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Persistence of invariant tori in singularly perturbed systems and the averaging method

Persistência de toros invariantes em sistemas singularmente perturbados e o método da média

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“ Exaudi orationem meam, Domine, et deprecationem meam;
auribus percipe lacrimas meas.
Ne sileas, quoniam advena ego sum apud te, et peregrinus sicut omnes patres mei.
(Biblia Sacra Vulgata, Psalmi 38, 13)

Resumo

Essa dissertação é focada no estudo de condições suficientes para a existência de toros invariantes no espaço de fase estendido de sistemas padrão da Teoria da Média, também chamada de Teoria *Averaging*. A demonstração dos resultados está fortemente ancorada na teoria de variedades invariantes normalmente hiperbólicas. Serão apresentados conceitos básicos dessas duas teorias, necessários para a compreensão completa da demonstração que iremos desenvolver. O percurso da demonstração também nos incentivou a apresentar uma discussão sobre a existência de folheações invariantes pelo fluxo de vizinhanças de ciclos limites atratores hiperbólicos no plano.

Palavras-chave: Teoria Averaging; Teoria da Média; Variedades invariantes normalmente hiperbólicas; Toros invariantes.

Abstract

This dissertation centers on results providing sufficient conditions for the existence of invariant tori in the extended phase space of systems in the standard form according to the averaging theory. The proof of the results relies strongly on the theory of normally hyperbolic invariant manifolds. Fundamental concepts of those two theories will be presented, as they will be necessary for full comprehension of the proof we will present for the main result. The path to proving this result has also motivated us to introduce a brief exposition of results regarding the existence of foliations of neighbourhoods of attracting hyperbolic limit cycles that are invariant under the flow of the field.

Keywords: Averaging theory; Normally hyperbolic invariant manifolds; Invariant tori.

Contents

Introduction

An important method in the study of nonlinear oscillating systems which are affected by small perturbations is the *Averaging Theory.* While its origins can be traced back to names such as Clairaut, Laplace and Lagrange, this theory was rigorously formalised only later, in the 20th century - see (FATOU, 1928) and (BOGOLIUBOV; MITROPOLSKY, 1961), for instance. The theory is mainly applied to provide long-time asymptotic estimates for solutions of non-autonomous differential equations given in the following standard form:

$$\dot{x} = \sum_{i=1}^{N} \varepsilon^i F_i(t, x) + \varepsilon^{N+1}\tilde{F}(t, x, \varepsilon), \tag{1}$$

where $\tilde{F} : \mathbb{R} \times D \times [0, \varepsilon_0]$ and each $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ are smooth functions, T-periodic in t. The set D is an open subset of $\mathbb{R}^n$ and $\varepsilon_0 > 0$ is assumed to be small. The estimates provided by the Averaging Theory are related to solutions of an autonomous system, named the *truncated average equation*, which has the form

$$\dot{\xi} = \sum_{i=1}^{N} \varepsilon^i g_i(\xi). \tag{2}$$

As the name indicates, this equation is obtained by truncating a different equation. In fact, one fundamental result of the Averaging Theory is precisely that there is a change of coordinates under which system (1) is transformed into

$$\dot{\xi} = \sum_{i=1}^{N} \varepsilon^i g_i(\xi) + \varepsilon^{N+1} G(t, \xi, \varepsilon). \tag{3}$$

We remark that g_1 is in general the time average of $F_1(t, x)$, that is,

$$g_1(\xi) = \frac{1}{T}\int_0^T F_1(s, \xi)\, ds.$$

Motivated by this identity, each g_i is named *averaged function of order i* of (1). This theory has found great success when applied to investigate invariant manifolds of differential systems (HALE, 1961). In particular, it has been extensively used to study periodic solutions, for example (BUICĂ; LLIBRE, 2004). Examples of results regarding the relation between simple zeros of the first-order averaged function, g_1, and isolated T-periodic solutions of (1) can be found in (HALE, 1961), (HALE, 1980), and (VERHULST, 1996).

Let $\ell \in \{1, 2, ..., k\}$ be the first index for which g_i is not identically zero. The main result of this work is concerned with sufficient conditions for the existence of invariant tori in the extended phase space of systems of the form (1) in $\mathbb{R}^2$ that satisfy the following hypothesis:

Hypothesis A: The differential system $\xi' = g_\ell(\xi)$ has an attractive hyperbolic limit cycle.

The main result of this work is stated in Theorem A of next chapter, and basically claims that a system of the form (1) in $\mathbb{R}^2$ that satisfies Hypothesis A has an invariant torus in the extended phase space.

This dissertation is divided in four chapters. In the first one, we will properly state the main result of this work, and show how it relates to the Averaging Theory. In the second chapter, we will state some preliminary results concerning methods that we will employ to prove the main result, such as the theory of normally hyperbolic invariant manifolds. In the third one, we will prove a fundamental proposition that will be used for the proof of our result. This proposition requires some discussion about the existence of invariant foliations of neighbourhoods of hyperbolic limit cycles, which will be present in the same section. The last chapter will contain the proof of our main result. This proof will be divided in several lemmas, which will be proved in the same section.

1 Main Theorem

Let $N \in \mathbb{N}$ such that $N \geqslant 1$. Consider the system

$$\dot{x} = \sum_{i=1}^{N} \varepsilon^i F_i(t,x) + \varepsilon^{N+1}\tilde{F}(t,x,\varepsilon), \tag{1.1}$$

where $\tilde{F} : \mathbb{R} \times D \times [0,\varepsilon_0] \to \mathbb{R}^2$ and $F_i : \mathbb{R} \times D \to \mathbb{R}^2$, for $i = 1,2,...,N$, are smooth functions, T-periodic in t. The set D is an open subset of $\mathbb{R}^2$ and $\varepsilon_0 > 0$ is small. System (1.1) is said to be in *standard form.* We will make use of the following theorem, proved in (SANDERS; VERHULST; MURDOCK, 2007):

Theorem 1. *There is a T-periodic transformation,*

$$x = U(t,\xi,\varepsilon) = \xi + \sum_{i=1}^{N} \varepsilon^i u_i(t,\xi),$$

under which system (1.1) is transformed into

$$\dot{\xi} = \sum_{i=1}^{N} \varepsilon^i g_i(\xi) + \varepsilon^{N+1} G(t,\xi,\varepsilon). \tag{1.2}$$

Let $\ell \in \{1,2,...,N\}$ be the first index for which g_i is not identically zero. Then, system (1.2) can be written as

$$\dot{\xi} = \varepsilon^\ell g_\ell(\xi) + \varepsilon^{\ell+1}\tilde{G}(t,\xi,\varepsilon). \tag{1.3}$$

We assume that Hypothesis A is valid for system (1.1) and we apply a time rescaling of the form $s = \varepsilon^\ell t$. We will denote the derivative of ξ with respect to the new time s by ξ'. System (1.3) becomes:

$$\xi' = g_\ell(\xi) + \varepsilon\tilde{G}(s/\varepsilon^\ell,\xi,\varepsilon). \tag{1.4}$$

Note that, since $\tilde{F}_i$ and U are T-periodic in t, $\tilde{G}$ is periodic in s. The procedures that follows are based on a idea from Novaes and Cândido, which is being developed in a paper yet to be published . We apply an additional change of variables, which is a combination of a translation of the origin to the interior of the limit cycle present in Hypothesis A and a change to coordinates (ρ,σ) similar to polar coordinates with respect to which the limit cycle is given as a graph of a function of the angular variable, σ. Thus, system (1.4) may be rewritten:

$$\begin{aligned} \rho' &= f(\rho,\sigma) + \varepsilon R(\rho,\sigma,s/\varepsilon^\ell,\varepsilon), \\ \sigma' &= g(\rho,\sigma) + \varepsilon S(\rho,\sigma,s/\varepsilon^\ell,\varepsilon). \end{aligned} \tag{1.5}$$

The functions f, g, R, and S are smooth, and 2π-periodic in the angular variable σ. Moreover, $t \mapsto R(\rho, \sigma, t, \varepsilon)$ and $t \mapsto S(\rho, \sigma, t, \varepsilon)$ are periodic, and we will denote this period by $2\pi m/\Omega$, where $m \in \mathbb{N}$. We remark that solutions of the truncated system

$$\xi' = g_\ell(\xi) \tag{1.6}$$

correspond naturally to solutions of

$$\begin{aligned} \rho' &= f(\rho, \sigma), \\ \sigma' &= g(\rho, \sigma). \end{aligned} \tag{1.7}$$

In order to study (1.5), we introduce a new angular variable modulo $2\pi m\varepsilon^\ell/\Omega$, denoted by τ, and the autonomous system

$$\begin{aligned} \rho' &= f(\rho, \sigma) + \varepsilon R(\rho, \sigma, \tau/\varepsilon^\ell, \varepsilon), \\ \sigma' &= g(\rho, \sigma) + \varepsilon S(\rho, \sigma, \tau/\varepsilon^\ell, \varepsilon), \\ \tau' &= 1. \end{aligned} \tag{1.8}$$

Solutions of (1.8) correspond naturally to solutions of (1.5). Thus, the search for invariant tori in (1.5) can be performed via the study of (1.8). To study this system, we will consider it the result of a singular perturbation effected on the system

$$\begin{aligned} \rho' &= f(\rho, \sigma), \\ \sigma' &= g(\rho, \sigma), \\ \tau' &= 1. \end{aligned} \tag{1.9}$$

The result we seek to prove is:

Theorem A: Consider a differential system in the form (1.1). Suppose that $g_i = 0$ for $i = 1, 2, ..., \ell - 1$, where $0 < \ell \leqslant N$. If Hypothesis A holds, that is, the truncated equation $\xi' = g_\ell(\xi)$ has an attracting hyperbolic limit cycle, then system (1.1) has a normally hyperbolic invariant torus in the extended phase space.

The invariant torus appears if we consider the time variable, t, as an angular variable of the system (1.1), and treat it as an autonomous system in a space of dimension three. This can be done because we are working with functions which are periodic in t. The correspondence of systems discussed above allows us to state this result in terms of the transformed system (1.8). In this new setting, Hypothesis A is restated as follows:

Hypothesis B: The system

$$\begin{aligned} \rho' &= f(\rho, \sigma) \\ \sigma' &= g(\rho, \sigma), \end{aligned} \tag{1.10}$$

has an attracting hyperbolic limit cycle Γ that is the graph of a function of the angular variable σ.

The following result will take the place of Theorem A in our new setting:

Theorem B: Let k be an integer such that $2 \leqslant k \leqslant r$. Suppose that Hypothesis B is satisfied. Then, if $|\varepsilon|$ is sufficiently small, system (1.8) has a k-normally hyperbolic invariant manifold that is the graph of a function of the angular variables σ and τ.

The definition of a k-normally hyperbolic manifold will be given in the next chapter. We remark for now that the fact that the invariant manifold is a graph of a function of the angular variables is actually a stronger conclusion than we had envisioned in Theorem A. It will be proved that Theorem B is valid, so that Theorem A must also be valid.

As mentioned above, the proof of Theorem B will be done by studying a singular perturbation effected on system (1.9). To study this perturbation, we will make use of the theory of normally hyperbolic invariant manifolds and of a continuation method, following Chicone in (CHICONE; LIU, 1999/00). First, we will introduce briefly the theory of normally hyperbolic invariant manifolds. This is done in the next section, where other useful results will be discussed.

2 Preliminaries

We begin this section with a brief introduction of the theory of normally hyperbolic invariant manifolds. Afterwards, we present results related to the Arzelà-Ascoli theorem and results derived from the Grönwall inequality that will be used later on in this work.

2.1 Normally hyperbolic invariant manifolds

We shall present the definition of normally hyperbolic invariant manifolds and a fundamental theorem regarding those objects proved by Fenichel in (FENICHEL, 1971/72). Afterwards, we discuss some conditions that guarantee normal hyperbolicity for a special class of invariant manifolds with which we shall work. Our exposition is based mainly in (FENICHEL, 1971/72), (HIRSCH; PUGH; SHUB, 1977) and (WIGGINS, 1994).

2.1.1 Hyperbolic splittings

Let f be a C^1 vector field on $\mathbb{R}^n$, with flow ϕ^t. Let also M be a compact, connected C^1 manifold invariant under f. We suppose that M is properly embedded in $\mathbb{R}^n$, that is, each point in M has a neighbourhood U and coordinates (x, y) for U such that $M \cap U = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{n-d} : y = 0\}$.

We define $T\mathbb{R}^n|_M := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : a \in M\}$, the tangent bundle of $\mathbb{R}^n$ restricted to M, and we denote the tangent bundle of the manifold M by TM, as is customary. Moreover, the symbol $\oplus$ represents the Whitney sum of vector bundles. Let $T\mathbb{R}^n|_M = TM \oplus N^s \oplus N^u$ be a continuous splitting such that $TM \oplus N^s$ and $TM \oplus N^u$ are invariant under $D\phi^t$ for all t, that is, the fibre corresponding to an arbitrary $m \in M$ of one of those bundles is carried by $D\phi^t$ to the fibre corresponding to $\phi^t(m)$ of the same bundle.

Define π, π^s and π^u the projections on TM, N^s and N^u, respectively. For each $m \in M$, define

$$\nu^s(m) = \limsup_{t\to\infty} \|\pi^s D\phi^t(\phi^{-t}(m))|_{N^s}\|^{1/t} \quad \text{and}$$
$$\nu^u(m) = \limsup_{t\to-\infty} \|\pi^u D\phi^t(\phi^{-t}(m))|_{N^u}\|^{1/t}.$$

We say that this splitting is hyperbolic if $\nu^s(m) < 1$ and $\nu^u(m) < 1$ for all $m \in M$.

2.1.2 Definition of k-normally hyperbolic invariant manifolds

Suppose that M, f and ϕ^t are as above, and assume that we have a hyperbolic splitting $T\mathbb{R}^n = TM \oplus N^s \oplus N^u$. For each $m \in M$, define:

$$\sigma^s(m) = \limsup_{t\to\infty} \frac{\log \|D(\phi^{-t}|_M)(m)\|}{-\log \|\pi^s D\phi^t(\phi^{-t}(m))|_{N^s}\|} \quad \text{and}$$
$$\sigma^u(m) = \limsup_{t\to-\infty} \frac{\log \|D(\phi^{-t}|_M)(m)\|}{-\log \|\pi^u D\phi^t(\phi^{-t}(m))|_{N^u}\|}.$$

Let $k \in \mathbb{N}^*$. We say that M is k-normally hyperbolic if $\sigma^s(m) < 1/k$ and $\sigma^u(m) < 1/k$ for all $m \in M$. If $k = 1$, we say simply that M is normally hyperbolic.

The numbers ν^s, ν^u, σ^s and σ^u are called the generalised Lyapunov type numbers. An interesting result, proved in (FENICHEL, 1971/72), is that they are independent of the choice of metric on $T\mathbb{R}^n|_M$. They generally depend on the splitting.

A different definition of k-normally hyperbolic invariant manifolds is provided in (HIRSCH; PUGH; SHUB, 1977). If M is assumed to be a C^1 compact manifold, this definition is equivalent to the one presented above.

2.1.3 Fenichel's Theorem

The following theorem was proved by Fenichel in (FENICHEL, 1971/72). It ensures the persistence of normally hyperbolic invariant manifolds under regular perturbations.

Theorem 2. *Let f be a C^r vector field on $\mathbb{R}^n$, $r \geqslant 1$. Let M be a compact, connected C^r manifold properly embedded in $\mathbb{R}^n$ and invariant under f. Suppose that M is k-normally hyperbolic. Then, for any vector field g in some C^1 neighbourhood of f, there is a C^r manifold M_g invariant under g and C^r diffeomorphic to M.*

Remark: The same result is proved in (HIRSCH; PUGH; SHUB, 1977), albeit in a different setting. In this reference, it is explicitly proved that M_g is near M. In particular, if M and M_g are both given as graphs, respectively of the functions h_1 and h_2 of two angular variables, then, for each $r > 0$, there is $\epsilon > 0$ such that $|f - g|_{C^1} < \epsilon$ implies that $|h_1 - h_2|_{C^0} < r$, a result we will make use of later on.

2.1.4 Existence of invariant normal bundles

Once again, let M and f be as in Section 2.1.1. In (FENICHEL, 1971/72) Fenichel also proved, under specific conditions, the existence of a normal bundle over M

that is invariant under $D\phi^t$. With the same notation as before, we define:

$$\rho^s(m) = \limsup_{t\to\infty} \left(\|D(\phi^{-t}|_M)(m)\|\|\pi^s D\phi^t(\phi^{-t}(m))|_{N^s}\|\right)^{1/t} \quad \text{and}$$
$$\rho^u(m) = \limsup_{t\to-\infty} \left(\|D(\phi^{-t}|_M)(m)\|\|\pi^u D\phi^t(\phi^{-t}(m))|_{N^u}\|\right)^{-1/t}.$$

The theorem proved in that reference is the following:

Theorem 3. *If $\rho^s(m) < 1$ and $\rho^u(m) < 1$ for all $m \in M$, there are bundles I^s, and I^u in $T\mathbb{R}^n|_M$, homeomorphic to N^s and N^u and invariant under $D\phi^t$ for all t. $I^s \oplus I^u$ is transversal to TM.*

2.1.5 Normal hyperbolicity via contraction rates

In this section, we introduce sufficient conditions for an invariant manifold to be normally hyperbolic in the particular case of 2-dimensional C^1 invariant manifold in $\mathbb{R}^3$. Those conditions were extracted from (CHICONE; LIU, 1999/00).

Let f be a C^1 vector field on $\mathbb{R}^3$, with flow ϕ^t. Let M be a compact connected properly embedded 2-dimensional C^1 manifold. Suppose that M is invariant under the vector field f. Let $T\mathbb{R}^3|_M = TM \oplus N$ be a splitting of $T\mathbb{R}^3|_M$ and suppose that the 1-dimensional normal bundle N is invariant under $D\phi^t$. For each $m \in M$, define $\gamma(s,m)$ to be solution to $x' = f(x)$ such that $\gamma(0,m) = m$.

Let a solution $\gamma(s,m)$ be fixed. Consider the first variational equation along this solution:

$$X' = df(\gamma(s,m)) \cdot X, \quad X \in \mathbb{R}^3. \tag{2.1}$$

Let $\Phi(s)$ be the principal fundamental matrix solution of (2.1), and let $X_1(s)$ and $X_2(s)$ be independent solutions of this equation that span the tangent space $T_{\gamma(s,m)}M$ for each $s \in \mathbb{R}$. Define $\theta(s)$ by

$$\theta(s) = \arccos\left(\frac{\langle X_1(s),\, X_2(s)\rangle}{|X_1(s)||X_2(s)|}\right), \tag{2.2}$$

and suppose that there are $k_\theta, K_\theta \in \mathbb{R}$ such that $0 < k_\theta \leqslant \theta(s) \leqslant K_\theta < \pi$ for all $s \geqslant 0$, independently of the choice of $m \in M$. This condition ensures that the vectors $X_1(s)$ and $X_2(s)$ do not approach parallelism even as $s \to \infty$.

Let X_0 be a vector in N_m, the fibre of the bundle N corresponding to the point $m \in M$. We define the following quantities:

$$\lambda_1(s) := \frac{|X_1(s)|}{|X_1(0)|}, \quad \lambda_2(s) := \frac{|X_2(s)|}{|X_2(0)|}, \quad \lambda_3(s) := \frac{|\Phi(s)X_0|}{|X_0|}. \tag{2.3}$$

We present now the main result of this section.

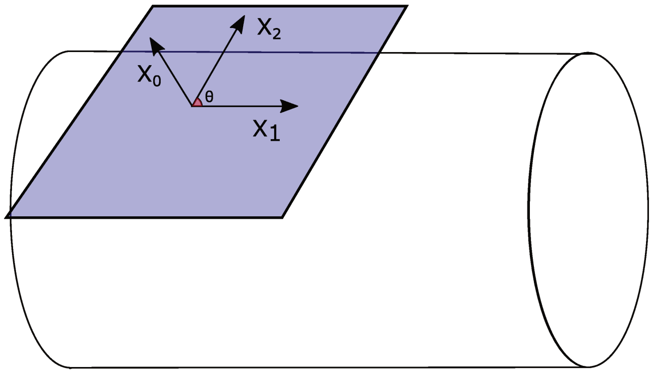

Figure 1 – Vectors X_1, X_2 and X_0, and angle θ.

Lemma 1. *Let k be a positive integer. Suppose that there are $\beta > 0$ and $c > 0$ independent of the choice of m such that the following conditions are satisfied for $s \geqslant 0$:*

$$\lambda_3(s) \leqslant ce^{-\beta s}, \quad \frac{\lambda_3(s)}{\lambda_1^k(s)} \leqslant ce^{-\beta s}, \quad \frac{\lambda_3(s)}{\lambda_2^k(s)} \leqslant ce^{-\beta s}. \tag{2.4}$$

Then, M is a k-normally hyperbolic invariant manifold.

Before proving this lemma, we establish some definitions and present some basic facts. First, we remark that, since N is invariant under $D\phi^t$ and the matrix of $D\phi^t(m)$ is exactly $\Phi(s)$, the quantity $\lambda_3(s)$ can be rewritten as

$$\lambda_3(s) = \|\pi_N D\phi^s(m)|_{N_m}\|.$$

Now, we may replace m by any point in M and the inequalities (2.4) will still hold. Therefore, we have

$$\|\pi_N D\phi^s(\phi^{-s}(m))|_{N_{\phi^{-s}(m)}}\| \leqslant ce^{-\beta s}, \quad \forall s \geqslant 0.$$

In order to simplify notation, we define

$$A^s(m) = D(\phi^{-s}|_M)(m) \quad \text{and} \quad B^s(m) = \pi_N D\phi^s(\phi^{-s}(m))|_{N_{\phi^{-s}(m)}}.$$

Let us also define

$$\nu(m) = \limsup_{s\to\infty} \|B^s(m)\|^{1/s} \quad \text{and}$$
$$\sigma(m) = \limsup_{s\to\infty} \frac{\log\|A^s(m)\|}{-\log\|B^s(m)\|}.$$

just as we defined ν^s and σ^s before. Observe that, if $\nu(m) < 1$, equivalent definitions for $\nu(m)$ and $\sigma(m)$ are

$$\nu(m) = \inf\{a : \|B^s(m)\|/a^s \to 0 \text{ as } s \to \infty\} \quad \text{and}$$
$$\sigma(m) = \inf\{\alpha : \|A^s(m)\|\|B^s(m)\|^\alpha \to 0 \text{ as } s \to \infty\}.$$

The following result, proved by Fenichel in (FENICHEL, 1971/72), will be used in our proof of Lemma 1.

Lemma 2 (Uniformity Lemma)**.**

1. *Suppose* $\|B^s(m)\|/a^s \to 0$ *as* $s \to \infty$ *for all* $m \in M$*. Then there are constants* $\hat{a} < a$ *and* c *such that* $\|B^s(m)\| < c\hat{a}$ *for all* $m \in M$ *and* $s \geqslant 0$*.*

2. *Under the hypotheses of item 1, suppose also that* $a \leqslant 1$ *and that* α *is such that* $\|A^s(m)\|\|B^s(m)\|^\alpha \to 0$ *as* $s \to \infty$ *for all* $m \in M$*. Then there are constants* $\hat{\alpha} < \alpha$ *and* $C > 0$ *such that* $\|A^s(m)\|\|B^s(m)\|^{\hat{\alpha}} < C$ *for all* $m \in M$ *and* $s \geqslant 0$*.*

3. *If the inequalities* $\nu(m) < a \leqslant 1$ *and* $\sigma(m) < \alpha$ *hold for all* $m \in M$*, then* $\|B^s(m)\| \to 0$ *and* $\|A^s(m)\|\|B^s(m)\|^\alpha \to 0$ *as* $s \to \infty$ *uniformly for* $m \in M$*.*

4. ν *and* σ *attain their suprema on* M*.*

We present below the proof of Lemma 1.

Proof (of Lemma 1)**:** By our remarks, we have:

$$\nu(m) \leqslant \limsup_{s\to\infty} (c\, e^{-\beta s})^{1/s} = e^{-\beta} < 1,$$

and thus the splitting $T\mathbb{R}^3|_M = TM \oplus N$ is hyperbolic.

We must prove that $\sigma(m) < 1/k$ for all $m \in M$. For a linear map T, define $\mu(T) = \inf\{|Tv| : |v| = 1\}$. We claim that there is $C > 0$, independent of m, such that

$$\mu(D\phi^s(m)|_{T_mM}) > C \cdot \|D\phi^s(m)|_{N_m}\|^{1/k}\, e^{\beta s/k},$$

for all $s \geqslant 0$ and all $m \in M$. Let $m \in M$ and take $v \in T_mM$ satisfying $|v| = 1$. Since $X_1(0)$ and $X_2(0)$ span T_mM, there are $a, b \in \mathbb{R}$ such that $aX_1(0) + bX_2(0) = v$. Considering that $|v| = 1$, we have that

$$a^2|X_1(0)|^2 + b^2|X_2(0)|^2 + 2ab|X_1(0)||X_2(0)| \cdot \cos\theta(0) = 1.$$

Now, we have $D\phi^s(m)|_{T_mM} \cdot v = aX_1(s) + bX_2(s)$. Therefore,

$$|D\phi^s(m)|_{T_mM} \cdot v|^2 = a^2|X_1(s)|^2 + b^2|X_2(s)|^2 + 2ab|X_1(s)||X_2(s)| \cdot \cos\theta(s).$$

Define $p := a|X_1(0)|$ and $q := b|X_2(0)|$. By (2.3) and (2.4), we have,

$$|D\phi^s(m)_{T_mM} \cdot v|^2 \geqslant \left(\frac{\lambda_3(s)}{c} e^{\beta s}\right)^{\frac{2}{k}} \left(p^2 + q^2 + 2pq\cos\theta(s)\right),$$

where $r(p,q) := p^2 + q^2 + 2pq\cos\theta(0) = 1$ and $\theta(s) \in [k_\theta, K_\theta]$. Consider the problem of finding the global minimum of the function $\psi(p,q,z) = p^2 + q^2 + 2pqz$ in the region

$$R := \{(p,q,z) \in \mathbb{R}^3 : r(p,q) = 1 \text{ and } -1 < \cos K_\theta \leqslant z \leqslant \cos k_\theta < 1\},$$

which is ensured to exist because R is compact. Observe that

$$\psi(p,q,z) = pq\left(\frac{p}{q} + \frac{q}{p} + 2z\right).$$

If $pq > 0$, the condition $\psi(p,q,z) > 0$ is equivalent to

$$\left(\frac{p}{q} + \frac{1}{\frac{p}{q}} + 2z\right) > 0,$$

which is true in the region R because the minimum of the function $t \mapsto \left(t + \frac{1}{t}\right)$ with $t > 0$ is equal to 2 and because $z > -1$. Similarly, if $pq < 0$, the condition $\psi(p,q,z) > 0$ is equivalent to

$$\left(\frac{p}{q} + \frac{1}{\frac{p}{q}} + 2z\right) < 0,$$

which is again true in R, because the maximum of the function $t \mapsto \left(t + \frac{1}{t}\right)$ with $t < 0$ is equal to -2 and because $z < 1$. Therefore, we have

$$k_R := \min\{\psi(p,q,z) : (p,q,z) \in R\} > 0.$$

Hence, we get

$$|D\phi^s(m)_{T_mM} \cdot v|^2 \geqslant k_R \left(\frac{\lambda_3(s)}{c} e^{\beta s}\right)^{\frac{2}{k}},$$

which means, by recalling the definition of $\lambda_3(s)$ in (2.3), that there is $C > 0$ satisfying

$$\mu(D\phi^s(m)|_{T_mM}) > C \cdot \|D\phi^s(m)|_{N_m}\|^{1/k} e^{\frac{\beta s}{k}}, \tag{2.5}$$

for all $s \geqslant 0$ and independently of the choice of $m \in M$.

Since (2.5) holds for all $m \in M$, we have

$$\mu(D\phi^s(\phi^{-s}(m))|_{T_{\phi^{-s}(m)}M}) > C \cdot \|D\phi^s(\phi^{-s}(m))|_{N_{\phi^{-s}(m)}}\|^{1/k} e^{\frac{\beta s}{k}}.$$

Since $A^s(m)$ is the inverse of $D\phi^s(\phi^{-s}(m))|_{T_{\phi^{-s}(m)}M}$, the last inequality may be rewritten as

$$\frac{1}{\|A^s(m)\|} > C \cdot \|B^s(m)\|^{1/k} e^{\frac{\beta s}{k}},$$

which implies that $\|A^s(m)\|\|B^s(m)\|^{1/k} \to 0$ as $s \to \infty$. Hence, $\sigma(m) \leqslant 1/k$.

At last, we must show that $\sigma(m)$ is, in fact, less than $1/k$. In order to do that, we recall Lemma 2. Since $\nu(m) < 1$, there is $a < 1$ such that $\|B^s(m)\|/a^s \to 0$ as $s \to \infty$. Furthermore, we know that $\|A^s(m)\|\|B^s(m)\|^{1/k} \to 0$ as $s \to \infty$. Therefore, there are $\hat{\alpha} < 1/k$ and $C > 0$ such that $\|A^s(m)\|\|B^s(m)\|^{\hat{\alpha}} < C$ for all $m \in M$ and $s \geqslant 0$. Let $r > 0$ be such that $0 < r < \dfrac{1}{k} - \hat{\alpha}$. Observe that

$$\|A^s(m)\|\|B^s(m)\|^{\hat{\alpha}+r} < C\|B^s(m)\|^r$$

for all $s \geqslant 0$. Since $\|B^s(m)\|^r \to 0$ as $s \to \infty$ for all $m \in M$, it follows promptly that $\|A^s(m)\|\|B^s(m)\|^{\hat{\alpha}+r} \to 0$ as $s \to \infty$. Therefore, we have

$$\sigma(m) = \inf\{\alpha \in \mathbb{R} : \|A^s(m)\|\|B^s(m)\|^{\alpha} \to 0 \text{ as } s \to \infty\} \leqslant \hat{\alpha} + r < \frac{1}{k},$$

as we wanted. □

2.2 Arzelà-Ascoli theorem and C^r norms

In this section, we will provide results regarding convergence in sets of continuously differentiable functions. Those results will be used later on in the proof of Theorem B. Let X be a compact metric space. Define $C(X)$ the space of real valued continuous functions from X. We begin with two definitions:

Definition 1. *A subset $\mathcal{F} \subset C(X)$ is uniformly bounded if there is $M > 0$ such that $|f(x)| < M$ for all $x \in X$ and all $f \in \mathcal{F}$.*

Definition 2. *A subset $\mathcal{F} \subset C(X)$ is uniformly equicontinuous if, for every $\epsilon > 0$, there is $\delta > 0$ such that $d(x_1, x_2) < \delta \implies |f(x_1) - f(x_2)| < \epsilon$ for all $f \in \mathcal{F}$.*

We state now the Arzelà-Ascoli theorem, which gives us conditions for a sequence of continuous functions to have a uniformly convergent subsequence. Its proof may be found in (DUNFORD; SCHWARTZ, 1988).

Theorem 4 (Arzelà-Ascoli theorem)**.** *Let X be a compact metric space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C(X)$. Define $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence that converges uniformly if, and only if, $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous.*

Definition 3. *A family $\mathcal{F}$ of real valued C^r functions is uniformly bounded in the C^r norm if there is $M > 0$ such that $\|f\|_{C^r} < M$.*

The next result concerns uniform convergence of a sequences of functions uniformly bounded in the C^2 norm. Such a result will be useful later on.

Proposition 1. *Let $U := [a, b] \times [c, d] \subset \mathbb{R}^2$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of C^2 real valued functions from U. If $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is uniformly bounded in the C^2 norm, then there is a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ that converges uniformly to a C^1 function.*

Proof: First, we remark that, since $\mathcal{F}$ is uniformly bounded in the C^2 norm, there is $M > 0$ such that $\|f_n\|_{C^2} < M$ for all $n \in \mathbb{N}$. In particular, $\|f_n\|_\infty < M$ for all n, which means that $\mathcal{F}$ is uniformly bounded in the sense of Definition 1. We will prove that $\mathcal{F}$ is also uniformly equicontinuous. In fact, take $\epsilon > 0$. Uniform boundedness in the C^2 norm also ensures that $\|Df_n\|_\infty < M$. Let $x_1, x_2 \in U$ be such that $|x_1 - x_2| < \epsilon/M$. By the mean value inequality,

$$|f_n(x_1) - f_n(x_2)| < M \cdot \frac{\epsilon}{M} = \epsilon,$$

for all $n \in \mathbb{N}$. Hence, $\mathcal{F}$ is uniformly equicontinuous. Because $\mathcal{F}$ is uniformly equicontinuous and uniformly bounded, we may apply the Arzelà-Ascoli theorem to get a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly to $f \in C(U)$.

It remains to be proved that f is a C^1 function. Define $g_k = \partial_1 f_{n_k}$, where ∂_1 denotes the partial derivative with respect to the first variable. Define also $\mathcal{G} = \{g_k : k \in \mathbb{N}\}$, which is clearly uniformly bounded considering that $\mathcal{F}$ is uniformly bounded in the C^2 norm. We remark also that uniform boundedness in the C^2 norm of $\mathcal{F}$ ensures that the derivative of each g_k is bounded by M in U. Therefore, each g_k is a Lipschitz continuous function with Lipschitz constant equal to M. The set $\mathcal{G}$ is thus uniformly equicontinuous. Once again, we apply the Arzelà-Ascoli theorem to get a subsequence $\{g_{k_l}\}_{l\in\mathbb{N}}$ that converges uniformly to $g \in C(U)$.

Define now $h_l = \partial_2 f_{n_{k_l}}$ and proceed as above to get a subsequence of $\{h_l\}_{l\in\mathbb{N}}$ that converges uniformly to $h \in C(U)$. Let I be the set of indices present in this subsequence. We know that $\{f_i\}_{i\in I}$ converges uniformly to f, $\{g_i\}_{i\in I}$ converges uniformly to g and $\{h_i\}_{i\in I}$ converges uniformly to h, since they are all subsequences of uniformly convergent sequences. For each $(s,t) \in U$, we have

$$f_i(s,t) = f_i(a,t) + \int_a^s \partial_1 f_i(\sigma,t)\, d\sigma = f_i(a,t) + \int_a^s g_i(\sigma,t)\, d\sigma.$$

Therefore, applying the limit, we get

$$f(s,t) = f(a,t) + \int_a^s g(\sigma,t)\, d\sigma,$$

and by taking the partial derivative with respect to s, we get

$$\partial_1 f(s,t) = g(s,t).$$

Similarly, we can show that $\partial_2 f(s,t) = h(s,t)$ for all $(s,t) \in U$. Hence, f has continuous partial derivatives over U, proving that f is C^1. □

2.3 Grönwall's inequality

We state a version of Grönwall's inequality, proved in (CHICONE, 2006), that will be used throughout this work.

Theorem 5 (Grönwall's inequality)**.** *Let $a < b$ and suppose that α, β and ψ are non-negative continuous functions defined on $[a, b]$. Suppose also that either α is a constant function, or that α is differentiable on $[a, b]$ with positive derivative, $\dot{\alpha}$. If, for all $t \in [a, b]$,*

$$\psi(t) \leqslant \alpha(t) + \int_0^t \beta(s)\, \psi(s)\, ds,$$

then

$$\psi(t) \leqslant \alpha(t)\, e^{\int_0^t \beta(s)\, ds}$$

for all $t \in [a, b]$.

We proceed to the proof of two lemmas that will be of use later and that are direct consequences of Grönwall's inequality.

Lemma 3. *Let $\dot{x} = F(x, \epsilon)$ be a smooth family of differential equations, where F is a C^r function, $r \geqslant 1$. Define $x(t, z, \epsilon)$ to be the solution to $\dot{x} = F(x, \epsilon)$ such that $x(0, z, \epsilon) = z$. Let $A, B > 0$ and let $C \subset \mathbb{R}^n$ be a compact set. Then, there is a constant $K > 0$ such that*

$$|x(t, z_2, \epsilon) - x(t, z_1, 0)| \leqslant K e^{K|t|}\big(|x(0, z_2, \epsilon) - x(0, z_1, 0)| + \epsilon|t|\big),$$

for all $t \in [0, A]$, $z_1, z_2 \in C$ and $\epsilon \in [0, B]$. The same is also true, possibly for a different constant $K' > 0$, for all $t \in [-A, 0]$, $z_1, z_2 \in C$ and $\epsilon \in [0, B]$.

Proof: Let $z_1, z_2 \in C$, $t \in [0, A]$ and $\epsilon \in [0, B]$. Since $t \mapsto x(t, z_2, \epsilon)$ is a solution to $\dot{x} = F(x, \epsilon)$ and $t \mapsto x(t, z_1, 0)$ is a solution to $\dot{x} = F(x, 0)$, we have

$$\begin{aligned} x(t, z_2, \epsilon) - x(t, z_1, 0) = x(0, z_2, \epsilon) - x(0, z_1, 0) \\ + \int_0^t \big(F(x(s, z_2, \epsilon), \epsilon) - F(x(s, z_1, 0), 0)\big)\, ds. \end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned} |x(t, z_2, \epsilon) - x(t, z_1, 0)| \leqslant |x(0, z_2, \epsilon) - x(0, z_1, 0)| \\ + \left|\int_0^t \big(F(x(s, z_2, \epsilon), \epsilon) - F(x(s, z_1, 0), 0)\big)\, ds\right|. \end{aligned} \tag{2.6}$$

By properties of the integral, we have

$$\begin{aligned} \left|\int_0^t \big(F(x(s, z_2, \epsilon), \epsilon) - F(x(s, z_1, 0), 0)\big)\, ds\right| \leqslant \\ \int_0^t \big|F(x(s, z_2, \epsilon), \epsilon) - F(x(s, z_1, 0), 0)\big|\, ds. \end{aligned}$$

Define $\tilde{C} := [0, A] \times C \times [0, B]$. The image $x(\tilde{C})$ is a compact set. We remark that F is Lipschitz continuous in $x(\tilde{C}) \times [0, B]$, because F is a C^r function and $x(\tilde{C}) \times [0, B]$ is compact. Therefore, there is $M > 0$ such that

$$\big|F\big(x(s, z_2, \epsilon), \epsilon\big) - F\big(x(s, z_1, 0), 0\big)\big| \leqslant M\big(|x(s, z_2, \epsilon) - x(s, z_1, 0)| + \epsilon\big).$$

. Therefore, from (2.6), we have:

$$|x(t,z_2,\epsilon)-x(t,z_1,0)| \leqslant |x(0,z_2,\epsilon)-x(0,z_1,0)| + \int_0^t M\big(|x(s,z_2,\epsilon)-x(s,z_1,0)|+\epsilon\big)\,ds.$$

Define $K=\max\{1,M\}$. We have

$$|x(t,z_2,\epsilon)-x(t,z_1,0)| \leqslant K\big(|x(0,z_2,\epsilon)-x(0,z_1,0)|+\epsilon\,|t|\big) + \int_0^t K\big(|x(s,z_2,\epsilon)-x(s,z_1,0)|\big)\,ds. \tag{2.7}$$

Let us define the following functions:

$$\begin{aligned}
\psi(t) &= |x(t,z_2,\epsilon)-x(t,z_1,0)|;\\
\alpha(t) &= K\big(|x(0,z_2,\epsilon)-x(0,z_1,0)|+\epsilon\,|t|\big);\\
\beta(t) &= K,
\end{aligned}$$

so that (2.7) may be rewritten as

$$\psi(t) \leqslant \alpha(t) + \int_0^t \beta(s)\,\psi(s)\,ds.$$

We remark that ψ, α and β are non-negative continuous functions, and that α is differentiable with $\dot\alpha(t)=\epsilon>0$ for all $t\in[0,A]$. By Grönwall's inequality, we have

$$|x(t,z_2,\epsilon)-x(t,z_1,0)| \leqslant Ke^{K|t|}\big(|x(0,z_2,\epsilon)-x(0,z_1,0)|+\epsilon|t|\big),$$

as wanted. We remark that the constant K depends on the set $\tilde C$.

In order to prove the lemma for $t\in[-A,0]$, we define $y(t,z,\epsilon)$ as the solution to $\dot y=-F(x,\epsilon)$ with initial value $y(0,z,\epsilon)=z$. We remark that $y(t,z,\epsilon)=x(-t,z,\epsilon)$. It suffices then to apply the result we have proved above to the family y with $t\in[0,A]$. □

Lemma 4. *Let $\dot x = f(x)+\widetilde F(x,\epsilon,\mu)$ such that f is a C^r function on $\mathbb{R}^n$, $n\in\mathbb{N}$, and $\tilde F$ is a continuous function on $\mathbb{R}^n\times\Delta$, where $r\geqslant 1$ and $\Delta\subset\mathbb{R}^2$ is a compact set. Suppose that $\tilde F(x,0,\mu)=0$ for all μ and all x. Define $x(t,z,\epsilon,\mu)$ to be the solution to $\dot x=f(x)+\tilde F(x,\epsilon,\mu)$ such that $x(0,z,\epsilon,\mu)=z$. Let $A>0$ and let $C\subset\mathbb{R}^n$ be a compact set. If there is $M>0$ such that $|\tilde F(x,\mu,\epsilon)|<M\epsilon$ for all $(x,\epsilon,\mu)\in x([-A,0]\times C\times\Delta)\times\Delta$, then there is a constant $K>0$ such that*

$$|x(t,z_2,\epsilon,\mu)-x(t,z_1,0,\mu)| \leqslant Ke^{K|t|}\big(|x(0,z_2,\epsilon,\mu)-x(0,z_1,0,\mu)|+\epsilon|t|\big),$$

for all $t\in[-A,0]$, $z_1,z_2\in C$ and $(\epsilon,\mu)\in\Delta$. The same is also true, possibly for a different constant $K'>0$, for all $t\in[0,A]$, $z_1,z_2\in C$ and $(\epsilon,\mu)\in\Delta$.

Proof: Let $z_1, z_2 \in C$, $t \in [-A, 0]$ and $(\epsilon, \mu) \in \Delta$. Since $\tilde{F}(x, 0, \mu) = 0$, we have

$$\begin{aligned} x(t, z_2, \epsilon, \mu) - x(t, z_1, 0, \mu) = & x(0, z_2, \epsilon, \mu) - x(0, z_1, 0, \mu) \\ & + \int_0^t f(x(s, z_1, \epsilon, \mu)) - f(x(s, z_2, 0, \mu))\, ds \\ & + \int_0^t \epsilon \tilde{F}(x(s, z_1, \epsilon, \mu), \epsilon, \mu)\, ds. \end{aligned}$$

Observe that the solution x is a continuous function. Therefore, the image $I := x([-A, 0] \times C \times \Delta)$ is a compact set. Since f is a C^1 function, there is $L > 0$ such that

$$|f(x(s, z_1, \epsilon, \mu)) - f(x(s, z_2, 0, \mu))| \leqslant L|x(s, z_1, \epsilon, \mu) - x(s, z_2, 0, \mu)|, \tag{2.8}$$

for all $s \in [-A, 0]$. Note that L does not depend on the choice of z_1, z_2, ϵ or μ. By hypothesis,

$$|\tilde{F}(x(s, z_1, \epsilon, \mu), \epsilon, \mu)| < M\epsilon, \tag{2.9}$$

for all $s \in [-A, 0]$. Once again, we remark that M does not depend on the choice of the other entries as long as they are in the sets established. By the triangle inequality, we have

$$\begin{aligned} |x(t, z_2, \epsilon, \mu) - x(t, z_1, 0, \mu)| \leqslant & |x(0, z_2, \epsilon, \mu) - x(0, z_1, 0, \mu)| + M\epsilon|t| \\ & + \int_0^t L|x(s, z_1, \epsilon, \mu) - x(s, z_2, 0, \mu)|\, ds. \end{aligned} \tag{2.10}$$

We apply Grönwall's lemma as done in Lemma 3, and find $K > 0$ such that

$$|x(t, z_2, \epsilon, \mu) - x(t, z_1, 0, \mu)| \leqslant Ke^{K|t|}(|x(0, z_2, \epsilon, \mu) - x(0, z_1, 0, \mu)| + \epsilon|t|).$$

□

3 Fundamental proposition

The main goal of this section is proving the following proposition, which will be crucial to further results.

Proposition 2 (Fundamental proposition)**.** *Consider a planar differential equation*

$$x' = f(x) \tag{3.1}$$

with an asymptotically stable limit cycle Γ *of period* $\omega > 0$*. Let* τ *be an angular variable modulo* $T > 0$*. Suppose that* Γ *is hyperbolic, that is,*

$$b := \int_0^\omega \operatorname{tr} Df(\phi^t(p))\, dt < 0. \tag{3.2}$$

Let M *be the invariant torus corresponding to* Γ *for the system*

$$x' = f(x), \quad \tau' = 1. \tag{3.3}$$

Then, there are a neighbourhood $N \subset \mathbb{R}^2 \times \mathbb{R}$ *of* M *and a constant* $C > 0$ *such that, for every smooth function* $g : N \to \mathbb{R}^2$*, with* $g(x, \tau + T) = g(x, \tau)$ *for all* $x \in \mathbb{R}^2$ *and all* $\tau \in \mathbb{R}$*, and for which the system*

$$x' = f(x) + g(x, \tau), \quad \tau' = 1 \tag{3.4}$$

has an invariant set $\tilde{M} \subset N$*, the following estimate holds:*

$$\sup\{d((x, \tau), M) : (x, \tau) \in \tilde{M}\} \leqslant C\|g\|_{C^0}$$

In order to prove the above proposition, we shall study invariant foliations of a neighbourhood of a hyperbolic limit cycle. A deeper exposition of the results presented below can be found in (CHICONE; LIU, 2004). The approach and the style of proofs in this section were greatly influenced by (TESCHL, 2012) and (PERKO, 2001).

3.1 Groundwork for the proof

Proposition 3. *Let* $\phi^s(x)$ *denote the flow of the system* $x' = f(x)$ *in* $\mathbb{R}^n$*, where* f *is a* C^1 *field. Let* $K \subset \mathbb{R}^n$ *be a compact, and* S *be a positive constant. Then, there is a constant* $C > 0$ *satisfying*

$$|\phi^s(x) - \phi^s(y)| \leqslant e^{Cs}|x - y|, \quad s \in [0, S],$$

provided that $\phi^s(x) \in K$ *and* $\phi^s(y) \in K$ *for all* $s \in [0, S]$*.*

Proof: Let B be a closed ball containing K. Observe that, for every $s \in [0, S]$:

$$|\phi^s(x) - \phi^s(y)| \leqslant |(\phi^s(x) - x) - (\phi^s(y) - y)| + |x - y|.$$

Therefore, we have

$$|\phi^s(x) - \phi^s(y)| \leqslant \left| \int_0^s f(\phi^t(x)) - f(\phi^t(y))\, dt \right| + |x - y|,$$

and then

$$|\phi^s(x) - \phi^s(y)| \leqslant \int_0^s |f(\phi^t(x)) - f(\phi^t(y))|\, dt + |x - y|.$$

Let $C := \max\{|f'(p)| : p \in B\}$. Since $\phi^t(x)$ and $\phi^t(y)$ are both in B, the mean value inequality ensures that

$$|f(\phi^t(x)) - f(\phi^t(y))| \leqslant C|\phi^t(x) - \phi^t(y)|, \quad t \in [0, s].$$

We thus have

$$|\phi^s(x) - \phi^s(y)| \leqslant \int_0^s C|\phi^t(x) - \phi^t(y)|\, dt + |x - y|.$$

We apply Grönwall's inequality and get

$$|\phi^s(x) - \phi^s(y)| \leqslant |x - y| e^{Cs},$$

as wanted. □

We proceed by proving an estimate of the distance between an attractive hyperbolic limit cycle and the flow of point near this cycle.

Lemma 5. *Let Γ be a ω-periodic asymptotically stable limit cycle for the planar system $x' = f(x)$. Let $\phi^s(x_0)$ be the flow of this equation with initial point x_0. Suppose that Γ is hyperbolic, that is, the quantity b, as defined in (3.2), is a negative number. Assume also that there is a neighbourhood V of Γ, contained in the stable manifold of Γ, with an invariant foliation with respect to $x' = f(x)$, and whose leaves are curves. Then, there is $C > 0$ such that $d(\phi^s(q), \Gamma) \leqslant Ce^{\frac{bs}{\omega}}$ for all $s \geqslant 0$, provided that $q \in V$.*

Proof: Let p be a point in Γ. Let $M_s(p)$ denote the leaf through p. We define $\tilde{q}$ the first crossing of the flow $\phi^s(q)$ and the curve $M_s(p)$. We also define $\tilde{s} \in [0, \omega)$ the time for which $\phi^{\tilde{s}}(q) = \tilde{q}$. Let s be a positive number. We will study the distance $d(\phi^s(q), \phi^{s-\tilde{s}}(p))$. First, we shall prove that there is $C_1 > 0$, independent of p and q such that

$$|\phi^s(q) - \phi^{s-\tilde{s}}(p)| \leqslant C_1|\phi^s(\tilde{q}) - \phi^s(p)|, \quad \text{for all } s \geqslant 0.$$

To do so, observe that, if we denote the flow of the system $x' = -f(x)$ by $\psi^s(x_0)$, we have

$$\phi^s(q) = \psi^{\tilde{s}}(\phi^s(\tilde{q}));$$

$$\phi^{s-\tilde{s}}(p) = \psi^{\tilde{s}}(\phi^s(p)).$$

Let K be a compact set such that $V \subset K$. Since V is contained in the stable manifold of Γ, we know that $\psi^t(\phi^s(\tilde{q}))$ and $\psi^t(\phi^s(p))$ are in K for every $t \in [0, \tilde{s}]$. By Proposition 3, there is $M > 0$ such that

$$|\phi^s(q) - \phi^{s-\tilde{s}}(p)| \leqslant e^{M\tilde{s}}|\phi^s(\tilde{q}) - \phi^s(p)|$$

Because $0 \leqslant \tilde{s} < \omega$, we know that $e^{M\tilde{s}} \leqslant e^{M\omega}$. We finish by defining $C_1 := e^{M\omega}$, which yields

$$|\phi^s(q) - \phi^{s-\tilde{s}}(p)| \leqslant C_1|\phi^s(\tilde{q}) - \phi^s(p)|. \tag{3.5}$$

We will now study the expression $|\phi^s(\tilde{q}) - \phi^s(p)|$. Divide s by ω, and let n be the quotient and r the remainder of this division. Since $\tilde{q} \in M_s(p)$ and the foliation is invariant, we have $\phi^\omega(\tilde{q}) \in M_s(\phi^\omega(p)) = M_s(p)$. Therefore,

$$\phi^s(\tilde{q}) = \phi^r(i \circ \pi^n(\tilde{q}));$$

$$\phi^s(p) = \phi^r(p),$$

where π is the Poincaré map defined on $M_s(p)$, and $i : M_s(p) \to \mathbb{R}^2$ is the natural embedding. Therefore, we have

$$|\phi^s(\tilde{q}) - \phi^s(p)| = |\phi^r(i \circ \pi^n(\tilde{q})) - \phi^r(p)|.$$

We know that $\phi^t(x) \in V$ for all $t \geqslant 0$ if $x \in V$. Therefore, since $r \in [0, \omega)$, we can apply Proposition 3 again and get $N > 0$ such that

$$|\phi^s(\tilde{q}) - \phi^s(p)| \leqslant e^{Nr} d(\pi^n(\tilde{q}), p) \leqslant e^{N\omega} d(\pi^n(\tilde{q}), p).$$

As before, we define $C_2 := e^N\omega$, and get

$$|\phi^s(\tilde{q}) - \phi^s(p)| \leqslant C_2 \cdot d(\pi^n(\tilde{q}), p). \tag{3.6}$$

We must now study the expression $d(\pi^n(\tilde{q}), p)$. We know that the derivative of the map π at p is given by $\pi'(p) = e^b$. Since $\|\pi'(p)\| < 1$, the Poincaré map π is C^1-conjugated to its derivative at its fixed point, p, as proved in (RODRIGUES; RUBIÓ, 2012). Therefore, there exist a compact neighbourhood U of p and a C^1 diffeomorphism $H : U' \to U$ such that $H(0) = p$ and $\pi^n(x) = H \circ (\pi'(p))^n \circ H^{-1}(x)$. *Let $N \in \mathbb{N}$ be such that $\pi^N(x) \in U$ for all $x \in M_s(p)$*. Let C_3 be such that $C_3 \geqslant e^{-bN} \cdot \sup\{d(x,p) : x \in M_s(p)\}$. If $n < N$, we have

$$d(\pi^n(\tilde{q}), p) \leqslant d(\tilde{q}, p) \leqslant e^{b\,(n-N)} d(\tilde{q}, p) \leqslant C_3 \cdot e^{bn}$$

On the other hand, if $n \geqslant N$, we have

$$d(\pi^n(\tilde{q}), p) = d(\pi^{n-N}(\pi^N(\tilde{q})), p) = d(H \circ (\pi'(p))^{n-N} \circ H^{-1}(\pi^N(\tilde{q})), H(0)).$$

Since H is a C^1 function on a compact set, there is $C_4 > 0$ such that $d(H(x), H(y)) \leqslant C_4 \cdot d(x, y)$ for all $x, y \in U'$. Therefore,

$$d(H \circ (\pi'(p))^{n-N} \circ H^{-1}(\pi^N(\tilde{q})), H(0)) \leqslant C_4 \cdot d(\pi'(p)^{n-N} \circ H^{-1}(\pi^N(\tilde{q})), 0).$$

Similarly, there is $C_5 > 0$ such that $d(H^{-1}(x), H^{-1}(y)) \leqslant C_5 \cdot d(x, y)$ for all $x, y \in U$. Thus,

$$C_4 \cdot d(\pi'(p)^{n-N} \circ H^{-1}(\pi^N(\tilde{q})), 0) \leqslant C_4 C_5 \cdot e^{b(n-N)} \cdot d(\pi^N(\tilde{q}), p).$$

Finally, we

$$C_4 C_5 \cdot e^{b(n-N)} \cdot d(\pi^N(\tilde{q}), p) \leqslant C_4 C_5 \cdot e^{-bN} d(\tilde{q}, p) \cdot e^{bn} \leqslant C_3 C_4 C_5 \cdot e^{bn}.$$

Hence, defining $C_6 = \max\{C_3 C_4 C_5, C_3\}$, we get, for all $n \in \mathbb{N}$:

$$d(\pi^n(\tilde{q}), p) \leqslant C_6 \cdot e^{bn} \tag{3.7}$$

Combining inequalities (3.5), (3.6) and (3.7), we get

$$|\phi^s(q) - \phi^{s-\tilde{s}}(p)| \leqslant C_1 C_2 C_6 \cdot e^{bn} = C_1 C_2 C_6 \cdot e^{\frac{bs}{\omega} - \frac{br}{\omega}}$$

Finally, defining $C = C_1 C_2 C_6 \cdot e^{-b}$, we get:

$$|\phi^s(q) - \phi^{s-\tilde{s}}(p)| \leqslant C e^{\frac{bs}{\omega}},$$

which concludes our proof. □

The following two results are non-autonomous versions of the stable manifold theorem, which will help us with the construction of a foliation of a neighbourhood of an attractive hyperbolic limit cycle that is invariant under the flow.

Proposition 4. *Let $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function that is T-periodic in the first variable. Suppose that*

$$\lim_{|x| \to 0} \frac{|g(t, x)|}{|x|} = 0 \quad \text{and} \quad \frac{\partial g}{\partial x}(t, 0) = 0.$$

Consider the system

$$x' = Ax + e^{\beta t} g(t, e^{-\beta t} x), \tag{3.8}$$

where A is a constant matrix and $\beta \geqslant 0$. Suppose 0 is a hyperbolic equilibrium point for the unperturbed system $x' = Ax$. Let $k > 0$ be the dimension of E^s, the stable subspace of A. Then, there exists a k-dimensional C^1-manifold W satisfying:

i. $0 \in W$;

ii. Let $x(t, x_0)$ be the solution to (3.8) with $x(0, x_0) = x_0$. If $x_0 \in W$,

$$\lim_{t \to \infty} |x(t, x_0)| = 0;$$

iii. W is tangent to E^s at 0.

Proof: Let $x_0 \in \mathbb{R}^n$ be close to 0. We will abbreviate $x(t, x_0)$ to $x(t)$ and define $h(t, x) = e^{\beta t} g(t, e^{-\beta t} x)$. By the method of variation of parameters, we have

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-r)A} h(r, x(r)) dr. \tag{3.9}$$

We will first find a necessary condition for $x(t)$ to be bounded for $t \geqslant 0$. Let E^s and E^u be the stable and unstable subspaces for the system $x' = Ax$, respectively. Define P^s the projection operator onto E^s and P^u the projection operator onto E^u. Also, abbreviate $x_u = P^u x_0$, $x_s = P^s x_0$, $h_u(t, x) = P^u h(t, x)$ and $h_s(t, x) = P^s h(t, x)$. Applying P^u to $x(t)$, we get

$$P^u x(t) = e^{tA} x_u + \int_0^t e^{(t-r)A} h_u(r, x(r)) dr,$$

which can be written

$$x_u = e^{-tA} P^u x(t) - \int_0^t e^{-rA} h_u(r, x(r)) dr.$$

Suppose that $x(t)$ remains bounded. Then, the integral on the right-hand side of the equation is absolutely convergent, and we can apply the limit and get

$$x_u = -\int_0^\infty e^{-rA} h_u(r, x(r)) dr.$$

Substituting back into (3.9), we have

$$x(t) = e^{tA} x_s + \int_0^t e^{(t-r)A} h_s(r, x(r)) dr - \int_t^\infty e^{(t-r)A} h_u(r, x(r)) dr. \tag{3.10}$$

Therefore, if $x(t)$ remains bounded, it must satisfy (3.10). We will now prove that this integral equation admits solutions. In fact, we will prove that, provided we take x_s small enough, we can choose x_u, depending on x_s, such that $x(t)$ is a solution to (3.10).

Let then x_s be a point in E^s. We will study $(u_j(t, x_s))_{j \in \mathbb{N}}$, the sequence of functions defined for $t \geqslant 0$ given by

$$u_0(t, x_s) = 0$$

and

$$\begin{aligned} u_{j+1}(t, x_s) = e^{tA} x_s &+ \int_0^t e^{(t-r)A} h_s(r, u_j(r, x_s)) dr \\ &- \int_t^\infty e^{(t-r)A} h_u(r, u_j(r, x_s)) dr. \end{aligned} \tag{3.11}$$

This definition includes a improper integral, which might not be well defined. Thus, we must first show that, if we fix x_s small enough, then each u_j is a well defined function of t. In order to do that, take $\alpha > 0$ such that $\alpha < \min\{|\mathrm{Re}(\lambda_i)| : \lambda_i \text{ is an eigenvalue of } A\}$ and we will prove by induction that, provided x_s is small, the following inequality holds for all $j \in \mathbb{N}$

$$|u_j(t,x_s) - u_{j-1}(t,x_s)| \leqslant \frac{|x_s|e^{-\alpha t}}{2^{j-1}} \tag{3.12}$$

The case $j = 1$ is clearly valid. For the other cases, let us note that, given $v \in E^s$ and $w \in E^u$, if $\sigma + \alpha < \min\{|\mathrm{Re}(\lambda_i)| : \lambda_i \text{ is an eigenvalue of } A\}$, we have

$$|e^{tA}v| \leqslant e^{-(\alpha+\sigma)t}|v| \quad \text{for } t \geqslant 0$$

and

$$|e^{tA}w| \leqslant e^{\sigma t}|w| \quad \text{for } t \leqslant 0.$$

Since $\dfrac{\partial g}{\partial x}(t,0) = 0$, we know that, for each $\epsilon > 0$, there is $\delta > 0$ such that

$$|x|,|y| < \delta \implies \sup_{t\in[0,T]} |g(t,x) - g(t,y)| < \epsilon|x-y|. \tag{3.13}$$

Assume the induction hypothesis (3.12) is valid for $j = 1, ..., m$. Then, we have

$$\begin{aligned} |u_{m+1}(t,x_s) - u_m(t,x_s)| \leqslant &\int_0^t \left|e^{(t-r)A}\left(h_s(r,u_m(r,x_s)) - h_s(r,u_{m-1}(r,x_s))\right)\right| dr \\ &+ \int_t^\infty \left|e^{(t-r)A}\left(h_u(r,u_m(r,x_s)) - h_u(r,u_{m-1}(r,x_s))\right)\right| dr. \end{aligned}$$

Note that, with σ chosen as before, the first integral in last inequality is less than

$$\int_0^t e^{-(t-r)(\alpha+\sigma)}e^{\beta t}\left|g_s(r,e^{-\beta t}u_m(r,x_s)) - g_s(r,e^{-\beta t}u_{m-1}(r,x_s))\right| dr.$$

Also, the second integral is less than

$$\int_t^\infty e^{\sigma(t-r)}e^{\beta t}\left|g_u(r,e^{-\beta t}u_m(r,x_s)) - g_u(r,e^{-\beta t}u_{m-1}(r,x_s))\right| dr.$$

Let $\epsilon > 0$ be such that $\epsilon/\sigma < 1/4$, and let δ be such that condition (3.13) is true for this ϵ. If we take $|x_s| < \delta/2$, the induction hypothesis ensures that

$$|u_m(t,x_s)| < \delta \quad \text{and} \quad |u_{m-1}(t,x_s)| < \delta,$$

and then

$$\begin{aligned} |u_{m+1}(t,x_s) - u_m(t,x_s)| \leqslant &\int_0^t e^{-(t-r)(\alpha+\sigma)}\epsilon\,|u_m(r,x_s) - u_{m-1}(r,x_s)|\, dr \\ &+ \int_t^\infty e^{\sigma(t-r)}\epsilon\,|u_m(r,x_s) - u_{m-1}(r,x_s)|\, dr. \end{aligned}$$

Again considering the induction hypothesis, we have

$$\begin{aligned} \int_0^t e^{-(t-r)(\alpha+\sigma)}\epsilon\,|u_m(r,x_s) - u_{m-1}(r,x_s)|\, dr &\leqslant \frac{\epsilon|x_s|e^{-(\alpha+\sigma)t}}{2^{m-1}}\int_0^t e^{\sigma r}dr \\ &\leqslant \frac{\epsilon|x_s|e^{-\alpha t}}{\sigma 2^{m-1}}, \end{aligned}$$

and

$$\int_t^\infty e^{\sigma(t-r)}\epsilon\left|u_m(r,x_s)-u_{m-1}(r,x_s)\right|dr \leqslant \frac{\epsilon|x_s|e^{\sigma t}}{2^{m-1}}\int_t^\infty e^{-(\sigma+\alpha)r}dr$$
$$\leqslant \frac{\epsilon|x_s|e^{-\alpha t}}{\sigma 2^{m-1}}.$$

Finally, since $\epsilon/\sigma < 1/4$, we have

$$|u_{m+1}(t,x_s)-u_m(t,x_s)| \leqslant \frac{2\epsilon|x_s|e^{-\alpha t}}{\sigma 2^{m-1}} < \frac{|x_s|e^{-\alpha t}}{2^m},$$

as we wanted.

Now that we have proven (3.12), we can show that u_j is well defined if x_s is chosen as above. We begin by

$$u_j(t,x_s) = u_0(t,x_s) + \sum_{i=1}^{j}\left(u_i(t,x_s)-u_{i-1}(t,x_s)\right),$$

to which we can apply the triangle inequality and (3.12) to get

$$|u_j(t,x_s)| \leqslant \sum_{i=1}^{j}|u_i(t,x_s)-u_{i-1}(t,x_s)| \leqslant 2|x_s|e^{-\alpha t} \tag{3.14}$$

Thus, the improper integral appearing in the definition of each u_j is absolutely convergent, which means u_j is well defined.

We will now prove that (3.10) admits solution. In fact, the solution we seek is the limit of the sequence $(u_j(t,x_s))_{j\in\mathbb{N}}$. We must first show that this sequence has a limit. Let $m,n\in\mathbb{N}$, with $n>m$. Then,

$$|u_n(t,x_s)-u_m(t,x_s)| \leqslant \sum_{i=m+1}^{n}\left|u_i(t,x_s)-u_{i-1}(t,x_s)\right|.$$

If $N\in\mathbb{N}$ is such that $n>m>N$, we can apply (3.12) and get

$$|u_n(t,x_s)-u_m(t,x_s)| \leqslant |x_s|e^{-\alpha t}\sum_{i=N}^{\infty}\frac{1}{2^{i-1}} = \frac{|x_s|e^{-\alpha t}}{2^{N-2}}.$$

Therefore, if x_s is fixed and small enough, the sequence of functions $u_j(t,x_s)$ is a uniformly Cauchy sequence of continuous functions for $t\geqslant 0$. We know then that this sequence converges uniformly to a continuous function: $u(t,x_s)$. Considering that the convergence is uniform, we may take the limit of both sides of equation (3.11) and conclude that $u(t,x_s)$ is a solution to the integral equation (3.10). Taking the derivative of (3.10), we deduce that $u(t,x_s)$ is the unique solution of $x' = Ax + h(t,x)$ with initial value $u(0,x_s)$.

Finally, define $V\subset E^s$, the open ball centered at 0 with radius equal to $\delta/2$, where δ is chosen as before. We introduce the function $\xi : V\to E^u$ given by

$$\xi(x_s) = P^u u(0,x_s).$$

The set $W := \{a + \xi(a) : a \in V\}$ is a submanifold of $\mathbb{R}^n$, whose dimension equals the dimension of E^s. It is also clear that W satisfies property (i). By construction, the solution to the system $x' = Ax + h(t,x)$ beginning at $x_0 \in W$ approaches 0 as $t \to \infty$. In fact, (3.14) ensures that this solution approaches 0 at exponential rate α. Therefore, property (ii) is also proved.

At last, we will prove that W is tangent to E^s at 0. Let $i : \mathbb{R}^k \to E^s \subset \mathbb{R}^n$ be a linear parametrisation of the surface E^s. Define $\tilde{\xi} : \mathbb{R}^k \to \mathbb{R}^n$ the function given by $\tilde{\xi} = \xi \circ i$. By (3.10), the derivative of $\tilde{\xi}$ at $z \in \mathbb{R}^k$ is

$$D\tilde{\xi}(z)(\cdot) = P^u \left(i(\cdot) - \int_0^\infty e^{-rA} \frac{\partial h_u}{\partial x}(r, u(r, i(z))) \frac{\partial u}{\partial z}(r, i(z))\, dr \right).$$

If we take $z = 0$, we have $i(z) = 0$, by linearity of i. Thus, $u(r, i(0)) = 0$ for all $r \geqslant 0$. Since $\frac{\partial g_u}{\partial x}(r, 0) = 0$ by hypothesis, we have

$$D\tilde{\xi}(0)(\cdot) = P^u\,(i(\cdot)) = 0,$$

so that W is tangent to E^s at 0. □

Proposition 5. *Let $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function that is T-periodic in the first variable. Suppose that*

$$\lim_{|x| \to 0} \frac{|g(t,x)|}{|x|} = 0 \quad \textit{and} \quad \frac{\partial g}{\partial x}(t, 0) = 0.$$

Consider the system

$$x' = Ax + g(t,x), \tag{3.15}$$

where A is a constant matrix. Suppose 0 is an equilibrium point for the unperturbed system $x' = Ax$. Let $k > 0$ be the dimension of E^s, the stable subspace of A. Then, there exists a k-dimensional C^1-manifold W satisfying:

i. $0 \in W$;

ii. Let $x(t, x_0)$ be the solution to (3.15) with $x(0, x_0) = x_0$. If $x_0 \in W$,

$$\lim_{t \to \infty} |x(t, x_0)| = 0;$$

iii. W is tangent to E^s at 0.

Proof: Let $\beta > 0$ be such that $\beta < \min\{|Re(\lambda_i)| : \lambda_i$ is an eigenvalue of $A\}$. Define $y(t) = e^{\beta t}\, x(t)$. Observe that $y(t)$ satisfies the following differential equation:

$$y = (A + \beta\, I)\, y + e^{\beta t} g(t, e^{-\beta t}).$$

Furthermore, our choice of β ensures that all eigenvalues of $A + \beta\, I$ have non-zero real part and that the stable subspace of $A + \beta I$ is equal to E^s. Applying Proposition 4 to this system, we get a k-dimensional C^1-manifold, W_y satisfying:

i. $0 \in W_y$;

ii. If $y_0 \in W_y$, then $\lim\limits_{t\to\infty} |y(t, y_0)| = 0$;

iii. W_y is tangent to E^s.

It suffices then to notice that, since $x(t) = e^{-\beta t} y(t)$, it follows:

$$\lim_{t\to\infty} |y(t, y_0)| = 0 \implies \lim_{t\to\infty} |x(t, y_0)| = 0.$$

Thus, we conclude our proof by defining $W := W_y$. □

The next two results are dedicated to the construction of a foliation of a neighbourhood of an attractive hyperbolic limit cycle that is invariant under the flow. This construction will be of great importance in the proof of our fundamental proposition.

Proposition 6. *Let Γ be a ω-periodic asymptotically stable hyperbolic limit cycle for the planar system $x' = f(x)$. Let b be defined as in (3.2). For each $p \in \Gamma$, there is a 1-dimensional C^1-manifold, $M_s(p)$, contained in the stable manifold of Γ, satisfying:*

i. $p \in M_s(p)$*;*

ii. $\lim\limits_{t\to\infty} |\phi^s(x) - \phi^s(p)| = 0$ *for all* $x \in M_s(p)$*;*

iii. $M_s(p)$ *intersects* Γ *transversally at* p*.*

Proof: Let p be a point in Γ. We will study the difference $y(s) := \phi^s(\tilde{p}) - \phi^s(p)$, where $\tilde{p}$ is a point near p. We have:

$$y'(s) = \frac{d}{ds}\left(\phi^s(\tilde{p}) - \phi^s(p)\right) = f(\phi^s(\tilde{p})) - f(\phi^s(p)).$$

Define $h : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(s, x) = f(\phi^s(p) + x) - f(\phi^s(p)) - Df(\phi^s(p))(x)$$

By Taylor's theorem, we know that

$$\lim_{x\to 0} \frac{|h(s, x)|}{|x|} = 0.$$

Furthermore, $h(s + \omega, x) = h(s, x)$ for all $s \in \mathbb{R}$ and all $x \in \mathbb{R}^2$. Define $A(s) = Df(\phi^s(p))$. Observe that y satisfies the following differential equation:

$$y' = A(s)\, y + h(s, y), \tag{3.16}$$

where $A(s)$ is ω-periodic and h satisfies all the hypotheses in Proposition 5. Define $\Phi(s)$ as the fundamental matrix solution at $s = 0$ for the linear system

$$y' = A(s)\, y. \tag{3.17}$$

By Floquet's theorem, there are an ω-periodic matrix function $P(s)$ and a matrix B such that

$$\Phi(s) = P(s)\, e^{sB}.$$

Then, we have

$$A(s)\, P(s)e^{sB} = \Phi'(s) = P'(s)e^{sB} + P(s)\, Be^{sB}.$$

so that, since e^{sB} is invertible,

$$A(s)P(s) = P'(s) + P(s)B \tag{3.18}$$

for all $s \in \mathbb{R}$. Let us put $z(s) := P^{-1}(s)\, y(s)$. Then, we may write

$$y'(s) = P'(s)\, z(s) + P(s)\, z'(s),$$

and, by equation (3.16), we have

$$P'(s)z(s) + P(s)z'(s) = A(s)P(s)z(s) + h(s, y).$$

Using equation (3.18), we conclude that z satisfies the differential equation

$$z' = Bz + g(s, z), \tag{3.19}$$

where $g(s, z) = P^{-1}(s) \cdot h(s, P(s)\, z)$. Observe that g also satisfies all the conditions required in Proposition 5.

In order to study this system, let us note that

$$\det \Phi(\omega) = \det P(\omega)\ \det e^{\omega B} = \det e^{\omega B},$$

because $P(\omega) = P(0) = I$. By Liouville's formula, we have

$$\det \Phi(\omega) = \exp \int_0^{\omega} \operatorname{tr} A(t)dt = e^b.$$

Furthermore, $s \mapsto f(\phi^s(p))$ is clearly a solution for (3.17). Therefore, we have

$$\Phi(\omega)f(p) = f(\phi^{\omega}(p)) = f(p),$$

and $f(p)$ is eigenvector of $\Phi(\omega) = e^{\omega B}$ associated to the eigenvalue 1. We know then, since the determinant of a matrix is equal to the product of its eigenvalues, that $e^{\omega B}$ has another eigenvalue equal to e^b. Thus, the eigenvalues of the matrix B are 0 and $\dfrac{b}{\omega} < 0$. Therefore, system (3.19) satisfies all the hypotheses needed to apply Proposition 5. Applying it, we get a 1-dimensional C^1-manifold, W_z, satisfying all the properties presented in the proposition.

We define $M_s(p) = \{p + z : z \in W_z\}$. Let us prove that this 1-dimensional C^1-manifold satisfies all the properties asserted. Property (i) is clear because $0 \in W_z$. Now, let $x \in M_s(p)$. Then, we know that $x - p \in W_z$. Following the same steps and using the same notation as above, we know that there are an ω-periodic matrix function, P, and a constant matrix, B, such that

$$\Phi(s) = P(s)e^{sB}$$

and $z(s) := P(s)(\phi^s(x) - \phi^s(p))$ satisfies differential equation (3.19). Since $\Phi(0) = I$, we also have $P(0) = I$. Hence, we have

$$z(0) = \phi^0(x) - \phi^0(p) = x - p \in W_z.$$

By unicity of solution, it follows, by the properties of W_z, that

$$\lim_{s\to\infty} |P(s) \cdot (\phi^s(x) - \phi^s(p))| = 0.$$

It suffices then to notice that P is a continuous and periodic function, and therefore that it is bounded, to see that property *(ii)* holds.

In order to prove *(iii)*, observe that W_z is tangent at 0 to the eigenspace of B corresponding to the eigenvalue $\frac{b}{\omega}$. Thus, it intersects transversally the eigenspace corresponding to the other eigenvalue 0. But the eigenvector corresponding to 0 is $f(p)$, which is tangent to Γ at p. Therefore, W_z must be transverse to Γ at p. ☐

Proposition 7. *Let Γ be a ω-periodic asymptotically stable hyperbolic limit cycle for the planar system $x' = f(x)$. Then, there is a neighbourhood V of Γ, contained in the stable manifold of Γ, foliated by curves that cross Γ once. This foliation is also invariant by the flow, that is, if $M_s(p)$ is the leaf passing through $p \in \Gamma$, we have:*

$$\phi^t(M_s(p)) \subset M_s(\phi^t(p)),$$

for all $t \geqslant 0$.

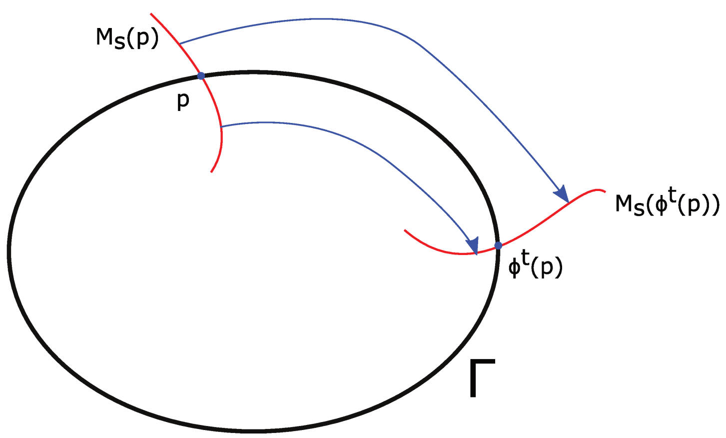

Figure 2 – Foliation built in Proposition 7.

Proof: Let p_0 be a point in Γ. We build $M_s(p_0)$ as in Proposition 6. Observe that, for every other point $p' \in \Gamma$, there is a unique $T_{p'} \in (0, \omega)$ such that $\phi^{T_{p'}}(p_0) = p'$. We define

$$M_s(p') := \phi^{T_{p'}}(M_s(p_0)).$$

By construction, each $M_s(p')$ is a curve intersecting Γ only at p'. We also define the neighbourhood $V = \bigcup_{p' \in \Gamma} M_s(p')$ of Γ.

Let $p \in \Gamma$. Suppose that q is a point in $M_s(p)$. We will prove first that

$$\lim_{t \to \infty} |\phi^t(q) - \phi^t(p)| = 0.$$

By the definition of $M_s(p)$, there is $q_0 \in M_s(p_0)$ such that $\phi^{T_p}(q_0) = q$. Hence, for all $t \in \mathbb{R}$,

$$|\phi^t(q) - \phi^t(p)| = |\phi^{t+T_p}(q_0) - \phi^{t+T_p}(p_0)|.$$

By property (ii) in Proposition 6, we have

$$\lim_{t \to \infty} |\phi^t(q) - \phi^t(p)| = \lim_{t \to \infty} |\phi^{t+T_p}(q_0) - \phi^{t+T_p}(p_0)| = 0,$$

as wanted.

Now, let t be a positive real number. We will prove that $\phi^t(q) \in M_s(\phi^t(p))$. Since $\phi^t(q) \in V$, there is $p' \in \Gamma$ such that $\phi^t(q) \in M_s(p')$. Therefore, we have

$$\lim_{s \to \infty} |\phi^s(\phi^t(q)) - \phi^s(p')| = 0.$$

Defining $p'' = \phi^{-t}(p')$, we have

$$\lim_{s \to \infty} |\phi^{s+t}(q) - \phi^{s+t}(p'')| = 0,$$

or, equivalently,

$$\lim_{s \to \infty} |\phi^s(q) - \phi^s(p'')| = 0.$$

Since $q \in M_s(p)$, we also have

$$\lim_{s \to \infty} |\phi^s(q) - \phi^s(p)| = 0.$$

By the triangle inequality, we have, for all $s \in \mathbb{R}$,

$$|\phi^s(p) - \phi^s(p'')| \leqslant |\phi^s(p) - \phi^s(q)| + |\phi^s(q) - \phi^s(p'')|.$$

Hence, we have

$$\lim_{s \to \infty} |\phi^s(p) - \phi^s(p'')| = 0,$$

which, since $p, p'' \in \Gamma$, can only be true if $p = p''$. Therefore, $p' = \phi^t(p)$, as wanted. ☐

Corollary 1. *Let b be defined as in (3.2). There is $C > 0$ such that, if $q \in V$, the following inequality holds for all $s \geqslant 0$:*

$$d(\phi^s(q), \Gamma) \leqslant C e^{\frac{bs}{\omega}}.$$

Proof: We need only apply Lemma 5 to the neighbourhood V obtained in the last proposition. ☐

3.2 Proof of the fundamental proposition

We begin by building two families of C^1 plane curves, which we will call S_r^+ and S_r^-, with $r \in (0, 1]$. The following properties will be satisfied for those families:

i. The curves S_r^+ lie in the exterior of Γ, and the curves S_r^- lie in the interior of Γ. Each S_r^+ (or S_r^-) together with Γ encloses an annulus;

ii. The curves S_r^+ and S_r^- are transverse to the vector field f;

iii. There is a constant $C_0 > 0$, independent of r, such that $\sup\{d(x, \Gamma) : x \in S_r^\pm\} \leqslant C_0 r$;

iv. If we define $C_f^\pm(r) := \min\{\langle f(x), n(x)\rangle : x \in S_r^\pm\}$, where $n(x)$ is the inward (resp. outward) unit normal vector to S_r^+ (resp. S_r^-) at x, then $C_f^\pm(r) \geqslant C_1 r$ for some constant C_1, independent of r.

The construction of those two families and the verification of the properties stated above will be done in sections 3.2.1 and 3.2.2 below. The proof of the proposition itself will be presented afterwards, in section 3.2.3.

3.2.1 Construction of S_r^+ and S_r^-

By Proposition 7, there is a neighbourhood V of Γ, contained in the stable manifold of Γ, with an invariant foliation with respect to the system (3.1). Each leaf of this foliation is a curve that crosses Γ once. We denote by $M_s(p)$ the leaf that passes through $p \in \Gamma$. Let $t \mapsto x(t, \xi)$ be the solution of (3.1) with $x(0, \xi) = \xi$. Also, let $\Phi(t, \xi)$ denote the principal matrix fundamental solution at $t = 0$ of the linearized system along this solution. This means that Φ satisfies the following differential equation

$$X' = Df(x(t, \xi)) \cdot X, \tag{3.20}$$

and that $\Phi(0, \xi) = I$.

In order to build S_r^+, we will first build a closed C^1 curve, S, contained in the exterior of Γ. Fix a point $q_1 \in M_s(p)$ that lies in the exterior of Γ, and let $q_0 := x(\omega, q_1)$, the point where q_1 first returns to $M_s(p)$. Note that the time needed for q_1 to return is indeed ω, because $M_s(p)$ is invariant with respect to (3.1). Let $q : [0, 1] \to M_s(p) \subset \mathbb{R}^2$ be a smooth function such that $q(0) = q_0$, $q(1) = q_1$, and the derivative of q, including the left-hand and right-hand derivatives at the end points of $[0, 1]$, does not vanish. We also require that q satisfies the following property:

$$\dot{q}(0^+) = \Phi(\omega, q_1)\dot{q}(1^-). \tag{3.21}$$

This requirement can be met because, since the foliation is invariant with respect to (3.1), we have

$$\Phi(\omega, q_1)T_{q_1}M^s(p) = T_{q_0}M^s(p).$$

Let $t : [0,1] \to [0,\omega]$ denote the linear transformation $t(\lambda) = \lambda\omega$, and define the curve S parametrically by $\lambda \mapsto x(t(\lambda), q(\lambda))$. Let us note that S is closed. In fact, we need only see that

$$x(t(0), q(0)) = x(0, q_0) = q_0,$$
$$x(t(1), q(1)) = x(\omega, q_1) = q_0.$$

Let us prove that S is C^1. In order to do so, define $T(\lambda)$ to be the tangent vector to S at the point $x(t(\lambda), q(\lambda))$, where $0 < \lambda < 1$. We have

$$T(\lambda) = \frac{d}{ds}x(t(s), q(s))\Big|_{s=\lambda} = \partial_t x(t(\lambda), q(\lambda))\,\omega + \partial_\xi x(t(\lambda), q(\lambda))\,\dot{q}(\lambda). \tag{3.22}$$

Since x satisfies equation (3.1), we have $\partial_t x(t(\lambda), q(\lambda)) = f(x(t(\lambda), q(\lambda)))$. Moreover, the derivative $\partial_\xi x(t,q)$ satisfies (3.20) and $\partial_\xi x(0,q) = I$ for all q. Hence, $\partial_\xi x(t(\lambda), q(\lambda)) = \Phi(t(\lambda), q(\lambda))$. It follows

$$T(\lambda) = \omega\; f(x(t(\lambda), q(\lambda))) + \Phi(t(\lambda), q(\lambda))\,\dot{q}(\lambda). \tag{3.23}$$

To prove that S is C^1, we need only show that $T(0^+) = T(1^-)$. Indeed, by (3.21),

$$T(0^+) = \omega\, f(x(0, q_0)) + \Phi(0, q_0)\,\dot{q}(0^+) = \omega\, f(q_0) + \dot{q}(0^+);$$
$$T(1^-) = \omega\, f(x(\omega, q_1)) + \Phi(\omega, q_1)\,\dot{q}(1^-) = \omega\, f(q_0) + \dot{q}(0^+).$$

S is, thus, a closed C^1 curve.

Let ϕ^s denote the flow associated with the system $x' = f(x)$. Let $r \in (0,1]$. The curve S_r^+ is defined by $S_r^+ := \phi^s(S)$, where $s = (\omega/b)\ln r$. The construction of S_r^- is done in like manner.

3.2.2 Verification of properties

Property (i) is clearly valid. In order to prove (ii), let us first show that the curve S is transverse to f. Observe that, since $\dot{q}(\lambda)$ is tangent to $M_s(p)$ at $q(\lambda)$, this vector is not parallel to $f(q(\lambda))$. Furthermore, $f(x(t,q))$ is always a solution to the linearized system $x' = Df(x(t,q)) \cdot x$ along the solution $x(t,q)$. Therefore, since $\Phi(t,q)$ is the fundamental matrix solution at for this system,

$$\Phi(t(\lambda), q(\lambda))\; f(q(\lambda)) = f(x(t(\lambda), q(\lambda))).$$

Since the determinant of $\Phi(t(\lambda), q(\lambda))$ is non-zero, it follows that $\Phi(t(\lambda), q(\lambda))\,\dot{q}(\lambda)$ and $\Phi(t(\lambda), q(\lambda))\,f(t(\lambda), q(\lambda))$ are linearly independent. By (3.23), $T(\lambda)$ is nowhere parallel to $f(t(\lambda), q(\lambda))$. Thus, S is everywhere transverse to f.

Now, let us prove that f is transverse to S_r^+. For each $Q \in S_r^+$, there is $P \in S$ such that $\phi^s(P) = Q$. By definition of S_r^+, we have $T_Q S_r^+ = D\phi^s(P)\, T_P S$. Once again, use that $f(\phi^t(P))$ is the solution to the linearized system $x' = Df(x(t,P)) \cdot x$ with initial value $f(P)$. This means that $f(Q) = D\phi^s(P)\, f(P)$. Since $f(P)$ is transverse to $T_P S$, it follows that $T_Q S_r^+$ is also transverse to $f(Q)$, and property (ii) is proved.

To prove property (iii), observe that, by corollary 1, there is $C_0 > 0$ such that, if $x_0 \in V$, we have $d(\phi^s(x_0), \Gamma) \leqslant C_0 e^{\frac{bs}{\omega}}$. By construction, $S \subset V$. Applying the inequality to $P \in S$, we get

$$d(Q, \Gamma) = d(\phi^s(P), \Gamma) \leqslant C_0 e^{\frac{bs}{\omega}} = C_0 r,$$

and C_0 depends only on the neighbourhood V.

Finally, let us prove property (iv). Once more, for each $Q \in S_r^+$, let $P \in S$ be such that $\phi^s(P) = Q$. Additionally, denote by λ the number satisfying $x(t(\lambda), q(\lambda)) = P$. Define the vector $T(Q) \in T_Q S_r^+$ to be, with an abuse of notation, $T(Q) = D\phi^s(P)\, T(\lambda)$. Define $n(Q)$ to be the inward unit normal vector to S_r^+ at Q.

Since $T(Q)$ and $n(Q)$ are orthogonal, we have $\langle f(Q), n(Q)\rangle\, |T(Q)| = |f(Q) \times T(Q)|$. By definition of $C_f^+(r)$, if $r \in (0,1]$, then

$$C_f^+(r) = \min_{Q \in S_r} \left\{ \frac{|f(Q) \times T(Q)|}{|T(Q)|} \right\}.$$

Because $D\phi^t(P)$ is a solution to (3.20) with $\xi = P$ and $D\phi^0(P) = I$, we have $D\phi^t(P) = \Phi(t, P)$. Then, by (3.23),

$$\begin{aligned} T(Q) &= D\phi^s(P)\, T(\lambda) = \omega\, \Phi(s,P)\, f(P) + \Phi(s,P)\, \Phi(t(\lambda), q(\lambda))\, \dot{q}(\lambda) \\ &= \omega\, f(Q) + \Phi(s + t(\lambda), q(\lambda))\, \dot{q}(\lambda). \end{aligned} \tag{3.24}$$

Hence, $|f(Q) \times T(Q)| = |f(Q) \times \Phi(s + t(\lambda), q(\lambda))\, \dot{q}(\lambda)|$. Next, we will make use of the following auxiliary lemma, which will be proved later on in order not to interrupt the flow of the proof.

Lemma 6 (Auxiliary lemma)**.** *Under the same hypotheses and notation from the proof of Proposition 2, if q_1 is chosen sufficiently close to $p \in M_s(p)$, there is $K \geqslant 1$ such that*

$$\frac{1}{K} \exp \int_0^t \operatorname{tr} A(\tau) d\tau \leqslant |\Phi(t, q(\lambda))\, \dot{q}(\lambda)| \leqslant K \exp \int_0^t \operatorname{tr} A(\tau) d\tau,$$

for all $t \geqslant 0$, where $A(\tau) = Df(\phi^\tau(p))$.

By the lemma above, if we choose q_1 sufficiently close to $p \in M_s(p)$, there is a number $K \geqslant 1$ such that

$$\frac{1}{K} \exp \int_0^{s+t(\lambda)} \operatorname{tr} A(\tau) d\tau \leqslant |\Phi(s + t(\lambda), q(\lambda))\, \dot{q}(\lambda)| \leqslant K \exp \int_0^{s+t(\lambda)} \operatorname{tr} A(\tau) d\tau,$$

where $A(t) := Df(\phi^t(p))$. The proof of last inequality is given in Lemma 6 in order not to interrupt the flow of the proof.

Let $v(Q)$ denote the unit tangent vector at Q to the stable fiber through Q. By (3.24), we have:

$$|f(Q) \times T(Q)| = |f(Q) \times \Phi(s + t(\lambda), q(\lambda))\, \dot{q}(\lambda)|.$$

Since $\Phi(s + t(\lambda), q(\lambda))\, \dot{q}(\lambda)$ is tangent to the stable fiber through Q, we have:

$$|f(Q) \times T(Q)| \geqslant \frac{1}{K}|f(Q) \times v(Q)| \exp \int_0^{s+t(\lambda)} \operatorname{tr} A(\tau) d\tau,$$

Furthermore, again by (3.24), there is a constant $C_2 > 0$ such that

$$\begin{aligned} |T(Q)| &\leqslant \omega|f(Q)| + |\Phi(s + t(\lambda), q(\lambda))\dot{q}(\lambda)| \\ &\leqslant \omega|f(Q)| + K \exp \int_0^{s+t(\lambda)} \operatorname{tr} A(\tau) d\tau \\ &\leqslant \omega|f(Q)| + C_2. \end{aligned}$$

Observe that $|f|$ and $|f(Q) \times v(Q)|$ are bounded below over the compact set whose boundary is given by $S_1^+ \cup S_1^-$. Since Q must be in this set, using the above estimates, we get that there is $C_3 > 0$ such that

$$\begin{aligned} \frac{|f(Q) \times T(Q)|}{|T(Q)|} &\geqslant \frac{|f(Q) \times v(Q)|}{K(\omega|f(Q)| + C_2)} \exp \int_0^{s+t(\lambda)} \operatorname{tr} A(\tau) d\tau \\ &\geqslant C_3 \exp \int_0^{s} \operatorname{tr} A(\tau) d\tau. \end{aligned}$$

Let m be a nonnegative integer and ρ be a number in $[0, \omega)$ such that $s = m\omega + \rho$. Then there are constants $C_1 > 0$ and $C_4 > 0$ such that

$$\begin{aligned} C_3 \exp \int_0^{s} \operatorname{tr} A(\tau) d\tau &= C_3 \exp \int_0^{m\omega+\rho} \operatorname{tr} A(\tau) d\tau \\ &\geqslant C_4 \exp \int_0^{m\omega} \operatorname{tr} A(\tau) d\tau \\ &= C_4 e^{bm} = C_4 e^{\frac{bs-b\rho}{\omega}} \geqslant C_1 r, \end{aligned}$$

proving that $C_f^+(r) \geqslant C_1 r$.

3.2.3 Proof of the fundamental proposition

We begin by "suspending" the curves S_r^+ and S_r^-, generating the tori $B_r^\pm = \{(x, \tau) : x \in S_r^\pm\} \subset \mathbb{R}^2 \times \mathbb{R}$. We then define N as the annular region bounded by B_1^+ and B_1^-. M is the torus generated by the "suspension" of Γ. Observe that $f_s(x, \tau) := (f(x), 1)$ is the vector field present in equation (3.3). The following properties of $B_r^\pm$ follow directly from the properties we established for $S_r^\pm$:

i. Each B_r^+ lies in the exterior of M, and each B_r^- lies in the interior of M;

ii. B_r^+ and B_r^- are transverse to the vector field f_s;

iii. There is a constant $C_0 > 0$, independent of r, such that $\sup\{d((x,\tau), M) : x \in S_r^\pm, \tau \in \mathbb{R}\} \leqslant C_0 r$;

iv. If we define $C_{f_s}^\pm(r) := \min\{\langle f_s(x,\tau), n_s(x,\tau)\rangle : (x,\tau) \in B_r^\pm\}$, where we have defined $n_s(x,\tau) = (n(x), 0)$, the inward (resp. outward) unit normal vector to B_r^+ (resp. B_r^-) at (x,τ), then $C_{f_s}^\pm(r) \geqslant C_1 r$ for some constant C_1, independent of r.

Let $g : N \to \mathbb{R}^2$ be such that $g(x,\tau) = g(x, \tau + T)$ for all $\tau \in \mathbb{R}$, and for which the system (3.4) has an invariant set $\tilde{M} \subset N$. The periodicity of g will ensure that g is well defined for the angular variable τ.

We divide the proof in two cases, depending on the value of the quantity $\|g\|_{C^0} := \sup\{|g(x,\tau)| : (x,\tau) \in N\}$. First, we suppose that $\|g\|_{C^0} > C_1$. In this case, we have

$$\sup\{d((x,\tau), M) : (x,\tau) \in \tilde{M}\} \leqslant \sup\{d((x,\tau), M) : (x,\tau) \in N\} \leqslant C_0,$$

by property (iii). Thus, we can write

$$\sup\{d((x,\tau), M) : x \in \tilde{M}\} \leqslant C_0 = C_1 \cdot \frac{C_0}{C_1} < \|g\|_{C^0} \cdot \left(\frac{C_0}{C_1} + 1\right),$$

which proves the proposition in this case.

Now, let us assume that $\|g\|_{C^0} \leqslant C_1$. Then, there is $r_0 \in (0,1]$ such that $\|g\|_{C^0} = C_1 r_0$. Let $(x,\tau) \in B_r^\pm$. Let us denote the vector field appearing in system (3.4) by $\tilde{f}_s(x,\tau) = f_s(x,\tau) + (g(x,\tau), 0)$. We have

$$\langle \tilde{f}_s(x,\tau),\, n_s(x,\tau)\rangle \geqslant C_{f_s}^\pm(r) - \|g\|_{C_0} \geqslant C_1(r - r_0),$$

by property (iv). Thus, for $r > r_0$, we have

$$\langle \tilde{f}_s(x,\tau),\, n_s(x,\tau)\rangle > 0.$$

Therefore, if $r > r_0$, the set $B_r^+ \cup B_r^-$ is the boundary of a positively invariant set for the flow of (3.4). To conclude the proof, we will prove that this ensures that $\tilde{M}$ must be contained in the region bounded by $B_{r_0}^+$ and $B_{r_0}^-$. Indeed, provided that this is true, we have

$$\sup\{d((x,\tau), M) : x \in \tilde{M}\} \leqslant C_0 r_0 = \frac{C_0}{C_1}\|g\|_{C^0} < \|g\|_{C^0}\left(\frac{C_0}{C_1} + 1\right).$$

We begin by defining the equivalence relation $\sim$ in the space $\mathbb{R}^2 \times \mathbb{R}$:

$$(x,\tau) \sim (x', \tau') \iff x = x' \text{ and } \exists\, n \in \mathbb{Z} \text{ such that } \tau' = \tau + nT.$$

Since g is T-periodic in τ, system (3.4) and the sets M and $\tilde{M}$ are well defined in $\mathbb{R}^2 \times \mathbb{R}/\sim$. In this new space, however, the region bounded by $B_r^+ \cup B_r^-$ is compact. We will prove that the orbit of $\tilde{f}_s$ through every point $y = (x, \tau) \in \text{Ext}(B_{r_0}^+)$ intersects $\text{Ext}(B_1^+)$. Let $\phi(t, y)$ be the flow of $\tilde{f}_s$. Define $\gamma_-(y) := \{\phi(t, y) : t \leqslant 0\}$. Assume that $\gamma_-(y) \subset \overline{\text{Int}(B_1^+)}$, which is compact. Then, the α-limit set $\alpha(y)$ is a non-empty, compact and connected set contained in $\overline{\text{Int}(B_1^+)} \backslash \overline{\text{Int}(B_{r_0}^+)}$. We will prove that this assumption leads us to a contradiction.

Define $L := \{r \in (r_0, 1] : \alpha(y) \cap \overline{\text{Ext}(B_r^+)} = \varnothing\}$. Suppose first that L is empty. Then, there is $z \in \alpha(x) \cap \overline{\text{Ext}(B_1^+)}$. Since $\alpha(y) \subset \overline{\text{Int}(B_1^+)}$, we have: $z \in B_1^+$. Thus, there is $\epsilon > 0$ such that $\phi(-\epsilon, z) \in \text{Ext}(B_1^+)$, because $\langle \tilde{f}_s(z), n_s(z) \rangle > 0$. Therefore, $z \in \gamma_-(y) \cap \text{Ext}(B_1^+)$, which is a contradiction.

Now, suppose L is not empty. Define $i := \inf L$. Assume that $i = r_0$. Let $r > r_0$ be such that $y \in B_r^+$. By assumption, there is $\eta < 0$ such that $\phi(\eta, y) \in \overline{\text{Int}(B_{r'}^+)}$, where $r_0 < r' < r$. But this contradicts the fact that $B_{r'}^+ \cup B_{r'}^-$ is the boundary of a positively invariant region. Thus, we have $i > r_0$.

We now consider two cases. If $i \in L$, since $\alpha(y)$ and $\overline{\text{Ext}(B_i^+)}$ are closed sets, there is a neighbourhood U of $\overline{\text{Ext}(B_i^+)}$ such that $U \cap \alpha(y) = \varnothing$. But this means there is $l \in (r_0, i)$ such that $\alpha(y) \cap \overline{\text{Ext}(B_l^+)} = \varnothing$, which is a contradiction. On the other hand, if $i \notin L$, we can take $z \in \alpha(y) \cap \overline{\text{Ext}(B_i^+)}$. Suppose that $z \in \text{Ext}(B_i^+)$. Then, there would be $j > i$ such that $z \in \alpha(y) \cap \overline{\text{Ext}(B_j^+)}$, that is, $j \notin L$. Observe that if $l \in L$ and $l' \geqslant l$, then $l' \in L$. This means that, if $l < j$, then l cannot be in L. Since i is the greatest lower bound of L, this is a contradiction. The only possibility then is that $z \in \alpha(y) \cap B_i^+$. But then again there is $\epsilon > 0$ such that $\phi(-\epsilon, z) \in B_j^+$, where $j > i$. Since $\alpha(y)$ is invariant, we have $j \notin L$. Once more, this is a contradiction with the fact that i is the infimum.

We have thus proved that $\gamma_-(y) \cap \text{Ext}(B_1^+) \neq \varnothing$ if $y \in \text{Ext}(B_{r_0}^+)$. By hypothesis, the invariant set $\tilde{M}$ is contained in $N \subset \overline{\text{Int}(B_1^+)}$. Therefore, we must have $\tilde{M} \subset \overline{\text{Int}(B_{r_0}^+)}$. We can also prove that $\tilde{M} \subset \overline{\text{Ext}(B_{r_0}^-)}$ in the same way.

3.2.4 Proof of the auxiliary lemma

We present the proof of the auxiliary lemma used in section 3.2.2.

Proof: Let $t \geqslant 0$. Let m be a positive integer and $\rho \in [0, \omega)$ be such that $t = m\omega + \rho$. We have, by properties of the flow:

$$\phi^t(x) = \phi^\rho(\phi^{m\omega}(x)),$$

for all $x \in \mathbb{R}^2$. Therefore, by differentiating with respect to x, we have:

$$\Phi(t, x) = \Phi(\rho, \phi^{m\omega}(x)) \cdot \Phi(m\omega, x).$$

If we take $x = q(\lambda)$ and apply both sides to $\dot{q}(\lambda)$, we get:

$$|\Phi(t, q(\lambda)) \cdot \dot{q}(\lambda)| = |\Phi(\rho, \phi^{m\omega}(q(\lambda))) \cdot \Phi(m\omega, q(\lambda)) \cdot \dot{q}(\lambda)|.$$

Thus, we have

$$|\Phi(t,q(\lambda))\cdot\dot{q}(\lambda)| = \Big|\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)\Big|\Big|\Phi(\rho,\phi^{m\omega}(q(\lambda)))\cdot\left(\frac{\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)}{|\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)|}\right)\Big|.$$

Let us define the function $\alpha : [0,\omega]\times S^1 \to \mathbb{R}$ by

$$\alpha(\rho,v) = |\Phi(\rho,p)\cdot v|.$$

Observe that α is a continuous function defined on compact set; hence, it has a maximum and a minimum value on that set. Moreover, $\alpha(\rho,v)>0$, because, by Liouville's formula, we have

$$\det(\Phi(\rho,p))>0, \qquad \forall\rho\in[0,\omega].$$

then there are $l,L>0$ such that

$$l<\alpha(\rho,v)<L. \tag{3.25}$$

Let $\epsilon>0$. By continuity, if z is sufficiently close to p, we have

$$|\Phi(\rho,z)\cdot v-\Phi(\rho,p)\cdot v|<\epsilon.$$

Therefore, by taking q_1 sufficiently close to p, we get:

$$\Big|\Phi(\rho,\phi^{m\omega}(q(\lambda)))\cdot\left(\frac{\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)}{|\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)|}\right)-\Phi(\rho,p)\cdot\left(\frac{\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)}{|\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)|}\right)\Big|<\frac{l}{2}.$$

Combining last inequality with (3.25), we get:

$$\frac{l}{2}<\Big|\Phi(\rho,\phi^{m\omega}(q(\lambda)))\cdot\left(\frac{\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)}{|\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)|}\right)\Big|<L+\frac{l}{2}.$$

Since $q(\lambda)$ is a parametrisation of a submanifold of $M_s(p)$, we have

$$\phi^{m\omega}(q(\lambda)) = \pi^m(q(\lambda)),$$

where π denotes the Poincaré map on $M_s(p)$. Taking the derivative with respect to λ, we get

$$\Big|\Phi(m\omega,q(\lambda))\cdot\dot{q}(\lambda)\Big| = \Big|(\pi^m)'(q(\lambda))\Big|\Big|\dot{q}(\lambda)\Big|.$$

□

4 Proof of the main theorem

In this chapter, we shall prove Theorem B. This theorem is related to systems of the following form:

$$\begin{aligned} \rho' &= f(\rho,\sigma) + \mu R(\rho,\sigma,\tau/\mu^\ell,\mu) \\ \sigma' &= g(\rho,\sigma) + \mu S(\rho,\sigma,\tau/\mu^\ell,\mu) \\ \tau' &= 1, \end{aligned} \tag{4.1}$$

where $\ell \geqslant 2$ is a natural number, where $|\mu| > 0$, where f, g, R and S are C^r functions ($r \geqslant 2$) that are all 2π-periodic functions of the angular variable σ, and where $t \mapsto R(\rho,\sigma,t,\mu)$ and $t \mapsto S(\rho,\sigma,t,\mu)$ are $2\pi m/\Omega$-periodic, $m \in \mathbb{N}$. We view τ as an angular variable modulo $2\pi m\mu^\ell/\Omega$, and we use s for the independent variable.

In order to study system (4.1) and prove the theorem, we follow Chicone and Liu in (CHICONE; LIU, 1999/00). Consider the unperturbed system

$$\begin{aligned} \rho' &= f(\rho,\sigma) \\ \sigma' &= g(\rho,\sigma). \end{aligned} \tag{4.2}$$

We recall Hypothesis B, stated in chapter 2.

Hypothesis B: System (4.2) has an attracting hyperbolic limit cycle Γ that is the graph of a function of the angular variable σ.

For the convenience of the reader, we restate Theorem B, changing the notation to that which we will adopt throughout this chapter:

Theorem B: Let k be an integer such that $2 \leqslant k \leqslant r$. Suppose that Hypothesis B is satisfied. Then, if $|\mu|$ is sufficiently small, system (4.1) has a k-normally hyperbolic invariant manifold that is the graph of a function of the angular variables σ and τ.

We remark that it suffices to prove the case $\mu > 0$, since the case $\mu < 0$ is easily transformed to the first case by redefining R and S. We will study $E^{\epsilon,\mu}$, the auxiliary family of differential systems given by

$$E^{\epsilon,\mu} : \begin{cases} \rho' = f(\rho,\sigma) + \epsilon R(\rho,\sigma,\tau/\mu^\ell,\mu) \\ \sigma' = g(\rho,\sigma) + \epsilon S(\rho,\sigma,\tau/\mu^\ell,\mu) \\ \tau' = 1, \end{cases}$$

Note that system (4.1) coincides with $E^{\mu,\mu}$. Furthermore, Hypothesis B guarantees that the suspended system

$$\begin{aligned} \rho' &= f(\rho,\sigma) \\ \sigma' &= g(\rho,\sigma) \\ \tau' &= 1, \end{aligned} \tag{4.3}$$

where τ is seen as an angular variable modulo $2\pi m\mu^{\ell}/\Omega$, has a normally hyperbolic torus that is a graph over the angular variables σ and τ.

4.1 Outline of the proof: the method of continuation

In order to prove theorem B, we apply a method of continuation, by defining, for each $\mu > 0$, the set A^{μ}, which is the maximal interval with left endpoint at $\epsilon = 0$ such that $E^{\epsilon,\mu}$ has a k-normally hyperbolic invariant manifold, $k \geqslant 2$. We will prove that, for $\mu > 0$ sufficiently small, the set $A^{\mu} \cap [0,\mu]$ is a non-empty, connected subset of $[0,\mu]$ which is also relatively open and closed, and thus $A^{\mu} \cap [0,\mu] = [0,\mu]$. Then, Theorem B follows by taking $\epsilon = \mu \in A^{\mu}$.

First, notice that $A^{\mu} \cap [0,\mu]$ is indeed connected, because it is an interval by definition. It is also non-empty, because $0 \in A^{\mu} \cap [0,\mu]$ by hypothesis. Moreover, Theorem 2 ensures that, if $\epsilon \in A^{\mu}$, there is an open interval containing ϵ that is contained in A^{μ}. Hence, the set $A^{\mu} \cap [0,\mu]$ is relatively open. Theorem B is thus a direct consequence of the following proposition:

Proposition 8. *If $\mu > 0$ is chosen sufficiently small and if $\epsilon_* < \mu$ is the least upper bound of a relatively open interval with left endpoint at 0 in A^{μ}, then $\epsilon_* \in A^{\mu}$.*

The proof of last proposition will be divided in three lemmas.

Lemma L1: With the hypotheses and notation of this section, the system $E^{\epsilon_*,\mu}$ has an invariant manifold $M(\epsilon_*,\mu)$ given as the graph of a C^1 function of the angular variables.

Lemma L2: If $M(\epsilon_*,\mu)$ is the invariant manifold in Lemma L1, then it has an invariant normal bundle.

Lemma L3: If $M(\epsilon_*,\mu)$ is the invariant manifold in Lemma L1, then it is k-normally hyperbolic. In particular, $M(\epsilon_*,\mu)$ is C^k and $\epsilon_* \in A^{\mu}$.

The rest of this work will be dedicated to proving the three lemmas above. In section 4.2, we will build the objects needed to render the concepts presented in 2.1 applicable. In section 4.3, we set out to prove technical lemmas that will be used later on. Finally, in section 4.4, we effectively prove Lemmas L1, L2 and L3.

For the remainder of this chapter, let us assume that $\mu > 0$ and $\epsilon \geqslant 0$ are fixed, and that system (4.3) has an invariant torus, $M(\epsilon, \mu)$, given as the graph of h^ϵ, a C^1 function of the angular variables.

4.2 Normal splitting

The proof of Lemma L3 consists in showing that an invariant manifold is k-normally hyperbolic. In order to do this, we will study quantities that arise when we have an invariant splitting of the tangent bundle of the ambient space, as seen in section 2.1.5. We begin this section by effectively constructing a pair $X_1(s)$ and $X_2(s)$ as in that section for the case of the vector field given by system (4.3) and the invariant torus $M(\epsilon, \mu)$, invariant under this field.

Let us define new functions

$$\begin{aligned} F(\rho, \sigma, \tau, \mu, \epsilon) &= f(\rho, \sigma) + \epsilon R(\rho, \sigma, \tau/\mu^\ell, \mu) \\ G(\rho, \sigma, \tau, \mu, \epsilon) &= g(\rho, \sigma) + \epsilon S(\rho, \sigma, \tau/\mu^\ell, \mu) \end{aligned} \tag{4.4}$$

and suppose that the invariant torus $M(\epsilon, \mu)$ is given as the graph of the function $(\sigma, \tau) \mapsto h^\epsilon(\sigma, \tau)$. First, let us find two vector fields that span the tangent space $T_\xi M(\epsilon, \mu)$ at every point $\xi \in M(\epsilon, \mu)$. The vector field

$$\mathcal{X}_1^\epsilon(\sigma, \tau) = \begin{pmatrix} F(\rho, \sigma, \tau, \mu, \epsilon) \\ G(\rho, \sigma, \tau, \mu, \epsilon) \\ 1 \end{pmatrix} \tag{4.5}$$

is clearly tangent to $M(\epsilon, \mu)$, because it is a vector field under which $M(\epsilon, \mu)$ is invariant.

The curve given by $\sigma \mapsto (h^\epsilon(\sigma, \tau), \sigma, \tau)$ for fixed τ is on $M(\epsilon, \mu)$. Therefore, a vector tangent to this curve is always tangent to $M(\epsilon, \mu)$. One such vector is obtained by the derivative of the function defining the curve:

$$\mathcal{X}_2^\epsilon(\sigma, \tau) = \begin{pmatrix} h_\sigma^\epsilon(\sigma, \tau) \\ 1 \\ 0 \end{pmatrix}. \tag{4.6}$$

Observe that, for each (σ, τ), the vectors $\mathcal{X}_1^\epsilon(\sigma, \tau)$ and $\mathcal{X}_2^\epsilon(\sigma, \tau)$ are linearly independent vectors in $T_\xi M(\epsilon, \mu)$, where $\xi := (h^\epsilon(\sigma, \tau), \sigma, \tau)$. Thus, these vectors span the fiber $T_\xi M(\epsilon, \mu)$ of the tangent bundle of $M(\epsilon, \mu)$.

Now, we will determine contraction rates for the flow on the invariant torus $M(\epsilon, \mu)$ by studying the solutions of the first variational equational for system (4.3). Let

$$s \mapsto \gamma^\epsilon(s, q) =: (h^\epsilon(\sigma^\epsilon(s, q), \tau(s)), \sigma^\epsilon(s, q), \tau(s)) \tag{4.7}$$

be the solution to system (4.3) with $\gamma^\epsilon(0,q) = (h^\epsilon(q,0),q,0)$. This solution is on the invariant torus. The first variational equation along the solution $s \mapsto \gamma^\epsilon(s,q)$ is defined as

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} F_\rho & F_\sigma & F_\tau \\ G_\rho & G_\sigma & G_\tau \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{4.8}$$

where the argument of each function is $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)$. Solutions to the variational equation are given by the next proposition, and are directly related to the vector fields $\mathcal{X}_1^\epsilon$ and $\mathcal{X}_2^\epsilon$.

Proposition 9. *Two independent solutions to the variational equation (4.8) along the solution (4.7), are given by, with a slight abuse of notation,*

$$\begin{aligned} X_1(s) &:= \mathcal{X}_1^\epsilon(\gamma^\epsilon(s,q)) \\ X_2(s) &:= y^\epsilon(s,q)\mathcal{X}_2^\epsilon(\gamma^\epsilon(s,q)), \end{aligned}$$

where

$$y^\epsilon(t,q) := \exp\left(\int_0^t (G_\rho h_\sigma^\epsilon + G_\sigma)\right) ds.$$

In the last integral, the arguments of the functions G_ρ and G_σ are given by the expression $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)$, and the argument of h_σ^ϵ is $(\sigma^\epsilon(s,q),\tau(s))$. Moreover, $X_1(s)$ and $X_2(s)$ span the tangent space of the invariant torus $M(\epsilon,\mu)$ at each point along the solution (4.7).

Proof: For $X_1(s)$, we define, to simplify notation

$$\gamma^\epsilon(s,q) =: (\gamma_1^\epsilon(s,q),\gamma_2^\epsilon(s,q),\gamma_3^\epsilon(s,q))$$

We will omit the arguments μ and ϵ from the functions F and G and their derivatives for simplicity. They are assumed to remain unchanged throughout the proof. Then, since $\gamma^\epsilon(s,q)$ is a solution to (4.3), we have

$$\begin{aligned} (\gamma_1^\epsilon)'(s,q) &= F(\gamma^\epsilon(s,q)) \\ (\gamma_2^\epsilon)'(s,q) &= G(\gamma^\epsilon(s,q)) \\ (\gamma_3^\epsilon)'(s,q) &= 1 \end{aligned}$$

Applying the derivative to $X_1(s)$, we get

$$\begin{pmatrix} F_\rho(\gamma^\epsilon(s,q))(\gamma_1^\epsilon(s,q))' + F_\sigma(\gamma^\epsilon(s,q))(\gamma_2^\epsilon(s,q))' + F_\tau(\gamma^\epsilon(s,q))(\gamma_3^\epsilon(s,q))' \\ G_\rho(\gamma^\epsilon(s,q))(\gamma_1^\epsilon(s,q))' + G_\sigma(\gamma^\epsilon(s,q))(\gamma_2^\epsilon(s,q))' + G_\tau(\gamma^\epsilon(s,q))(\gamma_3^\epsilon(s,q))' \\ 0 \end{pmatrix}$$

This can be written as

$$\begin{pmatrix} F_\rho & F_\sigma & F_\tau \\ G_\rho & G_\sigma & G_\tau \\ 0 & 0 & 0 \end{pmatrix} X_1(s)$$

where the argument of each function in the square matrix is $\gamma^\epsilon(s,q)$ as we wanted.

In order to prove that $X_2(s)$ is also a solution, we will first prove that

$$X_2(s) = \frac{\partial \gamma^\epsilon}{\partial q}(s,q). \tag{4.9}$$

In fact, we have

$$\frac{\partial \gamma^\epsilon}{\partial q}(s,q) = \left(h^\epsilon_\sigma(\sigma^\epsilon(s,q),\tau(s)) \frac{\partial \sigma^\epsilon}{\partial q}(s,q), \frac{\partial \sigma^\epsilon}{\partial q}(s,q), 0 \right) = \frac{\partial \sigma^\epsilon}{\partial q}(s,q) \mathcal{X}^\epsilon_2(\gamma^\epsilon(s,q)).$$

Moreover, we have, by changing order of derivatives

$$\begin{aligned} \frac{\partial}{\partial s}\left(\frac{\partial \sigma^\epsilon}{\partial q}\right)(s,q) &= \frac{\partial}{\partial q}\Big(G(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)), \sigma^\epsilon(s,q), \tau(s)) \Big) = \\ &= G_\rho(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)), \sigma^\epsilon(s,q),\tau(s)) \cdot h^\epsilon_\sigma(\sigma^\epsilon(s,q),\tau(s)) \cdot \frac{\partial \sigma^\epsilon}{\partial q}(s,q) \\ &+ G_\sigma(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)), \sigma^\epsilon(s,q),\tau(s)) \cdot \frac{\partial \sigma^\epsilon}{\partial q}(s,q). \end{aligned}$$

Thus, the partial derivative of σ^ϵ with respect to q satisfies the same initial value problem as $y^\epsilon(s,q)$, which proves (4.9).

Now, observe that

$$\frac{\partial}{\partial s}\left(\frac{\partial \gamma^\epsilon}{\partial q}\right)(s,q) = \frac{\partial}{\partial q}\left(\frac{\partial \gamma^\epsilon}{\partial s}\right)(s,q) = \frac{\partial}{\partial q}\Big(F(\gamma^\epsilon(s,q), G(\gamma^\epsilon(s,q), 1 \Big).$$

Applying the derivative, we get

$$\frac{\partial}{\partial s}\left(\frac{\partial \gamma^\epsilon}{\partial q}\right)(s,q) = \begin{pmatrix} F_\rho & F_\sigma & F_\tau \\ G_\rho & G_\sigma & G_\tau \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial \gamma^\epsilon}{\partial q}(s,q),$$

where the argument of each function in the matrix is once again $\gamma^\epsilon(s,q)$. By (4.9), we know that $X_2(s)$ is a solution to the variational equation.

Finally, since $\mathcal{X}^\epsilon_1(\gamma^\epsilon(s,q))$ and $\mathcal{X}^\epsilon_2(\gamma^\epsilon(s,q))$ are linearly independent and $y^\epsilon(s,q)$ is a positive function, the vectors $X_1(s)$ and $X_2(s)$ span the tangent space $T_{\gamma^\epsilon(s,q)} M(\epsilon,\mu)$. □

We remark also that the angle condition stated in section 2.1.5 is satisfied by $X_1(s)$ and $X_2(s)$ in this case. In order to prove this, observe that the angle $\theta(s)$ between

$X_1(s)$ and $X_2(s)$ is the angle between $\mathcal{X}_1^\epsilon(\gamma^\epsilon(s,q))$ and $\mathcal{X}_2^\epsilon(\gamma^\epsilon(s,q))$. The vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ are continuous over the compact $M(\epsilon,\mu)$, therefore the angle between them over $M(\epsilon,\mu)$ attains its minimum and maximum value in this manifold. Let us name k_θ and K_θ the minimum and maximum values, respectively, of the angle between $\mathcal{X}_1(\xi)$ and $\mathcal{X}_2(\xi)$ with $\xi \in M(\epsilon,\mu)$. Since $\gamma^\epsilon(s,q) \in M(\epsilon,\mu)$ regardless of $s \geqslant 0$ and $q \in M(\epsilon,\mu)$, we have $k_\theta < \theta(s) < K_\theta$, for all possible choices of s and q. Furthermore, the inequality $-\pi < k_\theta < K_\theta < \pi$ holds. In fact, if it were otherwise, we would have that the fields $\mathcal{X}_1$ and $\mathcal{X}_2$ are parallel at a point $\xi \in M(\epsilon,\mu)$, which is absurd, because they were proved to span the fiber $T_\xi M(\epsilon,\mu)$.

We have constructed X_1 and X_2 as needed. We will now assume that $M(\epsilon,\mu)$ is normally hyperbolic in order to obtain necessary conditions for it to be so. Define $\Phi^\epsilon(s)$ to be the principal fundamental matrix solution of (4.8) at $s = 0$. By Theorem 3, there is a normal bundle over $M(\epsilon,\mu)$ that is invariant under $\Phi^\epsilon(s)$. Since the codimension of M is one, the dimension of the fibres of the normal bundle is also one. Let $\mathcal{X}_0^\epsilon$ be a continuous nonzero section of the normal bundle. We define the following family

$$\mathcal{L}^s = \{(\rho,\sigma,\tau) \in \mathbb{R}^3 : s \in \mathbb{R}\}.$$

We remark that $\{\mathcal{L}^s\}_{s\in\mathbb{R}}$ is a foliation of the ambient space that is invariant under the flow of system (4.3). For each $\xi \in M(\epsilon,\mu)$, let $\mathcal{L}(\xi)$ denote the leaf of this foliation that passes through ξ. The invariance of this foliation under the flow ensures that

$$D\phi^s(\xi) \cdot T_\xi\mathcal{L}(\xi) = T_{\phi^s(\xi)}\mathcal{L}(\phi^s(\xi)). \tag{4.10}$$

We define

$$X_0(s) := \mathcal{X}_0^\epsilon(\gamma^\epsilon(s,q)),$$

and

$$X_2^\perp(s) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot X_2(s).$$

It is clear that $\{X_1(s), X_2(s), X_2^\perp(s)\}$ is a basis for the tangent space of the ambient space at the point $\gamma^\epsilon(s,q)$. It is also clear that $X_2(s)$ and $X_2^\perp(s)$ span $T_{\gamma^\epsilon(s,q)}\mathcal{L}(\gamma^\epsilon(s,q))$ for all $s \geqslant 0$. Furthermore, (4.10) guarantees that

$$\Phi^\epsilon(s)X_2^\perp(0) \in T_{\gamma^\epsilon(s,q)}\mathcal{L}(\gamma^\epsilon(s,q)),$$

or, equivalently, that there are $a(s)$ and $b(s)$ such that

$$\Phi^\epsilon(s)X_2^\perp(0) = \alpha_1(s)X_2(s) + \alpha_2(s)X_2^\perp(s).$$

We will prove that $X_0(s)$ in the span of $X_2(s)$ and $X_2^{\perp}(s)$ for all $s \geqslant 0$. There are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$X_0(0) = \alpha_1 X_1(0) + \alpha_2 X_2(0) + \alpha_3 X_2^{\perp}(0).$$

Therefore, since $X_1(s)$ and $X_2(s)$ are solutions to the variational equation (4.8), we have

$$\Phi^{\epsilon}(s)X_0(0) = \alpha_1 X_1(s) + \alpha_2 X_2(s) + \alpha_3\big(a(s)X_2(s) + b(s)X_2^{\perp}(s)\big),$$

and, by considering the third entry of $\Phi^{\epsilon}(s)X_0(0)$, we have

$$|\Phi^{\epsilon}(s)X_0(0)| \geqslant |\alpha_1|,$$

for all $s \geqslant 0$. Since $M(\epsilon,\mu)$ is normally hyperbolic, there must be $\beta > 0$ and $c > 0$ such that

$$|\Phi^{\epsilon}(s)X_0(0)| \leqslant |X_0(0)| \cdot ce^{-\beta s},$$

for all $s \geqslant 0$. This is possible only if $\alpha_1 = 0$, so that the vector $\Phi^{\epsilon}(s)X_0(0)$ is indeed in the span of the set $\{X_2(s), X_2^{\perp}(s)\}$.

4.2.1 A formula for $\lambda_3(s)$

We have just proved that a necessary condition for $M(\epsilon,\mu)$ to be normally hyperbolic is: there is a normal bundle over $M(\epsilon,\mu)$, invariant under the linearized flow, that is always tangent to the leaves of the foliation $\{\mathcal{L}^s\}_{s\in\mathbb{R}}$. In this section, we assume that such a normal bundle, $\mathcal{X}_0^{\epsilon}$, is already constructed, and find a convenient formula for the quantity λ_3 as defined in section 2.1.5. The fact that $X_2(s)$ and $X_2^{\perp}(s)$ span $T_{\gamma^{\epsilon}(s,q)}\mathcal{L}(\gamma^{\epsilon}(s,q))$ for all $s \geqslant 0$ does not rely on $M(\epsilon,\mu)$ being normally hyperbolic. Thus, there are $a(s)$ and $b(s)$ such that

$$X(s) := \Phi^{\epsilon}(s)X_2^{\perp}(0) = a(s)X_2(s) + b(s)X_2^{\perp}(s). \tag{4.11}$$

We will compute formulas regarding $a(s)$ and $b(s)$. We define

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B(s) = \begin{pmatrix} F_\rho & F_\sigma & F_\tau \\ G_\rho & G_\sigma & G_\tau \\ 0 & 0 & 0 \end{pmatrix},$$

where the argument of each function in $B(s)$ is once again given by the expression $(h^{\epsilon}(\sigma^{\epsilon}(s,q),\tau(s)), \sigma^{\epsilon}(s,q), \tau(s), \mu, \epsilon)$. Since $\Phi^{\epsilon}(s)$ is the principal matrix fundamental solution at $t = 0$ of the linearized system along $\gamma^{\epsilon}(s,q)$, $X(s)$ satisfies the differential equation $X'(s) = B(s) \cdot X(s)$. Therefore, we have:

$$\big(a(s)X_2(s) + b(s)X_2^{\perp}(s)\big)' = B(s) \cdot \big(a(s)X_2(s) + b(s)X_2^{\perp}(s)\big) \tag{4.12}$$

Applying the usual rules of differentiation, the left side of the last equation may be rewritten as:

$$a'(s)X_2(s) + a(s)X_2'(s) + b'(s)X_2^{\perp}(s) + b(s)(X_2^{\perp})'(s)).$$

Since $X_2(s)$ is a solution to the variational equation, we have $X_2'(s) = B(s)X_2(s)$. Moreover, $(X_2^{\perp}(s))' = (R \cdot X_2(s))' = R \cdot (B(s)X_2(s))$. Hence, equation (4.12) is equivalent to

$$a'(s)X_2(s) + b'(s)X_2^{\perp}(s) = b(s)\big(B(s)X_2^{\perp}(s) - R \cdot B(s)X_2(s)\big). \tag{4.13}$$

By taking the inner product of both sides of last equation with $X_2^{\perp}(s)$, we get

$$b'\|X_2^{\perp}(s)\|^2 = b(s)\big(\langle B(s)X_2^{\perp}(s), X_2^{\perp}(s)\rangle - \langle R \cdot B(s)X_2(s), X_2^{\perp}(s)\rangle\big). \tag{4.14}$$

We remark that, since R is an isometry, we have

$$\langle R \cdot B(s)X_2(s), X_2^{\perp}(s)\rangle = \langle R \cdot B(s)X_2(s), RX_2(s)\rangle = \langle B(s)X_2(s), X_2(s)\rangle,$$

and equation (4.14) may be rewritten:

$$\begin{aligned} b'\|X_2^{\perp}(s)\|^2 = b(s)\big(&\langle B(s)X_2^{\perp}(s), X_2^{\perp}(s)\rangle \\ &+ \langle B(s)X_2(s), X_2(s)\rangle - 2\langle B(s)X_2(s), X_2(s)\rangle\big) \end{aligned} \tag{4.15}$$

In order to simplify notation, we will omit the arguments of the functions in the next calculations.

$$\begin{aligned} \langle BX_2^{\perp}, X_2^{\perp}\rangle &= (y^{\epsilon})^2\big\langle(-F_\rho + F_\sigma h_\sigma^{\epsilon}, -G_\rho + G_\sigma h_\sigma^{\epsilon}, 0), (-1, h_\sigma^{\epsilon}, 0)\big\rangle \\ &= (y^{\epsilon})^2\big(F_\rho - F_\sigma h_\sigma^{\epsilon} - G_\rho h_\sigma^{\epsilon} + G_\sigma (h_\sigma^{\epsilon})^2\big) \\ \langle BX_2, X_2\rangle &= (y^{\epsilon})^2\big\langle(F_\rho h_\sigma^{\epsilon} + F_\sigma, G_\rho h_\sigma^{\epsilon} + G_\sigma, 0), (h_\sigma^{\epsilon}, 1, 0)\big\rangle \\ &= (y^{\epsilon})^2\big(F_\rho (h_\sigma^{\epsilon})^2 + F_\sigma h_\sigma^{\epsilon} + G_\rho h_\sigma^{\epsilon} + G_\sigma\big) \end{aligned}$$

Therefore, we have

$$\begin{aligned} \langle BX_2^{\perp}, X_2^{\perp}\rangle + \langle BX_2, X_2\rangle &= (y^{\epsilon})^2\Big(F_\rho\big(1 + (h_\sigma^{\epsilon})^2\big) + G_\sigma\big(1 + (h_\sigma^{\epsilon})^2\big)\Big) \\ &= \|X_2\|^2 \cdot \operatorname{tr} B. \end{aligned}$$

We also have

$$2\langle BX_2, X_2\rangle = 2\langle X_2', X_2\rangle = \frac{d}{ds}\Big(\langle X_2, X_2\rangle\Big) = \frac{d}{ds}\|X_2\|^2,$$

so that equation (4.15) is equivalent to

$$\begin{aligned} b'(s) &= b(s) \cdot \left(\operatorname{tr} B(s) - \frac{1}{\|X_2(s)\|^2} \cdot \frac{d}{ds}\|X_2(s)\|^2\right) \\ &= b(s) \cdot \left(\operatorname{tr} B(s) - \frac{d}{ds}\left(\ln \|X_2(s)\|^2\right)\right). \end{aligned}$$

Since $X_2^\perp(0) = \Phi^\epsilon(0)X_2^\perp(0) = a(0)X_2(0) + b(0)X_2^\perp(0)$, we must have $b(0) = 1$. Therefore, by integrating, we get:

$$b(s) = \frac{\|X_2(0)\|^2}{\|X_2(s)\|^2} \cdot \exp\left(\int_0^s \operatorname{tr} B(t)\, dt\right). \tag{4.16}$$

We have found a formula for $b(s)$. We move to the task of finding a formula related to $a(s)$. We will actually obtain an initial value problem to which $a(s)$ must be a solution. First, we take the inner product of both sides of equation (4.13) with the vector $X_2(s)$:

$$a'(s)\|X_2(s)\|^2 = b(s)\big(\langle B(s)X_2^\perp(s), X_2(s)\rangle - \langle R \cdot B(s)X_2(s), X_2(s)\rangle\big).$$

Since R is an isometry:

$$\langle R \cdot B(s)X_2(s), X_2(s)\rangle = \langle R^2 \cdot B(s)X_2(s), RX_2(s)\rangle.$$

Observe that

$$R^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that, since the last component of $B(s)X_2(s)$ is zero, we have

$$\langle R \cdot B(s)X_2(s), X_2(s)\rangle = \langle -B(s)X_2(s), X_2^\perp(s)\rangle.$$

Therefore, we have

$$a'(s) = \frac{b(s)}{\|X_2(s)\|^2}\big(\langle B(s)X_2^\perp(s), X_2(s)\rangle + \langle B(s)X_2(s), X_2^\perp(s)\rangle\big). \tag{4.17}$$

To completely determine the initial value problem, we must only observe that $a(0)X_2(0) + b(0)X_2^\perp(0) = \Phi^\epsilon(0)X_2^\perp(0) = X_2^\perp(0)$ implies that $a(0) = 0$. By assumption, $X_0(s)$ is in the span of $X_2(s)$ and $X_2^\perp(s)$. Therefore, by choosing an appropriate section of the normal bundle $\mathcal{X}_0^\epsilon$, there is a smooth function $s \mapsto \alpha(s)$ such that

$$X_0(s) = \alpha(s)X_2(s) + X_2^\perp(s). \tag{4.18}$$

We emphasize that we have normalized the coefficient of $X_2^\perp(s)$ by choosing the section of $\mathcal{X}_0^\epsilon$ appropriately. Since the normal bundle is invariant under Φ^ϵ, there is also a smooth function $s \mapsto \lambda(s)$ such that

$$\Phi^\epsilon(s)X_0(0) = \lambda(s)X_0(s) \tag{4.19}$$

By combining identities (4.18) and (4.19), we get

$$\Phi^\epsilon(s)\big(\alpha(0)X_2(0) + X_2^\perp(0)\big) = \lambda(s)\,\alpha(s)X_2(s) + \lambda(s)X_2^\perp(s).$$

Considering the definitions of Φ^ϵ and X_2, and applying identity (4.11), we have

$$\alpha(0)X_2(s) + a(s)X_2(s) + b(s)X_2^\perp(s) = \lambda(s)\,\alpha(s)X_2(s) + \lambda(s)X_2^\perp(s).$$

Hence, since $X_2(s)$ and $X_2^\perp(s)$ are linearly independent, we obtain two new identities: $\lambda(s)\alpha(s) = \alpha(0) + a(s)$ and $b(s) = \lambda(s)$. We remark that, by (4.18), we have

$$|X_0(s)| = \sqrt{\alpha^2(s)+1}\,|X_2(s)|$$

Finally, considering the definition of $\lambda_3(s)$, given in display (2.3) in section 2.1.5, we find a formula for $\lambda_3(s)$ in the case studied

$$\begin{aligned}\lambda_3(s) = \lambda(s)\frac{|X_0(s)|}{|X_0(0)|} &= \left(\frac{|X_2(0)|^2}{|X_2(s)|^2}\exp\left(\int_0^s \operatorname{tr} B(t)dt\right)\right)\frac{\sqrt{\alpha^2(s)+1}\,|X_2(s)|}{\sqrt{\alpha^2(0)+1}\,|X_2(0)|}\\ &= \frac{|X_2(0)|}{|X_2(s)|}\sqrt{\frac{\alpha^2(s)+1}{\alpha^2(0)+1}}\exp\left(\int_0^s \operatorname{tr} B(t)\,dt\right). \qquad (4.20)\end{aligned}$$

This formula will be used later to verify the inequalities on display (2.4).

4.3 Derivative estimates

In this section, we prove some technical lemmas regarding estimates for the size derivatives of h^ϵ. Those estimates will be useful later when we turn our attention to the proof of Lemmas L1, L2 and L3. The proofs presented here are heavily based on the proofs found in (CHICONE; LIU, 1999/00).

Lemma 7. *Suppose that $\mu > 0$ and $\epsilon_* > 0$ and that $E^{\epsilon,\mu}$ has an invariant manifold given as the graph of the function h^ϵ of the angular variables for $0 \leqslant \epsilon < \epsilon_* \leqslant \mu$. If there is a constant $C_1 > 0$ such that the following estimates hold for all angles σ and τ:*

$$|h^\epsilon(\sigma,\tau) - h^0(\sigma,\tau)| < C_1\epsilon, \qquad |h^\epsilon_\sigma(\sigma,0) - h^0_\sigma(\sigma,0)| < C_1\epsilon,$$

then for μ sufficiently small there is a constant $C_2 > 0$ such that

$$|h^\epsilon_\sigma(\sigma,\tau) - h^0_\sigma(\sigma,\tau)| < C_2\epsilon.$$

Proof: Let $s \mapsto \phi(s,(\rho,\sigma,\tau),\epsilon)$ be the solution to $E^{\epsilon,\mu}$ with initial conditions given by $\phi(0,(\rho,\sigma,\tau),\epsilon) = (\rho,\sigma,\tau)$. We remark that, if γ^ϵ is defined as in (4.7), then

$$\gamma^\epsilon(s,q) = \phi(s,(h^\epsilon(q,0),q,0),\epsilon).$$

For each pair of angles, (p,τ), with $0 \leqslant p < 2\pi$ and $0 \leqslant \tau < 2\pi m\mu^\ell/\omega$, there is a unique angle q^ϵ defined by the equation

$$(h^\epsilon(q^\epsilon,0),q^\epsilon,0) = \phi(-\tau,(h^\epsilon(p,\tau),p,\tau),\epsilon). \qquad (4.21)$$

We apply Lemma 3 to the family of solutions $\phi(s,(\rho,\sigma,\tau),\epsilon)$, choosing $t=-\tau$, $z_2 = (h^\epsilon(p,\tau),p,\tau)$ and $z_1=(h^0(p,\tau),p,\tau)$. Therefore, there is $K>0$ such that

$$|\phi(-\tau,(h^\epsilon(p,\tau),p,\tau),\epsilon)-\phi(-\tau,(h^0(p,\tau),p,\tau),0)| \leqslant \\ Ke^{K\tau}\big(|(h^\epsilon(p,\tau),p,\tau)-(h^0(p,\tau),p,\tau)|+\epsilon\tau\big). \tag{4.22}$$

Since $\tau\in[0,\,2\pi m\mu^\ell/\omega]$, there is $K_1>0$, independent of the value of τ in this interval, such that

$$Ke^{K\tau}\big(|(h^\epsilon(p,\tau),p,\tau)-(h^0(p,\tau),p,\tau)|+\epsilon\tau\big)\leqslant \\ K_1\big(|(h^\epsilon(p,\tau),p,\tau)-(h^0(p,\tau),p,\tau)|+\epsilon\big) \tag{4.23}$$

Combining (4.23) and (4.22), we get

$$|\phi(-\tau,(h^\epsilon(p,\tau),p,\tau),\epsilon)-\phi(-\tau,(h^0(p,\tau),p,\tau),0)| \leqslant \\ K_1\big(|(h^\epsilon(p,\tau),p,\tau)-(h^0(p,\tau),p,\tau)|+\epsilon\big) \tag{4.24}$$

We will use the 1-norm for convenience, as all norms are equivalent in the space considered. Hence, we have

$$|(h^\epsilon(p,\tau),p,\tau)-(h^0(p,\tau),p,\tau)| = |h^\epsilon(p,\tau)-h^0(p,\tau)|,$$

and

$$|\phi(-\tau,(h^\epsilon(p,\tau),p,\tau),\epsilon)-\phi(-\tau,\,(h^0(p,\tau),p,\tau),0)| = \\ |(h^\epsilon(q^\epsilon,0),q^\epsilon,0)-(h^0(q^0,0),q^0,0)| = \big(|h^\epsilon(q^\epsilon,0)-h_0(q^0,0)|+|q^\epsilon-q^0|\big)$$

Therefore, (4.24) is equivalent to

$$|h^\epsilon(q^\epsilon,0)-h_0(q^0,0)|+|q^\epsilon-q^0|\leqslant K_1\big(|h^\epsilon(p,\tau)-h^0(p,\tau)|+\epsilon\big).$$

By hypothesis, $|h^\epsilon(p,\tau)-h^0(p,\tau)|<C_1\epsilon$. Hence, we have

$$|h^\epsilon(q^\epsilon,0)-h_0(q^0,0)|+|q^\epsilon-q^0|\leqslant K_1(C_1+1)\epsilon.$$

In particular, there is $K_2>0$ such that

$$|q^\epsilon-q^0|<K_2\,\epsilon. \tag{4.25}$$

We remark that

$$\begin{aligned}\gamma^\epsilon(\tau,q^\epsilon)=\phi(\tau,(h^\epsilon(q^\epsilon,0),q^\epsilon,0),\epsilon) &= \phi\big(\tau,\phi(-\tau,(h^\epsilon(p,\tau),p,\tau),\epsilon),\epsilon\big)\\ &=\phi\big(0,(h^\epsilon(p,\tau),p,\tau),\epsilon\big)\\ &=(h^\epsilon(p,\tau),p,\tau)\end{aligned}$$

We modify the notation introduced in Proposition 9 as to include the initial angle q and the parameter ϵ:

$$X_2^\epsilon(s,q) := y^\epsilon(s,q)\mathcal{X}_2^\epsilon(\gamma^\epsilon(s,q)) = y^\epsilon(s,q)\begin{pmatrix} h_\sigma^\epsilon(\sigma^\epsilon(q,s),\tau(s)) \\ 1 \\ 0 \end{pmatrix}.$$

Considering the solution of the first variational equation for $E^{\epsilon,\mu}$ along the solution $\gamma^\epsilon(s,q^\epsilon)$, we have, at $s=0$ and at $s=\tau$,

$$X_2^\epsilon(0,q^\epsilon) = \begin{pmatrix} h_\sigma^\epsilon(q^\epsilon,0) \\ 1 \\ 0 \end{pmatrix}, \quad X_2^\epsilon(\tau,q^\epsilon) = y^\epsilon(\tau,q^\epsilon)\begin{pmatrix} h_\sigma^\epsilon(p,\tau) \\ 1 \\ 0 \end{pmatrix}$$

Since these are solutions to the family of variational equations (4.8), parameterised by ϵ, we can once more apply Lemma 3 to the family of solutions and get $K_3>0$ such that

$$|X_2^\epsilon(\tau,q^\epsilon) - X_2^0(\tau,q^0)| \leqslant K_3 e^{K_3\tau}\big(|X_2^\epsilon(0,q^\epsilon) - X_2^0(0,q^0)| + \epsilon\,\tau\big).$$

Since $\tau \in [0, 2\pi m\mu^\ell/\omega]$, there is $K_4>0$ such that

$$|X_2^\epsilon(\tau,q^\epsilon) - X_2^0(\tau,q^0)| \leqslant K_4\big(|X_2^\epsilon(0,q^\epsilon) - X_2^0(0,q^0) + \epsilon\big),$$

which can be expanded to:

$$|y^\epsilon(\tau,q^\epsilon)\,h_\sigma^\epsilon(p,t) - y^0(\tau,q^0)\,h_\sigma^0(p,\tau)| + |y^\epsilon(\tau,q^\epsilon) - y^0(\tau,q^0)| \leqslant \\ K_4(|h_\sigma^\epsilon(q^\epsilon,0) - h_\sigma^0(q^0,0)| + \epsilon). \tag{4.26}$$

By hypothesis, we know that $|h^\epsilon(q^\epsilon,0) - h^0(q^\epsilon,0)| < C_1\epsilon$, so that, applying the triangle inequality, we get

$$|h_\sigma^\epsilon(q^\epsilon,0) - h_\sigma^0(q^0,0)| \leqslant C_1\epsilon + |h_\sigma^0(q^\epsilon,0) - h_\sigma^0(q^0,0)|.$$

The function h_σ^0 is a C^{k-1} function defined on a compact set, with $k \geqslant 2$. Therefore, it is Lipschitz continuous. Thus, there is $L>0$ such that

$$|h_\sigma^0(q^\epsilon,0) - h_\sigma^0(q^0,0)| \leqslant L|q^\epsilon - q^0| \leqslant L\cdot K_2\epsilon,$$

the last inequality being a consequence of (4.25). Hence, we get

$$|h_\sigma^\epsilon(q^\epsilon,0) - h_\sigma^0(q^0,0)| \leqslant C_1\epsilon + L\cdot K_2\epsilon.$$

Define $K_5 = C_1 + L\cdot K_2 + 1$. We have, considering (4.26),

$$|y^\epsilon(\tau,q^\epsilon)\,h_\sigma^\epsilon(p,t) - y^0(\tau,q^0)\,h_\sigma^0(p,\tau)| + |y^\epsilon(\tau,q^\epsilon) - y^0(\tau,q^0)| \leqslant K_5\epsilon. \tag{4.27}$$

We remark that both summands on the left-hand side of the last inequality are consequently bounded above by $K_5\epsilon$. The first of those summands may be rewritten as

$$\left|\left(y^\epsilon(\tau,q^\epsilon)h_\sigma^\epsilon(p,\tau) - y^\epsilon(\tau,q^\epsilon)h_\sigma^0(p,\tau)\right) - \left(y^0(\tau,q^0)h_\sigma^0(p,\tau) - y^\epsilon(\tau,q^\epsilon)h_\sigma^0(p,\tau)\right)\right|.$$

The last expression must be bounded above by $K_5\epsilon$. By applying the reverse triangle inequality, we obtain

$$|y^\epsilon(\tau,q^\epsilon)|\,|h_\sigma^\epsilon(p,\tau) - h_\sigma^0(p,\tau)| - |h_\sigma^0(p,\tau)|\,|y^\epsilon(\tau,q^\epsilon) - y^0(\tau,q^0)| \leqslant K_5\epsilon$$

Considering that $|y^\epsilon(\tau,q^\epsilon) - y^0(\tau,q^0)|$, the second summand appearing in (4.27), is bounded above by $K_5\epsilon$, we get:

$$|y^\epsilon(\tau,q^\epsilon)|\,|h_\sigma^\epsilon(p,\tau) - h_\sigma^0(p,\tau)| \leqslant K_5\left(|h_\sigma^0(p,\tau)| + 1\right)\epsilon$$

Since h_σ^0 is a continuous function defined on a compact set, it is bounded. Therefore, there is $K_6 > 0$ such that

$$|y^\epsilon(\tau,q^\epsilon)|\,|h_\sigma^\epsilon(p,\tau) - h_\sigma^0(p,\tau)| \leqslant K_6\epsilon \tag{4.28}$$

Moreover, we have, by once more applying the reverse triangle inequality and considering (4.27),

$$\left|y^\epsilon(\tau,q^\epsilon)\right| \geqslant \left|y^0(\tau,q^0)\right| - \left|y^0(\tau,q^0) - y^\epsilon(\tau,q^\epsilon)\right| \geqslant |y^0(\tau,q^0)| - K_5\epsilon$$

Since $\epsilon \in [0,\mu]$, we have

$$|y^\epsilon(\tau,q^\epsilon)| \geqslant |y^0(\tau,q^0)| - K_5\mu$$

Since y^0 is a strictly positive continuous function and $(\tau,q^\epsilon) \in [0, 2\pi m\mu^\ell/\omega] \times [0,2\pi]$, we have

$$|y^0(\tau,q^0)| \geqslant \min\left\{y^0(\tau',q') : (\tau',q') \in [0,2\pi m\mu^\ell/\omega] \times [0,2\pi]\right\} > 0,$$

so that, by choosing $\mu > 0$ sufficiently small, there must be $K_7 > 0$ such that

$$|y^\epsilon(\tau,q^\epsilon)| \geqslant |y^0(\tau,q^0)| - K_5\mu > K_7. \tag{4.29}$$

Combining (4.28) and (4.29), we get

$$K_6\epsilon \geqslant |y^\epsilon(\tau,q^\epsilon)|\,|h_\sigma^\epsilon(p,\tau) - h_\sigma^0(p,\tau)| > K_7|h_\sigma^\epsilon(p,\tau) - h_\sigma^0(p,\tau)|,$$

so that, finally,

$$|h_\sigma^\epsilon(p,\tau) - h_\sigma^0(p,\tau)| < \frac{K_6}{K_7}\epsilon, \tag{4.30}$$

and the result follows simply by defining $C_2 = K_6/K_7$. □

Lemma 8. *Let $m : \mathbb{R} \times [0, 2\pi] \to M_2(\mathbb{R})$ be a continuous matrix function and $\lambda : [0, 2\pi] \to \mathbb{R}$ a continuous function. Let also T_0 be a positive number. Suppose that there is $M_0 > 0$ such that*

$$|m_{21}(s, q)\lambda(q) + m_{22}(s, q)| \geqslant M_0,$$

for all $s \in \mathbb{R}$ and all $q \in [0, 2\pi]$. Then there is $r > 0$ such that, for $T \in [T_0, T_0 + 1)$,

$$|d_{21}\xi + d_{22}| > \frac{2}{3}M_0, \quad \textit{and} \quad |\det D - \det m(T, q)| < \frac{1}{8}M_0^2,$$

whenever ξ, a real number, $D = (d_{ij})$, a 2×2 real matrix and $q \in [0, 2\pi]$ are such that

$$|\xi - \lambda(q)| < r, \quad \textit{and} \quad |D - m(T, q)| < r.$$

Proof: Let $q \in [0, 2\pi]$ and $T \in [T_0, T_0 + 1]$ be given. Note that the function $(u, v, w) \mapsto |uv + w|$ is continuous. Therefore, there is $r_1 > 0$ such that, if

$$|\xi - \lambda(q)| < r_1 \quad \text{and} \quad |D - m(T, q)|_\infty < r_1,$$

then

$$\left|d_{21}\xi + d_{22} - \big(m_{21}(T, q)\lambda(q) + m_{22}(T, q)\big)\right| < \frac{M_0}{3}.$$

By the reverse triangle inequality, we get:

$$\big||d_{21}\xi + d_{22}| - |m_{21}(T, q)\lambda(q) + m_{22}(T, q)|\big| < \frac{M_0}{3},$$

which implies that

$$|m_{21}(T, q)\lambda(q) + m_{22}(T, q)| - \frac{M_0}{3} < |d_{21}\xi + d_{22}|.$$

Since $|m_{21}(T, q)\lambda(q) + m_{22}(T, q)| \geqslant M_0$ by hypothesis, last inequality also implies that

$$|d_{21}\xi + d_{22}| > \frac{2}{3}M_0.$$

The determinant is also a continuous function. Thus, there is $r_2 > 0$ such that, if

$$|D - m(T, q)|_\infty < r_2,$$

then

$$|\det D - \det m(T, q)| < \frac{1}{16}M_0^2.$$

For each pair (T, q) given, we get different values of r_1 and r_2. We define $r(T, q) = \min\{r_1, r_2\}$.

Since $[T_0, T_0+1] \times [0, 2\pi]$ is a compact set and m is continuous, the product of images

$$K := m([T_0, T_0+1] \times [0, 2\pi]) \times \lambda([0, 2\pi])$$

is also a compact set. Let $B((m, \alpha), \rho) \subset M_2(\mathbb{R}) \times \mathbb{R}$ denote the open ball centered at (m, α) of radius ρ in the norm

$$\begin{aligned} |\cdot| : M_2(\mathbb{R}) \times \mathbb{R} &\to \mathbb{R}^+ \\ (m, \alpha) &\longmapsto \max\{|m|_\infty, |\alpha|\} \end{aligned}$$

Note that

$$\mathcal{C} := \left\{ B\left((m(T,q), \lambda(q)), \frac{r(T,q)}{2} \right) : (T,q) \in [T_0, T_0+1] \times [0, 2\pi] \right\},$$

is an open cover of K. Therefore, $\mathcal{C}$ admits a finite subcover,

$$\mathcal{S} = \left\{ B\left((m(T_i,q_i), \lambda(q_i)), \frac{r(T_i,q_i)}{2} \right) : i = 1, 2, ..., n \right\}.$$

We define

$$r = \min_{i=1,\dots,n} \frac{r(T_i, q_i)}{2}.$$

Let $\tilde{q} \in [0, 2\pi]$ and $\tilde{T} \in [T_0, T_0+1)$. Suppose that ξ and D are such that

$$|\xi - \lambda(\tilde{q})| < r, \quad |D - m(\tilde{T}, \tilde{q})|_\infty < r. \tag{4.31}$$

Since $\mathcal{S}$ covers K, there is $k \in \{1, ..., n\}$ such that

$$|\lambda(\tilde{q}) - \lambda(q_k)| < \frac{r(T_k, q_k)}{2}, \quad |m(\tilde{T}, \tilde{q}) - m(T_k, q_k)|_\infty < \frac{r(T_k, q_k)}{2}. \tag{4.32}$$

In particular, by definition of $r(T, q)$, we have

$$|\det m(\tilde{T}, \tilde{q}) - \det m(T_k, q_k)| < \frac{1}{16} M_0^2 \tag{4.33}$$

Combining (4.31) and (4.32) and applying the triangle inequality, we get

$$|\xi - \lambda(q_k)| < r + \frac{r(T_k, q_k)}{2} \leqslant r(T_k, q_k),$$

and

$$|D - M(T_k, q_k)|_\infty < r + \frac{r(T_k, q_k)}{2} \leqslant r(T_k, q_k).$$

Therefore, by definition of $r(T_k, q_k)$, we have

$$|d_{21}\xi + d_{22}| > \frac{2}{3} M_0,$$

and

$$|\det D - \det m(T_k, q_k)| < \frac{1}{16} M_0^2. \tag{4.34}$$

Finally, combining (4.33) and (4.34) and applying once more the triangle inequality, we get

$$|\det D - \det m(\tilde{T}, \tilde{q})| < \frac{1}{8} M_0^2,$$

thus proving the lemma. □

Lemma 9. *Suppose there is $C > 0$ such that*

$$d(x, M(0, \mu)) \leqslant C\epsilon, \quad \forall x \in M(\epsilon, \mu).$$

Then there is $\tilde{C} > 0$ such that

$$|h^\epsilon - h^0|_{C^0} < \tilde{C}\epsilon$$

Proof: Let $x \in M(\epsilon, \mu)$. By the definition of $M(\epsilon, \mu)$, it follows that x has the form $x = (h^\epsilon(\sigma, \tau), \sigma, \tau)$, where $(\sigma, \tau) \in K := [0, 2\pi] \times [0, 2\pi m \mu^\ell / \omega]$. By definition, we have

$$d((h^\epsilon(\sigma, \tau), \sigma, \tau), M(0, \mu)) = \inf_{(\xi, \eta) \in K} d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \eta), \xi, \eta)\big).$$

Observe that

$$d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \eta), \xi, \eta)\big) = \big|\big(h^\epsilon(\sigma, \tau) - h^0(\xi, \eta), \sigma - \xi, \tau - \eta\big)\big|.$$

Since h^0 is constant with respect to its second argument, we have:

$$h^0(\xi, \eta) = h^0(\xi, \tau).$$

Therefore, we have

$$\begin{aligned} d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \eta), \xi, \eta)\big) &= \big|\big(h^\epsilon(\sigma, \tau) - h^0(\xi, \eta), \sigma - \xi, \tau - \eta\big)\big| \\ &\geqslant \big|\big(h^\epsilon(\sigma, \tau) - h^0(\xi, \tau), \sigma - \xi, 0\big)\big| \\ &= d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \tau), \xi, \tau)\big) \end{aligned}$$

for all $(\xi, \eta) \in K$. Thus, we get

$$\inf_{\xi \in [0, 2\pi]} d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \tau), \xi, \tau)\big) \leqslant \inf_{(\xi, \eta) \in K} d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \eta), \xi, \eta)\big)$$

But the reverse inequality holds by definition of infimum. Hence, the following equality holds:

$$\inf_{\xi \in [0, 2\pi]} d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \tau), \xi, \tau)\big) = \inf_{(\xi, \eta) \in K} d\big((h^\epsilon(\sigma, \tau), \sigma, \tau), (h^0(\xi, \eta), \xi, \eta)\big)$$

Considering that the image of $\xi \mapsto (h^0(\xi,\tau),\xi,\tau)$ is compact, there is $\tilde{\sigma} \in [0,2\pi]$ such that

$$\inf_{\xi\in[0,2\pi]} d\big((h^\epsilon(\sigma,\tau),\sigma,\tau),(h^0(\xi,\tau),\xi,\tau)\big) = d\big((h^\epsilon(\sigma,\tau),\sigma,\tau),(h^0(\tilde{\sigma},\tau),\tilde{\sigma},\tau)\big)$$

By hypothesis, we have

$$d\big((h^\epsilon(\sigma,\tau),\sigma,\tau),(h^0(\tilde{\sigma},\tau),\tilde{\sigma},\tau)\big) < C\epsilon.$$

In particular, $|\sigma-\tilde{\sigma}| < C\epsilon$ and $|h^\epsilon(\sigma,\tau)-h^0(\tilde{\sigma},\tau)| < C\epsilon$. Since h^0 is a C^1 function defined on a compact set, it is Lipschitz continuous. Let us denote its Lipschitz constant by L. We have then

$$|h^0(\sigma,\tau)-h^0(\tilde{\sigma},\tau)| \leqslant L|\sigma-\tilde{\sigma}| < LC\epsilon.$$

By the triangle inequality, we get

$$|h^\epsilon(\sigma,\tau)-h^0(\sigma,\tau)| \leqslant |h^\epsilon(\sigma,\tau)-h^0(\tilde{\sigma},\tau)| + |h^0(\tilde{\sigma},\tau)-h^0(\sigma,\tau)| < C(L+1)\epsilon.$$

the lemma is proved by taking $\tilde{C} = C(L+1)$. □

Lemma 10. *Suppose Hypothesis B holds for the unperturbed system (4.2). There is $\mu_0 > 0$ such that, for each $\mu \in (0,\mu_0]$, there is $C > 0$ such that, if ϵ_* is as in Proposition 8, then*

$$|h^\epsilon - h^0|_{C^0} < C\epsilon, \quad |h^\epsilon_\sigma - h^0_\sigma|_{C_0} \leqslant C\epsilon, \quad |h^\epsilon_\tau|_{C^0} \leqslant C\epsilon,$$

for $0 \leqslant \epsilon < \epsilon_*$.

Proof: Choose a bounded neighbourhood N of the graph of h^0 and a constant $C_0 > 0$ as in Proposition 2. Let $r_0 > 0$ be so small that, if $|h^\epsilon - h^0|_{C_0} < r_0$, then the graph of h^ϵ is in N. Note that, if R and S are the perturbation terms present in (4.3), then

$$K := \sup\left\{|R(\rho,\sigma,\tau/\mu^\ell,\mu)| + |S(\rho,\sigma,\tau/\mu^\ell,\mu)| : (\rho,\sigma,\tau)\in N,\ 0<\mu^\ell<\frac{\Omega}{2\pi m}\right\}$$

is finite. If we choose $\mu > 0$ so small that $0 < \mu^\ell < \Omega/2\pi m$ and $C_0K\mu < r_0$, then, by Proposition 2,

$$d(x, M(0,\mu)) \leqslant C_0K\epsilon, \quad \forall\, x\in M(\epsilon,\mu) \tag{4.35}$$

as long as $|h^\epsilon - h^0|_{C^0} < r_0$. Therefore, (4.35) holds for $\epsilon \in [0,\epsilon_*)$. In fact, let B be the maximal interval with left endpoint at $\epsilon = 0$ such that $|h^\epsilon - h^0|_{C_0} < r_0$ for all ϵ in B. The set $B \cap [0,\epsilon_*)$ is clearly connected, open and non-empty. Suppose $\alpha := \sup\big(B\cap[0,\epsilon_*)\big) < \epsilon_*$. Because $B\cap[0,\epsilon_*)$ is open, we would have $|h^\alpha - h^0|_{C^0} \geqslant r_0 > C_0K\mu > C_0K\alpha$, which would mean that h^ϵ is not continuous with respect to its parameter at $\epsilon = \alpha$. Indeed, $|h^\alpha - h^0|_{C^0} < C_0K\mu$ for all $\epsilon \in B\cap[0,\epsilon_*)$. In fact, if ϵ is sufficiently small, since $M(\epsilon,\mu)$ is C^r close to $M(0,\mu)$, $|h^\epsilon - h^0|_{C_0}$ must be less than r_0.

By Lemma 9, the following inequality holds for all $\epsilon \in [0, \epsilon_*)$:

$$|h^\epsilon - h^0|_{C^0} < \tilde{C}\epsilon,$$

where $\tilde{C} > 0$ is a constant.

We will now show that, if μ is sufficiently small, then there is a constant $C > 0$ such that $|h^\epsilon_\sigma - h^0_\sigma|_{C^0} \leqslant C\epsilon$. By Lemma 7, it suffices to find C such that, for all $q \in [0, 2\pi]$,

$$|h^\epsilon_\sigma(q,0) - h^0_\sigma(q,0)| \leqslant C\epsilon.$$

In order to prove that this inequality holds, we first note, by recalling Proposition 9, that the function given by $s \mapsto (h^\epsilon_\sigma(\sigma^\epsilon(s,q), \tau(s))y^\epsilon(s,q), y^\epsilon(s,q))$ is a solution of the following system:

$$\begin{aligned} u' &= (f_\rho + \epsilon R_\rho)u + (f_\sigma + \epsilon R_\sigma)v \\ v' &= (g_\rho + \epsilon S_\rho)u + (g_\sigma + \epsilon S_\sigma)v, \end{aligned} \tag{4.36}$$

where the argument of the functions are given by the solutions of (4.3), $\gamma^\epsilon(s,q)$. In particular, for $\epsilon \in [0, \epsilon_*)$, the argument of the derivatives of f and g is given by the expression $(h^\epsilon(\sigma^\epsilon(s,q), \tau(s)), \sigma^\epsilon(s,q))$ and the argument of the derivatives of S and R is given by the expression $(h^\epsilon(\sigma^\epsilon(s,q), \tau(s)), \sigma^\epsilon(s,q), \tau(s), \mu, \epsilon)$. Let $\Psi^\epsilon(s,q) := (\psi^\epsilon_{ij}(s,q))_{2\times 2}$ be the principal fundamental matrix solution of (4.36) at $s = 0$. We remark that, for $\mu > 0$ fixed, ϵ and q are parameters of the C^{r-1} family of differential systems given by (4.36). The smoothness of this family is ensured by the fact that $\gamma^\epsilon(s,q)$, as defined in (4.7), is itself a solution of a C^r system. Thus, we have

$$\begin{pmatrix} h^\epsilon_\sigma(\sigma^\epsilon(s,q), \tau(s))y^\epsilon(s,q) \\ y^\epsilon(s,q) \end{pmatrix} = \Psi^\epsilon(s,q) \begin{pmatrix} h^\epsilon_\sigma(q,0) \\ 1 \end{pmatrix},$$

so that

$$h^\epsilon_\sigma(\sigma^\epsilon(s,q), \tau(s)) = \frac{\psi^\epsilon_{11}(s,q)h^\epsilon_\sigma(q,0) + \psi^\epsilon_{12}(s,q)}{\psi^\epsilon_{21}(s,q)h^\epsilon_\sigma(q,0) + \psi^\epsilon_{22}(s,q)}. \tag{4.37}$$

If $s \mapsto (\rho(s), \sigma(s))$ is a solution of the unperturbed system (4.2), then

$$s \mapsto \big(f(h^0(\sigma(s),0), \sigma(s)), g(h^0(\sigma(s),0), \sigma(s))\big)$$

is a solution of the corresponding variational equation (4.36) with $\epsilon = 0$. Since h^0 is constant with respect to τ, we have: $\rho(s) = h^0(\sigma(s), 0)$ on the invariant manifold. By differentiating this equation with respect to s, we get

$$f(h^0(\sigma,0), \sigma(s)) = h^0_\sigma(\sigma(s),0)\, g(h^0(\sigma,0), \sigma(s)).$$

Therefore, the function $s \mapsto \big(h^0_\sigma(\sigma(s),0)\, g(h^0(\sigma,0),\sigma(s)), g(h^0(\sigma(s),0),\sigma(s))\big)$ is a solution of the variational equation (4.36). Using the fundamental matrix solution Ψ^ϵ, we find

$$\psi^0_{21}(s,q)h^0_\sigma(q,0) + \psi^0_{22}(s,q) = \frac{g\big(h^0(\sigma(s,q),0),\sigma(s,q)\big)}{g\big(h^0(q,0),\sigma(s,q)\big)},$$

where $\sigma(s,q)$ denotes the solution $\sigma(s)$ with initial point given by q. Since σ' does not vanish, there is $M_0 > 0$ such that $|\psi^0_{21}(s,q)h^0_\sigma(q,0) + \psi^0_{22}(s,q)| \geqslant M_0$ for all $s \in \mathbb{R}$ and $q \in [0,2\pi]$. We remark that M_0 is independent of the choice of μ.

By hypothesis, the unperturbed normally hyperbolic invariant manifold that exists for the system $E^{0,\mu}$ is the suspension of an attracting hyperbolic limit cycle. The characteristic multiplier of this limit cycle is, thus, negative. Therefore, by Liouville's formula and by continuity of Ψ^0 with respect to the parameter q, there is $T_0 > 0$ such that $\det \Psi^0(t,q) \leqslant (1/4)M_0^2$ for all $t \geqslant T_0$ and all $q \in [0,2\pi]$. We remark that T_0 does not depend on the choice of μ. If we restrict μ so that $0 < \mu^\ell < \Omega/2\pi m$, there is a positive integer n such that

$$T_0 \leqslant \frac{2\pi m n \mu^\ell}{\Omega} < T_0 + 1.$$

For definiteness, let us define $n = n(\mu)$ as the smallest such integer for a given μ. We define also

$$T := T(\mu) = \frac{2\pi m n \mu^\ell}{\Omega}. \tag{4.38}$$

We remark that, while the value of T may vary as μ is made sufficiently small to satisfy new requirements, the final value of T is an integer multiple of the period of the perturbation terms in $E^{\epsilon,\mu}$. It is also bounded above and below and approaches T_0 as μ decreases toward zero.

Applying Lemma 8, choose $r > 0$ sufficiently small so that, for $T_0 \leqslant T < T_0 + 1$, the following inequalities hold:

$$|d_{21}\xi + d_{22}| > \frac{2}{3}M_0, \quad |\det D - \det \Psi^0(T,q)| < \frac{1}{8}M_0^2, \tag{4.39}$$

whenever ξ, a real number, D, a 2×2 real matrix and $q \in [0,2\pi]$ are such that

$$|\xi - h^0_\sigma(q)| < r, \quad |D - \Psi^0(T,q)| < r.$$

Note that r is independent of the choice of μ.

Let $p \in [0,2\pi]$. If $\mu > 0$ is so that $0 < \mu^\ell < \Omega/2\pi m$ and $\tilde{C}\mu < r_0$, then T is well defined. If we then set $s = T$ in (4.37), we get

$$h^\epsilon_\sigma(p,0) = \frac{\psi^\epsilon_{11}(T,q^\epsilon)h^\epsilon_\sigma(q^\epsilon,0) + \psi^\epsilon_{12}(T,q^\epsilon)}{\psi^\epsilon_{21}(T,q^\epsilon)h^\epsilon_\sigma(q^\epsilon,0) + \psi^\epsilon_{22}(T,q^\epsilon)}, \tag{4.40}$$

where q^ϵ is defined by

$$p = \sigma^\epsilon(T, q^\epsilon). \tag{4.41}$$

Define $\mu_1 = \min\{\sqrt[\ell]{\Omega/2\pi m}, \frac{r_0}{\tilde{C}}\}$. We apply Lemma 4 to solutions $\gamma^\epsilon(t,p)$ and $\gamma^0(t,p)$, taking $A = T_0 + 1$, $\Delta = \{(\epsilon,\mu) \in \mathbb{R}^2 : \mu \in [0,\mu_1] \text{ and } \epsilon \in [0,\mu]\}$ and $\mathcal{C}$ a compact set such that

$$\bigcup_{\mu\in(0,\mu_1]} \left(\bigcup_{\epsilon\in[0,\epsilon*)} M(\epsilon,\mu) \right) \subset \mathcal{C}.$$

to find $C_1 > 0$ such that

$$|h^\epsilon(q^\epsilon, -T) - h^0(q^0, -T)| + |q^\epsilon - q^0| \leqslant C_1(|h^\epsilon(p,0) - h^0(p,0)| + \epsilon).$$

Note that, in principle, we are not allowed to simply take $\mu = 0$ in system (4.3), and thus it should not be possible to apply Lemma 4 as we did. This problem is tackled by observing that, in this case, the function $\tilde{F}$ as appearing in the statement of Lemma 4 should be taken to be the continuous extension of the perturbation terms to the set Δ. The only point in Δ at which the perturbation terms were not defined is $(0,0)$, and we extend to this point by defining $\tilde{F}(x,0,0) := 0$. Moreover, since T is an integer multiple of the period of $2\pi m\mu^\ell/\Omega$, we have $h^\epsilon(q^\epsilon, T) = h^\epsilon(q^\epsilon, 0)$. Therefore, the following inequality holds:

$$|h^\epsilon(q^\epsilon, 0) - h^0(q^0, 0)| + |q^\epsilon - q^0| \leqslant C_1(|h^\epsilon(p,0) - h^0(p,0)| + \epsilon).$$

Considering the estimate (4.35), we conclude that there is $C_2 > 0$ such that

$$|q^\epsilon - q^0| \leqslant C_2\epsilon \tag{4.42}$$

Consider the principal fundamental matrix solution of (4.36), $\Psi^\epsilon(s,q)$. Observe that ϵ and q can be seen as parameters of a C^{r-1} family of differential equations, given by (4.36). Applying Lemma 4 to the solutions $\Psi^\epsilon(s,q^\epsilon)$ and $\Psi^0(s,q^0)$, we find $C_3 > 0$ such that

$$|\Psi^\epsilon(T,q^\epsilon) - \Psi^0(T,q^0)| \leqslant C_3(|q^\epsilon - q^0| + \epsilon). \tag{4.43}$$

Combining (4.42) and (4.43), we conclude that there is $C_5 > 0$

$$|\Psi^\epsilon(T,q^\epsilon) - \Psi^0(T,q^0)| \leqslant C_5\epsilon. \tag{4.44}$$

We will prove now that there is a constant $C_d > 0$ such that, for $0 \leqslant \epsilon < \epsilon_*$, if $|h^\epsilon_\sigma - h^0_\sigma|_{C^0} \leqslant r$, then $|h^\epsilon_\sigma - h^0_\sigma|_{C^0} \leqslant C_d\epsilon$. Henceforth, we will omit the second entry to the functions h^ϵ and h^0 when it is equal to zero. As before, fix $p \in [0,2\pi]$ and define q^ϵ according to (4.41). By (4.40), we have

$$|h^\epsilon_\sigma(p) - h^0_\sigma(p)| = \left| \frac{\psi^\epsilon_{1,1}(T,q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{1,2}(T,q^\epsilon)}{\psi^\epsilon_{2,1}(T,q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{2,2}(T,q^\epsilon)} - \frac{\psi^0_{1,1}(T,q^0)h^0_\sigma(q^0) + \psi^0_{1,2}(T,q^0)}{\psi^0_{2,1}(T,q^0)h^0_\sigma(q^0) + \psi^0_{2,2}(T,q^0)} \right|$$

Define

$$I := \left|\frac{\psi^\epsilon_{1,1}(T,q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{1,2}(T,q^\epsilon)}{\psi^\epsilon_{2,1}(T,q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{2,2}(T,q^\epsilon)} - \frac{\psi^\epsilon_{1,1}(T,q^\epsilon)h^0_\sigma(q^\epsilon) + \psi^\epsilon_{1,2}(T,q^\epsilon)}{\psi^\epsilon_{2,1}(T,q^\epsilon)h^0_\sigma(q^\epsilon) + \psi^\epsilon_{2,2}(T,q^\epsilon)}\right|$$
$$II := \left|\frac{\psi^\epsilon_{1,1}(T,q^\epsilon)h^0_\sigma(q^\epsilon) + \psi^\epsilon_{1,2}(T,q^\epsilon)}{\psi^\epsilon_{2,1}(T,q^\epsilon)h^0_\sigma(q^\epsilon) + \psi^\epsilon_{2,2}(T,q^\epsilon)} - \frac{\psi^0_{1,1}(T,q^0)h^0_\sigma(q^0) + \psi^0_{1,2}(T,q^0)}{\psi^0_{2,1}(T,q^0)h^0_\sigma(q^0) + \psi^0_{2,2}(T,q^0)}\right|$$

By the triangle inequality, we have

$$|h^\epsilon_\sigma(p) - h^0_\sigma(p)| \leqslant I + II \tag{4.45}$$

We will estimate the quantities I and II separately. Define $u : \mathbb{R} \times M_2(\mathbb{R}) \to \mathbb{R}$ by

$$u(\xi, D) = \frac{d_{11}\xi + d_{12}}{d_{21}\xi + d_{22}}.$$

Observe that

$$I = |u(h^\epsilon_\sigma(q^\epsilon), \Psi^\epsilon(T,q^\epsilon)) - u(h^0_\sigma(q^\epsilon), \Psi^\epsilon(T,q^\epsilon))|$$
$$II = |u(h^0_\sigma(q^\epsilon), \Psi^\epsilon(T,q^\epsilon)) - u(h^0_\sigma(q^0), \Psi^0(T,q^0))|$$

Moreover, we have

$$u_\xi(\xi, D) = \frac{\det D}{(d_{21}\xi + d_{22})^2}.$$

We apply the mean value theorem to the function $\xi \mapsto u(\xi, D)$ and conclude that there is ξ between $h^\epsilon_\sigma(q^\epsilon)$ and $h^0_\sigma(q^\epsilon)$.

$$I \leqslant \left|\frac{\det \Psi^\epsilon(T,q^\epsilon)}{(\psi^\epsilon_{21}(T,q^\epsilon)\xi + \psi^\epsilon_{22}(T,q^\epsilon))^2}\right| |h^\epsilon_\sigma(q^\epsilon) - h^0_\sigma(q^\epsilon)|$$

Since we assumed that $|h^\epsilon_\sigma - h^0_\sigma|_{C^0} \leqslant r$, we have $|\xi - h^0_\sigma(q^\epsilon)| \leqslant r$. There is $\mu_2 > 0$ such that, if $\mu \in (0, \mu_2]$, then

$$|\Psi^\epsilon(T,q^\epsilon) - \Psi^0(T,q^\epsilon)| < r$$

Therefore, by (4.39), we have

$$\frac{1}{|\psi^\epsilon_{21}(T,q^\epsilon)\xi + \psi^\epsilon_{22}(T,q^\epsilon)|} < \frac{3}{2M_0}, \tag{4.46}$$

and

$$|\det \Psi^\epsilon(T,q^\epsilon) - \det \Psi^0(T,q^\epsilon)| < \frac{1}{8}M_0^2. \tag{4.47}$$

Considering that $T \geqslant T_0$, we have $|\det \Psi^0(T,q^\epsilon)| < (1/4)M_0^2$. Hence, applying the triangle inequality to (4.47), we get:

$$|\det \Psi^\epsilon(T,q^\epsilon)| < \frac{3}{8}M_0^2 \tag{4.48}$$

Estimates (4.46) and (4.48) combined ensure that

$$I \leqslant \frac{27}{32}|h_\sigma^\epsilon(q^\epsilon) - h_\epsilon^0(q^\epsilon)| \leqslant \frac{27}{32}|h_\sigma^\epsilon - h_\sigma^0|_{C^0}. \tag{4.49}$$

In order to estimate II, let us note that the function u is Lipschitz continuous on the set

$$\{(h_\sigma^0(q), \Psi^\epsilon(T,q)) \in \mathbb{R} \times M_2(\mathbb{R}) : q \in [0, 2\pi],\ \epsilon \in [0, \epsilon_*),\ \mu \in (0, \mu_2]\}.$$

Thus, there is $L_1 > 0$ such that

$$II \leqslant L_1\big(|h_\sigma^0(q^\epsilon) - h_\sigma^0(q^0)| + |\Psi^\epsilon(T, q^\epsilon) - \Psi^0(T, q^0)|\big).$$

Since h^0 is Lipschitz on $[0, 2\pi]$, there is $L > 0$ such that

$$II \leqslant L\big(|q^\epsilon - q^0| + |\Psi^\epsilon(T, q^\epsilon) - \Psi^0(T, q^0)|\big)$$

By (4.43), there is $C_4 > 0$ such that

$$II \leqslant C_4\epsilon. \tag{4.50}$$

Combining (4.45), (4.49) and (4.50), we get:

$$|h_\sigma^\epsilon(p) - h_\sigma^0(p)| \leqslant \frac{27}{32}|h_\sigma^\epsilon - h_\sigma^0|_{C^0} + C_4\epsilon.$$

Thus, we have finally, by subtracting the first term on the right-hand side from both sides of the inequality, and considering that $|h_\sigma^\epsilon - h_\sigma^0|_{C^0} \geqslant |h_\sigma^\epsilon(p) - h_\sigma^0(p)|$ for all $p \in [0, 2\pi]$,

$$\frac{5}{32}|h_\sigma^\epsilon(p) - h_\sigma^0(p)| \leqslant C_4\epsilon.$$

Thus, we have, for some $C_d > 0$:

$$|h_\sigma^\epsilon - h_\sigma^0|_{C^0} \leqslant C_d\epsilon. \tag{4.51}$$

We have thus proved that, if $|h_\sigma^\epsilon - h_\sigma^0|_{C^0} \leqslant r$, then $|h_\sigma^\epsilon - h_\sigma^0|_{C^0} \leqslant C_d\epsilon$. Let us require in addition to all other requirements made on μ that $\mu < r/C_d$. Define

$$\epsilon_0 = \sup\{\tilde{\epsilon} \in [0, \epsilon_*) : |h_\sigma^\epsilon - h_\sigma^0|_{C^0} \leqslant r \text{ for } \epsilon \in [0, \tilde{\epsilon}]\}.$$

We will show that $\epsilon_0 = \epsilon_*$, so that (4.51) holds for all $\epsilon \in [0, \epsilon_*)$. Suppose not, that is, suppose that $\epsilon_0 < \epsilon_*$. For $\epsilon < \epsilon^0$, $|h_\sigma^\epsilon - h_\sigma^0|_{C^0} \leqslant r$, therefore $|h_\sigma^\epsilon - h_\sigma^0|_{C^0} \leqslant C_d\epsilon$. By definition of ϵ_*, the graph of h^{ϵ_0} is normally hyperbolic. Taking the limit as $\epsilon \to \epsilon_0$, we have

$$|h_\sigma^{\epsilon_0} - h_\sigma^0|_{C^0} \leqslant C_d\epsilon \leqslant C_d\mu < C_d\left(\frac{r}{C_d}\right) = r.$$

This is a contradiction to the fact that ϵ_0 is the supremum. Thus, we must have $\epsilon_0 = \epsilon_*$. Hence, inequality (4.51) holds for all $\epsilon \in [0, \epsilon_*)$.

Finally, we will estimate h^ϵ_τ. Since the graph of h^ϵ is invariant, then

$$\begin{aligned} h^\epsilon_\sigma(\sigma,\tau)\sigma' + h^\epsilon_\tau &= f(h^\epsilon(\sigma,\tau),\sigma) + \epsilon R(h^\epsilon(\sigma,\tau),\sigma,\tau/\mu^\ell,\mu) \\ \sigma' &= g(h^\epsilon(\sigma,\tau),\sigma) + \epsilon S(h^\epsilon(\sigma,\tau),\sigma,\tau/\mu^\ell,\mu) \end{aligned} \tag{4.52}$$

Let us define the function H by $h^\epsilon(\sigma,\tau) = h^0(\sigma,\tau) + \epsilon H(\sigma,\tau,\epsilon)$. Henceforth, we will omit the arguments of the functions if they are the same as above, for simplicity. We use the definition of H in the second equation (4.52) and get

$$\sigma' = g(h^0 + \epsilon H, \sigma) + \epsilon S(h^0 + \epsilon H, \sigma, \tau/\mu^\ell, \mu).$$

By substituting in the first equation of (4.52), we get

$$\begin{aligned} &\big(g(h^0+\epsilon H,\sigma) + \epsilon S(h^0+\epsilon H,\sigma,\tau/\mu^\ell,\mu)\big)H_\sigma + H_\tau \\ &\qquad = \frac{1}{\epsilon}\big(f(h^0+\epsilon H,\sigma) - h^0_\sigma g(h^0+\epsilon H,\sigma)\big) - h^0_\sigma S + R \end{aligned} \tag{4.53}$$

By the fact that (4.51) holds for $\epsilon \in [0,\epsilon_*)$, we have

$$|H_\sigma|_{C^0} = \frac{1}{\epsilon}|h^\epsilon_\sigma - h^0_\sigma|_{C^0} \leqslant C_d.$$

Moreover, since $f(h^0,\sigma) = h^0_\sigma g(h^0,\sigma)$ and the function defined by $\rho \mapsto f(\rho,\sigma) - h^0_\sigma\, g(\rho,\sigma)$ is Lipschitz on compact sets, we have

$$|f(h^0+\epsilon H,\sigma) - h^0_\sigma g(h^0+\epsilon H,\sigma)| \leqslant \overline{L}\,\epsilon |H|,$$

where $\overline{L} > 0$ is a Lipschitz constant. Therefore, by (4.53), we have

$$|H_\tau| \leqslant \big|g(h^0+\epsilon H,\sigma) + \epsilon S(h^0+\epsilon H,\sigma,\tau/\mu^\ell,\mu)\big|C_d + \overline{L}|H| + |h^0_\sigma S| + |R|.$$

By continuity of all the functions present in the right-hand side of last inequality, and considering that the domains of those functions are compact sets, we conclude that there is $C_\tau > 0$, independent of ϵ, such that

$$|H_\tau|_{C^0} \leqslant C_\tau,$$

that is

$$|h^\epsilon_\tau|_{C^0} \leqslant C_\tau \epsilon.$$

By defining $C = \max\{\tilde{C}, C_d, C_\tau\}$, the lemma is proved. □

Lemma 11. *Let S_τ denote the derivative of S with respect to its third entry. For all $n \geqslant 2$ the following equality holds:*

$$\begin{aligned} \frac{\epsilon^2}{\mu^\ell}\frac{SS_\tau}{G^2} &= \sum_{k=2}^{n} \frac{\epsilon^k}{k}\frac{d}{ds}\left(\frac{S^k}{G^k}\right) - (S_\rho h^\epsilon_\sigma G + S_\rho h^\epsilon_\tau + S_\sigma G)\sum_{k=2}^{n}\epsilon^k \frac{S^{k-1}}{G^k} \\ &\quad + (G_\rho h^\epsilon_\sigma G + G_\rho h^\epsilon_\tau + G_\sigma G)\sum_{k=2}^{n}\epsilon^k\frac{S^k}{G^{k+1}} + \frac{\epsilon^{n+1}}{\mu^\ell}\frac{S^n S_\tau}{G^{n+1}}, \end{aligned} \tag{4.54}$$

where the argument of the function G and its derivatives is given by the expression $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)$, the argument of h^ϵ and its derivatives is given by $(\sigma^\epsilon(s,q),\tau(s))$ and the argument of S and its derivatives is given by the expression $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s)/\mu^\ell,\mu)$.

Proof: We will omit the arguments of the functions in order to simplify notation. The proof is done by induction. For $n=2$, we simply differentiate S^2/G^2 with respect to s:

$$\begin{aligned}\frac{d}{ds}\left(\frac{S^2}{G^2}\right) =&\frac{2S(S_\rho h^\epsilon_\sigma G+S_\rho h^\epsilon_\tau+S_\sigma G)}{G^2}+\frac{2}{\mu^\ell}\frac{SS_\tau}{G^2}\\ &-\frac{2S^2(G_\rho h^\epsilon_\sigma G+G_\rho h^\epsilon_\tau+G_\sigma G)}{G^3}-\frac{2\epsilon}{\mu^\ell}\frac{S^2S_\tau}{G^3}.\end{aligned}$$

Rearranging the terms and multiplying by $\epsilon^2/2$, we get

$$\begin{aligned}\frac{\epsilon^2}{\mu^\ell}\frac{SS_\tau}{G^2} =&\frac{\epsilon^2}{2}\frac{d}{ds}\left(\frac{S^2}{G^2}\right)-\epsilon^2 S\frac{(S_\rho h^\epsilon_\sigma G+S_\rho h^\epsilon_\tau+S_\sigma G)}{G^2}\\ &+\epsilon^2S^2\frac{(G_\rho h^\epsilon_\sigma G+G_\rho h^\epsilon_\tau+G_\sigma G)}{G^3}+\frac{\epsilon^3}{\mu^\ell}\frac{S^2S_\tau}{G^3},\end{aligned}$$

as wanted. Suppose that (4.54) is valid for some $n\geqslant 2$. We must prove that it is also valid for $n+1$ in this case. We differentiate S^{n+1}/G^{n+1} with respect to s:

$$\begin{aligned}\frac{d}{ds}\left(\frac{S^{n+1}}{G^{n+1}}\right) =&\frac{(n+1)S^n(S_\rho h^\epsilon_\sigma G+S_\rho h^\epsilon_\tau+S_\sigma G)}{G^{n+1}}+\frac{(n+1)}{\mu^\ell}\frac{S^nS_\tau}{G^{n+1}}\\ &-\frac{(n+1)S^{n+1}(G_\rho h^\epsilon_\sigma G+G_\rho h^\epsilon_\tau+G_\sigma G)}{G^{n+2}}-\frac{(n+1)\epsilon}{\mu^\ell}\frac{S^{n+1}S_\tau}{G^{n+2}}.\end{aligned}$$

Rearranging and multiplying by $\epsilon^{n+1}/(n+1)$, we get:

$$\begin{aligned}\frac{\epsilon^{n+1}}{\mu^\ell}\frac{S^nS_\tau}{G^{n+1}} =&\frac{\epsilon^{n+1}}{(n+1)}\frac{d}{ds}\left(\frac{S^{n+1}}{G^{n+1}}\right)-\epsilon^{n+1}S^n\frac{(S_\rho h^\epsilon_\sigma G+S_\rho h^\epsilon_\tau+S_\sigma G)}{G^{n+1}}\\ &+\epsilon^{n+1}S^{n+1}\frac{(G_\rho h^\epsilon_\sigma G+G_\rho h^\epsilon_\tau+G_\sigma G)}{G^{n+2}}+\frac{\epsilon^{n+2}}{\mu^\ell}\frac{S^{n+1}S_\tau}{G^{n+2}},\end{aligned}$$

By substituting this expression in (4.54), we get

$$\begin{aligned}\frac{\epsilon^2}{\mu^\ell}\frac{SS_\tau}{G^2} =&\sum_{k=2}^{n+1}\frac{\epsilon^k}{k}\frac{d}{ds}\left(\frac{S^k}{G^k}\right)-(S_\rho h^\epsilon_\sigma G+S_\rho h^\epsilon_\tau+S_\sigma G)\sum_{k=2}^{n+1}\epsilon^k\frac{S^{k-1}}{G^k}\\ &+(G_\rho h^\epsilon_\sigma G+G_\rho h^\epsilon_\tau+G_\sigma G)\sum_{k=2}^{n+1}\epsilon^k\frac{S^k}{G^{k+1}}+\frac{\epsilon^{n+2}}{\mu^\ell}\frac{S^{n+1}S_\tau}{G^{n+2}},\end{aligned}$$

which is the formula for $n+1$. The lemma is thus proved. □

Lemma 12. *Let $y^\epsilon(s,q)$ be defined as in Propositon 9, that is*

$$y^\epsilon(s,q):=\exp\left(\int_0^s(G_\rho h^\epsilon_\sigma+G_\sigma)\right)dt.$$

where the argument G_ρ and G_σ is $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)$, and the argument of h^ϵ_σ is $(\sigma^\epsilon(s,q),\tau(s))$. If the function $s \mapsto G(h^\epsilon(\sigma^\epsilon(s),\tau(s)),\sigma^\epsilon(s),\tau(s),\mu,\epsilon)$ has no zeros for all $s \geqslant 0$, and if $\mu > 0$ is sufficiently small, then there is $C > 0$ such that

$$y^\epsilon(s,q) \geqslant e^{-C}\, e^{-C\epsilon s}.$$

Proof: Note that

$$\begin{aligned}\frac{d}{ds}\ln\left|G(h^\epsilon(\sigma^\epsilon(s),\tau(s)),\sigma^\epsilon(s),\tau(s),\mu,\epsilon)\right| &= \frac{1}{G}(G_\rho h^\epsilon_\sigma(\sigma^\epsilon)' + G_\rho h^\epsilon_\tau + G_\sigma(\sigma^\epsilon)' + G_\tau)\\ &= G_\rho h^\epsilon_\sigma + G_\sigma + \frac{G_\rho h^\epsilon_\tau}{G} + \frac{G_\tau}{G}.\end{aligned}$$

where the arguments omitted in the expressions on the right-hand side of the equation are $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)$ for the function G and its derivatives, $(\sigma^\epsilon(s,q),\tau(s))$ for h^ϵ and its derivatives, and s for $(\sigma^\epsilon)'$. Henceforth, whenever the arguments of those functions are omitted, they are to be understood as stated. By integrating and rearranging the terms, we get:

$$\begin{aligned}\int_0^s (G_\rho h^\epsilon_\sigma + G_\sigma)\,dt = \ln&\left|\frac{G(h^\epsilon(\sigma^\epsilon(s),\tau(s)),\sigma^\epsilon(s),\tau(s),\mu,\epsilon)}{G(h^\epsilon(\sigma^\epsilon(0),\tau(0)),\sigma^\epsilon(0),\tau(0),\mu,\epsilon)}\right|\\ &- \int_0^s \frac{G_\rho h^\epsilon_\tau}{G}\,dt - \int_0^s \frac{G_\tau}{G}\,dt.\end{aligned}$$

Recall the definition of G, found in (4.4). We observe that, if S is the function present in this definition, we have

$$\begin{aligned}\frac{d}{ds}\left(\frac{S}{G}\right) = &\frac{S_\rho h^\epsilon_\sigma(\sigma^\epsilon)' + S_\rho h^\epsilon_\tau + S_\sigma(\sigma^\epsilon)' + \frac{1}{\mu^\ell}S_\tau}{G}\\ &- \frac{S(G_\rho h^\epsilon_\sigma(\sigma^\epsilon)' + G_\rho h^\epsilon_\tau + G_\sigma(\sigma^\epsilon)' + \frac{\epsilon}{\mu^\ell}S_\tau)}{G^2},\end{aligned}$$

where the argument of the function S and its derivatives are given by the expression $(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s)/\mu^\ell,\mu)$. We remark that S_τ is a symbol representing the partial derivative of S with respect to its third entry. Since $G_\tau = \dfrac{\epsilon}{\mu^\ell}S_\tau$ and $(\sigma^\epsilon)' = G$, we find that

$$\begin{aligned}\frac{G_\tau}{G} = \frac{\epsilon}{\mu^\ell}\left(\frac{S_\tau}{G}\right) &= \epsilon\cdot\frac{S(G_\rho h^\epsilon_\sigma G + G_\rho h^\epsilon_\tau + G_\sigma G)}{G^2} + \frac{\epsilon^2}{\mu^\ell}\frac{SS_\tau}{G^2}\\ &- \epsilon\cdot\frac{S_\rho h^\epsilon_\sigma G + S_\rho h^\epsilon_\tau + S_\sigma G}{G} + \epsilon\,\frac{d}{ds}\left(\frac{S}{G}\right)\end{aligned}$$

By integrating, we get

$$\begin{aligned}\int_0^s \frac{G_\tau}{G}\,dt = \epsilon \int_0^s &\frac{SG_\rho h^\epsilon_\sigma G + SG_\rho h^\epsilon_\tau + SG_\sigma G}{G^2}\,dt\\ &- \epsilon \int_0^s \frac{S_\rho h^\epsilon_\sigma G + S_\rho h^\epsilon_\tau + S_\sigma G}{G}\,dt\\ &+ \epsilon \cdot \frac{S(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\frac{\tau(s)}{\mu^\ell},\mu)}{G(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)}\\ &- \epsilon \cdot \frac{S(h^\epsilon(\sigma^\epsilon(0,q),\tau(0)),\sigma^\epsilon(0,q),\frac{\tau(0)}{\mu^\ell},\mu)}{G(h^\epsilon(\sigma^\epsilon(0,q),\tau(0)),\sigma^\epsilon(0,q),\tau(0),\mu,\epsilon)}\\ &+ \frac{\epsilon^2}{\mu^\ell}\int_0^s \frac{SS_\tau}{G^2}\,dt\end{aligned}$$

Observe that, for $\mu_0 > 0$, the set $\mathcal{H} := \{h^\epsilon : \epsilon \in [0,\epsilon_*(\mu)),\ \mu \in (0,\mu_0]\}$ is uniformly bounded. The sets $\mathcal{H}_\sigma$ and $\mathcal{H}_\tau$, defined similarly to $\mathcal{H}$, are also uniformly bounded. Therefore, there is $M > 0$ such that $|h^\epsilon| < M$, $|h^\epsilon_\sigma| < M$ and $|h^\epsilon_\tau| < M$ for all $\mu \in (0,\mu_0]$ and all $\epsilon \in [0,\epsilon_*(\mu))$. Moreover, S and its derivatives are bounded on $[-M,M]\times\mathbb{R}\times\mathbb{R}\times[0,\mu_0]$. Thus, the integrands of the first and second integrals on the right-hand side of last equation are bounded, and there is $K_1 > 0$ such that

$$\epsilon\int_0^s \left|\frac{SG_\rho h^\epsilon_\sigma G + SG_\rho h^\epsilon_\tau + SG_\sigma G}{G^2}\right| dt \leqslant K_1\epsilon s,$$

and

$$\epsilon\int_0^s \left|\frac{S_\rho h^\epsilon_\sigma G + S_\rho h^\epsilon_\tau + S_\sigma G}{G}\right| dt \leqslant K_1\epsilon s.$$

Moreover, since S and G are bounded, since μ is small, and since $\epsilon \in [0,\mu]$, there is $K_2 > 0$ such that

$$\epsilon\left|\frac{S(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\frac{\tau(s)}{\mu^\ell},\mu)}{G(h^\epsilon(\sigma^\epsilon(s,q),\tau(s)),\sigma^\epsilon(s,q),\tau(s),\mu,\epsilon)}\right| \leqslant K_2,$$

and

$$\epsilon\left|\frac{S(h^\epsilon(\sigma^\epsilon(0,q),\tau(0)),\sigma^\epsilon(0,q),\frac{\tau(0)}{\mu^\ell},\mu)}{G(h^\epsilon(\sigma^\epsilon(0,q),\tau(0)),\sigma^\epsilon(0,q),\tau(0),\mu,\epsilon)}\right| \leqslant K_2.$$

In order to estimate the last term, we apply Lemma 11 with $n = c$ and integrate to get

$$\begin{aligned}\frac{\epsilon^2}{\mu^\ell}\int_0^s \frac{SS_\tau}{G^2}dt = \sum_{k=2}^{c}\frac{\epsilon^k}{k}\int_0^s \frac{d}{dt}\left(\frac{S^k}{G^k}\right)dt - (S_\rho h^\epsilon_\sigma G + S_\rho h^\epsilon_\tau + S_\sigma G)\sum_{k=2}^{c}\epsilon^k\int_0^s \frac{S^{k-1}}{G^k}dt\\ + (G_\rho h^\epsilon_\sigma G + G_\rho h^\epsilon_\tau + G_\sigma G)\sum_{k=2}^{c}\epsilon^k\int_0^s \frac{S^k}{G^{k+1}}dt + \frac{\epsilon^{c+1}}{\mu^\ell}\int_0^s \frac{S^cS_\tau}{G^{c+1}}dt\end{aligned}$$

Similarly as above, the first term on right-hand side is bounded by a positive constant K_3. The second and third terms are bounded by $K_4\epsilon s$, where and K_4 is also a positive constant. For last term, because $\epsilon \leqslant \mu$, there is $K_5 > 0$ such that

$$\frac{\epsilon^{c+1}}{\mu^\ell}\int_0^s \left|\frac{S^cS_\tau}{G^{c+1}}\right| dt \leqslant K_5\epsilon s. \tag{4.55}$$

Defining $C_1 = \max\{K_i : i = 1, 2, ..., 5\}$, we have:

$$\left|\int_0^s \frac{G_\tau}{G}\, dt\right| \leqslant C_1 + C_1 \epsilon s$$

We remark that C_1 is independent of the choice of ϵ and μ, provided that μ is sufficiently small. By continuity of the functions and compactness of the domain, there is also $C_2 > 0$ such that

$$\ln \left|\frac{G(h^\epsilon(\sigma^\epsilon(s), \tau(s)), \sigma^\epsilon(s), \tau(s), \mu, \epsilon)}{G(h^\epsilon(\sigma^\epsilon(0), \tau(0)), \sigma^\epsilon(0), \tau(0), \mu, \epsilon)}\right| \leqslant C_2 \tag{4.56}$$

for all $s \geqslant 0$. Finally, we remark that, by Lemma 10, there is $C_\tau > 0$ such that $|h^\epsilon_\tau| \leqslant C_\tau \epsilon$. Thus, we have

$$\left|\int_0^s \frac{G_\rho h^\epsilon_\tau}{G}\, dt\right| \leqslant \int_0^s \left|\frac{G_\rho C_\tau}{G}\right| \epsilon\, dt \leqslant C_3 \epsilon s,$$

where C_3 is a positive constant. Defining $C = \max\{C_i : i = 1, 2, 3\}$, we have

$$\left|\int_0^s (G_\rho h^\epsilon_\sigma + G_\sigma)\, dt\right| \leqslant C + C\epsilon s,$$

so that

$$e^{-C} e^{-C\epsilon s} \leqslant y^\epsilon(s, q) \leqslant e^C e^{C\epsilon s},$$

thus proving the lemma. □

Lemma 13. *With the hypotheses of Lemma 10, there are $\mu_0 > 0$ and $C > 0$ such that, if $\mu \in (0, \mu_0]$, then $|h^\epsilon_{\sigma\sigma}|_{C^0} \leqslant C$.*

Proof: Recall that h^ϵ_σ satisfies equation (4.37), that is,

$$h^\epsilon_\sigma(\sigma^\epsilon(s, q), \tau(s)) = \frac{\psi^\epsilon_{11}(s, q) h^\epsilon_\sigma(q, 0) + \psi^\epsilon_{12}(s, q)}{\psi^\epsilon_{21}(s, q) h^\epsilon_\sigma(q, 0) + \psi^\epsilon_{22}(s, q)}.$$

Let $y^\epsilon(s, q)$ be defined as before, that is

$$y^\epsilon(s, q) := \exp\left(\int_0^t (G_\rho h^\epsilon_\sigma + G_\sigma)\right) ds.$$

By Lemma 12, there are $\mu_1 > 0$ and $C_1 > 0$ such that, if $\mu \in (0, \mu_1]$, then

$$y^\epsilon(s, q) \geqslant e^{-C_1} e^{-C_1 \epsilon s}.$$

Similarly to what was done in Lemma 10, we choose $T_0 > 0$ such that, if $t \geqslant T_0$, then

$$\det \Psi^0(t, q) \leqslant \frac{M_0^2}{4} e^{-C_1} e^{-C_1 \mu_1 (T_0 + 1)},$$

for all $q \in [0, 2\pi]$. Let μ_2 be defined as

$$\mu_2 = \min\left\{\mu_1, \sqrt[c]{\frac{\Omega}{2\pi m}}\right\},$$

and define $T = T(\mu) \in [T_0, T_0 + 1)$ as in the proof of Lemma 10 for all $\mu \in (0, \mu_2]$. Let $p \in [0, 2\pi]$ and define q^ϵ by $p = \sigma(T, q^\epsilon)$, as before. By equation (4.37),

$$h^\epsilon_\sigma(p) = \frac{\psi^\epsilon_{11}(T, q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{12}(T, q^\epsilon)}{\psi^\epsilon_{21}(T, q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{22}(T, q^\epsilon)},$$

where we have omitted the second argument of h^ϵ_σ if it is zero. By differentiating both sides with respect to p, we get:

$$\begin{aligned} h^\epsilon_{\sigma\sigma}(p) = {} & \frac{\frac{dq^\epsilon}{dp}}{\left(\psi^\epsilon_{21}(T, q^\epsilon)h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{22}(T, q^\epsilon)\right)^2}\left(\left(\psi^\epsilon_{21}\frac{d\psi^\epsilon_{11}}{dq} - \psi^\epsilon_{11}\frac{d\psi^\epsilon_{21}}{dq}\right)\left(h^\epsilon_\sigma(q^\epsilon)\right)^2\right. \\ & + \left(\psi^\epsilon_{22}\frac{d\psi^\epsilon_{11}}{dq} - \psi^\epsilon_{12}\frac{d\psi^\epsilon_{21}}{dq} + \psi^\epsilon_{21}\frac{d\psi^\epsilon_{12}}{dq} - \psi^\epsilon_{11}\frac{d\psi^\epsilon_{22}}{dq}\right)h^\epsilon_\sigma(q^\epsilon) \\ & \left. + \left(\psi^\epsilon_{11}\psi^\epsilon_{22} - \psi^\epsilon_{12}\psi^\epsilon_{21}\right)h^\epsilon_{\sigma\sigma}(q^\epsilon) + \psi^\epsilon_{22}\frac{d\psi^\epsilon_{12}}{dq} - \psi^\epsilon_{12}\frac{d\psi^\epsilon_{22}}{dq}\right), \end{aligned} \tag{4.57}$$

where the functions ψ^ϵ_{ij} are evaluated at (T, q^ϵ). We also differentiate the relation $p = \sigma^\epsilon(T, q^\epsilon)$ with respect to p, and get

$$\frac{dq^\epsilon}{dp} = \frac{1}{\sigma^\epsilon_q(T, q^\epsilon)}.$$

Let $\gamma^\epsilon(s, q)$ be defined as before. Note that the partial derivative of $\gamma^\epsilon(s, q)$ with respect to q, which we will denote by $Y(s)$, is the solution of the first variational equation along $\gamma^\epsilon(s, q)$ with initial condition $Y(0) = (h^\epsilon_\sigma(q, 0), 1, 0)$. By Proposition 9, we must have $Y(s) = X_2(s)$, an identity we had already shown to be true in (4.9). By differentiating $\gamma^\epsilon(s, q)$ directly, we conclude that the second entry of $Y(s)$ is equal to $\sigma^\epsilon_q(s, q)$. Since the second entry of $X_2(s)$ is equal to $y^\epsilon(s, q)$, we have

$$\sigma^\epsilon_q(T, q) = y^\epsilon(T, q).$$

Thus, for $\mu \in (0, \mu_2]$, there is, by Lemma 12, $C_y > 0$ such that

$$\left|\frac{dq^\epsilon}{dp}\right| = \frac{1}{y^\epsilon(T, q^\epsilon)} < e^{C_1}e^{C_1\epsilon T} \leqslant e^{C_1}e^{C_1\mu_2(T_0+1)}.$$

Applying Lemma 8, let $r > 0$ be such that, for $T \in [T_0, T_0 + 1)$,

$$|d_{21}\xi + d_{22}| > \frac{2}{3}M_0, \quad |\det D - \det \Psi^0(T, q)| < \frac{M_0^2}{8}e^{-C_1}e^{-C_1\mu_2(T_0+1)},$$

whenever ξ, a real number, $D = (d_{ij})$, a 2×2 real matrix and $q \in [0, 2\pi]$ are such that

$$|\xi - h^0(q)| < r, \quad |D - \Psi^0(T, q)| < r.$$

Let $\mu > 0$ be sufficiently small so that $|h^\epsilon_\sigma - h^0_\sigma| < r$, $|\Psi^\epsilon(T,q) - \Psi^0(T_0,q)| < r$ for all $q \in [0,2\pi]$ and the partial derivative Ψ^ϵ_q is uniformly bounded. Then, for all q and all $T = T(\mu)$, we have

$$|\psi^\epsilon_{21}(T,q)h^\epsilon_\sigma(q) + \psi^\epsilon_{22}(T,q)|^2 > \frac{4}{9}M_0^2,$$

and

$$|\det \Psi^\epsilon(T,q) - \det \Psi^0(T_0,q)| < \frac{M_0^2}{8}e^{-C_1}e^{-C_1\mu_2(T_0+1)}.$$

Since

$$\det \Psi^0(T_0,q) \leqslant \frac{M_0^2}{4}e^{-C_1}e^{-C_1\mu_1(T_0+1)} \leqslant \frac{M_0^2}{4}e^{-C_1}e^{-C_1\mu_2(T_0+1)},$$

for all $q \in [0,2\pi]$, we have

$$|\det \Psi^\epsilon(T,q^\epsilon)| < \frac{3M_0^2}{8}e^{-C_1}e^{-C_1\mu_2(T_0+1)}.$$

Therefore, we have

$$\left|\frac{\frac{dq^\epsilon}{dp}}{(\psi^\epsilon_{21}h^\epsilon_\sigma(q^\epsilon) + \psi^\epsilon_{22})^2}(\psi^\epsilon_{11}\psi^\epsilon_{22} - \psi^\epsilon_{12}\psi^\epsilon_{21})\right| < \frac{27}{32}$$

To estimate the $|h_{\sigma\sigma}|$, we take the absolute value of each side of (4.57), apply the triangle inequality to the right-hand side, take the supremum over q^ϵ on the right-hand side and take the supremum on the right-hand side over p, to get

$$|h^\epsilon_{\sigma\sigma}| \leqslant K_1 + \frac{27}{32}|h^\epsilon_{\sigma\sigma}|,$$

where K_1 is a positive constant. Therefore, we have

$$|h^\epsilon_{\sigma\sigma}| \leqslant \frac{32}{5}K_1.$$

The lemma is proved by defining μ_0 small enough to satisfy all requirements made hitherto and defining $C = (32/5)K_1$. □

Lemma 14. *With the hypotheses of Lemma 10, there are $\mu_1 > 0$ and $C_1 > 0$ such that, if $\mu \in (0,\mu_1]$, then $|h^\epsilon_{\sigma\tau}|_{C^0} \leqslant C_1$ and $|h^\epsilon_{\tau\tau}|_{C^0} \leqslant C_1$*

Proof: We recall the definition of H from Lemma 10 and consider equation (4.53). By differentiating both sides with respect to σ, we get

$$AH_\sigma + BH_{\sigma\sigma} + H_{\tau\sigma} = \frac{1}{\epsilon}C + D, \tag{4.58}$$

where A, B, C and D are functions of $(\sigma, \tau, \epsilon, \mu)$ given by

$$
\begin{aligned}
A =&(h^0_\sigma + \epsilon H_\sigma)g_\rho + g_\sigma + \epsilon(h^0_\sigma + \epsilon H_\sigma)S_\rho + \epsilon S_\sigma \\
B =&g + \epsilon S \\
C =&(h^0_\sigma + \epsilon H_\sigma)f_\rho + f_\sigma - h^0_{\sigma\sigma} g - (h^0_\sigma + \epsilon H_\sigma)h^0_\sigma g_\rho - h^0_\sigma g_\sigma \\
D =&- h^0_{\sigma\sigma} S - ((h^0_\sigma + \epsilon H_\sigma)S_\rho + S_\sigma)h^0_\sigma + (h^0_\sigma + \epsilon H_\sigma)R_\rho + R_\sigma.
\end{aligned}
$$

In the definitions above, the argument of f, g and their derivatives is given by the expression $(h^0(\sigma) + \epsilon H(\sigma, \tau, \epsilon, \mu), \sigma)$, and the argument of R, S and their derivatives is given by $(h^0(\sigma) + \epsilon H(\sigma, \tau, \epsilon, \mu), \sigma, \tau/\mu^\ell, \mu)$.

Considering Lemma 13, there are $\mu_0 >$ and $K_0 > 0$ such that, if $\mu \in (0, \mu_0]$, then

$$
|H_{\sigma\sigma}| \leqslant \frac{K_0}{\epsilon}, \tag{4.59}
$$

for all $\epsilon \in [0, \epsilon_*)$. Taking into account Lemmas 10 and 13, functions AH_σ, B, C and D are all uniformly bounded. Therefore, applying the triangle inequality to (4.58) after some rearranging, we get $K_2 > 0$ and $K_3 > 0$ such that

$$
|H_{\tau\sigma}| \leqslant \frac{K_2}{\epsilon} + K_3.
$$

Thus, we have

$$
|h^\epsilon_{\tau\sigma}| = \epsilon|H_{\tau\sigma}| \leqslant K_2 + K_3\epsilon \leqslant K_2 + K_3\mu_0 \leqslant C_1,
$$

where $C_1 > 0$ is a constant. The proof for $|h^\epsilon_{\tau\tau}|$ is analogous. □

4.4 Proof of the main lemmas

In this section, we present the proofs of Lemmas L1, L2, and L3. The section is divided in three parts, each one containing the proof of one of the lemmas.

4.4.1 Proof of Lemma L1

Let $\mu_0 > 0$ be given by Lemma 13. Let $\mu \in (0, \mu_0]$ and choose $C > 0$ be as in Lemma 10. As before, let ϵ_* be such that, for $\epsilon \in [0, \epsilon_*)$, system $E^{\epsilon,\mu}$ has a k-normally hyperbolic invariant manifold, $k \geqslant 2$, that is the graph of a C^k function h^ϵ of the angular variables. By Lemmas 10 and 13, the set $\mathcal{S} := \{h^\epsilon : 0 \leqslant \epsilon < \epsilon_*\}$ is uniformly bounded in the C^2 norm. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $\epsilon_n \to \epsilon_*$ as $n \to \infty$. By Proposition 1, we can extract from this sequence a subsequence (ϵ_k) that converges uniformly to a C^1 function, $h^{\epsilon *}$.

We show now that the graph of $h^{\epsilon *}$ is invariant for $E^{\epsilon *,\mu}$. In order to do so, let $s \mapsto (\rho^\epsilon(s,q,w), \sigma^\epsilon(s,q), \tau(s,w))$ denote the solution to $E^{\epsilon,\mu}$ such that $\sigma^\epsilon(0,q) = q$, $\rho^\epsilon(0,q,w) = h^\epsilon(q,w)$ and $\tau(0,w) = w$. If $s \in \mathbb{R}$, then, using the continuity of the flow with respect to parameters, we have: $\sigma^{\epsilon_k}(s,q) \to \sigma^{\epsilon *}(s,q)$, $\rho^{\epsilon_k}(s,q,w) \to \rho^{\epsilon *}(s,q,w)$ and $\tau^{\epsilon_k}(s,w) \to \tau^{\epsilon *}(s,w)$. We note that the identity $\rho^{\epsilon_k}(s,q,w) = h^{\epsilon_k}(\sigma^{\epsilon_k}(s,q), \tau^{\epsilon_k}(s,w))$ becomes $\rho^{\epsilon *}(s,q,w) = h^{\epsilon *}(\sigma^{\epsilon *}(s,q), \tau^{\epsilon *}(s,w))$ by passing to the limit as $k \to \infty$. Thus, it follows that the graph of $h^{\epsilon *}$ is an invariant set for $E^{\epsilon *,\mu}$. □

4.4.2 Proof of Lemma L2

Let $\mu > 0$ be chosen as in Lemma L1. By definition, ϵ_* is such that, for $0 \in [0, \epsilon_*)$, the system $E^{\epsilon,\mu}$ has a k-normally hyperbolic invariant manifold that is the graph of a C^k function, h^ϵ, of the angular variables. By Lemma L1, $E^{\epsilon *,\mu}$ has a C^1 invariant manifold, $M(\epsilon_*, \mu)$, given as the graph of the function h^ϵ_* of the angular variables. We must show that $M(\epsilon_*, \mu)$ has an invariant normal bundle.

We show first that it suffices to build a normal bundle over the curve

$$q \mapsto (h^{\epsilon *}(q,0), q, 0) \tag{4.60}$$

that is invariant with respect to some iterate of the stroboscopic linearized Poincaré map, P_s. In fact, the line tangent to the invariant torus is invariant under P_s. Also, for every two dimensional linear map with an invariant line, if some iterate of this map has two distinct invariant lines, then so does the map itself. Since P_s is two-dimensional, if we construct an invariant normal bundle over the curve (4.60) for some of its iterates, then P_s will also have an invariant normal bundle. Finally, if there is an invariant normal bundle over the curve, then an invariant normal bundle over the torus is easily constructed by moving the vector in the given normal bundle forward by the linearized flow.

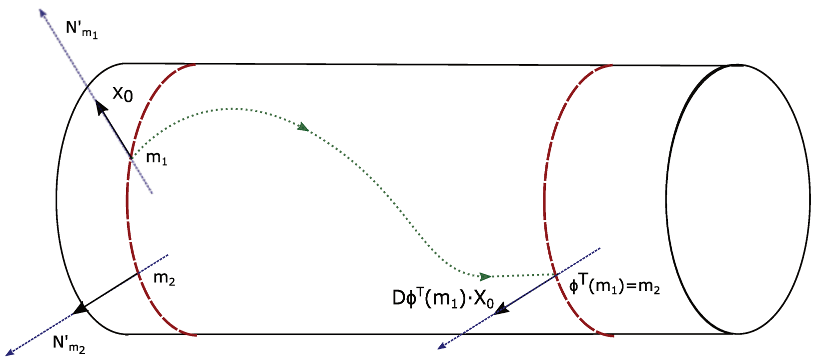

Figure 3 – Building of the normal bundle.

Let us now construct a normal bundle over curve (4.60). Consider the function space

$$\mathcal{F} := \{\alpha : S^1 \to \mathbb{R} : \alpha \in C^0\}. \tag{4.61}$$

For each $\alpha \in \mathcal{F}$, define a vector in the tangent space of $(h^{\epsilon *}(q), q, 0)$ as follows:

$$X_0(q) := \alpha(q) X_2(q) + X_2(q)^{\perp}, \tag{4.62}$$

where $X_2(q) := \mathcal{X}_2(h^{\epsilon *}(q), q, 0)$ and $\mathcal{X}_2$ is defined as in section 4.2. We will show that there is $\alpha \in \mathcal{F}$ such that X_0 generates an invariant normal bundle over the curve (4.60).

Let n be an integer, $T := 2\pi m n \mu^{\ell}/\Omega$ and $p = \sigma^{\epsilon *}(T, q)$. Let also $\lambda^{\epsilon *}$, $a^{\epsilon *}$ and $b^{\epsilon *}$ be defined as in section 4.2. Note that X_0 generates an invariant normal bundle if, and only if,

$$\lambda^{\epsilon *}(T, q) X_0(p) = (\alpha(q) + a^{\epsilon *}(T, q)) X_2(p) + b^{\epsilon *}(T, q) X_2(p)^{\perp}. \tag{4.63}$$

By substituting $X_0(p)$ on the left-hand side, using (4.62), and considering the coefficients of $X_2(p)$ and $X_2(p)^{\perp}$, we find that (4.63) is equivalent to

$$\alpha(p) = \frac{\alpha(q) + a^{\epsilon *}(T, q)}{b^{\epsilon *}(T, q)}$$

Let $T < 0$ be defined analogously to what we have done in (4.38), with the property that, for $\epsilon \in [0, \epsilon_*)$, there is η such that

$$b^{\epsilon}(T) := \sup\{b^{\epsilon}(T, q) : q \in S^1\} > \eta > 1.$$

Using the fact that h^{ϵ} tends to $h^{\epsilon *}$ as $\epsilon \to \epsilon_*$ from below, we have

$$b^{\epsilon *}(T) := \sup\{b^{\epsilon *}(T, q) : q \in S^1\} \geqslant \eta > 1.$$

Let us define the operator $\Lambda : \mathcal{F} \to \mathcal{F}$ by

$$(\Lambda \alpha)(p) = \frac{\alpha(q) + a^{\epsilon *}(T, q)}{b^{\epsilon *}(T, q)}.$$

We remark that a fixed point of Λ corresponds to the function that provide the invariant normal bundle we seek. Note that

$$|(\Lambda\alpha_2)(p) - (\Lambda\alpha_1)(p)| = \left|\frac{\alpha_2(p) - \alpha_1(p)}{b^{\epsilon_*}(T,q)}\right| \leqslant \frac{1}{\eta}|\alpha_2(p) - \alpha_1(p)|.$$

Thus, Λ is a contraction on the complete metric space $\mathcal{F}$. By the Banach fixed-point theorem, there is a unique fixed point of Λ, as we wanted to show. □

4.4.3 Proof of Lemma L3

We will show that the C^1 invariant manifold $M(\epsilon_*, \mu)$, given as the graph of h^{ϵ_*}, is k-normally hyperbolic under the assumption that this manifold has an invariant normal splitting. In order to do so, we will make use of the inequalities (2.4), deduced earlier. Recall formula (4.20) for $\lambda_3(s)$. Let us suppose that a bounded neighbourhood N is chosen for the graph of h^0 and that μ is sufficiently small so that the invariant manifold $M(\epsilon_*, \mu)$ is in N. By Lemma 10, the following inequality holds for $\epsilon \in [0, \epsilon_*)$:

$$|h^\epsilon - h^0| < C\epsilon. \tag{4.64}$$

Let $s \mapsto \gamma^{\epsilon_*}(s,q,\tau) = \big(h^{\epsilon_*}(\sigma^{\epsilon_*}(s,q,\tau), s+\tau), \sigma^{\epsilon_*}(s,q,\tau), s+\tau\big)$ be the solution of (4.3) corresponding ϵ_* with the initial condition $\big(h^{\epsilon_*}(q,\tau), q, \tau\big)$ and let the B be defined as in section 4.2.1. We change the notation of the argument of B slightly to include the solution along which the variational equation is built. Note that:

$$\operatorname{tr} B(\gamma^{\epsilon_*}(t,q,\tau)) = f_\rho + g_\sigma + \epsilon(R_\rho + S_\sigma).$$

Let ω be the minimal period of the unperturbed system (4.2). Let $(\tilde{q}, \tilde{\tau})$ be any pair of the angular variables. We define

$$b := \int_0^\omega \operatorname{tr} B(\gamma^0(s,\tilde{q},\tilde{\tau}))\, ds.$$

The quantity b is a Floquet exponent of the periodic orbit, thus it is independent of the choice of the angular variables. By an application of Lemma 3, there is a constant $C_1 > 0$ such that, for $s \in [0,\omega]$,

$$|\gamma^{\epsilon_*}(s,\tilde{q},\tilde{\tau}) - \gamma^0(s,\tilde{q},\tilde{\tau})| \leqslant C_1\epsilon_*.$$

Therefore, there is a constant $C_2 > 0$ such that

$$|\operatorname{tr} B(\gamma^{\epsilon_*}(s,\tilde{q},\tilde{\tau})) - \operatorname{tr} B(\gamma^0(s,\tilde{q},\tilde{\tau}))| \leqslant C_2\epsilon_*.$$

Thus, by an application of the reverse triangle inequality, we get

$$\int_0^\omega \operatorname{tr} B(\gamma^{\epsilon_*}(s,\tilde{q},\tilde{\tau}))\, ds \leqslant \int_0^\omega \operatorname{tr} B(\gamma^{\epsilon_*}(s,\tilde{q},\tilde{\tau}))\, ds + C_2\epsilon_*\omega = b + C_2\epsilon_*\omega.$$

Any $s \geqslant 0$ can be expressed in the form $s = l\omega + r$, where l is an integer and $r \in [0, \omega)$. For $k = 1, 2, ..., l$, we define

$$q_k := \sigma^{\epsilon_*}(k\omega, q, \tau), \quad \tau_k := k\omega + \tau.$$

We have

$$\begin{aligned}\int_0^s \operatorname{tr} B(\gamma^{\epsilon_*}(t,q,\tau))\,dt &= \sum_{k=0}^{l-1} \int_{k\omega}^{(k+1)\omega} \operatorname{tr} B(\gamma^{\epsilon_*}(t,q,\tau))\,dt + \int_{l\omega}^{l\omega+r} \operatorname{tr} B(\gamma^{\epsilon_*}(t,q,\tau))\,dt \\ &= \sum_{k=0}^{l-1} \int_0^{\omega} \operatorname{tr} B(\gamma^{\epsilon_*}(t,q_k,\tau_k))\,dt + \int_0^{r} \operatorname{tr} B(\gamma^{\epsilon_*}(t,q_l,\tau_l))\,dt \\ &\leqslant l(b + C_2\epsilon_*\omega) + C_3 \leqslant \left(\frac{b}{\omega} + C_2\epsilon_*\right) s + C_4,\end{aligned}$$

where C_3 and C_4 are positive constants. Furthermore, by Lemma 12, there are constants $C_5 > 0$ and $C_6 > 0$ such that

$$|X_2(s)| \geqslant C_6 e^{-\mu C_5 s}. \tag{4.65}$$

Recall formula (4.20) and observe that the function α appearing in it must be bounded. In fact, it suffices to compare the definition of α with how we built the invariant manifold in Lemma L2 and notice that α must be periodic. Thus, combining all estimates above and formula (4.20), there is a constant $c > 0$ such that

$$\lambda_3(s) \leqslant c e^{(\frac{b}{\omega} + c\mu)s}. \tag{4.66}$$

If we take μ sufficiently small, we have $-\beta := \dfrac{b}{\omega} + c\mu < 0$. Therefore, by (4.66), we have

$$\lambda_3(s) \leqslant c e^{-\beta s}, \tag{4.67}$$

for all $s \geqslant 0$.

By considering its components, we see that $|X_1(s)|$ is uniformly bounded from below by 1. Thus, if k is a positive integer, then there is $c_1 > 0$ such that

$$\frac{\lambda_3(s)}{\lambda_1^k(s)} \leqslant c_1 e^{-\beta s}. \tag{4.68}$$

By combining 4.65 and 4.67, we also get

$$\frac{\lambda_3(s)}{\lambda_2^k(s)} \leqslant \frac{c_1}{C_6^k} e^{-s(\beta - \mu k c_1)}.$$

Thus, if $\mu > 0$ is sufficiently small, there are constants $c_2 > 0$ and $\beta_1 > 0$ such that

$$\frac{\lambda_3(s)}{\lambda_2^k(s)} \leqslant c_2 e^{-\beta_1 s}. \tag{4.69}$$

Therefore, by Lemma 1, $M(\epsilon_*, \mu)$ is k-normally hyperbolic. $\square$

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