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Fusion Products of Modules Over Current Algebras

Produtos de Fusão de Módulos Sobre Álgebras de Corrente

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"It's the questions we can't answer that teach us the most.
They teach us how to think.
If you give a man an answer, all he gains is a little fact.
But give him a question and he'll look for his own answers."
— Patrick Rothfuss, The Wise Man's Fear

"São as perguntas que não sabemos responder que mais nos ensinam.
Elas nos ensinam a pensar.
Se você dá uma resposta a um homem, tudo o que ele ganha é um fato qualquer.
Mas, se você lhe der uma pergunta, ele procurará suas próprias respostas."
— Patrick Rothfuss, O Temor do Sábio

Resumo

Dada uma álgebra de Lie complexa simples de dimensão finita $\mathfrak{g}$, consideramos a correspondente álgebra de correntes $\mathfrak{g}[t]$ e a subjacente categoria dos $\mathfrak{g}[t]$-módulos graduados de dimensão finita. O presente trabalho é motivado pelo conceito de produto de fusão de certos objetos nesta categoria. Em particular, pela conjectura de E. Feigin afirmando que tais produtos não dependem dos parâmetros espectrais escolhidos para sua definição. Em casos especiais, tal conjectura foi demonstrada ao estabelecer isomorfismo entre o produto de fusão dado e um módulo para o qual se conhece geradores e relações (que não dependem dos mencionados parâmetros). Fazemos aqui uma revisão de dois artigos representativos deste fato: [29] e [45]. Nestes artigos o ponto central é uma conjectura a respeito da realização de módulos de Weyl truncados como produto de fusão de certos módulos irredutíveis. Casos particulares são demonstrados ao explorar o relacionamento entre diversas classes de objetos da categoria, como módulos de Demazure, de Chari-Venkatesh e de Kirillov-Reshetikhin. Terminamos o trabalho com uma breve discussão dos resultados de [23], que traz uma abordagem usando bases de Gröbner para provar a conjectura de E. Feigin.

Palavras-chave: Produtos de fusão, Módulos de Weyl, Módulos de Demazure, Módulos de Chari-Venkatesh.

Abstract

Given a simple finite-dimensional Lie algebra $\mathfrak{g}$, we consider the underlying current algebra $\mathfrak{g}[t]$ as well as the category of finite-dimensional graded $\mathfrak{g}[t]$-modules. The present work is motivated by the concept of fusion products of certain objects in this category, specially by E. Feigin's conjecture stating that such products are independent of the spectral parameters which are chosen for their definitions. Particular cases of this conjecture were established by describing an isomorphism between the given fusion product and a module for which a presentation in terms of generators and relations independent of the mentioned parameters is known. We review two papers which are examples of this fact: [29] and [45]. The main focus of these papers is on a conjecture concerning the realization of truncated Weyl modules as fusion products of certain simple modules. They prove some particular cases by exploring the relationship between several classes of objects in the category such as Demazure, Chari-Venkatesh, and Kirillov-Reshetikhin modules. We end the text with a brief discussion of the results of [23] which proposes a Gröbner bases approach for proving Feigin's Conjecture.

Keywords: Fusion Products, Weyl Modules, Demazure Modules, Chari-Venkatesh Modules.

Contents

Introduction

Let $\mathfrak{g}$ be a complex Lie algebra. The current algebra $\mathfrak{g}[t]$ can be defined as the tensor product $\mathfrak{g} \otimes A$, where $A = \mathbb{C}[t]$ or, equivalently, as the algebra of polynomial maps $\mathbb{C} \to \mathfrak{g}$. One can easily check that $\mathfrak{g}[t]$ can be equiped with a Lie algebra structure. Moreover, both $\mathfrak{g}[t]$ and its universal enveloping algebra $U(\mathfrak{g}[t])$ inherit a grading that comes from the grading of $\mathbb{C}[t]$. The category of finite-dimensional graded representations of current algebras $\mathcal{G}$ has been intensely studied mainly because of its conection with the theory of representations of quantum groups. In addition, the homological properties of this category is one of its important aspects, because of the similarity with the BGG category $\mathcal{O}$ for the simple Lie algebra. One can also consider the truncated case, that is, the Lie algebra $\mathfrak{g}[t]_N := \mathfrak{g} \otimes (\mathbb{C}[t]/(t^N\mathbb{C}[t]))$. The pullback of the canonical projection of $\mathfrak{g}[t]$ onto $\mathfrak{g}[t]_N$ gives a way to define a $\mathfrak{g}[t]$-module structure in any $\mathfrak{g}[t]_N$-module, so that the category of graded finite-dimensional representations over $\mathfrak{g}[t]_N$ is, actualy, a subcategory of $\mathcal{G}$ denoted by $\mathcal{G}_N$.

The Weyl modules are certain objects in the category of representations of quantum affine and affine Lie algebras that were first defined by Chari and Pressley in [13]. The definition was motivated by a phenomenon from modular representation theory of algebraic groups where simple modules in characteristic zero give rise to non-simple modules in positive characteristic by a process of base field change. The modules thus obtained are known as Weyl modules. This same phenomenon occurs in the context of quantum affine algebras [13] for the limit $q = 1$ of simple modules. By now, the notion of Weyl modules has been intensively studied in a broader range of contexts such as for algebras of the form $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is either a symmetrizable Kac-Moody algebra or a super Lie algebra and A is an associative commutative unital algebra, for quantum affine algebras at roots of unity, and for hyper loop algebras [1, 2, 3, 5, 6, 9, 28, 29, 30, 36, 44].

In the category of finite-dimensional representations for the current algebra $\mathfrak{g}[t]$, the local Weyl modules are universal objects in the following sense. Given a triangular decomposition of the semi-simple Lie algebra $\mathfrak{g}$, say $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h}$ is the Cartan subalgebra, consider the induced triangular decomposition of $\mathfrak{g}[t]$ given by

$$\mathfrak{g}[t] = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t],$$

where $\mathfrak{n}^\pm[t] := \mathfrak{n}^\pm \otimes \mathbb{C}[t]$ and $\mathfrak{h}[t] := \mathfrak{h} \otimes \mathbb{C}[t]$. Now, given an integral dominant weight λ, one can consider the Verma module $M(\lambda)$ associated to this triangular decomposition of $\mathfrak{g}[t]$. It turns out that $M(\lambda)$ has some finite-dimensional quotients and the local Weyl modules are the largest ones. Thus, any finite-dimensional quotient of $M(\lambda)$ is also a quotient of one of the associated Weyl modules. Moreover, for each integral dominant weight λ, there exists a unique (up to isomorphism) graded Weyl module, usualy denoted by $W(\lambda)$. Since this work is focused on graded modules, we refer to $W(\lambda)$ as Weyl modules. The same construction can be made for truncated current algebras and their Weyl modules are called truncated Weyl modules, usually denoted by $W_N(\lambda)$. Demazure and Chari-Venkatesh modules are other examples of finite-dimensional graded quotients of $M(\lambda)$ which have been intensively studied (see, for instance, [16, 11, 27, 33, 45, 49]).

One interesting construction in the category of finite-dimensional graded modules for current algebras, which has received growing interest in the last years, is that of fusion products introduced by B. Feigin and S. Loktev in [21]. Given $a_1, \cdots, a_n \in \mathbb{C}$ and a family of cyclic objects $V_1, \cdots, V_n$ in $\mathcal{G}$ generated by $v_1, \cdots, v_n$, respectively, consider a twisting on the action of $\mathfrak{g}[t]$ on each V_i given by

$$(x \otimes f(t))v_i = (x \otimes f(t + a_i))v_i.$$

Denote each V_i with the twisted action as $V_i^{a_i}$. One easily checks that $V_i^{a_i}$ is not a graded $\mathfrak{g}[t]$-module and it is well-known that if the complex parameters a_i are pairwise distinct, then the tensor product $V_1^{a_1} \otimes \cdots \otimes V_n^{a_n}$ is also cyclic and generated by $v_1 \otimes \cdots \otimes v_n$. Thus, one can define a filtration on this tensor product and, then, the associated graded $\mathfrak{g}[t]$-module is called fusion product. It was conjectured in [21] that the fusion product is independent of the complex parameters, which motivates the notation $V_1 * \cdots * V_n$. Although many works have proved this conjecture to be true for specific cases (see [11, 15, 16, 20, 21, 27, 33, 42, 43, 49]), the general case remains an open problem. However, a very interesting approach has been taken by J. Flake, G. Fourier and V. Levandovskyy in [23] where they proved that the existence of a Grobner basis for a certain ideal of the universal enveloping algebra $U(\mathfrak{g}[t])$ implies the conjecture made by E. Feigin about the defining relations for the fusion product. In the same work, they gave a new proof to the $\mathfrak{sl}_2$ case, which might be easier to generalize to $\mathfrak{sl}_n$ than the current proof made in [20]. Their approach also leads to a proof for the conjecture on Schur positivity stated in [10], which can be rewritten as a conjecture about a realization for truncated Weyl modules as fusion products of irreducible objects in $\mathcal{G}$.

Fusion products are a usefull tool to study the objects in $\mathcal{G}$. For instance, it is known that for the integral dominant weight $\lambda = \lambda_1 + \cdots + \lambda_m$, then the local Weyl module $W(\lambda)$ can be writen as the following fusion product:

$$W(\lambda) = W(\lambda_1) * \cdots * W(\lambda_m).$$

This fact was proved in [27] for $\mathfrak{g}$ simply laced and then in [43] for the general case. This decomposition makes it easier to answer structural questions, such as dimension and character, by reducing the problem to study local Weyl modules associated to fundamental weights. This fact motivates the conjecture mentioned in the end of the last paragraph about truncated Weyl modules, which can be explained as follows. Consider the set

$$P^+(\lambda, N) = \{\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N) \in (P^+)^N \ : \ \lambda = \lambda_1 + \cdots + \lambda_N\}.$$

A partial order on $P^+(\lambda, N)$ was considered in [10] and the maximal elements can be computed following an algorithm described in [25].

Conjecture. *Let* $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N)$ *be a maximal element of* $P^+(\lambda, N)$*. If* $N \leqslant |\lambda|$*, then* $W_N(\lambda) \cong ev_0V(\lambda_1) * \cdots * ev_0V(\lambda_N)$*.*

The level-ℓ Demazure modules $D(\ell, \lambda)$ are well-understood objects in $\mathcal{G}$ that are related to many objects in $\mathcal{G}$. The characters of these modules were computed in [35] and [39] and then used in [11] to prove the isomorphism between Weyl modules and level-1 Demazure modules as representations for the affine Lie algebra of $\mathfrak{sl}_{n+1}$. Althought these modules are well-understood, the independence of parameters for the fusion products of Demazure modules remains an open problem. However, many works have given positive answers for fusion products of Demazure modules of same level (see [15, 16, 27, 43, 49]) and in [45] for $D(1, \lambda)^{*m} * ev_0V(\theta)^{*n}$. In particular, a positive answer for the above conjecture for $\mathfrak{g}$ simply laced follows as a corollary of the main result of [45]. It is also known, in the case $\mathfrak{g} = \mathfrak{sl}_2$, the necessary and sufficient conditions for a truncated Weyl module be isomorphic to a Demazure module. This fact relies on the theory of Chari-Venkatesh (CV) modules and results from [29] and [16].

The Chari-Venkatesh modules were first defined in [16]. These objects are graded quotients of local Weyl modules by certain submodules that are related to some families $\{\xi_\alpha\}_{\alpha \in R^+}$ of partitions. It was proved in [16] that given $\lambda \in P^+$, every level-ℓ Demazure module is isomorphic to the CV module related to the λ-compatible partition

$$\xi_{\ell,\lambda}(\alpha) = ((\ell d_\alpha)^{(s_\alpha - 1)}, m_\alpha),$$

where $d_\alpha = \dfrac{2}{(\alpha, \alpha)}$ and the numbers m_α and s_α appear in the definition of Demazure module as we explain later.

This work is focused on reviewing the results from [29] and [45]. The main results of the former are about truncated Weyl modules, namely Theorem 4.1.2 that proves that every truncated Weyl module is isomorphic to a certain CV module and Theorem 4.1.1 that gives a positive answer to the above conjecture in the case that λ is a multiple of a minuscle weight and $\mathfrak{g}$ is simply laced. Moreover, Proposition 4.3.1 ([29, Proposition 2.5.2]) takes a further step towards proving the conjecture without any hypothesis on λ or $\mathfrak{g}$. On the other hand, the main result of [45] is Theorem 5.2.3 ([45, Theorem 1]), that shows the existence of a short exact sequence of certain quotients of Demazure modules and also the existence of a realization for the fusion product of Demazure modules of different levels as one of these quotients. The main consequences of this results are Theorem 5.3.2 and

Corollary 5.2.3, where the first generalizes [16, Theorem 5, items (i) and (ii)][1] and the second shows the existence of the isomorphism

$$W_N(k\theta) \cong \begin{cases} W(\theta)^{*N-k} * ev_0V(\theta)^{2k-N}, & k \leqslant N < 2k \\ W(k\theta), & N \geqslant 2k \end{cases}$$

when $\mathfrak{g}$ is simply laced, $k \geqslant 0$ and $|k\theta| \leqslant N \leqslant \lambda(h_\nu)$. Clearly, for[2] $k = N$, this corollary gives a positive answer for the conjecture on truncated Weyl modules above.

The text is organized as follows. In Chapter 1, we give a brief review of simple Lie algebras, the category $\mathcal{O}$, and current algebras. In Chapter 2, we study the Weyl modules, truncated Weyl modules, Chari-Venkatesh modules, and Demazure modules. In Chapter 3, we define the fusion products, review the main results for these modules and the conjecture about truncated Weyl modules. In Chapter 4, we review [29] and define the Kirillov-Reshetikhin modules. In Chapter 5, we review [45]. Next, in Chapter 6, we specialize the discussion to $\mathfrak{sl}_2$ to study the Demazure flags and chains of inclusions of truncated Weyl modules. Finally, in Chapter 7, we give an example of the direction we can pursue in the future, that is, we review the necessary background on Grobner basis and also state the main results of [23].

1 [16, Theorem 5] apears as Theorem 3.2.10 and is strongly used to prove Theorem 4.1.2.

2 Which means $N = |k\theta|$.

1 Preliminaries

In this chapter, we review the fundamental definitions and results involving Lie algebras, categories, representations and modules over a Lie algebra. All the tensor products and vector spaces are over the complex field and we omit the proofs, since they can be easily found in [7, 32, 19].

1.1 Lie algebras

Definition 1.1.1. *A Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a bilinear map*

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (x,y) &\longmapsto [x,y], \end{aligned}$$

called Lie bracket, which satisfies

(i) $[x,x] = 0, \quad \textit{for all} \quad x \in \mathfrak{g}$

(ii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0, \quad \textit{for all} \quad x,y,z \in \mathfrak{g}.$

It is important to observe that (i) can be used to show that the Lie bracket is antisymmetric, while (ii) is called Jacobi identity. From now on, we set $\mathbb{K} = \mathbb{C}$.

Definition 1.1.2. *Let $\mathfrak{g}$ be a Lie algebra. A Lie subalgebra of $\mathfrak{g}$ is a subvector space $\mathfrak{h}$ of $\mathfrak{g}$ such that $[x,y] \in \mathfrak{h}$, for all $x,y \in \mathfrak{h}$.*

Example 1.1.3. *The algebra of $n \times n$ matrices over $\mathbb{C}$ can be equipped with a Lie algebra structure. In this case, we denote $\mathfrak{gl}_n$ and the Lie bracket is given by*

$$[A,B] = AB - BA, \quad \textit{for all} \quad A,B \in \mathfrak{gl}_n.$$

A Lie bracket defined this way is called commutator. In addition, let

$$\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \, : \, tr(A) = 0\}.$$

Then $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$.

Example 1.1.4. *The last example can be generalized to an arbitrary associative algebra with the Lie bracket as the commutator. If $A = End_{\mathbb{C}}(V)$, then A as a Lie algebra is denoted by $\mathfrak{gl}(V)$ and called general Lie algebra.*

Definition 1.1.5. *Let I be a subspace of a Lie algebra $\mathfrak{g}$. I is called ideal of $\mathfrak{g}$ if*

$$[x, y] \in I \quad \textit{for all} \quad x \in I,\ y \in \mathfrak{g}.$$

Observe that any ideal of $\mathfrak{g}$ is bilateral, since $[x, y] = -[y, x]$.

The immediate examples of ideals are $\{0\}$ and $\mathfrak{g}$. Besides that, the following plays an important role.

Example 1.1.6. *The center of a Lie algebra $\mathfrak{g}$ is an ideal given by*

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \,:\, [x, y] = 0 \textit{ for all } y \in \mathfrak{g}\}.$$

If $Z(\mathfrak{g}) = \mathfrak{g}$, then $\mathfrak{g}$ is called abelian Lie algebra.

Let $\mathfrak{g}/I$ be the quotient space. If I is an ideal of $\mathfrak{g}$, then $\mathfrak{g}/I$ can be regarded as a Lie algebra with the Lie bracket being

$$[x + I, y + I] = [x, y] + I, \quad \text{for all} \quad x, y \in \mathfrak{g}.$$

Definition 1.1.7. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a Lie algebra homomorphism if it preserves Lie brackets, i.e. for all $x, y \in \mathfrak{g}_1$ we have*

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

Example 1.1.8. *Given a Lie algebra $\mathfrak{g}$, the mapping*

$$\begin{array}{rcl} ad : \mathfrak{g} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}) \\ x & \longmapsto & ad(x) \end{array},$$

such that $ad(x)(y) = [x, y]$ for $x, y \in \mathfrak{g}$, is called adjoint homomorphism

Example 1.1.9. *Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. Then $\ker \phi$ is an ideal of $\mathfrak{g}_1$ and there exists a Lie algebra homomorphism*

$$\mathfrak{g}_1 \twoheadrightarrow \mathfrak{g}_1 / \ker \phi$$

mapping x onto $x + \ker \phi$ for all $x \in \mathfrak{g}_1$.

Theorem 1.1.10. *(i) Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. Then $Im\phi$ is a Lie subalgebra of $\mathfrak{g}_2$ and there exists the isomorphism*

$$\mathfrak{g}_1 / \ker \phi \cong Im\phi;$$

(ii) If I and J are ideals of a Lie algebra $\mathfrak{g}$, then

$$\frac{(I + J)}{J} \cong \frac{I}{(I \cap J)};$$

(iii) If the ideals I and J are such that $I \subseteq J$, then J/I is an ideal and

$$\frac{(\mathfrak{g}/I)}{(J/I)} \cong \frac{\mathfrak{g}}{J}.$$

As usual, a Lie algebra homomorphism ϕ is called monomorphism (epimorphism) if it is injective (surjective) and is called isomorphism if it is bijective. Futhermore, if $\phi : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism, it is called automorphism.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be subspaces of a Lie algebra $\mathfrak{g}$. The subspace spanned by

$$\{[x, y] \,:\, x \in \mathfrak{g}_1,\ y \in \mathfrak{g}_2\}$$

is denoted by $[\mathfrak{g}_1, \mathfrak{g}_2]$. Now, for all $n \geqslant 1$ and $k \geqslant 0$, define

$$\mathfrak{g}^1 := \mathfrak{g}, \quad \mathfrak{g}^n := [\mathfrak{g}^{n-1}, \mathfrak{g}]$$

and

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}].$$

This way we define the lower central series and the derived series of $\mathfrak{g}$, respectively, by

$$\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots$$

and

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots .$$

Definition 1.1.11. *A Lie algebra $\mathfrak{g}$ is said to be nilpotent (solvable) if its lower central series (derived series) stabilizes at $\{0\}$ for some $n \geqslant 1$ ($k \geqslant 0$).*

One easily checks that every nilpotent Lie algebra is solvable. It is well-known that every finite-dimensional Lie algebra has a unique solvable ideal containing every solvable ideal of $\mathfrak{g}$. This ideal is called **radical** of $\mathfrak{g}$ and is denoted by $rad(\mathfrak{g})$. In addition, $\mathfrak{g}/rad(\mathfrak{g})$ does not contain any non-trivial solvable ideal. From now on, we consider all the Lie algebras being finite-dimensional.

Definition 1.1.12. *(i) A Lie algebra $\mathfrak{g}$ is said to be semisimple if $rad(\mathfrak{g}) = 0$;*

(ii) It is said to be simple if its only ideals are $\{0\}$ and $\mathfrak{g}$.

Note that if $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is also semisimple.

Definition 1.1.13. *(i) Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The set*

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \,:\, [h, x] \in \mathfrak{h} \text{ for all } h \in \mathfrak{h}\}$$

is called normalizer of $\mathfrak{h}$.

(ii) *A nilpotent Lie subalgebra* $\mathfrak{h}$ *is called Cartan subalgebra of* $\mathfrak{g}$ *if*

$$N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}.$$

Note that neither the existence nor uniqueness of a Cartan subalgebra is guaranteed by its definition. However, since our ground field is the complex numbers and $\mathfrak{g}$ is finite-dimensional, the Cartan subalgebras will aways exist. In addition, we have the following.

Theorem 1.1.14. *If* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *are two Cartan subalgebras of a Lie algebra* $\mathfrak{g}$*, then there exists an automorphism* ϕ *such that* $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$*.*

Definition 1.1.15. *Let* $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ *be a bilinear form given by*

$$\langle x, y \rangle := tr(ad(x)ad(y)).$$

Then $\langle \cdot, \cdot \rangle$ *is called Killing form of* $\mathfrak{g}$*.*

The Killing is symmetric, since the trace function is commutative, and is associative in the following sense

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle.$$

The Killing form is **non-degenerate** when $\langle x, y \rangle$=0 for all $y \in \mathfrak{g}$ implies $x = 0$. The next result is often called Cartan's criterions

Theorem 1.1.16. *(First and second Cartan's Criterions)*

(i) *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $\langle x, y \rangle = 0$ *for all* $x \in \mathfrak{g}$ *and* $y \in \mathfrak{g}^2$*.*

(ii) $\mathfrak{g}$ *is semisimple if and only if its Killing form is non-denerate.*

Another equivalence for semisimplicity is the following.

Theorem 1.1.17. *A Lie algebra is semisimple if and only if it is isomorphic to a direct sum of simple Lie algebras.*

1.2 Representations and irreducible objects

Now we recall the definition of an important Lie algebra homomorphism called representation.

Definition 1.2.1. *A representation of* $\mathfrak{g}$ *is a Lie algebra homomorphism*

$$\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

where V *is a vector space. Moreover, the dimension of* ϕ *is defined as* $\dim(V)$*.*

Representations provide a way to examine a Lie algebra as a subalgebra of the endomorphisms of a vector space. From now on, let us consider finite-dimensional representations.

Example 1.2.2. *The adjoint homomorphism* $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a representation of* $\mathfrak{g}$ *called adjoint representation. For* $\mathfrak{g}$ *simple we have* $\ker(ad) = 0$*, which is equivalent to say that the adjoint representation is injective. Therefore, every simple finite-dimensional Lie algebra is isomorphic to a subalgebra of* $\mathfrak{gl}(\mathfrak{g})$*.*

The last statement of Example 1.2.2 can be generalized due to Ado's theorem, which states that every finite-dimensional Lie algebra over a field of characteristic zero can be seen as a subalgebra of $\mathfrak{gl}(V)$ for some vector space V.

In light of Theorem 1.1.17, the classification of semisimple Lie algebras is reduced to classify simple Lie algebras, that is, subalgebras of $\mathfrak{gl}(V)$ by the last example. The first step is the following theorem due to Engel.

Theorem 1.2.3. *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if, for all* $x \in \mathfrak{g}$*,* $ad(x)$ *is a nilpotent linear map.*

As a corollary of Theorem 1.2.3, a Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ being nilpotent implies that there exists a basis of V such that all elements of $\mathfrak{g}$ are represented by strictly upper triangular matrices. Now, one can ask the sufficient conditions for each element of $\mathfrak{g}$ be represented as an upper diagonal matrix.

Theorem 1.2.4. *(Lie's Theorem) Let* V *be a finite-dimensional vector space. If* $\mathfrak{g}$ *is a solvable Lie subalgebra of* $\mathfrak{gl}(V)$*, then there exists a basis of* V *such that the elements of* $\mathfrak{g}$ *are all represented as upper triangular matrices.*

Given a representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, the next step is decomposing V into a direct sum of centain subspaces, but first we introduce some concepts related to modules over $\mathfrak{g}$.

Definition 1.2.5. *A module over* $\mathfrak{g}$ *(also called* $\mathfrak{g}$*-module) is a vector space* V *in which is possible to define an action of* $\mathfrak{g}$ *given by*

$$\begin{aligned} \mathfrak{g} \times V &\longrightarrow V \\ (x, v) &\longmapsto xv \end{aligned}$$

satisfying that, for all $x, y \in \mathfrak{g}$*,* $v, u \in V$ *and* $a \in \mathbb{C}$*,*

(i) $(x + ay)v = xv + a(yv)$*;*

(ii) $x(v + au) = xv + a(xu)$*;*

(iii) $[x, y]v = x(yv) - y(xv)$.

Definition 1.2.6. *A subspace W of a $\mathfrak{g}$-module V is called $\mathfrak{g}$-submodule if $xw \in W$ for all $w \in W$ and $x \in \mathfrak{g}$.*

Example 1.2.7. *Let V_1 and V_2 be modules over a Lie algebra $\mathfrak{g}$.*

1. *The direct sum $V_1 \oplus V_2$ is a $\mathfrak{g}$-module where the action of $\mathfrak{g}$ is given by*

$$x(v_1 + v_2) = xv_1 + xv_2 \quad \text{for all} \quad x \in \mathfrak{g} \text{ and } v_i \in V_i, \text{ with } i = 1, 2;$$

2. *The tensor product $V_1 \otimes V_2$ is a $\mathfrak{g}$-module where the action of any $x \in \mathfrak{g}$ in a homogeneous tensor $v_1 \otimes v_2$ is given by*

$$x(v_1 \otimes v_2) = (xv_1) \otimes v_2 + v_1 \otimes (xv_2). \tag{1.2.1}$$

3. *Given a $\mathfrak{g}$-module V, the trivial subspace $\{0\}$ and V are $\mathfrak{g}$-submodules of V. A $\mathfrak{g}$-module such that the only submodules are $\{0\}$ and V is called irreducible.*

Remark 1.2.8. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. One can define an action of $\mathfrak{g}$ on the vector space V by $xv := \rho(x)v$. On the other hand, if V is a $\mathfrak{g}$-module, then the Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\rho(x)v := xv$ defines a representation of $\mathfrak{g}$. One can easily check that this defines an equivalence between the categories of $\mathfrak{g}$-modules and representations of $\mathfrak{g}$.*

Definition 1.2.9. (i) *A $\mathfrak{g}$-module V is called indecomposable if there are no two non-trivial submodules V_1 and V_2 such that $V = V_1 \oplus V_2$.*

(ii) *V is called completely-reducible if it is a direct sum of irreducible submodules.*

For each $\lambda \in \mathfrak{g}^*$ and a $\mathfrak{g}$-module V, define the subspace

$$V_\lambda := \{v \in V \,:\, \text{for each } x \in \mathfrak{g}, \exists n \geqslant 1 \text{ s.t. } (x - \lambda(x))^n v = 0\}.$$

Theorem 1.2.10. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra. If V is a finite-dimension $\mathfrak{g}$-module, then V_λ is a $\mathfrak{g}$-submodule of V and*

$$V = \bigoplus_{\lambda \in \mathfrak{g}^*} V_\lambda.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of a Lie algebra $\mathfrak{g}$. One can consider $\mathfrak{g}$ as a $\mathfrak{h}$-module with the action of $\mathfrak{h}$ being the adjoint representation. The last theorem implies that

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda. \tag{1.2.2}$$

Moreover, it is well-known that $\mathfrak{g}_0 = \mathfrak{h}$. The elements $\lambda \in \mathfrak{h}^*\backslash\{0\}$ and $\mathfrak{g}_\lambda \neq 0$ are called roots, the set of all roots is denoted by Φ and $\mathfrak{g}_\lambda$ is called root space if λ is a root. When $\mathfrak{g}$ is semi-simple, the decomposition (1.2.2) has many properties as recorded in the following theorem.

Theorem 1.2.11. *Let $\mathfrak{g}$ be semi-simple. Then the following properties hold:*

(i) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$, *which implies*

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \begin{cases} \subset \mathfrak{g}_{\lambda+\mu}, & \text{if } \lambda+\mu \in \Phi \\ \subset \mathfrak{h}, & \text{if } \lambda = -\mu \\ = 0, & \text{if } \lambda+\mu \neq 0 \text{ and } \lambda+\mu \notin \Phi. \end{cases}$$

(ii) *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is non-degenerated;*

(iii) *$\mathfrak{h}$ is abelian;*

(iv) $\dim(\mathfrak{g}_\lambda) = 1$ *for all $\lambda \in \Phi$;*

(v) *If $\lambda \in \Phi$, then the only scalar multiples of λ in Φ are $\pm\lambda$;*

(vi) *If $\lambda \neq -\mu$, then the root space $\mathfrak{g}_\lambda$ is orthogonal to $\mathfrak{g}_\mu$ with respect to the Killing form;*

(vii) *Φ generates $\mathfrak{h}^*$;*

(viii) *Since $\mathfrak{h}$ is not semi-simple, the Killing form of $\mathfrak{h}$ is degenerated.*

Note that the second item implies that there exists a Lie algebra isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ given by the assignment $h \mapsto h^*$ such that $h^*(x) = \langle h, x\rangle$ for all $x \in \mathfrak{h}$.

Fix a root sistem Φ of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. Now, for each $\lambda \in \Phi$, define the element t_λ satisfying

$$\lambda(x) = \langle t_\lambda, x\rangle \quad \text{for all} \quad x \in \mathfrak{h}.$$

One easily checks that the elements t_λ generates $\mathfrak{h}$ and hence it is possible to define a bilinear form in Φ given by

$$(\lambda, \mu) := \langle t_\lambda, t_\mu\rangle = \lambda(t_\mu) = \mu(t_\lambda), \quad \text{for all} \quad \lambda, \mu \in \Phi.$$

Now, for every $\lambda \in \Phi$, one can check that $(\lambda, \lambda) = \langle t_\lambda, t_\lambda\rangle > 0$ and that given $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_{-\lambda}$, we have $[x, y] = (x, y)t_\lambda$. It follows that the element $h_\lambda := \dfrac{2t_\lambda}{(\lambda, \lambda)}$ can be defined. Moreover, given $x_\lambda^+ \in \mathfrak{g}_\lambda$, there exists an unique $x_\lambda^- \in \mathfrak{g}_{-\lambda}$ such that $[x_\lambda^+, x_\lambda^-] = h_\lambda$. Therefore the subalgebra generated by $\{x_\lambda^-, h_\lambda, x_\lambda^+\}$ is denoted by $\mathfrak{sl}_\lambda$. Futhermore, the assignment

$$x_\lambda^- \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h_\lambda \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad x_\lambda^+ \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

defines the isomorphism $\mathfrak{sl}_\lambda \cong \mathfrak{sl}_2$.

Choose a basis $\{\alpha_1 \cdots, \alpha_l\}$ of $\mathfrak{h}$ such that $\lambda_i \in \Phi$ for all $i = 1, \cdots, l$. Thus any $\beta \in \Phi$ can be uniquely writen as

$$\beta = \sum_{i=1}^{l} c_i \alpha_i.$$

One can check that $c_i \in \mathbb{Q}$ for all $i = 1, \cdots, l$ and hence the $\mathbb{Q}$-vector subspace of $\mathfrak{h}$ spanned by $t_{\alpha_1}, \cdots, t_{\alpha_l}$, say $\mathfrak{h}_\mathbb{Q}$ has dimension over $\mathbb{Q}$ equals $\dim(\mathfrak{h})$. Moreover, the form $(\cdot, \cdot)$ over $\mathfrak{h}_\mathbb{Q}$ is positive definite and if we extend the base field from $\mathbb{Q}$ to $\mathbb{R}$, then the form can be extended canonically to $\mathfrak{h}_\mathbb{R} := \mathbb{R} \otimes_\mathbb{Q} \mathfrak{h}_\mathbb{Q}$. In addition, the form over $\mathfrak{h}_\mathbb{R}$ is also positive definite, that is, $\mathfrak{h}_\mathbb{R}$ is an Euclidean space.

Theorem 1.2.12. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ. Then*

(i) Φ spans $\mathfrak{h}_\mathbb{R}$ and $0 \notin \Phi$;

(ii) Define $\sigma_\alpha(\beta) := \beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$. If $\alpha, \beta \in \Phi$, then $\sigma_\alpha(\beta) \in \Phi$.;

(iii) If $\alpha, \beta \in \Phi$, then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

(iv) Let $\mathfrak{h}^*_\mathbb{R}$ be the image of $\mathfrak{h}_\mathbb{R}$ by the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. Define a bilinear form in $\mathfrak{h}^*_\mathbb{R}$ by

$$(h_1^*, h_2^*) := \langle h_1, h_2 \rangle \in \mathbb{R}, \quad \text{for all} \quad h_1, h_2 \in \mathfrak{h}_\mathbb{R}.$$

Then $(\cdot, \cdot)$ is positive definite, that is, $\mathfrak{h}^*_\mathbb{R}$ is an Euclidean space.

1.3 Abstract root systems

Let V be an euclidean space and $(\cdot, \cdot)$ its inner product. Each $\alpha \in V \backslash \{0\}$ defines a reflection σ_α with respect to the orthogonal hyperplane $P_\alpha := \{\beta \in V \,:\, (\beta, \alpha) = 0\}$. Since $\sigma_\alpha(\alpha) = -\alpha$ and $\sigma_\alpha(\beta) = \beta$, for all $\beta \in P_\alpha$, it follows that

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

The number $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$ is denoted by $\langle \beta, \alpha \rangle$

Definition 1.3.1. *A subset Φ of the euclidean space V is a root system in V if*

(i) Φ is finite, generates V, and $0 \notin \Phi$;

(ii) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$;

(iii) If $\alpha \in \Phi$, then σ_α leaves Φ invariant;

(iv) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Let Φ be a root system of V. The subgroup $\mathcal{W}$ of $GL(V)$ generated by elements σ_α, $\alpha \in \Phi$, is called **Weyl group** of Φ. $\mathcal{W}$ is finite and permutes the set Φ. For this reason, the Weyl group of Φ is a subgroup of the symmetric group on Φ.

Definition 1.3.2. *A subset Δ of Φ is called a base if Δ is a basis of V and every $\beta \in \Phi$ is such that*

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha, \text{ where either } c_\alpha \in \mathbb{Z}_{\geqslant 0},\ \forall \alpha \in \Delta, \text{ or } c_\alpha \in \mathbb{Z}_{\leqslant 0},\ \forall \alpha \in \Delta.$$

The elements in Δ are called simple roots.x

Note that if $c_\alpha \geqslant 0$ ($c_\alpha \leqslant 0$), for all $\alpha \in \Delta$, then β is called positive (negative). The set of positive roots is denoted by Φ^+ and the set of negative roots is denoted Φ^-. In addition, $\Phi^+ \cap \Phi^- = \varnothing$, $\Phi^+ = -\Phi^-$ and $\Phi = \Phi^+ \cup \Phi^-$.

Theorem 1.3.3. *Every root system Φ has a base.*

Definition 1.3.4. *(i) Given a root $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$, the height $ht\beta$ is the number*

$$ht\beta := \sum_{\alpha \in \Delta} c_\alpha;$$

(ii) The rank of Φ is the cardinality of Δ;

(iii) Let $w \in \mathcal{W}$. Denote by $l(w)$ the length of w, which is the smallest number such that w is writen as product of simple reflections and $l(1) = 0$;

(iv) Define $n(\sigma)$, $\sigma \in \mathcal{W}$, as the number of roots $\alpha \in \Phi^+$ such that $\sigma(\alpha) \in \Phi^-$.

Theorem 1.3.5. *Let Δ be a basis of a root system Φ.*

(i) If Δ' is another basis of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathcal{W}$;

(ii) For every root α there exists some $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$;

(iii) $\mathcal{W}$ is generated by $\{\sigma_\alpha\}_{\alpha \in \Delta}$;

(iv) If $\sigma(\Delta) = \Delta$ with $\sigma \in \mathcal{W}$, then $\sigma = 1$.

Proposition 1.3.6. *Let $\mathcal{W}$ be the Weyl group of a root system Φ. Then*

(i) For all $\sigma \in \mathcal{W}$, $l(\sigma) = n(\sigma)$;

(ii) There exists a unique $\sigma_0 \in \mathcal{W}$ such that $l(\sigma_0) = |\Phi^+|$. Moreover, $\sigma_0(\Phi^+) = \Phi^-$ and $\sigma_0^2 = 1$;

(iii) For all $\sigma \in \mathcal{W}$, $l(\sigma) \leqslant |\Phi^+|$.

1.4 The Cartan Matrix

Fix a root system Φ with rank n, a Weyl group $\mathcal{W}$, and a basis Δ for Φ. For an ordered basis of V of simple roots, say $\{\alpha_1, \cdots, \alpha_n\}$, the matrix $(c_{ij})_{n\times n}$ such that $c_{ij} = \langle \alpha_i, \alpha_j \rangle$ is called **Cartan matrix** of Φ.

Proposition 1.4.1. *The Cartan matrix $C = (c_{ij})_{n\times n}$ satisfies*

(i) $c_{ii} = 2$ *with* $i = 1, \cdots, n$*;*

(ii) $c_{ij} \in \{0, -1, -2, -3\}$ *for all* $i, j \in \{1, \cdots, n\}$ *and* $i \neq j$*;*

(iii) If c_{ij} *is equal -2 or -3, then* $c_{ji} = -1$*. Moreover,* $c_{ij} = 0$ *if and only if* $c_{ji} = 0$*.*

Although the Cartan matrix depends on the chosen ordering on the basis, this matrix does not depend on the choice of the basis, i.e. the Cartan matrix is unique up to reordering. The simple Lie algebras are in a one-to-one correspondence with the Cartan matrices, which, in turn, can be identified with certain graphs called Dynkin diagrams. Those diagrams can be explained as follows: The vertices are labelled by the simple roots of the root system. Between two vertices α and β, there are $d_{\alpha\beta}$ lines, where

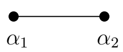
$$d_{\alpha\beta} = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

Whenever $d_{\alpha\beta} > 1$, we add an arrow pointing from the longer to the the shorter root.

Example 1.4.2. *1. Consiter the Cartan matrix given by*

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

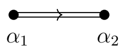
The Dynkin diagram of this matrix is said to be of type A_2 and is given by



2. Consiter the Cartan matrix given by

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

The Dynkin diagram of this matrix is said to be of type B_2 and is given by



Definition 1.4.3. *The root system Φ is irreducible if its Dynkin diagram is connected. Otherwise, Φ is reducible.*

Definition 1.4.4. *Let Φ and Φ' be root systems in the Euclidean spaces V and V', respectively. We say that Φ and Φ' are isomorphic if there is a vector space isomorphism $\varphi : V \to V'$ such that*

(i) $\varphi(\Phi) = \Phi'$;

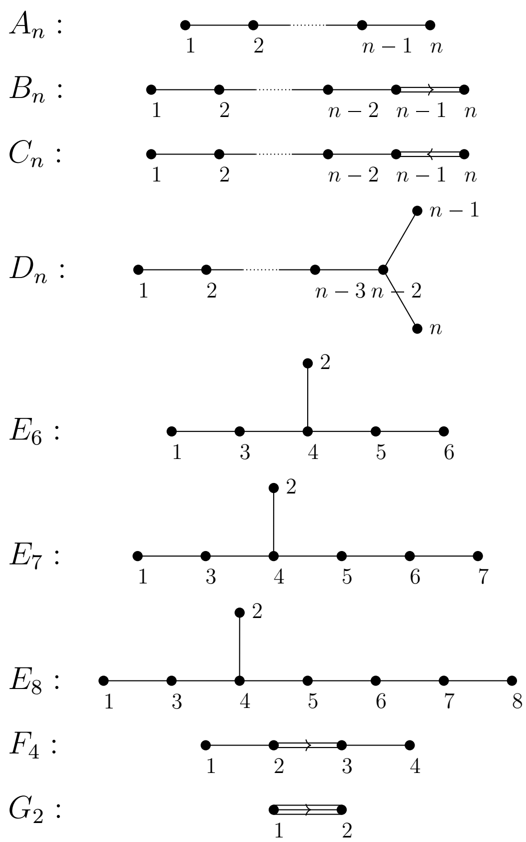
(ii) For any two roots $\alpha, \beta \in \Phi$, $(\alpha, \beta) = (\varphi(\alpha), \varphi(\beta))$.

Proposition 1.4.5. *If Φ is a root system of an Euclidean space V, then there exists pairwise orthogonal subspaces $V_1, \cdots, V_r$ with irreducible root systems $\Phi_1, \cdots, \Phi_r$ such that Φ can be uniquely decomposed as*

$$\Phi = \bigcup_{i=1}^{r} \Phi_i.$$

In addition, $V = V_1 \oplus \cdots \oplus V_r$

Theorem 1.4.6. *If Φ is an irreducible root system of rank n, its Dynkin diagram must be one of the following.*



It is finally possible to classify all root systems.

Theorem 1.4.7. *There exists an one-to-one correspondence (up to isomorphism) between the Dynkin diagrams of Theorem 1.4.6 and irreducible root systems.*

Theorem 1.4.8. *If two root systems are isomorphic, then their Dynkin diagrams are the same.*

1.5 Classification of semi-simple Lie algebras

Recall that the subalgebra $\mathfrak{h}_{\mathbb{R}}$ is an Euclidean space. Observe that the root system of $\mathfrak{h}^*_{\mathbb{R}}$ is also a root system in the sense of Definition 1.3.1. Moreover, one can check that if $\mathfrak{h}'$ is another Cartan subalgebra of $\mathfrak{g}$, then the root systems of $\mathfrak{h}^*_{\mathbb{R}}$ and $\mathfrak{h}'^*_{\mathbb{R}}$ are isomorphic. Hence, one can define the following.

Definition 1.5.1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A root system for $\mathfrak{g}$ is a root system for the vector space $\mathfrak{h}^*_{\mathbb{R}}$. Furthermore, the Dynkin diagram of $\mathfrak{g}$ is the Dynkin diagram of a root system of $\mathfrak{h}^*_{\mathbb{R}}$.*

Proposition 1.5.2. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ. If $\mathfrak{g}$ is simple, then Φ is irreducible.*

Theorem 1.5.3. *Let $\mathfrak{g}$ be a semi-simple Lie algebra.*

(i) The Dynkin diagram of a semisimple Lie algebra $\mathfrak{g}$ is connected if and only if $\mathfrak{g}$ is simple. Moreover, any two Lie algebras with the same Dynkin diagrams are isomorphic.

(ii) $\mathfrak{g}$ is the direct sum of simple Lie algebras $\mathfrak{g}_1, \cdots, \mathfrak{g}_r$ if and only if the Dynkin diagrams of $\mathfrak{g}$ is union of the diagrams of $\mathfrak{g}_i$ for all $i = 1, \cdots, r$.

To complete the classification of simple Lie algebras, it remains to prove that every irreducible root system arises as the root system of some simple Lie algebra. This is a result due to Serre. We shall review the concept of generators and relations before stating Serre's Theorem.

Given a complex vector space V, consider the following vector spaces

$$T^i(V) := V \otimes V \otimes \cdots \otimes V \quad (i \text{ times}).$$

Definition 1.5.4. *The tensor algebra $T(V)$ is the vector space*

$$T(V) := \bigoplus_{i=0}^{\infty} T^i(V)$$

with the associative product

$$xy := (x_1 \otimes \cdots \otimes x_r)(y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_m \in T^{r+m}(V),$$

for all homogeneous tensors $x \in T^r(V)$ and $y \in T^m(V)$ and $r, m \in \mathbb{Z}_{\geqslant 0}$.

Now, let $\mathfrak{g}$ be a Lie algebra and J be the ideal of $T(\mathfrak{g})$ generated by

$$xy - yx - [x, y], \quad \text{for all} \quad x, y \in \mathfrak{g}.$$

Definition 1.5.5. *The universal enveloping algebra of* $\mathfrak{g}$ *is the quotient*

$$U(\mathfrak{g}) := T(\mathfrak{g})/J.$$

Note that $T^1(\mathfrak{g}) = \mathfrak{g}$, so that we have an inclusion $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$. In addition, given π_J the canonical projection of $T(\mathfrak{g})$ onto $U(\mathfrak{g})$, there exists a Lie algebra homomorphism $i : \mathfrak{g} \to U(\mathfrak{g})$ given by the composition of the projection π_J, the inclusion of $T^1(\mathfrak{g})$, and the identification $\mathfrak{g} \cong T^1(\mathfrak{g})$.

Proposition 1.5.6. *The universal enveloping algebra* $U(\mathfrak{g})$ *of any Lie algebra* $\mathfrak{g}$ *satisfies the following universal property. If there exists another pair* (A, ϕ) *such that* A *is an associative algebra with unity,* 1_A*, regarded as a Lie algebra with the commutator and* $\phi : \mathfrak{g} \to A$ *is a Lie algebra homomorphism, then there exists a unique homomorphism of associative algebras* $\varphi : U(\mathfrak{g}) \to A$ *(sending* $1_{U(\mathfrak{g})}$ *to* 1_A*) such that the following diagram commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[d, "\phi"'] & U(\mathfrak{g}) \arrow[dl, dashed, "\exists!\varphi"] \\ A & \end{tikzcd}$$

Theorem 1.5.7. *(Poincaré-Birkhoff-Witt) Let* $\mathfrak{g}$ *be a Lie algebra and* J *be a set of indexes. If* $\{x_i : i \in J\}$ *is an ordered basis of* $\mathfrak{g}$*, then* $1_{U(\mathfrak{g})}$ *together with the elements*

$$x_{i_1}^{\ell_1} x_{i_2}^{\ell_2} \cdots x_{i_r}^{\ell_r},$$

with $r \in \mathbb{Z}_{>0}$*,* $i_1 < i_2 < \cdots < i_r$*, and* $\ell_i \in \mathbb{Z}_{>0}$ *for* $i = 1, \cdots, r$ *form a basis for* $U(\mathfrak{g})$*.*

This theorem is known as Poincaré-Birkhoff-Witt (PBW) Theorem.

Corollary 1.5.8. *1. The Lie algebra homomorphism* $\mathfrak{g} \to U(\mathfrak{g})$ *is injective, that is,* $\mathfrak{g}$ *is identified with a subalgebra of* $U(\mathfrak{g})$*. In particular, if* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$*, then* $U(\mathfrak{h})$ *is a subalgebra of* $U(\mathfrak{g})$*.*

2. Let $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be two Lie algebras with universal enveloping algebras* $U(\mathfrak{g}_1)$ *and* $U(\mathfrak{g}_2)$*, repectively. Then*

$$U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2).$$

3. The category of left modules over a Lie algebra $\mathfrak{g}$ *is equivalent to the category of left modules over* $U(\mathfrak{g})$*.*

The last ingredient to prove Serre's Theorem is the free Lie algebras. These algebras are defined by the following universal property.

Definition 1.5.9. *Let* $FL(X)$ *be a Lie algebra containing a set* X*.* $FL(X)$ *is said to be free over* X *if, given any mapping* $\phi : X \to \mathfrak{g}$*, there exists a unique Lie algebra homomorphism* $\varphi : FL(X) \to \mathfrak{g}$ *extending* ϕ*.*

The existence of the free Lie algebra can be explained as follows. Let V be a complex vector space with basis X and $T(V)$ be its tensor algebra regarded as a Lie algebra. Consider the Lie subalgebra $\mathfrak{g}$ of $T(V)$ generated by X. Given any function

$$\phi : X \to L,$$

extend ϕ to a linear map $V \to L$ and then to a Lie algebra homomorphism $\Psi : T(V) \to U(L)$. The restriction $\psi = \Psi|_{\mathfrak{g}} : \mathfrak{g} \to L$ is the desired extension of ϕ. One can easily check the uniqueness of ψ.

Definition 1.5.10. *Let $X = \{x_i : i \in I\}$. If $FL(X)$ is a free Lie algebra on X and $R = \{f_j : j \in J\} \subset FL(X)$ generates an ideal $\mathcal{R}$, then the Lie algebra given by generators x_i, $i \in I$, and relations $f_j = 0$, $j \in J$, is the quotient*

$$FL(X, R) := FL(X)/\mathcal{R}.$$

Proposition 1.5.11. *The universal enveloping algebra of $FL(X)$ is isomorphic to the tensor algebra $T(V)$ where V is a vector space having X as a basis.*

Proposition 1.5.12. *Under the conditions of Definition 1.5.10, if $R' \subset R$, then $FL(X, R)$ is a quotient of $FL(X, R')$.*

Finally, the next theorem, due to Serre, classifies all the simple Lie algebras.

Theorem 1.5.13. *(Serre) Let Φ be a root system with basis $\Delta = \alpha_1, \cdots, \alpha_n$ and $C = (c_{ij})_{n\times n}$ be its Cartan matrix. Consider the complex Lie algebra $\mathfrak{g}$ generated by elements $x_i^{\pm}, h_i$, $i = 1, \cdots, n$ with relations*

(i) $[h_i, h_j] = 0, \quad$ *for all* $\quad i, j = 1, \cdots, n;$

(ii) $[x_i^+, x_j^-] = \delta_{ij} h_i \quad$ *for all* $\quad i, j = 1, \cdots, n;$

(iii) $[h_i, x_j^{\pm}] = \pm c_{ij} x_j^{\pm} \quad$ *for all* $\quad i, j = 1, \cdots, n;$

(iv) $(ad(x_i^{\pm}))^{1-c_{ij}}(x_j^{\pm}) = 0 \quad$ *for all* $\quad i = 1, \cdots, n,\ i \neq j.$

Then $\mathfrak{g}$ has finite-dimension, is semi-simple and its Cartan subalgebra $\mathfrak{h}$ is spanned by $\{h_j : j = 1, \cdots, n\}$ with root system isomorphic to Φ.

Remark 1.5.14. *Theorem 1.4.6 describes all the possible root systems for simple Lie algebras. Moreover, a Lie algebra $\mathfrak{g}$ is said to be of type $X \in \{A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\}$ if its Dynkin diagram is of type X.*

1.6 The category $\mathcal{O}$

From now on, the set of roots of a Lie algebra is denoted by R (and the positive roots by R^+). We now recall some properties of the category of integrable representations over a semi-simple Lie algebra $\mathfrak{g}$, which is denoted by $\mathcal{O}^{int}$. This is a full[1] subcategory of the Bernstein-Gelfand-Gelfand's category $\mathcal{O}$.

Definition 1.6.1. *Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a fixed triangular decomposition of $\mathfrak{g}$. The category $\mathcal{O}$ is the full subcategory of $Mod(\mathfrak{g})$[2], whose objects are $\mathfrak{g}$-modules satisfying*

(i) $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ *where* $V_\alpha = \{v \in V : hv = \alpha(h)v, \text{ for all } h \in \mathfrak{h}\}$*;*

(ii) V is finitely-generated;

(iii) V is locally $\mathfrak{n}^+$-finite, i.e. for each $v \in V$, the $\mathfrak{n}^+$-module $U(\mathfrak{n}^+)v$ is finite-dimensional.

The category $\mathcal{O}$ is an abelian and noetherian category, and closed under submodules, quotients and direct sums. Some additional facts about this category can be summarized as follows.

Proposition 1.6.2. *If $V, W \in Obj\mathcal{O}$, then*

1. *If V is finite-dimensional, then $V \otimes W \in Obj\mathcal{O}$;*
2. *V and W are finitely-generated $U(\mathfrak{n}^-)$-modules.*

The spaces V_α in Definition (1.6.1) are called **weight spaces** of V, $\alpha \in \mathfrak{h}^*$ are called **weights** and the non-zero elements $v \in V_\alpha$ are called **weight vectors**. Note that 1.2.2 is a weight space decomposition. If $\mathfrak{g}_\mu$ is a weight space of $\mathfrak{g}$, then

$$\mathfrak{g}_\mu V_\alpha \subseteq V_{\alpha+\mu}, \quad \alpha \in \mathfrak{h}^*, \, \mu \in R. \tag{1.6.1}$$

Moreover, a non-zero vector $v \in V$ is called **highest weight vector of weight** λ if $v \in V_\lambda$ and $\mathfrak{n}^+ v = 0$. One can easily check any object V in $\mathcal{O}$ has at least one highest weight vector.

Definition 1.6.3. *A $U(\mathfrak{g})$-module (or a $\mathfrak{g}$-module) V is a highest weight module with highest weight λ if there exists a highest weight vector $v \in V_\lambda$ such that $V = U(\mathfrak{g})v$.*

Lemma 1.6.4. *Let V be a $\mathfrak{sl}_2$-module and $v \in V$ be a weight vector of weight*[3] $m \in \mathbb{Z}_{\geqslant 0}$*. If $x^+ v = 0$, then $(x^-)^{m+1} v = 0$.*

---

1 A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is a full subcategory if for any $X, Y \in Obj(\mathcal{S})$, every morphism in $Hom_{\mathcal{C}}(X, Y)$ is also a morphism in $Hom_{\mathcal{S}}(X, Y)$ (that is, the inclusion functor is full).

2 The category of left modules over a Lie algebra $\mathfrak{g}$.

3 If $\mathfrak{g} = \mathfrak{sl}_2$, then P^+ can be identified with $\mathbb{Z}_{\geqslant 0}$.

Remark 1.6.5. *The PBW Theorem and (1.6.1) implies that if V is a highest weight module of highest weight λ, then V contains a unique maximal proper submodule and, consequently, a unique irreducible quotient. Moreover, if $V_\mu \neq 0$, then $\mu \leqslant \lambda$, since $V = U(\mathfrak{n}^-)v$.*

Definition 1.6.6. *Let $\lambda \in \mathfrak{h}^*$, the Verma module $M(\lambda)$ is the $\mathfrak{g}$-module generated by $v \neq 0$ with the following defining relations*

$$\mathfrak{n}^+ v = 0, \quad hv = \lambda(h)v, \quad \textit{for all} \quad h \in \mathfrak{h}.$$

Note that the Verma module $M(\lambda)$ is the universal highest weight module, that is all highest weight modules with highest weight λ are quotient of $M(\lambda)$. By using PBW Theorem, one can check that $M(\lambda)$ is an object in $\mathcal{O}$. In addition, as mentioned before, $M(\lambda)$ has a unique irreducible quotient denoted by $V(\lambda)$.

Finally, the subcategory $\mathcal{O}^{int}$ can be constructed.

Definition 1.6.7. *A $\mathfrak{g}$-module V is said to be integrable if for any $v \in V$ and $i \in I$, there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that*

$$(x_i^-)^m v = (x_i^+)^m v = 0.$$

The category $\mathcal{O}^{int}$ is the full subcategory of $\mathcal{O}$ whose objects are integrable $\mathfrak{g}$-modules.

From now on, we use the following notation. Let ω_i, for $i = 1, \cdots, n$, be the fundamental weights corresponding to a set of simple roots $\Delta = \{\alpha_1, \cdots, \alpha_n\}$. Consider the sets

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \quad Q^+ = \sum_{i=1}^{n} \mathbb{Z}_{\geqslant 0}\alpha_i, \quad P = \sum_{i=1}^{n} \mathbb{Z}\omega_i, \quad \text{and} \quad P^+ \sum_{i=1}^{n} \mathbb{Z}_{\geqslant 0}\omega_i.$$

For our purposes, the following are the main results involving $\mathcal{O}^{int}$.

Theorem 1.6.8. *Let $\lambda \in \mathfrak{h}^*$ and $I = \{1, \cdots, n\}$ where n is the rank of $\mathfrak{g}$. Given a highest weight vector $v \in M(\lambda)$, the module $V(\lambda)$ is integrable if and only in $\lambda \in P^+$. In this case, $V(\lambda)$ is the quotient of $M(\lambda)$ by the submodule generated by $(x_i^-)^{\lambda(h_i)+1}v$ for all $i \in I$, and $V(\lambda)$ is finite-dimensional.*

Theorem 1.6.9. *If $V \in \mathcal{O}^{int}$ is simple, then $V \cong V(\lambda)$ for some $\lambda \in P^+$. In addition, every object in $\mathcal{O}^{int}$ is a direct sum of simple submodules and, in particular, $\mathcal{O}^{int}$ consists of finite-dimensional $\mathfrak{g}$-modules.*

Given $V, W \in Obj\mathcal{O}^{int}$, then, equipped with the same action as in item 2 of Example 1.2.7, the tensor product $V \otimes W$ is an object of $\mathcal{O}^{int}$. In addition,

$$V_\mu \otimes W_\nu \subseteq (V \otimes W)_{\mu+\nu}.$$

Proposition 1.6.10. *Let $\lambda, \mu \in P^+ \backslash \{0\}$. Then $V(\lambda) \otimes V(\mu)$ is not simple.*

Example 1.6.11. *Let $\mathfrak{g} = \mathfrak{sl}_2$. The standard basis for $\mathfrak{g}$ is $\{x^-, h, x^+\}$ with*

$$[x^+, x^-] = h, \quad [h, x^\pm] = \pm 2x^\pm.$$

In this case, a Cartan subalgebra is $\mathfrak{h} = span_{\mathbb{C}}\{h\}$, which can be identified with $\mathbb{C}$ and the set of multiple of fundamental weights P can be identified with $\mathbb{Z}$. The Verma module $M(\lambda)$ is generated by $\{v_0, v_1, v_2, \cdots\}$ and the structure of $\mathfrak{sl}_2$-module is described by the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "a_{n+2}"] & v_{n+1} \arrow[l, bend left, "1"] \arrow[loop above, "\lambda-2(n+1)"] \arrow[r, "a_{n+1}"] & v_n \arrow[l, bend left, "1"] \arrow[loop above, "\lambda-2n"] \arrow[r, "a_n"] & \cdots \arrow[l, bend left, "1"] \arrow[r, "a_3"] & v_2 \arrow[l, bend left, "1"] \arrow[loop above, "\lambda-4"] \arrow[r, "a_2"] & v_1 \arrow[l, bend left, "1"] \arrow[loop above, "\lambda-2"] \arrow[r, "a_1"] & v_0 \arrow[l, bend left, "1"] \arrow[loop above, "\lambda"] \arrow[r, "0"] & 0
\end{tikzcd}$$

where $a_i = i(\lambda - i + 1)$ denotes the action of a certain element of $\mathfrak{g}$. The left arrows represents the action of x^+, the right arrows are the action of x^-, and the laces are the action of h, while the numbers over and under the arrows are the coefficients of the action of $x^\pm$ and h on each v_i, $i \in I$, e.g.

$$v_j \xrightarrow{k} v_{j-1} \quad \textit{denotes} \quad x^+ v_j = k v_{j-1}.$$

1.7 Current Algebra

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and A be an associative and commutative algebra. Consider the vector space $\mathfrak{g} \otimes A$ and the following bilinear function:

$$\begin{array}{rcl} [\cdot, \cdot] : (\mathfrak{g} \otimes A) \times (\mathfrak{g} \otimes A) & \longrightarrow & \mathfrak{g} \otimes A \\ (x \otimes a, y \otimes b) & \longmapsto & [x, y] \otimes ab \end{array}.$$

One easily checks that this equips $\mathfrak{g} \otimes A$ with a Lie algebra structure. If A has identity, say 1_A, then the subspace $\mathfrak{g} \otimes 1_A$ is a subalgebra isomorphic to $\mathfrak{g}$. If A is $\mathbb{Z}_{\geqslant 0}$-graded, that is $A = \bigoplus_{s \geqslant 0} A[s]$, then $\mathfrak{g} \otimes A$ is also graded and can be written as

$$\mathfrak{g} \otimes A = \bigoplus_{s \geqslant 0} \mathfrak{g} \otimes A[s],$$

When $A = \mathbb{C}[t]$, the graded Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ is called the **current algebra** over $\mathfrak{g}$ and is denoted by $\mathfrak{g}[t]$. If $A = \mathbb{C}[t]/t^N \mathbb{C}[t]$, for $N \in \mathbb{N}$, the algebra $\mathfrak{g} \otimes A$ is denoted by $\mathfrak{g}[t]_N$ and is called the **truncated current algebra of nilpotence index** N. In addition, $\mathfrak{g}[t]_N$ can be seen as the graded quotient $\mathfrak{g}[t] \cong \mathfrak{g}[t]/(\mathfrak{g} \otimes t^N \mathbb{C}[t])$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition , note that

$$\mathfrak{g}[t] = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t] \tag{1.7.1}$$

where $\mathfrak{n}^{\pm}[t] = \mathfrak{n}^{\pm} \otimes \mathbb{C}[t]$ and $\mathfrak{h}[t] = \mathfrak{h} \otimes \mathbb{C}[t]$. Given $a \in \mathbb{C}$, there is a Lie algebra homomorphism $ev_a : \mathfrak{g}[t] \longrightarrow \mathfrak{g}$, called **evaluation map**, which sends $x \otimes f$ to $f(a)x$. If V is a $\mathfrak{g}$-module, the evaluation map allows us to define a $\mathfrak{g}[t]$-module structure on V by pulling-back the action of $\mathfrak{g}$, that is,

$$(x \otimes f)v := f(a)xv \quad \text{for each} \quad a \in \mathbb{C}.$$

A $\mathfrak{g}[t]$-module, defined this way is called an **evaluation module** and is denoted by $\mathrm{ev}_a V$. One easily checks that $ev_a V$ is simple if, and only if, V is simple.

Given $\lambda \in P^+$, an irreducible $\mathfrak{g}$-module of highest weight λ is denoted by $V(\lambda)$, which is generated by an element v_λ with the following defining relations:

$$\mathfrak{n}^+ v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1} v_\lambda = 0, \quad \text{for all } \alpha \in R^+,\ h \in \mathfrak{h}.$$

Note that v_λ regarded as an element of $ev_a V(\lambda)$ satisfies

$$\mathfrak{n}^+[t]v_\lambda = 0, \quad (h \otimes t^r)v_\lambda = a^r \lambda(h) v_\lambda \quad \text{for all } h \in \mathfrak{h},\ r \in \mathbb{Z}_{\geqslant 0}. \tag{1.7.2}$$

Since v_λ is a generator of $ev_a V(\lambda)$, (1.7.2) says that it is a highest weight module with respect to the triangular decomposition (1.7.1). Furthermore, the parameter $a \in \mathbb{C}$ being zero is equivalent both to $ev_a V(\lambda)$ being graded and $ev_a V(\lambda)$ factors to a $\mathfrak{g}[t]_N$-module, for $N > 1$.

We record a well-known fact about the tensor product of evaluation modules (see [41, Corollary 2.1.10] and references therein): given $k \geqslant 0$, $\lambda_1, \cdots, \lambda_k \in P^+ \backslash \{0\}$, and distinct $a_1, \cdots, a_k \in \mathbb{C}\backslash\{0\}$, then

$$ev_{a_1}V(\lambda_1) \otimes \cdots \otimes ev_{a_k}V(\lambda_k) \text{ is irreducible if and only if } a_i \neq a_j \text{ for } i \neq j. \tag{1.7.3}$$

In addition, every irreducible finite-dimensional $\mathfrak{g}[t]$-module is isomorphic to a tensor product of this form.

The category where the objects are graded finite-dimensional $\mathfrak{g}[t]$-modules and the morphisms are grade preserving $\mathfrak{g}[t]$-module homomorphisms is denoted by $\mathcal{G}$. It is also possible to consider the analogous category for the truncated case, which is denoted by $\mathcal{G}_N$. Note that every objects in $\mathcal{G}_N$ can be seen as an object in $\mathcal{G}$ by pulling-back the canonical epimorphism of Lie algebras $\mathfrak{g}[t] \to \mathfrak{g}[t]_N$. Given V an object of $\mathcal{G}$, its s-th graded piece is denoted by $V[s]$ and the sum of its positive graded pieces is denoted by

$$V_+ := \bigoplus_{s>0} V[s].$$

Observe that there exists an endofunctor of the category of $\mathbb{Z}$-graded vector spaces called grade shift functor given by

$$\tau_s(V[k]) = V[k-s], \tag{1.7.4}$$

which induces an endofunctor of $\mathcal{G}$ that changes the grading of an object while the action of $\mathfrak{g}[t]$ remains the same. Let $\lambda \in P^+$ and $s \in \mathbb{Z}$, set

$$V(\lambda, s) = \tau_s(ev_0 V(\lambda)). \tag{1.7.5}$$

Now, for some $s \in \mathbb{Z}$, consider the grade shifting of some irreducible element $ev_0V(\lambda)$ in $\mathcal{G}_N$. By (1.7.3), it is isomorphic to $ev_{a_1}V(\lambda_1) \otimes \cdots \otimes ev_{a_k}V(\lambda_k)$, for some choice of distinct $a_1, \cdots, a_k$ and $\lambda_1 + \cdots + \lambda_k = \lambda$. In addition, by using (1.7.5), one can check that any simple object in $\mathcal{G}_N$ is isomorphic to a unique element of the form $V(\mu, s)$ for some $s \in \mathbb{Z}$ and some $\mu \in P^+$.

Let $U(\mathfrak{g}[t])$ be the universal enveloping algebra of the current algebra $\mathfrak{g}[t]$. The grading on $\mathfrak{g}[t]$ induces one on $U(\mathfrak{g}[t])$ whose subspace of grade is

$$U^s(\mathfrak{g}[t]) := span\{(x_1 \otimes t^{r_1}) \cdots (x_k \otimes t^{r_k}) \ : \ k \geqslant 1, \ x_i \in \mathfrak{g}, \ r_i \in \mathbb{Z}_{\geqslant 0}, \sum_{i=1}^{k} r_i = s\}. \tag{1.7.6}$$

A $\mathfrak{g}[t]$-module is said to be **cyclic** if there exists an element $v \in V$ such that $V = U(\mathfrak{g}[t])v$. In this case, by (1.7.6), there exists a filtration F_rV, $r \in \mathbb{Z}_{\geqslant 0}$, given by

$$F_rV := \sum_{0 \leqslant s \leqslant r} U^s(\mathfrak{g}[t])v. \tag{1.7.7}$$

In addition, if we set $F_{-1}V = \{0\}$, then the associated graded vector space

$$gr(V) = \bigoplus_{r \geqslant 0} F_rV/F_{r-1}V, \tag{1.7.8}$$

can be viewed as a cyclic $\mathfrak{g}[t]$-module generated by $v + F_{-1}V$ with the action of $\mathfrak{g}[t]$ as follows

$$(x \otimes t^s)(w + F_{r-1}V) = (x \otimes t^s)w + F_{r+s-1}V, \quad \forall x \in \mathfrak{g}, \quad w \in F_rV, \quad r, s \in \mathbb{Z}_{\geqslant 0}. \tag{1.7.9}$$

Let ζ_z, $x \in \mathbb{C}$, be the automorphism of $\mathfrak{g}[t]$ induced by the mapping $t \mapsto t + z$. Denote by V^z the pullback of V by ζ_z. The action of $\mathfrak{g}[t]$ on V^z becomes

$$(x \otimes t^s)v = (x \otimes (t+z)^s)\, v, \quad x \in \mathfrak{g}, \quad v \in V, \quad s \in \mathbb{Z}_{\geqslant 0}. \tag{1.7.10}$$

Let W be a $\mathfrak{g}$-module. If $V = ev_0W$, one easily checks that $V^z \cong ev_zW$.

For $r, s \in \mathbb{Z}_{\geqslant 0}$, consider the set

$$S(r,s) = \left\{ (b_p)_{p \geqslant 0} \ : \ b_p \in \mathbb{Z}_{\geqslant 0}, \ \sum_{p \geqslant 0} b_p = r, \ \sum_{p \geqslant 0} p b_p = s \right\},$$

which is finite, because $b_p = 0$ whenever $p > s$, and empty for $r = 0$ and $s > 0$. For $k \geqslant 0$, we also consider the set

$${}_kS(r,s) = \{(b_p)_{0 \leqslant p \leqslant s} \in S(r,s) \ : \ b_p = 0 \text{ if } p < k\}$$

Given $x \in \mathfrak{g}$, $r, s \in \mathbb{Z}_{\geqslant 0}$, we can define the elements $\mathbf{x}(r,s), {}_k\mathbf{x}(r,s) \in U(\mathfrak{g}[t])$, respectively, by

$$\mathbf{x}(r,s) := \sum_{(b_p)\in S(r,s)} (x \otimes 1)^{(b_0)}(x \otimes t)^{(b_1)} \cdots (x \otimes t^s)^{(b_s)}$$

and

$${}_k\mathbf{x}(r,s) := \sum_{(b_p)\in {}_kS(r,s)} (x \otimes t^k)^{(b_k)}(x \otimes t^{k+1})^{(b_{k+1})} \cdots (x \otimes t^s)^{(b_s)}$$

where $y^{(b)}$ denotes $y^b/b!$, for any $y \in \mathfrak{g}[t]$ and any integer $b \geqslant 0$. One easily checks that

$${}_k\mathbf{x}_\alpha^-(r, kr) = (x_\alpha^- \otimes t^k)^{(r)}. \tag{1.7.11}$$

The following result has been proved in [31] and it will be used in the next sections to find relations for some $\mathfrak{g}[t]$-modules.

Lemma 1.7.1. *Given $s \in \mathbb{N}$, $r \in \mathbb{Z}_{\geqslant 0}$ and $\alpha \in R^+$, then*

$$(x_\alpha^+ \otimes t)^{(s)}(x_\alpha^- \otimes 1)^{(s+r)} - (-1)^s \mathbf{x}_\alpha^-(r,s) \in U(\mathfrak{g}[t])\mathfrak{n}^+[t] \oplus U(\mathfrak{n}^-[t])t\mathfrak{h}[t].$$

For $k \in \mathbb{Z}_{\geqslant 0}$, consider the monomorphism $\tau_k : \mathfrak{g}[t] \to \mathfrak{g}[t]$ induced by $t \mapsto t^k$. It follows from the above lemma that

$$(x_\alpha^+ \otimes t^k)^{(s)}(x_\alpha^- \otimes 1)^{(s+r)} - (-1)^s \tau_k \mathbf{x}_\alpha^-(r,s) \in U(\mathfrak{g}[t])\mathfrak{n}^+[t] \oplus U(\mathfrak{n}^-[t])\mathfrak{h}[t]_+. \tag{1.7.12}$$

2 Modules Over Current Algebras

2.1 Weyl Modules

We now recall the notion of local Weyl modules over the algebra $\mathfrak{g} \otimes A$ where A is either $A = \mathbb{C}[t]$ or $A = \mathbb{C}[t]/t^N\mathbb{C}[t]$. For a general unital associatiave algebra A, see [9]. Let $\omega \in (\mathfrak{h}[t])^*$ and consider the Verma module $M(\omega)$, i.e. the module generated by a vector v satisfying the following defining relations:

$$\mathfrak{n}^+[t]v = 0 \quad \text{and} \quad hv = \omega(h)v, \text{ for all } h \in \mathfrak{h}[t].$$

It turns out that $M(\omega)$ admits nonzero finite-dimensional quotients if and only if there exists $k \geqslant 0$, $\lambda_1, \cdots, \lambda_k \in P^k \backslash \{0\}$, and pairwise distinct $a_1, \cdots, a_k \in \mathbb{C}$ such that

$$\omega(h \otimes t^r) = \sum_{j=1}^{k} a_j^r \lambda(h) \quad \text{for all } h \in \mathfrak{h},\ r \in \mathbb{Z}_{\geqslant 0}. \tag{2.1.1}$$

In this case, $\omega|_{\mathfrak{h}} \in P^+$ and, then, the local Weyl module $W(\omega)$ can be defined as the quotient of $M(\omega)$ by the submodule generated by

$$(x_i^-)^{\omega(h_i)+1}v, \quad \text{for all} \quad i \in I.$$

$W(\omega)$ is finite-dimensional and every finite-dimensional quotient of $M(\omega)$ is also a quotient of $W(\omega)$. Note that $M(\omega)$ is graded if and only if $\omega(\mathfrak{h}[t]_+) = 0$. Since we are interested in the case when $M(\omega)$ is graded, so that $W(\omega)$ is also graded, from now on, we will use the following definition for Weyl modules.

Definition 2.1.1. *Given $\lambda \in P^+$ the local Weyl module $W(\lambda)$ is the cyclic $\mathfrak{g}[t]$-module generated by an element w_λ, with the following defining relations:*

$$\mathfrak{n}^+[t]w_\lambda = 0, \quad (h \otimes t^s)w_\lambda = \lambda(h)\delta_{s,0}w_\lambda = 0, \quad s \geqslant 0, \quad h \in \mathfrak{h}, \tag{2.1.2}$$

$$(x_\alpha^- \otimes 1)^{\lambda(h_\alpha)+1}w_\lambda = 0, \quad \alpha \in R^+. \tag{2.1.3}$$

Relation (2.1.2) implies $\left(U(\mathfrak{g}[t])\mathfrak{n}^+[t] \oplus U(\mathfrak{n}^-[t])\mathfrak{h}[t]_+\right) w_\lambda = 0 \in W(\lambda)$ and then

$$\left(x_\alpha^- \otimes t^{\lambda(h_\alpha)}\right) w_\lambda = 0, \quad \alpha \in R^+. \tag{2.1.4}$$

Indeed, if $s = \lambda(h_\alpha)$ and $r = 1$ in the Lemma 1.7.1, then $(-1)^s\mathbf{x}_\alpha^-(r,s)w_\lambda = 0$ with $(b_i) = 0$, for all $i \neq s$ and $(b_s) = 1$. Hence, $0 = \mathbf{x}_\alpha^-(r,s)w_\lambda = (x_\alpha^- \otimes t^{\lambda(h_\alpha)})w_\lambda$ as claimed.

One can consider Weyl modules in the category $\mathcal{G}_N$, it is denoted by $W_N(\lambda)$ and is defined as the $\mathfrak{g}[t]_N$-module generated by an element v satisfying relations (2.1.2) and

(2.1.3). Regarded as a $\mathfrak{g}[t]$-module, the generator v of $W_N(\lambda)$ must satisfy the following additional relation

$$(x_\alpha^- \otimes t^r)v = 0 \text{ for all } \alpha \in R^+ \text{ and } r \geqslant N. \tag{2.1.5}$$

In addition, the universal property of $W(\lambda)$ implies the existence of an epimorphism of $\mathfrak{g}[t]$-modules

$$\pi_N : W(\lambda) \twoheadrightarrow W_N(\lambda).$$

By (2.1.2) and (2.1.5), the kernel of π_N is generated by

$$(x \otimes t^N)w_\lambda, \quad x \in \mathfrak{n}^-. \tag{2.1.6}$$

Thus, the **truncated Weyl module** $W_N(\lambda)$ can be defined as the quotient of $W(\lambda)$ by $\ker(\pi_N)$. Observe that by (2.1.5), for all $M \leqslant N$, there exists a projection

$$\pi_{N,M} : W_N(\lambda) \twoheadrightarrow W_M(\lambda). \tag{2.1.7}$$

Another important well-known fact is that, for all $\alpha \in R^+$,

$$(x_\alpha \otimes t^s)v = 0, \text{ if } s \geqslant \lambda(h_\alpha), \tag{2.1.8}$$

and therefore, denoting the highest short root as η,

$$W_N(\lambda) \cong W(\lambda), \text{ if } N \geqslant \lambda(h_\eta). \tag{2.1.9}$$

Let θ be the highest root. If $\mathfrak{g}$ is simply laced, then all the roots have the same length and hence the previous isomorphism holds for all $N \geqslant \lambda(h_\theta)$.

The next lemma is a result from [45], which gives some additional relations in the case that $\lambda \in \mathbb{N}\theta$, that is, some scalar multiple of the highest root.

Lemma 2.1.2. *Let $k \in \mathbb{N}$. The following relations hold in the local Weyl module $W(k\theta)$:*

1. $(x_\theta^- \otimes 1)^{2k-1}(x_\theta^- \otimes t^{2k-1-i})w_{k\theta} = 0,$ *for all* $0 \leqslant i \leqslant k-1$;
2. $(x_\theta^- \otimes t^m)(x_\theta^- \otimes t^{m+1})w_{k\theta} \in \langle (x_\theta^- \otimes t^{m+2})w_{k\theta}\rangle,$ *for all* $m \geqslant k-1$.

Proof. Note (2.1.3) implies $(x_\theta^- \otimes 1)^{2k+1}w_{k\theta} = 0$. Hence, by (1.7.12), we get

$$\tau_{2k-1-i}\mathbf{x}_\theta^-(2k,1)w_{k\theta} = 0.$$

Since $\mathbf{x}_\theta^-(2k,1) = (x_\theta^- \otimes 1)^{(2k-1)}(x_\theta^- \otimes t)$, part (1) follows.

Part (2) is clear since

$$(x_\theta^- \otimes t^{r+s})w_{k\theta} \in \langle (x_\theta^- \otimes t^r)w_{k\theta}\rangle \quad \text{for any} \quad r,s \geqslant 0 \tag{2.1.10}$$

and

$$0 = \mathbf{x}_\theta^-(2,2m+1)w_{(k+1)\theta} = \sum_{j=0}^{m}(x_\theta^- \otimes t^{m-j})(x_\theta^- \otimes t^{m+1+j})w_{k\theta}. \tag{2.1.11}$$

□

2.2 Demazure Modules

Let $\lambda \in P^+$, $\alpha \in R^+$ with $\lambda(h_\alpha) > 0$ and $\ell, s_\alpha, m_\alpha \in \mathbb{Z}_{\geqslant 0}$ be the unique numbers such that

$$\lambda(h_\alpha) = (s_\alpha - 1)\ell d_\alpha + m_\alpha, \quad 0 < m_\alpha \leqslant \ell d_\alpha \tag{2.2.1}$$

and if $\lambda(h_\alpha) = 0$, set $s_\alpha = m_\alpha = 0$.

Definition 2.2.1. *The level-ℓ Demazure module $D(\ell, \lambda)$ is the graded quotient of $W(\lambda)$ by the submodule generated by the union of the following two sets:*

$$\{(x_\alpha^- \otimes t^{s_\alpha})w_\lambda : \alpha \in R^+ \text{ such that } d_\alpha > 1\}; \tag{2.2.2}$$

$$\{(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha+1}w_\lambda : \alpha \in R^+ \text{ such that } m_\alpha < \ell d_\alpha\}. \tag{2.2.3}$$

We denote by $\overline{w}_\lambda$, the image of w_λ in this quotient. In addition, set

$$D(\ell, \lambda, m) = \tau_m D(\ell, \lambda).$$

In particular, there exists the $\mathfrak{g}[t]$-modules isomorphism

$$D(1, \lambda) \cong W(\lambda)$$

which was proved in [13] for $\mathfrak{sl}_2$, in [11] for $\mathfrak{sl}_{r+1}$ and then in [27] for the simply laced case.

It is a well known fact that one can relate Demazure modules of different levels, say $D(\ell, \lambda)$ and $D(\ell', \lambda)$, by the epimorphism

$$D(\ell, \lambda) \twoheadrightarrow D(\ell', \lambda), \quad \text{for all} \quad \lambda \in P^+, \ell \leqslant \ell'. \tag{2.2.4}$$

In particular, for ℓ large enough, $D(\ell, \lambda) \cong ev_0 V(\lambda)$.

If $\lambda = k\theta,\ k \in \mathbb{N}$ and $\theta \in R^+$ being the highest root, the defining relations for $D(1, \lambda)$ can be rewriten as follows

Proposition 2.2.2. *[45, Proposition 1] Given $k \geqslant 1$, the level-1 Demazure module $D(1, k\theta)$ is the graded $\mathfrak{g}[t]$-module generated by an element $\overline{w}_{k\theta}$ with the following defining relations:*

1. $\mathfrak{n}^+[t]\overline{w}_{k\theta} = 0, \quad (h \otimes t^s)\overline{w}_{k\theta} = k\theta(h)\delta_{s,0}\overline{w}_{k\theta} = 0, \quad s \geqslant 0, \quad h \in \mathfrak{h}$;

2. $(x_\alpha^- \otimes 1)\overline{w}_{k\theta} = 0, \quad \alpha \in R^+, \quad (\theta, \alpha) = 0$;

3. $(x_\alpha^- \otimes 1)^{kd_\alpha+1}\overline{w}_{k\theta} = 0, \quad (x_\alpha^- \otimes t^k)\overline{w}_{k\theta} = 0, \quad \alpha \in R^+, \quad (\theta, \alpha) = 1$;

4. $(x_\theta^- \otimes 1)^{2k+1}\overline{w}_{k\theta} = 0.$

Proof. Note that the defining relations of $W(k\theta)$ imply that $(x_\alpha^- \otimes 1)^{kd_\alpha+1}\overline{w}_{k\theta} = 0$ as well as relations 1. and 4. Now, since θ is the highest root in R^+, the abstract theory of root systems says that (θ, α) is either 0 or 1 for every $\alpha \in R^+\backslash\{\theta\}$ and hence

$$k\theta(h_\alpha) = \begin{cases} 0, & \text{if } (\theta,\alpha) = 0 \\ kd_\alpha, & \text{if } (\theta,\alpha) = 1. \end{cases}$$

Therefore, relations determined by the set in (2.2.3) do not occur, since $m_\alpha \neq 0$. For $k\theta(h_\alpha) = 0$, we have $s_\alpha = 0$ and (2.2.2) becomes (2.). Otherwise, for all $\alpha \in R^+\backslash\{\theta\}$, (2.2.2) implies $(x_\alpha^- \otimes t^k)\overline{w}_{k\theta} = 0$.
Note that the opposite is also true, i.e. if $(x_\alpha^- \otimes t^k)\overline{w}_{k\theta} = 0$ holds, then relations determined by (2.2.2) hold. In fact, since $k\theta(h_\alpha) = kd_\alpha$, then $s_\alpha = k+1$ and by (2.1.10), we have

$$0 = (x_\alpha^- \otimes t^{k+1})\overline{w}_{k\theta} = (x_\alpha^- \otimes t^{s_\alpha})\overline{w}_{k\theta}.$$

□

Remark 2.2.3. *The Demazure module $D(1,\theta)$, as $\mathfrak{g}$-module, is isomorphic to $V(\theta) \oplus \mathbb{C}$ and, in particular, we have*

$$\dim D(1,\theta) = \dim V(\theta) + 1. \tag{2.2.5}$$

The next lemma, which was proved in [45], gives some additional relations for $D(1, k\theta)$ and is crucial to prove Theorem 5.1.2.

Lemma 2.2.4. *Let $k \geqslant 1$ and $0 \leqslant i < k$. The following relations hold in the module $D(1, k\theta)$:*

1. $(x_\alpha^- \otimes 1)^{(k-1)d_\alpha+1}(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta} = 0,$ *for all* $\alpha \in R^+$ *such that* $(\theta,\alpha) = 1$;
2. $(x_\alpha^- \otimes t^{k-1})(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta} \in \langle (x_\theta^- \otimes t^{2k-i})\overline{w}_{k\theta}\rangle$, *for all* $\alpha \in R^+$ *such that* $(\theta,\alpha) = 1$;
3. $(x_\theta^- \otimes t^{2k-2-i})(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta} \in \langle (x_\theta^- \otimes t^{2k-i})\overline{w}_{k\theta}\rangle$.

Proof. Let $\alpha \in R^+$ be such that $(\theta,\alpha) = 1$. Since θ is the highest root of $\mathfrak{g}$, $\theta - \alpha$ is also a root with $(\theta, \theta-\alpha) = 1$ and hence we can consider the element $(x_{\theta-\alpha}^- \otimes t^{2k-1-i})\overline{w}_{k\theta}$, which is zero for all $0 \leqslant i < k$. Indeed, since $2k-1-i > k$ for $0 \leqslant i < k$, by part 3. of Proposition 2.2.2, we have

$$(x_{\theta-\alpha}^- \otimes t^k)\overline{w}_{k\theta} = 0,$$

and it implies

$$(x_{\theta-\alpha}^- \otimes t^s)\overline{w}_{k\theta} = 0, \quad \text{for all} \quad s \geqslant k.$$

Now, 1. follows from Lemma 1.6.4, since $(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta}$ has weight $(k-1)\theta$,

$$(x_\alpha^+ \otimes 1)(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta} = (x_\theta^- \otimes t^{2k-1-i})(x_\alpha^+ \otimes 1)\overline{w}_{k\theta} + (x_{\theta-\alpha}^- \otimes t^{2k-1-i})\overline{w}_{k\theta} = 0,$$

and the subalgebra spanned by $(x_\alpha^- \otimes 1), (x_\alpha^+ \otimes 1)$ and $(h \otimes 1)$ is isomorphic to $\mathfrak{sl}_2$. For 2., note that $2k - i > k$ for all $0 \leqslant i < k$. Hence, by (2.1.5),

$$(x_\theta^- \otimes 1)^{3k-i}\overline{w}_{k\theta} = (x_\theta^- \otimes 1)^k (x_\theta^- \otimes 1)^{2k-i}\overline{w}_{k\theta} = 0.$$

Then, by putting $r = 2$ and $s = 2k - 1 - i$ in Lemma 1.7.1 and by (2.1.10), we have

$$\mathbf{x}_\theta^-(2, 2k-1-i)\overline{w}_{k\theta} \in \langle (x_\theta^- \otimes t^{2k-i})\overline{w}_{k\theta} \rangle. \tag{2.2.6}$$

Given $\alpha \in R^+$, one easily checks that $(x_{\theta-\alpha}^- \otimes 1)\left((x_\theta^- \otimes t^p)(x_\theta^- \otimes t^q)\overline{w}_{k\theta}\right)$ is equal to

$$\left((x_\theta^- \otimes t^p)(x_\theta^- \otimes t^q)(x_{\theta-\alpha}^- \otimes 1) + (x_\theta^- \otimes t^p)(x_\alpha^- \otimes t^q) + (x_\alpha^- \otimes t^p)(x_\theta^- \otimes t^q)\right)\overline{w}_{k\theta},$$

so that, together with (2.1.5), this implies

$$(x_\alpha^+ \otimes t^{k-1})(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta} = (x_{\theta-\alpha} \otimes 1)(x_\theta^- \otimes t^{k-1})(x_\theta^- \otimes t^{2k-1-i})\overline{w}_{k\theta} \tag{2.2.7}$$

which belongs to $\langle (x_\theta^- \otimes t^{2k-i})\overline{w}_{k\theta} \rangle$ as we wanted. Finally, 3. follows directly from part 2. of Lemma 2.1.2. □

2.3 Chari-Venkatesh Modules

Let $\mathscr{P}$ be the set of non-increasing monotonic sequences of nonnegative integers with finite-support. The elements of $\mathscr{P}$ are called partitions. Given any sequence $\mathrm{x} = (x_i)_{i \in \mathbb{Z}_{>0}}$ with finite support, we denote as $\underline{\mathrm{x}}$ the partition obtained from x by reordering its elements and set

$$l(\underline{\mathrm{x}}) = \max\{j \ : \ x_j \neq 0\} \quad |\underline{\mathrm{x}}| = \sum_{j \geqslant 1} x_j.$$

When $|\underline{\mathrm{x}}| = m$, $\underline{\mathrm{x}}$ is said to be a partition of m and the set of all partitions of m is denoted by $\mathscr{P}_m$. The integer $l(\underline{\mathrm{x}})$ is called the length of $\underline{\mathrm{x}}$ and it represents the number of nonzero elements in $\underline{\mathrm{x}}$.

Now, given $\lambda \in P^+$, a family of partitions $\xi = (\xi(\alpha))_{\alpha \in R^+}$ is said to be λ-compatible if

$$\lambda(h_\alpha) = \sum_{j \geqslant 1} \xi(\alpha)_j, \quad \forall \alpha \in R^+.$$

We will denote by $\mathscr{P}_\lambda$ the set of families of λ-compatible partitions.

Definition 2.3.1. *Given $\xi \in \mathscr{P}_\lambda$, the Chari-Venkatesh (or CV) module $V(\xi)$ is the quotient of $W(\lambda)$ by the submodule generated by*

$$\left\{(x_\alpha^+ \otimes t)^s (x_\alpha^- \otimes 1)^{s+r} w_\lambda \ : \ \alpha \in R^+, \ s, r \in \mathbb{N} \quad s.t. \quad s + r \geqslant 1 + rk + \sum_{j \geqslant k+1} \xi(\alpha)_j, \ k \in \mathbb{N}\right\}$$

These modules were defined for the first time in [16]. Note $V(\xi)$ is a graded quotient of $W(\lambda)$. We now recall a result that was implicitly shown in the proof of [16, Theorem 1], which gives some additional understanting about the CV modules.

Lemma 2.3.2. *Let $\lambda \in P^+$, $r \in \mathbb{N}$ and $(\xi(\alpha))_{\alpha\in R^+} \in \mathscr{P}_\lambda$. If $r \geqslant \xi(\alpha)_1$ then $\mathbf{x}_\alpha^-(r,s)w_\lambda = 0$ in $W(\lambda)$, for all $\alpha \in R^+$, $s,k \in \mathbb{N}$, $s + r \geqslant 1 + rk + \sum_{j\geqslant k+1} \xi(\alpha)_j$.*

Proof. Let $\alpha \in R^+$, $\xi = (\xi(\alpha))_{\alpha\in R^+} \in \mathscr{P}_\lambda$ such that $s + r \geqslant 1 + rk + \sum_{j\geqslant k+1} \xi(\alpha)_j$, for all $k \in \mathbb{N}$. If $r \geqslant \xi(\alpha)_1$, then $s + r \geqslant 1 + \sum_{j\geqslant k+1} \xi(\alpha)_j = 1 + \lambda(h_\alpha)$. By Lemma 1.7.1 and (2.1.3), we have

$$\mathbf{x}_\alpha^-(r,s)w_\lambda = 0,$$

since $(x_\alpha^+ \otimes t)^{(s)}(x_\alpha^- \otimes 1)^{(s+r)}w_\lambda = 0.$ □

Keeping the notation of the previous lemma, we denote by $V'(\xi)$ the quotient of $W(\lambda)$ by the submodule generated by

$$\mathbf{x}_\alpha^-(r,s)w_\lambda, \quad \forall \alpha \in R^+,\ \forall s,r \in \mathbb{N} \quad \text{such that} \quad s + r \geqslant 1 + rk + \sum_{j\geqslant k+1} \xi(\alpha)_j,\ \ k \in \mathbb{N}.$$

Futhermore, consider the quotient $V''(\xi)$ of $W(\lambda)$ by the submodule generated by

$$_k\mathbf{x}_\alpha^-(r,s)w_\lambda, \quad \forall \alpha \in R^+,\ \forall s,r,k \in \mathbb{N} \quad \text{such that} \quad s + r \geqslant 1 + rk + \sum_{j\geqslant k+1} \xi(\alpha)_j,\ \ k \in \mathbb{N}.$$

Lemma 2.3.2 implies:

Proposition 2.3.3. *The modules $V'(\xi)$ and $V''(\xi)$ are isomorphic to $V(\xi)$.*

We denote a generator of $V(\xi)$ with weight λ by v_ξ, where $\xi \in \mathscr{P}_\lambda$. The previous proposition together with (1.7.11) gives us

$$(x_\alpha^- \otimes t^k)^{(r)}v_\xi = 0, \quad \text{for all} \quad \alpha \in R^+ \text{ and } k, r > 0 \quad \text{such that} \quad r > \sum_{j>k} \xi(\alpha)_j. \tag{2.3.1}$$

If $k \geqslant l(\xi(\alpha))$, then $\sum_{j>k} \xi(\alpha)_j = 0$ and hence

$$(x_\alpha^- \otimes t^k)v_\xi = 0, \quad \text{for all} \quad \alpha \in R^+ \text{ and } k \geqslant l(\xi(\alpha)). \tag{2.3.2}$$

For $\lambda \in P^+$, let $\{\lambda\}$ and 1^λ denote the following partitions in $\mathscr{P}_\lambda$

$$\{\lambda\} = (\lambda(h_\alpha))_{\alpha\in R^+}, \quad \text{and} \quad 1^\lambda := (1^{(\lambda(h_\alpha))})_{\alpha\in R^+},$$

where $(1^{(\lambda(h_\alpha))})$ denotes the partition consisting of $\lambda(h_\alpha)$ copies of 1. The next proposition gives another way to describe the CV modules $V(\{\lambda\})$ and $V(1^\lambda)$.

Proposition 2.3.4. *The following holds for all $\lambda \in P^+$:*

$$V(\{\lambda\}) \cong_{\mathfrak{g}[t]} ev_0 V(\lambda) \quad \textit{and} \quad V(1^\lambda) \cong_{\mathfrak{g}[t]} W(\lambda).$$

Proof. Since $V(\{\lambda\})$ is a quotient of $W(\lambda)$ by the submodule generated by

$$\mathbf{x}_\alpha^-(1,1)w_\lambda = (x_\alpha^- \otimes t)w_\lambda, \quad \alpha \in R^+,$$

there exists an epimorphism of $\mathfrak{g}[t]$-modules mapping $v_{\{\lambda\}}$ to v_λ. Now, equation (2.3.2) says that for all $\alpha \in R^+$ and $k \geqslant l(\lambda(h_\alpha)) = 1$, we have

$$(x_\alpha^- \otimes t^k)v_{\{\lambda\}} = 0.$$

Therefore, $\mathfrak{g} \otimes t\mathbb{C}[t]v_{\{\lambda\}} = 0$ and this gives us the existence of an inverse epimorphism and the first isomorphism follows.

Since $(1^\lambda)_1 = 1$ for all $\alpha \in R^+$, by Lemma 2.3.2, we can identify the element $w_\lambda \in W(\lambda)$ with v_{1^λ} and therefore, we have the second isomorphism. □

In [16, Theorem 2], it was shown a way to identify CV modules and level-ℓ Demazure modules as follows:
Given $\xi_{\ell,\lambda} \in \mathscr{P}_\lambda$ defined by

$$\xi_{\ell,\lambda}(\alpha) := ((\ell d_\alpha)^{(s_\alpha - 1)}, m_\alpha) \tag{2.3.3}$$

for $\alpha \in R^+$ and s_α, m_α being as in the definition of level-ℓ Demazure modules. Then, there is an isomorphism of graded $\mathfrak{g}[t]$-modules

$$D(\ell,\lambda) \cong V(\xi_{\ell,\lambda}). \tag{2.3.4}$$

3 Fusion Products

3.1 Fusion product of cyclic $\mathfrak{g}[t]$-modules

Let $V_1, \cdots, V_m$ be cyclic $\mathfrak{g}$-modules with cyclic vectors $v_1, \cdots, v_m$. For each choice of $\mathcal{A} = (a_1, a_2, \cdots, a_m) \in \mathbb{C}^m$, we can consider the $\mathfrak{g}[t]$-module

$$\mathcal{V}(\mathcal{A}) = ev_{a_i}(V_1) \otimes \cdots \otimes ev_{a_m}(V_m).$$

In that case, the action of $\mathfrak{g}[t]$ on any homogeneous tensor, say $w_1 \otimes \cdots \otimes w_m$ is given by

$$(x \otimes t^k)(w_1 \otimes \cdots \otimes w_m) := \sum_{i=1}^{m} a_i^k \ w_1 \otimes \cdots \otimes xw_i \otimes \cdots \otimes w_m. \tag{3.1.1}$$

Proposition 3.1.1. *Let $\mathcal{A} = (a_1, \cdots, a_m)$. If a_i are pairwise distinct numbers, then the $\mathfrak{g}[t]$-module $\mathcal{V}(\mathcal{A})$ is generated by $v_1 \otimes \cdots \otimes v_m$.*

Proof. We want to prove

$$\mathcal{V}(\mathcal{A}) = U(\mathfrak{g})v_1 \otimes \cdots \otimes U(\mathfrak{g})v_m = U(\mathfrak{g}[t])(v_1 \otimes \cdots \otimes v_m).$$

First, note that the action described in (3.1.1) implies that an element of the form $(x \otimes t^k)(u_1 \otimes \cdots \otimes u_m) \in U(\mathfrak{g}[t])(v_1 \otimes \cdots \otimes v_m)$ can be writen as a linear combination of elements in $U(\mathfrak{g})v_1 \otimes \cdots \otimes U(\mathfrak{g})v_m$. In addition, for a fixed $x \in \mathfrak{g}$ and an homogeneous tensor $u_1 \otimes \cdots \otimes u_m \in \mathcal{V}(\mathcal{A})$, consider the subspaces

$$W = span\{(x \otimes t^0)(u_1 \otimes \cdots \otimes u_m), (x \otimes t)(u_1 \otimes \cdots \otimes u_m), \cdots, (x \otimes t^{m-1})(u_1 \otimes \cdots \otimes u_m)\}$$

and

$$\tilde{W} = span\{xu_1 \otimes \cdots \otimes u_m \,, u_1 \otimes xu_2 \otimes \cdots \otimes u_m \,, \cdots, u_1 \otimes u_2 \otimes \cdots \otimes xu_m\}.$$

Note that the vectors of the definition of W can be written as linear combinations of those in the definition of $\tilde{W}$ using the scalars of the columns of the following matrix.

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{m-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^{m-1} \end{pmatrix}.$$

Since $a_i \neq a_j$, for all $j \neq i$ and $i, j = 1, \cdots, m$, it follows that

$$\det(A) = \prod_{1 \leqslant i < j \leqslant m} (a_j - a_i) \neq 0.$$

It follows that any homogeneous tensor $u_1 \otimes \cdots \otimes xu_i \otimes \cdots \otimes u_m$, for $i = 1, \cdots, m$, can be written as linear combination of $(x \otimes t^j)(u_1 \otimes \cdots \otimes u_m)$ for all $x \in \mathfrak{g}$ and $j = 0, \cdots, m-1$. In particular, the previous procedure can be done on the vector $v_1 \otimes \cdots \otimes v_m$. One can check that some iterations of the previous process implies, first, that any element

$$v_1 \otimes \cdots \otimes X_i v_i \otimes \cdots \otimes v_m, \quad X_i \in U(\mathfrak{g}), \quad i = 1, \cdots, m$$

is a linear combination of elements in $U(\mathfrak{g}[t])(v_1 \otimes \cdots \otimes v_m)$ and then that the same holds for

$$X_1 v_1 \otimes X_2 v_2 \otimes \cdots \otimes X_i v_i \otimes \cdots \otimes X_m v_m, \quad \text{for any } X_i \in U(\mathfrak{g}), \quad i = 1, \cdots, m.$$

Therefore,

$$U(\mathfrak{g})v_1 \otimes \cdots \otimes U(\mathfrak{g})v_m \subseteq U(\mathfrak{g}[t])(v_1 \otimes \cdots \otimes v_m).$$

□

Let $a_1, \cdots, a_m \in \mathbb{C}$ be as in the last proposition and consider the filtration on $\mathcal{V}(\mathcal{A})$ as in (1.7.7) for $v = v_1 \otimes \cdots \otimes v_m$.

Definition 3.1.2. *Let $V_1, \cdots, V_m$ be cyclic objects in $\mathcal{G}$ with cyclic vectors $v_1, \cdots, v_m$ and $a_1, \cdots, a_m \in \mathbb{C}$ with $a_j \neq a_i$ for all $j \neq i$. The fusion product $V_1^{a_1} * \cdots * V_m^{a_m}$ is the graded $\mathfrak{g}[t]$-module $gr(\mathcal{V}) = \bigoplus_{i \in I} F_r\mathcal{V}/F_{r-1}\mathcal{V}$ with cyclic vector $v_1 \otimes \cdots \otimes v_m \in F_{-1}\mathcal{V}$.*

3.2 Conjectures on fusion products

We denote a generator of the fusion product as $v_1 * \cdots * v_m$. Note that it depends on the parameters $a_1, \cdots, a_m$. However, the following was conjetured in [21].

Conjecture 3.2.1. *Let $a_1, \cdots, a_m \in \mathbb{C}$ pairwise distinct and $V_1, \cdots, V_m$ cyclic objects in $\mathcal{G}$. Then*

1. *The fusion product $V_1^{a_1} * \cdots * V_m^{a_m}$ is independent of the parameters $a_1, \cdots, a_m$.*
2. *The fusion product of any finite collection of cyclic elements in $\mathcal{G}$ is associative up to isomorphism.*

This motivates us to shorten the notation and write $V_1 * \cdots * V_m$. Conjecture 3.2.1 has been proved for some particular cases. This is the case for V_j being certain quotients of a Weyl module with its cyclic generator being a highest weight vector (see [11, 20, 21, 27, 33, 42]). In particular, this conjecture was proved for Demazure modules of the same level (see [15, 16, 27, 43, 49]) and for $D(1, \theta)^{*j} * ev_0 V(\theta)^{*m}$ as we will see later. Many of these works have proved Conjecture 3.2.1 by finding an isomorphism between certain fusion products and $\mathfrak{g}[t]$-modules whose defining relations are known. Another important fact about fusion products is the following isomorphism of $\mathfrak{g}[t]$-modules

$$W(\lambda_1) * \cdots * W(\lambda_m) \cong W(\lambda) \quad \text{if} \quad \lambda = \lambda_1 + \cdots + \lambda_m, \tag{3.2.1}$$

which was first proved in [11] for $\mathfrak{g} = \mathfrak{sl}_n$, then in [27] for the simply laced case, and, finally, for the general case in [42]. Since each λ_j can be writen as a sum of $l_i = \lambda(h_i)$ copies of ω_i, then, (3.2.1) is equivalent to the isomorphism of $\mathfrak{g}[t]$-modules

$$W(\lambda) \cong W(\omega_1)^{*\lambda(h_1)} * \cdots * W(\omega_n)^{*\lambda(h_n)}, \quad \text{for all} \quad \lambda \in P^+. \tag{3.2.2}$$

Let $\lambda \in P^+$, and $N \in \mathbb{Z}_{>0}$. Define the set

$$P^+(\lambda, N) := \{\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N) \in (P^+)^N \ : \ \lambda_1 + \cdots + \lambda_N = \lambda\}. \tag{3.2.3}$$

Let us define a partial order on $P^+(\lambda, N)$. For this, given $\alpha \in R^+$ and $1 \leqslant s \leqslant N$, consider the following number

$$r_{\alpha,s}(\boldsymbol{\lambda}) = \min\{(\lambda_{i_1} + \cdots + \lambda_{i_r})(h_\alpha) \ \ 1 \leqslant i_1 \leqslant \cdots \leqslant i_s \leqslant N\} \tag{3.2.4}$$

and define

$$\boldsymbol{\mu} \leqslant \boldsymbol{\lambda} \quad \Leftrightarrow \quad r_{\alpha,s}(\mu) \leqslant r_{\alpha,s}(\lambda), \quad \text{for all} \quad \alpha \in R^+, 1 \leqslant s \leqslant N. \tag{3.2.5}$$

By following the algorithm described in [25], we can compute the maximal element in $P^+(\lambda, N)$ as follows.

If $\lambda = \sum_{i=1}^{n} a_i\omega_i$, consider the unique nonnegative integers p_i and r_i, $i \in I$ such that

$$|\lambda| = p_i N + r_i, \quad \text{for} \quad 0 \leqslant r_i < N.$$

Also, for $i \in I$ and $1 \leqslant j \leqslant N$, set

$$m^{i,j} = \begin{cases} p_i + 1, & \text{if } j \leqslant r_i, \\ p_i, & \text{if } j > r_i. \end{cases}$$

Hence $m^{i,j} \geqslant m^{i+1,j}$ for all i, j. It follows that $\lambda_j := \sum_{i=1}^{n} m^{i,j}(\omega_i - \omega_{i-1}) \in P^+$. A maximal element in $P^+(\lambda, N)$ is

$$\boldsymbol{\lambda}^{\max} := (\lambda_1, \cdots, \lambda_N).$$

In addition, it was also proved in [25, Lemma 3.1] that the maximal elements are in the same orbit under the action of the symmetric group S_N. One of the main results in [29] is about the next conjecture. It was stated in [10, 26, 33] and generalizes (3.2.1).

Conjecture 3.2.2. *Let $N \in \mathbb{Z}_{>0}, \lambda \in P^+$, and suppose $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ is a maximal element of $P^+(\lambda, N)$. If $N \leqslant |\lambda|$, $W_N(\lambda) \cong V(\lambda_1) * \cdots * V(\lambda_N)$ as graded $\mathfrak{g}[t]$-modules.*

This conjecture has been proved in [45] for $\mathfrak{g}$ simply laced, λ being a natural multiple of the highest root θ and $N = |\lambda|$ (Theorem 5.2.3), in [33] for $\lambda = N\mu + \nu$ where $\mu \in P^+_{\mathrm{sym}}$ and ν minuscle[1] and in [26] for $\mathfrak{g}$ of type A, $N = 2$, and $\lambda = m\omega_i,\ m \geqslant 0$. The following is [45, Lemma 5].

Lemma 3.2.3. *Let $V_1, \cdots, V_m$ be finite-dimensional cyclic graded $\mathfrak{g}[t]$-modules generated by $v_1, \cdots, v_m$, respectively, and $s_1, \cdots, s_m \in \mathbb{Z}_{\geqslant 0}$. If $(x \otimes t^{s_i})v_i = 0$, for some $x \in \mathfrak{g}$ and for all $1 \leqslant i \leqslant m$, then $(x \otimes t^{s_1+\cdots+s_m})(v_1 * \cdots * v_m) = 0$.*

Proof. Let $a_1, \cdots, a_m \in \mathbb{C}$ be distinct complex numbers, $v_1 * \cdots * v_m$ be the generator of $V_1^{a_1} * \cdots * V_m^{a_m}$ and consider the polynomial $f(t) = \prod_{j=1}^{m}(t-a_j)^{s_j}$. Note that the element $(x \otimes (t^S - f(t)))(v_1 * \cdots * v_m)$ belongs to $F_{S-1}(V_1 \otimes \cdots \otimes V_m)$, where $S = s_1 + \cdots + s_m$. Hence

$$(x \otimes t^S)(v_1 * \cdots * v_m) = (x \otimes f(t))(v_1 * \cdots * v_m). \tag{3.2.6}$$

Write $p_{j,k}(t) = t + a_j - a_k \in \mathbb{C}[t]$ for all $j, k = 1, \cdots m$. By (1.7.10), we have

$$\begin{aligned}(x \otimes f(t))(v_1 \otimes \cdots \otimes v_m) &= \sum_{j=1}^{m} v_1 \otimes \cdots \otimes \left(x \otimes \left(\prod_{k=1}^{m}(p_{j,k}(t))^{s_k}\right)\right) v_j \otimes \cdots \otimes v_m \\ &= \sum_{j=1}^{m} v_1 \otimes \cdots \otimes \left(x \otimes t^{s_j}\left(\prod_{k=1,k\neq j}^{m}(p_{j,k}(t))^{s_k}\right)\right) v_j \otimes \cdots \otimes v_m \\ &= 0\end{aligned}$$

Futhermore, by (3.2.6), we have $(x \otimes t^S)(v_1 * \cdots * v_m) = (x \otimes f(t))(v_1 * \cdots * v_m) = 0$ □

The following result is [29, Proposition 3.2.2.].

Proposition 3.2.4. *Given $l \in \mathbb{Z}_{>0}$, for each $1 \leqslant i \leqslant l$, let V_i be a quotient of $W(\lambda_i)$ for some $\lambda_i \in P^+$ and $v_i \in (V_i)_{\lambda_i}\backslash\{0\}$. Suppose*

$$(\mathfrak{n}^- \otimes t^{N_i}\mathbb{C}[t])v_i = 0, \tag{3.2.7}$$

$N_i \in \mathbb{Z}_{>0}$, $N = N_1 + \cdots + N_l$ and $\lambda = \lambda_1 + \cdots + \lambda_l$. Then, for any choice of distinct $a_1, \cdots, a_l \in \mathbb{C}$, there exists an epimorphism of graded $\mathfrak{g}[t]$-modules

$$W_N(\lambda) \twoheadrightarrow ev_{a_1}V_1 * \cdots * ev_{a_l}V_l. \tag{3.2.8}$$

Proof. Let v_i be a highest weight vector of V_i for each $i = 1, \cdots, l$, so that $v = v_1 * \cdots * v_l$ is the cyclic generator of $ev_{a_1}V_1 * \cdots * ev_{a_l}V_l$. One easily checks that

$$\mathfrak{n}^+[t]v = \mathfrak{h}[t]_+ v = 0. \tag{3.2.9}$$

[1] $\lambda \in P^+$ is said to be minuscule if $\{\mu \in P^+ : \mu < \lambda\} = \varnothing$.

Moreover, for all $h \in \mathfrak{h}$, we have $(h \otimes 1)v = \lambda(h)v$ and $(x_i^- \otimes 1)^{\lambda(h_i)+1}v = 0,$ for all $i \in I$. Note that the epimorphism in the statement exists if $(x_\alpha^- \otimes t^N)v = 0$ for all $\alpha \in R^+$, but this follows directly from Lemma 3.2.3. □

In [16] it was shown the existence of an isomorphism between certain CV modules and fusion products of irreducible $\mathfrak{g}[t]$-modules under specific conditions. Since we know a presentation in terms of generators and relations for CV modules, this isomorphism gives us one more situation where the fusion product is independet of complex parameters.

Set $\mathfrak{g} = \mathfrak{sl}_2$. In this case, we identify P with $\mathbb{Z}$. Denote by $x^\pm$ and h the elements $x_1^\pm$ and h_1, respectively, and the set $\mathscr{P}_\lambda$, for $\lambda \in P^+$, becomes the set of partitions of λ. Given a partition $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_i \geqslant 0)$, we define the partitions $\xi^\pm$ as follows. If $i = 1$, then $\xi^+ = \xi$ and ξ^- is empty. If $i > 1$, then $\xi^- := (\xi_1^- \geqslant \cdots \geqslant \xi_{i-2}^- \geqslant \xi_{i-1}^- \geqslant 0)$, where

$$\xi_j^- := \begin{cases} \xi_j, & \text{if } j \leqslant i-1, \\ \xi_{i-1} - \xi_i, & \text{if } r = i-1, \\ 0, & \text{if } j \geqslant i, \end{cases}$$

and $\xi^+ = (\xi_1^+ \geqslant \cdots \geqslant \xi_{i-1}^+ \geqslant \xi_i^+ \geqslant 0)$ is the unique partition whose parts are associated to the sequence $(\xi_1, \cdots, \xi_{i-2}, \xi_{i-1}+1, \xi_i - 1)$.

Theorem 3.2.5. *[16, Theorem 5]. Let $\xi \in \mathscr{P}_\lambda$, $r = l(\xi)$. Then*

(i) For $r > 1$, there exists a short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \tau_{(r-1)\xi_r} V(\xi^-) \to V(\xi) \to V(\xi^+) \to 0.$$

(ii) For any choice of distinct complex parametes $a_1, \cdots, a_r$, there exists an isomorphism of graded $\mathfrak{g}[t]$-modules

$$V(\xi) \cong ev_{a_1}V(\xi_1) * \cdots * ev_{a_r}V(\xi_r). \tag{3.2.10}$$

Theorem 3.2.5 has a corollary that is a key to prove Theorem 4.1.2 bellow. In order to state this result, we have to introduce another important λ-compatible partition. Fix $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ and let q_α and p_α be the integers such that

$$\lambda(h_\alpha) = Nq_\alpha + p_\alpha, \quad \text{with} \quad 0 \leqslant p_\alpha < N \quad \text{for each} \quad \alpha \in R^+. \tag{3.2.11}$$

If $N = \infty$, then $q_\alpha = 0$ and $p_\alpha = \lambda(h_\alpha)$. Now, consider the element $\xi_N^\lambda \in \mathscr{P}_\lambda$ given by

$$\xi_N^\lambda(\alpha) = ((q_\alpha + 1)^{(p_\alpha)}, q_\alpha^{(N-p_\alpha)}). \tag{3.2.12}$$

In the case $\mathfrak{g} = \mathfrak{sl}_2$, we write $p_1 = p$ and $q_1 = q$.

Corollary 3.2.6. *Let $\mathfrak{g} = \mathfrak{sl}_2$. The fusion product $V(q+1)^{*p} * V(q)^{*N-p}$ is isomorphic to the quotient of $W(\lambda)$ by the submodule generated by*

$$(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} w_\lambda \quad \text{for all} \quad r, s > 0, \quad s.t. \quad s+r > rk + q(N-k)_+ + (p-k)_+ \tag{3.2.13}$$

with $k > 0$ and $n_+ = n$ if $m \geqslant 0$ and $m_+ = 0$ if $m < 0$.

Proof. Let ξ_N^λ be as in (3.2.12). By Theorem 3.2.5, there exists an isomorphism of graded $\mathfrak{g}[t]$-modules

$$V(q+1)^{*p} * V(q)^{*N-p} \cong V(\xi_N^\lambda). \tag{3.2.14}$$

We have to show

$$(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} v_{\xi_N^\lambda} = 0$$

for s and r as in (3.2.13). It follows from (3.2.12) and from the definition of CV module that

$$(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} v_{\xi_N^\lambda} = 0, \quad \text{for all} \quad s, r > 0 \quad \text{such that} \quad s + r > rk + \sum_{i>k} \xi_i, \text{ for } k > 0.$$

Thus, we have to check the inequality for $k < p$ and $k \geqslant p$. Note that

$$rk + \sum_{i \geqslant k+1} \xi_i = rk + p(q+1) + q(N-p),$$

so that, if $k < p$, then

$$s + r > rk + p + qN > rk + q(N-k) + (p-k) = rk + q(N-k)_+ + (p-k)_+,$$

and if $k \geqslant p$, then

$$s + r > rk + p + qN > rk + qN - qk = rk + q(N-k) = rk + q(N-k)_+.$$

Therefore, $s + r > rk + \sum_{i \geqslant k+1} \xi_i$ if and only if $s + r > rk + q(N-k)_+ + (p-k)_+$. $\square$

4 The case of minuscle weights

In this chapter, we study the main results of [29], namely Theorem 4.1.1, Theorem 4.1.2 and Proposition 4.3.1, which are [29, Theorem 2.3.2, Theorem 2.4.1, Proposition 2.5.2] respectively. In particular, the first of these results proves Conjecture 3.2.2 for minuscle weights and $\mathfrak{g}$ simply laced, while the proposition is a step forward to prove Theorem 4.1.1 for general $\mathfrak{g}$.

4.1 Truncated Weyl modules viewed as CV modules

Theorem 4.1.1. *If $\mathfrak{g}$ is simply laced and ω_i is minuscle, then Conjecture 3.2.2 holds for $\lambda = k\omega_i$ for all $k \in \mathbb{Z}_{\geqslant 0}$.*

The proof of this theorem relies on some additional results about CV modules and another class of objects in $\mathcal{G}$ called Kirillov-Reshetikhin (KR) modules. The following theorem was proved in [33, Theorem 4.2] for $\mathfrak{g} = \mathfrak{sl}_2$ and then in [29] for the general case.

Theorem 4.1.2. *The modules $V(\xi_N^\lambda)$ and $W_N(\lambda)$ are isomorphic graded $\mathfrak{g}[t]$-modules.*

Proof. In order to shorten the notation, only in this proof, we denote $\xi = \xi_N^\lambda$, the cyclic generator of $V(\xi)$ by v_ξ and the cyclic generator of $W_N(\lambda)$ by w. By (2.3.2), we have

$$(x_\alpha^- \otimes t^N)v_\xi = 0, \quad \text{for all} \quad \alpha \in R^+.$$

Therefore, there exists an epimorphism of $\mathfrak{g}[t]$-modules

$$W_N(\lambda) \twoheadrightarrow V(\xi).$$

Denote by w the cyclic generator of $W_N(\lambda)$. We want to find the following sequence

$$V''(\xi) \twoheadrightarrow W_N(\lambda) \twoheadrightarrow V(\xi),$$

since, by Proposition 2.3.3, $V(\xi)$ and $V''(\xi)$ are isomorphic. In other words, it suffices to show

$${}_k\mathbf{x}_\alpha^-(r,s)w = 0, \quad \text{for all} \quad \alpha \in R^+ \text{ and } s,r,k \in \mathbb{N} \quad \text{s.t.} \quad s + r > rk + \sum_{j>k} \xi(\alpha)_j, \;\; k \in \mathbb{N}.$$

In order to do this, let us prove it for $\mathfrak{g} = \mathfrak{sl}_2$ and after this, by considering the subalgebra $\mathfrak{sl}_\alpha[t]$ we are done. There are two cases to consider: $N \leqslant \lambda(h_\eta)$ and $N > \lambda(h_\eta)$. When $N \leqslant \lambda(h_\eta)$, by (2.1.9), the theorem follows from Proposition 2.3.4.

Note that, by Proposition 3.2.4 and (3.2.14), we have the epimorphism of $\mathfrak{g}[t]$-modules

$$W_N(\lambda) \twoheadrightarrow V(q+1)^{*p} * V(q)^{*(N-p)} \cong V(\xi).$$

Therefore, by Corollary 3.2.6, it suffices to show

$$(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} w = 0, \quad \forall r, s > 0, \text{ such that } s+r > rk + q(N-k)_+ + (p-k)_+, \tag{4.1.1}$$

where p and q are as in (3.2.11) and $k > 0$. Now, suppose that s, r and k satisfy (4.1.1). Hence, we have to check that $(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} w = 0$ for $r > q$ and $r \leqslant q$.

First, for $r > q$, note that it suffices to show that $s + r \geqslant \lambda(h_\eta)$. If $k \geqslant p$, then the term $(p-k)_+ = 0$. Hence

$$s + r \geqslant rk + q(N-k)_+ \geqslant (q+1)k + q(N-k)_+ \geqslant k + q(k + (N-k)_+) \geqslant p + qN = \lambda(h_\zeta),$$

and it is done. Now, if $k < p$, then $(N-k)+ = N-k$ and $(p-k)_+ = p-k$. It follows that

$$s + r > rk + q(N-k) + p - k = rk + p + qN - k(q-1) = rq - k(q-1) + \lambda(h_\zeta) \geqslant \lambda(h_\zeta).$$

Second, suppose $r \leqslant q$ and observe that it gives the following inequality

$$s + r > Nr.$$

In fact, if $k < N$, then $(N-k)_+ = N-k$ and hence

$$s + r > rk + qN - qk + (p-k)_+ \geqslant rk + rN - rk + (p-k)_+ \geqslant rN.$$

While if $N \leqslant k$, then

$$s + r > rk \geqslant rN. \tag{4.1.2}$$

Now, observe that (4.1.2) implies ${}_k\mathbf{x}^-(r,s)w = 0$ which completes the proof, since

$$(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} w = 0 \quad \text{if and only if} \quad {}_k\mathbf{x}^-(r,s)w = 0$$

by Proposition 2.3.3. Indeed, let us check that

$$(x^- \otimes t^k)^{(b_k)} (x^- \otimes t^{k+1})^{(b_{k+1})} \cdots (x^- \otimes t^s)^{(b_s)} w = 0, \quad \text{for all} \quad (b_m) \in {}_kS(r,s). \tag{4.1.3}$$

One easily checks that (4.1.3) holds for (b_m) having some term $b_i > 0$ with $i \geqslant N$ and also for a general (b_m) with $k \geqslant N$. It remains to show that (4.1.3) holds for $k < N$ and (b_m) being a sequence such that $b_i = 0$ for all $i \geqslant N$. If we assume the contrary, it follows that

$$s = \sum_{k \leqslant i < N} i b_i \leqslant (N-1) \sum_{k \leqslant i < N} b_i \leqslant Nr - r.$$

However, this contradicts (4.1.2) and, therefore, (4.1.3) holds. $\square$

4.2 Kirillov-Reshetikhin modules

The idea of KR modules was first explored in the context of finite-dimensional irreducible modules for quantum affine algebras. Nowadays, for this context, these modules are sometimes referred to as quantum KR modules. The graded limits of those modules began to be studied after several years and then started to be called (graded) KR modules. For more information about the relation between the quantum case and our context see [12, 40, 41, 42] and references therein. In this work, we use the term KR modules to mean the modules defined as follows.

Definition 4.2.1. *Let $m \in \mathbb{Z}_{\geqslant}0$ and $i \in I$. The Kirillov-Reshetikhin module (or KR modules) $KR(m\omega_i)$ is the quotient of $W(m\omega_i)$ by the submodule generated by*

$$(x_i^- \otimes t)v \quad \textit{with} \quad v \in W(m\omega_i)_{m\omega_i} \backslash \{0\}.$$

It is well known (see [8, 12, 27]) that under the hypothesis of Theorem 4.1.1 we have

$$KR(m\omega_i) \cong V(m\omega_i), \ \forall m \geqslant 0. \tag{4.2.1}$$

Let $N > 0$. Consider the N-tuples $\mathbf{i} = (i_1, \cdots, i_N) \in I^N$ and $\boldsymbol{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N_{>0}$. For some choice of complex parameters, set $S_i(\mathbf{i}) = \{j \ : i_j = i\}$ and

$$KR_{\mathbf{i}}(\boldsymbol{m}) = KR(m_1\omega_{i_1}) * \cdots * KR(m_N\omega_{i_N}) \tag{4.2.2}$$

$$V_{\mathbf{i}}(\boldsymbol{m}) = V(m_1\omega_{i_1}) * \cdots * V(m_N\omega_{i_N}). \tag{4.2.3}$$

If for all $1 \leqslant j \leqslant N$ we have $i_j = i$ for some $i \in I$, we simplify notation and write $KR_i(\boldsymbol{m})$ instead of $KR_{\mathbf{i}}(\boldsymbol{m})$ as well as $V_i(\boldsymbol{m})$ instead of $V_{\mathbf{i}}(\boldsymbol{m})$. The following theorem, one of the main results of [29], gives a presentation to $KR_i(\boldsymbol{m})$ in terms of generators and relations. The only proof we know relies on the quantum setting, reason why we shall not review the proof.

Theorem 4.2.2 ([42, Theorem B])**.** *For every $N > 0$, $\mathbf{i} \in I^N$, and $\boldsymbol{m} \in \mathbb{Z}^N_{\geqslant 0}$, the module $KR_{\mathbf{i}}(\boldsymbol{m})$ is isomorphic to the quotient of $W(\lambda)$ by the submodule generated by*

$$x^-_{\alpha_i}(r,s)w_\lambda \quad \textit{for all} \quad i \in I, \ r > 0, \ s + r > \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\}.$$

□

As observed in [42, Remark 3.4(b)], this theorem gives a way to relate CV modules and $KR_{\mathbf{i}}(\boldsymbol{m})$ as in Corollary 4.2.3 bellow. But first, given $\mathbf{i}$ and $\boldsymbol{m}$ as before, let

$$\underline{\boldsymbol{m}}_i = (m_j)_{j \in S_i(\boldsymbol{i})}, \quad \text{for all} \quad i \in I$$

and $\xi_{\boldsymbol{i}}^{\boldsymbol{m}} \in \mathscr{P}_\lambda$ be given by

$$\xi_{\boldsymbol{i}}^{\boldsymbol{m}}(\alpha) = \begin{cases} \underline{\boldsymbol{m}}_i, & \text{if } \alpha = \alpha_i \text{ for some } i \in I \\ (1^{\lambda(h_\alpha)}), & \text{otherwise.} \end{cases}$$

Corollary 4.2.3. *There is a* $\mathfrak{g}[t]$*-module isomorphism* $V(\xi_{\boldsymbol{i}}^{\boldsymbol{m}}) \cong KR_{\boldsymbol{i}}(\boldsymbol{m})$.

Proof. As in Theorem 4.1.2, we write $\xi = \xi_{\boldsymbol{i}}^{\boldsymbol{m}}$. Let u_λ and v_ξ be the cyclic generators of $KR_{\boldsymbol{i}}(\boldsymbol{m})$ and $V(\xi)$, respectively. The desired isomorphism follows from the existence of the epimorphisms

$$KR_{\boldsymbol{i}}(\boldsymbol{m}) \twoheadrightarrow V(\xi) \tag{4.2.4}$$

and

$$V(\xi) \twoheadrightarrow KR_{\boldsymbol{i}}(\boldsymbol{m}). \tag{4.2.5}$$

In light of Proposition 2.3.3, we have to show that

$$\mathbf{x}_\alpha^-(r,s)u_\lambda = 0, \quad \forall \alpha \in R^+, \ \forall s, r \in \mathbb{N} \quad \text{such that} \quad s + r \geqslant rk + \sum_{j>k} \xi(\alpha)_j \tag{4.2.6}$$

for (4.2.4) and also, for some $k \geqslant 0$ that

$$\mathbf{x}_{\alpha_i}^-(r,s)v_\xi = 0 \quad \text{for all} \quad i \in I, \ r > 0, \ s + r > \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\} \tag{4.2.7}$$

for (4.2.5). First, if α is not a simple root, then $\xi(\alpha)_1 = 1 \leqslant r$ and it is exactly Lemma 2.3.2 and hence (4.2.6) is satisfied.

Now, for $\alpha = \alpha_i$, with $i \in I$, let

$$\underline{\boldsymbol{m}}_i = (m_{i,1}, \cdots, m_{i,N_i}),$$

for some $N_i \in \mathbb{Z}_{\geqslant 0}$, so that $\xi(\alpha)_j = m_{i,j}$ for $1 \leqslant j \leqslant N_i$. It follows that

$$rk + \sum_{j>k} \xi(\alpha_i)_j \geqslant \sum_{j=1}^{N_i} \min\{r, m_{i,j}\} = \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\}. \tag{4.2.8}$$

Thus (4.2.6) holds, by Theorem 4.2.2 and (4.2.8).

Fix i, r and s as in (4.2.7). Note that if there exists $k > 0$ such that $s + r > rk + \sum_{j>k} \xi(\alpha_i)_j$, then, by definition of $V(\xi)$, we have $\mathbf{x}_{\alpha_i}^- v_\xi = 0$ and, by (4.2.8), (4.2.7) holds. Observe that such k exists, because otherwise we would have

$$s + r \leqslant rk + \sum_{j>k} \xi_j(\alpha_i) = rk + \sum_{j>k} m_{i,j} \quad \text{for all} \quad k > 0.$$

Therefore, taking $k = \max\{j \ : m_{i,j} \leqslant r\}$, it follows that

$$s + r \leqslant rk + \sum_{j>k} m_{i,j} = \sum_{j=1}^{N_i} \min\{r, m_{i,j}\},$$

which contradicts the choice of i, r and s. $\square$

This result allows us to prove the following corollary, which is one of the main tools to prove Theorem 4.1.1.

Corollary 4.2.4. *For all $m, N \in \mathbb{N}$, $i \in I$, and $\boldsymbol{m} = \xi_N^{m\omega_i}(\alpha_i)$ there exists an epimorphism of graded $\mathfrak{g}[t]$-modules*

$$KR_i(\boldsymbol{m}) \twoheadrightarrow W_N(m\omega_i).$$

Proof. By Theorem 4.1.2 and Corollary 4.2.3 we have

$$W_N(\omega_i) \cong V(\xi_N^{m\omega_i}) \quad \text{and} \quad V(\xi_{\boldsymbol{i}}^{\boldsymbol{m}}) \cong KR_i(\boldsymbol{m}),$$

for $\boldsymbol{i} = (i^{(N)})$. Hence, this proof comes down to constructing an epimorphism

$$V(\xi_{\boldsymbol{i}}^{\boldsymbol{m}}) \twoheadrightarrow V(\xi_N^{m\omega_i}),$$

which is equivalent to showing that

$$x_\alpha^-(r,s)v_{\xi_N^{m\omega_i}} = 0 \quad \text{for all} \quad \alpha \in R^+,\ r, s > 0 \quad \text{such that} \quad s + r \geqslant 1 + rk + \sum_{j>k} \xi_{\boldsymbol{i}}^{\boldsymbol{m}}(\alpha)_j$$

Note that we have only to consider nonsimple α because $\xi_{\boldsymbol{i}}^{\boldsymbol{m}}(\alpha_i) = \xi_N^{m\omega_i}(\alpha_i)$ for all $i \in I$. In that case, we have $\xi_{\boldsymbol{i}}^{\boldsymbol{m}}(\alpha)_1 = 1 \leqslant r$ and the proof follows by Lemma 2.3.2. □

Proof of Theorem 4.1.1. The previous corollary together with Proposition 3.2.4 gives the sequence of epimorphisms

$$KR_i(\boldsymbol{m}) \twoheadrightarrow W_N(m\omega_i) \twoheadrightarrow V_i(\boldsymbol{m}).$$

Under the hypothesis of the theorem, that is $\mathfrak{g}$ simply laced and ω_i minuscle, by (4.2.1), it follows that $KR_i(\boldsymbol{m}) \cong V_i(\boldsymbol{m})$.

Now set $\xi_N^{m\omega_i}(\alpha_i) = (m_1, \cdots, m_l)$ and $\boldsymbol{\lambda} = (m_1\omega_1, \cdots, m_l\omega_l)$. It only remains to prove that $\boldsymbol{\lambda}$ is a maximal element of $P^+(m\omega_i, N)$. In [10, Lemma 3.3], it was proved that showing that element $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_s) \in P^+(m\omega_i, N)$ is maximal is equivalent to proving

$$\max_{1\leqslant j\leqslant k} \gamma_j(h_i) - \min_{1\leqslant j\leqslant k} \gamma_j(h_i) \leqslant 1.$$

However, by definition of $\xi_N^{m\omega_i}(\alpha_i)$, we have $(m_1 - m_l) \leqslant 1$ and hence $\boldsymbol{\lambda}$ is maximal. □

4.3 A further step to generalize Conjecture 3.2.2

Observe that the hypotheses on $\mathfrak{g}$ and ω_i were strongly used in the proof of Theorem 4.1.1 and, then, it is natural to ask what happens in a more general context. In this direction, a further step was proved in [29, Proposition 2.5.2]. However, in order to understant this result, we have to define the following.

For $i \in I$ and $k \geqslant 0$, given $\Lambda = \sum_{i\in I} a_i\alpha_i \in Q$ set $ht_i(\Lambda) = a_i$ and

$$R_{i,k}^+ := \{\alpha \in R^+ \ : \ ht_i(\alpha) = k\}.$$

If ω_i is minuscle, the set $R^+_{i,k}$ is empty for $k \geqslant 2$. Otherwise, by inspecting the root system, one can prove that it has a unique minimal element with respect the standard partial order in P. Let $u_{m\omega_i}$ a highest-weight generator of $KR_i(\boldsymbol{m})$ and write $\beta = \min R^+_{i,k}$, for $k \geqslant 2$. We denote as $T_i(\boldsymbol{m})$ the quotient of $KR_i(\boldsymbol{m})$ by the submodule generated by

$$(x^-_\beta \otimes t^N)u_{m\omega_i}. \tag{4.3.1}$$

If $\boldsymbol{m} = \xi^{m\omega_i}_N(\alpha_i)$ for some $m, N \in \mathbb{Z}$, by Corollary 4.2.4 and (2.1.6) it follows that there exists an epimorphism

$$T_i(\boldsymbol{m}) \twoheadrightarrow W_N(m\omega_i),$$

while by Proposition 3.2.4 we have the epimorphism $W_N(m\omega_i) \twoheadrightarrow V_i(\boldsymbol{m})$, so that

$$T_i(\boldsymbol{m}) \twoheadrightarrow W_N(m\omega_i) \twoheadrightarrow V_i(\boldsymbol{m}). \tag{4.3.2}$$

The following is [29, Propostion 2.5.2]

Proposition 4.3.1. *If $\boldsymbol{m} = \xi^{m\omega_i}_N(\alpha_i)$ for some $i \in I$, $m \geqslant 0$, $N > 0$, then*

$$T_i(\boldsymbol{m}) \cong W_N(m\omega_i).$$

Proof. Let u be the image of $u_{m\omega_i}$ in $T_i(\boldsymbol{m})$. In light of (4.3.2), we have to prove the existence of a $\mathfrak{g}[t]$-modules epimorphism

$$W_N(m\omega_i) \twoheadrightarrow T_i(\boldsymbol{m}), \tag{4.3.3}$$

which is equivalent to showing that

$$(x^-_\beta \otimes t^N)u = 0 \quad \text{for all} \quad \beta \in R^+. \tag{4.3.4}$$

Let us prove (4.3.4) for $ht_i(\beta) = k$ equals 0 and 1. If $k = 0$, then $(x^-_\beta \otimes 1)u = 0$ and hence (4.3.4) holds.

Now, for $k = 1$, i.e. $\beta \in R^+_{i,1}$, we can assume without loss of generality that $l(\boldsymbol{m}) = r \leqslant N$ and write $u_{m\omega_i} = u_1 * \cdots * u_r$ with u_j being a highest weight vector if $KR_i(\boldsymbol{m})\backslash\{0\}$ for all $1 \leqslant j \leqslant r$. We prove this case by induction on $ht(\beta)$. First, recall that by definition of KR modules we have

$$(x^-_{\alpha_i} \otimes t)u_j = 0 \quad \text{for all} \quad 1 \leqslant j \leqslant r.$$

Thus, if $ht(\beta) = 1$, then $\beta = \alpha_i$ and hence

$$(x^-_{\alpha_i} \otimes t^r)u_{m\omega_i} = 0,$$

which imples that

$$(x^-_\beta \otimes t^N)u_{m\omega_i} = 0.$$

Now, for $ht(\beta) > 1$ and $\beta = \beta_1 + \gamma$, with $\gamma \in R^+$ and $\beta_1 \in R^+_{i,0}$, we have

$$(x_{\beta_1} \otimes 1)u_{m\omega_i} = 0,$$

which, together with the inductive hypothesis, implies $(x_\gamma \otimes t^r)u_{m\omega_i} = 0$. Therefore, since

$$x^-_\beta \otimes t^r = b[x^-_{\beta_1} \otimes 1, x^-_\gamma \otimes t^r], \quad \text{for some} \quad b \in \mathbb{C},$$

(4.3.4) holds for this current case.

Finally, for the inductive step, suppose $\beta \in R^+_{i,k}$ for $k \geqslant 2$. Note that, by definition of $T_i(\boldsymbol{m})$, (4.3.4) holds for

$$\alpha = \beta_k := \min R^+_{i,k} \backslash \{0\}, \quad k \geqslant 2.$$

It follows that there exists $m \geqslant 1$ and γ_j, with $1 \leqslant j \leqslant m$ such that

$$\beta_k + \sum_{j=1}^{n} \gamma_j \in R^+, \quad \forall 1 \leqslant n \leqslant m \quad \text{and} \quad \beta = \beta_k + \sum_{j=1}^{m} \gamma_j. \tag{4.3.5}$$

Since $(x^-_{\gamma_j} \otimes 1)u_{m\omega_i} = 0$ for all $1 \leqslant j \leqslant m$, we have

$$(x_\beta \otimes t^N)u = b[(x^-_{\gamma_m} \otimes 1), \cdots [(x^-_{\gamma_1} \otimes 1), (x^-_{\beta_k} \otimes t^N)] \cdots]u = 0,$$

for some $b \in \mathbb{C}$ and for all $\beta \in R^+$. □

5 The case of multiples of highest root

In this chapter, we review the proof of [45, Theorem 1]. There are some strong implications of this result, namely it shows the conjecture about the independence of parameters for the fusion product of $k-i$ copies of $D(1,\theta)$ and i copies of $ev_0V(\theta)$ and proves Conjecture 3.2.2 for $\mathfrak{g}$ simply laced and λ being the highest root. Besides that, the quotients on its statement are better interpreted in terms of truncated algebras as we see later.

5.1 Quotients of Demazure modules

Given $k \geqslant 1$ and $0 \leqslant i \leqslant k$, consider the following quotient:

$$\mathbb{V}_{i,k} := D(1,k\theta)/\langle (x_\theta^- \otimes t^{2k-i})\overline{w}_{k\theta}\rangle, \tag{5.1.1}$$

and denote by $v_{i,k}$ the image of $\overline{w}_{k\theta}$ in $\mathbb{V}_{i,k}$. Moreover, later in this section, we show that, for $\mathfrak{g}$ simply laced, $\mathbb{V}_{i,k}$ can be seen as a truncated Weyl modules.

By Proposition 2.2.2, the following result is proved

Proposition 5.1.1. *$\mathbb{V}_{i,k}$ is the cyclic graded $\mathfrak{g}[t]$-module generated by $v_{i,k}$ with the following defining relations:*

1. $\mathfrak{n}^+[t]v_{i,k} = 0$;
2. $(h \otimes t^s)v_{i,k} = k\theta(h)\delta_{s,0}v_{i,k} = 0 \quad s \geqslant 0, \quad h \in \mathfrak{h}$;
3. $(x_\alpha^- \otimes 1)v_{i,k} = 0, \quad \alpha \in R^+, \quad (\theta,\alpha) = 0$;
4. $(x_\alpha^- \otimes 1)^{kd_\alpha+1}v_{i,k} = 0, \quad (x_\alpha^- \otimes t^k)v_{i,k} = 0, \quad \alpha \in R^+, \quad (\theta,\alpha) = 1$;
5. $(x_\theta^- \otimes 1)^{2k+1}v_{i,k} = 0, \quad (x_\theta^- \otimes t^{2k-i})v_{i,k} = 0.$

Using (5.1.1), [45, Theorem 1] can be rewriten as

Theorem 5.1.2. *Let $k \geqslant 1$. For $0 \leqslant i \leqslant k$ we have:*

(i) a short exact sequence of $\mathfrak{g}[t]$-modules,

$$0 \to \tau_{2k-1-i}\mathbb{V}_{i,k-1} \xrightarrow{\phi^-} \mathbb{V}_{i,k} \xrightarrow{\phi^+} \mathbb{V}_{i+1,k} \to 0$$

(ii) an isomorphism of $\mathfrak{g}[t]$-modules,

$$\mathbb{V}_{i,k} \cong D(1,\theta)^{*(k-i)} * ev_0V(\theta)^{*i}.$$

Before proving the theorem, let us construct the homomorphisms $\phi^{\pm}$. First observe that

$$(x_\theta^- \otimes t^{2k-i})v_{i+1,k} = 0,$$

by (5.) and (2.1.10), then $v_{i+1,k}$ satisfies the same relations as $v_{i,k}$. Thus the map ϕ^+ can be simply defined as

$$\begin{array}{rcl} \phi^+ : \mathbb{V}_{i,k} & \longrightarrow & \mathbb{V}_{i+1,k} \\ v_{i,k} & \longmapsto & v_{i+1,k} \end{array},$$

which is surjective and $\ker(\phi^+) = \langle (x_\theta^- \otimes t^{2k-1-i})v_{i,k} \rangle$. Indeed, note that $(x_\theta^- \otimes t^{2k-1-i})v_{i,k}$ is non-zero in $\mathbb{V}_{i,k}$, but

$$(x_\theta^- \otimes t^{2k-1-i})v_{i,k} \overset{\phi^+}{\longmapsto} (x_\theta^- \otimes t^{2k-1-i})v_{i+1,k} = 0 \text{ (in } \mathbb{V}_{i+1,k}).$$

In addition, $\ker(\phi^+)$ is determined by Proposition 5.1.1 and its only non-zero elements are $(x_\theta^- \otimes t^{2k-1-i+s})v_{i,k}$, for $s \geqslant 0$ (cf. (2.1.10)). Since we know $\ker(\phi^+)$, the map ϕ^- can be found.

Proposition 5.1.3. *There exists an epimorphism of $\mathfrak{g}[t]$-modules*

$$\phi^- : \tau_{2k-1-i}\mathbb{V}_{i,k-1} \to \ker(\phi^+)$$

sending $v_{i,k-1}$ to $(x_\theta^- \otimes t^{2k-1-i})v_{i,k}$.

Proof. If there exists such morphism of $\mathfrak{g}[t]$-modules, it is surjective, because $v_{i,k-1}$ is mapped to a generator of $\ker(\phi^+)$. In addition, it exists if $(x_\theta^- \otimes t^{2k-1-i})v_{i,k}$ satisfies the defining relations of $\mathbb{V}_{i,k-1}$.

Relations (4.) and (5.) follow imediately from Lemmas 2.1.2 and 2.2.4, while relation (2.) follows from the definition of $D(1,\lambda)$, and (3.) comes from the same argument as item 3. of Proposition 2.2.2. As for relation (1.), the cases $\alpha = \theta$ or $\alpha \neq \theta$ and $(\theta, \alpha) = 0$ are imediate.

Now, if $(\theta, \alpha) = 1$, then $\theta - \alpha$ is also a root and $(\theta, \theta - \alpha) = 1$, hence, by relation (4.) for $\mathbb{V}_{i,k}$, we get $(x_{\theta-\alpha}^- \otimes t^s)v_{i,k} = 0$ for all $s \geqslant k$. Therefore, by equation (2.2.7) we have

$$(x_\alpha^+ \otimes t^s)(x_\theta^- \otimes t^{2k-1-i})v_{i,k+1} = 0.$$

□

The existence of the epimorphisms $\phi^{\pm}$ gives the following inequality

$$\dim \mathbb{V}_{i,k} \leqslant \dim \mathbb{V}_{i,k-1} + \dim \mathbb{V}_{i+1,k} \tag{5.1.2}$$

5.2 Proof of Theorem 5.2.3

In order to prove Theorem 5.1.2, we need to prove the reverse inequality. So, let us check the existence of the map

$$\begin{array}{rcl}\psi : \mathbb{V}_{i,k} & \longrightarrow & D(1,\theta)^{*(k-i)} * \mathrm{ev}_0 V(\theta)^{*i} \\ v_{i,k} & \longmapsto & \overline{w}_\theta^{*(k-i)} * v_\theta^{*i}\end{array}. \tag{5.2.1}$$

Note that $\phi(v_{i,k})$ satisfies the relations of Proposition 5.1.1. Indeed, the definition of $D(1,\theta)$ and $ev_0V(\theta)$ gives, respectively, $(x_\alpha^+ \otimes t^s)\overline{w}_\theta = 0$ and $(x_\alpha^+ \otimes t^s)v_\theta = 0$, for all $s \geqslant 0,\ \alpha \in R^+$. Then, by Lemma 3.2.3, it follows that

$$(x_\alpha^+ \otimes t^s)(\overline{w}_\theta^{*(k-i)} * v_\theta^{*i}) = (x_\alpha^+ \otimes t^s)\psi(v_{i,k}) = 0, \quad \text{for all} \quad s \geqslant 0, \quad \alpha \in R^+,$$

which gives us relation 1.. Now, since $\overline{w}_\theta$ and v_θ are eigenvectors for the action of $(h \otimes t^s)$ with eigenvalue $\delta_{s,0}\theta(h)$ and by the formula

$$(h \otimes t^s)(u_1 \otimes u_2 \otimes \cdots \otimes u_{k-i}) = \sum_{j=1}^{k-i} u_1 \otimes \cdots \otimes (h \otimes t^s)u_j \otimes \cdots \otimes u_{k-i},$$

relation 2. follows, i.e.

$$(h \otimes t^s)(\overline{w}_\theta^{\otimes k-i} \otimes v_\theta^{\otimes i}) = \delta_{s,0}k\theta(h)(\overline{w}_\theta^{\otimes k-i} \otimes v_\theta^{\otimes i}), \quad \text{for all} \quad s \geqslant 0, \quad h \in \mathfrak{h}.$$

Relations 3., 4. and 5. are given by Lemma 3.2.3 together with the relations

$$(x_\alpha^- \otimes t)\overline{w}_\theta = (x_\theta^- \otimes t^2)\overline{w}_\theta = 0 = (x_\theta^- \otimes t)v_\theta = (x_\alpha^- \otimes t)v_\theta, \quad \text{for all} \quad \alpha \in R^+\backslash\{\theta\}.$$

The existence of (5.2.1), gives us the reverse inequality

$$\dim \mathbb{V}_{i,k} \geqslant (\dim D(1,\theta))^{k-i}(\dim V(\theta))^i. \tag{5.2.2}$$

5.2.0.1 Proof of Theorem 5.1.2

Since we have already proved the existence of ϕ^- and ϕ^+, the previous mentioned sequence is exact and part (i) is proved. We want to show the $\mathfrak{g}[t]$-module isomorphism

$$\mathbb{V}_{i,k} \cong D(1,\theta)^{*(k-i)} * ev_0V(\theta)^{*i}, \quad \text{for all} \quad k \geqslant 1 \text{ and } 0 \leqslant i \leqslant k. \tag{5.2.3}$$

Let us proceed by induction on k considering that the case $i = k$ for all $k \geqslant 1$ has been proved in [22, Corollary 2]. Since this result uses some tools that are outside the scope of this work such as PBW filtrations, the proof of this case is omitted.

The case $k = 1$ follows directly from item 3. of Proposition 2.2.2, while for $k = 2$ we have to prove it in the cases i equals 0 and 1, that is

$$\mathbb{V}_{0,2} \cong D(1,\theta)^{*2} \quad \text{and} \quad \mathbb{V}_{1,2} \cong D(1,\theta) * ev_0V(\theta).$$

If $i = 1$, by Proposition 5.1.1, one easily checks that relations on $v_{1,1}$ are exactly the defining relations of $v_\theta \in ev_0V(\theta)$. Therefore, there is a $\mathfrak{g}[t]$-modules isomorphism $\mathbb{V}_{1,1} \cong ev_0V(\theta)$. That isomorphism together with Remark 2.2.3 and (5.1.2) gives us

$$\begin{aligned}\dim \mathbb{V}_{1,2} &\leqslant \dim ev_0V(\theta) + \dim \mathbb{V}_{2,2}\\ &= \dim ev_0V(\theta) + (\dim ev_0V(\theta))^2\\ &= \dim ev_0V(\theta) + (\dim D(1,\theta) - 1)\dim ev_0V(\theta)\\ &= \dim ev_0V(\theta)\dim D(1,\theta).\end{aligned}$$

By (5.2.2) the equality holds and hence we have the isomorphism we wanted. If $i = 0$, in light of part (3.) of Proposition 2.2.2 and (2.1.10), observe that the submodule generated by $(x_\theta \otimes t^2)\overline{w}_\theta$ is zero in $D(1,\theta)$. It follows that $\mathbb{V}_{0,1} \cong D(1,\theta)$ and together with Remark 2.2.3 and (5.1.2), we have

$$\begin{aligned}\dim \mathbb{V}_{0,2} &\leqslant \dim \mathbb{V}_{0,1} + \dim \mathbb{V}_{1,2}\\ &= \dim D(1,\theta) + \dim D(1,\theta)\dim ev_0V(\theta)\\ &= \dim D(1,\theta)(1 + \dim ev_0V(\theta))\\ &= \dim D(1,\theta)(1 + \dim D(1,\theta) - 1)\\ &= (\dim D(1,\theta))^2.\end{aligned}$$

Hence, by (5.2.2) this case follows. Now, suppose $k \geqslant 2$ and that the theorem holds for $k-1$, that is, for every $0 \leqslant i \leqslant k-1$, we have

$$\mathbb{V}_{i,k-1} \cong D(1,\theta)^{*(k-1-i)} * ev_0V(\theta)^{*i}. \tag{5.2.4}$$

In order to prove that this also holds for k, let us proceed by induction on $k - i \geqslant 1$.

For $k - i = 1$, by (5.1.2) we have

$$\begin{aligned}\dim \mathbb{V}_{k-1,k} &\leqslant \dim \mathbb{V}_{k-1,k-1} + \dim \mathbb{V}_{k,k}\\ &= (\dim ev_0V(\theta))^{k-1} + (\dim ev_0V(\theta))^{k}\\ &= (\dim ev_0V(\theta))^{k-1}(1 + \dim ev_0V(\theta))\\ &= (\dim ev_0V(\theta))^{k-1}\dim D(1,\theta)\end{aligned}$$

and this case is proved, by (5.2.2).

Now, for $k - i \geqslant 1$, suppose that (5.2.3) holds for $k - 1 - i$, that is, for all $k \geqslant 2$, we have

$$\mathbb{V}_{i+1,k} \cong D(1,\theta)^{*(k-1-i)} * ev_0V(\theta)^{*i}. \tag{5.2.5}$$

Note that (5.2.4) and (5.2.5) imply

$$\dim \mathbb{V}_{i+1,k} + \dim \mathbb{V}_{i,k-1} = (\dim D(1,\theta))^{k-1-i}(\dim ev_0V(\theta))^{i}(\dim ev_0V(\theta) + 1).$$

Therefore, by (2.2.3), it follows that

$$\dim \mathbb{V}_{i+1,k} + \dim \mathbb{V}_{i,k-1} = (\dim D(1,\theta))^{k-i}(\dim ev_0V(\theta))^{i}. \tag{5.2.6}$$

Finally, using (5.1.2) for $\mathbb{V}_{i,k}$, we get

$$\begin{aligned}\dim \mathbb{V}_{i,k} &\leqslant \dim \mathbb{V}_{i,k-1} + \dim \mathbb{V}_{i+1,k} \\ &= (\dim D(1,\theta))^{k-i}(\dim ev_0 V(\theta))^i\end{aligned}$$

and the theorem follows, since we have (5.2.2). □

Using part (ii) of Theorem 5.1.2, item (i) can be rephrased as a statement about fusion products as follows.

Corollary 5.2.1. *Given $k \geqslant 1$ and $0 \leqslant i \leqslant k$, there exists a short exact sequence of $\mathfrak{g}[t]$-modules,*

$$\begin{aligned}0 \to \tau_{2k-1-i}\left(D(1,\theta)^{*(k-1-i)} * ev_0 V(\theta)^{*i}\right) &\to D(1,\theta)^{*(k-i)} * ev_0 V(\theta)^{*i} \\ &\to D(1,\theta)^{*(k-i)} * ev_0 V(\theta)^{*(i+1)} \to 0.\end{aligned}$$

By using item (ii), we obtain

Corollary 5.2.2. *Given $m, N \geqslant 0$, we have the isomorphism of $\mathfrak{g}[t]$-modules*

$$D(1,\theta)^{*m} * ev_0 V(\theta)^{*N} \cong \mathbb{V}_{N,m+N}.$$

Let us remember that for $\mathfrak{g}$ simply laced, we have $W(\lambda) \cong D(1,\lambda)$ as $\mathfrak{g}[t]$-modules. It follows that

$$W_N(k\theta) \cong \mathbb{V}_{2k-N,k}.$$

Now, using Corollary 5.2.2, we get

Corollary 5.2.3. *Let $\mathfrak{g}$ be a simply laced Lie algebra. Given $k, N \geqslant 1$, we have the following isomorphism of $\mathfrak{g}[t]$-modules.*

$$W_N(k\theta) \cong \begin{cases} W(\theta)^{*N-k} * ev_0 V(\theta)^{2k-N}, & k \leqslant N < 2k \\ W(k\theta), & N \geqslant 2k. \end{cases}$$

5.3 Results for CV modules

Until the end of this section, $\mathfrak{g}$ is a simply laced Lie algebra. In this case, Theorem 5.1.2 can be restated in terms of CV modules as follows.

Given $k \geqslant 1$, $0 \leqslant i \leqslant k$ and $\lambda \in \{(k-1)\theta, k\theta\}$, consider the following λ-compatible partitions:

$$\boldsymbol{\xi}_i^- := (\xi_i^-(\alpha))_{\alpha\in R^+}, \quad \text{where } \xi_i^-(\alpha) = \begin{cases} (1^{((k-1)\theta(h_\alpha))}), & \alpha \neq \theta, \\ (2^{(i)}, 1^{(2(k-1-i))}), & \alpha = \theta, \end{cases}$$

$$\boldsymbol{\xi}_i := (\xi_i(\alpha))_{\alpha\in R^+}, \quad \text{where } \xi_i(\alpha) = \begin{cases} (1^{(k\theta(h_\alpha))}), & \alpha \neq \theta, \\ (2^{(i)}, 1^{(2(k-i))}), & \alpha = \theta, \end{cases}$$

$$\boldsymbol{\xi}_i^+ := (\xi_i^+(\alpha))_{\alpha\in R^+}, \quad \text{where } \xi_i^+(\alpha) = \begin{cases} (1^{(k\theta(h_\alpha))}), & \alpha \neq \theta, \\ (2^{(i+1)}, 1^{(2(k-i))}), & \alpha = \theta. \end{cases}$$

Note that the length of those partitions are

$$l(\xi_i^-(\alpha)) = \begin{cases} (k-1)\theta(h_\alpha), & \alpha \neq \theta, \\ 2k-2-i, & \alpha = \theta, \end{cases}$$
$$l(\xi_i(\alpha)) = \begin{cases} k\theta(h_\alpha), & \alpha \neq \theta, \\ 2k-i, & \alpha = \theta, \end{cases}$$
$$l(\xi_i^+(\alpha)) = \begin{cases} k\theta(h_\alpha), & \alpha \neq \theta, \\ 2k-1-i, & \alpha = \theta, \end{cases}$$

and hence, by (2.3.2), the following relations hold in $V(\xi_i^-)$, $V(\xi_i)$ and $V(\xi_i^+)$ respectively:

$$(x_\theta^- \otimes t^{2k-2-i})v_{\xi_i^-} = 0, \quad (x_\theta^- \otimes t^{2k-i})v_{\xi_i} = 0, \quad \text{and } (x_\theta^- \otimes t^{2k-1-i})v_{\xi_i^+} = 0.$$

Futhermore, it is clear that the generators in $\mathbb{V}_{i,k-1}$, $\mathbb{V}_{i,k}$ and $\mathbb{V}_{i+1,k}$ satisfy the relations of $v_{\xi_i^-} \in V(\xi_i^-)$, $v_{\xi_i} \in V(\xi_i)$ and $v_{\xi_i^+} \in V(\xi_i^+)$, respectively. Therefore, we have proved the following:

Lemma 5.3.1. *For $\mathfrak{g}$ simply laced, there exists the isomorphisms of graded $\mathfrak{g}[t]$-modules*

$$V(\xi_i^-) \cong \mathbb{V}_{i,k-1}, \quad V(\xi_i) \cong \mathbb{V}_{i,k}, \quad \textit{and} \quad V(\xi_i^+) \cong \mathbb{V}_{i+1,k}.$$

Finally, by Theorem 5.1.2, Lemma 5.3.1 and Proposition 2.3.4, the next theorem is proved.

Theorem 5.3.2. *Let $\mathfrak{g}$ be a simply laced Lie algebra. Given $k \geqslant 1$ and $0 \leqslant i \leqslant k$, there exists*

1. *a short exact sequence of $\mathfrak{g}[t]$-modules,*

$$0 \to \tau_{2k-1-i} V(\boldsymbol{\xi}_i^-) \to V(\boldsymbol{\xi}_i) \to V(\boldsymbol{\xi}_i^+) \to 0$$

2. *an isomorphism of $\mathfrak{g}[t]$-modules,*

$$V(\boldsymbol{\xi}_i) \cong V(1^{\theta(h_\alpha)})^{*(k-i)} * V(\{\theta\})^{*i}.$$

6 Demazure Flags

In this chapter, we review some results concerning the concept of Demazure flags.

6.1 Demazure flags for $\mathfrak{sl}_2$

Given $\ell \in \mathbb{Z}_{\geqslant 0}$, $\lambda \in P^+$, set

$$D(\ell, \lambda, m) := \tau_m D(\ell, \lambda), \quad \text{for} \quad m \in \mathbb{Z}. \tag{6.1.1}$$

Definition 6.1.1. *(i) A $\mathfrak{g}[t]$-module V admits a level-ℓ Demazure flag if there exist $k > 0$, $\lambda_1, \cdots, \lambda_k \in P^+$, $m_1, \cdots, m_k \in \mathbb{Z}$ and a sequence of inclusions*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k = V,$$

such that $V_i/V_{i-1} \cong D(\ell, \lambda_i, m_i)$ for all $1 \leqslant i \leqslant k$.

(ii) Let $\mathbb{V}$ be a Demazure flag of V. The multiplicity of a Demazure module D in V is the number

$$[\mathbb{V} : D] := \#\{1 \leqslant i \leqslant k \ : \ V_i/V_{i-1} \cong D\}.$$

It was proved in [14, Lemma 2.1] that the multiplicity is independent of the choice of the flag, so that, for a fixed ℓ, it is denoted as $[V : D]$ instead of $[\mathbb{V} : D]$. Define the generating function

$$[V : D](t) := \sum_{m \in \mathbb{Z}} [V : \tau_m D] t^m \in \mathbb{Z}[t, t^{-1}].$$

Since a Demazure module D is isomorphic as non-graded $\mathfrak{g}[t]$-modules to $\tau_m D$, it can be also interesting to compute the ungraded multiplicity of D in V as follows:

$$[V : D](1) := \sum_{m \in \mathbb{Z}} [V : \tau_m D].$$

We now recall the definition of level-ℓ Demazure module. Given $\lambda \in P^+$, $\alpha \in R^+$ and $\ell \in \mathbb{Z}_{\geqslant 0}$, consider the integers s_α and m_α such that

$$\lambda(h_\alpha) = (s_\alpha - 1)\ell d_\alpha + m_\alpha, \quad 0 < m_\alpha \leqslant \ell d_\alpha. \tag{6.1.2}$$

Let $\xi_{\ell,\lambda}$ be a λ-compatible partition given by

$$\xi_{\ell,\lambda}(\alpha) = ((\ell d_\alpha)^{s_\alpha - 1}, m_\alpha), \quad \alpha \in R^+.$$

Consider the isomorphism

$$V(\xi_{\ell,\lambda}) \cong D(\ell, \lambda). \tag{6.1.3}$$

Let $\mathfrak{g} = \mathfrak{sl}_2$, so that $q = q_{\alpha_1}$ and $p = p_{\alpha_1}$. Identify $\mathbb{Z}$ with P^+ by the assignment $\lambda \mapsto \lambda(h_1)$ consider the partition $\xi_N^\lambda = ((q+1)^{(p)}, q^{(N-p)})$. The following result is [40, Theorem 3.3.2].

Theorem 6.1.2. *There exists a $\mathfrak{g}[t]$-module isomorphism*

$$W_N(\lambda) \cong \begin{cases} D(q,\lambda), & \text{if } N \text{ divides } \lambda \\ D(q+1,\lambda), & \text{if } p \in \{N-1, \lambda\}. \end{cases}$$

Proof. By using Theorem 4.1.2, it suffices to show that ξ_N^λ and $\xi_{\ell,\lambda}$ are equal in each case.

If N divides λ, then $p = 0$ and $\xi_N^\lambda = (q^{N-1}, q)$. Therefore

$$\lambda = Nq = (N-1)q + q.$$

If $p = N-1$, then $\xi_N^\lambda = ((q+1)^{(N-1)}, q) = \xi_{q+1,\lambda}$ and by (6.1.3), this case is proved. Now, note that $p = \lambda$ if and only if $N > \lambda$. Thus, if $N > \lambda$, then $q = 0$ and $\xi_N^\lambda = (1^{(p)}) = (1^{(p-1)}, 1) = \xi_{1,\lambda}$. Hence, by (6.1.3), $V(\xi_N^\lambda) \cong D(1,\lambda) = D(q+1,\lambda)$. $\square$

The next result, proved in [14], gives sufficient conditions for $W_N(\lambda)$ not to be a Demazure module in some cases. Its proof relies on the existence of a short exact sequence of CV modules

$$0 \to \tau_{(s-1)\xi_s} V(\xi^-) \to V(\xi) \to V(\xi^+) \to 0, \tag{6.1.4}$$

wich is given by Theorem 3.2.5.

The following lemma is easily checked (cf. [14, Lemma 2.4]).

Lemma 6.1.3. *If U is a submodule of a $\mathfrak{g}[t]$-module V such that both U and V/U admit a level-ℓ Demazure flag, then so does V.* $\square$

The following result is [14, Lemma 3.7]

Lemma 6.1.4. *Suppose that $r, \ell \in \mathbb{Z}_{>0}$, ω is a fundamental weight and ξ is a partition. If $D(\ell, r\omega)$ is a quotient of $V(\xi)$, then $\ell > \xi_1$.*

Proof. Let v_ξ and $\overline{w}_{r\omega}$ be generators of $V(\xi)$ and $D(\ell, r\omega)$, respectively. Suppose that $D(\ell, r\omega)$ is a quotient of $V(\xi)$. It follows that $r = |\xi|$ and the projection of $V(\xi)$ onto $D(\ell, r\omega)$ maps $v_\xi \mapsto \overline{w}_{r\omega}$. By (2.1.10) and the defining relations of $V(\xi)$ (more precisely the relation obtained from (2.1.3)) we have

$$(x^- \otimes t)^{1+|\xi|-\xi_1} v_\xi = 0$$

and hence

$$(x^- \otimes t)^{1+|\xi|-\xi_1}\overline{w}_{r\omega} = 0.$$

However, it follows from the definition of level-ℓ Demazure module that $(x^- \otimes t)^{r+1}\overline{w}_{r\omega} \neq 0$ in $D(\ell, r\omega)$ whenever $r < \max\{0, |\xi| - \ell\}$. Therefore,

$$|\xi| - \xi_1 \geqslant |\xi| - \ell,$$

which implies that $\ell \geqslant \xi_1$. $\square$

Theorem 6.1.5. *([14, Theorem 3.3]) Let $\xi \in \mathscr{P}_\lambda$, for some $\lambda \in P^+$. Then $V(\xi)$ admits a level-ℓ Demazure flag if and only if $\ell \geqslant \xi_1$. In particular, $D(\ell, \lambda)$ admits a level-ℓ' Demazure flag if and only if $\ell' \geqslant \ell$.* $\square$

Proof. Define a partial order on the set of all partitions $\mathscr{P}$ as follows. Given two partitions $\xi, \xi' \in \mathscr{P}$ with a and b parts, respectively, we say that $\xi \geqslant \xi'$ if and only if $a \geqslant b$ and if $a = b$, then $\xi_a > \xi'_b$. Recall that if $\xi \in \mathscr{P}$ has a parts, then ξ^- has $a - 2$ parts, ξ^+ has a parts and $\xi_a > \xi_a^+$. Therefore,

$$\xi > \xi^+ > \xi^-, \quad \text{for all} \quad \xi \in \mathscr{P}. \tag{6.1.5}$$

We proceed by induction on the number of parts of an arbtrary partition ξ to prove that if $\ell > \xi_1$, then $V(\xi)$ admits a level-ℓ Demazure flag. Let ξ be a partition with number of parts equals 1 with $\ell \geqslant \xi_1$. In this case, it suffices to consider $\xi = \{\ell\}$, in which case we are done by Proposition 2.3.4. Now, suppose that we have proved this statement for any partition with $i - 1$ parts and that ξ is a partition with i parts. The inductive hypothesis can be applied to $\xi^\pm$ by (6.1.5) provided that $m > \xi_1^+$. Therefore, both $\tau_{(i-1)\xi_i}V(\xi^-)$ and $V(\xi^+)$ admit a level-ℓ Demazure flag. Since $V(\xi^+)$ is isomorphic to the quotient of $V(\xi)$ by $\tau_{(i-1)\xi_i}V(\xi^-)$, by (6.1.4), it follows from Lemma 6.1.3 that $V(\xi)$ admits a level-ℓ Demazure flag.

In the case that $\ell < \xi_1^+$, we have $\ell = \xi_1$ and $\xi = (\ell^{(i-1)}, \xi_i)$. It follows that $\xi = \xi_{\ell,(i-1)\ell+\xi_i}$ and hence

$$V(\xi) \cong D(\ell, (i-1)\ell + \xi_i),$$

and there is nothing to prove.

It remains to prove that if $V(\xi)$ has a level-ℓ Demazure flag, then $\ell > \xi_1$, but this follows from Lemma 6.1.4. $\square$

One easily checks that Theorems 4.1.2 and 6.1.5 implies

Corollary 6.1.6. *The module $W_N(\lambda)$ admits a level-ℓ Demazure flag if and only if*

$$\ell \geqslant \begin{cases} q, & \text{if } N \text{ divides } \lambda \\ q+1 & \text{otherwise.} \end{cases}$$

$\square$

Given $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_i \geqslant 0)$ we write $\xi^- = (\xi_1^- \geqslant \cdots \geqslant \xi_{j-2}^- \geqslant \xi_{j-1}^- \geqslant 0)$, where

$$\xi_j^- := \begin{cases} \xi_j, & \text{if } j \leqslant i-1, \\ \xi_{i-1} - \xi_i, & \text{if } r = i-1, \\ 0 & \text{if } j \geqslant i \end{cases}$$

and $\xi^+ = (\xi_1^+ \geqslant \cdots \geqslant \xi_{i-1}^+ \geqslant \xi_i^+ \geqslant 0)$ is the unique partition associated to the sequence $(\xi_1 , \cdots , \xi_{i-2} , \xi_{i-1} + 1 , \xi_i - 1)$.

One easily checks that

$$\xi^+ \in \mathscr{P}_\lambda \quad \text{and} \quad \xi^- \in \mathscr{P}_{\lambda - 2\xi_i}. \tag{6.1.6}$$

In addition, note that

$$\text{If } \xi = \xi_N^\xi \quad \text{and} \quad p \notin \{0, N-1\}, \quad \text{then} \quad \xi_1^\pm = q+1. \tag{6.1.7}$$

Therefore, by Corollary 6.1.6 and (2.2.4), in order to show that $W_N(\lambda)$ is not a Demazure module it is sufficient to show that the length of its level-$(q+1)$ Demazure flag is bigger than 1, so that

$$V_0 \subsetneq V_1 \subseteq W_N(\lambda) \quad \text{with} \quad V_1/V_0 \cong D(q+1, \lambda_1, m_1).$$

This is, indeed, the case, since by (6.1.4) and Theorem 4.1.2 we have the isomorphism $(\tau_{(s-1)(\xi_N^\lambda)_s} V((\xi_N^\lambda)^-)) \oplus V((\xi_N^\lambda)^+) \cong W_N(\lambda)$. This isomorphism implies that the length of its level-$(q+1)$ Demazure flag is the sum of lengths of level-$(q+1)$ Demazure flags of $V((\xi_N^\lambda)^\pm)$, wich shows that $W_N(\lambda)$ is not a Demazure module for such λ and N.

6.2 Existence of length 2 Demazure flags

Next we study some examples from [29] of Demazure flags for truncated Weyl modules. From these examples, a characterization of Weyl modules admitting a Demazure flag of length 2 can be deduced.

Example 6.2.1. *Let $p = N-2 \neq \lambda$. Note that*

$$\xi_N^\lambda = ((q+1)^{(N-2)}, q^{(2)}), \quad (\xi_N^\lambda)^- = ((q+1)^{(N-2)}, 0), \quad \text{and} \quad (\xi_N^\lambda)^+ = ((q+1)^{(N-1)}, q-1).$$

Now, by (2.3.4), it follows that

$$V((\xi_N^\lambda)^-) \cong D(q+1, \lambda - 2q) \quad \text{and} \quad V((\xi_N^\lambda)^+) \cong D(q+1, \lambda),$$

which, together with Theorem 6.1.4, gives the following short exact sequence

$$0 \to D(q+1, \lambda - 2q, (N-1)q) \to W_N(\lambda) \to D(q+1, \lambda) \to 0. \tag{6.2.1}$$

Denote by soc(M) the socle of a $\mathfrak{g}[t]$-module, i.e. the largest semisimple submodule of M. Let us compute the socle of $W_N(\lambda)$ in the case $\lambda = 4$ and $N = 3$. In this case, note that $p = q = 1$, so that (6.2.1) becomes

$$0 \to V(2,2) \to W_3(4) \to D(2,4) \to 0,$$

since $D(2,2,2) \cong V(2,2)$. Now, by definition of Demazure module, there exist the isomorphisms $D(2,0) \cong ev_0V(0)$ and $D(2,1) \cong ev_0V(1)$, so that we have the short exact sequences

$$V(0,2) \to D(2,4) \to D(3,4) \quad \text{and} \quad V(2,1) \to D(3,4) \to V(4,0).$$

Thus $D(2,4) \cong V(0,2) \oplus D(3,4)$, $D(3,4) \cong V(2,1) \oplus V(4,0)$ and hence

$$W_3(4) \cong V(2,2) \oplus V(0,2) \oplus V(2,1) \oplus V(4,0).$$

Therefore, soc$(W_3(4)) \cong W_3(4)[2] \cong V(2,2) \oplus V(0,2)$.
This example shows us that while all Weyl modules have simple socle, truncated Weyl modules may have non simple socle.

Example 6.2.2. *Suppose $p = N - 3 \neq \lambda$. In this case, we have*

$$\xi_N^\lambda = ((q+1)^{(N-3)}, q^{(3)}), \quad (\xi_N^\lambda)^- = ((q+1)^{(N-3)}, q), \quad \text{and } (\xi_N^\lambda)^+ = ((q+1)^{(N-2)}, q, q-1).$$

Note that $(\xi_N^\lambda)^- = ((q+1)^{(N-3)}, q) = \xi_{\ell,\lambda'}$, where $\lambda' = \lambda - 2q$. Hence, by (6.1.3),

$$V((\xi_N^\lambda)^-) = V(\xi_{\ell,\lambda'}) \cong D(q+1, \lambda - 2q).$$

For $q = 1$ it follows that we have a length-2 Demazure flag

$$0 \to D(2, \lambda - 2, N-1) \to W_N(\lambda) \to D(2,\lambda) \to 0. \tag{6.2.2}$$

Otherwise, if $q \neq 1$, (6.1.3) implies that $V((\xi_N^\lambda)^+)$ admits a length-2 Demazure flag

$$0 \to D(q+1, \lambda - 2q, (N-1)(q-1)) \to V((\xi_N^\lambda)^+) \to D(q+1,\lambda) \to 0,$$

which means that $W_N(\lambda)$ does not admit a length-2 Demazure flag.

Now, let $\lambda = 5$ and $N = 4$, so that $p = q = 1$ (in particular, $p = N - 3$). Hence, (6.2.2) becomes

$$0 \to D(2,3,3) \to W_4(5) \to D(2,5) \to 0,$$

which implies that $W_4(5) \cong D(2,3,3) \oplus D(2,5)$. One easily checks, using (6.1.4), (3.2.10), and Proposition 2.3.4, that

$$D(2,3,3) \cong V(1,4) \oplus V(3,3)$$

and

$$D(2,5) \cong V(5,0) \oplus V(3,1) \oplus V(3,2) \oplus V(1,2) \oplus V(1,4).$$

We claim that $soc(W_4(5))$ *is simple, so that it is isomorphic to* $V(1,4)$*. In order to prove that claim, note that* $soc(W_4(5))$ *is simple if and only if there exists* $v \in W_4(5)[3]_1$ *such that*

$$(h \otimes t^r)v = (x^+ \otimes t^r)v = 0 \quad \text{for all} \quad h \in \mathfrak{h}[t] \text{ and } r \geqslant 0$$

where $W_N(5)_k$ *denotes the subspace whose elements have weight* k*. Now, consider* $v \in W_4(5)_5$ *being nonzero. Note that* $W_4(5)[3]_1 = span\{(x^- \otimes 1)(x^- \otimes t^3)v,\ (x^- \otimes t^2)(x^- \otimes t^1)v\}$*. In particular, observe that* $(x^- \otimes t^r)w = 0$ *for* $w \in W_4(5)[3]_1$*,* $r \geqslant 0$ *if and only if*

$$w = k((x^- \otimes t)(x^- \otimes t^2) - (x^- \otimes 1)(x^- \otimes t^3))v, \quad \text{for some} \quad k \in \mathbb{C}.$$

It follows that $(h \otimes t)((x^- \otimes t)(x^- \otimes t^2) - (x^- \otimes 1)(x^- \otimes t^3))v = -2(x^- \otimes t^2)(x^- \otimes t^2)v$ *and hence we are left to check that* $-2(x^- \otimes t^2)(x^- \otimes t^2)v$ *is nonzero. But this is clear, since* $W_4(5)[4]_1 = span\{(x^- \otimes t^2)(x^- \otimes t^2)v,\ (x^- \otimes t)(x^- \otimes t^3)v\}$ *is non-trivial and*

$$0 = \mathbf{x}^-_{\alpha_1}(2,4)v = ((x^- \otimes t^2)(x^- \otimes t^2) + 2(x^- \otimes t)(x^- \otimes t^3))v.$$

The next Proposition can be easily deduced from Examples 6.2.1 and 6.2.2.

Proposition 6.2.3. *Let* $p \notin \{0, N-1\}$*. The level-*$(q+1)$ *Demazure flag of* $W_N(\lambda)$ *has length 2 if and only if either* $p = N-2$ *or* $p = N-3$ *and* $q = 1$*.*

Proof. The first paragraphs of Examples 6.2.1 and 6.2.2 prove that if $p \in \{N-3, N-2\}$ and $q = 1$, then $W_N(\lambda)$ admits a level-$(q+1)$ Demazure flag of legth 2.

The comment right after (6.2.2) shows that if $W_N(\lambda)$ has a length 2 Demazure flag, then $q = 1$. Moreover, one can check that for $p = N - (3+i)$ with $i \geqslant 1$, the CV module $V((\xi_N^\lambda)^+)$ is not a Demazure module, so that, by Theorem 3.2.5, $W_N(\lambda)$ does not admit a level-$(q+1)$ Demazure flag. □

In order to understand the implications of the simplicity of the socle of a $\mathfrak{g}[t]$-module, let us recall the definitions of the radical, socle and grading series.

Given a $\mathfrak{g}[t]$-module W, a semisimple filtration for W is a sequence of inclusions

$$0 = F_0W \subseteq \cdots \subseteq F_kW = W \tag{6.2.3}$$

such that the quotient $F_iW/F_{i-1}W$ is semisimple for all $0 \leqslant i \leqslant k$. The **radical** of W, denoted by $rad(W)$ is the intersection of all submodules of W such that the corresponding quotient is semisimple. The quotient $W/rad(W)$, which is called head of W, allows us to define the radical series as follows:

Write $rad_0(W) = W$ and then define

$$rad_k(W) := rad(rad_{k-1}(W)) \quad \text{for} \quad k \geqslant 1.$$

inductively. Hence, we have a semisimple filtration for W

$$0 \subset rad_k(W) \subset \cdots \subset rad(W) \subset rad_0(W) = W$$

called **radical series**.

The socle of a $\mathfrak{g}[t]$-module W can be defined as the sum of all simple submodules of W. Now, write $soc_0(W) = 0$ and define

$$soc_k(W) := soc\left(\frac{W}{soc_{k-1}W}\right) = \frac{soc_k(W)}{soc_{k-1}(W)}, \quad \text{for} \quad k \geqslant 1.$$

Therefore, there exists a semisimple filtration for W

$$0 = soc_0(W) \subset soc(W) \subset soc_2(W) \subset \cdots \subset W,$$

which is called **socle series**.

Finally, if W is a $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module, the filtration

$$F_kW = \bigoplus_{k \leqslant s} W[s]$$

is called **grading series**.

Remark 6.2.4. *One easily checks that $W_N(\lambda)$ has simple head. On the other hand, it was proved in [34, Lemma 2.3] that if the head of a finite-dimensional module over a $\mathbb{C}$-algebra is simple, then the grading and radical series coincide. Moreover, the same result states that if the socle of that module is simple, then the socle and grading series coincide.*

Although Example 6.2.1 shows that $W_N(\lambda)$ may have non-simple socle and [34, Lemma 2.3] does not guarantee the coincidence for a non-simple case, Example 6.2.1 shows that the coincidence is possible in that case.

Given $n, m, \ell \in \mathbb{Z}_{\geqslant 0}$, consider the following λ-compatible partition

$$\xi^{\ell}_{s,r} = ((\ell+1)^{(s)}, \ell^{(r)})$$

and write $\lambda^{\ell}_{s,r} = \ell(s+r) + s$. One easily checks that

$$\xi^{\lambda}_{N} = \xi^{q}_{p,N-p}, \tag{6.2.4}$$

which is equivalent to

$$V(\xi^{\ell}_{s,r}) \cong W_{s+r}(\lambda^{\ell}_{s,r}). \tag{6.2.5}$$

In particular,

$$V(\xi^{\ell-1}_{N,0}) = V(\xi^{\ell}_{0,N}) \cong D(\ell, N\ell) \cong W_N(N\ell). \tag{6.2.6}$$

Now, if we write $\lambda = \lambda^1_{s,r}$ and $N = s + r$ with $r \neq 0$, Theorem 6.1.4 immediately implies

$$0 \to \tau_{N-1} W_{N-1-\delta_{p,N-1}}(\lambda - 2) \to W_N(\lambda) \to W_{N-1}(\lambda) \to 0. \tag{6.2.7}$$

Given $\mu \in P^+$, consider the function

$$\gamma^{\ell}_{s,r}(\mu, t) = [V(\xi^{\ell}_{s,r}) : D(\ell + 1, \mu)](t). \tag{6.2.8}$$

These functions were studied in [4, 14]. In [4] the autors have greatly extended the results of [14], namely they proved that the numerical multiplicity for the case $\ell = 1$ is a racional number given by

$$\gamma^1_{0,\lambda}(\lambda - 2k, t) = [W(\lambda) : D(2, \lambda - 2k)](t) = t^{\lceil \frac{\lambda}{2} \rceil} \begin{bmatrix} \lfloor \frac{\lambda}{2} \rfloor \\ k \end{bmatrix}_t \tag{6.2.9}$$

for all $0 \leqslant k \leqslant \lfloor \dfrac{\lambda}{2} \rfloor$, where

$$\begin{bmatrix} m \\ k \end{bmatrix}_t = \prod_{j=0}^{k-1} \frac{1 - t^{m-j}}{1 - t^{k-j}} \quad \text{for} \quad 0 \leqslant k \leqslant m. \tag{6.2.10}$$

However, these functions are far from being completely understood.

From now on we consider $s > 0$. Observe that if r is equal to 0 or 1, then $\xi^{\ell}_{s,r} = \xi_{\ell+1, \lambda^{\ell}_{s,r}}$. Therefore, by (6.1.3),

$$V(\xi^{\ell}_{s,r}) \cong D(\ell + 1, \lambda^{\ell}_{s,r}).$$

In particular, if $r = 0, 1$, then

$$\gamma^{\ell}_{s,r}(\mu, t) = \delta_{\lambda^{\ell}_{s,r}, \mu}.$$

Now, for $r > 1$, by (6.2.7), it follows that

$$\gamma^1_{s,r}(\mu, t) = \gamma^1_{s-1,r+2}(\mu, t) - \gamma^1_{s-1,r}(\mu, t) t^{s+r}. \tag{6.2.11}$$

Hence, together with (6.2.9), there is a recursive procedure to compute $\gamma^1_{s,r}(\mu, t)$. As we will see next, one can reach an alternative formula for $\gamma^1_{s,r}(\mu, t)$.

Let $\xi = (\xi_1, \cdots, \xi_k) \in \mathscr{P}_\lambda$ for some $k \in \mathbb{Z}_{\geqslant 0}$. By removing the largest part of ξ, we obtain a $(\lambda - \xi_1)$-compatible partition given by

$$\xi^* = (\xi_2, \cdots, \xi_k).$$

In particular,

$$(\xi^{\ell}_{s,r})^* = \xi^{\ell}_{s-1,r} \quad \text{for some} \quad s \geqslant 0. \tag{6.2.12}$$

It is also important to consider the following equality proved in [14, Lemma 3.8].

$$[V(\xi)\ :\ D(\xi_1,\mu)](t) = t^{\frac{\lambda-\mu}{2}}[V(\xi^*)\ :\ D(\xi_1,\mu-\xi_1)](t), \tag{6.2.13}$$

which says that if $\xi_1 > \mu$, then

$$[V(\xi),:\ D(\xi_1,\mu)] = 0.$$

By (6.2), one easily checks that if we iterate (6.2.13), then

$$\gamma^{\ell}_{s,r}(\mu,t) = t^{\frac{s}{2}(\lambda^{\ell}_{s,r}-\mu)}\gamma^{\ell}_{0,r}(\mu-s(\ell+1),t). \tag{6.2.14}$$

In particular, by (6.2.4), we have

$$[W_N(\lambda) : D(q+1,\mu)](t) = t^{\frac{p}{2}(\lambda-\mu)}\ [D(q,q(N-p)) : D(q+1,\mu-p(q+1))](t). \tag{6.2.15}$$

The following result is [29, Corollary 4.2.7].

Proposition 6.2.5. *Let $\lambda \in P^+$. Given $N > 1$ such that $N \leqslant \lambda < 2N$, then, for all $0 \leqslant k \leqslant \lambda/2$, we have*

$$[W_N(\lambda) : D(2,\lambda-2k)](t) = \begin{cases} t^{k\lceil\frac{\lambda}{2}\rceil}\begin{bmatrix} N-\lceil\frac{\lambda}{2}\rceil \\ k\end{bmatrix}_t, & \text{if } k \leqslant N-\lceil\dfrac{\lambda}{2}\rceil, \\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let $\lambda,\mu \in P^+$ be such that $\lambda = N+p$ for $0 \leqslant p < N$ and $\mu = \lambda - 2k$. Note that

$$[W_N(\lambda) : D(2,\mu)](t) = \gamma^1_{p,N-p}(\mu,t).$$

By (6.2.14), we have

$$\begin{aligned}\gamma^1_{p,N-p}(\mu,t) &= t^{\frac{p}{2}(\lambda-\mu)}\gamma^1_{0,N-p}(\mu-2p,t)\\ &= t^{pk}\gamma^1_{0,N-p}(N+p-2k-2p,t)\\ &= t^{pk}\gamma^1_{0,\lambda-2p}(\lambda-2p-2k,t),\end{aligned}$$

which, together with (6.2.9), implies

$$\gamma^1_{p,N-p}(\mu,t) = t^{pk}[W(\lambda-2p)\ :\ D(2,\lambda-2p-2k)](t) = t^{k\lfloor\frac{\lambda-2p}{2}\rfloor}\begin{bmatrix}\lceil\frac{\lambda-2p}{2}\rceil \\ k\end{bmatrix}_t = t^{(k\lfloor\frac{\lambda}{2}\rfloor-p)}\begin{bmatrix}\lceil\frac{\lambda}{2}\rceil-p \\ k\end{bmatrix}_t$$

for $k \leqslant N - \lceil\dfrac{\lambda}{2}\rceil$, and, since $N-\lambda = -p$, this case is done.

Otherwise, i.e. $2k > \lambda - 2p = 2N-\lambda$, we have $\gamma^1_{p,N-p}(\mu,t) = 0$. □

It follows from [14, Section 3.8] that, for every partition ξ,

$$[V(\xi)\ :\ D(\ell,\mu)](t) \neq 0$$

implies that $|\xi| - 2\mu \in 2\mathbb{Z}_{\geqslant 0}$, so that the generating function

$$L_\xi^\ell(x,t) = \sum_{k=0}^{\lfloor \frac{|\xi|}{2} \rfloor} [V(\xi) : D(\ell, |\xi| - 2k)](t)\ x^k$$

allows us to compute the length of the level-ℓ Demazure flag, which is given by

$$L_\xi^\ell := L_\xi^\ell(1,1).$$

Therefore, letting $\xi = \xi_N^\lambda$, if λ and N are as in Proposition 6.2.5, then

$$L_\xi^2 = \sum_{k=0}^{N-\lceil \frac{\lambda}{2} \rceil} \binom{N - \lceil \frac{\lambda}{2} \rceil}{k} = 2^{N - \lceil \frac{\lambda}{2} \rceil}.$$

6.3 Chains of inclusions

In the context of local Weyl modules, the discussion about chains of inclusions is mostly related with bases for $W(\lambda)$, more precisely, the stability of these bases (see [46, 47, 48] for more details). However, by following [29], we study only the existence of chains of inclusions for truncated Weyl modules.

Let $\lambda \in P^+$ and identify P^+ with $\mathbb{Z}_{\geqslant 0}$. Proposition 2.3.4 together with Theorem 3.2.5 implies the inclusion

$$\tau_{\lambda-1} W(\lambda - 2) \hookrightarrow W(\lambda).$$

One easily checks that the image of this inclusion is the submodule generated by $(x \otimes t^k) w_\lambda$ for all $k \geqslant \lambda - 1$ and $x \in \mathfrak{n}^-$. Therefore, by (2.1.6) there exists the following short exact sequence

$$0 \to \tau_{\lambda-1} W(\lambda-2) \to W(\lambda) \to W_{\lambda-1}(\lambda) \to 0. \tag{6.3.1}$$

If we write $N = \lambda$ and $M = \lambda - 1$ and consider the projection (2.1.7), then we have, up to grade shift, the isomorphism $W(\lambda-2) \cong W_{\lambda-2}(\lambda-2)$. Hence, (6.3.1) can be rewriten as

$$0 \to \tau_{\lambda-1} W_{\lambda-2}(\lambda-2) \to W_\lambda(\lambda) \to W_{\lambda-1}(\lambda) \to 0. \tag{6.3.2}$$

One interesting approach is characterizing all chains of inclusions of truncated Weyl modules, e.g. if $\lambda = Nq + p$ for $0 \leqslant p < N$ and either $p < N - 1$ or $q = 1$, then Theorem 3.2.5 implies

$$\tau_{(N-1)q} W_{N-2}(\lambda - 2q) \hookrightarrow W_N(\lambda).$$

Note that the quotient of $W_N(\lambda)$ by the submodule generated by

$$(x \otimes t^{(N-1)q}) w_\lambda, \quad \text{with} \quad x \in \mathfrak{n}^-$$

is a truncated Weyl module if and only if $q = 1$, which results exactly in (6.2.7). Although for $q > 1$ and $p = N - 1$ Theorem 3.2.5 does not result in an inclusion of truncated Weyl modules as before, a second application does. This inclusion is given by

$$\tau_{N-2} \tau_{(N-1)q} W_{N-2}(\lambda - 2(q+1)) \hookrightarrow W_N(\lambda).$$

Recall that $\pi_{N,M}$ denotes the projection (2.1.7). If $M = N - 1$, we write π_N instead of $\pi_{N-1,N}$. In addition, it follows from (2.1.6) that the kernel of π_N is the submodule of $W_N(\lambda)$ generated by $(x^- \otimes t^{N-1})v$, for v being the highest weight generator of $W_N(\lambda)$. Observe that

$$\pi_{N,M} = \pi_{M+1} \circ \cdots \circ \pi_N, \quad \text{for all} \quad M < N.$$

Moreover, $(x^- \otimes t^{N-1})v$ satisfies the same relations as the cyclic generator of $W_N(\lambda - 2)$ and hence we have the graded $\mathfrak{g}[t]$-module epimorphism

$$\varphi_N : \tau_{N-1} W_N(\lambda - 2) \twoheadrightarrow \ker(\pi_N).$$

If we write $\delta_N(\lambda) = \dim(W_N(\lambda))$, Theorems 4.1.2 and 3.2.5 imply

$$\delta_N(\lambda) = (q+2)^p (q+1)^{N-p}.$$

Since $\delta_N(\lambda) = \delta_{N-1}(\lambda) + \dim\ker(\pi_N)$, one easily checks that φ_N is an isomorphism if and only if

$$\delta_N(\lambda) - \delta_N(\lambda - 2) = \delta_{N-1}(\lambda). \tag{6.3.3}$$

Note that $\xi_N^{\lambda-2}$ and ξ_{N-1}^{λ} are given by

$$\xi_N^{\lambda-2} = \begin{cases} ((q+1)^{(p-2)}, q^{(N-p+2)}), & \text{if } p \geqslant 2, \\ (q^{(N-2+p)}, (q-1)^{2-p}), & \text{if } p = 0, 1 \end{cases}$$

and

$$\xi_{N-1}^{\lambda} = ((q+q'+1)^{(p')}, (q+q')^{(N-1-p')})$$

with $p + q = (N-1)q' + p'$, $0 \leqslant p' < N - 1$. Moverover, for $N = 2$ we have $p = 0, 1$, so that

$$\delta_2(\lambda) = \begin{cases} (q+1)^2, & \text{for } p = 0 \\ (q+2)(q+1), & \text{for } p = 1, \end{cases}$$

$$\delta_2(\lambda - 2) = \begin{cases} q^2, & \text{for } p = 0 \\ q(q+1), & \text{for } p = 1, \end{cases}$$

$$\delta_1(\lambda) = \begin{cases} 2q+1, & \text{for } p = 0 \\ 2(q+1), & \text{for } p = 1. \end{cases}$$

Therefore, for $N = 2$ (6.3.3) aways holds and hence the following short sequence is exact

$$0 \to \tau_1 W_2(\lambda) \to W_2(\lambda) \to V(\lambda) \to 0.$$

Observe that if $N > 2$ and $q = 1$, then $N - 1 - \delta_{p,N-1} \geqslant \lambda - 2$ if and only if $p = 0, 1$, so that (6.2.7) implies that φ_N is injective if and only if λ is N or $N + 1$. For this reason, we can assume $q > 1$. The following example is the smallest case where φ is not injective.

Example 6.3.1. *Let $N = 3$ and $\lambda = 6$. Since we have assumed $q > 1$, then $q = 2$ and hence*

$$\delta_3(6) = 27, \quad \delta_2(6) = 16 \quad \textit{and} \quad \delta_3(4) = 12.$$

It follows that (6.3.3) does not hold and hence ϕ_3 is not injective. Note that $W_3(6) \cong D(2,6)$ by Theorem 6.1.2 and, in particular, $W_3(6)$ has simple socle.

Let us prove by contradiction that φ is not injective. Suppose that φ_3 is injective. It follows that $W_3(6)$ contains a submodule isomorphic to $\tau_2 W_3(4)$, but, by Example 6.2.1, soc$(W_3(4))$ is not simple, which is a contradiction. Hence φ_3 is non-injective. The same example says that there exists a short exact sequence

$$0 \to V(0,2) \to W_3(4) \xrightarrow{\varphi_3} \ker(\pi_3) \to 0,$$

since $\dim(\ker(\pi_3)) = \delta_3(6) - \delta_2(6) < \delta_3(4)$. *Moreover, one can check that* $\ker(\pi_3)$ *is not a CV module just because it would imply that* $\ker(\pi_3) \cong V(\xi)$, *for ξ being a partition of 4, but* $\dim(V(\xi)) \neq 11$ *by Theorem 3.2.5.*

However, one easily checks that there exists an inclusion of truncated Weyl modules in $W_3(6)$, which is exactly the inclusion of soc$(W_3(6)) \cong V(2,2) \cong \tau_2 W_1(2)$ given by Theorem 3.2.5, that is

$$0 \to \tau_2 W_1(2) \to W_3(6) \to V(\xi) \to 0,$$

where $\xi = (3,2,1)$. One can check that this inclusion is, in fact, the only inclusion of truncated Weyl modules for $W_3(6)$.

7 Final remarks and further steps

In this chapter, we state some results of [23] about the conjecture on the defining relations for fusion products as an example of direction that we can pursue in the future. From now on, $\mathfrak{g}$ is of type A.

7.1 Defining relations for fusion products

Let $\lambda_1, \lambda_2 \in P^+$ and $\lambda = \lambda_1 + \lambda_2$. Denote by $I(\lambda_1, \lambda_2)$ the left ideal of $U(\mathfrak{g}[t])$ generated by

$$\mathfrak{n}^+[t], \quad h_\alpha \otimes 1 - \lambda(h_\alpha), \quad \mathfrak{h} \otimes t\mathbb{C}[t]$$

and

$$(x_\alpha^- \otimes 1)^{\lambda(h_\alpha)+1}, \quad (x_\alpha^- \otimes t)^{\min\{\lambda_1(h_\alpha), \lambda_2(h_\alpha)\}+1}, \quad \mathfrak{n}^- \otimes t^2\mathbb{C}[t],$$

with $\alpha \in R^+$. We have the following conjecture (see [23, Conjecture 2.7] and references therein).

Conjecture 7.1.1. *If $a_1 \neq a_2 \in \mathbb{C}$, then there exists an isomorphism of graded $\mathfrak{g}[t]$-modules*

$$ev_{a_1}V(\lambda_1) * ev_{a_2}V(\lambda_2) \cong U(\mathfrak{g}[t])/I(\lambda_1, \lambda_2).$$

This conjecture has been proved for $\mathfrak{sl}_2$ in [20] and for $\mathfrak{sl}_n$ with certain conditions on λ_1 and λ_2. In particular, for $\mathfrak{sl}_n$, many cases involving multiples of fundamental weights were discussed in [26].

Conjecture 7.1.1 can be reformulated into the language of Gröbner bases as we will see later. Denote by $V(\lambda_1, \lambda_2, a_1, a_2)$ the fusion product $ev_{a_1}V(\lambda_1) * ev_{a_2}V(\lambda_2)$ and note that the automorphism $x \otimes t \mapsto x \otimes (t-a)$ induces an automorphism ϕ_a of $U(\mathfrak{g}[t])$. Now, we have

$$\phi_a^* V(\lambda_1, \lambda_2, a_1, a_2) \cong V(\lambda_1, \lambda_2, 0, a_2 - a_1), \tag{7.1.1}$$

which reduces the analysis of fusion products with parameters in $\mathbb{C} \times \mathbb{C}$ to the ones with parameters in $\{0\} \times (\mathbb{C}\backslash\{0\})$.

Lemma 7.1.2. *[23, Lemma 2.6] Let $\lambda_1, \lambda_2 \in P^+$, $a \in \mathbb{C}\backslash\{0\}$. Then $V(\lambda_1, \lambda_2, 0, a)$ is isomorphic to the quotient of $U(\mathfrak{g}[t])$ by the left ideal $I_a(\lambda_1, \lambda_2)$ generated by*

$$\mathfrak{n}^+[t], \quad h \otimes 1 - (\lambda_1 + \lambda_2)(h), \quad h \otimes t - a\lambda_2(h), \quad \textit{for all } h \in \mathfrak{h}$$

and

$$x \otimes t^2 - ax \otimes t, \quad (x_\alpha^- \otimes 1)^{(\lambda_1+\lambda_2)(h_\alpha)+1}, \quad (x_\alpha^- \otimes t)^{\lambda_2(h_\alpha)\}+1}, \quad (x_\alpha^- \otimes (t-a))^{\lambda_1(h_\alpha)+1}.$$

for all $x \in \mathfrak{g}$, $\alpha \in R^+$. □

For each choice of admissible order in $U(\mathfrak{g}[t])$, the tools from the theory of Gröbner bases can be used in the family of left ideals $I_a(\lambda_1, \lambda_2)$. The following was conjectured in [23].

Conjecture 7.1.3. *$I_a(\lambda_1, \lambda_2)$ is a flat family of left ideals (over $\mathbb{C}[a]$) in $U(\mathfrak{g}[t])$, i.e. the quotient $U(\mathfrak{g}[t])/I_0(\lambda_1, \lambda_2)$ is a special fiber of a Gröbner degeneration. In addition, there exists a monomial ordering on $U(\mathfrak{g}[t])$ such that the leading term ideals of $I_a(\lambda_1, \lambda_2)$ and $I(\lambda_1, \lambda_2)$ coincide.*

Now, we have a sufficient condition for Conjecture 7.1.1 to be true. The following result is [23, Theorem 2.8].

Theorem 7.1.4. *Conjecture 7.1.3 implies Conjecture 7.1.1.*

Proof. Note that Conjecture 7.1.3 implies

$$\dim U(\mathfrak{g}[t])/I_a(\lambda_1, \lambda_2) = \dim U(\mathfrak{g}[t])/I(\lambda_1, \lambda_2). \tag{7.1.2}$$

On the other hand, denoting by $v_1 * v_2$ the cyclic generator of $V(\lambda_1, \lambda_2, a_1, a_2)$, one easily checks that $v_1 * v_2$ satisfies the same relations as the ones determined by the ideal $I(\lambda_1, \lambda_2)$. Hence we have a $\mathfrak{g}[t]$-module epimophism

$$U(\mathfrak{g}[t])/I(\lambda_1, \lambda_2) \twoheadrightarrow V(\lambda_1, \lambda_2, a_1, a_2),$$

which is actualy an isomorphism, by (7.1.1), (7.1.2), and Lemma 7.1.2. □

7.2 Gröbner basis

Before defining the Gröbner bases, we need to introduce some notation.

Let $\leqslant$ be a total ordering on the monoid $(\mathbb{Z}^n_{\geqslant 0}, +)$. Then $\leqslant$ is called **admissible** if $a \leqslant b$ implies $a + c \leqslant b + c$ for all $a, b, c \in \mathbb{Z}^n_{\geqslant 0}$. Let L be an algebra over a field K generated by $x_1, \cdots, x_n$. The set of **standard monomials** of L is

$$Mon(L) := \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \ : \ \alpha_i \in \mathbb{Z}_{\geqslant 0}\}.$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$. It is also usefull to introduce the notation

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Given an admissible ordering $\leqslant$ on $\mathbb{Z}^n_{\geqslant 0}$, one can check that any $f \in (span_K Mon(A)) \backslash \{0\}$ is uniquely represented by $\sum_{\alpha \in \mathbb{Z}^{\kappa}{}_{\geqslant 0}} b_\alpha x^\alpha$ for some $b_\alpha \in K$ with $a_\alpha \neq 0$ only for finitely many

α. With respect to $\leqslant$, for $f \in (span_K Mon(A))\backslash\{0\}$, we denote the **leading exponent**, the **leading coefficient**, and the **leading monomial** of f respectively by

$$\begin{aligned} lexp_{\leqslant}(f) &:= \max_{\leqslant}\{\alpha \in \mathbb{Z}^n_{\geqslant 0} \,:\, b_\alpha \neq 0\}; \\ lc_{\leqslant}(f) &:= b_{lexp_{\leqslant}(f)} \in K\backslash\{0\}; \\ lm_{\leqslant}(f) &:= x^{lexp_{\leqslant}(f)} \in Mon(A). \end{aligned}$$

Definition 7.2.1. *Let $n \in \mathbb{N}$, $1 \leqslant i < j \neq n$. Consider nonzero scalars $b_{ij} \in K$, and polynomials $p_{ij} \in K[x_1, \cdots, x_n]$. Given an admissible ordering $\leqslant$ in $\mathbb{Z}^n_{\geqslant 0}$ such that for $1 \leqslant i < j \leqslant n$ either $p_{ij} = 0$ or $lexp(p_{ij}) \leqslant lexp(x_i x_j)$ holds, the K-algebra*

$$A := K\langle x_1, \cdots, x_n \,:\, x_j x_i = b_{ij} x_i x_j + p_{ij} \text{ for all } 1 \leqslant i < j \leqslant n\rangle$$

is called a G-algebra if $Mon(A)$ is a K-basis of A. Moreover, we say that A is a Lie type G-algebra if all $b_{ij} = 1$.

The G-algebras can be equivalently defined via algebraic relations involving b_{ij} and p_{ij} as explained in [38]. These algebras are left and right noetherian domains and are known as algebras of solvable type and as PBW algebras. One easily checks that, given a finite-dimensional Lie algebra $\mathfrak{g}$, then $U(\mathfrak{g})$ is a G-algebra of Lie type.

Definition 7.2.2. *Let A be a G-algebra with a fixed monomial ordering $\succ$ and let I be a left ideal of A. A subset $G \subset I$ is a left Gröbner basis of I, if for all $f \in I$, there exists $h \in G$ such that $lexp(h) \leqslant lexp(f)$ componentwise.*

Remark 7.2.3. *In the context of commutative rings, a Gröbner basis of an ideal can be computed via Buchberger's algorithm (see [18, Buchberger's algorithm 15.9, page 333]). Although an ideal can have more than one Gröbner basis, this ideal has a unique reduced Gröbner basis, whose elements are minimal in some sense. Moreover, for non-commutative rings, one can prove that it is always possible to find a finite left Gröbner basis for every ideal of a G-algebra. These bases are usually constructed using the generalized Buchberger's algorithm (see [37, 38]).*

The next step is to study the effect of certain degenerations for current algebras. First, consider the polynomial $p = t^r - \sum_{i=0}^{m-1} b_i t^i \in \mathbb{C}[t]$ where[1] $b_i \in \mathbb{C}$. Now, given a basis $\{x_1, \cdots, x_n\}$ of $\mathfrak{g}$, one can check that the algebra $U(\mathfrak{g} \otimes \mathbb{C}[t]/\langle p\rangle)$ is a G-algebra, which admits a Gröbner basis given by

$$\{x_i \otimes t^j \,:\, 0 < i \leqslant n \text{ and } 0 \leqslant j \leqslant r\}.$$

Next, for $a \in \mathbb{C}$, consider the ideal $I_a \subset U(\mathfrak{g}[t])$ generated by $\mathfrak{g} \otimes (t^2 - at)$. One easily checks the isomorphism

$$U(\mathfrak{g}[t])/I_a \cong U\left((\mathfrak{g} \otimes \mathbb{C}[t])/\langle \mathfrak{g} \otimes (t^2 - at)\rangle\right) \cong U(\mathfrak{g} \otimes (\mathbb{C}[t]/\langle t^2 - at\rangle)).$$

[1] One can also consider $b_i \in \mathbb{C}[a_1, \cdots, a_q]$ for the indeterminants a_i being central with respect to $\mathfrak{g}$.

Consider a monomial admissible ordering $\succ$ on the homogeneous monomials $\{(x \otimes t^k) \in U(\mathfrak{g}[t]) \,:\, k \in \mathbb{Z}_{\geqslant 0}\}$ compatible with the natural grading on $\mathbb{C}[t]$. In $\langle t^2 - at \rangle \subset \mathbb{C}[t]$, note that the ideal of leading terms with respect to $\succ$ is independent on a and thus $\mathbb{C}[t]/\langle t^2 - at \rangle$ defines a flat family. This implies that the ideal of leading terms of I_a is generated by $\mathfrak{g} \otimes t^2$, which does not depend on a and hence the quotient $U(\mathfrak{g}[t])/I_a$ is a Gröbner degeneration, i.e. this quotient is a flat family of $\mathbb{C}[a]$-modules.

Geometrically, a Gröbner degeneration defines a linear bundle. In the case of $U(\mathfrak{g}[t])/I_a$, the quotient $U(\mathfrak{g}[t])/I_0$ is a special fiber. If we consider the left ideal $\overline{I_a}$ of $U(\mathfrak{g}[t])/I_a$ consisting of all elements whose leading terms are independent of a, we have the following.

Theorem 7.2.4. *[23, Theorem 3.3] The quotient $(U(\mathfrak{g}[t])/I_a)/\overline{I_a}$ is a flat family of $\mathbb{C}[a]$-modules, i.e. $(U(\mathfrak{g}[t])/\langle t^2 \rangle)/\overline{I_0}$ is the special fiber of a Gröbner degeneration.* □

7.3 Gröbner basis for fusion products

Now, we recall a conjecture on Schur positivity, which was stated for the first time in [17]. Let $\lambda \in P^+$, consider the same set as in (3.2.3), that is

$$P^+(\lambda, 2) = \{(\lambda_1, \lambda_2) \in P^+ \times P^+ \,:\, \lambda_1 + \lambda_2 = \lambda\}$$

with a partial order defined by

$$(\lambda_1, \lambda_2) \succ (\mu_1, \mu_2) \text{ if and only if } (\lambda_1 - \lambda_2)(h_\alpha) \leqslant (\mu_1 - \mu_2)(h_\alpha), \text{ for all } \alpha \in R^+,$$

which is equivalent to (3.2.4).

Let us review the notion of character of a $\mathfrak{g}$-module consider the free abelian group consisting of the isomorphism classes $[V]$ of finite-dimensional representations V of $\mathfrak{g}$ and divide it by the relation

$$[V] = [V'] + [V''],$$

whenever $V = V' \oplus V''$. This quotient is called representation ring and is denoted by $R(\mathfrak{g})$, whose product is given by

$$[V] \cdot [W] := [V \otimes W].$$

We now recall the definition of $\mathbb{Z}[P]$, the group ring of P, which is the free $\mathbb{Z}$-module with basis $\{e^\lambda \,:\, \lambda \in P\}$. The addition on $\mathbb{Z}[P]$ is denoted by $e^\lambda + e^\mu$ and its product is given by $e^\lambda e^\mu := e^{\lambda+\mu}$. One can check that there exists a ring homomorphism

$$Char : R(\mathfrak{g}) \longrightarrow \mathbb{Z}[P]$$

sending $[V]$ to $\sum_{\lambda \in P} \dim V_\lambda e^\lambda$. In addition, the image of $Char$ is contained in the ring $\mathbb{Z}[P]^{\mathcal{W}}$ of invariants in $\mathbb{Z}[P]$ under the action of the Weyl group $\mathcal{W}$.

Theorem 7.3.1. *[24, Theorem 23.24]*

(i) The representation ring $R(\mathfrak{g})$ is a polynomial ring on the variables $\Gamma_1, \cdots, \Gamma_n$, where Γ_i is the class in $R(\mathfrak{g})$ of the irreducible representation with highest weight ω_i, for $i = 1, \cdots, n$.

(ii) The homomorphism Char : $R(\mathfrak{g}) \longrightarrow \mathbb{Z}[P]^{\mathcal{W}}$ is an isomorphism.

Since the symmetric group S_{n+1} is the Weyl group of $\mathfrak{sl}_{n+1}$, the previous theorem implies that $\mathbb{Z}[P]^{S_{n+1}}$ is a polynomial ring on the variables $Char(V(\omega_1)), \cdots, Char(V(\omega_n))$ consisting of symmetric polynomials.

The Schur polynomials are a generalization of the symmetric elementary polynomials that form a basis of the ring of polynomials in n variables up to the relation $x_1 \cdots x_n - 1$. If write $s_\lambda = \mathrm{Char}(V(\lambda))$, one can check that the Schur polynomial of the partition λ coincides with s_λ

Definition 7.3.2. *A symmetric polynomial in n variables is called Schur positive if it is a non-negative linear combination of Schur polynomials.*

The following is the previously mentioned conjecture on Schur positivity

Conjecture 7.3.3. *Let $\lambda \in P^+$. If $(\lambda_1, \lambda_2) \succ (\mu_1, \mu_2)$ in $(P^+(\lambda, 2), \succ)$, then $s_{\lambda_1}s_{\lambda_2} - s_{\mu_1}s_{\mu_1}$ is Schur positive.*

About this conjecture, it has been proved in [10] the following:

(i) If $(\lambda_1, \lambda_2) \succ (\mu_1, \mu_2)$ in $P^+(\lambda, 2)$, then

$$\dim V(\lambda_1) \otimes V(\lambda_2) \geqslant \dim V(\mu_1) \otimes V(\mu_2)$$

(ii) The conjecture is true for $\mathfrak{sl}_3$.

Now, one of the main results of [23] is related to both the conjecture on Schur positivity and Conjecture 3.2.2.

Theorem 7.3.4. *[23, Theorem 4.8] Conjecture 7.1.3 implies Conjecture 7.3.3 and also Conjecture 3.2.2 for $N = 2$.* □

Finally, for $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda \geqslant \mu$ being two dominant weights, it was computed in [23, Theorem 5.7] a Gröbner basis for the ideal $I_a(\lambda, \mu)$ as follows:

First, fix $a \in \mathbb{C}$ and non-negative integers $\lambda \geqslant \mu$. Again, we write

$$x_1^{\pm} = x^{\pm} \quad \text{and} \quad h_1 = h.$$

Now, consider the element

$$F_i(\lambda,\mu) = \sum_{k=0}^{i} c_{ki}(-a)^{i-k}(x^-\otimes t)^k(x^-\otimes 1)^{m_i-k},$$

where

$$m_i = \lambda+\mu+1-i \quad \text{and} \quad c_{ji} = \binom{m_i}{j}\binom{\mu-j}{\mu-i}.$$

The commutator relations between $\{(x^\pm\otimes t^j),(h\otimes t^j)\ :\ j=0,1\}$ and $F_i(\lambda,\mu)$ were computed in [23, Lemma 5.2] and [23, Lemma 5.4].

Second, denote by $\succ$ the monomial admissible ordering on both $\{x^-\otimes 1, x^-\otimes t\}$ and $\{x^\pm\otimes t^j, h\otimes t^j\ :\ j=0,1\}$ satisfying $(x^-\otimes t)\succ(x^-\otimes 1)$.

Theorem 7.3.5. *[23, Theorem 5.7] If $\lambda\geqslant\mu\in\mathbb{Z}_{\geqslant 0}$ are dominant weights for $\mathfrak{sl}_2$ and $\succ$ is the monomial admissible ordering chosen as above, then the following is a Gröbner basis with respect to $\succ$ of the left ideal $I_a(\lambda,\mu)$.*

$$\{(x^+\otimes 1),(x^+\otimes t),(h\otimes 1)-(\lambda+\mu),(h\otimes t)-\mu a\}\cup\{F_i(\lambda,\mu)\ :\ i=0,\cdots\mu)\}.$$

□

Remark 7.3.6. *By [23, Proposition 5.8], Conjecture 7.1.3 holds for $\mathfrak{sl}_2$ and hence, by Theorem 7.1.4, Conjecture 7.1.1 holds for $\mathfrak{sl}_2$, which is already known due to [20]. However, this approach might allow a generalization to $\mathfrak{sl}_n$.*

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