UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

THIAGO GINEZ VELANGA MOREIRA

Multipolynomials between Banach spaces: designs of a theory

Multipolinômios entre espaços de Banach: esboços de uma teoria

Campinas

2019

Thiago Ginez Velanga Moreira

Multipolynomials between Banach spaces: designs of a theory

Multipolinômios entre espaços de Banach: esboços de uma teoria

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Supervisor: Marcelo Martins dos Santos

Co-supervisor: Daniel Marinho Pellegrino

Este exemplar corresponde à versão final da Tese defendida pelo aluno Thiago Ginez Velanga Moreira e orientada pelo Prof. Dr. Marcelo Martins dos Santos.

Campinas

2019

Agência(s) de fomento e nº(s) de processo(s): FAPERO, 01133100023-0000.41/2014; CAPES
ORCID: https://orcid.org/0000-0001-7788-5212

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

M813m Moreira, Thiago Ginez Velanga, 1983-
Multipolynomials between Banach spaces : designs of a theory / Thiago Ginez Velanga Moreira. – Campinas, SP : [s.n.], 2019.

Orientador: Marcelo Martins dos Santos.
Coorientador: Daniel Marinho Pellegrino.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Polinômios. 2. Banach, Espaços de. 3. Análise funcional. 4. Análise funcional não-linear. 5. Análise matemática. 6. Operadores absolutamente somantes. I. Santos, Marcelo Martins dos, 1961-. II. Pellegrino, Daniel Marinho. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações para Biblioteca Digital

Título em outro idioma: Multipolinômios entre espaços de Banach : esboços de uma teoria
Palavras-chave em inglês:
Polynomials
Banach spaces
Functional analysis
Nonlinear functional analysis
Mathematical analysis
Absolutely summing operators
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Marcelo Martins dos Santos
Geraldo Márcio de Azevedo Botelho
Sahibzada Waleed Noor
Nilson da Costa Bernardes Junior
Daniela Mariz Silva Vieira
Data de defesa: 15-02-2019
Programa de Pós-Graduação: Matemática

Tese de Doutorado defendida em 15 de fevereiro de 2019 e aprovada

pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). MARCELO MARTINS DOS SANTOS

Prof(a). Dr(a). GERALDO MÁRCIO DE AZEVEDO BOTELHO

Prof(a). Dr(a). SAHIBZADA WALEED NOOR

Prof(a). Dr(a). NILSON DA COSTA BERNARDES JUNIOR

Prof(a). Dr(a). DANIELA MARIZ SILVA VIEIRA

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Dedicated to the memory of Jorge Mujica (1946-2017).

Acknowledgements

Professor Daniel M. Pellegrino, from whom I wouldn't be capable of measuring how much I have learned. Professor Jorge Mujica, who even during a brief period of coexistence, will always be among my best references. IMECC-UNICAMP staff, professors, and friends which I had the pleasure to know and accompany.

I am directly indebted to the members of the Examination Committee for their essential contributions which have improved the quality of the text and the mathematical level of this thesis.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001; and Fundação de Amparo à Pesquisa do Estado de Rondônia - Brasil (FAPERO) - Grant no. 41/2016.

"Não vos amoldeis às estruturas deste mundo,
mas transformai-vos pela renovação da mente,
a fim de distinguir qual é a vontade de Deus:
o que é bom, o que Lhe é agradável, o que é perfeito."
(Bíblia Sagrada, Romanos 12, 2)

Resumo

O conteúdo desta tese se divide em duas partes. A primeira, compreendendo os capítulos 1 e 2, apresenta os fundamentos das clássicas teorias multilinear e polinomial. Além disso, exibe várias linhas de pesquisa dentro de tais contextos, que foram separadamente e isoladamente estudadas até o momento. A segunda parte, o capítulo 3, é projetada para oferecer uma abordagem generalizada e unificada dos tópicos da primeira parte. Espera-se que o método do capítulo 3 possa ser amplamente aplicado em investigações futuras onde alguma estrutura multilinear ou polinomial esteja envolvida.

Palavras-chave: espaço normado, espaço de Banach, aplicação multilinear, polinômio homogêneo, multipolinômio, ideal de operadores, hiper-ideal, coerência, compatibilidade, desigualdade de Bohnenblust–Hille, operador absolutamente somante, cotipo.

Abstract

The contents of this thesis are divided into two parts. The first, comprising chapters 1 and 2, presents the fundamentals of the multilinear and polynomial classical theories. Also, it exhibits diverse research lines within such settings, which have so far been studied separately and in isolation. The second part, chapter 3, is designed to offer an extended and unified approach of the topics of the first part. It is expected that the methods of chapter 3 may widely be applied in future investigations where a multilinear or polynomial framework is involved.

Keywords: normed space, Banach space, multilinear mapping, homogeneous polynomial, multipolynomial, operator ideal, hyper-ideal, coherence, compatibility, Bohnenblust–Hille inequality, absolutely summing operator, cotype.

Contents

Introduction

Linear Functional Analysis emerged in the '30s after the publication of Stefan Banach's monograph (BANACH, 1932). The investigation of polynomials and multilinear mappings between Banach spaces is the first natural step when moving from linear to Nonlinear Functional Analysis. In this sense, polynomials and multilinear mappings have been exhaustively examined by numerous different viewpoints.

Overall, research lines in multilinear and polynomial scenarios are independent and, albeit very close to each other, have some subtle differences. An effort in unifying them seems to be an exciting task. A key to such a purpose is the concept of multipolynomial. This notion seems to have been first explored by I. Chernega and A. Zagorodnyuk in (CHERNEGA; ZAGORODNYUK, 2009) (with different terminology). It is a somewhat natural extension of the notions of multilinear mapping and homogeneous polynomial. Recall that a map $A : E_1 \times \cdots \times E_m \to F$ is m-linear if it is linear in each variable; now, a map $P : E_1 \times \cdots \times E_m \to F$ is called an $(n_1, \ldots, n_m)$-homogeneous polynomial if it is n_j-homogeneous in the respective coordinate $(1 \leqslant j \leqslant m)$. When $n_1 = \cdots = n_m = 1$ one recovers the notion of an m-linear mapping, and when $m = 1$ one recovers the notion of an n_1-homogeneous polynomial. Thus, multipolynomials encompass the notions of multilinear mapping and homogeneous polynomial as "extreme" cases.

Chapter 1 recalls the basics of the multilinear and polynomial classical theories. It is translated to multipolynomials in section 3.1, with the advantage of bringing its unity. Mostly in these parts, just normed spaces are required but warning the reader to the use of Banach spaces whenever necessary. Chapter 2 exhibits several research lines, within the multilinear and polynomial settings, which have been studied in separated ways so far. Finally, chapter 3 extends and unifies that whole previous theory to multipolynomials. Still in this chapter, section 3.6 pushes the summing theory further and generalizes earlier works concerning absolutely summing multilinear/polynomial mappings in Banach spaces with unconditional Schauder basis.

We may say that chapter 3 mostly draws theory sketches which emerge to future investigations, and it suggests, as A. Pietsch once did (PIETSCH, 1984), a *modus vivendi.*

1 Preliminaries

This chapter is devoted to briefly recalling some results from the basic theory of multilinear mappings and homogeneous polynomials that will be useful in this thesis. One can find a complete explanation in the classical books (MUJICA, 1986) and (DINEEN, 1999).

Throughout the whole thesis the letter $\mathbb{K}$ will stand either for the field $\mathbb{R}$ of all real numbers or for the field $\mathbb{C}$ of all complex numbers. $\mathbb{N}$ will denote the set of all strictly positive integers, whereas the set $\mathbb{N} \cup \{0\}$ will be indicated by $\mathbb{N}_0$. When dealing with basic theories (we mean, the present chapter and section 3.1), the letters $E, E_1, \ldots, E_m$, and F shall represent normed spaces over the same field $\mathbb{K}$. In all other parts of the text, unless stated otherwise, they will always denote Banach spaces.

1.1 Multilinear mappings

For each $m \in \mathbb{N}$, we recall that a mapping $A : E_1 \times \cdots \times E_m \to F$ is said to be *m-linear* if, for each $1 \leqslant j \leqslant m$, the mapping

$$A\left(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_m\right) : E_j \to F$$

is linear for all fixed $x_i \in E_i$ with $i \neq j$.

For each $m \in \mathbb{N}$, we shall denote by $\mathcal{L}_a(E_1, \ldots, E_m; F)$ the vector space of all m-linear mappings from the cartesian product $E_1 \times \cdots \times E_m$ into F, whereas we shall denote by $\mathcal{L}(E_1, \ldots, E_m; F)$ the subspace of all continuous members of $\mathcal{L}_a(E_1, \ldots, E_m; F)$. For each $A \in \mathcal{L}_a(E_1, \ldots, E_m; F)$ we define

$$\|A\| := \sup\left\{\|A\left(x_1, \ldots, x_m\right)\| : x_j \in E_j, \max_j \|x_j\| \leqslant 1\right\}.$$

When $F = \mathbb{K}$ we shall write $\mathcal{L}_a(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}_a(E_1, \ldots, E_m)$ and $\mathcal{L}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m)$.

Similarly, when $E_1 = \cdots = E_m = E$ we shall write $\mathcal{L}_a(^m E; F)$ and $\mathcal{L}(^m E; F)$. In this case, we shall denote by $\mathcal{L}_a^s(^m E; F)$ the subspace of all $A \in \mathcal{L}_a(^m E; F)$ which are symmetric. When $m = 1$, as usual, we shall write $\mathcal{L}_a(^1 E; F) = \mathcal{L}_a(E; F)$ and $\mathcal{L}(^1 E; F) = \mathcal{L}(E; F)$. When $F = \mathbb{K}$ then, for short, we shall write $\mathcal{L}_a(^m E; \mathbb{K}) = \mathcal{L}_a(^m E)$, $\mathcal{L}(^m E; \mathbb{K}) = \mathcal{L}(^m E)$, $\mathcal{L}_a^s(^m E; \mathbb{K}) = \mathcal{L}_a^s(^m E)$, etc. Finally, when $m = 1$ and $F = \mathbb{K}$ we shall write $\mathcal{L}(E) = E'$.

Proposition 1.1.1. *For each $A \in \mathcal{L}_a(E_1, \ldots, E_m; F)$ the following conditions are equivalent:*

(i) *A is continuous;*

(ii) *A is continuous at the origin;*

(iii) *There exists a constant $c \geqslant 0$ such that*

$$\|A(x_1, \ldots, x_m)\| \leqslant c\,\|x_1\| \cdots \|x_m\|,$$

for all $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$;

(iv) *$\|A\| < \infty$;*

(v) *A is uniformly continuous on bounded subsets of $E_1 \times \cdots \times E_m$;*

(vi) *A is bounded on every ball with finite radius;*

(vii) *A is bounded on some ball;*

(viii) *A is bounded on some ball with center at the origin.*

For each $A \in \mathcal{L}(E_1, \ldots, E_m; F)$, we have the straightforward properties:

- $\|A(x_1, \ldots, x_m)\| \leqslant \|A\|\,\|x_1\| \cdots \|x_m\|$, for all $x_j \in E_j$, $j = 1, \ldots, m$;
- $\|A\| = \inf\{c \geqslant 0 : \|A(x_1, \ldots, x_m)\| \leqslant c\,\|x_1\| \cdots \|x_m\|, \forall x_j \in E_j, j = 1, \ldots, m\}$.

One may increase the list in Proposition 1.1.1 provided we add Banach spaces in the assumptions.

Proposition 1.1.2. *If $E_1, \ldots, E_m$ are Banach spaces, then $A \in \mathcal{L}_a(E_1, \ldots, E_m; F)$ is continuous if and only if A is separately continuous in each variable.*

One can readily see that if (A_j) is a sequence in $\mathcal{L}_a(E_1, \ldots, E_m; F)$ such that the limit $A(x) = \lim_{j\to\infty} A_j(x)$ exists for every $x = (x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$, then $A \in \mathcal{L}_a(E_1, \ldots, E_m; F)$. If $E_1 = \cdots = E_m$ and each A_j is symmetric, then A is symmetric. If $E_1, \ldots, E_m$ are Banach spaces and each A_j is continuous, then A is continuous as well.

Proposition 1.1.3. *If F is a Banach space, then $\mathcal{L}(E_1, \ldots, E_m; F)$ is a Banach space under the norm $A \mapsto \|A\|$.*

The *Uniform Boundedness Principle* (UBP), as well as its corollary *Banach–Steinhaus Theorem* (BST), can be naturally extended to m-linear mappings as follows:

Theorem 1.1.4 (Uniform Boundedness Principle)**.** *Let $E_1, \ldots, E_m$ be Banach spaces, F be a normed space and let $\{A_i\}_{i\in I}$ be a family in $\mathcal{L}(E_1, \ldots, E_m; F)$. The following conditions are equivalent:*

(i) *For every $x = (x_1, \dots, x_m) \in E_1 \times \cdots \times E_m$ there exists $C_x < \infty$ such that*

$$\sup_{i \in I} \|A_i(x)\| < C_x;$$

(ii) *The family $\{A_i\}_{i \in I}$ is norm bounded, that is,*

$$\sup_{i \in I} \|A_i\| < \infty.$$

Corollary 1.1.5 (Banach–Steinhaus Theorem)**.** *Let $E_1, \dots, E_m$ be Banach spaces, F be a normed space and let (A_j) be a sequence in $\mathcal{L}(E_1, \dots, E_m; F)$ such that $(A_j(x_1, \dots, x_m))$ is convergent in F for every $(x_1, \dots, x_m) \in E_1 \times \cdots \times E_m$. If we define*

$$A : E_1 \times \cdots \times E_m \to F$$

by

$$A(x_1, \dots, x_m) := \lim_{j \to \infty} A_j(x_1, \dots, x_m),$$

then $A \in \mathcal{L}(E_1, \dots, E_m; F)$.

If, in addition, F in BST-hypotheses is complete, then (A_j) converges to A uniformly on compact subsets of $E_1 \times \cdots \times E_m$.

Remark 1.1.6. *We refer to (SANDBERG, 1985) and (BERNARDINO, 2009) as a couple of references to the multilinear UBP and BST. The first one contains a little bit unnatural proof and uses the linear UBP. The last one presents a quite simple argument which does not need to invoke the linear UBP and, when $m = 1$, recovers the classical proof of the linear case.*

For each $n \in \mathbb{N}$ and each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!.$$

Let $A \in \mathcal{L}_a(^m E; F)$. Then for each $(x_1, \dots, x_n) \in E^n$ and each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = m$ we write

$$A x_1^{\alpha_1} \cdots x_n^{\alpha_n} = A(\underbrace{x_1, \dots, x_1}_{\alpha_1}, \dots, \underbrace{x_n, \dots, x_n}_{\alpha_n}).$$

- (Leibniz Formula) If $A \in \mathcal{L}_a^s(^m E; F)$, then for all $x_1, \dots, x_n \in E$ we have

  $$A(x_1 + \cdots + x_n)^m = \sum \frac{m!}{\alpha!} A x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

  where the summation is taken over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ such that $|\alpha| = m$.

- (Polarization Formula) If $A \in \mathcal{L}_a^s(^mE;F)$, then for all $x_0, \ldots, x_m \in E$ we have

$$A(x_1, \ldots, x_m) = \frac{1}{m!2^m} \sum_{\varepsilon_k = \pm 1} \varepsilon_1 \cdots \varepsilon_m A(x_0 + \varepsilon_1 x_1 + \cdots + \varepsilon_m x_m)^m .$$

Definition 1.1.7. *Given $A \in \mathcal{L}_a(^mE)$ and $B \in \mathcal{L}_a(^nE)$ their tensor product $A \otimes B \in \mathcal{L}_a(^{m+n}E)$ is defined by*

$$(A \otimes B)(x_1, \ldots, x_{m+n}) = A(x_1, \ldots, x_m)B(x_{m+1}, \ldots, x_{m+n}), \quad \forall\, x_1, \ldots, x_{m+n} \in E.$$

If A and B are continuous, then it is clear that $A \otimes B$ is continuous as well. When E is finite dimensional, another well-known formula comes into play:

- Let $\{e_1, \ldots, e_d\}$ be a basis for E and let $\xi_1, \ldots, \xi_d$ denote the corresponding coordinate functionals. Then each $A \in \mathcal{L}_a(^mE;F)$ can be uniquely represented as a sum

$$A = \sum_{j_1, \ldots, j_m = 1}^{d} c_{j_1 \cdots j_m} \xi_{j_1} \otimes \cdots \otimes \xi_{j_m} \tag{1.1}$$

  where each $c_{j_1 \cdots j_m} \in F$. We conclude that $\mathcal{L}_a(^mE;F) = \mathcal{L}(^mE;F)$.

1.2 Homogeneous polynomials

We recall that if E and F are vector spaces, a map $P : E \to F$ is called an *m-homogeneous polynomial* if there exists an m-linear mapping

$$A : E^m \to F$$

such that

$$P(x) = A(x, \ldots, x)$$

for every $x \in E$. The vector space of all m-homogeneous polynomials from E into F is denoted by $\mathcal{P}_a(^mE;F)$. We shall represent by $\mathcal{P}(^mE;F)$ the subspace of all continuous members of $\mathcal{P}_a(^mE;F)$. For each $P \in \mathcal{P}_a(^mE;F)$ we define

$$\|P\| := \sup\{\|P(x)\| : x \in E, \|x\| \leqslant 1\}.$$

When $F = \mathbb{K}$ then, for short, we shall write $\mathcal{P}_a(^mE;\mathbb{K}) = \mathcal{P}_a(^mE)$ and $\mathcal{P}(^mE;\mathbb{K}) = \mathcal{P}(^mE)$.

Theorem 1.2.1. *For each $A \in \mathcal{L}_a(^mE;F)$ let $\widehat{A} \in \mathcal{P}_a(^mE;F)$ be defined by $\widehat{A}(x) = Ax^m$ for every $x \in E$. Then:*

(i) *The mapping*

$$\wedge : \mathcal{L}_a^s(^mE;F) \to \mathcal{P}_a(^mE;F)$$

is a linear isomorphism. We denote the inverse of this mapping by $^\vee$.

(ii) *We have the inequalities*

$$\left\|\widehat{A}\right\| \leqslant \|A\| \leqslant \frac{m^m}{m!}\left\|\widehat{A}\right\|, \tag{1.2}$$

for every $A \in \mathcal{L}^s(^mE;F)$.

In particular, the mapping $A \mapsto \widehat{A}$ induces a topological isomorphism between $\mathcal{L}^s(^mE;F)$ and $\mathcal{P}(^mE;F)$. Its inverse is again denoted by $^\vee$. Since $\mathcal{L}^s(^mE;F)$ is a closed subspace of $\mathcal{L}(^mE;F)$, it follows from Proposition 1.1.3 and the isomorphism above that

Corollary 1.2.2. *If* F *is a Banach space, then* $\mathcal{P}(^mE;F)$ *is a Banach space under the norm* $P \mapsto \|P\|$.

The next lemma is useful in characterizing continuous polynomials.

Lemma 1.2.3. *Let* $P \in \mathcal{P}_a(^mE;F)$. *If* P *is bounded by* c *on an open ball* $B(a;r)$, *then* P *is bounded by* $cm^m/m!$ *on the ball* $B(0;r)$.

Proposition 1.2.4. *For each* $P \in \mathcal{P}_a(^mE;F)$ *the following conditions are equivalent:*

(i) P *is continuous;*

(ii) P *is continuous at the origin;*

(iii) *There exists a constant* $c \geqslant 0$ *such that*

$$\|P(x)\| \leqslant c\|x\|^m,$$

for all $x \in E$;

(iv) $\|P\| < \infty$;

(v) P *is uniformly continuous on bounded subsets of* E;

(vi) P *is bounded on every ball with finite radius;*

(vii) P *is bounded on some ball;*

(viii) P *is bounded on some ball with center at the origin.*

For each $P \in \mathcal{P}(^mE;F)$, we have the straightforward properties:

- $\|P(x)\| \leqslant \|P\|\,\|x\|^m$, for every $x \in E$;
- $\|P\| = \inf\{c \geqslant 0 : \|P(x)\| \leqslant c\|x\|^m, \forall x \in E\}$.

Before we present the polynomial versions of UBP and BST, it is convenient noting that if (P_j) is a sequence in $\mathcal{P}_a({}^mE;F)$ such that the limit $P(x) = \lim_{j\to\infty} P_j(x)$ exists for every $x \in E$, then $P \in \mathcal{P}_a({}^mE;F)$.

Next lemma is helpful in proving UBP future versions.

Lemma 1.2.5. *Let U be an open subset of E, and let $\{f_i\}_{i\in I}$ be a family of continuous mappings from U into F. If the family $\{f_i\}_{i\in I}$ is pointwise bounded on U, then there is an open set $V \subset U$ where the family $\{f_i\}_{i\in I}$ is uniformly bounded.*

Theorem 1.2.6 (Uniform Boundedness Principle)**.** *Let E be a Banach space, F be a normed space and let $\{P_i\}_{i\in I}$ be a family in $\mathcal{P}({}^mE;F)$. The following conditions are equivalent:*

(i) *For every $x \in E$ there exists $C_x < \infty$ such that*

$$\sup_{i\in I} \|P_i(x)\| < C_x;$$

(ii) *The family $\{P_i\}_{i\in I}$ is norm bounded, that is,*

$$\sup_{i\in I} \|P_i\| < \infty.$$

Corollary 1.2.7 (Banach–Steinhaus Theorem)**.** *Let E be a Banach space, F be a normed space and let (P_j) be a sequence in $\mathcal{P}({}^mE;F)$ such that $(P_j(x))$ is convergent in F for every $x \in E$. If we define*

$$P : E \to F$$

by

$$P(x) := \lim_{j\to\infty} P_j(x),$$

then $P \in \mathcal{P}({}^mE;F)$.

If, in addition, F in BST-hypotheses is complete, then (P_j) converges to P uniformly on compact subsets of E.

If E is finite dimensional, let $\{e_1, \ldots, e_d\}$ be a basis for E and let $\xi_1, \ldots, \xi_d$ denote the corresponding coordinate functionals. Then each $P \in \mathcal{P}_a({}^mE;F)$ can be uniquely represented as a sum

$$P = \sum c_\alpha \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}, \tag{1.3}$$

where each $c_\alpha \in F$ and where the summation is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ such that $|\alpha| = m$. We conclude that $\mathcal{P}_a({}^mE;F) = \mathcal{P}({}^mE;F)$.

2 Multilinear mappings versus homogeneous polynomials

Each section of this chapter introduces a well-known research line in the multilinear and polynomial settings. We shall present them separately and often in that order.

2.1 Ideals

Polynomials and multilinear mappings have been exhaustively investigated by quite different viewpoints. While polynomials are suitable to the investigation of holomorphic mappings, multilinear mappings are commonly explored in the context of the extension of the operator ideals' theory to the nonlinear setting. The notion of ideals of linear operators between Banach spaces is due to Albrecht Pietsch (PIETSCH, 1978). The natural extension to multilinear mappings and polynomials was designed by Pietsch some years later in (PIETSCH, 1984). Nowadays, ideals of polynomials and multilinear mappings are explored by several authors in many and diverse directions (see, for instance, (ACHOUR et al., 2016; BERRIOS; BOTELHO, 2016; BERTOLOTO; BOTELHO; JATOBÁ, 2015; BOTELHO; PELLEGRINO; RUEDA, 2007; BOTELHO; PELLEGRINO, 2006b; BOTELHO et al., 2006; BOTELHO; PELLEGRINO, 2005; FLORET; GARCÍA, 2003)). In this section, we shall parallelly confront the basics of such (apparently separated) ideals' theories.

Recall that a continuous linear operator $u : E \to F$ is said to have *finite rank* if $\dim u(E) < \infty$. One can readily see that an operator $u \in \mathcal{L}(E;F)$ has finite rank if, and only if, there exist $\varphi_1, \ldots, \varphi_n \in E'$ and $b_1, \ldots, b_n \in F$ such that

$$u(x) = \sum_{i=1}^{n} \varphi_i(x) b_i,$$

for every $x \in E$.

Let us begin by recalling the classical definition of linear operator ideal.

Definition 2.1.1 ((PIETSCH, 1978))**.** *An **operator ideal** is a class $\mathcal{I}$ of continuous linear operators between Banach spaces such that for all Banach space E and F, its components*

$$\mathcal{I}(E;F) := \mathcal{L}(E;F) \cap \mathcal{I}$$

satisfy:

(Oa) *$\mathcal{I}(E;F)$ is a linear subspace of $\mathcal{L}(E;F)$ which contains the finite rank operators;*

(Ob) *The ideal property: if $u \in \mathcal{I}(E;F)$, $v \in \mathcal{L}(G;E)$, and $t \in \mathcal{L}(F;H)$, then*

$$t \circ u \circ v \in \mathcal{I}(G;H).$$

Moreover, $\mathcal{I}$ is said to be a ***(quasi-) normed operator ideal*** *if there exists a map $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \to [0,\infty)$ satisfying:*

(O1) *$\|\cdot\|_{\mathcal{I}}$ restricted to $\mathcal{I}(E;F)$ is a (quasi-) norm, for all Banach spaces E and F;*

(O2) *$\|id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} : id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}} = 1$;*

(O3) *if $u \in \mathcal{I}(E;F)$, $v \in \mathcal{L}(G;E)$ and $t \in \mathcal{L}(F;H)$, then*

$$\|t \circ u \circ v\|_{\mathcal{I}} \leqslant \|t\| \, \|u\|_{\mathcal{I}} \, \|v\| .$$

When all the components $\mathcal{I}(E;F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{I}}$ above, then $\mathcal{I}$ is called a ***(quasi-) Banach operator ideal****.*

An operator ideal $\mathcal{I}$ is said to be *closed* if all components $\mathcal{I}(E;F)$ are closed subspaces of $(\mathcal{L}(E;F), \|\cdot\|)$, where $\|\cdot\|$ is the usual operator norm.

The theory of operator ideal is extensively presented in (PIETSCH, 1978). We now give a list of examples.

$\mathcal{L}$**:** Ideal of continuous operators;

$\mathcal{F}$**:** Ideal of finite rank operators;

$\overline{\mathcal{I}}$**:** The closure (with the usual operator norm) of an operator ideal $\mathcal{I}$;

$\mathcal{A}$**:** Ideal of approximable operators;

$\mathcal{V}$**:** Ideal of completely continuous operators;

$\mathcal{K}$**:** Ideal of compact operators;

$\mathcal{W}$**:** Ideal of weakly compact operators;

$\mathcal{N}_p$**:** Ideal of p-nuclear operators;

$\mathcal{I}_p$**:** Ideal of p-integral operators;

Π_p**:** Ideal of absolutely p-summing operators.

The operator ideals $(\mathcal{L}, \|\cdot\|)$, $(\overline{\mathcal{I}}, \|\cdot\|)$, $(\mathcal{A}, \|\cdot\|)$, $(\mathcal{V}, \|\cdot\|)$, $(\mathcal{K}, \|\cdot\|)$ and $(\mathcal{W}, \|\cdot\|)$ are closed (therefore, Banach). Further, for any fixed $1 \leqslant p < \infty$, $(\mathcal{N}_p, \nu_p)$, $(\mathcal{I}_p, \iota_p)$, and (Π_p, π_p) are Banach operator ideals; $(\mathcal{N}_1, \nu_1)$ is the smallest of all Banach operator ideals (see, also, (DIESTEL; JARCHOW; TONGE, 1995) for details).

A multilinear mapping $A \in \mathcal{L}(E_1, \ldots, E_m; F)$ is said to be of *finite type* if there exist $n \in \mathbb{N}$, $\varphi_{i1} \in E_1', \ldots, \varphi_{im} \in E_m'$ and $b_i \in F$ $(1 \leqslant i \leqslant n)$ such that

$$A(x_1, \ldots, x_m) = \sum_{i=1}^{n} \varphi_{i1}(x_1) \cdots \varphi_{im}(x_m) b_i,$$

for every $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$. We shall denote by $\mathcal{L}_f(E_1, \ldots, E_m; F)$ the subspace of all finite-type members of $\mathcal{L}(E_1, \ldots, E_m; F)$.

Definition 2.1.2 (see, e.g., (FLORET; GARCÍA, 2003))**.** *For each positive integer m, let $\mathcal{L}_m$ denote the class of all continuous m-linear mappings between Banach spaces. An **ideal of multilinear mappings** $\mathcal{M}$ is a subclass of the class $\mathcal{L} = \cup_{m=1}^{\infty} \mathcal{L}_m$ of all continuous multilinear mappings between Banach spaces such that for a positive integer m, Banach spaces $E_1, \ldots, E_m$ and F, the components*

$$\mathcal{M}(E_1, \ldots, E_m; F) := \mathcal{L}(E_1, \ldots, E_m; F) \cap \mathcal{M}$$

satisfy:

(Ma) *$\mathcal{M}(E_1, \ldots, E_m; F)$ is a linear subspace of $\mathcal{L}(E_1, \ldots, E_m; F)$ which contains the m-linear mappings of finite type;*

(Mb) *The ideal property: if $A \in \mathcal{M}(E_1, \ldots, E_m; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \ldots, m$, and $t \in \mathcal{L}(F; H)$, then*

$$t \circ A \circ (u_1, \ldots, u_m) \in \mathcal{M}(G_1, \ldots, G_m; H).$$

*Moreover, $\mathcal{M}$ is said to be a **(quasi-) normed ideal of multilinear mappings** if there exists a map $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to [0, \infty)$ satisfying:*

(M1) *$\|\cdot\|_{\mathcal{M}}$ restricted to $\mathcal{M}(E_1, \ldots, E_m; F)$ is a (quasi-) norm, for all $m \in \mathbb{N}$ and Banach spaces $E_1, \ldots, E_m$ and F;*

(M2) *$\|id_m : \mathbb{K}^m \to \mathbb{K} : id_m(\lambda_1, \ldots, \lambda_m) = \lambda_1 \cdots \lambda_m\|_{\mathcal{M}} = 1$, for all $m \in \mathbb{N}$;*

(M3) *If $A \in \mathcal{M}(E_1, \ldots, E_m; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \ldots, m$, and $t \in \mathcal{L}(F; H)$, then*

$$\|t \circ A \circ (u_1, \ldots, u_m)\|_{\mathcal{M}} \leqslant \|t\| \, \|A\|_{\mathcal{M}} \, \|u_1\| \cdots \|u_m\|.$$

When all the components $\mathcal{M}(E_1,\ldots,E_m;F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{M}}$ above, $\mathcal{M}$ is said to be a ***(quasi-) Banach ideal of multilinear mappings****. For a fixed ideal of multilinear mappings $\mathcal{M}$ and a positive integer $m\in\mathbb{N}$, the class*

$$\mathcal{M}_m := \bigcup_{E_1,\ldots,E_m,F} \mathcal{M}\left(E_1,\ldots,E_m;F\right)$$

is called an ***ideal of m-linear mappings****.*

An ideal of multilinear mappings $\mathcal{M}$ is said to be *closed* if all components $\mathcal{M}(E_1,\ldots,E_m;F)$ are closed subspaces of $(\mathcal{L}(E_1,\ldots,E_m;F),\|\cdot\|)$, where $\|\cdot\|$ is the usual multilinear norm.

As examples, we may cite some natural extensions of operator ideals.

$\mathcal{L}$: Ideal of continuous multilinear mappings;

$\mathcal{L}_f$: Ideal of finite-type multilinear mappings;

$\overline{\mathcal{M}}$: The closure (with the usual sup norm) of an ideal of multilinear mappings $\mathcal{M}$;

$\mathcal{L}_A$: Ideal of approximable multilinear mappings;

$\mathcal{L}_{as(p;q)}$: Ideal of absolutely $(p;q)$-summing multilinear mappings;

$\mathcal{L}_{ms(p;q)}$: Ideal of multiple $(p;q)$-summing multilinear mappings.

The ideals of multilinear mappings $(\mathcal{L},\|\cdot\|)$, $(\overline{\mathcal{M}},\|\cdot\|)$, and $(\mathcal{L}_A,\|\cdot\|)$ are closed (therefore, Banach). The classes $\mathcal{L}_{as(p;q)}$ and $\mathcal{L}_{ms(p;q)}$ are Banach ideals for $p\geqslant 1$, and p-Banach ideals for $0<p<1$ (see section 2.5 for more details).

A homogeneous polynomial $P\in\mathcal{P}(^mE;F)$ is said to be of *finite type* if there exist $\varphi_1,\ldots,\varphi_n\in E'$ and $b_1,\ldots,b_n\in F$ such that

$$P\left(x\right)=\sum_{i=1}^{n}\varphi_i\left(x\right)^m b_i,$$

for every $x\in E$. We shall denote by $\mathcal{P}_f(^mE;F)$ the subspace of all finite-type members of $\mathcal{P}(^mE;F)$.

Definition 2.1.3 (see, e.g., (FLORET, 2002))**.** *For each positive integer m, let $\mathcal{P}_m$ denote the class of all continuous m-homogeneous polynomials between Banach spaces. A* ***polynomial ideal*** *$\mathcal{Q}$ (or* ***ideal of homogeneous polynomials****) is a subclass of the class $\mathcal{P}=\cup_{m=1}^{\infty}\mathcal{P}_m$ of all continuous homogeneous polynomials between Banach spaces such that for all $m\in\mathbb{N}$ and all Banach spaces E and F, the components*

$$\mathcal{Q}\left(^mE;F\right) := \mathcal{P}\left(^mE;F\right)\cap\mathcal{Q}$$

satisfy:

(Pa) *$\mathcal{Q}(^mE;F)$ is a linear subspace of $\mathcal{P}(^mE;F)$ which contains the finite-type m-homogeneous polynomials;*

(Pb) *The ideal property: if $u \in \mathcal{L}(G;E)$, $P \in \mathcal{Q}(^mE;F)$, and $t \in \mathcal{L}(F;H)$, then*

$$t \circ P \circ u \in \mathcal{Q}\left(^mG;H\right).$$

Moreover, $\mathcal{Q}$ is said to be a ***(quasi-) normed polynomial ideal*** *if there exists a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \to [0,\infty)$ satisfying:*

(P1) *$\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}\left(^mE;F\right)$ is a (quasi-) norm, for all $m \in \mathbb{N}$ and all Banach spaces E and F;*

(P2) *$\|id_m : \mathbb{K} \to \mathbb{K} : id_m(\lambda) = \lambda^m\|_{\mathcal{Q}} = 1$, for all $m \in \mathbb{N}$;*

(P3) *If $u \in \mathcal{L}(G;E)$, $P \in \mathcal{Q}\left(^mE;F\right)$ and $t \in \mathcal{L}(F;H)$, then*

$$\|t \circ P \circ u\|_{\mathcal{Q}} \leqslant \|t\| \, \|P\|_{\mathcal{Q}} \, \|u\|^m .$$

When all the components $\mathcal{Q}(^mE;F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{Q}}$ above, then $\mathcal{Q}$ is called a ***(quasi-) Banach polynomial ideal****. For a fixed polynomial ideal $\mathcal{Q}$ and a positive integer $m \in \mathbb{N}$, the class*

$$\mathcal{Q}_m := \bigcup_{E,F} \mathcal{Q}\left(^mE;F\right)$$

is called an ***ideal of m-homogeneous polynomials****.*

An ideal of homogeneous polynomials $\mathcal{Q}$ is said to be *closed* if all components $\mathcal{Q}(^mE;F)$ are closed subspaces of $(\mathcal{P}(^mE;F), \|\cdot\|)$, where $\|\cdot\|$ is the usual polynomial norm.

We give some examples.

$\mathcal{P}$: Ideal of continuous homogeneous polynomials;

$\mathcal{P}_f$: Ideal of finite-type homogeneous polynomials;

$\overline{\mathcal{Q}}$: The closure (with the usual sup norm) of an ideal of homogeneous polynomials $\mathcal{Q}$;

$\mathcal{P}_A$: Ideal of approximable homogeneous polynomials;

$\mathcal{P}_{as(p;q)}$: Ideal of absolutely $(p;q)$-summing homogeneous polynomials;

The polynomial ideals $(\mathcal{P}, \|\cdot\|)$, $(\overline{\mathcal{Q}}, \|\cdot\|)$, and $(\mathcal{P}_A, \|\cdot\|)$ are closed (therefore, Banach). The class $\mathcal{P}_{as(p;q)}$ is a Banach ideal for $p \geqslant 1$, and p-Banach ideal for $0 < p < 1$ (see section 2.5 for more details).

2.2 Hyper-ideals and two-sided ideals

Recently, in the papers (BOTELHO; TORRES, 2015) and (BOTELHO; TORRES, 2016; BOTELHO; TORRES, 2018), the authors introduced and developed the notions of hyper-ideals of multilinear mappings and homogeneous polynomials between Banach spaces. While the well-studied concepts of multilinear-mapping ideal (multi-ideals), as well as polynomial ideal, relies on the composition with linear operators (the so-called ideal property), the notion proposed by the authors, called now as hyper-ideal property, considers in (BOTELHO; TORRES, 2015) composition with multilinear mappings and, under the polynomial viewpoint, considers in (BOTELHO; TORRES, 2016; BOTELHO; TORRES, 2018) composition with homogeneous polynomials. Historically speaking, the hyper-ideal property has already been studied individually for some specific classes, see, e.g., (DEFANT; POPA; SCHWARTING, 2010; POPA, 2012; POPA, 2014), and then (BOTELHO; TORRES, 2015; BOTELHO; TORRES, 2016; BOTELHO; TORRES, 2018) started the systematic study of the classes satisfying this stronger condition.

Definition 2.2.1 ((BOTELHO; TORRES, 2015))**.** *A **hyper-ideal of multilinear mappings** is a subclass $\mathcal{H}$ of the class of all continuous multilinear mappings between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces $E_1, \ldots, E_n$ and F, the components*

$$\mathcal{H}(E_1, \ldots, E_n; F) := \mathcal{L}(E_1, \ldots, E_n; F) \cap \mathcal{H}$$

satisfy:

(ha) *$\mathcal{H}(E_1, \ldots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ which contains the n-linear mappings of finite type;*

(hb) *The hyper-ideal property: given natural numbers n and $1 \leqslant m_1 < \cdots < m_n$ and Banach spaces $G_1, \ldots, G_{m_n}, E_1, \ldots, E_n, F$ and H, if $B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n), A \in \mathcal{H}(E_1, \ldots, E_n; F)$ and $t \in \mathcal{L}(F; H)$, then*

$$t \circ A \circ (B_1, \ldots, B_n) \in \mathcal{H}(G_1, \ldots, G_{m_n}; H).$$

*Moreover, $\mathcal{H}$ is said to be a **(quasi-) normed hyper-ideal of multilinear mappings** if there exists a map $\|\cdot\|_{\mathcal{H}} : \mathcal{H} \to [0, \infty)$ satisfying:*

(h1) *$\|\cdot\|_{\mathcal{H}}$ restricted to $\mathcal{H}(E_1, \ldots, E_n; F)$ is a (quasi-) norm, for all $n \in \mathbb{N}$ and all Banach spaces $E_1, \ldots, E_n$ and F;*

(h2) *$\|I_n : \mathbb{K}^n \to \mathbb{K}, I_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{\mathcal{H}} = 1$, for all $n \in \mathbb{N}$;*

(h3) *The hyper-ideal inequality: if $B_1 \in \mathcal{L}(G_1, \ldots, G_{m_1}; E_1), \ldots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \ldots, G_{m_n}; E_n), A \in \mathcal{H}(E_1, \ldots, E_n; F)$ and $t \in \mathcal{L}(F; H)$, then*

$$\|t \circ A \circ (B_1, \ldots, B_n)\|_{\mathcal{H}} \leqslant \|t\| \, \|A\|_{\mathcal{H}} \, \|B_1\| \cdots \|B_n\|.$$

When all the components $\mathcal{H}(E_1, \ldots, E_n; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{H}}$ above, then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is called a ***(quasi-) Banach hyper-ideal of multilinear mappings.***

It is plain that every (normed, quasi-normed, Banach, quasi-Banach) hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) multi-ideal.

Definition 2.2.2 ((BOTELHO; TORRES, 2016))**.** *A* ***polynomial hyper-ideal*** *is a subclass $\mathcal{Q}$ of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces E and F, the components*

$$\mathcal{Q}\left({}^{n}E; F\right) := \mathcal{P}\left({}^{n}E; F\right) \cap \mathcal{Q}$$

satisfy:

(pa) *$\mathcal{Q}({}^{n}E; F)$ is a linear subspace of $\mathcal{P}({}^{n}E; F)$ which contains the n-homogeneous polynomials of finite type;*

(pb) *The hyper-ideal property: given $m, n \in \mathbb{N}$ and Banach spaces E, F, G and H, if $Q \in \mathcal{P}({}^{m}G; E), P \in \mathcal{Q}({}^{n}E; F)$ and $t \in \mathcal{L}(F; H)$, then*

$$t \circ P \circ Q \in \mathcal{Q}\left({}^{mn}G; H\right).$$

If there exist a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \to [0, \infty)$ and a sequence $(C_j)_{j=1}^{\infty}$ of real numbers with $C_j \geqslant 1$ for every $j \in \mathbb{N}$ and $C_1 = 1$, such that:

(p1) *$\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}({}^{n}E; F)$ is a (quasi-) norm, for all $n \in \mathbb{N}$ and all Banach spaces E and F;*

(p2) *$\|I_n : \mathbb{K} \to \mathbb{K}, I_n(\lambda) = \lambda^n\|_{\mathcal{Q}} = 1$, for all $n \in \mathbb{N}$;*

(p3) *The hyper-ideal inequality: if $Q \in \mathcal{P}({}^{m}G; E), P \in \mathcal{Q}({}^{n}E; F)$ and $t \in \mathcal{L}(F; H)$, then*

$$\|t \circ P \circ Q\|_{\mathcal{Q}} \leqslant C_m^n \|t\| \|P\|_{\mathcal{Q}} \|Q\|^n,$$

then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a ***(quasi-) normed polynomial*** *$(C_j)_{j=1}^{\infty}$**-hyper-ideal**. When all the components $\mathcal{Q}({}^{n}E; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{Q}}$ above, then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a* ***(quasi-) Banach polynomial*** *$(C_j)_{j=1}^{\infty}$**-hyper-ideal**.*

When $C_j = 1$ for every $j \in \mathbb{N}$, we simply say that $\mathcal{Q}$ is a *(quasi-) normed/ (quasi-) Banach polynomial hyper-ideal.* When the hyper-ideal property (and inequality) holds for every $n \in \mathbb{N}$, but only for $m = 1$, we recover the concept of (quasi-) normed/ (quasi-) Banach polynomial ideal (remember that $C_1 = 1$).

Definition 2.2.3 ((BOTELHO; TORRES, 2018)). *A **polynomial two-sided ideal** is a subclass $\mathcal{Q}$ of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces E and F, the components*

$$\mathcal{Q}\left({}^{n}E;F\right) := \mathcal{P}\left({}^{n}E;F\right) \cap \mathcal{Q}$$

satisfy:

(ts-a) *$\mathcal{Q}({}^{n}E;F)$ is a linear subspace of $\mathcal{P}({}^{n}E;F)$ which contains the n-homogeneous polynomials of finite type;*

(ts-b) *The two-sided ideal property: given $m,n,r \in \mathbb{N}$ and Banach spaces E,F,G and H, if $Q \in \mathcal{P}({}^{m}G;E), P \in \mathcal{Q}({}^{n}E;F)$ and $R \in \mathcal{P}({}^{r}F;H)$, then*

$$R \circ P \circ Q \in \mathcal{Q}\left({}^{mnr}G;H\right).$$

If there exist a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \to [0,\infty)$ and a sequence $(C_j,K_j)_{j=1}^{\infty}$ of pairs of real numbers with $C_j, K_j \geqslant 1$ for every $j \in \mathbb{N}$ and $C_1 = K_1 = 1$, such that:

(ts-1) *$\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}({}^{n}E;F)$ is a (quasi-) norm, for all $n \in \mathbb{N}$ and all Banach spaces E and F;*

(ts-2) *$\|I_n : \mathbb{K} \to \mathbb{K}, I_n(\lambda) = \lambda^n\|_{\mathcal{Q}} = 1$, for all $n \in \mathbb{N}$;*

(ts-3) *The two-sided ideal inequality: if $Q \in \mathcal{P}({}^{m}G;E), P \in \mathcal{Q}({}^{n}E;F)$ and $R \in \mathcal{P}({}^{r}F;H)$, then*

$$\|R \circ P \circ Q\|_{\mathcal{Q}} \leqslant K_r C_m^{rn} \|R\| \|P\|_{\mathcal{Q}}^{r} \|Q\|^{rn},$$

*then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a **(quasi-) normed polynomial** $(C_j,K_j)_{j=1}^{\infty}$**-two-sided ideal**. When all the components $\mathcal{Q}({}^{n}E;F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{Q}}$ above, then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a **(quasi-) Banach polynomial** $(C_j,K_j)_{j=1}^{\infty}$**-two-sided ideal**.*

When $C_j = K_j = 1$ for every $j \in \mathbb{N}$, we simply say that $\mathcal{Q}$ is a *(quasi-) normed/(quasi-) Banach polynomial two-sided ideal*. The condition $C_1 = K_1 = 1$ guarantees that every (normed, quasi-normed, Banach, quasi-Banach) polynomial $(C_j,K_j)_{j=1}^{\infty}$-two-sided ideal is a (normed, quasi-normed, Banach, quasi-Banach) polynomial $(C_j)_{j=1}^{\infty}$-hyper-ideal; and that, as we mentioned before, every (normed, quasi-normed, Banach, quasi-Banach) polynomial $(C_j)_{j=1}^{\infty}$-hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) polynomial ideal.

2.3 Coherence and compatibility

The extension of an operator ideal to the multilinear and polynomial settings is not always a trivial question. For example, the absolutely-summing operator ideal has at least eight possible extensions to higher degrees (see, for example, (BOTELHO; PELLEGRINO; RUEDA, 2007; CALISKAN; PELLEGRINO, 2007; DIMANT, 2003; MATOS, 2003a; MATOS, 2003b; PELLEGRINO; SANTOS, 2011; PELLEGRINO; SANTOS; SEOANE-SEPÚLVEDA, 2012; PÉREZ-GARCÍA, 2005) and references therein). The almost-summing operator ideal is another example which has several different possible extensions to the setting of multilinear and polynomial ideals (BOTELHO; BRAUNSS; JUNEK, 2001; PELLEGRINO, 2003b; PELLEGRINO; RIBEIRO, 2012). Motivated by questions about finding a more suitable and less artificial extension of a given operator ideal, being able to preserve its main properties and essence, several concepts like ideal closed for scalar multiplication (*csm*) and ideal closed under differentiation (*cud*) were first introduced in (BOTELHO; PELLEGRINO, 2005) (see also (BOTELHO et al., 2006) for related notions). With the same aim of filtering good polynomial extensions of a given operator ideal, and exclusively directed toward polynomial ideals, D. Carando et al. introduced the notions of coherent sequence and compatible ideal in (CARANDO; DIMANT; MURO, 2009) (see also in (CARANDO; DIMANT; MURO, 2012a; CARANDO; DIMANT; MURO, 2012b)). In this section, we are mainly interested in these last concepts. We recall them after fixing the following notation:

- Remember that if $P \in \mathcal{P}(^m E; F)$, then $\check{P}$ denotes the unique symmetric m-linear mapping associated to P.
- If $P \in \mathcal{P}(^m E; F)$ and $a \in E$, then P_{a^k} is the $(m-k)$-homogeneous polynomial in $\mathcal{P}(^{m-k} E; F)$ defined by

$$P_{a^k}(x) := \check{P}(a, \ldots, a, x, \ldots, x).$$

Next, we recall the definitions of coherent and compatible polynomial ideals (our notation essentially follows (CARANDO; DIMANT; MURO, 2009)). Nevertheless, it is convenient to be aware that such notions are just the notions of ideals *cud* and *csm* (with different terminology) from (BOTELHO; PELLEGRINO, 2005).

Definition 2.3.1 (Compatible polynomial ideals (CARANDO; DIMANT; MURO, 2009))**.** *Let $\mathcal{I}$ be a normed ideal of linear operators. The normed ideal of n-homogeneous polynomials $\mathcal{U}_n$ is **compatible** with $\mathcal{I}$ if there exist positive constants α_1 and α_2 such that for all Banach spaces E and F, the following conditions hold:*

(cp 1) *For each $P \in \mathcal{U}_n(^n E; F)$ and $a \in E$, the mapping $P_{a^{n-1}}$ belongs to $\mathcal{I}(E; F)$ and*

$$\|P_{a^{n-1}}\|_{\mathcal{I}} \leqslant \alpha_1 \|P\|_{\mathcal{U}_n} \|a\|^{n-1}.$$

(cp 2) *For each $P \in \mathcal{I}(E;F)$ and $\gamma \in E'$, the mapping $\gamma^{n-1}P$ belongs to $\mathcal{U}_n({}^nE;F)$ and*

$$\left\|\gamma^{n-1}P\right\|_{\mathcal{U}_n} \leqslant \alpha_2 \left\|\gamma\right\|^{n-1} \left\|P\right\|_{\mathcal{I}}.$$

Definition 2.3.2 (Coherent polynomial ideals (CARANDO; DIMANT; MURO, 2009))**.** *Consider a sequence $(\mathcal{U}_k)_{k=1}^N$, where for each k, $\mathcal{U}_k$ is a normed ideal of k-homogeneous polynomials and N is eventually infinite. The sequence $(\mathcal{U}_k)_{k=1}^N$ is a **coherent sequence of polynomial ideals** if there exist positive constants β_1 and β_2 such that for all Banach spaces E and F, the following conditions hold for $k = 1, \ldots, N-1$:*

(ch 1) *For each $P \in \mathcal{U}_{k+1}({}^{k+1}E;F)$ and $a \in E$, the mapping P_a belongs to $\mathcal{U}_k({}^kE;F)$ and*

$$\left\|P_a\right\|_{\mathcal{U}_k} \leqslant \beta_1 \left\|P\right\|_{\mathcal{U}_{k+1}} \left\|a\right\|.$$

(ch 2) *For each $P \in \mathcal{U}_k({}^kE;F)$ and $\gamma \in E'$, the mapping γP belongs to $\mathcal{U}_{k+1}({}^{k+1}E;F)$ and*

$$\left\|\gamma P\right\|_{\mathcal{U}_{k+1}} \leqslant \beta_2 \left\|\gamma\right\| \left\|P\right\|_{\mathcal{U}_k}.$$

The philosophy brought by the above concepts is about to be able to transit between different levels of homogeneity of a given polynomial ideal, preserving the interconnection and the spirit of the original level ($n = 1$). Motivated by the fact that an operator ideal $\mathcal{I}$ can always be extended (at least in an abstract sense) not only to polynomials but also to the multilinear settings (see (BOTELHO, 2005/06)), D. Pellegrino and J. Ribeiro (PELLEGRINO; RIBEIRO, 2014) proposed a significant new approach to coherence and compatibility which simultaneously deals with multilinear and polynomial ideals by considering pairs $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^{\infty}$, where $\mathcal{U}_k$ is a (quasi-) normed ideal of k-homogeneous polynomials and $\mathcal{M}_k$ is a (quasi-) normed ideal of k-linear mappings. In the next definitions, we recall how it was done (we mainly follow the notation in (PELLEGRINO; RIBEIRO, 2014)).

Definition 2.3.3 (Compatible pair of ideals (PELLEGRINO; RIBEIRO, 2014))**.** *Let $\mathcal{I}$ be a normed operator ideal and $N \in (\mathbb{N}\backslash\{1\}) \cup \{\infty\}$. A sequence $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{I}$, is **compatible** with $\mathcal{I}$ if there exist positive constants $\alpha_1, \alpha_2, \alpha_3$ such that for all Banach spaces $E, E_1, \ldots, E_n$ and F, the following conditions hold for all $n \in \{2, \ldots, N\}$:*

(cp-i) *For each $k \in \{1, \ldots, n\}$, $A \in \mathcal{M}_n(E_1, \ldots, E_n; F)$, and $a_j \in E_j$ for all $j \in \{1, \ldots, n\}\backslash\{k\}$, the mapping $A(a_1, \ldots, a_{k-1}, \cdot, a_{k+1}, \ldots, a_n)$ belongs to $\mathcal{I}(E_k;F)$ and*

$$\left\|A\left(a_1, \ldots, a_{k-1}, \cdot, a_{k+1}, \ldots, a_n\right)\right\|_{\mathcal{I}} \leqslant \alpha_1 \left\|A\right\|_{\mathcal{M}_n} \left\|a_1\right\| \cdots \left\|a_{k-1}\right\| \left\|a_{k+1}\right\| \cdots \left\|a_n\right\|.$$

(cp-ii) *For each $P \in \mathcal{U}_n({}^nE;F)$ and $a \in E$, the mapping $P_{a^{n-1}}$ belongs to $\mathcal{I}(E;F)$ and*

$$\left\|P_{a^{n-1}}\right\|_{\mathcal{I}} \leqslant \alpha_2 \max\left\{\left\|\check{P}\right\|_{\mathcal{M}_n}, \left\|P\right\|_{\mathcal{U}_n}\right\} \left\|a\right\|^{n-1}.$$

(cp-iii) *For each* $A \in \mathcal{I}(E_n; F)$ *and* $\gamma_j \in E_j'$ *for* $j = 1, \ldots, n-1$, *the mapping* $\gamma_1 \cdots \gamma_{n-1} A$ *belongs to* $\mathcal{M}_n(E_1, \ldots, E_n; F)$ *and*

$$\|\gamma_1 \cdots \gamma_{n-1} A\|_{\mathcal{M}_n} \leqslant \alpha_3 \|\gamma_1\| \cdots \|\gamma_{n-1}\| \|A\|_{\mathcal{I}}.$$

(cp-iv) *For each* $P \in \mathcal{I}(E; F)$ *and* $\gamma \in E'$, *the mapping* $\gamma^{n-1} P$ *belongs to* $\mathcal{U}_n({}^n E; F)$.

(cp-v) P *belongs to* $\mathcal{U}_n({}^n E; F)$ *if, and only if,* $\check{P}$ *belongs to* $\mathcal{M}_n({}^n E; F)$.

Definition 2.3.4 (Coherent pair of ideals (PELLEGRINO; RIBEIRO, 2014))**.** *Let* $\mathcal{I}$ *be a normed operator ideal and* $N \in \mathbb{N} \cup \{\infty\}$. *A sequence* $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, *with* $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{I}$, *is* ***coherent*** *if there exist positive constants* $\beta_1, \beta_2, \beta_3$ *such that for all Banach spaces* $E, E_1, \ldots, E_{k+1}$ *and* F, *the following conditions hold for all* $k = 1, \ldots, N-1$:

(ch-i) *For each* $A \in \mathcal{M}_{k+1}(E_1, \ldots, E_{k+1}; F)$ *and* $a_j \in E_j$ *for* $j = 1, \ldots, k+1$, *the mapping* $A(\cdot, \ldots, \cdot, a_j, \cdot, \ldots, \cdot)$ *belongs to* $\mathcal{M}_k(E_1, \ldots, E_{j-1}, E_{j+1}, \ldots E_{k+1}; F)$ *and*

$$\|A(\cdot, \ldots, \cdot, a_j, \cdot, \ldots, \cdot)\|_{\mathcal{M}_k} \leqslant \beta_1 \|A\|_{\mathcal{M}_{k+1}} \|a_j\|.$$

(ch-ii) *For each* $P \in \mathcal{U}_{k+1}({}^{k+1} E; F)$ *and* $a \in E$, *the mapping* P_a *belongs to* $\mathcal{U}_k({}^k E; F)$ *and*

$$\|P_a\|_{\mathcal{U}_k} \leqslant \beta_2 \max\left\{\left\|\check{P}\right\|_{\mathcal{M}_{k+1}}, \|P\|_{\mathcal{U}_{k+1}}\right\} \|a\|.$$

(ch-iii) *For each* $A \in \mathcal{M}_k(E_1, \ldots, E_k; F)$ *and* $\gamma \in E_{k+1}'$, *the mapping* γA *belongs to* $\mathcal{M}_{k+1}(E_1, \ldots, E_{k+1}; F)$ *and*

$$\|\gamma A\|_{\mathcal{M}_{k+1}} \leqslant \beta_3 \|\gamma\| \|A\|_{\mathcal{M}_k}.$$

(ch-iv) *For each* $P \in \mathcal{U}_k({}^k E; F)$ *and* $\gamma \in E'$, *the mapping* γP *belongs to* $\mathcal{U}_{k+1}({}^{k+1} E; F)$.

(ch-v) P *belongs to* $\mathcal{U}_k({}^k E; F)$ *if, and only if,* $\check{P}$ *belongs to* $\mathcal{M}_k({}^k E; F)$.

2.4 Bohnenblust–Hille inequalities

In this section, we present the famous Bohnenblust–Hille inequalities (BOHNENBLUST; HILLE, 1931) for homogeneous polynomials and multilinear forms. The theory of Bohnenblust–Hille inequalities has been exhaustively investigated in recent years (see, for instance (ALBUQUERQUE et al., 2014; BAYART; PELLEGRINO; SEOANE-SEPÚLVEDA, 2014; CARO; ALARCÓN; SERRANO-RODRÍGUEZ, 2017; ALARCÓN, 2013; PELLEGRINO; TEIXEIRA, 2018; SANTOS; VELANGA, 2017), and the references therein).

For each pair of sequences $x = (x_j)_{j=1}^{\infty}$ and $\alpha = (\alpha_j)_{j=1}^{\infty}$ in $\mathbb{K}$ and $\mathbb{N}_0$, respectively, such that $|\alpha| := \sum_{j=1}^{\infty} \alpha_j < \infty$, we shall write $x^{\alpha} := \prod_j x_j^{\alpha_j}$. Under such notation, it follows from Leibniz formula that each continuous m-homogeneous polynomial $P : c_0 \to \mathbb{K}$ can be uniquely represented as a sum

$$P(x) = \sum c_{\alpha}(P) x^{\alpha},$$

for every $x \in c_0$, where $c_{\alpha}(P) \in \mathbb{K}$ and where the summation is taken over all sequences α such that $|\alpha| = m$

The Bohnenblust–Hille inequality for homogeneous polynomials (BOHNENBLUST; HILLE, 1931) asserts that

Theorem 2.4.1 (Polynomial Bohnenblust–Hille inequality)**.** *Let m be a positive fixed integer. The following assertions are equivalent:*

(i) *There exists a constant $C_{\mathbb{K},m} \geqslant 1$ such that*

$$\left(\sum_{|\alpha|=m} |c_{\alpha}(P)|^p \right)^{\frac{1}{p}} \leqslant C_{\mathbb{K},m} \|P\|$$

for all continuous m-homogeneous polynomial $P : c_0 \to \mathbb{K}$;

(ii)

$$p \geqslant \frac{2m}{m+1}.$$

We also have the Bohnenblust–Hille inequality for multilinear forms (BOHNENBLUST; HILLE, 1931):

Theorem 2.4.2 (Multilinear Bohnenblust–Hille inequality)**.** *Let m be a positive fixed integer. The following assertions are equivalent:*

(i) *There exists a constant $C_{\mathbb{K},m} \geqslant 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} |A(e_{i_1},\ldots,e_{i_m})|^p \right)^{\frac{1}{p}} \leqslant C_{\mathbb{K},m} \|A\|$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$;

(ii)

$$p \geqslant \frac{2m}{m+1}.$$

2.5 Absolutely summing mappings

The basics of the linear theory of absolutely summing operators can be found in the classical book (DIESTEL; JARCHOW; TONGE, 1995). Its extension to the multilinear setting was sketched by Albrecht Pietsch in (PIETSCH, 1984) and it was rapidly developed thereafter in several nonlinear environments.

Let $0 < p < \infty$. The vector space of all sequences $(x_j)_{j=1}^{\infty}$ in E such that $\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty}\|x_j\|^p)^{1/p} < \infty$ will be denoted by $\ell_p(E)$. We will also denote by $\ell_p^w(E)$ the vector space formed by the sequences $(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty}$ in $\ell_p(\mathbb{K})$ for every continuous linear functional $\varphi : E \to \mathbb{K}$. The function $\|\cdot\|_{w,p}$ in $\ell_p^w(E)$ defined by $\|(x_j)_{j=1}^{\infty}\|_{w,p} = \sup_{\varphi \in B_{E'}}(\sum_{j=1}^{\infty}|\varphi(x_j)|^p)^{1/p}$ is a p-norm for $p < 1$, and a norm for $p \geqslant 1$. In any case, they are complete metrizable linear spaces. The case $p = \infty$ is the case of the bounded sequences and in $\ell_\infty(E)$ we use the sup norm.

Let us begin by recalling the notions of absolutely summing homogeneous polynomials and multilinear mappings. These notions date back to the works of A. Pietsch (PIETSCH, 1984) and Alencar-Matos (ALENCAR; MATOS, 1989).

Definition 2.5.1. *Let $0 < p, q, q_1, \ldots, q_m$. A continuous m-homogeneous polynomial $P : E \to F$ is **absolutely** $(p; q)$-**summing** (or $(p; q)$-**summing**) if $(P(x_j))_{j=1}^{\infty} \in \ell_p(F)$ for all $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$. A continuous m-linear mapping $A : E_1 \times \cdots \times E_m \to F$ is **absolutely** $(p; q_1, \ldots, q_m)$-**summing** (or $(p; q_1, \ldots, q_m)$-**summing**) if $(A(x_j^{(1)}, \ldots, x_j^{(m)}))_{j=1}^{\infty} \in \ell_p(F)$ for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$, $k = 1, \ldots, m$.*

The vector space of all absolutely $(p; q)$-summing m-homogeneous polynomials from E into F is denoted by $\mathcal{P}_{as(p;q)}(^mE; F)$ ($\mathcal{P}_{as(p;q)}(^mE)$ if $F = \mathbb{K}$). Analogously, the vector space of all absolutely $(p; q_1, \ldots, q_m)$-summing m-linear mappings from $E_1 \times \cdots \times E_m$ into F is denoted by $\mathcal{L}_{as(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F)$ ($\mathcal{L}_{as(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m)$ if $F = \mathbb{K}$). When $q_1 = \cdots = q_m = q$, we simply write $\mathcal{L}_{as(p;q)}(E_1, \ldots, E_m; F)$.

We have $\mathcal{P}_{as(p;q)}(^mE; F) = \{0\}$ (resp. $\mathcal{L}_{as(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F) = \{0\}$) if $1/p > m/q$ (resp. $1/p > 1/q_1 + \cdots + 1/q_m$). So, in order to avoid trivialities, we must suppose $p \geqslant q/m$ (resp. $1/p \leqslant 1/q_1 + \cdots + 1/q_m$) in the polynomial (resp. m-linear) case.

As in the linear case, we have characterization theorems by means of inequalities.

Theorem 2.5.2 ((MATOS, 1996))**.** *Let $P \in \mathcal{P}(^mE; F)$. The following statements are equivalent:*

(i) *P is absolutely $(p; q)$-summing;*

(ii) *There exists a constant* $C > 0$ *such that*

$$\left(\sum_{j=1}^{n} \| P(x_j) \|^p \right)^{\frac{1}{p}} \leqslant C \left\| (x_j)_{j=1}^{n} \right\|_{w,q}^{m}$$

for all $n \in \mathbb{N}$ *and* $x_j \in E$, $j = 1, \ldots, n$.

(iii) *There exists a constant* $C > 0$ *such that*

$$\left(\sum_{j=1}^{\infty} \| P(x_j) \|^p \right)^{\frac{1}{p}} \leqslant C \left\| (x_j)_{j=1}^{\infty} \right\|_{w,q}^{m} \tag{2.1}$$

for every $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$.

The infimum of the $C > 0$ for which inequality (2.1) always holds is denoted by $\| \cdot \|_{as(p;q)}$ and defines a norm (resp. p-norm) on $\mathcal{P}_{as(p;q)}(^m E; F)$ for the case $p \geqslant 1$ (resp. $p < 1$). In any case, we thus obtain complete topological metrizable spaces.

Theorem 2.5.3 ((MATOS, 1996))**.** *Let* $A \in \mathcal{L}(E_1, \ldots, E_m; F)$. *The following statements are equivalent:*

(i) *A is absolutely* $(p; q_1, \ldots, q_m)$*-summing;*

(ii) *There exists a constant* $C > 0$ *such that*

$$\left(\sum_{j=1}^{n} \left\| A\left(x_j^{(1)}, \ldots, x_j^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leqslant C \prod_{k=1}^{m} \left\| \left(x_j^{(k)} \right)_{j=1}^{n} \right\|_{w,q_k}$$

for all $n \in \mathbb{N}$ *and* $x_1^{(k)}, \ldots, x_n^{(k)} \in E_k$, $k = 1, \ldots, m$.

(iii) *There exists a constant* $C > 0$ *such that*

$$\left(\sum_{j=1}^{\infty} \left\| A\left(x_j^{(1)}, \ldots, x_j^{(m)} \right) \right\|^p \right)^{\frac{1}{p}} \leqslant C \prod_{k=1}^{m} \left\| \left(x_j^{(k)} \right)_{j=1}^{\infty} \right\|_{w,q_k} \tag{2.2}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$, $k = 1, \ldots, m$.

The infimum of the $C > 0$ for which inequality (2.2) always holds is denoted by $\| \cdot \|_{as(p;q_1,\ldots,q_m)}$ and defines a norm (resp. p-norm) on $\mathcal{L}_{as(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F)$ for the case $p \geqslant 1$ (resp. $p < 1$). In any case, we thus obtain complete topological metrizable spaces.

Next, we present a stronger concept. It was independently introduced in (MATOS, 2003a) and (PÉREZ-GARCÍA; VILLANUEVA, 2003).

Definition 2.5.4. *A continuous m-linear mapping $A : E_1 \times \cdots \times E_m \to F$ is said to be **multiple** $(p; q_1, \ldots, q_m)$-**summing** if there exists $C \geqslant 0$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left\| A\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right) \right\|^p \right)^{\frac{1}{p}} \leqslant C \prod_{k=1}^{m} \left\| \left(x_j^{(k)}\right)_{j=1}^{\infty} \right\|_{w,q_k} \tag{2.3}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$, $k = 1, \ldots, m$. In this case we write $\mathcal{L}_{ms(p;q_1,\ldots,q_m)}(E_1, \ldots$ $\ldots, E_m; F)$.

If $q_k > p$ for some $k = 1, \ldots, m$, we have $\mathcal{L}_{ms(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F) = \{0\}$. So, we must suppose $q_k \leqslant p$ for every $k = 1, \ldots, m$. The infimum of the $C > 0$ for which inequality (2.3) always holds is denoted by $\|\cdot\|_{ms(p;q_1,\ldots,q_m)}$ and defines a norm (resp. p-norm) on $\mathcal{L}_{ms(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F)$ for the case $p \geqslant 1$ (resp. $p < 1$). In any case, we thus obtain complete topological metrizable spaces.

One can see that

$$\mathcal{L}_{ms(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F) \subseteq \mathcal{L}_{as(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F)$$

and

$$\|A\|_{as(p;q_1,\ldots,q_m)} \leqslant \|A\|_{ms(p;q_1,\ldots,q_m)}, \qquad \forall\, A \in \mathcal{L}_{ms(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F).$$

3 Multipolynomials: a unified approach

This chapter mostly extends and unifies the previous two. In section 3.6 we go further with multipolynomial summing theory to generalize recent advances concerning absolutely summing multilinear/polynomial mappings between Banach spaces with unconditional Schauder basis.

Let us start with the following natural definition:

Definition 3.0.1. *Let $m \in \mathbb{N}$ and $(n_1, \ldots, n_m) \in \mathbb{N}^m$. A mapping $P : E_1 \times \cdots \times E_m \to F$ is said to be an $(n_1, \ldots, n_m)$-homogeneous polynomial if, for each $1 \leqslant j \leqslant m$, the mapping*

$$P\left(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_m\right) : E_j \to F$$

is an n_j-homogeneous polynomial for all fixed $x_i \in E_i$ with $i \neq j$.

When $m = 1$ and $n_1 = 1$, it is just the concept of an linear operator; when $m = 1$ and $n_1 > 1$, we have an homogeneous polynomial and, finally, when $m > 1$ and $n_1 = \cdots = n_m = 1$, we recover the concept of an m-linear mapping. This kind of map is called a *multipolynomial*. Sometimes those particular cases will be called *extreme cases*. We shall denote by $\mathcal{P}_a({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ the vector space of all $(n_1, \ldots, n_m)$-homogeneous polynomials from the cartesian product $E_1 \times \cdots \times E_m$ into F, whereas we shall denote by $\mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ the subspace of all continuous members of $\mathcal{P}_a({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$. For each $P \in \mathcal{P}_a({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ we define

$$\|P\| := \sup\left\{\|P\left(x_1, \ldots, x_m\right)\| : x_j \in E_j, \max_j \|x_j\| \leqslant 1\right\}.$$

When $E_1 = \cdots = E_m = E$ we shall write $\mathcal{P}_a({}^{n_1,\ldots,n_m}E; F)$ and $\mathcal{P}({}^{n_1,\ldots,n_m}E; F)$; if $n_1 = \cdots = n_m = n$ we use $\mathcal{P}_a({}^{n,\overset{m}{\ldots},n}E; F)$ instead. Finally, when $F = \mathbb{K}$ then, for short, we shall utilize the notations $\mathcal{P}_a({}^{n_1}E_1, \ldots, {}^{n_m}E_m)$, $\mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m)$, etc.

The concept of multipolynomials was firstly conceived by I. Chernega and A. Zagorodnyuk in (CHERNEGA; ZAGORODNYUK, 2009) and was rediscovered in (VELANGA, 2018), in the current notation/language, as an attempt to unify the theories of multilinear mappings and homogeneous polynomials between Banach spaces. An illustration of how it works can also be seen in (BOTELHO; TORRES; VELANGA, 2018).

The basics of the theories of homogeneous polynomials and multilinear mappings, as well as several topics in such settings, can be translated to multipolynomials with the advantage of having a unified and elegant approach. The present chapter is dedicated to developing this subject.

Modus vivendi: Every concept separately dealt with in the multilinear or polynomial setting should be designed such that it extends and unifies the whole theory by the procedure to be outlined in this chapter.

3.1 Basic theory

We begin by extending Lemma 1.2.3.

Lemma 3.1.1. *Let $P \in \mathcal{P}_a(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$. If P is bounded by c on an open ball $B((a_1, \ldots, a_m); r)$ then P is bounded by $cn_1^{n_1}/n_1! \cdots n_m^{n_m}/n_m!$ on the ball $B((0, \ldots, 0); r)$.*

Proof. Let $(x_1, \ldots, x_m) \in B((0, \ldots, 0); r)$. We prove this by induction on m. If $m = 1$ it is just Lemma 1.2.3. Suppose that the result holds for $m - 1$, then the multipolynomial

$$P\Bigg(\underbrace{\cdot, \ldots, \cdot}_{m-1}, y\Bigg) \in \mathcal{P}_a\left(^{n_1}E_1, \ldots, ^{n_{m-1}}E_{m-1}; F\right)$$

is bounded by c on the ball $B((a_1, \ldots, a_{m-1}); r)$, for all $y \in B(a_m; r)$. The induction hypothesis implies that

$$P\Bigg(\underbrace{\cdot, \ldots, \cdot}_{m-1}, y\Bigg)$$

is bounded by $cn_1^{n_1}/n_1! \cdots n_{m-1}^{n_{m-1}}/n_{m-1}!$ on the ball $B((0, \ldots, 0); r)$, whenever $y \in B(a_m; r)$. Applying the polarization formula to $\check{P}_{(x_1, \ldots, x_{m-1}, \cdot)}$, with $x_0 = a_m$ and $x_1 = \cdots = x_{n_m} = x_m/n_m$, we get

$$\begin{aligned}
&\left\| P\left(x_1, \ldots, x_m\right) \right\| \\
&= n_m^{n_m} \left\| \check{P}_{(x_1, \ldots, x_{m-1}, \cdot)} \left(\frac{x_m}{n_m}\right)^{n_m} \right\| \\
&= n_m^{n_m} \left\| \frac{1}{n_m! 2^{n_m}} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_{n_m} \check{P}_{(x_1, \ldots, x_{m-1}, \cdot)} \left(a_m + \left(\varepsilon_1 + \cdots + \varepsilon_{n_m}\right) \frac{x_m}{n_m}\right)^{n_m} \right\| \\
&\leqslant \frac{n_m^{n_m}}{n_m! 2^{n_m}} \sum_{\varepsilon_j = \pm 1} \left\| \check{P}_{(x_1, \ldots, x_{m-1}, \cdot)} \left(a_m + \left(\varepsilon_1 + \cdots + \varepsilon_{n_m}\right) \frac{x_m}{n_m}\right)^{n_m} \right\| \\
&= \frac{n_m^{n_m}}{n_m! 2^{n_m}} \sum_{\varepsilon_j = \pm 1} \left\| P\left(\underbrace{\cdot, \ldots, \cdot}_{m-1}, a_m + \left(\varepsilon_1 + \cdots + \varepsilon_{n_m}\right) \frac{x_m}{n_m}\right)\left(x_1, \ldots, x_{m-1}\right) \right\| \\
&\leqslant \frac{n_m^{n_m}}{n_m! 2^{n_m}} 2^{n_m} c \frac{n_1^{n_1}}{n_1!} \cdots \frac{n_{m-1}^{n_{m-1}}}{n_{m-1}!}.
\end{aligned}$$

It follows that P is bounded by $cn_1^{n_1}/n_1! \cdots n_m^{n_m}/n_m!$ on the ball $B((0, \ldots, 0); r)$, and the proof is complete. □

Continuous multipolynomials can be described as follows:

Theorem 3.1.2. *For each $P \in \mathcal{P}_a({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ the following conditions are equivalent:*

(i) *P is continuous;*

(ii) *P is continuous at the origin;*

(iii) *There exists a constant $c \geqslant 0$ such that*

$$\|P(x_1, \ldots, x_m)\| \leqslant c\,\|x_1\|^{n_1} \cdots \|x_m\|^{n_m},$$

for all $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$;

(iv) *$\|P\| < \infty$;*

(v) *P is uniformly continuous on bounded subsets of $E_1 \times \cdots \times E_m$;*

(vi) *P is bounded on every ball with finite radius;*

(vii) *P is bounded on some ball;*

(viii) *P is bounded on some ball with center at the origin.*

Proof. The implications $(i) \Rightarrow (ii)$ and $(vi) \Rightarrow (vii)$ are obvious.

$(ii) \Rightarrow (iii)$: Suppose P continuous at the origin. Then, there exists $\delta > 0$ such that

$$(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m,\ \|(x_1, \ldots, x_m)\| < \delta \Rightarrow \|P(x_1, \ldots, x_m)\| < 1.$$

The inequality in (iii) is obvious if $x_i = 0$ for some $i = 1, \ldots, m$. So, we can assume $x_i \neq 0$ for every $i = 1, \ldots, m$. Then,

$$\left\|\left(\frac{\delta x_1}{2\,\|x_1\|}, \ldots, \frac{\delta x_m}{2\,\|x_m\|}\right)\right\| = \frac{\delta}{2} < \delta$$

and thus

$$\begin{aligned}\|P(x_1, \ldots, x_m)\| &= \left(\frac{2}{\delta}\right)^{n_1+\cdots+n_m} \|x_1\|^{n_1} \cdots \|x_m\|^{n_m} \left\|P\left(\frac{\delta x_1}{2\,\|x_1\|}, \ldots, \frac{\delta x_m}{2\,\|x_m\|}\right)\right\| \\ &< \left(\frac{2}{\delta}\right)^{n_1+\cdots+n_m} \|x_1\|^{n_1} \cdots \|x_m\|^{n_m}.\end{aligned}$$

It gives us (iii) with $c = (2/\delta)^{n_1+\cdots+n_m}$.

$(iii) \Rightarrow (iv)$: If (iii) is true then we have

$$\|P(x_1, \ldots, x_m)\| \leqslant c\,\|x_1\|^{n_1} \cdots \|x_m\|^{n_m} \leqslant c,$$

for all $x_1 \in E_1, \ldots, x_m \in E_m$, with $\|x_1\|, \ldots, \|x_m\| \leqslant 1$. It shows that $\|P\| \leqslant c$.

$(iv) \Rightarrow (v)$: Let $a = (a_1, \ldots, a_m)$, $x = (x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$ with

$$\max_i \|x_i\| \leqslant r$$

and

$$\max_i \|a_i\| \leqslant r.$$

Then

$$\begin{aligned} \|P(a_1, \ldots, a_{i-1}, \cdot, x_{i+1}, \ldots, x_m)\| &\leqslant \|a_1\|^{n_1} \cdots \|a_{i-1}\|^{n_{i-1}} \|x_{i+1}\|^{n_{i+1}} \cdots \|x_m\|^{n_m} \|P\| \\ &\leqslant r^{n_1+\cdots+n_{i-1}+n_{i+1}+\cdots+n_m} \|P\|. \end{aligned}$$

From (1.2) we get

$$\begin{aligned} \left\|\check{P}_{(a_1,\ldots,a_{i-1},\cdot,x_{i+1},\ldots,x_m)}\right\| &\leqslant \frac{n_i^{n_i}}{n_i!} \|P(a_1, \ldots, a_{i-1}, \cdot, x_{i+1}, \ldots, x_m)\| \\ &\leqslant \frac{n_i^{n_i}}{n_i!} r^{n_1+\cdots+n_{i-1}+n_{i+1}+\cdots+n_m} \|P\|, \end{aligned}$$

for every $i = 1, \ldots, m$. Now, we can write

$$\begin{aligned} &\|P(x) - P(a)\| \\ &\leqslant \sum_{i=1}^{m} \|P(a_1, \ldots, a_{i-1}, x_i, \ldots, x_m) - P(a_1, \ldots, a_i, x_{i+1}, \ldots, x_m)\| \\ &= \sum_{i=1}^{m} \left\|\check{P}_{(a_1,\ldots,a_{i-1},\cdot,x_{i+1},\ldots,x_m)}(x_i)^{n_i} - \check{P}_{(a_1,\ldots,a_{i-1},\cdot,x_{i+1},\ldots,x_m)}(a_i)^{n_i}\right\| \\ &= \sum_{i=1}^{m} \left\| \begin{array}{l} \check{P}_{(a_1,\ldots,a_{i-1},\cdot,x_{i+1},\ldots,x_m)}(x_i - a_i, x_i, \ldots, x_i) + \cdots \\ \cdots + \check{P}_{(a_1,\ldots,a_{i-1},\cdot,x_{i+1},\ldots,x_m)}(a_i, \ldots, a_i, x_i - a_i) \end{array} \right\| \\ &\leqslant \sum_{i=1}^{m} n_i \left\|\check{P}_{(a_1,\ldots,a_{i-1},\cdot,x_{i+1},\ldots,x_m)}\right\| \|x_i - a_i\| r^{n_i-1} \\ &\leqslant \sum_{i=1}^{m} n_i \left(\frac{n_i^{n_i}}{n_i!} r^{n_1+\cdots+n_{i-1}+n_{i+1}+\cdots+n_m} \|P\|\right) \|x - a\| r^{n_i-1} \\ &\leqslant \left(\sum_{i=1}^{m} \frac{n_i^{n_i+1}}{n_i!}\right) r^{n_1+\cdots+n_m-1} \|P\| \|x - a\|, \end{aligned}$$

and the uniform continuity of P on bounded subsets of $E_1 \times \cdots \times E_m$ follows.

$(v) \Rightarrow (i)$: Let us show that P is continuous at an arbitrary point $a \in E = E_1 \times \cdots \times E_m$. Given $\varepsilon > 0$ it follows from (v) that there exist $\delta_0 > 0$ such that, for every $x, y \in B_E(0; \|a\| + 1)$,

$$\|x - y\| < \delta_0 \Rightarrow \|P(x) - P(y)\| < \varepsilon.$$

Defining $\delta = \min\{\delta_0, 1\}$ we get

$$x \in E, \quad \|x - a\| < \delta \Rightarrow \|P(x) - P(a)\| < \varepsilon,$$

and thus P is continuous at the point a.

$(iii) \Rightarrow (vi)$: Let B be a ball with center at $(a_1, \ldots, a_m) \in E_1 \times \cdots \times E_m$ and radius $r > 0$. For every $(x_1, \ldots, x_m) \in B$ the hypothesis (iii) gives us a constant $c \geqslant 0$ such that

$$\|P(x_1, \ldots, x_m)\| \leqslant c\|x_1\|^{n_1} \cdots \|x_m\|^{n_m} \leqslant c(r + \|a_1\|)^{n_1} \cdots (r + \|a_m\|)^{n_m},$$

and so P is bounded on B.

$(vii) \Rightarrow (viii)$: It follows immediately from Lemma 3.1.1.

$(viii) \Rightarrow (iv)$: Suppose that there exist $r > 0$ and $c \geqslant 0$ such that

$$\|P(x_1, \ldots, x_m)\| \leqslant c, \qquad \forall (x_1, \ldots, x_m) \in B_{E_1 \times \cdots \times E_m}((0, \ldots, 0); r).$$

Thus, given $x_1 \in E_1, \ldots, x_m \in E_m$, with $\|x_1\|, \ldots, \|x_m\| \leqslant 1$, we have $((r/2)x_1, \ldots$ $\ldots, (r/2)x_m) \in B_{E_1 \times \cdots \times E_m}((0, \ldots, 0); r)$ and hence

$$\|P(x_1, \ldots, x_m)\| = \left(\frac{2}{r}\right)^{n_1 + \cdots + n_m} \left\|P\left(\frac{r}{2}x_1, \ldots, \frac{r}{2}x_m\right)\right\| \leqslant c\left(\frac{2}{r}\right)^{n_1 + \cdots + n_m}.$$

□

Proposition 3.1.3. *For each $P \in \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ we have the following:*

(i) $\|P(x_1, \ldots, x_m)\| \leqslant \|P\|\|x_1\|^{n_1} \cdots \|x_m\|^{n_m}, \quad \forall x_j \in E_j,\ j = 1, \ldots, m.$

(ii) $\|P\| = \inf\{c \geqslant 0 : \|P(x_1, \ldots, x_m)\| \leqslant c\|x_1\|^{n_1} \cdots \|x_m\|^{n_m},\ \forall x_j \in E_j,\ j = 1, \ldots, m\}.$

As in the multilinear case, one might increase the list above provided we add Banach spaces in the hypothesis.

Proposition 3.1.4. *If $E_1, \ldots, E_m$ are Banach spaces, then $P \in \mathcal{P}_a({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ is continuous if and only if P is separately continuous in each variable.*

Proof. We will do it for $m = 2$. To prove the non-trivial assertion, let us assume that P is separately continuous. Consider the family $\{P(x, \cdot)\}_{x \in B_{E_1}} \subseteq \mathcal{P}({}^{n_2}E_2; F)$. Given $y \in E_2$,

$$\|P(x, \cdot)\, y\| \leqslant \|P(\cdot, y)\| \|x\|^{n_1} \leqslant \|P(\cdot, y)\| =: C_y < \infty,$$

for every $x \in B_{E_1}$. By Theorem 1.2.6, $\sup_{x \in B_{E_1}} \|P(x, \cdot)\| =: C < \infty$. Thus,

$$\|P(x, y)\| = \|P(x, \cdot)\, y\| \leqslant \|P(x, \cdot)\| \leqslant C,$$

for all $x \in E_1$, $y \in E_2$ with $\|x\|, \|y\| \leqslant 1$. Hence, $\|P\| < \infty$ and the proof is done. □

if (P_j) is a sequence in $\mathcal{P}_a(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ such that the limit $P(x) = \lim P_j(x)$ exists for every $x = (x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$, then $P \in \mathcal{P}_a(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$. If $E_1 = \cdots = E_m$ and each P_j is symmetric (see definition in subsection 3.1.2), then P is symmetric. If $E_1, \ldots, E_m$ are Banach spaces and each P_j is continuous, then P is continuous as well. A direct route to prove the latter assertion is by using Proposition 3.1.4 and Corollary 1.2.7. An alternative way follows in the form of the coming multipolynomial BST (Corollary 3.1.7).

Proposition 3.1.5. *If F is a Banach space, then $\mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ is a Banach space under the norm $P \mapsto \|P\|$.*

Proof. Let (P_j) be a Cauchy sequence in $\mathcal{P}(^{n_1}E_1, ^{n_2}E_2; F)$. Given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$j, k \geqslant n_0 \quad \Rightarrow \quad \|P_j - P_k\| < \varepsilon.$$

Then for each $(x, y) \in E_1 \times E_2$ and $j, k \geqslant n_0$ we have that

$$\|P_j(x, y) - P_k(x, y)\| = \|(P_j - P_k)(x, y)\| \leqslant \|P_j - P_k\| \|x\|^{n_1} \|y\|^{n_2} \leqslant \varepsilon \|x\|^{n_1} \|y\|^{n_2}, \quad (3.1)$$

and it follows that $(P_j(x, y))$ is a Cauchy sequence in F. Since F is complete, we have the well-defined mapping

$$\begin{array}{rccc} P: & E_1 \times E_2 & \to & F \\ & (x, y) & \mapsto & \lim_{j\to\infty} P_j(x, y) \end{array}, \quad (3.2)$$

which, as we already noted, belongs to $\mathcal{P}_a(^{n_1}E_1, ^{n_2}E_2; F)$. Furthermore, since (P_j) is a Cauchy sequence in $\mathcal{P}_a(^{n_1}E_1, ^{n_2}E_2; F)$ there is a constant $c > 0$ such that $\|P_j\| \leqslant c$, for every j. Then it follows from (3.2) that $\|P\| \leqslant c$, and P is therefore continuous, by Theorem 3.1.2. Finally, letting $k \to \infty$ in (3.1) we obtain

$$\|(P_j - P)(x, y)\| \leqslant \varepsilon \|x\|^{n_1} \|y\|^{n_2},$$

for all $(x, y) \in E_1 \times E_2$ and $j \geqslant n_0$. It shows that $\lim_{j\to\infty}\|P_j - P\| = 0$ and completes the proof. □

Next, we extend the UBP and the BST to multipolynomials.

Theorem 3.1.6 (Uniform Boundedness Principle)**.** *Let $E_1, \ldots, E_m$ be Banach spaces, F be a normed space and let $\{P_i\}_{i\in I}$ be a family in $\mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$. The following conditions are equivalent:*

(i) *For every $x = (x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$ there exists $C_x < \infty$ such that*

$$\sup_{i\in I} \|P_i(x)\| < C_x.$$

(ii) *The family $\{P_i\}_{i\in I}$ is norm bounded, that is,*

$$\sup_{i\in I}\|P_i\| < \infty.$$

Proof. If (i) holds, it follows from Lemma 1.2.5, with $U = E = E_1 \times \cdots \times E_m$, that there exist a ball $B_E(a;r) \subset E$ and a constant c such that

$$\|P_i\left(x\right)\| \leqslant c, \qquad \forall\, x \in B_E\left(a;r\right) \quad \text{and} \quad \forall\, i \in I.$$

By Lemma 3.1.1,

$$\|P_i\left(x\right)\| \leqslant c\frac{n_1^{n_1}}{n_1!}\cdots\frac{n_m^{n_m}}{n_m!}, \qquad \forall\, x \in B_E\left(0;r\right) \quad \text{and} \quad \forall\, i \in I$$

and, by the multipolynomial homogeneity,

$$\|P_i\| \leqslant \left(\frac{2}{r}\right)^{n_1+\cdots+n_m} c\frac{n_1^{n_1}}{n_1!}\cdots\frac{n_m^{n_m}}{n_m!}, \quad \forall\, i \in I.$$

It shows (ii). The other implication follows immediately from Proposition 3.1.3. □

Corollary 3.1.7 (Banach–Steinhaus Theorem)**.** *Let $E_1, \ldots, E_m$ be Banach spaces, F be a normed space and let (P_j) be a sequence in $\mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ such that $(P_j(x_1, \ldots, x_m))$ is convergent in F for all $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$. If we define*

$$P : E_1 \times \cdots \times E_m \to F$$

by

$$P(x_1, \ldots, x_m) := \lim_{j\to\infty} P_j(x_1, \ldots, x_m),$$

then $P \in \mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$.

Proof. It is clear that $P \in \mathcal{P}_a(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$. For each $x = (x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$, the sequence $(P_j(x))$ is convergent and, therefore, bounded. By Theorem 3.1.6, there exists uma constante $c > 0$ such that $\sup_{j\in\mathbb{N}}\|P_j\| \leqslant c$. Thus,

$$\|P_j\left(x\right)\| \leqslant \|P_j\|\,\|x_1\|^{n_1}\cdots\|x_m\|^{n_m} \leqslant c\,\|x_1\|^{n_1}\cdots\|x_m\|^{n_m},$$

for all $x \in E_1 \times \cdots \times E_m$ and $j \in \mathbb{N}$. Taking $j \to \infty$ completes the proof. □

If, in addition, F in BST-hypotheses is complete, we can also conclude that (P_j) converges to P uniformly on compact subsets of $E_1 \times \cdots \times E_m$. Precisely, we have the

Corollary 3.1.8. *Let $E_1, \ldots, E_m$ and F be Banach spaces and let $P, P_1, P_2, \ldots,$ be multipolynomials in $\mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ such that $\lim_{j\to\infty}P_j(x_1, \ldots, x_m) = P(x_1, \ldots, x_m)$ for every $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$. Then, (P_j) converges to P uniformly on compact subsets of $E_1 \times \cdots \times E_m$.*

Proof. By Theorem 3.1.6, $\sup_{n\in\mathbb{N}}\|P_n\| := c < \infty$. It suffices to prove that

$$\sup_{x\in K}\|P_n(x) - P(x)\| \longrightarrow 0 \tag{3.3}$$

for every compact subset $K \subseteq E = E_1 \times \cdots \times E_m$. Indeed, if (3.3) is not true, there exists a compact subset $K \subseteq E$ such that the sequence $\sup_{x\in K}\|P_n(x) - P(x)\|$ does not converge to zero. That is, there exists an $\varepsilon_0 > 0$ with the following property:

$$\forall\, k \in \mathbb{N} \quad \exists\, n_k \in \mathbb{N}: \quad n_k \geqslant k \quad \text{and} \quad \sup_{x\in K}\|P_{n_k}(x) - P(x)\| > \varepsilon_0.$$

It yields a sequence $(x_k) = (x_k^{(1)}, \ldots, x_k^{(m)})$ in K such that $\|P_{n_k}(x_k) - P(x_k)\| > \varepsilon_0$, for every $k \in \mathbb{N}$. Since K is compact, (x_k) has a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim x_{k_j} = a \in K$. Thus,

$$\begin{aligned}\varepsilon_0 &< \left\|P_{n_{k_j}}\left(x_{k_j}\right) - P\left(x_{k_j}\right)\right\| \\ &\leqslant \left\|P_{n_{k_j}}\left(x_{k_j}\right) - P_{n_{k_j}}(a)\right\| + \left\|P_{n_{k_j}}(a) - P(a)\right\| + \left\|P\left(x_{k_j}\right) - P(a)\right\|.\end{aligned}$$

Since

$$\left\|P_{n_{k_j}}\left(x_{k_j}\right) - P_{n_{k_j}}(a)\right\| \leqslant c\left\|x_{k_j}^{(1)} - a_1\right\|^{n_1} \cdots \left\|x_{k_j}^{(m)} - a_m\right\|^{n_m},$$

we obtain $\varepsilon_0 \leqslant 0$, after taking $j \to \infty$, which is absurd. ☐

Let us establish some notation that will be required from now on. For fixed $m, n_1, \ldots, n_m$ positive integers, we shall write $M := \sum_{j=1}^m n_j$. For each $m, d \in \mathbb{N}$ we shall denote by $\mathbb{M}_{m\times d}(\mathbb{N}_0)$ the set of all $m \times d$ matrices with entries in $\mathbb{N}_0$. Given $\alpha = (\alpha_{ij})_{ij} \in \mathbb{M}_{m\times d}(\mathbb{N}_0)$ and a fixed $1 \leqslant j_0 \leqslant d$, we define $|\alpha_{ij_0}| := \sum_{i=1}^m \alpha_{ij_0}$, that is, the summation of the j_0-th column $(\alpha_{1j_0}, \ldots, \alpha_{mj_0})$ of α. For its rows $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{id})$, $1 \leqslant i \leqslant m$, we set $|\alpha_i| := \sum_{j=1}^d \alpha_{ij}$ and $\alpha_i! := \alpha_{i1}! \cdots \alpha_{id}!$. If, for each i with $1 \leqslant i \leqslant m$, $\lambda_i := (\lambda_{i1}, \ldots, \lambda_{id}) \in \mathbb{K}^d$, we shall write $\lambda_i^{\alpha_i} := \lambda_{i1}^{\alpha_{i1}} \cdots \lambda_{id}^{\alpha_{id}}$. More generally, if λ and α are infinite-column matrices in $\mathbb{M}_{m\times\infty}(\mathbb{K})$ and $\mathbb{M}_{m\times\infty}(\mathbb{N}_0)$, respectively, such that $|\alpha_i| = n_i$ for each row i with $1 \leqslant i \leqslant m$, then we shall set $\lambda_i^{\alpha_i} := \prod_j \lambda_{ij}^{\alpha_{ij}}$. Finally, given $\varepsilon_n \in \{1, -1\}$ with $n \in \mathbb{N}$, we put

$$\varepsilon_{i,j} := \sum_{k=1}^{\alpha_{ij}} \varepsilon_{M-(n_i+\cdots+n_m)+|\alpha_i|-(\alpha_{ij}+\cdots+\alpha_{id})+k}$$

for each pair $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, d\}$. For convenience, we also define $\varepsilon_{i,j} = 0$ whenever $\alpha_{ij} = 0$.

With this in mind, let $P \in \mathcal{P}_a({}^{n_1,\ldots,n_m}E; F)$. Then for all $x_1, \ldots, x_d \in E$ and $\lambda_{i1}, \ldots, \lambda_{id} \in \mathbb{K}$, $1 \leqslant i \leqslant m$, one can inductively combine Leibniz and polarization formulas to yield

$$\begin{aligned}&P\left(\sum_{j=1}^d \lambda_{1j}x_j, \ldots, \sum_{j=1}^d \lambda_{mj}x_j\right) \\ &= \frac{1}{2^M}\sum\sum_{\varepsilon_k=\pm1} \frac{\lambda_1^{\alpha_1}\cdots\lambda_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}\varepsilon_1\cdots\varepsilon_M P\left(\sum_{j=1}^d \varepsilon_{1,j}x_j, \ldots, \sum_{j=1}^d \varepsilon_{m,j}x_j\right)\end{aligned} \tag{3.4}$$

where the summation is taken over all matrices $\alpha \in \mathbb{M}_{m\times d}(\mathbb{N}_0)$ such that $|\alpha_i| = n_i$, for each i with $1 \leqslant i \leqslant m$.

Equation 3.4 shows that if E is finite dimensional with a basis $(e_1, \ldots, e_d)$, let $\xi_1, \ldots, \xi_d$ denote the corresponding coordinate functionals, then each $P \in \mathcal{P}_a({}^{n_1,\ldots,n_m}E; F)$ can be uniquely represented as a sum

$$P = \sum c_\alpha \left(\xi_1^{\alpha_{11}} \cdots \xi_d^{\alpha_{1d}}\right) \otimes \cdots \otimes \left(\xi_1^{\alpha_{m1}} \cdots \xi_d^{\alpha_{md}}\right) \tag{3.5}$$

where $c_\alpha \in F$ and where the summation is taken over all matrices $\alpha \in \mathbb{M}_{m\times d}(\mathbb{N}_0)$ such that $|\alpha_i| = n_i$, for each i with $1 \leqslant i \leqslant m$. In particular, $\mathcal{P}_a({}^{n_1,\ldots,n_m}E; F) = \mathcal{P}({}^{n_1,\ldots,n_m}E; F)$.

Equation 3.5 unifies previous well-known formulas. Indeed, when $n = 1$ we have Equation 1.1. Putting $m = 1$, and then $n = m$, we have Equation 1.3.

If E is an infinite dimensional Banach space with a Schauder basis (e_n) and coordinate functionals (e_n^*), let $e^*(x) := (e_n^*(x))_{n\in\mathbb{N}}$ denote the coordinates of x, for every $x \in E$. An application of Equation 3.4 shows that each $P \in \mathcal{P}({}^{n_1,\ldots,n_m}E; F)$ can be uniquely represented as a sum

$$P(x_1, \ldots, x_m) = \sum c_\alpha e^*(x_1)^{\alpha 1} \cdots e^*(x_m)^{\alpha_m}, \tag{3.6}$$

for all $x_1, \ldots, x_m \in E$, where $c_\alpha \in F$ and where the summation is taken over all matrices $\alpha \in \mathbb{M}_{m\times\infty}(\mathbb{N}_0)$ such that $|\alpha_i| = n_i$, for each i with $1 \leqslant i \leqslant m$.

3.1.1 Every multipolynomial is a polynomial

Next, we show that the class of homogeneous polynomials encompasses distinct classes of nonhomogeneous polynomials. To be explicit, we shall prove that every multipolynomial is a homogeneous polynomial. As corollaries, we expose some apparently overlooked properties in the literature. For instance, multilinear mappings are specific cases of polynomials.

Theorem 3.1.9. *Let E and F be vector spaces over $\mathbb{K}$. Let $\{e_i\}_{i\in I}$ be a Hamel basis for E and let ξ_i denote the corresponding coordinate functionals. Then, each $P \in \mathcal{P}_a({}^{n_1,\ldots,n_m}E; F)$ can be uniquely represented as a sum*

$$\begin{aligned} &P(x_1, \ldots, x_m) \\ &= \sum_{i_1,\ldots,i_M\in I} c_{i_1\cdots i_M} \prod_{j=1}^{m} \left(\prod_{r_j=1}^{n_j} \xi_{i_{M-(n_j+\cdots+n_m)+r_j}}\right)(x_j), \end{aligned}$$

where $c_{i_1\cdots i_M} \in F$ and where all but finitely many summands are zero.

Proof. For simplicity, let us do the proof for $m = 2$. The proof of the case $m = 2$ makes clear that the other cases are similar. Every $x \in E$ can be uniquely represented as a sum

$x = \sum_{i \in I} \xi_i(x) e_i$ where almost all of the scalars $\xi_i(x)$ (i.e., all but a finite set) are zero. So, we can write

$$P(x_1, x_2) = \sum_{i_1, \ldots, i_{n_1} \in I} \left(\xi_{i_1} \cdots \xi_{i_{n_1}}\right)(x_1) \overset{\vee}{P}_{(\cdot, x_2)} \left(e_{i_1}, \ldots, e_{i_{n_1}}\right).$$

Since

$$\overset{\vee}{P}_{(\cdot, x_2)} \left(e_{i_1}, \ldots, e_{i_{n_1}}\right) = \frac{1}{n_1! 2^{n_1}} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_{n_1} \overset{\vee}{P}_{\left(\sum_{k=1}^{n_1} \varepsilon_k e_{i_k}, \cdot\right)} x_2^{n_2},$$

repeat the process for $\overset{\vee}{P}_{\left(\sum_{k=1}^{n_1} \varepsilon_k e_{i_k}, \cdot\right)}$ and the proof is done with

$$c_{i_1 \cdots i_M} = \frac{1}{n_1! n_2! 2^M} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_M P \left(\sum_{k=1}^{n_1} \varepsilon_k e_{i_k}, \sum_{k=1}^{n_2} \varepsilon_{n_1 + k} e_{i_{n_1 + k}} \right),$$

for every $i_1, \ldots, i_M \in I$. □

Corollary 3.1.10. *Let E and F be vector spaces over $\mathbb{K}$. Then,*

$$\mathcal{P}_a\left({}^{n_1, \ldots, n_m} E; F\right) \subset \mathcal{P}_a\left({}^M E^m; F\right). \tag{3.7}$$

Proof. Indeed, the map $A : (\underbrace{E \times \cdots \times E}_{m})^M \to F$ defined by

$$\begin{aligned} &A\left(\left(x_{11}, \ldots, x_{1m}\right), \ldots, \left(x_{M1}, \ldots, x_{Mm}\right)\right) \\ &= \sum_{i_1, \ldots, i_M} c_{i_1 \cdots i_M} \textstyle\prod_{j=1}^{m} \prod_{r_j=1}^{n_j} \xi_{i_{M-(n_j + \cdots + n_m) + r_j}} \left(x_{[M-(n_j + \cdots + n_m) + r_j] j}\right) \end{aligned}$$

is an M-linear mapping which is equal to P on the diagonal. □

In other words, every $(n_1, \ldots, n_m)$-homogeneous polynomial is an M-homogeneous polynomial.

Remark 3.1.11. *It is worth noting that (k, m)-linear mappings, introduced by I. Chernega and A. Zagorodnyuk in (CHERNEGA; ZAGORODNYUK, 2009, Definition 3.1), are km-homogeneous polynomials. It suffices to observe that $\mathcal{L}_a({}^k_m E; F) = \mathcal{P}_a({}^{m, \overset{k}{\ldots}, m} E; F)$ and apply Corollary 3.1.10.*

If $n_1 = \cdots = n_m = 1$, then Corollary 3.1.10 also implies the following:

Corollary 3.1.12. *Let E and F be vector spaces over $\mathbb{K}$. Then every m-linear mapping in $\mathcal{L}_a({}^m E; F)$ is an m-homogeneous polynomial in $\mathcal{P}_a({}^m (E^m); F)$.*

Some applications are in order:

- When $m = 1$, inclusion (3.7) trivially becomes equality, but it is always strict when $m > 1$. For instance, when $n_1 = \cdots = n_m = 1$, it is clear that there exists a homogeneous polynomial in $\mathcal{P}_a({}^m(E^m); F)$ which is not an m-linear mapping in $\mathcal{L}_a({}^m E; F)$. If $n_i > 1$ for some i with $1 \leqslant i \leqslant m$, let us say $m = 2$ and $n_2 = 2$, the mapping
$$P : \ell_2 \times \ell_2 \longrightarrow \mathbb{K}, \quad P((x_j),(y_j)) = \textstyle\sum_j x_j^3$$
belongs to $\mathcal{P}({}^3(\ell_2 \times \ell_2))$, with $\check{P}((a,b),(c,d),(w,z)) = \sum_j a_j c_j w_j$, but $P \notin \mathcal{P}({}^{1,2}\ell_2)$, by Equation 3.6. Analogously,
$$Q : \ell_2 \times \ell_2 \longrightarrow \mathbb{K}, \quad Q((x_j),(y_j)) = \textstyle\sum_j x_j^2 y_j$$
is another instance in $\mathcal{P}({}^3(\ell_2 \times \ell_2))$ which is not in $\mathcal{P}({}^{1,2}\ell_2)$.

- The previous results show, in particular, that (algebraically speaking) multilinear mappings are homogeneous polynomials. So, at first glance, one may wonder why the theory of multilinear mappings is investigated separately? The point is that this algebraic identification does not catch analytical information. For instance, the estimate (see Proposition 3.1.3 for the corresponding multipolynomial inequality)
$$\|A(x_1,\ldots,x_m)\| \leqslant \|A\| \, \|(x_1,\ldots,x_m)\|^m \,,$$
is far less precise than
$$\|A(x_1,\ldots,x_m)\| \leqslant \|A\| \, \|x_1\| \cdots \|x_m\| \,. \tag{3.8}$$
In this sense, when dealing with quantitative, computational or statistical problems and applications, such as (to cite some) the search for optimal constants in Hardy–Littlewood and Bohnenblust–Hille inequalities, Gale–Berlekamp games, and applications for multilinear forms (see (ALARCÓN, 2013; ALBUQUERQUE et al., 2018; ARAÚJO; PELLEGRINO, 2019; PELLEGRINO; TEIXEIRA, 2018; JÚNIOR, 2018)), the above identification is useless. However, Corollary 3.1.10 says that qualitative results, especially topological properties, e.g., uniform boundedness principle and Banach–Steinhaus theorem, can be inherited from polynomials.

3.1.2 A polarization formula

For each $m, n \in \mathbb{N}$, we shall denote by $\mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E; F)$ the subspace of all $P \in \mathcal{P}_a({}^{n,\overset{m}{\ldots},n}E; F)$ which are symmetric, that is, such that
$$P\left(x_{\sigma(1)},\ldots,x_{\sigma(m)}\right) = P\left(x_1,\ldots,x_m\right)$$
for all $x_1,\ldots,x_m \in E$ and for any permutation σ of the set $\{1,\ldots,m\}$. Note that if $n_i \neq n_j$ for some $1 \leqslant i \neq j \leqslant m$, then multi-homogeneity and symmetry imply that $\mathcal{P}_a^s({}^{n_1,\ldots,n_m}E; F) = \{0\}$.

Definition 3.1.13. *Let m and n be positive integers. Let $M \subset \mathbb{M}_{m\times(m+1)}(\mathbb{N}_0)$ be the subset of $m \times (m+1)$ matrices α such that its 0th column is zero and $\sum_{j=1}^m \alpha_{ij} = n = \sum_{i=1}^m \alpha_{ij}$, for all $i, j \in \{1, \ldots, m\}$. We define the remainder function $R_n : E^m \to F$ as follows:*

$$R_n(x_1, \ldots, x_m) = \sum_{\alpha\in M\setminus D} \sum_{\varepsilon_k=\pm 1} \frac{\varepsilon_1\cdots\varepsilon_{mn}}{\alpha_1!\cdots\alpha_m!} P\left(\sum_{j=1}^m \varepsilon_{1,j}x_j, \ldots, \sum_{j=1}^m \varepsilon_{m,j}x_j\right),$$

where

$$D = \{\alpha \in M : \forall j \in \{1,\ldots,m\}\ \exists i \in \{1,\ldots,m\} \text{ s.t. } \alpha_{ij} = n\}.$$

In other words, D can be seen as the set of all $m!$ row-permutation matrices of the diagonal matrix $(d_{ij})_{ij} = n$.

Next, we extend the polarization formula to multipolynomials.

Theorem 3.1.14. *Let $P \in \mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E; F)$. Then for all $x_0, \ldots, x_m \in E$ we have*

$$\begin{aligned} P(x_1, \ldots, x_m) &= \frac{1}{m!(n!2^n)^m} \sum_{\varepsilon_k=\pm 1} \varepsilon_1\cdots\varepsilon_{mn} P\left(x_0 + \sum_{k=1}^n \varepsilon_k x_1 + \cdots + \sum_{k=1}^n \varepsilon_{(m-1)n+k} x_m\right)^m \\ &- \frac{1}{m!2^{mn}} R_n(x_1, \ldots, x_m). \end{aligned}$$

Proof. By Equation 3.4 we have that

$$\begin{aligned} &P\left(x_0 + \sum_{k=1}^n \delta_k x_1 + \cdots + \sum_{k=1}^n \delta_{(m-1)n+k} x_m\right)^m \\ &= \frac{1}{2^{mn}} \sum \sum_{\varepsilon_k=\pm 1} \frac{\prod_{j=1}^m \left(\sum_{k_j=1}^n \delta_{(j-1)n+k_j}\right)^{|\alpha_{ij}|}}{\alpha_1!\cdots\alpha_m!} \varepsilon_1\cdots\varepsilon_{mn} P\left(\sum_{j=0}^m \varepsilon_{1,j}x_j, \ldots, \sum_{j=0}^m \varepsilon_{m,j}x_j\right) \end{aligned}$$

where the summation is taken over all matrices $\alpha \in \mathbb{M}_{m\times(m+1)}(\mathbb{N}_0)$ such that $\alpha_{i0} + \cdots + \alpha_{im} = n$, for each i with $1 \leqslant i \leqslant m$. Thus, if $|\alpha_{ij_0}| > n$ for some column $(\alpha_{1j_0}, \ldots, \alpha_{mj_0})$ with $1 \leqslant j_0 \leqslant m$, then there must exist $1 \leqslant j_1 \neq j_0 \leqslant m$ such that $|\alpha_{ij_1}| < n$. Otherwise, we would have $\sum_{i,j=1}^m \alpha_{ij} > mn$, which is absurd. Since for each $j = 1, \ldots, m$ we have

$$\sum_{\delta_k=\pm 1} \delta_{(j-1)n+1}\cdots\delta_{jn} \left(\sum_{k_j=1}^n \delta_{(j-1)n+k_j}\right)^{|\alpha_{ij}|} = \begin{cases} 0, & \text{if } |\alpha_{ij}| < n \\ n!2^n, & \text{if } |\alpha_{ij}| = n \end{cases},$$

it follows that

$$\begin{aligned} &\sum_{\delta_k=\pm 1} \delta_1\cdots\delta_{mn} P\left(x_0 + \sum_{k=1}^n \delta_k x_1 + \cdots + \sum_{k=1}^n \delta_{(m-1)n+k} x_m\right)^m \\ &= (n!)^m \left[\begin{array}{c} \sum_{\alpha\in D} \sum_{\varepsilon_k=\pm 1} \frac{\varepsilon_1\cdots\varepsilon_{mn}}{\alpha_1!\cdots\alpha_m!} P\left(\sum_{j=1}^m \varepsilon_{1,j}x_j, \ldots, \sum_{j=1}^m \varepsilon_{m,j}x_j\right) \\ +R_n(x_1, \ldots, x_m) \end{array}\right]. \end{aligned}$$

Since P is symmetric and $\#D = m!$, we get

$$\sum_{\alpha\in D}\sum\nolimits_{\varepsilon_k=\pm1}\frac{\varepsilon_1\cdots\varepsilon_{mn}}{\alpha_1!\cdots\alpha_m!}P\left(\sum_{j=1}^{m}\varepsilon_{1,j}x_j,\ldots,\sum_{j=1}^{m}\varepsilon_{m,j}x_j\right)$$
$$= m!\left(\frac{n!2^n}{n!}\right)^m P\left(x_1,\ldots,x_m\right),$$

and the desired result follows. □

Corollary 3.1.15. *Let $A \in \mathcal{L}_a^s({}^mE;F)$. Then for all $x_0,\ldots,x_m \in E$ we have*

$$A\left(x_1,\ldots,x_m\right) = \frac{1}{m!2^m}\sum_{\varepsilon_k=\pm1}\varepsilon_1\cdots\varepsilon_m A\left(x_0+\varepsilon_1x_1+\cdots+\varepsilon_mx_m\right)^m.$$

Proof. Choose $n = 1$ in Theorem 3.1.14 and observe that since $D = M$ the remainder-function R_1 must be zero. □

If $n > 1$, the pointwise-polynomial nature of a multipolynomial $P \in \mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E;F)$ is an obstacle to obtain, in general, an *exact* polarization formula, that is, the one with null remainder-function. The next results characterize the class of such mappings as a proper subspace of $\mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E;F)$.

Proposition 3.1.16. *For each $A \in \mathcal{L}_a^s({}^{mn}E;F)$ let $\Psi A \in \mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E;F)$ be defined by*

$$\Psi A\left(x_1,\ldots,x_m\right) = Ax_1^n\cdots x_m^n$$

for every $x_1,\ldots,x_m \in E$. Then the mapping

$$\Psi : \mathcal{L}_a^s({}^{mn}E;F) \to \mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E;F)$$

is a linear isomorphism onto its range $\operatorname{Im}\Psi$*. Moreover, for each $P \in \mathcal{P}_a^s({}^{n,\overset{m}{\ldots},n}E;F)$, we have the following equivalent conditions:*

(a) *$P \in \operatorname{Im}\Psi$;*

(b) *For all $x_0,\ldots,x_m \in E$ we have the* ***exact polarization formula***

$$P\left(x_1,\ldots,x_m\right)$$
$$= \frac{1}{(mn)!2^{mn}}\sum_{\varepsilon_k=\pm1}\varepsilon_1\cdots\varepsilon_{mn}P\left(x_0+\sum_{k=1}^{n}\varepsilon_kx_1+\cdots+\sum_{k=1}^{n}\varepsilon_{(m-1)n+k}x_m\right)^m.$$

Proof. By Corollary 3.1.15, we get the 1st and $(a) \Rightarrow (b)$ statements. By Corollary 3.1.10, there exists a unique $\check{P} \in \mathcal{L}_a^s({}^{mn}(E^m);F)$ which is equal to P on its diagonal. Now, it suffices to consider $A \in \mathcal{L}_a^s({}^{mn}E;F)$ defined by

$$A\left(x_1,\ldots,x_{mn}\right) = \check{P}\left(\left(x_1,\ldots,x_1\right),\ldots,\left(x_{mn},\ldots,x_{mn}\right)\right),$$

and notice that

$$\check{P}\left(\left(\left(\sum_{k=1}^{n}\varepsilon_k\right)(x_1,\ldots,x_1)+\cdots+\left(\sum_{k=1}^{n}\varepsilon_{(m-1)n+k}\right)(x_m,\ldots,x_m)\right)^{mn}\right.$$
$$=P\left(\sum_{k=1}^{n}\varepsilon_k x_1+\cdots+\sum_{k=1}^{n}\varepsilon_{(m-1)n+k}x_m\right)^{m}.$$

□

Example 3.1.17. *Let $E=\mathbb{R}^2$, $F=\mathbb{K}=\mathbb{R}$ and let $\{e_1,e_2\}$ be the canonical basis of E. By Equation 3.5, with $m=n=2$, we have that the mapping*

$$P\left((x_1,x_2),(y_1,y_2)\right)=x_1x_2y_1y_2$$

belongs to $\mathcal{P}_a^s({}^{n,n}E;F)$ but $P\notin \operatorname{Im}\Psi$. Indeed, one can quickly check that such a P cannot satisfy the exact polarization formula. For instance, take $x_0=0$, $x=e_1$, and $y=e_2$.

Remark 3.1.18. *By the above proposition and example, we conclude with a correction to the important paper (CHERNEGA; ZAGORODNYUK, 2009, p. 200–201). Namely, the canonical isomorphism indicated therein cannot occur between $\mathcal{L}_a^s({}^{km}E;F)$ onto the whole vector space $\mathcal{L}_a^s({}^{k}_{m}E;F)$ of all symmetric (k,m)-linear mappings (or, with our notation, onto $\mathcal{P}_a^s({}^{m,\overset{k}{\ldots},m}E;F)$). Finally, to fill the gap where the exact polarization formula does not work, one can use Theorem 3.1.14.*

3.2 Ideals

This section extends section 2.1 to multipolynomials and gives detailed examples.

Let $m\in\mathbb{N}$, $(n_1,\ldots,n_m)\in\mathbb{N}^m$ and let E_j,F_j,G,H $(1\leqslant j\leqslant m)$ be normed spaces over $\mathbb{K}$. Given $s_j\in\mathcal{L}_a(E_j;F_j)$, $P\in\mathcal{P}_a({}^{n_1}F_1,\ldots,{}^{n_m}F_m;G)$ and $t\in\mathcal{L}_a(G;H)$, one can quickly check that $P\circ(s_1,\ldots,s_m)\in\mathcal{P}_a({}^{n_1}E_1,\ldots,{}^{n_m}E_m;G)$ and $t\circ P\in\mathcal{P}_a({}^{n_1}F_1,\ldots$ $\ldots,{}^{n_m}F_m;H)$.

A multipolynomial $P\in\mathcal{P}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$ is said to be of *finite type* if there exist $n\in\mathbb{N}$, $\varphi_{i1}\in E_1',\ldots,\varphi_{im}\in E_m'$ and $b_i\in F$ $(1\leqslant i\leqslant n)$ such that

$$P(x_1,\ldots,x_m)=\sum_{i=1}^{n}\varphi_{i1}(x_1)^{n_1}\cdots\varphi_{im}(x_m)^{n_m}b_i,$$

for every $(x_1,\ldots,x_m)\in E_1\times\cdots\times E_m$. We shall denote by $\mathfrak{F}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$ the subspace of all finite type members of $\mathcal{P}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$.

Definition 3.2.1. *For each $m\in\mathbb{N}$ and multi-degree $(n_1,\ldots,n_m)\in\mathbb{N}^m$, let $\mathcal{P}_m^{(n_1,\ldots,n_m)}$ denote the class of all continuous $(n_1,\ldots,n_m)$-homogeneous polynomials between Banach spaces. A* ***multipolynomial ideal*** *$\mathfrak{U}$ is a subclass of the class $\mathfrak{P}:=\cup_{m=1}^{\infty}(\cup_{(n_1,\ldots,n_m)\in\mathbb{N}^m}^{\infty}$*

$\mathcal{P}_m^{(n_1,\dots,n_m)}$) *of all continuous multipolynomials between Banach spaces such that for all* $m \in \mathbb{N}$, *multi-degree* $(n_1,\dots,n_m) \in \mathbb{N}^m$ *and all Banach spaces* $E_1,\dots,E_m$ *and* F, *the components*

$$\mathfrak{U}\left({}^{n_1}E_1,\dots,{}^{n_m}E_m;F\right) := \mathcal{P}\left({}^{n_1}E_1,\dots,{}^{n_m}E_m;F\right) \cap \mathfrak{U}$$

satisfy:

(Ua) $\mathfrak{U}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$ *is a linear subspace of* $\mathcal{P}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$ *which contains the* $(n_1,\dots,n_m)$*-homogeneous polynomials of finite type;*

(Ub) *The ideal property: if* $P \in \mathfrak{U}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$, $u_j \in \mathcal{L}(G_j;E_j)$ *for* $j = 1,\dots,m$, *and* $t \in \mathcal{L}(F;H)$, *then*

$$t \circ P \circ (u_1,\dots,u_m) \in \mathfrak{U}\left({}^{n_1}G_1,\dots,{}^{n_m}G_m;H\right).$$

Moreover, $\mathfrak{U}$ *is said to be a* ***(quasi-) normed multipolynomial ideal*** *if there exists a map* $\|\cdot\|_{\mathfrak{U}} : \mathfrak{U} \to [0,\infty)$ *satisfying:*

(U1) $\|\cdot\|_{\mathfrak{U}}$ *restricted to* $\mathfrak{U}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$ *is a (quasi-) norm, for all* $m \in \mathbb{N}$, *multi-degree* $(n_1,\dots,n_m) \in \mathbb{N}^m$ *and all Banach spaces* $E_1,\dots,E_m$ *and* F;

(U2) $\|id_m^{(n_1,\dots,n_m)} : \mathbb{K}^m \to \mathbb{K} : id_m^{(n_1,\dots,n_m)}(\lambda_1,\dots,\lambda_m) = \lambda_1^{n_1}\cdots\lambda_m^{n_m}\|_{\mathfrak{U}} = 1$, *for all* $m \in \mathbb{N}$ *and* $(n_1,\dots,n_m) \in \mathbb{N}^m$;

(U3) *If* $P \in \mathfrak{U}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$, $u_j \in \mathcal{L}(G_j;E_j)$ *for* $j = 1,\dots,m$, *and* $t \in \mathcal{L}(F;H)$, *then*

$$\|t \circ P \circ (u_1,\dots,u_m)\|_{\mathfrak{U}} \leqslant \|t\| \, \|P\|_{\mathfrak{U}} \, \|u_1\|^{n_1} \cdots \|u_m\|^{n_m}.$$

When all the components $\mathfrak{U}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$ *are complete under the (quasi-) norm* $\|\cdot\|_{\mathfrak{U}}$ *above, then* $\mathfrak{U}$ *is called a* ***(quasi-) Banach multipolynomial ideal****. For a fixed multipolynomial ideal* $\mathfrak{U}$, *a positive integer* $m \in \mathbb{N}$, *and a multi-degree* $(n_1,\dots,n_m) \in \mathbb{N}^m$, *the class*

$$\mathfrak{U}_m^{(n_1,\dots,n_m)} := \bigcup_{E_1,\dots,E_m,F} \mathfrak{U}\left({}^{n_1}E_1,\dots,{}^{n_m}E_m;F\right)$$

is called an ***ideal of*** $(n_1,\dots,n_m)$***-homogeneous polynomials****.*

A multipolynomial ideal $\mathfrak{U}$ is said to be *closed* if all components $\mathfrak{U}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F)$ are closed subspaces of $(\mathcal{P}({}^{n_1}E_1,\dots,{}^{n_m}E_m;F),\|\cdot\|)$, where $\|\cdot\|$ is the usual multipolynomial norm.

Some basic remarks are in order:

- As particular cases or, more precisely, as extreme cases, every ideal of multilinear mappings (which includes the ideals of linear operators) as well as every polynomial ideal already established in the literature is a multipolynomial ideal. They will be called *extreme multipolynomial ideals.*

- Condition (Ua) easily leads $\mathfrak{F}$ right to the smallest multipolynomial ideal, as we shall see.

- From now on the symbol $\|\cdot\|$ will always denote the usual sup norm (uniform norm) and a class of multipolynomials to which no specific norm has been assigned is supposed to be endowed with it.

Proposition 3.2.2. *Regardless of the normed multipolynomial ideal* $(\mathfrak{U}, \|\cdot\|_{\mathfrak{U}})$*, we have* $\|P\| \leqslant \|P\|_{\mathfrak{U}}$ *for any* P *in* $\mathfrak{U}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$*.*

Proof. Given a component $\mathfrak{U}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ and $P \in \mathfrak{U}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$, let $x_j \in E_j$ $(1 \leqslant j \leqslant m)$ and define

$$\begin{array}{rccc} id_{\mathbb{K}} \otimes x_j : & \mathbb{K} & \to & E_j \\ & \lambda & \mapsto & \lambda x_j \end{array}$$

which belongs to $\mathcal{L}(\mathbb{K}; E_j)$ and $\|id_{\mathbb{K}} \otimes x_j\| = \|x_j\|$, for each $j = 1, \ldots, m$. Also,

$$\varphi \circ P \circ (id_{\mathbb{K}} \otimes x_1, \ldots, id_{\mathbb{K}} \otimes x_m) = (\varphi \circ P)(x_1, \ldots, x_m)\, id_m^{(n_1, \ldots, n_m)},$$

for every $\varphi \in F'$. Then

$$\begin{aligned} |(\varphi \circ P)(x_1, \ldots, x_m)| &= |(\varphi \circ P)(x_1, \ldots, x_m)| \left\| id_m^{(n_1, \ldots, n_m)} \right\|_{\mathfrak{U}} \\ &= \left\| (\varphi \circ P)(x_1, \ldots, x_m)\, id_m^{(n_1, \ldots, n_m)} \right\|_{\mathfrak{U}} \\ &= \|\varphi \circ P \circ (id_{\mathbb{K}} \otimes x_1, \ldots, id_{\mathbb{K}} \otimes x_m)\|_{\mathfrak{U}} \\ &\leqslant \|\varphi\| \, \|P\|_{\mathfrak{U}} \, \|x_1\|^{n_1} \cdots \|x_m\|^{n_m}, \end{aligned}$$

for every $\varphi \in F'$. It follows from Hanh-Banach's theorem that

$$\|P(x_1, \ldots, x_m)\| = \sup_{\varphi \in B_{F'}} |(\varphi \circ P)(x_1, \ldots, x_m)| \leqslant \|P\|_{\mathfrak{U}} \, \|x_1\|^{n_1} \cdots \|x_m\|^{n_m},$$

for every $x_j \in E_j$ $(1 \leqslant j \leqslant m)$. □

Recall that the family of normed operator ideals has a natural partial order (DIESTEL; JARCHOW; TONGE, 1995, p. 135). One can easily extend that to multipolynomials as follows. Given two normed multipolynomial ideals $(\mathfrak{a}, \|\cdot\|_{\mathfrak{a}})$ and $(\mathfrak{b}, \|\cdot\|_{\mathfrak{b}})$, we shall define

$$(\mathfrak{a}, \|\cdot\|_{\mathfrak{a}}) \subset (\mathfrak{b}, \|\cdot\|_{\mathfrak{b}}) \tag{3.9}$$

if, and only if, regardless of the choice of $m \in \mathbb{N}$, multi-degree $(n_1, \ldots, n_m) \in \mathbb{N}^m$, and Banach spaces $E_1, \ldots, E_m$ and F we have

$$\mathfrak{a}\left({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F\right) \subset \mathfrak{b}\left({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F\right),$$

and

$$\|P\|_{\mathfrak{b}} \leqslant \|P\|_{\mathfrak{a}}$$

for all $P \in \mathfrak{a}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$. Naturally,

$$(\mathfrak{a}, \|\cdot\|_{\mathfrak{a}}) = (\mathfrak{b}, \|\cdot\|_{\mathfrak{b}})$$

means that both relations $(\mathfrak{a}, \|\cdot\|_{\mathfrak{a}}) \subset (\mathfrak{b}, \|\cdot\|_{\mathfrak{b}})$ and $(\mathfrak{b}, \|\cdot\|_{\mathfrak{b}}) \subset (\mathfrak{a}, \|\cdot\|_{\mathfrak{a}})$ hold simultaneously.

We now give a list of several examples which will be studied next.

$\mathfrak{P}$: Ideal of continuous multipolynomials;

$\mathfrak{F}$: Ideal of finite-type multipolynomials;

$\overline{\mathfrak{U}}$: The closure of a multipolynomial ideal $\mathfrak{U}$;

$\mathfrak{P}_{\mathcal{A}}$: Ideal of approximable multipolynomials;

$\mathfrak{P}_{\mathbf{as}}$: Ideal of absolutely summing multipolynomials;

$\mathfrak{P}_{\mathbf{ms}}$: Ideal of multiple summing multipolynomials.

Let us now check each of the above examples.

3.2.1 Ideal of continuous multipolynomials

Proposition 3.2.3. *The class $(\mathfrak{P}, \|\cdot\|)$ of all continuous multipolynomials is a Banach multipolynomial ideal.*

Proof. Conditions (Ua), (Ub), and (U2) are immediate.

(U1): The map $\|\cdot\|_{\mathfrak{P}} : \mathfrak{P} \to [0, \infty)$, defined by $\|P\|_{\mathfrak{P}} := \|P\|$ for every $P \in \mathfrak{P}$, coincides with $\|\cdot\|$ on each $\mathfrak{P}$-component, which is a Banach space by Proposition 3.1.5.

(U3): For $P \in \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ $(1 \leqslant j \leqslant m)$, and $t \in \mathcal{L}_1(F; H)$, we have

$$\begin{aligned}
\|(t \circ P \circ (u_1, \ldots, u_m))(x_1, \ldots, x_m)\| &= \|t\left(P\left(u_1(x_1), \ldots, u_m(x_m)\right)\right)\| \\
&\leqslant \|t\| \, \|P\left(u_1(x_1), \ldots, u_m(x_m)\right)\| \\
&\leqslant \|t\| \, \|P\| \, \|u_1(x_1)\|^{n_1} \cdots \|u_m(x_m)\|^{n_m} \\
&\leqslant \|t\| \, \|P\| \left(\|u_1\| \, \|x_1\|\right)^{n_1} \cdots \left(\|u_m\| \, \|x_m\|\right)^{n_m},
\end{aligned}$$

for every $x_j \in G_j$ $(1 \leqslant j \leqslant m)$. Thus,

$$\|t \circ P \circ (u_1, \ldots, u_m)\| \leqslant \|t\| \, \|P\| \, \|u_1\|^{n_1} \cdots \|u_m\|^{n_m}$$

as desired. □

$(\mathfrak{P}, \|\cdot\|)$ is maximal (with respect to the partial order (3.9)). Indeed, by Definition 3.2.1 and Proposition 3.2.2 we have $(\mathfrak{U}, \|\cdot\|_{\mathfrak{U}}) \subset (\mathfrak{P}, \|\cdot\|)$ for all normed multipolynomial ideal $(\mathfrak{U}, \|\cdot\|_{\mathfrak{U}})$.

3.2.2 Ideal of finite-type multipolynomials

Proposition 3.2.4. *The class $(\mathfrak{F}, \|\cdot\|)$ of all finite-type multipolynomials is a normed multipolynomial ideal.*

Proof. (Ua) is obvious. If $P \in \mathfrak{F}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ $(1 \leqslant j \leqslant m)$, and $t \in \mathcal{L}(F; H)$, then

$$t \circ P \circ (u_1, \ldots, u_m) \in \mathcal{P}\left(^{n_1}G_1, \ldots, ^{n_m}G_m; H\right).$$

Besides,

$$\begin{aligned} t \circ P \circ (u_1, \ldots, u_m)(x_1, \ldots, x_m) &= t \circ P\left(u_1(x_1), \ldots, u_m(x_m)\right) \\ &= t\left(\sum_{i=1}^{k} \varphi_i^{(1)}\left(u_1(x_1)\right)^{n_1} \cdots \varphi_i^{(m)}\left(u_m(x_m)\right)^{n_m} b_i\right) \\ &= \sum_{i=1}^{k} \varphi_i^{(1)}\left(u_1(x_1)\right)^{n_1} \cdots \varphi_i^{(m)}\left(u_m(x_m)\right)^{n_m} t(b_i) \\ &= \sum_{i=1}^{k} \left(\varphi_i^{(1)} \circ u_1\right)(x_1)^{n_1} \cdots \left(\varphi_i^{(m)} \circ u_m\right)(x_m)^{n_m} t(b_i), \end{aligned}$$

where $(\varphi_i^{(j)} \circ u_j) \in G'_j$ and $t(b_i) \in H$. Thus,

$$t \circ P \circ (u_1, \ldots, u_m) \in \mathfrak{F}\left(^{n_1}G_1, \ldots, ^{n_m}G_m; H\right).$$

We have shown (Ub). The remaining axioms are inherited by the $\mathfrak{F}$-components from the corresponding $(\mathfrak{P}, \|\cdot\|)$-components, and the proof follows. □

$\mathfrak{F}$ is the smallest multipolynomial ideal, but $(\mathfrak{F}, \|\cdot\|)$ is not a Banach multipolynomial ideal. In fact, the linear $\mathfrak{F}$-component is $\mathcal{F}$, the ideal of finite rank linear operators, which is not a Banach ideal (DIESTEL; JARCHOW; TONGE, 1995, p. 131-132).

3.2.3 The closure of a multipolynomial ideal

Given a multipolynomial ideal $\mathfrak{U}$, we shall define

$$\overline{\mathfrak{U}}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right) := \overline{\mathfrak{U}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right)}^{\|\cdot\|},$$

for all $m \in \mathbb{N}$, multi-degree $(n_1,\ldots,n_m) \in \mathbb{N}^m$ and Banach spaces $E_1,\ldots,E_m$ and F. We shall denote by $\overline{\mathfrak{U}}$ the *closure* of $\mathfrak{U}$.

Example 3.2.5. *The so-called **approximable multipolynomials** are defined to be the members of*

$$\mathfrak{P}_A\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right) := \overline{\mathfrak{F}}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right).$$

Proposition 3.2.6. *If $\mathfrak{U}$ is a multipolynomial ideal then $(\overline{\mathfrak{U}},\|\cdot\|)$ is a Banach multipolynomial ideal.*

Proof. Since

$$\begin{aligned}\mathfrak{F}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right) &\subset \mathfrak{U}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right)\\ &\subset \overline{\mathfrak{U}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right)}^{\|\cdot\|}\\ &\subset \mathcal{P}\left({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F\right),\end{aligned}$$

and the closure of every linear subspace is a linear subspace, we obtain (Ua). If $P \in \overline{\mathfrak{U}}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$, $u_j \in \mathcal{L}(G_j;E_j)$ $(1 \leqslant j \leqslant m)$, and $t \in \mathcal{L}(F;H)$, then there exists a sequence (P_k) of multipolynomials in $\mathfrak{U}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$ such that $\lim_{k\to\infty}\|P_k - P\| = 0$ and

$$t \circ P_k \circ (u_1,\ldots,u_m) \in \mathfrak{U}\left({}^{n_1}G_1,\ldots,{}^{n_m}G_m;H\right).$$

It follows from Proposition 3.2.3 that

$$\begin{aligned}\|t \circ P_k \circ (u_1,\ldots,u_m) - t \circ P \circ (u_1,\ldots,u_m)\| &= \|t \circ (P_k - P) \circ (u_1,\ldots,u_m)\|\\ &\leqslant \|t\|\,\|P_k - P\|\,\|u_1\|^{n_1}\cdots\|u_m\|^{n_m},\end{aligned}$$

and (Ub) finishes together with (U3) after letting $k \to \infty$. The $\overline{\mathfrak{U}}$-components are closed under the uniform norm induced by its corresponding Banach $\mathfrak{P}$-components and therefore, are complete. The remaining axioms are inherited from $(\mathfrak{P},\|\cdot\|)$. ☐

In particular, $\mathfrak{P}_A$ is a Banach multipolynomial ideal. It is straightforward to see that $\overline{\mathfrak{U}}$ is the smallest closed multipolynomial ideal containing $\mathfrak{U}$ and $\mathfrak{P}_A$ is the smallest closed multipolynomial ideal.

3.2.4 Ideal of absolutely summing multipolynomials

Definition 3.2.7. *Let $0 < p, q_1, \ldots, q_m$. A continuous $(n_1, \ldots, n_m)$-homogeneous polynomial $P : E_1 \times \cdots \times E_m \to F$ is said to be* ***absolutely*** $(p; q_1, \ldots, q_m)$-***summing*** *(or $(p; q_1, \ldots, q_m)$-**summing**) if*

$$\left(P\left(x_j^{(1)}, \ldots, x_j^{(m)}\right)\right)_{j=1}^{\infty} \in \ell_p(F),$$

provided that $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$, $k = 1, \ldots, m$.

The vector space of all absolutely $(p; q_1, \ldots, q_m)$-summing $(n_1, \ldots, n_m)$-homogeneous polynomials from $E_1 \times \cdots \times E_m$ into F is denoted by $\mathcal{P}_{as(p;q_1,\ldots,q_m)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ ($\mathcal{P}_{as(p;q_1,\ldots,q_m)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m)$ if $F = \mathbb{K}$). When $q_1 = \cdots = q_m = q$, we simply write $\mathcal{P}_{as(p;q)}({}^{n_1}E_1, \ldots, {}^{n_1}E_m; F)$.

Lemma 3.2.8. *If $1/p > n_1/q_1 + \cdots + n_m/q_m$, then $\mathcal{P}_{as(p;q_1,\ldots,q_m)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F) = \{0\}$.*

Proof. Suppose there exists a multipolynomial $P \in \mathcal{P}_{as(p;q_1,\ldots,q_m)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ which is not zero. It follows from the hypothesis that

$$p < \frac{1}{\frac{n_1}{q_1} + \cdots + \frac{n_m}{q_m}} =: q,$$

and therefore, there exists a sequence of scalars $(\lambda_j) \in \ell_q \backslash \ell_p$ and a vector $(x_1, \ldots, x_m) \in E_1 \times \cdots \times E_m$ such that $P(x_1, \ldots, x_m) \neq 0$. Since $(\lambda_j^{q/q_k} x_k)_j \in \ell_{q_k}(E_k) \subset \ell_{q_k}^w(E_k)$ $(1 \leqslant k \leqslant m)$, we have

$$\begin{aligned}
(\lambda_j P(x_1, \ldots, x_m))_{j=1}^{\infty} &= \left(\lambda_j^{q\left(\frac{n_1}{q_1} + \cdots + \frac{n_m}{q_m}\right)} P(x_1, \ldots, x_m)\right)_{j=1}^{\infty} \\
&= \left(\lambda_j^{\frac{qn_1}{q_1}} \cdots \lambda_j^{\frac{qn_m}{q_m}} P(x_1, \ldots, x_m)\right)_{j=1}^{\infty} \\
&= \left(P\left(\lambda_j^{\frac{q}{q_1}} x_1, \ldots, \lambda_j^{\frac{q}{q_m}} x_m\right)\right)_{j=1}^{\infty} \in \ell_p(F),
\end{aligned}$$

and thus $(\lambda_j) \in \ell_p$, which is absurd. □

From now on, in order to avoid trivialities, we will suppose $1/p \leqslant n_1/q_1 + \cdots + n_m/q_m$.

Proposition 3.2.9. *A multipolynomial $P \in \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ is absolutely $(p; q_1, \ldots, q_m)$-summing if, and only if, the correspondence*

$$\widehat{P} : \left(\left(x_j^{(1)}\right)_{j=1}^{\infty}, \ldots, \left(x_j^{(m)}\right)_{j=1}^{\infty}\right) \mapsto \left(P\left(x_j^{(1)}, \ldots, x_j^{(m)}\right)\right)_{j=1}^{\infty}$$

induces a continuous $(n_1, \ldots, n_m)$-homogeneous polynomial

$$\ell_{q_1}^w(E_1) \times \cdots \times \ell_{q_m}^w(E_m) \to \ell_p(F).$$

Proof. To prove the non-trivial assertion, it is apparent that every multipolynomial $P \in \mathcal{P}_{as(p;q_1,\dots,q_m)}(^{n_1}E_1,\dots,^{n_m}E_m;F)$ induces the mapping

$$\begin{array}{cccc} \widehat{P}: & \ell_{q_1}^{w}(E_1)\times\cdots\times\ell_{q_m}^{w}(E_m) & \to & \ell_p(F) \\ & \left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\dots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right) & \mapsto & \left(P\left(x_j^{(1)},\dots,x_j^{(m)}\right)\right)_{j=1}^{\infty} \end{array}$$

which we claim to be an $(n_1,\dots,n_m)$-homogeneous polynomial. To see this, note that for each $i=1,\dots,m$, the mapping

$$\widehat{P}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\dots,\overset{i\text{-th}}{\cdot},\dots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)^{\vee}:\ell_q^w(E_i)\times\cdots\times\ell_q^w(E_i)\to\ell_p(F)$$

defined by

$$\left(\left(y_j^{(1)}\right)_{j=1}^{\infty},\dots,\left(y_j^{(n_i)}\right)_{j=1}^{\infty}\right)\mapsto\left(\overset{\vee}{P}_{\left(x_j^{(1)},\dots,\overset{i\text{-th}}{\cdot},\dots,x_j^{(m)}\right)}\left(y_j^{(1)},\dots,y_j^{(n_i)}\right)\right)_{j=1}^{\infty} \tag{3.10}$$

is n_i-linear and symmetric which coincides with $\widehat{P}((x_j^{(1)})_{j=1}^{\infty},\dots,\overset{i\text{-th}}{\cdot},\dots,(x_j^{(m)})_{j=1}^{\infty})$ on the diagonal. To see that (3.10) is well defined, note that

$$\begin{aligned}&\overset{\vee}{P}_{\left(x_j^{(1)},\dots,\overset{i\text{-th}}{\cdot},\dots,x_j^{(m)}\right)}\left(y_j^{(1)},\dots,y_j^{(n_i)}\right)\\ &=\frac{1}{n_i!2^{n_i}}\sum_{\varepsilon_k=\pm1}\varepsilon_1\cdots\varepsilon_{n_i}P\left(x_j^{(1)},\dots,\varepsilon_1y_j^{(1)}+\cdots+\varepsilon_{n_i}y_j^{(n_i)},\dots,x_j^{(m)}\right)\end{aligned}$$

and therefore,

$$\left(\overset{\vee}{P}_{\left(x_j^{(1)},\dots,\overset{i\text{-th}}{\cdot},\dots,x_j^{(m)}\right)}\left(y_j^{(1)},\dots,y_j^{(n_i)}\right)\right)_{j=1}^{\infty}\in\ell_p(F),$$

provided that P is $(p;q_1,\dots,q_m)$-summing. We have shown that $\widehat{P}$ is an $(n_1,\dots,n_m)$-homogeneous polynomial. Besides, if the mapping (3.10) is continuous for each $i=1,\dots,m$ then $\widehat{P}$ is separately continuous by Theorem 1.2.1 and therefore, continuous by Proposition 3.1.4. So, we only need to prove the continuity of (3.10), which will be done by closed graph theorem (see, e.g., (DEFANT; FLORET, 1993, Ex 1.11)). In fact, let $(y_n)_{n=1}^{\infty}$ be a sequence of vectors

$$y_n=\left(\left(y_{n,j}^{(1)}\right)_{j=1}^{\infty},\dots,\left(y_{n,j}^{(n_i)}\right)_{j=1}^{\infty}\right)\in\left(\ell_q^w(E_i)\right)^{n_i}$$

such that

$$\lim_{n\to\infty}y_n=y:=\left(\left(y_j^{(1)}\right)_{j=1}^{\infty},\dots,\left(y_j^{(n_i)}\right)_{j=1}^{\infty}\right)\in\left(\ell_q^w(E_i)\right)^{n_i}$$

and

$$\lim_{n\to\infty}\widehat{P}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\dots,\overset{i\text{-th}}{\cdot},\dots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)^{\vee}(y_n)=z:=(z_j)_{j=1}^{\infty}\in\ell_p(F).$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\left(y_{n,j}^{(1)}\right)_{j=1}^{\infty} &= \left(y_{j}^{(1)}\right)_{j=1}^{\infty}\in \ell_q^w\left(E_i\right)\\ &\vdots\\ \lim_{n\to\infty}\left(y_{n,j}^{(n_i)}\right)_{j=1}^{\infty} &= \left(y_{j}^{(n_i)}\right)_{j=1}^{\infty}\in \ell_q^w\left(E_i\right),\end{aligned}$$

and

$$\lim_{n\to\infty}\left(\overset{\vee}{P}_{\left(x_j^{(1)},\ldots,\overset{i\text{-th}}{\cdot},\ldots,x_j^{(m)}\right)}\left(y_{n,j}^{(1)},\ldots,y_{n,j}^{(n_i)}\right)\right)_{j=1}^{\infty} = z = (z_j)_{j=1}^{\infty}\in \ell_p\left(F\right).$$

It follows from $\ell_q^w(E_i)$ and $\ell_p(F)$ norm definitions that

$$\lim_{n\to\infty}\left(y_{n,j}^{(1)},\ldots,y_{n,j}^{(n_i)}\right) = \left(y_{j}^{(1)},\ldots,y_{j}^{(n_i)}\right)\in E_i^{n_i}$$

and

$$\lim_{n\to\infty}\overset{\vee}{P}_{\left(x_j^{(1)},\ldots,\overset{i\text{-th}}{\cdot},\ldots,x_j^{(m)}\right)}\left(y_{n,j}^{(1)},\ldots,y_{n,j}^{(n_i)}\right) = z_j\in F,$$

for each $j\in\mathbb{N}$. Since $\overset{\vee}{P}_{\left(x_j^{(1)},\ldots,\overset{i\text{-th}}{\cdot},\ldots,x_j^{(m)}\right)}$ is continuous, it follows from the uniqueness of the limit that

$$z_j = \overset{\vee}{P}_{\left(x_j^{(1)},\ldots,\overset{i\text{-th}}{\cdot},\ldots,x_j^{(m)}\right)}\left(y_{j}^{(1)},\ldots,y_{j}^{(n_i)}\right)$$

for every $j\in\mathbb{N}$ and thus,

$$\begin{aligned} z &= \left(\overset{\vee}{P}_{\left(x_j^{(1)},\ldots,\overset{i\text{-th}}{\cdot},\ldots,x_j^{(m)}\right)}\left(y_{j}^{(1)},\ldots,y_{j}^{(n_i)}\right)\right)_{j=1}^{\infty}\\ &=: \widehat{P}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\ldots,\overset{i\text{-th}}{\cdot},\ldots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)^{\vee}(y),\end{aligned}$$

and the proof follows. □

Corollary 3.2.10. *The mapping* $P\mapsto\widehat{P}$ *is an one-to-one linear operator from* $\mathcal{P}_{as(p;q_1,\ldots,q_m)}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$ *to* $\mathcal{P}({}^{n_1}(\ell_{q_1}^w(E_1)),\ldots,{}^{n_m}(\ell_{q_m}^w(E_m));\ell_p(F))$ *with closed range.*

Proof. One can readily see that the mapping $P\mapsto\widehat{P}$ is an one-to-one linear operator. To establish the last assertion, let $R\in\mathcal{P}({}^{n_1}(\ell_{q_1}^w(E_1)),\ldots,{}^{n_m}(\ell_{q_m}^w(E_m));\ell_p(F))$ be a multipolynomial in the closure of this operator range. Then, there exists a sequence $(\widehat{P_n})$, with $P_n\in\mathcal{P}_{as(p;q_1,\ldots,q_m)}({}^{n_1}E_1,\ldots,{}^{n_m}E_m;F)$, such that $\|\widehat{P_n}-R\|\to 0$. In particular, we have the pointwise convergence and therefore, with the aid of the 1-st projection map π_1, we get

$$\begin{aligned}\lim_{n\to\infty}P_n\left(x_1,\ldots,x_m\right) &= \lim_{n\to\infty}\pi_1\left(P_n\left(x_1,\ldots,x_m\right),0,0,\ldots\right)\\ &= \lim_{n\to\infty}\pi_1\left(\widehat{P_n}\left(\left(x_1,0,0,\ldots\right),\ldots,\left(x_m,0,0,\ldots\right)\right)\right)\\ &= \pi_1\left(\lim_{n\to\infty}\widehat{P_n}\left(\left(x_1,0,0,\ldots\right),\ldots,\left(x_m,0,0,\ldots\right)\right)\right)\\ &= \pi_1\left(R\left(\left(x_1,0,0,\ldots\right),\ldots,\left(x_m,0,0,\ldots\right)\right)\right).\end{aligned}$$

An application of Corollary 3.1.7 allows us to define a multipolynomial $P \in \mathcal{P}({}^{n_1}E_1, \dots$ $\dots, {}^{n_m}E_m; F)$ by

$$P(x_1, \dots, x_m) := \lim_{n\to\infty} P_n(x_1, \dots, x_m).$$

We claim that $P \in \mathcal{P}_{as(p;q_1,\dots,q_m)}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ and $\widehat{P} = R$. Indeed, for every positive integer N,

$$\begin{aligned}
\left(\sum_{j=1}^{N}\left\|P\left(x_j^{(1)},\dots,x_j^{(m)}\right)\right\|^p\right)^{\frac{1}{p}} &= \lim_{n\to\infty}\left(\sum_{j=1}^{N}\left\|P_n\left(x_j^{(1)},\dots,x_j^{(m)}\right)\right\|^p\right)^{\frac{1}{p}} \\
&= \lim_{n\to\infty}\left\|\widehat{P_n}\left(\left(x_j^{(1)}\right)_{j=1}^{N},\dots,\left(x_j^{(m)}\right)_{j=1}^{N}\right)\right\|_p \\
&= \left\|R\left(\left(x_j^{(1)}\right)_{j=1}^{N},\dots,\left(x_j^{(m)}\right)_{j=1}^{N}\right)\right\|_p \\
&\leqslant \|R\|\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{N}\right\|_{w,q_k}^{n_k} \\
&\leqslant \|R\|\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q_k}^{n_k}
\end{aligned}$$

provided that $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^{w}(E_k)$ $(1 \leqslant k \leqslant m)$ and therefore, $P \in \mathcal{P}_{as(p;q_1,\dots,q_m)}({}^{n_1}E_1, \dots$ $\dots, {}^{n_m}E_m; F)$. We prove the last claim by using the i-th projection map π_i, for each $i \in \mathbb{N}$. In fact, if $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^{w}(E_k)$ $(1 \leqslant k \leqslant m)$ then

$$\begin{aligned}
\pi_i\left(R\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\dots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)\right) &= \pi_i\left(\lim_{n\to\infty}\widehat{P_n}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\dots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)\right) \\
&= \lim_{n\to\infty}\pi_i\left(\left(P_n\left(x_j^{(1)},\dots,x_j^{(m)}\right)\right)_{j=1}^{\infty}\right) \\
&= \lim_{n\to\infty}P_n\left(x_i^{(1)},\dots,x_i^{(m)}\right) \\
&= P\left(x_i^{(1)},\dots,x_i^{(m)}\right) \\
&= \pi_i\left(\widehat{P}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\dots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right)\right),
\end{aligned}$$

for every $i \in \mathbb{N}$. It shows that $\widehat{P} = R$ and completes the proof. □

As in the classical framework, we have a characterization which plays a prominent role in the theory. For instance, it will lead us to define a norm on the space of absolutely summing multipolynomials.

Corollary 3.2.11. *Let $P \in \mathcal{P}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$. The following statements are equivalent:*

(i) *P is absolutely $(p; q_1, \dots, q_m)$-summing;*

(ii) *There exists a constant $C > 0$ such that*

$$\left(\sum_{j=1}^{n}\left\|P\left(x_j^{(1)},\ldots,x_j^{(m)}\right)\right\|^{p}\right)^{\frac{1}{p}} \leqslant C\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{n}\right\|_{w,q_k}^{n_k} \tag{3.11}$$

for all $n \in \mathbb{N}$ and $x_1^{(k)},\ldots,x_n^{(k)} \in E_k$, $k = 1,\ldots,m$.

(iii) *There exists a constant $C > 0$ such that*

$$\left(\sum_{j=1}^{\infty}\left\|P\left(x_j^{(1)},\ldots,x_j^{(m)}\right)\right\|^{p}\right)^{\frac{1}{p}} \leqslant C\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q_k}^{n_k} \tag{3.12}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^{w}(E_k)$, $k = 1,\ldots,m$.

The infimum of the $C > 0$ for which inequality (3.12) always holds is denoted by $\pi_{as(p;q_1,\ldots,q_m)}^{(n_1,\ldots,n_m)}$ and defines a norm on $\mathcal{P}_{as(p;q_1,\ldots,q_m)}(^{n_1}E_1,\ldots,^{n_m}E_m;F)$.

Proof. The implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(ii) \Rightarrow (iii)$: For each $k = 1,\ldots,m$, let $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^{w}(E_k)$. Then we have

$$\left(\sum_{j=1}^{n}\left|\varphi\left(x_j^{(k)}\right)\right|^{q_k}\right)^{\frac{1}{q_k}} \leqslant \left(\sum_{j=1}^{\infty}\left|\varphi\left(x_j^{(k)}\right)\right|^{q_k}\right)^{\frac{1}{q_k}} \leqslant \left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q_k},$$

for all $\varphi \in B_{E_k'}$ and every $n \in \mathbb{N}$. Therefore,

$$\left\|\left(x_j^{(k)}\right)_{j=1}^{n}\right\|_{w,q_k} = \sup_{\varphi\in B_{E_k'}}\left(\sum_{j=1}^{n}\left|\varphi\left(x_j^{(k)}\right)\right|^{q_k}\right)^{\frac{1}{q_k}} \leqslant \left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q_k},$$

for every $n \in \mathbb{N}$ $(1 \leqslant k \leqslant m)$. By (ii) we get

$$\begin{aligned}\left(\sum_{j=1}^{n}\left\|P\left(x_j^{(1)},\ldots,x_j^{(m)}\right)\right\|^{p}\right)^{\frac{1}{p}} &\leqslant C\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{n}\right\|_{w,q_k}^{n_k}\\ &\leqslant C\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q_k}^{n_k},\end{aligned}$$

and (iii) follows after letting $n \to \infty$.

$(i) \Rightarrow (iii)$: Since, by Proposition 3.2.9, the induced mapping

$$\widehat{P} : \ell_{q_1}^{w}(E_1) \times \cdots \times \ell_{q_m}^{w}(E_m) \to \ell_p(F)$$

defined by

$$\widehat{P}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty},\ldots,\left(x_j^{(m)}\right)_{j=1}^{\infty}\right) = \left(P\left(x_j^{(1)},\ldots,x_j^{(m)}\right)\right)_{j=1}^{\infty},$$

is a continuous $(n_1, \ldots, n_m)$-homogeneous polynomial, we have

$$\left(\sum_{j=1}^{\infty}\left\|P\left(x_j^{(1)}, \ldots, x_j^{(m)}\right)\right\|^p\right)^{\frac{1}{p}} = \left\|\widehat{P}\left(\left(x_j^{(1)}\right)_{j=1}^{\infty}, \ldots, \left(x_j^{(m)}\right)_{j=1}^{\infty}\right)\right\|_p \leqslant \left\|\widehat{P}\right\| \prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q_k}^{n_k} \tag{3.13}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^{w}(E_k)$ $(1 \leqslant k \leqslant m)$.

To prove the last part, let $C > 0$ be any constant for which inequality (3.12) holds. Then it follows from the definition of $\widehat{P}$ that $\|\widehat{P}\| \leqslant C$, for all such constants. It follows from (3.13) that the infimum is attained by $\|\widehat{P}\| = \pi_{as(p;q_1,\ldots,q_m)}^{(n_1,\ldots,n_m)}(P)$. All the norm axioms follow immediately from this last equation combined with Corollary 3.2.10. The proof has been completed. □

To establish completeness, we recall a useful result.

Lemma 3.2.12. *Let E be a vector space, let F be a Banach space and let $T : E \to F$ be an one-to-one linear operator with closed range. Then the map*

$$\begin{array}{rccc} \|\cdot\|_E : & E & \to & \mathbb{R} \\ & x & \mapsto & \|Tx\| \end{array}$$

defines a norm on E which makes $(E, \|\cdot\|_E)$ complete.

Proof. One can readily see that the mapping $\|\cdot\|_E$ defines a norm on E. Note that if (x_n) is a Cauchy sequence in $(E, \|\cdot\|_E)$, then (Tx_n) is a Cauchy sequence in F. Since F is complete, there exists $y \in F$ for which (Tx_n) converges. Since $T(E)$ is closed in F, there is $x \in E$ such that $y = Tx$ and therefore, (x_n) converges to $x \in E$ because

$$\|x_n - x\|_E = \|T(x_n - x)\| = \|Tx_n - y\|.$$

□

Corollary 3.2.13. $\mathcal{P}_{as(p;q_1,\ldots,q_m)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ *is a Banach space under the norm* $\pi_{as(p;q_1,\ldots,q_m)}^{(n_1,\ldots,n_m)}$.

Proof. It is an immediate consequence of Corollary 3.2.10, Lemma 3.2.12, and the equation

$$\left\|\widehat{P}\right\| = \pi_{(p,q_1,\ldots,q_m)}^{(n_1,\ldots,n_m)}(P), \quad \forall\, P \in \mathcal{P}_{as(p;q_1,\ldots,q_m)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F), \tag{3.14}$$

proved in the last part of Corollary 3.2.11. □

Corollary 3.2.14. *The mapping $P \mapsto \widehat{P}$ is an isometric isomorphism from the Banach space $(\mathcal{P}_{as(p;q_1,\ldots,q_m)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F), \pi_{as(p;q_1,\ldots,q_m)}^{(n_1,\ldots,n_m)})$ onto its range.*

Proof. It follows immediately from Corollary 3.2.10 combined with the isometry (3.14) and the open mapping theorem. □

Proposition 3.2.15. *If $P \in \mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ is a finite-type multipolynomial then P is absolutely $(p; q_1, \ldots, q_m)$-summing.*

Proof. It suffices to show that the map

$$\begin{array}{ccc} E_1 \times \cdots \times E_m & \to & F \\ (x_1, \ldots, x_m) & \mapsto & \varphi_1(x_1)^{n_1} \cdots \varphi_m(x_m)^{n_m} b \end{array}$$

is absolutely $(p; q_1, \ldots, q_m)$-summing, for all $\varphi_k \in E_k'$ $(1 \leqslant k \leqslant m)$ and $b \in F$. Indeed, we can assume $\varphi_k \neq 0$ for every $k = 1, \ldots, m$ and, since $1/(q_1/n_1) + \cdots + 1/(q_m/n_m) \geqslant 1/p$, it follows from Hlder's inequality (see Corollary A.0.3) that

$$\begin{aligned}
&\left(\sum_{j=1}^{n} \left\| \varphi_1\left(x_j^{(1)}\right)^{n_1} \cdots \varphi_m\left(x_j^{(m)}\right)^{n_m} b \right\|^p\right)^{\frac{1}{p}} \\
&= \|\varphi_1\|^{n_1} \cdots \|\varphi_m\|^{n_m} \|b\| \left(\sum_{j=1}^{n} \left| \frac{\varphi_1}{\|\varphi_1\|}\left(x_j^{(1)}\right)^{n_1} \cdots \frac{\varphi_m}{\|\varphi_m\|}\left(x_j^{(m)}\right)^{n_m} \right|^p\right)^{\frac{1}{p}} \\
&\leqslant \|\varphi_1\|^{n_1} \cdots \|\varphi_m\|^{n_m} \|b\| \left(\sum_{j=1}^{n} \left| \frac{\varphi_1}{\|\varphi_1\|}\left(x_j^{(1)}\right)^{n_1} \right|^{\frac{q_1}{n_1}}\right)^{\frac{n_1}{q_1}} \cdots \left(\sum_{j=1}^{n} \left| \frac{\varphi_m}{\|\varphi_m\|}\left(x_j^{(m)}\right)^{n_m} \right|^{\frac{q_m}{n_m}}\right)^{\frac{n_m}{q_m}} \\
&= \|\varphi_1\|^{n_1} \cdots \|\varphi_m\|^{n_m} \|b\| \left(\sum_{j=1}^{n} \left| \frac{\varphi_1}{\|\varphi_1\|}\left(x_j^{(1)}\right) \right|^{q_1}\right)^{\frac{n_1}{q_1}} \cdots \left(\sum_{j=1}^{n} \left| \frac{\varphi_m}{\|\varphi_m\|}\left(x_j^{(m)}\right) \right|^{q_m}\right)^{\frac{n_m}{q_m}} \\
&\leqslant \|\varphi_1\|^{n_1} \cdots \|\varphi_m\|^{n_m} \|b\| \left\| \left(x_j^{(1)}\right)_{j=1}^{n} \right\|_{w,q_1}^{n_1} \cdots \left\| \left(x_j^{(m)}\right)_{j=1}^{n} \right\|_{w,q_m}^{n_m}
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x_1^{(k)}, \ldots, x_n^{(k)} \in E_k$ $(1 \leqslant k \leqslant m)$. The desired result is now a consequence of Corollary 3.2.11. □

Proposition 3.2.16. *For fixed $0 < p, q < \infty$, the class $(\mathfrak{P}_{as(p;q)}, \pi_{as(p;q)})$ of all absolutely $(p; q)$-summing multipolynomials is a Banach multipolynomial ideal.*

Proof. It has been established in the previous proposition that $\mathcal{P}_{as(p;q)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ is a linear subspace of $\mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ containing the $(n_1, \ldots, n_m)$-homogeneous polynomials of finite type. If $P \in \mathcal{P}_{as(p;q)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ $(1 \leqslant j \leqslant m)$, and $t \in \mathcal{L}(F;H)$, then

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty}\left\|t\circ P\circ(u_1,\dots,u_m)\left(x_j^{(1)},\dots,x_j^{(m)}\right)\right\|^p\right)^{\frac{1}{p}}\\
&\leqslant \|t\|\left(\sum_{j=1}^{\infty}\left\|P\left(u_1\left(x_j^{(1)}\right),\dots,u_m\left(x_j^{(m)}\right)\right)\right\|^p\right)^{\frac{1}{p}}\\
&\leqslant \|t\|\,\pi_{as(p;q)}^{(n_1,\dots,n_m)}(P)\left\|\left(u_1\left(x_j^{(1)}\right)\right)_{j=1}^{\infty}\right\|_{w,q}^{n_1}\cdots\left\|\left(u_m\left(x_j^{(m)}\right)\right)_{j=1}^{\infty}\right\|_{w,q}^{n_m}\\
&\leqslant \|t\|\,\pi_{as(p;q)}^{(n_1,\dots,n_m)}(P)\,\|u_1\|^{n_1}\cdots\|u_m\|^{n_m}\left\|\left(x_j^{(1)}\right)_{j=1}^{\infty}\right\|_{w,q}^{n_1}\cdots\left\|\left(x_j^{(m)}\right)_{j=1}^{\infty}\right\|_{w,q}^{n_m},
\end{aligned}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_q^w(G_k)$, $(1 \leqslant k \leqslant m)$. It follows from Corollary 3.2.11 that

$$t\circ P\circ(u_1,\dots,u_m)\in\mathcal{P}_{as(p;q)}\left({}^{n_1}G_1,\dots,{}^{n_m}G_m;H\right),$$

and

$$\pi_{as(p;q)}^{(n_1,\dots,n_m)}(t\circ P\circ(u_1,\dots,u_m))\leqslant\|t\|\,\pi_{as(p;q)}^{(n_1,\dots,n_m)}(P)\,\|u_1\|^{n_1}\cdots\|u_m\|^{n_m}.$$

We have shown conditions (Ub) and (U3). Completeness follows from Corollary 3.2.13. One can readily see from (3.11) that

$$1=\left\|id_m^{(n_1,\dots,n_m)}\right\|\leqslant\pi_{as(p;q)}^{(n_1,\dots,n_m)}\left(id_m^{(n_1,\dots,n_m)}\right).$$

To prove the reverse inequality, let $(\lambda_j^{(k)})_{j=1}^{\infty} \in \ell_q^w(\mathbb{K})$ $(1 \leqslant k \leqslant m)$. It follows from Hlder's inequality that, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\left(\sum_{j=1}^{n}\left|id_m^{(n_1,\dots,n_m)}\left(\lambda_j^{(1)},\dots,\lambda_j^{(m)}\right)\right|^p\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^{n}\left|\left(\lambda_j^{(1)}\right)^{n_1}\cdots\left(\lambda_j^{(m)}\right)^{n_m}\right|^p\right)^{\frac{1}{p}}\\
&\leqslant\left(\sum_{j=1}^{n}\left|\left(\lambda_j^{(1)}\right)^{n_1}\right|^{\frac{q}{n_1}}\right)^{\frac{n_1}{q}}\cdots\left(\sum_{j=1}^{n}\left|\left(\lambda_j^{(m)}\right)^{n_m}\right|^{\frac{q}{n_m}}\right)^{\frac{n_m}{q}}\\
&\leqslant\left\|\left(\lambda_j^{(1)}\right)_{j=1}^{\infty}\right\|_q^{n_1}\cdots\left\|\left(\lambda_j^{(m)}\right)_{j=1}^{\infty}\right\|_q^{n_m}\\
&=\left\|\left(\lambda_j^{(1)}\right)_{j=1}^{\infty}\right\|_{w,q}^{n_1}\cdots\left\|\left(\lambda_j^{(m)}\right)_{j=1}^{\infty}\right\|_{w,q}^{n_m}.
\end{aligned}$$

Letting $n \to \infty$ we get

$$\left(\sum_{j=1}^{\infty}\left|id_m^{(n_1,\dots,n_m)}\left(\lambda_j^{(1)},\dots,\lambda_j^{(m)}\right)\right|^p\right)^{\frac{1}{p}}\leqslant\prod_{k=1}^{m}\left\|\left(\lambda_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q}^{n_k}.$$

Hence $\pi_{as(p;q)}^{(n_1,\dots,n_m)}(id_m^{(n_1,\dots,n_m)}) \leqslant 1$ and (U2) follows. □

3.2.5 Ideal of multiple summing multipolynomials

Let J be a countable set. We intend to introduce the following vector-valued function spaces. For $1 \leqslant p < \infty$ and a Banach space E, we shall define $\ell_p^J(E)$ to be the set of all functions $f : J \to E$ such that $\sum_{j\in J}\|f(j)\|^p$ is finite. It follows from Minkowski's inequality that $\ell_p^J(E)$ is a vector space and

$$\|f\|_p := \left(\sum_{j\in J}\|f(j)\|^p\right)^{\frac{1}{p}}$$

is a norm on $\ell_p^J(E)$. An effortless adaptation of the usual proof that ℓ_p is a Banach space rapidly leads to the conclusion that $\ell_p^J(E)$ is a Banach space. The classical vector-valued sequence spaces $\ell_p(E)$ is obtained by taking $J = \mathbb{N}$.

Next, we present a quite more demanding definition than Definition 3.2.7.

Definition 3.2.17. *Let $0 < p, q_1, \ldots, q_m$. A continuous $(n_1, \ldots, n_m)$-homogeneous polynomial $P : E_1 \times \cdots \times E_m \to F$ is said to be **multiple** $(p; q_1, \ldots, q_m)$-**summing** if*

$$\left(P\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right)\right)_{j_1,\ldots,j_m=1}^{\infty} \in \ell_p^{\mathbb{N}^m}(F),$$

provided that $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$, $k = 1, \ldots, m$.

The vector space of all multiple $(p; q_1, \ldots, q_m)$-summing $(n_1, \ldots, n_m)$-homogeneous polynomials from $E_1 \times \cdots \times E_m$ into F is denoted by $\mathcal{P}_{ms(p;q_1,\ldots,q_m)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ ($\mathcal{P}_{ms(p;q_1,\ldots,q_m)}(^{n_1}E_1, \ldots, ^{n_m}E_m)$ if $F = \mathbb{K}$). When $q_1 = \cdots = q_m = q$, we simply write $\mathcal{P}_{ms(p;q)}(^{n_1}E_1, \ldots, ^{n_1}E_m; F)$.

The previous absolutely-summing theory can be easily translated to multiple summing multipolynomials. For instance, we have the following key results.

If $q_k/n_k > p$ for some $k = 1, \ldots, m$, we have $\mathcal{P}_{ms(p;q_1,\ldots,q_m)}(E_1, \ldots, E_m; F) = \{0\}$. So, we must suppose $q_k/n_k \leqslant p$ for every $k = 1, \ldots, m$.

Proposition 3.2.18. *Let $P \in \mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$. The following statements are equivalent:*

(i) *P is absolutely $(p; q_1, \ldots, q_m)$-summing;*

(ii) *There exists a constant $C > 0$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}\left\|P\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right)\right\|^p\right)^{\frac{1}{p}} \leqslant C\prod_{k=1}^{m}\left\|\left(x_j^{(k)}\right)_{j=1}^{n}\right\|_{w,q_k}^{n_k}$$

for every $n \in \mathbb{N}$ and all $x_1^{(k)}, \ldots, x_n^{(k)} \in E_k$, $k = 1, \ldots, m$.

(iii) *There exists a constant $C > 0$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left\| P\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right) \right\|^p \right)^{\frac{1}{p}} \leqslant C \prod_{k=1}^{m} \left\| \left(x_j^{(k)}\right)_{j=1}^{\infty} \right\|_{w,q_k}^{n_k} \tag{3.15}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$, $k = 1, \ldots, m$.

The infimum of the $C > 0$ for which inequality (3.15) always holds is denoted by $\pi_{ms(p;q_1,\ldots,q_m)}^{(n_1,\ldots,n_m)}$ and defines a norm on $\mathcal{P}_{ms(p;q_1,\ldots,q_m)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$.

Proposition 3.2.19. *For fixed $0 < p, q < \infty$, the class $(\mathfrak{P}_{ms(p;q)}, \pi_{ms(p;q)})$ of all multiple $(p;q)$-summing multipolynomials is a Banach multipolynomial ideal.*

With respect to the order (3.9), we have $(\mathfrak{P}_{ms(p;q)}, \pi_{ms(p;q)}) \subset (\mathfrak{P}_{as(p;q)}, \pi_{as(p;q)})$. Indeed, if P is multiple $(p;q)$-summing then

$$\begin{aligned} \sum_{j=1}^{n} \left\| P\left(x_j^{(1)}, \ldots, x_j^{(m)}\right) \right\|^p &\leqslant \sum_{j_1,\ldots,j_m=1}^{\infty} \left\| P\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right) \right\|^p \\ &\leqslant \pi_{ms(p,q)}^{(n_1,\ldots,n_m)}(P) \prod_{k=1}^{m} \left\| \left(x_j^{(k)}\right)_{j=1}^{\infty} \right\|_{w,q}^{n_k}. \end{aligned}$$

Hence, P is absolutely $(p;q)$-summing with

$$\pi_{as(p;q)}^{(n_1,\ldots,n_m)}(P) \leqslant \pi_{ms(p;q)}^{(n_1,\ldots,n_m)}(P), \quad \forall P \in \mathcal{P}_{ms(p;q)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F).$$

3.3 Hyper-ideals

This section aims to invoke the multipolynomials again to generalize and propose a unified approach to section 2.2.

Definition 3.3.1. *A **hyper-ideal of multipolynomials** (or **multipolynomial hyper-ideal**) is a subclass $\mathfrak{H}$ of the class of all continuous multipolynomials between Banach spaces such that for all $n \in \mathbb{N}$, multi-degree $(k_1, \ldots, k_n) \in \mathbb{N}^n$ and all Banach spaces $E_1, \ldots, E_n$ and F, the components*

$$\mathfrak{H}\left(^{k_1}E_1, \ldots, ^{k_n}E_n; F\right) := \mathcal{P}\left(^{k_1}E_1, \ldots, ^{k_n}E_n; F\right) \cap \mathfrak{H}$$

satisfy:

(Ha) *$\mathfrak{H}(^{k_1}E_1, \ldots, ^{k_n}E_n; F)$ is a linear subspace of $\mathcal{P}(^{k_1}E_1, \ldots, ^{k_n}E_n; F)$ which contains the $(k_1, \ldots, k_n)$-homogeneous polynomials of finite type;*

(Hb) *The hyper-ideal property: given natural numbers* n, $1 \leqslant m_1 < \cdots < m_n$ *and* $r_1, \ldots, r_{m_n}, k_1, \ldots, k_n$ *and* r *and Banach spaces* $G_1, \ldots, G_{m_n}, E_1, \ldots, E_n, F$ *and* H, *if* $Q_1 \in \mathcal{P}(^{r_1}G_1, \ldots, ^{r_{m_1}}G_{m_1}; E_1), \ldots, Q_n \in \mathcal{P}(^{r_{m_{n-1}+1}}G_{m_{n-1}+1}, \ldots, ^{r_{m_n}}G_{m_n}; E_n)$, $P \in \mathfrak{H}(^{k_1}E_1, \ldots$ $\ldots, ^{k_n}E_n; F)$ *and* $R \in \mathcal{P}(^rF; H)$, *then*

$$R \circ P \circ (Q_1, \ldots, Q_n) \in \mathfrak{H}\left(^{r_1 k_1 r}G_1, \ldots, ^{r_{m_n} k_n r}G_{m_n}; H\right).$$

If there exist a map $\|\cdot\|_{\mathfrak{H}} : \mathfrak{H} \to [0, \infty)$ *and a sequence* $(C_j, K_j)_{j=1}^{\infty}$ *of pairs of real numbers with* $C_j, K_j \geqslant 1$ *for every* $j \in \mathbb{N}$ *and* $C_1 = K_1 = 1$, *such that:*

(H1) $\|\cdot\|_{\mathfrak{H}}$ *restricted to* $\mathfrak{H}(^{k_1}E_1, \ldots, ^{k_n}E_n; F)$ *is a (quasi-) norm, for all* $n \in \mathbb{N}$, *multi-degree* $(k_1, \ldots, k_n) \in \mathbb{N}^n$ *and all Banach spaces* $E_1, \ldots, E_n$ *and* F;

(H2) $\|I_n^{(k_1,\ldots,k_n)} : \mathbb{K}^n \to \mathbb{K}, I_n^{(k_1,\ldots,k_n)}(\lambda_1, \ldots, \lambda_n) = \lambda_1^{k_1} \cdots \lambda_n^{k_n}\|_{\mathfrak{H}} = 1$, *for all* $n \in \mathbb{N}$ *and* $(k_1, \ldots, k_n) \in \mathbb{N}^n$;

(H3) *The hyper-ideal inequality: if* $Q_1 \in \mathcal{P}(^{r_1}G_1, \ldots, ^{r_{m_1}}G_{m_1}; E_1), \ldots, Q_n \in \mathcal{P}(^{r_{m_{n-1}+1}}G_{m_{n-1}+1}, \ldots, ^{r_{m_n}}G_{m_n}; E_n)$, $P \in \mathfrak{H}(^{k_1}E_1, \ldots, ^{k_n}E_n; F)$ *and* $R \in \mathcal{P}(^rF; H)$, *then*

$$\begin{aligned}&\|R \circ P \circ (Q_1, \ldots, Q_n)\|_{\mathfrak{H}} \\ &\leqslant K_r \left(C_{r_1} \cdots C_{r_{m_1}}\right)^{rk_1} \cdots \left(C_{r_{m_{n-1}+1}} \cdots C_{r_{m_n}}\right)^{rk_n} \|R\| \, \|P\|_{\mathfrak{H}}^r \, \|Q_1\|^{rk_1} \cdots \|Q_n\|^{rk_n},\end{aligned}$$

then $(\mathfrak{H}, \|\cdot\|_{\mathfrak{H}})$ *is called a* ***(quasi-) normed multipolynomial*** $(C_j, K_j)_{j=1}^{\infty}$***-hyper-ideal****. When all the components* $\mathfrak{H}(^{k_1}E_1, \ldots, ^{k_n}E_n; F)$ *are complete under the (quasi-) norm* $\|\cdot\|_{\mathfrak{H}}$ *above, then* $(\mathfrak{H}, \|\cdot\|_{\mathfrak{H}})$ *is called a* ***(quasi-) Banach multipolynomial*** $(C_j, K_j)_{j=1}^{\infty}$***-hyper-ideal****.*

When $C_j = K_j = 1$ for every $j \in \mathbb{N}$, we simply say that $\mathfrak{H}$ is a *(quasi-) normed/(quasi-) Banach multipolynomial hyper-ideal*.

Note that Definition 3.3.1 recovers the multilinear and polynomial cases. Indeed, setting $n = 1 = m_1$ we get Definition 2.2.3, if $r > 1$; and Definition 2.2.2, if $r = 1$. In the other end, setting $k_1 = \cdots = k_n = r_1 = \cdots = r_{m_n} = r = 1$ we recover Definition 2.2.1. Finally, it is plain that every (normed, quasi-normed, Banach, quasi-Banach) multipolynomial hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) multipolynomial ideal.

3.4 Coherence and compatibility

The pair-of-ideals notions from section 2.3 seems to have been the first attempt to obtain an evaluating method for extending ideals addressed to polynomials and multilinear mappings, simultaneously. Let us see a route to a more general and unified view of it.

Definition 3.4.1 (Compatible multipolynomial ideals). *Let $\mathcal{I}$ be a normed operator ideal. The (quasi-) normed ideal of $(n_1, \ldots, n_m)$-homogeneous polynomials $\mathcal{U}_m^{(n_1,\ldots,n_m)}$ is* ***compatible*** *with $\mathcal{I}$ if there exist positive constants α_1 and α_2 such that for all Banach spaces $E_1, \ldots, E_m$, the following conditions hold:*

(CP 1) *For each $k \in \{1, \ldots, m\}$, $P \in \mathcal{U}_m^{(n_1,\ldots,n_m)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$, $a_k \in E_k$ and $x_j \in E_j$ for all $j \in \{1, \ldots, m\} \backslash \{k\}$, the mapping $P_{a_k^{n_k-1}}(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_m)$ belongs to $\mathcal{I}(E_k; F)$ and*

$$\left\| P_{a_k^{n_k-1}}(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_m) \right\|_{\mathcal{I}}$$
$$\leqslant \alpha_1 \|P\|_{\mathcal{U}_m^{(n_1,\ldots,n_m)}} \|x_1\|^{n_1} \cdots \|x_{k-1}\|^{n_{k-1}} \|a_k\|^{n_k-1} \|x_{k+1}\|^{n_{k+1}} \cdots \|x_m\|^{n_m}.$$

(CP 2) *For each $P \in \mathcal{I}(E_m; F)$ and $\gamma_j \in E_j'$ for $j = 1, \ldots, m$, the mapping $\gamma_1^{n_1} \cdots \cdots \gamma_{m-1}^{n_{m-1}} \gamma_m^{n_m-1} P$ belongs to $\mathcal{U}_m^{(n_1,\ldots,n_m)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ and*

$$\left\| \gamma_1^{n_1} \cdots \gamma_{m-1}^{n_{m-1}} \gamma_m^{n_m-1} P \right\|_{\mathcal{U}_m^{(n_1,\ldots,n_m)}} \leqslant \alpha_2 \|\gamma_1\|^{n_1} \cdots \|\gamma_{m-1}\|^{n_{m-1}} \|\gamma_m\|^{n_m-1} \|P\|_{\mathcal{I}}.$$

(CP 3) *For each $P \in \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$, $k \in \{1, \ldots, m\}$ and $x_j \in E_j$ for all $j \in \{1, \ldots, m\} \backslash \{k\}$, $P(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_m)$ belongs to $\mathcal{U}_1^{(n_k)}({}^{n_k}E_k; F)$ if, and only if, $\check{P}_{(x_1,\ldots,x_{k-1},\cdot,x_{k+1},\ldots,x_m)}$ belongs to $\mathcal{U}_{n_k}^{(1,\ldots,1)}({}^1E_k, \ldots, {}^1E_k; F)$.*

We shall denote by $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ the *family of multipolynomial ideals* such that $\mathcal{U}_\alpha := \mathcal{U}_m^{(n_1,\ldots,n_m)}$ is a (quasi-) normed ideal of α-homogeneous polynomials, for each multi-index $\alpha = (n_1, \ldots, n_m) \in I = \bigcup_{m=1}^{\infty} \mathbb{N}^m$.

Definition 3.4.2 (Coherent multipolynomial ideals). *Let $\mathcal{I}$ be a normed operator ideal. A family $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of multipolynomial ideals, with $\mathcal{U}_1^{(1)} = \mathcal{I}$, is* ***coherent*** *if there exist positive constants $\beta_1, \beta_2, \beta_3, \beta_4$ such that for all Banach spaces $E, E_1, \ldots, E_{k+1}$ and F, the following conditions hold for all $k \in \mathbb{N}$ and for all multi-index $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$.*

(CH 1) *For each $j \in \{1, \ldots, k\}$, $P \in \mathcal{U}_k^{(n_1,\ldots,n_j+1,\ldots,n_k)}({}^{n_1}E_1, \ldots, {}^{n_j+1}E_j, \ldots, {}^{n_k}E_k; F)$, $a_j \in E_j$ and $x_i \in E_i$ for all $i \in \{1, \ldots, k\} \backslash \{j\}$, the n_j-homogeneous polynomial $P_{a_j}(x_1, \ldots \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_k)$ belongs to $\mathcal{U}_1^{(n_j)}({}^{n_j}E_j; F)$ and*

$$\left\| P_{a_j}(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_k) \right\|_{\mathcal{U}_1^{(n_j)}}$$
$$\leqslant \beta_1 \|P\|_{\mathcal{U}_k^{(n_1,\ldots,n_j+1,\ldots,n_k)}} \|x_1\|^{n_1} \cdots \|x_{j-1}\|^{n_{j-1}} \|a_j\| \|x_{j+1}\|^{n_{j+1}} \cdots \|x_k\|^{n_k}.$$

(CH 2) *For each $P \in \mathcal{U}_{k+1}^{(n_1,\ldots,n_{k+1})}({}^{n_1}E_1, \ldots, {}^{n_{k+1}}E_{k+1}; F)$ and $a_j \in E_j$ for $j = 1, \ldots, k+1$, the multipolynomial $P(\cdot, \ldots, \cdot, a_j, \cdot, \ldots, \cdot)$ belongs to $\mathcal{U}_k^{(n_1,\ldots,n_{j-1},n_{j+1},\ldots,n_{k+1})}({}^{n_1}E_1, \ldots \ldots, {}^{n_{j-1}}E_{j-1}, {}^{n_{j+1}}E_{j+1}, \ldots, {}^{n_{k+1}}E_{k+1}; F)$ and*

$$\left\| P(\cdot, \ldots, \cdot, a_j, \cdot, \ldots, \cdot) \right\|_{\mathcal{U}_k^{(n_1,\ldots,n_{j-1},n_{j+1},\ldots,n_{k+1})}} \leqslant \beta_2 \|P\|_{\mathcal{U}_{k+1}^{(n_1,\ldots,n_{k+1})}} \|a_j\|^{n_j}.$$

(CH 3) *For each* $j \in \{1, \dots, k\}$, $P \in \mathcal{U}_k^{(n_1,\dots,n_j,\dots,n_k)}({}^{n_1}E_1, \dots, {}^{n_j}E_j, \dots, {}^{n_k}E_k; F)$ *and* $\gamma_j \in E'_j$, *the mapping* $\gamma_j P$ *belongs to* $\mathcal{U}_k^{(n_1,\dots,n_j+1,\dots,n_k)}({}^{n_1}E_1, \dots, {}^{n_j+1}E_j, \dots, {}^{n_k}E_k; F)$ *and*

$$\|\gamma_j P\|_{\mathcal{U}_k^{(n_1,\dots,n_j+1,\dots,n_k)}} \leqslant \beta_3 \|\gamma_j\| \|P\|_{\mathcal{U}_k^{(n_1,\dots,n_j,\dots,n_k)}} .$$

(CH 4) *For each* $P \in \mathcal{U}_k^{(n_1,\dots,n_k)}({}^{n_1}E_1, \dots, {}^{n_k}E_k; F)$ *and* $\gamma \in E'_{k+1}$, *the mapping* γP *belongs to* $\mathcal{U}_{k+1}^{(n_1,\dots,n_k,1)}({}^{n_1}E_1, \dots, {}^{n_k}E_k, {}^{1}E_{k+1}; F)$ *and*

$$\|\gamma P\|_{\mathcal{U}_{k+1}^{(n_1,\dots,n_k,1)}} \leqslant \beta_4 \|\gamma\| \|P\|_{\mathcal{U}_k^{(n_1,\dots,n_k)}} .$$

(CH 5) *For each* $P \in \mathcal{P}({}^{n_1}E_1, \dots, {}^{n_k}E_k; F)$, $j \in \{1, \dots, k\}$ *and* $x_i \in E_i$ *for all* $i \in \{1, \dots, k\} \backslash \{j\}$, $P(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_k)$ *belongs to* $\mathcal{U}_1^{(n_j)}({}^{n_j}E_j; F)$ *if, and only if,* $\check{P}_{(x_1,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_k)}$ *belongs to* $\mathcal{U}_{n_j}^{(1,\dots,1)}({}^{1}E_j, \dots, {}^{1}E_j; F)$.

Definition 3.4.1, with $m = 1$, recovers Definition 2.3.1. If we fix $k = 1$ and makes n_1 vary in $\{1, \dots, N-1\}$, then items (CH 1) and (CH 3) of Definition 3.4.2 recover Definition 2.3.2. Note that, in any case, one may go even further toward the apparently overlooked multilinear setting. Indeed, it suffices to set $m > 1$ and $n_1 = \cdots = n_m = 1$.

One can extract a compatible (or coherent) pair of ideals from a given family of multipolynomial ideals.

Proposition 3.4.3. *Let* $N \in (\mathbb{N}\backslash\{1\}) \cup \{\infty\}$. *If a family* $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ *of multipolynomial ideals is compatible with the operator ideal* $\mathcal{I} := \mathcal{U}_1^{(1)}$ *(resp. coherent), then the sequence of pairs of ideals* $(\mathcal{U}_n, \mathcal{M}_n)_{n=1}^N$ *is compatible with* $\mathcal{I}$ *(resp. coherent).*

Proof. Recall that $\mathcal{U}_n = \mathcal{U}_1^{(n)}$ and $\mathcal{M}_n = \mathcal{U}_n^{(1,\dots,1)}$, for every $n \in \mathbb{N}$. Let us treat the compatible case (the coherent case is analogous). Let $E, E_1, \dots, E_n$ and F be Banach spaces for a fixed (but arbitrary) $n \in \{2, \dots, N\}$. Applying the hypothesis with $\alpha = (1, \dots, 1) \in \mathbb{N}^n$ then (cp-i) and (cp-iii) respectively follow from (CP 1) and (CP 2). Applying the hypothesis with $\alpha = n$ then (cp-ii) and (cp-iv) respectively follow from (CP 1) and (CP 2). Finally, (cp-v) follows from (CP 3). □

3.5 Bohnenblust–Hille inequality

Let us see how to extend (and unify) section 2.4 theorems to multipolynomials. Considering the canonical basis of c_0, it follows from Equation 3.6 that every continuous $(n_1, \dots, n_m)$-homogeneous polynomial $P : c_0 \times \cdots \times c_0 \to \mathbb{K}$ can be written as

$$P(x_1, \dots, x_m) = \sum c_\alpha (P)\, x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

for all $x_1, \dots, x_m \in c_0$, where $c_\alpha(P) \in \mathbb{K}$ and where the summation is taken over all matrices $\alpha \in \mathbb{M}_{m\times\infty}(\mathbb{N}_0)$ such that $|\alpha_i| = n_i$, for each i with $1 \leqslant i \leqslant m$.

Theorem 3.5.1 (Multipolynomial Bohnenblust–Hille inequality)**.** *Let $n_1, \ldots, n_m$ and m be fixed positive integers (recall that $M := \sum_{j=1}^{m} n_j$). The following assertions are equivalent:*

(i) *There is a constant $C_{\mathbb{K},M} \geqslant 1$ such that*

$$\left(\sum_{|\alpha_1|=n_1,\ldots,|\alpha_m|=n_m} |c_\alpha (P)|^p \right)^{\frac{1}{p}} \leqslant C_{\mathbb{K},M} \|P\|$$

for all continuous $(n_1, \ldots, n_m)$-homogeneous polynomial $P : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

(ii)

$$p \geqslant \frac{2M}{M+1}.$$

Proof. *(ii)* $\Rightarrow$ *(i)*: It suffices to prove the assertion for

$$p_0 = \frac{2M}{M+1}.$$

Let $Q : c_0 \to \mathbb{K}$ be the M-homogeneous polynomial given by

$$Q(z) := P\left((z_j)_{j\in\mathbb{N}_1}, \ldots, (z_j)_{j\in\mathbb{N}_m} \right),$$

where $\mathbb{N} = \mathbb{N}_1 \cup \cdots \cup \mathbb{N}_m$ is a disjoint union with $\operatorname{card}(\mathbb{N}_j) = \operatorname{card}(\mathbb{N})$, for $j = 1, \ldots, m$. Note that since we are dealing with the sup norm we have

$$\|Q\| \leqslant \|P\|$$

and

$$\sum_{|\beta|=M} |c_\beta (Q)|^p = \sum_{|\alpha_1|=n_1,\ldots,|\alpha_m|=n_m} |c_\alpha (P)|^p,$$

for all p. By the polynomial Bohnenblust–Hille inequality there exists a constant $C_{\mathbb{K},M} \geqslant 1$ such that

$$\begin{aligned} \left(\sum_{|\alpha_1|=n_1,\ldots,|\alpha_m|=n_m} |c_\alpha (P)|^p \right)^{\frac{1}{p}} \leqslant C_{\mathbb{K},M} \|P\| &= \left(\sum_{|\beta|=M} |c_\beta (Q)|^{p_0} \right)^{\frac{1}{p_0}} \\ &\leqslant C_{\mathbb{K},M} \|Q\| \\ &\leqslant C_{\mathbb{K},M} \|P\|. \end{aligned}$$

(i) $\Rightarrow$ *(ii)*: Let

$$\begin{aligned} T_r &: c_0 \times \cdots \times c_0 \to \mathbb{K} \\ T_r\left(x^{(1)}, \ldots, x^{(M)}\right) &= \sum_{i_1,\ldots,i_M=1}^{r} \pm x_{i_1}^{(1)} \cdots x_{i_M}^{(M)} \end{aligned}$$

be the M-linear mapping given by the Kahane-Salem-Zygmund inequality (see, (ALBUQUERQUE et al., 2014, Lemma 6.1)). Define

$$P_r : \overbrace{c_0 \times \cdots \times c_0}^{m} \to \mathbb{K}$$

by

$$P_r\left(x^{(1)}, \ldots, x^{(m)}\right) = T_r\left(\underbrace{\left(x_j^{(1)}\right)_{j\in\mathbb{N}_1^{(1)}}, \ldots, \left(x_j^{(1)}\right)_{j\in\mathbb{N}_{n_1}^{(1)}}}_{n_1}, \ldots, \underbrace{\left(x_j^{(m)}\right)_{j\in\mathbb{N}_1^{(m)}}, \ldots, \left(x_j^{(m)}\right)_{j\in\mathbb{N}_{n_m}^{(m)}}}_{n_m}\right),$$

where

$$\begin{aligned} \mathbb{N} &= \mathbb{N}_1^{(1)} \bigcup \cdots \bigcup \mathbb{N}_{n_1}^{(1)} \\ &\vdots \\ \mathbb{N} &= \mathbb{N}_1^{(m)} \bigcup \cdots \bigcup \mathbb{N}_{n_m}^{(m)} \end{aligned}$$

are disjoint unions with $\operatorname{card}(\mathbb{N}_k^{(i)}) = \operatorname{card}(\mathbb{N})$, for $i = 1, \ldots, m$ and $k = 1, \ldots, n_i$. Note that P_r is a continuous $(n_1, \ldots, n_m)$-homogeneous polynomial and $\|P_r\| \leqslant \|T_r\|$. Moreover,

$$\sum_{|\alpha_1|=n_1,\ldots,|\alpha_m|=n_m} |c_\alpha(P_r)|^p = r^M$$

for all p. Since

$$\|T_r\| \leqslant K_M r^{\frac{M+1}{2}},$$

by (i) we conclude that

$$\left(r^M\right)^{\frac{1}{p}} \leqslant C_{\mathbb{K},M} K_M r^{\frac{M+1}{2}},$$

for every positive integer r. Thus

$$\frac{M}{p} \leqslant \frac{M+1}{2}$$

and the proof is done. □

To see how to unify Theorem 2.4.1 and Theorem 2.4.2, it suffices applying this-section theorem with $m = 1$ and $n_1 = \cdots = n_m = 1$, respectively.

3.6 Absolutely summing multipolynomials

In this section, we generalize to multipolynomials previous results of (BOTELHO; PELLEGRINO, 2006a) and (PELLEGRINO, 2003a; PELLEGRINO, 2004) concerning absolutely summing polynomials and multilinear mappings, inspired by techniques from the famous paper "Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications" by J. Lindenstrauss and A. Pełczyński (LINDENSTRAUSS; PELCZYŃSKI, 1968).

Remark 3.6.1. *The results presented here have been published in (VELANGA, 2019).*

3.6.1 Preliminary results

Let us begin by introducing material that will be needed later. A well-known result due to A. Defant and J. Voigt states that every scalar-valued m-linear mapping is absolutely $(1;1)$-summing (see (ALENCAR; MATOS, 1989, Theorem 3.10)). The polynomial version is also valid. We start by extending that to multipolynomials.

Lemma 3.6.2. *Every* $(n_1,\ldots,n_m)$*-homogeneous polynomial* $P : E_1 \times \cdots \times E_m \to \mathbb{K}$ *is absolutely* $(1;1)$*-summing.*

Proof. Let $P : c_0 \times \cdots \times c_0 \to \mathbb{K}$ be an $(n_1,\ldots,n_m)$-homogeneous polynomial. Define the $M := (n_1 + \cdots + n_m)$-homogeneous polynomial $Q : c_0 \to \mathbb{K}$ by

$$Q(x) := P\left(\left(x_{(i-1)m+1}\right)_{i\in\mathbb{N}}, \ldots, \left(x_{(i-1)m+m}\right)_{i\in\mathbb{N}}\right).$$

Note that, since we are dealing with the sup norm, we have

$$\|Q\| \leqslant \|P\|$$

and, since Q is a scalar-valued M-homogeneous polynomial, it follows from the theorem of Defant-Voigt and the open mapping theorem that there exists a constant $C > 0$ such that

$$\sum_{j=1}^{\infty} \left| P\left(\left(x^{(j)}_{(i-1)m+1}\right)_{i\in\mathbb{N}}, \ldots, \left(x^{(j)}_{(i-1)m+m}\right)_{i\in\mathbb{N}}\right)\right| = \sum_{j=1}^{\infty} \left|Q(x^{(j)})\right| \leqslant C\,\|P\|\,\left\|(x^{(j)})_{j=1}^{\infty}\right\|_{w,1}^{M},$$

whenever $(x^{(j)})_{j=1}^{\infty} \in \ell_1^w(c_0)$. In particular,

$$\sum_{j=1}^{\infty} |P(e_j,\ldots,e_j)| = \sum_{j=1}^{\infty} \left|Q(z^{(j)})\right|$$

where, for each $j \in \mathbb{N}$,

$$\left(z_i^{(j)}\right)_{i\in\mathbb{N}} := \begin{cases} 1, & \text{if } (j-1)\,m + 1 \leqslant i \leqslant jm \\ 0, & \text{otherwise} \end{cases}.$$

Since

$$\left\|(z^{(j)})_{j=1}^{\infty}\right\|_{w,1} = 1,$$

then

$$\sum_{j=1}^{\infty} |P(e_j,\ldots,e_j)| \leqslant C\,\|P\|.$$

Applying the isometric isomorphism from $\mathcal{L}(c_0;E_k)$ onto $\ell_1^w(E_k)$, for each $k = 1,\ldots,m$, (see (DIESTEL; JARCHOW; TONGE, 1995, Proposition 2.2)) we are led to the desired conclusion. □

Recall that if $2 \leqslant q \leqslant \infty$ and $(r_j)_{j=1}^{\infty}$ are the Rademacher functions, then E has *cotype* q if there exists $C_q(E) \geqslant 0$ such that, for every $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E$,

$$\left(\sum_{j=1}^{k} \|x_j\|^q\right)^{1/q} \leqslant C_q(E) \left(\int_0^1 \left\|\sum_{j=1}^{k} r_j(t)\, x_j\right\|^2 dt\right)^{1/2}.$$

To cover the case $q = \infty$, we replace $(\sum_{j=1}^{k} \|x_j\|^q)^{1/q}$ with $\max_{j \leqslant k} \|x_j\|$. We denote $\inf\{q; E \text{ has cotype } q\}$ by $\cot E$.

The following result, also known as Maurey–Talagrand's theorem, gives the main connection between cotype and absolutely summing operators.

Theorem 3.6.3 ((TALAGRAND, 1992))**.** *If E has finite cotype q, then the identity operator $id_E : E \to E$ is $(q;1)$-summing. The converse is true, except for $q = 2$.*

By exploiting the notion of cotype, we prove other coincidence results.

Proposition 3.6.4. *Let $m \in \mathbb{N}$ and $(n_1, \ldots, n_m) \in \mathbb{N}^m$.*

(i) *If E_j has cotype $q_j < \infty$ for each $j = 1, \ldots, m$, then*

$$\mathcal{P}_{as(s;1)}\left({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F\right) = \mathcal{P}\left({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F\right),$$

for every F and every $s > 0$ such that $1/s \leqslant n_1/q_1 + \cdots + n_m/q_m$.

(ii) *If F has cotype $q < \infty$, then*

$$\mathcal{P}_{as(q;1)}\left({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F\right) = \mathcal{P}\left({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F\right),$$

for every $E_1, \ldots, E_m$.

Proof. (i): By Proposition 3.1.3, Hölder's inequality, and Maurey–Talagrand's theorem we have that

$$\begin{aligned}
\left(\sum_{j=1}^{n} \left\|P\left(x_j^{(1)}, \ldots, x_j^{(m)}\right)\right\|^s\right)^{\frac{1}{s}} &= \|P\| \left(\sum_{j=1}^{n} \left\|x_j^{(1)}\right\|^{n_1 s} \cdots \left\|\left(x_j^{(m)}\right)\right\|^{n_m s}\right)^{\frac{1}{s}} \\
&\leqslant \|P\| \left(\sum_{j=1}^{n} \left\|x_j^{(1)}\right\|^{q_1}\right)^{\frac{n_1}{q_1}} \cdots \left(\sum_{j=1}^{n} \left\|x_j^{(m)}\right\|^{q_m}\right)^{\frac{n_m}{q_m}} \\
&\leqslant \|P\| \, \|id_{E_1}\|_{as(q_1;1)}^{n_1} \cdots \|id_{E_m}\|_{as(q_m;1)}^{n_m} \prod_{k=1}^{m} \left\|\left(x_j^{(k)}\right)_{j=1}^{n}\right\|_{w,1}^{n_k}
\end{aligned}$$

for all $n \in \mathbb{N}$, $x_1^{(k)}, \ldots, x_n^{(k)} \in E_k$, with $k = 1, \ldots, m$. Then, P is absolutely $(s;1)$-summing.

(ii): By Lemma 3.6.2, it is straightforward that every multipolynomial P in $\mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ is such that

$$\left(P\left(x_j^{(1)}, \ldots, x_j^{(m)}\right)\right)_{j=1}^{\infty} \in \ell_1^w(F),$$

whenever $(x_j^{(k)})_{j=1}^{\infty} \in \ell_1^w(E_k)$, $k = 1, \ldots, m$. After applying Corollary 3.2.11, the open mapping theorem provides a constant $C \geqslant 0$ such that

$$\left\| \left(P\left(x_j^{(1)}, \ldots, x_j^{(m)} \right) \right)_{j=1}^{\infty} \right\|_{w,1} \leqslant C \|P\| \prod_{k=1}^{m} \left\| \left(x_j^{(k)} \right)_{j=1}^{\infty} \right\|_{w,1}^{n_k},$$

for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_1^w(E_k)$, $k = 1, \ldots, m$. Therefore, it follows from Maurey–Talagrand's theorem that

$$\begin{aligned} \left(\sum_{j=1}^{\infty} \left\| P\left(x_j^{(1)}, \ldots, x_j^{(m)} \right) \right\|^q \right)^{\frac{1}{q}} &= \left(\sum_{j=1}^{\infty} \left\| id_F \left(P\left(x_j^{(1)}, \ldots, x_j^{(m)} \right) \right) \right\|^q \right)^{\frac{1}{q}} \\ &\leqslant \|id_F\|_{as(q;1)} \left\| \left(P\left(x_j^{(1)}, \ldots, x_j^{(m)} \right) \right)_{j=1}^{\infty} \right\|_{w,1} \\ &\leqslant \|id_F\|_{as(q;1)} C \|P\| \prod_{k=1}^{m} \left\| \left(x_j^{(k)} \right)_{j=1}^{\infty} \right\|_{w,1}^{n_k} \end{aligned}$$

for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_1^w(E_k)$, $k = 1, \ldots, m$. We have shown that P is absolutely $(q;1)$-summing. □

In particular, we extract the coincidence results for the class of polynomials/-multilinear mappings due to G. Botelho.

Corollary 3.6.5 ((BOTELHO, 1997, Theorem 2.2)). *Let $m \in \mathbb{N}$.*

(i) *If E has cotype $mq < \infty$, then*

$$\mathcal{P}_{as(q;1)}\left({}^m E; F\right) = \mathcal{P}\left({}^m E; F\right), \text{ for every } F.$$

(ii) *If F has cotype q, then*

$$\mathcal{P}_{as(q;1)}\left({}^m E; F\right) = \mathcal{P}\left({}^m E; F\right), \text{ for every } E.$$

Proof. Apply Proposition 3.6.4 with $m = 1$. □

Corollary 3.6.6 ((BOTELHO, 1997, Theorem 2.5)). *Let $m \in \mathbb{N}$.*

(i) *If E_j has cotype $q_j < \infty$ for each $j = 1, \ldots, m$, then*

$$\mathcal{L}_{as(s;1)}\left(E_1, \ldots, E_m; F\right) = \mathcal{L}\left(E_1, \ldots, E_m; F\right),$$

for every F and every $s > 0$ such that $1/s \leqslant 1/q_1 + \cdots + 1/q_m$.

(ii) *If F has cotype q, then*

$$\mathcal{L}_{as(q;1)}\left(E_1, \ldots, E_m; F\right) = \mathcal{L}\left(E_1, \ldots, E_m; F\right),$$

for every $E_1, \ldots, E_m$.

Proof. Apply Proposition 3.6.4 with $n_1 = \cdots = n_m = 1$. □

3.6.2 Main result

Let $m, n_1, \ldots, n_m$ be natural numbers and let $E_1, \ldots, E_m$ be infinite dimensional Banach spaces with normalized unconditional Schauder basis $(x_n^{(k)})_{n\in\mathbb{N}}$, $k = 1, \ldots, m$. We define

$$\eta = \eta\left(n_1, \ldots, n_m, E_1, \ldots, E_m, \left(x_n^{(1)}\right)_{n\in\mathbb{N}}, \ldots, \left(x_n^{(m)}\right)_{n\in\mathbb{N}}\right)$$

by

$$\eta = \inf\left\{t : \left(\prod_{k=1}^{m}\left(a_j^{(k)}\right)^{n_k}\right)_{j\in\mathbb{N}} \in \ell_t \text{ whenever } x_k = \sum_{j=1}^{\infty} a_j^{(k)} x_j^{(k)} \in E_k,\ k = 1, \ldots, m\right\}$$

and investigate the following general question:

Problem 3.6.7. *If $\mathcal{P}_{as(q,1)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F) = \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$, then how does η behave?*

The techniques used to solve this kind of problem date back to the seminal paper (LINDENSTRAUSS; PELCZYŃSKI, 1968) where J. Lindenstrauss and A. Pełczyński provide a beautiful theorem stating that if E is an infinite dimensional Banach space with an unconditional Schauder basis and every bounded linear operator from E into an infinite dimensional Banach space F is absolutely $(1;1)$-summing, then E is isomorphic to $\ell_1(\Gamma)$ and F is a Hilbert space.

Firstly, recall that a Banach space F *finitely factors* the formal inclusion $\ell_p \hookrightarrow \ell_\infty$ for $0 < \delta < 1$ if for every $n \in \mathbb{N}$ there exist $y_1, \ldots, y_n \in F$ such that

$$(1-\delta)\left\|a\right\|_\infty \leqslant \left\|\sum_{j=1}^{n} a_j y_j\right\| \leqslant \left\|a\right\|_p, \text{ for all } a = (a_j)_{j=1}^n \in \ell_p^n.$$

Note that $1 - \delta \leqslant \|y_j\| \leqslant 1$, for all j.

For the extreme cases, the answers to the Problem 3.6.7 are already known. More precisely, when $m = 1$ and $n_1 = 1$, we recover the linear setting which as we comment has a solution in (LINDENSTRAUSS; PELCZYŃSKI, 1968). When $m = 1$ and $n_1 = m > 1$, D. Pellegrino has shown the following:

Theorem 3.6.8 ((PELLEGRINO, 2003a, Theorem 5))**.** *Let E and F be infinite dimensional Banach spaces. Suppose that E has an unconditional Schauder basis. If F finitely factors the formal inclusion $\ell_p \hookrightarrow \ell_\infty$ for some δ and $\mathcal{P}_{as(q;1)}({}^mE; F) = \mathcal{P}({}^mE; F)$, then*

(a) *$\eta \leqslant pq/(p-q)$, if $q < p$;*

(b) *$\eta \leqslant q$, if $q \leqslant p/2$.*

Theorem 3.6.9 ((PELLEGRINO, 2004, Theorem 5))**.** *Let E be an infinite dimensional Banach space with an unconditional Schauder basis. If $\mathcal{P}_{as(q;1)}(^mE) = \mathcal{P}(^mE)$, then*

(a) $\eta \leqslant q/(1-q)$, *if* $q < 1$*;*

(b) $\eta \leqslant q$, *if* $q \leqslant 1/2$.

As to the other extreme case, that is, when $m > 1$ and $n_1 = \cdots = n_m = 1$, the following has been proved:

Theorem 3.6.10 ((PELLEGRINO, 2003a, Theorem 8))**.** *Let F be an infinite dimensional Banach space and let $E_1, \ldots, E_m$ denote infinite dimensional Banach spaces with unconditional Schauder basis. If F finitely factors the formal inclusion $\ell_p \hookrightarrow \ell_\infty$ for some δ and $\mathcal{L}_{as(q;1)}(E_1, \ldots, E_m; F) = \mathcal{L}(E_1, \ldots, E_m; F)$, then*

(a) $\eta \leqslant pq/(p-q)$, *if* $q < p$*;*

(b) $\eta \leqslant q$, *if* $q \leqslant p/2$.

Later, still dealing with the class of homogeneous polynomials ($m = 1$ and $n_1 = m > 1$), G. Botelho and D. Pellegrino obtained better estimates for η, improving Theorem 3.6.8 and Theorem 3.6.9, as we see below:

Lemma 3.6.11 ((BOTELHO; PELLEGRINO, 2006a, Lemma 2.1))**.** *Suppose that F satisfies the following condition:*

There exist $C_1, C_2 > 0$ and $p \geqslant 1$ such that for every $n \in \mathbb{N}$, there are $y_1, \ldots, y_n$ in F with $\|y_j\| \geqslant C_1$ for every j and

$$\left\| \sum_{j=1}^{n} a_j y_j \right\| \leqslant C_2 \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}}$$

for every $a_1, \ldots, a_n \in \mathbb{K}$.

In this case, if E has a normalized unconditional Schauder basis $(x_n)_{n\in\mathbb{N}}$, $q < p$, and $\mathcal{P}_{as(q;1)}(^mE; F) = \mathcal{P}(^mE; F)$, then $\eta \leqslant q$.

Next, we will extend Lemma 3.6.11 to multipolynomials which will provide a unified approach to Problem 3.6.7. Indeed, the next lemma recovers all the aforementioned results as particular extreme cases.

Lemma 3.6.12. *Suppose that F satisfies the following condition:*

There exist $C_1, C_2 > 0$ and $p \geqslant 1$ such that for every $n \in \mathbb{N}$, there are $y_1, \ldots, y_n$ in F with $\|y_j\| \geqslant C_1$ for every j and

$$\left\| \sum_{j=1}^{n} a_j y_j \right\| \leqslant C_2 \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}}$$

for every $a_1, \ldots, a_n \in \mathbb{K}$.

In this case, if E_k *has a normalized unconditional Schauder basis for each* $k = 1, \ldots, m$, $q < p$, *and* $\mathcal{P}_{as(q;1)}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F) = \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$, *then* $\eta \leqslant q$.

Proof. We follow the ideas of the original proof in (BOTELHO; PELLEGRINO, 2006a); it is done by an induction argument. Recall that $\|(z_j)_{j=1}^n\|_{w,1} = \max_{\varepsilon_j \in \{1,-1\}} \{\| \sum_{j=1}^n \varepsilon_j z_j \|\}$ for $\mathbb{K} = \mathbb{R}$; and $\|(z_j)_{j=1}^n\|_{w,1} \leqslant 2 \max_{\varepsilon_j \in \{1,-1\}} \{\| \sum_{j=1}^n \varepsilon_j z_j \|\}$ for $\mathbb{K} = \mathbb{C}$. By the coincidence hypothesis, there exists $K > 0$ such that the absolutely summing multipolynomial norm $\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(P) \leqslant K\|P\|$ for all $P \in \mathcal{P}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$. Let n be a fixed natural number and $\{\mu_j\}_{j=1}^n$ be such that $\sum_{j=1}^n |\mu_j|^s = 1$ with $s = p/q$. Define $P : E_1 \times \cdots \times E_m \to F$ by

$$P(x_1, \ldots, x_m) = \sum_{j=1}^n |\mu_j|^{\frac{1}{q}} \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} y_j,$$

where $x_k = \sum_{j=1}^\infty a_j^{(k)} x_j^{(k)}$ $(1 \leqslant k \leqslant m)$ and $y_1, \ldots, y_n$ are as in the assumptions. Since $(x_n^{(k)})_{n\in\mathbb{N}}$ is an unconditional basis, there exist $\varrho_k > 0$ satisfying

$$\left\| \sum_{j=1}^\infty \varepsilon_j a_j^{(k)} x_j^{(k)} \right\| \leqslant \varrho_k \left\| \sum_{j=1}^\infty a_j^{(k)} x_j^{(k)} \right\| = \varrho_k \|x_k\|, \text{ for any } \varepsilon_j = \pm 1.$$

Hence $\| \sum_{j=1}^n \varepsilon_j a_j^{(k)} x_j^{(k)} \| \leqslant \varrho_k \|x_k\|$ for all n and any $\varepsilon_j = \pm 1$. So, if $x_k = \sum_{j=1}^\infty a_j^{(k)} x_j^{(k)}$, we have $|a_j^{(k)}| \leqslant \varrho_k \|x_k\|$ for all j and then we get

$$\begin{aligned}
\|P(x_1, \ldots, x_m)\| &= \left\| \sum_{j=1}^n |\mu_j|^{\frac{1}{q}} \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} y_j \right\| \\
&\leqslant C_2 \left(\sum_{j=1}^n \left| |\mu_j|^{\frac{1}{q}} \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} \right|^p \right)^{\frac{1}{p}} \\
&\leqslant C_2 \prod_{k=1}^m (\varrho_k \|x_k\|)^{n_k} \left(\sum_{j=1}^n |\mu_j|^s \right)^{\frac{1}{p}} \\
&= C_2 \prod_{k=1}^m (\varrho_k \|x_k\|)^{n_k}.
\end{aligned}$$

We obtain $\|P\| \leqslant C_2 \prod_{k=1}^m \varrho_k^{n_k}$ and $\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(P) \leqslant K C_2 \prod_{k=1}^m \varrho_k^{n_k}$ and achieve the estimate below:

$$
\begin{aligned}
\left(\sum_{j=1}^{n}\left|\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m}|\mu_j|^{\frac{1}{q}}\,C_1\right|^q\right)^{\frac{1}{q}} &\leqslant \left(\sum_{j=1}^{n}\left\|\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m}|\mu_j|^{\frac{1}{q}}\,y_j\right\|^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^{n}\left\|P\left(a_j^{(1)}x_j^{(1)},\ldots,a_j^{(m)}x_j^{(m)}\right)\right\|^q\right)^{\frac{1}{q}} \\
&\leqslant \pi_{as(q;1)}^{(n_1,\ldots,n_m)}(P)\prod_{k=1}^{m}\left\|\left(a_j^{(k)}x_j^{(k)}\right)_{j=1}^{n}\right\|_{w,1}^{n_k} \\
&\leqslant KC_2\prod_{k=1}^{m}\varrho_k^{n_k}2^{n_k}\max_{\varepsilon_j\in\{1,-1\}}\left\{\left\|\sum_{j=1}^{n}\varepsilon_j a_j^{(k)}x_j^{(k)}\right\|\right\}^{n_k} \\
&\leqslant KC_2\prod_{k=1}^{m}\left(2\varrho_k^2\left\|x_k\right\|\right)^{n_k}. \qquad (3.16)
\end{aligned}
$$

Note that the last inequality holds whenever $\sum_{j=1}^{n}|\mu_j|^s = 1$. Hence, since $1/s + 1/(\frac{s}{s-1}) = 1$, we have

$$
\begin{aligned}
&\left(\sum_{j=1}^{n}\left|\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m}\right|^{\frac{s}{s-1}q}\right)^{1/\left(\frac{s}{s-1}\right)} \\
&= \sup\left\{\left|\sum_{j=1}^{n}\mu_j\left(a_j^{(1)}\right)^{n_1 q}\cdots\left(a_j^{(m)}\right)^{n_m q}\right| ; \sum_{j=1}^{n}|\mu_j|^s = 1\right\} \\
&\leqslant \sup\left\{\sum_{j=1}^{n}\left|\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m}|\mu_j|^{\frac{1}{q}}\right|^q ; \sum_{j=1}^{n}|\mu_j|^s = 1\right\}. \qquad (3.17)
\end{aligned}
$$

Then, by (3.16) and (3.17), it follows that

$$
\left(\sum_{j=1}^{n}\left|\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m}\right|^{\frac{s}{s-1}q}\right)^{1/\left(\frac{s}{s-1}\right)} \leqslant \left(C_1^{-1}KC_2\prod_{k=1}^{m}\left(2\varrho_k^2\left\|x_k\right\|\right)^{n_k}\right)^q,
$$

and then

$$
\left(\sum_{j=1}^{n}\left|\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m}\right|^{\frac{s}{s-1}q}\right)^{1/\left(\frac{s}{s-1}\right)q} \leqslant C_1^{-1}KC_2\prod_{k=1}^{m}\left(2\varrho_k^2\right)^{n_k}\prod_{k=1}^{m}\left\|x_k\right\|^{n_k}.
$$

Since $\frac{s}{s-1}q = pq/(p-q)$ and n is arbitrary, we have $\eta \leqslant pq/(p-q)$. Now, if $q \leqslant p/2$, define, for a fixed n, $S : E_1 \times \cdots \times E_m \to F$ by

$$
S\left(x_1,\ldots,x_m\right) = \sum_{j=1}^{n}\left(a_j^{(1)}\right)^{n_1}\cdots\left(a_j^{(m)}\right)^{n_m} y_j, \text{ where } x_k = \sum_{j=1}^{\infty} a_j^{(k)}x_j^{(k)}, \text{ for } k = 1,\ldots,m.
$$

Since $p \geqslant \frac{s}{s-1}q$, combining the preceding estimates, we obtain

$$\begin{aligned}
\|S(x_1,\ldots,x_m)\| &= \left\| \sum_{j=1}^{n} \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} y_j \right\| \\
&\leqslant C_2 \left(\sum_{j=1}^{n} \left| \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} \right|^p \right)^{\frac{1}{p}} \\
&\leqslant C_2 \left(\sum_{j=1}^{n} \left| \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} \right|^{\frac{s}{s-1}q} \right)^{1/\left(\frac{s}{s-1}\right)q} \\
&\leqslant C_1^{-1} K C_2^2 \prod_{k=1}^{m} \left(2\varrho_k^2\right)^{n_k} \prod_{k=1}^{m} \|x_k\|^{n_k}.
\end{aligned}$$

Thus, $\|S\| \leqslant C_1^{-1}KC_2^2\prod_{k=1}^{m}(2\varrho_k^2)^{n_k}$ and $\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(S) \leqslant C_1^{-1}K^2C_2^2\prod_{k=1}^{m}(2\varrho_k^2)^{n_k}$. Hence

$$\begin{aligned}
\sum_{j=1}^{n} \left| \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} C_1 \right|^q &\leqslant \sum_{j=1}^{n} \left\| \left(a_j^{(1)}\right)^{n_1} \cdots \left(a_j^{(m)}\right)^{n_m} y_j \right\|^q \\
&= \sum_{j=1}^{n} \left\| S\left(a_j^{(1)}x_j^{(1)}, \ldots, a_j^{(m)}x_j^{(m)}\right) \right\|^q \\
&\leqslant \left(\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(S) \prod_{k=1}^{m} 2^{n_k} \max_{\varepsilon_j \in \{1,-1\}} \left\{ \left\| \sum_{j=1}^{n} \varepsilon_j a_j^{(k)} x_j^{(k)} \right\| \right\}^{n_k} \right)^q \\
&\leqslant \left(C_1^{-1} K^2 C_2^2 \prod_{k=1}^{m} \left(4\varrho_k^3\right)^{n_k} \right)^q \prod_{k=1}^{m} \|x_k\|^{n_k q}.
\end{aligned}$$

Consequently, since n is arbitrary, we have $\sum_{j=1}^{\infty}|(a_j^{(1)})^{n_1}\cdots(a_j^{(m)})^{n_m}|^q < \infty$ whenever $x_k = \sum_{j=1}^{\infty} a_j^{(k)} x_j^{(k)} \in E_k$, for $k = 1, \ldots, m$, and $\eta \leqslant q$ if $q \leqslant p/2$.

Now we state the induction hypothesis. Suppose that we have:

(i) $\eta \leqslant \frac{pq}{jp-jq}$ and $(\sum_{i=1}^{\infty}|(a_i^{(1)})^{n_1}\cdots(a_i^{(m)})^{n_m}|^{\frac{pq}{jp-jq}})^{1/(\frac{pq}{jp-jq})} \leqslant A_j\prod_{k=1}^{m}\|x_k\|^{n_k}$, if $\frac{jp}{j+1} < q < p$,

(ii) $\eta \leqslant q$ and $(\sum_{i=1}^{\infty}|(a_i^{(1)})^{n_1}\cdots(a_i^{(m)})^{n_m}|^q)^{1/q} \leqslant B_j\prod_{k=1}^{m}\|x_k\|^{n_k}$, if $q \leqslant \frac{jp}{j+1}$,

where

- $A_1 = C_1^{-1}KC_2\prod_{k=1}^{m}(2\varrho_k^2)^{n_k}$,
- $B_1 = C_1^{-2}K^2C_2^2\prod_{k=1}^{m}(4\varrho_k^3)^{n_k}$,
- $A_j = C_1^{-1}KC_2A_{j-1}\prod_{k=1}^{m}(2\varrho_k)^{n_k}$ for $j \geqslant 2$,
- $B_j = C_1^{-2}K^2C_2^2A_{j-1}\prod_{k=1}^{m}(4\varrho_k^2)^{n_k}$ for $j \geqslant 2$.

Note that the case $j = 1$ is done. We assume that (i) and (ii) hold for j and prove that they hold for $j+1$. To prove (i), assume $\frac{(j+1)p}{j+2} < q < p$.

Fix n and let $\{\mu_i\}_{i=1}^n$ be such that $\sum_{i=1}^n |\mu_i|^{s_j} = 1$, where $s_j = \frac{p}{(j+1)q-jp}$. Defining P as at the beginning and putting $l_j = \frac{pq}{jp-jq}$ and $t_j = \frac{pq}{(j+1)q-jp}$, we have $\frac{1}{t_j} + \frac{1}{l_j} = \frac{1}{p}$ and so

$$\begin{aligned}
\|P(x_1,\ldots,x_m)\| &= \left\|\sum_{i=1}^n |\mu_i|^{\frac{1}{q}} \left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m} y_i\right\| \\
&\leqslant C_2 \left(\sum_{i=1}^n \left||\mu_i|^{\frac{1}{q}} \left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m}\right|^p\right)^{\frac{1}{p}} \\
&\leqslant C_2 \left(\sum_{i=1}^n \left||\mu_i|^{\frac{1}{q}}\right|^{t_j}\right)^{\frac{1}{t_j}} \left(\sum_{i=1}^n \left|\left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m}\right|^{l_j}\right)^{\frac{1}{l_j}} \\
&\leqslant C_2 \left(\sum_{i=1}^n |\mu_i|^{s_j}\right)^{\frac{1}{t_j}} \left(\sum_{i=1}^n \left|\left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m}\right|^{l_j}\right)^{\frac{1}{l_j}} \\
&\leqslant C_2 A_j \prod_{k=1}^m \|x_k\|^{n_k}.
\end{aligned}$$

We obtain $\|P\| \leqslant C_2 A_j$ and $\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(P) \leqslant K C_2 A_j$ and achieve the estimate below:

$$\begin{aligned}
\left(\sum_{i=1}^n \left|\left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m} |\mu_i|^{\frac{1}{q}} C_1\right|^q\right)^{\frac{1}{q}} &\leqslant \left(\sum_{i=1}^n \left\|\left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m} |\mu_i|^{\frac{1}{q}} y_i\right\|^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^n \left\|P\left(a_i^{(1)} x_i^{(1)},\ldots,a_i^{(m)} x_i^{(m)}\right)\right\|^q\right)^{\frac{1}{q}} \\
&\leqslant \pi_{as(q;1)}^{(n_1,\ldots,n_m)}(P) \prod_{k=1}^m \left\|\left(a_i^{(k)} x_i^{(k)}\right)_{i=1}^n\right\|_{w,1}^{n_k} \\
&\leqslant K C_2 A_j \prod_{k=1}^m 2^{n_k} \max_{\varepsilon_i \in \{1,-1\}} \left\{\left\|\sum_{i=1}^n \varepsilon_i a_i^{(k)} x_i^{(k)}\right\|\right\}^{n_k} \\
&\leqslant K C_2 A_j \prod_{k=1}^m \left(2\varrho_k \|x_k\|\right)^{n_k}. \qquad (3.18)
\end{aligned}$$

Since $\frac{1}{s_j} + 1/(\frac{s_j}{s_j-1}) = 1$, we have

$$\begin{aligned}
&\left(\sum_{i=1}^n \left|\left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m}\right|^{\frac{s_j}{s_j-1}q}\right)^{1/\left(\frac{s_j}{s_j-1}\right)} \\
&= \sup\left\{\left|\sum_{i=1}^n \mu_i \left(a_i^{(1)}\right)^{n_1 q} \cdots \left(a_i^{(m)}\right)^{n_m q}\right| ; \sum_{i=1}^n |\mu_i|^{s_j} = 1\right\} \\
&\leqslant \sup\left\{\sum_{i=1}^n \left|\left(a_i^{(1)}\right)^{n_1} \cdots \left(a_i^{(m)}\right)^{n_m} |\mu_i|^{\frac{1}{q}}\right|^q ; \sum_{i=1}^n |\mu_i|^{s_j} = 1\right\}. \qquad (3.19)
\end{aligned}$$

It is plain that (3.18) holds whenever $\sum_{i=1}^{n}|\mu_i|^{s_j} = 1$. Thus, by (3.18) and (3.19), it follows that

$$\left(\sum_{i=1}^{n}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}\right|^{\frac{s_j}{s_j-1}q}\right)^{1/\left(\frac{s_j}{s_j-1}\right)} \leqslant \left(C_1^{-1}KC_2A_j\prod_{k=1}^{m}\left(2\varrho_k\left\|x_k\right\|\right)^{n_k}\right)^q,$$

and then

$$\left(\sum_{i=1}^{n}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}\right|^{\frac{s_j}{s_j-1}q}\right)^{1/\left(\frac{s_j}{s_j-1}\right)q} \leqslant C_1^{-1}KC_2A_j\prod_{k=1}^{m}\left(2\varrho_k\right)^{n_k}\prod_{k=1}^{m}\left\|x_k\right\|^{n_k}.$$

Since $\frac{s_j}{s_j-1}q = \frac{pq}{(j+1)p-(j+1)q}$ and n is arbitrary, we have $\eta \leqslant \frac{pq}{(j+1)p-(j+1)q}$ and

$$\left(\sum_{i=1}^{\infty}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}\right|^{\frac{pq}{(j+1)p-(j+1)q}}\right)^{1/\left(\frac{pq}{(j+1)p-(j+1)q}\right)} \leqslant A_{j+1}\prod_{k=1}^{m}\left\|x_k\right\|^{n_k},$$

which proves (i) for $j+1$. To prove (ii), assume $q \leqslant \frac{(j+1)p}{j+2}$ and invoke, for a fixed n, S again. We have $\frac{pq}{(j+1)p-(j+1)q} \leqslant p$, so

$$\begin{aligned}
\left\|S\left(x_1,\ldots,x_m\right)\right\| &= \left\|\sum_{i=1}^{n}\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}y_i\right\| \\
&\leqslant C_2\left(\sum_{i=1}^{n}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}\right|^p\right)^{\frac{1}{p}} \\
&\leqslant C_2\left(\sum_{i=1}^{n}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}\right|^{\frac{pq}{(j+1)p-(j+1)q}}\right)^{1/\left(\frac{pq}{(j+1)p-(j+1)q}\right)} \\
&\leqslant C_2A_{j+1}\prod_{k=1}^{m}\left\|x_k\right\|^{n_k}.
\end{aligned}$$

Thus $\|S\| \leqslant C_2A_{j+1}$ and $\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(S) \leqslant KC_2A_{j+1}$ and then we get

$$\begin{aligned}
\sum_{i=1}^{n}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}C_1\right|^q &\leqslant \sum_{i=1}^{n}\left\|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}y_i\right\|^q \\
&= \sum_{i=1}^{n}\left\|S\left(a_i^{(1)}x_i^{(1)},\ldots,a_i^{(m)}x_i^{(m)}\right)\right\|^q \\
&\leqslant \left(\pi_{as(q;1)}^{(n_1,\ldots,n_m)}(S)\prod_{k=1}^{m}2^{n_k}\max_{\varepsilon_i\in\{1,-1\}}\left\{\left\|\sum_{i=1}^{n}\varepsilon_i a_i^{(k)}x_i^{(k)}\right\|\right\}^{n_k}\right)^q \\
&\leqslant \left(KC_2A_{j+1}\right)^q\prod_{k=1}^{m}\left(2\varrho_k\left\|x_k\right\|\right)^{n_kq}.
\end{aligned}$$

Consequently, since n is arbitrary, we have

$$\left(\sum_{i=1}^{\infty}\left|\left(a_i^{(1)}\right)^{n_1}\cdots\left(a_i^{(m)}\right)^{n_m}\right|^q\right)^{\frac{1}{q}} \leqslant B_{j+1}\prod_{k=1}^{m}\left\|x_k\right\|^{n_k}$$

whenever $x_k = \sum_{i=1}^{\infty} a_i^{(k)} x_i^{(k)} \in E_k$, $k = 1, \ldots, m$, proving (ii) for $j+1$. The induction argument is done. Finally, since $\lim_{j\to\infty} \frac{jp}{j+1} = p$, the proof is concluded. □

Theorem 3.6.13. *Let $E_1, \ldots, E_m$ be infinite dimensional Banach spaces with normalized unconditional Schauder basis and $\mathcal{P}_{as(q;1)}(^{n_1}E_1, \ldots, ^{n_m}E_m; F) = \mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$. Then $\eta \leqslant q$ if:*

(i) $q < 1$ *and* $\dim F < \infty$;

(ii) $q < \cot F$ *and* $\dim F = \infty$.

Proof. (i): It suffices to deal with the case $F = \mathbb{K}^n$. Since

$$\begin{aligned} &\mathcal{P}_{as(q;1)}(^{n_1}E_1, \ldots, ^{n_m}E_m; \mathbb{K}^n) = \mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; \mathbb{K}^n) \\ \Rightarrow\ &\mathcal{P}_{as(q;1)}(^{n_1}E_1, \ldots, ^{n_m}E_m; \mathbb{K}) = \mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; \mathbb{K}), \end{aligned}$$

we just need to consider $F = \mathbb{K}$. Applying Lemma 3.6.12 with $p = C_1 = C_2 = y_1 = \cdots = y_n = 1$, the proof is done. (ii): Since Maurey–Pisier's theorem (see (DIESTEL; JARCHOW; TONGE, 1995, p. 226)) asserts that F finitely factors $\ell_{\cot F} \hookrightarrow \ell_\infty$, it suffices to call on Lemma 3.6.12 with $p = \cot F$. □

Corollary 3.6.14 ((BOTELHO; PELLEGRINO, 2006a, Theorem 2.3))**.** *Let E be an infinite dimensional Banach spaces with a normalized unconditional Schauder basis and $\mathcal{P}_{as(q;1)}(^mE; F) = \mathcal{P}(^mE; F)$. Then $\eta \leqslant q$ if:*

(i) $q < 1$ *and* $\dim F < \infty$;

(ii) $q < \cot F$ *and* $\dim F = \infty$.

Proof. Apply Theorem 3.6.13 with $m = 1$. □

Corollary 3.6.15. *Let $E_1, \ldots, E_m$ be infinite dimensional Banach spaces with normalized unconditional Schauder basis and $\mathcal{L}_{as(q;1)}(E_1, \ldots, E_m; F) = \mathcal{L}(E_1, \ldots, E_m; F)$. Then $\eta \leqslant q$ if:*

(i) $q < 1$ *and* $\dim F < \infty$;

(ii) $q < \cot F$ *and* $\dim F = \infty$.

Proof. Apply Theorem 3.6.13 with $n_1 = \cdots = n_m = 1$. □

From now on we consider the canonical basis of the classical sequence spaces.

Corollary 3.6.16. *Let $m \in \mathbb{N}$ and $(n_1, \ldots, n_m) \in \mathbb{N}^m$.*

(i) *If* $1 \leqslant r_1, \ldots, r_m < \infty$ *and* $\dim F = \infty$*, we have*

$$\mathcal{P}_{as(q;1)}\left({}^{n_1}\ell_{r_1}, \ldots, {}^{n_m}\ell_{r_m}; F\right) = \mathcal{P}\left({}^{n_1}\ell_{r_1}, \ldots, {}^{n_m}\ell_{r_m}; F\right)$$

$\Rightarrow$

$$q \geqslant \min\left\{\frac{1}{\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}}, \cot F\right\}.$$

(ii) *If* $2 \leqslant r_1, \ldots, r_m < \infty$, $\dim F = \infty$ *and* F *has cotype* $\cot F$*, we have*

$$\mathcal{P}_{as(q;1)}\left({}^{n_1}\ell_{r_1}, \ldots, {}^{n_m}\ell_{r_m}; F\right) = \mathcal{P}\left({}^{n_1}\ell_{r_1}, \ldots, {}^{n_m}\ell_{r_m}; F\right)$$

$\Leftrightarrow$

$$q \geqslant \min\left\{\frac{1}{\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}}, \cot F\right\}.$$

(iii) *For* $2 \leqslant r_1, \ldots, r_m < \infty$*, we have*

$$\mathcal{P}_{as(q;1)}\left({}^{n_1}\ell_{r_1}, \ldots, {}^{n_m}\ell_{r_m}\right) = \mathcal{P}\left({}^{n_1}\ell_{r_1}, \ldots, {}^{n_m}\ell_{r_m}\right)$$

$\Leftrightarrow$

$$q \geqslant \min\left\{\frac{1}{\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}}, 1\right\}.$$

Proof. (i): If $q < \min\{1/(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}), \cot F\}$, then Theorem 3.6.13 and Hölder's inequality provide $1/(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}) = \eta \leqslant q$ (contradition).

(ii): Suppose that $q \geqslant \min\{1/(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}), \cot F\}$. If $q \geqslant 1/(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m})$, the result follows from Proposition 3.6.4-(i). If $q \geqslant \cot F$, then it follows from Proposition 3.6.4-(ii). The converse follows from (i).

(iii): Assume the coincidence hypothesis and suppose that

$$q < \min\left\{1/\left(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}\right), 1\right\}.$$

Then $\eta = 1/(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m}) > q$, which contradicts Theorem 3.6.13-(i). Conversely, if $q \geqslant 1/(\frac{n_1}{r_1} + \cdots + \frac{n_m}{r_m})$ apply Proposition 3.6.4-(i) and we are done. If $q \geqslant 1$, since $\ell_q \supseteq \ell_1$, the proof is now a consequence of Lemma 3.6.2. ☐

We recover the original Botelho–Pellegrino's polynomial version.

Corollary 3.6.17 ((BOTELHO; PELLEGRINO, 2006a, Corollary 2.2))**.** *Let* $m \in \mathbb{N}$*.*

(i) *If* $r \geqslant 1$, $\dim F = \infty$ *and* F *has cotype* $\cot F$*, we have*

$$\mathcal{P}_{as(q;1)}\left({}^{m}\ell_{r}; F\right) = \mathcal{P}\left({}^{m}\ell_{r}; F\right) \Rightarrow q \geqslant \min\left\{\frac{r}{m}, \cot F\right\}.$$

(ii) *If* $r \geqslant 2$, $\dim F = \infty$ *and* F *has cotype* $\cot F$*, we have*

$$\mathcal{P}_{as(q;1)}\left({}^m\ell_r; F\right) = \mathcal{P}\left({}^m\ell_r; F\right) \Leftrightarrow q \geqslant \min\left\{\frac{r}{m}, \cot F\right\}.$$

(iii) *For* $r \geqslant 2$*, we have* $\mathcal{P}_{as(q;1)}\left({}^m\ell_r\right) = \mathcal{P}\left({}^m\ell_r\right) \Leftrightarrow q \geqslant \min\left\{\frac{r}{m}, 1\right\}$.

Proof. Apply Corollary 3.6.16 with $m = 1$. □

Likewise, we naturally extract the multilinear version.

Corollary 3.6.18. *Let* $m \in \mathbb{N}$.

(i) *If* $1 \leqslant r_1, \ldots, r_m < \infty$, $\dim F = \infty$*, we have*

$$\mathcal{L}_{as(q;1)}\left(\ell_{r_1}, \ldots, \ell_{r_m}; F\right) = \mathcal{L}\left(\ell_{r_1}, \ldots, \ell_{r_m}; F\right)$$

$\Rightarrow$

$$q \geqslant \min\left\{\frac{1}{\frac{1}{r_1} + \cdots + \frac{1}{r_m}}, \cot F\right\}.$$

(ii) *If* $2 \leqslant r_1, \ldots, r_m < \infty$, $\dim F = \infty$ *and* F *has cotype* $\cot F$*, we have*

$$\mathcal{L}_{as(q;1)}\left(\ell_{r_1}, \ldots, \ell_{r_m}; F\right) = \mathcal{L}\left(\ell_{r_1}, \ldots, \ell_{r_m}; F\right)$$

$\Leftrightarrow$

$$q \geqslant \min\left\{\frac{1}{\frac{1}{r_1} + \cdots + \frac{1}{r_m}}, \cot F\right\}.$$

(iii) *For* $2 \leqslant r_1, \ldots, r_m < \infty$*, we have*

$$\mathcal{L}_{as(q;1)}\left(\ell_{r_1}, \ldots, \ell_{r_m}\right) = \mathcal{L}\left(\ell_{r_1}, \ldots, \ell_{r_m}\right)$$

$\Leftrightarrow$

$$q \geqslant \min\left\{\frac{1}{\frac{1}{r_1} + \cdots + \frac{1}{r_m}}, 1\right\}.$$

Proof. Apply Corollary 3.6.16 $n_1 = \cdots = n_m = 1$. □

Bibliography

ACHOUR, D.; RUEDA, P.; SÁNCHEZ-PÉREZ, E. A.; YAHI, R. Lipschitz operator ideals and the approximation property. *J. Math. Anal. Appl.*, v. 436, n. 1, p. 217–236, 2016. ISSN 0022-247X. Disponível em: <https://doi.org/10.1016/j.jmaa.2015.11.050>. Citado na página 18.

ALARCÓN, D. Núñez. On the growth of the optimal constants of the multilinear Bohnenblust-Hille inequality. *Linear Algebra Appl.*, v. 439, n. 8, p. 2494–2499, 2013. ISSN 0024-3795. Disponível em: <https://doi.org/10.1016/j.laa.2013.06.017>. Citado 2 vezes nas páginas 28 and 43.

ALBUQUERQUE, N.; ARAÚJO, G.; CAVALCANTE, W.; NOGUEIRA, T.; NÚNEZ, D.; PELLEGRINO, D.; RUEDA, P. On summability of multilinear operators and applications. *Ann. Funct. Anal.*, v. 9, n. 4, p. 574–590, 2018. ISSN 2008-8752. Disponível em: <https://doi.org/10.1215/20088752-2018-0026>. Citado na página 43.

ALBUQUERQUE, N.; BAYART, F.; PELLEGRINO, D.; SEOANE-SEPÚLVEDA, J. B. Sharp generalizations of the multilinear Bohnenblust-Hille inequality. *J. Funct. Anal.*, v. 266, n. 6, p. 3726–3740, 2014. ISSN 0022-1236. Disponível em: <https://doi.org/10.1016/j.jfa.2013.08.013>. Citado 2 vezes nas páginas 28 and 66.

ALENCAR, R.; MATOS, M. C. Some classes of multilinear mappings between Banach spaces. *Publicaciones del Departamento de Análisis Matemático 12, Universidad Complutense Madrid*, p. 1–34, 1989. Citado 2 vezes nas páginas 30 and 67.

ARAÚJO, G.; PELLEGRINO, D. A Gale-Berlekamp permutation-switching problem in higher dimensions. *European J. Combin.*, v. 77, p. 17–30, 2019. ISSN 0195-6698. Disponível em: <https://doi.org/10.1016/j.ejc.2018.10.007>. Citado na página 43.

BANACH, S. *Théorie des opérations linéaires*. [S.l.]: Monografie Matematyczne, 1, Warsaw, 1932. Citado na página 11.

BAYART, F.; PELLEGRINO, D.; SEOANE-SEPÚLVEDA, J. B. The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$. *Adv. Math.*, v. 264, p. 726–746, 2014. ISSN 0001-8708. Disponível em: <https://doi.org/10.1016/j.aim.2014.07.029>. Citado na página 28.

BERNARDINO, A. T. A simple natural approach to the uniform boundedness principle for multilinear mappings. *Proyecciones*, v. 28, n. 3, p. 203–207, 2009. ISSN 0716-0917. Disponível em: <https://doi.org/10.4067/S0716-09172009000300001>. Citado na página 14.

BERRIOS, S.; BOTELHO, G. Ideal topologies and corresponding approximation properties. *Ann. Acad. Sci. Fenn. Math.*, v. 41, n. 1, p. 265–285, 2016. ISSN 1239-629X. Disponível em: <https://doi.org/10.5186/aasfm.2016.4117>. Citado na página 18.

BERTOLOTO, F.; BOTELHO, G.; JATOBÁ, A. Ideal properties of the Dunford integration operator. *Math. Nachr.*, v. 288, n. 11-12, p. 1207–1215, 2015. ISSN 0025-584X. Disponível em: <https://doi.org/10.1002/mana.201400181>. Citado na página 18.

BOHNENBLUST, H. F.; HILLE, E. On the absolute convergence of Dirichlet series. *Ann. of Math. (2)*, v. 32, n. 3, p. 600–622, 1931. ISSN 0003-486X. Disponível em: <https://doi.org/10.2307/1968255>. Citado 2 vezes nas páginas 28 and 29.

BOTELHO, G. Cotype and absolutely summing multilinear mappings and homogeneous polynomials. *Proc. Roy. Irish Acad. Sect. A*, v. 97, n. 2, p. 145–153, 1997. ISSN 0035-8975. Citado na página 69.

______. Ideals of polynomials generated by weakly compact operators. *Note Mat.*, v. 25, n. 1, p. 69–102, 2005/06. ISSN 1123-2536. Citado na página 27.

BOTELHO, G.; BRAUNSS, H.-A.; JUNEK, H. Almost p-summing polynomials and multilinear mappings. *Arch. Math. (Basel)*, v. 76, n. 2, p. 109–118, 2001. ISSN 0003-889X. Disponível em: <https://doi.org/10.1007/s000130050550>. Citado na página 26.

BOTELHO, G.; BRAUNSS, H.-A.; JUNEK, H.; PELLEGRINO, D. Holomorphy types and ideals of multilinear mappings. *Studia Math.*, v. 177, n. 1, p. 43–65, 2006. ISSN 0039-3223. Disponível em: <https://doi.org/10.4064/sm177-1-4>. Citado 2 vezes nas páginas 18 and 26.

BOTELHO, G.; PELLEGRINO, D. Two new properties of ideals of polynomials and applications. *Indag. Math. (N.S.)*, v. 16, n. 2, p. 157–169, 2005. ISSN 0019-3577. Disponível em: <https://doi.org/10.1016/S0019-3577(05)80019-9>. Citado 2 vezes nas páginas 18 and 26.

______. Absolutely summing polynomials on Banach spaces with unconditional basis. *J. Math. Anal. Appl.*, v. 321, n. 1, p. 50–58, 2006. ISSN 0022-247X. Disponível em: <https://doi.org/10.1016/j.jmaa.2005.08.017>. Citado 5 vezes nas páginas 66, 71, 72, 77, and 78.

______. On symmetric ideals of multilinear mappings between Banach spaces. *J. Aust. Math. Soc.*, v. 81, n. 1, p. 141–148, 2006. ISSN 1446-7887. Disponível em: <https://doi.org/10.1017/S1446788700014683>. Citado na página 18.

BOTELHO, G.; PELLEGRINO, D.; RUEDA, P. On composition ideals of multilinear mappings and homogeneous polynomials. *Publ. Res. Inst. Math. Sci.*, v. 43, n. 4, p. 1139–1155, 2007. ISSN 0034-5318. Disponível em: <http://projecteuclid.org/euclid.prims/1201012383>. Citado 2 vezes nas páginas 18 and 26.

BOTELHO, G.; TORRES, E. R. Hyper-ideals of multilinear operators. *Linear Algebra Appl.*, v. 482, p. 1–20, 2015. ISSN 0024-3795. Disponível em: <https://doi.org/10.1016/j.laa.2015.05.012>. Citado na página 23.

______. Polynomial ideals from a nonlinear viewpoint. 2016. Disponível em: <https://arxiv.org/abs/1603.00246>. Citado 2 vezes nas páginas 23 and 24.

______. Two-sided polynomial ideals on Banach spaces. *J. Math. Anal. Appl.*, v. 462, n. 1, p. 900–914, 2018. ISSN 0022-247X. Disponível em: <https://doi.org/10.1016/j.jmaa.2017.12.054>. Citado 2 vezes nas páginas 23 and 25.

BOTELHO, G.; TORRES, E. R.; VELANGA, T. Linearization of multipolynomials and applications. *Arch. Math. (Basel)*, v. 110, n. 6, p. 605–615, 2018. ISSN 0003-889X. Disponível em: <https://doi.org/10.1007/s00013-018-1161-5>. Citado na página 33.

CARANDO, D.; DIMANT, V.; MURO, S. Coherent sequences of polynomial ideals on Banach spaces. *Math. Nachr.*, v. 282, n. 8, p. 1111–1133, 2009. ISSN 0025-584X. Disponível em: <https://doi.org/10.1002/mana.200610791>. Citado 2 vezes nas páginas 26 and 27.

______. Every Banach ideal of polynomials is compatible with an operator ideal. *Monatsh. Math.*, v. 165, n. 1, p. 1–14, 2012. ISSN 0026-9255. Disponível em: <https://doi.org/10.1007/s00605-010-0255-3>. Citado na página 26.

______. Holomorphic functions and polynomial ideals on Banach spaces. *Collect. Math.*, v. 63, n. 1, p. 71–91, 2012. ISSN 0010-0757. Disponível em: <https://doi.org/10.1007/s13348-010-0025-5>. Citado na página 26.

CARO, N.; ALARCÓN, D. Núñez; SERRANO-RODRÍGUEZ, D. M. On the generalized Bohnenblust-Hille inequality for real scalars. *Positivity*, v. 21, n. 4, p. 1439–1455, 2017. ISSN 1385-1292. Disponível em: <https://doi.org/10.1007/s11117-017-0478-9>. Citado na página 28.

CALISKAN, E.; PELLEGRINO, D. M. On the multilinear generalizations of the concept of absolutely summing operators. *Rocky Mountain J. Math.*, v. 37, n. 4, p. 1137–1154, 2007. ISSN 0035-7596. Disponível em: <https://doi.org/10.1216/rmjm/1187453101>. Citado na página 26.

CHERNEGA, I.; ZAGORODNYUK, A. Generalization of the polarization formula for nonhomogeneous polynomials and analytic mappings on Banach spaces. *Topology*, v. 48, n. 2-4, p. 197–202, 2009. ISSN 0040-9383. Disponível em: <https://doi.org/10.1016/j.top.2009.11.019>. Citado 4 vezes nas páginas 11, 33, 42, and 46.

DEFANT, A.; FLORET, K. *Tensor norms and operator ideals*. [S.l.]: North-Holland Publishing Co., Amsterdam, 1993. v. 176. xii+566 p. (North-Holland Mathematics Studies, v. 176). ISBN 0-444-89091-2. Citado na página 53.

DEFANT, A.; POPA, D.; SCHWARTING, U. Coordinatewise multiple summing operators in Banach spaces. *J. Funct. Anal.*, v. 259, n. 1, p. 220–242, 2010. ISSN 0022-1236. Disponível em: <https://doi.org/10.1016/j.jfa.2010.01.008>. Citado na página 23.

DIESTEL, J.; JARCHOW, H.; TONGE, A. *Absolutely summing operators*. Cambridge University Press, Cambridge, 1995. v. 43. xvi+474 p. (Cambridge Studies in Advanced Mathematics, v. 43). ISBN 0-521-43168-9. Disponível em: <https://doi.org/10.1017/CBO9780511526138>. Citado 6 vezes nas páginas 20, 30, 48, 50, 67, and 77.

DIMANT, V. Strongly p-summing multilinear operators. *J. Math. Anal. Appl.*, v. 278, n. 1, p. 182–193, 2003. ISSN 0022-247X. Disponível em: <https://doi.org/10.1016/S0022-247X(02)00710-2>. Citado na página 26.

DINEEN, S. *Complex analysis on infinite-dimensional spaces*. Springer-Verlag London, Ltd., London, 1999. xvi+543 p. (Springer Monographs in Mathematics). ISBN 1-85233-158-5. Disponível em: <https://doi.org/10.1007/978-1-4471-0869-6>. Citado na página 12.

FLORET, K. On ideals of n-homogeneous polynomials on Banach spaces. In: *Topological algebras with applications to differential geometry and mathematical physics (Athens, 1999)*. [S.l.]: Univ. Athens, Athens, 2002. p. 19–38. Citado na página 21.

FLORET, K.; GARCÍA, D. On ideals of polynomials and multilinear mappings between Banach spaces. *Arch. Math. (Basel)*, v. 81, n. 3, p. 300–308, 2003. ISSN 0003-889X. Disponível em: <https://doi.org/10.1007/s00013-003-0439-3>. Citado 2 vezes nas páginas 18 and 20.

HARDY, G. H.; LITTLEWOOD, J. E.; PÓLYA, G. *Inequalities*. [S.l.]: Cambridge, at the University Press, 1952. xii+324 p. 2d ed. Citado na página 86.

HÖLDER, O. Über einen Mittelwertsatz. *Göttinger Nachrichten*, p. 38–47, 1889. Citado na página 86.

JÚNIOR, F. V. C. The optimal multilinear Bohnenblust-Hille constants: a computational solution for the real case. *Numer. Funct. Anal. Optim.*, v. 39, n. 15, p. 1656–1668, 2018. ISSN 0163-0563. Disponível em: <https://doi.org/10.1080/01630563.2018.1489414>. Citado na página 43.

LINDENSTRAUSS, J.; PELCZYŃSKI, A. Absolutely summing operators in L_p-spaces and their applications. *Studia Math.*, v. 29, p. 275–326, 1968. ISSN 0039-3223. Disponível em: <https://doi.org/10.4064/sm-29-3-275-326>. Citado 2 vezes nas páginas 66 and 70.

MATOS, M. C. Absolutely summing holomorphic mappings. *An. Acad. Brasil. Ciênc.*, v. 68, n. 1, p. 1–13, 1996. ISSN 0001-3765. Citado 2 vezes nas páginas 30 and 31.

______. Fully absolutely summing and Hilbert-Schmidt multilinear mappings. *Collect. Math.*, v. 54, n. 2, p. 111–136, 2003. ISSN 0010-0757. Citado 2 vezes nas páginas 26 and 31.

______. Nonlinear absolutely summing mappings. *Math. Nachr.*, v. 258, p. 71–89, 2003. ISSN 0025-584X. Disponível em: <https://doi.org/10.1002/mana.200310087>. Citado na página 26.

MUJICA, J. *Complex analysis in Banach spaces*. [S.l.]: North-Holland Publishing Co., Amsterdam, 1986. v. 120. xii+434 p. (North-Holland Mathematics Studies, v. 120). Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática [Mathematical Notes], 107. ISBN 0-444-87886-6. Citado na página 12.

PELLEGRINO, D. Cotype and absolutely summing homogeneous polynomials in L_p spaces. *Studia Math.*, v. 157, n. 2, p. 121–131, 2003. ISSN 0039-3223. Disponível em: <https://doi.org/10.4064/sm157-2-2>. Citado 3 vezes nas páginas 66, 70, and 71.

______. Strongly almost summing holomorphic mappings. *J. Math. Anal. Appl.*, v. 287, n. 1, p. 244–252, 2003. ISSN 0022-247X. Disponível em: <https://doi.org/10.1016/S0022-247X(03)00548-1>. Citado na página 26.

______. On scalar-valued nonlinear absolutely summing mappings. *Ann. Polon. Math.*, v. 83, n. 3, p. 281–288, 2004. ISSN 0066-2216. Disponível em: <https://doi.org/10.4064/ap83-3-10>. Citado 2 vezes nas páginas 66 and 71.

PELLEGRINO, D.; RIBEIRO, J. On almost summing polynomials and multilinear mappings. *Linear Multilinear Algebra*, v. 60, n. 4, p. 397–413, 2012. ISSN 0308-1087. Disponível em: <https://doi.org/10.1080/03081087.2011.603315>. Citado na página 26.

______. On multi-ideals and polynomial ideals of Banach spaces: a new approach to coherence and compatibility. *Monatsh. Math.*, v. 173, n. 3, p. 379–415, 2014. ISSN 0026-9255. Disponível em: <https://doi.org/10.1007/s00605-013-0534-x>. Citado 2 vezes nas páginas 27 and 28.

PELLEGRINO, D.; SANTOS, J. Absolutely summing multilinear operators: a panorama. *Quaest. Math.*, v. 34, n. 4, p. 447–478, 2011. ISSN 1607-3606. Disponível em: <https://doi.org/10.2989/16073606.2011.640459>. Citado na página 26.

PELLEGRINO, D.; SANTOS, J.; SEOANE-SEPÚLVEDA, J. B. Some techniques on nonlinear analysis and applications. *Adv. Math.*, v. 229, n. 2, p. 1235–1265, 2012. ISSN 0001-8708. Disponível em: <https://doi.org/10.1016/j.aim.2011.09.014>. Citado na página 26.

PELLEGRINO, D.; TEIXEIRA, E. V. Towards sharp Bohnenblust-Hille constants. *Commun. Contemp. Math.*, v. 20, n. 3, p. 1750029, 33, 2018. ISSN 0219-1997. Disponível em: <https://doi.org/10.1142/S0219199717500298>. Citado 2 vezes nas páginas 28 and 43.

PÉREZ-GARCÍA, D. Comparing different classes of absolutely summing multilinear operators. *Arch. Math. (Basel)*, v. 85, n. 3, p. 258–267, 2005. ISSN 0003-889X. Disponível em: <https://doi.org/10.1007/s00013-005-1125-4>. Citado na página 26.

PÉREZ-GARCÍA, D.; VILLANUEVA, I. Multiple summing operators on Banach spaces. *J. Math. Anal. Appl.*, v. 285, n. 1, p. 86–96, 2003. ISSN 0022-247X. Disponível em: <https://doi.org/10.1016/S0022-247X(03)00352-4>. Citado na página 31.

PIETSCH, A. *Operator ideals*. [S.l.]: VEB Deutscher Verlag der Wissenschaften, Berlin, 1978. v. 16. 451 p. (Mathematische Monographien [Mathematical Monographs], v. 16). Citado 2 vezes nas páginas 18 and 19.

______. Ideals of multilinear functionals (designs of a theory). In: *Proceedings of the second international conference on operator algebras, ideals, and their applications in theoretical physics (Leipzig, 1983)*. [S.l.]: Teubner, Leipzig, 1984. (Teubner-Texte Math., v. 67), p. 185–199. Citado 3 vezes nas páginas 11, 18, and 30.

POPA, D. Nuclear multilinear operators with respect to a partition. *Rend. Circ. Mat. Palermo (2)*, v. 61, n. 3, p. 307–319, 2012. ISSN 0009-725X. Disponível em: <https://doi.org/10.1007/s12215-012-0091-5>. Citado na página 23.

______. Composition results for strongly summing and dominated multilinear operators. *Cent. Eur. J. Math.*, v. 12, n. 10, p. 1433–1446, 2014. ISSN 1895-1074. Disponível em: <https://doi.org/10.2478/s11533-014-0423-0>. Citado na página 23.

ROGERS, L. J. An extension of a certain theorem in inequalities. *Messenger of Math.*, v. 17, p. 145–150, 1888. Citado na página 86.

SANDBERG, I. W. Multilinear maps and uniform boundedness. *IEEE Trans. Circuits and Systems*, v. 32, n. 4, p. 332–336, 1985. ISSN 0098-4094. Disponível em: <https://doi.org/10.1109/TCS.1985.1085724>. Citado na página 14.

SANTOS, J.; VELANGA, T. On the Bohnenblust-Hille inequality for multilinear forms. *Results Math.*, v. 72, n. 1-2, p. 239–244, 2017. ISSN 1422-6383. Disponível em: <https://doi.org/10.1007/s00025-016-0628-6>. Citado na página 28.

TALAGRAND, M. Cotype and $(q, 1)$-summing norm in a Banach space. *Invent. Math.*, v. 110, n. 3, p. 545–556, 1992. ISSN 0020-9910. Disponível em: <https://doi.org/10.1007/BF01231344>. Citado na página 68.

VELANGA, T. Ideals of polynomials between Banach spaces revisited. *Linear Multilinear Algebra*, v. 66, n. 11, p. 2328–2348, 2018. ISSN 0308-1087. Disponível em: <https://doi.org/10.1080/03081087.2017.1394963>. Citado na página 33.

______. Absolutely summing multipolynomials. *Results Math*, v. 74, n. 1, p. 1–17, 2019. ISSN 1422-6383. Disponível em: <https://doi.org/10.1007/s00025-018-0947-x>. Citado na página 66.

APPENDIX A – Hölder's inequality and its extensions

The Hölder's inequality, as stated in Proposition A.0.1, was first proved by L. J. Rogers (ROGERS, 1888). The Hölder's proof appeared in a less symmetrical form a little later in (HÖLDER, 1889). A thorough discussion with its analogs and extensions can be found, for instance, in the classical book (HARDY; LITTLEWOOD; PÓLYA, 1952). We remark some of those extensions which has been helpful in this thesis.

Proposition A.0.1 (Hölder's inequality)**.** *Let $n \in \mathbb{N}$ and $p, q > 1$ be such that $1/p + 1/q = 1$. Then*

$$\sum_{j=1}^{n} |a_j b_j| \leqslant \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}$$

regardless of the choice of the scalars $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Corollary A.0.2. *Let $n \in \mathbb{N}$ and $p, q, s > 0$ be such that $1/p + 1/q = 1/s$. Then*

$$\left(\sum_{j=1}^{n} |a_j b_j|^s\right)^{\frac{1}{s}} \leqslant \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}$$

regardless of the choice of the scalars $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Proof. In fact,

$$\frac{s}{p}, \frac{s}{q} < \frac{s}{p} + \frac{s}{q} = 1$$

implies that p/s, $q/s > 1$ such that

$$\frac{1}{\frac{p}{s}} + \frac{1}{\frac{q}{s}} = 1.$$

An application of Proposition A.0.1 completes the proof. □

The next extension can also be deduced from (HARDY; LITTLEWOOD; PÓLYA, 1952, Theorem 12).

Corollary A.0.3. *Let $n \in \mathbb{N}$ and $p_1, \ldots, p_m, s > 0$ be such that $1/p_1 + \cdots + 1/p_m = 1/s$. Then*

$$\left(\sum_{j=1}^{n} \left|x_j^{(1)} \cdots x_j^{(m)}\right|^s\right)^{\frac{1}{s}} \leqslant \left(\sum_{j=1}^{n} \left|x_j^{(1)}\right|^{p_1}\right)^{\frac{1}{p_1}} \cdots \left(\sum_{j=1}^{n} \left|x_j^{(m)}\right|^{p_m}\right)^{\frac{1}{p_m}} \tag{A.1}$$

regardless of the choice of the scalars $x_j^{(1)}, \ldots, x_j^{(m)}$, where $j = 1, \ldots, n$.

Proof. we proceed by induction on m. There is nothing to do when $m = 1$. If inequality (A.1) holds for $m - 1$, then it follows from Corollary A.0.2 that

$$\left(\sum_{j=1}^{n} \left| x_j^{(1)} \cdots x_j^{(m-1)} x_j^{(m)} \right|^s \right)^{\frac{1}{s}}$$
$$\leqslant \left(\sum_{j=1}^{n} \left| x_j^{(1)} \cdots x_j^{(m-1)} \right|^{\frac{1}{\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}}}} \right)^{\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}}} \left(\sum_{j=1}^{n} \left| x_j^{(m)} \right|^{p_m} \right)^{\frac{1}{p_m}}. \tag{A.2}$$

The induction hypothesis with $s = 1/(1/p_1 + \cdots + 1/p_{m-1})$ implies that

$$\left(\sum_{j=1}^{n} \left| x_j^{(1)} \cdots x_j^{(m-1)} \right|^{\frac{1}{\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}}}} \right)^{\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}}}$$
$$\leqslant \left(\sum_{j=1}^{n} \left| x_j^{(1)} \right|^{p_1} \right)^{\frac{1}{p_1}} \cdots \left(\sum_{j=1}^{n} \left| x_j^{(m-1)} \right|^{p_{m-1}} \right)^{\frac{1}{p_{m-1}}} \tag{A.3}$$

Now, replace (A.2) with (A.3), and we are done. □

Remark A.0.4. *The Hölder's inequality (as well as its extensions) is also true for $p, q > 1$ such that $1/p + 1/q \geqslant 1$. Indeed, there exist $p' \geqslant p$ and $q' \geqslant q$ such that $1/p' + 1/q' = 1$. Then, it follows from Hölder's inequality and the canonical inclusion between the ℓ_p norms spaces that*

$$\sum_{j=1}^{n} |a_j b_j| \leqslant \left(\sum_{j=1}^{n} |a_j|^{p'} \right)^{\frac{1}{p'}} \left(\sum_{j=1}^{n} |b_j|^{q'} \right)^{\frac{1}{q'}}$$
$$\leqslant \left(\sum_{j=1}^{n} |a_j|^{p} \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^{q} \right)^{\frac{1}{q}},$$

as desired.

APPENDIX B – Index