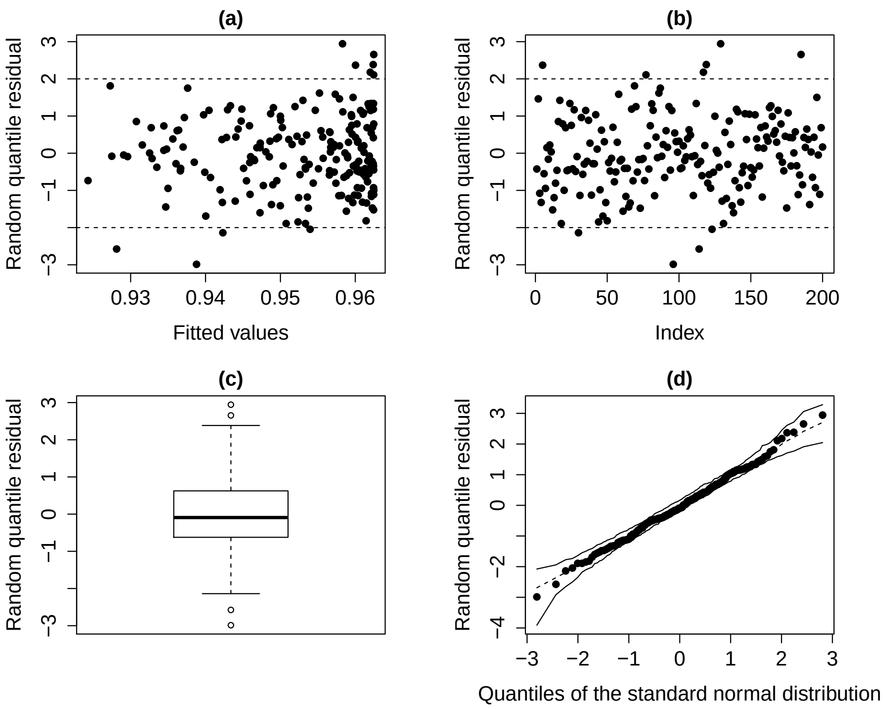

Figure 133 – Residual plots for the ZA-SGtBS2 regression model.

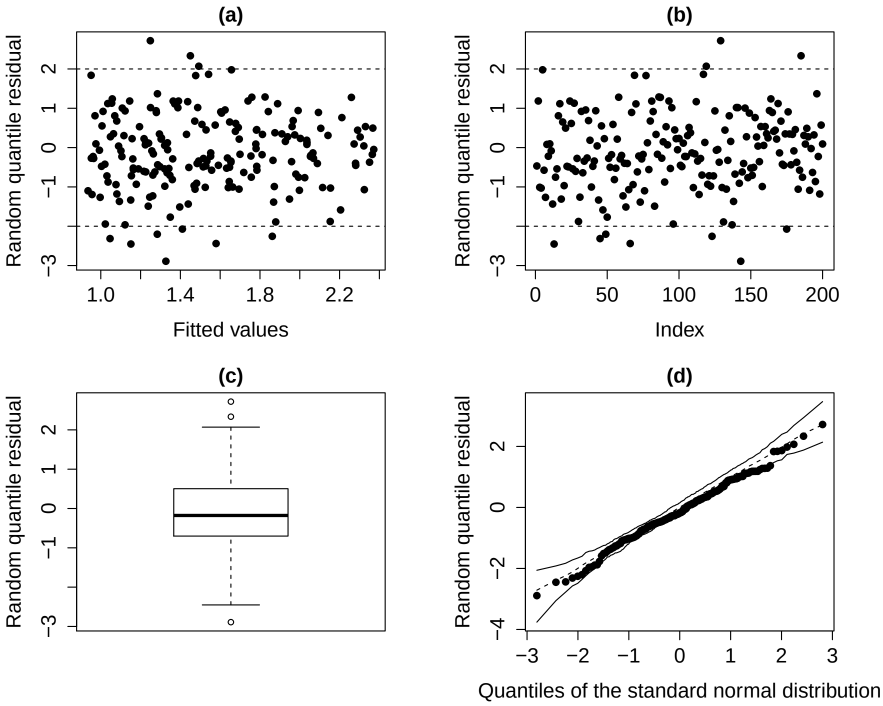

Figure 134 – Residual plots for the ZA-SGtBS1 regression model.

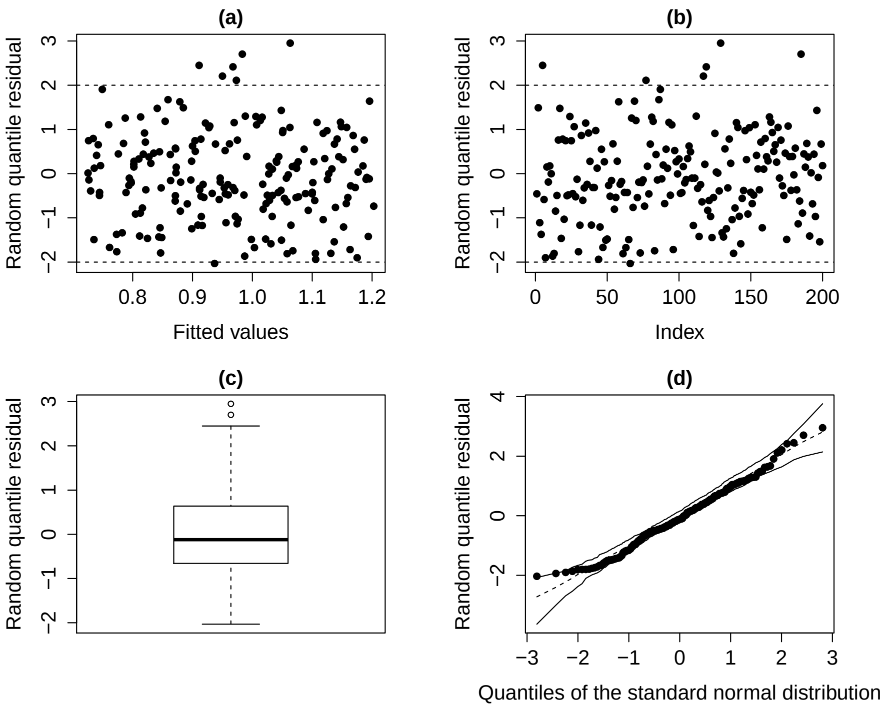

Figure 135 – Residual plots for the ZA-StBS regression model.

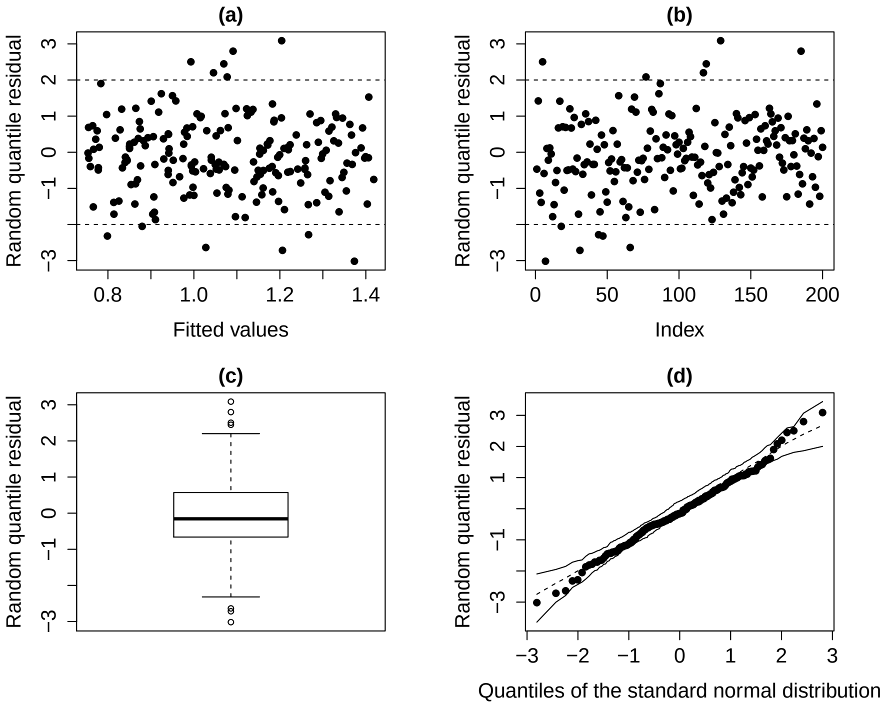

Figure 136 – Residual plots for the ZA-SSLBS1 regression model.

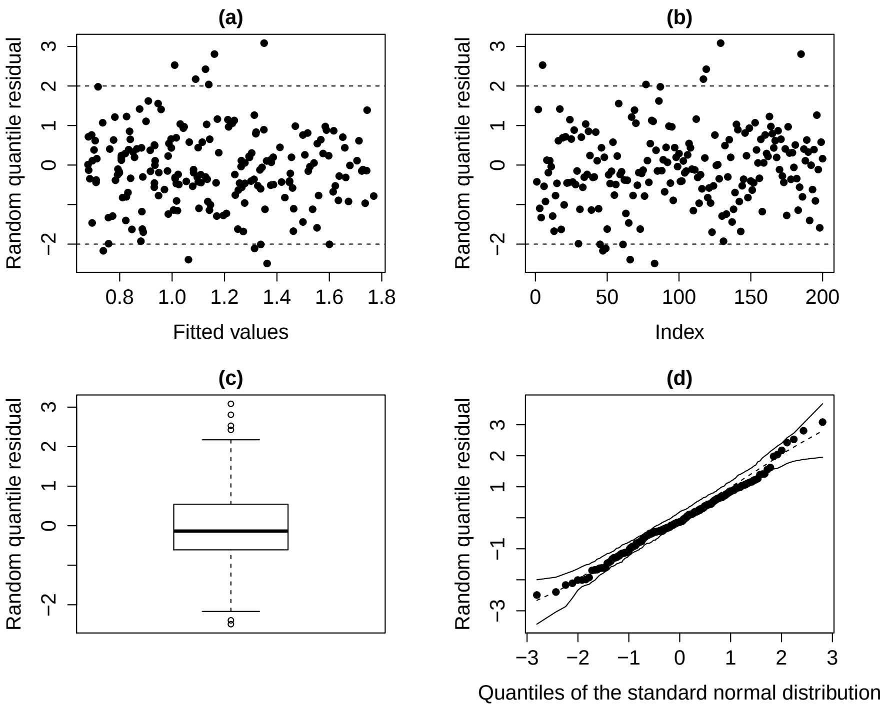

Figure 137 – Residual plots for the ZA-SSLBS2 regression model.

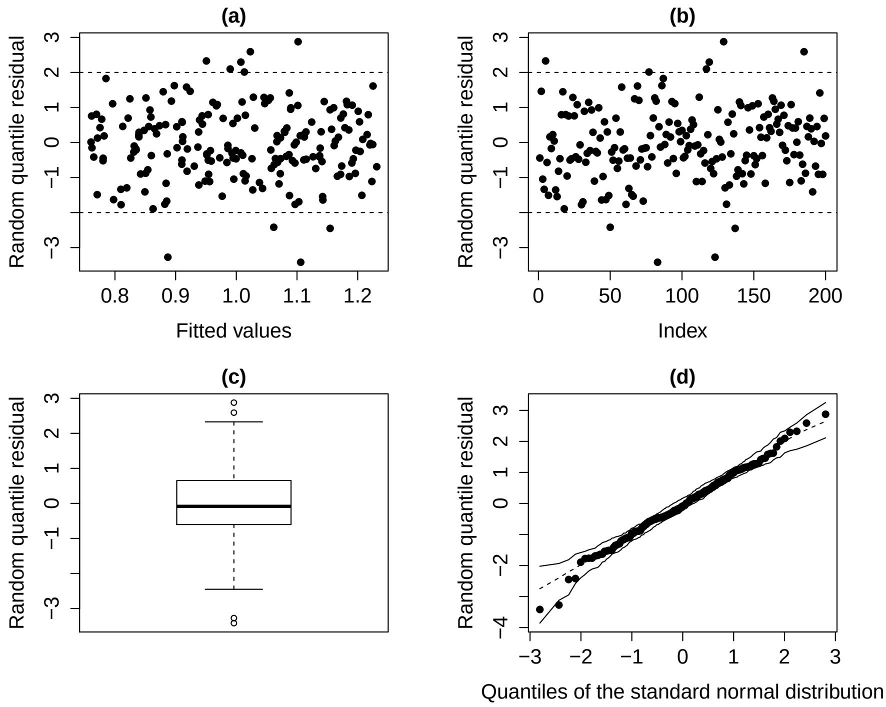

Figure 138 – Residual plots for the ZA-SCNBS regression model.

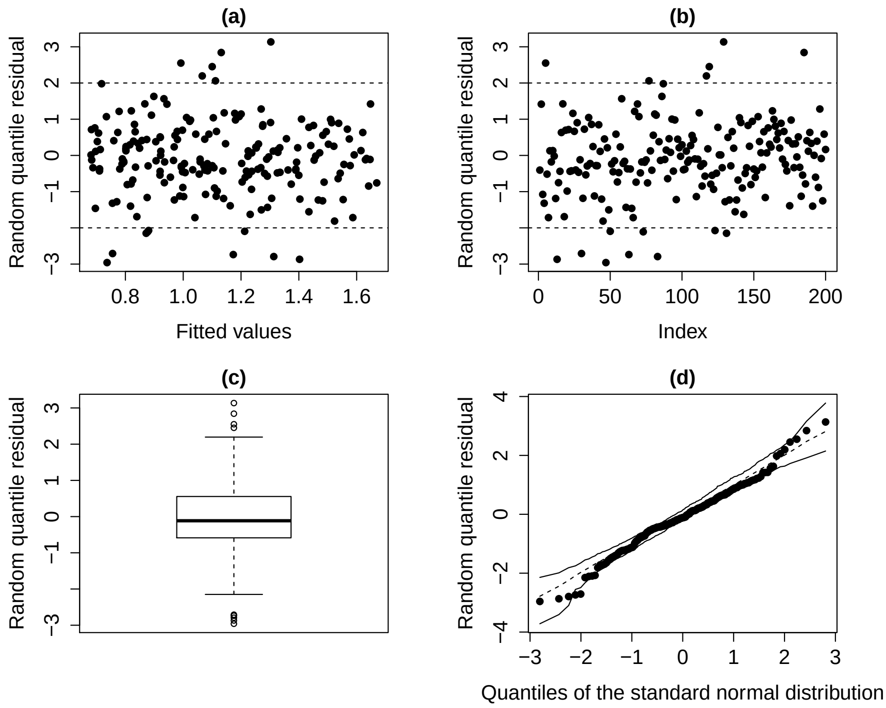

Figure 139 – Residual plots for the ZA-SNBS regression model.

Simulated observations from ZA-StBS regression model

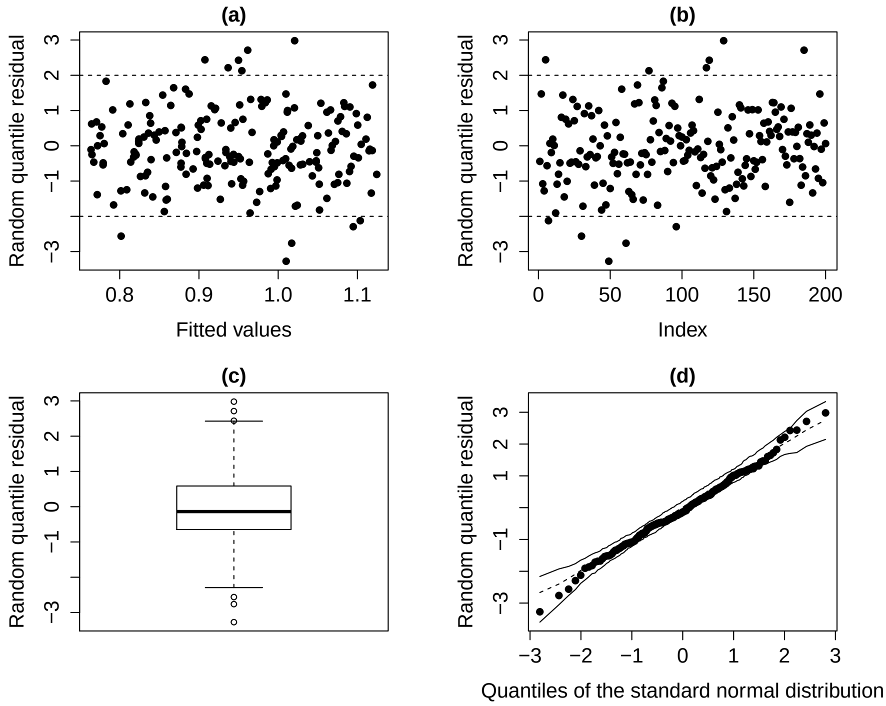

Figure 140 – Residual plots for the ZA-StBS regression model.

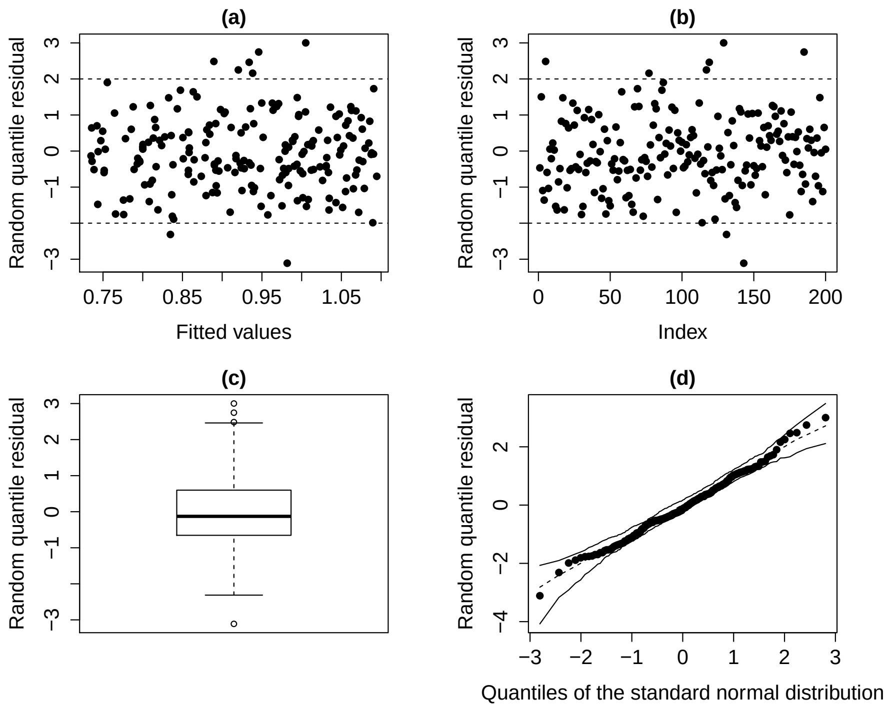

Figure 141 – Residual plots for the ZA-SGtBS1 regression model.

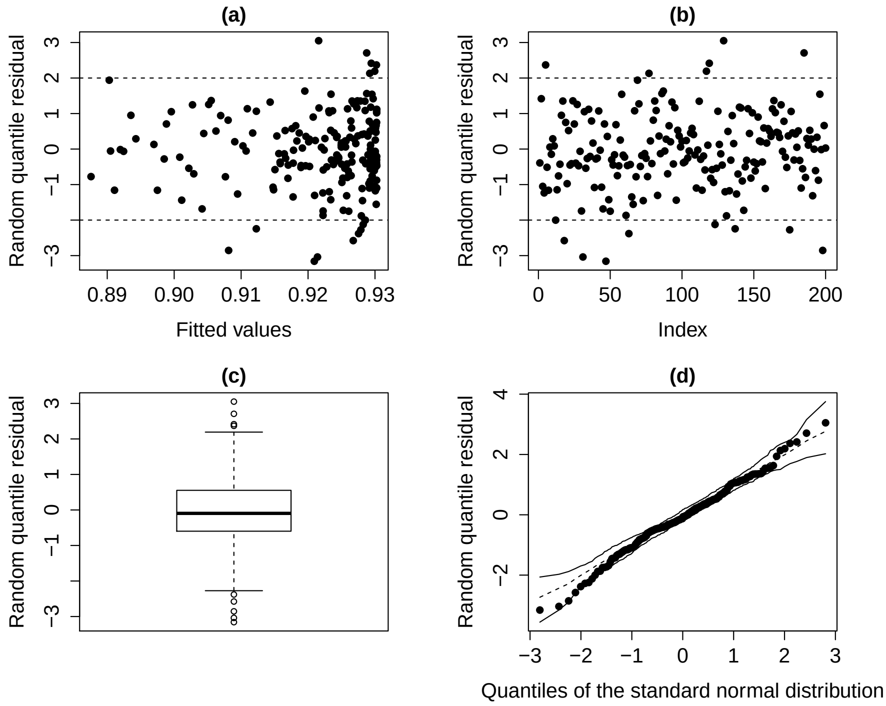

Figure 142 – Residual plots for the ZA-SGtBS2 regression model.

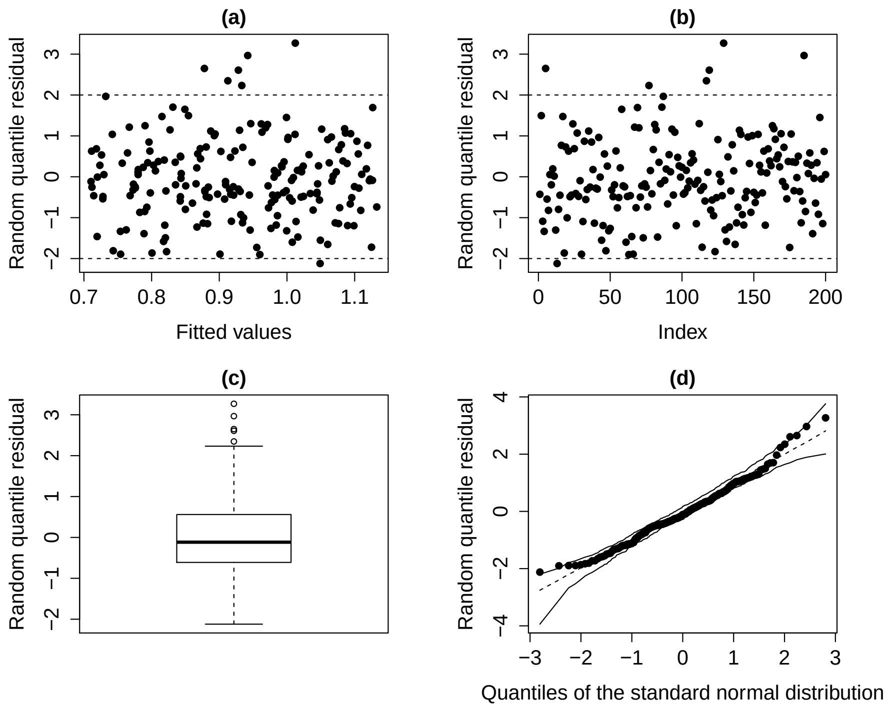

Figure 143 – Residual plots for the ZA-SSLBS1 regression model.

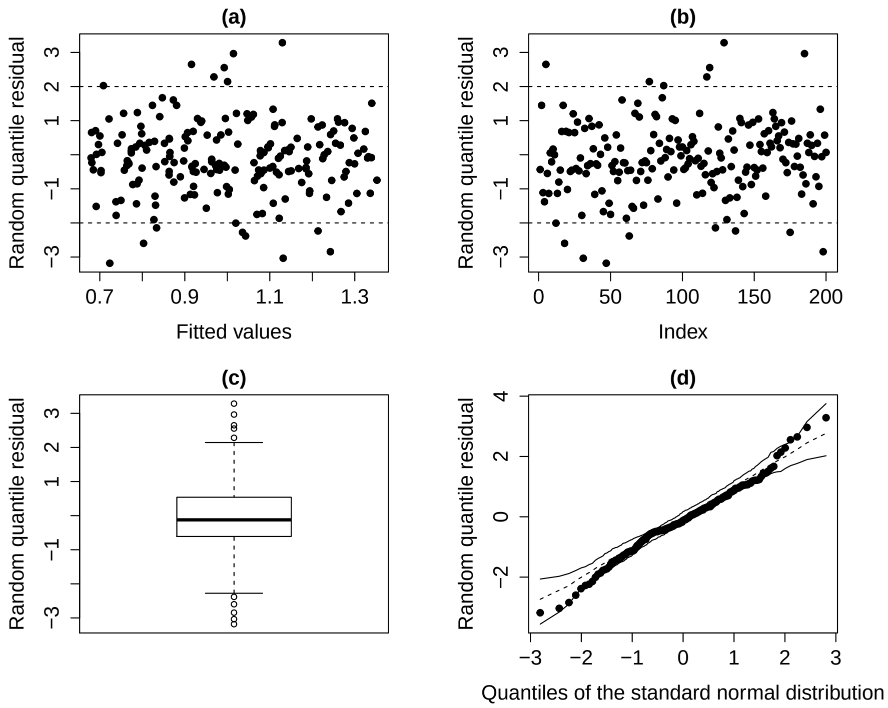

Figure 144 – Residual plots for the ZA-SSLBS2 regression model.

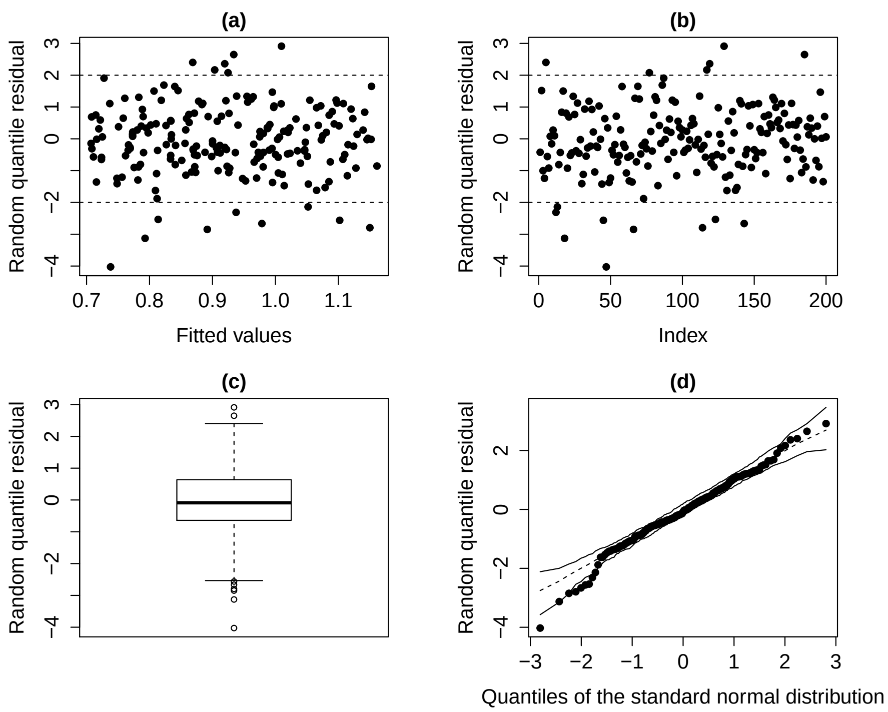

Figure 145 – Residual plots for the ZA-SCNBS regression model.

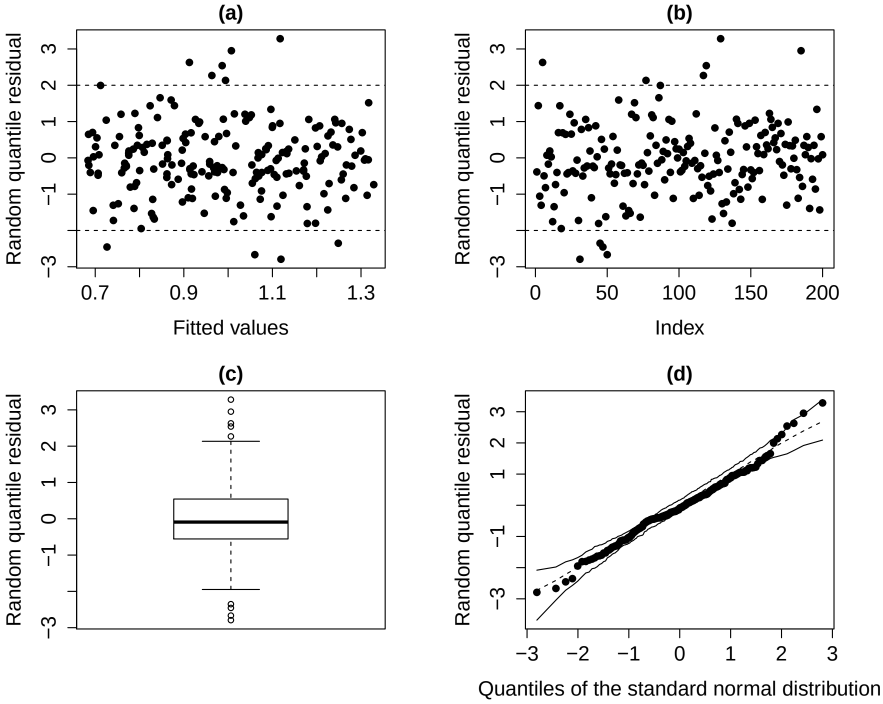

Figure 146 – Residual plots for the ZA-SNBS regression model.

Simulated observations from ZA-SSLBS regression model

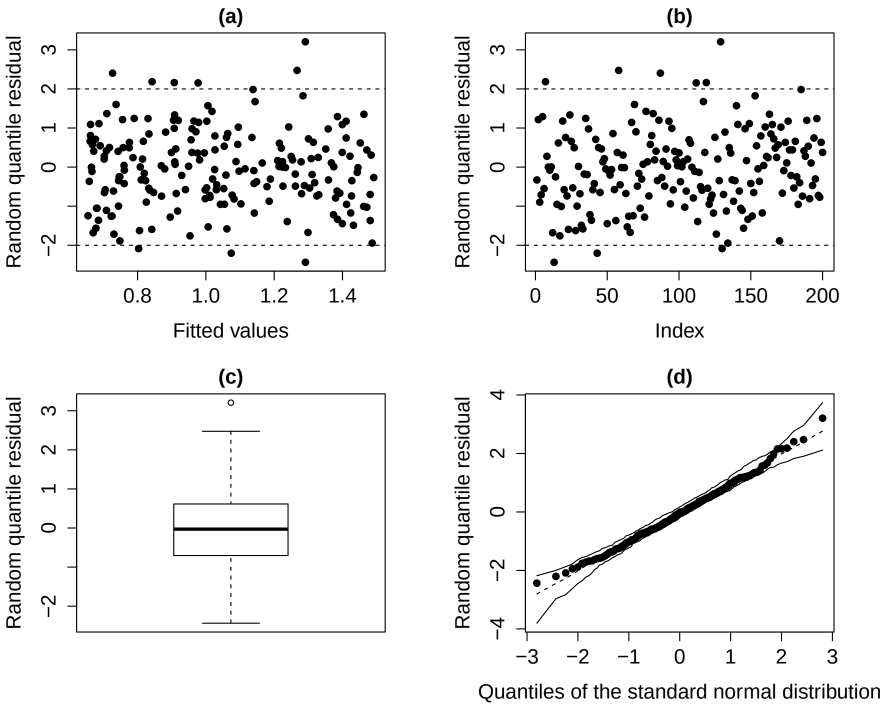

Figure 147 – Residual plots for the ZA-SSLBS1 regression model.

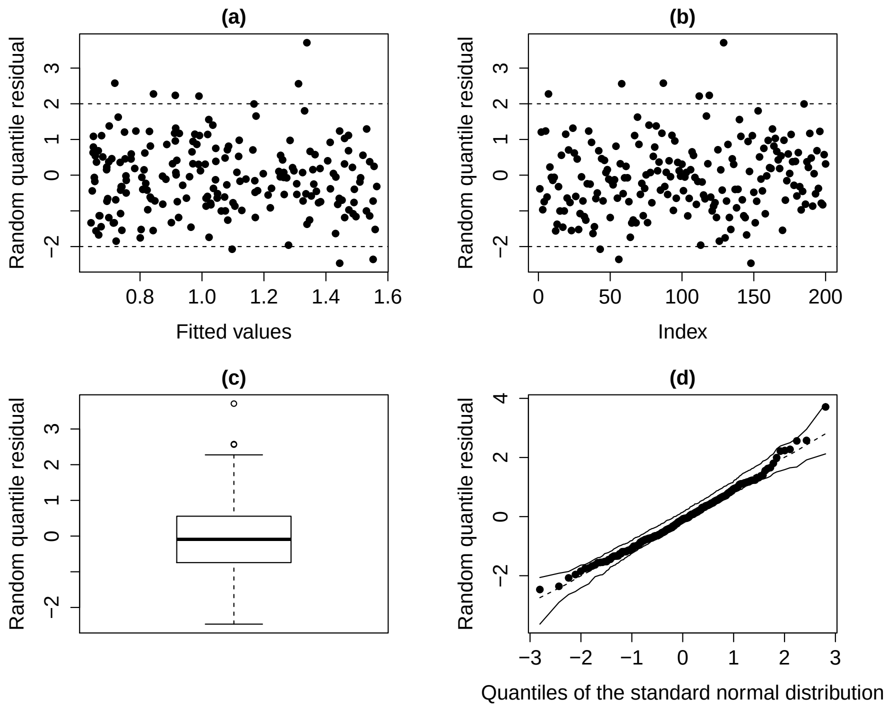

Figure 148 – Residual plots for the ZA-SSLBS2 regression model.

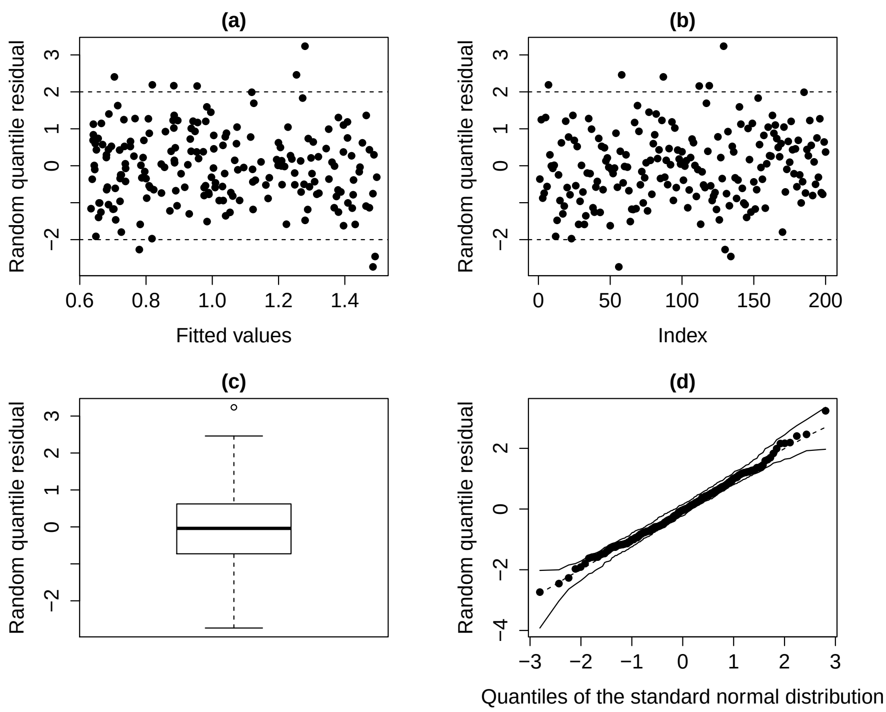

Figure 149 – Residual plots for the ZA-SGtBS1 regression model.

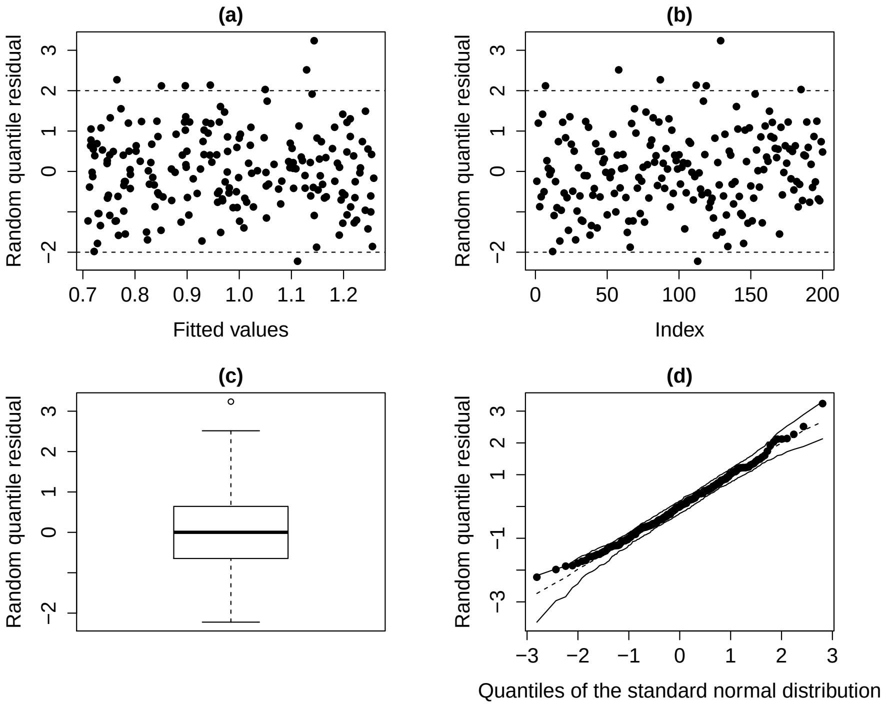

Figure 150 – Residual plots for the ZA-SGtBS2 regression model.

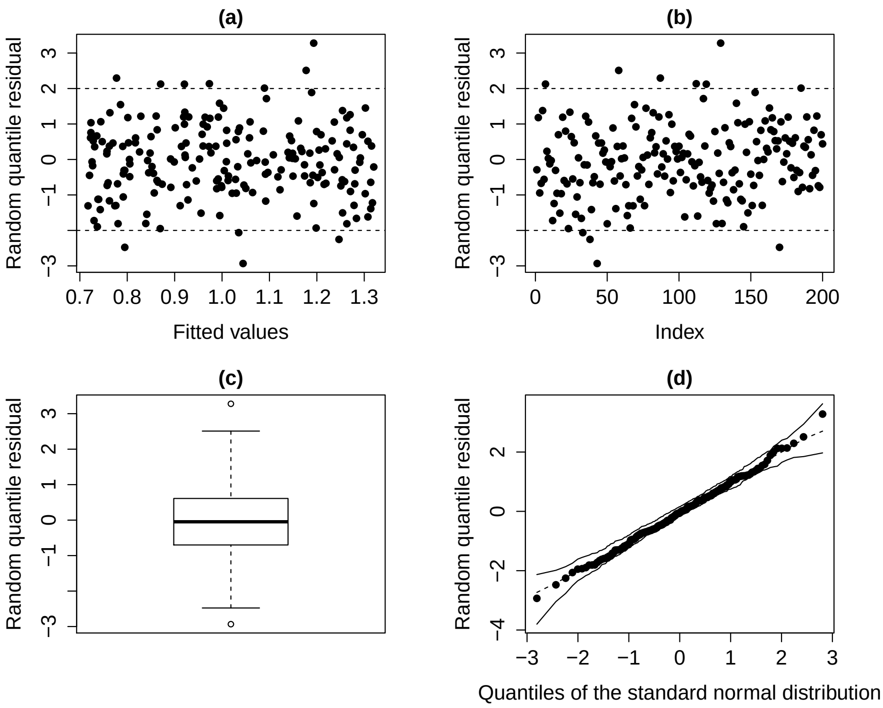

Figure 151 – Residual plots for the ZA-StBS regression model.

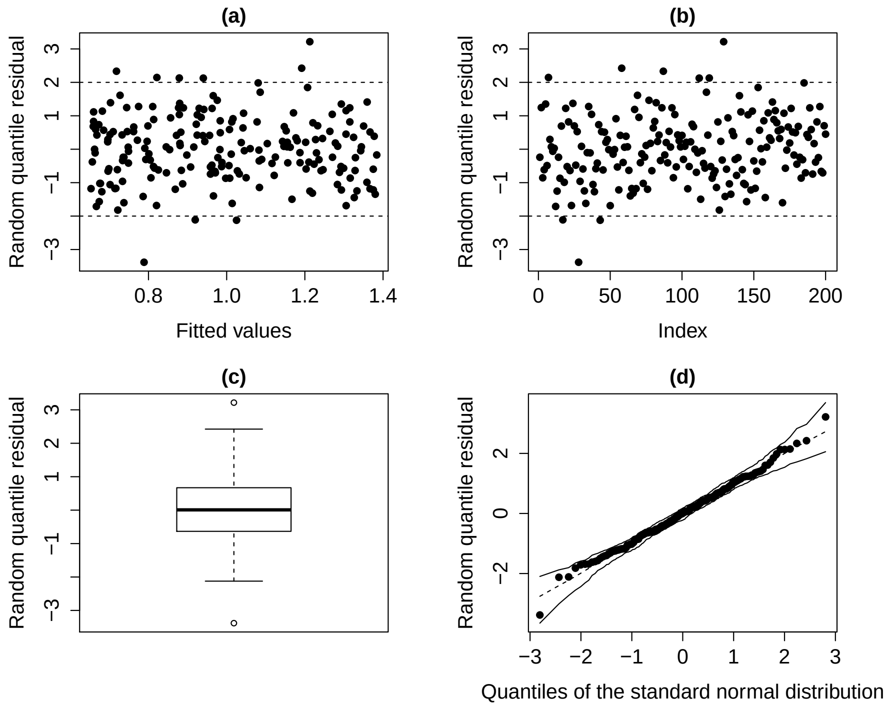

Figure 152 – Residual plots for the ZA-SCNBS regression model.

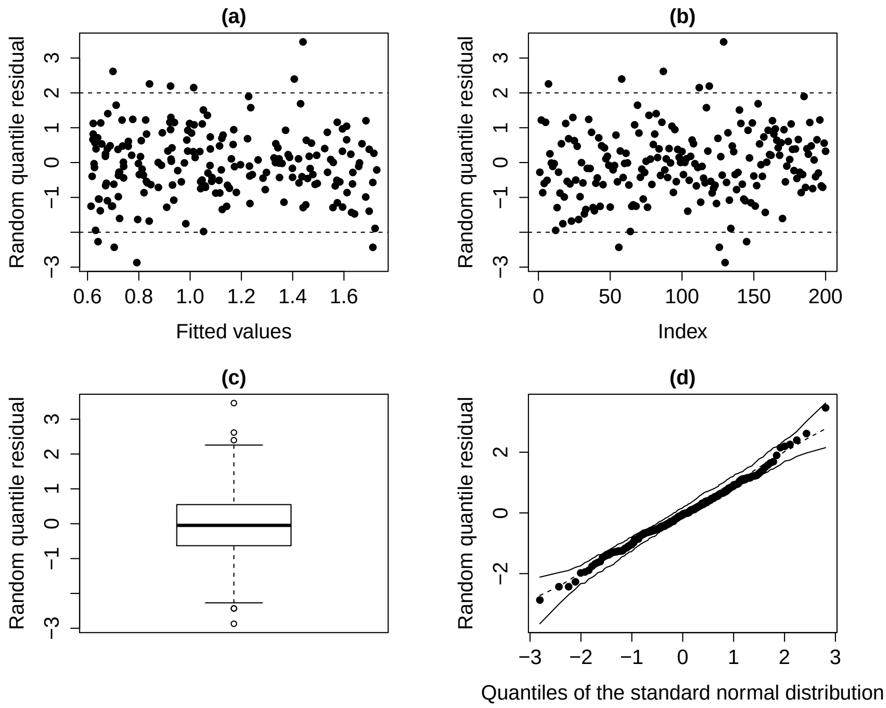

Figure 153 – Residual plots for the ZA-SNBS regression model.

Simulated observations from ZA-SCNBS regression model

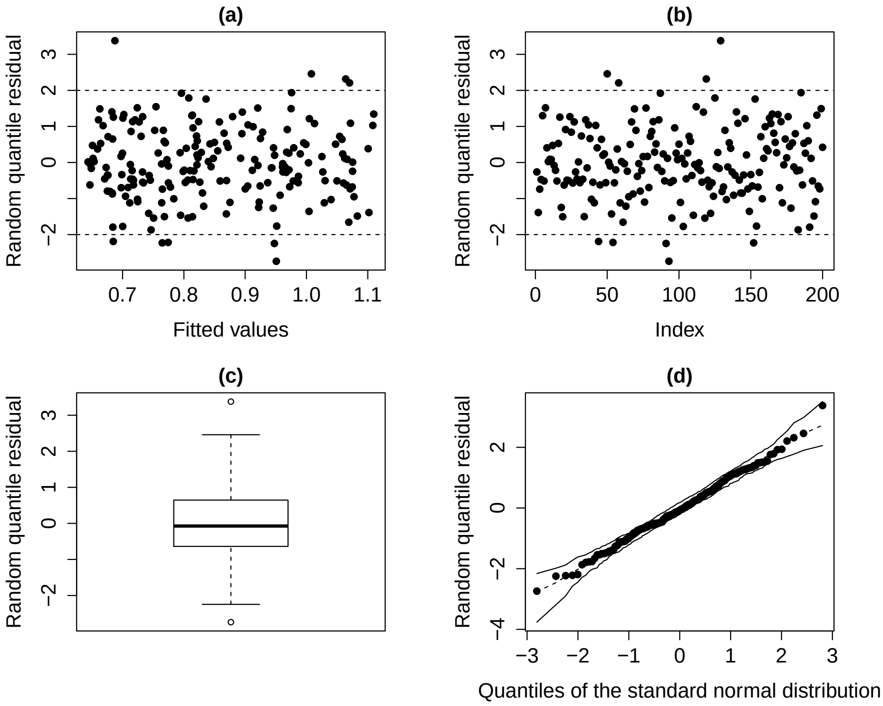

Figure 154 – Residual plots for the ZA-SCNBS regression model.

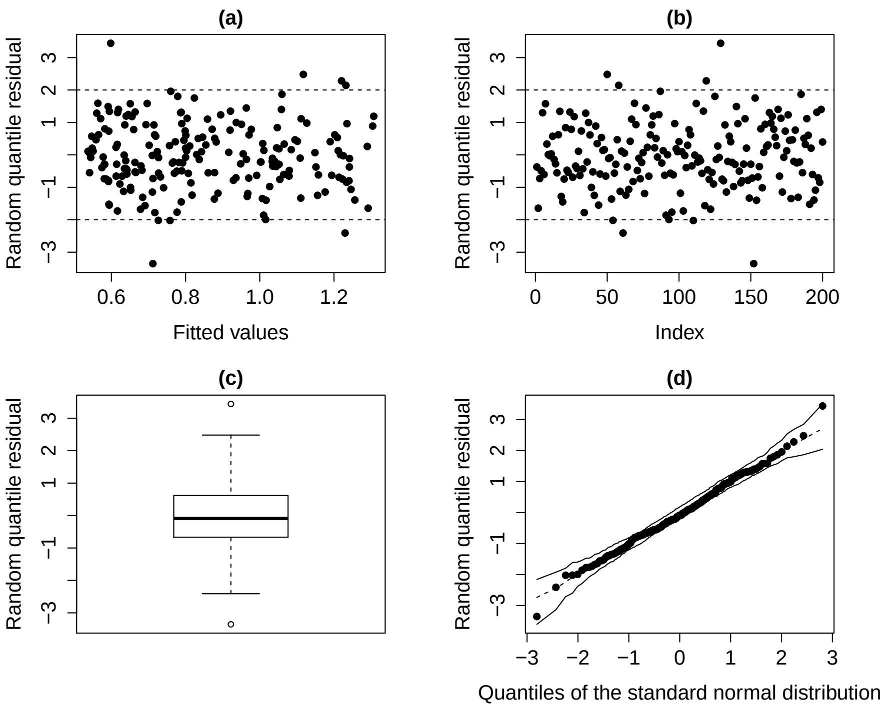

Figure 155 – Residual plots for the ZA-SGtBS1 regression model.

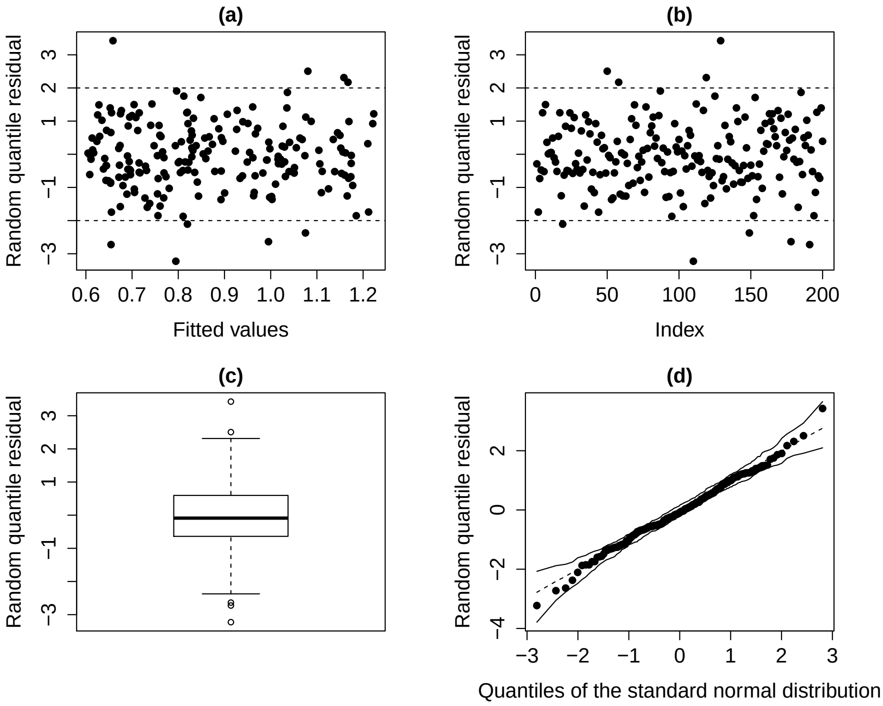

Figure 156 – Residual plots for the ZA-SGtBS2 regression model.

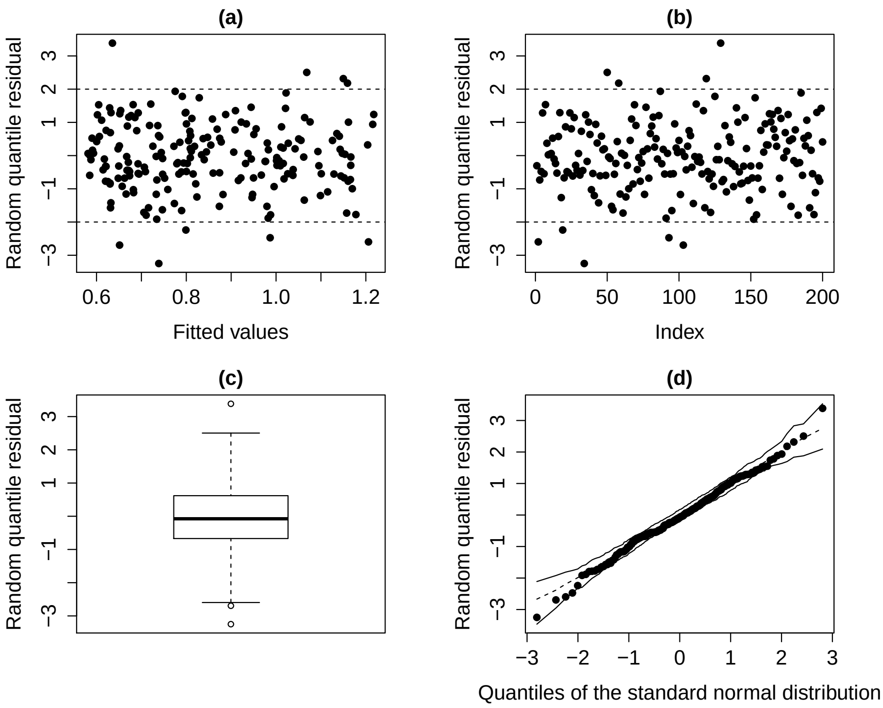

Figure 157 – Residual plots for the ZA-StBS regression model.

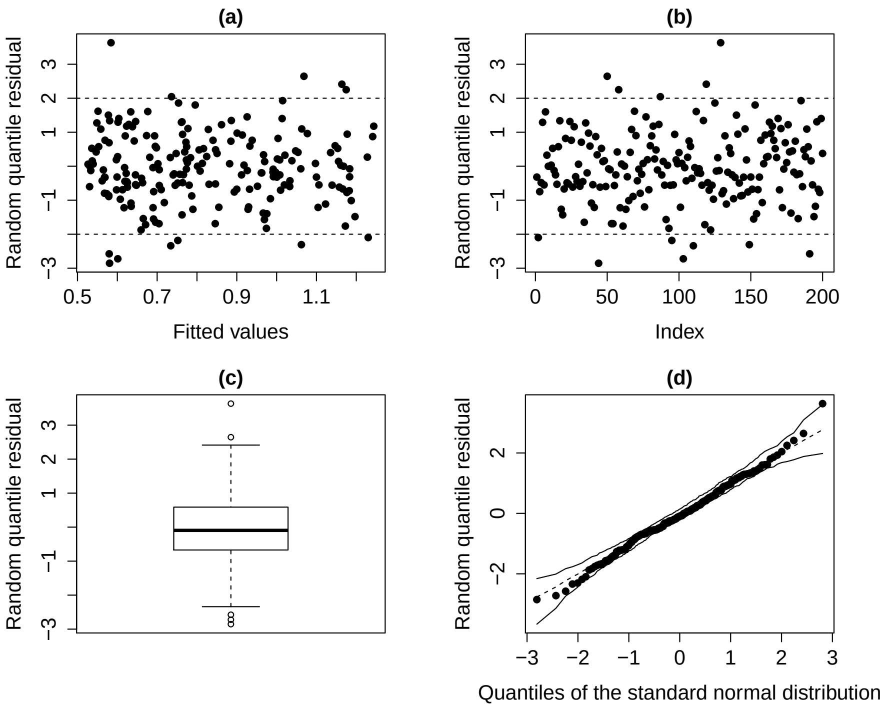

Figure 158 – Residual plots for the ZA-SSLBS1 regression model.

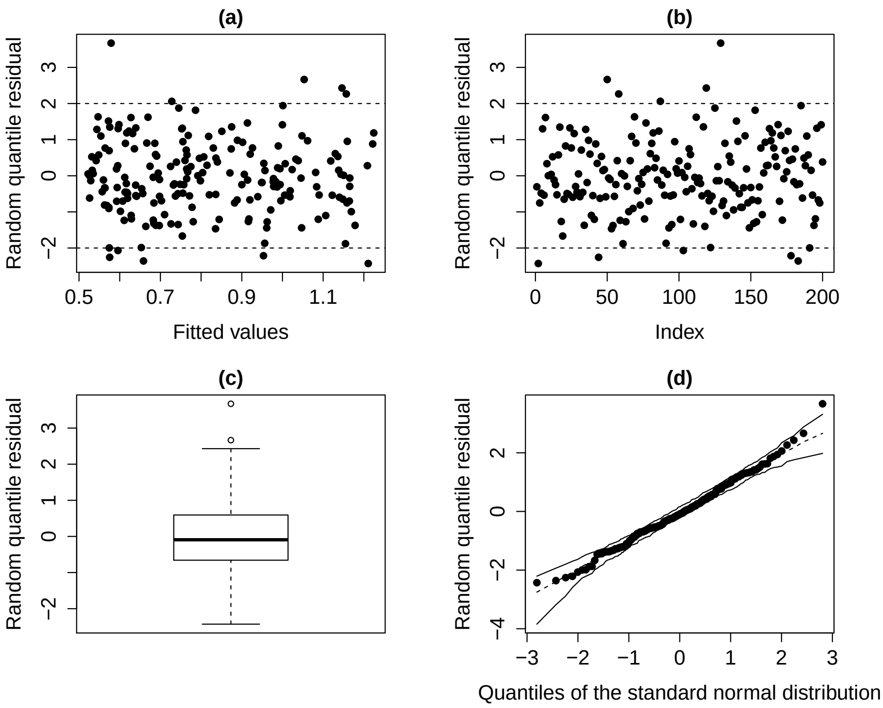

Figure 159 – Residual plots for the ZA-SSLBS2 regression model.

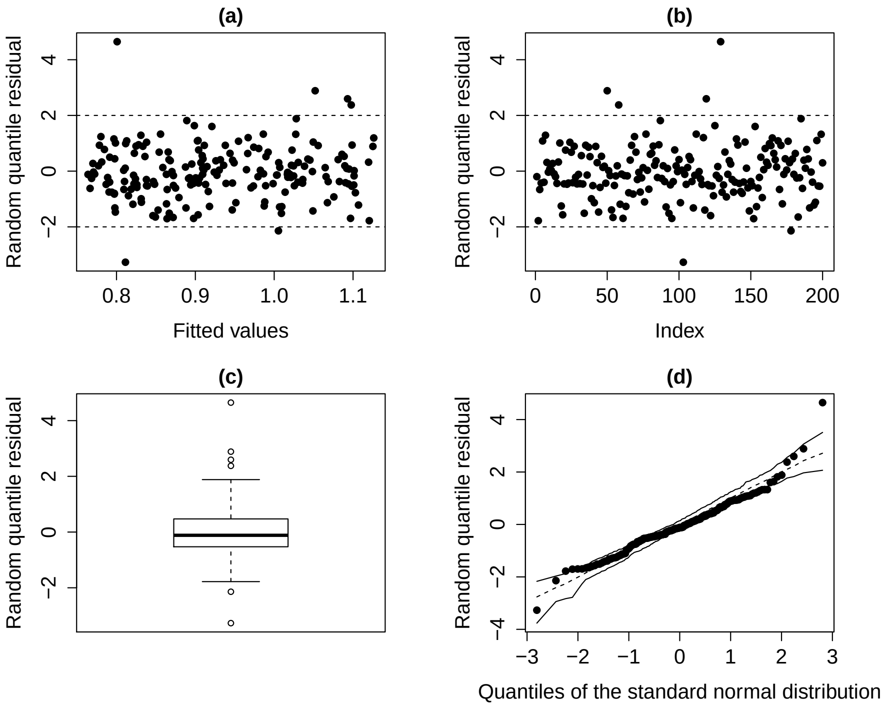

Figure 160 – Residual plots for the ZA-SNBS regression model.

D.3 Behavior of the K-L divergence

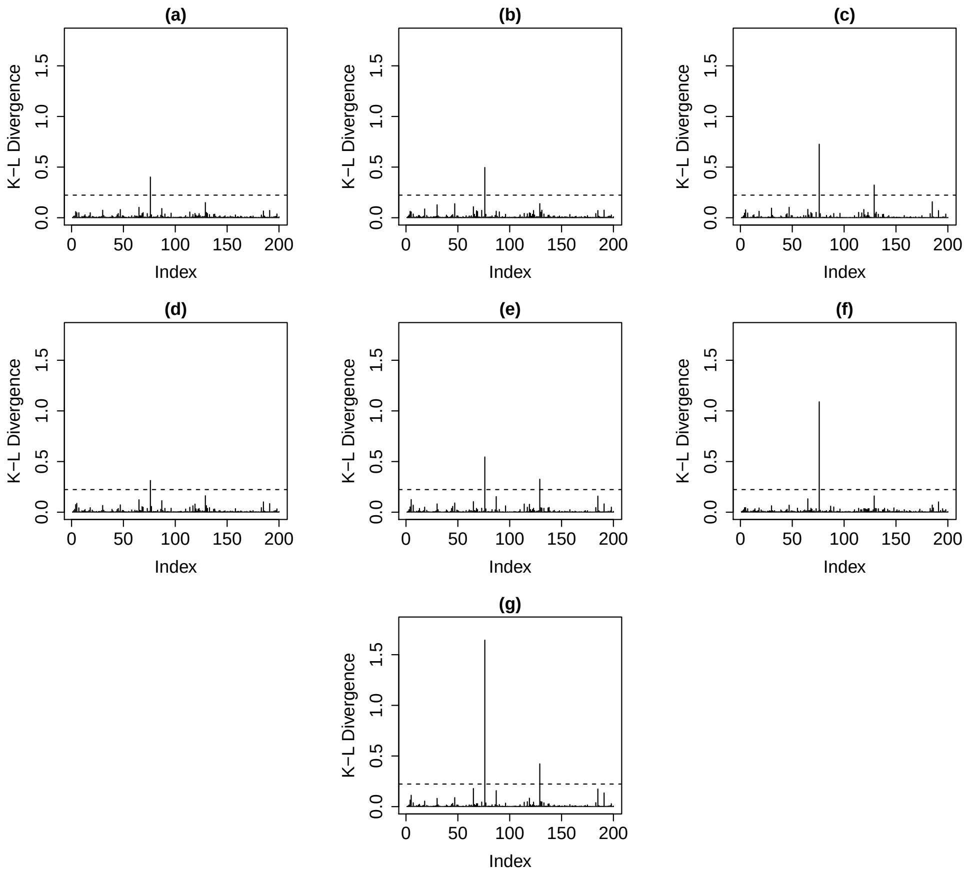

Figure 161 – K-L divergence when we generated the data set from ZA-SGtBS1 and fitted the following models: (a) ZA-StBS, (b) ZA-SGtBS1, (c) ZA-SGtBS2, (d) ZA-SSLBS1, (e) ZA-SSLBS2, (f) ZA-SCNBS, and (g) ZA-SNBS.

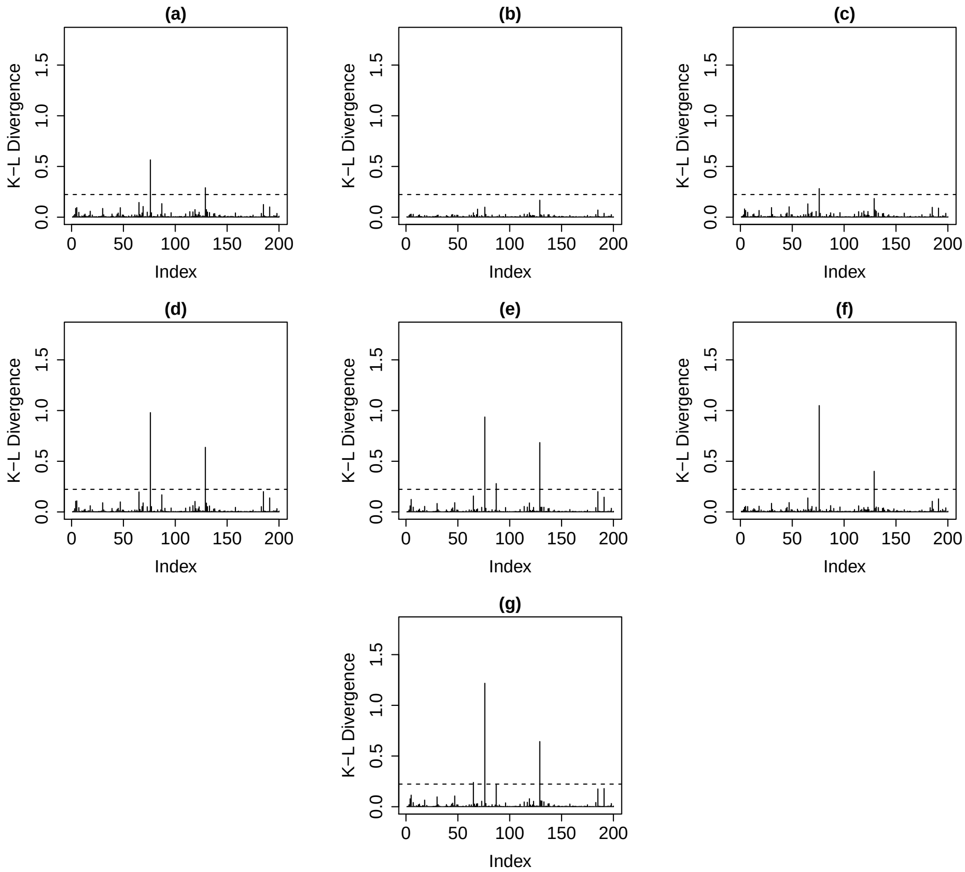

Figure 162 – K-L divergence when we generated the data set from ZA-SGtBS2 and fitted the following models: (a) ZA-StBS, (b) ZA-SGtBS1, (c) ZA-SGtBS2, (d) ZA-SSLBS1, (e) ZA-SSLBS2, (f) ZA-SCNBS, and (g) ZA-SNBS.

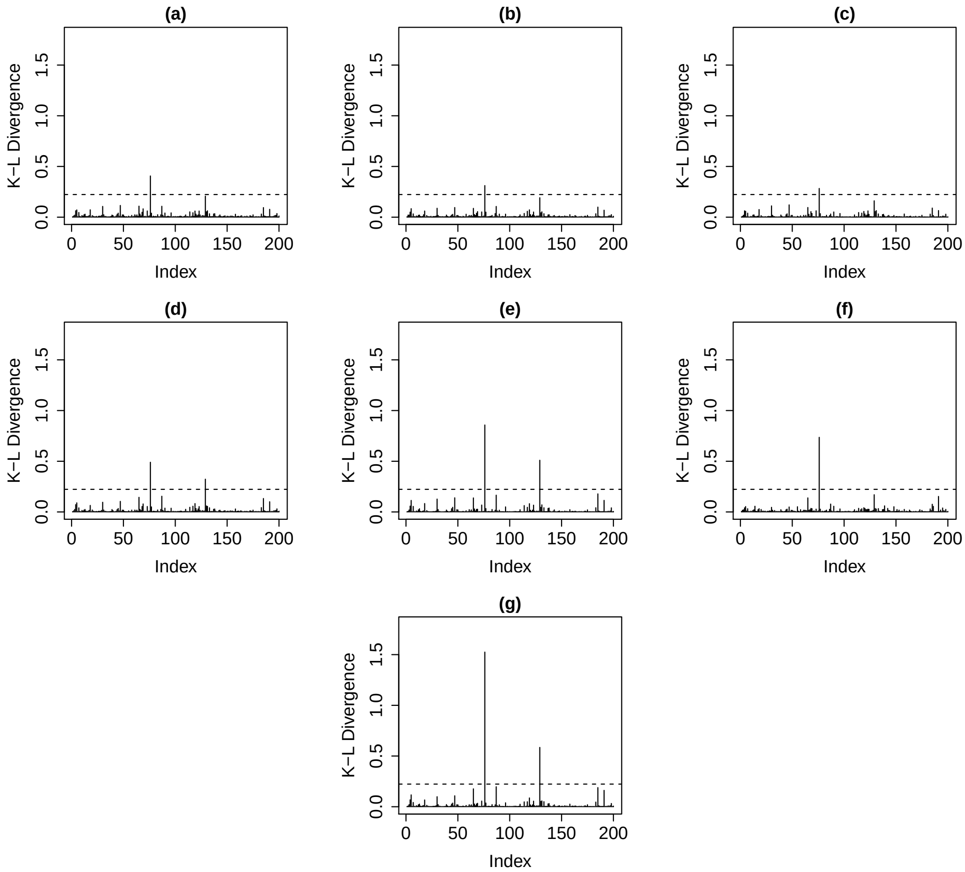

Figure 163 – K-L divergence when we generated the data set from ZA-StBS and fitted the following models: (a) ZA-StBS, (b) ZA-SGtBS1, (c) ZA-SGtBS2, (d) ZA-SSLBS1, (e) ZA-SSLBS2, (f) ZA-SCNBS, and (g) ZA-SNBS.

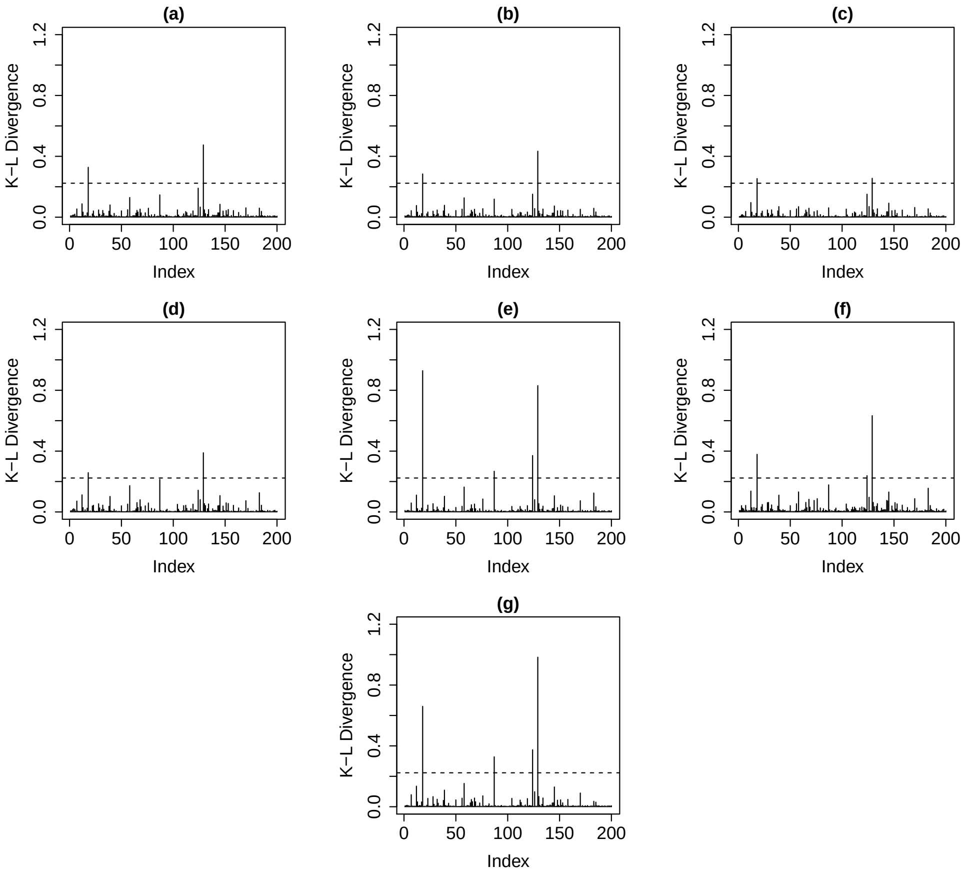

Figure 164 – K-L divergence when we generated the data set from ZA-SSLBS and fitted the following models: (a) ZA-StBS, (b) ZA-SGtBS1, (c) ZA-SGtBS2, (d) ZA-SSLBS1, (e) ZA-SSLBS2, (f) ZA-SCNBS, and (g) ZA-SNBS.

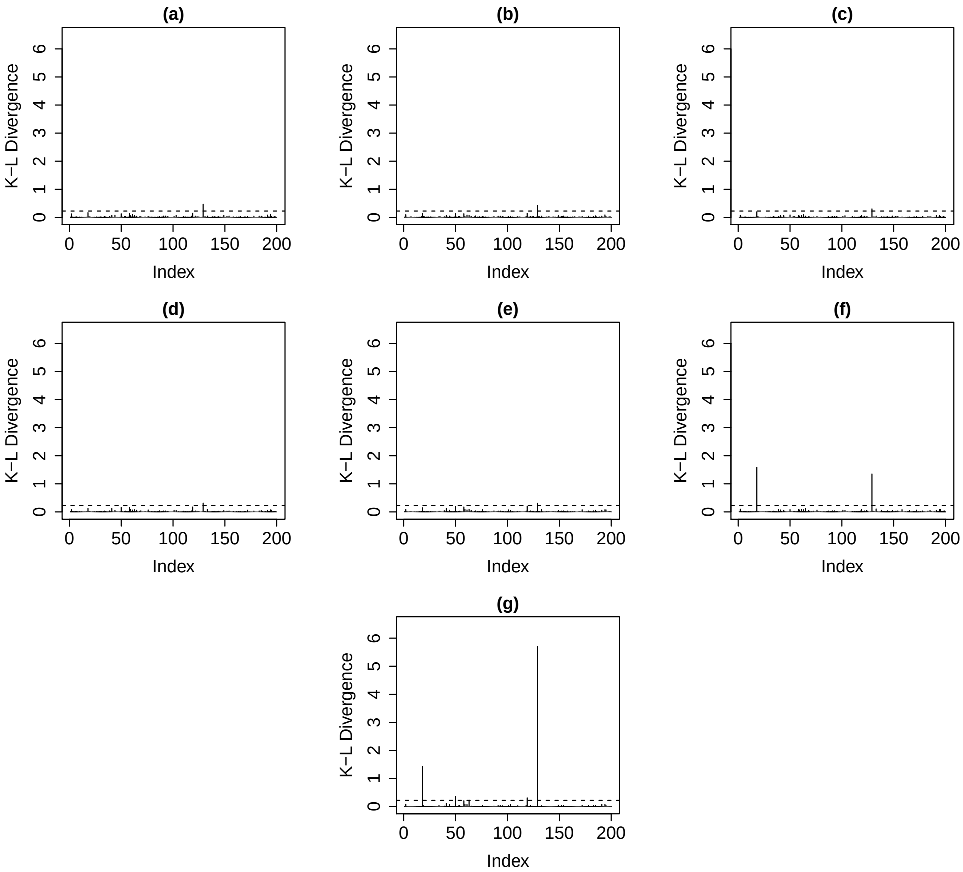

Figure 165 – K-L divergence when we generated the data set from ZA-SCNBS and fitted the following models: (a) ZA-StBS, (b) ZA-SGtBS1, (c) ZA-SGtBS2, (d) ZA-SSLBS1, (e) ZA-SSLBS2, (f) ZA-SCNBS, and (g) ZA-SNBS.

D.4 Statistics of model comparison

Table 130 – Averaged criteria for the simulation study.

True underlying model: ZA-StBS regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-StBS	458.471	484.857	1321.272	-224.870
ZA-SGtBS1	458.321	484.707	1321.358	**-224.622**
ZA-SGtBS2	**457.679**	**480.767**	**1325.304**	-224.973
ZA-SSLBS1	460.771	487.157	1328.650	-226.420
ZA-SSLBS2	461.883	488.270	1331.987	-227.266
ZA-SCNBS	459.386	489.071	1316.991	-225.042
ZA-SNBS	468.187	491.275	1355.662	-232.343
True underlying model: ZA-SGtBS1 regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-StBS	**364.907**	**391.293**	1040.082	**-178.126**
ZA-SGtBS1	365.439	391.825	1042.181	-178.379
ZA-SGtBS2	370.177	393.265	1062.893	-181.664
ZA-SSLBS1	366.964	393.350	1046.956	-179.284
ZA-SSLBS2	368.139	394.525	1051.499	-180.216
ZA-SCNBS	366.003	395.688	**1036.968**	-178.478
ZA-SNBS	374.671	397.759	1075.250	-185.384
True underlying model: ZA-SGtBS2 regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-StBS	231.747	258.134	**640.745**	-111.670
ZA-SGtBS1	233.315	259.702	646.741	-111.974
ZA-SGtBS2	**230.129**	**253.218**	642.355	**-111.282**
ZA-SSLBS1	233.753	260.140	647.907	-112.947
ZA-SSLBS2	234.873	261.260	651.302	-113.975
ZA-SCNBS	233.026	262.710	638.039	-112.043
ZA-SNBS	240.643	263.731	673.148	-118.557
True underlying model: ZA-SSLBS regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-StBS	456.108	482.494	1313.538	-223.679
ZA-SGtBS1	455.736	482.123	1315.387	**-223.218**
ZA-SGtBS2	**455.068**	**478.156**	1317.212	-223.469
ZA-SSLBS1	455.639	482.026	1312.537	-223.613
ZA-SSLBS2	456.472	482.858	1316.660	-224.298
ZA-SCNBS	457.512	487.196	**1311.949**	-223.743

Table 130 (continued).

ZA-SNBS	455.414	478.502	1317.595	-224.932
True underlying model: ZA-SCNBS regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-StBS	426.437	452.824	1224.711	-209.164
ZA-SGtBS1	426.741	453.128	1226.303	-209.197
ZA-SGtBS2	426.681	**449.769**	1232.504	-209.658
ZA-SSLBS1	428.791	455.178	1232.006	-210.673
ZA-SSLBS2	429.494	455.880	1234.773	-211.141
ZA-SCNBS	**425.246**	454.931	**1214.253**	**-208.889**
ZA-SNBS	443.301	466.389	1281.004	-221.483

Table 131 – Percentage of times that the correct model was selected.

Model	EAIC	EBIC	DIC	LPML
ZA-StBS	10%	10%	10%	10%
ZA-SGtBS1	0%	0%	0%	20%
ZA-SGtBS2	70%	90%	30%	50%
ZA-SSLBS	0%	0%	20%	30%
ZA-SCNBS	60%	0%	80%	50%

D.5 Posterior predictive checking

Table 132 – Posterior predictive checking for the ZA-SSBS regression model.

True underlying model: ZA-SGtBS1							
	ZA-SGtBS1	ZA-SGtBS2	ZA-StBS	ZA-SSLBS1	ZA-SSLBS2	ZA-SCNBS	ZA-SNBS
p-value	.196	.346	.320	.257	.196	.057	.062
True underlying model: ZA-SGtBS2							
	ZA-SGtBS2	ZA-SGtBS1	ZA-StBS	ZA-SSLBS1	ZA-SSLBS2	ZA-SCNBS	ZA-SNBS
p-value	.350	.640	.332	.335	.215	.353	.105
True underlying model: ZA-StBS							
	ZA-StBS	ZA-SGtBS1	ZA-SGtBS2	ZA-SSLBS1	ZA-SSLBS2	ZA-SCNBS	ZA-SNBS
p-value	.308	.269	.238	.227	.130	.268	.045
True underlying model: ZA-SSLBS							
	ZA-SSLBS1	ZA-SSLBS2	ZA-SGtBS1	ZA-SGtBS2	ZA-StBS	ZA-SCNBS	ZA-SNBS
p-value	.384	.173	.187	.259	.283	.190	.060
True underlying model: ZA-SCNBS							
	ZA-SCNBS	ZA-SGtBS1	ZA-SGtBS2	ZA-StBS	ZA-SSLBS1	ZA-SSLBS2	ZA-SNBS
p-value	.348	.087	.083	.224	.018	.015	$< .001$

D.6 Results of the statistical analysis of bilirubin concentration

Table 133 – Bayesian estimates for the ZA-SGtBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.954	.051	[1.867; 2.058]
β_1	-.212	.066	[-.341; -.085]
β_2	-.054	.009	[-.070; -.037]
ψ_0	.433	.068	[.324; .585]
ψ_2	.294	.039	[.218; .355]
ζ_0	-2.531	.256	[-3.062; -2.126]
ζ_1	.038	.041	[-.038; 0.115]
γ_1	.525	.114	[.336; .799]
γ_2	-.951	.053	[-.994; -.814]
γ_3	-.942	.050	[-.986; -.807]
γ_4	-.948	.045	[-.990; -.828]
γ_5	-.949	.049	[-.992; -.810]
γ_6	-.935	.069	[-.988; -.736]
γ_7	-.891	.119	[-.985; -.549]
γ_8	-.735	.172	[-.954; -.283]
γ_9	-.882	.106	[-.986; -.605]
ν_{11}	7.471	.883	[5.865; 8.892]
ν_{12}	12.193	2.169	[8.107; 15.020]
ν_{13}	9.813	1.272	[7.754; 12.661]
ν_{14}	9.744	1.440	[7.223; 12.411]
ν_{15}	10.665	1.792	[7.840; 14.551]
ν_{16}	12.636	2.355	[8.827; 18.240]
ν_{17}	18.128	4.421	[10.930; 26.976]
ν_{18}	33.973	9.605	[18.318; 53.413]
ν_{19}	40.074	12.385	[19.447; 66.373]

Table 134 – Bayesian estimates for the ZA-StBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.895	.036	[1.829; 1.969]
β_1	-.132	.068	[-.265; -.027]
β_2	-.044	.008	[-.062; -.030]
ψ_0	-1.518	.082	[-1.657; -1.381]
ψ_2	.159	.021	[.122; .203]
ζ_0	-1.106	.086	[-1.254; -.974]
ζ_1	-.174	.029	[-.233; -.118]
γ_1	-.590	.097	[-.756; -.449]
γ_2	-.930	.054	[-.985; -.790]
γ_3	-.939	.049	[-.990; -.804]
γ_4	-.960	.051	[-.993; -.817]
γ_5	-.947	.046	[-.991; -.820]
γ_6	-.961	.037	[-.993; -.857]
γ_7	-.852	.158	[-.992; -.371]
γ_8	-.769	.232	[-.985; -.007]
γ_9	-.875	.113	[-.987; -.576]
ν_1	4.298	.305	[4.005; 5.160]
ν_2	19.236	12.478	[6.097; 58.290]
ν_3	16.109	7.219	[6.451; 33.944]
ν_4	18.192	8.459	[7.992; 41.778]
ν_5	18.416	9.869	[6.678; 46.402]
ν_6	22.740	11.034	[8.073; 48.030]
ν_7	21.540	10.204	[7.748; 46.884]
ν_8	26.789	12.602	[9.180; 56.477]
ν_9	23.504	12.014	[7.836; 52.146]

Table 135 – Bayesian estimates for the ZA-SSLBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	2.025	.058	[1.927; 2.140]
β_1	-.453	.060	[-.586; -.356]
β_2	-.072	.010	[-.091; -.053]
ψ_0	-.650	.181	[-1.112; -.459]
ψ_1	.021	.021	[-.007; .066]
ζ_0	-1.360	.176	[-1.831; -1.167]
ζ_1	-.124	.030	[-.165; -.070]
γ_1	.406	.119	[.245; .674]
γ_2	-.993	.010	[-.995; -.978]
γ_3	-.922	.062	[-.987; -.753]
γ_4	-.953	.037	[-.990; -.861]
γ_5	-.940	.047	[-.991; -.810]
γ_6	-.970	.031	[-.994; -.883]
γ_7	-.854	.126	[-.981; -.515]
γ_8	-.674	.186	[-.919; -.227]
γ_9	-.833	.122	[-.989; -.513]
ν_1	3.618	1.982	[2.127; 9.739]
ν_2	16.521	4.297	[7.094; 22.390]
ν_3	12.090	6.801	[3.799; 29.781]
ν_4	10.561	5.340	[3.692; 22.692]
ν_5	8.800	4.636	[3.162; 19.841]
ν_6	9.772	4.905	[3.601; 21.490]
ν_7	7.965	4.164	[3.012; 18.492]
ν_8	10.587	5.836	[3.706; 25.702]
ν_9	5.527	3.002	[2.361; 14.363]

Table 136 – Bayesian estimates for the ZA-SCNBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.717	.038	[1.644; 1.786]
β_1	-.106	.034	[-.173; -.026]
β_2	.169	.010	[.149; .188]
ψ_0	-1.494	.059	[-1.613; -1.387]
ψ_2	.141	.013	[.114; .162]
ζ_0	-2.024	.045	[-2.104; -1.935]
ζ_1	-.026	.023	[-.071; .010]
γ_1	-.295	.070	[-.417; -.142]
γ_2	-.953	.034	[-.989; -.869]
γ_3	-.932	.090	[-.984; -.641]
γ_4	-.933	.038	[-.983; -.825]
γ_5	-.912	.099	[-.991; -.610]
γ_6	-.992	.001	[-.993; -.990]
γ_7	-.841	.124	[-.958; -.495]
γ_8	-.827	.124	[-.958; -.547]
γ_9	-.861	.059	[-.929; -.729]
ν_{11}	.153	.017	[.110; .182]
ν_{12}	.504	.199	[.114; .869]
ν_{13}	.099	.085	[.041; .299]
ν_{14}	.050	.026	[.016; .106]
ν_{15}	.179	.078	[.077; .357]
ν_{16}	.055	.009	[.035; .066]
ν_{17}	.086	.035	[.024; .157]
ν_{18}	.048	.024	[.012; .101]
ν_{19}	.041	.028	[.011; .117]
ν_{21}	.960	.041	[.838; .998]
ν_{22}	.744	.130	[.538; .988]
ν_{23}	.119	.096	[.031; .390]
ν_{24}	.024	.013	[.007; .054]
ν_{25}	.059	.022	[.024; .097]
ν_{26}	.013	.002	[.008; .015]
ν_{27}	.012	.005	[.003; .023]
ν_{28}	.004	.002	[.001; .009]
ν_{29}	.003	.002	[.001; .008]

Table 137 – Bayesian estimates for the ZA-SNBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.747	.131	[1.470; 1.899]
β_1	.087	.438	[-.301; .858]
β_2	-.024	.021	[-.050; .015]
ψ_0	-1.054	.386	[-1.378; .093]
ψ_2	.108	.057	[-.056; .161]
ζ_0	-1.889	.629	[-2.452; -.559]
ζ_1	-.072	.103	[-.284; .032]
γ_1	-.589	.147	[-.986; -.453]
γ_2	-.839	.201	[-.988; -.314]
γ_3	-.917	.050	[-.977; -.779]
γ_4	-.922	.052	[-.980; -.777]
γ_5	-.919	.050	[-.973; -.785]
γ_6	-.961	.034	[-.992; -.863]
γ_7	-.905	.088	[-.988; -.668]
γ_8	-.779	.194	[-.975; -.253]
γ_9	-.870	.119	[-.991; -.559]

Table 138 – Bayesian estimates for the ZA-BS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.747	.069	[1.619; 1.892]
β_1	.046	.102	[-.156; .221]
β_2	-.039	.014	[-.066; -.011]
ψ_0	-1.032	.094	[-1.209; -.853]
ψ_2	.107	.017	[.074; .140]
ζ_0	-2.845	.221	[-3.291; -2.432]
ζ_1	.070	.034	[.005; .135]

APPENDIX E – Results of Chapter 5

In this section, we present in detail the results related to the marginal means, variances and covariance of the mixed CSSBS regression models. Also, we present all results of the parameter recovery study. Furthermore, we present the results of the statistical analysis of the cholesterol data set.

E.1 Results related to the marginal means, variances and covariance

Let $T_{ij}|\boldsymbol{b}_i, \boldsymbol{\Omega} \sim \text{CSSBS}(\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}), i = 1, \ldots, n, j = 1, \ldots, k_i$, where $\boldsymbol{\Omega} = (\boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, μ_{ij} and ϕ_{ij} are defined in Equation (5.1). The hierarchical structure of the CSSBS regression models with random-effects is given by

$$\begin{aligned} T_{ij}|\boldsymbol{b}_i, \boldsymbol{\Omega} &\sim \text{CSSBS}(\mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}) \\ \boldsymbol{b}_i|\boldsymbol{\Sigma}_b &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_b). \end{aligned}$$

By using results from conditional distributions and the expressions presented in Equation (5.3), we have that

$$\begin{aligned} \mathbb{E}(T_{ij}) &= \mathbb{E}\left[\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top\boldsymbol{b}\right\}\right] \\ &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\mathbb{E}\left[\boldsymbol{z}_{ij}^\top\boldsymbol{b}\right] \\ &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\} M_b(\boldsymbol{z}_{ij}). \end{aligned}$$

$$\begin{aligned} \mathbb{V}(T_{ij}) &= \mathbb{V}(\mathbb{E}(T_{ij}|\boldsymbol{b})) + \mathbb{E}[\mathbb{V}(T_{ij}|\boldsymbol{b})] \\ &= \mathbb{E}\left(\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + 2\boldsymbol{z}_{ij}^\top\boldsymbol{b}\right\}\right) - \left[\mathbb{E}\left(\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top\boldsymbol{b}\right\}\right)\right]^2 + c\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\} M_b(2\boldsymbol{z}_{ij}) \\ &= \exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\} M_b(2\boldsymbol{z}_{ij}) - \exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}[M_b(\boldsymbol{z}_{ij})]^2 + c\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\} M_b(2\boldsymbol{z}_{ij}) \\ &= \exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\left\{(1+c)M_b(2\boldsymbol{z}_{ij}) - [M_b(\boldsymbol{z}_{ij})]^2\right\}. \end{aligned}$$

$$\begin{aligned} \text{Cov}(T_{ij}, T_{ij'}) &= \text{Cov}[\mathbb{E}(T_{ij}|\boldsymbol{b}), \mathbb{E}(T_{ij'}|\boldsymbol{b})] + \underbrace{\mathbb{E}[\text{Cov}(T_{ij}, T_{ij'}|\boldsymbol{b})]}_{=0} \\ &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\}\text{Cov}\left[\exp\left\{\boldsymbol{z}_{ij}^\top\boldsymbol{\beta}\right\}\exp\left\{\boldsymbol{z}_{ij'}^\top\boldsymbol{\beta}\right\}\right] \\ &= \exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\}\left[M_b(\boldsymbol{z}_{ij} + \boldsymbol{z}_{ij'}) - M_b(\boldsymbol{z}_{ij})M_b(\boldsymbol{z}_{ij'})\right]. \end{aligned}$$

E.2 Results of the recovery parameter study

Mixed SGtBS1 regression model

Table 139 – Results of the simulation study for the mixed SGtBS1 regression model ($\nu_1 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.355	-.423	-.350	-.620	-.681	-.663
	SD	.260	.167	.205	.208	.134	.285
	LCI	1.096	1.006	.883	.629	.689	.457
β_0	CP	1.000	1.000	.800	.800	1.000	.600
	Bias	.145	.077	.150	-.120	-.181	-.163
	RMSE	.298	.184	.254	.240	.225	.328
	AVRB	.290	.154	.301	.239	.362	.326
	Mean	1.069	1.035	.927	.956	1.037	1.027
	SD	.104	.142	.093	.097	.033	.101
	LCI	.471	.482	.441	.288	.340	.273
β_1	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	.069	.035	-.073	-.044	.037	.027
	RMSE	.125	.146	.118	.106	.050	.104
	AVRB	.069	.035	.073	.044	.037	.027
	Mean	-.800	-.737	-.561	-.805	-.510	-.933
	SD	.416	.446	.624	.261	.260	.165
	LCI	1.592	1.858	1.699	1.155	1.523	.842
ψ_0	CP	.800	1.000	.800	1.000	.8	1.000
	Bias	.200	.263	.439	.195	.490	.067
	RMSE	.461	.518	.763	.326	.555	.178
	AVRB	.200	.263	.439	.195	.490	.067
	Mean	.720	.644	.584	.383	.234	.404
	SD	.386	.199	.274	.415	.124	.267
	LCI	1.596	1.538	1.452	1.326	1.172	.798
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.220	.144	.084	-.117	-.266	-.096
	RMSE	.444	.245	.286	.431	.294	.283
	AVRB	.439	.287	.168	.234	.532	.191
	Mean	-.469	-.035	.403	-.628	.079	.739
	SD	.378	.212	.364	.076	.257	.071
	LCI	.587	1.012	.552	.537	.587	.464
γ	CP	.800	.800	.400	.800	.200	1.000

Table 139 (continued).

	Bias	.331	-.035	-.397	.172	.079	-.061
	RMSE	.502	.215	.539	.188	.269	.093
	AVRB	.413	-	.496	.215	-	.076
	Mean	6.556	6.88	8.515	5.883	6.75	5.113
	SD	1.391	2.165	5.121	1.683	1.406	.416
	LCI	7.689	11.043	10.854	4.341	8.961	2.846
ν_1	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	1.556	1.880	3.515	.883	1.75	.113
	RMSE	2.087	2.867	6.212	1.901	2.244	.431
	AVRB	.311	.376	.703	.177	.350	.023
	Mean	4.726	4.774	4.879	4.150	4.164	4.228
	SD	.987	.870	1.114	.329	.447	.488
	LCI	3.835	3.915	3.880	2.373	2.369	2.387
σ^2	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.726	.774	.879	.150	.164	.228
	RMSE	1.226	1.164	1.419	.362	.476	.538
	AVRB	.182	.193	.220	.038	.041	.057

Table 140 – Results of the simulation study for the mixed SGtBS1 regression model ($\nu_1 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.343	-.433	-.354	-.620	-.677	-.658
	SD	.256	.175	.222	.211	.139	.290
	LCI	1.096	1.006	.883	.629	.689	.457
	CP	1.000	1.000	.800	.800	1.000	.600
	Bias	.157	.067	.146	-.120	-.177	-.158
	RMSE	.300	.187	.266	.243	.226	.330
	AVRB	.314	.135	.293	.240	.355	.316
β_1	Median	1.074	1.039	.933	.956	1.036	1.025
	SD	.109	.145	.094	.098	.035	.096
	LCI	.471	.482	.441	.288	.340	.273
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	.074	.039	-.067	-.044	.036	.025
	RMSE	.132	.150	.116	.108	.050	.099
	AVRB	.074	.039	.067	.044	.036	.025
ψ_0	Median	-.849	-.801	-.582	-.811	-.537	-.944
	SD	.420	.431	.659	.288	.267	.171
	LCI	1.592	1.858	1.699	1.155	1.523	.842
	CP	.800	1.000	.800	1.000	.800	1.000
	Bias	.151	.199	.418	.189	.463	.056
	RMSE	.446	.475	.781	.345	.535	.180
	AVRB	.151	.199	.418	.189	.463	.056
ψ_1	Median	.721	.645	.570	.398	.244	.393
	SD	.394	.194	.258	.435	.123	.267
	LCI	1.596	1.538	1.452	1.326	1.172	.798
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.221	.145	.070	-.102	-.256	-.107
	RMSE	.452	.242	.268	.447	.284	.288
	AVRB	.441	.289	.139	.205	.513	.214
γ	Median	-.491	-.027	.411	-.649	.100	.757
	SD	.384	.216	.349	.100	.252	.076
	LCI	.587	1.012	.552	.537	.587	.464
	CP	.800	.800	.400	.800	.200	1.000
	Bias	.309	-.027	-.389	.151	.100	-.043
	RMSE	.492	.217	.523	.181	.271	.088
	AVRB	.386	-	.486	.189	-	.054

Table 140 (continued).

	Median	6.069	5.763	7.827	5.692	6.191	4.953
	SD	1.535	1.162	4.956	1.734	1.160	.346
	LCI	7.689	11.043	10.854	4.341	8.961	2.846
ν_1	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	1.069	.763	2.827	.692	1.191	-.047
	RMSE	1.870	1.390	5.706	1.867	1.663	.349
	AVRB	.214	.153	.565	.138	.238	.009
	Median	4.608	4.635	4.765	4.087	4.098	4.172
	SD	.973	.837	1.107	.326	.436	.485
	LCI	3.835	3.915	3.880	2.373	2.369	2.387
σ^2	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.608	.635	.765	.087	.098	.172
	RMSE	1.147	1.05	1.346	.337	.446	.515
	AVRB	.152	.159	.191	.022	.025	.043

Table 141 – Results of the simulation study for the mixed SGtBS1 regression model ($\nu_1 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.343	-.451	-.365	-.626	-.677	-.660
	SD	.263	.205	.230	.214	.137	.288
	LCI	1.096	1.006	.883	.629	.689	.457
	CP	1.000	1.000	.800	.800	1.000	.600
	Bias	.157	.049	.135	-.126	-.177	-.160
	RMSE	.306	.211	.267	.248	.224	.330
	AVRB	.314	.097	.270	.252	.353	.320
β_1	Mode	1.075	1.037	.929	.956	1.036	1.027
	SD	.106	.143	.094	.097	.034	.099
	LCI	.471	.482	.441	.288	.340	.273
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	.075	.037	-.071	-.044	.036	.027
	RMSE	.129	.148	.117	.107	.050	.103
	AVRB	.075	.037	.071	.044	.036	.027
ψ_0	Mode	-.922	-.956	-.550	-.813	-.593	-.947
	SD	.429	.309	.831	.322	.323	.173
	LCI	1.592	1.858	1.699	1.155	1.523	.842
	CP	.800	1.000	.800	1.000	.800	1.000
	Bias	.078	.044	.45	.187	.407	.053
	RMSE	.436	.312	.946	.372	.520	.181
	AVRB	.078	.044	.450	.187	.407	.053
ψ_1	Mode	.735	.640	.540	.427	.248	.395
	SD	.397	.186	.256	.452	.124	.275
	LCI	1.596	1.538	1.452	1.326	1.172	.798
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.235	.140	.040	-.073	-.252	-.105
	RMSE	.462	.233	.259	.458	.281	.295
	AVRB	.469	.280	.081	.146	.504	.210
γ	Mode	-.480	-.010	.407	-.645	.102	.754
	SD	.388	.225	.354	.096	.262	.073
	LCI	.587	1.012	.552	.537	.587	.464
	CP	.800	.800	.400	.800	.200	1.000
	Bias	.32	-.010	-.393	.155	.102	-.046
	RMSE	.503	.225	.529	.183	.281	.086
	AVRB	.400	-	.491	.194	-	.057

Table 141 (continued).

	Mode	4.769	4.458	5.594	5.416	4.895	4.646
	SD	.309	.204	1.873	1.746	1.191	.275
	LCI	7.689	11.043	10.854	4.341	8.961	2.846
ν	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	-.231	-.542	.594	.416	-.105	-.354
	RMSE	.386	.579	1.965	1.795	1.196	.448
	AVRB	.046	.108	.119	.083	.021	.071
	Mode	4.269	4.441	4.548	3.990	3.975	4.082
	SD	.847	.799	1.169	.349	.392	.504
	LCI	3.835	3.915	3.880	2.373	2.369	2.387
σ^2	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.269	.441	.548	-.010	-.025	.082
	RMSE	.888	.913	1.291	.349	.393	.510
	AVRB	.067	.110	.137	.003	.006	.021

Table 142 – Results of the simulation study for the mixed SGtBS1 regression model ($\nu_1 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.303	-.454	-.349	-.408	-.542	-.534
	SD	.389	.221	.227	.489	.547	.391
	LCI	.170	.651	.196	.136	.348	.162
	CP	$< .001$	1.000	$< .001$	$< .001$	.400	$< .001$
	Bias	.197	.046	.151	.092	-.042	-.034
	RMSE	.436	.225	.273	.498	.549	.393
	AVRB	.394	.092	.302	.183	.083	.068
β_1	Mean	.997	1.037	.985	1.019	.985	.974
	SD	.021	.036	.033	.044	.050	.025
	LCI	.123	.167	.143	.089	.117	.105
	CP	1.000	.800	1.000	.600	.600	.800
	Bias	-.003	.037	-.015	.019	-.015	-.026
	RMSE	.021	.052	.036	.048	.052	.036
	AVRB	.003	.037	.015	.019	.015	.026
ψ_0	Mean	-.659	-1.736	-1.512	-.807	-2.122	-1.024
	SD	.752	.561	.535	1.067	.883	.817
	LCI	1.307	1.514	1.709	.623	1.312	.771
	CP	.800	.400	.600	.200	.400	.400
	Bias	.341	-.736	-.512	.193	-1.122	-.024
	RMSE	.826	.925	.74	1.085	1.428	.817
	AVRB	.341	.736	.512	.193	1.122	.024
ψ_1	Mean	.608	.717	.520	.647	.597	.522
	SD	.305	.517	.225	.403	.249	.405
	LCI	1.205	1.442	1.063	.791	1.188	.758
	CP	.800	.800	1.000	.400	1.000	.600
	Bias	.108	.217	.020	.147	.097	.022
	RMSE	.324	.561	.226	.429	.267	.406
	AVRB	.217	.434	.040	.294	.195	.045
γ	Mean	-.855	.002	.791	-.825	.023	.740
	SD	.105	.177	.104	.085	.248	.049
	LCI	.311	.588	.414	.221	.445	.348
	CP	.600	.400	1.000	.800	$< .001$	1.000
	Bias	-.055	.002	-.009	-.025	.023	-.060
	RMSE	.118	.177	.104	.089	.249	.078
	AVRB	.068	-	.011	.031	-	.075

Table 142 (continued).

ν_1	Mean	53.017	22.820	22.318	51.010	16.048	40.374
	SD	38.936	18.048	8.625	29.136	14.149	34.662
	LCI	34.279	28.488	30.980	22.571	14.749	19.779
	CP	.800	.400	.600	.200	.400	.400
	Bias	23.017	-7.180	-7.682	21.010	-13.952	10.374
	RMSE	45.231	19.424	11.550	35.921	19.871	36.181
	AVRB	.767	.239	.256	.700	.465	.346
σ^2	Mean	3.784	3.816	3.791	3.991	3.985	4.027
	SD	.675	.735	.638	.191	.151	.139
	LCI	3.038	3.197	2.996	2.249	2.212	2.225
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.216	-.184	-.209	-.009	-.015	.027
	RMSE	.708	.758	.671	.191	.151	.141
	AVRB	.054	.046	.052	.002	.004	.007

Table 143 – Results of the simulation study for the mixed SGtBS1 regression model ($\nu_1 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.302	-.486	-.353	-.410	-.543	-.533
	SD	.388	.241	.228	.488	.549	.398
	LCI	.170	.651	.196	.136	.348	.162
	CP	< .001	1.000	< .001	< .001	.400	< .001
	Bias	.198	.014	.147	.090	-.043	-.033
	RMSE	.436	.241	.271	.496	.550	.399
	AVRB	.395	.028	.294	.181	.085	.065
β_1	Median	.996	1.037	.980	1.018	.984	.974
	SD	.022	.036	.032	.046	.052	.025
	LCI	.123	.167	.143	.089	.117	.105
	CP	1.000	.800	1.000	.600	.600	.800
	Bias	-.004	.037	-.020	.018	-.016	-.026
	RMSE	.022	.052	.037	.049	.055	.036
	AVRB	.004	.037	.020	.018	.016	.026
ψ_0	Median	-.613	-1.716	-1.572	-.820	-2.108	-1.017
	SD	.749	.661	.630	1.078	.866	.810
	LCI	1.307	1.514	1.709	.623	1.312	.771
	CP	.800	.400	.600	.200	.400	.400
	Bias	.387	-.716	-.572	.180	-1.108	-.017
	RMSE	.843	.975	.851	1.093	1.406	.811
	AVRB	.387	.716	.572	.180	1.108	.017
ψ_1	Median	.574	.725	.500	.637	.599	.523
	SD	.329	.504	.205	.403	.240	.387
	LCI	1.205	1.442	1.063	.791	1.188	.758
	CP	.800	.800	1.000	.400	1.000	.600
	Bias	.074	.225	< .001	.137	.099	.023
	RMSE	.337	.552	.205	.426	.259	.388
	AVRB	.148	.450	.001	.275	.199	.045
γ	Median	-.874	.006	.815	-.835	.019	.751
	SD	.097	.143	.106	.083	.241	.050
	LCI	.311	.588	.414	.221	.445	.348
	CP	.600	.400	1.000	.800	< .001	1.000
	Bias	-.074	.006	.015	-.035	.019	-.049
	RMSE	.122	.143	.107	.090	.242	.070
	AVRB	.092	-	.019	.043	-	.061

Table 143 (continued).

ν_1	Median	53.994	22.688	19.975	50.783	15.659	39.428
	SD	39.486	18.484	8.485	29.584	12.897	35.505
	LCI	34.279	28.488	30.98	22.571	14.749	19.779
	CP	$< .001$	$< .001$	.200	$< .001$	.400	$< .001$
	Bias	48.994	17.688	14.975	45.783	10.659	34.428
	RMSE	62.925	25.584	17.212	54.510	16.732	49.456
	AVRB	9.799	3.538	2.995	9.157	2.132	6.886
σ^2	Median	3.683	3.708	3.688	3.941	3.936	3.970
	SD	.655	.715	.615	.192	.144	.136
	LCI	3.038	3.197	2.996	2.249	2.212	2.225
	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.317	-.292	-.312	-.059	-.064	-.030
	RMSE	.728	.772	.689	.201	.157	.140
	AVRB	.079	.073	.078	.015	.016	.008

Table 144 – Results of the simulation study for the mixed SGtBS1 regression model ($\nu_1 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.305	-.499	-.349	-.408	-.545	-.534
	SD	.388	.238	.226	.488	.549	.392
	LCI	.170	.651	.196	.136	.348	.162
	CP	$<.001$	1.000	$<.001$	$<.001$	.400	$<.001$
	Bias	.195	.001	.151	.092	-.045	-.034
	RMSE	.435	.238	.271	.497	.550	.394
	AVRB	.391	.002	.301	.184	.089	.069
β_1	Mode	.996	1.037	.984	1.020	.984	.975
	SD	.022	.035	.032	.045	.051	.025
	LCI	.123	.167	.143	.089	.117	.105
	CP	1.000	.800	1.000	.600	.600	.800
	Bias	-.004	.037	-.016	.020	-.016	-.025
	RMSE	.022	.051	.036	.049	.053	.036
	AVRB	.004	.037	.016	.020	.016	.025
ψ_0	Mode	-.584	-1.673	-1.502	-.821	-2.049	-1.009
	SD	.765	.836	.836	1.076	.868	.839
	LCI	1.307	1.514	1.709	.623	1.312	.771
	CP	.800	.400	.600	.200	.400	.400
	Bias	.416	-.673	-.502	.179	-1.049	-.009
	RMSE	.871	1.073	.975	1.091	1.362	.839
	AVRB	.416	.673	.502	.179	1.049	.009
ψ_1	Mode	.510	.744	.466	.612	.603	.525
	SD	.336	.490	.135	.414	.211	.386
	LCI	1.205	1.442	1.063	.791	1.188	.758
	CP	.800	.800	1.000	.400	1.000	.600
	Bias	.010	.244	-.034	.112	.103	.025
	RMSE	.336	.548	.139	.429	.235	.386
	AVRB	.021	.489	.068	.225	.206	.050
γ	Mode	-.865	.006	.805	-.827	.019	.748
	SD	.096	.146	.096	.085	.247	.047
	LCI	.311	.588	.414	.221	.445	.348
	CP	.600	.400	1.000	.800	$<.001$	1.000
	Bias	-.065	.006	.005	-.027	.019	-.052
	RMSE	.116	.146	.096	.089	.247	.07
	AVRB	.082	-	.007	.033	-	.065

Table 144 (continued).

	Mode	56.346	17.999	14.868	50.408	13.260	37.360
	SD	40.415	11.623	6.820	29.961	12.028	37.522
	LCI	34.279	28.488	30.980	22.571	14.749	19.779
ν	CP	$< .001$	$< .001$	.200	$< .001$	.400	$< .001$
	Bias	51.346	12.999	9.868	45.408	8.260	32.360
	RMSE	65.344	17.438	11.995	54.402	14.591	49.549
	AVRB	10.269	2.600	1.974	9.082	1.652	6.472
	Mode	3.517	3.577	3.525	3.866	3.844	3.877
	SD	.635	.744	.558	.205	.115	.150
	LCI	3.038	3.197	2.996	2.249	2.212	2.225
σ^2	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.483	-.423	-.475	-.134	-.156	-.123
	RMSE	.798	.856	.733	.244	.194	.194
	AVRB	.121	.106	.119	.033	.039	.031

Mixed SGtBS2 regression model

Table 145 – Results of the simulation study for the mixed SGtBS2 regression model ($\nu_1 = 15, \nu_2 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.440	-.377	-.372	-.442	-.414	-.421
	SD	.209	.218	.184	.200	.212	.223
	LCI	1.081	1.343	1.077	.680	.853	.883
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.060	.123	.128	.058	.086	.079
	RMSE	.218	.250	.224	.209	.229	.237
	AVRB	.120	.245	.255	.117	.173	.157
β_1	Mean	1.012	.921	.948	1.040	.960	.977
	SD	.210	.115	.171	.161	.057	.130
	LCI	.639	.768	.682	.476	.564	.527
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.012	-.079	-.052	.040	-.040	-.023
	RMSE	.210	.139	.179	.166	.070	.132
	AVRB	.012	.079	.052	.040	.040	.023
γ	Mean	-.700	.018	.727	-.778	-.039	.680
	SD	.159	.334	.066	.107	.202	.146
	LCI	.578	.861	.602	.382	.551	.431
	CP	1.000	.600	1.000	1.000	.400	.600
	Bias	.100	.018	-.073	.022	-.039	-.120
	RMSE	.188	.334	.099	.109	.205	.189
	AVRB	.125	-	.091	.027	-	.151
ν_1	Mean	17.396	15.742	15.587	17.449	17.969	12.965
	SD	5.857	3.690	1.008	8.868	6.853	3.245
	LCI	45.017	40.514	31.351	29.044	30.268	21.099
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	2.396	.742	.587	2.449	2.969	-2.035
	RMSE	6.328	3.764	1.167	9.200	7.469	3.830
	AVRB	.160	.049	.039	.163	.198	.136
ν_2	Mean	5.841	5.505	5.309	5.747	6.252	4.232
	SD	1.912	1.354	.391	3.117	2.721	1.295
	LCI	16.815	15.494	11.864	11.068	11.633	7.797
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.841	.505	.309	.747	1.252	-.768

Table 145 (continued).

	RMSE	2.089	1.446	.498	3.205	2.995	1.505
	AVRB	.168	.101	.062	.149	.250	.154
	Mean	4.961	4.992	4.882	3.992	4.095	4.284
	SD	.966	1.215	1.109	.493	.597	.449
	LCI	4.151	4.207	4.063	2.269	2.377	2.522
σ^2	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.961	.992	.882	-.008	.095	.284
	RMSE	1.363	1.568	1.417	.493	.605	.531
	AVRB	.240	.248	.220	.002	.024	.071

Table 146 – Results of the simulation study for the mixed SGtBS2 regression model ($\nu_1 = 15, \nu_2 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.440	-.374	-.356	-.433	-.412	-.419
	SD	.217	.223	.182	.222	.200	.232
	LCI	1.081	1.343	1.077	.680	.853	.883
β_0	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.060	.126	.144	.067	.088	.081
	RMSE	.225	.256	.232	.231	.219	.246
	AVRB	.119	.252	.287	.134	.177	.163
	Median	1.005	.927	.949	1.044	.959	.981
	SD	.212	.115	.173	.162	.062	.127
	LCI	.639	.768	.682	.476	.564	.527
β_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.005	-.073	-.051	.044	-.041	-.019
	RMSE	.212	.136	.180	.167	.075	.128
	AVRB	.005	.073	.051	.044	.041	.019
	Median	-.726	.025	.759	-.800	-.027	.693
	SD	.162	.327	.077	.111	.194	.155
	LCI	.578	.861	.602	.382	.551	.431
γ	CP	1.000	.600	1.000	1.000	.400	.600
	Bias	.074	.025	-.041	$< .001$	-.027	-.107
	RMSE	.178	.328	.087	.111	.196	.188
	AVRB	.092	-	.052	.001	-	.134
	Median	13.268	12.648	12.929	15.825	16.241	11.387
	SD	3.108	1.765	.453	7.864	6.73	2.679
	LCI	45.017	40.514	31.351	29.044	30.268	21.099
ν_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-1.732	-2.352	-2.071	.825	1.241	-3.613
	RMSE	3.558	2.941	2.120	7.907	6.843	4.498
	AVRB	.115	.157	.138	.055	.083	.241
	Median	4.290	4.306	4.278	5.095	5.566	3.611
	SD	.877	.818	.249	2.752	2.613	1.010
	LCI	16.815	15.494	11.864	11.068	11.633	7.797
ν_2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.710	-.694	-.722	.095	.566	-1.389
	RMSE	1.128	1.073	.763	2.754	2.673	1.717

Table 146 (continued).

	AVRB	.142	.139	.144	.019	.113	.278
	Median	4.827	4.870	4.726	3.940	4.049	4.228
	SD	.934	1.182	1.076	.485	.582	.434
	LCI	4.151	4.207	4.063	2.269	2.377	2.522
σ^2	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.827	.870	.726	-.060	.049	.228
	RMSE	1.247	1.467	1.298	.489	.584	.490
	AVRB	.207	.217	.182	.015	.012	.057

Table 147 – Results of the simulation study for the mixed SGtBS2 regression model ($\nu_1 = 15, \nu_2 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.427	-.369	-.363	-.442	-.412	-.422
	SD	.228	.220	.176	.211	.200	.247
	LCI	1.081	1.343	1.077	.680	.853	.883
β_0	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.073	.131	.137	.058	.088	.078
	RMSE	.239	.256	.224	.219	.218	.259
	AVRB	.145	.262	.274	.116	.176	.156
	Mode	1.009	.928	.946	1.042	.958	.979
	SD	.212	.113	.172	.164	.059	.129
	LCI	.639	.768	.682	.476	.564	.527
β_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.009	-.072	-.054	.042	-.042	-.021
	RMSE	.212	.134	.180	.170	.073	.131
	AVRB	.009	.072	.054	.042	.042	.021
	Mode	-.728	.029	.754	-.789	-.030	.688
	SD	.147	.325	.064	.104	.195	.145
	LCI	.578	.861	.602	.382	.551	.431
γ	CP	1.000	.600	1.000	1.000	.400	.600
	Bias	.072	.029	-.046	.011	-.030	-.112
	RMSE	.163	.326	.078	.104	.198	.184
	AVRB	.090	-	.057	.014	-	.140
	Mode	8.999	8.539	9.163	9.332	13.010	8.652
	SD	1.485	1.169	.764	1.705	5.346	.631
	LCI	45.017	40.514	31.351	29.044	30.268	21.099
ν_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-6.001	-6.461	-5.837	-5.668	-1.990	-6.348
	RMSE	6.182	6.566	5.887	5.919	5.705	6.379
	AVRB	.400	.431	.389	.378	.133	.423
	Mode	2.600	2.736	2.523	2.860	4.084	2.653
	SD	.300	.613	.147	.702	1.981	.339
	LCI	16.815	15.494	11.864	11.068	11.633	7.797
ν_2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-2.400	-2.264	-2.477	-2.140	-.916	-2.347
	RMSE	2.419	2.346	2.481	2.252	2.182	2.371

Table 147 (continued).

	AVRB	.480	.453	.495	.428	.183	.469
	Mode	4.673	4.568	4.481	3.866	3.983	4.139
	SD	.830	1.068	1.058	.483	.560	.422
	LCI	4.151	4.207	4.063	2.269	2.377	2.522
σ^2	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	.673	.568	.481	-.134	-.017	.139
	RMSE	1.068	1.210	1.162	.501	.561	.445
	AVRB	.168	.142	.120	.034	.004	.035

Table 148 – Results of the simulation study for the mixed SGtBS2 regression model ($\nu_1 = \nu_2 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.290	-.322	-.481	-.503	-.467	-.520
	SD	.243	.265	.547	.228	.290	.372
	LCI	1.158	1.389	1.289	.632	.915	.917
β_0	CP	1.000	1.000	.600	.800	1.000	.800
	Bias	.210	.178	.019	-.003	.033	-.020
	RMSE	.321	.319	.548	.228	.292	.372
	AVRB	.419	.356	.037	.005	.066	.041
	Mean	.953	.780	1.070	1.024	.887	.970
	SD	.294	.367	.201	.295	.263	.212
	LCI	.970	1.194	1.070	.698	.790	.739
β_1	CP	.800	.800	1.000	.800	1.000	1.000
	Bias	-.047	-.220	.070	.024	-.113	-.030
	RMSE	.298	.427	.213	.296	.286	.214
	AVRB	.047	.220	.070	.024	.113	.030
	Mean	-.711	-.002	.795	-.821	-.007	.710
	SD	.207	.294	.071	.108	.421	.169
	LCI	.583	.821	.518	.416	.636	.471
γ	CP	1.000	.400	1.000	.800	$<$ .001	1.000
	Bias	.089	-.002	-.005	-.021	-.007	-.090
	RMSE	.225	.294	.071	.110	.421	.191
	AVRB	.111	-	.007	.026	-	.113
	Mean	11.185	10.808	12.257	14.448	11.379	16.053
	SD	2.901	2.029	2.877	5.794	1.500	9.340
	LCI	22.961	33.857	25.508	31.368	23.192	44.275
ν_1	CP	.600	.600	.200	.800	.400	.600
	Bias	-18.815	-19.192	-17.743	-15.552	-18.621	-13.947
	RMSE	19.037	19.299	17.975	16.597	18.682	16.785
	AVRB	.627	.640	.591	.518	.621	.465
	Mean	10.211	10.072	11.264	12.909	10.322	14.962
	SD	3.013	1.994	2.979	5.977	1.766	10.384
	LCI	24.308	37.851	26.674	31.938	24.672	46.923
ν_2	CP	.600	.600	.200	.800	.400	.600
	Bias	-19.789	-19.928	-18.736	-17.091	-19.678	-15.038
	RMSE	20.017	20.027	18.971	18.106	19.757	18.275

Table 148 (continued).

	AVRB	.660	.664	.625	.570	.656	.501
	Mean	4.732	4.803	4.786	4.196	4.176	4.300
	SD	.983	1.006	1.046	1.029	.750	.866
	LCI	3.970	4.322	4.163	2.572	2.465	2.577
σ^2	CP	.800	.800	.800	.800	1.000	.800
	Bias	.732	.803	.786	.196	.176	.300
	RMSE	1.226	1.288	1.309	1.047	.771	.916
	AVRB	.183	.201	.196	.049	.044	.075

Table 149 – Results of the simulation study for the mixed SGtBS2 regression model ($\nu_1 = \nu_2 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.271	-.329	-.496	-.484	-.472	-.515
	SD	.258	.270	.546	.261	.292	.382
	LCI	1.158	1.389	1.289	.632	.915	.917
	CP	1.000	1.000	.600	.800	1.000	.800
	Bias	.229	.171	.004	.016	.028	-.015
	RMSE	.345	.320	.546	.261	.293	.382
	AVRB	.458	.342	.009	.032	.056	.029
β_1	Median	.933	.781	1.068	1.057	.888	.966
	SD	.262	.364	.216	.324	.260	.211
	LCI	.970	1.194	1.070	.698	.790	.739
	CP	.800	.800	1.000	.800	1.000	1.000
	Bias	-.067	-.219	.068	.057	-.112	-.034
	RMSE	.270	.425	.227	.329	.283	.214
	AVRB	.067	.219	.068	.057	.112	.034
γ	Median	-.739	-.001	.832	-.843	-.001	.725
	SD	.225	.270	.074	.105	.419	.169
	LCI	.583	.821	.518	.416	.636	.471
	CP	1.000	.400	1.000	.800	$< .001$	1.000
	Bias	.061	-.001	.032	-.043	-.001	-.075
	RMSE	.233	.270	.081	.114	.419	.185
	AVRB	.076	-	.040	.054	-	.094
ν_1	Median	9.502	8.455	10.149	11.916	9.764	11.049
	SD	2.162	1.451	1.599	4.726	.781	3.316
	LCI	22.961	33.857	25.508	31.368	23.192	44.275
	CP	.600	.600	.200	.800	.400	.600
	Bias	-20.498	-21.545	-19.851	-18.084	-20.236	-18.951
	RMSE	20.611	21.594	19.915	18.692	20.251	19.239
	AVRB	.683	.718	.662	.603	.675	.632
ν_2	Median	8.512	7.457	9.024	10.416	8.638	9.479
	SD	2.353	1.216	1.552	4.914	.981	3.383
	LCI	24.308	37.851	26.674	31.938	24.672	46.923
	CP	.600	.600	.200	.800	.400	.600
	Bias	-21.488	-22.543	-20.976	-19.584	-21.362	-20.521
	RMSE	21.617	22.575	21.034	20.191	21.384	20.798

Table 149 (continued).

	AVRB	.716	.751	.699	.653	.712	.684
	Median	4.606	4.667	4.658	4.118	4.117	4.239
	SD	.957	.970	1.012	1.006	.722	.865
	LCI	3.970	4.322	4.163	2.572	2.465	2.577
σ^2	CP	.800	.800	.800	.800	1.000	.800
	Bias	.606	.667	.658	.118	.117	.239
	RMSE	1.132	1.177	1.207	1.013	.732	.898
	AVRB	.152	.167	.164	.029	.029	.060

Table 150 – Results of the simulation study for the mixed SGtBS2 regression model ($\nu_1 = \nu_2 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.276	-.342	-.490	-.472	-.475	-.513
	SD	.244	.278	.556	.253	.287	.388
	LCI	1.158	1.389	1.289	.632	.915	.917
β_0	CP	1.000	1.000	.600	.800	1.000	.800
	Bias	.224	.158	.010	.028	.025	-.013
	RMSE	.331	.320	.556	.254	.288	.388
	AVRB	.448	.316	.021	.056	.050	.026
	Mode	.936	.781	1.044	1.056	.888	.965
	SD	.264	.362	.227	.321	.262	.213
	LCI	.970	1.194	1.070	.698	.790	.739
β_1	CP	.800	.800	1.000	.800	1.000	1.000
	Bias	-.064	-.219	.044	.056	-.112	-.035
	RMSE	.272	.423	.231	.326	.285	.216
	AVRB	.064	.219	.044	.056	.112	.035
	Mode	-.721	-.006	.824	-.842	-.001	.722
	SD	.227	.258	.060	.095	.416	.165
	LCI	.583	.821	.518	.416	.636	.471
γ	CP	1.000	.400	1.000	.800	$< .001$	1.000
	Bias	.079	-.006	.024	-.042	-.001	-.078
	RMSE	.241	.258	.064	.103	.416	.182
	AVRB	.098	-	.030	.052	-	.098
	Mode	7.421	6.241	8.128	9.096	7.807	8.213
	SD	2.236	.849	1.399	3.129	1.364	2.214
	LCI	22.961	33.857	25.508	31.368	23.192	44.275
ν_1	CP	.600	.600	.200	.800	.400	.600
	Bias	-22.579	-23.759	-21.872	-20.904	-22.193	-21.787
	RMSE	22.689	23.774	21.917	21.137	22.235	21.899
	AVRB	.753	.792	.729	.697	.740	.726
	Mode	6.148	5.338	7.456	7.458	6.208	7.049
	SD	2.677	.544	1.993	2.539	.954	1.882
	LCI	24.308	37.851	26.674	31.938	24.672	46.923
ν_2	CP	.600	.600	.200	.800	.400	.600
	Bias	-23.852	-24.662	-22.544	-22.542	-23.792	-22.951
	RMSE	24.002	24.668	22.632	22.684	23.811	23.028

Table 150 (continued).

	AVRB	.795	.822	.751	.751	.793	.765
	Mode	4.346	4.297	4.356	4.028	4.027	4.129
	SD	.840	.672	.779	.996	.638	.862
	LCI	3.970	4.322	4.163	2.572	2.465	2.577
σ^2	CP	.800	.800	.800	.800	1.000	.800
	Bias	.346	.297	.356	.028	.027	.129
	RMSE	.909	.734	.856	.996	.638	.871
	AVRB	.087	.074	.089	.007	.007	.032

Mixed StBS regression model

Table 151 – Results of the simulation study for the mixed StBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.481	-.445	-.416	-.389	-.453	-.591
	SD	.337	.264	.190	.352	.216	.178
	LCI	1.296	1.286	1.334	.837	.912	.876
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.019	.055	.084	.111	.047	-.091
	RMSE	.337	.270	.208	.369	.221	.200
	AVRB	.037	.111	.168	.222	.095	.182
β_1	Mean	1.079	.945	.880	.757	.904	1.204
	SD	.137	.128	.106	.232	.301	.285
	LCI	.954	1.013	.979	.680	.690	.678
	CP	1.000	1.000	1.000	.400	.800	.400
	Bias	.079	-.055	-.120	-.243	-.096	.204
	RMSE	.158	.139	.160	.336	.316	.350
	AVRB	.079	.055	.120	.243	.096	.204
ψ_0	Mean	-.979	-.885	-.973	-.858	-.934	-.925
	SD	.242	.073	.106	.138	.234	.180
	LCI	1.034	.971	.908	.750	.732	.651
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.021	.115	.027	.142	.066	.075
	RMSE	.243	.137	.110	.198	.243	.196
	AVRB	.021	.115	.027	.142	.066	.075
ψ_1	Mean	.519	.606	.591	.288	.413	.395
	SD	.543	.228	.106	.248	.279	.237
	LCI	1.635	1.365	1.235	1.121	1.060	.845
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.019	.106	.091	-.212	-.087	-.105
	RMSE	.543	.251	.140	.327	.292	.259
	AVRB	.037	.211	.183	.424	.175	.210
γ	Mean	-.564	.046	.600	-.649	-.049	.629
	SD	.215	.347	.110	.108	.357	.176
	LCI	.803	.859	.820	.607	.761	.610
	CP	.800	.600	1.000	1.000	.400	.800
	Bias	.236	.046	-.200	.151	-.049	-.171
	RMSE	.319	.350	.228	.186	.361	.245

Table 151 (continued).

	AVRB	.295	-	.250	.189	-	.213
	Mean	6.784	11.094	6.643	7.441	6.483	6.340
	SD	1.771	3.594	1.741	2.161	1.251	1.598
	LCI	13.070	37.351	12.354	16.152	9.726	9.565
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.784	6.094	1.643	2.441	1.483	1.340
	RMSE	2.514	7.075	2.394	3.261	1.941	2.086
	AVRB	.357	1.219	.329	.488	.297	.268
	Mean	4.331	4.496	4.523	4.346	4.406	4.293
	SD	1.008	.947	1.253	.496	.502	.641
	LCI	3.717	3.753	3.884	2.516	2.590	2.546
σ^2	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.331	.496	.523	.346	.406	.293
	RMSE	1.061	1.069	1.357	.605	.646	.705
	AVRB	.083	.124	.131	.086	.102	.073

Table 152 – Results of the simulation study for the mixed StBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.476	-.447	-.427	-.386	-.458	-.590
	SD	.333	.265	.186	.357	.217	.183
	LCI	1.296	1.286	1.334	.837	.912	.876
β_0	CP	1.000	1.000	1.000	.8	1.000	1.000
	Bias	.024	.053	.073	.114	.042	-.090
	RMSE	.334	.270	.200	.374	.221	.203
	AVRB	.047	.106	.146	.228	.084	.180
	Median	1.077	.944	.886	.762	.905	1.200
	SD	.131	.133	.103	.229	.301	.283
	LCI	.954	1.013	.979	.680	.690	.678
β_1	CP	1.000	1.000	1.000	.400	.800	.400
	Bias	.077	-.056	-.114	-.238	-.095	.200
	RMSE	.152	.144	.153	.330	.316	.346
	AVRB	.077	.056	.114	.238	.095	.200
	Median	-.982	-.890	-.978	-.860	-.934	-.919
	SD	.247	.077	.105	.141	.232	.187
	LCI	1.034	.971	.908	.75	.732	.651
ψ_0	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.018	.110	.022	.140	.066	.081
	RMSE	.247	.134	.107	.199	.241	.204
	AVRB	.018	.110	.022	.140	.066	.081
	Median	.514	.602	.590	.287	.406	.390
	SD	.555	.230	.117	.244	.278	.240
	LCI	1.635	1.365	1.235	1.121	1.06	.845
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.014	.102	.090	-.213	-.094	-.110
	RMSE	.555	.252	.147	.324	.294	.264
	AVRB	.028	.205	.180	.426	.188	.219
	Median	-.590	.051	.637	-.665	-.055	.648
	SD	.218	.346	.086	.116	.354	.178
	LCI	.803	.859	.820	.607	.761	.610
γ	CP	.800	.600	1.000	1.000	.400	.800
	Bias	.210	.051	-.163	.135	-.055	-.152
	RMSE	.303	.350	.184	.178	.358	.234
	AVRB	.263	-	.204	.169	-	.190

Table 152 (continued).

	Median	5.617	7.605	5.503	6.061	5.678	5.619
	SD	.996	1.849	.800	1.100	.744	1.03
	LCI	13.070	37.351	12.354	16.152	9.726	9.565
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.617	2.605	.503	1.061	.678	.619
	RMSE	1.172	3.195	.945	1.529	1.007	1.202
	AVRB	.123	.521	.101	.212	.136	.124
	Median	4.198	4.369	4.386	4.290	4.339	4.222
	SD	.999	.928	1.206	.492	.488	.626
	LCI	3.717	3.753	3.884	2.516	2.59	2.546
σ^2	CP	1.000	1.000	.8	1.000	1.000	1.000
	Bias	.198	.369	.386	.290	.339	.222
	RMSE	1.019	.998	1.266	.571	.594	.664
	AVRB	.050	.092	.097	.073	.085	.056

Table 153 – Results of the simulation study for the mixed StBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.478	-.453	-.444	-.381	-.462	-.588
	SD	.324	.269	.183	.364	.218	.183
	LCI	1.296	1.286	1.334	.837	.912	.876
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.022	.047	.056	.119	.038	-.088
	RMSE	.325	.273	.192	.384	.222	.203
	AVRB	.044	.094	.112	.239	.075	.176
β_1	Mode	1.075	.944	.889	.761	.904	1.206
	SD	.131	.138	.102	.229	.303	.287
	LCI	.954	1.013	.979	.680	.690	.678
	CP	1.000	1.000	1.000	.400	.800	.400
	Bias	.075	-.056	-.111	-.239	-.096	.206
	RMSE	.151	.149	.150	.330	.318	.353
	AVRB	.075	.056	.111	.239	.096	.206
ψ_0	Mode	-.987	-.893	-.977	-.854	-.934	-.920
	SD	.250	.084	.105	.142	.235	.190
	LCI	1.034	.971	.908	.750	.732	.651
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.013	.107	.023	.146	.066	.080
	RMSE	.250	.136	.107	.203	.245	.206
	AVRB	.013	.107	.023	.146	.066	.080
ψ_1	Mode	.517	.597	.588	.292	.404	.383
	SD	.553	.248	.131	.241	.274	.240
	LCI	1.635	1.365	1.235	1.121	1.06	.845
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.017	.097	.088	-.208	-.096	-.117
	RMSE	.553	.266	.157	.318	.29	.267
	AVRB	.034	.195	.175	.415	.192	.233
γ	Mode	-.621	.055	.664	-.669	-.052	.651
	SD	.172	.343	.056	.111	.354	.171
	LCI	.803	.859	.82	.607	.761	.610
	CP	.800	.600	1.000	1.000	.400	.800
	Bias	.179	.055	-.136	.131	-.052	-.149
	RMSE	.249	.347	.147	.172	.358	.227
	AVRB	.224	-	.170	.164	-	.187

Table 153 (continued).

ν	Mode	4.640	4.996	4.518	4.848	4.720	4.780
	SD	.451	.558	.309	.780	.342	.677
	LCI	13.070	37.351	12.354	16.152	9.726	9.565
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.360	-.004	-.482	-.152	-.280	-.220
	RMSE	.577	.558	.573	.795	.442	.712
	AVRB	.072	.001	.096	.030	.056	.044
σ^2	Mode	4.010	4.157	4.172	4.168	4.188	4.097
	SD	1.018	.822	1.112	.472	.420	.564
	LCI	3.717	3.753	3.884	2.516	2.590	2.546
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.010	.157	.172	.168	.188	.097
	RMSE	1.018	.837	1.125	.501	.460	.572
	AVRB	.003	.039	.043	.042	.047	.024

Table 154 – Results of the simulation study for the mixed StBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.210	-.313	-.376	-.532	-.470	-.390
	SD	.314	.409	.245	.126	.157	.200
	LCI	1.332	1.283	1.164	.717	.917	.834
	CP	.800	.800	1.000	1.000	1.000	1.000
	Bias	.290	.187	.124	-.032	.030	.110
	RMSE	.427	.450	.274	.130	.159	.228
	AVRB	.579	.373	.249	.065	.060	.219
β_1	Mean	.827	.945	1.169	1.081	1.038	1.076
	SD	.185	.425	.151	.110	.225	.155
	LCI	.798	.903	.767	.467	.607	.529
	CP	.800	.800	1.000	1.000	.800	.800
	Bias	-.173	-.055	.169	.081	.038	.076
	RMSE	.253	.428	.226	.136	.228	.173
	AVRB	.173	.055	.169	.081	.038	.076
ψ_0	Mean	-1.144	-1.228	-1.040	-1.188	-1.122	-1.022
	SD	.385	.223	.246	.260	.177	.060
	LCI	1.001	.956	.864	.726	.686	.589
	CP	.800	.800	1.000	.800	.800	1.000
	Bias	-.144	-.228	-.040	-.188	-.122	-.022
	RMSE	.411	.319	.249	.321	.215	.064
	AVRB	.144	.228	.040	.188	.122	.022
ψ_1	Mean	.594	.802	.399	.570	.564	.357
	SD	.675	.409	.307	.480	.363	.081
	LCI	1.584	1.547	1.219	1.169	.997	.799
	CP	.600	.800	1.000	.600	.800	1.000
	Bias	.094	.302	-.101	.070	.064	-.143
	RMSE	.682	.509	.323	.485	.369	.164
	AVRB	.187	.605	.202	.140	.128	.285
γ	Mean	-.664	-.146	.690	-.822	.064	.793
	SD	.182	.239	.208	.095	.368	.080
	LCI	.586	.965	.667	.362	.655	.441
	CP	1.000	.600	.800	1.000	.400	1.000
	Bias	.136	-.146	-.110	-.022	.064	-.007
	RMSE	.227	.280	.235	.097	.374	.081
	AVRB	.169	-	.138	.027	-	.009

Table 154 (continued).

	Mean	20.599	17.988	19.120	18.615	22.405	25.947
	SD	4.254	4.299	8.154	5.349	6.046	10.993
	LCI	70.844	58.476	60.792	60.467	80.443	78.523
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-9.401	-12.012	-10.880	-11.385	-7.595	-4.053
	RMSE	10.318	12.758	13.596	12.579	9.708	11.716
	AVRB	.313	.400	.363	.379	.253	.135
	Mean	4.437	4.494	4.557	4.070	4.091	4.042
	SD	.995	.847	.742	.602	.747	.549
	LCI	3.660	3.757	3.845	2.396	2.419	2.319
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.437	.494	.557	.070	.091	.042
	RMSE	1.087	.981	.928	.606	.752	.550
	AVRB	.109	.123	.139	.017	.023	.011

Table 155 – Results of the simulation study for the mixed StBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.217	-.317	-.336	-.530	-.476	-.378
	SD	.298	.412	.222	.106	.152	.214
	LCI	1.332	1.283	1.164	.717	.917	.834
	CP	.800	.800	1.000	1.000	1.000	1.000
	Bias	.283	.183	.164	-.030	.024	.122
	RMSE	.411	.451	.276	.110	.154	.246
	AVRB	.566	.367	.328	.059	.049	.244
β_1	Median	.827	.943	1.167	1.082	1.041	1.073
	SD	.197	.432	.155	.108	.223	.142
	LCI	.798	.903	.767	.467	.607	.529
	CP	.800	.800	1.000	1.000	.800	.800
	Bias	-.173	-.057	.167	.082	.041	.073
	RMSE	.262	.435	.227	.135	.226	.160
	AVRB	.173	.057	.167	.082	.041	.073
ψ_0	Median	-1.138	-1.234	-1.025	-1.189	-1.121	-1.016
	SD	.378	.221	.275	.263	.175	.060
	LCI	1.001	.956	.864	.726	.686	.589
	CP	.800	.800	1.000	.800	.800	1.000
	Bias	-.138	-.234	-.025	-.189	-.121	-.016
	RMSE	.402	.322	.277	.324	.212	.062
	AVRB	.138	.234	.025	.189	.121	.016
ψ_1	Median	.582	.809	.379	.576	.558	.365
	SD	.658	.405	.308	.483	.361	.076
	LCI	1.584	1.547	1.219	1.169	.997	.799
	CP	.600	.800	1.000	.600	.800	1.000
	Bias	.082	.309	-.121	.076	.058	-.135
	RMSE	.663	.510	.331	.489	.365	.155
	AVRB	.163	.619	.243	.153	.116	.271
γ	Median	-.681	-.130	.737	-.844	.063	.814
	SD	.194	.233	.242	.095	.369	.093
	LCI	.586	.965	.667	.362	.655	.441
	CP	1.000	.600	.800	1.000	.400	1.000
	Bias	.119	-.130	-.063	-.044	.063	.014
	RMSE	.227	.267	.251	.105	.375	.094
	AVRB	.149	-	.079	.055	-	.018

Table 155 (continued).

ν	Median	13.969	12.558	13.612	12.700	15.618	19.734
	SD	3.629	2.972	5.771	2.642	4.178	11.228
	LCI	70.844	58.476	60.792	60.467	80.443	78.523
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-16.031	-17.442	-16.388	-17.300	-14.382	-10.266
	RMSE	16.437	17.694	17.374	17.500	14.976	15.214
	AVRB	.534	.581	.546	.577	.479	.342
σ^2	Median	4.328	4.371	4.423	4.012	4.035	3.987
	SD	.979	.818	.731	.606	.739	.530
	LCI	3.66	3.757	3.845	2.396	2.419	2.319
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.328	.371	.423	.012	.035	-.013
	RMSE	1.032	.898	.845	.606	.740	.530
	AVRB	.082	.093	.106	.003	.009	.003

Table 156 – Results of the simulation study for the mixed StBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.224	-.324	-.323	-.516	-.476	-.374
	SD	.300	.421	.222	.113	.154	.209
	LCI	1.332	1.283	1.164	.717	.917	.834
	CP	.800	.800	1.000	1.000	1.000	1.000
	Bias	.276	.176	.177	-.016	.024	.126
	RMSE	.407	.456	.284	.114	.156	.244
	AVRB	.551	.352	.354	.031	.047	.253
β_1	Mode	.828	.944	1.165	1.082	1.038	1.076
	SD	.197	.433	.158	.107	.226	.146
	LCI	.798	.903	.767	.467	.607	.529
	CP	.800	.800	1.000	1.000	.800	.800
	Bias	-.172	-.056	.165	.082	.038	.076
	RMSE	.262	.437	.229	.135	.229	.165
	AVRB	.172	.056	.165	.082	.038	.076
ψ_0	Mode	-1.133	-1.241	-1.022	-1.195	-1.119	-1.02
	SD	.369	.217	.274	.256	.177	.064
	LCI	1.001	.956	.864	.726	.686	.589
	CP	.800	.800	1.000	.800	.800	1.000
	Bias	-.133	-.241	-.022	-.195	-.119	-.020
	RMSE	.392	.325	.275	.322	.213	.067
	AVRB	.133	.241	.022	.195	.119	.020
ψ_1	Mode	.526	.818	.362	.591	.553	.368
	SD	.655	.399	.295	.479	.355	.072
	LCI	1.584	1.547	1.219	1.169	.997	.799
	CP	.600	.800	1.000	.600	.800	1.000
	Bias	.026	.318	-.138	.091	.053	-.132
	RMSE	.655	.511	.325	.488	.359	.151
	AVRB	.052	.637	.276	.182	.107	.264
γ	Mode	-.679	-.118	.730	-.833	.062	.810
	SD	.191	.226	.231	.088	.375	.081
	LCI	.586	.965	.667	.362	.655	.441
	CP	1.000	.600	.800	1.000	.400	1.000
	Bias	.121	-.118	-.070	-.033	.062	.010
	RMSE	.226	.255	.241	.094	.380	.082
	AVRB	.151	-	.087	.041	-	.012

Table 156 (continued).

	Mode	9.729	8.950	8.173	8.306	10.665	9.866
	SD	3.515	2.542	2.847	1.058	2.544	4.280
	LCI	70.844	58.476	60.792	60.467	80.443	78.523
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-20.271	-21.050	-21.827	-21.694	-19.335	-20.134
	RMSE	20.574	21.202	22.012	21.720	19.502	20.584
	AVRB	.676	.702	.728	.723	.645	.671
	Mode	4.185	4.090	4.220	3.955	3.958	3.898
	SD	.940	.616	.702	.633	.756	.515
	LCI	3.660	3.757	3.845	2.396	2.419	2.319
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.185	.090	.220	-.045	-.042	-.102
	RMSE	.958	.623	.735	.635	.757	.525
	AVRB	.046	.022	.055	.011	.011	.025

Mixed SSLBS regression model

Table 157 – Results of the simulation study for the mixed SSLBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.468	-.607	-.634	-.467	-.563	-.591
	SD	.471	.423	.406	.125	.119	.250
	LCI	1.258	1.220	1.112	.787	.890	.847
	CP	1.000	.800	.800	1.000	1.000	.800
	Bias	.032	-.107	-.134	.033	-.063	-.091
	RMSE	.472	.436	.427	.130	.134	.266
	AVRB	.064	.213	.268	.067	.126	.181
β_1	Mean	.912	1.109	1.040	.826	.980	1.096
	SD	.167	.154	.211	.177	.167	.215
	LCI	.770	.883	.900	.658	.640	.595
	CP	.800	1.000	1.000	.800	1.000	.800
	Bias	-.088	.109	.040	-.174	-.020	.096
	RMSE	.189	.189	.215	.248	.169	.236
	AVRB	.088	.109	.040	.174	.020	.096
ψ_0	Mean	-1.106	-1.093	-1.065	-.977	-1.026	-1.006
	SD	.173	.181	.245	.286	.088	.156
	LCI	1.050	.937	.830	.663	.716	.571
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.106	-.093	-.065	.023	-.026	-.006
	RMSE	.203	.203	.254	.287	.092	.157
	AVRB	.106	.093	.065	.023	.026	.006
ψ_1	Mean	.710	.702	.540	.417	.572	.552
	SD	.181	.348	.354	.413	.093	.060
	LCI	1.616	1.431	1.054	.988	.931	.716
	CP	1.000	1.000	.8	.8	1.000	1.000
	Bias	.210	.202	.040	-.083	.072	.052
	RMSE	.277	.403	.356	.421	.118	.079
	AVRB	.421	.404	.081	.165	.144	.103
γ	Mean	-.695	.030	.697	-.790	.141	.755
	SD	.105	.263	.068	.098	.253	.136
	LCI	.582	.951	.604	.412	.551	.412
	CP	1.000	.800	1.000	1.000	.200	1.000
	Bias	.105	.030	-.103	.010	.141	-.045
	RMSE	.148	.265	.123	.098	.289	.144

Table 157 (continued).

	AVRB	.131	-	.129	.013	-	.056
ν	Mean	7.399	7.746	6.958	5.691	7.313	6.926
	SD	2.520	1.630	2.038	1.298	2.003	1.905
	LCI	13.812	16.188	15.542	8.384	15.528	11.961
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	2.399	2.746	1.958	.691	2.313	1.926
	RMSE	3.479	3.193	2.826	1.470	3.060	2.709
	AVRB	.480	.549	.392	.138	.463	.385
σ^2	Mean	4.167	4.130	4.192	4.093	4.116	4.318
	SD	.576	.793	.645	.827	.790	.721
	LCI	3.556	3.466	3.454	2.409	2.417	2.583
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.167	.130	.192	.093	.116	.318
	RMSE	.600	.803	.673	.832	.799	.788
	AVRB	.042	.033	.048	.023	.029	.080

Table 158 – Results of the simulation study for the mixed SSLBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.461	-.618	-.629	-.465	-.563	-.605
	SD	.466	.419	.409	.125	.117	.250
	LCI	1.258	1.220	1.112	.787	.890	.847
	CP	1.000	.800	.800	1.000	1.000	.800
	Bias	.039	-.118	-.129	.035	-.063	-.105
	RMSE	.468	.436	.429	.130	.133	.271
	AVRB	.079	.236	.258	.069	.126	.211
β_1	Median	.914	1.121	1.034	.827	.984	1.086
	SD	.168	.163	.207	.184	.167	.200
	LCI	.770	.883	.900	.658	.640	.595
	CP	.800	1.000	1.000	.800	1.000	.800
	Bias	-.086	.121	.034	-.173	-.016	.086
	RMSE	.189	.203	.210	.253	.168	.218
	AVRB	.086	.121	.034	.173	.016	.086
ψ_0	Median	-1.112	-1.102	-1.066	-.967	-1.017	-1.010
	SD	.179	.188	.249	.281	.084	.152
	LCI	1.050	.937	.830	.663	.716	.571
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.112	-.102	-.066	.033	-.017	-.010
	RMSE	.211	.214	.257	.283	.086	.153
	AVRB	.112	.102	.066	.033	.017	.010
ψ_1	Median	.702	.722	.540	.401	.568	.556
	SD	.155	.348	.360	.428	.093	.060
	LCI	1.616	1.431	1.054	.988	.931	.716
	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	.202	.222	.040	-.099	.068	.056
	RMSE	.255	.413	.362	.440	.115	.082
	AVRB	.405	.444	.080	.198	.136	.112
γ	Median	-.716	.025	.723	-.814	.139	.771
	SD	.110	.257	.081	.110	.249	.149
	LCI	.582	.951	.604	.412	.551	.412
	CP	1.000	.800	1.000	1.000	.200	1.000
	Bias	.084	.025	-.077	-.014	.139	-.029
	RMSE	.138	.258	.111	.111	.285	.152
	AVRB	.105	-	.096	.017	-	.037

Table 158 (continued).

	Median	6.494	6.508	5.787	5.430	6.103	6.107
	SD	2.238	1.337	1.829	1.445	2.198	1.742
	LCI	13.812	16.188	15.542	8.384	15.528	11.961
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	1.494	1.508	.787	.430	1.103	1.107
	RMSE	2.691	2.016	1.991	1.508	2.459	2.063
	AVRB	.299	.302	.157	.086	.220	.221
	Median	4.053	4.027	4.066	4.026	4.06	4.26
	SD	.552	.786	.639	.817	.789	.714
	LCI	3.556	3.466	3.454	2.409	2.417	2.583
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.053	.027	.066	.026	.060	.260
	RMSE	.555	.786	.642	.818	.791	.760
	AVRB	.013	.007	.017	.006	.015	.065

Table 159 – Results of the simulation study for the mixed SSLBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.448	-.640	-.622	-.470	-.568	-.611
	SD	.457	.406	.409	.134	.118	.244
	LCI	1.258	1.22	1.112	.787	.890	.847
	CP	1.000	.800	.800	1.000	1.000	.800
	Bias	.052	-.140	-.122	.030	-.068	-.111
	RMSE	.459	.430	.427	.137	.136	.268
	AVRB	.104	.281	.243	.061	.136	.222
β_1	Mode	.911	1.126	1.032	.83	.984	1.089
	SD	.169	.162	.206	.179	.164	.206
	LCI	.770	.883	.900	.658	.640	.595
	CP	.800	1.000	1.000	.800	1.000	.800
	Bias	-.089	.126	.032	-.170	-.016	.089
	RMSE	.191	.206	.208	.247	.165	.225
	AVRB	.089	.126	.032	.170	.016	.089
ψ_0	Mode	-1.114	-1.109	-1.070	-.966	-1.016	-1.010
	SD	.194	.190	.254	.271	.089	.153
	LCI	1.050	.937	.830	.663	.716	.571
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.114	-.109	-.070	.034	-.016	-.010
	RMSE	.225	.219	.264	.273	.090	.153
	AVRB	.114	.109	.070	.034	.016	.010
ψ_1	Mode	.679	.747	.534	.413	.566	.556
	SD	.125	.347	.367	.432	.092	.058
	LCI	1.616	1.431	1.054	.988	.931	.716
	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	.179	.247	.034	-.087	.066	.056
	RMSE	.218	.426	.369	.441	.113	.081
	AVRB	.358	.493	.068	.174	.132	.112
γ	Mode	-.716	.027	.725	-.803	.140	.764
	SD	.105	.248	.067	.099	.252	.134
	LCI	.582	.951	.604	.412	.551	.412
	CP	1.000	.800	1.000	1.000	.200	1.000
	Bias	.084	.027	-.075	-.003	.140	-.036
	RMSE	.135	.249	.100	.099	.288	.139
	AVRB	.105	-	.094	.004	-	.045

Table 159 (continued).

	Mode	4.166	3.919	3.365	4.251	4.262	4.419
	SD	1.720	1.429	.965	2.243	1.747	1.436
	LCI	13.812	16.188	15.542	8.384	15.528	11.961
ν	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.834	-1.081	-1.635	-.749	-.738	-.581
	RMSE	1.911	1.792	1.899	2.365	1.897	1.549
	AVRB	.167	.216	.327	.150	.148	.116
	Mode	3.836	3.899	3.844	3.859	3.895	4.192
	SD	.426	.908	.616	.733	.685	.732
	LCI	3.556	3.466	3.454	2.409	2.417	2.583
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.164	-.101	-.156	-.141	-.105	.192
	RMSE	.457	.914	.635	.746	.693	.757
	AVRB	.041	.025	.039	.035	.026	.048

Table 160 – Results of the simulation study for the mixed SSLBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.575	-.663	-.497	-.435	-.519	-.562
	SD	.486	.305	.359	.280	.222	.277
	LCI	.848	1.173	1.047	.630	.800	.785
β_0	CP	.800	1.000	.800	.600	1.000	1.000
	Bias	-.075	-.163	.003	.065	-.019	-.062
	RMSE	.492	.346	.359	.288	.223	.284
	AVRB	.150	.326	.006	.129	.037	.123
	Mean	.973	1.123	.856	.803	.993	1.151
	SD	.262	.159	.222	.088	.189	.075
	LCI	.639	.801	.683	.501	.591	.514
β_1	CP	.800	1.000	.600	.600	1.000	1.000
	Bias	-.027	.123	-.144	-.197	-.007	.151
	RMSE	.264	.201	.265	.216	.189	.169
	AVRB	.027	.123	.144	.197	.007	.151
	Mean	-1.035	-1.071	-1.051	-1.149	-1.042	-1.114
	SD	.217	.209	.219	.188	.111	.150
	LCI	.716	.825	.727	.607	.624	.567
ψ_0	CP	1.000	1.000	.800	.800	1.000	.800
	Bias	-.035	-.071	-.051	-.149	-.042	-.114
	RMSE	.220	.221	.225	.240	.119	.188
	AVRB	.035	.071	.051	.149	.042	.114
	Mean	.522	.574	.462	.652	.520	.458
	SD	.493	.413	.411	.318	.198	.244
	LCI	1.290	1.361	1.101	1.001	.887	.725
ψ_1	CP	.800	.800	.800	.800	1.000	1.000
	Bias	.022	.074	-.038	.152	.020	-.042
	RMSE	.494	.420	.413	.353	.199	.247
	AVRB	.044	.148	.075	.305	.039	.084
	Mean	-.862	-.169	.777	-.82	.147	.800
	SD	.100	.141	.095	.079	.265	.089
	LCI	.389	.776	.534	.314	.489	.359
γ	CP	1.000	.600	1.000	.800	$< .001$	1.000
	Bias	-.062	-.169	-.023	-.020	.147	$< .001$
	RMSE	.117	.220	.098	.082	.303	.089
	AVRB	.077	-	.029	.025	-	$< .001$

Table 160 (continued).

	Mean	34.893	38.988	20.903	22.401	31.747	13.592
	SD	16.941	6.443	9.606	16.738	8.756	11.499
	LCI	53.950	108.805	42.895	50.869	81.714	34.569
ν	CP	.800	1.000	.800	.600	1.000	.400
	Bias	4.893	8.988	-9.097	-7.599	1.747	-16.408
	RMSE	17.633	11.059	13.229	18.382	8.928	20.036
	AVRB	.163	.300	.303	.253	.058	.547
	Mean	4.242	4.098	4.038	4.060	3.989	4.073
	SD	.871	1.058	.969	.799	.635	.696
	LCI	3.449	3.384	3.336	2.346	2.361	2.388
σ^2	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.242	.098	.038	.060	-.011	.073
	RMSE	.904	1.062	.969	.801	.635	.700
	AVRB	.061	.024	.010	.015	.003	.018

Table 161 – Results of the simulation study for the mixed SSLBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.568	-.663	-.479	-.439	-.513	-.549
	SD	.490	.312	.360	.279	.215	.283
	LCI	.848	1.173	1.047	.630	.800	.785
	CP	.800	1.000	.800	.600	1.000	1.000
	Bias	-.068	-.163	.021	.061	-.013	-.049
	RMSE	.495	.352	.361	.286	.216	.287
	AVRB	.137	.327	.041	.122	.027	.098
β_1	Median	.973	1.122	.856	.801	.993	1.155
	SD	.275	.153	.218	.084	.192	.096
	LCI	.639	.801	.683	.501	.591	.514
	CP	.800	1.000	.600	.600	1.000	1.000
	Bias	-.027	.122	-.144	-.199	-.007	.155
	RMSE	.276	.196	.262	.217	.193	.182
	AVRB	.027	.122	.144	.199	.007	.155
ψ_0	Median	-1.041	-1.063	-1.051	-1.153	-1.036	-1.116
	SD	.227	.208	.213	.186	.112	.143
	LCI	.716	.825	.727	.607	.624	.567
	CP	1.000	1.000	.800	.800	1.000	.800
	Bias	-.041	-.063	-.051	-.153	-.036	-.116
	RMSE	.231	.218	.219	.241	.117	.185
	AVRB	.041	.063	.051	.153	.036	.116
ψ_1	Median	.531	.576	.461	.662	.519	.445
	SD	.513	.415	.391	.302	.200	.233
	LCI	1.290	1.361	1.101	1.001	.887	.725
	CP	.800	.800	.800	.800	1.000	1.000
	Bias	.031	.076	-.039	.162	.019	-.055
	RMSE	.514	.422	.393	.343	.201	.239
	AVRB	.062	.151	.078	.325	.038	.109
γ	Median	-.904	-.179	.810	-.834	.144	.818
	SD	.101	.101	.093	.081	.266	.090
	LCI	.389	.776	.534	.314	.489	.359
	CP	1.000	.600	1.000	.800	$< .001$	1.000
	Bias	-.104	-.179	.010	-.034	.144	.018
	RMSE	.145	.206	.094	.088	.302	.092
	AVRB	.130	-	.012	.043	-	.023

Table 161 (continued).

	Median	35.317	31.975	18.183	19.505	26.292	11.244
	SD	20.622	9.900	8.415	14.030	7.706	8.371
	LCI	53.950	108.805	42.895	50.869	81.714	34.569
ν	CP	.800	1.000	.800	.600	1.000	.400
	Bias	5.317	1.975	-11.817	-10.495	-3.708	-18.756
	RMSE	21.296	10.095	14.506	17.521	8.552	20.540
	AVRB	.177	.066	.394	.350	.124	.625
	Median	4.123	3.996	3.914	3.998	3.944	4.014
	SD	.839	1.034	.924	.799	.629	.683
	LCI	3.449	3.384	3.336	2.346	2.361	2.388
σ^2	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.123	-.004	-.086	-.002	-.056	.014
	RMSE	.848	1.034	.928	.799	.631	.683
	AVRB	.031	.001	.022	$< .001$	.014	.003

Table 162 – Results of the simulation study for the mixed SSLBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.586	-.673	-.455	-.436	-.511	-.544
	SD	.517	.326	.363	.280	.214	.280
	LCI	.848	1.173	1.047	.630	.800	.785
β_0	CP	.800	1.000	.800	.600	1.000	1.000
	Bias	-.086	-.173	.045	.064	-.011	-.044
	RMSE	.524	.369	.366	.288	.215	.283
	AVRB	.171	.346	.091	.129	.022	.087
	Mode	.970	1.120	.856	.800	.991	1.152
	SD	.263	.151	.214	.086	.192	.085
	LCI	.639	.801	.683	.501	.591	.514
β_1	CP	.800	1.000	.600	.600	1.000	1.000
	Bias	-.030	.120	-.144	-.200	-.009	.152
	RMSE	.265	.193	.258	.218	.192	.174
	AVRB	.030	.120	.144	.200	.009	.152
	Mode	-1.033	-1.066	-1.049	-1.154	-1.035	-1.115
	SD	.240	.208	.217	.186	.112	.145
	LCI	.716	.825	.727	.607	.624	.567
ψ_0	CP	1.000	1.000	.800	.800	1.000	.800
	Bias	-.033	-.066	-.049	-.154	-.035	-.115
	RMSE	.242	.218	.222	.241	.117	.185
	AVRB	.033	.066	.049	.154	.035	.115
	Mode	.500	.571	.458	.665	.518	.446
	SD	.553	.415	.380	.295	.201	.233
	LCI	1.290	1.361	1.101	1.001	.887	.725
ψ_1	CP	.800	.800	.800	.800	1.000	1.000
	Bias	$< .001$	.071	-.042	.165	.018	-.054
	RMSE	.553	.421	.383	.338	.202	.239
	AVRB	$< .001$	.141	.084	.329	.037	.108
	Mode	-.885	-.191	.806	-.827	.145	.807
	SD	.083	.080	.083	.076	.263	.087
	LCI	.389	.776	.534	.314	.489	.359
γ	CP	1.000	.600	1.000	.800	$< .001$	1.000
	Bias	-.085	-.191	.006	-.027	.145	.007
	RMSE	.119	.207	.083	.081	.300	.087
	AVRB	.106	-	.007	.034	-	.008

Table 162 (continued).

	Mode	31.497	19.890	9.149	8.709	12.397	5.364
	SD	23.311	17.434	3.378	3.623	9.595	1.054
	LCI	53.950	108.805	42.895	50.869	81.714	34.569
ν	CP	.800	1.000	.800	.600	1.000	.400
	Bias	1.497	-10.110	-20.851	-21.291	-17.603	-24.636
	RMSE	23.359	20.154	21.123	21.597	20.048	24.658
	AVRB	.050	.337	.695	.710	.587	.821
	Mode	3.943	3.799	3.727	3.940	3.904	3.930
	Sd	.821	.926	.886	.799	.651	.656
	LCI	3.449	3.384	3.336	2.346	2.361	2.388
σ^2	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.057	-.201	-.273	-.060	-.096	-.070
	RMSE	.823	.948	.928	.801	.658	.660
	AVRB	.014	.050	.068	.015	.024	.017

Mixed SCNBS regression model

Table 163 – Results of the simulation study for the mixed SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.493	-.499	-.570	-.451	-.501	-.505
	SD	.307	.225	.236	.349	.315	.406
	LCI	1.539	1.548	1.553	.987	1.003	1.089
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.007	.001	-.070	.049	-.001	-.005
	RMSE	.307	.225	.246	.352	.315	.406
	AVRB	.015	.003	.139	.098	.002	.011
β_1	Mean	.969	1.029	1.032	.953	1.124	1.097
	SD	.295	.157	.161	.169	.193	.299
	LCI	1.701	1.722	1.692	1.023	1.085	1.115
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.031	.029	.032	-.047	.124	.097
	RMSE	.297	.160	.164	.175	.229	.314
	AVRB	.031	.029	.032	.047	.124	.097
ψ_0	Mean	.915	.906	.849	.796	.742	.940
	SD	.403	.206	.237	.362	.191	.140
	LCI	1.366	1.642	1.717	1.363	1.336	.973
	CP	$< .001$	$< .001$	$< .001$	.200	$< .001$	$< .001$
	Bias	1.915	1.906	1.849	1.796	1.742	1.940
	RMSE	1.956	1.917	1.865	1.832	1.752	1.945
	AVRB	1.915	1.906	1.849	1.796	1.742	1.940
ψ_1	Mean	.422	.476	.077	.580	.651	.490
	SD	.444	.147	.297	.094	.220	.151
	LCI	1.538	1.320	1.134	1.088	.933	.687
	CP	1.000	1.000	.600	1.000	1.000	1.000
	Bias	-.078	-.024	-.423	.080	.151	-.010
	RMSE	.451	.149	.517	.123	.266	.151
	AVRB	.156	.049	.845	.160	.302	.019
γ	Mean	-.630	-.099	.505	-.610	-.001	.617
	SD	.147	.187	.179	.116	.252	.074
	LCI	.831	1.011	.886	.565	.508	.632
	CP	1.000	.800	1.000	.800	.200	.800
	Bias	.170	-.099	-.295	.190	-.001	-.183

Table 163 (continued).

	RMSE	.225	.211	.345	.222	.252	.197
	AVRB	.213	-	.368	.237	-	.229
	Mean	.451	.483	.545	.516	.518	.486
	SD	.027	.018	.052	.087	.047	.030
	LCI	.738	.784	.744	.811	.790	.815
ν_1	CP	$< .001$	$< .001$	.600	.600	.200	.400
	Bias	-.449	-.417	-.355	-.384	-.382	-.414
	RMSE	.45	.417	.359	.394	.385	.415
	AVRB	.499	.463	.395	.427	.424	.460
	Mean	.559	.574	.476	.623	.585	.662
	SD	.217	.078	.113	.129	.107	.070
	LCI	.623	.784	.746	.730	.746	.680
ν_2	CP	.200	.200	.200	.200	$< .001$	$< .001$
	Bias	.459	.474	.376	.523	.485	.562
	RMSE	.507	.481	.393	.539	.497	.567
	AVRB	4.586	4.745	3.763	5.228	4.853	5.622
	Mean	4.376	4.913	4.935	3.777	3.731	3.662
	SD	.654	.950	1.350	.779	.813	.903
	LCI	3.968	4.521	4.509	2.453	2.526	2.368
σ^2	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.376	.913	.935	-.223	-.269	-.338
	RMSE	.755	1.318	1.642	.810	.857	.964
	AVRB	.094	.228	.234	.056	.067	.084

Table 164 – Results of the simulation study for the mixed SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.494	-.500	-.556	-.456	-.502	-.510
	SD	.303	.230	.243	.351	.317	.399
	LCI	1.539	1.548	1.553	.987	1.003	1.089
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.006	$< .001$	-.056	.044	-.002	-.010
	RMSE	.303	.230	.249	.354	.317	.399
	AVRB	.012	$< .001$	.112	.088	.004	.019
β_1	Median	.987	1.022	1.019	.952	1.118	1.102
	SD	.276	.172	.173	.174	.191	.316
	LCI	1.701	1.722	1.692	1.023	1.085	1.115
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.013	.022	.019	-.048	.118	.102
	RMSE	.277	.173	.174	.180	.225	.332
	AVRB	.013	.022	.019	.048	.118	.102
ψ_0	Median	.938	.980	.920	.861	.808	.985
	SD	.399	.146	.234	.299	.153	.121
	LCI	1.366	1.642	1.717	1.363	1.336	.973
	CP	$< .001$	$< .001$	$< .001$	.200	$< .001$	$< .001$
	Bias	1.938	1.980	1.920	1.861	1.808	1.985
	RMSE	1.979	1.985	1.934	1.885	1.814	1.989
	AVRB	1.938	1.980	1.920	1.861	1.808	1.985
ψ_1	Median	.405	.486	.082	.566	.645	.492
	SD	.462	.153	.296	.080	.222	.156
	LCI	1.538	1.320	1.134	1.088	.933	.687
	CP	1.000	1.000	.600	1.000	1.000	1.000
	Bias	-.095	-.014	-.418	.066	.145	-.008
	RMSE	.471	.154	.512	.104	.265	.156
	AVRB	.19	.027	.836	.132	.289	.016
γ	Median	-.673	-.081	.509	-.626	-.014	.630
	SD	.143	.194	.195	.120	.247	.072
	LCI	.831	1.011	.886	.565	.508	.632
	CP	1.000	.800	1.000	.800	.200	.800
	Bias	.127	-.081	-.291	.174	-.014	-.170
	RMSE	.191	.211	.351	.211	.248	.184

Table 164 (continued).

	AVRB	.158	-	.364	.217	-	.212
ν_1	Median	.442	.484	.547	.529	.511	.492
	SD	.030	.027	.059	.125	.075	.040
	LCI	.738	.784	.744	.811	.790	.815
	CP	< .001	< .001	.600	.600	.200	.400
	Bias	-.458	-.416	-.353	-.371	-.389	-.408
	RMSE	.459	.417	.358	.391	.396	.410
	AVRB	.509	.462	.392	.412	.433	.454
ν_2	Median	.556	.564	.462	.612	.572	.667
	SD	.229	.093	.130	.182	.139	.089
	LCI	.623	.784	.746	.730	.746	.680
	CP	.200	.200	.200	.200	< .001	< .001
	Bias	.456	.464	.362	.512	.472	.567
	RMSE	.510	.473	.384	.544	.492	.574
	AVRB	4.559	4.639	3.617	5.123	4.721	5.670
σ^2	Median	4.250	4.754	4.788	3.718	3.675	3.606
	SD	.638	.938	1.329	.787	.800	.890
	LCI	3.968	4.521	4.509	2.453	2.526	2.368
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.250	.754	.788	-.282	-.325	-.394
	RMSE	.685	1.203	1.545	.836	.864	.973
	AVRB	.063	.188	.197	.070	.081	.098

Table 165 – Results of the simulation study for the mixed SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.484	-.500	-.547	-.457	-.499	-.517
	SD	.313	.238	.260	.360	.310	.389
	LCI	1.539	1.548	1.553	.987	1.003	1.089
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.016	$<.001$	-.047	.043	.001	-.017
	RMSE	.313	.238	.265	.362	.310	.389
	AVRB	.031	$<.001$	.093	.086	.003	.034
β_1	Mode	1.019	1.000	.984	.960	1.114	1.116
	SD	.265	.197	.203	.176	.186	.344
	LCI	1.701	1.722	1.692	1.023	1.085	1.115
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.019	$<.001$	-.016	-.040	.114	.116
	RMSE	.266	.197	.204	.181	.218	.363
	AVRB	.019	$<.001$	.016	.040	.114	.116
ψ_0	Mode	.964	1.044	.962	1.006	.889	1.019
	SD	.407	.090	.300	.057	.114	.087
	LCI	1.366	1.642	1.717	1.363	1.336	.973
	CP	$<.001$	$<.001$	$<.001$	.200	$<.001$	$<.001$
	Bias	1.964	2.044	1.962	2.006	1.889	2.019
	RMSE	2.006	2.046	1.985	2.007	1.892	2.021
	AVRB	1.964	2.044	1.962	2.006	1.889	2.019
ψ_1	Mode	.393	.502	.089	.552	.637	.491
	SD	.486	.161	.297	.074	.221	.157
	LCI	1.538	1.32	1.134	1.088	.933	.687
	CP	1.000	1.000	.600	1.000	1.000	1.000
	Bias	-.107	.002	-.411	.052	.137	-.009
	RMSE	.498	.161	.507	.090	.260	.157
	AVRB	.214	.005	.822	.103	.275	.017
γ	Mode	-.693	-.062	.506	-.624	-.006	.632
	SD	.126	.204	.206	.117	.247	.072
	LCI	.831	1.011	.886	.565	.508	.632
	CP	1.000	.800	1.000	.800	.200	.800
	Bias	.107	-.062	-.294	.176	-.006	-.168
	RMSE	.166	.213	.359	.211	.247	.183

Table 165 (continued).

	AVRB	.134	-	.367	.220	-	.210
ν_1	Mode	.441	.484	.554	.547	.498	.497
	SD	.031	.035	.063	.143	.107	.047
	LCI	.738	.784	.744	.811	.79	.815
	CP	$< .001$	$< .001$	.600	.600	.200	.400
	Bias	-.459	-.416	-.346	-.353	-.402	-.403
	RMSE	.460	.418	.352	.381	.417	.406
	AVRB	.510	.462	.385	.393	.447	.448
ν_2	Mode	.556	.556	.449	.602	.559	.668
	SD	.239	.116	.155	.218	.167	.098
	LCI	.623	.784	.746	.730	.746	.680
	CP	.200	.200	.200	.200	$< .001$	$< .001$
	Bias	.456	.456	.349	.502	.459	.568
	RMSE	.515	.471	.382	.547	.489	.577
	AVRB	4.560	4.563	3.494	5.018	4.593	5.685
σ^2	Mode	4.082	4.451	4.305	3.609	3.519	3.558
	SD	.635	.986	1.152	.824	.675	.914
	LCI	3.968	4.521	4.509	2.453	2.526	2.368
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.082	.451	.305	-.391	-.481	-.442
	RMSE	.641	1.085	1.191	.913	.829	1.015
	AVRB	.021	.113	.076	.098	.120	.111

Table 166 – Results of the simulation study for the mixed SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.477	-.566	-.640	-.460	-.513	-.580
	SD	.241	.252	.210	.243	.270	.330
	LCI	1.263	1.369	1.284	.749	.857	.744
β_0	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.023	-.066	-.140	.040	-.013	-.080
	RMSE	.242	.261	.253	.246	.271	.340
	AVRB	.047	.133	.280	.081	.027	.159
	Mean	.846	.955	1.128	.928	1.125	1.079
	SD	.145	.248	.148	.204	.167	.143
	LCI	1.032	1.088	.974	.576	.727	.671
β_1	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.154	-.045	.128	-.072	.125	.079
	RMSE	.211	.252	.196	.216	.209	.163
	AVRB	.154	.045	.128	.072	.125	.079
	Mean	-1.298	-1.193	-1.174	-1.357	-1.274	-1.126
	SD	.253	.416	.375	.276	.261	.040
	LCI	1.417	1.973	1.693	1.171	1.214	.908
ψ_0	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.298	-.193	-.174	-.357	-.274	-.126
	RMSE	.391	.459	.414	.451	.378	.133
	AVRB	.298	.193	.174	.357	.274	.126
	Mean	.601	.63	.157	.704	.494	.432
	SD	.421	.373	.337	.254	.303	.064
	LCI	1.676	1.537	1.227	1.139	.967	.885
ψ_1	CP	1.000	1.000	.600	.800	1.000	1.000
	Bias	.101	.130	-.343	.204	-.006	-.068
	RMSE	.433	.395	.481	.326	.303	.093
	AVRB	.202	.260	.686	.409	.013	.136
	Mean	-.606	-.169	.483	-.650	.281	.574
	SD	.019	.251	.145	.258	.402	.185
	LCI	.782	.963	.838	.541	.61	.671
γ	CP	1.000	.600	.800	.800	.200	.800
	Bias	.194	-.169	-.317	.150	.281	-.226
	RMSE	.195	.303	.349	.298	.490	.292

Table 166 (continued).

	AVRB	.242	-	.396	.187	-	.282
ν_1	Mean	.309	.414	.378	.256	.290	.214
	SD	.100	.122	.164	.086	.139	.057
	LCI	.506	.679	.554	.415	.433	.387
	CP	.600	.800	.600	.800	.600	1.000
	Bias	.209	.314	.278	.156	.190	.114
	RMSE	.231	.337	.323	.178	.235	.128
	AVRB	2.086	3.14	2.779	1.565	1.896	1.140
ν_2	Mean	.203	.331	.242	.191	.177	.183
	SD	.160	.192	.199	.109	.060	.065
	LCI	.318	.633	.395	.290	.249	.225
	CP	1.000	.800	.800	.600	.800	.600
	Bias	.103	.231	.142	.091	.077	.083
	RMSE	.190	.300	.244	.142	.097	.105
	AVRB	1.032	2.306	1.417	.910	.770	.826
σ^2	Mean	4.676	4.734	4.858	3.715	3.819	3.803
	SD	.643	.968	1.175	.794	.859	.835
	LCI	4.056	4.053	4.200	2.250	2.228	2.259
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.676	.734	.858	-.285	-.181	-.197
	RMSE	.933	1.215	1.455	.843	.878	.858
	AVRB	.169	.184	.214	.071	.045	.049

Table 167 – Results of the simulation study for the mixed SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.454	-.571	-.634	-.456	-.516	-.569
	SD	.23	.25	.206	.238	.261	.339
	LCI	1.263	1.369	1.284	.749	.857	.744
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.046	-.071	-.134	.044	-.016	-.069
	RMSE	.234	.260	.246	.242	.261	.346
	AVRB	.091	.143	.268	.089	.032	.138
β_1	Median	.844	.951	1.135	.932	1.122	1.079
	SD	.144	.251	.138	.206	.160	.147
	LCI	1.032	1.088	.974	.576	.727	.671
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.156	-.049	.135	-.068	.122	.079
	RMSE	.213	.255	.193	.217	.201	.167
	AVRB	.156	.049	.135	.068	.122	.079
ψ_0	Median	-1.299	-1.108	-1.147	-1.372	-1.255	-1.118
	SD	.254	.378	.414	.264	.277	.047
	LCI	1.417	1.973	1.693	1.171	1.214	.908
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.299	-.108	-.147	-.372	-.255	-.118
	RMSE	.392	.393	.439	.456	.376	.127
	AVRB	.299	.108	.147	.372	.255	.118
ψ_1	Median	.598	.633	.157	.698	.502	.440
	SD	.409	.389	.340	.258	.301	.061
	LCI	1.676	1.537	1.227	1.139	.967	.885
	CP	1.000	1.000	.600	.800	1.000	1.000
	Bias	.098	.133	-.343	.198	.002	-.060
	RMSE	.420	.411	.483	.325	.301	.086
	AVRB	.195	.267	.686	.397	.003	.120
γ	Median	-.622	-.161	.501	-.666	.284	.576
	SD	.025	.259	.160	.284	.401	.218
	LCI	.782	.963	.838	.541	.61	.671
	CP	1.000	.600	.800	.800	.200	.800
	Bias	.178	-.161	-.299	.134	.284	-.224
	RMSE	.180	.305	.339	.314	.492	.312

Table 167 (continued).

	AVRB	.223	-	.374	.167	-	.279
ν_1	Median	.296	.410	.382	.247	.273	.199
	SD	.102	.135	.177	.087	.132	.055
	LCI	.506	.679	.554	.415	.433	.387
	CP	.600	.800	.600	.800	.600	1.000
	Bias	.196	.310	.282	.147	.173	.099
	RMSE	.221	.338	.333	.171	.217	.113
	AVRB	1.960	3.095	2.820	1.471	1.728	.986
ν_2	Median	.195	.305	.227	.173	.167	.174
	SD	.157	.202	.202	.083	.057	.061
	LCI	.318	.633	.395	.290	.249	.225
	CP	1.000	.800	.800	.600	.800	.600
	Bias	.095	.205	.127	.073	.067	.074
	RMSE	.184	.288	.239	.111	.088	.096
	AVRB	.950	2.049	1.274	.734	.671	.742
σ^2	Median	4.535	4.614	4.706	3.657	3.769	3.746
	SD	.632	.929	1.149	.780	.856	.822
	LCI	4.056	4.053	4.200	2.250	2.228	2.259
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.535	.614	.706	-.343	-.231	-.254
	RMSE	.827	1.114	1.349	.852	.886	.861
	AVRB	.134	.154	.176	.086	.058	.064

Table 168 – Results of the simulation study for the mixed SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.444	-.586	-.633	-.453	-.522	-.570
	SD	.212	.247	.189	.227	.264	.339
	LCI	1.263	1.369	1.284	.749	.857	.744
	CP	1.000	1.000	1.000	1.000	1.000	.800
	Bias	.056	-.086	-.133	.047	-.022	-.070
	RMSE	.220	.261	.231	.231	.265	.346
	AVRB	.113	.171	.266	.093	.044	.140
β_1	Mode	.840	.956	1.140	.936	1.121	1.083
	SD	.143	.241	.131	.207	.160	.150
	LCI	1.032	1.088	.974	.576	.727	.671
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.160	-.044	.140	-.064	.121	.083
	RMSE	.215	.245	.192	.217	.200	.171
	AVRB	.160	.044	.140	.064	.121	.083
ψ_0	Mode	-1.304	-1.006	-1.125	-1.382	-1.260	-1.113
	SD	.265	.303	.511	.261	.344	.049
	LCI	1.417	1.973	1.693	1.171	1.214	.908
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.304	-.006	-.125	-.382	-.260	-.113
	RMSE	.403	.303	.526	.462	.431	.123
	AVRB	.304	.006	.125	.382	.260	.113
ψ_1	Mode	.623	.642	.163	.682	.500	.447
	SD	.465	.419	.347	.271	.297	.070
	LCI	1.676	1.537	1.227	1.139	.967	.885
	CP	1.000	1.000	.600	.800	1.000	1.000
	Bias	.123	.142	-.337	.182	$< .001$	-.053
	RMSE	.481	.443	.483	.327	.297	.088
	AVRB	.246	.283	.674	.365	.001	.107
γ	Mode	-.640	-.154	.519	-.662	.293	.577
	SD	.028	.269	.162	.278	.400	.225
	LCI	.782	.963	.838	.541	.610	.671
	CP	1.000	.600	.800	.800	.200	.800
	Bias	.160	-.154	-.281	.138	.293	-.223
	RMSE	.163	.310	.324	.310	.495	.316

Table 168 (continued).

	AVRB	.200	-	.351	.173	-	.278
ν_1	Mode	.290	.407	.382	.248	.276	.205
	SD	.083	.130	.177	.080	.130	.053
	LCI	.506	.679	.554	.415	.433	.387
	CP	.600	.800	.600	.800	.600	1.000
	Bias	.190	.307	.282	.148	.176	.105
	RMSE	.207	.333	.333	.168	.219	.117
	AVRB	1.897	3.069	2.817	1.477	1.759	1.048
ν_2	Mode	.193	.304	.236	.174	.173	.179
	SD	.139	.199	.204	.078	.058	.063
	LCI	.318	.633	.395	.290	.249	.225
	CP	1.000	.800	.800	.600	.800	.600
	Bias	.093	.204	.136	.074	.073	.079
	RMSE	.167	.285	.246	.108	.093	.102
	AVRB	.928	2.043	1.360	.741	.73	.795
σ^2	Mode	4.366	4.348	4.505	3.570	3.703	3.674
	SD	.676	.766	1.071	.733	.877	.814
	LCI	4.056	4.053	4.200	2.250	2.228	2.259
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.366	.348	.505	-.430	-.297	-.326
	RMSE	.769	.841	1.184	.850	.926	.877
	AVRB	.092	.087	.126	.107	.074	.081

E.3 Behavior of the residuals

Simulated observations from mixed SGtBS1 regression model

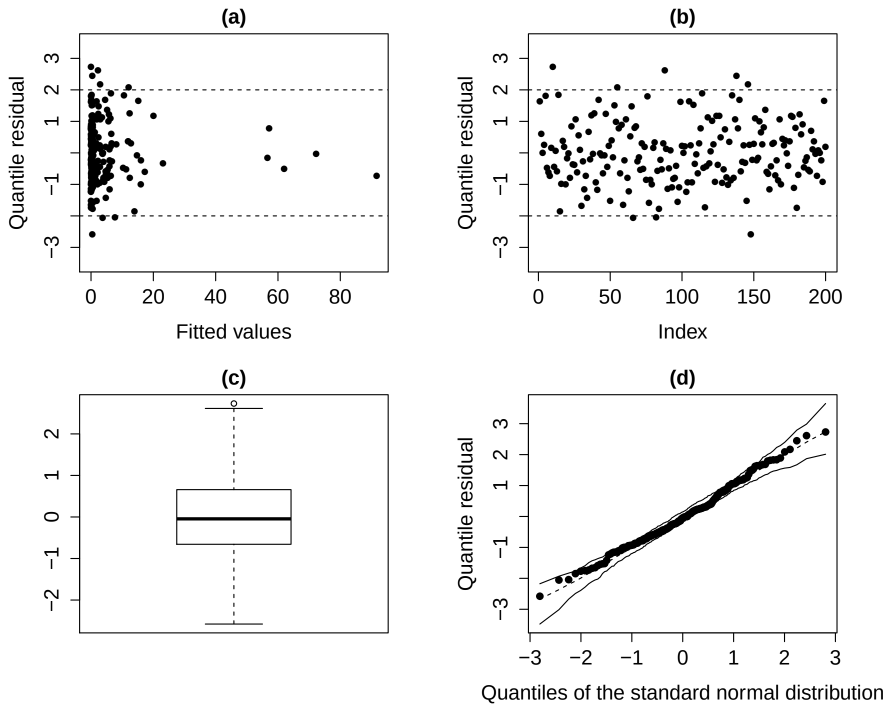

Figure 166 – Residual plots for the mixed SGtBS1 regression model.

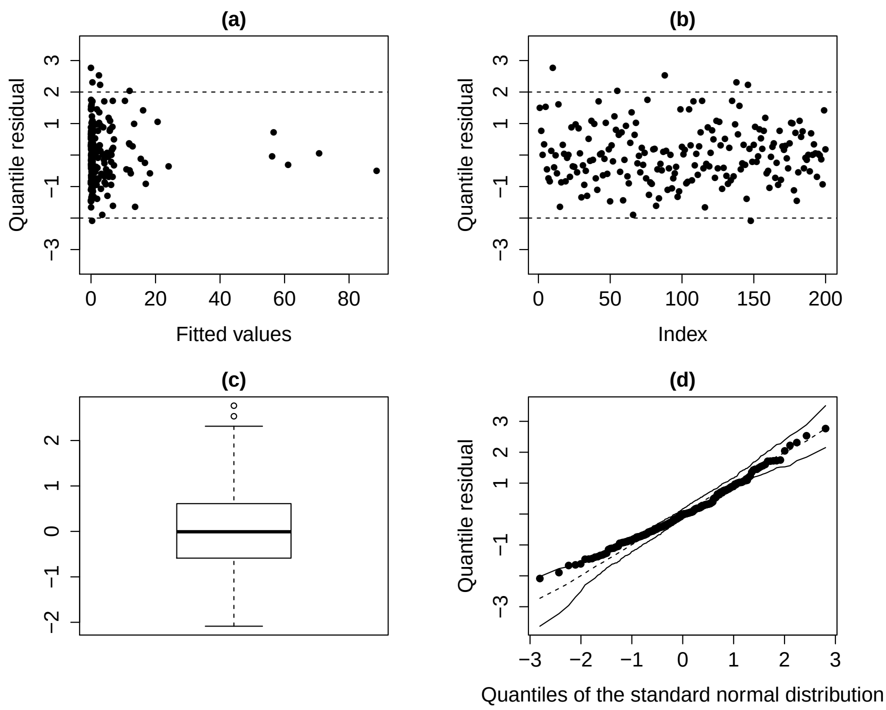

Figure 167 – Residual plots for the mixed SGtBS2 regression model.

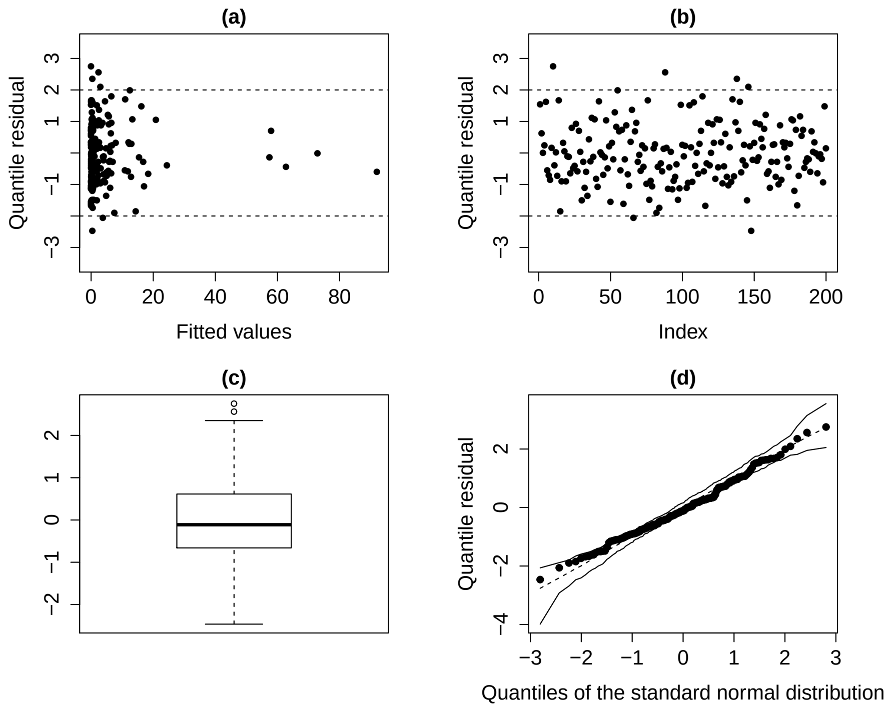

Figure 168 – Residual plots for the mixed StBS regression model.

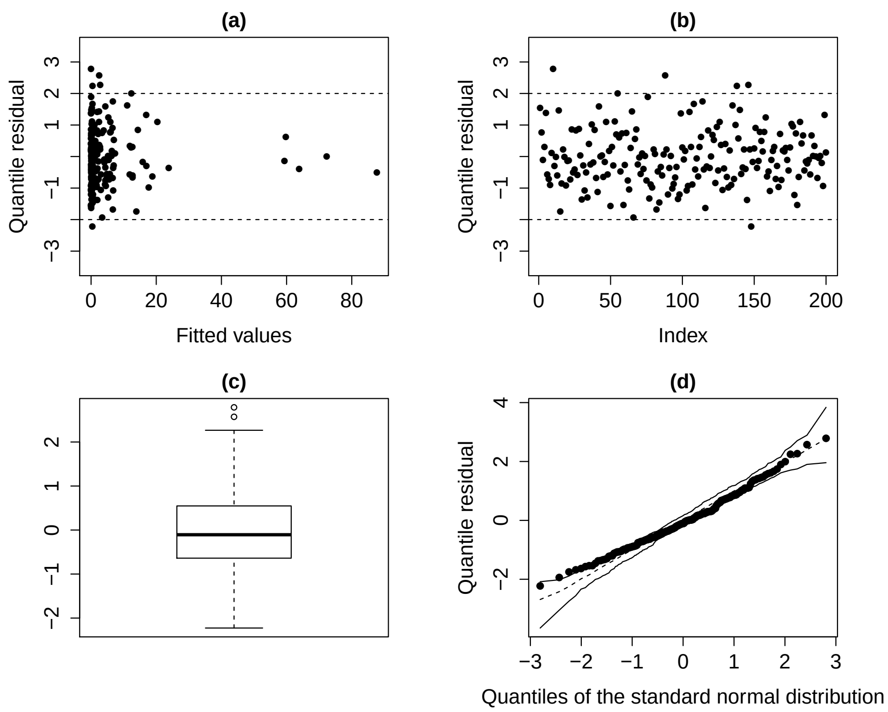

Figure 169 – Residual plots for the mixed SSLBS1 regression model.

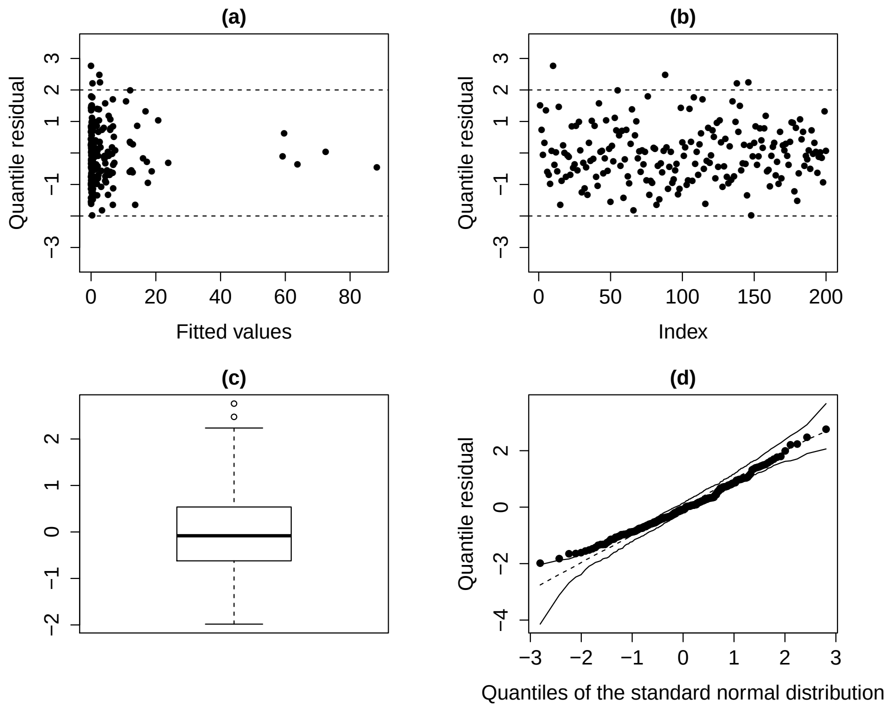

Figure 170 – Residual plots for the mixed SSLBS2 regression model.

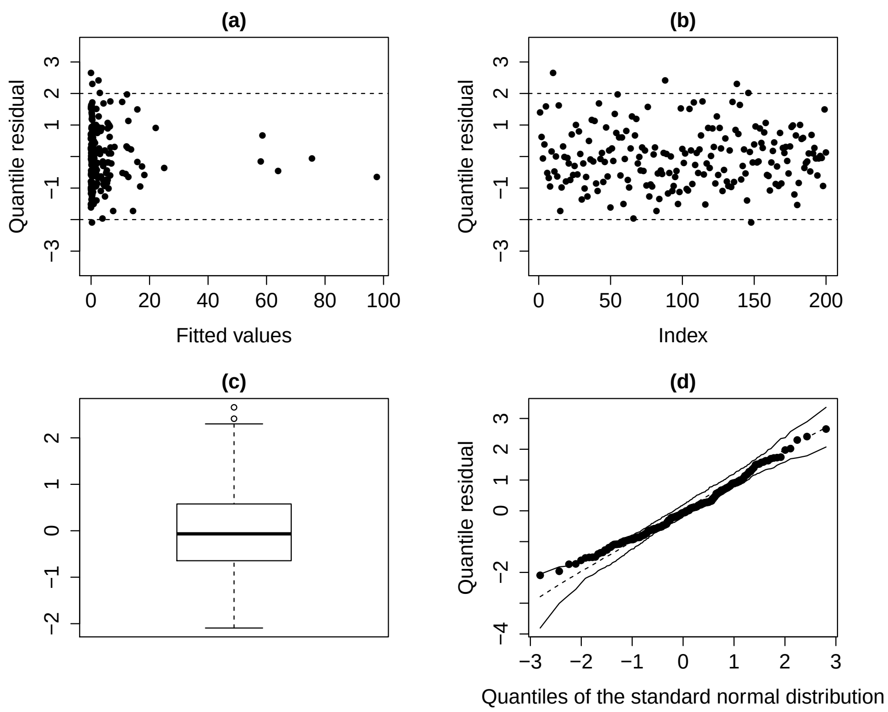

Figure 171 – Residual plots for the mixed SCNBS regression model.

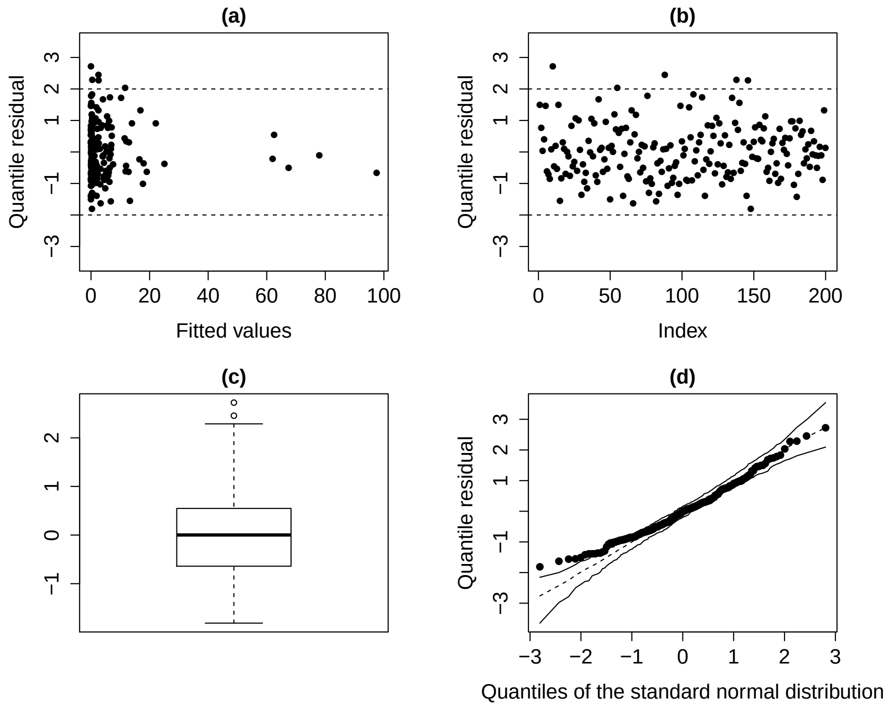

Figure 172 – Residual plots for the mixed SNBS regression model.

Simulated observations from mixed SGtBS2 regression model

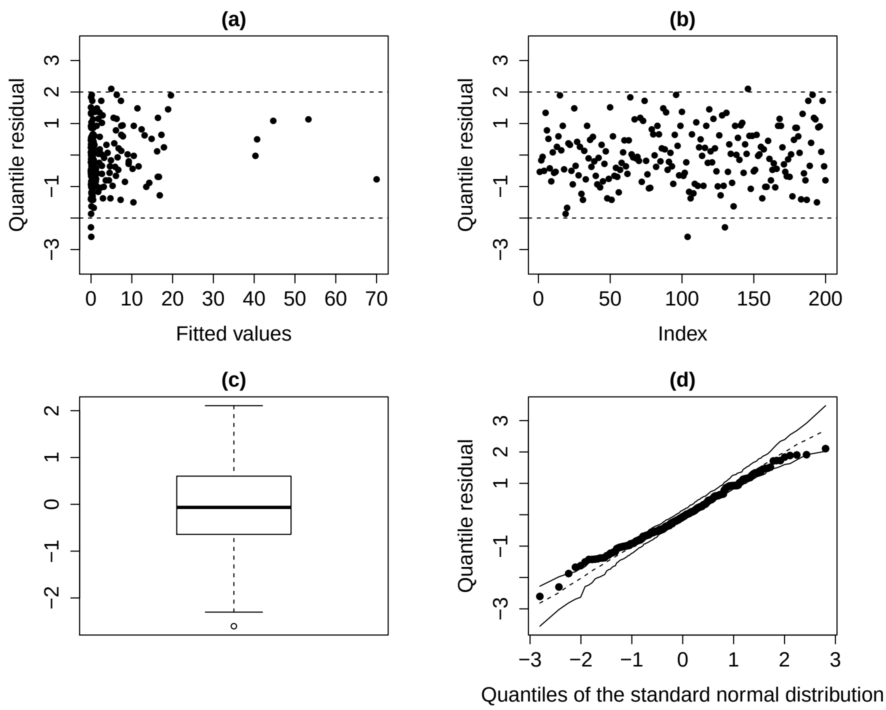

Figure 173 – Residual plots for the mixed SGtBS2 regression model.

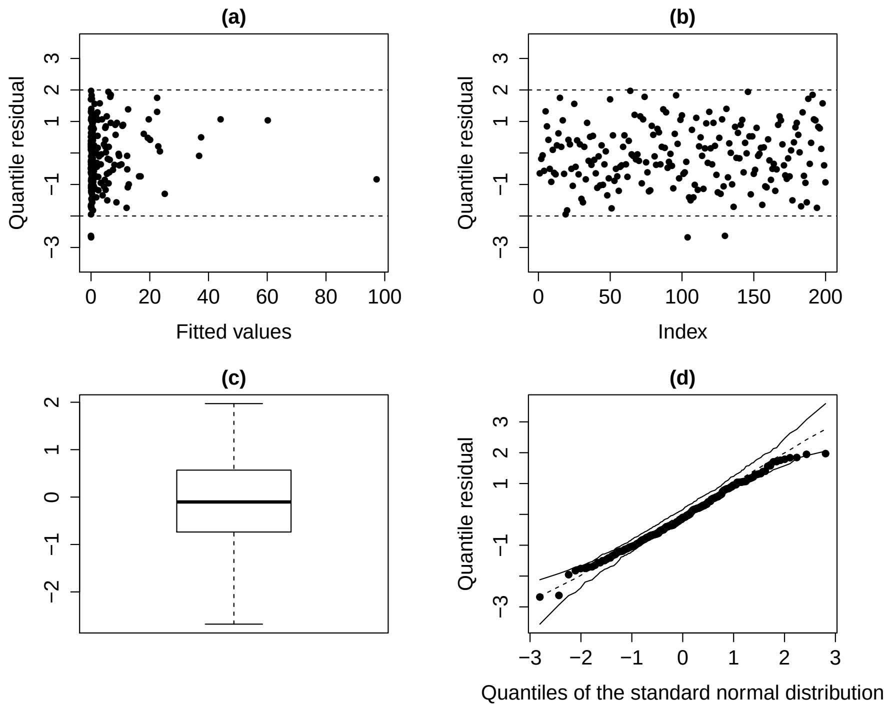

Figure 174 – Residual plots for the mixed SGtBS1 regression model.

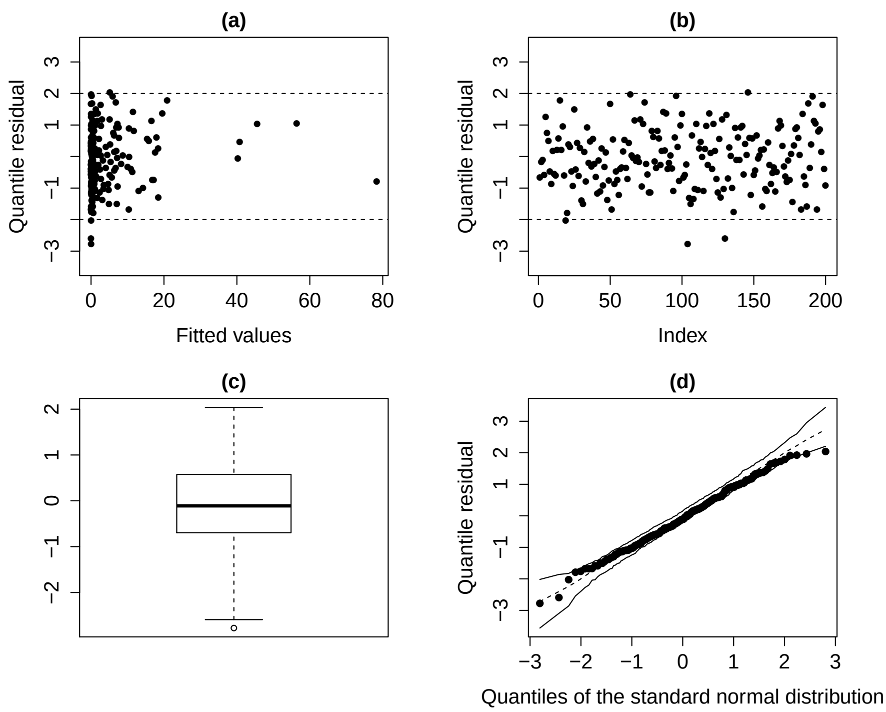

Figure 175 – Residual plots for the mixed StBS regression model.

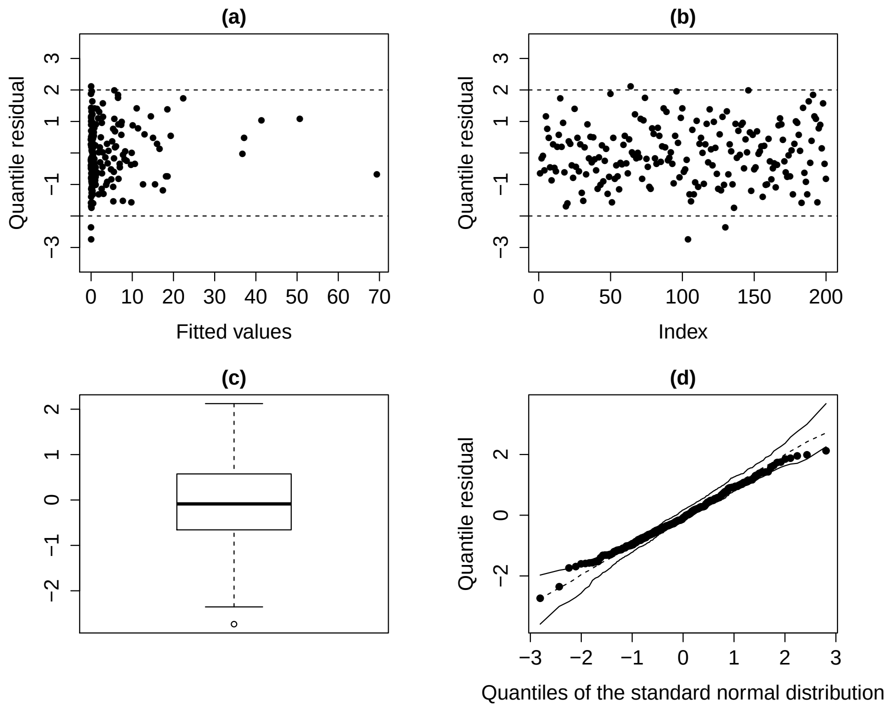

Figure 176 – Residual plots for the mixed SSLBS1 regression model.

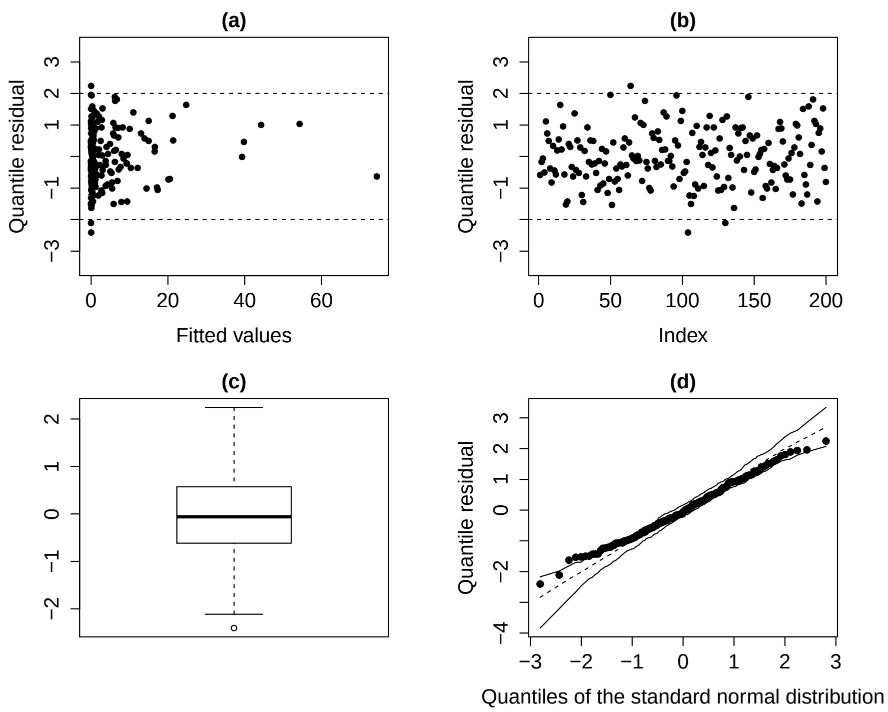

Figure 177 – Residual plots for the mixed SSLBS2 regression model.

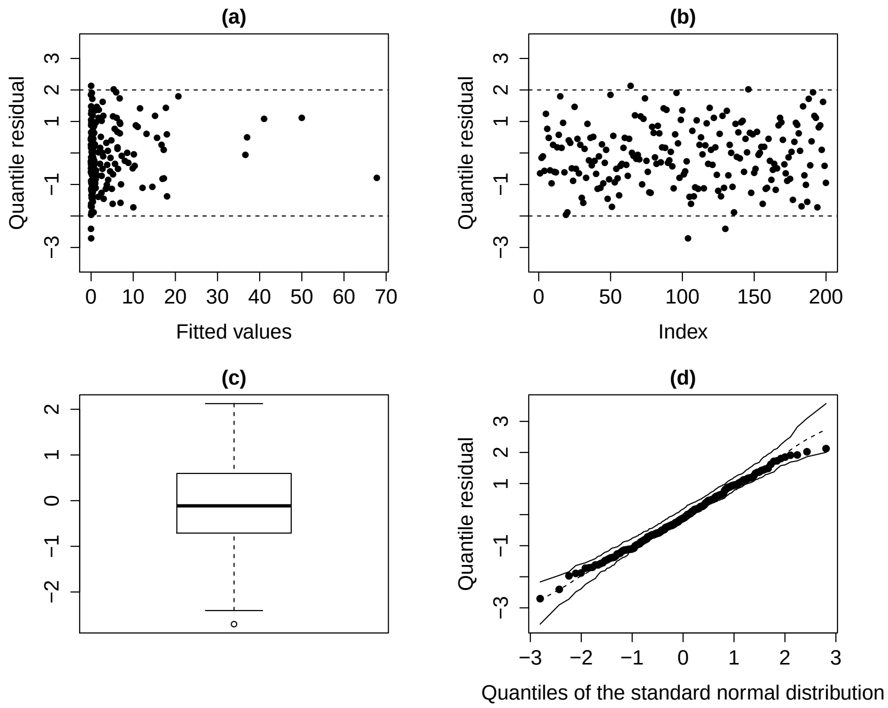

Figure 178 – Residual plots for the mixed SCNBS regression model.

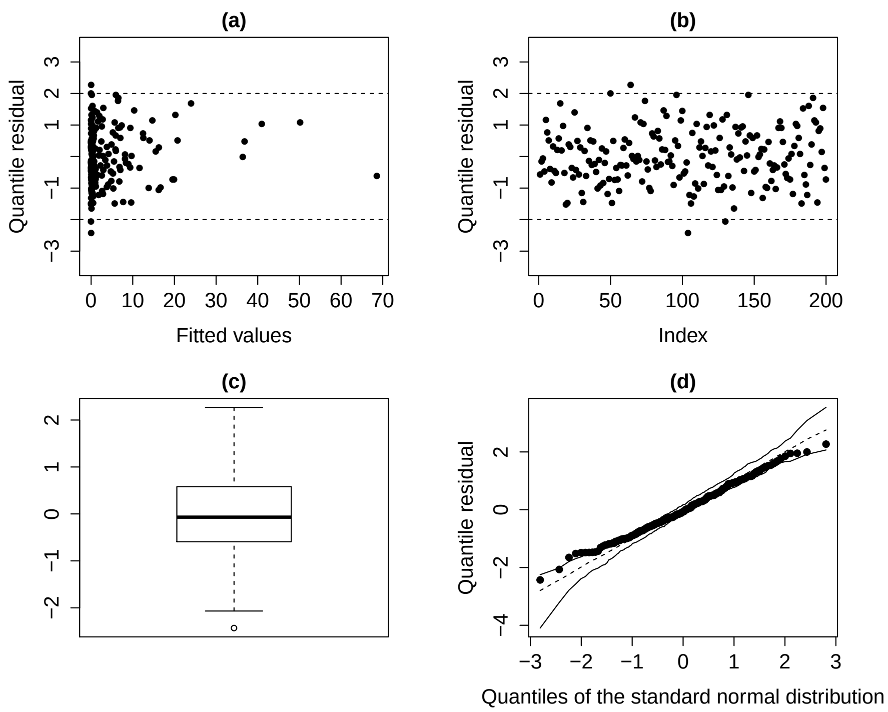

Figure 179 – Residual plots for the mixed SNBS regression model.

Simulated observations from mixed StBS regression model

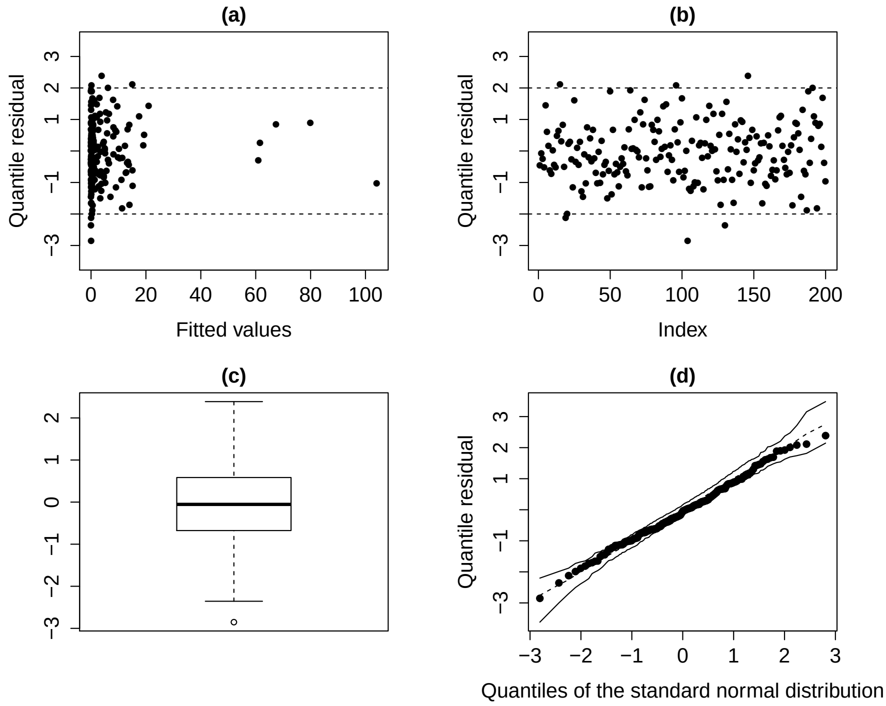

Figure 180 – Residual plots for the mixed StBS regression model.

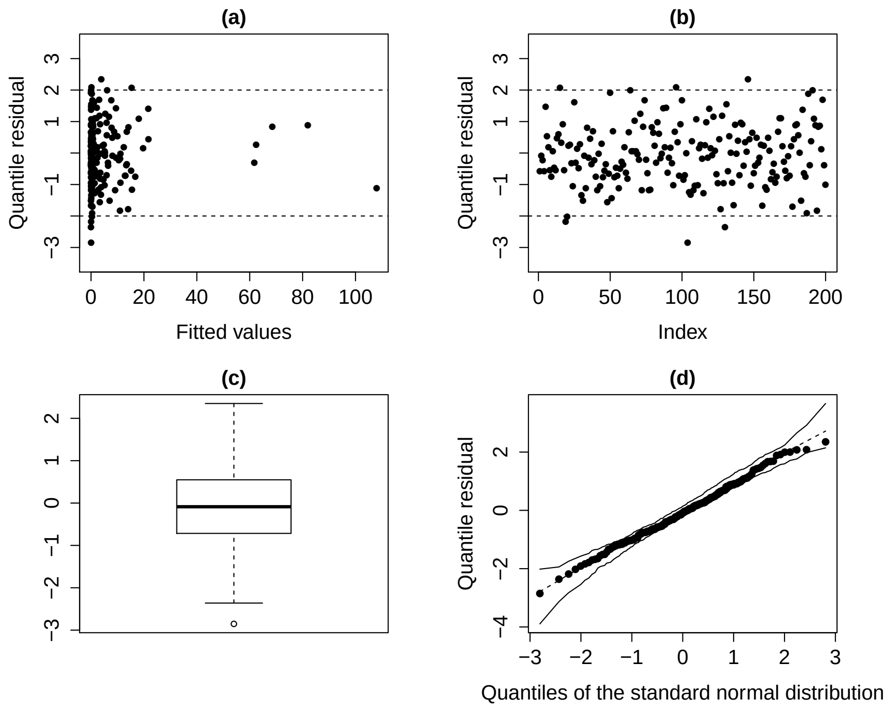

Figure 181 – Residual plots for the mixed SGtBS1 regression model.

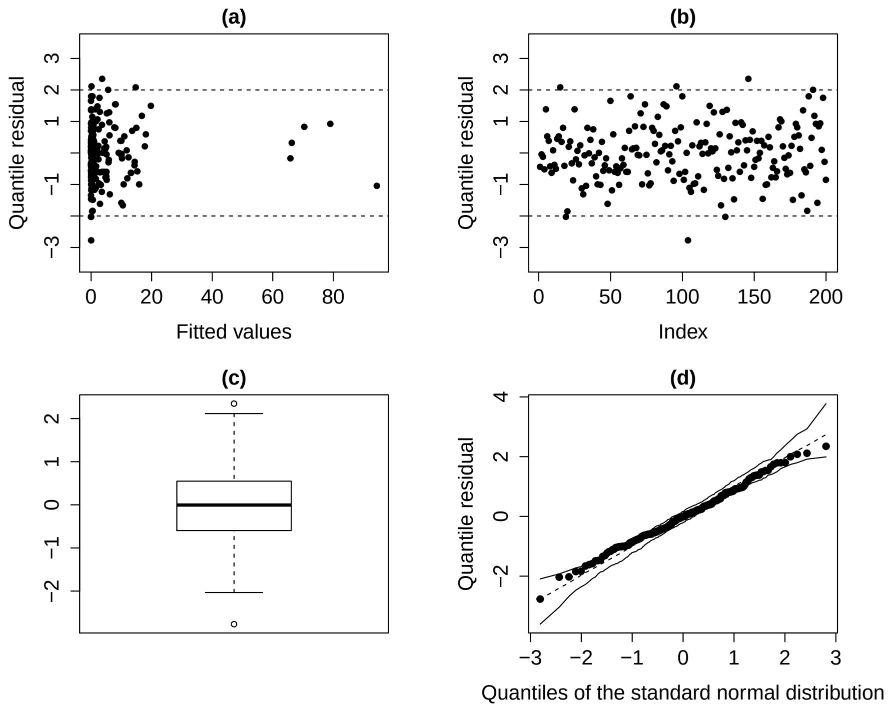

Figure 182 – Residual plots for the mixed SGtBS2 regression model.

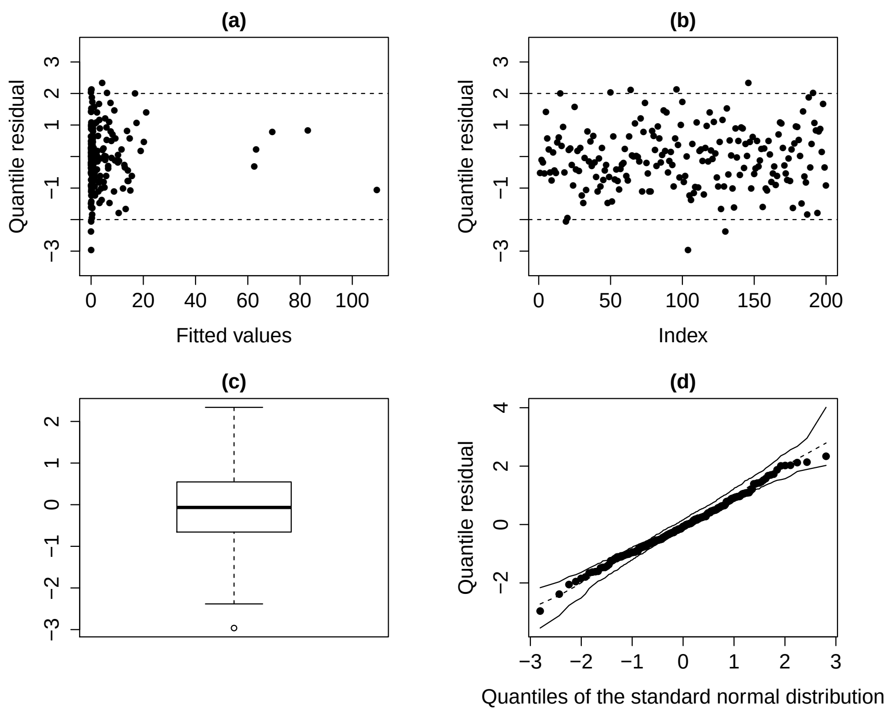

Figure 183 – Residual plots for the mixed SSLBS1 regression model.

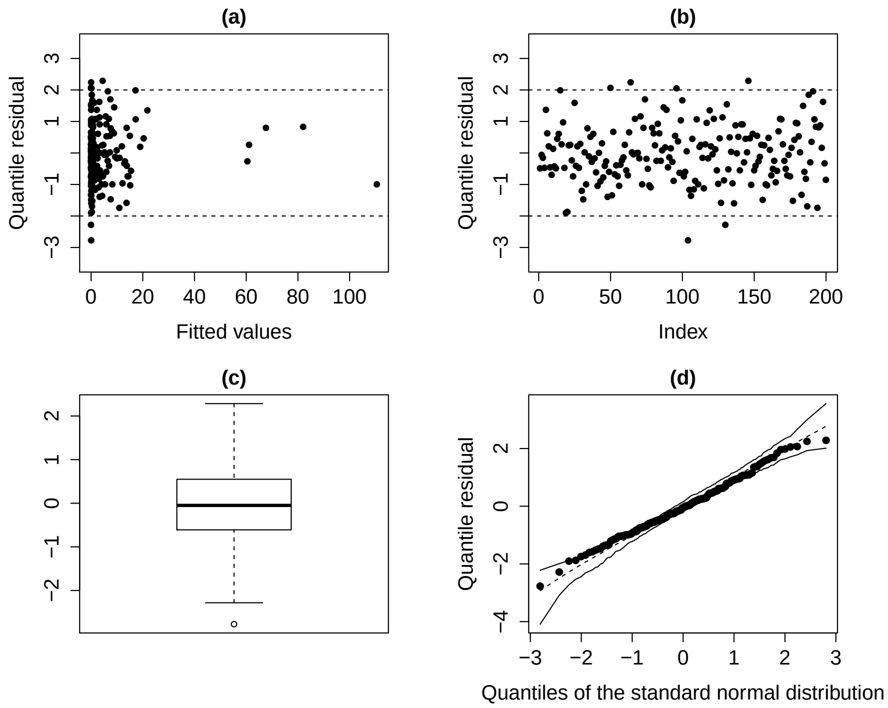

Figure 184 – Residual plots for the mixed SSLBS2 regression model.

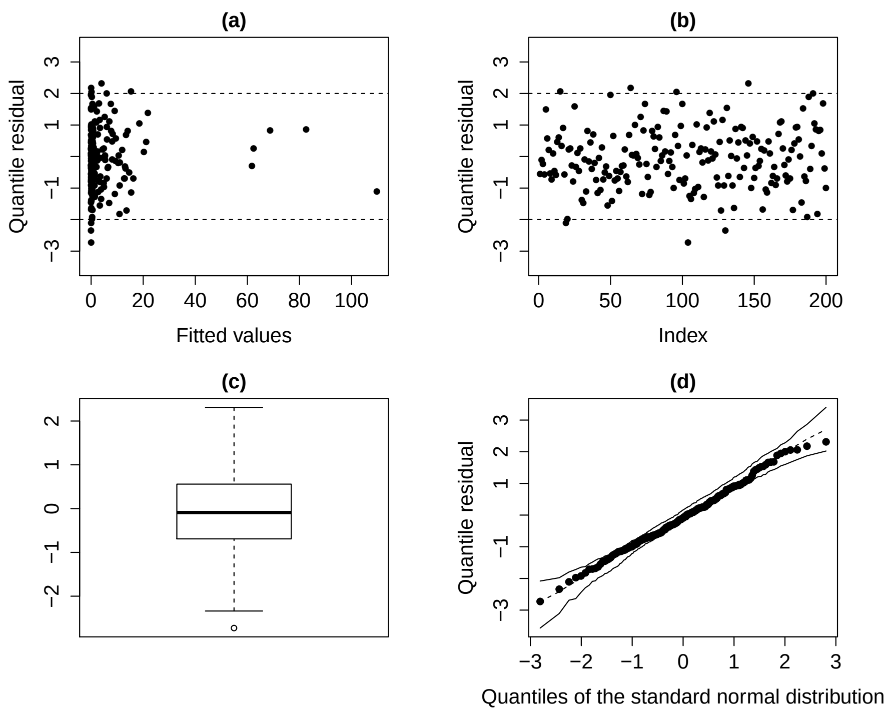

Figure 185 – Residual plots for the mixed SCNBS regression model.

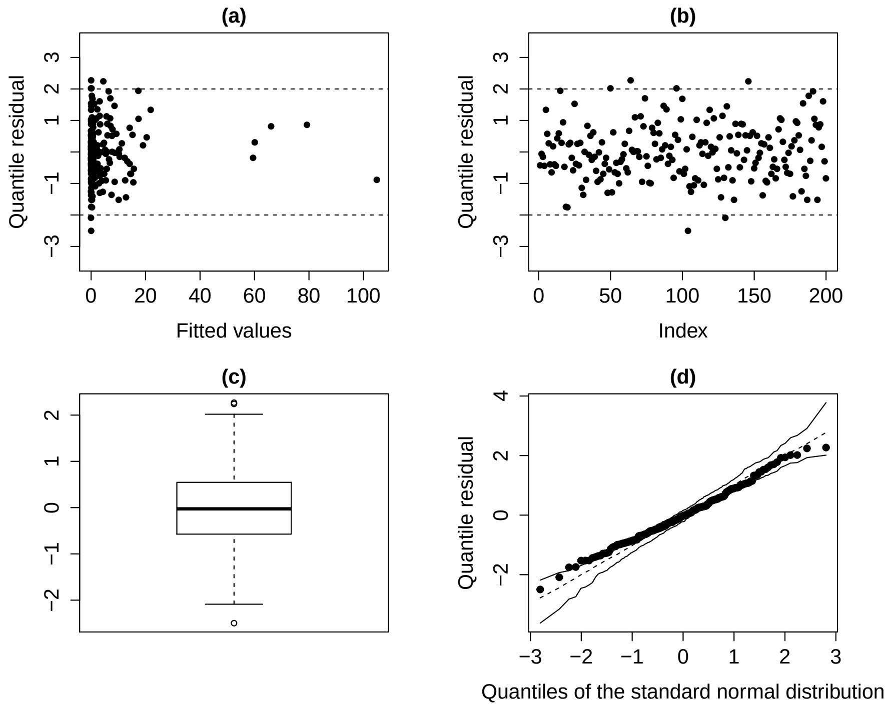

Figure 186 – Residual plots for the mixed SNBS regression model.

Simulated observations from mixed SSLBS regression model

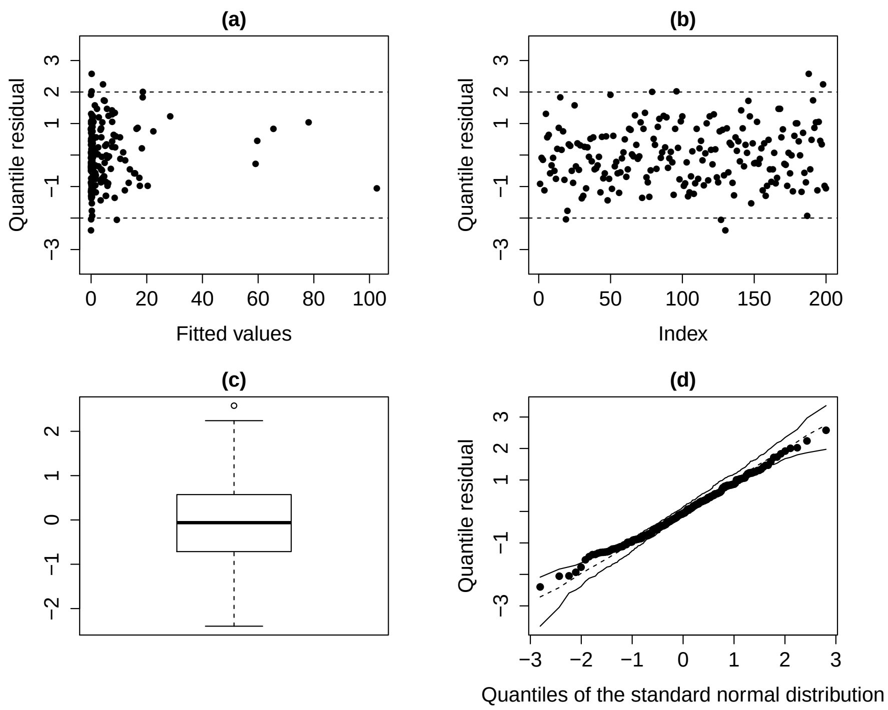

Figure 187 – Residual plots for the mixed SSLBS1 regression model.

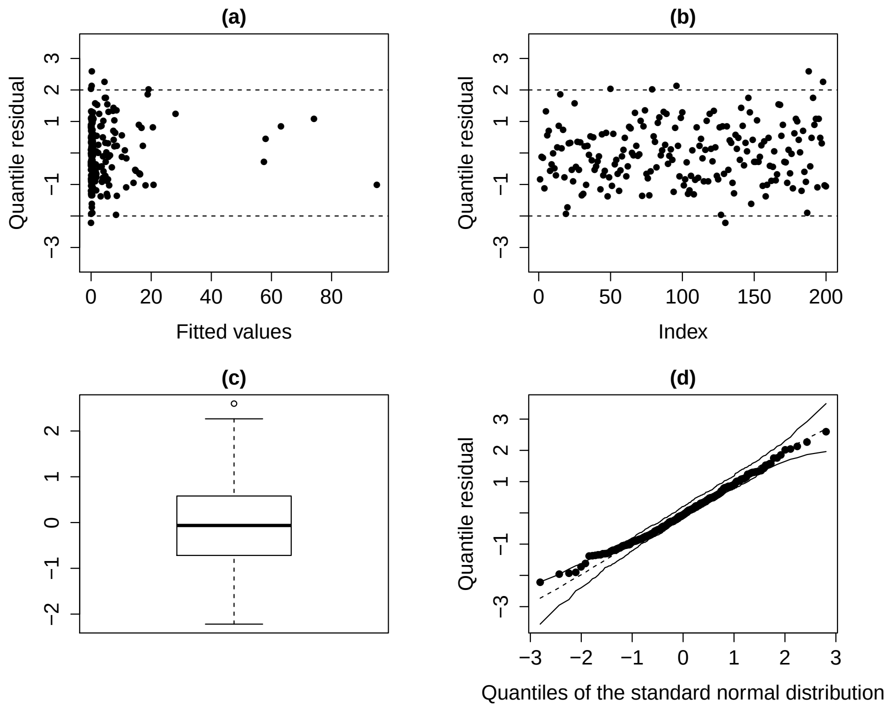

Figure 188 – Residual plots for the mixed SSLBS2 regression model.

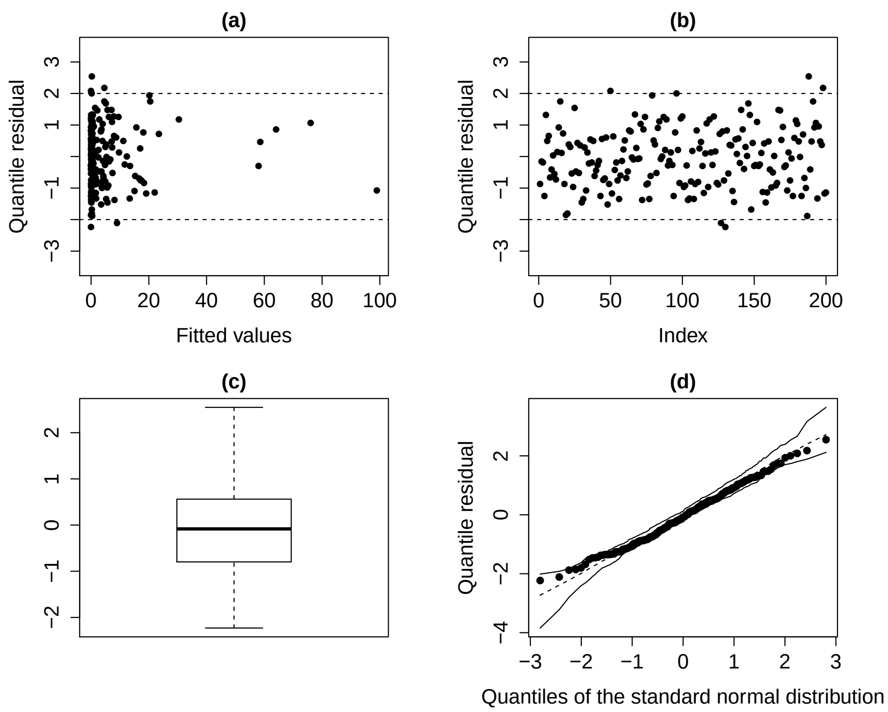

Figure 189 – Residual plots for the mixed SGtBS1 regression model.

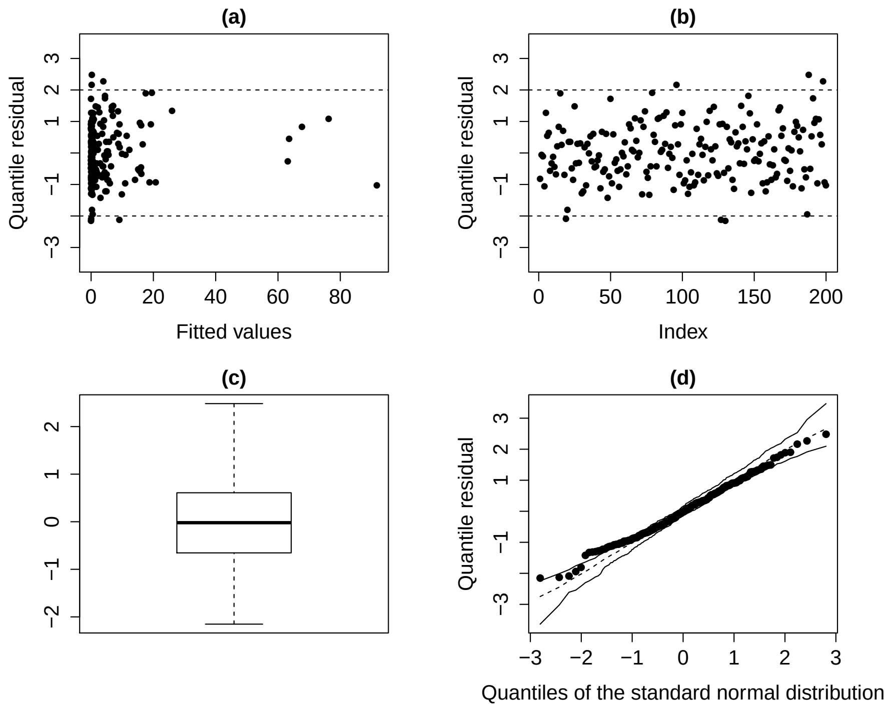

Figure 190 – Residual plots for the mixed SGtBS2 regression model.

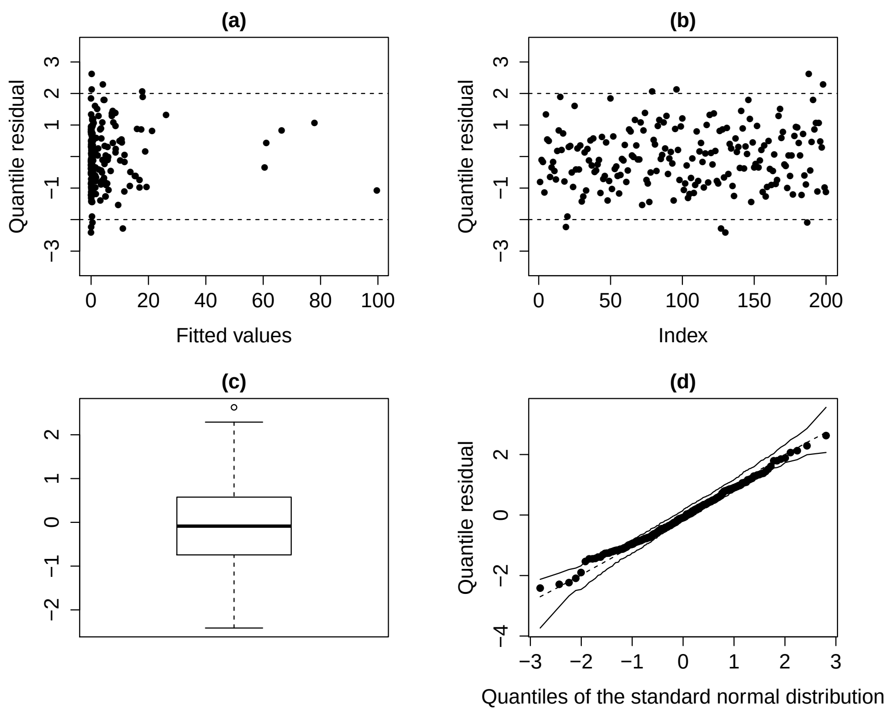

Figure 191 – Residual plots for the mixed StBS regression model.

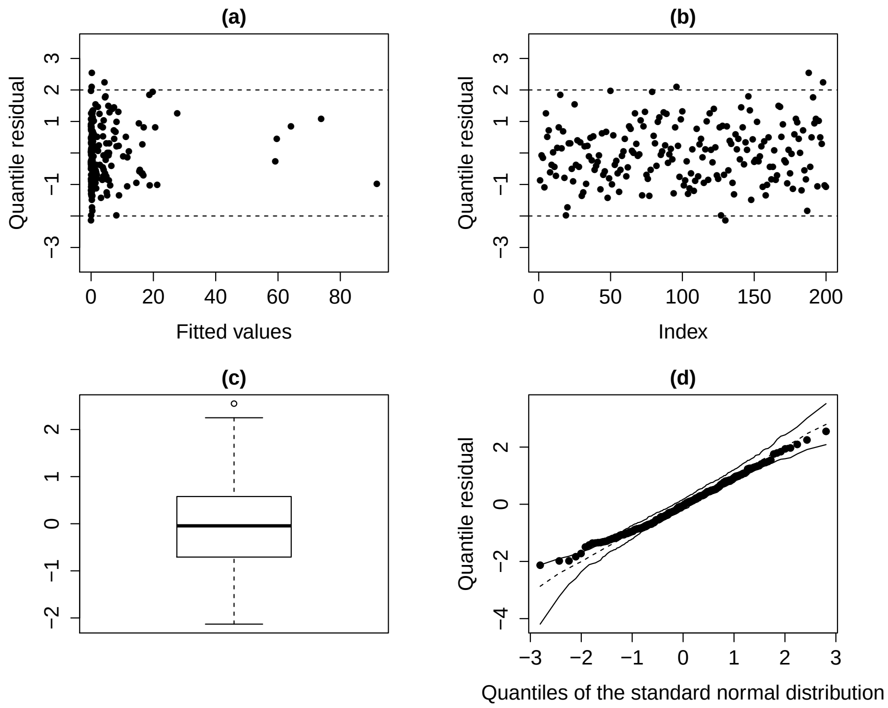

Figure 192 – Residual plots for the mixed SCNBS regression model.

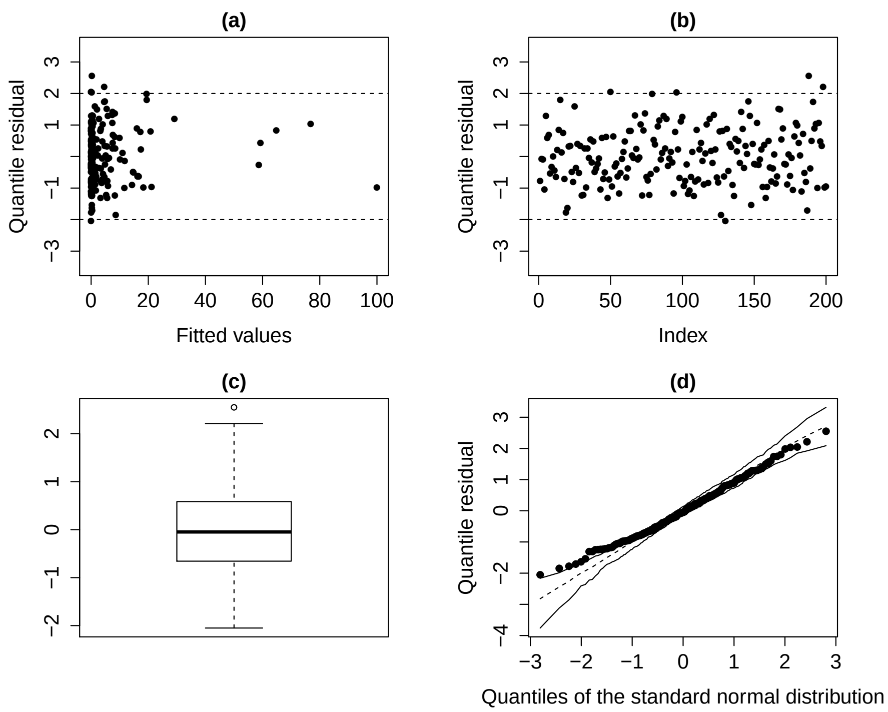

Figure 193 – Residual plots for the mixed SNBS regression model.

Simulated observations from mixed SCNBS regression model

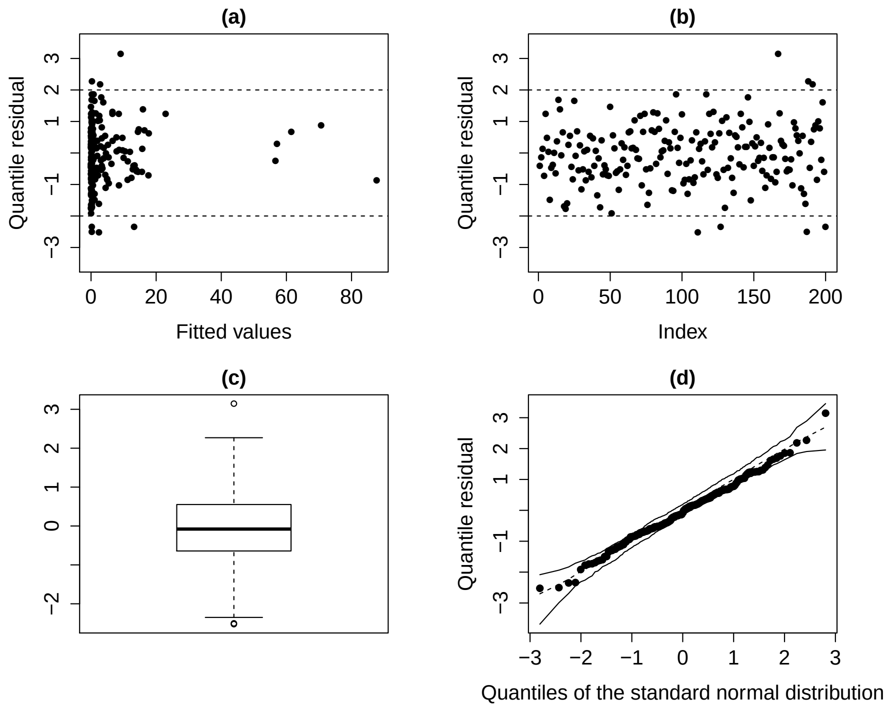

Figure 194 – Residual plots for the mixed SSLBS1 regression model.

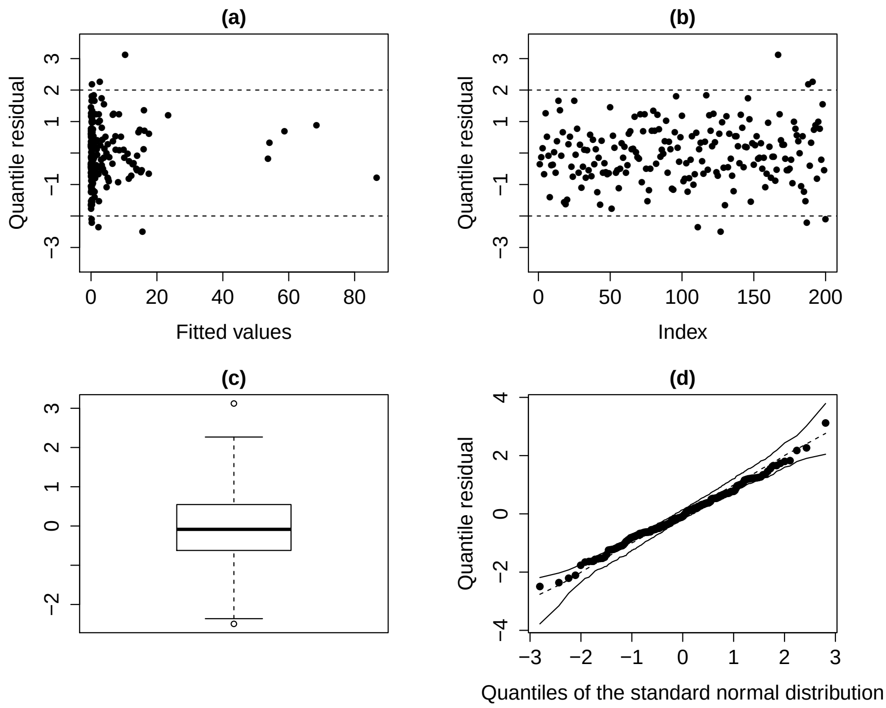

Figure 195 – Residual plots for the mixed SSLBS2 regression model.

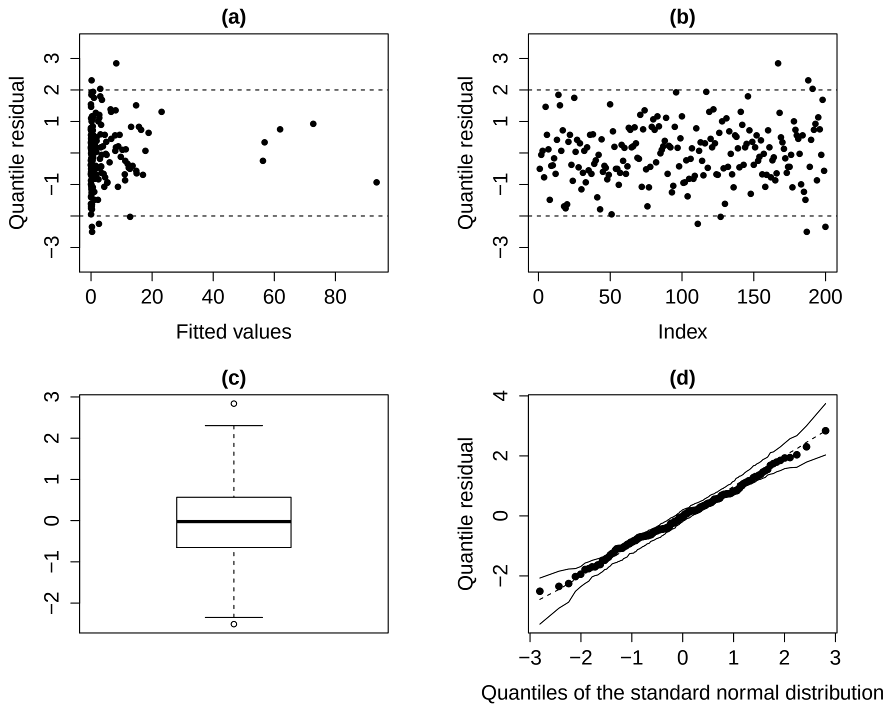

Figure 196 – Residual plots for the mixed SGtBS1 regression model.

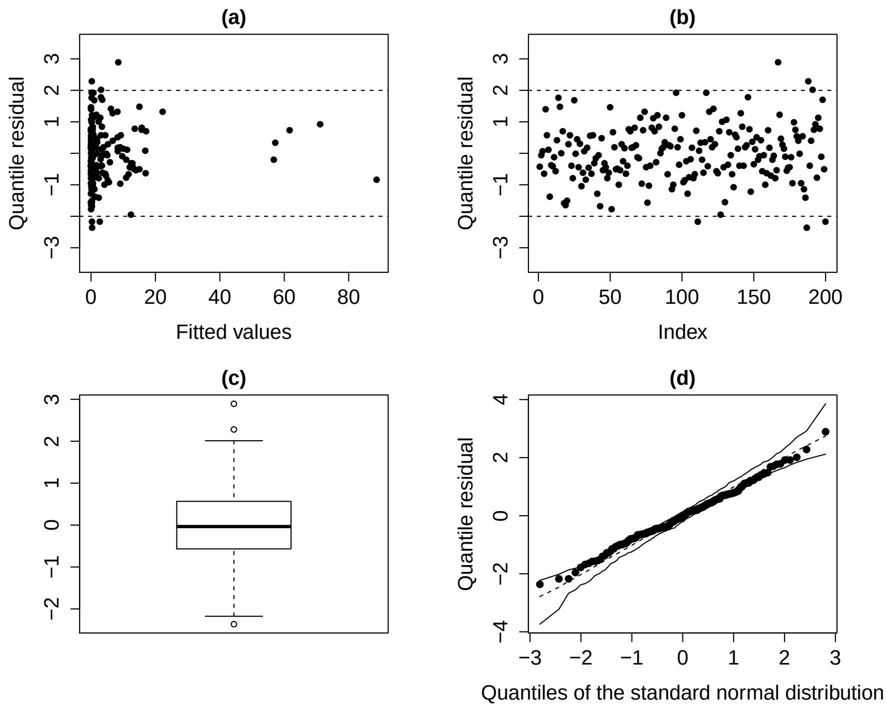

Figure 197 – Residual plots for the mixed SGtBS2 regression model.

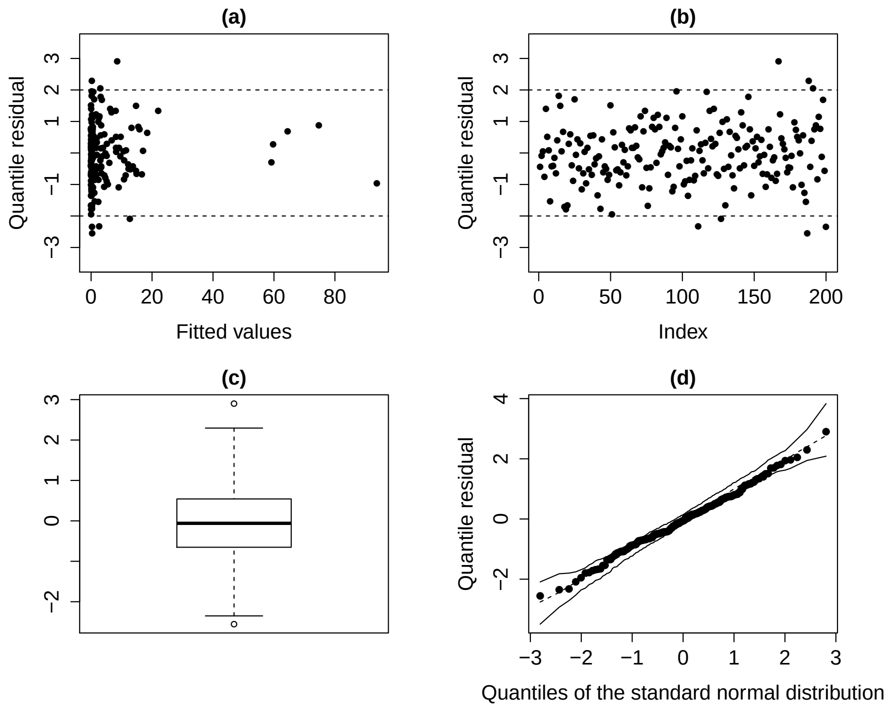

Figure 198 – Residual plots for the mixed StBS regression model.

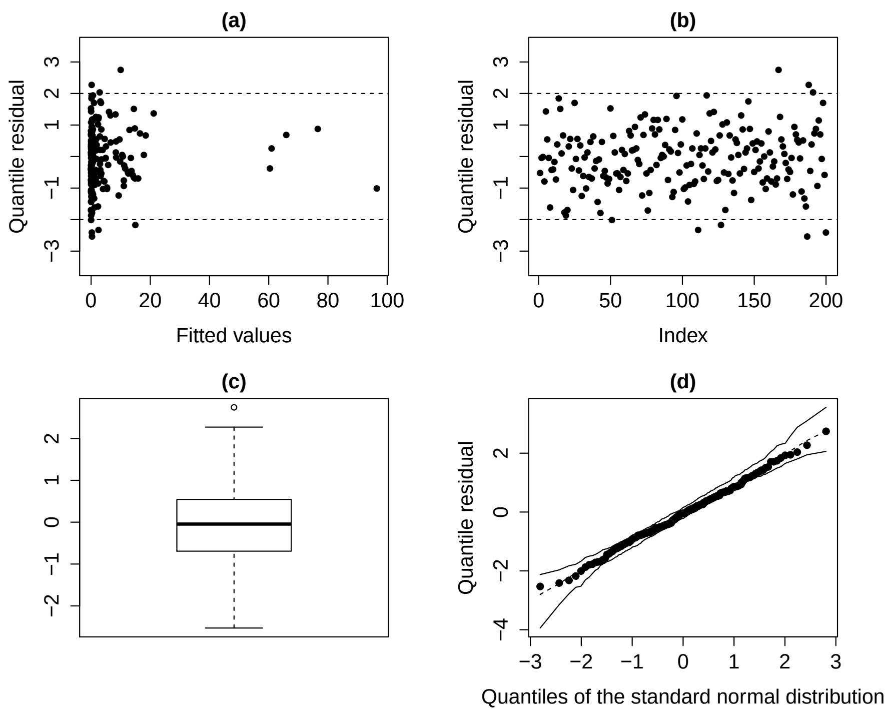

Figure 199 – Residual plots for the mixed SCNBS regression model.

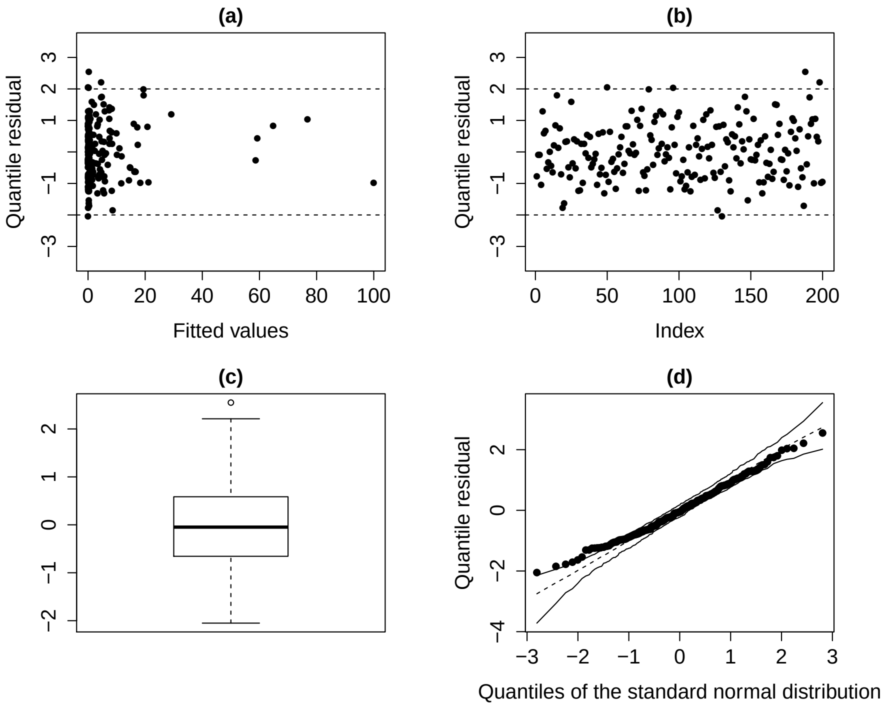

Figure 200 – Residual plots for the mixed SNBS regression model.

E.4 Behavior of the K-L divergence

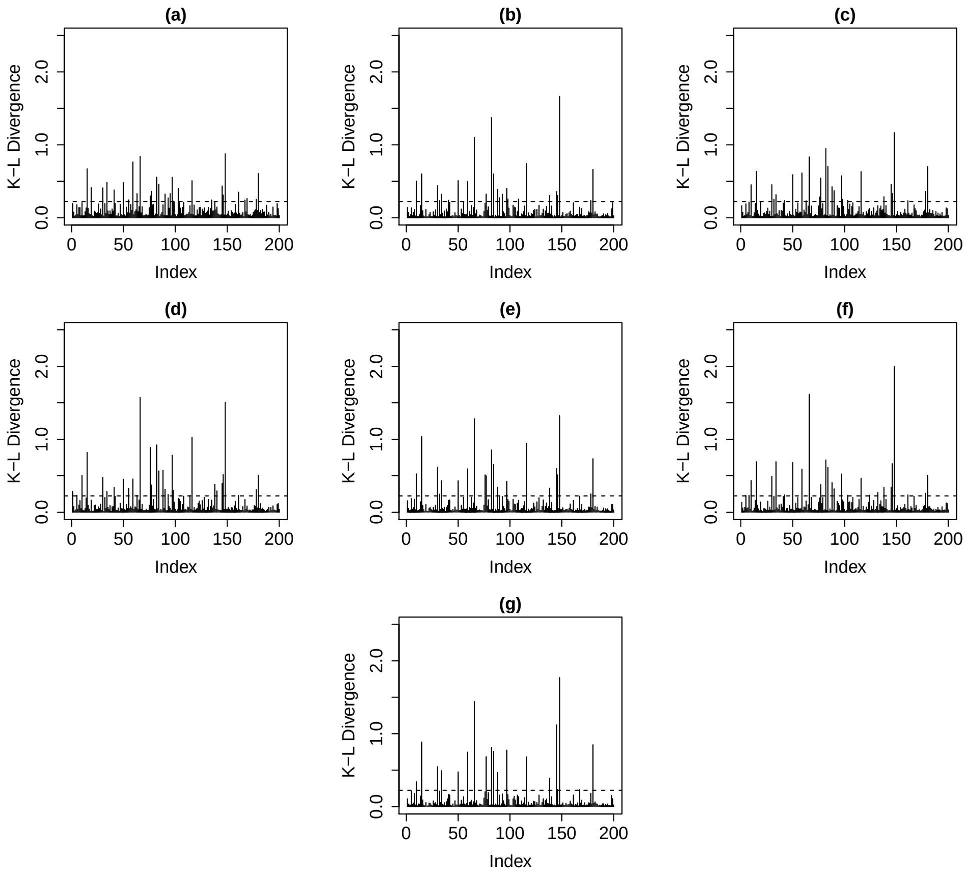

Figure 201 – K-L divergence when we generated the data set from mixed SGtBS1 and fitted the following models: (a) mixed SGtBS1, (b) mixed SGtBS2, (c) mixed StBS, (d) mixed SSLBS1, (e) mixed SSLBS2, (f) mixed SCNBS, and (g) mixed SNBS.

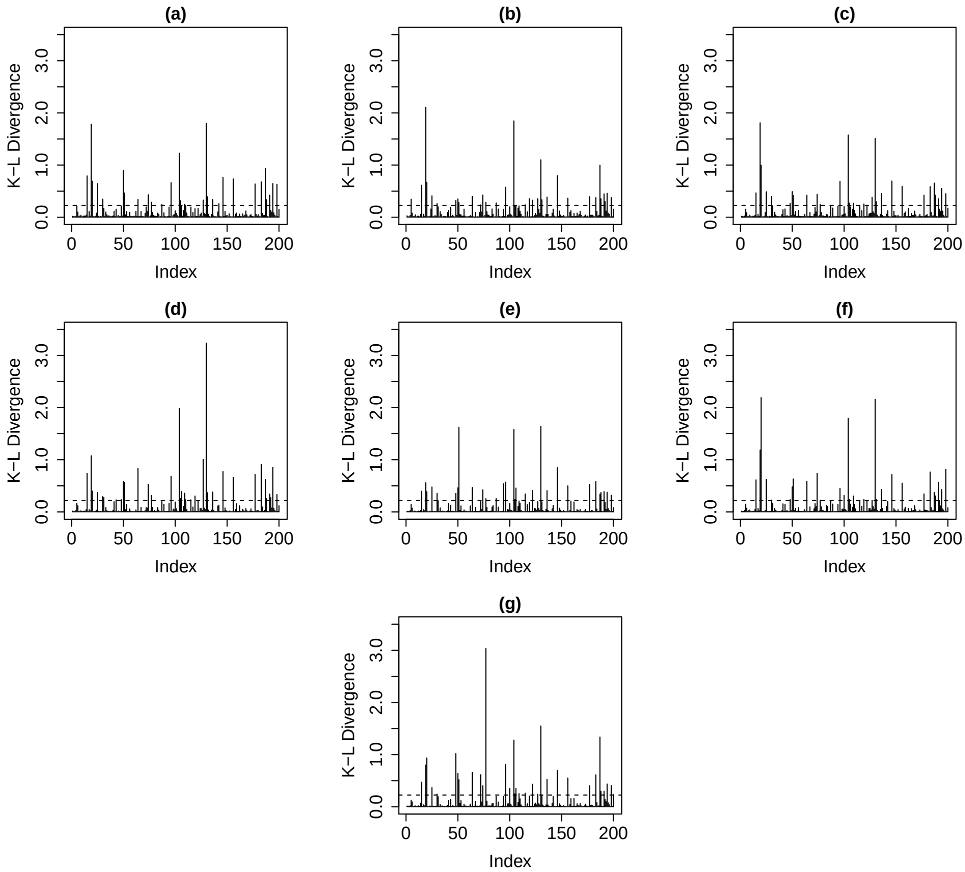

Figure 202 – K-L divergence when we generated the data set from mixed SGtBS2 and fitted the following models: (a) mixed SGtBS1, (b) mixed SGtBS2, (c) mixed StBS, (d) mixed SSLBS1, (e) mixed SSLBS2, (f) mixed SCNBS, and (g) mixed SNBS.

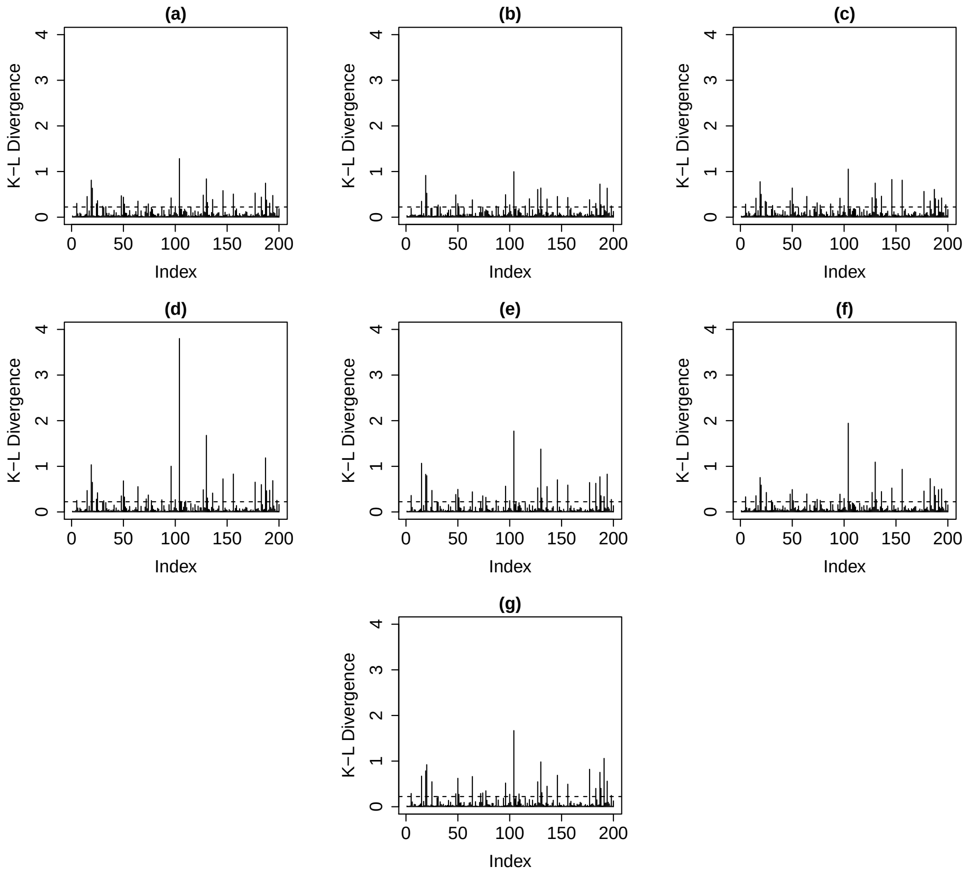

Figure 203 – K-L divergence when we generated the data set from mixed StBS and fitted the following models: (a) mixed SGtBS1, (b) mixed SGtBS2, (c) mixed StBS, (d) mixed SSLBS1, (e) mixed SSLBS2, (f) mixed SCNBS, and (g) mixed SNBS.

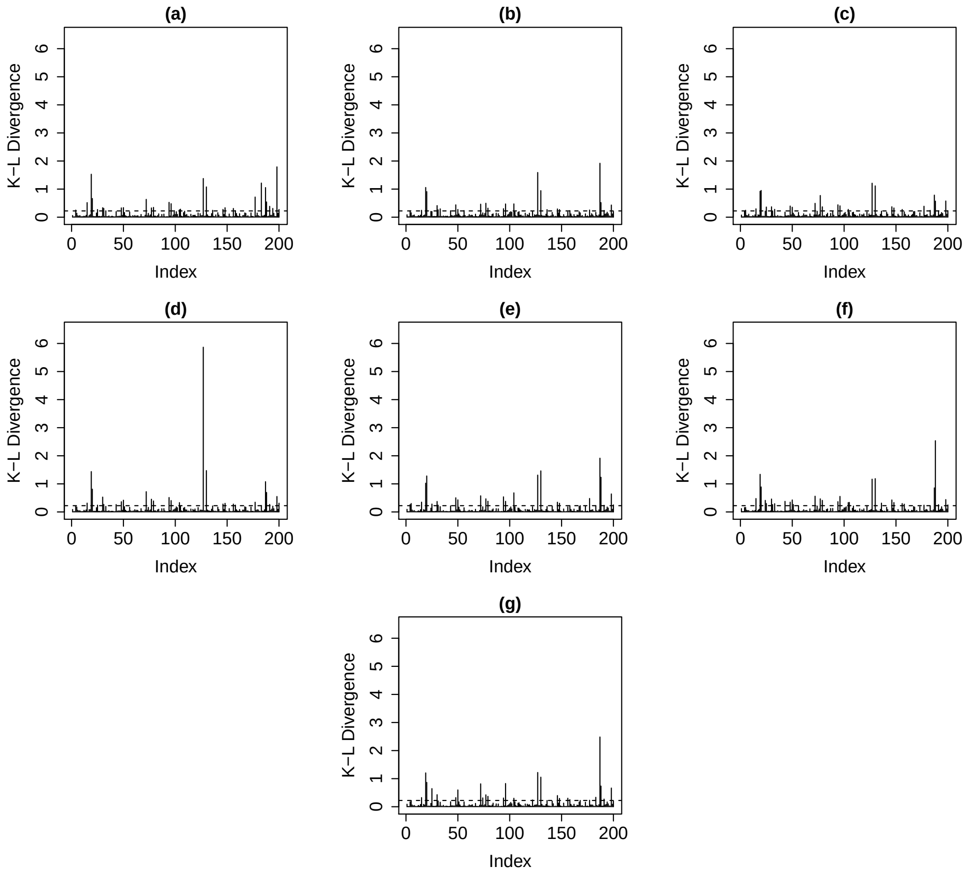

Figure 204 – K-L divergence when we generated the data set from the mixed SSLBS and fitted the following models: (a) mixed SGtBS1, (b) mixed SGtBS2, (c) mixed StBS, (d) mixed SSLBS1, (e) mixed SSLBS2, (f) mixed SCNBS, and (g) mixed SNBS.

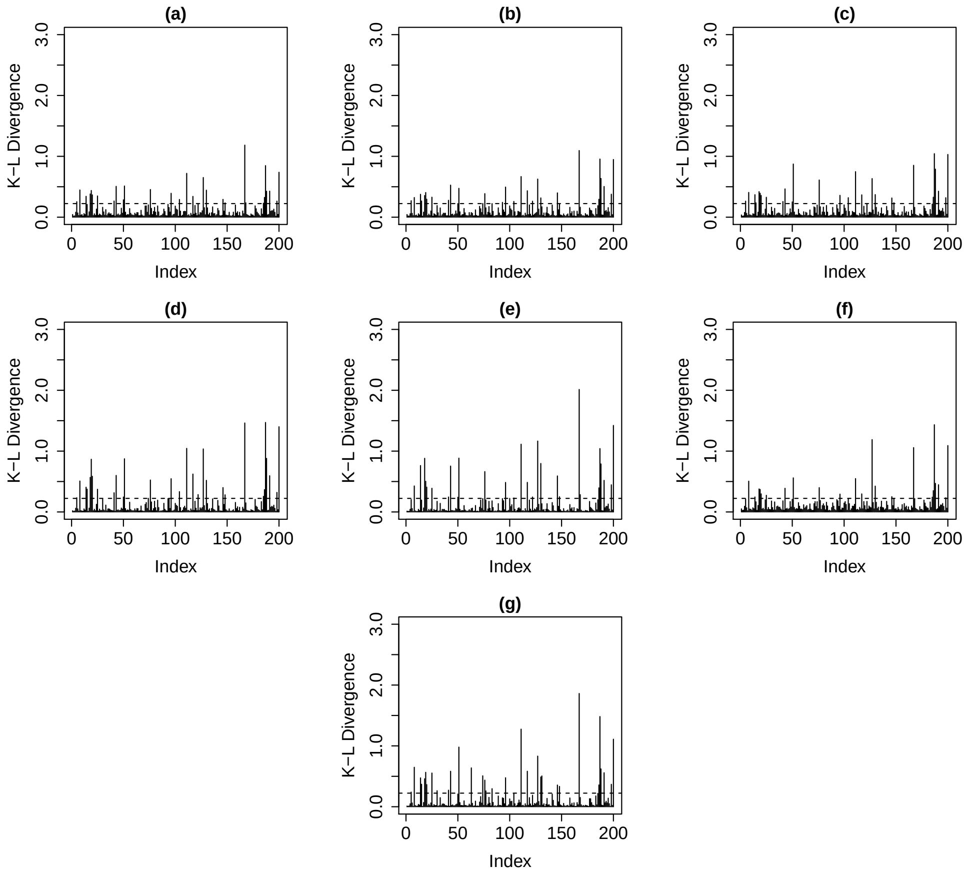

Figure 205 – K-L divergence when we generated the data set from the mixed SCNBS and fitted the following models: (a) mixed SGtBS1, (b) mixed SGtBS2, (c) mixed StBS, (d) mixed SSLBS1, (e) mixed SSLBS2, (f) mixed SCNBS, and (g) mixed SNBS.

E.5 Statistics of model comparison

Table 169 – Averaged criteria for the simulation study.

True underlying model: mixed SGtBS1 regression model				
Model	EAIC	EBIC	DIC	LPML
SGtBS1	331.294	352.423	901.900	-190.787
SGtBS2	345.596	363.204	955.609	-199.016
StBS	331.524	352.652	903.122	-191.108
SSLBS1	334.745	355.874	914.685	-193.337
SSLBS2	336.848	357.977	921.139	-194.562
SCNBS	330.487	355.137	896.091	-191.756
SNBS	344.870	362.477	954.508	-199.699
True underlying model: mixed SGtBS2 regression model				
Model	EAIC	EBIC	DIC	LPML
SGtBS1	141.078	162.207	340.754	-93.289
SGtBS2	141.573	159.180	351.345	-94.152
StBS	141.411	162.540	341.138	-94.080
SSLBS1	137.481	158.610	331.050	-94.644
SSLBS2	135.456	156.585	328.393	-92.689
SCNBS	140.129	164.779	337.604	-91.146
SNBS	133.694	151.301	334.175	-88.729
True underlying model: mixed StBS regression model				
Model	EAIC	EBIC	DIC	LPML
SGtBS1	462.939	484.068	1299.697	-255.607
SGtBS2	464.941	482.548	1315.006	-257.032
StBS	463.437	484.566	1301.001	-256.370
SSLBS1	465.149	486.278	1307.700	-258.655
SSLBS2	464.959	486.088	1308.274	-258.587
SCNBS	462.262	486.912	1295.448	-255.646
SNBS	466.659	484.267	1323.798	-259.169
True underlying model: mixed SSLBS regression model				
Model	EAIC	EBIC	DIC	LPML
SGtBS1	467.984	489.112	1315.201	-257.151
SGtBS2	468.559	486.166	1326.340	-257.617
StBS	467.343	488.472	1311.991	-256.764
SSLBS1	464.203	485.331	1305.296	-256.706
SSLBS2	462.045	483.174	1301.884	-254.992
SCNBS	466.108	490.758	1307.337	-255.869

Table 169 (continued).

SNBS	460.888	478.496	1307.443	-253.908
True underlying model: mixed SCNBS regression model				
Model	EAIC	EBIC	DIC	LPML
SGtBS1	393.060	414.189	1090.437	-220.875
SGtBS2	395.662	413.270	1107.385	-221.719
StBS	393.023	414.152	1090.933	-220.880
SSLBS1	396.145	417.274	1102.471	-224.712
SSLBS2	396.896	418.024	1104.441	-224.346
SCNBS	387.920	412.570	1071.676	-219.664
SNBS	408.865	426.472	1150.619	-232.32

Table 170 – Percentage of times that the correct model was selected.

Model	EAIC	EBIC	DIC	LPML
StBS	10%	10%	30%	20%
SGtBS1	10%	20%	20%	20%
SGtBS2	0%	0%	0%	0%
SSLBS	0%	0%	50%	0%
SCNBS	60%	60%	60%	50%

E.6 Posterior predictive checking

Table 171 – Posterior predictive checking for the mixed CSSBS regression model.

True underlying model: SGtBS1							
	SGtBS1	SGtBS2	StBS	SSLBS1	SSLBS2	SCNBS	SNBS
p-value	.583	.360	.579	.385	.543	.737	.493
True underlying model: SGtBS2							
	SGtBS2	SGtBS1	StBS	SSLBS1	SSLBS2	SCNBS	SNBS
p-value	.614	.598	.692	.631	.721	.790	.623
True underlying model: StBS							
	StBS	SGtBS1	SGtBS2	SSLBS1	SSLBS2	SCNBS	SNBS
p-value	.598	.676	.658	.561	.576	.818	.571
True underlying model: SSLBS							
	SSLBS1	SSLBS2	SGtBS1	SGtBS2	StBS	SCNBS	SNBS
p-value	.637	.581	.688	.623	.606	.734	.578
True underlying model: SCNBS							
	SCNBS	SGtBS1	SGtBS2	StBS	SSLBS1	SSLBS2	SNBS
p-value	.797	.460	.382	.484	.379	.402	.395

E.7 Results of the statistical analysis of cholesterol levels

Table 172 – Bayesian estimates for the mixed SGtBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	-.080	.004	[-.085; -.070]
β_1	.574	$< .001$	[.574; .575]
β_{21}	.028	.005	[.018; .039]
β_{22}	.138	.004	[.130; .144]
ψ_0	-2.067	.015	[-2.086; -2.039]
ψ_1	.015	.005	[.006; .024]
ψ_{21}	3.194	.016	[3.166; 3.230]
ψ_{22}	.310	.013	[.295; .342]
γ	-.943	.018	[-.963; -.911]
ν_1	18.477	.056	[18.360; 18.570]
σ^2	20.131	2.482	[15.939; 25.310]

Table 173 – Bayesian estimates for the mixed StBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.486	.001	[.484; .488]
β_1	-.096	.001	[-.098; -.095]
β_{21}	.331	.002	[.328; .333]
β_{22}	-.197	.005	[-.204; -.190]
ψ_0	-3.103	.020	[-3.136; -3.078]
ψ_1	-.059	.004	[-.064; -.053]
ψ_{21}	-.632	.004	[-.639; -.628]
ψ_{22}	-.224	.003	[-.227; -.220]
γ	.989	0.001	[.988; .992]
ν	132.283	66.387	[51.186; 285.540]
σ^2	.809	0.105	[.626; 1.051]

Table 174 – Bayesian estimates for the mixed SSLBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.415	.004	[.410; .422]
β_1	.267	$< .001$	[.267; .268]
β_{21}	.036	.002	[.033; .040]
β_{22}	.049	.004	[.043; .056]
ψ_0	-.008	.002	[-.013; -.004]
ψ_1	.596	.003	[.592; .602]
ψ_{21}	.621	.004	[.616; .628]
ψ_{22}	-.061	.004	[-.068; -.053]
γ	-.990	.001	[-.991; -.988]
ν	16.782	.070	[16.660; 16.900]
σ^2	5.465	.688	[4.303; 6.920]

Table 175 – Bayesian estimates for the mixed SSLBS2 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.277	.001	[.275; .280]
β_1	.659	$<$ .001	[.658; .659]
β_{21}	.011	.002	[.008; .014]
β_{22}	-.007	.007	[-.018; .006]
ψ_0	-1.267	.004	[-1.276; -1.262]
ψ_1	.455	.002	[0.451; 0.458]
ψ_{21}	1.414	.006	[1.408; 1.424]
ψ_{22}	.194	.004	[0.189; 0.203]
γ	-.991	.001	[-.992; -.990]
ν	2.798	.014	[2.781; 2.823]
σ^2	25.919	3.040	[20.767; 32.681]

Table 176 – Bayesian estimates for the mixed SCNBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	-.958	.004	[-.963; -.951]
β_1	.223	$<$ .001	[.223; .224]
β_{21}	-.321	.009	[-.335; -.305]
β_{22}	.586	.013	[.563; .603]
ψ_0	-3.093	.027	[-3.123; -3.043]
ψ_1	-.007	.003	[-.012; .001]
ψ_{21}	.767	.016	[.738; .786]
ψ_{22}	-2.899	.297	[-3.510; -2.347]
γ	-.978	.003	[-.982; -.970]
ν_1	.005	.001	[.002; .008]
ν_2	.257	.007	[.242; .265]
σ^2	5.632	.708	[4.405; 7.122]

APPENDIX F – Results of Chapter 6

In this section, we present in detail the results related to the marginal means, variances and covariance of the mixed ZA-SSBS regression models. Furthermore, we present the results of the statistical analysis of the bilirubin concentration data set.

F.1 Results related to the marginal means, variances and covariance

Let $T_{ij}|\boldsymbol{b}_i, \boldsymbol{\Omega} \sim \text{ZA-SSBS}(p_{ij}, \mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}), i = 1, \ldots, n, j = 1, \ldots, k_i$, where $\boldsymbol{\Omega} = (\boldsymbol{\beta}^\top, \boldsymbol{\psi}^\top, \boldsymbol{\zeta}^\top, \gamma, \boldsymbol{\nu}^\top)^\top$, p_{ij}, μ_{ij}, and ϕ_{ij} are defined in Equation (6.1). The hierarchical structure of the mixed ZA-SSBS regression models is given by

$$\begin{aligned} T_{ij}|\boldsymbol{b}_i, \boldsymbol{\Omega} &\sim \text{ZA-SSBS}(p_{ij}, \mu_{ij}, \phi_{ij}, \gamma, \boldsymbol{\nu}) \\ \boldsymbol{b}_i|\boldsymbol{\Sigma}_b &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_b). \end{aligned}$$

By using results from conditional distributions and the expressions presented in Equation (6.3), we have that

$$\begin{aligned} \mathbb{E}(T_{ij}) &= \mathbb{E}\left[(1-p_{ij})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{z}_{ij}^\top\boldsymbol{b}\right\}\right] \\ &= (1-p_{ij})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\mathbb{E}\left[\boldsymbol{z}_{ij}^\top\boldsymbol{b}\right] \\ &= (1-p_{ij})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}M_b(\boldsymbol{z}_{ij}). \end{aligned}$$

$$\begin{aligned} \mathbb{V}(T_{ij}) &= \mathbb{V}(\mathbb{E}(T_{ij}|\boldsymbol{b})) + \mathbb{E}[\mathbb{V}(T_{ij}|\boldsymbol{b})] \\ &= (1-p_{ij})^2\left\{\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}M_b(2\boldsymbol{z}_{ij}) - \exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}[M_b(\boldsymbol{z}_{ij})]^2\right\} \\ &\quad + (1-p_{ij})(p_{ij}+c)\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}M_b(2\boldsymbol{z}_{ij}) \\ &= (1-p_{ij})\exp\left\{2\boldsymbol{x}_{ij}^\top\boldsymbol{\beta}\right\}\left\{(1-p_{ij})\left[M_b(2\boldsymbol{z}_{ij}) - \{M_b(\boldsymbol{z}_{ij})\}^2\right] + (p+c)M_b(2\boldsymbol{z}_{ij})\right\}. \end{aligned}$$

$$\begin{aligned} \text{Cov}(T_{ij}, T_{ij'}) &= \text{Cov}[\mathbb{E}(T_{ij}|\boldsymbol{b}), \mathbb{E}(T_{ij'}|\boldsymbol{b})] + \underbrace{\mathbb{E}[\text{Cov}(T_{ij}, T_{ij'}|\boldsymbol{b})]}_{=0} \\ &= (1-p_{ij})(1-p_{ij'})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\}\text{Cov}\left[\exp\left\{\boldsymbol{z}_{ij}^\top\boldsymbol{\beta}\right\}\exp\left\{\boldsymbol{z}_{ij'}^\top\boldsymbol{\beta}\right\}\right] \\ &= (1-p_{ij})(1-p_{ij'})\exp\left\{\boldsymbol{x}_{ij}^\top\boldsymbol{\beta} + \boldsymbol{x}_{ij'}^\top\boldsymbol{\beta}\right\}[M_b(\boldsymbol{z}_{ij} + \boldsymbol{z}_{ij'}) - M_b(\boldsymbol{z}_{ij})M_b(\boldsymbol{z}_{ij'})]. \end{aligned}$$

F.2 Results of the recovery parameter

Mixed ZA-SGtBS1 regression model

Table 177 – Results of the simulation study for the mixed ZA-SGtBS1 regression model ($\nu_1 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.509	-.312	-.458	-.638	-.555	-.518
	Sd	.365	.135	.212	.315	.307	.306
	LCI	.815	.960	1.115	.539	.756	.642
β_0	CP	1.000	1.000	1.000	.600	.400	.600
	Bias	-.009	.188	.042	-.138	-.055	-.018
	REQM	.365	.231	.216	.344	.311	.307
	AVRB	.017	.376	.084	.275	.110	.036
	Mean	.952	.968	1.159	1.014	.993	.926
	Sd	.207	.233	.194	.103	.105	.118
	LCI	.513	.603	.637	.334	.443	.446
β_1	CP	.800	.800	.800	.800	1.000	.800
	Bias	-.048	-.032	.159	.014	-.007	-.074
	REQM	.212	.235	.251	.104	.106	.140
	AVRB	.048	.032	.159	.014	.007	.074
	Mean	-.818	-.813	-1.044	-.429	-.760	-.311
	Sd	.512	.401	.251	.967	.152	1.156
	LCI	1.676	1.559	1.699	1.278	1.166	1.153
ψ_0	CP	.800	.800	1.000	.400	1.000	.400
	Bias	.182	.187	-.044	.571	.240	.689
	REQM	.543	.442	.255	1.123	.284	1.345
	AVRB	.182	.187	.044	.571	.240	.689
	Mean	.870	.789	.822	.510	.323	.327
	Sd	.351	.441	.372	.615	.508	.537
	LCI	2	1.804	2.064	1.294	1.389	1.387
ψ_1	CP	1.000	1.000	1.000	.600	.800	.800
	Bias	.370	.289	.322	.010	-.177	-.173
	REQM	.510	.527	.493	.615	.538	.564
	AVRB	.739	.577	.645	.021	.355	.345
	Mean	-2.721	-2.731	-2.660	-2.459	-2.461	-2.397
	Sd	.472	.317	.395	.567	.541	.529
	LCI	1.678	1.810	1.526	1.182	1.160	1.056
ζ_0	CP	.800	1.000	1.000	.600	.600	.600

Table 177 (continued).

	Bias	-.221	-.231	-.160	.041	.039	.103
	REQM	.522	.392	.426	.568	.543	.538
	AVRB	.089	.092	.064	.016	.016	.041
	Mean	1.020	1.079	1.002	.744	.752	.703
	Sd	.924	.583	.735	.771	.726	.695
	LCI	2.600	2.827	2.579	1.938	1.858	1.669
ζ_1	CP	.800	1.000	1.000	.800	.800	.600
	Bias	.220	.279	.202	-.056	-.048	-.097
	REQM	.950	.646	.762	.773	.728	.702
	AVRB	.275	.348	.253	.070	.060	.122
	Mean	-.448	.027	.381	-.678	-.005	.711
	Sd	.297	.253	.215	.096	.231	.049
	LCI	.842	1.046	.964	.536	.641	.547
γ	CP	.400	.600	.600	1.000	.400	1.000
	Bias	.352	.027	-.419	.122	-.005	-.089
	REQM	.461	.254	.471	.155	.232	.102
	AVRB	.440	-	.524	.153	-	.112
	Mean	7.709	6.784	5.899	9.119	5.811	12.644
	Sd	3.189	2.077	1.235	5.171	1.271	16.030
	LCI	8.466	7.206	7.372	9.541	4.327	9.62
ν_1	CP	.600	1.000	1.000	.400	1.000	.600
	Bias	2.709	1.784	.899	4.119	.811	7.644
	REQM	4.184	2.738	1.528	6.611	1.507	17.759
	AVRB	.542	.357	.180	.824	.162	1.529
	Mean	3.831	3.796	3.871	4.085	4.08	4.049
	Sd	.963	.926	.884	.455	.353	.300
	LCI	3.218	3.110	3.198	2.343	2.286	2.348
σ^2	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	-.169	-.204	-.129	.085	.080	.049
	REQM	.978	.948	.894	.463	.362	.304
	AVRB	.042	.051	.032	.021	.020	.012

Table 178 – Results of the simulation study for the mixed ZA-SGtBS1 regression model ($\nu_1 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.540	-.284	-.451	-.640	-.555	-.485
	SD	.404	.167	.234	.319	.303	.316
	LCI	.815	.960	1.115	.539	.756	.642
β_0	CP	1.000	1.000	1.000	.600	.400	.600
	Bias	-.040	.216	.049	-.140	-.055	.015
	RMSE	.406	.273	.239	.348	.308	.316
	AVRB	.080	.431	.097	.279	.110	.030
	Median	.949	.974	1.158	1.018	.987	.922
	SD	.211	.217	.199	.105	.105	.124
	LCI	.513	.603	.637	.334	.443	.446
β_1	CP	.800	.800	.800	.800	1.000	.800
	Bias	-.051	-.026	.158	.018	-.013	-.078
	RMSE	.217	.219	.254	.107	.106	.146
	AVRB	.051	.026	.158	.018	.013	.078
	Median	-.836	-.843	-1.064	-.432	-.740	-.309
	SD	.507	.442	.265	.954	.152	1.182
	LCI	1.676	1.559	1.699	1.278	1.166	1.153
ψ_0	CP	.800	.800	1.000	.400	1.000	.400
	Bias	.164	.157	-.064	.568	.260	.691
	RMSE	.533	.469	.273	1.11	.301	1.369
	AVRB	.164	.157	.064	.568	.260	.691
	Median	.872	.810	.855	.502	.300	.340
	SD	.358	.410	.381	.637	.516	.532
	LCI	2.000	1.804	2.064	1.294	1.389	1.387
ψ_1	CP	1.000	1.000	1.000	.600	.800	.800
	Bias	.372	.310	.355	.002	-.200	-.160
	RMSE	.516	.514	.520	.637	.553	.555
	AVRB	.744	.621	.709	.003	.399	.320
	Median	-2.706	-2.724	-2.662	-2.461	-2.462	-2.381
	SD	.422	.326	.400	.548	.556	.514
	LCI	1.678	1.810	1.526	1.182	1.160	1.056
ζ_0	CP	.800	1.000	1.000	.600	.600	.600
	Bias	-.206	-.224	-.162	.039	.038	.119
	RMSE	.470	.396	.432	.55	.557	.528
	AVRB	.082	.090	.065	.016	.015	.048

Table 178 (continued).

	Median	.999	1.113	1.007	.739	.759	.696
	SD	.851	.580	.734	.769	.726	.670
	LCI	2.600	2.827	2.579	1.938	1.858	1.669
ζ_1	CP	.800	1.000	1.000	.800	.800	.600
	Bias	.199	.313	.207	-.061	-.041	-.104
	RMSE	.874	.659	.763	.772	.727	.678
	AVRB	.248	.391	.259	.076	.051	.129
	Median	-.477	-.030	.393	-.691	.003	.731
	SD	.308	.228	.252	.093	.221	.061
	LCI	.842	1.046	.964	.536	.641	.547
γ	CP	.400	.600	.600	1.000	.400	1.000
	Bias	.323	-.030	-.407	.109	.003	-.069
	RMSE	.447	.229	.479	.143	.221	.092
	AVRB	.404	-	.509	.136	-	.086
	Median	6.995	6.438	5.458	8.348	5.659	12.689
	SD	2.540	2.173	.983	4.419	1.294	16.586
	LCI	8.466	7.206	7.372	9.541	4.327	9.620
ν_1	CP	.600	1.000	1.000	.400	1.000	.600
	Bias	1.995	1.438	.458	3.348	.659	7.689
	RMSE	3.229	2.605	1.085	5.544	1.453	18.281
	AVRB	.399	.288	.092	.670	.132	1.538
	Median	3.727	3.698	3.752	4.029	4.017	3.988
	SD	.941	.923	.860	.453	.336	.292
	LCI	3.218	3.110	3.198	2.343	2.286	2.348
σ^2	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	-.273	-.302	-.248	.029	.017	-.012
	RMSE	.980	.971	.895	.454	.336	.292
	AVRB	.068	.075	.062	.007	.004	.003

Table 179 – Results of the simulation study for the mixed ZA-SGtBS1 regression model ($\nu_1 = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.554	-.282	-.439	-.641	-.556	-.473
	SD	.415	.171	.246	.319	.305	.314
	LCI	.815	.960	1.115	.539	.756	.642
	CP	1.000	1.000	1.000	.600	.400	.600
	Bias	-.054	.218	.061	-.141	-.056	.027
	RMSE	.419	.277	.254	.349	.310	.315
	AVRB	.107	.437	.123	.281	.112	.053
β_1	Mode	.949	.972	1.157	1.014	.990	.925
	SD	.208	.215	.200	.104	.106	.121
	LCI	.513	.603	.637	.334	.443	.446
	CP	.800	.800	.800	.800	1.000	.800
	Bias	-.051	-.028	.157	.014	-.010	-.075
	RMSE	.214	.217	.254	.105	.107	.143
	AVRB	.051	.028	.157	.014	.010	.075
ψ_0	Mode	-.933	-.872	-1.114	-.468	-.729	-.310
	SD	.421	.487	.313	.887	.155	1.213
	LCI	1.676	1.559	1.699	1.278	1.166	1.153
	CP	.800	.800	1.000	.400	1.000	.400
	Bias	.067	.128	-.114	.532	.271	.690
	RMSE	.427	.503	.333	1.034	.312	1.395
	AVRB	.067	.128	.114	.532	.271	.690
ψ_1	Mode	.888	.778	1.015	.488	.284	.400
	SD	.376	.423	.315	.696	.521	.459
	LCI	2.000	1.804	2.064	1.294	1.389	1.387
	CP	1.000	1.000	1.000	.600	.800	.800
	Bias	.388	.278	.515	-.012	-.216	-.100
	RMSE	.540	.506	.604	.696	.564	.470
	AVRB	.776	.556	1.030	.025	.432	.200
ζ_0	Mode	-2.751	-2.713	-2.670	-2.460	-2.471	-2.360
	SD	.324	.359	.421	.539	.581	.502
	LCI	1.678	1.810	1.526	1.182	1.160	1.056
	CP	.800	1.000	1.000	.600	.600	.600
	Bias	-.251	-.213	-.170	.040	.029	.140
	RMSE	.410	.417	.454	.541	.582	.521
	AVRB	.100	.085	.068	.016	.012	.056

Table 179 (continued).

	Mode	1.003	1.219	1.022	.663	.781	.656
	SD	.729	.591	.769	.907	.754	.637
	LCI	2.600	2.827	2.579	1.938	1.858	1.669
ζ_1	CP	.800	1.000	1.000	.800	.800	.600
	Bias	.203	.419	.222	-.137	-.019	-.144
	RMSE	.756	.724	.800	.917	.755	.653
	AVRB	.254	.524	.277	.171	.023	.180
	Mode	-.541	-.075	.422	-.692	.001	.730
	SD	.274	.244	.315	.098	.220	.055
	LCI	.842	1.046	.964	.536	.641	.547
γ	CP	.400	.600	.600	1.000	.400	1.000
	Bias	.259	-.075	-.378	.108	.001	-.070
	RMSE	.378	.256	.492	.146	.220	.089
	AVRB	.324	-	.473	.135	-	.087
	Mode	5.971	5.240	4.763	7.934	5.462	11.920
	SD	2.103	1.120	.797	4.143	1.382	15.465
	LCI	8.466	7.206	7.372	9.541	4.327	9.620
ν_1	CP	.600	1.000	1.000	.400	1.000	.600
	Bias	.971	.240	-.237	2.934	.462	6.920
	RMSE	2.316	1.145	.831	5.077	1.457	16.942
	AVRB	.194	.048	.047	.587	.092	1.384
	Mode	3.540	3.553	3.612	3.920	3.924	3.885
	SD	.914	.957	.867	.439	.326	.290
	LCI	3.218	3.110	3.198	2.343	2.286	2.348
σ^2	CP	.800	.800	.800	1.000	1.000	1.000
	Bias	-.460	-.447	-.388	-.080	-.076	-.115
	RMSE	1.024	1.056	.950	.446	.335	.312
	AVRB	.115	.112	.097	.020	.019	.029

Table 180 – Results of the simulation study for the mixed ZA-SGtBS1 regression model ($\nu_1 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.435	-.439	-.359	-.361	-.515	-.566
	SD	.459	.242	.219	.308	.277	.403
	LCI	.287	.678	.346	.189	.525	.340
β_0	CP	.400	.800	.600	$< .001$	.800	.200
	Bias	.065	.061	.141	.139	-.015	-.066
	RMSE	.463	.250	.260	.338	.277	.408
	AVRB	.130	.122	.282	.278	.030	.132
	Mean	.999	1.004	.994	.976	.989	1.022
	SD	.033	.037	.041	.042	.026	.056
	LCI	.174	.180	.203	.112	.138	.132
β_1	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.001	.004	-.006	-.024	-.011	.022
	RMSE	.033	.037	.042	.048	.028	.060
	AVRB	.001	.004	.006	.024	.011	.022
	Mean	-.774	-1.912	-1.241	-1.266	-1.299	-.750
	SD	1.128	.511	.858	1.115	.744	.347
	LCI	.942	1.815	1.365	.690	1.108	.878
ψ_0	CP	.400	.400	.400	$< .001$	.400	.800
	Bias	.226	-.912	-.241	-.266	-.299	.250
	RMSE	1.150	1.046	.891	1.147	.802	.428
	AVRB	.226	.912	.241	.266	.299	.250
	Mean	.253	.799	.404	.430	.791	.462
	SD	.541	.666	.340	.343	.164	.192
	LCI	1.126	1.811	1.608	.589	1.162	.972
ψ_1	CP	.800	.800	1.000	.600	.800	.800
	Bias	-.247	.299	-.096	-.070	.291	-.038
	RMSE	.594	.730	.354	.350	.334	.196
	AVRB	.494	.598	.192	.141	.581	.077
	Mean	-2.486	-2.877	-2.686	-2.015	-2.145	-1.936
	SD	.704	.649	.799	.064	.164	.316
	LCI	1.279	1.799	1.179	.653	1.087	.755
ζ_0	CP	.600	.800	.600	.200	1.000	.400
	Bias	.014	-.377	-.186	.485	.355	.564
	RMSE	.704	.750	.820	.489	.391	.647

Table 180 (continued).

	AVRB	.006	.151	.074	.194	.142	.226
ζ_1	Mean	.497	1.119	.816	-.112	.044	-.370
	SD	1.123	1.070	1.206	.184	.290	.655
	LCI	1.876	3.000	1.788	.912	1.810	1.165
	CP	.400	.800	.400	.200	.800	.400
	Bias	-.303	.319	.016	-.912	-.756	-1.170
	RMSE	1.163	1.117	1.206	.930	.809	1.341
	AVRB	.379	.399	.020	1.140	.945	1.462
γ	Mean	-.758	-.071	.672	-.778	.049	.725
	SD	.112	.335	.187	.093	.168	.116
	LCI	.501	.796	.576	.501	.508	.397
	CP	1.000	.400	1.000	1.000	.400	1.000
	Bias	.042	-.071	-.128	.022	.049	-.075
	RMSE	.120	.343	.227	.095	.175	.139
	AVRB	.053	-	.160	.028	-	.094
ν_1	Mean	47.269	22.291	29.258	28.727	34.449	39.305
	SD	45.247	21.678	18.842	17.659	27.560	10.030
	LCI	26.809	31.289	21.995	13.353	29.225	30.707
	CP	.200	.400	.200	.200	.400	.800
	Bias	17.269	-7.709	-.742	-1.273	4.449	9.305
	RMSE	48.431	23.008	18.856	17.705	27.917	13.681
	AVRB	.576	.257	.025	.042	.148	.310
σ^2	Mean	4.257	4.163	4.170	3.920	3.871	3.965
	SD	.088	.131	.184	.232	.278	.326
	LCI	3.385	3.315	3.373	2.202	2.170	2.272
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.257	.163	.170	-.080	-.129	-.035
	RMSE	.271	.209	.251	.246	.307	.328
	AVRB	.064	.041	.042	.020	.032	.009

Table 181 – Results of the simulation study for the mixed ZA-SGtBS1 regression model ($\nu_1 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.437	-.463	-.364	-.361	-.505	-.576
	SD	.463	.302	.227	.318	.264	.434
	LCI	.287	.678	.346	.189	.525	.340
	CP	.400	.800	.600	$< .001$	.800	.200
	Bias	.063	.037	.136	.139	-.005	-.076
	RMSE	.468	.304	.264	.347	.264	.441
	AVRB	.126	.074	.273	.278	.009	.152
β_1	Median	1.000	1.003	.991	.976	.988	1.020
	SD	.033	.035	.039	.042	.026	.060
	LCI	.174	.180	.203	.112	.138	.132
	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	$< .001$	.003	-.009	-.024	-.012	.020
	RMSE	.033	.035	.040	.048	.029	.063
	AVRB	$< .001$	.003	.009	.024	.012	.020
ψ_0	Median	-.802	-1.905	-1.218	-1.253	-1.287	-.709
	SD	1.160	.542	.885	1.091	.736	.328
	LCI	.942	1.815	1.365	.690	1.108	.878
	CP	.400	.400	.400	$< .001$	.400	.800
	Bias	.198	-.905	-.218	-.253	-.287	.291
	RMSE	1.177	1.055	.912	1.119	.790	.439
	AVRB	.198	.905	.218	.253	.287	.291
ψ_1	Median	.265	.803	.376	.442	.777	.457
	SD	.550	.658	.308	.339	.176	.189
	LCI	1.126	1.811	1.608	.589	1.162	.972
	CP	.800	.800	1.000	.600	.800	.800
	Bias	-.235	.303	-.124	-.058	.277	-.043
	RMSE	.598	.724	.332	.344	.328	.194
	AVRB	.470	.607	.249	.116	.554	.087
ζ_0	Median	-2.439	-2.869	-2.698	-2.020	-2.146	-1.936
	SD	.706	.625	.804	.069	.148	.320
	LCI	1.279	1.799	1.179	.653	1.087	.755
	CP	.600	.800	.600	.200	1.000	.400
	Bias	.061	-.369	-.198	.480	.354	.564
	RMSE	.709	.726	.828	.485	.384	.648

Table 181 (continued).

	AVRB	.024	.148	.079	.192	.142	.226
ζ_1	Median	.446	1.169	.758	-.121	.050	-.380
	SD	1.086	1.015	1.207	.140	.272	.648
	LCI	1.876	3.000	1.788	.912	1.810	1.165
	CP	.400	.800	.400	.200	.800	.400
	Bias	-.354	.369	-.042	-.921	-.750	-1.180
	RMSE	1.142	1.080	1.207	.931	.798	1.346
	AVRB	.442	.461	.052	1.151	.938	1.475
γ	Median	-.789	-.045	.692	-.816	.044	.741
	SD	.111	.353	.214	.090	.152	.127
	LCI	.501	.796	.576	.501	.508	.397
	CP	1.000	.400	1.000	1.000	.400	1.000
	Bias	.011	-.045	-.108	-.016	.044	-.059
	RMSE	.111	.356	.240	.091	.158	.140
	AVRB	.013	-	.136	.020	-	.073
ν_1	Median	46.722	21.059	29.403	28.114	33.246	38.469
	SD	45.620	21.717	19.812	17.086	25.761	10.265
	LCI	26.809	31.289	21.995	13.353	29.225	30.707
	CP	.200	.400	.200	.200	.400	.800
	Bias	16.722	-8.941	-.597	-1.886	3.246	8.469
	RMSE	48.589	23.485	19.821	17.189	25.964	13.308
	AVRB	.557	.298	.020	.063	.108	.282
σ^2	Median	4.147	4.051	4.036	3.868	3.817	3.906
	SD	.098	.142	.188	.236	.279	.314
	LCI	3.385	3.315	3.373	2.202	2.170	2.272
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.147	.051	.036	-.132	-.183	-.094
	RMSE	.177	.151	.191	.271	.333	.328
	AVRB	.037	.013	.009	.033	.046	.023

Table 182 – Results of the simulation study for the mixed ZA-SGtBS1 regression model ($\nu_1 = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.440	-.463	-.362	-.362	-.504	-.538
	SD	.466	.299	.221	.309	.263	.459
	LCI	.287	.678	.346	.189	.525	.340
β_0	CP	.400	.800	.600	$<$ .001	.800	.200
	Bias	.060	.037	.138	.138	-.004	-.038
	RMSE	.470	.301	.261	.339	.263	.461
	AVRB	.121	.073	.275	.276	.007	.075
	Mode	.999	1.005	.995	.976	.989	1.021
	SD	.033	.037	.041	.043	.025	.056
	LCI	.174	.180	.203	.112	.138	.132
β_1	CP	1.000	1.000	1.000	.800	1.000	.800
	Bias	-.001	.005	-.005	-.024	-.011	.021
	RMSE	.033	.038	.041	.049	.028	.060
	AVRB	.001	.005	.005	.024	.011	.021
	Mode	-.869	-1.862	-1.202	-1.221	-1.276	-.706
	SD	1.244	.617	.934	1.036	.727	.347
	LCI	.942	1.815	1.365	.690	1.108	.878
ψ_0	CP	.400	.400	.400	$<$ .001	.400	.800
	Bias	.131	-.862	-.202	-.221	-.276	.294
	RMSE	1.251	1.06	.955	1.059	.777	.455
	AVRB	.131	.862	.202	.221	.276	.294
	Mode	.265	.832	.329	.443	.776	.431
	SD	.559	.647	.283	.348	.192	.204
	LCI	1.126	1.811	1.608	.589	1.162	.972
ψ_1	CP	.800	.800	1.000	.600	.800	.800
	Bias	-.235	.332	-.171	-.057	.276	-.069
	RMSE	.607	.727	.331	.353	.336	.216
	AVRB	.471	.663	.342	.115	.552	.137
	Mode	-2.328	-2.858	-2.684	-2.010	-2.148	-1.930
	SD	.641	.561	.820	.052	.135	.314
	LCI	1.279	1.799	1.179	.653	1.087	.755
ζ_0	CP	.600	.800	.600	.200	1.000	.400
	Bias	.172	-.358	-.184	.490	.352	.570
	RMSE	.664	.666	.840	.493	.377	.650

Table 182 (continued).

	AVRB	.069	.143	.074	.196	.141	.228
ζ_1	Mode	.344	1.169	.675	-.136	.025	-.375
	SD	1.079	.903	1.186	.115	.331	.659
	LCI	1.876	3.000	1.788	.912	1.810	1.165
	CP	.400	.800	.400	.200	.800	.400
	Bias	-.456	.369	-.125	-.936	-.775	-1.175
	RMSE	1.171	.976	1.192	.943	.843	1.347
	AVRB	.570	.462	.157	1.170	.969	1.469
γ	Mode	-.781	-.014	.684	-.807	.044	.734
	SD	.098	.373	.219	.083	.158	.117
	LCI	.501	.796	.576	.501	.508	.397
	CP	1.000	.400	1.000	1.000	.400	1.000
	Bias	.019	-.014	-.116	-.007	.044	-.066
	RMSE	.100	.373	.248	.083	.164	.134
	AVRB	.024	-	.145	.008	-	.082
ν_1	Mode	49.228	13.070	27.197	26.539	32.503	30.070
	SD	49.281	11.558	19.969	16.108	22.159	15.092
	LCI	26.809	31.289	21.995	13.353	29.225	30.707
	CP	.200	.400	.200	.200	.400	.800
	Bias	19.228	-16.930	-2.803	-3.461	2.503	.070
	RMSE	52.900	20.499	20.165	16.476	22.300	15.092
	AVRB	.641	.564	.093	.115	.083	.002
σ^2	Mode	3.948	3.857	3.840	3.784	3.725	3.817
	SD	.178	.204	.192	.238	.278	.300
	LCI	3.385	3.315	3.373	2.202	2.170	2.272
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.052	-.143	-.160	-.216	-.275	-.183
	RMSE	.185	.249	.250	.322	.391	.352
	AVRB	.013	.036	.040	.054	.069	.046

Mixed ZA-StBS regression model

Table 183 – Results of the simulation study for the mixed ZA-StBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.736	-.655	-.582	-.627	-.617	-.655
	Sd	.304	.205	.154	.191	.176	.181
	LCI	1.152	1.364	1.367	.800	.924	.918
β_0	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.236	-.155	-.082	-.127	-.117	-.155
	RMSE	.385	.257	.175	.229	.211	.238
	AVRB	.471	.311	.164	.255	.234	.310
	Mean	1.274	1.287	1.104	1.098	.986	.939
	Sd	.320	.162	.222	.235	.226	.228
	LCI	1.092	1.491	1.535	.778	.948	1.116
β_1	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.274	.287	.104	.098	-.014	-.061
	RMSE	.422	.330	.246	.255	.226	.236
	AVRB	.274	.287	.104	.098	.014	.061
	Mean	-1.192	-.913	-1.261	-1.050	-.858	-.952
	Sd	.271	.142	.209	.216	.117	.191
	LCI	1.130	1.219	1.284	.840	.796	.905
ψ_0	CP	.800	1.000	1.000	1.000	.800	1.000
	Bias	-.192	.087	-.261	-.050	.142	.048
	RMSE	.333	.167	.334	.221	.184	.197
	AVRB	.192	.087	.261	.050	.142	.048
	Mean	.873	.511	1.055	.640	.372	.523
	Sd	.552	.242	.453	.505	.230	.349
	LCI	1.881	2.122	2.080	1.274	1.357	1.371
ψ_1	CP	.800	1.000	.800	.800	1.000	1.000
	Bias	.373	.011	.555	.140	-.128	.023
	RMSE	.666	.242	.716	.524	.263	.350
	AVRB	.746	.022	1.110	.281	.257	.047
	Mean	-2.249	-2.313	-2.343	-2.690	-2.668	-2.698
	Sd	.288	.340	.305	.498	.470	.478
	LCI	1.632	1.606	1.595	1.283	1.324	1.211
ζ_0	CP	1.000	.800	.800	.800	.800	.800
	Bias	.251	.187	.157	-.190	-.168	-.198
	RMSE	.382	.388	.343	.533	.499	.517

Table 183 (continued).

	AVRB	.100	.075	.063	.076	.067	.079
ζ_1	Mean	.277	.408	.433	.89	.854	.912
	Sd	.176	.244	.26	.853	.761	.753
	LCI	2.694	2.73	2.781	2.068	2.106	1.925
	CP	1.000	1.000	1.000	.800	.800	.800
	Bias	-.523	-.392	-.367	.090	.054	.112
	RMSE	.552	.461	.450	.858	.763	.762
	AVRB	.654	.490	.459	.112	.067	.140
γ	Mean	-.587	-.021	.491	-.624	.111	.628
	Sd	.190	.298	.298	.142	.253	.091
	LCI	.796	1.121	.907	.629	.629	.642
	CP	.800	.800	1.000	1.000	.200	1.000
	Bias	.213	-.021	-.309	.176	.111	-.172
	RMSE	.285	.298	.429	.227	.276	.195
	AVRB	.266	-	.387	.220	-	.216
ν	Mean	7.369	9.337	7.715	6.615	10.606	7.438
	Sd	1.449	5.477	2.630	1.011	5.244	1.191
	LCI	15.716	29.487	19.017	14.224	27.484	15.169
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	2.369	4.337	2.715	1.615	5.606	2.438
	RMSE	2.777	6.986	3.780	1.906	7.676	2.714
	AVRB	.474	.867	.543	.323	1.121	.488
σ^2	Mean	4.133	4.023	4.052	3.851	3.816	3.793
	Sd	.867	.995	1.147	.390	.329	.332
	LCI	3.610	3.479	3.514	2.354	2.213	2.241
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.133	.023	.052	-.149	-.184	-.207
	RMSE	.877	.995	1.148	.418	.377	.391
	AVRB	.033	.006	.013	.037	.046	.052

Table 184 – Results of the simulation study for the mixed ZA-StBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.735	-.652	-.589	-.629	-.623	-.655
	Sd	.306	.194	.139	.194	.183	.184
	LCI	1.152	1.364	1.367	.800	.924	.918
β_0	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.235	-.152	-.089	-.129	-.123	-.155
	RMSE	.386	.246	.165	.232	.220	.241
	AVRB	.469	.304	.179	.257	.247	.309
	Median	1.271	1.290	1.107	1.098	.990	.942
	Sd	.324	.163	.218	.237	.230	.233
	LCI	1.092	1.491	1.535	.778	.948	1.116
β_1	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.271	.290	.107	.098	-.010	-.058
	RMSE	.423	.333	.243	.256	.231	.240
	AVRB	.271	.290	.107	.098	.010	.058
	Median	-1.184	-.918	-1.255	-1.063	-.860	-.956
	Sd	.280	.139	.216	.226	.122	.198
	LCI	1.130	1.219	1.284	.840	.796	.905
ψ_0	CP	.800	1.000	1.000	1.000	.800	1.000
	Bias	-.184	.082	-.255	-.063	.140	.044
	RMSE	.335	.161	.334	.235	.185	.202
	AVRB	.184	.082	.255	.063	.140	.044
	Median	.880	.523	1.055	.642	.381	.530
	Sd	.533	.218	.461	.502	.235	.346
	LCI	1.881	2.122	2.080	1.274	1.357	1.371
ψ_1	CP	.800	1.000	.800	.800	1.000	1.000
	Bias	.380	.023	.555	.142	-.119	.030
	RMSE	.654	.219	.722	.522	.264	.347
	AVRB	.76	.045	1.110	.284	.239	.060
	Median	-2.233	-2.300	-2.327	-2.679	-2.651	-2.695
	Sd	.262	.338	.294	.487	.473	.475
	LCI	1.632	1.606	1.595	1.283	1.324	1.211
ζ_0	CP	1.000	.800	.800	.800	.800	.800
	Bias	.267	.200	.173	-.179	-.151	-.195
	RMSE	.374	.392	.341	.519	.496	.513
	AVRB	.107	.08	.069	.072	.060	.078

Table 184 (continued).

ζ_1	Median	.302	.411	.400	.884	.845	.916
	Sd	.159	.229	.245	.849	.742	.749
	LCI	2.694	2.730	2.781	2.068	2.106	1.925
	CP	1.000	1.000	1.000	.800	.800	.800
	Bias	-.498	-.389	-.400	.084	.045	.116
	RMSE	.523	.451	.469	.853	.743	.758
	AVRB	.623	.486	.500	.105	.057	.145
γ	Median	-.610	-.007	.500	-.632	.109	.646
	Sd	.207	.306	.342	.159	.238	.105
	LCI	.796	1.121	.907	.629	.629	.642
	CP	.800	.800	1.000	1.000	.200	1.000
	Bias	.190	-.007	-.300	.168	.109	-.154
	RMSE	.281	.306	.455	.231	.262	.186
	AVRB	.237	-	.375	.210	-	.192
ν	Median	5.867	6.679	5.986	5.426	7.933	6.130
	Sd	.726	2.611	1.278	.468	2.895	.618
	LCI	15.716	29.487	19.017	14.224	27.484	15.169
	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	.867	1.679	.986	.426	2.933	1.130
	RMSE	1.130	3.104	1.614	.633	4.121	1.288
	AVRB	.173	.336	.197	.085	.586	.226
σ^2	Median	4.023	3.906	3.942	3.791	3.775	3.738
	Sd	.842	.959	1.121	.383	.323	.329
	LCI	3.610	3.479	3.514	2.354	2.213	2.241
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.023	-.094	-.058	-.209	-.225	-.262
	RMSE	.842	.964	1.122	.437	.394	.421
	AVRB	.006	.023	.014	.052	.056	.066

Table 185 – Results of the simulation study for the mixed ZA-StBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.741	-.643	-.590	-.628	-.630	-.658
	Sd	.308	.188	.134	.193	.183	.185
	LCI	1.152	1.364	1.367	.800	.924	.918
β_0	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	-.241	-.143	-.090	-.128	-.130	-.158
	RMSE	.391	.236	.162	.231	.224	.244
	AVRB	.482	.285	.181	.256	.259	.317
	Mode	1.268	1.305	1.112	1.093	.990	.940
	Sd	.329	.165	.213	.239	.232	.232
	LCI	1.092	1.491	1.535	.778	.948	1.116
β_1	CP	.800	1.000	1.000	1.000	1.000	1.000
	Bias	.268	.305	.112	.093	-.010	-.060
	RMSE	.424	.347	.240	.257	.232	.239
	AVRB	.268	.305	.112	.093	.010	.060
	Mode	-1.173	-.916	-1.255	-1.066	-.860	-.956
	Sd	.292	.138	.214	.228	.126	.205
	LCI	1.130	1.219	1.284	.840	.796	.905
ψ_0	CP	.800	1.000	1.000	1.000	.800	1.000
	Bias	-.173	.084	-.255	-.066	.140	.044
	RMSE	.339	.162	.333	.238	.188	.209
	AVRB	.173	.084	.255	.066	.140	.044
	Mode	.944	.538	1.032	.643	.387	.546
	Sd	.482	.184	.494	.502	.237	.347
	LCI	1.881	2.122	2.08	1.274	1.357	1.371
ψ_1	CP	.800	1.000	.800	.800	1.000	1.000
	Bias	.444	.038	.532	.143	-.113	.046
	RMSE	.655	.188	.726	.522	.262	.350
	AVRB	.888	.076	1.063	.286	.227	.092
	Mode	-2.213	-2.281	-2.297	-2.678	-2.628	-2.691
	Sd	.214	.326	.275	.474	.466	.477
	LCI	1.632	1.606	1.595	1.283	1.324	1.211
ζ_0	CP	1.000	.800	.800	.800	.800	.800
	Bias	.287	.219	.203	-.178	-.128	-.191
	RMSE	.358	.393	.342	.507	.483	.514
	AVRB	.115	.088	.081	.071	.051	.076

Table 185 (continued).

	Mode	.387	.438	.263	.771	.833	.952
	Sd	.134	.190	.245	1.026	.735	.704
	LCI	2.694	2.730	2.781	2.068	2.106	1.925
ζ_1	CP	1.000	1.000	1.000	.800	.800	.800
	Bias	-.413	-.362	-.537	-.029	.033	.152
	RMSE	.434	.409	.590	1.026	.736	.720
	AVRB	.516	.452	.671	.036	.041	.190
	Mode	-.623	.004	.498	-.634	.108	.651
	Sd	.223	.314	.375	.156	.240	.096
	LCI	.796	1.121	.907	.629	.629	.642
γ	CP	.800	.800	1.000	1.000	.2	1.000
	Bias	.177	.004	-.302	.166	.108	-.149
	RMSE	.284	.314	.482	.228	.263	.177
	AVRB	.221	-	.378	.208	-	.186
	Mode	4.567	4.998	4.877	4.605	6.005	4.791
	Sd	.133	1.013	.757	.263	1.771	.683
	LCI	15.716	29.487	19.017	14.224	27.484	15.169
ν	CP	1.000	1.000	1.000	1.000	.800	1.000
	Bias	-.433	-.002	-.123	-.395	1.005	-.209
	RMSE	.453	1.013	.767	.475	2.036	.714
	AVRB	.087	$< .001$	.025	.079	.201	.042
	Mode	3.760	3.725	3.867	3.722	3.700	3.643
	Sd	.789	1.003	1.148	.373	.313	.313
	LCI	3.610	3.479	3.514	2.354	2.213	2.241
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.240	-.275	-.133	-.278	-.300	-.357
	RMSE	.824	1.040	1.156	.465	.434	.475
	AVRB	.060	.069	.033	.069	.075	.089

Table 186 – Results of the simulation study for the mixed ZA-StBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.554	-.483	-.289	-.521	-.509	-.525
	Sd	.401	.354	.420	.215	.198	.327
	LCI	1.187	1.223	1.083	.772	.923	.952
	CP	.800	1.000	.800	1.000	1.000	.800
	Bias	-.054	.017	.211	-.021	-.009	-.025
	RMSE	.405	.355	.470	.216	.198	.328
	AVRB	.109	.034	.423	.043	.018	.051
β_1	Mean	1.044	.974	.790	1.057	.995	.944
	Sd	.220	.186	.220	.168	.085	.133
	LCI	.963	1.174	1.204	.649	.837	.883
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.044	-.026	-.210	.057	-.005	-.056
	RMSE	.224	.187	.304	.178	.086	.144
	AVRB	.044	.026	.210	.057	.005	.056
ψ_0	Mean	-1.020	-1.137	-.997	-1.055	-.946	-.998
	Sd	.264	.230	.250	.138	.160	.148
	LCI	1.237	1.153	1.174	.733	.820	.773
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.020	-.137	.003	-.055	.054	.002
	RMSE	.264	.268	.250	.148	.168	.149
	AVRB	.020	.137	.003	.055	.054	.002
ψ_1	Mean	.204	.448	.233	.434	.280	.394
	Sd	.350	.375	.422	.246	.309	.299
	LCI	2.040	1.966	1.944	1.186	1.364	1.277
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.296	-.052	-.267	-.066	-.220	-.106
	RMSE	.458	.379	.499	.255	.379	.317
	AVRB	.592	.104	.534	.131	.441	.213
ζ_0	Mean	-2.388	-2.423	-2.329	-2.422	-2.510	-2.505
	Sd	.404	.409	.350	.657	.613	.582
	LCI	1.538	1.690	1.602	1.076	1.310	1.224
	CP	1.000	1.000	.800	.600	.800	.800
	Bias	.112	.077	.171	.078	-.010	-.005
	RMSE	.420	.416	.390	.662	.613	.583
	AVRB	.045	.031	.069	.031	.004	.002

Table 186 (continued).

ζ_1	Mean	.614	.660	.538	.525	.646	.653
	Sd	.494	.483	.420	.962	.849	.830
	LCI	2.565	2.819	2.627	1.743	2.170	1.996
	CP	1.000	1.000	1.000	.6	.800	.800
	Bias	-.186	-.140	-.262	-.275	-.154	-.147
	RMSE	.528	.503	.495	1.001	.863	.843
	AVRB	.233	.175	.327	.344	.192	.184
γ	Mean	-.697	.201	.729	-.738	.024	.710
	Sd	.167	.173	.092	.155	.144	.100
	LCI	.646	1.178	.734	.495	.805	.518
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.103	.201	-.071	.062	.024	-.090
	RMSE	.196	.265	.116	.167	.146	.134
	AVRB	.129	-	.089	.078	-	.112
ν	Mean	19.312	16.958	16.465	19.119	23.010	21.443
	Sd	6.765	7.424	4.794	7.084	6.054	5.051
	LCI	67.181	60.139	59.34	58.325	79.072	65.446
	CP	1.000	.800	1.000	.800	1.000	1.000
	Bias	-10.688	-13.042	-13.535	-10.881	-6.99	-8.557
	RMSE	12.649	15.007	14.359	12.984	9.248	9.937
	AVRB	.356	.435	.451	.363	.233	.285
σ^2	Mean	3.976	4.184	4.039	4.221	4.257	4.227
	Sd	.911	1.027	.944	.885	.865	.601
	LCI	3.42	3.634	3.41	2.472	2.453	2.478
	CP	.800	.800	1.000	.800	.800	1.000
	Bias	-.024	.184	.039	.221	.257	.227
	RMSE	.911	1.043	.945	.913	.902	.643
	AVRB	.006	.046	.010	.055	.064	.057

Table 187 – Results of the simulation study for the mixed ZA-StBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Median	-.559	-.485	-.282	-.528	-.520	-.529
	Sd	.392	.343	.379	.246	.198	.331
	LCI	1.187	1.223	1.083	.772	.923	.952
β_0	CP	.800	1.000	.800	1.000	1.000	.800
	Bias	-.059	.015	.218	-.028	-.020	-.029
	RMSE	.396	.343	.437	.247	.199	.333
	AVRB	.118	.031	.436	.057	.040	.058
	Median	1.041	.977	.787	1.066	.996	.944
	Sd	.218	.188	.238	.165	.079	.138
	LCI	.963	1.174	1.204	.649	.837	.883
β_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.041	-.023	-.213	.066	-.004	-.056
	RMSE	.222	.189	.319	.178	.079	.149
	AVRB	.041	.023	.213	.066	.004	.056
	Median	-1.024	-1.140	-1.010	-1.043	-.940	-.998
	Sd	.267	.235	.275	.130	.160	.147
	LCI	1.237	1.153	1.174	.733	.820	.773
ψ_0	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.024	-.140	-.010	-.043	.060	.002
	RMSE	.268	.273	.275	.137	.171	.147
	AVRB	.024	.140	.010	.043	.060	.002
	Median	.215	.452	.258	.416	.280	.379
	Sd	.373	.374	.428	.240	.304	.322
	LCI	2.040	1.966	1.944	1.186	1.364	1.277
ψ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.285	-.048	-.242	-.084	-.220	-.121
	RMSE	.469	.377	.492	.254	.375	.344
	AVRB	.570	.095	.485	.167	.441	.242
	Median	-2.373	-2.404	-2.322	-2.407	-2.498	-2.501
	Sd	.391	.408	.339	.654	.608	.581
	LCI	1.538	1.69	1.602	1.076	1.31	1.224
ζ_0	CP	1.000	1.000	.800	.600	.800	.800
	Bias	.127	.096	.178	.093	.002	-.001
	RMSE	.411	.419	.382	.660	.608	.581
	AVRB	.051	.038	.071	.037	.001	$< .001$

Table 187 (continued).

ζ_1	Median	.612	.652	.529	.532	.645	.637
	Sd	.514	.487	.426	.945	.848	.833
	LCI	2.565	2.819	2.627	1.743	2.170	1.996
	CP	1.000	1.000	1.000	.600	.800	.800
	Bias	-.188	-.148	-.271	-.268	-.155	-.163
	RMSE	.547	.509	.505	.982	.861	.849
	AVRB	.235	.185	.339	.335	.193	.204
γ	Median	-.727	.205	.770	-.743	.045	.731
	Sd	.164	.197	.094	.157	.126	.105
	LCI	.646	1.178	.734	.495	.805	.518
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.073	.205	-.030	.057	.045	-.069
	RMSE	.180	.284	.098	.167	.133	.126
	AVRB	.091	-	.038	.072	-	.086
ν	Median	12.825	11.364	11.390	13.616	15.426	15.413
	Sd	3.740	4.326	3.920	3.572	3.102	3.092
	LCI	67.181	60.139	59.34	58.325	79.072	65.446
	CP	1.000	.800	1.000	.800	1.000	1.000
	Bias	-17.175	-18.636	-18.610	-16.384	-14.574	-14.587
	RMSE	17.577	19.132	19.019	16.768	14.900	14.911
	AVRB	.572	.621	.620	.546	.486	.486
σ^2	Median	3.855	4.067	3.920	4.155	4.206	4.173
	Sd	.898	.996	.912	.868	.862	.592
	LCI	3.420	3.634	3.410	2.472	2.453	2.478
	CP	.800	.800	1.000	.800	.800	1.000
	Bias	-.145	.067	-.080	.155	.206	.173
	RMSE	.910	.998	.916	.881	.886	.617
	AVRB	.036	.017	.020	.039	.051	.043

Table 188 – Results of the simulation study for the mixed ZA-StBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.563	-.486	-.293	-.529	-.528	-.523
	Sd	.388	.325	.354	.249	.199	.334
	LCI	1.187	1.223	1.083	.772	.923	.952
	CP	.800	1.000	.800	1.000	1.000	.800
	Bias	-.063	.014	.207	-.029	-.028	-.023
	RMSE	.393	.325	.410	.251	.201	.335
	AVRB	.125	.028	.413	.058	.055	.046
β_1	Mode	1.038	.979	.774	1.063	.997	.947
	Sd	.219	.191	.236	.164	.078	.142
	LCI	.963	1.174	1.204	.649	.837	.883
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.038	-.021	-.226	.063	-.003	-.053
	RMSE	.222	.192	.327	.176	.078	.151
	AVRB	.038	.021	.226	.063	.003	.053
ψ_0	Mode	-1.023	-1.149	-1.014	-1.043	-.939	-1.004
	Sd	.274	.242	.267	.131	.158	.147
	LCI	1.237	1.153	1.174	.733	.820	.773
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.023	-.149	-.014	-.043	.061	-.004
	RMSE	.275	.284	.267	.137	.169	.147
	AVRB	.023	.149	.014	.043	.061	.004
ψ_1	Mode	.257	.456	.325	.393	.283	.363
	Sd	.430	.354	.425	.205	.297	.348
	LCI	2.040	1.966	1.944	1.186	1.364	1.277
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.243	-.044	-.175	-.107	-.217	-.137
	RMSE	.494	.357	.460	.231	.368	.374
	AVRB	.487	.087	.349	.213	.433	.275
ζ_0	Mode	-2.308	-2.385	-2.347	-2.411	-2.487	-2.489
	Sd	.357	.419	.329	.627	.615	.572
	LCI	1.538	1.690	1.602	1.076	1.310	1.224
	CP	1.000	1.000	.800	.600	.800	.800
	Bias	.192	.115	.153	.089	.013	.011
	RMSE	.406	.434	.363	.633	.615	.572
	AVRB	.077	.046	.061	.036	.005	.004

Table 188 (continued).

ζ_1	Mode	.609	.630	.561	.544	.644	.625
	Sd	.577	.522	.44	.911	.887	.853
	LCI	2.565	2.819	2.627	1.743	2.170	1.996
	CP	1.000	1.000	1.000	.600	.800	.800
	Bias	-.191	-.170	-.239	-.256	-.156	-.175
	RMSE	.607	.549	.500	.946	.901	.870
	AVRB	.239	.212	.299	.320	.195	.219
γ	Mode	-.734	.175	.782	-.745	.046	.726
	Sd	.143	.250	.077	.151	.129	.099
	LCI	.646	1.178	.734	.495	.805	.518
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.066	.175	-.018	.055	.046	-.074
	RMSE	.157	.305	.079	.161	.137	.123
	AVRB	.082	-	.023	.069	-	.092
ν	Mode	7.269	6.631	7.032	9.443	8.155	9.023
	Sd	1.303	3.263	1.756	1.479	1.728	2.353
	LCI	67.181	60.139	59.34	58.325	79.072	65.446
	CP	1.000	.800	1.000	.800	1.000	1.000
	Bias	-22.731	-23.369	-22.968	-20.557	-21.845	-20.977
	RMSE	22.769	23.595	23.035	20.61	21.914	21.109
	AVRB	.758	.779	.766	.685	.728	.699
σ^2	Mode	3.619	3.798	3.718	4.071	4.138	4.089
	Sd	.831	.866	.797	.861	.908	.583
	LCI	3.42	3.634	3.410	2.472	2.453	2.478
	CP	.800	.800	1.000	.800	.800	1.000
	Bias	-.381	-.202	-.282	.071	.138	.089
	RMSE	.914	.889	.846	.863	.918	.589
	AVRB	.095	.05	.070	.018	.035	.022

Mixed ZA-SSLBS regression model

Table 189 – Results of the simulation study for the mixed ZA-SSLBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.470	-.523	-.551	-.470	-.453	-.379
	SD	.247	.121	.149	.226	.338	.298
	LCI	.960	1.190	1.131	.678	.901	.910
β_0	CP	.800	1.000	1.000	1.000	.800	.800
	Bias	.030	-.023	-.051	.030	.047	.121
	RMSE	.249	.123	.157	.228	.342	.321
	AVRB	.059	.046	.101	.061	.094	.243
	Mean	.934	1.074	1.020	.979	.948	.800
	SD	.377	.085	.315	.194	.277	.070
	LCI	.878	1.186	1.299	.531	.832	.969
β_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.066	.074	.020	-.021	-.052	-.200
	RMSE	.383	.113	.316	.195	.281	.212
	AVRB	.066	.074	.020	.021	.052	.200
	Mean	-.975	-1.017	-1.029	-.908	-.929	-.951
	SD	.475	.070	.269	.326	.231	.233
	LCI	.971	.961	.898	.712	.871	.841
ψ_0	CP	.800	1.000	1.000	.800	1.000	.800
	Bias	.025	-.017	-.029	.092	.071	.049
	RMSE	.476	.072	.270	.339	.242	.238
	AVRB	.025	.017	.029	.092	.071	.049
	Mean	.338	.601	.488	.227	.342	.347
	SD	.559	.123	.474	.226	.362	.297
	LCI	1.549	1.606	1.548	.969	1.166	1.197
ψ_1	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.162	.101	-.012	-.273	-.158	-.153
	RMSE	.582	.160	.474	.354	.396	.334
	AVRB	.324	.202	.025	.545	.317	.306
	Mean	-2.158	-2.440	-2.438	-2.325	-2.574	-2.449
	SD	.796	.605	.549	.600	.623	.683
	LCI	1.593	1.675	1.603	.801	1.197	1.194
ζ_0	CP	.600	1.000	.800	.600	.600	.400
	Bias	.342	.060	.062	.175	-.074	.051
	RMSE	.866	.607	.552	.625	.627	.685

Table 189 (continued).

	AVRB	.137	.024	.025	.070	.029	.020
ζ_1	Mean	.055	.585	.606	.472	.885	.664
	SD	1.282	.792	.815	.774	.691	.818
	LCI	2.367	2.770	2.501	1.295	1.979	1.992
	CP	.600	1.000	.800	.600	.800	.800
	Bias	-.745	-.215	-.194	-.328	.085	-.136
	RMSE	1.483	.821	.838	.841	.696	.830
	AVRB	.932	.269	.243	.410	.107	.17
γ	Mean	-.751	.125	.705	-.861	.074	.777
	SD	.167	.508	.151	.046	.207	.027
	LCI	.627	.864	.726	.324	.596	.525
	CP	1.000	.400	1.000	1.000	.400	1.000
	Bias	.049	.125	-.095	-.061	.074	-.023
	RMSE	.174	.523	.179	.076	.220	.035
	AVRB	.062	-	.119	.076	-	.028
ν	Mean	7.730	7.948	5.475	6.355	6.265	5.538
	SD	4.380	2.285	1.946	4.708	1.74	2.746
	LCI	14.370	14.856	10.388	6.610	13.961	8.308
	CP	.800	1.000	1.000	.600	1.000	1.000
	Bias	2.730	2.948	.475	1.355	1.265	.538
	RMSE	5.161	3.730	2.003	4.899	2.152	2.798
	AVRB	.546	.590	.095	.271	.253	.108
σ^2	Mean	4.340	4.424	4.136	4.055	3.932	3.825
	SD	1.235	1.068	.953	.678	.577	.613
	LCI	3.710	3.765	3.469	2.274	2.325	2.223
	CP	1.000	1.000	.800	.800	.800	.800
	Bias	.340	.424	.136	.055	-.068	-.175
	RMSE	1.281	1.149	.963	.681	.581	.638
	AVRB	.085	.106	.034	.014	.017	.044

Table 190 – Results of the simulation study for the mixed ZA-SSLBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.476	-.523	-.552	-.459	-.452	-.382
	SD	.223	.111	.133	.231	.340	.311
	LCI	.960	1.190	1.131	.678	.901	.910
	CP	.800	1.000	1.000	1.000	.800	.800
	Bias	.024	-.023	-.052	.041	.048	.118
	RMSE	.224	.114	.143	.234	.344	.333
	AVRB	.049	.046	.104	.081	.096	.236
β_1	Median	.939	1.068	1.037	.977	.948	.808
	SD	.384	.079	.333	.186	.277	.075
	LCI	.878	1.186	1.299	.531	.832	.969
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.061	.068	.037	-.023	-.052	-.192
	RMSE	.389	.104	.335	.187	.282	.206
	AVRB	.061	.068	.037	.023	.052	.192
ψ_0	Median	-.969	-1.017	-1.051	-.921	-.916	-.956
	SD	.475	.074	.269	.326	.237	.236
	LCI	.971	.961	.898	.712	.871	.841
	CP	.800	1.000	1.000	.800	1.000	.800
	Bias	.031	-.017	-.051	.079	.084	.044
	RMSE	.476	.076	.274	.335	.252	.240
	AVRB	.031	.017	.051	.079	.084	.044
ψ_1	Median	.354	.610	.508	.226	.338	.361
	SD	.558	.125	.482	.241	.364	.273
	LCI	1.549	1.606	1.548	.969	1.166	1.197
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.146	.110	.008	-.274	-.162	-.139
	RMSE	.577	.166	.482	.365	.398	.307
	AVRB	.293	.219	.016	.548	.325	.278
ζ_0	Median	-2.167	-2.428	-2.409	-2.307	-2.554	-2.432
	SD	.820	.614	.491	.598	.613	.681
	LCI	1.593	1.675	1.603	.801	1.197	1.194
	CP	.600	1.000	.800	.600	.600	.400
	Bias	.333	.072	.091	.193	-.054	.068
	RMSE	.885	.618	.499	.628	.615	.684
	AVRB	.133	.029	.037	.077	.022	.027

Table 190 (continued).

ζ_1	Median	.038	.584	.590	.450	.870	.666
	SD	1.329	.802	.717	.769	.678	.818
	LCI	2.367	2.770	2.501	1.295	1.979	1.992
	CP	.600	1.000	.800	.600	.800	.800
	Bias	-.762	-.216	-.210	-.350	.070	-.134
	RMSE	1.532	.830	.747	.845	.682	.829
	AVRB	.953	.271	.262	.438	.087	.168
γ	Median	-.799	.128	.768	-.875	.078	.804
	SD	.178	.517	.105	.047	.192	.035
	LCI	.627	.864	.726	.324	.596	.525
	CP	1.000	.400	1.000	1.000	.400	1.000
	Bias	.001	.128	-.032	-.075	.078	.004
	RMSE	.178	.533	.110	.089	.207	.035
	AVRB	.001	-	.040	.094	-	.004
ν	Median	7.145	7.117	4.842	6.207	5.134	5.037
	SD	5.058	2.577	1.431	4.817	1.207	2.348
	LCI	14.370	14.856	10.388	6.610	13.961	8.308
	CP	.800	1.000	1.000	.600	1.000	1.000
	Bias	2.145	2.117	-.158	1.207	.134	.037
	RMSE	5.494	3.335	1.440	4.966	1.215	2.348
	AVRB	.429	.423	.032	.241	.027	.007
σ^2	Median	4.197	4.303	4.018	3.997	3.880	3.775
	SD	1.191	1.037	.942	.676	.578	.606
	LCI	3.710	3.765	3.469	2.274	2.325	2.223
	CP	1.000	1.000	.800	.800	.800	.800
	Bias	.197	.303	.018	-.003	-.120	-.225
	RMSE	1.207	1.081	.942	.676	.590	.646
	AVRB	.049	.076	.004	.001	.030	.056

Table 191 – Results of the simulation study for the mixed ZA-SSLBS regression model ($\nu = 5$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.457	-.523	-.554	-.459	-.444	-.383
	SD	.212	.108	.122	.247	.347	.317
	LCI	.960	1.190	1.131	.678	.901	.910
β_0	CP	.800	1.000	1.000	1.000	.800	.800
	Bias	.043	-.023	-.054	.041	.056	.117
	RMSE	.217	.110	.134	.251	.351	.337
	AVRB	.087	.046	.109	.081	.113	.233
	Mode	.934	1.054	1.030	.978	.940	.821
	SD	.379	.070	.332	.188	.280	.087
	LCI	.878	1.186	1.299	.531	.832	.969
β_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	-.066	.054	.030	-.022	-.060	-.179
	RMSE	.385	.088	.334	.190	.286	.199
	AVRB	.066	.054	.030	.022	.060	.179
	Mode	-.970	-1.019	-1.065	-.927	-.912	-.946
	SD	.479	.076	.262	.335	.236	.236
	LCI	.971	.961	.898	.712	.871	.841
ψ_0	CP	.800	1.000	1.000	.800	1.000	.800
	Bias	.030	-.019	-.065	.073	.088	.054
	RMSE	.480	.078	.270	.343	.252	.242
	AVRB	.030	.019	.065	.073	.088	.054
	Mode	.362	.633	.532	.231	.336	.404
	SD	.555	.129	.470	.242	.365	.227
	LCI	1.549	1.606	1.548	.969	1.166	1.197
ψ_1	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.138	.133	.032	-.269	-.164	-.096
	RMSE	.572	.185	.471	.362	.400	.247
	AVRB	.277	.265	.065	.538	.328	.192
	Mode	-2.236	-2.418	-2.337	-2.303	-2.536	-2.415
	SD	.904	.636	.346	.596	.606	.675
	LCI	1.593	1.675	1.603	.801	1.197	1.194
ζ_0	CP	.600	1.000	.800	.600	.600	.400
	Bias	.264	.082	.163	.197	-.036	.085
	RMSE	.942	.641	.382	.628	.607	.680
	AVRB	.106	.033	.065	.079	.014	.034

Table 191 (continued).

	Mode	.012	.603	.573	.427	.849	.670
	SD	1.170	.808	.647	.819	.661	.839
	LCI	2.367	2.77	2.501	1.295	1.979	1.992
ζ_1	CP	.600	1.000	.800	.600	.800	.800
	Bias	-.788	-.197	-.227	-.373	.049	-.130
	RMSE	1.410	.832	.686	.900	.663	.849
	AVRB	.985	.247	.284	.466	.061	.162
	Mode	-.791	.129	.817	-.870	.074	.799
	SD	.155	.529	.099	.044	.194	.034
	LCI	.627	.864	.726	.324	.596	.525
γ	CP	1.000	.400	1.000	1.000	.400	1.000
	Bias	.009	.129	.017	-.070	.074	-.001
	RMSE	.156	.544	.101	.083	.208	.034
	AVRB	.012	-	.022	.087	-	.001
	Mode	6.475	5.915	3.420	5.484	3.349	3.851
	SD	7.721	3.771	.591	3.577	.825	1.451
	LCI	14.370	14.856	10.388	6.610	13.961	8.308
ν	CP	.800	1.000	1.000	.600	1.000	1.000
	Bias	1.475	.915	-1.580	.484	-1.651	-1.149
	RMSE	7.861	3.880	1.687	3.610	1.846	1.851
	AVRB	.295	.183	.316	.097	.330	.230
	Mode	4.060	4.151	3.859	3.901	3.794	3.681
	SD	1.149	.956	.970	.674	.586	.585
	LCI	3.710	3.765	3.469	2.274	2.325	2.223
σ^2	CP	1.000	1.000	.800	.800	.800	.800
	Bias	.060	.151	-.141	-.099	-.206	-.319
	RMSE	1.151	.968	.980	.681	.621	.666
	AVRB	.015	.038	.035	.025	.051	.080

Table 192 – Results of the simulation study for the mixed ZA-SSLBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mean	-.674	-.563	-.460	-.713	-.618	-.447
	SD	.295	.401	.283	.111	.131	.228
	LCI	.994	1.182	1.077	.692	.733	.867
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.174	-.063	.040	-.213	-.118	.053
	RMSE	.342	.406	.286	.241	.176	.234
	AVRB	.348	.127	.079	.427	.237	.107
β_1	Mean	1.240	.987	.856	1.124	1.025	.870
	SD	.242	.400	.311	.128	.141	.381
	LCI	.971	1.097	1.095	.611	.751	.845
	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.240	-.013	-.144	.124	.025	-.130
	RMSE	.341	.400	.343	.178	.143	.403
	AVRB	.24	.013	.144	.124	.025	.130
ψ_0	Mean	-1.168	-1.048	-1.103	-1.160	-1.035	-.953
	SD	.134	.289	.154	.197	.067	.314
	LCI	.967	.920	.871	.539	.689	.699
	CP	1.000	.800	1.000	.800	1.000	.800
	Bias	-.168	-.048	-.103	-.160	-.035	.047
	RMSE	.215	.293	.185	.254	.076	.318
	AVRB	.168	.048	.103	.160	.035	.047
ψ_1	Mean	.630	.632	.528	.766	.568	.426
	SD	.313	.476	.315	.308	.089	.619
	LCI	1.643	1.621	1.509	.969	1.188	1.224
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.130	.132	.028	.266	.068	-.074
	RMSE	.339	.494	.317	.407	.112	.623
	AVRB	.259	.263	.057	.532	.136	.148
ζ_0	Mean	-2.768	-2.741	-2.679	-2.642	-2.784	-2.531
	SD	.425	.481	.438	.281	.440	.447
	LCI	1.974	1.852	1.541	.885	1.287	1.119
	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	-.268	-.241	-.179	-.142	-.284	-.031
	RMSE	.502	.538	.474	.315	.523	.448
	AVRB	.107	.096	.072	.057	.113	.012

Table 192 (continued).

ζ_1	Mean	.937	.916	.819	.906	1.153	.716
	SD	.657	.739	.650	.383	.557	.510
	LCI	3.180	2.864	2.448	1.504	1.996	1.647
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.137	.116	.019	.106	.353	-.084
	RMSE	.671	.748	.650	.398	.66	.516
	AVRB	.171	.146	.023	.133	.442	.105
γ	Mean	-.616	.068	.665	-.804	.098	.820
	SD	.234	.324	.158	.058	.194	.043
	LCI	.682	.829	.687	.466	.612	.347
	CP	1.000	.400	1.000	1.000	.800	1.000
	Bias	.184	.068	-.135	-.004	.098	.020
	RMSE	.297	.331	.208	.058	.218	.047
	AVRB	.230	-	.169	.005	-	.025
ν	Mean	23.471	25.448	18.635	17.713	23.582	15.414
	SD	13.395	8.399	8.924	12.386	9.427	4.517
	LCI	54.980	66.928	38.133	22.034	58.677	29.159
	CP	.600	.800	.600	.400	1.000	.600
	Bias	-6.529	-4.552	-11.365	-12.287	-6.418	-14.586
	RMSE	14.901	9.553	14.450	17.446	11.405	15.269
	AVRB	.218	.152	.379	.410	.214	.486
σ^2	Mean	4.324	4.361	4.208	3.927	3.792	3.774
	SD	1.214	1.180	1.133	.551	.434	.604
	LCI	3.670	3.694	3.470	2.280	2.202	2.185
	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.324	.361	.208	-.073	-.208	-.226
	RMSE	1.257	1.234	1.152	.556	.481	.645
	AVRB	.081	.090	.052	.018	.052	.057

Table 193 – Results of the simulation study for the mixed ZA-SSLBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.674	-.582	-.463	-.708	-.611	-.432
	SD	.294	.396	.297	.119	.122	.244
	LCI	.994	1.182	1.077	.692	.733	.867
	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.174	-.082	.037	-.208	-.111	.068
	RMSE	.341	.404	.299	.240	.164	.253
	AVRB	.347	.165	.073	.417	.221	.137
β_1	Median	1.240	.986	.851	1.123	1.028	.858
	SD	.231	.393	.310	.137	.134	.398
	LCI	.971	1.097	1.095	.611	.751	.845
	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.240	-.014	-.149	.123	.028	-.142
	RMSE	.333	.394	.344	.184	.137	.423
	AVRB	.240	.014	.149	.123	.028	.142
ψ_0	Median	-1.179	-1.051	-1.098	-1.168	-1.035	-.949
	SD	.137	.282	.144	.199	.073	.350
	LCI	.967	.92	.871	.539	.689	.699
	CP	1.000	.800	1.000	.800	1.000	.800
	Bias	-.179	-.051	-.098	-.168	-.035	.051
	RMSE	.226	.286	.174	.261	.081	.354
	AVRB	.179	.051	.098	.168	.035	.051
ψ_1	Median	.619	.637	.517	.790	.571	.404
	SD	.313	.486	.302	.315	.089	.688
	LCI	1.643	1.621	1.509	.969	1.188	1.224
	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.119	.137	.017	.290	.071	-.096
	RMSE	.335	.505	.302	.429	.114	.695
	AVRB	.238	.275	.034	.580	.143	.193
ζ_0	Median	-2.754	-2.715	-2.660	-2.634	-2.780	-2.515
	SD	.403	.481	.449	.270	.437	.438
	LCI	1.974	1.852	1.541	.885	1.287	1.119
	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	-.254	-.215	-.159	-.134	-.280	-.015
	RMSE	.476	.527	.476	.301	.519	.438
	AVRB	.102	.086	.064	.054	.112	.006

Table 193 (continued).

	Median	.936	.882	.780	.907	1.149	.684
	SD	.628	.751	.668	.393	.570	.533
	LCI	3.180	2.864	2.448	1.504	1.996	1.647
ζ_1	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.136	.082	-.020	.107	.349	-.116
	RMSE	.643	.755	.668	.407	.669	.546
	AVRB	.170	.103	.026	.134	.437	.145
	Median	-.640	.068	.695	-.827	.090	.831
	SD	.235	.318	.153	.056	.185	.046
	LCI	.682	.829	.687	.466	.612	.347
γ	CP	1.000	.400	1.000	1.000	.800	1.000
	Bias	.161	.068	-.105	-.027	.090	.031
	RMSE	.284	.325	.185	.063	.206	.055
	AVRB	.201	-	.131	.034	-	.039
	Median	21.387	21.656	16.988	16.715	20.305	13.694
	SD	11.577	6.645	7.764	12.535	8.844	3.511
	LCI	54.980	66.928	38.133	22.034	58.677	29.159
ν	CP	.600	.800	.600	.400	1.000	.600
	Bias	-8.613	-8.344	-13.012	-13.285	-9.695	-16.306
	RMSE	14.429	10.667	15.152	18.265	13.123	16.680
	AVRB	.287	.278	.434	.443	.323	.544
	Median	4.208	4.237	4.083	3.878	3.745	3.724
	SD	1.182	1.160	1.105	.533	.425	.594
	LCI	3.670	3.694	3.470	2.280	2.202	2.185
σ^2	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.208	.237	.083	-.122	-.255	-.276
	RMSE	1.200	1.184	1.108	.547	.495	.655
	AVRB	.052	.059	.021	.030	.064	.069

Table 194 – Results of the simulation study for the mixed ZA-SSLBS regression model ($\nu = 30$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.673	-.603	-.466	-.709	-.606	-.428
	SD	.292	.380	.301	.125	.118	.250
	LCI	.994	1.182	1.077	.692	.733	.867
β_0	CP	1.000	1.000	1.000	.800	.800	1.000
	Bias	-.173	-.103	.034	-.209	-.106	.072
	RMSE	.339	.393	.303	.243	.159	.260
	AVRB	.345	.205	.068	.417	.211	.144
	Mode	1.240	.988	.845	1.124	1.028	.848
	SD	.229	.391	.307	.136	.136	.419
	LCI	.971	1.097	1.095	.611	.751	.845
β_1	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.240	-.012	-.155	.124	.028	-.152
	RMSE	.332	.391	.344	.184	.139	.446
	AVRB	.240	.012	.155	.124	.028	.152
	Mode	-1.191	-1.049	-1.102	-1.174	-1.035	-.955
	SD	.139	.277	.142	.206	.070	.362
	LCI	.967	.920	.871	.539	.689	.699
ψ_0	CP	1.000	.800	1.000	.800	1.000	.800
	Bias	-.191	-.049	-.102	-.174	-.035	.045
	RMSE	.236	.282	.174	.269	.078	.365
	AVRB	.191	.049	.102	.174	.035	.045
	Mode	.604	.665	.496	.812	.573	.408
	SD	.316	.487	.277	.348	.094	.718
	LCI	1.643	1.621	1.509	.969	1.188	1.224
ψ_1	CP	1.000	1.000	1.000	.800	1.000	1.000
	Bias	.104	.165	-.004	.312	.073	-.092
	RMSE	.333	.514	.277	.468	.119	.724
	AVRB	.207	.329	.008	.625	.145	.185
	Mode	-2.730	-2.677	-2.613	-2.624	-2.772	-2.483
	SD	.384	.483	.460	.261	.432	.404
	LCI	1.974	1.852	1.541	.885	1.287	1.119
ζ_0	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	-.230	-.177	-.113	-.124	-.272	.017
	RMSE	.448	.514	.474	.289	.511	.404
	AVRB	.092	.071	.045	.050	.109	.007

Table 194 (continued).

ζ_1	Mode	.978	.731	.745	.919	1.162	.638
	SD	.587	.785	.662	.416	.582	.515
	LCI	3.180	2.864	2.448	1.504	1.996	1.647
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.178	-.069	-.055	.119	.362	-.162
	RMSE	.613	.788	.664	.433	.686	.540
	AVRB	.223	.087	.069	.149	.453	.202
γ	Mode	-.655	.065	.707	-.824	.089	.827
	SD	.205	.313	.128	.058	.185	.042
	LCI	.682	.829	.687	.466	.612	.347
	CP	1.000	.400	1.000	1.000	.800	1.000
	Bias	.145	.065	-.093	-.024	.089	.027
	RMSE	.252	.320	.158	.063	.205	.050
	AVRB	.182	-	.116	.030	-	.033
ν	Mode	10.778	7.779	13.167	15.718	15.327	16.770
	SD	13.975	4.882	11.354	12.544	7.314	15.408
	LCI	54.980	66.928	38.133	22.034	58.677	29.159
	CP	.600	.800	.600	.400	1.000	.600
	Bias	-19.222	-22.221	-16.833	-14.282	-14.673	-13.230
	RMSE	23.766	22.751	20.305	19.009	16.395	20.309
	AVRB	.641	.741	.561	.476	.489	.441
σ^2	Mode	4.145	4.079	3.960	3.791	3.665	3.641
	SD	1.229	1.174	1.108	.519	.411	.559
	LCI	3.670	3.694	3.470	2.280	2.202	2.185
	CP	.800	.800	.800	1.000	1.000	.800
	Bias	.145	.079	-.040	-.209	-.335	-.359
	RMSE	1.238	1.176	1.109	.559	.530	.664
	AVRB	.036	.020	.010	.052	.084	.090

Mixed ZA-SCNBS regression model

Table 195 – Results of the simulation study for the mixed ZA-SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.503	-.466	-.250	-.497	-.518	-.417
	SD	.430	.261	.639	.293	.184	.457
	LCI	1.495	1.722	1.550	.966	1.300	1.150
β_0	CP	1.000	1.000	.800	1.000	1.000	.800
	Bias	-.003	.034	.250	.003	-.018	.083
	RMSE	.430	.263	.686	.293	.185	.464
	AVRB	.005	.068	.501	.006	.036	.166
	Mean	1.078	1.411	1.091	.960	.910	.685
	SD	.502	.286	.776	.373	.289	.931
	LCI	1.781	2.408	2.487	1.198	1.853	1.545
β_1	CP	1.000	1.000	1.000	.800	1.000	.600
	Bias	.078	.411	.091	-.040	-.090	-.315
	RMSE	.508	.501	.782	.375	.302	.983
	AVRB	.078	.411	.091	.040	.090	.315
	Mean	.716	.666	.577	.372	.484	.864
	SD	.258	.353	.491	.805	.387	.323
	LCI	1.631	1.876	2.124	1.672	1.862	1.518
ψ_0	CP	.200	$< .001$	.600	.200	.400	$< .001$
	Bias	1.716	1.666	1.577	1.372	1.484	1.864
	RMSE	1.735	1.703	1.652	1.590	1.534	1.892
	AVRB	1.716	1.666	1.577	1.372	1.484	1.864
	Mean	.651	.921	.733	.468	.408	.310
	SD	.484	.661	.526	.518	.268	.538
	LCI	1.708	1.979	1.743	1.142	1.447	1.083
ψ_1	CP	1.000	1.000	1.000	.800	1.000	.400
	Bias	.151	.421	.233	-.032	-.092	-.190
	RMSE	.507	.783	.575	.519	.284	.571
	AVRB	.303	.841	.465	.064	.184	.381
	Mean	-2.160	-2.250	-2.197	-2.512	-2.669	-2.621
	SD	.166	.134	.212	.365	.441	.439
	LCI	1.494	1.586	1.318	1.144	1.230	1.219
ζ_0	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	.340	.250	.303	-.012	-.169	-.121

Table 195 (continued).

	RMSE	.378	.284	.370	.365	.472	.455
	AVRB	.136	.100	.121	.005	.068	.049
	Mean	.071	.139	.067	.850	1.105	1.026
	SD	.599	.473	.287	.433	.596	.625
	LCI	2.562	2.658	2.291	1.751	1.959	1.846
ζ_1	CP	.800	1.000	.800	1.000	.800	.800
	Bias	-.729	-.661	-.733	.050	.305	.226
	RMSE	.944	.813	.787	.436	.670	.665
	AVRB	.912	.826	.916	.062	.381	.283
	Mean	-.669	.118	.739	-.690	.144	.690
	SD	.284	.332	.136	.110	.123	.082
	LCI	.612	.972	.582	.576	.585	.650
γ	CP	.800	.800	1.000	1.000	.600	1.000
	Bias	.131	.118	-.061	.110	.144	-.110
	RMSE	.313	.353	.149	.155	.190	.137
	AVRB	.164	-	.076	.137	-	.137
	Mean	.515	.512	.543	.590	.542	.555
	SD	.051	.045	.080	.160	.052	.109
	LCI	.757	.778	.747	.646	.770	.724
ν_1	CP	.400	.600	.600	.800	.600	.600
	Bias	-.385	-.388	-.357	-.310	-.358	-.345
	RMSE	.389	.390	.366	.349	.362	.361
	AVRB	.428	.431	.396	.345	.398	.383
	Mean	.596	.572	.524	.468	.358	.536
	SD	.053	.091	.173	.260	.135	.117
	LCI	.761	.798	.711	.643	.544	.623
ν_2	CP	$< .001$	$< .001$	.200	.200	.400	$< .001$
	Bias	.496	.472	.424	.368	.258	.436
	RMSE	.499	.481	.458	.451	.291	.451
	AVRB	4.960	4.723	4.241	3.684	2.576	4.356
	Mean	4.212	4.052	4.276	4.115	4.454	4.366
	SD	.638	.587	.487	.410	.574	.613
	LCI	4.085	3.917	4.104	2.699	2.969	2.860
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.212	.052	.276	.115	.454	.366
	RMSE	.672	.589	.560	.425	.732	.714
	AVRB	.053	.013	.069	.029	.114	.091

Table 196 – Results of the simulation study for the mixed ZA-SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.490	-.461	-.248	-.519	-.520	-.428
	SD	.437	.254	.663	.295	.172	.464
	LCI	1.495	1.722	1.550	.966	1.300	1.150
	CP	1.000	1.000	.800	1.000	1.000	.800
	Bias	.010	.039	.252	-.019	-.020	.072
	RMSE	.438	.257	.709	.296	.173	.470
	AVRB	.020	.078	.504	.038	.039	.144
β_1	Median	1.083	1.428	1.086	.957	.914	.692
	SD	.510	.284	.775	.381	.281	.955
	LCI	1.781	2.408	2.487	1.198	1.853	1.545
	CP	1.000	1.000	1.000	.800	1.000	.600
	Bias	.083	.428	.086	-.043	-.086	-.308
	RMSE	.517	.514	.779	.384	.294	1.003
	AVRB	.083	.428	.086	.043	.086	.308
ψ_0	Median	.783	.726	.697	.438	.560	.930
	SD	.231	.368	.431	.861	.373	.319
	LCI	1.631	1.876	2.124	1.672	1.862	1.518
	CP	.200	$< .001$	.600	.200	.400	$< .001$
	Bias	1.783	1.726	1.697	1.438	1.560	1.930
	RMSE	1.798	1.765	1.751	1.676	1.604	1.956
	AVRB	1.783	1.726	1.697	1.438	1.560	1.930
ψ_1	Median	.653	.945	.715	.456	.399	.327
	SD	.480	.648	.513	.523	.267	.551
	LCI	1.708	1.979	1.743	1.142	1.447	1.083
	CP	1.000	1.000	1.000	.800	1.000	.400
	Bias	.153	.445	.215	-.044	-.101	-.173
	RMSE	.504	.786	.556	.525	.285	.577
	AVRB	.305	.890	.429	.087	.201	.345
ζ_0	Median	-2.175	-2.232	-2.184	-2.514	-2.659	-2.620
	SD	.189	.125	.202	.374	.444	.455
	LCI	1.494	1.586	1.318	1.144	1.230	1.219
	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	.325	.268	.316	-.014	-.159	-.120
	RMSE	.375	.295	.375	.374	.471	.470

Table 196 (continued).

	AVRB	.130	.107	.126	.005	.063	.048
ζ_1	Median	.091	.113	.074	.860	1.103	1.036
	SD	.619	.476	.307	.455	.606	.682
	LCI	2.562	2.658	2.291	1.751	1.959	1.846
	CP	.800	1.000	.800	1.000	.800	.800
	Bias	-.709	-.687	-.726	.060	.303	.236
	RMSE	.942	.836	.788	.459	.677	.721
	AVRB	.887	.859	.907	.074	.379	.295
γ	Median	-.691	.104	.762	-.710	.140	.709
	SD	.284	.370	.139	.120	.113	.079
	LCI	.612	.972	.582	.576	.585	.650
	CP	.800	.800	1.000	1.000	.600	1.000
	Bias	.109	.104	-.038	.090	.140	-.091
	RMSE	.304	.385	.144	.150	.180	.121
	AVRB	.136	-	.047	.112	-	.114
ν_1	Median	.524	.510	.543	.592	.555	.552
	SD	.068	.062	.099	.165	.075	.118
	LCI	.757	.778	.747	.646	.770	.724
	CP	.400	.600	.600	.800	.600	.600
	Bias	-.376	-.39	-.357	-.308	-.345	-.348
	RMSE	.382	.395	.371	.349	.354	.368
	AVRB	.418	.433	.397	.342	.384	.387
ν_2	Median	.606	.567	.502	.439	.344	.536
	SD	.061	.125	.195	.284	.126	.105
	LCI	.761	.798	.711	.643	.544	.623
	CP	$< .001$	$< .001$	.200	.200	.400	$< .001$
	Bias	.506	.467	.402	.339	.244	.436
	RMSE	.510	.483	.447	.442	.275	.449
	AVRB	5.060	4.669	4.022	3.389	2.439	4.365
σ^2	Median	4.067	3.925	4.142	4.037	4.392	4.291
	SD	.627	.574	.449	.395	.580	.608
	LCI	4.085	3.917	4.104	2.699	2.969	2.860
	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.067	-.075	.142	.037	.392	.291
	RMSE	.631	.579	.470	.397	.700	.674
	AVRB	.017	.019	.035	.009	.098	.073

Table 197 – Results of the simulation study for the mixed ZA-SCNBS regression model ($\nu_1 = .9, \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mode	-.464	-.455	-.269	-.565	-.518	-.448
	SD	.450	.238	.707	.275	.163	.482
	LCI	1.495	1.722	1.550	.966	1.300	1.150
β_0	CP	1.000	1.000	.800	1.000	1.000	.800
	Bias	.036	.045	.231	-.065	-.018	.052
	RMSE	.452	.242	.743	.282	.164	.485
	AVRB	.071	.089	.461	.131	.036	.105
	Mode	1.103	1.534	.915	.966	.914	.705
	SD	.534	.243	.746	.396	.281	1.023
	LCI	1.781	2.408	2.487	1.198	1.853	1.545
β_1	CP	1.000	1.000	1.000	.800	1.000	.600
	Bias	.103	.534	-.085	-.034	-.086	-.295
	RMSE	.543	.586	.751	.397	.294	1.065
	AVRB	.103	.534	.085	.034	.086	.295
	Mode	.842	.804	.835	.520	.677	.973
	SD	.228	.376	.396	.935	.340	.339
	LCI	1.631	1.876	2.124	1.672	1.862	1.518
ψ_0	CP	.200	$< .001$	.600	.200	.400	$< .001$
	Bias	1.842	1.804	1.835	1.520	1.677	1.973
	RMSE	1.856	1.842	1.878	1.784	1.711	2.001
	AVRB	1.842	1.804	1.835	1.520	1.677	1.973
	Mode	.638	.996	.696	.446	.407	.354
	SD	.471	.617	.472	.518	.242	.569
	LCI	1.708	1.979	1.743	1.142	1.447	1.083
ψ_1	CP	1.000	1.000	1.000	.800	1.000	.400
	Bias	.138	.496	.196	-.054	-.093	-.146
	RMSE	.491	.791	.511	.520	.259	.587
	AVRB	.276	.992	.393	.107	.186	.293
	Mode	-2.214	-2.211	-2.171	-2.520	-2.651	-2.627
	SD	.235	.103	.201	.382	.443	.484
	LCI	1.494	1.586	1.318	1.144	1.23	1.219
ζ_0	CP	1.000	1.000	.800	.800	1.000	1.000
	Bias	.286	.289	.329	-.020	-.151	-.127
	RMSE	.370	.307	.385	.382	.468	.501

Table 197 (continued).

	AVRB	.114	.116	.132	.008	.061	.051
	Mode	.135	.042	.093	.863	1.131	1.055
	SD	.674	.490	.361	.481	.663	.778
	LCI	2.562	2.658	2.291	1.751	1.959	1.846
ζ_1	CP	.800	1.000	.800	1.000	.800	.800
	Bias	-.665	-.758	-.707	.063	.331	.255
	RMSE	.947	.903	.794	.485	.741	.819
	AVRB	.831	.948	.883	.079	.414	.318
	Mode	-.716	.094	.772	-.712	.143	.719
	SD	.221	.391	.121	.108	.111	.080
	LCI	.612	.972	.582	.576	.585	.650
γ	CP	.800	.800	1.000	1.000	.600	1.000
	Bias	.084	.094	-.028	.088	.143	-.081
	RMSE	.236	.402	.124	.139	.181	.114
	AVRB	.105	-	.035	.110	-	.102
	Mode	.527	.515	.547	.602	.560	.551
	SD	.074	.074	.123	.169	.087	.122
	LCI	.757	.778	.747	.646	.770	.724
ν_1	CP	.400	.600	.600	.800	.600	.600
	Bias	-.373	-.385	-.353	-.298	-.340	-.349
	RMSE	.380	.392	.374	.343	.351	.370
	AVRB	.414	.428	.392	.331	.378	.388
	Mode	.620	.569	.464	.443	.332	.531
	SD	.078	.173	.201	.295	.110	.113
	LCI	.761	.798	.711	.643	.544	.623
ν_2	CP	$< .001$	$< .001$	.200	.200	.400	$< .001$
	Bias	.520	.469	.364	.343	.232	.431
	RMSE	.526	.500	.416	.452	.257	.445
	AVRB	5.203	4.688	3.644	3.425	2.322	4.307
	Mode	3.759	3.730	3.948	3.901	4.335	4.185
	SD	.504	.577	.384	.363	.650	.620
	LCI	4.085	3.917	4.104	2.699	2.969	2.860
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.241	-.270	-.052	-.099	.335	.185
	RMSE	.558	.637	.387	.376	.731	.647
	AVRB	.06	.067	.013	.025	.084	.046

Table 198 – Results of the simulation study for the mixed ZA-SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
	Mean	-.404	-.396	-.331	-.471	-.596	-.543
	SD	.342	.243	.497	.208	.165	.295
	LCI	1.247	1.214	1.183	1.023	.951	.974
β_0	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.096	.104	.169	.029	-.096	-.043
	RMSE	.356	.264	.525	.210	.191	.298
	AVRB	.193	.208	.338	.058	.192	.087
	Mean	.988	1.140	.964	.878	.988	.905
	SD	.371	.387	.727	.160	.095	.422
	LCI	1.142	1.465	1.383	.880	.997	1.105
β_1	CP	1.000	1.000	.800	1.000	1.000	.800
	Bias	-.012	.140	-.036	-.122	-.012	-.095
	RMSE	.372	.412	.728	.201	.095	.433
	AVRB	.012	.140	.036	.122	.012	.095
	Mean	-1.154	-1.476	-1.16	-1.006	-1.155	-.982
	SD	.253	.341	.472	.311	.204	.373
	LCI	1.619	1.666	1.470	1.012	1.172	.992
ψ_0	CP	1.000	.800	.800	1.000	1.000	.800
	Bias	-.154	-.476	-.160	-.006	-.155	.018
	RMSE	.296	.585	.499	.311	.256	.373
	AVRB	.154	.476	.16	.006	.155	.018
	Mean	.468	.752	.546	.324	.311	.257
	SD	.438	1.145	.843	.446	.412	.493
	LCI	1.954	1.922	1.686	1.458	1.472	1.435
ψ_1	CP	1.000	.600	.800	.800	1.000	.800
	Bias	-.032	.252	.046	-.176	-.189	-.243
	RMSE	.439	1.173	.844	.479	.453	.55
	AVRB	.063	.504	.092	.352	.378	.486
	Mean	-2.213	-2.287	-2.156	-2.650	-2.617	-2.541
	SD	.121	.118	.227	.474	.423	.442
	LCI	1.562	1.567	1.325	1.225	1.159	1.182
ζ_0	CP	1.000	1.000	.800	.800	.800	.800
	Bias	.287	.213	.344	-.150	-.117	-.041
	RMSE	.311	.244	.412	.497	.438	.444

Table 198 (continued).

	AVRB	.115	.085	.138	.060	.047	.016
	Mean	.097	.261	-.031	1.060	1.022	.901
	SD	.319	.421	.447	.688	.601	.620
	LCI	2.571	2.833	2.369	1.963	1.841	1.853
ζ_1	CP	1.000	1.000	.600	.600	.800	.600
	Bias	-.703	-.539	-.831	.260	.222	.101
	RMSE	.772	.684	.944	.735	.641	.628
	AVRB	.879	.674	1.039	.325	.277	.126
	Mean	-.696	.071	.740	-.530	-.048	.571
	SD	.078	.373	.167	.194	.244	.206
	LCI	.691	1.059	.544	.650	.994	.646
γ	CP	1.000	.600	1.000	.800	1.000	.800
	Bias	.104	.071	-.060	.270	-.048	-.229
	RMSE	.130	.379	.177	.333	.248	.308
	AVRB	.130	-	.075	.338	-	.286
	Mean	.269	.375	.368	.189	.242	.169
	SD	.088	.108	.066	.099	.124	.105
	LCI	.487	.572	.658	.318	.423	.304
ν_1	CP	.800	.400	.400	1.000	.800	1.000
	Bias	.169	.275	.268	.089	.142	.069
	RMSE	.190	.296	.276	.133	.188	.126
	AVRB	1.688	2.752	2.677	.894	1.415	.694
	Mean	.253	.271	.336	.150	.137	.124
	SD	.127	.246	.130	.073	.048	.080
	LCI	.601	.422	.711	.202	.208	.176
ν_2	CP	.800	.600	.600	.600	.800	.800
	Bias	.153	.171	.236	.050	.037	.024
	RMSE	.199	.299	.269	.089	.061	.084
	AVRB	1.531	1.709	2.356	.498	.373	.238
	Mean	4.200	4.063	4.202	4.238	4.389	4.377
	SD	.552	.659	.418	.634	.581	.544
	LCI	3.676	3.515	3.596	2.609	2.601	2.533
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.200	.063	.202	.238	.389	.377
	RMSE	.587	.662	.464	.677	.699	.661
	AVRB	.050	.016	.050	.059	.097	.094

Table 199 – Results of the simulation study for the mixed ZA-SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Median	-.406	-.383	-.350	-.478	-.595	-.550
	SD	.337	.256	.500	.203	.171	.292
	LCI	1.247	1.214	1.183	1.023	.951	.974
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.094	.117	.150	.022	-.095	-.050
	RMSE	.350	.281	.522	.204	.196	.296
	AVRB	.188	.235	.300	.044	.191	.099
β_1	Median	.988	1.131	.955	.882	.990	.902
	SD	.366	.398	.761	.154	.103	.423
	LCI	1.142	1.465	1.383	.880	.997	1.105
	CP	1.000	1.000	.800	1.000	1.000	.800
	Bias	-.012	.131	-.045	-.118	-.010	-.098
	RMSE	.366	.419	.762	.194	.103	.434
	AVRB	.012	.131	.045	.118	.010	.098
ψ_0	Median	-1.133	-1.438	-1.159	-.997	-1.138	-.973
	SD	.228	.343	.464	.309	.220	.361
	LCI	1.619	1.666	1.470	1.012	1.172	.992
	CP	1.000	.800	.800	1.000	1.000	.800
	Bias	-.133	-.438	-.159	.003	-.138	.027
	RMSE	.264	.556	.491	.309	.260	.362
	AVRB	.133	.438	.159	.003	.138	.027
ψ_1	Median	.496	.742	.565	.326	.310	.254
	SD	.431	1.148	.858	.440	.413	.496
	LCI	1.954	1.922	1.686	1.458	1.472	1.435
	CP	1.000	.600	.800	.800	1.000	.800
	Bias	-.004	.242	.065	-.174	-.190	-.246
	RMSE	.431	1.174	.860	.473	.455	.553
	AVRB	.007	.484	.131	.349	.379	.492
ζ_0	Median	-2.181	-2.268	-2.160	-2.649	-2.615	-2.531
	SD	.115	.112	.222	.474	.414	.442
	LCI	1.562	1.567	1.325	1.225	1.159	1.182
	CP	1.000	1.000	.800	.800	.800	.800
	Bias	.319	.232	.340	-.149	-.115	-.031
	RMSE	.339	.257	.406	.496	.430	.443

Table 199 (continued).

	AVRB	.127	.093	.136	.059	.046	.013
	Median	.072	.250	-.016	1.065	1.020	.917
	SD	.272	.425	.413	.689	.584	.606
	LCI	2.571	2.833	2.369	1.963	1.841	1.853
ζ_1	CP	1.000	1.000	.600	.600	.800	.600
	Bias	-.728	-.550	-.816	.265	.220	.117
	RMSE	.777	.695	.915	.739	.624	.618
	AVRB	.910	.688	1.021	.332	.274	.147
	Median	-.735	.074	.764	-.536	-.064	.574
	SD	.073	.359	.173	.213	.247	.224
	LCI	.691	1.059	.544	.65	.994	.646
γ	CP	1.000	.600	1.000	.800	1.000	.800
	Bias	.065	.074	-.036	.264	-.064	-.226
	RMSE	.098	.367	.177	.339	.255	.318
	AVRB	.082	-	.045	.330	-	.282
	Median	.247	.371	.334	.183	.223	.153
	SD	.088	.120	.065	.098	.129	.091
	LCI	.487	.572	.658	.318	.423	.304
ν_1	CP	.800	.400	.400	1.000	.800	1.000
	Bias	.147	.271	.234	.083	.123	.053
	RMSE	.171	.297	.243	.128	.178	.106
	AVRB	1.471	2.714	2.341	.827	1.234	.530
	Median	.208	.254	.300	.143	.129	.116
	SD	.120	.247	.128	.070	.045	.073
	LCI	.601	.422	.711	.202	.208	.176
ν_2	CP	.800	.600	.600	.600	.800	.800
	Bias	.108	.154	.200	.043	.029	.016
	RMSE	.161	.291	.238	.083	.054	.075
	AVRB	1.079	1.536	2.004	.435	.291	.163
	Median	4.067	3.956	4.066	4.179	4.339	4.320
	SD	.528	.658	.402	.616	.579	.548
	LCI	3.676	3.515	3.596	2.609	2.601	2.533
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	.067	-.044	.066	.179	.339	.320
	RMSE	.532	.660	.407	.641	.670	.635
	AVRB	.017	.011	.016	.045	.085	.080

Table 200 – Results of the simulation study for the mixed ZA-SCNBS regression model ($\nu_1 = \nu_2 = .1$).

		$n = 50$			$n = 100$		
		$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$	$\gamma = -.8$	$\gamma = 0$	$\gamma = .8$
β_0	Mode	-.419	-.368	-.375	-.483	-.607	-.579
	SD	.339	.289	.521	.196	.180	.289
	LCI	1.247	1.214	1.183	1.023	.951	.974
	CP	1.000	1.000	.800	1.000	1.000	1.000
	Bias	.081	.132	.125	.017	-.107	-.079
	RMSE	.349	.317	.536	.196	.209	.299
	AVRB	.162	.264	.250	.034	.213	.158
β_1	Mode	1.021	1.121	.964	.881	.992	.904
	SD	.300	.397	.792	.152	.105	.418
	LCI	1.142	1.465	1.383	.880	.997	1.105
	CP	1.000	1.000	.800	1.000	1.000	.800
	Bias	.021	.121	-.036	-.119	-.008	-.096
	RMSE	.301	.415	.793	.193	.105	.429
	AVRB	.021	.121	.036	.119	.008	.096
ψ_0	Mode	-1.042	-1.409	-1.159	-.991	-1.128	-.975
	SD	.251	.327	.445	.307	.231	.357
	LCI	1.619	1.666	1.470	1.012	1.172	.992
	CP	1.000	.800	.800	1.000	1.000	.800
	Bias	-.042	-.409	-.159	.009	-.128	.025
	RMSE	.254	.523	.473	.307	.264	.358
	AVRB	.042	.409	.159	.009	.128	.025
ψ_1	Mode	.565	.744	.626	.348	.314	.249
	SD	.379	1.134	.898	.437	.416	.506
	LCI	1.954	1.922	1.686	1.458	1.472	1.435
	CP	1.000	.600	.800	.800	1.000	.800
	Bias	.065	.244	.126	-.152	-.186	-.251
	RMSE	.385	1.160	.907	.463	.456	.565
	AVRB	.131	.487	.252	.304	.371	.503
ζ_0	Mode	-2.143	-2.240	-2.149	-2.651	-2.613	-2.529
	SD	.109	.067	.215	.469	.405	.444
	LCI	1.562	1.567	1.325	1.225	1.159	1.182
	CP	1.000	1.000	.800	.800	.800	.800
	Bias	.357	.260	.351	-.151	-.113	-.029
	RMSE	.373	.268	.411	.492	.421	.445

Table 200 (continued).

	AVRB	.143	.104	.140	.060	.045	.011
	Mode	-.002	.234	.008	1.076	1.047	.936
	SD	.171	.431	.377	.689	.552	.576
	LCI	2.571	2.833	2.369	1.963	1.841	1.853
ζ_1	CP	1.000	1.000	.600	.600	.800	.600
	Bias	-.802	-.566	-.792	.276	.247	.136
	RMSE	.820	.712	.878	.742	.605	.591
	AVRB	1.002	.708	.991	.345	.309	.170
	Mode	-.749	.089	.762	-.534	-.089	.568
	SD	.051	.356	.168	.209	.262	.223
	LCI	.691	1.059	.544	.650	.994	.646
γ	CP	1.000	.600	1.000	.800	1.000	.800
	Bias	.051	.089	-.038	.266	-.089	-.232
	RMSE	.072	.367	.172	.339	.277	.322
	AVRB	.064	-	.048	.333	-	.290
	Mode	.253	.369	.330	.185	.229	.159
	SD	.084	.112	.064	.096	.125	.090
	LCI	.487	.572	.658	.318	.423	.304
ν_1	CP	.800	.400	.400	1.000	.800	1.000
	Bias	.153	.269	.230	.085	.129	.059
	RMSE	.175	.291	.239	.128	.180	.108
	AVRB	1.529	2.687	2.298	.849	1.294	.588
	Mode	.198	.257	.288	.149	.135	.121
	SD	.088	.255	.113	.074	.045	.076
	LCI	.601	.422	.711	.202	.208	.176
ν_2	CP	.800	.600	.600	.600	.800	.800
	Bias	.098	.157	.188	.049	.035	.021
	RMSE	.132	.300	.219	.089	.057	.079
	AVRB	.979	1.571	1.879	.486	.353	.214
	Mode	3.817	3.765	3.799	4.095	4.264	4.256
	SD	.503	.467	.332	.593	.555	.592
	LCI	3.676	3.515	3.596	2.609	2.601	2.533
σ^2	CP	1.000	1.000	1.000	1.000	1.000	1.000
	Bias	-.183	-.235	-.201	.095	.264	.256
	RMSE	.535	.523	.388	.600	.614	.644
	AVRB	.046	.059	.050	.024	.066	.064

F.3 Behavior of the residuals

Simulated observations from mixed ZA-SGtBS1 regression model

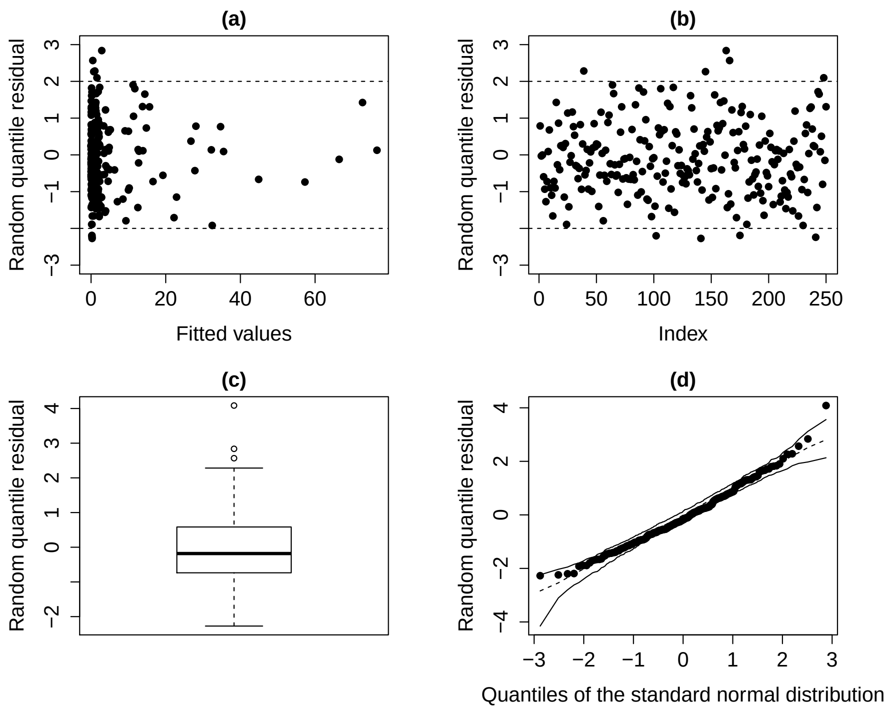

Figure 206 – Residual plots for the mixed ZA-SGtBS1 regression model.

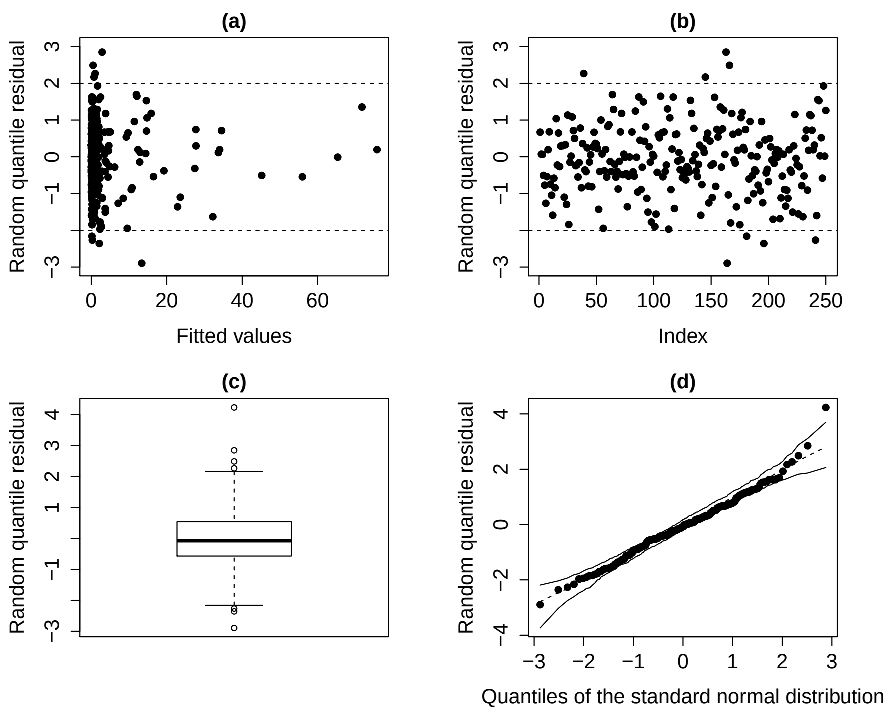

Figure 207 – Residual plots for the mixed ZA-StBS regression model.

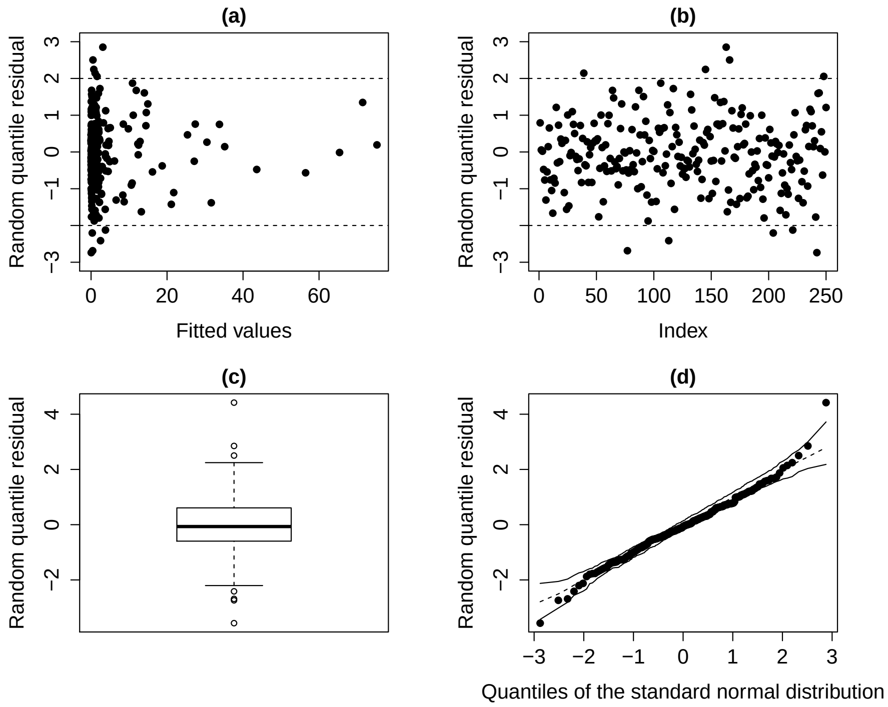

Figure 208 – Residual plots for the mixed ZA-SSLBS1 regression model.

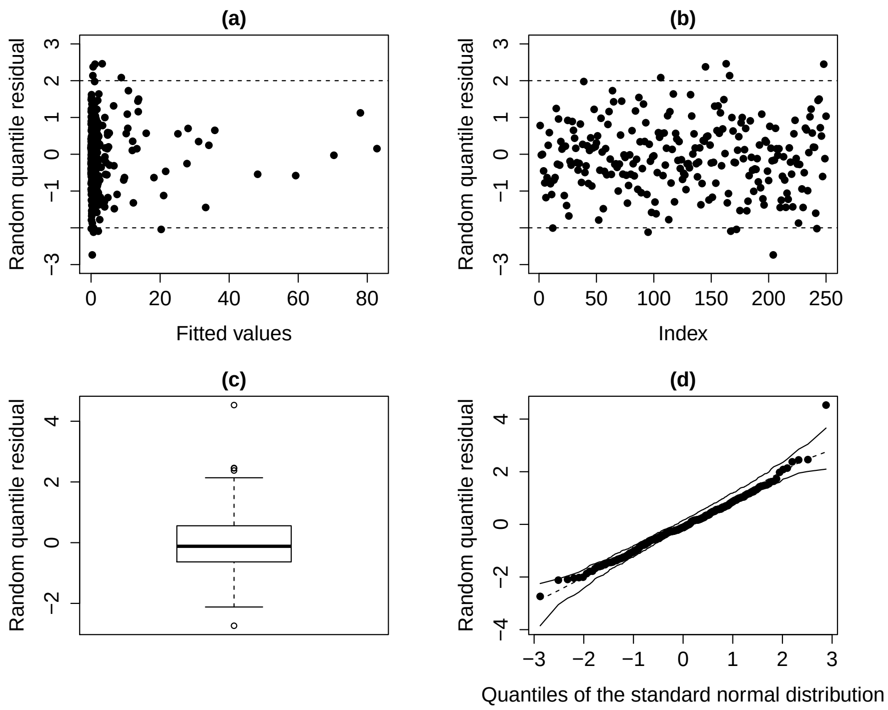

Figure 209 – Residual plots for the mixed ZA-SSLBS2 regression model.

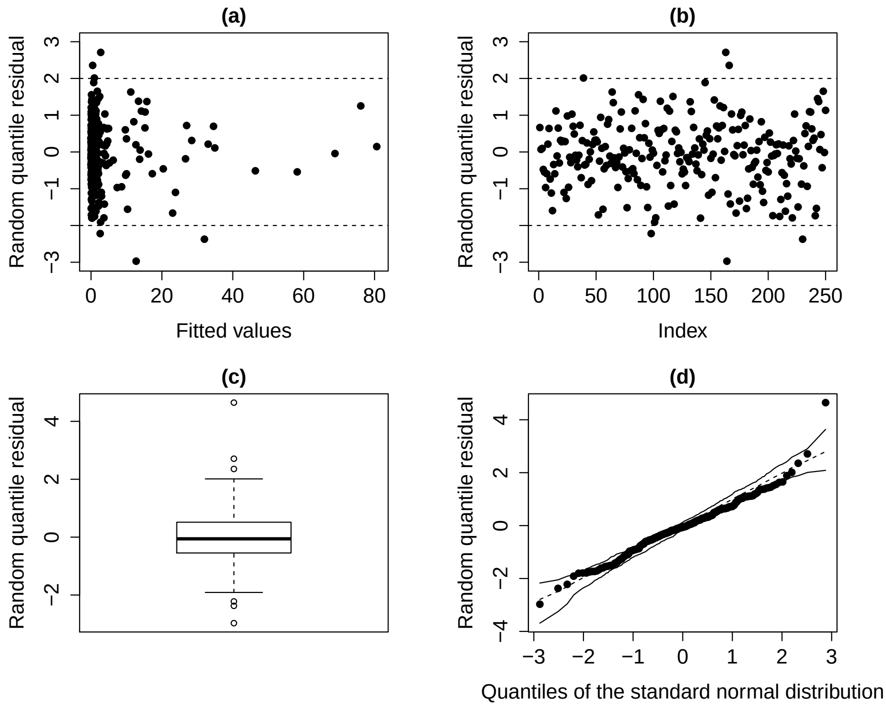

Figure 210 – Residual plots for the mixed ZA-SCNBS regression model.

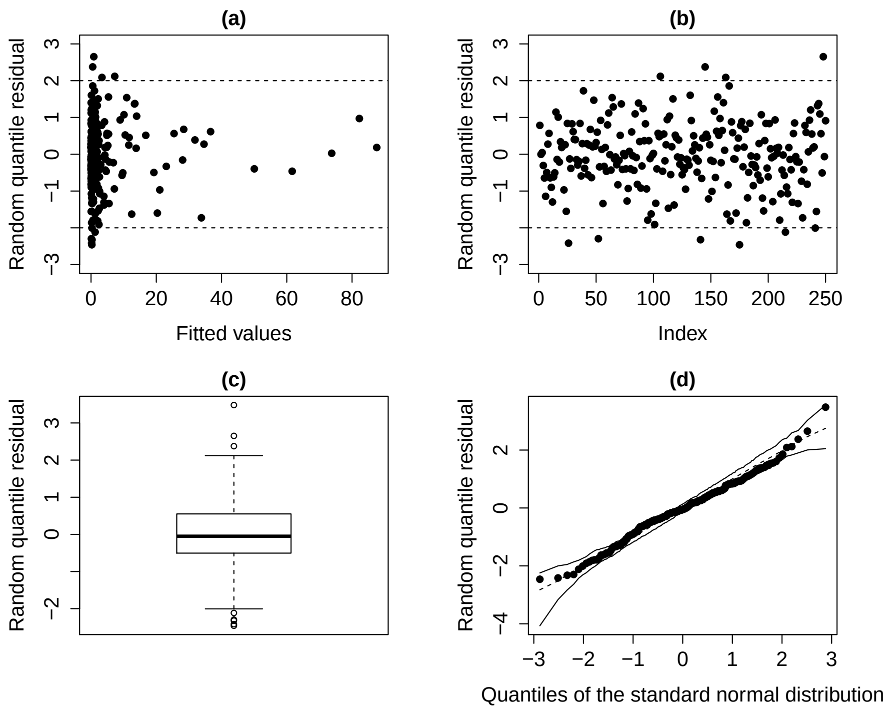

Figure 211 – Residual plots for the mixed ZA-SNBS regression model.

Simulated observations from mixed ZA-StBS regression model

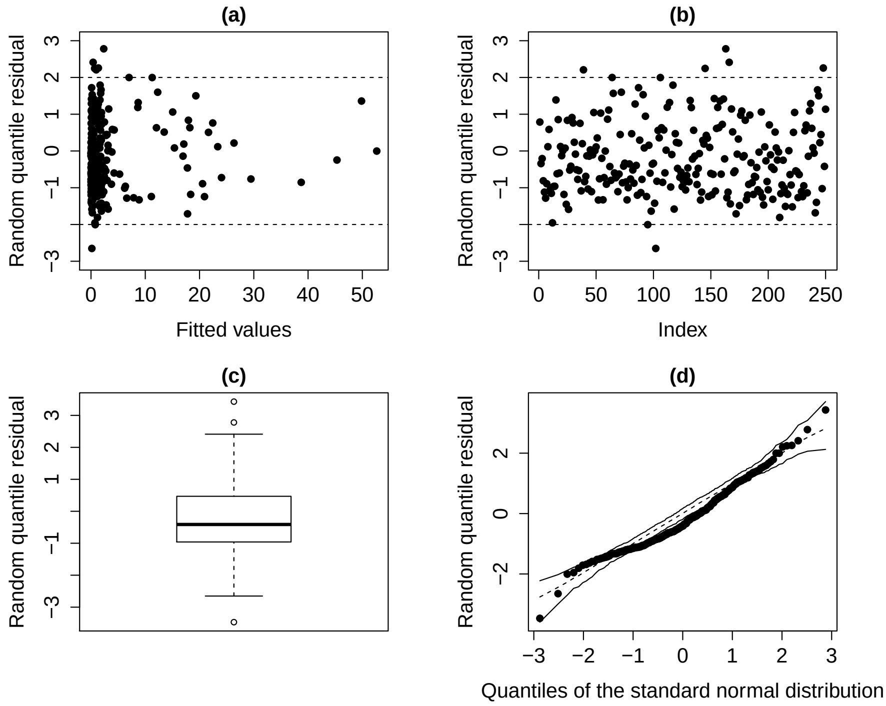

Figure 212 – Residual plots for the mixed ZA-SGtBS1 regression model.

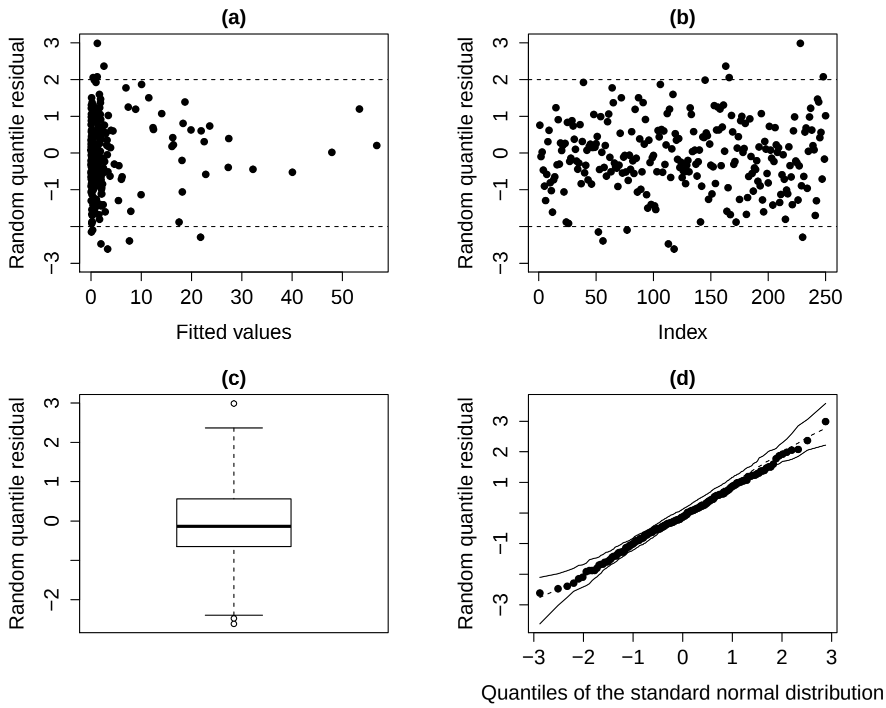

Figure 213 – Residual plots for the mixed ZA-StBS regression model.

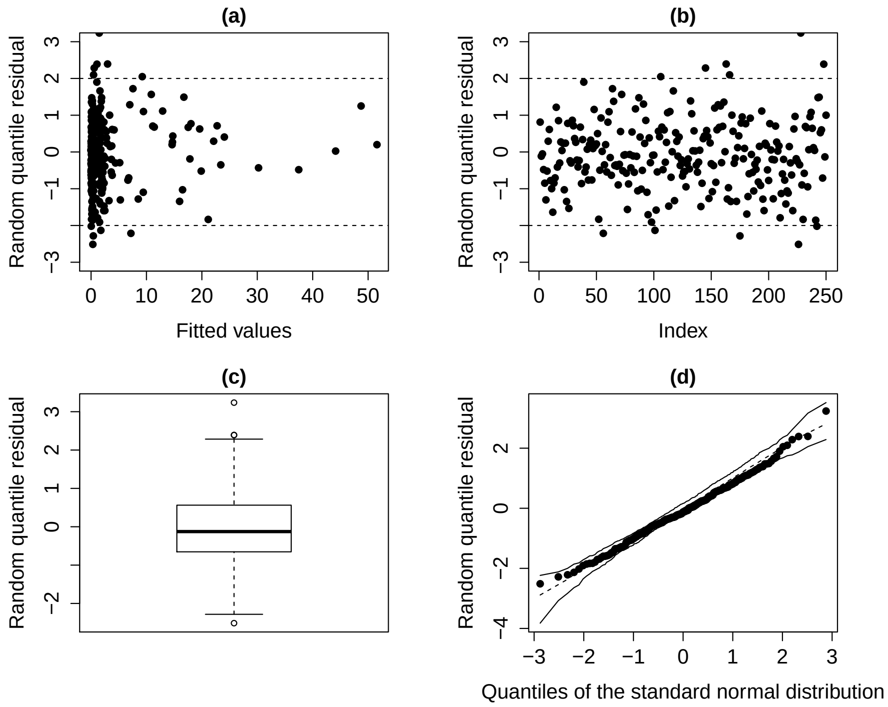

Figure 214 – Residual plots for the mixed ZA-SSLBS1 regression model.

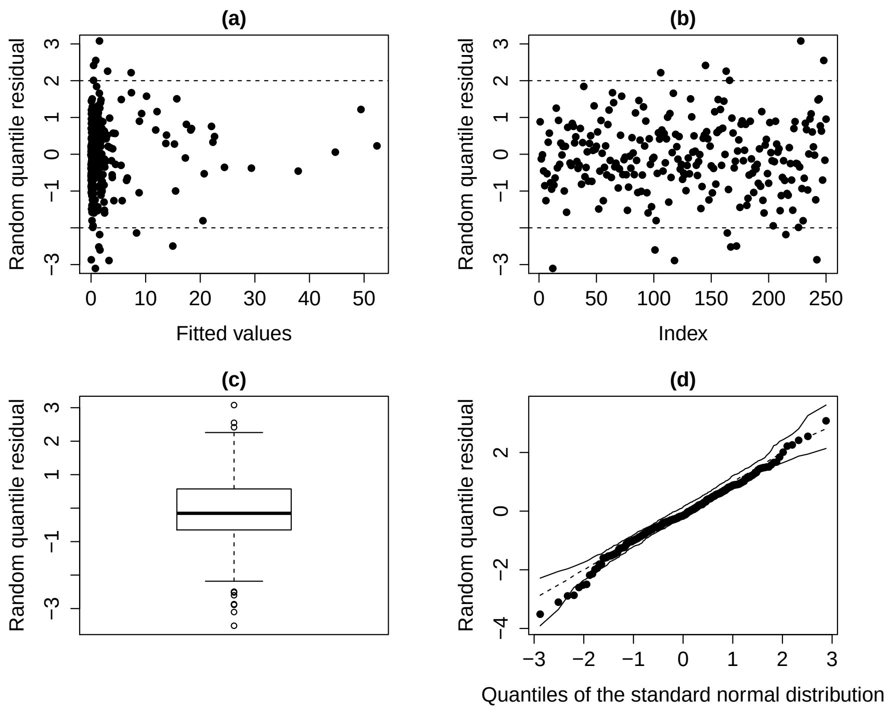

Figure 215 – Residual plots for the mixed ZA-SSLBS2 regression model.

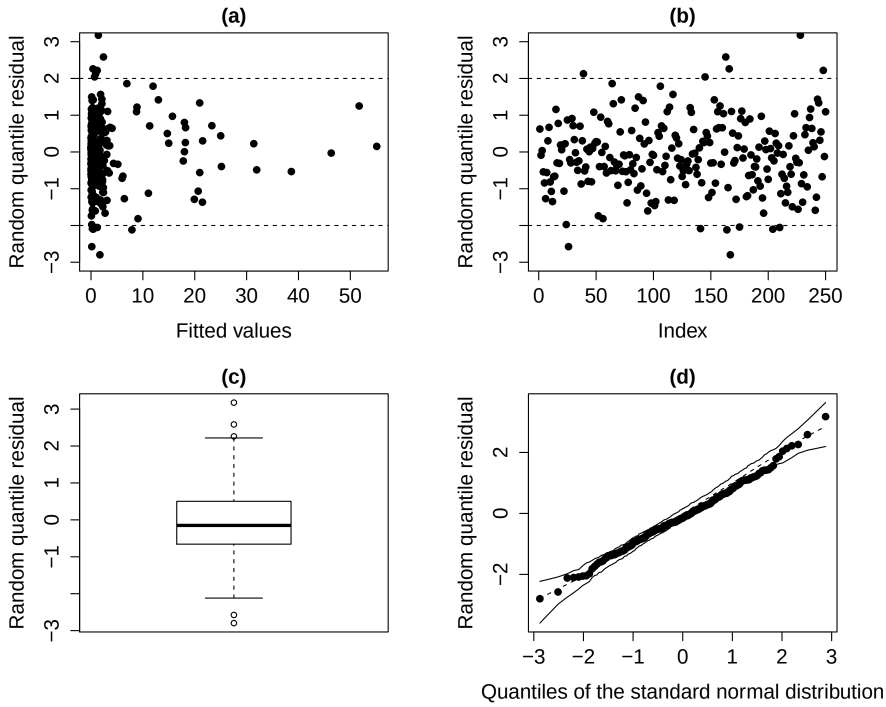

Figure 216 – Residual plots for the mixed ZA-SCNBS regression model.

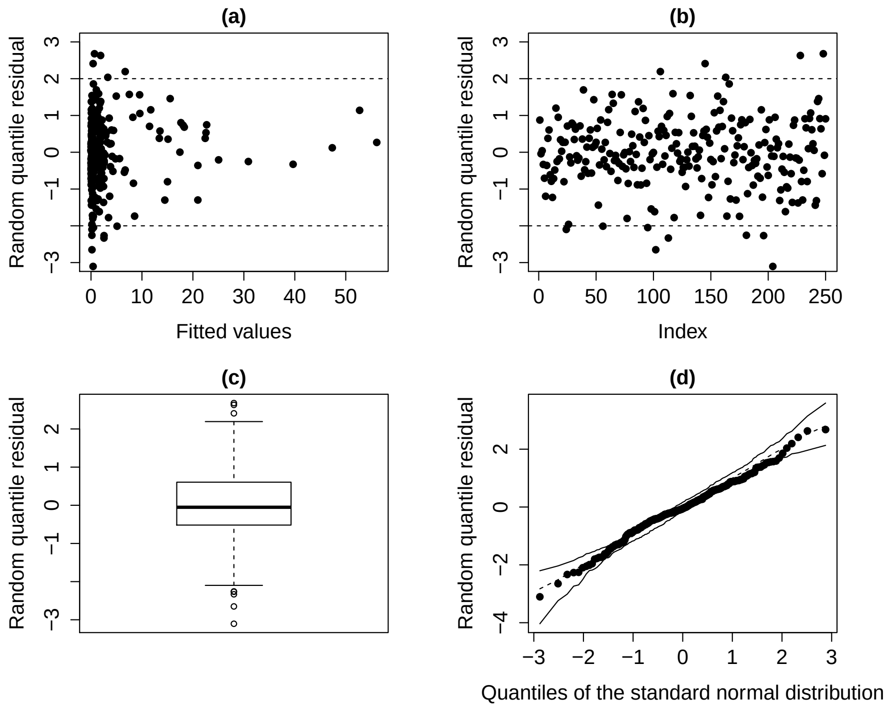

Figure 217 – Residual plots for the mixed ZA-SNBS regression model.

Simulated observations from mixed ZA-SSLBS regression model

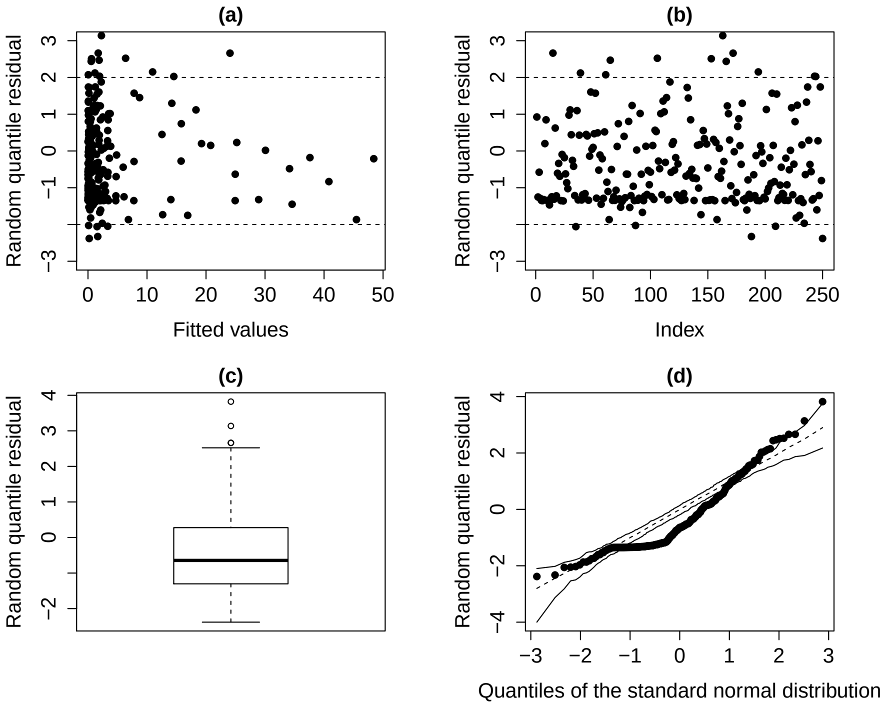

Figure 218 – Residual plots for the mixed ZA-SGtBS1 regression model.

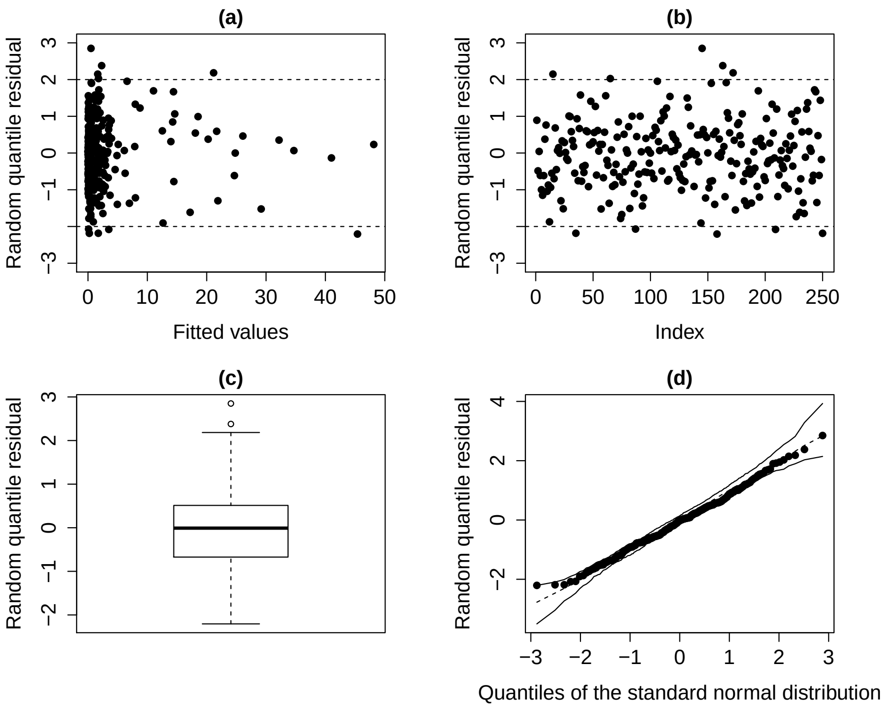

Figure 219 – Residual plots for the mixed ZA-StBS regression model.

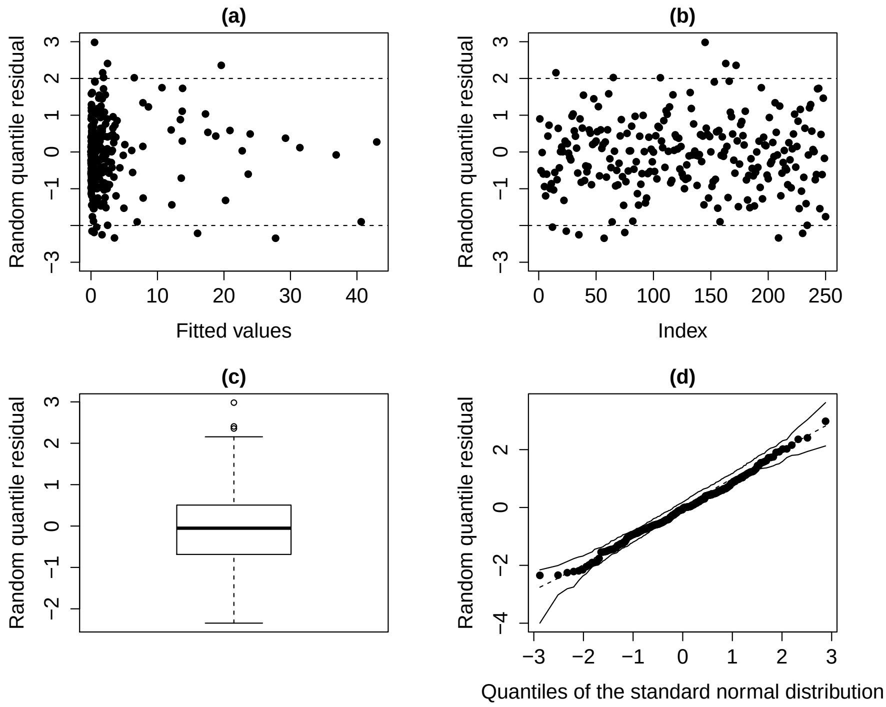

Figure 220 – Residual plots for the mixed ZA-SSLBS1 regression model.

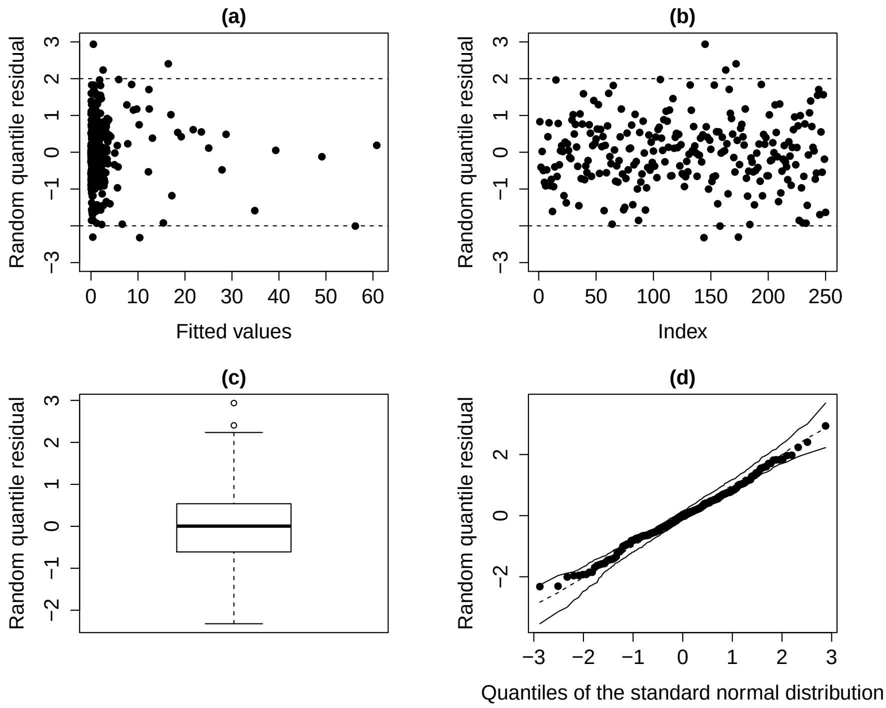

Figure 221 – Residual plots for the mixed ZA-SSLBS2 regression model.

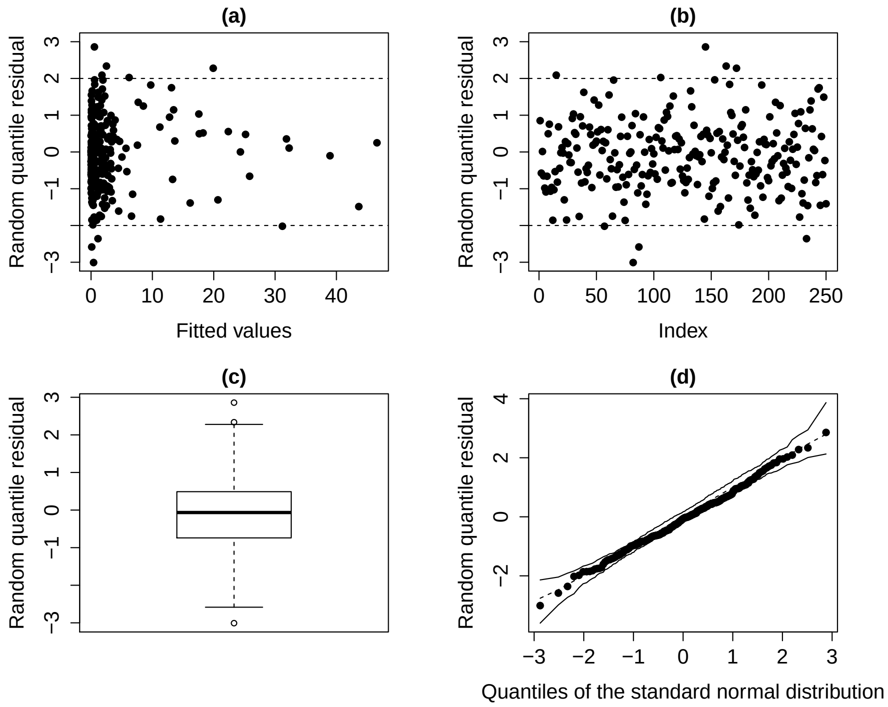

Figure 222 – Residual plots for the mixed ZA-SCNBS regression model.

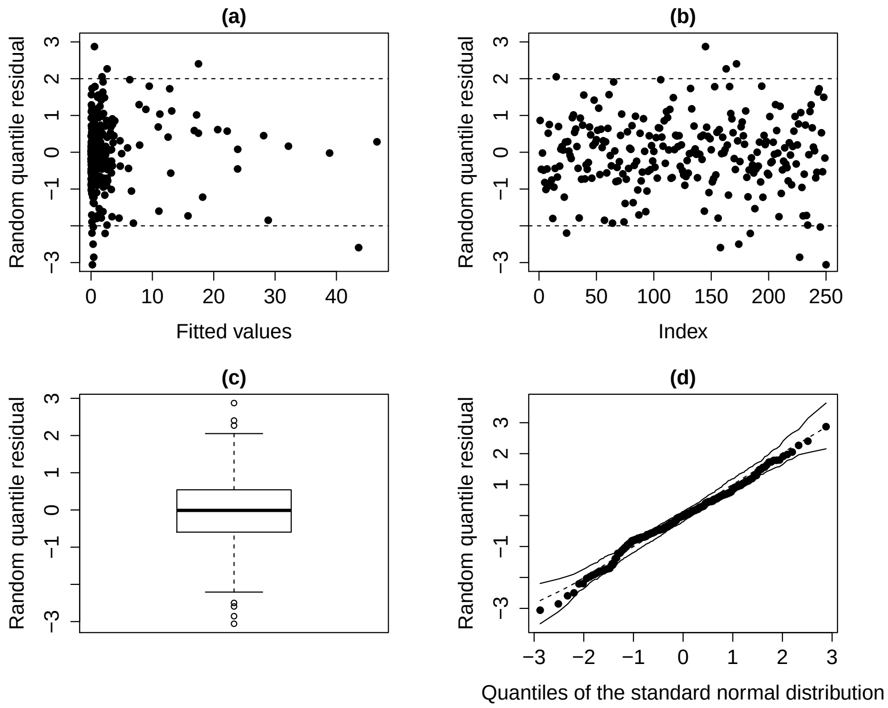

Figure 223 – Residual plots for the mixed ZA-SNBS regression model.

Simulated observations from mixed ZA-SCNBS regression model

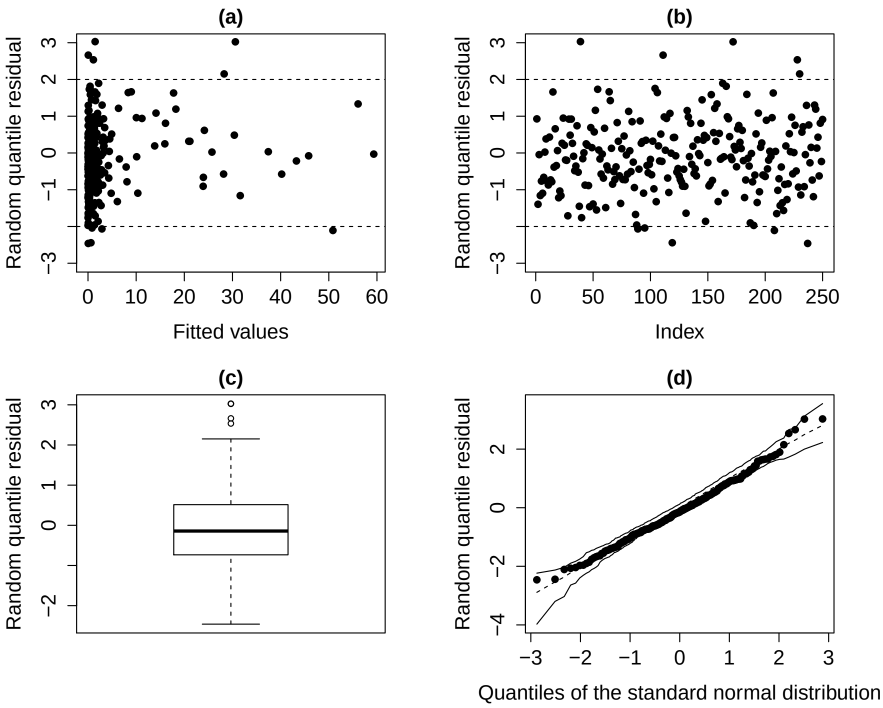

Figure 224 – Residual plots for the mixed ZA-SGtBS1 regression model.

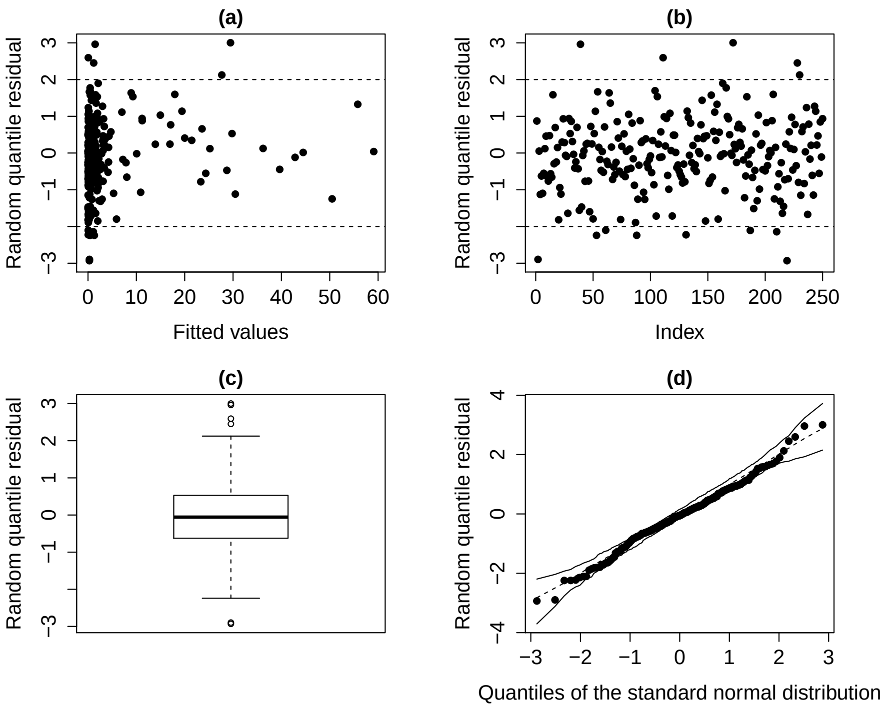

Figure 225 – Residual plots for the mixed ZA-StBS regression model.

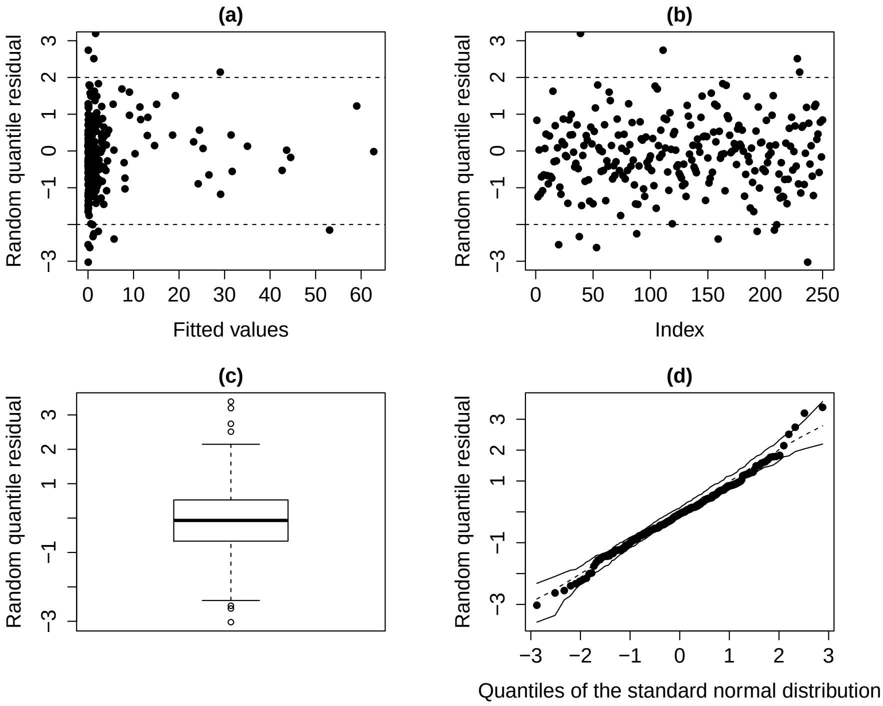

Figure 226 – Residual plots for the mixed ZA-SSLBS1 regression model.

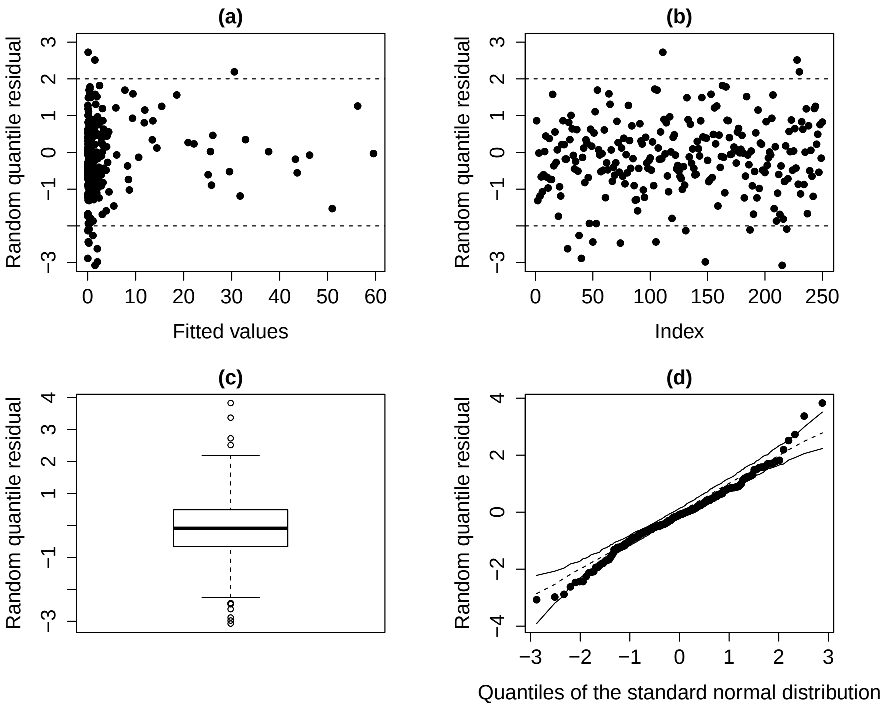

Figure 227 – Residual plots for the mixed ZA-SSLBS2 regression model.

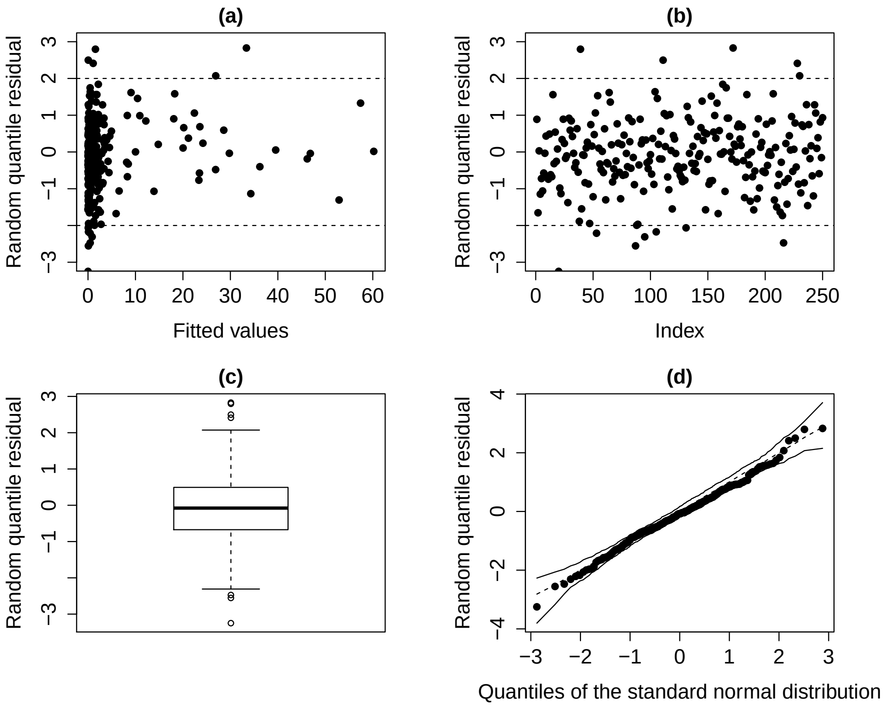

Figure 228 – Residual plots for the mixed ZA-SCNBS regression model.

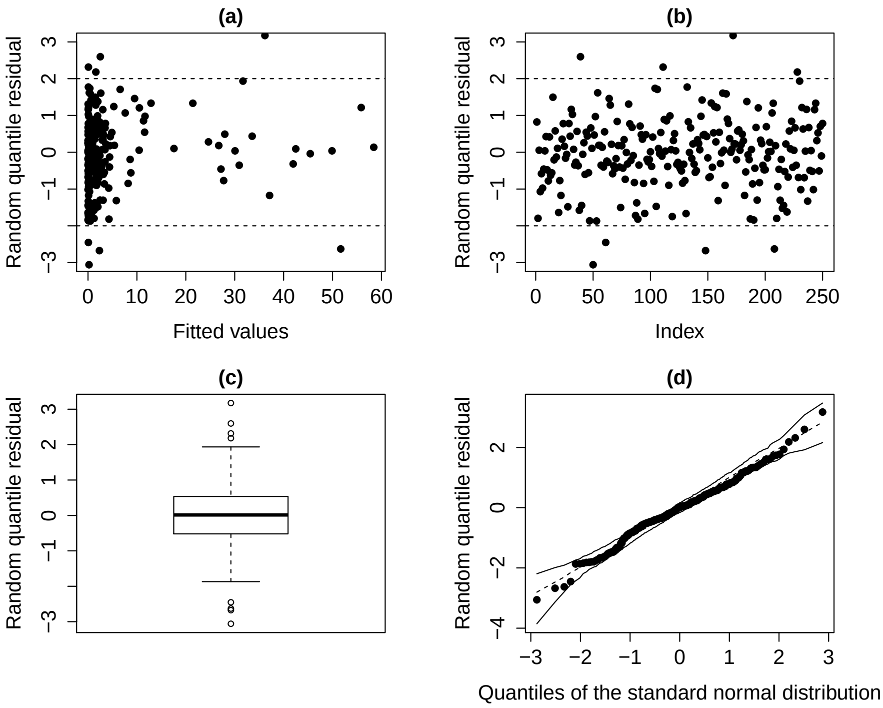

Figure 229 – Residual plots for the mixed ZA-SNBS regression model.

F.4 Behavior of the K-L divergence

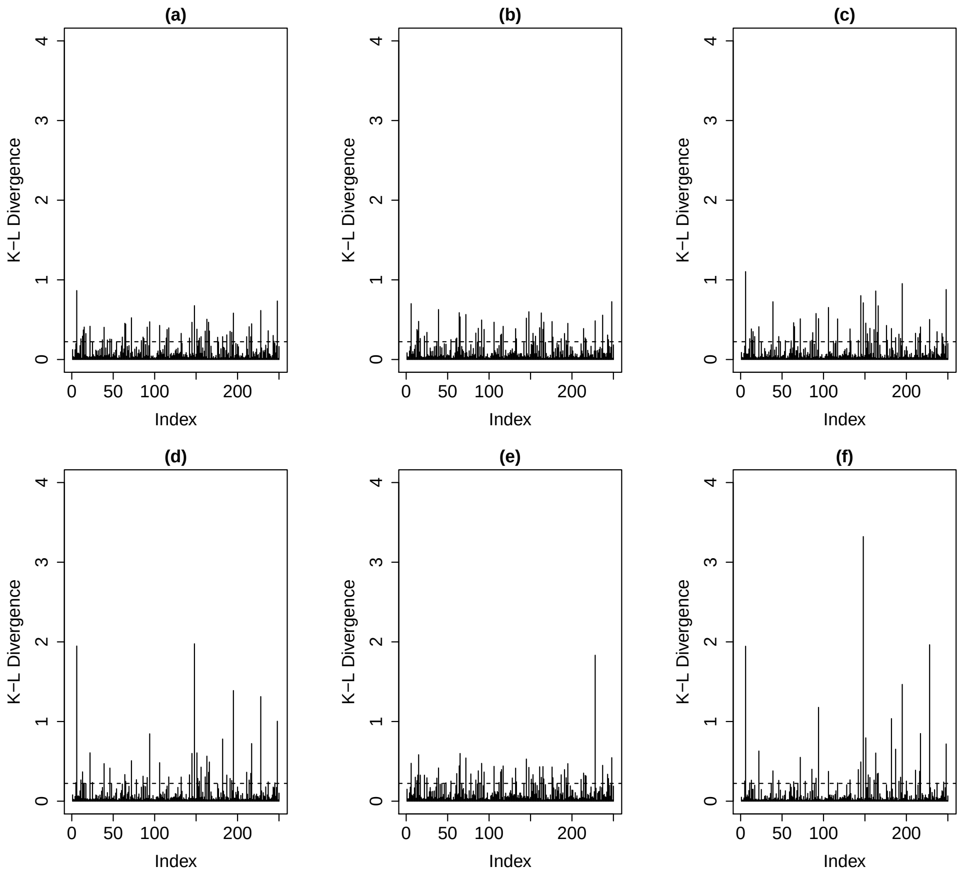

Figure 230 – K-L divergence when we generated the data set from mixed ZA-SGtBS1 and fitted the following models: (a) mixed ZA-SGtBS1, (b) mixed ZA-StBS, (c) mixed ZA-SSLBS1, (d) mixed ZA-SSLBS2, (e) mixed ZA-SCNBS, and (f) mixed ZA-SNBS.

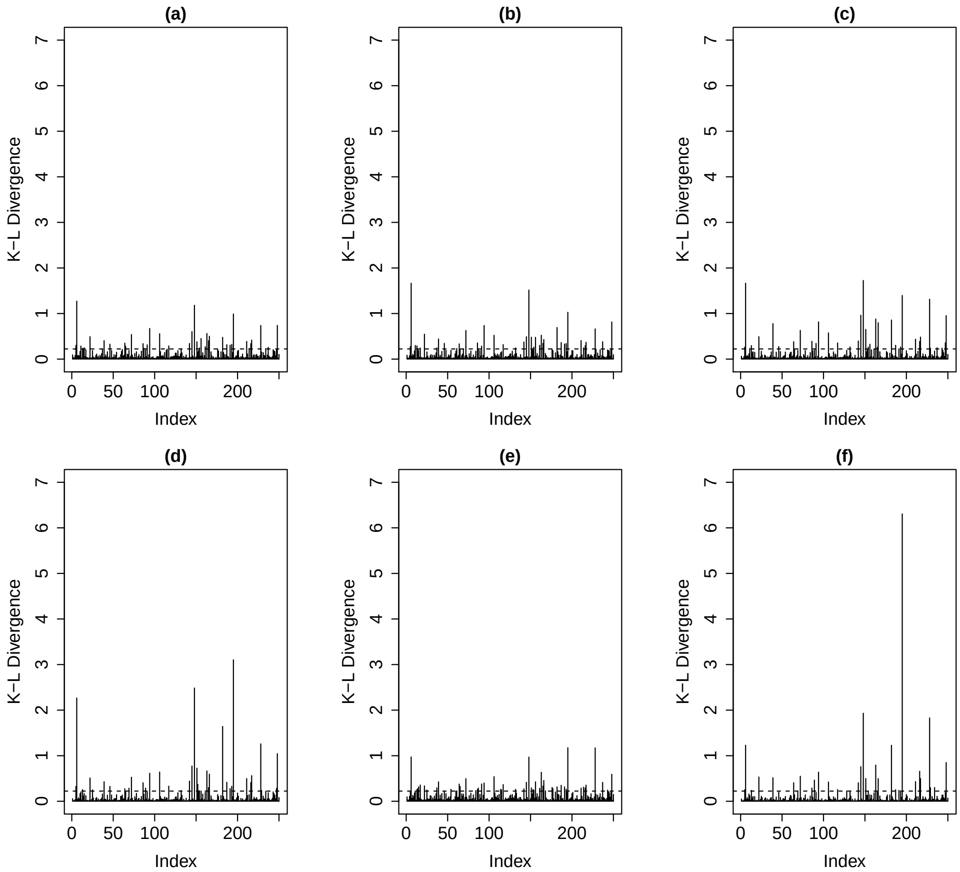

Figure 231 − K-L divergence when we generated the data set from mixed ZA-StBS and fitted the following models: (a) mixed ZA-SGtBS1, (b) mixed ZA-StBS, (c) mixed ZA-SSLBS1, (d) mixed ZA-SSLBS2, (e) mixed ZA-SCNBS, and (f) mixed ZA-SNBS.

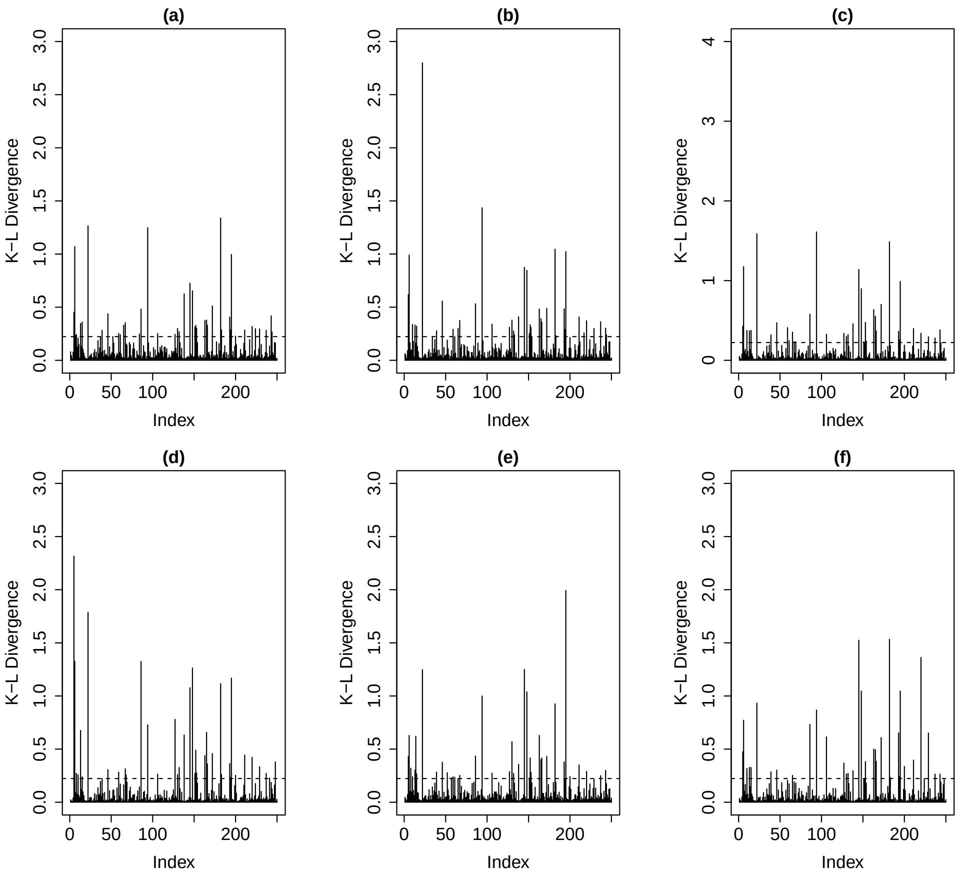

Figure 232 – K-L divergence when we generated the data set from mixed ZA-SSLBS and fitted the following models: (a) mixed ZA-SGtBS1, (b) mixed ZA-StBS, (c) mixed ZA-SSLBS1, (d) mixed ZA-SSLBS2, (e) mixed ZA-SCNBS, and (f) mixed ZA-SNBS.

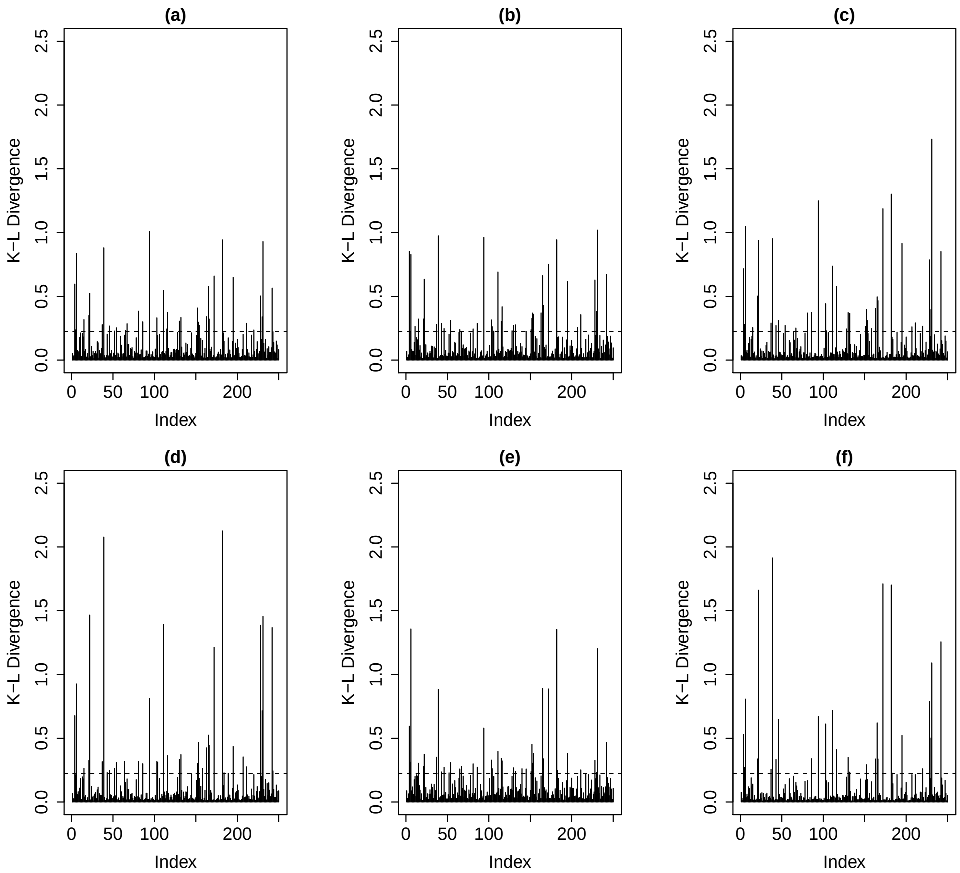

Figure 233 – K-L divergence when we generated the data set from mixed ZA-SCNBS and fitted the following models: (a) mixed ZA-SGtBS1, (b) mixed ZA-StBS, (c) mixed ZA-SSLBS1, (d) mixed ZA-SSLBS2, (e) mixed ZA-SCNBS, and (f) mixed ZA-SNBS.

F.5 Statistics of model comparison

Table 201 – Averaged criteria for the simulation study.

True underlying model: mixed ZA-SGtBS1 regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	459.071	487.243	1272.070	-254.552
ZA-StBS	456.282	484.454	1263.443	-253.283
ZA-SSLBS1	462.227	490.399	1283.496	-256.521
ZA-SSLBS2	464.452	492.624	1291.660	-257.593
ZA-SCNBS	455.108	486.801	1256.271	-251.975
ZA-SNBS	464.806	489.456	1303.143	-258.026
True underlying model: mixed ZA-StBS regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	573.284	601.456	1617.091	-310.245
ZA-StBS	573.778	601.949	1618.628	-310.754
ZA-SSLBS1	573.142	601.314	1618.418	-311.423
ZA-SSLBS2	572.781	600.952	1619.031	-309.972
ZA-SCNBS	568.974	600.667	1601.239	-307.366
ZA-SNBS	573.441	598.091	1632.927	-310.811
True underlying model: mixed ZA-SSLBS regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	601.558	629.729	1671.266	-563.757
ZA-StBS	572.654	600.826	1614.327	-309.985
ZA-SSLBS1	571.431	599.602	1613.021	-310.463
ZA-SSLBS2	569.489	597.661	1609.848	-309.140
ZA-SCNBS	571.206	602.899	1607.186	-308.203
ZA-SNBS	567.871	592.521	1614.509	-308.226
True underlying model: mixed ZA-SCNBS regression model				
Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	488.081	516.253	1362.850	-266.516
ZA-StBS	488.945	517.116	1363.541	-269.001
ZA-SSLBS1	492.429	520.601	1375.765	-272.150
ZA-SSLBS2	491.984	520.156	1375.598	-271.820
ZA-SCNBS	483.855	515.548	1346.787	-266.615
ZA-SNBS	497.565	522.215	1403.763	-273.137

Table 202 – Percentage of times that the correct model was selected.

Model	EAIC	EBIC	DIC	LPML
ZA-SGtBS1	30%	20%	30%	30%
ZA-StBS	0%	0%	0%	0%
ZA-SSLBS	10%	10%	40%	30%
ZA-SCNBS	50%	40%	70%	40%

F.6 Posterior predictive checking

Table 203 – Posterior predictive checking for the mixed ZA-SSBS regression model.

True underlying model: ZA-SGtBS1						
	ZA-SGtBS1	ZA-StBS	ZA-SSLBS1	ZA-SSLBS2	ZA-SCNBS	ZA-SNBS
p-value	.164	.091	.121	.247	.219	.209
True underlying model: ZA-StBS						
	ZA-StBS	ZA-SGtBS1	ZA-SSLBS1	ZA-SSLBS2	ZA-SCNBS	ZA-SNBS
p-value	.500	.439	.490	.440	.482	.375
True underlying model: SSLBS						
	ZA-SSLBS1	ZA-SSLBS2	ZA-SGtBS1	ZA-StBS	ZA-SCNBS	ZA-SNBS
p-value	.476	.418	.517	.428	.511	.319
True underlying model: SCNBS						
	ZA-SCNBS	ZA-SGtBS1	ZA-StBS	ZA-SSLBS1	ZA-SSLBS2	ZA-SNBS
p-value	.573	.186	.167	.353	.186	.241

F.7 Results of the statistical analysis of the bilirubin concentration

Table 204 – Bayesian estimates for the mixed ZA-SGtBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.484	.110	[1.244; 1.652]
β_1	.550	.062	[.404; .650]
β_2	-.080	.005	[-.088; -.069]
ψ_0	.538	.103	[.298; .705]
ψ_1	-.107	.014	[-.139; -.081]
ζ_0	-1.427	.171	[-1.705; -1.000]
ζ_1	-.116	.039	[-.213; -.036]
γ_1	.915	.040	[.841; .984]
γ_2	.844	.121	[.536; .968]
γ_3	-.640	.177	[-.942; -.273]
γ_4	-.818	.143	[-.975; -.444]
γ_5	-.854	.112	[-.980; -.575]
γ_6	-.805	.127	[-.970; -.501]
γ_7	-.787	.160	[-.955; -.320]
γ_8	-.740	.199	[-.953; -.136]
γ_9	-.894	.092	[-.975; -.620]
ν_1	6.120	.513	[5.219; 6.954]
ν_2	6.342	.573	[5.254; 7.605]
ν_3	19.764	4.990	[12.410; 31.664]
ν_4	32.221	7.253	[21.110; 47.953]
ν_5	22.667	3.995	[15.860; 30.986]
ν_6	18.809	3.703	[13.320; 27.040]
ν_7	14.299	2.749	[9.851; 20.390]
ν_8	14.112	3.147	[9.393; 21.490]
ν_9	5.382	.690	[4.315; 7.083]
σ^2	.868	.136	[.639; 1.163]

Table 205 – Bayesian estimates for the mixed ZA-StBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.916	.079	[.797; 1.004]
β_1	.058	.032	[-.004; .119]
β_2	-.074	.003	[-.081; -.069]
ψ_0	-1.665	.100	[-1.785; -1.516]
ψ_1	-.197	.011	[-.213; -.177]
ζ_0	-2.014	.054	[-2.097; -1.933]
ζ_1	-.015	.040	[-.071; .045]
γ_1	-.987	.006	[-.994; -.973]
γ_2	.694	.165	[.365; .941]
γ_3	-.470	.218	[-.799; -.018]
γ_4	-.809	.153	[-.993; -.422]
γ_5	-.667	.173	[-.941; -.270]
γ_6	.264	.244	[-.233; .674]
γ_7	-.530	.230	[-.972; -.210]
γ_8	-.841	.106	[-.970; -.580]
γ_9	-.694	.188	[-.971; -.288]
ν_1	4.628	.556	[4.024; 6.116]
ν_2	19.300	9.648	[6.000; 38.982]
ν_3	30.046	18.811	[8.404; 79.400]
ν_4	23.484	11.763	[7.079; 47.621]
ν_5	15.650	8.832	[5.287; 38.894]
ν_6	7.266	3.402	[4.169; 16.192]
ν_7	5.597	1.906	[4.033; 11.322]
ν_8	22.260	14.176	[4.806; 51.711]
ν_9	4.272	.316	[4.004; 5.204]
σ^2	1.350	.247	[.946; 1.896]

Table 206 – Bayesian estimates for the mixed SSLBS1 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.168	.058	[1.054; 1.242]
β_1	-.006	.040	[-.068; .080]
β_2	-.069	.003	[-.076; -.063]
ψ_0	-2.719	.073	[-2.839; -2.601]
ψ_1	.006	.013	[-.017; .028]
ζ_0	-2.679	.062	[-2.785; -2.564]
ζ_1	.036	.014	[.009; .069]
γ_1	.371	.015	.[344; .396]
γ_2	.253	.321	[-.419; .758]
γ_3	-.854	.092	[-.969; -.630]
γ_4	-.883	.088	[-.982; -.664]
γ_5	-.905	.072	[-.979; -.723]
γ_6	-.902	.088	[-.988; -.678]
γ_7	-.853	.118	[-.977; -.538]
γ_8	-.840	.145	[-.978; -.443]
γ_9	-.876	.117	[-.990; -.573]
ν_1	2.202	.102	[2.010; 2.357]
ν_2	4.825	3.436	[2.052; 14.489]
ν_3	7.930	4.558	[2.676; 20.624]
ν_4	11.686	6.524	[3.298; 28.045]
ν_5	12.037	6.347	[3.588; 26.530]
ν_6	7.352	4.749	[2.465; 19.876]
ν_7	3.483	1.571	[2.042; 7.854]
ν_8	9.886	5.848	[3.201; 25.621]
ν_9	2.308	.290	[2.010; 3.090]
σ^2	.895	.144	[.647; 1.207]

Table 207 – Bayesian estimates for the mixed ZA-SSLBS2 regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	.878	.045	[.818; .994]
β_1	.075	.082	[-.049; .226]
β_2	-.070	.009	[-.083; -.053]
ψ_0	-2.362	.173	[-2.765; -2.135]
ψ_1	-.026	.026	[-.073; .029]
ζ_0	-1.322	.067	[-1.493; -1.247]
ζ_1	-.159	.018	[-.199; -.130]
γ_1	.971	.022	[.919; .992]
γ_2	.464	.214	[.059; .865]
γ_3	-.856	.101	[-.982; -.610]
γ_4	-.923	.082	[-.993; -.709]
γ_5	-.909	.080	[-.990; -.698]
γ_6	-.898	.081	[-.989; -.673]
γ_7	-.907	.090	[-.987; -.640]
γ_8	-.745	.188	[-.959; -.270]
γ_9	-.806	.136	[-.975; -.472]
ν_1	2.131	.107	[2.005; 2.392]
ν_2	25.661	19.934	[2.354; 75.968]
ν_3	32.659	24.988	[7.096; 100.430]
ν_4	43.020	21.883	[12.309; 96.665]
ν_5	37.370	21.822	[8.004; 91.156]
ν_6	38.099	26.328	[5.480; 103.665]
ν_7	11.698	14.893	[2.125; 55.728]
ν_8	30.341	20.664	[7.137; 83.275]
ν_9	2.356	.349	[2.012; 3.262]
σ^2	1.224	.225	[.819; 1.748]

Table 208 – Bayesian estimates for the mixed ZA-SNBS regression model.

Parameter	PE	PSD	$CI_{95\%}$
β_0	1.082	.032	[1.032; 1.167]
β_1	.149	.054	[.045; .255]
β_2	-.065	.004	[-.072; -.057]
ψ_0	-2.419	.054	[-2.511; -2.323]
ψ_1	.053	.010	[.031; .070]
ζ_0	-2.396	.050	[-2.498; -2.298]
ζ_1	.016	.018	[-.017; .048]
γ_1	.231	.031	[.175; .289]
γ_2	-.226	.120	[-.435; .039]
γ_3	-.985	.009	[-.993; -.966]
γ_4	-.779	.185	[-.969; -.456]
γ_5	-.926	.098	[-.991; -.686]
γ_6	-.936	.046	[-.992; -.849]
γ_7	-.945	.030	[-.979; -.878]
γ_8	-.960	.027	[-.991; -.882]
γ_9	-.940	.031	[-.986; -.879]
σ^2	.840	.136	[.621; 1.158]