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# Fractional Derivatives: Generalizations

# Derivadas fracionárias: generalizações

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Daniela dos Santos de Oliveira

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*(Deuteronômio 31,6)*

# Resumo

Neste trabalho apresentamos generalizações para as derivadas fracionárias. Inicialmente discutimos, a partir de uma modificação do tipo Caputo nas derivadas fracionárias generalizadas, as chamadas derivadas fracionárias generalizadas do tipo Caputo. Discutimos algumas de suas propriedades e, como uma aplicação, apresentamos o teorema fundamental do cálculo fracionário envolvendo estes operadores de diferenciação fracionários. Após discutir as derivadas fracionárias generalizadas do tipo Caputo, apresentamos uma outra proposta para a generalização dos operadores de diferenciação fracionários. Esta generalização consiste em uma derivada fracionária do tipo Hilfer a qual está associada às integrais fracionários generalizadas propostas por Katugampola. Denominamos por Hilfer-Katugampola estas derivadas fracionárias. Discutimos algumas propriedades, bem como o problema de Cauchy envolvendo estes novos operadores de diferenciação. Por fim, de modo a generalizar ainda mais as derivadas fracionárias, propomos as derivadas $(k, \rho)$-fracionárias generalizadas. Esta formulação, mais geral do que as anteriores recupera, como casos particulares, as derivadas fracionárias mencionadas anteriormente e, ainda mais, recupera as derivadas fracionárias de Hilfer, Hilfer-Katugampola, Hilfer-Hadamard, $(k, \rho)$-fracionária, Riemann-Liouville generalizada, Katugampola generalizada, Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard; bem como, para particulares valores dos extremos de integração, esta também recupera as derivadas fracionárias de Weyl e Liouville.

**Palavras-chave**: Generalização para derivadas fracionárias. Modificação do tipo Caputo. Derivada fracionárias de Hilfer-Katugampola. Derivadas $(k, \rho)$-fracionárias generalizadas. Problema de Cauchy. Existência e unicidade.

# Abstract

In this thesis we present generalizations for fractional derivatives. Initially we discuss, by means of a Caputo-type modification of the generalized fractional derivatives, the so-called generalized Caputo-type fractional derivatives. We discuss some of their properties and, as an application, we present the fundamental theorem of fractional calculus involving these fractional differentiation operators. After discussing generalized Caputo-type fractional derivatives, we present another proposal for the generalization of fractional differentiation operators. This generalization consists of a Hilfer-type fractional derivative whose associated fractional integrals are the generalized fractional integrals proposed by Katugampola. We call these fractional derivatives Hilfer-Katugampola fractional derivatives. We discuss some properties, as well as a Cauchy problem involving these new fractional differentiation operators. Finally, in order to further generalize the fractional derivatives, we propose the generalized $(k,\rho)$-fractional derivatives. This formulation, more general than the previous ones, recovers, as particular cases, the fractional derivatives previously mentioned and, furthermore, recovers the fractional derivatives of Hilfer, Hilfer-Katugampola, Hilfer-Hadamard, $(k,\rho)$-fractional, Riemann-Liouville generalized, Katugampola generalized, Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard; moreover, for particular values of integration extremes, it also recovers the fractional derivatives of Weyl and Liouville.

**Keywords**: Generalization for fractional derivatives. Caputo-type modification. Hilfer-Katugampola fractional derivatives. Generalized $(k,\rho)$-fractional derivatives. Cauchy problem. Existence and uniqueness.

# Contents

# Introduction

The number of different fractional integration and differentiation operators has been increasing along the last few years. Classical fractional derivatives, among which we mention the Riemann-Liouville, Hadamard and Caputo derivatives [17, 38], have found many applications in physics and other fields of science. These non-integer order derivatives are usually expressed in terms of a corresponding fractional integral. Fractional integrals have also been widely used, for instance, in some recent generalizations involving Minkowski's inequality [58] and Grüss inequality [62]. As a fractional derivative is a non-local operation, and also admits a singular kernel, the classical fractional derivative incorporates the memory effect [42] when it is used to model physical processes[1] involving time evolution. In the early 2010's, several formulations of fractional derivatives have appeared in the literature [8, 33], distinct from the classical ones, as the new derivatives are defined by means of a limit process [32, 60, 59]. Furthermore, one has recently defined [9, 41] a fractional derivative and a corresponding fractional integral whose kernel can be a non-singular function such as a Mittag-Leffler function [64]. More recently, a new definition of fractional derivative with respect to a function was also presented [61]. Maybe the reason why there are so many approaches to fractional derivatives and integrals lies in the fact they do not have a classical geometrical interpretation as in ordinary differential calculus. However, using an adequate limit process, non-integer order derivatives can be transformed in integer order derivatives in Newton's sense and Leibniz's sense.

In this thesis our main interest is the study of fractional derivatives which are expressed in terms of a fractional integral with a singular kernel. There are three kernels to be considered, namely:

$$\text{(i)}\quad (x-t)^{\frac{\alpha}{k}-1} \qquad \text{(ii)}\quad t^{-1}\left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1} \qquad \text{(iii)}\quad t^{\rho-1}\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1},$$

where $k>0$ and $\rho>0$. When $\rho\to 1$ in (iii), we obtain the kernel given by (i) and when $\rho\to 0^+$ in (iii) we have an indeterminate form. Using $\ell$'Hôpital rule we obtain kernel (ii).

[1] By model of a physical process we mean a fractional system composed of a fractional differential equation and initial conditions and/or boundary conditions.

There are three different ways to define a fractional derivative. The first one consists in applying an integer order derivative on the left side of a fractional integral. The second one is to apply a fractional integral to the left side of an integer order derivative. Finally, the third way employs a differentiation operator of integer order acting between two fractional integrals, where one fractional integral is at the left side of this operator and the other is at the right side. The integer order differentiation operators considered here are:

$$\text{(a) } D^n \equiv \left(\frac{d}{dx}\right)^n, \qquad \text{(b) } \delta^n \equiv \left(x\frac{d}{dx}\right)^n, \qquad \text{(c) } \delta_\rho^n \equiv \left(x^{1-\rho}\frac{d}{dx}\right)^n,$$

where $n-1 < \alpha \leqslant n$ with $n \in \mathbb{N}$ and $\rho > 0$. Note that, when $\rho \to 1$ in (c), we obtain the operator defined in (a) and when $\rho \to 0^+$, we recover the operator given by (b). The derivatives of non-integer order with which we work throughout this thesis are combinations of fractional integrals with kernels given by (i), (ii) and (iii) with the integer order derivatives shown in (a), (b) and (c). Notice that when $n = 1$, we restrict the order of our derivatives to $0 < \alpha \leqslant 1$. We work mainly with fractional derivatives for which $k = 1$ in (i), (ii) e (iii).

Classical fractional derivatives —Riemann-Liouville, Hadamard and Caputo— are defined according to different schemes. The Riemann-Liouville derivative consists in applying the differentiation operator (a) to a fractional integral whose kernel is given by (i). The Hadamard derivative of arbitrary order applies the differentiation operator (b) to a fractional integral with kernel given by (ii). Finally, the Caputo derivative is defined according to the second scheme, that is, the differentiation operator given in (a) is used in the integrand of a fractional integral whose kernel is given by (i).

Let us consider the fractional derivatives defined according to the first way. In 2011, Katugampola [31] defined a generalized fractional integral whose kernel is the function (iii). We emphasize that, as kernel (iii) a generalization of kernels (i) and (ii), it is possible to recover, from this fractional integral, the Riemann-Liouville and Hadamard integrals. The same author [30], in 2014, proposed, from the generalized fractional integral, what he called a generalized fractional derivative which consists in applying operator (c) to the left of the generalized fractional integral.

Other authors employ the second way for the definition of fractional derivatives. In 2012, Jarad, Abdeljawad and Baleanu [27], motivated by the Caputo and Hadamard formulations, proposed the so-called Caputo-Hadamard fractional derivative. This definition consists in taking a Caputo-type modification of the Hadamard fractional derivative, namely, in introducing the differentiation operator into the integrand of fractional integration. Thus, one simply inserts operator (b) into the integrand of the Hadamard fractional integral, whose kernel, as previously mentioned, is given by (ii). Starting with this last proposal for fractional derivative, in 2016, Almeida, Malinowska and Odzijewicz [3] introduced the

Caputo-Katugampola fractional derivative. By means of a Caputo-type modification of the generalized fractional derivative, the integer order differentiation operator (c), with $n = 1$, was introduced into the integrand of the generalized fractional integral whose kernel is given by (iii). Using this definition, with an adequate choice of parameters, it is possible to obtain, as particular cases, the Caputo and Caputo-Hadamard fractional derivatives. However, in that paper the authors discuss only the case $0 < \alpha \leqslant 1$. Thus, in 2017, we proposed the generalized Caputo-type fractional derivative [48] of order $n-1 < \alpha \leqslant n$ with $n \in \mathbb{N}$ and, as particular cases, we recover the Caputo and Caputo-Hadamard fractional derivatives.

Finally, we mention the fractional derivatives defined by means of the third scheme. In 2000, Hilfer [22] proposed a non-integer order derivative which, for particular values of the order of derivation, recovers the formulations proposed by Riemann-Liouville and Caputo and, for particular values of the extreme of integration, also recovers the Weyl and the Liouville fractional derivatives [23]. This approach considers the differentiation operator (b), with $n = 1$, acting between two Riemann-Liouville fractional integrals. In 2012, Hilfer, Lucko and Tomoviski [24] proposed the generalized Riemann-Liouville fractional derivative. This definition follows the same reasoning that led to the Hilfer derivative, but without the restriction $n = 1$, that is, considering $n \in \mathbb{N}$. In the same year, Kassim, Furati and Tatar [28] defined the Hilfer-Hadamard fractional derivative, in which the differential operator (b), with $n = 1$, acts between two Hadamard fractional integrals. In order to generalize Hilfer and Hilfer-Hadamard fractional derivatives, in 2017, we proposed the Hilfer-Katugampola fractional derivative [47]. This definition uses the differentiation operator (c), with $n = 1$, acting between two generalized fractional integrals. Considering the differential operator (c), with $n = 1$, and the generalized fractional integral whose kernel contains function (iii), the derivative proposed by us recovers the Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, generalized and generalized Caputo-type derivatives, as well as the Weyl and Liouville fractional derivatives, for particular cases of the extremes of integration.

On the other hand, there are other ways to generalize fractional integrals and derivatives, for example, by inserting a new parameter. The fractional derivatives which we will mention in the sequence consider the kernels (i), (ii) and (iii) with $k > 0$. Thus, in 2012, Mubben and Habibullah [44], inserted the parameter $k > 0$ in a Riemann-Liouville fractional integral in order to obtain the so-called Riemann-Liouville $k$-fractional integral with kernel given by (i). To obtain the classical Riemann-Liouville fractional integral by means of this definition one just needs to consider $k = 1$. In 2015, Farid and Habibullah [16] introduced the Hadamard $k$-fractional integral, with kernel given by (ii). In the case $k = 1$ one recovers the Hadamard fractional integral. In 2016, Sarikaya et al. [56] proposed the $(k, \rho)$-fractional integral, whose kernel is given by (iii). As in the previous cases, one just needs to consider $k = 1$ to obtain the generalized fractional integral.

As the fractional derivatives that we study in this thesis are defined in terms of a respective fractional integral, it will be useful to define $k$-fractional derivatives. In 2013, Dorrego and Cerutti [15] defined the Hilfer $k$-fractional derivative, in which the differentiation operator (a), with $n = 1$, acts between two Riemann-Liouville $k$-fractional integrals. Recently, in 2017, Nisar et al. [46] defined the $(k, \rho)$-fractional derivative. In their definition, they consider operator (c), with $n = 1$, acting between two $(k, \rho)$-fractional integrals. In order to generalize the derivative of arbitrary order proposed by Nisar et al., we used operator (c) with $n \in \mathbb{N}$ and defined the so-called generalized $(k, \rho)$-fractional derivative [49]. In this definition we use operator (c), with $n \in \mathbb{N}$, acting between two $(k, \rho)$-fractional integrals.

This thesis is organized as follows. In Chapter 1 we present notations, definitions and the main results of real analysis which will be used throughout the work. Chapter 2 is dedicated to present the concepts and some properties of Hadamard fractional integrals and derivatives. Our main contribution in this chapter is to present a Leibniz-type rule for Hadamard fractional derivatives. In Chapter 3 we develop the theory and some properties of Caputo-Hadamard fractional derivatives; our greatest contribution is the proof of a Leibniz-type rule for these fractional differential operators. Chapter 4 is dedicated to an explanation of generalized Caputo-type fractional derivatives, which were originally proposed by us. In Chapter 5, we investigate the Hilfer-Katugampola fractional derivatives. Finally, Chapter 6 brings our main results, as generalized $(k, \rho)$-fractional derivatives constitute the most general case for fractional derivatives. Concluding remarks and future perspectives close the thesis, which contains also two appendices with some calculations used in the main text.

Chapter 1

# Preliminaries

It is necessary to introduce and discuss some concepts, which are not our main object of study, but will be useful in the next chapters. We present notations, theorems, properties, definitions and, also, the so-called special functions. The definitions involving left-sided integrals also hold for the right-sided integrals. These results are important to define the fractional integrals and derivatives which we call classics. The results presented in this chapter can be found in some textbooks and papers as [14, 13, 17, 20, 38, 45, 51, 55].

## 1.1 Notations, Definitions and Function Spaces

First, we fix the following notation which we use throughout the text.

**Notation 1.1.** $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *where* $\mathbb{N} = \{1, 2, \ldots\}$.

**Notation 1.2.** $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$, *where* $\mathbb{Z}^- = \{-1, -2, \ldots\}$.

**Notation 1.3.** *Let* $\alpha \in \mathbb{R}$, *then* $[\alpha]$ *denotes the integer part of* $\alpha$.

The Lipschitz condition on $f(x, \varphi)$ with respect to the second variable is defined as follows.

**Definition 1.1.** *Assume that* $f(x, \varphi)$ *is defined on the set* $(a, b] \times G$, $G \subset \mathbb{R}$. *A function* $f(x, \varphi)$ *satisfies Lipschitz condition with respect to* $\varphi$, *if for all* $x \in (a, b]$ *and for* $\varphi_1, \varphi_2 \in G$,

$$|f(x, \varphi_1) - f(x, \varphi_2)| \leqslant A|\varphi_1 - \varphi_2|,$$

*where* $A > 0$ *does not depend on* $x$.

The Dirichlet formula, used to interchange the order of integration in double integrals, is a particular case of the Fubini's theorem [55, p. 9] and is given by:

$$\int_a^b dx \int_a^x f(x, y)dy = \int_a^b dy \int_y^b f(x, y)dx. \tag{1.1}$$

**Definition 1.2.** [51] *If $f : [a, b] \to \mathbb{R}$ is continuous and if $g$ is integrable on $[a, x]$ and $g \geqslant 0$, then there exists a number $\xi_x$ in $[a, b]$ such that*

$$\int_a^x f(t)g(t)dt = f(\xi_x)\int_a^x g(t)dt. \tag{1.2}$$

Next we present the classical Banach fixed point theorem in a complete metric space.

**Theorem 1.1.** [38] *Let $(U, d)$ be a nonempty complete metric space; let $0 \leqslant \omega < 1$, and let $T : U \to U$ be the map such that, for every $u, v \in U$, the relation*

$$d(Tu, Tv) \leqslant \omega d(u, v), \quad (0 \leqslant \omega < 1) \tag{1.3}$$

*holds. Then the operator $T$ has a unique fixed point $u^* \in U$.*
*Furthermore, if $T^k$ $(k \in \mathbb{N})$ is the sequence of operators defined by*

$$T^1 = T \quad \textit{and} \quad T^k = TT^{k-1}, \quad (k \in \mathbb{N}\backslash\{1\}), \tag{1.4}$$

*then, for any $u_0 \in U$, the sequence $\{T^k u_0\}_{k=1}^{\infty}$ converges to the above fixed point $u^*$.*

The map $T : U \to U$ satisfying condition Eq.(1.3) is called a contractive map. Now we exhibit the adequate space functions which we use to define the fractional integrals [38].

**Definition 1.3.** *Let $\Omega = [a, b]$ $(-\infty \leqslant a < b \leqslant \infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $X_c^p(a, b)$ $(c \in \mathbb{R},\ 1 \leqslant p \leqslant \infty)$ the set of those complex-valued Lebesgue measurable functions $\varphi$ on $[a, b]$ for which $\|\varphi\|_{X_c^p} < \infty$, with*

$$\|\varphi\|_{X_c^p} = \left(\int_a^b |x^c\varphi(x)|^p \frac{dx}{x}\right)^{1/p}, \quad 1 \leqslant p < \infty \tag{1.5}$$

*and*

$$\|\varphi\|_{X_c^\infty} = \operatorname{ess\,sup}_{a \leqslant x \leqslant b} [x^c|\varphi(x)|], \tag{1.6}$$

*where* ess sup $|x^c\varphi(x)|$ *denotes the essential supremum of function $|x^c\varphi(x)|$. In particular, when $c = 1/p$, the space $X_c^p(a, b)$ coincides with the classical $L_p(a, b)$-space with*

$$\|\varphi\|_p = \left(\int_a^b |\varphi(x)|^p dx\right)^{1/p}, \quad 1 \leqslant p < \infty \tag{1.7}$$

*and*

$$\|\varphi\|_p = \operatorname{ess\,sup}_{a \leqslant x \leqslant b} |\varphi(x)|, \tag{1.8}$$

*where* ess sup $|\varphi(x)|$ *denotes the essential supremum of function $|\varphi(x)|$. We denote $L_1(a, b)$ by $L(a, b)$.*

**Definition 1.4.** *Let $[a,b]$ be a finite interval and let $AC[a,b]$ be the space of functions $\varphi$ which are absolutely continuous on $[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:*

$$\varphi(x) \in AC[a,b] \Longleftrightarrow \varphi(x) = c + \int_a^x f(t)dt, \quad f(t) \in L(a,b), \tag{1.9}$$

*and therefore an absolutely continuous function $\varphi(x)$ has a summable derivative $\varphi'(x) = f(x)$ almost everywhere on $[a,b]$. Thus* Eq.(1.9) *yields*

$$f(t) = \varphi'(t) \quad \textit{and} \quad c = \varphi(a). \tag{1.10}$$

**Definition 1.5.** *For $n \in \mathbb{N}$ we denote by $AC^n[a,b]$ the space of complex-valued functions $\varphi(x)$ which have continuous derivatives up to order $(n-1)$ on $[a,b]$ such that $\varphi^{(n-1)}(x) \in AC[a,b]$:*

$$AC^n[a,b] = \left\{\varphi : [a,b] \to \mathbb{C} : (D^{n-1}\varphi)(x) \in AC[a,b], D = \frac{d}{dx}\right\}, \tag{1.11}$$

*with $\mathbb{C}$ being the set of complex numbers. When $n = 1$, the space $AC^1[a,b] = AC[a,b]$.*

This space is characterize by assertion below, presented as a lemma.

**Lemma 1.1.** *The space $AC^n[a,b]$ consists of those and only those functions $\varphi(x)$ which can be represented in the form*

$$\varphi(x) = (\mathcal{I}^n_{a^+} f)(x) + \sum_{k=0}^{n-1} c_k (x-a)^k, \tag{1.12}$$

*where $f(t) \in L(a,b)$, $c_k$ $(k = 0, 1, \ldots, n-1)$ are arbitrary constants, and*

$$(\mathcal{I}^n_{a^+} f)(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt.$$

*Proof.* See Lemma 2.4 in [55]. □

It follows from Eq.(1.12) that

$$f(t) = \varphi^{(n)}(t), \qquad c_k = \frac{\varphi^{(k)}(a)}{k!} \quad (k = 0, 1, \ldots, n-1). \tag{1.13}$$

We also use a weighted modification of the space $AC^n\,[a,b]$ $(n \in \mathbb{N})$, in which the usual derivative $D = d/dx$ is replaced by the so-called $\delta$-derivative, defined by

$$\delta = xD = \left(x\frac{d}{dx}\right).$$

**Definition 1.6.** *Let $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$, the space $AC^n_{\delta_\rho}[a,b]$ with $-\infty < a < b < \infty$ consists of those complex-valued functions $\varphi$ which have continuous derivatives up to order $(n-1)$ on $[a,b]$ such that $\delta_\rho^{n-1}\varphi \in AC[a,b]$ is absolutely continuous on $[a,b]$:*

$$AC^{\,n}_{\delta_\rho}[a,b] = \left\{\varphi : [a,b] \to \mathbb{R} : \delta_\rho^{n-1}\varphi(x) \in AC\,[a,b], \delta_\rho = \left(x^{1-\rho}\frac{d}{dx}\right)\right\}.$$

*When $\rho \to 1$ we have $D = d/dx$, that is, $AC^n_{\delta_\rho}[a,b] = AC^n[a,b]$. On the other hand, in the case $\rho \to 0^+$ we have $\delta = x\,d/dx$, this is, $AC^n_{\delta_\rho}[a,b] = AC^n_\delta[a,b]$.*

**Definition 1.7.** *Let $\Omega = [a,b]\,(-\infty \leqslant a < b \leqslant \infty)$ and $m \in \mathbb{N}_0$. We denote by $C^m(\Omega)$ a space functions $f$ which are $m$ times continuously differentiable on $\Omega$ with the norm*

$$\|f\|_{C^m} = \sum_{k=0}^{m} \|f^{(k)}\|_C = \sum_{k=0}^{m} \max_{x\in\Omega} |f^{(k)}(x)|, \quad m \in \mathbb{N}_0.$$

*In particular, for $m = 0$, $C^0(\Omega)$ is the space of continuous functions $f$ on $\Omega$ with the norm*

$$\|f\|_C = \max_{x\in\Omega} |f(x)|.$$

## 1.2 Special Functions

We present the concept of $k$-$*$, where $*$ denoting gamma function, beta function or Pochhammer symbol in order to introduce the $k$-Mittag-Leffler function in Chapter 6, [14, 13, 20, 45]. In this section we only define the three- and two-parameters Mittag-Leffler functions [17]. The special functions and their properties are defined for complex numbers, but here we consider real values, only.

### 1.2.1 The $k$-Gamma and Incomplete Gamma Function

The Euler's gamma function $\Gamma(z)$, or gamma function only, generalizes the factorial and allows non-integer and complex values [6, 17, 38]. In order to generalize this function Díaz and Pariguan [13] defined the $k$-gamma function by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-\frac{t^k}{k}}\,dt, \quad \text{with} \quad x, k > 0, \tag{1.14}$$

satisfying the following relations, which can be easily proved.

1. $\Gamma_k(x+k) = x\Gamma_k(x)$;
2. $\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\dfrac{x}{k}\right)$;
3. $\Gamma_k(k) = 1$;
4. $\Gamma_k(x)\Gamma_k(k-x) = \dfrac{\pi}{\sin\left(\frac{\pi x}{k}\right)}$.

Recently, Mubeen and Rehman [45] discussed the following limit involving the $k$-gamma function, for $k > 0$,

$$\lim_{n\to\infty} \frac{\Gamma_k(a+nk)}{\Gamma_k(b+nk)}(nk)^{\frac{b}{k}-\frac{a}{k}} = 1, \tag{1.15}$$

where $a + nk, b + nk \in \mathbb{R}\backslash\{0, -k, -2k, \ldots\}$ and $n \in \mathbb{N}$. In the case $k \to 1$, we have $\Gamma_k(x) = \Gamma(x)$ and the well-known properties associated with the gamma function are recovered, [6].

On the other hand, the incomplete gamma function $\gamma(\nu, x)$ is defined for $\nu, x \in \mathbb{R}$ by the integral

$$\gamma(\nu, x) = \int_0^x t^{\nu-1} \mathrm{e}^{-t} dt, \quad \nu > 0. \tag{1.16}$$

### 1.2.2 The $k$-Pochhammer Symbol

Díaz and Pariguan [13] generalized the Pochhammer symbol by means of insertion of a new parameter, $k > 0$, in order to obtain the $k$-Pochhammer symbol given by

$$(x)_{n,k} = \begin{cases} 1, & \text{for} \quad n = 0 \\ x(x+k)\cdots(x+(n-1)k), & \text{for} \quad n \in \mathbb{N},\ x \in \mathbb{R}, \end{cases} \tag{1.17}$$

or, in terms of a quotient of $k$-gamma functions,

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{1.18}$$

For $k \to 1$ we recover the classical Pochhammer symbol, i.e., $(x)_{n,k} = (x)_n = \Gamma(x+n)/\Gamma(x)$, [17].

### 1.2.3 The $k$-Beta Function

The beta function, $B(x, y)$, is defined by the Euler integral of the first kind [38]. Thus, Díaz and Pariguan [13] generalized this function from the following integral

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt, \quad x > 0, \quad y > 0, \quad k > 0. \tag{1.19}$$

Notice that, when $k \to 1$ we have $B_k(x, y) = B(x, y)$. The $k$-beta function can be written, in terms of $k$-gamma functions and in terms of beta function, respectively, as follows

$$B_k(x, y) = \frac{\Gamma_k(x)\,\Gamma_k(y)}{\Gamma_k(x+y)} \qquad \text{and} \qquad B_k(x, y) = \frac{1}{k} B\left(\frac{x}{k}, \frac{y}{k}\right).$$

### 1.2.4 The Mittag-Leffler Functions

To be a generalization of the exponential function, the Mittag-Leffler functions play a very important role in the solution of linear fractional differential equations and integral equations, [12, 25, 35, 37, 38]. Such functions allows complex values for the parameters, but here we consider real values only.

**Definition 1.8.** [50] *The three-parameters Mittag-Leffler function*

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \quad z \in \mathbb{R}, \tag{1.20}$$

*is defined for* $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $\alpha, \beta, \gamma > 0$ *and* $(\gamma)_n$ *being the Pochhammer symbol.*

Notice that, when $\gamma = 1$, Eq.(1.20) becomes the two-parameters Mittag-Leffler function [63], i.e.,

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad z \in \mathbb{R}, \quad \alpha > 0, \quad \beta > 0. \tag{1.21}$$

On the other hand, if $\beta = \gamma = 1$ in Eq.(1.20), we have the one-parameter Mittag-Leffler function [43], as introduced by himself

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad z \in \mathbb{R}, \quad \alpha > 0. \tag{1.22}$$

The generalized Mittag-Leffler function $E_{\alpha,l,m}(x)$, introduced by Kilbas and Saigo [7, 36], is defined as follows:

**Definition 1.9.** *Let* $\alpha, l, m, x \in \mathbb{R}$ *and* $j \in \mathbb{N}_0$ *such that* $\alpha > 0$*,* $m > 0$ *and* $\alpha(jm+l) \notin \mathbb{Z}^-$*. The generalized Mittag-Leffler function is defined by*

$$E_{\alpha,l,m}(x) = \sum_{k=0}^{\infty} \left( \prod_{j=0}^{k-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) x^k, \quad \textit{with} \quad c_0 = 1 \quad \textit{and} \quad k \in \mathbb{N}. \tag{1.23}$$

**Property 1.1.** *Let* $x \in \mathbb{R}$*,* $\beta, \gamma, \delta \in \mathbb{R}$ *with* $\alpha > 0$*. We have,*

$$-\frac{1}{\Gamma(\gamma)} + E^{\delta}_{\beta,\gamma}(x) = x \sum_{k=0}^{\infty} \frac{(\delta)_{k+1}}{\Gamma(\beta k+\beta+\gamma)} \frac{x^k}{(k+1)!}. \tag{1.24}$$

## 1.3 Stirling Functions of the Second Kind

In this section we present Stirling functions of the second king $S(\alpha, k)$ with nonnegative $\alpha \geqslant 0$, [4]. These functions will be fundamental to prove Leibniz-type rule for Hadamard and Caputo-Hadamard fractional derivatives.

**Theorem 1.2.** *Let $\alpha \geqslant 0$ and $k \in \mathbb{N}_0$. Then the Stirling function of the second kind $S(\alpha, k)$ has the following explicit representation*

$$S(\alpha, k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}j^{\alpha}, \qquad (\alpha > 0\,, k \in \mathbb{N}). \tag{1.25}$$

*In particular, we have*

$$S(\alpha, 0) = 0,\ (\alpha > 0); \quad S(0, k) = 0,\ (k \in \mathbb{N}); \quad S(0, 0) = 1. \tag{1.26}$$

*Proof.* See [4]. □

## 1.4 Mellin Transform

In this section we define the Mellin transform as well as its inverse and we present some properties involving this integral transform [38]. The parameter associated with the Mellin transform, $s$, can be complex, but here we consider only the real case, this is, $s \in \mathbb{R}$.

**Definition 1.10.** *Let $\varphi(x)$ be a real function defined on $(0, \infty)$ and $s \in \mathbb{R}$, the parameter associated with the integral transform. The Mellin transform is defined by means of the following integral*

$$(\mathcal{M}\varphi)(s) = \mathcal{M}[\varphi(x)] = \int_0^{\infty} x^{s-1}\varphi(x)dx \tag{1.27}$$

*and its respective inverse by*

$$[(\mathcal{M}g)(s)]^{-1} = g(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s}[(\mathcal{M}g)(s)]ds$$

*with $\gamma > 0$, if the integrals exist.*

The direct and inverse Mellin transforms are inverse to each other for "sufficiently good" functions $\varphi$ and $g$:

$$(\mathcal{M}^{-1}\mathcal{M}\varphi) = \varphi(x) \qquad \text{and} \qquad (\mathcal{M}\mathcal{M}^{-1}g) = g(x). \tag{1.28}$$

### 1.4.1 Properties

We present some important properties involving the Mellin transform.

1. Let $c_1$ and $c_2$ are arbitrary constants, then

$$\mathcal{M}[c_1\,\varphi(x) + c_2\,g(x)] = c_1\,(\mathcal{M}\varphi)(s) + c_2\,(\mathcal{M}g)(s). \tag{1.29}$$

2. Let $a > 0$, then

$$(\mathcal{M}\, x^a\varphi)(s) = (\mathcal{M}\varphi)(s+a) \tag{1.30}$$

3. Let $n \in \mathbb{N}$, $\delta^n = \left(x\dfrac{d}{dx}\right)^n$, then

$$(\mathcal{M}\delta^n\varphi)(s) = (-s)^n(\mathcal{M}\varphi)(s). \tag{1.31}$$

## 1.5 Fractional Integrals and Fractional Derivatives

In this section, we define the Riemann-Liouville fractional integrals which is necessary to define the Riemann-Liouville, Caputo, Hilfer and generalized Riemann-Liouville fractional derivatives [23, 24, 38, 55]. The Riemann-Liouville fractional integral is defined by means of analytic continuation of the $n$th integral of Cauchy given as follows.

**Theorem 1.3.** *Let $n \in \mathbb{N}$ and $a \geqslant 0$. The $n$th classical integral is given by*

$$\begin{aligned}
(\mathcal{I}^n_{a^+}\varphi)(x) &= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} \varphi(t_n)\, dt_n \\
&= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)\, dt, \quad (x>a).
\end{aligned} \tag{1.32}$$

*Notice that, $(n-1)! = \Gamma(n)$.*

### 1.5.1 Riemann-Liouville Fractional Derivatives

Generalizing the result of Theorem 1.3, for $\alpha \in \mathbb{R}$, we obtain the definition for the Riemann-Liouville fractional integrals [38].

**Definition 1.11.** *Let $\Omega = [a, b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals, $\mathcal{I}^\alpha_{a^+}\varphi$ and $\mathcal{I}^\alpha_{b^-}\varphi$ of order $\alpha \in \mathbb{R}$, the left- and right-sided are defined, for $\varphi \in L_p(a, b)$, respectively, by*

$$(\mathcal{I}^\alpha_{a^+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad \alpha>0, \quad x>a, \tag{1.33}$$

and

$$(\mathcal{I}^\alpha_{b^-}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \quad \alpha>0, \quad b>x.$$

The Riemann-Liouville fractional derivatives are defined from the Riemann-Liouville fractional integrals as follow [38].

**Definition 1.12.** *Let $\alpha \geqslant 0$ and $n = [\alpha] + 1$ with $n \in \mathbb{N}$. Also let $\varphi \in AC^n[a,b]$ with $0 < a < b < \infty$. Then, the Riemann-Liouville fractional derivatives of order $\alpha$, the left- and right-sided, are defined, respectively, by*

$$\begin{aligned}({}_{RL}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) &= \left(\frac{d}{dx}\right)^n (\mathcal{I}^{n-\alpha}_{a^+}\varphi)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \left\{\int_a^x (x-t)^{n-\alpha-1}\varphi(t)dt\right\}\end{aligned} \tag{1.34}$$

and

$$\begin{aligned}({}_{RL}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) &= \left(-\frac{d}{dx}\right)^n (\mathcal{I}^{n-\alpha}_{b^-}\varphi)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \left\{\int_x^b (t-x)^{n-\alpha-1}\varphi(t)dt\right\}.\end{aligned} \tag{1.35}$$

*In particular, when $\alpha = n \in \mathbb{N}_0$, we have*

$$({}_{RL}\mathcal{D}^{0}_{a^+}\varphi)(x) = ({}_{RL}\mathcal{D}^{0}_{b^-}\varphi)(x) = \varphi(x);$$
$$({}_{RL}\mathcal{D}^{n}_{a^+}\varphi)(x) = \varphi^{(n)}(x) \quad \text{and} \quad ({}_{RL}\mathcal{D}^{n}_{b^-}\varphi)(x) = (-1)^n\varphi^{(n)}(x) \quad (n \in \mathbb{N})$$

*where $\varphi^{(n)}(x)$ is the ordinary derivative of $\varphi(x)$ of order $n$.*

### 1.5.2 Caputo Fractional Derivatives

Now we present the definition of the Caputo fractional derivatives [17, 38].

**Definition 1.13.** *Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$, and let $({}_{RL}\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$ and $({}_{RL}\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$ be the Riemann-Liouville fractional derivatives of order $\alpha\,(\alpha \geqslant 0)$ defined by Eq.(1.34) and Eq.(1.35), respectively. The fractional derivatives $({}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$ and $({}_{*}\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$ of order $\alpha \in \mathbb{R}\,(\alpha \geqslant 0)$ on $[a,b]$ are defined via the above Riemann-Liouville fractional derivatives by*

$$({}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) = \left({}_{RL}\mathcal{D}^{\alpha}_{a^+}\left[\varphi(t) - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(a)}{k!}(t-a)^k\right]\right)(x) \tag{1.36}$$

and

$$({}_{*}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) = \left({}_{RL}\mathcal{D}^{\alpha}_{a^+}\left[\varphi(t) - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(b)}{k!}(b-t)^k\right]\right)(x) \tag{1.37}$$

*respectively, where*

$$n = [\alpha] + 1 \quad \text{for} \quad \alpha \notin \mathbb{N}_0; \quad n = \alpha \quad \text{for} \quad \alpha \in \mathbb{N}_0. \tag{1.38}$$

*These derivatives are called left-sided and right-sided Caputo fractional derivatives of order $\alpha$, respectively, and are defined for $\varphi \in AC^n[a,b]$.*

The definition from the Riemann-Liouville fractional integrals is more restrictive than Definition 1.13 because the function $\varphi$ must be in the same space as the Riemann-Liouville fractional integrals. On the other hand, another way to define the Caputo fractional derivatives is given by the following theorem.

**Theorem 1.4.** *Let $\alpha \geqslant 0$ and let $n$ be given Eq.(1.38). If $\varphi \in AC^n[a,b]$, then the Caputo fractional derivatives $({}_*\mathcal{D}^\alpha_{a^+}\varphi)(x)$ and $({}_*\mathcal{D}^\alpha_{b^-}\varphi)(x)$ exist almost everywhere on $[a,b]$.*

(a) *If $\alpha \notin \mathbb{N}_0$, $({}_*\mathcal{D}^\alpha_{a^+}\varphi)(x)$ and $({}_*\mathcal{D}^\alpha_{b^-}\varphi)(x)$ are represented by*

$$({}_*\mathcal{D}^\alpha_{a^+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\varphi^{(n)}(t)\,dt}{(x-t)^{\alpha-n+1}} = (I^{n-\alpha}_{a^+}D^n\varphi)(x), \quad x > a \tag{1.39}$$

and

$$({}_*\mathcal{D}^\alpha_{b^-}\varphi)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{\varphi^{(n)}(t)\,dt}{(t-x)^{\alpha-n+1}} = (-1)^n(I^{n-\alpha}_{b^-}D^n\varphi)(x) \quad x < b$$

*respectively, where $D = d/dx$ and $n = [\alpha] + 1$.*

(b) *If $\alpha = n \in \mathbb{N}_0$, then $({}_*\mathcal{D}^n_{a^+}\varphi)(x)$ and $({}_*\mathcal{D}^n_{b^-}\varphi)(x)$ are represented by*

$$({}_*\mathcal{D}^n_{a^+}\varphi)(x) = \varphi^{(n)}(x) \quad \text{and} \quad ({}_*\mathcal{D}^n_{b^-}\varphi)(x) = (-1)^n\varphi^{(n)}(x) \quad (n \in \mathbb{N}).$$

*In particular, for $n = 0$ we recover the functions $\varphi(x)$,*

$$({}_*\mathcal{D}^0_{a^+}\varphi)(x) = ({}_*\mathcal{D}^0_{b^-}\varphi)(x) = \varphi(x).$$

### 1.5.3 Hilfer Fractional Derivatives

We now present the definition of Hilfer fractional derivatives which is associated with the Riemann-Liouville fractional integrals [22].

**Definition 1.14.** *The Hilfer fractional derivatives of order $0 < \alpha < 1$ and type $0 \leqslant \beta \leqslant 1$ with respect to $x$ is defined by*

$$(\mathcal{D}^{\alpha,\beta}_{a^\pm}\varphi)(x) = \left(\pm\,\mathcal{I}^{\beta(1-\alpha)}_{a^\pm}\frac{d}{dx}\,\mathcal{I}^{(1-\beta)(1-\alpha)}_{a^\pm}\varphi\right)(x) \tag{1.40}$$

*for functions in which the expression on the right-hand side exists.*

### 1.5.4 Generalized Riemann-Liouville Fractional Derivatives

In order to obtain more general derivatives than the one proposed by Hilfer, that is, a fractional derivatives of order $\alpha \in \mathbb{R}^+$ with $n-1 < \alpha \leqslant n$, where $n \in \mathbb{N}$, Hilfer, Luchko and Tomovski [24] proposed the generalized Riemann-Liouville fractional derivatives, which is associated with the Riemann-Liouville fractional integrals.

**Definition 1.15.** *Let $\alpha, \beta \in \mathbb{R}$ such that $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, $0 \leqslant \beta \leqslant 1$, where $\alpha$ is the order and $\beta$ is the type of generalized Riemann-Liouville fractional derivatives, then*

$$({}^{n}\mathcal{D}_{a^{\pm}}^{\alpha,\beta}\varphi)(x) = \left( \pm \mathcal{I}_{a^{\pm}}^{\beta(n-\alpha)} \frac{d^n}{dx^n} \mathcal{I}_{a^{\pm}}^{(1-\beta)(n-\alpha)} \varphi \right)(x) \tag{1.41}$$

*for functions in which the expression on the right-hand side exists.*

# Chapter 2

# Hadamard Fractional Integrals and Fractional Derivatives

We have already mentioned that our interest is in the study of fractional differentiation operators which are defined by means of a correspondent fractional integral. The integrals of arbitrary order presented in this chapter are called Hadamard fractional integrals and was introduced, in 1892, by Hadamard [21]. We emphasize that there is a more general definition, in the literature, the so-called Hadamard-type fractional integral and was introduced by Butzer, Kilbas and Trujillo [5], from the insertion of a term in the integrand and this depends of a new parameter. However,in this work, we consider only the Hadamard fractional integrals. To do so, we start the $n$th integral theorem, after that, as an analytical continuation of this theorem, we obtain the Hadamard integrals of arbitrary order. Let us, throughout this chapter, discuss some properties involving these integration operators.

After presenting the fractional integrals it is possible to define the fractional derivatives associated with these operators. In a similar way to fractional integrals there are the so-called Hadamard-type fractional derivatives which was introduced by Butzer, Kilbas and Trujillo [5], and that generalize the Hadamard fractional derivatives by means of insertion of two terms that depend of a new parameter, the same one considered in the Hadamard-type fractional integrals. However, in this thesis, we consider only the Hadamard fractional derivatives and, in similar way to the fractional integrals, we present some properties involving these differentiation operators. Both the fractional integrals and derivatives presented in this chapter allow $\alpha \in \mathbb{C}$, but we restrict to the cases where $\alpha \in \mathbb{R}$. The contents of this chapter are based essentially on the works of Baleanu et al. [4], Kilbas [34] and Kilbas et al. [38].

## 2.1 The $n$th Hadamard Integral

In this section we discuss the $n$th ($n \in \mathbb{N}$) integral [5]. By means of analytic continuation of this result it is possible to define the Hadamard fractional integrals.

**Theorem 2.1.** *Let $n \in \mathbb{N}$ and $a \geqslant 0$. The $n$th Hadamard integral is given by*

$$\begin{aligned}(\mathcal{J}^n_{a^+}\varphi)(x) &= \int_a^x \frac{dt_1}{t_1}\int_a^{t_1}\frac{dt_2}{t_2}\cdots\int_a^{t_{n-1}}\varphi(t)\,\frac{dt}{t}\\ &= \frac{1}{(n-1)!}\int_a^x\left(\ln\frac{x}{t}\right)^{n-1}\varphi(t)\,\frac{dt}{t},\quad (x>a). \qquad (2.1)\end{aligned}$$

*Proof.* We prove by mathematical induction. We consider Eq.(2.1) is true for $n=1$, so

$$(\mathcal{J}^1_{a^+}\varphi)(x) = \int_a^x \varphi(t)\,\frac{dt}{t}.$$

Suppose that Eq.(2.1) is true for $n = 1, 2, \ldots, k$, this is,

$$(\mathcal{J}^k_{a^+}\varphi)(x) = \frac{1}{(k-1)!}\int_a^x\left(\ln\frac{x}{t}\right)^{k-1}\varphi(t)\frac{dt}{t}. \qquad (2.2)$$

We need to show that if Eq.(2.2) holds for $n=k$, then it must also holds for $n=k+1$. Require to prove that

$$(\mathcal{J}^{k+1}_{a^+}\varphi)(x) = \frac{1}{k!}\int_a^x\left(\ln\frac{x}{t}\right)^{k}\varphi(t)\,\frac{dt}{t}.$$

From the semigroup property for the Hadamard fractional integrals $(\mathcal{J}^\alpha_{a^+}\mathcal{J}^\beta_{a^+}\varphi)(x) = (\mathcal{J}^{\alpha+\beta}_{a^+}\varphi)(x)$, which we present in this chapter, Property 2.2 and, using Eq.(1.1), which can be found in Chapter 1, we can write

$$\begin{aligned}(\mathcal{J}^{k+1}_{a^+}\varphi)(x) &= (\mathcal{J}^k_{a^+}\mathcal{J}^1_{a^+}\varphi)(x)\\ &\underbrace{=}_{\text{Eq.(2.2)}} \frac{1}{(k-1)!}\int_a^x\left(\ln\frac{x}{t}\right)^{k-1}\left\{\int_a^t\varphi(u)\,\frac{du}{u}\right\}\frac{dt}{t}\\ &\underbrace{=}_{\text{Eq.(1.1)}} \frac{1}{(k-1)!}\int_a^x\varphi(u)\left\{\int_u^x\left(\ln\frac{x}{t}\right)^{k-1}\frac{dt}{t}\right\}\frac{du}{u}.\end{aligned}$$

Considering the change of variable $z = \left(\ln\dfrac{x}{t}\right)$, in the last equation, we have

$$\begin{aligned}(\mathcal{J}^{k+1}_{a^+}\varphi)(x) &= \frac{1}{(k-1)!}\int_a^x\varphi(u)\,\frac{du}{u}\left\{\int_0^{\left(\ln\frac{x}{u}\right)} z^{k-1}dz\right\}\\ &= \frac{1}{k(k-1)!}\int_a^x\left(\ln\frac{x}{u}\right)^k\varphi(u)\,\frac{du}{u},\end{aligned}$$

this is,

$$(\mathcal{J}^{k+1}_{a^+}\varphi)(x) = \frac{1}{k!}\int_a^x\left(\ln\frac{x}{u}\right)^k\varphi(u)\,\frac{du}{u}.$$

This concludes the proof of Theorem 2.1. $\square$

## 2.2 Hadamard Fractional Integrals

Generalizing the result of Theorem 2.1, for $\alpha \in \mathbb{R}$, we obtain the definition for Hadamard fractional integrals, [21]. The formulation of these integrals is essential for the definition of Hadamard fractional derivatives, presented also in this chapter.

**Definition 2.1.** *Let $\alpha > 0$ and let $(a, b)$ be a limited or ilimited interval of the half-axis $\mathbb{R}^+$ and $c \leqslant 0$. The Hadamard fractional integrals, of order $\alpha$, the left-sided and right-sided, are defined for $\varphi \in X_c^p(a, b)$, respectively, by*

$$(\mathcal{J}_{a+}^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \varphi(t) \, \frac{dt}{t}, \qquad (x > a) \tag{2.3}$$

and

$$(\mathcal{J}_{b-}^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \varphi(t) \, \frac{dt}{t}, \qquad (b > x). \tag{2.4}$$

*When $a = 0$ and $b \to \infty$, we have*

$$(\mathcal{J}_{0+}^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \varphi(t) \, \frac{dt}{t}, \qquad (x > 0) \tag{2.5}$$

and

$$(\mathcal{J}_{-}^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \left(\ln \frac{t}{x}\right)^{\alpha-1} \varphi(t) \, \frac{dt}{t}, \qquad (x > 0). \tag{2.6}$$

*We denote $\mathcal{J}^0 = I$, where $I$ is the identity operator, we obtain $(\mathcal{J}_{a+}^0\varphi)(x) = \varphi(x)$.*

The integrals Eq.(2.3), Eq.(2.4), Eq.(2.5) and Eq.(2.6) are called Hadamard fractional integrals of order $\alpha$ and, as we have already mentioned, the integral given by Eq.(2.5) was proposed by Hadamard [21].

## 2.3 Hadamard Fractional Operators in the Space $X_c^p(a, b)$

The next theorem shows that the Hadamard fractional integration operator $\mathcal{J}_{a+}^{\alpha}$ is bounded in $X_c^p(a, b)$, [34].

**Theorem 2.2.** *Let $\alpha > 0$, $1 \leqslant p \leqslant \infty$, $0 < a < b < \infty$ and $c \leqslant 0$. Thus, the operator $\mathcal{J}_{a+}^{\alpha}$ is bounded in $X_c^p(a, b)$, this is,*

$$\|\mathcal{J}_{a+}^{\alpha}\varphi\|_{X_c^p} \leqslant K \, \|\varphi\|_{X_c^p} \ , \tag{2.7}$$

*where*

$$K = \frac{1}{\Gamma(\alpha+1)} \left(\ln \frac{b}{a}\right)^{\alpha} \tag{2.8}$$

*when $c = 0$, and*

$$K = \frac{1}{\Gamma(\alpha)}(-c)^{-\alpha} \, \gamma \left[\alpha, -c\left(\ln \frac{b}{a}\right)\right] \tag{2.9}$$

*when $c < 0$ with $\gamma(\nu, x)$ being the incomplete gamma function defined by Eq.(1.16).*

## 2.4 Properties Involving the Hadamard Fractional Integrals

Next, we present and prove some properties involving the Hadamard fractional integrals, the left-sided, but we omitted the cases involving the right-sided fractional integrals because they are proved similarly. The first property refers to the fractional integrals of order $\alpha$ of the function $\left(\ln \dfrac{t}{a}\right)^{\beta-1}$ and the second property refers to the semigroup property. On the other hand, the third property verified that the Hadamard fractional integrals of order $\alpha$ of the power function $t^{\beta}$ yield the same function, apart from a constant multiplication factor. We also present an proposition given in [4] for the Hadamard fractional integrals.

Finally, we present the same particular cases, for this proposition, involving the confluent hypergeometric function and the Mittag-Leffler functions.

**Property 2.1.** *Let $\alpha, \beta \in \mathbb{R}$, if $\alpha, \beta > 0$, where $\alpha > 1-\beta$ and $0 < a < b < \infty$, then we have*

$$\left(\mathcal{J}^{\alpha}_{a^+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\ln \frac{x}{a}\right)^{\beta+\alpha-1}. \tag{2.10}$$

and

$$\left(\mathcal{J}^{\alpha}_{b^-}\left(\ln \frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\ln \frac{b}{x}\right)^{\beta+\alpha-1}. \tag{2.11}$$

*Proof.* From the definition, Eq.(2.3), we can write

$$\left(\mathcal{J}^{\alpha}_{a^+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1}\left(\ln \frac{t}{a}\right)^{\beta-1}\frac{dt}{t}.$$

Introducing the chance of variable,

$$\tau = \frac{\left(\ln \frac{t}{a}\right)}{\left(\ln \frac{x}{a}\right)} \qquad \Rightarrow \qquad \frac{dt}{t} = \left(\ln \frac{x}{a}\right)d\tau \qquad \text{and} \qquad (1-\tau)\left(\ln \frac{x}{a}\right) = \left(\ln \frac{x}{t}\right),$$

in order to obtain the following expression

$$\left(\mathcal{J}^{\alpha}_{a^+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{1}{\Gamma(\alpha)}\left(\ln \frac{x}{a}\right)^{\alpha+\beta-1}\underbrace{\int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1}d\tau}_{B(\alpha,\beta)}.$$

Thus, we have

$$\left(\mathcal{J}^{\alpha}_{a^+}\left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\ln \frac{x}{a}\right)^{\alpha+\beta-1}.$$

$\square$

The next property yields the semigroup property associated the Hadamard fractional integrals which states that $(\mathcal{J}^{\alpha}_{a^+}\mathcal{J}^{\beta}_{a^+}\varphi)(x) = (\mathcal{J}^{\alpha+\beta}_{a^+}\varphi)(x)$.

**Property 2.2.** *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta > 0$ and $1 \leqslant p \leqslant \infty$. Thus, for $\varphi \in X^p_c(a,b)$ and $c \leqslant 0$, the semigroup property for Hadamard fractional integrals is given by*

$$(\mathcal{J}^{\alpha}_{a^+}\mathcal{J}^{\beta}_{a^+}\varphi)(x) = (\mathcal{J}^{\alpha+\beta}_{a^+}\varphi)(x). \tag{2.12}$$

and

$$(\mathcal{J}^{\alpha}_{b^-}\mathcal{J}^{\beta}_{b^-}\varphi)(x) = (\mathcal{J}^{\alpha+\beta}_{b^-}\varphi)(x). \tag{2.13}$$

*If $a = 0$ and $b \to \infty$, follows that*

$$(\mathcal{J}^{\alpha}_{0^+}\mathcal{J}^{\beta}_{0^+}\varphi)(x) = (\mathcal{J}^{\alpha+\beta}_{0^+}\varphi)(x). \tag{2.14}$$

and

$$(\mathcal{J}^{\alpha}_{-}\mathcal{J}^{\beta}_{-}\varphi)(x) = (\mathcal{J}^{\alpha+\beta}_{-}\varphi)(x). \tag{2.15}$$

*Proof.* See [34]. □

In the following property we obtain the Hadamard fractional integral of the power function, [4].

**Property 2.3.** *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha > 0$.*

(a) *If $\beta > 0$, then*

$$(\mathcal{J}^{\alpha}_{0^+}t^{\beta})(x) = \beta^{-\alpha}x^{\beta}. \tag{2.16}$$

(b) *If $\beta < 0$, then*

$$(\mathcal{J}^{\alpha}_{-}t^{\beta})(x) = (-\beta)^{-\alpha}x^{\beta}. \tag{2.17}$$

## 2.5 An Interesting Proposition for Hadamard Fractional Integrals

In this section we present the item (1) of Proposition 1.2 proposed in [4], but we consider the case when $\mu = 0$ in order to obtain the result involving the Hadamard fractional integral. This proposition discusses the Hadamard fractional integral of a given convergent power series that also results in a convergent power series.

**Proposition 2.1.** *Let $\alpha \in \mathbb{R}$ such that $\varphi(x)$ be a convergent power series, this is,*

$$\varphi(x) = \sum_{k=0}^{\infty} a_k\, x^k, \qquad (a_k \in \mathbb{R}). \tag{2.18}$$

*If $\alpha > 0$, the Hadamard fractional integral, with $a \to 0$, $(\mathcal{J}_{0^+}^{\alpha}\varphi)(x)$ is represented, also, by a convergent power series, given by*

$$(\mathcal{J}_{0^+}^{\alpha}\varphi)(x) = \sum_{k=1}^{\infty} k^{-\alpha} a_k\, x^k. \tag{2.19}$$

*The series given by Eq.(2.18) and Eq.(2.19) are convergent and the radii of convergence coincide.*

### 2.5.1 Particular Cases

Under the hypothesis of Proposition 2.1, we propose in this thesis the following particular cases involving the Hadamard fractional integral of order $\alpha$.

1. We consider the following convergent power series, $\varphi(x)$, represented by

$$\varphi(x) = \left(\sum_{k=1}^{\infty} k^{\alpha} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}\right).$$

Substituting this series in Eq.(2.19), we obtain the expression

$$\begin{aligned} \left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty} \frac{k^{\alpha}(a)_k}{k!(c)_k} t^k\right)\right](x) &= \sum_{k=1}^{\infty} k^{-\alpha}\left(\frac{k^{\alpha}(a)_k}{k!(c)_k}\right) x^k \\ &= \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{x^k}{k!} - 1 \\ &= {}_1F_1(a;c;x) - 1, \end{aligned} \tag{2.20}$$

where ${}_1F_1(a;c;x)$ is the confluent hypergeometric function. We note that the right-hand side of the above equation do not depend on the parameter $\alpha$.

2. Assuming

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{k^{\alpha}(\rho)_k}{\Gamma(\beta k + \gamma)} \frac{x^k}{k!}$$

in Eq.(2.19), we obtain the following result

$$\begin{aligned} \left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty} \frac{k^{\alpha}(\rho)_k}{\Gamma(\beta k + \gamma)} \frac{t^k}{k!}\right)\right](x) &= \left(\sum_{k=1}^{\infty} k^{-\alpha} \frac{k^{\alpha}(\rho)_k}{k!\Gamma(\beta k + \gamma)}\right) x^k \\ &= \sum_{k=1}^{\infty} \frac{(\rho)_k}{\Gamma(\beta k + \gamma)} \frac{x^k}{k!}. \end{aligned}$$

Considering a change of index $k \to k+1$, follows that

$$\begin{aligned}\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{t^k}{k!}\right)\right](x) &= \sum_{k=0}^{\infty}\frac{(\rho)_{k+1}}{\Gamma(\beta k+\beta+\gamma)}\frac{x^{k+1}}{(k+1)!}\\ &= x\sum_{k=0}^{\infty}\frac{(\rho)_{k+1}}{\Gamma(\beta k+\beta+\gamma)}\frac{x^{k}}{(k+1)!}\end{aligned}$$

and, by Property 1.1, we can write

$$\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{t^k}{k!}\right)\right](x) = -\frac{1}{\Gamma(\gamma)}+E_{\beta,\gamma}^{\rho}(x). \tag{2.21}$$

When $\rho=1$, we obtain $(1)_k=k!$, this is,

$$\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}t^k}{\Gamma(\beta k+\gamma)}\right)\right](x) = -\frac{1}{\Gamma(\gamma)}+E_{\beta,\gamma}(x)=xE_{\beta,\beta+\gamma}(x). \tag{2.22}$$

When $\rho=\gamma=1$, we have

$$\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}t^k}{\Gamma(\beta k+1)}\right)\right](x) = -1+E_{\beta}(x)=xE_{\beta,\beta+1}(x).$$

When $\beta=1$, the last equation, takes the following form

$$\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}(\rho)_k}{\Gamma(k+\gamma)}\frac{t^k}{k!}\right)\right](x) = -\frac{1}{\Gamma(\gamma)}+E_{1,\gamma}^{\rho}(x). \tag{2.23}$$

3. Knowing that,

$$E_{1,\gamma}^{\rho}(x)=\frac{1}{\Gamma(\gamma)}{}_1F_1(\rho;\gamma;x),$$

we can rewrite Eq.(2.23) as

$$\begin{aligned}\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}(\rho)_k}{\Gamma(k+\gamma)}\frac{t^k}{k!}\right)\right](x) &= -\frac{1}{\Gamma(\gamma)}+\frac{1}{\Gamma(\gamma)}{}_1F_1(\rho;\gamma;x)\\ &= \frac{1}{\Gamma(\gamma)}[{}_1F_1(\rho;\gamma;x)-1].\end{aligned}$$

By means of Eq.(2.20), follows that

$$\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}(\rho)_k}{\Gamma(k+\gamma)}\frac{t^k}{k!}\right)\right](x) = \frac{1}{\Gamma(\gamma)}\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}(\rho)_k}{k!(\gamma)_k}x^k\right).$$

4. Taking the derivative of integer order, $d/dx$, in both sides of Eq.(2.22), we obtain the following expression

$$\begin{aligned}\frac{d}{dx}\left\{\left[\mathcal{J}_{0^+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}t^k}{\Gamma(\beta k+\gamma)}\right)\right](x)\right\} &= \frac{d}{dx}\left\{-\frac{1}{\Gamma(\gamma)}+E_{\beta,\gamma}(x)\right\}\\ &= \frac{d}{dx}\left\{xE_{\beta,\beta+\gamma}(x)\right\},\end{aligned}$$

this is,

$$\begin{aligned}
\frac{d}{dx}\left\{\left[\mathcal{J}_{0+}^{\alpha}\left(\sum_{k=1}^{\infty}\frac{k^{\alpha}t^{k}}{\Gamma(\beta k+\gamma)}\right)\right](x)\right\} &= \frac{d}{dx}[E_{\beta,\gamma}(x)] \\
&= E_{\beta,\beta+\gamma}(x)+x\frac{d}{dx}E_{\beta,\beta+\gamma}(x).
\end{aligned}$$

This formula yields a recurrence relation involving the Mittag-Leffler function, [53, p. 23].

## 2.6 Mellin Transform of the Hadamard Fractional Integrals

In this section, we present the Mellin transform of Hadamard fractional integrals [38].

**Lemma 2.1.** *Let $\alpha > 0$ and a function $\varphi(x)$ be such that its Mellin transform $(\mathcal{M}\varphi)(s)$ exits for $s \in \mathbb{R}$.*

(a) *If $s < 0$ and $(\mathcal{M}\mathcal{J}_{0+}^{\alpha}\varphi)(s)$ exists, then*

$$(\mathcal{M}\,\mathcal{J}_{0+}^{\alpha}\varphi)(s) = (-s)^{-\alpha}(\mathcal{M}\varphi)(s). \tag{2.24}$$

(b) *If $s > 0$ and $(\mathcal{M}\mathcal{J}_{-}^{\alpha}\varphi)(s)$ exists, then*

$$(\mathcal{M}\,\mathcal{J}_{-}^{\alpha}\varphi)(s) = s^{-\alpha}(\mathcal{M}\varphi)(s).$$

## 2.7 Hadamard Fractional Derivatives

In this section, we present the Hadamard fractional dervatives [5, 34, 38]. As previously mentioned, these derivatives are defined by means of Hadamard fractional integrals.

**Definition 2.2.** *Let $\alpha > 0$ and $n = [\alpha] + 1$ where $n \in \mathbb{N}$ and $[\alpha]$ be the integer part of $\alpha$. Also let $\varphi \in AC_{\delta}^{n}[a,b]$ with $0 < a < b < \infty$. Then, the Hadamard fractional derivatives of order $\alpha$, the left-sided and right-sided, are defined, respectively, by*

$$(\mathcal{D}_{a+}^{\alpha}\varphi)(x) = \delta^{n}(\mathcal{J}_{a+}^{n-\alpha}\varphi)(x), \qquad (x > a) \tag{2.25}$$

$$= \left(x\frac{d}{dx}\right)^{n}\left\{\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\left(\ln\frac{x}{t}\right)^{n-\alpha+1}\varphi(t)\,\frac{dt}{t}\right\} \tag{2.26}$$

and

$$(\mathcal{D}_{b-}^{\alpha}\varphi)(x) = (-\delta)^{n}(\mathcal{J}_{b-}^{n-\alpha}\varphi)(x), \qquad (b > x) \tag{2.27}$$

$$= \left(-x\frac{d}{dx}\right)^{n}\left\{\frac{1}{\Gamma(n-\alpha)}\int_{x}^{b}\left(\ln\frac{t}{x}\right)^{n-\alpha+1}\varphi(t)\,\frac{dt}{t}\right\}. \tag{2.28}$$

*When $a = 0$ and $b \to \infty$, we have*

$$(\mathcal{D}^{\alpha}_{0+}\varphi)(x) = \delta^n(\mathcal{J}^{n-\alpha}_{0+}\varphi)(x), \qquad (x > 0) \tag{2.29}$$

and

$$(\mathcal{D}^{\alpha}_{-}\varphi)(x) = (-\delta)^n(\mathcal{J}^{n-\alpha}_{-}\varphi)(x), \qquad (x > 0). \tag{2.30}$$

*For $\alpha = n \in \mathbb{N}_0$, we have*

$$(\mathcal{D}^{n}_{a+}\varphi)(x) = \delta^n\varphi(x) \quad \text{and} \quad (\mathcal{D}^{n}_{b-}\varphi)(x) = (-1)^n\delta^n\varphi(x). \tag{2.31}$$

*In particular, when $n = 0$, we obtain*

$$(\mathcal{D}^{0}_{a+}\varphi)(x) = \varphi(x) \quad \text{and} \quad (\mathcal{D}^{0}_{b-}\varphi)(x) = \varphi(x), \tag{2.32}$$

*that is, it returns the function itself.*

## 2.8 Properties Involving the Hadamard Fractional Derivatives

This section is dedicated to present and prove some properties involving the Hadamard fractional derivatives and their respective fractional integration operators. We present the following results considering the left-sided Hadamard differentiation and integration operators, in analogous way defined for the right-sided operators, but here we will not do it. Such results can ben found in [38]. We start presenting the theorem that guarantees the linearity these non-integer order differential operators.

**Theorem 2.3.** *Let $\alpha \geqslant 0$ and $n = [\alpha]+1$ with $n \in \mathbb{N}$. If $\varphi \in AC^n_\delta[a,b]$ with $0 < a < b < \infty$, then*

$$(\mathcal{D}^{\alpha}_{a+}(\varphi + g))(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x) + (\mathcal{D}^{\alpha}_{a+}g)(x) \tag{2.33}$$

and

$$(\mathcal{D}^{\alpha}_{b-}(\varphi + g))(x) = (\mathcal{D}^{\alpha}_{b-}\varphi)(x) + (\mathcal{D}^{\alpha}_{b-}g)(x). \tag{2.34}$$

*Proof.* The result follows by the fact that the integration operators are linear. □

The following theorem presents the composition of the integration operator $\mathcal{J}^{\alpha}_{a+}$ with the differentiation operator $\mathcal{D}^{\beta}_{a+}$.

**Theorem 2.4.** *Let $\alpha, \beta \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $0 < a < b < \infty$ such that $\alpha > \beta > 0$ and $\varphi \in X^p_c(a,b)$ with $c \leqslant 0$. Thus, we have the following result,*

$$\mathcal{D}^{\beta}_{a+}(\mathcal{J}^{\alpha}_{a+}\varphi)(x) = (\mathcal{J}^{\alpha-\beta}_{a+}\varphi)(x)$$

*On the other hand, for $\beta = m \in \mathbb{N}$, follows*

$$\mathcal{D}^{m}_{a+}(\mathcal{J}^{\alpha}_{a+}\varphi)(x) = (\mathcal{J}^{\alpha-m}_{a+}\varphi)(x).$$

The following theorem gives the Hadamard fractional derivative of function $\left(\ln \frac{t}{a}\right)^{\beta-1}$.

**Property 2.4.** *Let $\alpha, \beta \in \mathbb{R}$, $0 < a < b < \infty$ such that $\alpha > 0$, $\beta > n$ and $n = [\alpha] + 1$, the*

$$\left(\mathcal{D}_{a+}^{\alpha} \left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{x}{a}\right)^{\beta-\alpha-1}. \tag{2.35}$$

and

$$\left(\mathcal{D}_{b-}^{\alpha} \left(\ln \frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{b}{x}\right)^{\beta-\alpha-1}. \tag{2.36}$$

*In particular, if $\beta = 1$ and $\alpha \geqslant 0$, then the Hadamard fractional derivative of a constant, in general, not equal to zero:*

$$(\mathcal{D}_{a+}^{\alpha} 1) = \frac{1}{\Gamma(1-\alpha)} \left(\ln \frac{x}{a}\right)^{-\alpha}$$

*and*

$$(\mathcal{D}_{b-}^{\alpha} 1) = \frac{1}{\Gamma(1-\alpha)} \left(\ln \frac{b}{x}\right)^{-\alpha}.$$

In what follows we present, as a corollary, conditions that must satisfy a function whose Hadamard fractional derivative is zero. We present only the result involving the left-sided differential operator, the corresponding result for the right-sided operator can be derived analogously.

**Corollary 2.1.** *Let $\alpha > 0$, $n = [\alpha] + 1$ and $0 < a < b < \infty$. The equation $(\mathcal{D}_{a+}^{\alpha}\varphi)(x) = 0$ is valid if, only if,*

$$\varphi(x) = \sum_{j=1}^{n} c_j \left(\ln \frac{x}{a}\right)^{\alpha-j},$$

*where $c_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants. In particular, when $0 < \alpha \leqslant 1$, the relation $(\mathcal{D}_{a+}^{\alpha}\varphi)(x) = 0$ holds if, and only if, $\varphi(x) = c\left(\ln \frac{x}{a}\right)^{\alpha-1}$ with any $c \in \mathbb{R}$.*

The next property gives the Hadamard fractional derivative, the left-sided, of the power function, this is, $(\mathcal{D}_{0+}^{\alpha} t^{\beta})(x)$.

**Property 2.5.** *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \geqslant 0$ and $\beta > 0$. Then, the Hadamard fractional derivative of power function, $t^{\beta}$, with $a \to 0$, is given by*

$$(\mathcal{D}_{0+}^{\alpha} t^{\beta})(x) = \beta^{\alpha} x^{\beta}. \tag{2.37}$$

*Proof.* See [4]. □

The following lemma shows that the Hadamard fractional derivative is the left-sided inverse of Hadamard fractional integral.

**Lemma 2.2.** *Let $\alpha > 0$ and $n = [\alpha] + 1$ with $n \in \mathbb{N}$. Also let $\varphi \in X_c^p(a,b)$ and $c \leqslant 0$. If $0 < a < b < \infty$ and $1 \leqslant p \leqslant \infty$, then*

$$(\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}\varphi)(x) = \varphi(x).$$

*Proof.* From the definition Eq.(2.25), we can write

$$(\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}\varphi)(x) = (\delta^n\mathcal{J}_{a+}^{n-\alpha}\mathcal{J}_{a+}^{\alpha}\varphi)(x).$$

By means of semigroup property for the Hadamard fractional integrals, Eq.(2.12), follows

$$(\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}\varphi)(x) = (\delta^n\mathcal{J}_{a+}^{n}\varphi)(x).$$

Considering the $n$th integral, given by Eq.(2.1) and differentiating under the integral sign, we obtain

$$\begin{aligned}(\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}\varphi)(x) &= \left(x\frac{d}{dx}\right)^n\left\{\frac{1}{\Gamma(n)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-1}\varphi(t)\frac{dt}{t}\right\}\\ &= \left(x\frac{d}{dx}\right)^{n-1}\int_a^x\left(x\frac{\partial}{\partial x}\right)\left(\ln\frac{x}{t}\right)^{n-1}\varphi(t)\frac{dt}{t}\\ &= \left(x\frac{d}{dx}\right)^{n-1}\left\{\frac{1}{\Gamma(n-1)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-2}\varphi(t)\frac{dt}{t}\right\}\\ &= (\delta^{n-1}\mathcal{J}_{a+}^{n-1}\varphi)(x).\end{aligned}$$

Continuing this process of differentiation for $(n-1)$ times, in order to obtain

$$(\mathcal{D}_{a+}^{\alpha}\mathcal{J}_{a+}^{\alpha}\varphi)(x) = (\delta^{n-n}\mathcal{J}_{a+}^{n-n}\varphi)(x) = \varphi(x).$$

□

We apply the Stirling functions of the second kind $S(\alpha,k)$, as defined in Theorem 1.2, to express Hadamard fractional integrals and derivatives [4].

**Theorem 2.5.** *Let $\varphi(x)$, defined for $x > 0$, be an arbitrarily often differentiable function such that its Taylor series converges, and let $\alpha \in \mathbb{R}$.*

(a) *When $\alpha \geqslant 0$, the Hadamard fractional derivative, $\mathcal{D}_{0+}^{\alpha}\varphi$, is given by Eq.(2.25) if, and only if, there holds for $x > 0$ the relation*

$$(\mathcal{D}_{0+}^{\alpha}\varphi)(x) = \sum_{i=0}^{\infty} S(\alpha,i)\,x^i\,\varphi^{(i)}(x). \tag{2.38}$$

(b) *When $\alpha > 0$, the Hadamard fractional integral, $(\mathcal{J}^{\alpha}_{0+}\varphi)(x)$, is given by* Eq.(2.3)*, if, and only if, there holds for $x > 0$ the relation*

$$(\mathcal{J}^{\alpha}_{0+}\varphi)(x) = \sum_{i=0}^{\infty} S(-\alpha, i)\, x^i\, \varphi^{(i)}(x). \tag{2.39}$$

We omitted the proof of Theorem 2.5, because is analogous to Theorem 3.5, considering Caputo-Hadamard fractional derivative, which we will prove in the next chapter.

## 2.9 Leibniz-Type Rule for Hadamard Fractional Derivatives

In this section, we propose the Leibniz-type rule for Hadamard fractional derivatives. Therefore, we apply the results presented, previously, in this chapter, as: Stirling function of the second kind and the Hadamard fractional derivative of power function.

**Theorem 2.6.** *Let $\alpha \in \mathbb{R}$ and $\varphi, g$ differentiable functions defined for $x > 0$. The Leibniz-type rule for Hadamard fractional derivative takes the following form*

$$(\mathcal{D}^{\alpha}_{0+}\varphi\, g)(x) = \sum_{i=0}^{\infty} S(\alpha, i)\, x^i\, D^i[\varphi(x)g(x)]. \tag{2.40}$$

*Proof.* From Eq.(2.38) of Theorem 2.5 and considering $\varphi(x) \to \varphi(x)g(x)$, we obtain the Eq.(2.40), where the Leibniz rule for integer order derivatives is used, this is,

$$D^i[\varphi(x)g(x)] = \sum_{m=0}^{i} \binom{i}{m} [D^{i-m}\varphi(x)][D^m g(x)]. \tag{2.41}$$

The proof of Theorem 2.6 is analogous to the right-sided Hadamard fractional derivative. □

In what follows, we present an example for the Leibniz-type rule involving the Hadamard fractional derivative, with $a \to 0$. After that, we compare the obtained result with the Theorem 2.6 and using Eq.(2.37).

**Example 2.1.** *Consider $\varphi(x) = x^2$, $g(x) = x^3$ and* Eq.(2.40)*. Note that, the product $\varphi(x)g(x)$ is 5-times differentiable, i.e., $n = 5$. Then,*

$$(\mathcal{D}^{\alpha}_{0+}t^2\, t^3)(x) = \sum_{i=0}^{5} S(\alpha, i)\, x^i \sum_{m=0}^{i} \binom{i}{m} [D^{i-m}\, x^2][D^m\, x^3],$$

*or*

$$\begin{aligned}
(\mathcal{D}^{\alpha}_{0+} t^2\, t^3)(x) &= S(\alpha,1)\, x \sum_{m=0}^{1} \binom{1}{m} [D^{1-m}\, x^2][D^m\, x^3] \\
&+ S(\alpha,2)\, x^2 \sum_{m=0}^{2} \binom{2}{m} [D^{2-m}\, x^2][D^m\, x^3] \\
&+ S(\alpha,3)\, x^3 \sum_{m=0}^{3} \binom{3}{m} [D^{3-m}\, x^2][D^m\, x^3] \\
&+ S(\alpha,4)\, x^4 \sum_{m=0}^{4} \binom{4}{m} [D^{4-m}\, x^2][D^m\, x^3] \\
&+ S(\alpha,5)\, x^5 \sum_{m=0}^{5} \binom{5}{m} [D^{5-m}\, x^2][D^m\, x^3],
\end{aligned}$$

*Remember that $S(\alpha,0)=0$. After that, we rearrange the expression in order to obtain*

$$(\mathcal{D}^{\alpha}_{0+} t^2\, t^3)(x) = S(\alpha,1)\, 5x^5 + S(\alpha,2)\, 20x^5 + S(\alpha,3)\, 60x^5 + S(\alpha,4)\, 120x^5 + S(\alpha,5)\, 120x^5$$

*Expanding the sums involving $S(\alpha,k)$ with $k=1,\ldots,5$, follows*

$$\begin{aligned}
(\mathcal{D}^{\alpha}_{0+} t^2\, t^3)(x) &= 5x^5 + (-2+2^\alpha)10x^5 + (3 - 3\cdot 2^\alpha + 3^\alpha)10^5 \\
&+ (-4 + 6\cdot 2^\alpha - 4\cdot 3^\alpha + 4^\alpha)5x^5 \\
&+ (5 - 10\cdot 2^\alpha + 10\cdot 3^\alpha - 5\cdot 4^\alpha + 5^\alpha)x^5.
\end{aligned}$$

*Simplifying the expression, we can write*

$$(\mathcal{D}^{\alpha}_{0+} t^5)(x) = 5^\alpha\, x^5. \tag{2.42}$$

Note that, using Eq.(2.37), with $\beta = 5$, we obtain, exactly, Eq.(2.42).

## 2.10 Mellin Transform of the Hadamard Fractional Derivatives

In this section we present the Mellin transform of Hadamard fractional derivative of order $\alpha$ [38].

**Lemma 2.3.** *Let $\alpha > 0$ and a function $\varphi(x)$ such that its Mellin transform $(\mathcal{M}\varphi)(s)$ exists for $s \in \mathbb{R}$.*

(a) *If $s < 0$ and $(\mathcal{M}\,\mathcal{D}^{\alpha}_{0+}\varphi)(s)$ exists, then*

$$(\mathcal{M}\,\mathcal{D}^{\alpha}_{0+}\varphi)(s) = (-s)^\alpha (\mathcal{M}\varphi)(s). \tag{2.43}$$

(b) *If $s > 0$ and $(\mathcal{M}\,\mathcal{D}^{\alpha}_{-}\varphi)(s)$ exists, then*

$$(\mathcal{M}\,\mathcal{D}^{\alpha}_{-}\varphi)(s) = s^\alpha (\mathcal{M}\varphi)(s). \tag{2.44}$$

## 2.11 Hilfer-Hadamard Fractional Derivatives

Similar to the Hilfer derivatives and the generalized Riemann-Liouville fractional derivatives, it is possible to define Hilfer-Hadamard fractional derivatives, which is associated with the Hadamard fractional integrals, [28].

**Definition 2.3.** *The Hilfer-Hadamard fractional derivatives of order* $0 < \alpha < 1$ *and type* $0 \leqslant \beta \leqslant 1$ *with respect to* $x$ *are defined by*

$$({}_{\mathcal{H}}\mathcal{D}^{\alpha,\beta}_{a^{\pm}}\varphi)(x) = \left( \pm\, \mathcal{J}^{\beta(1-\alpha)}_{a^{\pm}} \left( x\frac{d}{dx} \right) \mathcal{J}^{(1-\beta)(1-\alpha)}_{a^{\pm}} \varphi \right)(x) \tag{2.45}$$

*for functions in which the expression on the right hand side exists.*

# Chapter 3

# Caputo-Hadamard Fractional Derivatives

Recently, in 2012, Jarad, Abdeljawad and Baleanu [27], introduced a new formulation for the fractional derivatives, where the argument is basically the same as the one used to define the Caputo fractional derivatives. For this formulation is proposed a Caputo-type modification in the Hadamard fractional derivatives obtaining the so-called Caputo-Hadamard fractional derivatives. We emphasize, a Caputo-type modification mean the differentiation operator introduced in the integrand of fractional integration. From this recent formulation, in 2016, Almeida [2] established a new definition for the Caputo-Hadamard fractional derivatives by considering the order of these derivatives as variable. In this thesis, our interest is in the approach proposed by [27].

This chapter is dedicated to present the definition for the Caputo-Hadamard fractional derivatives by means of its relation with the Hadamard fractional derivatives. Therefore, further in this Chapter, we shall presents this same definition, similar to the definition of Hadamard fractional derivatives, however with an inversion of order: we change differentiation and integration operators. We present some properties involving the Caputo-Hadamard fractional derivatives as well as the fundamental theorem of fractional calculus. We use the particular case of a lemma proposed in [19] in order to obtain a generalization for another result presented in this same work. We propose to write the Caputo-Hadamard fractional derivatives of a convergent power series as an another convergent power series. We discuss particular cases involving this result, proposed by us. Finally, we present the Mellin transform for the fractional differentiation operators discussed in this chapter. We emphasize that this chapter, in addition our contribution, is based on the following works [19, 27, 38].

## 3.1 Caputo-Hadamard Fractional Derivatives

We define the Caputo-Hadamard fractional derivatives of order $\alpha$ by means of its relation with the Hadamard fractional derivatives [19, 27].

**Definition 3.1.** *Let $(\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$ and $(\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$ the Hadamard fractional derivatives of order $\alpha \in \mathbb{R}\,(\alpha \geqslant 0)$ defined by Eq.(2.25) and Eq.(2.27), respectively, with $0 < a < b < \infty$. The Caputo-Hadamard fractional derivatives of order $\alpha \in \mathbb{R}\,(\alpha \geqslant 0)$, $({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$ and $({}^{C}\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$ are defined via Hadamard fractional derivatives, with $\varphi \in \mathrm{AC}^{n}_{\delta}[a,b]$, the left- and right-sided, by*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) = \left(\mathcal{D}^{\alpha}_{a^+}\left[\varphi(t) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{k!}\left(\ln\frac{t}{a}\right)^k\right]\right)(x) \tag{3.1}$$

*and*

$$({}^{C}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) = \left(\mathcal{D}^{\alpha}_{b^-}\left[\varphi(t) - \sum_{k=0}^{n-1}\frac{(-1)^k\delta^k\varphi(b)}{k!}\left(\ln\frac{b}{t}\right)^k\right]\right)(x), \tag{3.2}$$

*respectively, where*

$$n = [\alpha] + 1 \quad \textit{for} \quad \alpha \notin \mathbb{N}, \qquad n = \alpha \quad \textit{for} \quad \alpha \in \mathbb{N}. \tag{3.3}$$

*In particular, when $0 < \alpha < 1$, we have*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi(x) = \mathcal{D}^{\alpha}_{a^+}[\varphi(t) - \varphi(a)])(x) \tag{3.4}$$

*and*

$$({}^{C}\mathcal{D}^{\alpha}_{b^-})\varphi(x) = \mathcal{D}^{\alpha}_{b^-}[\varphi(t) - \varphi(b)])(x). \tag{3.5}$$

It is possible rewrite the Definition 3.1 using the Property 2.4 by means of the following lemma.

**Lemma 3.1.** *Let $\alpha \in \mathbb{R}\,(\alpha \geqslant 0)$ and $n = [\alpha] + 1$, if $\varphi \in \mathrm{AC}^{n}_{\delta}[a,b]$ and the Hadamard and Caputo-Hadamard fractional derivatives, the left- and right-sided, $({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$, $({}^{C}\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$, $(\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$ and $(\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$ exist, respectively, then*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a^+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{\Gamma(k-\alpha+1)}\left(\ln\frac{x}{a}\right)^{k-\alpha} \tag{3.6}$$

and

$$({}^{C}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) = (\mathcal{D}^{\alpha}_{b^-}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(b)}{\Gamma(k-\alpha+1)}\left(\ln\frac{b}{x}\right)^{k-\alpha}. \tag{3.7}$$

*In particular, when $0 < \alpha < 1$, we have*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a^+}\varphi)(x) - \frac{\varphi(a)}{\Gamma(1-\alpha)}\left(\ln\frac{x}{a}\right)^{-\alpha} \tag{3.8}$$

and

$$({}^C\mathcal{D}^{\alpha}_{b-}\varphi)(x) = (\mathcal{D}^{\alpha}_{b-}\varphi)(x) - \frac{\varphi(b)}{\Gamma(1-\alpha)}\left(\ln\frac{b}{x}\right)^{-\alpha}. \tag{3.9}$$

*Proof.* As the Hadamard differetiation operators are linear, using the results in Theorem 2.3, from Eq.(3.1), we can write

$$({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{k!}\left(\mathcal{D}^{\alpha}_{a+}\left[\left(\ln\frac{t}{a}\right)^k\right]\right)(x). \tag{3.10}$$

Using Eq.(2.35), we obtain

$$({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{\Gamma(k-\alpha+1)}\left(\ln\frac{x}{a}\right)^{k-\alpha}.$$

□

The Caputo-Hadamard fractional derivatives, Eq.(3.1) and Eq.(3.2), coincide with the Hadamard fractional derivatives, Eq.(2.25) and Eq.(2.27) in the cases:

$$({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x) \quad \text{and} \quad ({}^C\mathcal{D}^{\alpha}_{b-}\varphi)(x) = (\mathcal{D}^{\alpha}_{b-}\varphi)(x),$$

if $\varphi(a) = \delta\,\varphi(a) = \cdots = \delta^{n-1}\,\varphi(a) = 0$ and $\varphi(b) = \delta\,\varphi(b) = \cdots = \delta^{n-1}\,\varphi(b) = 0$, respectively, with $n = [\alpha]+1$. In particular, when $0<\alpha<1$, we have

$$({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x), \quad \text{and} \quad ({}^C\mathcal{D}^{\alpha}_{b-}\varphi)(x) = (\mathcal{D}^{\alpha}_{b-}\varphi)(x),$$

when $\varphi(a)=0$ and $\varphi(b)=0$, respectively. If $\alpha = n \in \mathbb{N}$ and the derivative $\delta^n\,\varphi(x)$ of order $n$ exits, then $({}^C\mathcal{D}^{n}_{a+}\varphi)(x)$ coincides with $\delta^n\,\varphi(x)$, while $({}^C\mathcal{D}^{n}_{b-}\varphi)(x)$ coincides with $\delta^n\,\varphi(x)$, with exactness to a constant $(-1)^n$,

$$({}^C\mathcal{D}^{n}_{a+}\varphi)(x) = \delta^n\,\varphi(x) \quad \text{and} \quad ({}^C\mathcal{D}^{n}_{b-}\varphi)(x) = (-1)^n\,\delta^n\,\varphi(x) \quad \text{with} \quad n\in\mathbb{N}. \tag{3.11}$$

It is possible to define the Caputo-Hadamard fractional derivatives by means of the Hadamard fractional integrals, Eq.(2.3) and Eq.(2.4). The details of the proof can be seen in [27].

**Theorem 3.1.** *Let $\alpha\in\mathbb{R}\,(\alpha\geqslant 0)$ and let $n$ be given by Eq.(3.3). If $\varphi\in AC^n_\delta[a,b]$ and $0<a<b<\infty$, then the Caputo-Hadamard fractional derivatives, $({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x)$ and $({}^C\mathcal{D}^{\alpha}_{b-}\varphi)(x)$ exist on $[a,b]$.*

(a) *If $\alpha\notin\mathbb{N}$, $({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x)$ and $({}^C\mathcal{D}^{\alpha}_{b-}\varphi)(x)$ are represented by*

$$({}^C\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\delta^n\varphi(t)\frac{dt}{t} = (\mathcal{J}^{n-\alpha}_{a+}\delta^n\varphi)(x) \tag{3.12}$$

and

$$({}^{C}\mathcal{D}^{\alpha}_{b-}\varphi)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{n-\alpha-1}\delta^n\varphi(t)\frac{dt}{t} = (-1)^n(\mathcal{J}^{n-\alpha}_{b-}\delta^n\varphi)(x), \tag{3.13}$$

*respectively, where* $\delta = \left(t\dfrac{d}{dt}\right)$.

(b) *If* $\alpha = n \in \mathbb{N}_0$, *the* $({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x)$ *and* $({}^{C}\mathcal{D}^{\alpha}_{b-}\varphi)(x)$ *are represented by* Eq.(3.11). *In particular,*

$$({}^{C}\mathcal{D}^{0}_{a+}\varphi)(x) = ({}^{C}\mathcal{D}^{0}_{b-}\varphi)(x) = \varphi(x). \tag{3.14}$$

## 3.2 Properties

This section is dedicated to present some properties involving the Caputo-Hadamard fractional derivatives. First, we present the theorem that guarantee the linearity for the Caputo-Hadamard fractional differentiation operators.

**Theorem 3.2.** *Let* $\alpha \in \mathbb{R}\,(\alpha \geqslant 0)$, $n = [\alpha] + 1$ *with* $n \in \mathbb{N}$ *and* $\varphi, g \in \mathrm{AC}^n_\delta[a,b]$, $0 < a < b < \infty$, *then*

$$({}^{C}\mathcal{D}^{\alpha}_{a+}(\varphi + g))(x) = ({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x) + ({}^{C}\mathcal{D}^{\alpha}_{a+}g)(x) \tag{3.15}$$

and

$$({}^{C}\mathcal{D}^{\alpha}_{b-}(\varphi + g))(x) = ({}^{C}\mathcal{D}^{\alpha}_{b-}\varphi)(x) + ({}^{C}\mathcal{D}^{\alpha}_{b-}g)(x). \tag{3.16}$$

*Proof.* The result follows by the fact that, the integration operators are linear. □

The following assertion, similar to the Theorem 2.4, also holds [19].

**Theorem 3.3.** *Let* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha > \beta > 0$. *If* $0 < a < b < \infty$, $1 \leqslant p \leqslant \infty$ *and* $\varphi \in L^p(a,b)$, *we have*

$$({}^{C}\mathcal{D}^{\beta}_{a+}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = (\mathcal{J}^{\alpha-\beta}_{a+}\varphi)(x) \qquad \text{and} \qquad ({}^{C}\mathcal{D}^{\beta}_{b-}\mathcal{J}^{\alpha}_{b-}\varphi)(x) = (\mathcal{J}^{\alpha-\beta}_{b-}\varphi)(x).$$

*In particular, if* $\beta = m \in \mathbb{N}$, *then*

$$({}^{C}\mathcal{D}^{m}_{a+}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = (\mathcal{J}^{\alpha-m}_{a+}\varphi)(x) \qquad \text{and} \qquad ({}^{C}\mathcal{D}^{m}_{b-}\mathcal{J}^{\alpha}_{b-}\varphi)(x) = (\mathcal{J}^{\alpha-m}_{b-}\varphi)(x).$$

The following property is similar to the Property 2.4, this is, we present the Caputo-Hadamard fractional derivatives for the function $\left(\ln\dfrac{t}{a}\right)^{\beta-1}$.

**Property 3.1.** *Let $\alpha, \beta \in \mathbb{R}$ such that $\beta > \alpha > 0$ and $0 < a < b < \infty$, then*

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\ln\frac{x}{a}\right)^{\beta-\alpha-1}, \tag{3.17}$$

$$\left({}^{C}\mathcal{D}^{\alpha}_{b^-}\left(\ln\frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\ln\frac{b}{x}\right)^{\beta-\alpha-1}. \tag{3.18}$$

*Proof.* From the definition of Caputo-Hadamard fractional derivatives of order $\alpha$, the left-sided, Eq.(3.12), we can write

$$\begin{aligned}\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) &= \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\left(t\frac{d}{dt}\right)^n\left(\ln\frac{t}{a}\right)^{\beta-1}\frac{dt}{t}\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\left(t\frac{d}{dt}\right)^{n-1}\left[\left(t\frac{d}{dt}\right)\left(\ln\frac{t}{a}\right)^{\beta-1}\right]\frac{dt}{t}\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\left(t\frac{d}{dt}\right)^{n-1}\left[(\beta-1)\left(\ln\frac{t}{a}\right)^{\beta-2}\right]\frac{dt}{t}.\end{aligned}$$

After to derive $(n-1)$ times, follows

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{(\beta-1)!}{(\beta-n-1)!\,\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\left(\ln\frac{t}{a}\right)^{\beta-n-1}\frac{dt}{t}.$$

Taking the same change of variable as in Property 2.1, we obtain

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-n)\,\Gamma(n-\alpha)}\left(\ln\frac{x}{a}\right)^{\beta-\alpha-1}\underbrace{\int_0^1(1-\tau)^{n-\alpha-1}\tau^{\beta-n-1}d\tau}_{B(n-\alpha,\,\beta-n)},$$

which completes the proof. □

The Caputo-Hadamard fractional differentiation is an operation inverse to the Hadamard fractional integration from the left [27].

**Lemma 3.2.** *Let $\alpha > 0$, $n = [\alpha] + 1$ and $\varphi \in C[a,b]$. If $\alpha \neq 0$ or $\alpha \in \mathbb{N}$, then*

$${}^{C}\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}\varphi(x) = \varphi(x) \qquad \textit{and} \qquad {}^{C}\mathcal{D}^{\alpha}_{b^-}\mathcal{J}^{\alpha}_{b^-}\varphi(x) = \varphi(x). \tag{3.19}$$

*Proof.* From the definition of Caputo-Hadamard fractional derivative of order $\alpha$, the left-sided, given by Eq.(3.1), we can write

$${}^{C}\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}\varphi(x) = \mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}\varphi(x) - \sum_{k=0}^{n-1}\frac{\delta^k[\mathcal{J}^{\alpha}_{a^+}\varphi(a)]}{k!}\left(\ln\frac{x}{a}\right)^k.$$

Since $\mathcal{J}^{\alpha}_{a^+}\varphi(a) = \mathcal{J}^{\alpha}_{a^+}\varphi(x)\big|_{x=a}$, we can write

$$\mathcal{J}^{\alpha}_{a^+}\varphi(a) = \left\{\frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\varphi(t)\frac{dt}{t}\right\}\Bigg|_{x=a} = \frac{1}{\Gamma(\alpha)}\int_a^a \left(\ln\frac{a}{t}\right)^{\alpha-1}\varphi(t)\frac{dt}{t} = 0.$$

Thus,

$$^{C}\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}\varphi(x) = \varphi(x),$$

which yields the first formula in Eq.(3.19). The second one is proved similarly. □

The next statement is known as the semigroup property for Caputo-Hadamard derivatives [19].

**Theorem 3.4.** *Let $\alpha, \beta \in \mathbb{R}$, $\varphi \in C^{m+n}_{\delta}[a,b]$, such that $0 < a < b < \infty$ and $\alpha, \beta \geqslant 0$, then*

$$({}^{C}\mathcal{D}^{\alpha}_{a^+}\, {}^{C}\mathcal{D}^{\beta}_{a^+}\,\varphi)(x) = ({}^{C}\mathcal{D}^{\alpha+\beta}_{a^+}\,\varphi)(x),$$

*where $n-1 < \alpha \leqslant n$ and $m-1 < \beta \leqslant m$.*

In this same paper, [19], the authors proposed a generalization for the previous theorem. We present the theorem and its proof as follow.

**Lemma 3.3.** *Let $\varphi \in C^{n}_{\delta}[a,b]$ with $0 < a < b < \infty$, then*

$$({}^{C}\mathcal{D}^{\alpha_1}_{a^+}\,{}^{C}\mathcal{D}^{\alpha_2}_{a^+}\cdots{}^{C}\mathcal{D}^{\alpha_m}_{a^+}\,\varphi)(x) = ({}^{C}\mathcal{D}^{\sum_{j=1}^{m}\alpha_j}_{a^+}\,\varphi)(x), \tag{3.20}$$

*where $\alpha_j \geqslant 0$, $n_j < \alpha_j \leqslant n_j$ and $\displaystyle\sum_{j=1}^{m}\alpha_j \leqslant n$, $\forall j = \{1, 2, \ldots, m\}$.*

*Proof.* The proof follows by means of mathematical induction and using Theorem 3.4. The expression Eq.(3.20) is true for $m = 1$. Suppose that Eq.(3.20) is true for $m = k$, this is,

$$({}^{C}\mathcal{D}^{\alpha_1}_{a^+}\,{}^{C}\mathcal{D}^{\alpha_2}_{a^+}\cdots{}^{C}\mathcal{D}^{\alpha_k}_{a^+}\,\varphi)(x) = ({}^{C}\mathcal{D}^{\sum_{j=1}^{k}\alpha_j}_{a^+}\,\varphi)(x). \tag{3.21}$$

We need to show that if Eq.(3.20) holds for $m = k$, then it must also holds for $m = k+1$. In fact, we have

$$\begin{aligned}({}^{C}\mathcal{D}^{\alpha_1}_{a^+}\,{}^{C}\mathcal{D}^{\alpha_2}_{a^+}\cdots{}^{C}\mathcal{D}^{\alpha_{k+1}}_{a^+}\,\varphi)(x) &\underbrace{=}_{\text{Eq.(3.21)}} ({}^{C}\mathcal{D}^{\sum_{j=1}^{k}\alpha_j}_{a^+}\,{}^{C}\mathcal{D}^{\alpha_{k+1}}_{a^+}\,\varphi)(x)\\ &\underbrace{=}_{\text{Theorem 3.4}} ({}^{C}\mathcal{D}^{\left(\sum_{j=1}^{k}\alpha_j\right)+\alpha_{k+1}}_{a^+}\,\varphi)(x) = ({}^{C}\mathcal{D}^{\sum_{j=1}^{k+1}\alpha_j}_{a^+}\,\varphi)(x).\end{aligned}$$

□

Analogous to Property 2.5, the Caputo-Hadamard fractional derivative of the power function $t^{\beta}$ yields the same power function apart from a constant factor.

**Property 3.2.** *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \geqslant 0$ and $\beta > 0$. Then, the Caputo-Hadamard fractional derivative of the power function $t^{\beta}$ with $a \to 0$, is given by*

$$({}^{C}\mathcal{D}^{\alpha}_{0+} t^{\beta})(x) = \beta^{\alpha}\, x^{\beta}. \tag{3.22}$$

*Proof.* Consider Eq.(3.6) the relationship between $\mathcal{D}^{\alpha}_{0+}$ and ${}^{C}\mathcal{D}^{\alpha}_{0+}$ and $\varphi(t) = t^{\beta}$. Thus, we can write

$$({}^{C}\mathcal{D}^{\alpha}_{0+} t^{\beta})(x) = (\mathcal{D}^{\alpha}_{a+} t^{\beta})(x) - \sum_{k=0}^{n-1} \frac{\delta^{k}\varphi(a)}{\Gamma(k-\alpha+1)} \left(\ln\frac{x}{a}\right)^{k-\alpha},$$

where $\delta^{k}\varphi(a) = \left[\left(t\dfrac{d}{dt}\right)^{k} t^{\beta}\right]_{t=a} = [\beta^{k}\, t^{\beta}]_{t=0} = 0$. This means that the Caputo-Hadamard fractional derivative of the power function coincides with the Hadamard fractional derivative, for the case $a \to 0$, this is, by Property 2.5 we have

$$({}^{C}\mathcal{D}^{\alpha}_{0+} t^{\beta})(x) = \beta^{\alpha}\, x^{\beta}.$$

□

The following theorem is dedicated to express the Caputo-Hadamard fractional derivative in terms of infinite series involving the $S(\alpha, k)$.

**Theorem 3.5.** *Let $\varphi(x)$ with $x > 0$, be an arbitrary often differentiable function such that its Taylor series converges, and let $\alpha \in \mathbb{R}$. When $\alpha \leqslant 0$, the Caputo-Hadamard fractional derivative ${}^{C}\mathcal{D}^{\alpha}_{0+}\varphi$ is given by Eq.(3.12) if, and only if, there holds for $x > 0$ the relation*

$$({}^{C}\mathcal{D}^{\alpha}_{0+}\varphi)(x) = \sum_{i=0}^{\infty} S(\alpha, i)\, x^{i}\, \varphi^{(i)}(x). \tag{3.23}$$

*Proof.* Let $\alpha \geqslant 0$, and recall the relationship between $\mathcal{D}^{\alpha}_{0+}$ and ${}^{C}\mathcal{D}^{\alpha}_{0+}$ given by Eq.(3.6). Thus, we can write

$$({}^{C}\mathcal{D}^{\alpha}_{0+}\varphi)(x) = (\mathcal{D}^{\alpha}_{0+}\varphi)(x) - \sum_{k=0}^{n-1} \frac{\delta^{k}\varphi(a)}{\Gamma(k-\alpha+1)} \left(\ln\frac{x}{a}\right)^{k-\alpha}. \tag{3.24}$$

where $n = [\alpha] + 1$. By the hypothesis of the theorem, $\varphi(x)$ is a differentiable function, i.e., fix $x > 0$, for any $a \in [0, x]$ and any $y > 0$ we have, using the binomial formula, that

$$\varphi(a) = \sum_{i=0}^{\infty} \frac{\varphi^{(i)}(y)}{i!} \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} y^{i-j}\, a^{j}|_{a=0} = 0. \tag{3.25}$$

For any fixed $y > 0$, Eq.(3.24) is a convergence power series because it coincides with the Taylor series Eq.(2.38), being convergent by the condition of the theorem, this is,

$$({}^{C}\mathcal{D}^{\alpha}_{0+}\varphi)(x) = \sum_{i=0}^{\infty} S(\alpha, i)\, x^{i}\, \varphi^{(i)}(x).$$

When $\alpha = 0$, in the last expression, we obtain

$$({}^C\mathcal{D}^0_{0+}\varphi)(x) \quad = \quad \underbrace{S(0,0)}_{=1}\,\varphi(x) + \sum_{i=1}^{\infty} \underbrace{S(0,i)}_{=0}\, x^i\, \varphi^{(i)}(x). \tag{3.26}$$

that is, $(\mathcal{D}^0_{0+}\varphi)(x) = \varphi(x)$. □

## 3.3 Leibniz-Type Rule for Caputo-Hadamard Fractional Derivative

We propose a Leibniz-type rule for the Caputo-Hadamard fractional derivative.

**Theorem 3.6.** *Let $\alpha \in \mathbb{R}$ and $\varphi, g$ differentiable function defined for $x > 0$. Then, a Leibniz-type rule involving the Caputo-Hadamard fractional derivative is given by*

$$({}^C\mathcal{D}^\alpha_{0+}\varphi\, g)(x) = \sum_{i=0}^{\infty} S(\alpha, i)\, x^i\, D^i[\varphi(x)g(x)], \tag{3.27}$$

*where $D^i[\varphi(x)g(x)]$ is the Leibniz rule for integer order derivatives in accordance with* Eq.(2.41)*. In the case $\varphi g$ is $n$-times differentiable, we have*

$$({}^C\mathcal{D}^\alpha_{0+}\varphi\, g)(x) = \sum_{i=0}^{n} S(\alpha, i)\, x^i \sum_{m=0}^{i} \binom{i}{m} [D^{i-m}\,\varphi(x)][D^m\, g(x)]. \tag{3.28}$$

*Proof.* By means of Theorem 3.5 and considering $\varphi(x) \to \varphi(x)g(x)$ and using Leibniz rule for integer order derivatives, we obtain

$$D^i[\varphi(x)g(x)] = \sum_{m=0}^{i} \binom{i}{m} [D^{i-m}\varphi(x)][D^m g(x)].$$

this is, follows immediately Eq.(3.27). □

## 3.4 Fundamental Theorem of Fractional Calculus

The concept of Hadamard fractional integrals and the Caputo-Hadamard fractional derivatives were presented. By means of these concepts we present the fundamental theorem of fractional calculus. The proof can be found in [19].

**Theorem 3.7.** *Let $\alpha \in \mathbb{R}\,(\alpha \geqslant 0)$, $n = [\alpha] + 1$, $\varphi \in \mathrm{AC}^n_\delta[a, b]$ and $0 < a < b < \infty$.*

(i) *If $\Phi(x) = \mathcal{J}^\alpha_{a+}\varphi(x)$ or $\Phi(x) = \mathcal{J}^\alpha_{b-}\varphi(x)$, $\forall\, x \in [a, b]$, then*

$${}^C\mathcal{D}^\alpha_{a+}\Phi(x) = \varphi(x) \qquad \text{and} \qquad {}^C\mathcal{D}^\alpha_{b-}\Phi(x) = \varphi(x). \tag{3.29}$$

(ii) *We also have*

$${}_a\mathcal{J}^\alpha_b({}^C\mathcal{D}^\alpha_{a+})\Phi(x) = \Phi(b) - \Phi(a), \quad \text{and} \quad {}_a\mathcal{J}^\alpha_b({}^C\mathcal{D}^\alpha_{b-})\Phi(x) = \Phi(a) - \Phi(b). \tag{3.30}$$

## 3.5 A New Generalization

In what follow we present Lemma 5 as in [19] considering a particular case.

**Lemma 3.4.** *If $\alpha \in \mathbb{R}$ with $\alpha \geqslant 0$, $n = [\alpha] + 1$ and $k, m \in \mathbb{N}$, $\varphi(x) \in AC^n_\delta[a,b]$, $0 < a < b < \infty$, then*

$$[(\mathcal{J}^\alpha_{a+})^k \, ({}^C\mathcal{D}^\alpha_{a+}\varphi]^m)(x) = \frac{[({}^C\mathcal{D}^\alpha_{a+})^m \, \varphi](\xi)}{\Gamma(\alpha k + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k}, \qquad \xi \in (a,x) \tag{3.31}$$

or

$$[(\mathcal{J}^\alpha_{b-})^k \, ({}^C\mathcal{D}^\alpha_{b-})^m \varphi](x) = \frac{[({}^C\mathcal{D}^\alpha_{b-})^m \, \varphi](\xi)}{\Gamma(\alpha k + 1)} \left(\ln \frac{b}{x}\right)^{\alpha k}, \qquad \xi \in (x,b). \tag{3.32}$$

Also, we discuss a generalization of Lemma 9 proposed in [19].

**Lemma 3.5.** *Let $\alpha \in \mathbb{R}$, $k, m \in \mathbb{N}$, $\varphi(x) \in AC^n_\delta[a,b]$ and $0 < a < b < \infty$. Then,*

$$[(\mathcal{J}^\alpha_{a+})^{k+m} \, ({}^C\mathcal{D}^\alpha_{a+})^{k+m} \varphi](x) = \frac{[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi](\xi)}{\Gamma(\alpha k + \alpha m + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k + \alpha m}, \qquad \xi \in (a,x). \tag{3.33}$$

*Proof.* From the semigroup property for Hadamard fractional integral, the left-sided, Eq.(2.12), we can write

$$(\mathcal{J}^\alpha_{a+})^k = \underbrace{\mathcal{J}^\alpha_{a+} \, \mathcal{J}^\alpha_{a+} \, \cdots \mathcal{J}^\alpha_{a+}}_{k-times},$$

this is, $(\mathcal{J}^\alpha_{a+})^k = \mathcal{J}^{\alpha k}_{a+}$. Using the definition of Hadamard fractional integral of order $\alpha$, Eq.(2.3) and finally using the mean value theorem for integrals, Definition 1.2, we obtain

$$\begin{aligned}
[(\mathcal{J}^\alpha_{a+})^{k+m} \, ({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi](x) &= [(\mathcal{J}^\alpha_{a+})^m \, (\mathcal{J}^\alpha_{a+})^k \, ({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi](x) \\
&= (\mathcal{J}^\alpha_{a+})^m \, \frac{1}{\Gamma(\alpha k)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha k - 1} \left[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi(t)\right] \frac{dt}{t} \\
&\underbrace{=}_{\text{Definition 1.2}} (\mathcal{J}^\alpha_{a+})^m \, \frac{\left[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi\right](\xi)}{\Gamma(\alpha k)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha k - 1} \frac{dt}{t}, \\
&= (\mathcal{J}^\alpha_{a+})^m \, \frac{\left[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi\right](\xi)}{\Gamma(\alpha k + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k} \\
&= \frac{\left[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi\right](\xi)}{\Gamma(\alpha k + 1)} (\mathcal{J}^\alpha_{a+})^m \left(\ln \frac{x}{a}\right)^{\alpha k}, \quad \xi \in (a,x).
\end{aligned}$$

Using Eq.(2.10), we have

$$\begin{aligned}
[(\mathcal{J}^\alpha_{a+})^{k+m} \, ({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi](x) &= \frac{\left[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi\right](\xi)}{\cancel{\Gamma(\alpha k + 1)}} \, \frac{\cancel{\Gamma(\alpha k + 1)}}{\Gamma(\alpha k + \alpha m + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k + \alpha m} \\
&= \frac{\left[({}^C\mathcal{D}^\alpha_{a+})^{k+m} \, \varphi\right](\xi)}{\Gamma(\alpha k + \alpha m + 1)} \left(\ln \frac{x}{a}\right)^{\alpha(k+m)}, \qquad \xi \in (a,x),
\end{aligned}$$

which completes the proof. □

We mention two particular cases for the previous result.

- If $m = 0$ in Eq.(3.33), we have
$$[(\mathcal{J}^{\alpha}_{a^+})^k ({}^C\mathcal{D}^{\alpha}_{a^+})^k \varphi](x) = \frac{[({}^C\mathcal{D}^{\alpha}_{a^+})^k \varphi](\xi)}{\Gamma(\alpha k + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k}, \qquad \xi \in (a, x),$$
this is, we have the result of Lemma 3.4 with $m = k$.

- If $m = 1$ in Eq.(3.33), we obtain
$$[(\mathcal{J}^{\alpha}_{a^+})^{k+1} ({}^C\mathcal{D}^{\alpha}_{a^+})^{k+1} \varphi](x) = \frac{[({}^C\mathcal{D}^{\alpha}_{a^+})^{k+1} \varphi](\xi)}{\Gamma(\alpha k + \alpha + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k + \alpha}, \qquad \xi \in (a, x)$$
and considering $k \to k - 1$ follows that
$$[(\mathcal{J}^{\alpha}_{a^+})^k ({}^C\mathcal{D}^{\alpha}_{a^+})^k \varphi](x) = \frac{[({}^C\mathcal{D}^{\alpha}_{a^+})^k \varphi](\xi)}{\Gamma(\alpha k + 1)} \left(\ln \frac{x}{a}\right)^{\alpha k},$$
where $k = \{1, 2, \ldots\}$, i.e., we recover the result of Lemma 9 proposed in [19].

## 3.6 An Interesting Proposition

In this section, we prove the statement: the Caputo-Hadamard fractional derivatives of a convergent power series is a convergent power series.

**Proposition 3.1.** *Let $\alpha \in \mathbb{R}$ and $\varphi(x)$ a convergent power series, where*
$$\varphi(x) = \sum_{k=0}^{\infty} a_k\, x^k, \qquad (a_k \in \mathbb{R}). \tag{3.34}$$
*If $\alpha \geqslant 0$, then the Caputo-Hadamard fractional derivative of order $\alpha$, the left-sided, with $a = 0$, $({}^C\mathcal{D}^{\alpha}_{0^+}\varphi)(x)$, is represented by a convergent power series*
$$({}^C\mathcal{D}^{\alpha}_{0^+}\varphi)(x) = \sum_{k=1}^{\infty} k^{\alpha} a_k\, x^k. \tag{3.35}$$
*The radii of convergent of series in Eq.(3.34) and Eq.(3.35) coincide.*

*Proof.* From the Caputo-Hadamard fractional derivatives of $\alpha$, we have
$$\begin{aligned}
\left[{}^C\mathcal{D}^{\alpha}_{0^+}\left(\sum_{k=0}^{\infty} a_k\, t^k\right)\right](x) &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \left(\ln \frac{x}{t}\right)^{n-\alpha-1} \left(t\frac{d}{dt}\right)^n \left[\sum_{k=0}^{\infty} a_k\, t^k\right] \frac{dt}{t} \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^x \left(\ln \frac{x}{t}\right)^{n-\alpha-1} \sum_{k=0}^{\infty} a_k \left(t\frac{d}{dt}\right)^{n-1} \left[\left(t\frac{d}{dt}\right) t^k\right] \frac{dt}{t} \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^x \left(\ln \frac{x}{t}\right)^{n-\alpha-1} \sum_{k=1}^{\infty} a_k \left(t\frac{d}{dt}\right)^{n-1} [k\, t^k] \frac{dt}{t}.
\end{aligned}$$

After to derive $(n-1)$ times, we obtain

$$\begin{aligned}\left[{}^C\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=0}^{\infty} a_k\, t^k\right)\right](x) &= \frac{1}{\Gamma(n-\alpha)}\int_0^x \left(\ln\frac{x}{t}\right)^{n-\alpha-1}\left[\sum_{k=1}^{\infty} a_k\, k^n\, t^k\right]\frac{dt}{t}\\ &= \frac{1}{\Gamma(n-\alpha)}\sum_{k=1}^{\infty} a_k\, k^n \int_0^x \left(\ln\frac{x}{t}\right)^{n-\alpha-1} t^k\,\frac{dt}{t}.\end{aligned}$$

Introducing the change of variable, $u = \left(\ln\dfrac{x}{t}\right)$, follows that

$$\left[{}^C\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=0}^{\infty} a_k\, t^k\right)\right](x) = \frac{1}{\Gamma(n-\alpha)}\sum_{k=1}^{\infty} a_k\, k^n\, x^k \int_0^{\infty} u^{n-\alpha-1}\mathrm{e}^{-uk}du.$$

Further, with a new change of variable, $\xi = uk$, we obtain

$$\begin{aligned}\left[{}^C\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=0}^{\infty} a_k\, t^k\right)\right](x) &= \frac{1}{\Gamma(n-\alpha)}\sum_{k=1}^{\infty}\frac{a_k\, k^n\, x^k}{k^{n-\alpha}}\underbrace{\int_0^{\infty}\xi^{n-\alpha-1}\,\mathrm{e}^{-\xi}d\xi}_{\Gamma(n-\alpha)},\\ &= \sum_{k=1}^{\infty} k^{\alpha}\, a_k\, x^k, \qquad (n-\alpha) > 0.\end{aligned}$$

which conclude the proof. □

### 3.6.1 Particular Cases

The following particular case can be derived from Proposition 3.1.

1. Let

$$\varphi(x) = \left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{x^k}{k!}\right),$$

then replacing, this expression, in Eq.(3.35), we can write

$$\left[{}^C\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{t^k}{k!}\right)\right](x) = \sum_{k=1}^{\infty} k^{\alpha}\left(\frac{k^{-\alpha}(\rho)_k}{k!\Gamma(\beta k+\gamma)}\right)x^k = \sum_{k=1}^{\infty}\left(\frac{(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{x^k}{k!}\right).$$

Introducing the change of index $k \to k+1$, follows that

$$\begin{aligned}\left[{}^C\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{t^k}{k!}\right)\right](x) &= \sum_{k=0}^{\infty}\frac{(\rho)_{k+1}}{\Gamma(\beta k+\beta+\gamma)}\frac{x^{k+1}}{(k+1)!}\\ &= x\sum_{k=0}^{\infty}\frac{(\rho)_{k+1}}{\Gamma(\beta k+\beta+\gamma)}\frac{x^k}{(k+1)!},\end{aligned}$$

and by Property 1.1, we can write

$$\left[{}^C\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(\rho)_k}{\Gamma(\beta k+\gamma)}\frac{t^k}{k!}\right)\right](x) = -\frac{1}{\Gamma(\gamma)} + E^{\rho}_{\beta,\gamma}(x),$$

where $E^{\rho}_{\beta,\gamma}(x)$ is the three-parameters Mittag-Leffler function. For $\rho = 1$, we obtain

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}t^{k}}{\Gamma(\beta k+\gamma)}\right)\right](x) = -\frac{1}{\Gamma(\gamma)} + E_{\beta,\gamma}(x) = xE_{\beta,\beta+\gamma}(x), \tag{3.36}$$

where $E_{\beta,\gamma}(x)$ is the two-parameter Mittag-Leffler function. For $\rho = \gamma = 1$, we have

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}t^{k}}{\Gamma(\beta k+1)}\right)\right](x) = -1 + E_{\beta}(x) = xE_{\beta,\beta+1}(x).$$

Taking $x \to (\lambda x^2)$ in Eq.(3.36), where $\lambda \in \mathbb{R}$, we obtain

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(\lambda t^2)^{k}}{\Gamma(\beta k+\gamma)}\right)\right](x) = -\frac{1}{\Gamma(\gamma)} + E_{\beta,\gamma}(\lambda x^2). \tag{3.37}$$

Considering $\lambda = -1$, $\beta = 2$ and $\gamma = 1$ in Eq.(3.37), follows

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(-t^2)^{k}}{\Gamma(2k+1)}\right)\right](x) = -\frac{1}{\Gamma(1)} + E_{2,1}(-x^2) = \cos(x) - 1.$$

On the other hand, for $\lambda = -1$ and $\beta = \gamma = 2$ in Eq.(3.37), we have

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(-t^2)^{k}}{\Gamma(2k+2)}\right)\right](x) = -\frac{1}{\Gamma(2)} + E_{2,2}(-x^2) = \frac{\sin(x)}{x} - 1.$$

Considering $\lambda = 1$, $\beta = 2$ and $\gamma = 1$ in Eq.(3.37), we obtain

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(t^2)^{k}}{\Gamma(2k+1)}\right)\right](x) = -\frac{1}{\Gamma(1)} + E_{2,1}(x^2) = \cosh(x) - 1.$$

Finally, for $\lambda = 1$ and $\beta = \gamma = 2$ in Eq.(3.37), follows that

$$\left[{}^{C}\mathcal{D}^{\alpha}_{0+}\left(\sum_{k=1}^{\infty}\frac{k^{-\alpha}(t^2)^{k}}{\Gamma(2k+2)}\right)\right](x) = -\frac{1}{\Gamma(2)} + E_{2,2}(x^2) = \frac{\sinh(x)}{x} - 1.$$

## 3.7 Mellin Transform of the Caputo-Hadamard Fractional Derivative

In this section we present the result given by Mellin transform of the Caputo-Hadamard fractional derivatives, [27].

**Lemma 3.6.** *Let $\alpha \in \mathbb{R}$ such that $\alpha > 0$ and let a function $\varphi(x)$ be such that its Mellin transform $(\mathcal{M}\varphi)(s)$ exists for $s \in \mathbb{R}$.*

(a) *If $s < 0$ and $(\mathcal{M}\,\delta^n\varphi)(s)$ exists, then*

$$(\mathcal{M}\,{}^{C}\mathcal{D}^{\alpha}_{0+}\varphi)(s) = (-s)^{\alpha}(\mathcal{M}\varphi)(s). \tag{3.38}$$

(b) *If $s > 0$ and $(\mathcal{M}\,\delta^n\varphi)(s)$ exists, then*

$$(\mathcal{M}\,{}^{C}\mathcal{D}^{\alpha}_{-}\varphi)(s) = s^{\alpha}(\mathcal{M}\varphi)(s). \tag{3.39}$$

# Chapter 4

# Generalized Caputo-Type Fractional Derivatives

Here our main goal is to introduce a new fractional differentiation operator. For this end, we first present some relevant results. This chapter is part of a paper that was accepted for publication [48] and we presented an application in [57]. In 2011, Katugampola [30] proposed the generalized fractional integrals. These fractional integration operators recover both the Riemann-Liouville and Hadamard fractional integrals. In 2014, the same author [31] introduced the generalized fractional derivatives. These fractional differentiation operators admit, as particular cases, both the Riemann-Liouville and Hadamard fractional derivatives. We propose by means of a Caputo modification in the generalized fractional derivatives to define a new differentiation operator of arbitrary order which contains, as particular cases, the derivatives of arbitrary order in the sense of Caputo and Caputo-Hadamard. We call this new fractional derivative by generalized Caputo-type fractional derivatives. We emphasize that, Almeida et al. [3] presented a new type of fractional operator, Caputo-Katugampola derivative, which recovers the Caputo and Caputo-Hadamard fractional derivatives. However, in that paper the authors discuss only the case $0 < \alpha < 1$, while we discuss the general case $\alpha \in \mathbb{R}$ with $\alpha > 0$. Recently, also, Jarad et al. [26] proposed a fractional derivative which recovers the Caputo and Caputo-Hadamard fractional derivatives.

In section 4.1 we present the theorem for the $n$th generalized integral so that, by means of analytic continuation it is possible to define the generalized fractional integrals. In section 4.2 we revisit the results proposed by Katugampola. In section 4.3 we present our definition for fractional derivatives and some properties associated with these operators. In section 4.4, we show that there exists a relation between the generalized fractional derivatives and generalized Caputo-type fractional derivatives. Finally, in section 4.5 we present the fundamental theorem of fractional calculus associated with these new operators.

## 4.1 The $n$th Generalized Integral

In the following theorem we present the $n$th generalized integral [30].

**Theorem 4.1.** *Let $n \in \mathbb{N}$, $\rho \neq 0$ and $\varphi \in X_c^p(a,b)$ where $c \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, then the $n$th generalized integral, the left sided, is defined by*

$$({}^\rho\mathcal{J}_{a+}^n\varphi)(x) = \int_a^x t_1^{\rho-1}\,dt_1 \int_a^{t_1} t_2^{\rho-1}\,dt_2 \cdots \int_a^{t_{n-1}} t_n^{\rho-1}\,\varphi(t_n)dt_n = \frac{\rho^{1-n}}{\Gamma(n)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n}}\,\varphi(t)\,dt.$$

We note that when $\rho \to 1$, we obtain the $n$th integral of Cauchy, Theorem 1.3. On the other hand, when $\rho \to 0^+$ and, from the $\ell$'Hôpital rule, we obtain the $n$th integral of Hadamard, Theorem 2.1.

## 4.2 Generalized Fractional Integrals and Derivatives

The generalized fractional integral was introduced by Katugampola [30] in order to generalize the Riemann-Liouville and Hadamard fractional integrals. In that paper, he defines the generalized fractional derivatives associated with the generalized integral operators in such a way that the differential operators generalize the Riemann-Liouville and Hadamard fractional derivatives [31]. Both generalized fractional integral and fractional derivative are defined for $\alpha \in \mathbb{C}$; however, in this thesis we discuss only the case $\alpha \in \mathbb{R}$.

**Definition 4.1.** [30] *Let $\alpha, \rho, c \in \mathbb{R}$ with $\alpha > 0$ and $\rho > 0$. The generalized fractional integrals $({}^\rho\mathcal{J}_{a+}^\alpha\varphi)(x)$ (left-sided) and $({}^\rho\mathcal{J}_{b-}^\alpha\varphi)(x)$ (right-sided), with $\varphi \in X_c^p(a,b)$, are defined by*

$$({}^\rho\mathcal{J}_{a+}^\alpha\varphi)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho - t^\rho)^{1-\alpha}}dt, \qquad x > a \tag{4.1}$$

and

$$({}^\rho\mathcal{J}_{b-}^\alpha\varphi)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}\,\varphi(t)}{(t^\rho - x^\rho)^{1-\alpha}}dt, \qquad x < b. \tag{4.2}$$

Similarly, we introduce the generalized fractional derivatives corresponding to the fractional integrals, Eq.(4.1) and Eq.(4.2).

**Definition 4.2.** [31] *Let $\alpha \in \mathbb{R}$ such that $\alpha \notin \mathbb{N}$, $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$. The generalized fractional derivatives, $({}^\rho\mathcal{D}_{a+}^\alpha\varphi)(x)$ and $({}^\rho\mathcal{D}_{b-}^\alpha\varphi)(x)$, the left- and righ-sided, are defined by*

$$\begin{aligned} ({}^\rho\mathcal{D}_{a+}^\alpha\varphi)(x) &= \delta_\rho^n({}^\rho\mathcal{J}_{a+}^{n-\alpha}\varphi)(x) \\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(x^{1-\rho}\frac{d}{dx}\right)^n \int_a^x \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho - t^\rho)^{1-n+\alpha}}dt, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}({}^{\rho}\mathcal{D}^{\alpha}_{b-}\varphi)(x) &= (-1)^n\delta^n_{\rho}({}^{\rho}\mathcal{J}^{n-\alpha}_{b-}\varphi)(x)\\ &= \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^n\int_x^b\frac{t^{\rho-1}\,\varphi(t)}{(x^{\rho}-t^{\rho})^{1-n+\alpha}}dt, \end{aligned}\tag{4.4}$$

*respectively, if the integrals exist,* $\delta^n_{\rho} = \left(x^{1-\rho}\dfrac{d}{dx}\right)^n$ *and* $\varphi \in AC^n_{\delta_\rho}[a,b]$.

**Theorem 4.2.** *Let* $\alpha > 0$, $\beta > 0$, $1 \leqslant p \leqslant \infty$, $0 < a < b < \infty$ *and* $\rho, c \in \mathbb{R}$, $\rho \geqslant c$. *Then, for* $\varphi \in X^p_c(a,b)$ *the semigroup property is valid, i.e.*

$$({}^{\rho}\mathcal{J}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\beta}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{J}^{\alpha+\beta}_{a+}\varphi)(x).$$

*Proof.* See [31]. □

**Lemma 4.1.** *Let* $x > a$, ${}^{\rho}\mathcal{J}^{\alpha}_{a+}$ *and* ${}^{\rho}\mathcal{D}^{\alpha}_{a+}$, *as defined in* Eq.(4.1) *and* Eq.(4.3), *respectively. Then, for* $\alpha \geqslant 0$ *and* $\xi > 0$, *we have*

$$\begin{aligned}\left[{}^{\rho}\mathcal{J}^{\alpha}_{a+}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\xi-1}\right](x) &= \frac{\Gamma(\xi)}{\Gamma(\alpha+\xi)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\xi-1},\\ \left[{}^{\rho}\mathcal{D}^{\alpha}_{a+}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\right](x) &= 0, \quad 0<\alpha<1.\end{aligned}$$

*Proof.* See [3]. □

**Lemma 4.2.** *Let* $0 < \alpha < 1$, $0 \leqslant \gamma < 1$. *If* $\varphi \in C_{\gamma}[a,b]$ *and* ${}^{\rho}\mathcal{J}^{1-\alpha}_{a+}\varphi \in C^1_{\gamma}[a,b]$, *then*

$$({}^{\rho}\mathcal{J}^{\alpha}_{a+}\,{}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \varphi(x) - \frac{({}^{\rho}\mathcal{J}^{1-\alpha}_{a+}\varphi)(a)}{\Gamma(\alpha)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}, \qquad \textit{for all } x \in (a,b).$$

*Proof.* The proof uses integration by parts, with $u = (x^{\rho}-t^{\rho})^{\alpha-1}$ and $dv = \dfrac{d}{dt}({}^{\rho}\mathcal{J}^{1-\alpha}_{a+}\varphi)(t)dt$. □

**Lemma 4.3.** *Let* $\alpha > 0$, $0 \leqslant \gamma < 1$ *and* $\varphi \in C_{\gamma}[a,b]$. *Then,*

$$({}^{\rho}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = \varphi(x), \qquad \textit{for all } x \in (a,b)$$

*Proof.* See [30]. □

In what follows, we use a lemma and a property to prove the relation between the generalized fractional derivatives and the generalized Caputo-type fractional derivatives according to Definition 4.3.

**Lemma 4.4.** *Let $n \in \mathbb{N}$, $\rho > 0$ and $\varphi(t)$ as in Definition 1.3, such that*

$$({}^{\rho}\mathcal{J}^{n}_{a+}\varphi)(x) = \frac{\rho^{1-n}}{\Gamma(n)} \int_a^x \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho - t^\rho)^{1-n}} dt$$

*and* $\delta^n_\rho = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$, *then*

$$({}^{\rho}\mathcal{J}^{n}_{a+}\delta^n_\rho\varphi)(x) = \varphi(x) - \sum_{k=0}^{n-1} \frac{\delta^k_\rho\,\varphi(a)}{k!} \left(\frac{x^\rho - a^\rho}{\rho}\right)^k. \tag{4.5}$$

*Proof.* From Theorem 4.1, we have

$$({}^{\rho}\mathcal{J}^{n}_{a+}\,\delta^n_\rho\varphi)(x) = \frac{\rho^{1-n}}{\Gamma(n)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n}} \left(t^{1-\rho}\frac{d}{dt}\right)^n \varphi(t)dt.$$

Applying integrating by parts formula,

$$u = (x^\rho - t^\rho)^{n-1} \quad \text{and} \quad dv = t^{\rho-1}\left(t^{1-\rho}\frac{d}{dt}\right)^n \varphi(t)dt$$

we get

$$\begin{aligned} ({}^{\rho}\mathcal{J}^{n}_{a+}\delta^n_\rho\varphi)(x) &= -\frac{\rho^{1-n}}{\Gamma(n)}(x^\rho - t^\rho)^{n-1}\,\delta^{n-1}_\rho\varphi(a) \\ &+ \frac{\rho^{2-n}}{\Gamma(n-1)} \int_a^x (x^\rho - t^\rho)^{n-2}\,t^{\rho-1}\left(t^{1-\rho}\frac{d}{dt}\right)^{n-1} \varphi(t)\,dt. \end{aligned}$$

Integrating by parts $n-1$ times, follows that,

$$\begin{aligned} ({}^{\rho}\mathcal{J}^{n}_{a+}\delta^n_\rho\varphi)(x) &= \int_a^x \frac{d}{dt}\varphi(t)\,dt - \sum_{k=1}^{n-1} \frac{\delta^k_\rho\varphi(a)}{k!}\left(\frac{x^\rho - a^\rho}{\rho}\right)^k \\ &= \varphi(x) - \varphi(a) - \sum_{k=1}^{n-1} \frac{\delta^k_\rho\varphi(a)}{k!}\left(\frac{x^\rho - a^\rho}{\rho}\right)^k. \end{aligned}$$

From the last expression, we obtain Eq.(4.5). □

**Property 4.1.** *Let $\alpha, \rho \in \mathbb{R}$, $n = [\alpha] + 1$, $k \in \mathbb{N}_0$ and $\rho > 0$. If $\alpha > 0$ and $0 < a < b < \infty$, then*

$$\left({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}(t^\rho - a^\rho)^k\right)(x) = \frac{\Gamma(k+1)\rho^{\alpha-n}}{\Gamma(k+n-\alpha+1)}\,(x^\rho - a^\rho)^{k+n-\alpha} \tag{4.6}$$

and

$$\delta^n_\rho[(x^\rho - a^\rho)^{k+n-\alpha}] = \frac{\Gamma(k+n-\alpha+1)}{\Gamma(k-\alpha+1)}\,\rho^n(x^\rho - a^\rho)^{k-\alpha}. \tag{4.7}$$

*Proof.* First, we prove Eq.(4.6). From Eq.(4.1), we can write

$$\left({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}(t^\rho - a^\rho)^k\right)(x) = \frac{\rho^{1-(n-\alpha)}}{\Gamma(n-\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-(n-\alpha)}}(t^\rho - a^\rho)^k dt.$$

Introducing the change of variable, $u = \dfrac{t^\rho - a^\rho}{x^\rho - a^\rho}$, we find that

$$\begin{aligned}\left({}^\rho\mathcal{J}_{a^+}^{n-\alpha}(t^\rho - a^\rho)^k\right)(x) &= \frac{\rho^{1-(n-\alpha)}}{\Gamma(n-\alpha)}\int_0^1 \frac{[(x^\rho - a^\rho)u]^k}{[x^\rho - a^\rho - (x^\rho - a^\rho)u]^{1-n+\alpha}}\frac{(x^\rho - a^\rho)}{\rho}du\\ &= \frac{\rho^{1-(n-\alpha)}}{\Gamma(n-\alpha)}.(x^\rho - a^\rho)^{k+n-\alpha}\underbrace{\int_0^1 u^k(1-u)^{n-\alpha-1}du}_{B(k+1,n-\alpha)},\end{aligned}$$

this is,

$$\left({}^\rho\mathcal{J}_{a^+}^{n-\alpha}(t^\rho - a^\rho)^k\right)(x) = \frac{\Gamma(k+1)}{\Gamma(k+n-\alpha+1)}\rho^{\alpha-n}(x^\rho - \alpha^\rho)^{k+n-\alpha}. \tag{4.8}$$

Now we prove Eq.(4.7), this is,

$$\begin{aligned}\delta_\rho^n[(x^\rho - a^\rho)^{k+n-\alpha}] &= \left(x^{1-\rho}\frac{d}{dx}\right)^n[(x^\rho - a^\rho)^{k+n-\alpha}]\\ &= \left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}\left(x^{1-\rho}\frac{d}{dx}\right)[(x^\rho - a^\rho)^{k+n-\alpha}]\\ &= \left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}[x^{1-\rho}(k+n-\alpha)(x^\rho - a^\rho)^{k+n-\alpha-1}\rho\, x^{\rho-1}]\\ &= \rho\,(x^\rho - a^\rho)^{k+n-\alpha-1}\left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}[(x^\rho - a^\rho)^{k+n-\alpha-1}]\\ &= \cdots\rho^n(k+n-\alpha)\cdots(k+n-\alpha-(n-1))(x^\rho - a^\rho)^{k+n-\alpha-n}\\ &= \frac{(k+n-\alpha)\cdots(k-\alpha+1)\Gamma(k-\alpha+1)}{\Gamma(k-\alpha+1)}\rho^n\,(x^\rho - a^\rho)^{k-\alpha}.\end{aligned}$$

Therefore, we obtain

$$\delta_\rho^n[(x^\rho - a^\rho)^{k+n-\alpha}] = \frac{\Gamma(k+n-\alpha+1)}{\Gamma(k-\alpha+1)}\rho^n(x^\rho - a^\rho)^{k-\alpha}.$$

□

## 4.3 Generalized Caputo-Type Fractional Derivatives

In this section, we introduce the generalized Caputo-type fractional derivatives by means of a Caputo-type modification of the generalized fractional derivatives. After that, we present a theorem showing that, from two adequate limits, these generalized Caputo-type fractional derivatives recovers both Caputo and Caputo-Hadamard fractional derivatives.

**Definition 4.3.** *Let $\alpha, \rho \in \mathbb{R}$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$. The left-sided and the right-sided generalized Caputo-type fractional derivatives are defined, for $0 \leqslant a < x < b \leqslant \infty$ and $\varphi \in AC^n_{\delta_\rho}[a,b]$, by*

$$\begin{aligned}({}^\rho_*\mathcal{D}^\alpha_{a^+}\varphi)(x) &= ({}^\rho\mathcal{J}^{n-\alpha}_{a^+}\delta^n_\rho\varphi)(x) &(4.9)\\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)\,dt, &(4.10)\end{aligned}$$

and

$$
\begin{aligned}
({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) &= (-1)^n({}^{\rho}\mathcal{J}^{n-\alpha}_{b-}\delta^n_{\rho}\varphi)(x) \\
&= \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_x^b \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)\,dt, \qquad (4.11)
\end{aligned}
$$

*respectively, if the integrals exist. If* $\alpha\in\mathbb{N}_0$*, then* $({}^{\rho}_{*}\mathcal{D}^{n}_{a+}\varphi)(x)$ *and* $({}^{\rho}_{*}\mathcal{D}^{n}_{b-}\varphi)(x)$ *are represented by*

$$
({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \delta^n_{\rho}\varphi(x) \quad \textit{and} \quad ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) = (-1)^n\delta^n_{\rho}\varphi(x). \qquad (4.12)
$$

*In particular, we have*

$$
({}^{\rho}_{*}\mathcal{D}^{0}_{a+}\varphi)(x) = ({}^{\rho}_{*}\mathcal{D}^{0}_{b-}\varphi)(x) = \varphi(x).
$$

The following theorem shows that, from the definition of the generalized Caputo-type fractional derivatives, it is possible to recover, as particular cases, both the Caputo and Caputo-Hadamard derivatives.

**Theorem 4.3.** *Let* $\alpha,\rho\in\mathbb{R}$*,* $\alpha>0$*,* $\alpha\notin\mathbb{N}$*,* $n=[\alpha]+1$ *and* $\rho>0$*. Then, for* $x>a$

$$
\lim_{\rho\to 1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\varphi^{(n)}(t)dt. \qquad (4.13)
$$

$$
\lim_{\rho\to 0^+}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{n-\alpha-1}\delta^n\varphi(t)dt. \qquad (4.14)
$$

*Proof.* First we show Eq.(4.13): Using Eq.(4.10) and by dominated convergence theorem [54], we can write

$$
\begin{aligned}
\lim_{\rho\to 1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= \lim_{\rho\to 1}\left\{\frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)dt\right\} \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\varphi^{(n)}(t)dt \\
&= ({}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x), \qquad (4.15)
\end{aligned}
$$

where $\varphi^{(n)}(t) = \left(\frac{d}{dt}\right)^n\varphi(t)$.

Now we show Eq.(4.14): Again, we use Eq.(4.10), ℓ'Hôpital rule and dominated convergence theorem [54], to obtain

$$
\begin{aligned}
\lim_{\rho\to 0^+}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= ({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x) \\
&= \lim_{\rho\to 0^+}\left\{\frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)dt\right\} \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x \lim_{\rho\to 0^+} t^{\rho-1}\left(\frac{x^\rho-t^\rho}{\rho}\right)^{n-\alpha-1}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)dt \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{n-\alpha-1}\delta^n\varphi(t)\frac{dt}{t}, \quad \text{onde} \quad \delta^n = \left(t\frac{d}{dt}\right)^n.
\end{aligned}
$$

The proof is valid also for the right-sided operator. □

The linearity of the differential operators ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$ and ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$ is ensured by the following theorem:

**Theorem 4.4.** *Let $\alpha, \rho \in \mathbb{R}$ and $\rho > 0$ such that $\alpha \notin \mathbb{N}$ and $\alpha > 0$. If $0 < a < b < \infty$, then*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}(\varphi + g))(x) = ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) + ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}g)(x). \tag{4.16}$$

*Proof.* The result follows from the fact that integral operators are linear. □

The composition of the fractional integration operators ${}^{\rho}\mathcal{J}^{\alpha}_{a+}$ and ${}^{\rho}\mathcal{J}^{\alpha}_{b-}$, with the fractional differentiation operators, ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$ and ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$, is given by the following result.

**Theorem 4.5.** *Let $\alpha, \rho \in \mathbb{R}$, $\alpha > 0$ and $\rho > 0$. If $0 < a < b < \infty$, then*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = \varphi(x) \qquad \textit{and} \qquad ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\,{}^{\rho}\mathcal{J}^{\alpha}_{b-}\varphi)(x) = \varphi(x). \tag{4.17}$$

*Proof.* Using, Eq.(4.1) and Eq.(4.10), we can write

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n+\alpha}}\delta^n_\rho\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}\varphi(\tau)}{(x^\rho - t^\rho)^{1-\alpha}}d\tau\right]dt. \tag{4.18}$$

Knowing that,

$$\int_a^t \frac{\tau^{\rho-1}\varphi(\tau)}{(x^\rho - t^\rho)^{1-\alpha}}d\tau = \frac{1}{\alpha\rho}\underbrace{\left\{\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho - \tau^\rho)^\alpha \varphi'(\tau)d\tau\right\}}_{A(t)} = \frac{1}{\alpha\rho}A(t) \tag{4.19}$$

see Appendix A. If we substitute Eq.(4.19) into Eq.(4.18), we obtain

$$\begin{gathered}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = \\ \frac{\rho^{1-n}}{\Gamma(n-\alpha)\Gamma(\alpha+1)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n+\alpha}}\,\delta^n_\rho\left\{\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho - \tau^\rho)^\alpha\varphi'(\tau)d\tau\right]\right\}dt.\end{gathered} \tag{4.20}$$

Using the following result which we prove in Appendix A,

$$\delta^n_\rho A(t) = \frac{\Gamma(\alpha+1)\,\rho^n}{\Gamma(\alpha-n+1)}\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-n} + \int_a^t (t^\rho - \tau^\rho)^{\alpha-n}\varphi'(\tau)d\tau\right], \tag{4.21}$$

we obtain, substituting in Eq.(4.20), the expression

$$\begin{aligned}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) &= \frac{\rho\,\varphi(a)}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}\int_a^x \frac{(t^\rho - a^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{1-n+\alpha}}t^{\rho-1}dt \\ &+ \frac{\rho}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{\alpha-n+1}}\left\{\int_a^t (t^\rho - \tau^\rho)^{\alpha-n}\varphi'(\tau)d\tau\right\}dt.\end{aligned} \tag{4.22}$$

We use Fubini's theorem and Dirichlet's formula, in order to change the order of the integrals, together with the result

$$\int_a^x \frac{(t^\rho - a^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{1-n+\alpha}}t^{\rho-1}dt = \frac{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}{\rho}, \tag{4.23}$$

see Appendix A. As one can rewrite Eq.(4.22) as follows

$$\begin{aligned}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) &= \varphi(a) + \frac{\rho}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}\int_a^x \varphi'(\tau)d\tau \int_\tau^x \frac{(t^\rho-\tau^\rho)^{\alpha-n}}{(x^\rho-t^\rho)^{\alpha-n+1}}t^{\rho-1}dt \\ &= \varphi(a) + \int_a^x \varphi'(\tau)d\tau.\end{aligned}$$

Using the fundamental theorem of calculus we obtain the first expression in Eq.(4.17). The second expression in Eq.(4.17) is proved similarly. □

The following theorem yields the compositions of the fractional integral operators, ${}^{\rho}\mathcal{J}^{\beta}_{a+}$ and ${}^{\rho}\mathcal{J}^{\beta}_{b-}$ with the fractional differential operators, ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$ and ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$.

**Theorem 4.6.** *Let $\alpha, \beta, \rho \in \mathbb{R}$ such that $\beta > \alpha$ and $\alpha > 0$. If $0 < a < b < \infty$, then, for $\rho > 0$,*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\beta}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{J}^{\beta-\alpha}_{a+}\varphi)(x) \qquad \textit{and} \qquad ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\,{}^{\rho}\mathcal{J}^{\beta}_{b-}\varphi)(x) = ({}^{\rho}\mathcal{J}^{\beta-\alpha}_{b-}\varphi)(x). \quad (4.24)$$

*Proof.* The proof is analogous to the proof of Theorem 4.5. □

We now discuss the following property involving the power function:

**Property 4.2.** *Sejam $\alpha, \beta, \rho \in \mathbb{R}$, $\alpha > 0$, $\rho > 0$, $(\beta - \alpha\rho) > 0$ and $\varphi(t) = t^\beta$. Taking the limit $a \to 0$, we get*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^\beta)(x) = \begin{cases} \dfrac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-\alpha+1\right)}\rho^\alpha x^{\beta-\alpha\rho}, & \alpha > 0, \quad \left(\alpha - \dfrac{\beta}{\rho}\right) \notin \mathbb{N} \\[2ex] 0, & \alpha > 0, \quad \left(\alpha - \dfrac{\beta}{\rho}\right) \in \mathbb{N}. \end{cases} \quad (4.25)$$

*Proof.* We consider Eq.(4.10) with $a \to 0$ and $\varphi(t) = t^\beta$. Hence, we can write

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^\beta)(x) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\lim_{a\to 0}\int_a^x \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^\beta\, dt, \qquad x > a. \quad (4.26)$$

with $n = [\alpha] + 1$ and $n = \{1, 2, \ldots\}$. Since

$$\begin{aligned}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^\beta &= \beta(\beta-\rho)(\beta-2\rho)\ldots(\beta-(n-1)\rho)\, t^{\beta-n\rho} \\ &= \rho^n\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)}t^{\beta-n\rho},\end{aligned}$$

we can substitute this expression into Eq.(4.26) in order to obtain

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^\beta)(x) = \frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)}\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)}\int_0^x \frac{t^{\beta+(1-n)\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}dt.$$

Further, with the change of variable $u = t^\rho/x^\rho$ we have

$$({}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}t^\beta)(x) = \frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)}\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)}\rho^{-1}x^{\beta-\alpha\rho}\underbrace{\int_0^1 u^{\frac{\beta}{\rho}-n}(1-u)^{n-\alpha-1}du}_{B\left(\frac{\beta}{\rho}-n+1,\, n-\alpha\right)}$$

where $B(\cdot,\cdot)$ is the beta function. Hence, we get

$$({}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}t^\beta)(x) = \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-\alpha+1\right)}\rho^{\alpha}x^{\beta-\alpha\rho}, \tag{4.27}$$

as desired. We emphasize that, if $\left(\alpha-\dfrac{\beta}{\rho}\right) \notin \mathbb{N}$ in Eq.(4.27), we then obtain the first expression in Eq.(4.25). On the other hand, if $\left(\alpha-\dfrac{\beta}{\rho}\right) \in \mathbb{N}$, we obtain the second expression in Eq.(4.25). □

Notice that, by taking the limit $\rho \to 1$, Eq.(4.25) becomes

$$\lim_{\rho\to 1}({}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}t^\beta)(x) = ({}_{*}\mathcal{D}_{0+}^{\alpha}t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},$$

that is, it coincides with a Caputo arbitrary order derivative of the power function. Similarly, taking the limit $\rho \to 0$, and using the result (see [40])

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \approx z^{\alpha-\beta} \quad \text{when} \quad z\to\infty \quad \text{and} \quad \alpha,\beta \geqslant 0,$$

we obtain

$$\lim_{\rho\to 0^+}({}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}t^\beta)(x) = ({}^{C}\mathcal{D}_{0+}^{\alpha}t^\beta)(x) = \left(\frac{\beta}{\rho}\right)^{\alpha}\rho^{\alpha}x^{\beta} = \beta^{\alpha}x^{\beta},$$

that is, the new derivative coincides with the fractional derivative in the Caputo-Hadamard sense, Eq.(3.22).

We now present and prove the semigroup property of the generalized Caputo-type fractional derivative ${}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}$. The result is also valid for the operator ${}_{*}^{\rho}\mathcal{D}_{b-}^{\alpha}$.

**Theorem 4.7.** *Let $\alpha,\beta,\rho \in \mathbb{R}$ such that $\alpha,\beta > 0$. If $0 < a < b < \infty$, then, for $\rho > 0$, we have*

$$({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}\,{}_{*}^{\rho}\mathcal{D}_{a+}^{\beta}\varphi)(x) = ({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha+\beta}\varphi)(x). \tag{4.28}$$

*Proof.* Considering $n = [\alpha]+1$ and $m = [\beta]+1$ and without loss of generality, we take $m \geqslant n$. Thus, $m = n+k$, $k \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and $\alpha+\beta \leqslant m+n$; Then, from the semigroup property of the generalized fractional integral operator [30], we have

$$\begin{aligned}
({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}_{*}\mathcal{D}^{\beta}_{a+}\varphi)(x) &= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}_{*}\mathcal{D}^{\beta}_{a+}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}\mathcal{J}^{m-\beta}_{a+}\,\delta^{m}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}\mathcal{J}^{n+k-\beta}_{a+}\delta^{n+k}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}\mathcal{J}^{n-\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\delta^{n+k}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,{}^{\rho}\mathcal{D}^{\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\delta^{n+k}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{-\beta}_{a+}\,{}^{\rho}\mathcal{J}^{\beta}_{a+}\,{}^{\rho}\mathcal{D}^{\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\delta^{n+k}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n-\alpha-\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\,\delta^{n+k}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{n+k-(\alpha+\beta)}_{a+}\,\delta^{n+k}_{\rho}\varphi)(x) \\
&= ({}^{\rho}\mathcal{J}^{m-(\alpha+\beta)}_{a+}\,\delta^{m}_{\rho}\varphi)(x) \\
&= ({}^{\rho}_{*}\mathcal{D}^{\alpha+\beta}_{a+}\varphi)(x).
\end{aligned}$$

□

## 4.4 Relation Between Generalized and generalized Caputo-Type Fractional Derivatives

In this section we present the relation between the generalized fractional derivatives and the generalized Caputo-type fractional derivatives and recover particular cases.

**Theorem 4.8.** *Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$. The relation between the generalized fractional derivatives and the generalized Caputo-type fractional derivatives is given by the expressions*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \sum_{k=0}^{n-1} \frac{\delta^{k}_{\rho}\,\varphi(a)}{\Gamma(k-\alpha+1)} \left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{k-\alpha} \tag{4.29}$$

and

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{b-}\varphi)(x) - \sum_{k=0}^{n-1} \frac{(-1)^{k}\delta^{k}_{\rho}\,\varphi(b)}{\Gamma(k-\alpha+1)} \left(\frac{b^{\rho}-x^{\rho}}{\rho}\right)^{k-\alpha}. \tag{4.30}$$

*In particular, for $0 < \alpha < 1$, Eq.(4.29) and Eq.(4.30) take the following form:*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \frac{\varphi(a)}{\Gamma(1-\alpha)} \left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{-\alpha}$$

and

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{b-}\varphi)(x) - \frac{\varphi(b)}{\Gamma(1-\alpha)} \left(\frac{b^{\rho}-x^{\rho}}{\rho}\right)^{-\alpha}.$$

*Proof.* We consider initially the generalized fractional derivative, the left-sided, given by

$$({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \delta^n_{\rho}({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\varphi)(x).$$

We write $\varphi(t)$ explicitly as given by Eq.(4.5) and using the results of Property 4.1, we have

$$\begin{aligned}
({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= \delta^n_{\rho}\left({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\left[({}^{\rho}\mathcal{J}^{n}_{a+}\delta^n_{\rho}\varphi)(t) + \sum_{k=0}^{n-1}\frac{\delta^k_{\rho}\,\varphi(a)}{k!}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^k\right]\right)(x).\\
&= (\delta^n_{\rho}{}^{\rho}\mathcal{J}^{n}_{a+}\,{}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^n_{\rho}\varphi)(x) + \delta^n_{\rho}\sum_{k=0}^{n-1}\frac{\delta^k_{\rho}\,\varphi(a)}{k!}\left({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\left[\frac{t^{\rho}-a^{\rho}}{\rho}\right]^k\right)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^n_{\rho}\varphi)(x) + \sum_{k=0}^{n-1}\delta^k_{\rho}\,\varphi(a)\frac{\rho^{\alpha-n-k}}{\Gamma(n-\alpha+k+1)}\,\delta^n_{\rho}[(x^{\rho}-a^{\rho})^{k+n-\alpha}]\\
&= ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) + \sum_{k=0}^{n-1}\frac{\delta^k_{\rho}\,\varphi(a)}{\Gamma(k-\alpha+1)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{k-\alpha}.
\end{aligned}$$

This last expression follows immediately from the Eq.(4.29). The proof of Eq.(4.30) is analogous. □

In [31], the Riemann-Liouville fractional derivatives are recovered, applying the limit $\rho\to 1$ to the generalized fractional differential operators, that is,

$$\lim_{\rho\to 1}({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}_{RL}\mathcal{D}^{\alpha}_{a+}\varphi)(x).$$

The Hadamard fractional derivatives are recovered in the limit $\rho\to 0$, that is, $\lim_{\rho\to 0}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x)$. However, in this case, our differentiation operator recovers, as limit cases, both Caputo and Caputo-Hadamard fractional derivatives.

- When $\rho\to 1$ in Eq.(4.29), we obtain

$$\lim_{\rho\to 1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}_{RL}\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. \tag{4.31}$$

- On the other hand, if $\rho\to 0$, again, using Eq.(4.29), we have

$$\lim_{\rho\to 0^+}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\delta^k\,\varphi(a)}{\Gamma(k-\alpha+1)}\left(\ln\frac{x}{a}\right)^{k-\alpha}. \tag{4.32}$$

Therefore, when $\rho\to 1$ we recover the relation between the fractional derivative as proposed by Caputo and the Riemann-Liouville fractional derivative [38, p. 91]. On the other hand, when $\rho\to 0$ we recover the relation between the Hadamard fractional derivative and the fractional derivative in the Caputo-Hadamard sense Eq.(3.6).

## 4.5 Fundamental Theorem of Fractional Calculus

In this section we present the fundamental theorem of fractional calculus associated with the generalized fractional integral and the generalized Caputo-type differential operator [11, 19].

**Theorem 4.9.** *Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha > 0$ and $\rho > 0$ with $n = [\alpha] + 1$ and $\varphi \in AC^n_\delta[a, b]$.*

(a) *If $\alpha \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$ and $\Phi(x) = ({}^\rho\mathcal{J}^\alpha_{a+}\varphi)(x)$ or $\Phi(x) = ({}^\rho\mathcal{J}^\alpha_{b-}\varphi)(x)$, for all $x \in [a, b]$, we obtain*

$$({}^\rho_*\mathcal{D}^\alpha_{a+}\Phi)(x) = \varphi(x) \qquad \textit{and} \qquad ({}^\rho_*\mathcal{D}^\alpha_{b-}\Phi)(x) = \varphi(x). \tag{4.33}$$

(b) *If $({}^\rho\mathcal{J}^{n-\alpha}_{a+}\varphi)(x) \in AC^n_\delta[a, b]$, then*

$$({}^\rho\mathcal{J}^\alpha_{a+}\ {}^\rho_*\mathcal{D}^\alpha_{a+}\Phi)(x) = \varphi(x) - \sum_{k=0}^{[\alpha]} \frac{\delta^k_\rho\, \varphi(a)}{k!}\left(\frac{x^\rho - a^\rho}{\rho}\right)^k \tag{4.34}$$

and

$$({}^\rho\mathcal{J}^\alpha_{b-}\ {}^\rho_*\mathcal{D}^\alpha_{b-}\Phi)(x) = \varphi(x) - \sum_{k=0}^{[\alpha]} \frac{(-1)^k\delta^k_\rho\, \varphi(b)}{k!}\left(\frac{b^\rho - x^\rho}{\rho}\right)^k.$$

*For $0 < \alpha < 1$, we have*

$$({}^\rho_a\mathcal{J}^\alpha_b\ {}^\rho_*\mathcal{D}^\alpha_{a+}\Phi)(x) = \Phi(x) - \Phi(a) \quad \text{and} \quad ({}^\rho_a\mathcal{J}^\alpha_b\ {}^\rho_*\mathcal{D}^\alpha_{b-}\Phi)(x) = \Phi(x) - \Phi(a). \tag{4.35}$$

*Proof.* (a) Eq.(4.33) follow immediately from Theorem 4.5.

(b) Let $\alpha \notin \mathbb{N}$. Using the definition in Eq.(4.9), we can write

$$({}^\rho\mathcal{J}^\alpha_{a+}\ {}^\rho_*\mathcal{D}^\alpha_{a+}\Phi)(x) = ({}^\rho\mathcal{J}^\alpha_{a+}\ {}^\rho\mathcal{J}^{n-\alpha}_{a+}\ \delta^n_\rho\,\Phi)(x) = ({}^\rho\mathcal{J}^n_{a+}\ \delta^n_\rho\,\Phi)(x).$$

Thus, Eq.(4.34) follows from Lemma 4.4. In particular, if $0 < \alpha < 1$, then

$$\begin{aligned} ({}^\rho\mathcal{J}^\alpha_{a+}\ \delta^\alpha_\rho\,\Phi)(x) &= \int_a^x t^{\rho-1}\left(t^{1-\rho}\frac{d}{dt}\right)\Phi(t)dt \\ &= \int_a^x \left(\frac{d}{dt}\Phi(t)\right)dt = \Phi(x) - \Phi(a). \end{aligned}$$

which is the classical fundamental theorem of calculus. On the other hand, for $\alpha \in \mathbb{N}$, when $\alpha = 1$, we obtain

$$({}^\rho_a\mathcal{J}^1_b\ {}^\rho_*\mathcal{D}^1_{a+}\Phi)(x) = \int_a^b \left(\frac{d}{dt}\Phi(t)\right)dt = \Phi(b) - \Phi(a).$$

$\square$

# Chapter 5

# Hilfer-Katugampola Fractional Derivatives

In 2000, Hilfer [22] proposed a type of non-integer order derivative, also defined from fractional integration and integer order differentiation, but this type of formulation differs from other already presented by the fact that it is defined by means of integer order derivative performing between fractional integrals. The Hilfer fractional derivatives interpolates the well-known Riemann-Liouville and Caputo fractional derivatives [18]. In this sense, in 2012, Kassim et al. [28] presented the Hilfer-Hadamard fractional derivative which interpolates Hadamard and Caputo-Hadamard fractional derivatives.

In this chapter we introduce the Hilfer-Katugampola fractional derivatives and it's part of our paper that was published online [47]. This new formulation is a Hilfer-type fractional differentiation operator, this is, an integer order derivative performing between generalized fractional integrals according to Katugampola [30]. This new fractional derivatives interpolates the Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, generalized and generalized Caputo-type fractional derivatives, as well as the Weyl and Liouville fractional derivatives for particular cases of integration extremes. More details on these fractional derivatives which we don't discuss in this thesis can be found in [17, 22, 23].

This chapter is organized as follows: in section 5.1, we present results that will be used in the remaining sections. In section 5.2, we define our derivative of non-integer order, the Hilfer-Katugampola fractional derivative, together with some of its properties. In section 5.3, we discuss the equivalence between an initial value problem and a Volterra integral equation. In section 5.4, we present and prove the existence and uniqueness theorem for the initial value problem presented in the previous section. As an application, in section 5.5 we discuss, using the method of successive approximations, the analytical solution of some fractional differential equations involving this differentiation operator.

## 5.1 Function Spaces

In order to introduce Hilfer-Katugampola fractional derivatives, in this section, we propose the function spaces and some results involving these spaces that are adequate for such definition.

**Definition 5.1.** *Let $\Omega = [a,b]\,(0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$ and the parameters $\rho > 0$ and $0 \leqslant \gamma < 1$.*

(1) *The weighted space $C_{\gamma,\rho}[a,b]$ of functions $g$ on $(a,b]$ is defined by*

$$C_{\gamma,\rho}[a,b] = \left\{ g : (a,b] \to \mathbb{R} : \left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma g(x) \in C[a,b] \right\}, \tag{5.1}$$

*where $0 \leqslant \gamma < 1$ and with the norm*

$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma g(x) \right\|_C = \max_{x\in\Omega} \left| \left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma g(x) \right|, \tag{5.2}$$

*where $C_{0,\rho}[a,b] = C[a,b]$.*

(2) *Let $\delta_\rho = \left(t^{\rho-1}\dfrac{d}{dt}\right)$. For $n \in \mathbb{N}$ we denote by $C^n_{\delta_\rho,\gamma}[a,b]$ the Banach space of functions $g$ which are continuously differentiable on $[a,b]$, with operator $\delta_\rho$, up to order $(n-1)$ and which have the derivative $\delta^n_\rho\, g$ of order $n$ on $(a,b]$ such that $\delta^n_\rho g \in C_{\gamma,\rho}[a,b]$, that is,*

$$C^n_{\delta_\rho,\gamma}[a,b] = \left\{ g : [a,b] \to \mathbb{R} : \delta^k_\rho\, g \in C[a,b], k = 0,1,\ldots,n-1, \delta^n_\rho\, g \in C_{\gamma,\rho}[a,b] \right\},$$

*where $n \in \mathbb{N}$, with the norms*

$$\|g\|_{C^n_{\delta_\rho,\gamma}} = \sum_{k=0}^{n-1} \|\delta^k_\rho\, g\|_C + \|\delta^n_\rho\, g\|_{C_{\gamma,\rho}}, \quad \|g\|_{C^n_{\delta_\rho}} = \sum_{k=0}^{n} \max_{x\in\Omega} |\delta^k_\rho\, g(x)|.$$

*For $n = 0$, we have*

$$C^0_{\delta_\rho,\gamma}[a,b] = C_{\gamma,\rho}[a,b].$$

**Lemma 5.1.** *Let $n \in \mathbb{N}_0$ and $\mu_1, \mu_2 \in \mathbb{R}$ such that*

$$0 \leqslant \mu_1 \leqslant \mu_2 < 1.$$

*Then,*

$$C^n_{\delta_\rho}[a,b] \longrightarrow C^n_{\delta_\rho,\mu_1}[a,b] \longrightarrow C^n_{\delta_\rho,\mu_2}[a,b],$$

*with*

$$\|f\|_{C^n_{\delta_\rho,\mu_2}[a,b]} \leqslant K_{\delta_\rho}\|f\|_{C^n_{\delta_\rho,\mu_1}[a,b]},$$

where

$$K_{\delta_\rho} = \min\left[1, \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu_2-\mu_1}\right], \quad \text{and} \quad a \neq 0.$$

In particular,

$$C[a,b] \longrightarrow C_{\mu_1,\rho}[a,b] \longrightarrow C_{\mu_2,\rho}[a,b],$$

with

$$\|f\|_{C_{\mu_2,\rho}[a,b]} \leqslant \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu_2-\mu_1} \|f\|_{C_{\mu_1,\rho}[a,b]}, \quad a \neq 0.$$

**Lemma 5.2.** *Let $0 \leqslant \gamma < 1$, $a < c < b$, $g \in C_{\gamma,\rho}[a,c]$, $g \in C[c,b]$ and $g$ continuous at $c$. Then, $g \in C_{\gamma,\rho}[a,b]$.*

**Lemma 5.3.** *For $\alpha > 0$, ${}^\rho\mathcal{J}^\alpha_{a+}$ maps $C[a,b]$ into $C[a,b]$.*

**Lemma 5.4.** *Let $\alpha > 0$ and $0 \leqslant \gamma < 1$. Then, ${}^\rho\mathcal{J}^\alpha_{a+}$ is bounded from $C_{\gamma,\rho}[a,b]$ into $C_{\gamma,\rho}[a,b]$.*

**Lemma 5.5.** *Let $\alpha > 0$ and $0 \leqslant \gamma < 1$. If $\gamma \leqslant \alpha$, then ${}^\rho\mathcal{J}^\alpha_{a+}$ is bounded from $C_{\gamma,\rho}[a,b]$ into $C[a,b]$.*

**Lemma 5.6.** *Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leqslant \gamma < 1$ and $\varphi \in C_{\gamma,\rho}[a,b]$. If $\alpha > \gamma$, then ${}^\rho\mathcal{J}^\alpha_{a+}\varphi$ is continuous on $[a,b]$ and*

$$({}^\rho\mathcal{J}^\alpha_{a+}\varphi)(a) = \lim_{x\to a^+}({}^\rho\mathcal{J}^\alpha_{a+}\varphi)(x) = 0.$$

*Proof.* Since $\varphi \in C_{\gamma,\rho}[a,b]$, then $\left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma \varphi(x)$ is continuous on $[a,b]$ and

$$\left|\left(\frac{x^\rho - a^\rho}{\rho}\right)^\gamma \varphi(x)\right| \leqslant M, \quad x \in [a,b],$$

for some positive constant $M$. Consequently,

$$|({}^\rho\mathcal{J}^\alpha_{a+}\varphi)(x)| \leqslant M\left[{}^\rho\mathcal{J}^\alpha_{a+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\gamma}\right](x)$$

and by Lemma 4.1, we can write

$$|({}^\rho\mathcal{J}^\alpha_{a+}\varphi)(x)| \leqslant M\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-\gamma}. \tag{5.3}$$

As $\alpha > \gamma$, the right-hand side of Eq.(5.3) goes to zero when $x \to a^+$. □

## 5.2 Hilfer-Katugampola Fractional Derivative

In this section, our main result, we introduce the Hilfer-Katugampola fractional derivative and discuss, from adequate parameters cases, other formulations for fractional derivatives.

**Definition 5.2.** *Let order $\alpha$ and type $\beta$ satisfy $n-1<\alpha\leqslant n$ and $0\leqslant\beta\leqslant 1$, with $n\in\mathbb{N}$. The fractional derivative (left-sided/right-sided), with respect to $x$, with $\rho>0$ of a function $\varphi\in C_{1-\gamma,\rho}[a,b]$, is defined by*

$$\begin{aligned}({}^\rho\mathcal{D}^{\alpha,\beta}_{a\pm}\varphi)(x) &= \left(\pm\,{}^\rho\mathcal{J}^{\beta(n-\alpha)}_{a\pm}\left(t^{\rho-1}\frac{d}{dt}\right)^n {}^\rho\mathcal{J}^{(1-\beta)(n-\alpha)}_{a\pm}\varphi\right)(x)\\ &= \left(\pm\,{}^\rho\mathcal{J}^{\beta(n-\alpha)}_{a\pm}\delta^n_\rho\,{}^\rho\mathcal{J}^{(1-\beta)(n-\alpha)}_{a\pm}\varphi\right)(x),\end{aligned}$$

*where $\mathcal{J}$ is the generalized fractional integral given in Definition 4.1. In this thesis we consider the case $n=1$ only, because the Hilfer derivative and the Hilfer-Hadamard derivative are discussed with $0<\alpha<1$.*

We present and discuss our new results involving the Hilfer-Katugampola fractional derivative using only the left-sided operator. An analogous procedure can be developed using the right-sided operator. The following property shows that it is possible to write operator ${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}$ in terms of the operator given in Definition 4.2.

**Property 5.1.** *The operator ${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}$ can be written as*

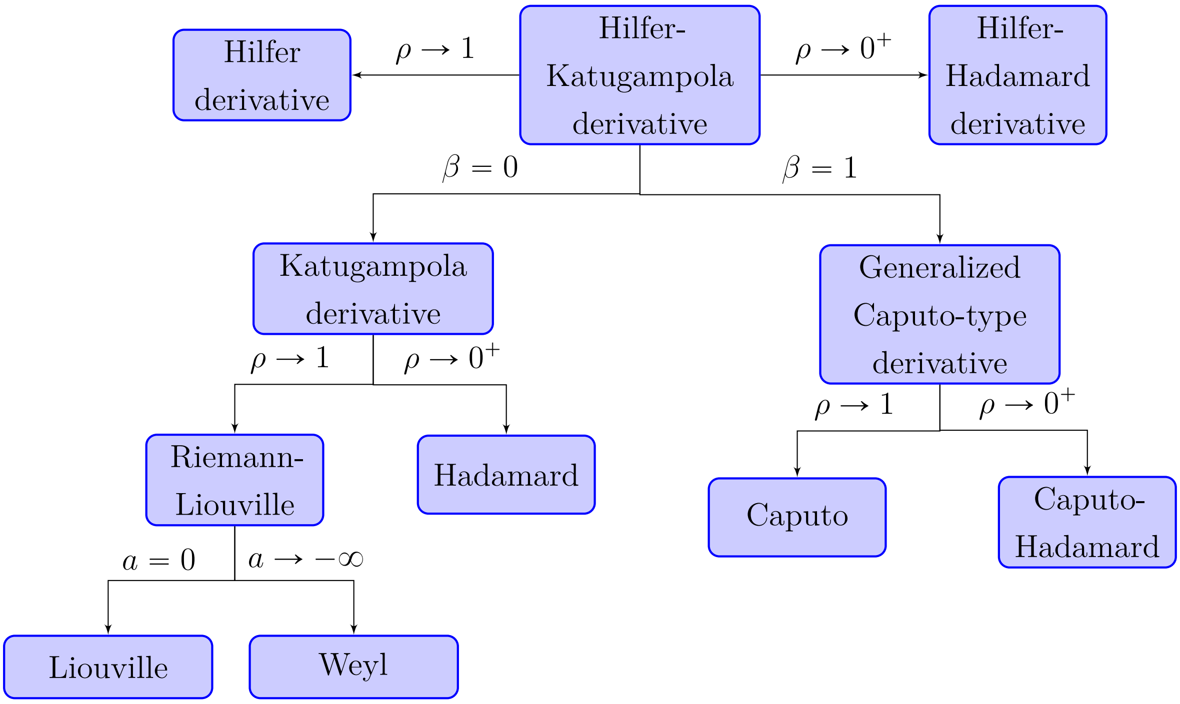
$${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} = {}^\rho\mathcal{J}^{\beta(1-\alpha)}_{a^+}\delta_\rho\,{}^\rho\mathcal{J}^{1-\gamma}_{a^+} = {}^\rho\mathcal{J}^{\beta(1-\alpha)}_{a^+}\,{}^\rho\mathcal{D}^{\gamma}_{a^+},\quad \gamma=\alpha+\beta(1-\alpha).$$

*Proof.* From definition of the generalized fractional integral, we have

$$\begin{aligned}({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi)(x) &= {}^\rho\mathcal{J}^{\beta(1-\alpha)}_{a^+}\left(x^{1-\rho}\frac{d}{dx}\right)\left\{\frac{\rho^{1-(1-\beta)(1-\alpha)}}{\Gamma[(1-\beta)(1-\alpha)]}\int_a^x\frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-(1-\beta)(1-\alpha)}}\varphi(t)dt\right\}\\ &= \left[{}^\rho\mathcal{J}^{\beta(1-\alpha)}_{a^+}\frac{\rho^{1+\alpha+\beta-\alpha\beta}}{\Gamma[(1-\beta)(1-\alpha)-1]}\int_a^x\frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1+\alpha+\beta-\alpha\beta}}\varphi(t)dt\right](x)\\ &= ({}^\rho\mathcal{J}^{\beta(1-\alpha)}_{a^+}\,{}^\rho\mathcal{D}^{\gamma}_{a^+}\varphi)(x),\end{aligned}$$

where operator $\mathcal{D}$ is the generalized fractional derivative given in Definition 4.2. □

**Property 5.2.** *The fractional derivative ${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho\to 1$) [23], Hilfer-Hadamard ($\rho\to 0^+$) [29], generalized ($\beta=0$) [31], generalized Caputo-type ($\beta=1$) [48], Riemann-Liouville ($\beta=0,\rho\to 1$) [38], Hadamard ($\beta=0,\rho\to 0^+$) [38], Caputo ($\beta=1,\rho\to 1$) [38], Caputo-Hadamard ($\beta=1,\rho\to 0^+$) [19], Liouville ($\beta=0,\rho\to 1,a=0$) [38] and Weyl ($\beta=0,\rho\to 1,a\to-\infty$) [22]. This fact is illustrated in the diagram below.*




**Property 5.3.** *We consider the following parameters $\alpha, \beta, \gamma, \mu$ satisfying*

$$\gamma = \alpha + \beta(1-\alpha), \qquad 0 < \alpha, \beta, \gamma < 1, \qquad 0 \leqslant \mu < 1.$$

*Thus, we define the spaces* $C^{\alpha,\beta}_{1-\gamma,\mu}[a,b] = \{\varphi \in C_{1-\gamma,\rho}[a,b], {}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi \in C_{\mu,\rho}[a,b]\}$ *and* $C^{\gamma}_{1-\gamma,\rho}[a,b] = \{\varphi \in C_{1-\gamma,\rho}[a,b], {}^\rho\mathcal{D}^{\gamma}_{a+}\varphi \in C_{1-\gamma,\rho}[a,b]\}$ *where* $C_{\mu,\rho}[a,b]$ *and* $C_{1-\gamma,\rho}[a,b]$ *are weighted spaces of continuous functions on* $(a,b]$ *defined by* item (2) in Definition 5.1. Since ${}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi = {}^\rho\mathcal{J}^{\gamma(1-\alpha)}_{a+}\,{}^\rho\mathcal{D}^{\gamma}_{a+}\varphi$, it follows from Lemma 5.4

$$C^{\gamma}_{1-\gamma}[a,b] \subset C^{\alpha,\beta}_{1-\gamma}[a,b].$$

**Lemma 5.7.** *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $\gamma = \alpha + \beta(1-\alpha)$. If $\varphi \in C^{\gamma}_{1-\gamma}[a,b]$, then*

$$ {}^\rho\mathcal{J}^{\gamma}_{a+}\,{}^\rho\mathcal{D}^{\gamma}_{a+}\varphi = {}^\rho\mathcal{J}^{\alpha}_{a+}\,{}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi \tag{5.4}$$

*and*

$$ {}^\rho\mathcal{D}^{\gamma}_{a+}\,{}^\rho\mathcal{J}^{\alpha}_{a+}\varphi = {}^\rho\mathcal{D}^{\beta(1-\alpha)}_{a+}\varphi. \tag{5.5}$$

*Proof.* We first prove Eq.(5.4). Using Theorem 4.2 and Property 5.1, we can write

$$ {}^\rho\mathcal{J}^{\gamma}_{a+}\,{}^\rho\mathcal{D}^{\gamma}_{a+}\varphi = {}^\rho\mathcal{J}^{\gamma}_{a+}\,{}^\rho\mathcal{J}^{-\beta(1-\alpha)}_{a+}\,{}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi = {}^\rho\mathcal{J}^{\alpha+\beta-\alpha\beta}_{a+}\,{}^\rho\mathcal{J}^{-\beta+\alpha\beta}_{a+}\,{}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi = {}^\rho\mathcal{J}^{\alpha}_{a+}\,{}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi.$$

To prove Eq.(5.5), we use Definition 5.2 and Theorem 4.2 to get

$$ {}^\rho\mathcal{D}^{\gamma}_{a+}\,{}^\rho\mathcal{J}^{\alpha}_{a+}\varphi = \delta_\rho\,{}^\rho\mathcal{J}^{1-\gamma}_{a+}\,{}^\rho\mathcal{J}^{\alpha}_{a+}\varphi = \delta_\rho\,{}^\rho\mathcal{J}^{1-\beta+\alpha\beta}_{a+}\varphi = \delta_\rho\,{}^\rho\mathcal{J}^{1-\beta(1-\alpha)}_{a+}\varphi = {}^\rho\mathcal{D}^{\beta(1-\alpha)}_{a+}\varphi.$$

$\square$

**Lemma 5.8.** *Let $\varphi \in L(a,b)$. If ${}^{\rho}\mathcal{D}_{a+}^{\beta(1-\alpha)}\varphi$ exists on $L(a,b)$, then*

$$ {}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi = {}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\,{}^{\rho}\mathcal{D}_{a+}^{\beta(1-\alpha)}\varphi. $$

*Proof.* From Lemma 4.1, Definition 4.2 and Definition 5.2, we obtain

$$ \begin{aligned} {}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi &= {}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\,{}^{\rho}\mathcal{D}_{a+}^{\gamma}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi = {}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\,\delta_{\rho}\,{}^{\rho}\mathcal{J}_{a+}^{1-\gamma}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi \\ &= {}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\,\delta_{\rho}\,{}^{\rho}\mathcal{J}_{a+}^{1-\beta(1-\alpha)}\varphi = {}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\,{}^{\rho}\mathcal{D}_{a+}^{\beta(1-\alpha)}\varphi. \end{aligned} $$

□

**Lemma 5.9.** *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $\gamma = \alpha + \beta(1-\alpha)$. If $\varphi \in C_{1-\gamma}[a,b]$ and ${}^{\rho}\mathcal{J}_{a+}^{1-\beta(1-\alpha)} \in C_{1-\gamma}^{1}[a,b]$, then ${}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi$ exists on $(a,b]$ and*

$$ {}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi = \varphi, \quad x \in (a,b]. \tag{5.6} $$

*Proof.* Using Lemma 4.2, Lemma 4.1 and Lemma 5.8, we obtain

$$ \begin{aligned} ({}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}\,{}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi)(x) &= ({}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\,{}^{\rho}\mathcal{D}_{a+}^{\beta(1-\alpha)}\varphi)(x) \\ &= \varphi(x) - \frac{({}^{\rho}\mathcal{J}_{a+}^{\beta(1-\alpha)}\varphi)(a)}{\Gamma(\alpha)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= \varphi(x), \quad x \in (a,b]. \end{aligned} $$

□

## 5.3 Equivalence Between the Generalized Cauchy problem and the Volterra Integral Equation

In this section, we consider the following nonlinear fractional differential equation

$$ ({}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}\varphi)(x) = f(x,\varphi(x)), \quad x > a > 0 \tag{5.7} $$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\rho > 0$, with the initial condition

$$ ({}^{\rho}\mathcal{J}_{a+}^{1-\gamma}\varphi)(a) = c, \quad \text{with} \quad \gamma = \alpha + \beta(1-\alpha), \quad c \in \mathbb{R}. \tag{5.8} $$

The following theorem yields the equivalence between the problem Eq.(5.7)-Eq.(5.8) and the Volterra integral equation, given by

$$ \varphi(x) \;=\; \frac{c}{\Gamma(\gamma)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^x \left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\alpha-1} t^{\rho-1} f(t,\varphi(t))\,dt. \tag{5.9} $$

**Theorem 5.1.** *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$. If $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot,\varphi(\cdot)) \in C_{1-\gamma}[a,b]$ for any $\varphi \in C_{1-\gamma}[a,b]$, then $\varphi$ satisfies Eq.(5.7)-Eq.(5.8) if, and only if, it satisfies Eq.(5.9).*

*Proof.* ($\Rightarrow$) Let $\varphi \in C^{\gamma}_{1-\gamma}[a,b]$ be a solution of the problem Eq.(5.7)-Eq.(5.8). We prove that $\varphi$ is also a solution of Eq.(5.9). From the definition of $C^{\gamma}_{1-\gamma}[a,b]$, Lemma 5.4 and using Definition 5.2, we have

$$ {}^{\rho}\mathcal{J}^{1-\gamma}_{a+}\varphi \in C[a,b] \quad \text{and} \quad {}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi = \delta_{\rho}\, {}^{\rho}\mathcal{J}^{1-\gamma}_{a+}\varphi \in C_{1-\gamma}[a,b]. $$

By Definition 5.1, it follows that ${}^{\rho}\mathcal{J}^{1-\gamma}_{a+}\varphi \in C^{1}_{1-\gamma}[a,b]$. Using Lemma 4.2, with $\alpha = \gamma$, and Eq.(5.8), we can write

$$ ({}^{\rho}\mathcal{J}^{\gamma}_{a+}\, {}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi)(x) = \varphi(x) - \frac{c}{\Gamma(\gamma)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}, \tag{5.10} $$

where $x \in (a,b]$. By hypothesis, ${}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi \in C_{1-\gamma}[a,b]$, using Lemma 5.7 with $\alpha = \gamma$ and Eq.(5.7), we have

$$ \begin{aligned} ({}^{\rho}\mathcal{J}^{\gamma}_{a+}\, {}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi)(x) &= ({}^{\rho}\mathcal{J}^{\alpha}_{a+}\, {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a+}\varphi)(x) \\ &= ({}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,\varphi(t)))(x). \end{aligned} \tag{5.11} $$

Comparing Eq.(5.10) and Eq.(5.11), we see that

$$ \varphi(x) = \frac{c}{\Gamma(\gamma)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + ({}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,\varphi(t)))(x), \tag{5.12} $$

with $x \in (a,b]$, that is, $\varphi(x)$ satisfies Eq.(5.9).

($\Leftarrow$) Let $\varphi \in C^{\gamma}_{1-\gamma}[a,b]$ satisfying Eq.(5.9). We prove that $\varphi$ also satisfies the problem Eq.(5.7)-Eq.(5.8). Apply operator ${}^{\rho}\mathcal{D}^{\gamma}_{a+}$ on both sides of Eq.(5.12). Then, from Lemma 4.1, Lemma 5.7 and Definition 5.2 we obtain

$$ ({}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{a+} f(t,\varphi(t)))(x). \tag{5.13} $$

By hypothesis, ${}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi \in C_{1-\gamma}[a,b]$; then, Eq.(5.13) implies that

$$ \begin{aligned} ({}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi)(x) &= \delta_{\rho}\, {}^{\rho}\mathcal{J}^{1-\beta(1-\alpha)}_{a+}\varphi)(x) \\ &= ({}^{\rho}\mathcal{D}^{\beta(1-\alpha)}_{a+}\varphi) \in C_{1-\gamma}[a,b]. \end{aligned} \tag{5.14} $$

As $f(\cdot,\varphi(\cdot)) \in C_{1-\gamma}[a,b]$ and from Lemma 5.4, follows

$$ {}^{\rho}\mathcal{J}^{1-\beta(1-\alpha)}_{a+} f \in C_{1-\gamma}[a,b]. \tag{5.15} $$

From Eq.(5.14), Eq.(5.15) and Definition 5.1, we obtain

$$ {}^{\rho}\mathcal{J}^{1-\beta(1-\alpha)}_{a+}\varphi \in C^{1}_{1-\gamma}[a,b]. $$

Applying operator ${}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{a+}$ on both sides of Eq.(5.14) and using Lemma 4.2, Lemma 4.1 and Definition 5.2, we have

$$ \begin{aligned} ({}^{\rho}\mathcal{J}^{\beta(1-\alpha)}_{a+}\, {}^{\rho}\mathcal{D}^{\gamma}_{a+}\varphi)(x) &= f(x,\varphi(x)) \\ &\quad + \frac{({}^{\rho}\mathcal{J}^{1-\beta(1-\alpha)}_{a+} f(t,\varphi(t)))(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= ({}^{\rho}\mathcal{D}^{\alpha,\beta}_{a+}\varphi)(x) = f(x,\varphi(x)), \end{aligned} $$

that is, Eq.(5.7) holds. Next, we prove that if $\varphi \in C^{\gamma}_{1-\gamma}[a,b]$ satisfies Eq.(5.9), it also satisfies Eq.(5.8). To this end, we multiply both sides of Eq.(5.12) by ${}^{\rho}\mathcal{J}^{1-\gamma}_{a+}$ and use Lemma 4.1 and Theorem 4.2 to get

$$({}^{\rho}\mathcal{J}^{1-\gamma}_{a+}\varphi)(x) = c + ({}^{\rho}\mathcal{J}^{1-\gamma(1-\alpha)}_{a+} f(t,\varphi(t)))(x). \tag{5.16}$$

Finally, taking $x \to a$ in Eq.(5.16), Eq.(5.8) follows. □

## 5.4 Existence and Uniqueness of Solution for the Cauchy Problem

In this section, we prove the existence and uniqueness of the solution for the problem Eq.(5.7)-Eq.(5.8) in the space $C^{\alpha,\beta}_{1-\gamma,\rho}[a,b]$ defined in Property 5.3, under the hypotheses of Theorem 5.1 and the Lipschitz condition on $f(\cdot,\varphi)$ with respect to the second variable, that is, $f(\cdot,\varphi)$ is bounded in a region $G \subset \mathbb{R}$ such that

$$\|f(x,\varphi_1) - f(x,\varphi_1)\|_{C_{1-\gamma,\rho}[a,b]} \leqslant A\|\varphi_1 - \varphi_2\|_{C_{1-\gamma,\rho}[a,b]}, \tag{5.17}$$

for all $x \in (a,b]$, and for all $\varphi_1, \varphi_2 \in G$, where $A > 0$ is constant.

**Theorem 5.2.** *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $\gamma = \alpha + \beta(1-\alpha)$. Let $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot,\varphi(\cdot)) \in C_{\mu,\rho}[a,b]$ for any $\varphi \in C_{\mu,\rho}[a,b]$ with $1-\gamma \leqslant \mu < 1 - \beta(1-\alpha)$ and satisfying the Lipschitz condition, Eq.(5.17), with respect to the second variable. Then, there exists a unique solution $\varphi$ for the problem Eq.(5.7)-Eq.(5.8) in the space $C^{\alpha,\beta}_{1-\gamma,\mu}[a,b]$.*

*Proof.* According to Theorem 5.1, we just have to prove that there exists a unique solution for the Volterra integral equation, Eq.(5.9). This equation can be written as

$$\varphi(x) = T\varphi(x),$$

where

$$T\varphi(x) = \varphi_0(x) + [{}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,\varphi(t))](x), \tag{5.18}$$

with

$$\varphi_0(x) = \frac{c}{\Gamma(\gamma)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\gamma-1}. \tag{5.19}$$

Thus, we divide the interval $(a,b]$ into subintervals on which operator $T$ is a contraction; we then use Banach fixed point theorem, Theorem 1.1. Note that $\varphi \in C_{1-\gamma,\rho}[a,x_1]$, where $a = x_0 < x_1 < \ldots < x_M = b$ and $C_{1-\gamma,\rho}[a,x_1]$ is a complete metric space with metric

$$d(\varphi_1,\varphi_2) = \|\varphi_1 - \varphi_2\|_{C_{1-\gamma,\rho}[a,x_1]} = \max_{x\in[a,x_1]}\left|\left(\frac{x^\rho - a^\rho}{\rho}\right)^{1-\gamma}[\varphi_1 - \varphi_2]\right|.$$

Choose $x_1 \in (a, b]$ such that the inequality

$$w_1 = \frac{A\,\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{{x_1}^{\rho}-a^{\rho}}{\rho}\right)^{\alpha} < 1, \tag{5.20}$$

where $A > 0$ is a constant, holds, as in Eq.(5.17). Thus, $\varphi_0 \in C_{1-\gamma,\rho}[a, x_1]$ and from Lemma 5.4, we have $T\varphi \in C_{1-\gamma,\rho}[a, x_1]$ and $T$ maps $C_{1-\gamma,\rho}[a, x_1]$ into $C_{1-\gamma,\rho}[a, x_1]$. Therefore, from Eq.(5.17), Eq.(5.18), Lemma 5.4 and for any $\varphi_1, \varphi_2 \in C_{1-\gamma,\rho}[a, x_1]$, we can write

$$\begin{aligned}
\|T\varphi_1 - T\varphi_2\|_{C_{1-\gamma}[a,x_1]} &= \|{}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,\varphi_1(t)) - {}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,\varphi_2(t))\|_{C_{1-\gamma,\rho}[a,x_1]} \\
&= \|{}^{\rho}\mathcal{J}^{\alpha}_{a+}[|f(t,\varphi_1(t)) - f(t,\varphi_2(t))|]\|_{C_{1-\gamma,\rho}[a,x_1]} \\
&\leqslant \left(\frac{{x_1}^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\|f(t,\varphi_1(t)) - f(t,\varphi_2(t))\|_{C_{1-\gamma,\rho}[a,x_1]} \\
&\leqslant A\left(\frac{{x_1}^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\|\varphi_1(t) - \varphi_2(t)\|_{C_{1-\gamma,\rho}[a,x_1]} \\
&\leqslant w_1\|\varphi_1(t) - \varphi_2(t)\|_{C_{1-\gamma,\rho}[a,x_1]}.
\end{aligned}$$

Since $f(x,\varphi(x)) \in C(a, x_1]$ for any $\varphi \in C(a, x_1]$, Lemma 5.3 implies that $({}^{\rho}\mathcal{J}^{\alpha}_{a+} f) \in C(a, x_1]$. By hypothesis Eq.(5.20) we can use the Banach fixed point to get a unique solution $\varphi^* \in C_{1-\gamma,\rho}[a, x_1]$ for Eq.(5.9) on the interval $(a, x_1]$. This solution $\varphi^*$ is obtained as a limit of a convergent sequence $T^k\varphi_0^*$:

$$\lim_{k\to\infty}\|T^k\varphi_0^* - \varphi^*\|_{C_{1-\gamma,\rho}[a,x_1]} = 0, \tag{5.21}$$

where $\varphi_0^*$ is any function in $C_{1-\gamma,\rho}[a, x_1]$ and

$$\begin{aligned}
(T^k\varphi_0^*)(x) &= (T\,T^{k-1}\varphi_0^*)(x) \\
&= \varphi_0(x) + [{}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,(T^{k-1}\varphi_0^*)(t))](x), \quad k \in \mathbb{N}.
\end{aligned}$$

We take $\varphi_0^*(x) = \varphi_0(x)$ with $\varphi_0(x)$ defined by Eq.(5.19). Denoting

$$\varphi_k(x) = (T^k\varphi_0^*)(x), \quad k \in \mathbb{N}, \tag{5.22}$$

then Eq.(5.22) admits the form

$$\varphi_k(x) = \varphi_0(x) + [{}^{\rho}\mathcal{J}^{\alpha}_{a+} f(t,\varphi_{k-1}(t))](x), \quad k \in \mathbb{N}.$$

On the other hand, Eq.(5.21) can be rewritten as

$$\lim_{k\to\infty}\|\varphi_k - \varphi^*\|_{C_{1-\gamma,\rho}[a,x_1]} = 0.$$

We consider the interval $[x_1, x_2]$, where $x_2 = x_1 + h_1$, $h_1 > 0$ and $x_2 < b$, then by Eq.(5.9), we can write

$$\begin{aligned}
\varphi(x) &= \frac{c}{\Gamma(\gamma)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^{x_1} t^{\rho-1}\left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\alpha-1} f(t,\varphi(t))dt \\
&+ \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x} t^{\rho-1}\left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\alpha-1} f(t,\varphi(t))dt \\
&= \varphi_{01}(x) + \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x} t^{\rho-1}\left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\alpha-1} f(t,\varphi(t))dt,
\end{aligned}$$

where $\varphi_{01}(x)$, defined by

$$\varphi_{01}(x) = \frac{c}{\Gamma(\gamma)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^{x_1} t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(t,\varphi(t))dt,$$

is a known function and $\varphi_{01}(x) \in C_{1-\gamma,\rho}[x_1, x_2]$. Using the same arguments as above, we conclude that there exists a unique solution $\varphi^* \in C_{1-\gamma,\rho}[x_1, x_2]$ for Eq.(5.9) on the interval $[x_1, x_2]$. The next interval to be considered is $[x_2, x_3]$, where $x_3 = x_2 + h_2$, $h_2 > 0$ and $x_3 < b$. Repeating this process, we conclude that there exists a unique solution $\varphi^* \in C_{1-\gamma,\rho}[a, b]$ for Eq.(5.9) on the interval $[a, b]$. We must show that such unique solution $\varphi^* \in C_{1-\gamma,\rho}[a, b]$ is also in $C^{\alpha,\beta}_{1-\gamma,\mu}[a, b]$. Thus, we need show that $({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi^*) \in C_{\mu,\rho}[a, b]$. We emphasize that $\varphi^*$ is the limit of the sequence $\varphi_k$, where $\varphi_k = T^k\varphi_0^* \in C_{1-\gamma,\rho}[a, b]$, that is,

$$\lim_{k\to\infty} \|\varphi_k - \varphi^*\|_{C_{1-\gamma,\rho}[a,b]} = 0, \tag{5.23}$$

for an adequate choice of $\varphi_0^*(x)$ on each subinterval $[a, x_1]$, ...,$[x_{M-1}, b]$. If $\varphi_0(x) \neq 0$, then we can admit $\varphi_0^*(x) = \varphi_0(x)$ and once $\mu \geqslant 1 - \gamma$, from Lipschitz condition, Definition 1.1, and by Lemma 5.1, we can write

$$\begin{aligned} \|{}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi_k - {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi^*\|_{C_{\mu,\rho}[a,b]} &= \|f(x,\varphi_k) - f(x,\varphi^*)\|_{C_{\mu,\rho}[a,b]} \\ &\leqslant A\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu-1+\gamma} \|\varphi_k - \varphi^*\|_{C_{1-\gamma,\rho}[a,b]}. \end{aligned} \tag{5.24}$$

By Eq.(5.23) and Eq.(5.24), we obtain

$$\lim_{k\to\infty} \|{}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi_k - {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi^*\|_{C_{\mu,\rho}[a,b]} = 0.$$

From this last expression, we have $({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi^*) \in C_{\mu,\rho}[a, b]$ if $({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi_k) \in C_{\mu,\rho}[a, b]$, $k = 1, 2, \ldots$ Since $({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi_k)(x) = f(x, \varphi_{k-1}(x))$, then by the previous argument, we obtain that $f(\cdot, \varphi^*(\cdot)) \in C_{\mu,\rho}[a, b]$ for any $\varphi^* \in C_{\mu,\rho}[a, b]$. Consequently, $\varphi^* \in C^{\alpha,\beta}_{1-\gamma,\mu}[a, b]$. □

## 5.5 Application: Particular Cases for Cauchy Problem

This section is devoted to explicit solutions to fractional differential equations associated with the Hilfer-Katugampola differential operator $({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi)(x)$ of order $0 < \alpha < 1$ and type $0 \leqslant \beta \leqslant 1$ in the space $C^{\alpha,\beta}_{1-\gamma,\rho}[a, b]$ defined in Property 5.3.

We consider the following Cauchy problem

$$({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi)(x) - \lambda\varphi(x) = f(x), \quad 0 < \alpha < 1, \quad 0 \leqslant \beta \leqslant 1, \tag{5.25}$$

$$({}^\rho\mathcal{J}^{1-\gamma}_{a^+})(a) = c, \quad \gamma = \alpha + \beta(1-\alpha), \tag{5.26}$$

where $c, \lambda \in \mathbb{R}$. We suppose that $f(x) \in C_{\mu,\rho}[a, b]$ with $0 \leqslant \mu < 1$ and $\rho > 0$. Then, by Theorem 5.1, the problem Eq.(5.25)-Eq.(5.26) is equivalent to solve the following integral equation

$$\begin{aligned}\varphi(x) &= \frac{c}{\Gamma(\gamma)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)}\int_a^x t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1}\varphi(t)dt \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^x t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(t)dt. \qquad (5.27)\end{aligned}$$

In order to solve Eq.(5.27), we use the method of successive approximations, that is,

$$\varphi_0(x) = \frac{c}{\Gamma(\gamma)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\gamma-1}, \qquad (5.28)$$

$$\begin{aligned}\varphi_k(x) &= \varphi_0(x) + \frac{\lambda}{\Gamma(\alpha)}\int_a^x t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1}\varphi_{k-1}(t)dt \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^x t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(t)dt, \quad (k \in \mathbb{N}). \qquad (5.29)\end{aligned}$$

Using Eq.(4.1), Eq.(5.28) and Lemma 4.1, we have the following expression for $\varphi_1(x)$:

$$\begin{aligned}\varphi_1(x) &= \varphi_0(x) + ({}^\rho\mathcal{J}^\alpha_{a+}\varphi_0)(x) + ({}^\rho\mathcal{J}^\alpha_{a+}f)(x) \\ &= c\sum_{j=1}^{2}\frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha j + \beta(1-\alpha)-1} + ({}^\rho\mathcal{J}^\alpha_{a+}f)(x). \qquad (5.30)\end{aligned}$$

Similarly, using Eq.(5.28), Eq.(5.29), Eq.(5.30) and Theorem 4.2, we get an expression for $\varphi_2(x)$, as follows:

$$\begin{aligned}\varphi_2(x) &= \varphi_0(x) + ({}^\rho\mathcal{J}^\alpha_{a+}\varphi_1)(x) + ({}^\rho\mathcal{J}^\alpha_{a+}f)(x) \\ &= c\sum_{j=1}^{3}\frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha j + \beta(1-\alpha)-1} \\ &+ \lambda({}^\rho\mathcal{J}^\alpha_{a+}\,{}^\rho\mathcal{J}^\alpha_{a+}f)(x) + ({}^\rho\mathcal{J}^\alpha_{a+}f)(x) \\ &= c\sum_{j=1}^{3}\frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha j + \beta(1-\alpha)-1} \\ &+ \int_a^x \sum_{j=1}^{2}\frac{\lambda^{j-1}}{\Gamma(\alpha j)}t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha j-1} f(t)dt.\end{aligned}$$

Continuing this process, the expression for $\varphi_k(x)$ is given by

$$\begin{aligned}\varphi_k(x) &= c\sum_{j=1}^{k+1}\frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha j + \beta(1-\alpha)-1} \\ &+ \int_a^x \sum_{j=1}^{k}\frac{\lambda^{j-1}}{\Gamma(\alpha j)}t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha j-1} f(t)dt.\end{aligned}$$

Taking the limit $k \to \infty$, we obtain the expression for $\varphi(x)$, that is,

$$\begin{aligned}\varphi(x) &= c\sum_{j=1}^{\infty}\frac{\lambda^{j-1}}{\Gamma(\alpha j + \beta(1-\alpha))}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha j + \beta(1-\alpha)-1} \\ &+ \int_a^x \sum_{j=1}^{\infty}\frac{\lambda^{j-1}}{\Gamma(\alpha j)}t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha j-1} f(t)dt.\end{aligned}$$

Replacing the index of summation in this last expression, $j \to j+1$, we have

$$\begin{aligned}\varphi(x) &= c\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(\alpha j+\gamma)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\alpha j+\gamma-1} \\ &+ \int_a^x\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(\alpha j+\alpha)}t^{\rho-1}\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\alpha j+\alpha-1} f(t)dt\end{aligned}$$

or, by two-parameters Mittag-Leffler function, we can rewrite the solution as

$$\begin{aligned}\varphi(x) &= c\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\gamma-1} E_{\alpha,\gamma}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\alpha}\right] \\ &+ \int_a^x t^{\rho-1}\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\alpha}\right] f(t)dt. \qquad (5.31)\end{aligned}$$

The function $f(x,\varphi)=\lambda\varphi(x)+f(x)$ satisfies the Lipschitz condition, Definition 1.1, for any $x_1, x_2 \in (a,b]$ and any $y \in G$, where $G$ is an open set on $\mathbb{R}$. If $\mu \geqslant 1-\gamma$, then by Theorem 5.2, the problem Eq.(5.25)-Eq.(5.26) has a unique solution given by Eq.(5.31) in the space $C^{\alpha,\beta}_{1-\gamma,\mu}[a,b]$. Note that, the problem Eq.(5.25)-Eq.(5.26), whose solution is given by Eq.(5.31), admits the following particular cases:

- If $\rho \to 1$ and $\beta = 0$, then $\gamma = \alpha$ and we have a problem involving the Riemann-Liouville fractional derivative; its solution is given by [38, p.224]

$$\varphi(x) = c\,(x-a)^{\alpha-1}E_{\alpha,\alpha}[\lambda(x-a)^\alpha] + \int_a^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\lambda(x-t)^\alpha]f(t)dt.$$

- For $\rho \to 1$ and $\beta = 1$ our derivative becomes the Caputo fractional derivative; the solution is given by [38, p.231].

$$\varphi(x) = c\,E_{\alpha}[\lambda(x-a)^\alpha] + \int_a^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\lambda(x-t)^\alpha]f(t)dt.$$

- Considering $\rho \to 0^+$ and $\beta = 0$, we have $\gamma = \alpha$ a Cauchy problem formulated with the Hadamard fractional derivative; whose solution is given by [38, p.235]

$$\varphi(x) = c\left(\ln\frac{x}{a}\right)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda\left(\ln\frac{x}{a}\right)^{\alpha}\right] + \int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda\left(\ln\frac{x}{t}\right)^{\alpha}\right] f(t)\frac{dt}{t}.$$

- Other particular cases arise when we vary the parameters as described in Property 5.2.

A special case occurs for $f(x)=0$ in Eq.(5.25); we then have the following problem

$$({}^\rho\mathcal{D}^{\alpha,\beta}_{a+}\varphi)(x) - \lambda\varphi(x) = 0, \quad 0<\alpha<1, \quad 0 \leqslant \beta \leqslant 1, \qquad (5.32)$$

$$({}^\rho\mathcal{J}^{1-\gamma}_{a+})(a) = c, \quad \gamma = \alpha+\beta(1-\alpha), \qquad (5.33)$$

with $\lambda \in \mathbb{R}$ and $a < x \leqslant b$. The solution is given by

$$\varphi(x) = c\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\gamma-1} E_{\alpha,\gamma}\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha}\right]. \tag{5.34}$$

Also, consider the following Cauchy problem:

$$({}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\varphi)(x) - \lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\xi}\varphi(x) = 0, \quad 0 < \alpha < 1, \quad 0 \leqslant \beta \leqslant 1, \tag{5.35}$$

$$({}^\rho\mathcal{J}^{1-\alpha}_{a^+})(a) = c, \quad c \in \mathbb{R}, \quad \rho > 0, \tag{5.36}$$

with $\lambda, \xi \in \mathbb{R}$, $a < x \leqslant b$ and $\xi > -\alpha$. We suppose $\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\xi}\varphi\right] \in C_{1-\alpha,\rho}[a,b]$. Then, by Theorem 5.1, the problem Eq.(5.35)-Eq.(5.36) is equivalent to the integral equation:

$$\begin{aligned}\varphi(x) &= \frac{c}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} \\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_a^x t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\xi}\varphi(t)dt.\end{aligned} \tag{5.37}$$

We apply the method of successive approximations to solve the integral equation Eq.(5.37), that is, we consider

$$\varphi_0(x) = \frac{c}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} \tag{5.38}$$

and

$$\varphi_k(x) = \varphi_0(x) + \frac{\lambda}{\Gamma(\alpha)}\int_a^x t^{\rho-1}\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\alpha-1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\xi}\varphi_{k-1}(t)dt. \tag{5.39}$$

For $k = 1$ and using Lemma 4.1, we have

$$\begin{aligned}\varphi_1(x) &= \varphi_0(x) + \lambda\left({}^\rho\mathcal{J}^{\alpha}_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\xi}\varphi_0\right)(x) \\ &= \frac{c}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} + \frac{c\lambda}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\xi)}{\Gamma(2\alpha+\xi)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{2\alpha+\xi-1}.\end{aligned} \tag{5.40}$$

For $k = 2$ and using again Lemma 4.1, we can write

$$\begin{aligned}\varphi_2(x) &= \varphi_0(x) + \lambda\left({}^\rho\mathcal{J}^{\alpha}_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\xi}\varphi_1\right)(x) \\ &= \varphi_0(x) + \frac{c\lambda}{\Gamma(\alpha)}\left({}^\rho\mathcal{J}^{\alpha}_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\xi-1}\right)(x) \\ &+ \frac{c\lambda^2}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\xi)}{\Gamma(2\alpha+\xi)}\left({}^\rho\mathcal{J}^{\alpha}_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{2\alpha+2\xi-1}\right)(x) \\ &= \frac{c}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1}\left\{1 + c_1\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha+\xi}\right] + c_2\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha+\xi}\right]^2\right\},\end{aligned}$$

where

$$c_1 = \frac{\Gamma(\alpha+\xi)}{\Gamma(2\alpha+\xi)} \qquad \text{and} \qquad c_2 = \frac{\Gamma(\alpha+\xi)}{\Gamma(2\alpha+\xi)} \frac{\Gamma(2\alpha+2\xi)}{\Gamma(3\alpha+2\xi)}. \tag{5.41}$$

Continuing this process, we obtain the expression for $\varphi_k(x)$, given by

$$\varphi_k(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha-1} \left\{ 1 + \sum_{j=1}^{k} c_j \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha+\xi} \right]^j \right\}, \tag{5.42}$$

where

$$c_j = \prod_{r=1}^{j} \frac{\Gamma[r(\alpha+\xi)]}{\Gamma[r(\alpha+\xi)+\alpha]}, \quad j \in \mathbb{N}.$$

Using Definition 1.9 we can write the solution of Eq.(5.42) as

$$\varphi_k(x) = \frac{c}{\Gamma(\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,1+\xi/\alpha,1+(\xi-1)/\alpha} \left[ \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\alpha+\xi} \right]. \tag{5.43}$$

If $\xi \geqslant 0$, then $f(x,\varphi) = \lambda \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\xi} \varphi(x)$ satisfies the Lipschitz condition, Eq.(5.17), for any $x_1, x_2 \in (a,b]$ and for all $\varphi_1, \varphi_2 \in G$, where $G$ is an open set on $\mathbb{R}$. If $\mu \geqslant 1-\gamma$, then by Theorem 5.2, there exists a unique solution to the problem Eq.(5.35)-Eq.(5.36), given by Eq.(5.43), in space $C^{\alpha,\beta}_{1-\gamma,\mu}[a,b]$. Note that the problem Eq.(5.35)-Eq.(5.36), whose solution is given by Eq.(5.43), admits the following particular cases:

- For $\rho \to 1$ and $\beta = 0$, we have formulation for this problem, as well as it solution considering the Riemann-Liouville fractional derivative [38, p.227], this is

$$\varphi(x) = \frac{c}{\Gamma(\alpha)} (x-a)^{\alpha-1} E_{\alpha,1+\xi/\alpha,1+(\xi-1)/\alpha}[\lambda(x-a)^{\alpha+\xi}].$$

- Consider $\rho \to 1$ and $\beta = 1$, we have formulation of the problem and it solution, considering the Caputo fractional derivative [38, p.233], is given by

$$\varphi(x) = c\, E_{\alpha,1+\xi/\alpha,\xi/\alpha}[\lambda(x-a)^{\alpha+\xi}].$$

- For $\rho \to 0^+$ and $\beta = 0$, we have formulation for this Cauchy problem and its solution considering the Hadamard fractional derivative that is given by [38, p.237]

$$\varphi(x) = \frac{c}{\Gamma(\alpha)} \left( \ln \frac{x}{a} \right)^{\alpha-1} E_{\alpha,1+\xi/\alpha,1+(\xi-1)/\alpha} \left[ \lambda \left( \ln \frac{x}{a} \right)^{\alpha+\xi} \right].$$

# Chapter 6

# Generalized $(k, \rho)$-Fractional Derivatives

In order to generalize the fractional integrals and derivatives some authors have inserted a new parameter in existing formulations. In 2012, Mubben and Habibullah [44] inserted the parameter $k > 0$ in Riemann-Liouville fractional integrals and called this generalization by $k$-Riemann-Liouville fractional integrals. In 2015, Farid and Habibullah [16], also inserted a new parameter in Hadamard fractional integrals, in order to obtain the Hadamard $k$-fractional integrals.

From the $k$-fractional integrals it was possible to define the $k$-fractional derivatives. In 2013, Dorrego and Cerutti [15] defined the Hilfer $k$-fractional derivatives by means of Riemann-Liouville $k$-fractional integrals. In the same year, Romero et al. [52] introduced the $k$-Riemann-Liouville fractional derivatives. More recently, in 2017, Nisar et al. [46] defined the $(k, \rho)$-fractional derivatives. However, in that paper the authors discuss only the case $0 < \alpha < 1$. In this chapter we proposed the $(k, \rho)$-fractional derivatives, but we discussed the general case $\alpha \in \mathbb{R}$ with $\alpha > 0$. This chapter is part of our paper that was accepted for publication [49].

This chapter is organized as follows: In section 6.1, we present some definitions aiming at our main result; in particular, the definition of $k$-Mittag-Leffler functions, the spaces in which we work and the $k$-fractional integrals in the senses of Riemann-Liouville and Hadamard. In section 6.3, we present some properties of the so-called $(k, \rho)$-fractional operator and in section 6.4, our main result, we introduce the generalized $(k, \rho)$-fractional derivative and we demonstrate that, using adequate parameters, we are able to recover a wide list of definitions of fractional derivatives. As an application, introduced in the previous section by means of theorems, we approach linear fractional differential equations by studying the Cauchy problem and discuss the existence and uniqueness of its solution and its dependence on initial conditions.

## 6.1 Generalizations of Special Functions and Fractional Integrals

In order to generalize such functions, Dorrego and Cerutti [14] defined the so-called $k$-Mittag-Leffler function as follows:

$$E^{\delta}_{k,\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\beta n + \gamma)} \frac{z^n}{n!}, \quad z \in \mathbb{R}, \quad \beta > 0, \quad \gamma > 0, \tag{6.1}$$

where $n \in \mathbb{N}$, $(\delta)_{n,k}$ is the $k$-Pochhammer symbol defined in Eq.(1.17) and $\Gamma_k(x)$ is the $k$-gamma function, Eq.(1.14). In the case $k = 1$ we recover the three-parameters Mittag-Leffler function, Eq.(1.20). Gupta and Parihar [20] defined the so-called $k$-new generalized Mittag-Leffler function using the following series:

$$E_{k,\xi,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\xi n + \sigma)}, \quad z \in \mathbb{R}, \quad \xi > 0, \quad \sigma > 0, \tag{6.2}$$

where $n \in \mathbb{N}$. Again, in order to recover the two-parameters Mittag-Leffler function introduced by Wiman, Eq.(1.21), one just has to consider $k = 1$.

## 6.2 $k$-Fractional Integrals

Mubeen and Habibullah [44] introduced the so-called $k$-Riemann-Liouville fractional integrals, a generalization of the Riemann-Liouville fractional integrals, obtained for $k = 1$. Such integral is defined here, for the left-sided only, as

$$({}_k\mathcal{I}^{\alpha}_{a^+}\varphi)(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} \varphi(t)dt, \quad \alpha > 0, \quad x > a, \tag{6.3}$$

where $\varphi \in L(a,b)$. For $k \to 1$, we have $\Gamma_k(\alpha) = \Gamma(\alpha)$ and ${}_k\mathcal{I}^{\alpha}_{a^+} = \mathcal{I}^{\alpha}_{a^+}$, where $\mathcal{I}^{\alpha}_{a^+}$ is the classical Riemann-Liouville fractional integral. Similarly, in order to generalize the Hadamard fractional integral, the $k$-Hadamard fractional integral [1] was introduced. The definition of this operator, for the left-sided only, is given by

$$({}_k\mathcal{J}^{\alpha}_{a^+}\varphi)(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{\alpha}{k}-1} \varphi(t)\frac{dt}{t}, \quad \alpha > 0, \quad x > a, \tag{6.4}$$

where $k > 0$ and $\varphi \in L(a,b)$. For $k \to 1$, we have ${}_k\mathcal{J}^{\alpha}_{a^+} \to \mathcal{J}^{\alpha}_{a^+}$, where $\mathcal{J}^{\alpha}_{a^+}$ is the Hadamard fractional integral.

Recently, Sarikaya et al. [56] proposed the so-called $(k,\rho)$-fractional integral which, at adequate limits, recovers the $k$-Riemann-Liouville and $k$-Hadamard fractional integrals. This operator is defined —left-sided only— by

$$({}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi)(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^x \left(\frac{x^{\rho} - t^{\rho}}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1}\varphi(t)dt, \quad \alpha > 0, \quad x > a, \tag{6.5}$$

with $n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $k>0$, $\rho>0$ and $\varphi\in L(a,b)$. When $k\to 1$, we have $\Gamma_k(\alpha)\to\Gamma(\alpha)$ and ${}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\to{}^{\rho}\mathcal{J}^{\alpha}_{a^+}$, where ${}^{\rho}\mathcal{J}^{\alpha}_{a^+}$ is the generalized fractional integral defined by Katugampola in [31]. When $\rho\to 1$, we obtain the $k$-Riemann-Liouville fractional integral, Eq.(6.3). On the other hand, considering $\rho\to 0^+$, we obtain the $k$-Hadamard fractional integral, Eq.(6.4).

## 6.3 Auxiliary Results

We now present some properties of the fractional integrals, defined in the previous section, in order to use them throughout this chapter. We start by presenting the semigroup property for the $(k,\rho)$-fractional operator and an application to the power function; both results are theorems can be found in [56].

**Theorem 6.1.** *Let $\alpha>0$, $\beta>0$, $k>0$, $\rho>0$ and $\varphi\in L_p(a,b)$, then*

$$({}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+}\varphi)(x)=({}^{\rho}_{k}\mathcal{J}^{\alpha+\beta}_{a^+}\varphi)(x)=({}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+}\,{}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi)(x).$$

**Theorem 6.2.** *Let $\alpha,\beta>0$ and $k,\rho>0$. Then, we have*

$$[{}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}(t^\rho-a^\rho)^{\frac{\beta}{k}-1}](x)=\frac{\Gamma_k(\beta)}{\rho^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+\beta)}(x^\rho-a^\rho)^{\frac{\alpha+\beta}{k}-1}.$$

The following lemma shows that the $(k,\rho)$-fractional operator is bounded in the space $L(a,b)$.

**Lemma 6.1.** [56] *Let $\varphi\in L(a,b)$; then, the $(k,\rho)$-Riemann-Liouville fractional integral of order $\alpha>0$ is bounded in the space $L(a,b)$, i.e.*

$$\|{}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi\|_1\leqslant M\|\varphi\|_1, \tag{6.6}$$

*where*

$$M=\frac{1}{\alpha\,\Gamma_k(\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\frac{\alpha}{k}}.$$

Recently, Nisar et al. [46] proposed the $(k,\rho)$-fractional derivative, which is associated with the $(k,\rho)$-fractional integral, Eq.(6.5).

**Definition 6.1.** [46] *Let $\mu,\nu,k\in\mathbb{R}$ such that $0<\mu<1$, $0\leqslant\nu\leqslant 1$ and $k>0$. The $(k,\rho)$-fractional derivative is defined by*

$$({}^{\rho}_{k}\mathcal{D}^{\mu,\nu}_{a^+}\varphi)(x)=\left({}^{\rho}_{k}\mathcal{J}^{\nu(k-\mu)}_{a^+}\left(x^{1-\rho}\frac{d}{dx}\right)(k\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(k-\mu)}_{a^+}\varphi)\right)(x), \tag{6.7}$$

*for functions for which the expression on the right hand side exists.*

## 6.4 Generalized $(k,\rho)$-Fractional Derivative

In this section we propose, as our main result of this chapter, a generalization for the fractional derivative proposed in [46]. The definition in that work considers the order of derivative to be $0<\mu<1$, but here we consider $\alpha\in\mathbb{R}^+$, with $n-1<\alpha\leqslant n$ and $n\in\mathbb{N}$. We call our definition by generalized $(k,\rho)$-fractional derivative. The fractional integrals associated with this differentiation operator is the $(k,\rho)$-fractional integral, Eq.(6.5). In this section we also prove some properties of this operator.

**Definition 6.2.** *Let $\alpha,\nu\in\mathbb{R}$ such that $n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $0\leqslant\nu\leqslant 1$, $\rho>0$ and $k>0$. We define the generalized $(k,\rho)$-fractional derivative by*

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) &= \left({}^{\rho}_{k}\mathcal{J}^{\nu(nk-\alpha)}_{a^+}\left(x^{1-\rho}\frac{d}{dx}\right)^n\left(k^n\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi\right)\right)(x) \qquad (6.8)\\ &= \left({}^{\rho}_{k}\mathcal{J}^{\nu(nk-\alpha)}_{a^+}\,\delta^n_\rho\left(k^n\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi\right)\right)(x), \qquad (6.9)\end{aligned}$$

*where* $\delta^n_\rho=\left(x^{1-\rho}\dfrac{d}{dx}\right)^n$.

With adequate choices of parameters in Definition 6.2, we recover some well-known operators of fractional differentiation, namely:

- if $n=1$, we obtain the $(k,\rho)$-fractional derivative Definition 6.1;
- if $k=1$ and $n=1$, we have the Hilfer-Katugampola fractional derivative proposed in Definition 5.2;
- if $k=1$ and $\rho=1$, we obtain the so-called generalized Riemann-Liouville fractional derivative Definition 1.15;
- if $k=1$, $\rho\to 0^+$ and $n=1$, we have the Hilfer-Hadamard fractional derivative Definition 2.3;
- if $k=1$, $\rho=1$ and $n=1$, we obtain the well-known Hilfer derivative Definition 1.14;
- if $k=1$, $\rho=1$ and $\nu=0$, we obtain the Riemann-Liouville fractional derivative Definition 1.12;
- if $k=1$, $\rho=1$ and $\nu=1$, we obtain the Caputo derivative Theorem 1.4;
- if $k=1$, $\rho\to 0^+$ and $\nu=0$, we obtain the Hadamard fractional derivative Definition 2.2;
- if $k=1$, $\rho\to 0^+$ and $\nu=1$, we have the Caputo-Hadamard fractional derivative Theorem 3.1;

- if $k = 1$ and $\nu = 0$, we have the generalized fractional derivative Definition 4.2;
- if $k = 1$ and $\nu = 1$, we obtain the generalized Caputo-type fractional derivative section 4.3.

It is also possible to recover, for particular extreme values of integration, the fractional derivative in the Liouville sense [38, p. 87] and in the Weyl [23] sense.
The generalized $(k,\rho)$-fractional derivative, ${}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}$, is the inverse operator of the $(k,\rho)$-fractional integral, ${}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}$. We prove this result by means of the following lemma.

**Lemma 6.2.** *Let $\alpha \in \mathbb{R}^*$ and $\rho > 0$, $k > 0$. If $1 \leqslant p \leqslant \infty$, then for $\varphi \in L_p(a,b)$ we have*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\ {}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi)(x) = \varphi(x). \tag{6.10}$$

*Proof.* In order to simplify the development and the notation, we define

$$\Psi = \frac{\nu(nk-\alpha)}{k} \quad \text{and} \quad \Phi = n - \Psi. \tag{6.11}$$

From Definition 6.2 and Theorem 6.1, we can write

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\ {}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi)(x) &= \left({}^{\rho}_{k}\mathcal{J}^{\nu(nk-\alpha)}_{a^+}\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)+\alpha}_{a^+}\varphi)\right)(x) \\ &= \frac{k^{n-2}\,\rho^{2-\Psi-\Phi}}{\Gamma_k[k\,\Psi]\Gamma_k[k\,\Phi]}\int_a^x (x^\rho - t^\rho)^{\Psi-1}t^{\rho-1}\delta^n_\rho\left[\int_a^t (t^\rho-u^\rho)^{\Phi-1}u^{\rho-1}\varphi(u)du\right]dt.\end{aligned} \tag{6.12}$$

Knowing that,

$$\int_a^t (t^\rho - u^\rho)^{\Phi-1}u^{\rho-1}\varphi(u)du = \frac{1}{\rho\,\Phi}\underbrace{\left\{\varphi(a)(t^\rho-a^\rho)^\Phi + \int_a^t (t^\rho-u^\rho)^\Phi\varphi'(u)du\right\}}_{F(t)}, \tag{6.13}$$

see Appendix B. Differentiating with the operator $\delta^n_\rho$

$$\delta^n_\rho F(t) = \frac{\Gamma(\Phi)\,\rho^{n-1}}{\Gamma(\Phi-n+1)}\left[\varphi(a)(t^\rho-a^\rho)^{\Phi-n} + \int_a^t (t^\rho-u^\rho)^{\Phi-n}\varphi'(u)du\right] \tag{6.14}$$

and after substituing the result Eq.(6.14) into Eq.(6.12), we have

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\ {}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi)(x) &= \frac{k^{n-2}\,\rho^{2-\Psi-\Phi}}{\Gamma_k[k\,\Psi]\Gamma_k[k\,\Phi]}\int_a^x (x^\rho-t^\rho)^{\Psi-1}t^{\rho-1} \\ &\times\frac{\Gamma(\Phi)\,\rho^{n-1}}{\Gamma(\Phi-n+1)}\left[\varphi(a)(t^\rho-a^\rho)^{\Phi-n} + \int_a^t (t^\rho-u^\rho)^{\Phi-n}\varphi'(u)du\right]dt.\end{aligned} \tag{6.15}$$

Using item 2 and rearranging the last expression to get

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\ {}^{\rho}_{k}\mathcal{J}^{\alpha}_{a^+}\varphi)(x) &= \frac{\rho}{k\,k^{\Psi-1}\Gamma[\Psi]\,k^{(1-\Psi)-1}\,\Gamma[1-\Psi]}\left\{\varphi(a)\int_a^x (x^\rho-t^\rho)^{\Psi-1}t^{\rho-1}(t^\rho-a^\rho)^{-\Psi}dt\right. \\ &+ \left.\int_a^x \varphi'(u)du\int_u^x (x^\rho-t^\rho)^{\Psi-1}t^{\rho-1}(t^\rho-u^\rho)^{-\Psi}dt\right\}.\end{aligned}$$

Introducing in the integral from $a$ to $x$ the change of variable $u = (t^\rho - a^\rho)/(x^\rho - a^\rho)$ and doing the same in the integral from $u$ to $x$, we have

$$({}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^\rho_k\mathcal{J}^{\alpha}_{a^+}\varphi)(x) = \frac{1}{\Gamma_k[k\Psi]\,\Gamma_k[k(1-\Psi)]}\left[\varphi(a) + \int_a^x \varphi'(u)du\right]\left\{\frac{1}{k}\int_0^1 (1-u)^{\Psi-1}u^{(1-\Psi)-1}du\right\}.$$

We then use the two expressions in item 2 to obtain

$$\begin{aligned}
({}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^\rho_k\mathcal{J}^{\alpha}_{a^+}\varphi)(x) &= \frac{1}{\Gamma_k[k\Psi]\,\Gamma_k[k(1-\Psi)]}\left[\varphi(a) + \int_a^x \varphi'(u)du\right]\frac{\Gamma_k[k\Psi]\,\Gamma_k[k(1-\Psi)]}{\Gamma_k[k]} \\
&= \varphi(a) + \int_a^x \varphi'(u)du.
\end{aligned}$$

Finally, we use the fundamental theorem of calculus, whence it immediately follows

$$({}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^\rho_k\mathcal{J}^{\alpha}_{a^+}\varphi)(x) = \varphi(x).$$

□

The following result yields the composition between the $(k,\rho)$-fractional integral and the generalized $(k,\rho)$-fractional derivative.

**Lemma 6.3.** *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha > \beta > 0$, $k, \rho > 0$, $n-1 < \alpha \leqslant n$ and $n \in \mathbb{N}$. If $1 \leqslant p \leqslant \infty$, then for $\varphi \in L_p(a,b)$, we have*

$$({}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^\rho_k\mathcal{J}^{\beta}_{a^+}\varphi)(x) = ({}^\rho_k\mathcal{J}^{\beta-\alpha}_{a^+}\varphi)(x). \tag{6.16}$$

*Proof.* The proof is analogous to previous Lemma 6.2. □

Again, in order to simplify the development and notation, we introduce the parameter $\Lambda$:

$$\Lambda = \frac{\nu(nk-\alpha)+\alpha}{k}. \tag{6.17}$$

**Lemma 6.4.** *Let $\alpha > 0$, $n = [\alpha]+1$, $n \in \mathbb{N}$. If $\varphi \in L_p(a,b)$ and $({}^\rho_k\mathcal{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi))(x) \in AC^n_\delta[a,b]$, then*

$$({}^\rho_k\mathcal{J}^{\alpha}_{a^+}\,{}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = \varphi(x) - \sum_{j=1}^{n}\frac{({}^\rho_k\mathcal{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a)}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j}. \tag{6.18}$$

*In particular, if $0 < \alpha < 1$, then*

$$({}^\rho_k\mathcal{J}^{\alpha}_{a^+}\,{}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = \varphi(x) - \frac{({}^\rho_k\mathcal{J}^{(1-\nu)(k-\alpha)-k(1-j)}_{a^+}\varphi)(a)}{\Gamma_k[\nu(k-\alpha)+\alpha-k(j-1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j}. \tag{6.19}$$

*Proof.* From Definition 6.2, we can write

$$\begin{aligned}
({}^{\rho}_{k}\mathcal{J}^{\alpha}_{a+}\ {}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a+}\varphi)(x) &= \left({}^{\rho}_{k}\mathcal{J}^{\alpha}_{a+}\ {}^{\rho}_{k}\mathcal{J}^{\nu(nk-\alpha)}_{a+}\delta^{n}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)\right)(x) \\
&= \left({}^{\rho}_{k}\mathcal{J}^{\nu(nk-\alpha)+\alpha}_{a+}\delta^{n}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)\right)(x) \\
&= \frac{\rho^{1-\Lambda}}{\Gamma_k[k\,\Lambda]}\int_a^x (x^{\rho}-t^{\rho})^{\Lambda-1}\,t^{\rho-1}\left\{\delta^{n}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(t)\right\}dt.
\end{aligned}$$

Integrating by parts the last expression, we obtain

$$\begin{aligned}
({}^{\rho}_{k}\mathcal{J}^{\alpha}_{a+}\ {}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a+}\varphi)(x) &= \frac{-\rho^{1-\Lambda}\,(x^{\rho}-a^{\rho})^{\Lambda-1}}{k^{\Lambda}\,\Gamma(\Lambda)}[\delta^{n-1}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(a)] \\
&+\frac{\rho^{2-\Lambda}}{k^{\Lambda}\,\Gamma(\Lambda-1)}\int_a^x (x^{\rho}-t^{\rho})^{\Lambda-1}\,t^{\rho-1}\delta^{n}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(t)dt.
\end{aligned}$$

Thus, integrating by parts $(n-1)$ times, we have

$$\begin{aligned}
({}^{\rho}_{k}\mathcal{J}^{\alpha}_{a+}\ {}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a+}\varphi)(x) &= -\sum_{j=0}^{n-1}\frac{\delta^{n-j-1}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(a)}{k^{j+1}\,\Gamma_k[k(\Lambda-j)]}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j-1} \\
&+ \frac{1}{k\,\Gamma_k[k(\Lambda-n)]}\int_a^x\left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\Lambda-n-1} t^{\rho-1}({}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(t)dt \\
&= -\sum_{j=0}^{n-1}\frac{\delta^{n-j-1}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(a)}{k^{j+1}\,\Gamma_k[k(\Lambda-j)]}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j-1} \\
&+ ({}^{\rho}_{k}\mathcal{J}^{\nu(kn-\alpha)+\alpha-nk}_{a+}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(x) \\
&= \varphi(x)-\sum_{j=1}^{n}\frac{\delta^{n-j}_{\rho}(k^{n}\,{}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a+}\varphi)(a)}{k^{j}\,\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j} \\
&= \varphi(x)-\sum_{j=1}^{n}\frac{({}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a+}\varphi)(a)}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j}.
\end{aligned}$$

□

Next, we show that the generalized $(k,\rho)$-fractional derivative of order $\alpha$ of the polynomial function $(t^{\rho}-a^{\rho})^{\Lambda-j}$ is null, i.e., $\left[{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a+}(t^{\rho}-a^{\rho})^{\Lambda-j}\right](x)=0$.

**Lemma 6.5.** *Let $\alpha,\nu\in\mathbb{R}$ such that $n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $0\leqslant\nu\leqslant 1$, $\rho>0$ and $k>0$. Then, for all $j=1,2,\dots,n$, we have*

$$\left[{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a+}(t^{\rho}-a^{\rho})^{\Lambda-j}\right](x)=0. \tag{6.20}$$

*Proof.* Again, in order to simplify the development and the notation in what follows, we use Eq.(6.17) and we define

$$\Omega=\frac{(1-\nu)(kn-\alpha)}{k}. \tag{6.21}$$

Thus, from Definition 6.2 and Eq.(6.5), we have

$$\left(k^n\, {}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a^+}\,(t^\rho-a^\rho)^{\Lambda-j}\right)(x)=\frac{k^n\,\rho^{1-\Omega}}{k\,\Gamma_k[k\,\Omega]}\int_a^x (x^\rho-t^\rho)^{\Omega-1}(t^\rho-a^\rho)^{\Lambda-j}\,t^{\rho-1}dt.$$

We introduce the change of variable $u=(t^\rho-a^\rho)/(x^\rho-a^\rho)$, and use the definition of $k$-beta function, Eq.(1.19), to obtain

$$\begin{aligned}\left(k^n\, {}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a^+}\,(t^\rho-a^\rho)^{\Lambda-j}\right)(x) &= \frac{k^n\,\rho^{-\Omega}}{\Gamma_k[k\Omega]}(x^\rho-a^\rho)^{n-j}\left\{\frac{1}{k}\int_0^1(1-u)^{\Omega-1}u^{\Lambda-j}du\right\}\\ &= \frac{k^n\,\rho^{-\Omega}\,\Gamma_k[k(\Lambda-j+1)]}{\Gamma_k[k(n-j-1)]}\,(x^\rho-a^\rho)^{n-j}.\end{aligned}$$

Next, we calculate $\delta^n_\rho(k^n\, {}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)}_{a^+}\,(t^\rho-a^\rho)^{\Lambda-j})(x)$, that is,

$$\begin{aligned}\left(x^{1-\rho}\frac{d}{dx}\right)^n(x^\rho-a^\rho)^{n-j} &= \left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}\left(x^{1-\rho}\frac{d}{dx}\right)(x^\rho-a^\rho)^{n-j}\\ &= \rho(n-j)\left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}(x^\rho-a^\rho)^{n-j-1}.\end{aligned}$$

Differentiating more $(n-1)$ times, we obtain

$$\left(x^{1-\rho}\frac{d}{dx}\right)^n(x^\rho-a^\rho)^{n-j}=\rho^n\,(n-j)(n-j-1)\cdots(2-j)(1-j)(x^\rho-a^\rho)^{-j}=0. \tag{6.22}$$

As $j=1,2,\ldots,n$, then for each $j$ there is one null term in the product given by Eq.(6.22); this completes the proof. □

Finally, we show the equivalence between the Cauchy problem and a Volterra integral equation of the second kind.

**Theorem 6.3.** *Let $\alpha>0$ and $n=[\alpha]+1$ where $n\in\mathbb{N}$. Let $G$ be an open set in $\mathbb{R}$ and $f:(a,b]\times G\to\mathbb{R}$ be a function such that $f(x,\varphi(x))\in L(a,b)$ for any $\varphi\in G$. If $\varphi\in L(a,b)$, then $\varphi$ satisfies the relations*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)=f(x,\varphi(x)), \tag{6.23}$$

$$({}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+)=b_j,\; b_j\in\mathbb{R},\;(j=1,2,\ldots,n), \tag{6.24}$$

*if, and only if, $\varphi$ satisfies the Volterra integral equation*

$$\varphi(x)=\sum_{j=1}^n\frac{b_j\,[(x^\rho-a^\rho)/\rho]^{\Lambda-j}}{\Gamma_k[k(\Lambda-j+1)]}+\frac{1}{k\,\Gamma_k(\alpha)}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}f(t,\varphi(t))\,dt, \tag{6.25}$$

*with $\Lambda$ defined in Eq.(6.17).*

*Proof.* ($\Rightarrow$) We consider $\varphi \in L(a,b)$ satisfying Eq.(6.23) and Eq.(6.24). As $\varphi \in L(a,b)$, then Eq.(6.23) exists and $({}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}\varphi)(x) \in L(a,b)$. Applying operator ${}_k^\rho\mathcal{J}_{a^+}^{\alpha}$ on both sides of Eq.(6.23) and using Lemma 6.4 and Eq.(6.24), we obtain

$$({}_k^\rho\mathcal{J}_{a^+}^{\alpha}\,{}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}\varphi)(x) = ({}_k^\rho\mathcal{J}_{a^+}^{\alpha}\,f(t,\varphi(t)))(x)$$

$$\varphi(x) - \sum_{j=1}^{n}\frac{({}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-j)}\varphi)(a)}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j} = ({}_k^\rho\mathcal{J}_{a^+}^{\alpha}\,f(t,\varphi(t)))(x)$$

$$\varphi(x) = \sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j} + ({}_k^\rho\mathcal{J}_{a^+}^{\alpha}\,f(t,\varphi(t)))(x).$$

From Lemma 6.1, the integral $({}_k^\rho\mathcal{J}_{a^+}^{\alpha}\,f(t,\varphi(t)))(x) \in L(a,b)$, thus Eq.(6.25) follows.

($\Leftarrow$) Assume that $\varphi \in L(a,b)$ satisfies Eq.(6.25). Applying operator ${}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}$ on both sides of Eq.(6.25), we obtain

$$({}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}\varphi)(x) = \sum_{j=1}^{n}\frac{b_j\,\rho^{j-\Lambda}}{\Gamma_k[k(\Lambda-j+1)]}\left[{}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}(t^\rho-a^\rho)^{\Lambda-j}\right](x) + ({}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}\,{}_k^\rho\mathcal{J}_{a^+}^{\alpha}\,f(t,\varphi(t)))(x).$$

From Lemma 6.2 and Lemma 6.5, Eq.(6.23) follows. Next, we prove the validity of Eq.(6.24). Therefore, we apply the operator ${}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-m)}$, with $m = 1,2,\ldots,n$, on both sides of Eq.(6.25), in order to obtain

$$\begin{aligned}
({}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-j)}\varphi)(x) &= \sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left[{}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-m)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\right]\\
&+ ({}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-m)}\,{}_k^\rho\mathcal{J}_{a^+}^{\alpha} f(t,\varphi(t)))(x)\\
&= \sum_{j=1}^{m}\frac{b_j}{\Gamma_k[k(m-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{m-j}\\
&+ ({}_k^\rho\mathcal{J}_{a^+}^{k(m-\nu n)+\alpha\nu} f(t,\varphi(t)))(x).
\end{aligned}$$

Letting $x \to a^+$, we finally have

$$({}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-j)}\varphi)(a^+) = b_m, \quad \text{with} \quad m = 1,2,\ldots,n.$$

$\square$

## 6.5 Linear Fractional Differential Equations

In this section we analyze some particular cases of function $f(x,\varphi(x))$ appearing in Theorem 6.3. We apply the method of successive approximations in order to obtain an analytical solution of the resulting linear fractional differential equations. Let us first consider $f(x,\varphi(x)) = \lambda\varphi(x)$ in Theorem 6.3.

**Theorem 6.4.** *Let $\alpha, \lambda \in \mathbb{R}^*$ such that $n-1 < \alpha \leqslant n$, where $n \in \mathbb{N}$. If $\varphi \in L(a,b)$, then the Cauchy problem*

$$({}_k^\rho\mathscr{D}_{a^+}^{\alpha,\nu}\varphi)(x) = \lambda\varphi(x) \tag{6.26}$$

$$({}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-j)}\varphi)(a^+) = b_j,\ b_j \in \mathbb{R},\ (j = 1,2,\ldots,n), \tag{6.27}$$

*admits a unique solution in the space $L(a,b)$, given by*

$$\varphi(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} E_{k,\alpha,k(\Lambda-j+1)}\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right], \tag{6.28}$$

*where $E_{k,\xi,\sigma}(\cdot)$ is defined in Eq.(6.2).*

*Proof.* According to Theorem 6.3, we just need to solve the Volterra integral equation, Eq.(6.25), with $f(t,\varphi(t)) = \lambda\varphi(t)$. As the Volterra integral equation of the second kind admits a unique solution [39], the uniqueness of Eq.(6.25) is guaranteed. In order to find the exact solution, we use the method of successive approximations, that is, we consider

$$\varphi_0 = \sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} \tag{6.29}$$

$$\varphi_i(x) = \varphi_0(x) + \frac{\lambda}{k\,\Gamma_k(\alpha)}\int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1}\varphi_{i-1}(t)\,dt. \tag{6.30}$$

We define a parameter $\Lambda_m$ by

$$\Lambda_m = \frac{\nu(nk-\alpha)+\alpha m}{k} \quad \text{with} \quad m = 1,2,\ldots,i+1. \tag{6.31}$$

For $m = 1$, we have $\Lambda_1 = \Lambda$ given by Eq.(6.17). Thus, from Eq.(6.29) and Eq.(6.30), we can write

$$\begin{aligned}\varphi_1(x) &= \varphi_0(x) + \frac{\lambda}{k\,\Gamma_k(\alpha)}\int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1}\varphi_0(t)\,dt \\ &= \varphi_0(x) + \sum_{j=1}^{n}\frac{\lambda\, b_j}{\Gamma_k[k(\Lambda-j+1)]}\left[{}_k^\rho\mathcal{J}_{a^+}^{\alpha}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\Lambda-j}\right](x).\end{aligned}$$

Using Theorem 6.2, we obtain

$$\begin{aligned}\varphi_1(x) &= \sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} + \sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda_2-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_2-j} \\ &= \sum_{j=1}^{n} b_j \sum_{m=1}^{2}\frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_m-j}.\end{aligned} \tag{6.32}$$

Similarly, using Eq.(6.29), Eq.(6.32) and Theorem 6.2, we obtain the expression for $\varphi_2(x)$, that is,

$$\begin{aligned}
\varphi_2(x) &= \varphi_0(x) + \frac{\lambda}{k\,\Gamma_k(\alpha)}\int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1}\varphi_1(t)\,dt \\
&= \varphi_0(x) + \lambda\sum_{j=1}^n b_j \sum_{m=1}^2 \frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m - j + 1)]}\left[{}^\rho_k\mathcal{J}^\alpha_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\Lambda_m - j}\right](x) \\
&= \sum_{j=1}^n b_j \sum_{m=1}^3 \frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m - j + 1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_m - j}.
\end{aligned}$$

Repeating this process, we obtain the expression for $\varphi_i(x)$, with $i \in \mathbb{N}$:

$$\varphi_i(x) = \sum_{j=1}^n b_j \sum_{m=1}^{i+1} \frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m - j + 1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_m - j}.$$

Taking $i \to \infty$, we obtain the explicit solution for $\varphi(x)$

$$\varphi(x) = \sum_{j=1}^n b_j \sum_{m=1}^{\infty} \frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m - j + 1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_m - j}.$$

Changing the summation index, $m \to m+1$, we have

$$\varphi(x) = \sum_{j=1}^n b_j \sum_{m=0}^{\infty} \frac{\lambda^{m}}{\Gamma_k[k(\Lambda_{m+1} - j + 1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_{m+1} - j}.$$

Moreover, we can rewrite this last expression in terms of $k$-new generalized Mittag-Leffler function, that is,

$$\varphi(x) = \sum_{j=1}^n b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda - j} E_{k,\alpha,k(\Lambda-j+1)}\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right]. \tag{6.33}$$

□

As another application, we consider $f(x,\varphi(x)) = \lambda({}^\rho_k\mathscr{D}^{\beta,\nu}_{a^+}\varphi)(x)$ in Theorem 6.3.

**Theorem 6.5.** *Let $\alpha, \beta \in \mathbb{R}$, $\alpha > \beta > 0$, $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Then, the Cauchy problem*

$$\begin{gathered}
({}^\rho_k\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = \lambda({}^\rho_k\mathscr{D}^{\beta,\nu}_{a^+}\varphi)(x) \\
({}^\rho_k\mathcal{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+) = b_j,\ b_j \in \mathbb{R},\ (j = 1, 2, \ldots, n),
\end{gathered}$$

*admits a unique solution in the space $L(a,b)$, given by*

$$\varphi(x) = \sum_{j=1}^n b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Theta - j} E_{k,\alpha-\beta,k(\Theta-j+1)}\left[\lambda\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha-\beta}{k}}\right],$$

*where* $\Theta = \dfrac{\alpha + \nu(nk - \alpha + \beta)}{k}$.

*Proof.* Suppose the solution $\varphi = ({}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+} g)(x) \in L(a,b)$, then

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+} g)(x) = \lambda({}^{\rho}_{k}\mathscr{D}^{\beta,\nu}_{a^+}\,{}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+} g)(x).$$

By Lemma 6.2, we can write

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+} g)(x) = \lambda g(x),$$

and by Lemma 6.3, we have

$$({}^{\rho}_{k}\mathcal{J}^{\beta-\alpha}_{a^+} g)(x) = \lambda g(x) \quad \text{or} \quad ({}^{\rho}_{k}\mathscr{D}^{\alpha-\beta,\nu}_{a^+} g)(x) = \lambda g(x).$$

We shall use the second expression. Thus, let $\Upsilon_m = \dfrac{(\alpha-\beta)m+\alpha+\beta+\nu(nk-\alpha+\beta)}{k}$; in the case $m=1$ we denote $\Upsilon_m = \Upsilon$. Taking $\alpha \to \alpha-\beta$ in Theorem 6.4, we can write

$$g(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Upsilon-j} E_{k,\alpha-\beta,k(\Upsilon-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha-\beta}{k}}\right]. \tag{6.34}$$

As $\varphi(x) = ({}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+} g)(x)$, we apply $({}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+})$ on both sides of Eq.(6.34), in order to obtain

$$({}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+} g)(x) = \sum_{j=1}^{n} b_j \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma_k[k(\Upsilon_m - j+1)]}\left[{}^{\rho}_{k}\mathcal{J}^{\beta}_{a^+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\Upsilon_m - j}\right](x).$$

Using Theorem 6.2 and rewriting the expression, we obtain

$$\varphi(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Theta-j} E_{k,\alpha-\beta,k(\Theta-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha-\beta}{k}}\right].$$

□

In the next theorem we consider a sequence of linear fractional differential equations of order $\alpha n$. This theorem generalizes the results presented in [10, 12].

**Theorem 6.6.** *Let $\alpha,\beta \in \mathbb{R}$, $\alpha > \beta > 0$, $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Then, the Cauchy problem*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha n,\nu}_{a^+}\varphi)(x) = \lambda\,\varphi(x) \tag{6.35}$$

$$({}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha n)-k(n-j)}_{a^+}\varphi)(a^+) = b_j,\ b_j \in \mathbb{R},\ (j=1,2,\ldots,n), \tag{6.36}$$

*admits a unique solution in the space $L(a,b)$, given by*

$$\varphi(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_n-j} E_{k,\alpha n,k(\Lambda_n-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha n}{k}}\right], \tag{6.37}$$

*where* $\Lambda_n = \dfrac{\nu(nk-\alpha n)+\alpha n}{k}$.

*Proof.* We consider $\alpha \to \alpha n$ in Theorem 6.4; we thus obtain the solution, Eq.(6.37). □

## 6.6 Dependence on Initial Conditions

In this section, we present the changes in a solution entailed by small changes in initial conditions. Consider Eq.(6.23) with the following changes in the initial conditions shown in Eq.(6.24):

$$({}_k^\rho\mathcal{J}_{a^+}^{(1-\nu)(kn-\alpha)-k(n-j)}\varphi)(a^+) = b_j + \eta_j, \quad b_j \in \mathbb{R}, \ (j = 1, 2, \ldots, n), \tag{6.38}$$

where $\eta_j$ $(j = 1, \ldots, n)$ are arbitrary constants.

**Theorem 6.7.** *Suppose that the hypotheses of Theorem 6.3 are satisfied. Let $\varphi(x)$ and $\tilde{\varphi}(x)$ be solutions of the initial value problems Eq.(6.23)-Eq.(6.24) and Eq.(6.23))-Eq.(6.38), respectively. Then,*

$$|\varphi(x) - \tilde{\varphi}(x)| \leqslant \sum_{j=1}^{n} |\eta_j| \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\nu(nk-\alpha)+\alpha}{k} - j} E_{k,\alpha,\alpha+\nu(nk-\alpha)-k(j-1)}\left[A\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right],$$

*with $x \in (a, b]$, where $E_{k,\xi,\sigma}(z)$ is the $k$-Mittag-Leffler function, Eq.(6.2).*

*Proof.* According to Theorem 6.4, we have

$$\varphi(x) = \lim_{i\to\infty} \varphi_i(x)$$

where $\varphi_0(x)$ is given by Eq.(6.29) and

$$\varphi_i(x) = \varphi_0(x) + \frac{1}{k\Gamma_k[\alpha]} \int_a^x \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} f(t, \varphi_{i-1}(t))dt. \tag{6.39}$$

We also have

$$\tilde{\varphi}(x) = \lim_{i\to\infty} \tilde{\varphi}_i(x), \tag{6.40}$$

$$\tilde{\varphi}_0(x) = \sum_{j=1}^{n} \frac{(b_j + \eta_j)}{\Gamma_k[k(\Lambda - j + 1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda - j}, \tag{6.41}$$

$$\tilde{\varphi}_i(x) = \tilde{\varphi}_0(x) + \frac{1}{k\Gamma_k[\alpha]} \int_a^x \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} f(t, \tilde{\varphi}_{i-1}(t))dt, \quad i = 1, 2, \ldots \tag{6.42}$$

From Eq.(6.29) and Eq.(6.41), we can write

$$|\varphi_0(x) - \tilde{\varphi}_0(x)| \leqslant \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma_k[k(\Lambda - j + 1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda - j}. \tag{6.43}$$

We finally consider Eq.(6.39) and Eq.(6.42) with $i=1$, the Lipschitz condition for $f(t,\varphi)$, Definition 1.1, the inequality Eq.(6.43) and Theorem 6.2, in order to obtain

$$\begin{aligned}
|\varphi_1(x)-\tilde{\varphi}_1(x)| &\leqslant \sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\\
&+ \frac{A}{k\Gamma_k[\alpha]}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}|f(t,\varphi_0(t))-f(t,\tilde{\varphi}_0(t))|dt\\
&\leqslant \sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\\
&+ \frac{A}{k\Gamma_k[\alpha]}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}|\varphi_0(t)-\tilde{\varphi}_0(t)|dt\\
&\leqslant \sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\\
&+ \frac{A}{k\Gamma_k[\alpha]}\sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\\
&= \sum_{j=1}^{n}|\eta_j|\sum_{m=1}^{2}\frac{A^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j},
\end{aligned}$$

where $\Lambda_m$ is given by Eq.(6.31). Thus, continuing this procedure, we obtain

$$|\varphi_i(x)-\tilde{\varphi}_i(x)| \leqslant \sum_{j=1}^{n}|\eta_j|\sum_{m=1}^{i+1}\frac{A^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j}.$$

Taking $i\to\infty$ and $m\to m+1$, it follows that

$$|\varphi(x)-\tilde{\varphi}(x)| \leqslant \sum_{j=1}^{n}|\eta_j|\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}E_{k,\alpha,k(\Lambda-j+1)}\left[A\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right].$$

$\square$

# Concluding Remarks and Future Perspectives

The fact that there exist much more than one definition for fractional derivatives makes choosing the adequate approach a crucial issue in solving a given problem. In order to overcome the problem of choosing the operator of fractional differentiation, we have developed generalizations. In this thesis we studied essentially five different formulations for fractional derivatives; for all those formulations, when the order of the derivative is an integer we recover the results of ordinary differential calculus. Before we could introduce our proposals for fractional derivatives, it was necessary to present some fundamental concepts. In Chapter 1 we presented classical concepts of real analysis. In Chapter 2 we showed how it is possible, starting from Hadamard integrals of arbitrary order, to present Hadamard fractional derivatives [21, 38]. Our main contribution, in this chapter, was the development a Leibniz-type rule involving Hadamard fractional derivatives. In Chapter 3, also starting from Hadamard integrals of arbitrary order, we presented the Caputo-Hadamard fractional derivatives [27, 19]. In this chapter, again, our main contribution was the proof of a Leibniz-type rule for these fractional differentiation operators. In Chapter 4 we proposed the generalized Caputo-type fractional derivatives. This new formulation was obtained by means of a Caputo modification in the generalized fractional derivatives and it recovers, as particular cases, the derivatives of arbitrary order in the sense of Caputo and Caputo-Hadamard. This chapter is part of a paper that was accepted for publication [48]. In Chapter 5 we presented an original proposal for fractional derivatives which we called Hilfer-Katugampola fractional derivatives. We demonstrated some theorems and properties involving this formulation, as well as the equivalence between a nonlinear initial value problem and a Volterra integral equation. We also discussed the existence and uniqueness of the solution for this initial value problem. Finally, we obtained the analytical solutions to some fractional differential equations using the method of successive approximations. This new derivative is much more general than that presented in Chapter 4 and the results presented in Chapter 5 are part of a paper that was published online [47]. Finally, Chapter 6 presented our main contribution for this thesis, because this is our most general proposal for fractional derivatives. We proposed the generalized $(k, \rho)$-fractional derivatives. We discussed the equivalence between a Cauchy problem using this operator of fractional differentiation, and a Volterra integral equation of the second kind; we also considered some particular cases of such a problem. Besides, we also proved that small changes on initial conditions entail small changes in the solution of the problem. The Chapter 6 is part of a paper that was accepted for publication [49].

A natural continuation of this work consists in proposing a new fractional integration operator which contains in its kernel a generalized Mittag-Leffler function, that is,

$$({}^{\rho}_{k}\mathscr{E}^{\mu,q}_{a^+,\beta,\gamma,\xi}\varphi)(x) = \frac{1}{k}\int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\gamma}{k}-1} t^{\rho-1} E^{\mu,q}_{k,\beta,\gamma,\xi}\left[\omega\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right]\varphi(t)dt.$$

We will investigate some properties involving this operator. After studying this operator we shall try to study a particular Cauchy problem, namely,

$$\begin{gathered} ({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = ({}^{\rho}_{k}\mathscr{E}^{\mu,q}_{a^+,\beta,\gamma,\xi}\varphi)(x), \\ ({}^{\rho}_{k}\mathcal{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+) = b_j,\ b_j \in \mathbb{R},\ (j = 1, 2, \ldots, n). \end{gathered}$$

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# Appendices

# APPENDIX A

# Auxiliary Results: Chapter 4

## A.1 Proof Eq.(4.19)

First we show that

$$\int_a^t \frac{\tau^{\rho-1}\,\varphi(\tau)}{(x^\rho - t^\rho)^{1-\alpha}} d\tau = \frac{1}{\alpha\rho}\left\{\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho - \tau^\rho)^\alpha \varphi'(\tau) d\tau\right\}.$$

For this end we need to integrate by parts the above expression with the choice $u = \varphi(\tau)$ and $du = \varphi'(\tau)d\tau$, in order to obtain

$$\begin{aligned}\int_a^t \frac{\tau^{\rho-1}\,\varphi(\tau)}{(x^\rho - t^\rho)^{1-\alpha}} d\tau &= -\frac{\varphi(\tau)}{\alpha\,\rho}(t^\rho - \tau^\rho)^\alpha \Big|_a^t + \frac{1}{\alpha\,\rho}\int_a^t (t^\rho - \tau^\rho)^\alpha \varphi'(\tau) d\tau \\ &= \frac{1}{\alpha\rho}\underbrace{\left\{\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho - \tau^\rho)^\alpha \varphi'(\tau) d\tau\right\}}_{A(t)}.\end{aligned}$$

## A.2 Proof Eq.(4.22)

Now we prove Eq.(4.21). We differentiate the expression $A(t)$ as above

$$\delta_\rho^n A(t) = \frac{\Gamma(\alpha+1)\,\rho^n}{\Gamma(\alpha-n+1)}\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho - \tau^\rho)^\alpha\,\varphi'(\tau) d\tau\right], \tag{A.1}$$

where $\delta_\rho^n = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$. Note that,

$$\left(t^{1-\rho}\frac{d}{dt}\right)\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-n} + \int_a^t (t^\rho - \tau^\rho)^{\alpha-n}\,\varphi'(\tau) d\tau\right] =$$
$$\alpha\rho\,\varphi(a)(t^\rho - a^\rho)^{\alpha-1} + \alpha\rho\int_a^t (t^\rho - \tau^\rho)^{\alpha-1}\varphi'(\tau) d\tau. \tag{A.2}$$

The proof is based on mathematical induction over $n$. The Eq.(A.1) holds for $n = 1$, then we obtain Eq.(A.2). Suppose that Eq.(A.1) is valid for $n = k$

$$\left(t^{1-\rho}\frac{d}{dt}\right)^k\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho-\tau^\rho)^\alpha\,\varphi'(\tau)d\tau\right] = \\ \frac{\Gamma(\alpha+1)\,\rho^k}{\Gamma(\alpha-k+1)}\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-k} + \int_a^t (t^\rho-\tau^\rho)^{\alpha-k}\,\varphi'(\tau)d\tau\right]. \tag{A.3}$$

We want to prove that

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{k+1}\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho-\tau^\rho)^\alpha\,\varphi'(\tau)d\tau\right] = \\ \frac{\Gamma(\alpha+1)\,\rho^{k+1}}{\Gamma(\alpha-k)}\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-k-1} + \int_a^t (t^\rho-\tau^\rho)^{\alpha-k-1}\,\varphi'(\tau)d\tau\right]. \tag{A.4}$$

Note that, we can write

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{k+1}\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho-\tau^\rho)^\alpha\,\varphi'(\tau)d\tau\right] = \\ \left(t^{1-\rho}\frac{d}{dt}\right)\underbrace{\left(t^{1-\rho}\frac{d}{dt}\right)^{k}\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho-\tau^\rho)^\alpha\,\varphi'(\tau)d\tau\right]}_{\text{induction hypothesis, Eq.(A.3)}}$$

or

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{k+1}\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho-\tau^\rho)^\alpha\,\varphi'(\tau)d\tau\right] = \\ \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\Gamma(\alpha+1)\,\rho^k}{\Gamma(\alpha-k+1)}\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-k} + \int_a^t (t^\rho-\tau^\rho)^{\alpha-k}\,\varphi'(\tau)d\tau\right] = \\ t^{1-\rho}\frac{\Gamma(\alpha+1)\,\rho^k}{\Gamma(\alpha-k+1)}\left(\frac{d}{dt}\right)\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-k} + \int_a^t (t^\rho-\tau^\rho)^{\alpha-k}\,\varphi'(\tau)d\tau\right].$$

We finally find

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{k+1}\left[\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho-\tau^\rho)^\alpha\,\varphi'(\tau)d\tau\right] = \\ \frac{\Gamma(\alpha+1)\,\rho^{k+1}}{\Gamma(\alpha-k)}\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-k-1} + \int_a^t (t^\rho-\tau^\rho)^{\alpha-k-1}\,\varphi'(\tau)d\tau\right].$$

## A.3 Proof Eq.(4.24)

We will prove Eq.(4.23),

$$\int_a^x \frac{(t^\rho - a^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{1-n+\alpha}}\,t^{\rho-1}dt = \frac{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}{\rho}.$$

Considering the change of variable $u = \dfrac{t^\rho - a^\rho}{x^\rho - a^\rho}$ in order to obtain

$$\begin{aligned}\int_a^x \frac{(t^\rho - a^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{1-n+\alpha}} t^{\rho-1} dt &= \frac{1}{\rho}\int_0^1 \frac{[(x^\rho - a^\rho)u]^{\alpha-n}}{[x^\rho - a^\rho - (x^\rho - a^\rho)u]^{\alpha-n+1}}(x^\rho - a^\rho)du \\ &= \frac{1}{\rho}\underbrace{\int_0^1 u^{\alpha-n}\,(1-u)^{n-\alpha-1} du}_{B(\alpha-n+1,\,n-\alpha)}.\end{aligned}$$

We immediately get Eq.(4.23).

# APPENDIX B

# Auxiliary Results: Chapter 6

## B.1 Proof Eq.(6.13)

We prove that

$$\int_a^t (t^\rho - u^\rho)^{\Phi-1} u^{\rho-1} \varphi(u) du = \frac{1}{\rho\,\Phi}\left\{\varphi(a)(t^\rho - a^\rho)^\Phi + \int_a^t (t^\rho - u^\rho)^\Phi \varphi'(u) du\right\},$$

this is, we prove Eq.(6.13). For this end we need to integrate by parts the above expression with the choice $w = \varphi(u)$ and $dv = (t^\rho - u^\rho)^{\Phi-1} u^{\rho-1} du$, in order to obtain

$$\begin{aligned}\int_a^t (t^\rho - u^\rho)^{\Phi-1} u^{\rho-1} \varphi(u) du &= -\frac{\varphi(u)}{\rho\,\Phi}(t^\rho - u^\rho)^\Phi \Big|_a^t + \frac{1}{\rho\,\Phi}\int_a^t (t^\rho - u^\rho)^\Phi \varphi'(u) du \\ &= \frac{1}{\rho\,\Phi}\underbrace{\left\{\varphi(a)(t^\rho - a^\rho)^\Phi + \int_a^t (t^\rho - u^\rho)^\Phi \varphi'(u) du\right\}}_{B(t)}.\end{aligned}$$

## B.2 Proof Eq.(6.14)

Differentiating the expression $B(t)$ with the operator $\delta_\rho^n$ follows

$$\delta_\rho^n B(t) = \frac{\Gamma(\Phi+1)\,\rho^n}{\Gamma(\Phi-n+1)}\left[\varphi(a)(t^\rho - a^\rho)^{\Phi-n} + \int_a^t (t^\rho - u^\rho)^{\Phi-n}\,\varphi'(u) du\right], \tag{B.1}$$

where $\delta_\rho^n = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$. Note that,

$$\begin{aligned}&\left(t^{1-\rho}\frac{d}{dt}\right)\left[\varphi(a)(t^\rho - a^\rho)^\Phi + \int_a^t (t^\rho - \tau^\rho)^\Phi\,\varphi'(u) du\right] = \\ &\rho\,\varphi(a)\,\Phi(t^\rho - a^\rho)^{\Phi-1} + \rho\,\Phi\int_a^t (t^\rho - u^\rho)^{\Phi-1}\varphi'(u) du.\end{aligned} \tag{B.2}$$

The proof is based on mathematical induction over $n$. The Eq.(B.1) is valid for $n = 1$, this is, considering $n = 1$ in Eq.(B.1) we obtain Eq.(B.2). Suppose that Eq.(B.1) is valid for

$n = m$, i.e.,

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{m}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-u^{\rho})^{\Phi}\varphi'(u)du\right]=$$
$$\frac{\Gamma(\Phi+1)\,\rho^{m}}{\Gamma(\Phi-m+1)}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-\tau^{\rho})^{\Phi}\,\varphi'(u)du\right]. \tag{B.3}$$

We want to prove that

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{m+1}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-u^{\rho})^{\Phi}\varphi'(u)du\right]=$$
$$\frac{\Gamma(\Phi+1)\,\rho^{m+1}}{\Gamma(\Phi-m)}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi-m-1}+\int_a^t (t^{\rho}-\tau^{\rho})^{\Phi-m-1}\,\varphi'(u)du\right]. \tag{B.4}$$

Note that, we can write

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{m+1}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-u^{\rho})^{\Phi}\varphi'(u)du\right]=$$
$$\left(t^{1-\rho}\frac{d}{dt}\right)\underbrace{\left(t^{1-\rho}\frac{d}{dt}\right)^{m}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\alpha}+\int_a^t (t^{\rho}-\tau^{\rho})^{\alpha}\,\varphi'(\tau)d\tau\right]}_{\text{induction hypothesis, Eq.(B.3)}}$$

or

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{m+1}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-u^{\rho})^{\Phi}\varphi'(u)du\right]=$$
$$\left(t^{1-\rho}\frac{d}{dt}\right)\frac{\Gamma(\Phi+1)\,\rho^{m}}{\Gamma(\Phi-m+1)}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-\tau^{\rho})^{\Phi}\,\varphi'(u)du\right]=$$
$$t^{1-\rho}\frac{\Gamma(\Phi+1)\,\rho^{m}}{\Gamma(\Phi-m+1)}\left(\frac{d}{dt}\right)\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-\tau^{\rho})^{\Phi}\,\varphi'(u)du\right].$$

We finally find

$$\left(t^{1-\rho}\frac{d}{dt}\right)^{m+1}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi}+\int_a^t (t^{\rho}-u^{\rho})^{\Phi}\varphi'(u)du\right]=$$
$$\frac{\Gamma(\Phi+1)\,\rho^{m+1}}{\Gamma(\Phi-m+1)}\left[\varphi(a)(t^{\rho}-a^{\rho})^{\Phi-m}+\int_a^t (t^{\rho}-u^{\rho})^{\Phi-m}\,\varphi'(u)du\right].$$