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EQUILIBRIUM STATES FOR CERTAIN PARTIALLY HYPERBOLIC SYSTEMS

ESTADOS DE EQUILÍBRIO PARA CERTOS SISTEMAS PARCIALMENTE HIPERBÓLICOS

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Advisor: José Régis Azevedo Varão Filho

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Resumo

Na primeira parte deste trabalho, estudamos certos difeomorfismos parcialmente hiperbólicos de $\mathbb{T}^d$ com folheações centrais bidimensionais compactas, para os quais mostramos que qualquer medida ergódica de máxima entropia é hiperbólica e existe no máximo um número finito (diferente de zero) delas. No caso de $\mathbb{T}^4$, podemos retirar a condição de compacidade para as folhas centrais e obter hiperbolicidade das medidas ergódicas de máxima entropia.

Também propomos estudar a desintegração de medidas ao longo de folheações centrais bidimensionais de uma classe de difeomorfismos parcialmente hiperbólicos de $\mathbb{T}^4$ isotópicos a um difeomorfismo de Anosov. Além disso, estudamos estados de equilíbrio ergódicos com relação a uma classe de potenciais, aproveitando as técnicas desenvolvidas para descrever a desintegração. Este é um trabalho em conjunto com Adriana Sánchez e Régis Varão.

Palavras-chaves: Estados de equlíbrio, Medidas de máxima entropia, Difeomorfismos parcialmente hiperbólicos, Medidas hiperbólicas, Desintegração de medidas.

Abstract

In the first part of this work, we study certain partially hyperbolic diffeomorphisms of $\mathbb{T}^d$ with compact two-dimensional center foliations, for which we show that any ergodic maximal entropy measure is hyperbolic and there exists at most a finite number (non-zero) of them. In the case of $\mathbb{T}^4$, we can remove the compactness condition for the center leaves and obtain hyperbolicity for ergodic maximal entropy measures.

We also propose to study the disintegration of measures along two-dimensional center foliations of a class of partially hyperbolic diffeomorphisms of $\mathbb{T}^4$ isotopic to an Anosov diffeomorphism. Moreover, we study ergodic equilibrium states with respect to a class of potentials, taking advantage of techniques developed for describing the disintegration. This is a joint work with Adriana Sánchez and Régis Varão.

Keywords: Equilibrium states, Maximal entropy measures, Partially hyperbolic diffeomorphisms, Hyperbolic measures, Disintegration of measures.

List of symbols

$A - B$	difference of two sets
$B(x, r)$	open ball centered at a point
$B_r^{\mathcal{F}}(x)$	open ball on the leaf at a point
$C^0(M, \mathbb{R})$	space of continuous functions
d_{C^0}	metric on the space of continuous functions
$\operatorname{diam} A$	diameter of a set
$dist_\sigma$	intrinsic metric in the corresponding leaf
$\operatorname{Diff}^r(M)$	space of C^r diffeomorphisms
$\operatorname{Diff}^\infty(M)$	space of C^∞ diffeomorphisms
$f \sim A$	isotopy between two diffeomorphisms
$\mathcal{F}^\sigma(x)$	leaf containing a point ($\sigma = s, c, u$)
$\mathcal{F}^\sigma_{loc}(x)$	local leaf containing a point ($\sigma = s, c, u$)
$\Gamma_\epsilon(x)$	bi-infinite Bowen ball at a point
$H_\mu(\mathcal{P})$	entropy of a partition
$H_1(M)$	first homology group
$HC(\mathcal{O})$	homoclinic class of a orbit
$h_\mu(f, \mathcal{P})$	metric entropy of a partition
$h_\mu(f)$	metric entropy

$h_\mu(f,\mathcal{F})$	partial entropy along a foliation
$h_{top}(f)$	topological entropy
$h^*_f(\epsilon)$	tail entropy at scale ϵ
Id	identity map
$\mathcal{O}_i \overset{h}{\sim} \mathcal{O}_j$	orbits homoclinically related
$\mathcal{M}(f)$	space of f-invariant probability measures
$\mathcal{M}_e(f)$	space of ergodic probability measures
$\mathcal{M}_h(f)$	space of hyperbolic ergodic probability measures
$\mathcal{P} \vee \mathcal{Q}$	sum of partitions
$\mathcal{P}^n$	iterated sum of a partition
$Per_h(f)$	set of hyperbolic periodic orbit of saddle type
$\mathsf{PH}(M)$	set of partially hyperbolic diffeomorphisms
supp μ	support of a measure
T_xM	tangent space at a point
TM	tangent bundle
$\mathbb{T}^d$	d-dimensional torus
$W^\sigma(\mathcal{O})$	stable/unstable manifold of a orbit ($\sigma = s,u$)
$W^\sigma_P(x)$	Pesin stable/unstable manifold ($\sigma = s,u$)
$W^\sigma_{loc}(x)$	local Pesin stable/unstable manifold ($\sigma = s,u$)
$\mathcal{X}_D$	characteristic function of a measurable subset
$\mathcal{X}_\mathcal{F}(x,f)$	the volume growth of a foliation at a point
$\mathcal{X}_\mathcal{F}(f)$	maximum volume growth rate of a foliation f-invariant

Contents

1 Introduction

The study of equilibrium states in dynamical systems was started by Sinai, Ruelle and Bowen [55, 54, 11] in the 1970s, taking advantage of techniques and results from statistical mechanics. Given a continuous map $f : M \to M$ over a compact metric space, an *equilibrium state* for a *continuous potential* $\phi : M \to \mathbb{R}$, is an f-invariant Borel probability measure μ that maximizes the quantity $h_\mu(f) + \int \phi d\mu$ among all f-invariant measures. In the case of uniformly hyperbolic systems, Bowen [10] showed existence and uniqueness of equilibrium states with respect to Hölder continuous potentials. Forty years later, Climenhaga and Thompson extended Bowen's techniques for a non-uniform setting [21] and these results have been applied for other classes of maps [16, 17, 27].

Another interesting class of non-hyperbolic systems very studied in smooth ergodic theory are *partially hyperbolic systems*, which are diffeomorphisms over compact manifolds with an invariant splitting of the tangent bundle in three subbundles E^s, E^c, E^u, such that vectors in E^s are contracted uniformly, vectors in E^u are expanded uniformly, and vectors in E^c lie in between those two, not quite as contracting nor as expanding, respectively (see 2.3.2 for a precise definition). In general the central bundle E^c may not be integrable [51], when it is, the partially hyperbolic diffeomorphism is *dynamically coherent*. There are many relevant works about equilibrium states for partially hyperbolic systems, we will cite some, Climenhaga, Pesin and Zelerowicz [20], Fisher and Oliveira [26] and Rios and Siqueira [48]; for a more complete study of equilibrium states for non-uniformly hyperbolic systems, see the survey of Climenhaga and Pesin [19].

When the potential is $\phi \equiv 0$, equilibrium states are called *maximal entropy measures*, such measures describe the complexity level of the whole system. In certain contexts, the existence of maximal entropy measures is not clear, however, it has already been proven for C^∞ diffeomorphisms on compact manifolds without boundary [42], and for robust classes of local diffeomorphisms [43]. Recently in [12], Buzzi, Crovisier and Sarig showed that C^∞ surface diffeomorphisms with positive topological entropy have at most finitely many maximal entropy measures and exactly one in the transitive case.

For the setting of partially hyperbolic systems, results in [24] imply that there exists at least one maximal entropy measure if the center bundle is one-dimensional. For $C^{1+\alpha}$ accessible partially hyperbolic over 3-manifolds having compact one-dimensional central leaves, a dichotomy was proved about the number of maximal entropy measures [50, 58]; Rocha and Tahzibi also obtained a similar dichotomy for partially hyperbolic diffeomorphisms defined on 3-torus with compact center leaves [49]. Certain partially hyperbolic systems with one-dimensional center bundles and isotopic to an Anosov diffeomorphism (*derived from Anosov* (DA)), have a unique maximal entropy measure [13, 28, 57].

There are very few works on equilibrium states for partially hyperbolic systems with central dimension greater than one. Among these studies, partially hyperbolic systems whose central bundles splits in a dominated way into one-dimensional subbundles, admit equilibrium states for any continuous potential [25], and partially hyperbolic systems with two-dimensional center bundles, produced by an isotopy from an Anosov (*mixed derived from Anosov*) have a unique maximal entropy measure [13].

1.1 Hyperbolic ergodic maximal entropy measures

The Lyapunov exponents are real numbers, which measure the exponential growth of the derivative of dynamical systems. For every $f \in \mathrm{Diff}^r(M)$ over a compact m-manifold, and an ergodic probability measure μ, by Oseledets' theorem there are m numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m$, and a splitting $T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_m(x)$ such that for μ-almost every $x \in M$ and for any $v \in TM - \{0\}$, we have $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda_i$, for some $1 \leqslant i \leqslant m$. The m-numbers λ_i are the *Lyapunov exponents* of (f, μ). An interesting question is to know under what conditions an ergodic maximal entropy is *hyperbolic*, that is, it has non-zero Lyapunov exponents.

In view of the lack of results about maximal entropy measures for partially hyperbolic systems with two-dimensional center bundles, in this work we propose a scenario in which every ergodic maximal entropy measure is hyperbolic and there is at most a non-zero finite number of them.

Let $A : \mathbb{T}^d \to \mathbb{T}^d$ be a linear Anosov admitting a dominated splitting of the form $E_A^{ss} \oplus E_A^{ws} \oplus E_A^{wu} \oplus E_A^{uu}$, with $E_A^c = E_A^{ws} \oplus E_A^{wu}$ and $\dim E_A^{ws} = E_A^{wu} = 1$. We consider the set of partially hyperbolic diffeomorphisms isotopic to A, all of them having the same dimension of stable and unstable bundle, that is,

$$\mathsf{PH}_{A,s,u}(\mathbb{T}^d) = \{f \in \mathsf{PH}(\mathbb{T}^d) : f \sim A,\ \dim E_f^s = \dim E_A^{ss},\ \dim E_f^u = \dim E_A^{uu}\}.$$

Here $f \sim A$ denotes an isotopy between f and A. To simplify notation we will denote $\mathsf{PH}_{A,s,u}(\mathbb{T}^d)$ as $\mathsf{PH}_A(\mathbb{T}^d)$, where the dimension of the bundles is implicitly understood.

Given $f \in \mathsf{PH}_A(\mathbb{T}^d)$ we know from [29, 40] there exists the Franks-Manning semiconjugacy $H : \mathbb{T}^d \to \mathbb{T}^d$. The map H varies continuously with f in the C^0 topology. This is a general fact which does not require f to be partially hyperbolic.

Now we consider $\mathsf{PH}^0_A(\mathbb{T}^d)$ to be the connected component of $\mathsf{PH}_A(\mathbb{T}^d)$ containing A. If A admits a foliation by tori $\mathbb{T}^2$ tangent to E^c_A, then every $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ has central foliation with all leaves compact (see [28]).

Let $A_c : \mathbb{T}^d/\mathcal{F}^c_A \to \mathbb{T}^d/\mathcal{F}^c_A$ be the corresponding factor to the linear Anosov A. In the previous setting, we obtain the following results:

Theorem A ([3]). *Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$ and let $k_0 := h_{top}(A_c)$. If μ is an ergodic measure such that $h_\mu(f) > k_0$, then*

1. *μ is hyperbolic, meaning that all its Lyapunov exponents are non-zero. In particular any maximal entropy measure is hyperbolic, provided that it exists.*
2. *For every $\epsilon > 0$, there exists a hyperbolic set $B_\epsilon \subset M$ such that $h_{top}(f|_{B_\epsilon}) > h_\mu(f) - \epsilon$.*

Theorem B ([3]). *Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$ admitting a dominated splitting of the form $E^s_f \oplus E^{c_1}_f \oplus E^{c_2}_f \oplus E^u_f$, where $E^{c_1}_f, E^{c_2}_f$ are one-dimensional. Then, f has a finite number (non-zero) of ergodic maximal entropy measures and all are hyperbolic.*

For $d = 4$ and $\dim E^{ws} = \dim E^{wu} = 1$. We can consider the center foliations of A compact or not, and obtain hyperbolicity of ergodic maximal entropy measures:

Theorem C ([4]). *If $f \in \mathsf{PH}^0_A(\mathbb{T}^4) \cap \mathrm{Diff}^\infty(\mathbb{T}^4)$, then every ergodic maximal entropy measure of f is hyperbolic.*

1.2 Disintegration and equilibrium states

In this work, we are also concerned with equilibrium states in a less explored context of partially hyperbolic diffeomorphisms with higher dimensional center foliation (i.e. two-dimensional or higher). Using disintegration techniques of measures along the center foliation, combined with a quotient process to face the higher dimension problem, we obtain similar results to those Crisostomo and Tahzibi [22].

For a dynamically coherent partially hyperbolic diffeomorphism, we say an invariant measure has *atomic disintegration* if there exists a full measure invariant subset which intersects each center leaf in at most a countable set. Furthermore, if each of these countable sets is finite with k points, we say the invariant measure is *virtually hyperbolic*.

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a DA partially hyperbolic diffeomorphism satisfying the following conditions:

A.1 f is dynamically coherent;

A.2 There exists a splitting $E^c = E^1 \oplus E^2$ where each E^i is a line-bundle and integrates to an f-invariant foliation $\mathcal{F}^i$ (non-compact), for $i = 1, 2$;

A.3 If $z, z' \in H^{-1}(x)$ and $z' \in \mathcal{F}^i(z)$ for some $1 \leqslant i \leqslant 2$, then $[z, z']_i \subset H^{-1}(x)$, where $[z, z']_i$ is the closed interval inside $\mathcal{F}^i(z)$ with end points z and z';

A.4 For each $x \in \mathbb{T}^d$, $H^{-1}(x)$ is a finite union of rectangles contained in a unique center leaf of $\mathcal{F}^c$;

A.5 $h(f, H^{-1}(x)) = 0$, for every $x \in \mathbb{T}^d$.

The rectangles are considered as in Section 4.1.

The previous assumptions are satisfied by the maps considered by Buzzi, Fisher, Sambarino and Vásquez [13] and by Carrasco, Lizana, Pujals and Vásquez [14]. In the spirit of [45, 46], we denote by C the set where H fails to be injective and we get a result about disintegration of measures along two-dimensional center foliations:

Theorem D ([5])**.** *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a DA partially hyperbolic diffeomorphism satisfying A.1, A.2, A.3 and A.4. Assume that f preserves the orientation for $\mathcal{F}^i$, $i = 1, 2$. Let μ be an ergodic probability for f:*

1. *If $\mu(C) = 0$, then (f, μ) is almost conjugate to an Anosov diffeomorphism.*

2. *If $\mu(C) = 1$, then C defines a partition such that μ is virtually hyperbolic.*

We also get a result about equilibrium states for the same class of potentials considered in [15, 22]:

Theorem E ([5])**.** *Let $f : \mathbb{T}^4 \to \mathbb{T}^4$ be a DA partially hyperbolic diffeomorphism satisfying A.1, A.2, A.3, A.4 and A.5. Assume that f preserves the orientation of $\mathcal{F}^i$, $i = 1, 2$. Let ϕ be a continuous potential such that (A, ϕ) has a unique equilibrium state with full support and define the potential $\varphi = \phi \circ H$. For every μ ergodic equilibrium state of f with respect to φ:*

1. *If $\mu(C) = 0$, then μ is the unique equilibrium state.*

2. *If $\mu(C) = 1$, then C defines a partition such that μ has atomic disintegration with a finite number of atoms. Moreover, if the semiconjugacy H sends center leaves of f to center leaves of A and one of the following conditions is satisfied*

   a) The center direction of A is expanding or contractive.

   b) $H(\mathcal{F}^i)$ is some invariant foliation of A, for each $i = 1, 2$.

Then, μ is virtually hyperbolic (one only atom per leaf) and is not a unique equilibrium state for φ.

1.3 Structure of the thesis

The organization of this thesis is as follows:

- In Chapter 2, we introduce some concepts and results about equilibrium states, homoclinic classes, hyperbolic measures, Pesin theory, partially hyperbolic diffeomorphisms and disintegration of measures.
- In Chapter 3, we study maximal entropy measures for partially hyperbolic diffeomorphisms isotopic to an Anosov diffeomorphism, where the isotopy is contained in the set of partially hyperbolic diffeomorphisms, and prove Theorems A, B, C.
- In Chapter 4, we study the disintegration of measures along center leaves and equilibrium states for a class of partially hyperbolic diffeomorphisms isotopic to an Anosov diffeomorphism and prove Theorems D, E.

2 Preliminaries

The goal of this chapter is to provide results of ergodic theory and partially hyperbolic systems that are required for the development of this work.

2.1 Equilibrium states

Let $(M, dist)$ be a compact metric space, $\mathfrak{B}$ be the Borel σ-algebra, and μ be a borelian probability measure. Let $f : M \to M$ be a measurable transformation, the probability measure μ is called **f-invariant** if $f_*\mu(B) = \mu(f^{-1}(B)) = \mu(B)$, for all $B \in \mathfrak{B}$.

We denote by $\mathcal{M}(f)$ the set of f-invariant Borel probability measures on M. By compactness of M, this set is non-empty compact set in the weak*-topology. The **support** of $\mu \in \mathcal{M}(f)$ is the set $\operatorname{supp}\mu$ formed by the points $x \in M$ such that $\mu(V) > 0$ for any neighborhood V of x.

Definition 2.1.1. *An f-invariant probability measure μ is said to be **ergodic** if the only measurable sets A with $f^{-1}(A) = A$ satisfy $\mu(A) = 0$ or $\mu(A) = 1$. We denote by $\mathcal{M}_e(f)$ the set of ergodic probability measures.*

Now, we define the notion of entropy with respect to a finite partition $\mathcal{P}$ of M.

Definition 2.1.2. *The entropy of $\mathcal{P}$ is defined by*

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P).$$

Definition 2.1.3. *We say that a partition $\mathcal{P}$ of M is **measurable** with respect to μ if there exist a measurable family $\{A_i\}_{i\in\mathbb{N}}$ and a measurable set C of full measure such that if $B \in \mathcal{P}$, then there exists a sequence $\{B_i\}_{i\in\mathbb{N}}$, where $B_i \in \{A_i, A_i^c\}$ such that $B \cap C = \bigcap_{i\in\mathbb{N}} B_i \cap C$.*

Before presenting the notion of entropy for a dynamical system (f, μ), we introduce some notations.

Let $\mathcal{P}^n := \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}) = \bigvee_{i=0}^{n-1}\{f^{-i}(P) : P \in \mathcal{P}\}$ for every $n \geqslant 1$. The elements of $\mathcal{P}^n$ are of the form $\mathcal{P}^n(x) = \mathcal{P}(x) \cap f^{-1}(\mathcal{P}(f(x))) \cap \ldots \cap f^{-n+1}(\mathcal{P}(f^{n-1}(x)))$.

Definition 2.1.4. *Let $f : M \to M$ be a measurable transformation preserving a probability measure μ. The metric entropy of f with respect to μ and a measurable partition $\mathcal{P}$ of M is defined by*

$$h_\mu(f, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^n) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

The ***metric entropy*** *of (f, μ) is defined by*

$$h_\mu(f) = \sup\{h_\mu(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } M\}.$$

The ergodic decomposition theorem asserts that every invariant measure is a convex combination of ergodic measures; in particular, it permits the reduction of the proof of many results to the case when the system is ergodic.

Theorem 2.1.5 (Ergodic decomposition)**.** *Let M be a complete separable metric space, $f : M \to M$ be a measurable transformation and μ an f-invariant probability measure. Then there exist a measurable set $M_0 \subset M$ with $\mu(M_0) = 1$, a partition $\mathcal{P}$ of M_0 into measurable subsets and a family $\{\mu_P : P \in \mathcal{P}\}$ of probability measures on M, satisfying*

1. *$\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;*
2. *$P \mapsto \mu_P(E)$ is measurable, for every measurable set $E \subset M$;*
3. *μ_P is invariant and ergodic for $\hat{\mu}$-almost every $P \in \mathcal{P}$;*
4. *$\mu(E) = \int \mu_P(E) d\hat{\mu}(P)$, for every measurable set $E \subset M$.*

Proof. See [59, Theorem 5.1.3]. □

We now present the well-known Jacobs' formula; it generalizes the affine property of the metric entropy for ergodic decomposition.

Theorem 2.1.6 (Jacobs)**.** *Let M be a complete separable metric space, $f : M \to M$ be a measurable transformation and μ be an invariant probability measure. If $\{\mu_P : P \in \mathcal{P}\}$ is the ergodic decomposition of μ, then $h_\mu(f) = \int h_{\mu_P}(f) d\hat{\mu}(P)$ (if one side is infinite, so is the other side).*

Proof. See [59, Theorem 9.6.2]. □

Let $f : M \to M$ be a continuous transformation over a compact metric space. Let $n \in \mathbb{N}$, $\epsilon > 0$ and $K \subset M$ a non-empty compact set. A subset $E \subseteq K$ is said to be (n, ϵ)-**separated**, if for $x, y \in E, x \neq y$, there exists $i \in \{0, 1, \dots, n-1\}$ such that $dist(f^i x, f^i y) \geqslant \epsilon$.

Definition 2.1.7. *The **topological entropy** of f on the non-empty compact set $K \subset M$ is defined by*

$$h(f, K) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup\{\#E : E \subseteq K \text{ is } (n, \epsilon)\text{-separated}\}.$$

We denote $h_{top}(f) := h(f, M)$.

The metric entropy describes the complexity of a dynamical system with respect to an invariant probability measure, and the topological entropy measures the complexity level of the whole system. The following result establishes a relationship between the two types of entropy.

Theorem 2.1.8 (Variational principle)**.** *If $f : M \to M$ is a continuous transformation over a compact metric space, then*

$$h_{top}(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}(f)\}.$$

Proof. See [59, Theorem 10.1]. □

Theorem 2.1.9 (Ledrappier-Walters Variational principle [35])**.** *Let M and N be compact metric spaces and $f : M \to M,\ g : N \to N,\ \pi : M \to N$ be continuous transformations such that π is surjective and $\pi \circ f = g \circ \pi$. Then*

$$\sup_{\mu : \pi_* \mu = \nu} h_\mu(f) = h_\nu(g) + \int_N h(f, \pi^{-1}(y)) d\nu(y).$$

Proof. See [35, Theorem 2.1]. □

Definition 2.1.10. *An f-invariant Borel probability measure μ is an **equilibrium state** for f with respect to a potential $\phi \in C^0(M, \mathbb{R})$ if it satisfies*

$$h_\mu(f) + \int \phi d\mu = \sup\{h_\nu(f) + \int \phi d\nu : \nu \in \mathcal{M}(f)\}.$$

*When $\phi \equiv 0$, any $\mu \in \mathcal{M}(f)$ such that $h_\mu(f) = \sup\{h_\nu(f) : \nu \in \mathcal{M}(f)\}$ is called **maximal entropy measure**.*

Remark 2.1.11. *In definition 2.1.10, we can change $\mathcal{M}(f)$ by $\mathcal{M}_e(f)$. This is a consequence of Theorem 2.1.6 and Theorem 2.1.5.*

Let $f : M \to M$ be a continuous transformation over a compact metric space, we define the entropy function that is denoted by $h : \mathcal{M}(f) \to [0, \infty)$ and defined by $h(\mu) := h_\mu(f)$.

Proposition 2.1.12. *If the entropy function h is upper semi-continuous, then f has an equilibrium state with respect to any potential $\phi \in C^0(M, \mathbb{R})$. Moreover, the set of equilibrium states for (f, ϕ) is compact and convex subset of $\mathcal{M}(f)$.*

Proof. See [59, Proposition 10.5.5 and Lemma 10.5.8]. □

Let $f : M \to M$ be a homeomorphism over a compact metric space, the **bi-infinite Bowen ball** around $x \in M$ of size $\epsilon > 0$ is the set

$$\Gamma_\epsilon(x) := \{y \in M : dist(f^n x, f^n y) < \epsilon \text{ for all } n \in \mathbb{Z}\}.$$

Definition 2.1.13. *We say that f is* ***expansive*** *if there is a constant $\epsilon > 0$ such that $\Gamma_\epsilon(x) = \{x\}$ for all $x \in M$. When f is not expansive, it is useful to consider the* ***tail entropy of f at scale*** *$\epsilon > 0$:*

$$h_f^*(\epsilon) := \sup_{x \in M} h(f, \Gamma_\epsilon(x)).$$

Definition 2.1.14. *We say that f is* ***h-expansive at scale*** *ϵ if $h_f^*(\epsilon) = 0$. When $\lim_{\epsilon \to 0} h_f^*(\epsilon) = 0$, we say that f is* ***asymptotically h-expansive***.

Theorem 2.1.15 (Misiurewicz [41])**.** *If $f : M \to M$ is asymptotically h-expansive, then the entropy function is upper semi-continuous.*

Proof. See [41, Theorem 4.2]. □

Corollary 2.1.16. *If $f : M \to M$ is asymptotically h-expansive, then f has an equilibrium state with respect to any potential $\phi \in C^0(M, \mathbb{R})$.*

2.2 Homoclinic classes and hyperbolic measures

For $r \geqslant 1$, we denote by $\mathrm{Diff}^r(M)$ the set of C^r diffeomorphism over a compact Riemannian manifold. Given $f \in \mathrm{Diff}^r(M), x \in M$, and $v \in T_x M - \{0\}$, the **Lyapunov exponent** of f at x in direction v is the exponential growth rate of Df along v, that is,

$$\lambda(x, v) := \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df_x^n(v)\|$$

in case both limits exist and coincide.

Definition 2.2.1. *Let $f \in \mathrm{Diff}^r(M)$. A* ***hyperbolic set*** *for f is a compact f-invariant set $\Lambda \subset M$ with a decomposition $T_x M = E^s(x) \oplus E^u(x)$ for all $x \in \Lambda$ such that for some $C > 0$ and $\lambda \in (0, 1)$, for all $x \in \Lambda$, $n \geqslant 0$, $v^s \in E^s(x)$ and $v^u \in E^u(x)$, we have*

$$\|Df_x^n(v^s)\| \leqslant C\lambda^n \|v^s\| \text{ and } \|Df_x^{-n}(v^u)\| \leqslant C\lambda^n \|v^u\|.$$

When $\Lambda = M$, f is called **Anosov diffeomorphism**. Given $x \in M$, define

$$W^u(x) := \{y \in M : \lim_{n\to\infty} dist(f^{-n}(x), f^{-n}(y)) = 0\},$$
$$W^s(x) := \{y \in M : \lim_{n\to\infty} dist(f^{n}(x), f^{n}(y)) = 0\}.$$

Recall that if Λ is a hyperbolic set, these sets are C^r sub-manifolds (see [31]), called **unstable** and **stable** manifold, respectively. The periodic orbit of x and period $n \geqslant 1$ for f is denoted by $\mathcal{O} := \mathcal{O}(x) = \{f^i(x) : i = 0, \ldots, n-1\}$.

Definition 2.2.2. *Let $f \in \mathrm{Diff}^r(M)$. The orbit of a f-periodic point x is **hyperbolic of saddle type** $\mathcal{O}$ if x has a positive Lyapunov exponent, a negative Lyapunov exponent and no zero Lyapunov exponents.*

We denote by $Per_h(f)$ the set of hyperbolic periodic orbit of saddle type. For $\mathcal{O} \in Per_h(f)$ we also define the unstable (stable) manifold for a whole orbit as

$$W^u(\mathcal{O}) = \bigcup_{x\in\mathcal{O}} W^u(x) \text{ and } W^s(\mathcal{O}) = \bigcup_{x\in\mathcal{O}} W^s(x).$$

Definition 2.2.3. *Two orbits $\mathcal{O}_i, \mathcal{O}_j \in Per_h(f)$ are called **homoclinically related** if for $i \neq j$*

$$W^u(\mathcal{O}_i)\pitchfork W^s(\mathcal{O}_j) \neq \varnothing, \quad W^u(\mathcal{O}_j)\pitchfork W^s(\mathcal{O}_i) \neq \varnothing.$$

We write $\mathcal{O}_i \overset{h}{\sim} \mathcal{O}_j$.

Definition 2.2.4. *The **homoclinic class** of $\mathcal{O}$ is the set*

$$HC(\mathcal{O}) = \overline{\{\hat{\mathcal{O}} \in Per_h(f) : \hat{\mathcal{O}} \overset{h}{\sim} \mathcal{O}\}}.$$

Let us present some results about Pesin theory that will be used in some parts this thesis. For further references, see the book by Barreira and Pesin [8] and Katok's paper [33].

Theorem 2.2.5 (Oseledets)**.** *Let $f \in \mathrm{Diff}^1(M)$ and $\mu \in \mathcal{M}(f)$. There exists a set $\mathcal{R}_f \subset M$ with $\mu(\mathcal{R}_f) = 1$, such that for every $\epsilon > 0$ it exists a measurable function $C_\epsilon : \mathcal{R}_f \to (1, \infty)$ with the following properties:*

1. *For any $x \in \mathcal{R}_f$ there are numbers $1 \leqslant l(x) \leqslant \dim M$, $l(x)$ Lyapunov exponents $\lambda_1(x) < \cdots < \lambda_{l(x)}(x)$ and a decomposition $T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{l(x)}(x)$;*
2. *$l(f(x)) = l(x)$, $\lambda_i(f(x)) = \lambda_i(x)$ and $Df_xE_i(x) = E_i(f(x))$, for every $i = 1, \cdots, l(x)$;*
3. *For every $v \in E_i(x) - \{0\}$ and $n \in \mathbb{Z}$*

$$C_\epsilon(x)^{-1}e^{n(\lambda_i(x)-\epsilon)}\|v\| \leqslant \|Df^n_x(v)\| \leqslant C_\epsilon(x)e^{n(\lambda_i(x)+\epsilon)}\|v\| \text{ and } \lambda(x,v) = \lambda_i(x);$$

4. *The angle between* $E_i(x)$ *and* $E_j(x)$ *is greater than* $C_\epsilon(x)^{-1}$, *if* $i \neq j$;

5. $C_\epsilon(f(x)) \leqslant e^\epsilon C_\epsilon(x)$.

Proof. See [8, Theorem 5.4.1]. □

Remark 2.2.6. *Every* $x \in \mathcal{R}_f$, *given by the previous theorem, is called a* ***regular point****. If* $\mu \in \mathcal{M}_e(f)$, *then the Lyapunov exponents and* $\dim E_i(x)$ *are constants* μ*-a.e.*

Given $\epsilon > 0$ and $l \in \mathbb{N}$ we define the **Pesin block**

$$\mathcal{R}_{\epsilon,l} = \{x \in \mathcal{R}_f : C_\epsilon(x) \leqslant l\}.$$

Note that Pesin blocks are not necessarily invariant. However $f(\mathcal{R}_{\epsilon,l}) \subset \mathcal{R}_{\epsilon,e^\epsilon l}$ and for each $\epsilon > 0$, we have that

$$\mathcal{R}_f = \bigcup_{l \in \mathbb{N}} \mathcal{R}_{\epsilon,l}.$$

Recall that every $\mathcal{R}_{\epsilon,l}$ is compact. For all $x \in \mathcal{R}_f$ we have

$$T_xM = \bigoplus_{\lambda_i(x)<0} E_i(x) \bigoplus E^0(x) \bigoplus_{\lambda_i(x)>0} E_i(x)$$

where $E^0(x)$ is the subspace generated by the vectors having zero Lyapunov exponents.

Definition 2.2.7. *An* f*-invariant probability measure* μ *is called* ***hyperbolic*** *if all its Lyapunov exponents are non-zero and there exist Lyapunov exponents with different signs. We denote* $\mathcal{M}_h(f)$ *the set of hyperbolic ergodic measures.*

Definition 2.2.8. *Let* μ *be a hyperbolic ergodic measure. We define the* ***stable index*** *or* s*-**index** (**unstable index** or* u*-**index**) of* μ *as the number of negative (positive) Lyapunov exponents. Here the exponents are counted with multiplicity.*

Definition 2.2.9. *For* $f \in \mathrm{Diff}^r(M)$ *with* $r > 1$, *the* ***stable Pesin manifold*** *of the point* $x \in \mathcal{R}_f$ *is*

$$W^s_P(x) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log dist(f^n(x), f^n(y)) < 0\}.$$

Similarly one defines the ***unstable Pesin manifold*** *as*

$$W^u_P(x) = \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log dist(f^{-n}(x), f^{-n}(y)) < 0\}.$$

Remark 2.2.10. *Stable and unstable Pesin manifolds of points in* $\mathcal{R}_f$ *are immersed submanifolds* [44, Section 4]. *The usual Pesin theory requires a* $C^{1+\alpha}$ *regularity* ($\alpha > 0$) *of the dynamics* [47], *in the case of* C^1 *regularity, we can use dominated Pesin theory developed in* [1, 6].

We denote by $W^s_{loc}(x)$ the connected component of $W^s_P(x) \cap B(x,r)$ containing x, where $B(x,r)$ denotes the Riemannian ball of center x and radius $r > 0$, which is sufficiently small but fixed.

Theorem 2.2.11 (Stable Pesin Manifold Theorem [44])**.** *Let $r > 1$ and $f \in \mathrm{Diff}^r(M)$ be a diffeomorphism preserving a smooth measure m. Then, for each $l > 1$ and small $\epsilon > 0$, if $x \in \mathcal{R}_{\epsilon,l}$:*

1. *$W^s_{loc}(x)$ is a disk such that $T_x W^s_{loc}(x) = \bigoplus_{\lambda_i(x)>0} E_i(x)$;*
2. *$x \mapsto W^s_{loc}(x)$ is continuous over $\mathcal{R}_{\epsilon,l}$ in the C^1-topology.*

Proof. See [44, Theorem 2.2.1]. □

In particular, the dimension of the disk $W^s_{loc}(x)$ equals the number of negative Lyapunov exponents of x respect to m. An analogous statement holds for the unstable Pesin manifold.

Let $\mathcal{O} \in Per_h(f)$ and $\mu \in \mathcal{M}_h(f)$. We write $\mathcal{O} \overset{h}{\sim} \mu$ when $W^u_P(x) \pitchfork W^s_P(\mathcal{O}) \neq \varnothing$ and $W^s_P(x) \pitchfork W^u_P(\mathcal{O}) \neq \varnothing$ for μ-almost every x as in Definition 2.2.2. The previous definition describes the homoclinic relation between μ and $\mathcal{O}$,

Katok's Horseshoe Theorem [33] gives an important characterization for hyperbolic ergodic measures with positive entropy.

Theorem 2.2.12 (Katok, [33])**.** *Suppose $f \in \mathrm{Diff}^r(M)$, $r > 1$. Let μ be an f-invariant ergodic and hyperbolic measure such that $h_\mu(f) > 0$. Then for every $\epsilon > 0$, there exists a hyperbolic set $B_\epsilon \subset M$ such that*

$$h_{top}(f|_{B_\epsilon}) > h_\mu(f) - \epsilon.$$

Proof. See [8, Theorem 15.6.1]. □

Corollary 2.2.13 (Katok, [34])**.** *Let $f \in \mathrm{Diff}^r(M)$ with $r > 1$, and $\mu \in \mathcal{M}_h(f)$. Then, there exists $\mathcal{O} \in Per_h(f)$ such that $\mathcal{O} \overset{h}{\sim} \mu$ and*

$$HC(\mathcal{O}) = \overline{\bigcup \{\operatorname{supp} \nu : \nu \in \mathcal{M}_h(f),\ \mathcal{O} \overset{h}{\sim} \nu\}}.$$

Proof. See [8, Corollary 15.4.9]. □

2.3 Partially hyperbolic diffeomorphisms

In this section M is a compact Riemannian manifold. We begin by presenting the definition of dominated splitting for a diffeomorphism over M.

Definition 2.3.1. *A diffeomorphism $f : M \to M$ has a* ***dominated splitting*** *if there are an invariant splitting $TM = E_1 \oplus \cdots \oplus E_k$, $k \geqslant 2$ (with no trivial subbundle), and an integer $l \geqslant 1$ such that for every $x \in M$, $i < j$, and unit vectors $u \in E_i(x)$ and $v \in E_j(x)$, one has*

$$\frac{\|Df_x^l(u)\|}{\|Df_x^l(v)\|} < \frac{1}{2}.$$

Definition 2.3.2. *A diffeomorphism $f : M \to M$ is called* ***partially hyperbolic*** *if the tangent bundle admits a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that there exists $N \in \mathbb{N}$ and $\lambda > 1$ verifying that for every $x \in M$ and unit vectors $v^\sigma \in E^\sigma(x)$ $(\sigma = s, c, u)$ we have*

- $\lambda\|Df_x^N(v^s)\| < \|Df_x^N(v^c)\| < \lambda^{-1}\|Df_x^N(v^u)\|$, *and*
- $\|Df_x^N(v^s)\| < \lambda^{-1} < \lambda < \|Df_x^N(v^u)\|$.

Remark 2.3.3. *For partially hyperbolic diffeomorphisms, it is a well-known fact that there are foliations $\mathcal{F}^\sigma$ tangent to the subbundles E^σ for $\sigma = s, u$. Not always the central bundle E^c may be tangent to an invariant foliation, but whenever such a foliation exists, it is denoted by $\mathcal{F}^c$.*

Definition 2.3.4. *A partially hyperbolic diffeomorphism $f : M \to M$ is called* ***dynamically coherent*** *if there exist invariant foliations $\mathcal{F}^{c\sigma}$ tangent to $E^{c\sigma} = E^c \oplus E^\sigma$ for $\sigma = s, u$, respectively.*

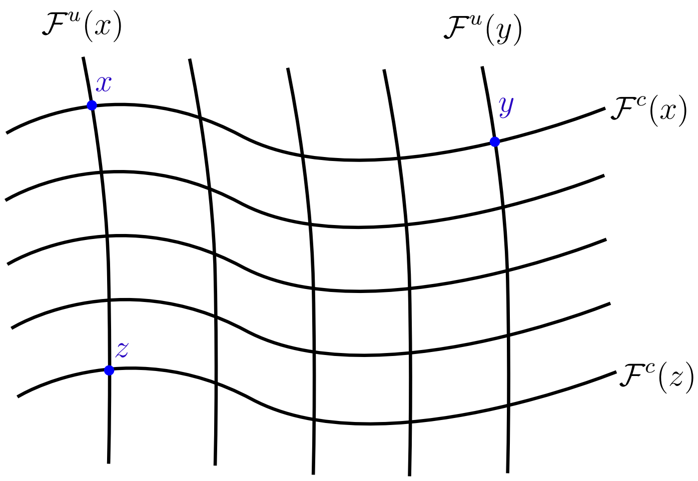


Figure 1 – Holonomies

Definition 2.3.5. *1. For any $f : M \to M$ partially hyperbolic diffeomorphism dynamically coherent and any two points x, y with $y \in \mathcal{F}^u(x)$, there exists a neighborhood U_x of x in $\mathcal{F}^c(x)$ and a homeomorphism $H^u_{x,y} : U_x \to \mathcal{F}^c(y)$ such that $H^u_{x,y}(x) = y$*

and $H^u_{x,y}(z) \in \mathcal{F}^u(z) \cap \mathcal{F}^c_{loc}(y)$. The homeomorphism, $H^u_{x,y}$, $x \in M$ and $y \in \mathcal{F}^u(x)$, is called ***local unstable holonomy****. Similarly, one may define* ***local stable holonomy*** *$H^s_{x,y}$ for $x \in M$ and $y \in \mathcal{F}^s(x)$.*

2. *We say that f admits global unstable holonomy if for any $y \in \mathcal{F}^u(x)$ the holonomy is defined globally $H^u_{x,y} : \mathcal{F}^c(x) \to \mathcal{F}^c(y)$. Similarly, we define the notion of global stable holonomy, and f admits* ***global holonomies*** *when it admits global stable and unstable holonomies (see Figure 1).*

Definition 2.3.6. *A foliation $\mathcal{F}$ is called* ***quasi-isometric*** *if there exists a constant $Q > 0$ such that, after lifting $\mathcal{F}$ to the universal cover, for any two points x, y in the same leaf, one has*

$$dist_{\mathcal{F}}(x, y) \leqslant Qdist(x, y)$$

where $dist_{\mathcal{F}}$ and $dist$ are respectively the distance along the leaf and the distance on the universal cover.

Definition 2.3.7. *A C^1 partially hyperbolic diffeomorphism $f : M \to M$ is called* ***derived from Anosov*** *(DA) if it is isotopic to an Anosov diffeomorphism. In the case $M = \mathbb{T}^d$, then f is isotopic to its action in the homology $A : H_1(\mathbb{T}^d) \to H_1(\mathbb{T}^d)$. We call A the linear part of f.*

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a diffeomorphism isotopic to a hyperbolic automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$. By a classical result due to Franks-Manning [29, 40] there exists a continuous surjection $H : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to the identity such that

$$A \circ H = H \circ f. \tag{2.3.1}$$

Moreover, its lift $\tilde{H}$ to $\mathbb{R}^d$ is a proper function that semiconjugates $\tilde{f}$ with $\tilde{A}$, and for some constant $K > 0$, we have

$$\|\tilde{H} - Id\|_{C^0} \leqslant K.$$

Remark 2.3.8. *Let $A : \mathbb{T}^d \to \mathbb{T}^d$ be a linear Anosov automorphism admitting a dominated splitting of the form $E^{ss}_A \oplus E^{ws}_A \oplus E^{wu}_A \oplus E^{uu}_A$. We denote as $E^s_A = E^{ss}_A \oplus E^{ws}_A$, $E^c_A = E^{ws}_A \oplus E^{wu}_A$ and $E^u_A = E^{wu}_A \oplus E^{uu}_A$. We denote by $\mathsf{PH}_{A,s,u}(\mathbb{T}^d) \subset \mathsf{PH}(\mathbb{T}^d)$ the subset of those which are isotopic to A and whose splitting verifies $\dim E^\sigma_f = \dim E^{\sigma\sigma}_A$ for $\sigma \in \{s, u\}$, and denote by $\mathsf{PH}^0_A(\mathbb{T}^d)$ to be the connected component of $\mathsf{PH}_{A,s,u}(\mathbb{T}^d)$ containing the linear Anosov A. Note that can also be characterized as the connected component containing a dynamically coherent and center-fibered partially hyperbolic diffeomorphism* (see [28, Subsection 1.3]).

Theorem 2.3.9 (Fisher-Potrie-Sambarino [28])**.** *Every $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ is dynamically coherent and different center leaves of f are sent by H to different center leaves of A.*

Proof. See [28, Theorem A]. ∎

2.4 Disintegration of measures

Let $(M, \mathfrak{B}, \mu)$ be a probability space and $\mathcal{P}$ a partition of M into measurable subsets. We consider $\pi : M \to M/\mathcal{P}$ the canonical projection that assigns to each point $x \in M$ the element $\mathcal{P}(x)$ of the partition that contains it. This projection map endows $M/\mathcal{P}$ with the structure of a probability space, as follows. We define $\hat{\mathfrak{B}} = \pi_* \mathfrak{B}$ and $\hat{\mu}$ by $\hat{\mu} = \pi_* \mu$. Then, $(M/\mathcal{P}, \hat{\mathfrak{B}}, \hat{\mu})$ is a probability space.

Definition 2.4.1. *A **disintegration** of μ with respect to $\mathcal{P}$ is a family $\{\mu_P\}_{P \in \mathcal{P}}$ of conditional probability measures on M such that, for every measurable set $E \subset M$:*

1. *$\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;*
2. *the function $\mathcal{P} \to \mathbb{R}$, defined by $P \mapsto \mu_P(E)$ is measurable;*
3. *$\mu(E) = \int \mu_P(E) d\hat{\mu}(P)$.*

*When it is clear which partition we are referring to, we say that the family $\{\mu_P\}$ **disintegrates** the measure μ.*

Proposition 2.4.2. *If $\{\mu_P\}_{P \in \mathcal{P}}$ and $\{\tilde{\mu}_P\}_{P \in \mathcal{P}}$ are disintegrations of μ with respect to $\mathcal{P}$, then $\mu_P = \tilde{\mu}_P$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$.*

Proof. See [59, Proposition 5.1.7]. □

The previous proposition asserts that disintegrations are essentially unique, when they exist. Consequently, for an invariant measure it follows that:

Corollary 2.4.3. *If $f : M \to M$ preserves μ and the partition $\mathcal{P}$, then $f_* \mu_P = \mu_{f(P)}$ $\hat{\mu}$-a.e.*

The next theorem guarantees the existence of disintegration with respect to a measurable partition.

Theorem 2.4.4 (Rokhlin's Disintegration [52])**.** *Let $\mathcal{P}$ be a measurable partition of a compact metric space M and μ a borelian probability measure. Then, μ admits some disintegration with respect to $\mathcal{P}$.*

Proof. See [59, Theorem 5.1.11]. □

In the context of partially hyperbolic dynamics the partition by leaves of a foliation, in general, may be non-measurable (see [18]). Thus, by disintegration of a measure along the leaves of a foliation we mean the disintegration on compact foliated boxes.

Definition 2.4.5. *We say that a foliation $\mathcal{F}$ has* ***atomic disintegration*** *with respect to a measure μ if the conditional measures on any foliated box are sum of Dirac measures. In other words, there exists a full measurable subset Z such that Z intersects all leaves in at most a countable set.*

Remark 2.4.6. *Even though the disintegration of a measure along a general foliation is defined in compact foliated boxes, it makes sense to say that a foliation $\mathcal{F}$ has a quantity $k \in \mathbb{N}$ of atoms per leaf. The meaning of "per leaf" should always be understood as a generic leaf, i.e. almost every leaf. That means that there is a set A of μ-full measure which intersects a generic leaf on exactly k points.*

Definition 2.4.7. *Let $f \in \mathsf{PH}(\mathbb{T}^d)$ be a dynamically coherence DA. We say that an f-invariant measure is* ***virtually hyperbolic*** *if there is a full measure set which intersects the center leaf in at most k points.*

Remark 2.4.8. *If μ is a virtually hyperbolic measure, then the conditional measures along center leaves are Dirac measures, and the central foliation is measurable with respect to μ, because the partition into central leaves is equivalent to the partition into points. Ponce, Tahzibi and Varão [45] studied examples of DA partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ with one-dimensional center foliation and volume measure virtually hyperbolic.*

3

Maximal entropy measures for diffeomorphisms isotopic to an Anosov diffeomorphism

Let us consider $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$, where $A : \mathbb{T}^d \to \mathbb{T}^d$ is a linear Anosov automorphism admitting a dominated splitting of the form $E^{ss}_A \oplus E^c_A \oplus E^{uu}_A$ with $E^c_A = E^{ws}_A \oplus E^{wu}_A$.

3.1 Known results

Our goal is to understand maximal entropy measures for partially hyperbolic diffeomorphisms under the above conditions. Before that, we present some useful results for developing the proofs of the main theorems on this chapter.

Theorem 3.1.1 (Fisher-Potrie-Sambarino [28])**.** *If $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ and $\dim E^c_f = 1$, then f has a unique maximal entropy measure.*

Proof. See [28, Corollary C]. □

Theorem 3.1.2 (Roldán [53])**.** *If $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ and $\dim E^c_f = 1$, then the unique maximal entropy measure of f is hyperbolic.*

Proof. See [53, Theorem A]. □

The following results can be found in [56]. Let $f : M \to M$ be a C^2 partially hyperbolic diffeomorphism over a compact manifold satisfying the following conditions:

H.1 f is dynamically coherent with all center leaves compact;

H.2 f admits global holonomies;

H.3 $f_c : M/\mathcal{F}^c \to M/\mathcal{F}^c$ is a transitive topological Anosov homeomorphism (see [39, Chapter IV]), where f_c is the induced dynamics satisfying $f_c \circ \pi = \pi \circ f$ and $\pi : M \to M/\mathcal{F}^c$ is the natural projection to the space of central leaves.

Definition 3.1.3. *Let $\{\mu_x^u\}$ be the conditional measures on local unstable plaques. In the above setting, the probability measure μ is called a **$Gibbs_\nu^u$-state** if $\pi_*\mu = \nu$ and for μ-almost every $x \in M$,*

$$\pi_*\mu_x^u = \nu_{\pi(x)}^u.$$

$Gibb_\nu^u(f)$ denotes the set of $Gibbs_\nu^u$-states of f.

Definition 3.1.4. *For every f-invariant probability μ, we say that a measurable partition ξ is μ**-adapted** (subordinated) to a foliation $\mathcal{F}$ if the following conditions are satisfied:*

1. *There is $r_0 > 0$ such that $\xi(x) \subset B_{r_0}^{\mathcal{F}}(x)$ for μ almost every x where $B_{r_0}^{\mathcal{F}}(x)$ is the ball inside of the leaf $\mathcal{F}(x)$;*
2. *$\xi(x)$ contains an open neighborhood of x inside $\mathcal{F}(x)$;*
3. *ξ is increasing; that is, for μ almost every $\xi(x) \subset f(\xi(f^{-1}(x)))$.*

Definition 3.1.5. *For every f-invariant probability μ, the **partial entropy** of f along the expanding foliation $\mathcal{F}^u$ is defined by*

$$h_\mu(f, \mathcal{F}^u) = H_\mu(f^{-1}\xi^u|\xi^u) = \int_M -\log \mu_z^u(f^{-1}\xi(z))d\mu(z),$$

where ξ^u is a partition μ-adapted to the foliation $\mathcal{F}^u$.

Theorem 3.1.6 (Tahzibi-Yang [56])**.** *Let f be a C^2 partially hyperbolic diffeomorphism satisfying H.1, H.2 and H.3. Suppose μ to be an f-invariant probability measure. Then, $h_\mu(f, \mathcal{F}^u) \leqslant h_{\pi_*\mu}(f_c)$ and equality occurs if and only if $\mu \in Gibb_{\pi_*\mu}^u(f)$.*

Proof. See [56, Theorem A]. □

3.2 Proof of Theorem A

Let $f \in \mathsf{PH}_A^0(\mathbb{T}^d)$ where $A : \mathbb{T}^d \to \mathbb{T}^d$ is a linear Anosov automorphism with a foliation by tori $\mathbb{T}^2$ tangent to $E_A^c = E_A^{ws} \oplus E_A^{wu}$, $\dim E_A^{ws} = \dim E_A^{wu} = 1$. By Theorem 2.3.9 f is dynamically coherent and H sends central leaves of f in central leaves of A. Moreover, in the proof of the main result in [28] it was guaranteed that the semiconjugacy H is injective on each strong stable and unstable leaves.

Let $f \in \mathsf{PH}_A^0(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$, then all central leaves of f are compact (see Theorem 2.3.9).

We recall that two transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbb{T}^d$ have a **global product structure** (GPS) if for any two points $x, y \in \mathbb{R}^d$ the leaves $\tilde{\mathcal{F}}_1(x)$ and $\tilde{\mathcal{F}}_2(y)$ intersect in a unique point. Now, we will recall some essential steps of the proof of following result, from [28, Section 2 and Section 3].

Lemma 3.2.1. *Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ be under above conditions. Then, f admits global holonomies, that is, for every $x, y \in \mathbb{T}^d$ with $y \in \mathcal{F}^u(x)$ and every $z \in \mathcal{F}^c(x)$ there is a unique w such that $w \in \mathcal{F}^u(z) \cap \mathcal{F}^c(y)$.*

Proof. Since $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ is dynamically coherent, by arguments of the proof in [28, Theorem 6.1], f satisfies the hypothesis of [28, Theorem 4.1]. Hence the Franks-Manning semiconjugacy H is injective along strong stable and unstable manifolds, and for any two points $y, z \in \mathbb{R}^d$ the leaves $\tilde{\mathcal{F}^u}(z)$ and $\tilde{\mathcal{F}^{cs}}(y)$ intersect in a unique point. Then, we obtain that $\mathcal{F}^{cs}$ has a global product structure with $\mathcal{F}^u$ and $\mathcal{F}^u(z) \cap \mathcal{F}^c(y)$ is a singleton set. □

The following proposition is contained in the proof of Corollary 2.1 from [56], where some results of Ledrappier-Young [36], [37] are used. Let us consider the topological quotient $\mathbb{T}^d/\mathcal{F}^c$ and the projection $\pi : \mathbb{T}^d \to \mathbb{T}^d/\mathcal{F}^c$ such that the transitive topological Anosov homeomorphism $f_c : \mathbb{T}^d/\mathcal{F}^c \to \mathbb{T}^d/\mathcal{F}^c$ satisfies $\pi \circ f = f_c \circ \pi$.

Proposition 3.2.2. *If μ is an ergodic probability with all the central exponents non-positive almost everywhere, then $h_\mu(f) = h_{\pi*\mu}(f_c)$.*

Proof. Let h_i be the entropy along the i-th Pesin unstable manifold W^i_P for $1 \leqslant i \leqslant u$ (see [37]). Here

$$W^i_P(x) = \{y \in \mathbb{T}^d : \limsup_{n\to\infty} \frac{1}{n} \log dist(f^{-n}(x), f^{-n}(y)) \leqslant -\lambda_i\}$$

and $\lambda_1 > \lambda_2 > \cdots > \lambda_u$ are the positive Lyapunov exponents of (f, μ). As f is a partially hyperbolic diffeomorphism with non-positive central Lyapunov exponents, it follows that W^u_P coincides with the unstable foliation $\mathcal{F}^u$. By Corollary 7.2.2 of [37] we have that $h_\mu(f) = h_u$ (h_u is the entropy along W^u_P), again using ideas of [37] we obtain that $h_u = h_\mu(f, \mathcal{F}^u)$, and consequently $h_\mu(f) = h_u = h_\mu(f, \mathcal{F}^u)$. On the other hand, Theorem 3.1.6 implies that $h_\mu(f) = h_\mu(f, \mathcal{F}^u) \leqslant h_{\pi*\mu}(f_c)$. As f_c is factor of f, we have that $h_\mu(f) = h_{\pi*\mu}(f_c)$. □

Theorem 3.2.3 (Theorem A)**.** *Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$. For some $0 < k_0 < h_{top}(A)$, if μ is an ergodic measure such that $h_\mu(f) > k_0$, then*

1. *μ is hyperbolic, meaning that all its Lyapunov exponents are non-zero. In particular any maximal entropy measure is hyperbolic, provided that it exists.*

2. *For every $\epsilon > 0$, there exists a hyperbolic set $B_\epsilon \subset M$ such that $h_{top}(f|_{B_\epsilon}) > h_\mu(f) - \epsilon$.*

Proof. Let λ^c_1, λ^c_2 be the central Lyapunov exponents of (f, μ). Define $k_0 := h_{top}(A_c)$ where $A_c : \mathbb{T}^d/\mathcal{F}^c_A \to \mathbb{T}^d/\mathcal{F}^c_A$ is the corresponding factor to the linear Anosov A, thus $h_{top}(A_c) < h_{top}(A)$. By Ruelle's inequality [34, Theorem S.2.13] we obtain that

$$k_0 < h_\mu(f) \leqslant \max\{\lambda^c_1, 0\} + \max\{\lambda^c_2, 0\} + \sum \lambda^+_{i,f} \tag{3.2.1}$$

where $\lambda^+_{i,f}$ are the positive (unstable) Lyapunov exponents of f. By [28, Theorem B] f is leaf conjugate to A, we can define a homeomorphism between central leaves and obtain that $h_{top}(f_c) = h_{top}(A_c)$. By contradiction suppose $\max\{\lambda^c_1, 0\} + \max\{\lambda^c_2, 0\} = 0$, that is, the central Lyapunov exponents are non-positive. From Proposition 3.2.2 it follows that

$$h_\mu(f) = h_\mu(f, \mathcal{F}^u) = h_{\pi_*\mu}(f_c) \leqslant h_{top}(f_c) = h_{top}(A_c) = k_0,$$

in contradiction with (3.2.1). Then, $\max\{\lambda^c_1, 0\}+\max\{\lambda^c_2, 0\} > 0$. Analogously, as $h_\mu(f^{-1}) = h_\mu(f)$ we have that $\max\{-\lambda^c_1, 0\} + \max\{-\lambda^c_2, 0\} > 0$. Therefore, μ is a hyperbolic measure. The second item follows from Katok's theorem 2.2.12, because $h_\mu(f) > 0$. The hyperbolicity for any ergodic maximal entropy measure, if it exists, is now immediate since $h_{top}(A_c) < h_{top}(A) \leqslant h_{top}(f)$. □

Using techniques from symbolic dynamics, it is proved in [12] that:

Lemma 3.2.4. *Let $f : \mathrm{Diff}^r(M)$ with $r > 1$, and $\mathcal{O}$ be a hyperbolic periodic orbit. Then, there is at most one ergodic hyperbolic maximal entropy measure homoclinically related to $\mathcal{O}$. Moreover, when such maximal entropy measure exists, its support coincides with $HC(\mathcal{O})$.*

Proof. This is explained in [12, Section 1.6]. See also [9] and [12, Corollary 3.3]. □

Corollary 3.2.5. *Let f be as in Theorem A.*

- *If $h(f, H^{-1}(x)) = 0$ for every $x \in \mathbb{T}^d$, then there exists a maximal entropy measure.*
- *If μ is an ergodic maximal entropy measure, then $\operatorname{supp}\mu$ coincides with the homoclinic class of some hyperbolic periodic orbit.*

Proof. When $h(f, H^{-1}(x)) = 0$ for every $x \in \mathbb{T}^d$, Ledrappier-Walters Principle 2.1.9 allows us to conclude that $h_{top}(A) = h_{top}(f)$ and that a lift of the Haar measure, μ, for A is a maximal entropy measure for f. For the second item, by Theorem 3.2.3 μ is a hyperbolic measure. Then, by Corollary 2.2.13 there exists $\mathcal{O} \in Per_h(f)$ such that $\mathcal{O} \overset{h}{\sim} \mu$, and from Lemma 3.2.4 follows that $\operatorname{supp}\mu = HC(\mathcal{O})$. □

Remark 3.2.6. *One useful reference is Roldán [53], in which we can find examples of partially hyperbolic diffeomorphisms $f : \mathbb{T}^4 \to \mathbb{T}^4$ with two-dimensional center bundle admitting high entropy measures, that is, measures μ such that $h_\mu(f) \geqslant h_{top}(A)$. Specifically, Theorem B (resp. Theorem C) of [53] are examples where there exists an open set U of $\mathrm{Diff}^1(\mathbb{T}^4)$ such that any $f \in U$ is an absolutely (resp. a pointwise) partially hyperbolic diffeomorphism with $\dim E^c_f = 2$, and any ergodic maximal entropy measure is hyperbolic.*

3.3 Proof of Theorem B

Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$. We want to understand the maximal entropy measures for f admitting a dominated splitting of the form $E^s_f \oplus E^{c_1}_f \oplus E^{c_2}_f \oplus E^u_f$, $\dim E^{c_1}_f = \dim E^{c_2}_f = 1$. From [25, Corollary 1.3] it follows that f has a maximal entropy measure.

Let $\Gamma_e(f)$ be the set of all ergodic maximal entropy measures of f, Theorem 3.2.3 implies that every $\mu \in \Gamma_e(f)$ is a hyperbolic measure of saddle type. Recall that $\mathcal{R}_f$ denotes the set of regular points.

Proposition 3.3.1. *Suppose $\mu \in \Gamma_e(f)$ and $\mathcal{O} \in Per_h(f)$. Then, the set*

$$H_{\mathcal{O}} := \{x \in \mathcal{R}_f : W^u_P(x) \pitchfork W^s_P(\mathcal{O}) \neq \varnothing \text{ and } W^s_P(x) \pitchfork W^u_P(\mathcal{O}) \neq \varnothing\},$$

is invariant and measurable. Moreover,

$$\forall \mu \in \mathcal{M}_e(f), \mu(H_{\mathcal{O}}) = 1 \iff \mu \in \mathcal{M}_h(f) \text{ and } \mu \overset{h}{\sim} \mathcal{O}.$$

Proof. For any $x \in H_{\mathcal{O}}$, by the properties of Pesin blocks ([12, Section 2.3]) there are positives integers $m, \tilde{m}$ and a Pesin block $\mathcal{R}_{\epsilon,n}$ such that

$$f^m(x), f^{\tilde{m}}(x) \in \mathcal{R}_{\epsilon,n},\ W^s_{loc}(f^m(x)) \pitchfork W^u_P(\mathcal{O}) \neq \varnothing \text{ and } W^u_{loc}(f^{-\tilde{m}}(x)) \pitchfork W^s_P(\mathcal{O}) \neq \varnothing.$$

For every $m, \tilde{m}, n$ the set of points x satisfying the condition above is measurable, since the local manifolds vary continuously for the C^1-topology on each Pesin block. Therefore, $H_{\mathcal{O}}$ is measurable. Since the union of Pesin blocks is invariant, it is not difficult to verify that $H_{\mathcal{O}}$ is f-invariant.

If μ is an ergodic hyperbolic measure such that $\mu \overset{h}{\sim} \mathcal{O}$, then by definition of the homoclinic relation, $\mu(H_{\mathcal{O}}) > 0$. From the invariance of $H_{\mathcal{O}}$ follows that $\mu(H_{\mathcal{O}}) = 1$. Conversely, if μ is an ergodic measure such that $\mu(H_{\mathcal{O}}) = 1$, from the properties of the Pesin blocks μ is a hyperbolic measure of saddle type and by definition of $H_{\mathcal{O}}$ we have that $\mu \overset{h}{\sim} \mathcal{O}$. □

Remark 3.3.2. *For $\mathcal{O} \in Per_h(f)$ the sets $H_{\mathcal{O}}$ have the following property:*

- *$H_{\mathcal{O}} = H_{\tilde{\mathcal{O}}}$ when $\mathcal{O}$ is homoclinically related to $\tilde{\mathcal{O}}$,*
- *$H_{\mathcal{O}} \cap H_{\tilde{\mathcal{O}}} = \varnothing$ when $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are not homoclinically related.*

Indeed, if there exists $x \in H_{\mathcal{O}} \cap H_{\tilde{\mathcal{O}}}$, then Inclination Lemma [12, Lemma 2.7] *implies that the stable manifolds of $\mathcal{O}$ and $\tilde{\mathcal{O}}$ contain discs that converge towards the stable manifold of x for the C^1-topology; the same argument is valid for unstable manifolds and these arguments imply the homoclinic relation between $\mathcal{O}$ and $\tilde{\mathcal{O}}$.*

Proposition 3.3.3. *Let f be a C^2 partially hyperbolic diffeomorphism and μ be a hyperbolic ergodic probability measure whose Oseledets decomposition $E^s_P \oplus E^u_P$ is dominated. Then μ is supported on a homoclinic class: there exists a sequence of hyperbolic periodic orbits $(\mathcal{O}_n)_{n\in\mathbb{N}}$ whose s-index is equal to $\dim E^s_P$, that are all homoclinically related, that converge towards the support of μ for the Hausdorff topology and such that the invariant measures supported on the $\mathcal{O}_n$ converge towards μ in the weak*-topology.*

Proof. We will argue as in the proof of [23, Proposition 1.4]. Let $\lambda_{E^s_P}$ be the Lyapunov exponent given by

$$\lambda_{E^s_P} = \lim_{n\to\infty} \frac{1}{n} \int \log \|Df^n|_{E^s_P}\| d\mu$$

and let $\epsilon > 0$ such that $\lambda_{E^s_P} + \epsilon < 0$. Consider an integer $n_0 \geqslant 1$ fixed and large enough so that for any $n \geqslant n_0$ we have

$$\left| \frac{1}{n} \int \log \|Df^n|_{E^s_P}\| d\mu - \lambda_{E^s_P} \right| \leqslant \frac{\epsilon}{2} \tag{3.3.1}$$

Here μ is ergodic for f, but not necessarily for f^{n_0}. Hence, μ decomposes as

$$\mu = \frac{1}{q}(\mu_1 + \ldots + \mu_q)$$

where $q \in \mathbb{N} - \{0\}$ divides n_0 and each μ_t is an ergodic f^{n_0}-invariant measure such that $\mu_{t+1} = f_*\mu_t$ for every t (mod q). Let $A_1 \cup \ldots \cup A_q$ be a measurable partition of $\mathbb{T}^d$ with respect to μ such that $f(A_t) = A_{t+1}$ for every t (mod q) and $\mu_t(A_t) = 1$. From 3.3.1 follows that there exists $t_0 \in \{1, \ldots, q\}$ such that

$$\frac{1}{n_0} \int \log \|Df^{n_0}|_{E^s_P}\| d\mu_{t_0} \leqslant \lambda_{E^s_P} + \frac{\epsilon}{2} \tag{3.3.2}$$

For $l \geqslant 1$ and μ_N-almost every x, one decomposes the segment of orbit with length l of x as $(x, f(x), \ldots, f^{j-1}(x))$, $(f^j(x), \ldots, f^{j+(r-1)n_0-1}(x))$ $(f^{j+(r-1)n_0}(x), \ldots, f^{l-1}(x))$ such that $j < n_0$, $j + rn_0 \geqslant l$ and all the points $f^j(x), f^{j+n_0}(x), \ldots, f^{j+rn_0}(x) \in A_{t_0}$. Then,

$$\|Df^l_x|_{E^s_P}\| \leqslant \|Df^j_x|_{E^s_P}\| \cdot (\|Df^{n_0}_{f^j(x)}|_{E^s_P}\| \cdots \|Df^{n_0}_{f^{j+(r-2)n_0}(x)}|_{E^s_P}\|) \cdot \|Df^{l-(j+(r-1)n_0)}_{f^{j+(r-1)n_0}(x)}|_{E^s_P}\|.$$

Hence, for μ-almost every point we have that

$$\log \|Df^l_x|_{E^s_P}\| \leqslant 2n_0 K_f + \sum_{s=0}^{r-2} \log \|Df^{n_0}_{f^{j+sn_0}(x)}|_{E^s_P}\|$$

where K_f is an upper bound for both $\log \|Df\|$ and $\log \|Df^{-1}\|$. Since $f^j(x)$ is a regular point for the dynamics (f^{n_0}, μ_{t_0}), we obtain that the average $\frac{1}{kn_0} \sum_{i=0}^{k-1} \log \|Df^{n_0}_{f^{j+in_0}(x)}|_{E^s_P}\|$ converges to $\frac{1}{n_0} \int \log \|Df^{n_0}|_{E^s_P}\| d\mu_{t_0}$. Thus,

$$\lim_{k\to\infty} \frac{1}{kl} \sum_{i=0}^{k-1} \log \|Df^l_{f^{il}(x)}|_{E^s_P}\| \leqslant 2\frac{n_0 K_f}{l} + \frac{1}{n_0} \int \log \|Df^{n_0}|_{E^s_P}\| d\mu_{t_0}.$$

Now, choosing $l > \dfrac{4n_0K_f}{\epsilon}$ and by the inequality (3.3.2) we obtain that

$$\frac{1}{lk}\sum_{i=0}^{k-1}\log\|Df^l_{f^{il}(x)}|_{E^s_P}\| < \lambda_{E^s_P} + \epsilon.$$

Define $-\rho = \lambda_{E^s_P} + \epsilon$, then there exists an integer $l \geqslant 1$ such that for μ-almost every x, the Birkhoff averages

$$\frac{1}{lk}\sum_{i=0}^{k-1}\log\|Df^l_{f^{il}(x)}|_{E^s_P}\|,\quad \frac{1}{lk}\sum_{i=0}^{k-1}\log\|Df^{-l}_{f^{-il}(x)}|_{E^u_P}\|$$

converge towards a number less than $-\rho$ when $k \to \infty$. Then, there exist a set B with $\mu(B) = 1$ and a constant $C > 0$ such that for every $x \in B$ and every $k \geqslant 0$ we have that

$$\prod_{i=0}^{k-1}|Df^l_{f^{il}(x)}|_{E^s_P}\| \leqslant Ce^{-k\rho},\quad \prod_{i=0}^{k-1}\|Df^{-l}_{f^{-il}(x)}|_{E^u_P}\| \leqslant Ce^{-k\rho}.$$

As μ is ergodic, μ-almost every point is recurrent. Now, using the previous inequality we obtain a segments of orbits $(x, \dots, f^m(x))$ in the support of μ such that $x, f^m(x) \in B$, the distance $d(x, f^m(x))$ is arbitrarily small and the non-invariant atomic measure $\frac{1}{m}\sum_{i=0}^{m-1}\delta_{f^i(x)}$ is arbitrarily close to μ. In particular for each $k = 0, \dots m$, we have that

$$\prod_{i=0}^{k-1}|Df^l_{f^{il}(x)}|_{E^s_P}\| \leqslant Ce^{-k\rho},\quad \prod_{i=0}^{k-1}\|Df^{-l}_{f^{-il}(x)}|_{E^u_P}\| \leqslant Ce^{-k\rho}. \tag{3.3.3}$$

This property and the domination $E^s_P \oplus E^u_P$ allow to apply Liao-Gan's shadowing Lemma [30], the segment of orbit $(x, \dots, f^m(x))$ is δ-shadowed (see [34, Definition 18.1.1]) by a periodic orbit $\mathcal{O} = \{y, \dots, f^m(y) = y\}$ where δ tends to 0 when $d(x, f^m(x))$ decreases. In particular, $\mathcal{O}$ is arbitrarily close to the support of μ for the Hausdorff topology and it supports a periodic measure arbitrarily close to the measure μ. Note that the segment of orbit $(y, \dots, f^m(y))$ satisfies an estimate like 3.3.3 with constants ρ', C' close to ρ, C. This way, we have that the orbit $\mathcal{O}$ is hyperbolic and has s-index $\dim E^s_P$. Repeating this argument, one obtains a sequence of such periodic points $(y_n)_{n\in\mathbb{N}}$ with s-index $\dim E^s_P$ which converge to x, whose orbits $(\mathcal{O}_n)_{n\in\mathbb{N}}$ converge toward to $\operatorname{supp}\mu$, and whose measures converge to μ in the weak*-topology. By the estimate 3.3.3, the size of the local stable and local unstable manifolds at y_n is uniform. Consequently, the periodic orbits $\mathcal{O}_n$ are homoclinically related for n large. □

Theorem 3.3.4 (Theorem B). *Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$ admitting a dominated splitting of the form $E^s_f \oplus E^{c_1}_f \oplus E^{c_2}_f \oplus E^u_f$, where $E^{c_1}_f, E^{c_2}_f$ are one-dimensional. Then, $\Gamma_e(f)$ is a non-empty finite set and all its elements are hyperbolic measures.*

Proof. $\Gamma_e(f)$ is a non-empty set and every $\mu \in \Gamma_e(f)$ is hyperbolic due to Theorem 3.2.3. Arguing by contradiction suppose there exists a sequence of measures $(\mu_i)_{i\in\mathbb{N}} \subset \Gamma_e(f)$.

By hypothesis of domination the Oseledets decomposition of μ_i defined by $E_i^s \oplus E_i^u$ is dominated, where $E_i^s = E_f^s \oplus E_f^{c_1}$ and $E_i^u = E_f^{c_2} \oplus E_f^u$ (for regular points of μ_i). Arguing as in the proof of Proposition 3.3.3, we obtain a sequence of orbits $(\mathcal{O}_i)_{i\in\mathbb{N}}$ such that $\mathcal{O}_i \in Per_h(f)$ and the size of the local stable (unstable) manifolds at a point is uniform (does not depend on measure). Taking an accumulation point of $(\mu_i)_{i\in\mathbb{N}}$, we have that the periodic orbits $\mathcal{O}_i$ are homoclinically related for i large. Consequently by Remark 3.3.2 there exists $j \neq i$ such that $H_{\mathcal{O}_i} = H_{\mathcal{O}_j}$. Hence, from Proposition 3.3.1 follows that $\mathcal{O}_i \overset{h}{\sim} \mu_j$ and $\mathcal{O}_j \overset{h}{\sim} \mu_j$ for $i \neq j$. This is a contradiction with Lemma 3.2.4. Therefore, $\Gamma_e(f)$ is a non-empty finite set. □

Let $\Gamma_e(f) = \{\mu_1, \mu_2, \ldots, \mu_k\}$ be the set of ergodic maximal entropy measures of f. The following results are consequences of the previous theorem.

Corollary 3.3.5. *Let f be as in Theorem 3.3.4. Then, every maximal entropy measure μ is of the form $\mu = \sum_{i=1}^{k} t_i\mu_i$ where $t_i \geqslant 0$, $\sum_{i=1}^{k} t_i = 1$.*

Proof. Let μ be a maximal entropy measure for f. By applying Jacobs' formula 2.1.6 to the ergodic decomposition of μ, we obtain that μ_P has maximal entropy for $\hat{\mu}$-a.e P, and from Theorem 3.3.4 there exists a finite number (non-zero) of them. Therefore, μ is a convex combination of ergodic maximal entropy measures. □

Corollary 3.3.6. *Let f be as in Theorem 3.3.4 and $(\nu_i)_{i\in\mathbb{N}}$ be a sequence of hyperbolic measures such that $\lim_{i\to\infty} h_{\nu_i}(f) = h_{top}(f)$. If ν_i converges to μ in the weak*-topology, then μ is a combination of elements of $\Gamma_e(f)$.*

Proof. Theorem 1 from [25] and Theorem 2.1.15 together imply that the function entropy is upper semi-continuous, then μ is a maximal entropy measure for f. Thus, by Corollary 3.3.5 one concludes the proof. □

3.4 Proof of Theorem C

We start by considering $f \in \mathsf{PH}(\mathbb{T}^d) \cap \mathrm{Diff}^2(\mathbb{T}^d)$. We define the *volume growth rate of the foliation* $\mathcal{F}^u$ by

$$\mathcal{X}_{\mathcal{F}^u}(x,f) = \limsup_{n\to\infty} \frac{1}{n} \log(Vol f^n(B_r^{\mathcal{F}^u}(x))),$$

where $B_r^{\mathcal{F}^u}(x)$ denotes the ball inside of the leaf $\mathcal{F}^u(x)$. Then, the *maximum volume growth rate of* $\mathcal{F}^u$ under f is defined by

$$\mathcal{X}_u(f) := \mathcal{X}_{\mathcal{F}^u}(f) = \sup_{x\in\mathbb{T}^d} \mathcal{X}_{\mathcal{F}^u}(x,f).$$

The following result provides a refined version of the Pesin-Ruelle inequality:

Theorem 3.4.1 (Hua-Saghin-Xia [32]). *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism. Let μ be an ergodic measure and $\lambda_i^c(\mu)$ the Lyapunov exponents to E^c. Then,*

$$h_\mu(f) \leqslant \mathcal{X}_u(f) + \sum_{\lambda_i^c > 0} \lambda_i^c(\mu).$$

Proof. See [32, Theorem 3.3]. □

Remark 3.4.2. *In the case of $\mathbb{T}^4$, the stable and unstable manifolds are one-dimensional, which is crucial to remove the compactness condition for the center leaves in Theorem 3.2.3.Hence, we obtain the same conclusion about hyperbolic measures.*

Let $A : \mathbb{T}^4 \to \mathbb{T}^4$ be a linear Anosov automorphism admitting a dominated splitting of the form $E_A^{ss} \oplus E_A^c \oplus E_A^{uu}$, $E_A^c = E_A^{ws} \oplus E_A^{wu}$ and let H be the Franks-Manning semiconjugacy between f and A.

Remark 3.4.3. *Let us denote by $[x] := H^{-1}(x)$ the class of $x \in \mathbb{T}^d$, and similarly for $\tilde{x} \in \mathbb{R}^d$ we write $[\tilde{x}] := \tilde{H}^{-1}(\tilde{x})$. For every $\tilde{x} \in \mathbb{R}^d$, each $[\tilde{x}]$ is a compact set whose diameter is uniformly bounded from above* $\operatorname{diam}([\tilde{x}]) \leqslant 2K$. *In particular, since $H \circ f^n = A^n \circ H$ for every $n \in \mathbb{Z}$ we obtain that*

$$\operatorname{diam}(\tilde{f}^n[\tilde{x}]) \leqslant 2K,$$

for every $n \in \mathbb{Z}$.

Theorem 3.4.4 (Theorem C). *Let $f \in \mathsf{PH}_A^0(\mathbb{T}^4) \cap \operatorname{Diff}^2(\mathbb{T}^4)$. For some $0 < k_0 < h_{top}(A)$, if μ is an ergodic measure such that $h_\mu(f) > k_0$, then μ is a hyperbolic measure.*

Proof. Let λ_1^c, λ_2^c be the central Lyapunov exponents of (f, μ) and $k_0 := \lambda_A^u$ the largest unstable Lyapunov exponent of A. Observe that $\mathcal{X}_u(f) = \mathcal{X}_u(\tilde{f})$, where $\tilde{f}$ is any lift of f to the universal cover, and $\mathcal{F}^u$ is an one-dimensional foliation, then the volume is the length. Take an strong unstable arc γ and since $\tilde{H} \circ \tilde{f}^n = \tilde{A}^n \circ H$ for every $n \in \mathbb{Z}$, we have that

$$\operatorname{diam} \tilde{f}^n(\gamma) \leqslant 2K + \operatorname{diam} \tilde{A}^n(\tilde{H}(\gamma)) \leqslant 2K + e^{n\lambda_A^u} \operatorname{diam} \tilde{H}(\gamma), \tag{3.4.1}$$

where K is a constant that bounds the distance between $\tilde{H}$ and the identity.

On the other hand, by [28, Proposition 7.1] $\mathcal{F}^s$ and $\mathcal{F}^u$ are quasi-isometric, then there is a constant $Q > 0$ such that $dist_{\mathcal{F}^u}(f^n(x), f^n(y)) \leqslant Q dist(f^n(x), f^n(y))$, and from (3.4.1) it follows that

$$dist_{\mathcal{F}^u}(f^n(x), f^n(y)) \leqslant Q e^{n\lambda_A^u} dist(\tilde{H}(x), \tilde{H}(y)) + 2QK.$$

Considering γ as an arc of $B_r^{\mathcal{F}^u}(x)$ and $C := \operatorname{diam} \tilde{H}(B_r^{\mathcal{F}^u}(x))$, we obtain that

$$\frac{1}{n} \log len(\tilde{f}^n(B_r^{\mathcal{F}^u}(\tilde{x}))) \leqslant \frac{1}{n} \log Q + \frac{1}{n} \log(C e^{n\lambda_A^u} + 2K) = \frac{1}{n} \log Q + \frac{1}{n} \log(C + \frac{2K}{e^{n\lambda_A^u}}) + \lambda_A^u.$$

Therefore, $\mathcal{X}_u(\tilde{f}) = \mathcal{X}_u(f) = \lambda_A^u$. Now, by Theorem 3.4.1 we have that

$$\lambda_A^u < h_\mu(f) \leqslant \mathcal{X}_u(f) + \max\{\lambda_1^c, 0\} + \max\{\lambda_2^c, 0\} \leqslant \lambda_A^u + \max\{\lambda_1^c, 0\} + \max\{\lambda_2^c, 0\}.$$

Then, $\max\{\lambda_1^c, 0\} + \max\{\lambda_2^c, 0\} > 0$. Analogously, as $h_\mu(f^{-1}) = h_\mu(f)$ and $\mathcal{F}^s$ is quasi-isometric, we have that $\max\{-\lambda_1^c, 0\} + \max\{-\lambda_2^c, 0\} > 0$, this concludes the proof. □

The proof above is inspired by ideas of [57]. We also obtain a class of examples of partially hyperbolic diffeomorphisms where every ergodic maximal entropy measure is hyperbolic.

Corollary 3.4.5. *If $f \in \mathsf{PH}_A^0(\mathbb{T}^4) \cap \mathrm{Diff}^\infty(\mathbb{T}^4)$, then every ergodic maximal entropy measure is hyperbolic.*

Proof. Since $f \in \mathrm{Diff}^\infty(\mathbb{T}^4)$, by a classical Newhouse's result [42] there is an ergodic maximal entropy measure μ for f . If $k_0 := \lambda_A^u$ is the largest unstable Lyapunov exponent of A, then $k_0 < h_{top}(A) \leqslant h_{top}(f) = h_\mu(f)$, and for Theorem 3.4.4 μ is a hyperbolic measure. □

Remark 3.4.6. *For find examples of diffeomorphisms satisfying the conditions of the theorems presented in this chapter, see [28] and reference therein.*

4

Equilibrium states for diffeomorphisms isotopic to an Anosov diffeomorphism

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a DA partially hyperbolic diffeomorphism isotopic to a linear Anosov automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$, and H a semiconjugacy between f and A. As we mentioned before the new results presented in this chapter are joint work with Adriana Sánchez and Régis Varão.

4.1 Known results

Our goal is to understand the disintegration of measures along the center foliation of certain DA partially hyperbolic diffeomorphisms with two-dimensional center bundle, and then to study equilibrium states for a particular class of potentials. Before that, we present some results used in the proofs of the main theorems on this chapter.

4.1.1 Mixed DA examples

Let $A : \mathbb{T}^d \to \mathbb{T}^d$ $(d \geqslant 4)$ be a linear Anosov map with dominated splitting $T\mathbb{T}^d = E^{ss}_A \oplus E^s_A \oplus E^u_A \oplus E^{uu}_A$ where $\dim E^s_A = \dim E^u_A = 1$, and the contraction/expansion rate satisfy $\lambda_{ss} < \lambda_s < 1 < \lambda_u < \lambda_{uu}$. For example, take any linear Anosov B_1 on $\mathbb{T}^2$ (center bundle) and take a linear Anosov B_2 on $\mathbb{T}^{d-2}$ such that it contracts and expands less than B_1, then $A = B_1 \times B_2$ is a linear Anosov map with the properties required. Notice that A is a strongly partially hyperbolic diffeomorphism when $E^c_A = E^s_A \oplus E^u_A$, and $\mathbb{T}^d$ has a normally hyperbolic foliation whose leaves are tori $\mathbb{T}^2$ tangent to E^c_A.

We will proceed as in the classical construction of derived from Anosov introduced by Mañé [38]. Let q and p be two different fixed points of A. Let $r > 0$ be small (to be determined later) and deform A inside $B(q, r)$ and $B(p, r)$, in the following way: in $B(q, r)$ we perform a pitchfork perturbation along E^s_A and on $B(p, r)$ we perform a pitchfork bifurcation along E^u_A in such a way that the foliation by tori $\mathbb{T}^2$ tangent to E^c_A it is preserved. In this way we obtain g that falls into Proposition 4.1 from [13]

Let η, r be as in shadowing theorem (see [13, Theorem 2.1]) and. We may assume that g satisfies the following properties:

- g is a strongly partially hyperbolic diffeomorphism with a dominated splitting $T\mathbb{T}^d = E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$ and each subbundle dominates the previous ones by a factor $a < 1$ with $\dim E^{cs} = \dim E^{cu} = 1$. These subbundles are C^0-close to the respective ones of A;
- $d_{C^0}(g, A) < r$;
- if $d(x, y) < 2\eta$, then $\dfrac{\|Dg|E^{ci}(x)\|}{\|Dg|E^{ci}(y)\|} < a^{-1/4}$, $i = s, u$;
- $Dg|_{E^{cs}(x)}$ is uniformly contracting outside $B(q, \rho)$ with rate λ_s;
- $Dg|_{E^{cu}(x)}$ is uniformly expanding outside $B(p, \rho)$ with rate λ_u.

The above conditions hold in a neighborhood of g (the last two, the rate expansion/contraction, will be close to λ_s and λ_u, respectively). By construction, g is dynamically coherent and $E^c = E^{cs} \oplus E^{cu}$ is uniquely integrable, and the same holds in a neighborhood of g. Moreover, this example is also robustly transitive by similar arguments as in [31], E^{cs}, E^{cu} are integrable and the stable and unstable manifolds of this periodic torus are dense.

Every small C^1 perturbation of g is called **mixed derived from Anosov**. We will denote by $W^{cs}_\gamma(x)$ the arc in the leaf $\mathcal{F}^{cs}(x)$ of size 2γ with x in the middle. We define $W^{cu}_\gamma(x)$ analogously. Lemma 5.1 from [13] asserts that for any point $x \in \mathbb{T}^d$ one and only one of the following holds:

- $H^{-1}(x)$ consist of a single point.
- $H^{-1}(x)$ is a segment tangent to E^{cs} of length less than 2η.
- $H^{-1}(x)$ is a segment tangent to E^{cu} of length less than 2η.
- $H^{-1}(x)$ is a square tangent to $E^c = E^{cs} \oplus E^{cu}$ such that
  - for each $y \in H^{-1}(x)$, we have that $W^{cs}_\gamma(y) \cap H^{-1}(x)$ is a center stable segment denoted by $J^{cs}(y)$, and similarly for E^{cu}; and
  - if y and z are in $H^{-1}(x)$, then $\varnothing \neq J^{cs}(y) \cap J^{cu}(z) \in H^{-1}(x)$.

The following proposition asserts that mixed derived from Anosov diffeomorphisms do not have entropy along the fibers.

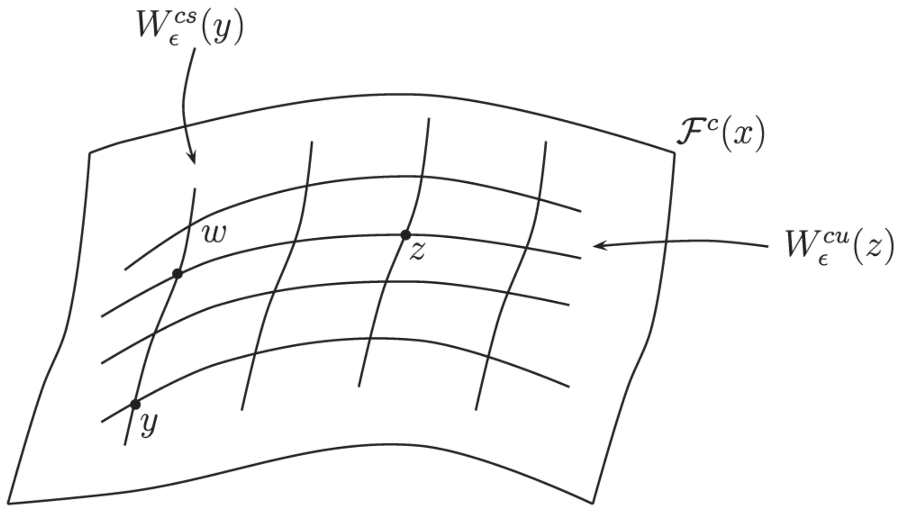


Figure 2 – [13, Figure 2].

Proposition 4.1.1. *$h(g, H^{-1}(x)) = 0$ for all $x \in \mathbb{T}^d$.*

Proof. See [13, Corollary 5.2]. □

4.1.2 DA with simple center bundle

The following definition was introduced in [14, Definition 1.4]:

Definition 4.1.2. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism over a compact boundaryless Riemannian manifold. We say that its center bundle E^c is **simple** if*

1. *$E^c = E^1 \oplus \cdots \oplus E^\ell$ with $\dim E^i = 1$, for every $i = 1, \ldots, l$.*
2. *For every $I \subset \{1, \ldots, l\}$ the bundle $\bigoplus_{i \in I} E^i$ integrates to an f-invariant foliation $\mathcal{F}^I$ (in particular, $E^c = E^{\{1,\ldots,l\}}$ is integrable). Furthermore, there is compatibility in the sense: $I \subset I' \Rightarrow \mathcal{F}^I$ subfoliates $\mathcal{F}^{I'}$.*

   *We say that E^c is **strongly simple** it is simple and furthermore*
3. *For every i, the lifts of $\mathcal{F}^i := \mathcal{F}^{\{i\}}, \mathcal{F}^{\{1,\ldots,\hat{i},\ldots,l\}}$ to the universal covering of M have global product structure inside each leaf of the lift of $\mathcal{F}^c$.*

Recall that $[x] := H^{-1}(x)$, for all $x \in \mathbb{T}^d$. In a recent work, Carrasco, Lizana, Pujals and Vásquez [14] proved that for certain DA partially hyperbolic diffeomorphisms of $\mathbb{T}^d$ with center bundle E^c strongly simple, the following properties hold for every $x \in \mathbb{T}^d$:

- $[x]$ is contained in a unique center leaf of $\mathcal{F}^c$.

- if $z, z' \in [x]$ and $z' \in \mathcal{F}^i(z)$ for some $1 \leqslant i \leqslant \ell$, then

$$[z, z']_i \subset [x],$$

where $[z, z']_i$ is the closed interval inside $\mathcal{F}^i(z)$ with end points z and z';

- the class $[x]$ is a rectangle in a single leaf of $\mathcal{F}^c$.

- $h(f, [x]) = 0$.

The *rectangles* mentioned above are compact sets obtained in the following inductive procedure. Let $z_0, \ldots, z_k$, with $1 \leqslant k \leqslant \ell$ for some $0 \leqslant l \leqslant d$, be points in $[x]$ such that $z_j \in \mathcal{F}^{i_j}(z_0)$. We construct the rectangle (of dimension k and corner z_0) by starting with $R_1 = [z_0, z_1]_{i_1} \subset \mathcal{F}^{i_1}(z_0)$ (see Figure 3). Taking $i_2 \neq i_1$ we can define R_2 as the trace inside $\mathcal{F}^c(z_0)$ of the set obtained by sliding R_1 along $[z_0, z_2]_{i_2} \subset \mathcal{F}^{i_2}(z_0)$, that is,

$$R_2 = \bigcup_{w \in [z_0, z_2]_{i_2}} [w, y(w)]_{i_1},$$

where $[w, y(w)]_{i_1}$ is the image of $[z_0, z_1]_{i_1}$ by the $\mathcal{F}^{i_2}$-holonomy. Continuing this way, we can define R_k as

$$R_k = \bigcup_{w \in [z_0, z_k]_{i_2}} R^{k-1}(w),$$

where $R^{k-1}(w)$ is a rectangle of dimension $k-1$ and corners $z_0, \ldots, z_{k-1}$ obtained as the image of R_{k-1} in the corresponding center manifold by the $\mathcal{F}^{i_k}$-holonomy sending z_0 in w.

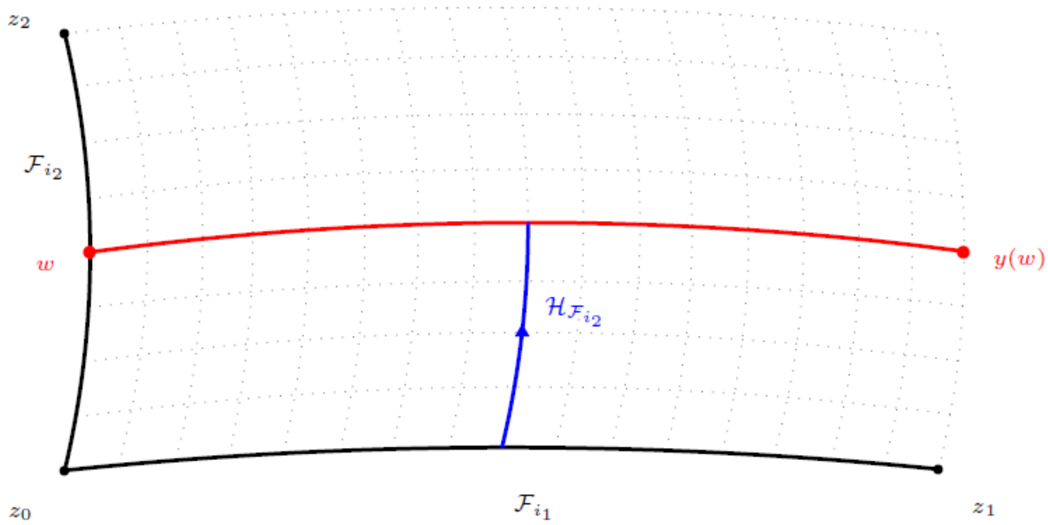


Figure 3 – Construction of a rectangle ($l = 2$) [5].

Theorem 4.1.3 (Carrasco-Lizana-Pujals-Vásquez [14])**.** *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a DA partially hyperbolic diffeomorphism. Assume further that the lifts of foliations $\mathcal{F}^{cs}, \mathcal{F}^u$ to $\mathbb{R}^d$ have GPS, and likewise $\mathcal{F}^{cu}, \mathcal{F}^s$. If E^c is strongly simple, then, $h_{top}(f) = h_{top}(A)$.*

Proof. See [14, Theorem A]. □

4.2 Proof of Theorem D

Consider the set where H fails to be injective

$$C := \{x \in \mathbb{T}^d : \#H^{-1}H(x) > 1\}. \tag{4.2.1}$$

Lemma 4.2.1. *C is an f-invariant set and $H^{-1}H(C) = C$.*

Proof. Notice that $C = \bigcup_{\#\mathcal{P}(x)>1} \mathcal{P}(x)$ where $\mathcal{P}(x) := H^{-1}H(x)$. For every $y \in C$ there exists $x \neq y$ such that $H(x) = H(y)$, by semiconjugacy we have

$$H(f(x)) = A(H(x)) = A(H(y)) = H(f(y))$$

and $f(y) \in \mathcal{P}(f(x))$. Then $f(y) \in C$, because $\#\mathcal{P}(f(x)) > 1$ for $x \neq y$. Hence $f(C) = C$.

For the other property, suppose $H^{-1}H(C) \nsubseteq C$. There exists $y \in H^{-1}H(C)$ such that $y \notin C$ with $H(y) = H(x)$ for $x \in C$. As $H(f(y)) = H(f(x))$ and $f(y) \notin C$ we have that $y = x$. This is a contradiction, and consequently it proves that $H^{-1}H(C) = C$. □

Remark 4.2.2. *We claim that C is a measurable set. One may check that by observing that simply reproducing ipsis litteris the proof of* [46, Lemma 3.2] *only changing $\mathbb{T}^3$ by $\mathbb{T}^d$ and $\mathbb{R}^3$ by $\mathbb{R}^d$ one obtains that $H(C)$ is a measurable set. Hence $C = H^{-1}H(C)$ is a measurable set. Moreover, for every $\mu \in \mathcal{M}_e(f)$, $\mu(C) = 0$ or $\mu(C) = 1$.*

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a DA partially hyperbolic diffeomorphism satisfying the following conditions:

- A.1 f is dynamically coherent;
- A.2 There exists a splitting $E^c = E^1 \oplus E^2$ where each E^i is a line-bundle and integrates to an f-invariant foliation $\mathcal{F}^i$ (non-compact), for $i = 1, 2$;
- A.3 If $z, z' \in H^{-1}(x)$ and $z' \in \mathcal{F}^i(z)$ for some $1 \leqslant i \leqslant 2$, then
$$[z, z']_i \subset H^{-1}(x),$$
where $[z, z']_i$ is the closed interval inside $\mathcal{F}^i(z)$ with end points z and z';
- A.4 For each $x \in \mathbb{T}^d$, $H^{-1}(x)$ is a finite union of rectangles contained in a unique center leaf of $\mathcal{F}^c$;
- A.5 $h(f, H^{-1}(x)) = 0$ for every $x \in \mathbb{T}^d$.

Remark 4.2.3. *Every mixed derived from Anosov $g : \mathbb{T}^d \to \mathbb{T}^d$ satisfies the assumptions* A.1, A.2, A.3, A.4 *and* A.5*. In particular, the center foliation $\mathcal{F}^c$ admits two invariant one-dimensional sub-foliations $\mathcal{F}^{cu}, \mathcal{F}^{cs}$ such that $H^{-1}(x) \cap \mathcal{F}^{cu}_{loc}(x)$ and $H^{-1}(x) \cap \mathcal{F}^{cs}_{loc}(x)$ are segments in the center foliation. Also, every derived from Anosov $f : \mathbb{T}^4 \to \mathbb{T}^4$ with strongly simple center bundle satisfies the Assumptions* A.1, A.2, A.3, A.4 *and* A.5.

Theorem 4.2.4 (Theorem D)**.** *Let $f : \mathbb{T}^4 \to \mathbb{T}^4$ be a DA partially hyperbolic diffeomorphism satisfying A.1, A.2, A.3 and A.4. Assume that f preserves the orientation for $\mathcal{F}^i$, $i = 1, 2$. Let μ be an ergodic probability for f:*

1. *If $\mu(C) = 0$, then (f, μ) is almost conjugate to an Anosov diffeomorphism.*

2. *If $\mu(C) = 1$, then C defines a partition such that μ has atomic disintegration with a finite number of atoms.*

Proof. If $\mu(C) = 0$, then H is a conjugacy μ-a.e. From now on we assume $\mu(C) = 1$ and let us prove that the partition determined by C has atomic disintegration. That is, consider the partition:

$$\mathcal{P} := \{\mathcal{P}(x) := H^{-1}H(x) : x \in C\}.$$

Let us prove that $\mathcal{P}$ is a measurable partition with respect to any measure considered. Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable basis for the topology of $\mathbb{T}^4$. Now for any point $x \in \mathbb{T}^4$ we have sets $B_i \equiv B_i(x) \in \{A_i, A_i^c\}$ such that $\{x\} = \bigcap_{i\in\mathbb{N}} B_i$. Since $\{H^{-1}(A_i)\}$ is a measurable set (because A_i is an open set and H is continuous) notice that

$$H^{-1}(x) = \bigcap_{i\in\mathbb{N}} H^{-1}(B_i).$$

Thus proving that $\mathcal{P}$ is a measurable partition. Moreover, it is easy to see that $\mathcal{P}$ is left invariant by f, that is, $f(\mathcal{P}(x)) = \mathcal{P}(f(x))$.

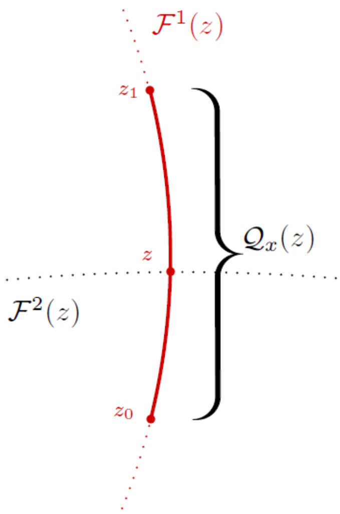
Assume, without loss of generality, that $\mathcal{F}^1$ is oriented and f preserves its orientation. We define another partition $\mathcal{Q}$ as the one whose elements are the connected components of the intersection of elements of $\mathcal{P}$ and $\mathcal{F}^1$ (see Figure 4). That is

$$\mathcal{Q} := \{Q(x) = \mathcal{F}^1(x) \cap \mathcal{P}(x) : x \in C\}.$$

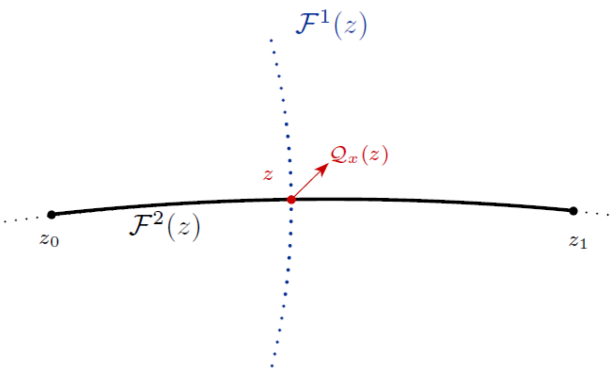
Recall that, by assumption A.4, $H^{-1}(z)$ is a finite union of rectangles in $\mathcal{F}^c$, so we can write for each $x \in C$

$$\mathcal{P}(x) = \bigcup_{j=1}^{n_x} R_j(x), \tag{4.2.2}$$

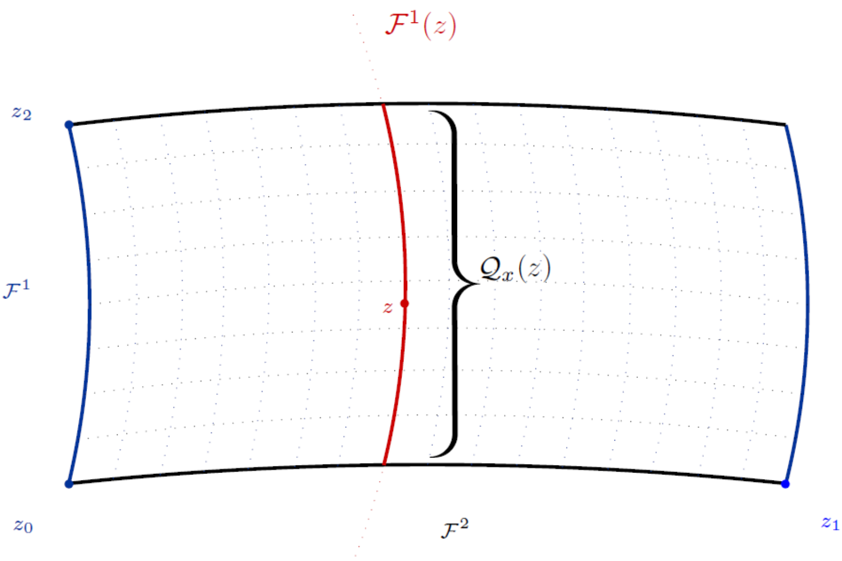
where n_x represents the number of rectangles in the class and $R_j(x)$ denotes a rectangle of dimension $1 \leqslant k_j = k_j(x) \leqslant 2$ with corners z_0, z_1, z_2. Moreover, assumption A.3 guarantees that $Q(x)$ has only one connected component, an interval or a point. Therefore, the foliation of each element of $\mathcal{P}$ by $\mathcal{F}^1$ is similar to a foliation by compact leaves. Thus, we can consider $\mathcal{Q}$ as a measurable partition. Indeed, any foliation with compact leaves can be considered as a measurable partition, see [7, Proposition 3.7]. Let us denote the conditional measures on $\mathcal{Q}$ by μ_x. It is easy to see that the partition $\mathcal{Q}$ is f-invariant and, therefore, $f_*\mu_x = \mu_{f(x)}$. Consider $\pi : C \to \widehat{C} := C/\mathcal{Q}$ the canonical projection that assigns



(A) R_j is a rectangle of dimension 1 contained in a $\mathcal{F}^1$-leaf



(B) R_j is a rectangle of dimension 1 contained in a $\mathcal{F}^2$-leaf



(C) R_j is a rectangle of dimension 2

Figure 4 – Partition [5].

to each point $x \in C$ the element $\mathcal{Q}(x)$ of the partition that contains it. Denote the quotient measure as $\hat{\mu} = \pi_* \mu$.

Lemma 4.2.5. *The measure μ has atomic disintegration with respect to the partition $\mathcal{Q}$.*

Proof. We want to show that the conditional measure μ_x is a countable linear combination of Dirac masses for $\hat{\mu}$-almost every $Q(x) \in \mathcal{Q}$. We will prove this by contradiction, assume there exists a set $\hat{\Lambda} \subset \mathcal{Q}$ with positive $\hat{\mu}$-measure such that for every $Q(x) \in \hat{\Lambda}$ the measure μ_x is not atomic. Moreover, by the invariance of the disintegration, $\hat{\Lambda}$ can be assumed to be invariant and, by the ergodicity, of full measure.

Let $Q = Q(x_0) \in \hat{\Lambda} \cap \operatorname{supp}(\hat{\mu})$ and let $\mathcal{B}$ be a foliated (by $\mathcal{F}^1$) box around Q. That is, some image of a topological embedding

$$\phi : D^3 \times D^1 \to \mathbb{T}^4,$$

where D^k is the closed unit disk in $\mathbb{R}^k$ and, such that every plaque $P_x = \phi(\{x\} \times D^1)$ is contained in a leaf of $\mathcal{F}^1$. Let us identify $\mathcal{B}$ with the product $D^3 \times D^1$ through the corresponding homeomorphism. Let $\hat{V}$ be an open neighborhood of Q small enough so it is contained in $\mathcal{B}$. Moreover, since $\tilde{H}^{-1}(\tilde{x})$ is uniformly bounded we can assume that $\mathcal{B}$ contains $\mathcal{P}(x)$ for every $x \in D^3$.

Consider the following map

$$\begin{aligned} \psi : D^3 \times [0,1] &\to \mathcal{B} \\ (x,t) &\mapsto (x, \theta_x(t)) \end{aligned}$$

where $(x, \theta_x(t))$ is defined as the highest point in the local leaf $Q(x) \subset \mathcal{B}$ such that $\mu_x([x, \theta_x(t)]_1) = t$.

Notice that ψ is an invertible map when restricted to its image. Moreover, since we are assuming a non-atomic disintegration, ψ^{-1} is a continuous map restricted to the second coordinate and a measurable map when restricted to the first coordinate. Maps of these type are known as Caratheodory functions and these are measurable maps ([2, Lemma 4.51]).

Consider the set $H_t^0 := \psi\left(D^3 \times [0,t]\right)$, which is measurable since ψ^{-1} is Caratheodory. Thus, the set $H_t = \bigcup_{n \in \mathbb{Z}} f^n(H_t^0)$ forms an invariant measurable set. Let $\hat{\mathcal{B}}$ be the set given by

$$\hat{\mathcal{B}} := \{Q(x) : x \in \mathcal{B}\}.$$

Remember that $\mathcal{B}$ is a foliated box around $Q = Q(x_0)$ then, $Q \in \hat{\mathcal{B}}$. Moreover, since $Q \in \operatorname{supp}(\hat{\mu})$, $\hat{\mathcal{B}}$ has positive $\hat{\mu}$-measure.

Notice that by the definition of ψ we have that if $0 < t < 1$

$$\mu(H_t) = \int \mu_x(H_t \cap Q(x)) d\hat{\mu}(Q(x)) \geqslant \int_{\hat{\mathcal{B}}} \mu_x(H_t^0 \cap Q(x)) d\hat{\mu}(Q(x)) \geqslant \int_{\hat{\mathcal{B}}} \mu_x([x, \theta_x(t)]_1) d\hat{\mu}(Q(x)),$$

and $\int_{\hat{\mathcal{B}}} \mu_x([x,\theta_x(t)]_1)d\hat{\mu}(Q(x)) = \hat{\mu}(\hat{\mathcal{B}})t > 0$, thus $\mu(H_t) > 0$. On the other hand, define $G_t^0 = \left(H_t^0\right)^c$ and the f-invariant set $G_t = \bigcup_{n\in\mathbb{Z}} f^n(G_t^0)$. In a similar way as before we have

$$\mu(G_t) \geqslant \hat{\mu}(\hat{\mathcal{B}})(1-t) > 0.$$

Therefore, by the ergodicity, both sets should have full measure. Although, this would imply that their intersection also should have full measure but we claim this is not the case. In fact, if it were true, for $\hat{\mu}$-almost every $Q(x)$

$$\mu_x(H_t \cap G_t \cap Q(x)) = 1.$$

But if ω belongs to $H_t \cap G_t \cap Q(x)$, without loss of generality, we may assume that for some $n \in \mathbb{N}$, $\omega \in f^{-n}(H_t^0) \cap G_t^0$. Hence, since f preserves orientation, it is easy to see that

$$t < \mu_\omega([0_\omega,\omega]_1) = (f^n)_*\mu_\omega\left(f^n([0_\omega,\omega]_1)\right) \leqslant \mu_{f^n(\omega)}\left(\left[0_{f^n(\omega)}, f^n(\omega)\right]_1\right) \leqslant t.$$

This is an absurd, which implies that the disintegration of μ is atomic for $\hat{\mu}$-almost every point. □

We have proved that μ has atomic disintegration with respect the partition $\mathcal{Q}$. We now want to see that there is a finite number of atoms on the disintegration considered. In order to do that we first need to prove the measurability of certain sets.

Consider $\mathcal{B}$ a foliated (by $\mathcal{F}^1$) box, as before, and identify $\mathcal{B}$ with the product $D^3 \times D^1$ through the corresponding homeomorphism.

Fix $\delta > 0$, and consider the set

$$H_\delta = \{x \in \mathcal{B} : \mu_x(\{x\}) \geqslant \delta\}.$$

Let us see that this is a measurable set. To do so, consider a countable basis $\mathcal{V}$ of the topology of $\mathbb{T}^4$. From Rokhlin's theorem 2.4.4 we know that the map $x \mapsto \mu_x(V)$ is measurable (up to measure zero) for any measurable set V. Therefore, by Lusin's theorem, given any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset D^3$ such that $\hat{\mu}(K_\varepsilon) > 1-\varepsilon$ and $x \mapsto \mu_x(V)$ is continuous on K_ε, for every $V \in \mathcal{V}$. In particular, $x \mapsto \mu_x$ is continuous with respect to the weak*-topology for any x in K_ε.

Let $\varepsilon > 0$ be fixed. For each $x \in C$, let $A(x)$ be the set of atoms of μ_x. It is clear that the set

$$\tilde{\Gamma}_\delta(x) := \{a \in A(x) : \mu_x(\{a\}) \geqslant \delta\}, \tag{4.2.3}$$

is finite, and hence compact. Furthermore, the definition of K_ε ensures that the function $x \mapsto \tilde{\Gamma}_\delta(x)$ is upper semi-continuous on $x \in K_\varepsilon$. Therefore,

$$\Gamma(\varepsilon,\delta) := \{(x,a) : x \in K_\varepsilon \text{ and } a \in \tilde{\Gamma}_\delta(x)\},$$

is a closed set. Then, $\bigcup_n \Gamma(1/n, \delta)$ is a (measurable) full measure subset of H_δ. Thus, H_δ is a measurable set (up to measure zero).

Consider $\{\mathcal{B}_k : k \in \mathbb{N}\}$ a countable cover of $\mathbb{T}^4$ by foliated boxes. Proceeding as before, we obtain the measurable sets H_δ^k of atoms of measure bigger or equal to δ in each foliated box $\mathcal{B}_k$. Therefore,

$$H_\delta^+ := \bigcup_{k\in\mathbb{N}} H_\delta^k = \{x \in C : \mu_x(\{x\}) \geqslant \delta\}$$

is also measurable.

Lemma 4.2.6. *$\hat{\mu}$-almost every $Q(x)$ contains only one atom.*

Proof. Let $x \in \mathcal{M}$ and $\delta \geqslant 0$. Consider the set H_δ^+ as before and notice that

$$\delta \leqslant \mu_x(\{x\}) \leqslant f_*\mu_x(\{f(x)\}) = \mu_{f(x)}(\{f(x)\}).$$

Therefore, H_δ^+ is invariant and, by ergodicity, it has measure zero or one. We know that $\mu(H_1^+) = 0$ and $\mu(H_0^+) = 1$. Let δ_0 be the discontinuity point of the function $\delta \mapsto \mu(H_\delta^+)$, for $\delta \in [0, 1]$. Hence $\mu(H_{\delta_0}^+) = 1$, that means the weight of the atoms are all equal to δ_0. Therefore there are $n = 1/\delta_0$ atoms on each element of the partition $\mathcal{Q}$.

Let us see that the disintegration of μ on $\mathcal{Q}$ has one atom per local leaf. Assume by contradiction that $n = 2$, as the case of finite atoms is similar. Let $a(x)$ and $b(x)$ be the two atoms of μ_x. Without loss of generality, let us assume that $a(x) < b(x)$, where "<" is the fixed order in $\mathcal{F}^1$. Consider

$$\hat{A} := \{a(x) : x \in C\} \text{ and } \hat{B} := \{b(x) : x \in C\},$$

the sets of first and second atoms respectively. Since f preserves the orientation in $\mathcal{F}^1$, it is easy to see that $\hat{A}$ and $\hat{B}$ are invariant sets.

Let $Q = Q(x_0) \in \operatorname{supp}\hat{\mu}$ and let $\hat{V}$ be an open neighborhood of Q. Consider the disjoint sets

$$B(a) := \bigcup_{Q(x)\in\hat{V}} \{x\} \times B(a(x)) \text{ and } B(b) := \bigcup_{Q(x)\in\hat{V}} \{x\} \times B(b(x)),$$

where $B(a(x))$ and $B(b(x))$ are two disjoint closed balls in $Q(x)$ around $a(x)$ and $b(x)$ respectively. Notice that, following the proof of the measurability of H_δ^+ by substituting the set $\tilde{\Gamma}_\delta(x)$ in (4.2.3) by $B(a(x))$, we can prove that $B(a)$ and $B(b)$ are both measurable sets. By the definition of $B(a)$ and $B(b)$, their saturation by $\mathcal{Q}$ coincide, that is, $\pi(B(a)) = \pi(B(b))$. Therefore, $B(a)$ and $B(b)$ have positive μ-measure.

Let us define the f-invariant sets

$$H(a) := \bigcup_{n\in\mathbb{Z}} f^n(B(a)) \text{ and } H(b) := \bigcup_{n\in\mathbb{Z}} f^n(B(b)).$$

We claim that $\mu(H(a) \cap H(b)) = 0$. In fact, if it is not true we have that

$$0 < \mu(H(a) \cap H(b)) = \int \mu_x(H(a) \cap H(b) \cap Q(x)) d\hat{\mu}.$$

Therefore, there must exist $\hat{\Lambda} \subset \hat{C}$ of positive $\hat{\mu}$-measure such that for every $Q(x) \in \hat{\Lambda}$, $\mu_x(H(a) \cap H(b) \cap Q(x)) > 0$. Hence, $a(x)$ or $b(x)$ must belong to the intersection of $H(a) \cap H(b)$. Without loss of generality, let us assume there exists $n \in \mathbb{Z}$ such that $a(x) \in f^n(B(b)) \cap B(a)$. Therefore, we have that $f^{-n}(a(x)) = b(y)$ for some $Q(y) \in \hat{V}$. However, this contradicts the invariance of $\hat{A}$, and our claim follows.

Now, by ergodicity of μ, the sets $H(a)$ and $H(b)$ should have full measure and have zero measure intersection. Absurd, therefore we have only one atom on $\mathcal{Q}(x)$ which proves our claim. □

Let us denote the atom found in Lemma 4.2.6 by $a(x)$, that is,

$$\mu_x = \delta_{a(x)}. \tag{4.2.4}$$

We now want to see that the disintegration of μ on $\mathcal{P}$ has only one atom in each connected component of every element of the partition. Consider $\widehat{C} := C/\mathcal{Q}$. Define $\hat{f} : \widehat{C} \to \widehat{C}$ by $\hat{f}(\hat{z}) := \widehat{f(z)}$, which satisfies $\pi \circ f = \hat{f} \circ \pi$.

Notice that by (4.2.2) we can identify $\widehat{\mathcal{P}(x)}$ with n_x connected components in the $\mathcal{F}^2$ foliation. This implies that the space $\widehat{C}$ has now a one dimensional foliation coming from this quotient. Consider the partition $\widehat{\mathcal{Q}}$ given by $\widehat{\mathcal{Q}} := \{\widehat{R(x)} : x \in C\}$, where $R(x)$ is the rectangle in $\mathcal{P}(x)$ containing x. Moreover, notice that $\widehat{R(x)}$ can be identified with the interval $[c_0(x), c_1(x)]_2$, where $c_0(x)$ and $c_1(x)$ are the corners of $R(x)$ in the same $\mathcal{F}^2$-leaf. Consequently, proceeding as before, the conditional measures $\hat{\eta}_x$ defined by the partition $\widehat{\mathcal{Q}}$ have at most one atom in each $\widehat{R(x)}$ that we denote by $\hat{a}(x)$. Thus,

$$\hat{\eta}_x = \delta_{\hat{a}(x)}. \tag{4.2.5}$$

Combining this with (4.2.4) we have that $a_j(x) \in \pi^{-1}(\hat{a}(x)) \cap R_j(x)$ is the only one atom per rectangle $R_j(x)$. □

Corollary 4.2.7. *Under the assumptions of Theorem 4.2.4. Let us assume that $\nu := H_*\mu$ has full support and the semiconjugacy H sends center leaves of f to center leaves of A. If one of the following conditions is satisfied:*

1. *The center direction of A is expanding or contractive.*
2. *$H(\mathcal{F}^i)$ is some invariant foliation of A, for each $i = 1, 2$.*

Then, μ is virtually hyperbolic with one only atom per center leaf (see 2.4.7).

Proof. We are left with the task of proving that if H sends center leaves of f to center leaves of A and if one of the conditions 1 or 2 are satisfied, then μ is virtually hyperbolic.

First, let us assume 1 is valid. Moreover, we assume that the center direction of A is expanding, otherwise we work with f^{-1}. By the proof of Theorem 4.2.4, there are at most countably many elements in $\mathcal{P}$ with positive measure, we get a full measurable subset $\mathcal{M} \subset \mathbb{T}^4$ which intersects each center leaf in at most countably many points. Furthermore, we claim that there are finitely many atoms of μ per (global) center leaf. In fact, this was proved in [22, Proposition 3.2]. Although they assume one dimensional center foliation, under our assumptions their proof could be applied. Let us recall the main steps.

Assume by contradiction that every full measurable subset of $\mathcal{M}$ intersects any typical center leaves in infinitely many points. Define $\nu = H_*\mu$ which is an invariant measure by the linear hyperbolic automorphism. Let R_i be the Markov partition for A (see [39, Chapter IV]), and consider the partition $\mathcal{Q} := \{\mathcal{F}^c_R(x) : x \in R_i \text{ for some } i\}$, where $\mathcal{F}^c_R(x)$ denotes the connected component of $\mathcal{F}^c(x) \cap R_i$ containing x. The partition $\mathcal{Q}$ is measurable and we denote ν_x the disintegration of ν along the elements of $\mathcal{Q}$. The assumption of full support of ν guarantees it gives zero mass to the boundary of the Markov partition.

As $H(\mathcal{M})$ intersects typical leaves in a countable number of points, ν_x must be atomic. Moreover, there exists a natural number $\alpha_0 \in \mathbb{N}$ such that ν_x contains exactly α_0 atoms for ν-almost every x (see [22, Lemma 3.3]). Hence, given a fixed $L \in \mathbb{R}_+$, there exists $N \in \mathbb{N}$ such that the number of atoms in any typical center plaque of diameter L is at most N. We are assuming that $H(\mathcal{M})$ intrinsically intersects center leaves in infinitely many points (or non-uniformly finite). Taking $D \subset \mathcal{F}^c(x)$ with more than N atoms. By backward contraction along central leaves by A, there exists $n > 0$ such that the diameter of $A^{-n}(D)$ is less than L. As ν is invariant and the disintegration is essentially unique, we get a center plaque with diameter less than L containing more than N atoms, which is absurd and establishes our claim.

We have proved that the number of atoms is finite and constant by ergodicity on almost every center leaf. The task is now to conclude that, since f preserves orientation, the number of atoms is one. In order to do this first consider the set of atoms in each $\mathcal{F}^1$-leaf. Proceeding as in the proof of Lemma 4.2.6, we can prove that there must be only one atom per $\mathcal{F}^1$-leaf. Now consider the space $\widetilde{C} := C/\sim$, where $x \sim y$ iff $y \in \mathcal{F}^1(y)$. The way we should see $\tilde{C}$ is as turning the center foliation (which is a plane) into a one-dimensional segment. Let us denote this new foliation as $\tilde{\mathcal{Q}}$. Notice that the disintegration of μ in the partition given by $\tilde{\mathcal{Q}}$ is exactly the quotient measure $\hat{\eta}^x$.

Since $\mathcal{F}^1$ has an orientation, we may define a transversal orientation by the following way: a vector $v \in T_x\mathcal{F}^c_{loc}(x)$ points in the positive direction if for any positive vector $w \in T_x\mathcal{F}^c_{loc}(x)$ we have $\omega_x(v, w) > 0$, where ω_x is the restriction of the volume form

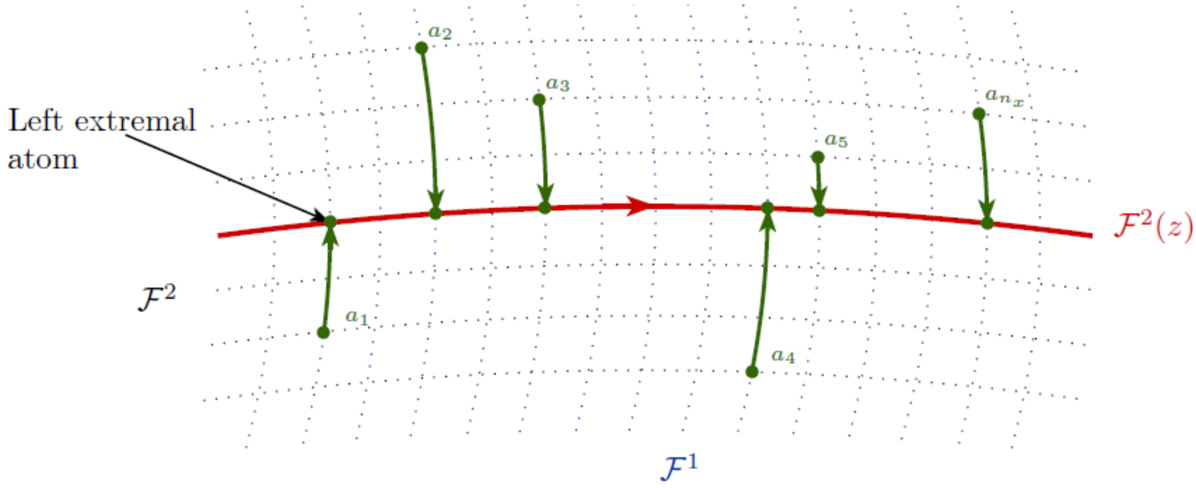


Figure 5 – Global atom in a center leaf [5].

to $\mathcal{F}^c_{loc}(x)$.

Now consider the extremal atoms per central leaf. By left extremal atom we consider the atom whose projection by the map $\pi : C \to \widetilde{C}$ is the left extreme one (see Figure 5). Since f preserves the orientation in $\mathcal{F}^1$, then f preserves the transversal orientation. Once again, proceeding as the proof of Lemma 4.2.6 we conclude that there is only one atom per global center leaf. Therefore, μ is virtually hyperbolic.

On the other hand, if 2 is valid then $H(\mathcal{F}^1)$ must coincides with E^s_A or E^u_A. Without loss of generality, let us assume $H(\mathcal{F}^1)$ coincides with E^u_A. By the proof of Theorem 4.2.4, the set $\mathcal{M}$ intersects each $\mathcal{F}^1$ leaf in at most countably many points. Proceeding as before, using the $\mathcal{F}^1$ foliation instead the center one, then μ is virtually hyperbolic. □

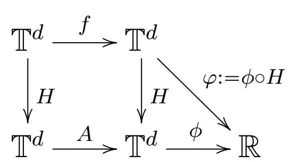
4.3 Proof of Theorem E

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a DA partially hyperbolic diffeomorphism satisfying A.5.

Remark 4.3.1. *Take μ any f-invariant measure and let $\nu = H_*\mu$. It is well-known that $h_\mu(f) \geqslant h_\nu(A)$. From Ledrappier-Walters variational principle 2.1.9 and assumption A.5 we have that*

$$h_\mu(f) = h_\nu(A). \tag{4.3.1}$$

Let $\phi \in C^0(\mathbb{T}^d, \mathbb{R})$ be a Hölder potential for A and consider the potential $\varphi := \phi \circ H \in C^0(\mathbb{T}^d, \mathbb{R})$ for f.



Proposition 4.3.2. *Under the above assumptions, the following properties hold:*

1. *If ν is an equilibrium state for (A, ϕ), then every $\mu \in \mathcal{M}(f)$ such that $H_*\mu = \nu$ is an equilibrium state for (f, φ);*

2. *If $\mu \in \mathcal{M}_e(f)$ and $\mu(C) = 0$, then μ is the unique equilibrium state.*

Proof. Let ν be an equilibrium state for (A, ϕ). By the Riezs theorem and the compactness of the set of Borel probability measures on $\mathbb{T}^d$, we can guarantee the existence of an f-invariant measure μ such that $\nu = H_*\mu$ (see for example [11, Lemma 4.3] for a similar construction). Moreover, by (4.3.1) we have that

$$\begin{aligned}
\sup\left\{h_\eta(f) + \int \varphi d\eta : \eta \in \mathcal{M}(f)\right\} &= \sup\left\{h_{H_*\eta}(A) + \int \phi dH_*\eta : \eta \in \mathcal{M}(f)\right\}, \\
&\leqslant \sup\left\{h_{\hat{\nu}}(A) + \int \phi d\hat{\nu} : \hat{\nu} \in \mathcal{M}(A)\right\}, \\
&\leqslant h_\nu(A) + \int \phi d\nu.
\end{aligned}$$

Therefore, any f-invariant measure μ satisfying that $\nu = H_*\mu$ is an equilibrium state for $(f, \varphi = \phi \circ H)$.

For proving the second statement, we will argue as in [22, Theorem A]. Let μ be an ergodic equilibrium state such that $\mu(C) = 0$ and ν be the unique equilibrium state of (A, ϕ). Assume that there exists η another equilibrium state for $(f, \phi \circ H)$, and by the uniqueness of ν we have that $H_*\mu = H_*\eta$. Let $\psi : \mathbb{T}^d \to \mathbb{R}$ be any continuous map. Since $H^{-1}H(C) = C$ follows that $\eta(C) = 0$. Therefore,

$$\int \psi d\mu = \int_{\mathbb{T}^d - C} \psi d\mu = \int_{\mathbb{T}^d - C} \psi \circ H^{-1} \circ H d\mu = \int_{\mathbb{T}^d - C} \psi \circ H^{-1} dH_*\mu = \int \psi d\eta.$$

Since ψ is arbitrary, this implies that $\mu = \eta$. □

Theorem 4.3.3 (Theorem E)**.** *Let $f : \mathbb{T}^4 \to \mathbb{T}^4$ be a DA partially hyperbolic diffeomorphism satisfying A.1, A.2, A.3, A.4 and A.5. Assume that f preserves the orientation of $\mathcal{F}^i$, $i = 1, 2$. Let ϕ be a continuous potential such that (A, ϕ) has a unique equilibrium state with full support and define the potential $\varphi = \phi \circ H$. For every μ ergodic equilibrium state of f with respect to φ:*

1. *If $\mu(C) = 0$, then μ is the unique equilibrium state;*

2. *If $\mu(C) = 1$, then C defines a partition such that μ has atomic disintegration with a finite number of atoms. Moreover, if the semiconjugacy H sends center leaves of f to center leaves of A and one of the following conditions is satisfied*

   a) The center direction of A is expanding or contractive.

b) $H(\mathcal{F}^i)$ is some invariant foliation of A, for each $i = 1, 2$.

Then, μ is virtually hyperbolic and it is not a unique equilibrium state φ.

Proof. **Case 1:** $\mu(C) = 0$. It follows directly from statement 2 of Proposition 4.3.2.

Case 2: $\mu(C) = 1$. Consider the partition:

$$\mathcal{P} := \{\mathcal{P}(x) := H^{-1}H(x) : x \in C\},$$

and denote by μ_x the conditional measure of μ supported on $\mathcal{P}(x)$. We proceed as in the proof of Theorem 4.2.4. Hence, we have that

$$\mu_x = \sum_{j=1}^{n_x} p_j(x)\delta_{a_j(x)},$$

for some $a_j(x) \in R_j(x)$. Moreover, if conditions 1 and 2 are satisfied, then μ is virtually hyperbolic. The only thing left to prove is the existence of another equilibrium state.

Lemma 4.3.4. *If H sends center leaves of f to center leaves of A and 2b is satisfied, then the set of extremal points of intervals $Q(x) = \mathcal{P}(x) \cap \mathcal{F}^1(x)$ forms a measurable set.*

Proof. Let us denote $\mathcal{F}^1_A$ the foliation of the center direction of A induced by the image of $\mathcal{F}^1$ by the semiconjugacy H. We will prove the measurability of the lower extremal points of $Q(x)$. The case of higher extremal points is similar.

Consider $\varphi : \mathbb{T}^4 \to \mathbb{T}^4$ the flow on $\mathbb{T}^4$ having constant speed one in $\mathcal{F}^1_A$. More precisely, we know that the leaves of $\mathcal{F}^1$ in the center foliation of A are straight lines and orientable by assumption. Define $\varphi(t, x)$ the unique point in the $\mathcal{F}^1_A(x)$ which has distance t inside this $\mathcal{F}^1_A$-leaf and in the positive direction from x.

Following the proof of [45, Lemma 3.2], we have that $H(C)$ is a measurable set. Therefore, $\varphi(-1/n, H(C))$ is a measurable set. Furthermore, since H is continuous, the set $H^{-1}(\varphi(-1/n, H(C)))$ is also measurable.

Consider $\hat{C} = C/\mathcal{Q}$ where $\mathcal{Q} := \{Q(x) := \mathcal{P}(x) \cap \mathcal{F}^1(x) : x \in C\}$. Let

$$\phi_n : \hat{C} \to H^{-1}(\varphi(-1/n, H(C))),$$

be the function given by the Measurable Choice Theorem [45, Theorem 2.11] applied to the product $\hat{C} \times \mathbb{T}^4$ and the measurable set $G = H^{-1}(\varphi(-1/n, H(C)))$.

Notice that fixing $Q(x) \in \hat{C}$ we have that $\phi_n(Q(x))$ is an increasing sequence. Therefore, we can define the function

$$\begin{aligned} \phi : &\hat{C} \to \mathbb{T}^4 \\ &Q(x) \mapsto \lim_{n\to\infty} \phi_n(Q(x)), \end{aligned}$$

and by its construction $\phi(Q(x))$ is the lower extreme of $Q(x)$. Notice that ϕ is a measurable function since it is the limit of measurable functions. Let π be the canonical projection and let $\hat{\mu} = \pi_*\mu$ the measure in the quotient space. By Lusin's theorem for any $n \in \mathbb{N}$ there exists a compact set $\hat{K}_n \subset \hat{C}$ such that $\hat{\mu}(\hat{K}_n^c) < 1/n$ and ϕ is a continuous function when restricted to $\hat{K}_n$. Therefore, $\phi(\hat{K}_n)$ is a compact set. Without loss of generality we may consider $\hat{C} = \bigcup_{n\in\mathbb{N}} \hat{K}_n$. Therefore,

$$\phi(\hat{C}) = \bigcup_{n\in\mathbb{N}} \phi(K_n),$$

is a measurable set. Thus, we have proven so far that the base of the intervals from $\mathcal{Q}$ forms a measurable set. □

We have seen that the center foliation is measure theoretically equivalent to the partition of $\mathbb{T}^4$ into points, hence measurable. Let us denote $(\hat{M}, \hat{\mu})$ the quotient space $\hat{M} := \mathbb{T}^4/\mathcal{F}^c$ equipped with the quotient measure. We denote by $\hat{f} : \hat{M} \to \hat{M}$ the induced map on the quotient space. Therefore, since μ if f-invariant then $\hat{\mu}$ is $\hat{f}$-invariant.

Notice that, by the virtual hyperbolicity proved above, every element $\hat{x} \in \hat{M}$ can be identified by the unique $\mathcal{Q}_x(z) \subset \mathcal{F}^c(x)$ where its atom belongs to. When $\mathcal{Q}_x(z)$ is a collapse interval inside a $\mathcal{F}^1$-leaf, we define $\mathcal{Q}(\hat{x}) := \mathcal{Q}_x(z)$. On the other hand, if $\mathcal{Q}_x(z)$ is a point, this means that the rectangle R_j containing the atom is one dimensional and contained in an $\mathcal{F}^2$-leaf. In this case, we define $\mathcal{Q}(\hat{x}) := R_j$ (see Figure 4).

Now, we will argue as in the final part of [22, Section 3]. Thus, we can write

$$\mu = \int \delta_{a(\hat{x})} d\hat{\mu},$$

where $a(\hat{x})$ is the atom inside the collapse interval $\mathcal{Q}(\hat{x})$. Choose $b(\hat{x}) \neq a(\hat{x})$ the left (or right) extreme point of $\mathcal{Q}(\hat{x})$. Let us define

$$\eta = \int \delta_{b(\hat{x})} d\hat{\mu},$$

which is well-defined because $\{b(\hat{x}) : \hat{x} \in \hat{M}\}$ is measurable by Lemma 4.3.4. We claim that this is an f-invariant ergodic measure satisfying $H_*\eta = H_*\mu$. In order to see this, consider any continuous map ψ and notice that

$$\begin{aligned} \int \psi \circ f d\eta &= \int\int \psi \circ f d\delta_{b(\hat{x})} d\hat{\mu} = \int \psi(f(b(\hat{x})))d\hat{\mu} = \int \psi(b(\hat{f}(\hat{x})))d\hat{\mu} \\ &= \int \psi(b(\hat{x}))d\hat{\mu} = \int \psi d\eta, \end{aligned}$$

where the third equality comes from the invariance of collapse intervals and that f preserves the orientation of the $\mathcal{F}^i$-foliations with $i = 1, 2$. The fourth equality is due to the $\hat{f}$-invariance of $\hat{\mu}$.

To see the ergodicity of η, consider any invariant subset D with positive η-measure. Since $\hat{\mu}$ is ergodic and $f(b(\hat{x})) = b(\hat{f}(\hat{x}))$, we have that the set $\{\hat{x} : \mathcal{X}_D(b(\hat{x})) = 1\}$ is $\hat{f}$-invariant. So the ergodicity of $\hat{\mu}$ guarantees it has full measure, which implies $\eta(D) = 1$.

Notice that, if $\varphi = \phi \circ H$, since $H(a(\hat{x})) = H(b(\hat{x}))$ then

$$\int \varphi d\eta = \int \varphi(b(\hat{x})) d\hat{\mu} = \int \varphi(a(\hat{x})) d\hat{\mu} = \int \varphi d\mu.$$

However, by the essential uniqueness of disintegration, we have that $\eta \neq \mu$. It remains to prove that $h_\eta(f) = h_\mu(f)$. But this is a direct consequence of the fact that (f, μ) and (f, η) are measure theoretically isomorphic by the map that sends $a(\hat{x})$ to $b(\hat{x})$. Thus, η is also an equilibrium state form (f, φ). □

Remark 4.3.5. *We remark that the results of Theorem 4.2.4 and Theorem 4.3.3 are valid for DA partially hyperbolic diffeomorphisms $f : \mathbb{T}^d \to \mathbb{T}^d$ with k-dimensional center bundle ($1 \leqslant k < d$), provided that they satisfy the assumptions A.1, A.4,A.5 and analogous assumptions to A.2, A.3.* The proof for the higher dimensional case follows in a similar way as in the two-dimensional case.

5 Final Considerations

1. Let $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ be as in the setting of Chapter 3. Is f transitive? Is there a unique maximal entropy measure when $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ is transitive?

2. In the setting of Theorem B, is it possible to obtain that there is exactly one maximal entropy measure?

3. An interesting problem is to produce an example of a transitive partially hyperbolic diffeomorphism (e.g. derived from Anosov) with more than one maximal entropy measure. Maybe, trying to take two periodic orbits and putting a horseshoe on them whose entropy exceeds the entropy of the system, it seems that it should be possible.

4. There is a lack of examples of partially hyperbolic diffeomorphisms with two-dimensional center bundle, where the maximal entropy measure (or even, some measure with large entropy), has one zero center Lyapunov exponent and the other is different from zero. In the literature there are two known examples, here we give a brief presentation:

   a) Take an example in $\mathbb{T}^3$ homotopic to $Anosov \times Id$ so that it is not extension by rotations but is accessible, where the main result from [50] guarantees that the maximal entropy measures have non-zero Lyapunov exponents, and multiply by identity on unitary circle $\mathbb{S}^1$.

   b) Take the example of Herman's cocycle over an irrational rotation in $\mathbb{T}^2$, where the action in the fibers is the linear cocycle which is explained in the Avila-Bochi Trieste notes (`http://www.mat.uc.cl/~jairo.bochi/docs/trieste.pdf`), and multiply by an Anosov diffeomorphism of $\mathbb{T}^2$.

   Are there examples more interesting than these two?

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