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DANIEL IDE FUTATA

**Classification of rank 2 Stable Vector Bundles
on $\mathbb{P}^3$ with odd degree**

**Classificação de Fibrados Vetoriais Estáveis de
posto 2 em $\mathbb{P}^3$ com grau ímpar**

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*"To show or to be shown?" is a question never, even not known by many to exist - The
Residents*

Resumo

Nesse trabalho nós estudamos fibrados vetoriais estáveis de posto 2 com primeira classe de Chern -1 analisando seus espectros. Além disso procuramos possíveis mônadas de acordo com cada espectro, seguindo o trabalho feito por Hartshorne e Rao em "Spectra and Monads of Stable Bundles".

Palavras-chave: Geometria Algébrica. Fibrados Vetoriais Estáveis. Mônadas.

Abstract

In this work we study rank 2 stable vector bundles with first Chern class -1 by analyzing their spectra, In addition we search for possible monads according with each spectrum, following the work done by Hartshorne and Rao in "Spectra and Monads of Stable Bundles".

Keywords: Algebraic Geometry. Stable Vector Bundles. Monads.

Contents

	Introduction	11
1	PRELIMINARIES	14
1.1	Basic definitions and results	14
1.2	Curves and Chern classes	17
1.3	Monads	20
2	LINEAR AND HORROCKS MONADS	23
2.1	Linear monads	23
2.2	Horrocks monads	28
3	TOWARD THE CLASSIFICATION	37
3.1	Spectrum and its properties	37
3.2	The Hartshorne-Serre correspondence	39
	BIBLIOGRAPHY	47

Introduction

Let $\mathcal{E}$ be a vector bundle of rank 2 on $\mathbb{P}^3$ and c_1 and c_2 denotes the first and second Chern classes of $\mathcal{E}$. Associated to $\mathcal{E}$ are its spectrum and its minimal monad.

Monads were introduced by Horrocks in 1964 [6, Horrocks]. It is defined to be a complex of vector bundles

$$\mathcal{V}_\bullet : \mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1 \xrightarrow{\beta} \mathcal{V}_2 \quad (1)$$

such that α is injective and β surjective as morphism of sheaves. We say $\mathcal{V}_\bullet$ is associated to $\mathcal{E}$ if its cohomology $\ker\beta/\text{im}\alpha$ is isomorphic to $\mathcal{E}$. A monad associated to $\mathcal{E}$ is minimal in the sense that it comes from a minimal free resolution of $M = H_*^1(\mathcal{E}) := \bigoplus_{k \in \mathbb{Z}} H^1(\mathcal{E}(k))$, the first cohomology module [18, Rao]. A method for constructing minimal monads is given by Rao in [18, Rao], essentially it depends on the minimal generators of the module M over $S := k[x_0, x_1, x_2, x_3]$ and their respective degrees:

Let

$$0 \rightarrow L_4 \rightarrow L_3 \rightarrow L_2 \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0 \quad (2)$$

be a minimal free resolution of M over S where $\text{rk}(L_0) = t$ and $\text{rk}(L_1) = s$. Then t is the number of minimal generators for M and s is the number of minimal relations among them. A minimal monad for $\mathcal{E}$ is given by

$$\tilde{L}_0^*(c_1) \rightarrow \tilde{L}_1 \rightarrow \tilde{L}_0 \quad (3)$$

Here $\tilde{L}$ means the sheafification of an S -module L .

[18, Rao] also gives two important results:

Theorem. *Let $M := H_*^1(\mathcal{E})$, then all rank two bundles on $\mathbb{P}^3$ that have M as their cohomology module have same Chern classes c_1 and c_2*

and

Proposition. *If M is the cohomology module of a stable rank 2 bundle on $\mathbb{P}^3$ with Chern classes c_1 and c_2 , then there is an irreducible subset of $\mathcal{M}(c_1, c_2)$ that parametrises those rank 2 bundles with first cohomology module equal to M*

Here $\mathcal{M}(c_1, c_2)$ denotes the coarse moduli space parametrising the stable rank 2 vector bundles with Chern classes c_1 and c_2 .

Inspired in the previous works of Horrocks [12, Horrocks] and Serre [22, Serre], Hartshorne proves the so called Hartshorne-Serre correspondence which states that there

is a bijective correspondence between nonsingular subcanonical varieties of codimension 2 in $\mathbb{P}^n$ and vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ of rank 2, $n \geq 3$. Moreover, the variety occurs as the zero-set of $\mathcal{E}$. The case $n = 3$ was later generalized by Hartshorne [10, Hartshorne] to give a correspondence between reflexive sheaves of rank 2 and more general curves.

The spectrum of $\mathcal{E}$ (stable with $c_1 = 0$) was first defined by Barth and Elençwajg in [3, Barth & Elençwajg], it is a sequence of integers $\{k_i\}_{i=1,\dots,c_2}$ satisfying specific conditions, the two principal conditions being:

- (1) $\{k_i\}$ is symmetric around 0;
- (2) $\{k_i\}$ is a connected sequence of integers;

In [10, Hartshorne] Hartshorne generalized the definition of spectrum to include reflexive sheaves $\mathcal{F}$ of rank 2 on $\mathbb{P}^3$, $c_1 = 0$ or -1 and $H^0(\mathcal{F}(-1)) = 0$.

In [11, Hartshorne & Rao] Hartshorne and Rao studied the possible spectra and minimal monads for $c_1 = 0$ and for low values of c_2 ($c_2 = 1, \dots, 8$). They constructed families of curves on $\mathbb{P}^3$, each family via Hartshorne-Serre correspondence gives a family of vector bundles. Together with the associated spectrum and minimal monad they summarized this data in a table [11, Hartshorne & Rao, Table 5.3].

This dissertation is structured in three chapters:

The first chapter contains the necessary concepts which will be necessary in the later chapters.

In chapter 2 we present the study of two papers, [13, Jardim] and [14, Jardim & Martins]. Which serves as an introduction to the study of monads and vector bundles using cohomology.

In chapter 3 we introduce the Hartshorne-Serre correspondence and the spectrum of a sheaf. Fixed a spectrum χ we then give a curve, which will correspond to a stable vector bundle $\mathcal{E}$ with spectrum χ and $c_1 = -1$, and finally we search for possible monads with cohomology $\mathcal{E}$.

We obtain a table of possible spectra depending on the values of c_2 , we also describe the following examples:

- A disjoint union of conics corresponds to a normalized vector bundle with only p_0 's in the spectrum

- A disjoint union of a plane curve of degree $2m - 2$ and a complete intersection of surfaces of degree m and $m - 1$ for $m \geq 2$ gives a normalized bundle with spectrum of the form $p_0 p_1 \dots p_{m-2}$

- The disjoint union of two plane curves of degree 4 and a complete intersection of surfaces of degree 3 and 2 gives a normalized bundle with spectrum $p_0^3 p_1^2$, this example

also shows that we can generate other bundles with higher second Chern class with only p_0 and p_1 appearing in the spectrum.

1 Preliminaries

In this chapter we will recall the necessary concepts and results about coherent sheaves, monads, curves and Chern classes, including the Hirzebruch-Riemann-Roch theorem, which we will need throughout this dissertation.

1.1 Basic definitions and results

X denotes a scheme, from now on whenever we write $\mathcal{F}^I$ for a sheaf $\mathcal{F}$ we mean the direct sum $\bigoplus_{i \in I} \mathcal{F}_i$ where $\mathcal{F}_i = \mathcal{F}$. If $r \in \mathbb{N}$ we will write $\mathcal{F}^r$ to denote the direct sum of r copies of $\mathcal{F}$. The base field k is an algebraically closed field of characteristic 0.

Definition 1.1.1. A sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$ is

(i) finitely generated, or of finite type if every point $x \in X$ has an open neighbourhood such that there is a surjective morphism

$$\mathcal{O}_X^n|_U \rightarrow \mathcal{F}|_U \quad (1.1)$$

with n finite

(ii) coherent if it is finitely generated and for every open U , every finite $p \in \mathbb{N}$ and every morphism

$$\mathcal{O}_X^p|_U \rightarrow \mathcal{F}|_U \quad (1.2)$$

of $\mathcal{O}_X|_U$ -modules has a finitely generated kernel.

(iii) finitely presented if there is an exact sequence of the form

$$\mathcal{O}_X^p \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{F} \rightarrow 0 \quad (1.3)$$

with p and n finite. Every finitely presented $\mathcal{O}_X$ -module is finitely generated.

(iv) quasi coherent if it is locally presentable, i.e. there is an open cover $\{U_i\}$ of X and for every i an exact sequence

$$\mathcal{O}_X^{I_i}|_{U_i} \rightarrow \mathcal{O}_X^{J_i}|_{U_i} \rightarrow \mathcal{F}|_{U_i} \rightarrow 0 \quad (1.4)$$

where I_i and J_i may be infinite.

If X is locally noetherian, those concepts become intertwined :

Lemma 1.1.2. [24, Stacks, 29.9.1] *Let X be a locally noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. The following are equivalent*

(i) $\mathcal{F}$ is coherent,

- (ii) $\mathcal{F}$ is a quasi-coherent, finite-type $\mathcal{O}_X$ -module,
- (iii) $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$ -module,
- (iv) for any affine open $\text{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \bar{M}$ with M a finite A -module,
- (v) there exists an affine open covering $X = \bigcup U_i$, $U_i = \text{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i} = \bar{M}_i$ with M_i a finite A_i -module.

We will always deal with locally noetherian ringed spaces. So (1.1.2) could be taken as a definition.

Another useful lemma for locally noetherian schemes is the following:

Lemma 1.1.3. [24, Stacks, 29.9.5] *Let X be a locally noetherian scheme. Let $\mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_X$ -modules. Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism and $x \in X$.*

- (i) *If $\mathcal{F}_x = 0$ then there is an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$,*
- (ii) *if $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective, then there is an open neighbourhood $U \subset X$ of x such that $\phi|_U$ is injective,*
- (iii) *if $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is surjective, then there is an open neighbourhood $U \subset X$ of x such that $\phi|_U$ is surjective,*
- (iv) *if $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is bijective, then there is an open neighbourhood $U \subset X$ of x such that $\phi|_U$ is an isomorphism.*

Lemma 1.1.4. *A coherent sheaf $\mathcal{F}$ on a locally noetherian scheme X with free stalks is a locally free sheaf.*

Proof. Being locally free is a local concept, so assume $X = \text{Spec}(A)$ and $\mathcal{F} = \bar{M}$. By Lemma (1.1.2) M is a finitely generated module over A . Then $\bar{M}$ is locally free if and only if M is projective. But a finitely generated module M is projective if and only if all its localizations $M_{\mathfrak{p}}$ are $A_{\mathfrak{p}}$ free modules. $\square$

From now on every scheme X will be locally noetherian, Lemma (1.1.4) gives a proposition which will be frequently used:

Proposition 1.1.5. *Let $\beta : \mathcal{F} \rightarrow \mathcal{G}$ be a surjective morphism of locally free sheaves, then $\ker \beta$ is a locally free sheaf.*

Proof. Let $x \in X$. $\mathcal{G}_x$ is free since $\mathcal{G}$ is locally free. In particular it is a projective module. Then the sequence of modules

$$0 \rightarrow \ker \beta_x \rightarrow \mathcal{F}_x \rightarrow \mathcal{G}_x \rightarrow 0 \quad (1.5)$$

splits. Hence $\mathcal{F}_x = \ker \beta_x \oplus \mathcal{G}_x$, and $\ker \beta_x$ is projective since $\mathcal{F}_x$ is free. A projective module over a local ring is free which implies that $\ker \beta_x$ is free. The kernel of a morphism between coherent sheaves on a locally noetherian scheme is also coherent, and a coherent sheaf with free stalks is locally free by Lemma (1.1.4). $\square$

Definition 1.1.6. Let $\mathcal{F}$ be a coherent sheaf on a projective variety X . We define its support to be the closed set $\text{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$ where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x . The singular locus of $\mathcal{F}$ is the closed set $\text{Sing}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \text{ is not a free } \mathcal{O}_{X,x} \text{-mod}\}$.

The next theorem tells us we can look at the sheaves $\mathcal{E}xt^p(\mathcal{F}, \mathcal{O}_X)$ to see where $\mathcal{F}$ fails to be free

Theorem 1.1.7. *The singular locus of $\mathcal{F}$ is $\bigcup_{p=1}^{\dim X} \text{Supp } \mathcal{E}xt^p(\mathcal{F}, \mathcal{O}_X)$*

Proof. [16, Okonek, Schneider & Spindler, Lemma 1.1.4] □

We are going to introduce some essential concepts of nonsingular varieties. They are going to be extensively used through the rest of this dissertation.

Definition 1.1.8 (Sheaf of relative differentials). Let $f : X \rightarrow Y$ be a morphism of schemes. Consider the diagonal morphism $\Delta : X \rightarrow X \times_Y X$, the diagonal morphism gives an isomorphism between X and $\Delta(X)$ so we can consider $\Delta(X)$ as a locally closed subscheme of $X \times_Y X$, i.e. a closed subscheme of an open set of $X \times_Y X$. Let $\mathcal{I}$ be the ideal sheaf of $\Delta(X)$. We define the sheaf of relative differentials of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X . It is a quasi-coherent sheaf of $\mathcal{O}_X$ -modules and if Y is noetherian and f is of finite type then $\Omega_{X/Y}$ is coherent.

Proposition 1.1.9 (The Euler sequence). *Let A be a ring, let $Y = \text{Spec}(A)$ and $X = \mathbb{P}_A^n$. Then there is an exact sequence of sheaves on X .*

$$0 \rightarrow \Omega_{X/Y} \rightarrow \mathcal{O}_X(-1)^{n+1} \rightarrow \mathcal{O}_X \rightarrow 0 \quad (1.6)$$

Proof. [8, Hartshorne, Theorem II.8.13] □

If X is a nonsingular variety of dimension n over k we define the **tangent sheaf** $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ and the **canonical sheaf** $\omega_X = \bigwedge^n \Omega_{X/k}$. The tangent sheaf is locally free of rank n and the canonical sheaf is invertible. If $Y \xrightarrow{i} X$ is a nonsingular subvariety we define the **conormal sheaf** of Y on X as $\mathcal{C}_{Y/X} = i^*(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the ideal sheaf of Y in X . Its dual $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}_{Y/X}, \mathcal{O}_Y)$ is the **normal sheaf** of Y in X . Both are locally free.

We need one more theorem from [8, Hartshorne]:

Theorem 1.1.10. *Let X be a nonsingular variety over k and $Y \subset X$ an irreducible closed subscheme defined by a sheaf of ideals $\mathcal{I}$. Then Y is nonsingular if and only if*

(i) $\Omega_{Y/X}$ is locally free and

(ii) the sequence $0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \rightarrow \Omega_{Y/k} \rightarrow 0$ is exact

Proof. [8, Hartshorne, theorem II.8.17] □

Proposition 1.1.11. *Let Y be a nonsingular variety of codimension r in a nonsingular variety X over k . Then $\omega_Y \cong \omega_X|_Y \otimes \bigwedge^r \mathcal{N}_{Y/X}$.*

Proof. Since Y is nonsingular 1.1.10 gives an exact sequence

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \rightarrow \Omega_{Y/k} \rightarrow 0 \quad (1.7)$$

Taking the highest exterior powers gives $\omega_X \otimes \mathcal{O}_Y \cong \omega_Y \otimes \bigwedge^r (\mathcal{I}/\mathcal{I}^2)$. Applying $\otimes \mathcal{N}_{Y/X}$ and commuting the highest exterior power with the dual yields $\omega_Y \cong \omega_X \otimes \bigwedge^r \mathcal{N}_{Y/X}$. □

Proposition 1.1.11 gives a way to compute the canonical sheaf of nonsingular varieties:

Example 1.1.12 (The canonical sheaf of a hypersurface). Let $X = \mathbb{P}_k^n$ for $n \geq 2$ and Y be a nonsingular hypersurface of degree d . Taking the highest exterior power from the Euler sequence gives $\omega_X \cong \mathcal{O}_X(-n-1)$, so by (1.1.11) $\omega_Y \cong \mathcal{O}_X(-n-1) \otimes \mathcal{N}_{Y/X}$. Since Y is a hypersurface of degree d we have the exact sequence $0 \rightarrow \mathcal{O}_X(-d) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$ so $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{I} \otimes \mathcal{O}_Y \cong \mathcal{O}(-d)|_Y$ and hence $\mathcal{N}_{Y/X} \cong \mathcal{O}_Y(d)$ which implies $\omega_Y \cong \mathcal{O}_Y(d-n-1)$.

1.2 Curves and Chern classes

In this section a curve will be an integral scheme of dimension 1, proper over k with regular local rings.

Definition 1.2.1. Let X be a projective scheme of dimension r over k and $\mathcal{F}$ be a coherent sheaf on X . The Euler characteristic of $\mathcal{F}$ is defined by

$$\chi^e(\mathcal{F}) = \sum_{i=0}^r (-1)^i h^i(X, \mathcal{F}) \quad (1.8)$$

Here $h^i(X, \mathcal{F})$ denotes the dimension of $H^i(X, \mathcal{F})$ as a k -vector space

Note that if $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is an exact sequence of coherent sheaves on X , then we have an exact sequence of finite dimensional k -vector spaces

$$0 \rightarrow H^0(X, \mathcal{F}_1) \rightarrow H^0(X, \mathcal{F}_2) \rightarrow H^0(X, \mathcal{F}_3) \rightarrow H^1(X, \mathcal{F}_1) \rightarrow \dots \quad (1.9)$$

which implies $\chi^e(\mathcal{F}_2) = \chi^e(\mathcal{F}_1) + \chi^e(\mathcal{F}_3)$. In other words, χ^e is additive.

Definition 1.2.2. Let X be a projective scheme of dimension r over k . We define the arithmetic genus $p_a(X)$ of X by:

$$p_a(X) = (-1)^r (\chi^e(\mathcal{O}_X) - 1) \quad (1.10)$$

If X is a curve we also define the geometric genus $p_g(X) = h^0(X, \omega_X)$, where ω_X is the canonical sheaf.

If X is a curve then $h^r(\mathcal{O}_X) = 0$ for $r \geq 2$ so $p_a(X) = (-1)(h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) - 1)$. Since X is integral it corresponds to a projective variety, hence $H^0(\mathcal{O}_X) = k$ which implies $p_a(X) = h^1(\mathcal{O}_X)$. Serre duality implies $H^0(X, \omega_X) \cong \text{Ext}^1(\omega_X, \omega_X)^* \cong \text{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)^* \cong H^1(X, \mathcal{O}_X)^*$ because ω_X is invertible, so $h^0(X, \omega_X) = h^1(X, \mathcal{O}_X)$. This leads to the definition of an important number

Definition 1.2.3. Let X be a curve, then $p_a(X) = p_g(X) = h^1(X, \mathcal{O}_X)$ is called the genus of X . We denote it by g .

Since we have only defined those numbers for projective schemes, it is important to notice that a curve X is necessarily projective since X is proper over k . [8, Hartshorne II.6.7].

We will briefly recall some basic definitions on divisors.

Let X a noetherian integral separated scheme which is regular on codimension one [8, Hartshorne, p.130]. A Weil divisor on X is an element of the free abelian group $\text{Div}(X)$ generated by prime divisors, in other words generated by closed integral subscheme Y of X . We can write a divisor $D = \sum_{i=1}^r n_i Y_i$ with $n_i \in \mathbb{Z}$. The degree of the divisor D is defined to be $\deg(D) = \sum_{i=1}^r n_i$. A divisor D is called effective if $n_i \geq 0$ for every i .

Since X is integral it has a generic point $\xi \in X$. The function field of an integral scheme is defined as the stalk at its generic point $K(X) = \mathcal{O}_{X, \xi}$. An element of $K(X)$ is called a rational function. A prime divisor Y is integral so we can consider the stalk at its generic point $\mathcal{O}_{X, \eta}$ which has quotient field $K(X)$ and it is a discrete valuation ring (this comes from the fact that a ring A is a discrete valuation ring if and only if it is noetherian, normal and it has only two prime ideals : 0 and m). So it makes sense to talk about the evaluation $v_Y(f)$ of $f \in K(X)^* = K(X) - \{0\}$ at Y . We define the divisor of f to be $\text{div}(f) = \sum v_Y(f) Y$, where $\sum$ runs over all prime divisors. It is well defined because any proper closed subset of X can contain only a finite amount of prime divisors. [8, Hartshorne, II.6.1]

Given two divisors D, D' we say D is linearly equivalent to D' if $D - D' = \text{div}(f)$ for some rational function f . The quotient group $\text{Div}(X)$ modulo linear equivalence is called the divisor class group $Cl(X)$ and it is isomorphic to the group $\text{Pic}(X)$ of invertible sheaves on X modulo isomorphism [8, Hartshorne, II.6.10], the image of $[D] \in \text{Div}(X)/\sim$ in $\text{Pic}(X)/\sim$ is denoted $\mathcal{L}(D)$. A divisor K corresponding to the invertible sheaf ω_X is said to be a canonical divisor.

The Riemann-Roch theorem was originally proven by Riemann in 1857 [19, Riemann] and later generalized by Roch in 1865 [20, Roch], it was first developed in

an analytic context. We present here a version of Riemann-Roch concerning curves in algebraic geometry.

Theorem 1.2.4. [8, Hartshorne, Riemann-Roch Theorem p.295] *Let D be a divisor on a curve X of genus g . Then*

$$h^0(\mathcal{L}(D)) - h^0(\mathcal{L}(K - D)) = \deg(D) + 1 - g \quad (1.11)$$

where K is a canonical divisor.

Theorem (1.2.4) is an important tool used to compute cohomology of invertible sheaves. The divisor $K - D$ corresponds to an invertible sheaf $\omega_X \otimes \mathcal{L}(D)^*$, using Serre duality we conclude $H^0(\omega_X \otimes \mathcal{L}(D)^*) \cong H^1(\mathcal{L}(D))^*$. Thus the Riemann-Roch theorem simply states $\chi^e(\mathcal{L}(D)) = \deg(D) + 1 - g$.

Example 1.2.5. Let $Y \xrightarrow{i} X = \mathbb{P}^3$ be a conic. It is of the form $V(f_1, f_2)$ where $f_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ and $f_2 \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ and it has degree $\deg(f_1)\deg(f_2) = 2$. The canonical sheaf ω_Y is isomorphic to $\mathcal{O}_Y(-1)$ by Proposition (1.1.12) thus K is linearly equivalent to $-L$ where L is the divisor corresponding to $i^*\mathcal{O}_X(1)$, it has degree $\deg(L) = 2$ because the intersection of a general hyperplane with Y has two points. Apply Theorem (1.2.4) we get: $h^0(\omega_Y) - h^0(\mathcal{O}_Y) = \deg(K) + 1 - g$. By definition $h^0(\omega_Y) = g$ so $g = (-\deg(H) + 2)/2 = 0$. Now let $r \in \mathbb{Z}$ and let $D = rL$. $\mathcal{L}(D) = \mathcal{O}_Y(r)$ so $h^0(\mathcal{O}_Y(r)) - h^1(\mathcal{O}_Y(r)) = 1 + 2r$.

We won't use theorem (1.2.4), but we will use one of the many generalizations, the Hirzebruch-Riemann-Roch. But first we need a few definitions.

Definition 1.2.6. (Chern class of a vector bundle) [8, Hartshorne, p.429] Let $\mathcal{E}$ be a locally free sheaf of rank r on a nonsingular quasi-projective variety X . For each $i = 0, \dots, r$ we define the i th Chern class $c_i(\mathcal{E}) \in A^i(X)$ ($A^i(X)$ is the i th Chow group) by the requirement $c_0(\mathcal{E}) = 1$ and

$$\sum_{i=0}^r (-1)^i \pi^* c_i(\mathcal{E}) \cdot \xi^{r-i} = 0 \quad (1.12)$$

in $A^r(\mathbb{P}(\mathcal{E}))$ where $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ is the projection from the associated projective bundle of $\mathcal{E}$ and $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ is the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

We will be most interested in the case $X = \mathbb{P}^n$. In this case $A^i(X) \cong \mathbb{Z}$ is generated by H^d where H is the hyperplane class associated to $\mathcal{O}_{\mathbb{P}^n}(1)$. So $A(X) = \bigoplus_{i \geq 0} A^i(X)$ is just $\mathbb{Z}[H]/H^{n+1}$ as a group since $A^{n+1}(X) = 0$. It is possible to define a product in $A(X)$, making it isomorphic to $\mathbb{Z}[H]/H^{n+1}$ as a ring. The product $A^d(X) \times A^e(X) \rightarrow A^{d+e}(X)$ can be seen as the general intersection of varieties. So we can identify $c_i(\mathcal{E})$ with an integer for every i and make computations with them as if they were in $\mathbb{Z}[H]/H^{n+1}$. The Chern polynomial $c_t(\mathcal{E})$ is just $1 + c_1(\mathcal{E})t + \dots + c_r(\mathcal{E})t^r$. The Chern polynomial is multiplicative in the sense that if $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$ is exact then $c_t(\mathcal{E}) = c_t(\mathcal{E}')c_t(\mathcal{E}'')$

At some point we will be talking about the Chern class of a coherent sheaf, this is possible because since c_t is multiplicative, it can be defined on the Grothendieck group of vector bundles on X [8, Hartshorne, p.435] and for a nonsingular variety X the Grothendieck groups of all vector bundles and of all coherent sheaves coincide [8, Hartshorne, III Ex. 6.9]. Using this isomorphism we can define the Chern class of a coherent sheaf on X .

Let $c_t(\mathcal{E}) = \prod_{i=1}^r (1 + a_i H)$ considering c_t in $A(X) \otimes \mathbb{Q}$ where a_i are formal symbols,

the exponential Chern character of $\mathcal{E}$ is $ch(\mathcal{E}) = \sum_{i=1}^r e^{a_i}$ where $e^x = 1 + x + \frac{x^2}{2} + \dots$ and the Todd class is defined to be $td(\mathcal{E}) = \prod_{i=1}^r \frac{a_i}{1 - e^{-a_i}}$ where $\frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots$

Theorem 1.2.7. (*Hirzebruch-Riemann-Roch*) For a locally free sheaf $\mathcal{E}$ of rank r on a nonsingular projective variety X of dimension n ,

$$\chi^e(\mathcal{E}) = \deg(ch(\mathcal{E}).td(\mathcal{T}_X))_n \quad (1.13)$$

where $()_n$ denotes the component of degree n in $A(X) \otimes \mathbb{Q}$

The Hirzebruch-Riemann-Roch theorem will be used in chapter 3 to develop the formula (3.8).

Let $\mathcal{E}$ be a vector bundle of rank 2 on $\mathbb{P}^3$. We have the following formulas:

- (i) $c_1(\mathcal{E}(r)) = c_1(\mathcal{E}) + 2r$
- (ii) $c_2(\mathcal{E}(r)) = c_2(\mathcal{E}) + c_1(\mathcal{E})r + r^2$
- (iii) $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$

If $\mathcal{E}$ has first Chern class $c_1 \in \{-1, 0\}$ we say $\mathcal{E}$ is **normalized**. Furthermore we can always normalize a vector bundle $\mathcal{E}$, in other words we can always find a $r \in \mathbb{Z}$ so that $c_1(\mathcal{E}(r)) \in \{-1, 0\}$.

We end this section with the notion of stability of torsion-free sheaves

Definition 1.2.8 (Stability of torsion-free sheaves). Let $\mathcal{F}$ be a torsion-free coherent sheaf on X , a normal projective variety with fixed very ample divisor H . We define the slope of $\mathcal{F}$ to be $\mu(\mathcal{F}) := \deg(c_1(\mathcal{F}))/rk(\mathcal{F})$ where we are considering c_1 as an element of $Pic(X)$ and $\deg$ is the degree with respect to H . $\mathcal{F}$ is **stable** if and only if $\mu(\mathcal{H}) < \mu(\mathcal{F})$ for all proper non-zero coherent subsheaves $\mathcal{H}$ of $\mathcal{F}$ (and is **semistable** if $\mu(\mathcal{H}) \leq \mu(\mathcal{F})$).

A normalized vector bundle $\mathcal{E}$ of rank 2 on $\mathbb{P}^3$ is stable if and only if $H^0(\mathcal{E}) = 0$ [16, Okonek, Schneider & Spindler, p. 167].

1.3 Monads

In this section we will define one of the main objects of this dissertation.

Definition 1.3.1. A monad $\mathcal{V}_\bullet$ on X is a complex of locally free sheaves on X

$$\mathcal{V}_\bullet : \mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1 \xrightarrow{\beta} \mathcal{V}_2 \quad (1.14)$$

such that α is injective and β is surjective as morphisms of sheaves. The sheaf $\mathcal{E} = \ker \beta / \operatorname{ima} \alpha$ is called the cohomology of $\mathcal{V}_\bullet$.

The degeneration locus of $\mathcal{V}_\bullet$ is the set $\Sigma_{\mathcal{V}_\bullet} = \{x \in X \mid \alpha(x) \text{ is not injective}\}$

Monads were introduced by Horrocks, who proved that every vector bundle on $\mathbb{P}^n$ is the cohomology of a minimal monad ([6, Horrocks] or [4, Barth]). In fact Horrocks shows that every rank 2 vector bundle on $\mathbb{P}^3$ is the cohomology of a monad $\mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1 \xrightarrow{\beta} \mathcal{V}_2$ where $\mathcal{V}_i$ are sums of line bundles. We are only interested in this particular situation. But before moving on we need an important theorem given by Beilinson.

Theorem 1.3.2. [16, Okonek, Schneider & Spindler, p. 240] For any locally free sheaf $\mathcal{E}$ on $\mathbb{P}^n$ there exists a spectral sequence $\mathcal{E}_r^{p,q}$ for $\mathcal{E}$ whose $\mathcal{E}_1$ -term is given by ($q = 0, \dots, n$ and $p = 0, -1, \dots, -n$):

$$\mathcal{E}_1^{p,q} = H^q(\mathcal{E}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p) \quad (1.15)$$

which converges to

$$\mathcal{E}^i = \begin{cases} \mathcal{E}, & \text{if } p + q = 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.16)$$

Theorem 1.3.3. [16, Okonek, Schneider & Spindler, p. 245] For any locally free sheaf $\mathcal{E}$ on $\mathbb{P}^n$ there exists a spectral sequence $\mathcal{E}_r^{p,q}$ for $\mathcal{E}$ whose $\mathcal{E}_1$ -term is given by ($q = 0, \dots, n$ and $p = 0, -1, \dots, -n$):

$$\mathcal{E}_1^{p,q} = H^q(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p) \quad (1.17)$$

which converges to

$$\mathcal{E}^i = \begin{cases} \mathcal{E}, & \text{if } p + q = 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.18)$$

In addition with theorems such as the Serre duality and Riemann-Roch the Beilinson spectral sequence is an important tool for characterizing vector bundles on $\mathbb{P}^n$.

We conclude the chapter giving an example on how to use the Beilinson spectral sequence. This result will also be used in chapter 2.

Lemma 1.3.4. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and $\{\mathcal{F}_r^{p,q}\}$ its Beilinson spectral sequence. If $\mathcal{F}_1^{p,q} = 0$ for $q \neq 1$ and for $q = 1$, $p \leq -3$ then the spectral sequence degenerates at the $\mathcal{F}_2$ -term and the monad

$$0 \rightarrow \mathcal{F}_\infty^{-2,1} \rightarrow \mathcal{F}_\infty^{-1,1} \rightarrow \mathcal{F}_\infty^{0,1} \rightarrow 0 \quad (1.19)$$

has $\mathcal{F}$ as its cohomology.

Proof. We have the following diagram on $\mathcal{F}_1$:

$$\begin{array}{ccccccc}
 & & & & q \uparrow & & \\
 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & 0 & \longrightarrow & \mathcal{F}_1^{-2,1} & \longrightarrow & \mathcal{F}_1^{-1,1} & \longrightarrow & \mathcal{F}_1^{0,1} \\
 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & p \rightarrow & & &
 \end{array}$$

The differentials $d_1^{pq} : \mathcal{F}_1^{p,q} \rightarrow \mathcal{F}_1^{p+1,q}$ give the complex

$$\mathcal{F}_1^{-2,1} \xrightarrow{\alpha} \mathcal{F}_1^{-1,1} \xrightarrow{\beta} \mathcal{F}_1^{0,1} \quad (1.20)$$

Let $\mathcal{K} = \ker \alpha$, $\mathcal{L} = \ker \beta / \text{im } \alpha$, $\mathcal{M} = \text{coker } \beta$, then the diagram of the $\mathcal{F}_2$ -term looks as follows :

$$\begin{array}{ccccccc}
 & & & & q \uparrow & & \\
 & 0 & & 0 & & 0 & & 0 \\
 & \searrow & & \searrow & & \searrow & & \\
 & 0 & & \mathcal{K} & & \mathcal{L} & & \mathcal{M} \\
 & & & \searrow & & \searrow & & \\
 & 0 & & 0 & & 0 & & 0 \\
 & & & & & & & p \rightarrow
 \end{array}$$

All the differentials $d_2^{pq} : \mathcal{F}_2^{p,q} \rightarrow \mathcal{F}_2^{p+2,q-1}$ vanish and so $\mathcal{K} = \mathcal{F}_\infty^{-2,1}$, $\mathcal{L} = \mathcal{F}_\infty^{-1,1}$, $\mathcal{M} = \mathcal{F}_\infty^{0,1}$. But by Beilinson's theorem $\mathcal{K} = \mathcal{M} = 0$ and $\mathcal{L} = \mathcal{F}$. In particular

$$0 \rightarrow \mathcal{F}_\infty^{-2,1} \rightarrow \mathcal{F}_\infty^{-1,1} \rightarrow \mathcal{F}_\infty^{0,1} \rightarrow 0 \quad (1.21)$$

is a monad whose cohomology is $\mathcal{F}$. □

2 Linear and Horrocks monads

This chapter contains the main goal of the first year of the master's degree: to study and understand the demonstrations from the first sections of [13, Jardim] and [14, Jardim & Martins] serving as an introduction to the study of monads, vector bundles and cohomology.

In the first section of [13, Jardim] it is studied the cohomology of linear monads, including a theorem that guarantees a given torsion-free sheaf to be linear provided some conditions on its cohomology.

In the first two sections of [14, Jardim & Martins] it is given a bijective correspondence between isomorphism classes of monads and collections of homogeneous elements of $H_*^1(\mathcal{E})$ and $H_*^1(\mathcal{E} \otimes \omega_X)$ where $\mathcal{E}$ is locally free and X is an ACM variety. It is also studied the cohomology functor that associates to each Horrocks monad its cohomology sheaf.

2.1 Linear monads

Let $\mathcal{V}_\bullet : \mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1 \xrightarrow{\beta} \mathcal{V}_2$ be a monad, it is worth noticing that its cohomology $\mathcal{E}$ is not always locally free, but conditions on α can be set to guarantee it. Next proposition states that $\mathcal{E}$ is locally free if and only if $\alpha(x)$ is injective for every x .

Proposition 2.1.1. *The degeneration locus of $\mathcal{V}_\bullet$ coincides with the singular locus of its cohomology $\mathcal{E}$. In other words $\Sigma_{\mathcal{V}_\bullet} = \text{Sing}(\mathcal{E}) = \text{Supp}(\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X))$*

Proof. Consider the short exact sequence :

$$0 \rightarrow \mathcal{V}_0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0 \quad (2.1)$$

where $\mathcal{K} = \ker \beta$. Applying the functor $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^n})$ the sequence yields:

$$0 \rightarrow \mathcal{E}^* \rightarrow \mathcal{K}^* \xrightarrow{\alpha^*} \mathcal{V}_0^* \rightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X) \rightarrow 0 \quad (2.2)$$

$\mathcal{K}$ being locally free by 1.1.5, and $\mathcal{V}_0$ is locally free by hypothesis which implies $\mathcal{E}xt^p(\mathcal{E}, \mathcal{O}_X) = 0$ for $p \geq 2$. Thus $\text{Sing}(\mathcal{E}) = \text{Supp}(\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X))$.

The first equality follows from the fact that surjectivity on the stalks happens if and only if the morphism of fibers is surjective.

Then $\alpha(x)$ is injective if and only if $\alpha^*(x)$ is surjective if and only if α_x^* is surjective if and only if $x \notin \Sigma_{\mathcal{V}_\bullet}$. $\square$

In 1977 [1, Atiyah, Hitchin, Drinfeld & Manin] introduced the notion of a mathematical instanton bundle on $\mathbb{P}^3$. Instantons are self-dual solutions of the Yang-Mills

equations in the compactified euclidean 4-space S^4 which corresponds to certain real algebraic bundles on $\mathbb{P}_{\mathbb{C}}^3$. In 1986 [17, Okonek & Spindler] generalized this idea to define mathematical instanton bundles on $\mathbb{P}^{2n+1}$.

Each instanton bundle appears as the cohomology of a monad of the form

$$\mathcal{O}_{\mathbb{P}^{2n+1}}(-1)^r \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^{2n+1}}^{2n+2r} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^{2n+1}}(1)^r \quad (2.3)$$

This motivates the following definition

Definition 2.1.2. [13, Jardim] A monad is said to be linear if it is of the form :

$$V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \quad (2.4)$$

Where V, W, U are finite dimensional vector spaces, and the product $V \otimes \mathcal{O}_X$ denotes the locally free sheaf of rank $\dim(V)$ with fiber V .

A coherent sheaf will be called linear if it can be represented as the cohomology of a linear monad.

The next theorem will give some necessary conditions for a sheaf $\mathcal{E}$ to be linear.

Theorem 2.1.3. [13, Jardim] If $\mathcal{E}$ is a linear sheaf, then:

- (i) for $n \geq 2$, $H^0(\mathcal{E}(k)) = H^0(\mathcal{E}^*(k)) = 0$, $\forall k \leq -1$
- (ii) for $n \geq 3$, $H^1(\mathcal{E}(k)) = 0$, $\forall k \leq -2$
- (iii) for $n \geq 4$, $H^p(\mathcal{E}(k)) = 0$, $2 \leq p \leq n-2$ and $\forall k$
- (iv) for $n \geq 3$, $H^{n-1}(\mathcal{E}(k)) = 0$, $\forall k \geq -n+1$
- (v) for $n \geq 2$, $H^n(\mathcal{E}(k)) = 0$ for $k \geq -n$
- (vi) for $n \geq 2$, $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^n}) = \text{coker } \alpha^*$ and $\mathcal{E}xt^p(\mathcal{E}(k), \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p \geq 2$ and all k

Proof. Since $\mathcal{E}$ is a linear sheaf, we can write it as the cohomology of a monad:

$$V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \quad (2.5)$$

First note that kernel $\mathcal{K} = \ker \beta$ is locally free (1.1.5), and there are two sequences for every integer k :

$$0 \rightarrow \mathcal{K}(k) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^n}(k) \rightarrow U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1) \rightarrow 0 \quad (2.6)$$

and

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(k-1) \rightarrow \mathcal{K}(k) \rightarrow \mathcal{E}(k) \rightarrow 0 \quad (2.7)$$

From the first sequence writing the long exact sequence of cohomology:

$$0 \rightarrow H^0(\mathcal{K}(k)) \rightarrow H^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(k)) \rightarrow H^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1)) \rightarrow \dots \quad (2.8)$$

If $k \leq -1$ then $H^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ so that $H^0(\mathcal{K}(k)) = 0$. Continuing the long sequence:

$$H^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1)) \rightarrow H^1(\mathcal{K}(k)) \rightarrow H^1(W \otimes \mathcal{O}_{\mathbb{P}^n}(k)) \rightarrow \dots \quad (2.9)$$

If $k \leq -2$ then $H^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1)) = 0$. Furthermore if $n \geq 2$ then $H^1(\mathcal{O}_{\mathbb{P}^n}(k)) = 0$ and therefore $H^1(W \otimes \mathcal{O}_{\mathbb{P}^n}(k)) = 0$, implying $H^1(\mathcal{K}(k)) = 0$.

Consider now $2 \leq p \leq n-1$ and the sequence

$$H^{p-1}(U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1)) \rightarrow H^p(\mathcal{K}(k)) \rightarrow H^p(W \otimes \mathcal{O}_{\mathbb{P}^n}(k)) \quad (2.10)$$

If $p-1, p \leq n-1$ then the extremities are 0, thus $H^p(\mathcal{K}(k)) = 0$.

Finally if $p = n$, $H^p(\mathcal{O}_{\mathbb{P}^n}(k)) = (\frac{1}{T_0 \dots T_n} \mathbb{C}[\frac{1}{T_0}, \dots, \frac{1}{T_n}])_k$, if $k \geq -n$ this group is 0 which implies $H^n(\mathcal{K}(k)) = 0$.

Now for the second sequence, again looking at the long exact sequence of cohomology:

$$H^p(V \otimes \mathcal{O}_{\mathbb{P}^n}(k-1)) \rightarrow H^p(\mathcal{K}(k)) \rightarrow H^p(\mathcal{E}(k)) \rightarrow H^{p+1}(V \otimes \mathcal{O}_{\mathbb{P}^n}(k-1)) \quad (2.11)$$

The extremities are zero when $p = 0$ and $k \leq -1$, $1 \leq p \leq n-2 \forall k$, $p = n$ and $k \geq -n$ (because we are on the projective space). Under these conditions we have $H^p(\mathcal{K}(k)) \simeq H^p(\mathcal{E}(k))$. All these results gives the first half of (i) through (v).

Dualizing both sequences yields:

$$0 \rightarrow U \otimes \mathcal{O}_{\mathbb{P}^n}(-k-1) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^n}(-k) \rightarrow \mathcal{K}^*(-k) \rightarrow 0 \quad (2.12)$$

$$0 \rightarrow \mathcal{E}^*(-k) \rightarrow \mathcal{K}^*(-k) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-k+1) \rightarrow \mathcal{E}xt^1(\mathcal{E}(k), \mathcal{O}_{\mathbb{P}^n}) \rightarrow \mathcal{E}xt^1(\mathcal{K}(k), \mathcal{O}_{\mathbb{P}^n}) \rightarrow \dots \quad (2.13)$$

Writing the long exact sequence for the first sequence :

$$H^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(-k)) \rightarrow H^0(\mathcal{K}^*(-k)) \rightarrow H^1(U \otimes \mathcal{O}_{\mathbb{P}^n}(-k-1)) \quad (2.14)$$

We conclude that $H^0(\mathcal{K}^*(k)) = 0$ for $k \leq -1$.

For the second sequence, $\mathcal{K}$ locally free implies $\mathcal{E}xt^1(\mathcal{K}(k), \mathcal{O}_{\mathbb{P}^n}) = 0$. Thus

$$0 \rightarrow \mathcal{E}^* \rightarrow \mathcal{K}^* \xrightarrow{\alpha^*} V \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^n}) \rightarrow 0 \quad (2.15)$$

So that $\text{coker } \alpha^* = \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^n})$, $H^0(\mathcal{E}(k)) = 0$ for $k \leq -1$ and $\mathcal{E}xt^p(\mathcal{E}(k), \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p \geq 2 \forall k$.

□

If $\mathcal{E}$ is a torsion-free sheaf, with some of the conditions on Theorem (2.1.3) we can guarantee it to be linear. Even further, we can explicitly give its monad:

Theorem 2.1.4. [13, Jardim] If $\mathcal{E}$ is a torsion-free sheaf on $\mathbb{P}^n$ satisfying:

- (i) for $n \geq 2$, $H^0(\mathcal{E}(-1)) = H^n(\mathcal{E}(-n)) = 0$;
- (ii) for $n \geq 3$, $H^1(\mathcal{E}(-2)) = H^{n-1}(\mathcal{E}(1-n)) = 0$;
- (iii) for $n \geq 4$, $H^p(\mathcal{E}(k)) = 0$, $2 \leq p \leq n-2$ and $\forall k \in \mathbb{Z}$;

then $\mathcal{E}$ is linear and can be represented as the cohomology of the monad:

$$H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^2(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^1) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow H^1(\mathcal{E}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \quad (2.16)$$

Proof. Let $H \subset \mathbb{P}^n$ be a hyperplane defined by a homogeneous polynomial f of degree 1. By a known result [24, Stacks Project Lemma 08A0] there is an exact sequence

$$0 \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(k-1) \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(k) \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(k)|_H \rightarrow 0 \quad (2.17)$$

By hypothesis $H^0(\mathcal{E}(-1)) = 0$, so for $k = -1$ the long exact sequence yields

$$0 \rightarrow H^0(\mathcal{E}(-2)) \rightarrow H^0(\mathcal{E}(-1)) = 0 \quad (2.18)$$

And thus $H^0(\mathcal{E}(-2)) = 0$. Using the same process, by induction results

$$H^0(\mathcal{E}(k)) = 0, \quad k \leq -1.$$

For $k = -n + 1$ we have

$$0 = H^n(\mathcal{E}(-n)) \rightarrow H^n(\mathcal{E}(-n+1)) \rightarrow H^n(\mathcal{E}(-n+1)|_H) \quad (2.19)$$

But $H^n(\mathcal{E}(-n+1)|_H) = 0$ because H^n vanishes on any Noetherian topological space of dimension $n-1$, and $H^i(\mathbb{P}^{n-1}, \mathcal{E}|_H) = H^i(\mathbb{P}^n, i_*\mathcal{E}|_H)$ [8, Hartshorne, Lemma III.2.10], hence $H^n(\mathcal{E}(k)) = 0$ for $k \geq -n$.

Since $H^0(\mathcal{E}(-1)) = H^1(\mathcal{E}(-2)) = 0$, the long sequence yields

$$0 = H^0(\mathcal{E}(-1)) \rightarrow H^0(\mathcal{E}(-1)|_H) \rightarrow H^1(\mathcal{E}(-2)) = 0 \quad (2.20)$$

So $H^0(\mathcal{E}(-1)|_H) = 0$ and therefore $H^0(\mathcal{E}(k)|_H) = 0$ for $k \leq -1$. Then for $k \leq -1$

$$0 = H^0(\mathcal{E}(k)|_H) \rightarrow H^1(\mathcal{E}(k-1)) \rightarrow H^1(\mathcal{E}(k)) \quad (2.21)$$

By induction $H^1(\mathcal{E}(k)) = 0$ for $k \leq -2$.

By hypothesis $H^n(\mathcal{E}(-n)) = H^{n-1}(\mathcal{E}(1-n)) = 0$ then $H^{n-1}(\mathcal{E}(1-n)|_H) = 0$.

We apply the Beilinson spectral sequence on $\mathcal{E}(-1)$ to obtain a spectral sequence with $\mathcal{E}_1^{p,q} = H^q(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{-p}(-p) \otimes \mathcal{O}_{\mathbb{P}^n}(p))$

Now, suppose we had

$$H^q(\mathcal{E}(-1) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for } q \neq 1 \text{ and for } q = 1, p \leq -3. \quad (2.22)$$

Lemma (1.3.4) gives a monad

$$\begin{aligned} 0 \rightarrow H^1(\mathcal{E}(-1) \otimes \Omega_{\mathbb{P}^n}^2(2)) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \\ \rightarrow H^1(\mathcal{E}(-1) \otimes \Omega_{\mathbb{P}^n}^1(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^1(\mathcal{E}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow 0 \end{aligned}$$

With $\mathcal{E}(-1)$ as cohomology. So twisting it by $\mathcal{O}_{\mathbb{P}^n}(1)$ gives a monad with cohomology $\mathcal{E}$. The only thing left to do is to prove (2.22). Consider the Euler sequence for p-forms

$$0 \rightarrow \Omega_{\mathbb{P}^n}^{-p}(-p) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus m} \rightarrow \Omega_{\mathbb{P}^n}^{-p-1}(-p) \rightarrow 0 \quad (2.23)$$

where $p = -1, \dots, -n$ and $m = \binom{n+1}{-p}$. and twist it by $\mathcal{E}(k)$

$$0 \rightarrow \mathcal{E}(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p) \rightarrow \mathcal{E}(k)^{\oplus m} \rightarrow \mathcal{E}(k) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p) \rightarrow 0 \quad (2.24)$$

We have the following vanishings

- $H^0(\mathcal{E}(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0$ for every p and $k \leq -1$
- $H^q(\mathcal{E}(-1) \otimes \Omega_{\mathbb{P}^n}^n(n)) = H^q(\mathcal{E}(-2)) = 0$ for every q
- $H^q(\mathcal{E}(-1)) = 0$ for every $q \neq 1$

$H^n(\mathcal{E}(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0$ for every p and $k \geq -n$. Indeed, by hypothesis $H^0(\mathcal{E}(-1)) = 0$, hence $H^0(\mathcal{E}(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0$ for every p and $k \leq -1$. The second equation follows from the fact that $\Omega_{\mathbb{P}^n}^n(n) \cong \mathcal{O}_{\mathbb{P}^n}(-1)$. $\square$

Now that we have established a characterization of a linear monad, it is natural to ask whether we can guarantee any further property. We already know $\mathcal{E}$ is locally free precisely when its degeneration locus is empty. We will prove that the degeneration locus of a monad is a subvariety.

Definition 2.1.5. Let M be a A -module where A is a ring, if M is finitely generated then it has a projective resolution $0 \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$ with $P_n \neq 0$. The homological dimension (or projective dimension) of M is defined to be the smallest number n . We can define a similar concept for coherent sheaves: let $\mathcal{F}$ be a coherent sheaf over $X = \mathbb{P}^n$ and $x \in X$. The stalk $\mathcal{F}_x$ is a finitely generated module over the local noetherian ring $\mathcal{O}_{X,x}$. Hence we can define the **homological dimension** $dh(\mathcal{F}_x)$ over $\mathcal{O}_{X,x}$ to be the minimal length of a projective (thus free) resolution of $\mathcal{F}_x$.

Let M be a module over a semi-local ring A and $\mathfrak{J}$ the Jacobson radical of A . An M -sequence is a sequence $(a_1, \dots, a_p)$, $a_i \in \mathfrak{J}$ satisfying

- For every integer i with $1 \leq i \leq p$ we have: a_i is not a zero divisor in $M/(a_1, \dots, a_{i-1})M$.
- The cohomological dimension $codh(\mathcal{F}_x)$ is defined as the maximal length of an $\mathcal{F}_x$ -sequence in $\mathcal{O}_{X,x}$. The detailed discussion can be found in [21, Serre, IV-14].

The integer $dh(\mathcal{F}_x)$ is the smallest number k so that for all finitely generated $\mathcal{O}_{X,x}$ -module M and all $i > k$ we have $Ext_{\mathcal{O}_{X,x}}^i(\mathcal{F}_x, M) = 0$. We briefly explain it.

Let $0 \rightarrow P_n \xrightarrow{d_n} P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \xrightarrow{d_0} \mathcal{F}_x$ be a minimal free resolution of $\mathcal{F}_x$. Let M be any $\mathcal{O}_{X,x}$ -module (it is sufficient to consider M finitely generated). Consider the exact sequences $0 \rightarrow K_i \rightarrow P_i \rightarrow K_{i-1} \rightarrow 0$ where $K_i = \ker d_i = \text{im } d_{i+1}$ and apply the functor $Hom_{\mathcal{O}_{X,x}}(-, M)$. Any $Ext_{\mathcal{O}_{X,x}}^i(P_j, M)$ vanishes for $i > 0$, thus setting $i = n - 1$ we obtain

$Ext_{\mathcal{O}_{X,x}}^1(K_{n-1}, M) \cong Ext_{\mathcal{O}_{X,x}}^2(K_{n-2}, M) = 0$ because $K_{n-1} \cong P_n$. This process implies $Ext_{\mathcal{O}_{X,x}}^i(K_{n-i}, M) \cong Ext_{\mathcal{O}_{X,x}}^{i+1}(K_{n-i-1}, M)$ until the last sequence $0 \rightarrow K_0 \rightarrow P_0 \rightarrow \mathcal{F}_x$. Hence $Ext_{\mathcal{O}_{X,x}}^{n+1}(\mathcal{F}_x, M) = 0$, while $Ext_{\mathcal{O}_{X,x}}^n(\mathcal{F}_x, M) \neq 0$ comes from the minimality of the resolution.

Lemma 2.1.6. [16, Okonek, Schneider & Spindler, Lemma 1.1.1] $dh(\mathcal{F}_x) \leq q$ if and only if for all $i > q$ we have $(Ext^i(\mathcal{F}, \mathcal{O}_X))_x = 0$ for all $x \in X$

Lemma 2.1.7. [13, Jardim] Let $\mathcal{E}$ be a linear sheaf, then

- (i) $\mathcal{E}$ is locally free if and only if its degeneration locus is empty
- (ii) $\mathcal{E}$ is reflexive if and only if its degeneration locus is a subvariety of codimension at least 3
- (iii) $\mathcal{E}$ is torsion free if and only if its degeneration locus is a subvariety of codimension at least 2

Proof. Let Σ be its degeneration locus. We have already proved (i).

To prove (ii) and (iii) we use the concept of the m -th singularity set of a coherent sheaf $\mathcal{F}$: $S_m(\mathcal{F}) = \{x \in \mathbb{P}^n | codh \mathcal{F}_x \leq m\} = \{x \in \mathbb{P}^n | dh(\mathcal{F}_x) \geq n - m\}$ where $dh(\mathcal{F}_x)$ denotes the homological dimension of $\mathcal{F}_x$ as an $\mathcal{O}_x$ -module. [16, Okonek, Schneider & Spindler, Lemma 1.1.3]. Lemma (2.1.6) tells us that

$$dh(\mathcal{F}_x) = d \iff \begin{cases} Ext^d(\mathcal{F}_x, \mathcal{O}_{X,x}) \neq 0 \\ Ext^p(\mathcal{F}_x, \mathcal{O}_{X,x}) = 0 \ \forall p > d \end{cases} \quad (2.25)$$

Back to our case, $x \in \Sigma$ implies $dh(\mathcal{E}_x) = 1$, and $x \notin \Sigma$ implies $dh(\mathcal{E}_x) = 0$. Then $S_0(\mathcal{E}) = \dots = S_{n-2}(\mathcal{E}) = \emptyset$ and $S_{n-1} = \Sigma$.

Recall that a coherent sheaf $\mathcal{F}$ is a k -th syzygy sheaf if there is an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X^{p_1} \rightarrow \mathcal{O}_X^{p_2} \rightarrow \dots \rightarrow \mathcal{O}_X^{p_k} \quad (2.26)$$

[23, Siu, Proposition 1.20] states that:

- (i) if $codim \Sigma \geq 2$ then $dim S_m(\mathcal{E}) \leq m - 1 \ \forall m < n$, hence $\mathcal{E}$ is a locally 1^{st} -syzygy sheaf
 - (ii) if $codim \Sigma \geq 3$ then $dim S_m(\mathcal{E}) \leq m - 2 \ \forall m < n$, hence $\mathcal{E}$ is a locally 2^{nd} -syzygy sheaf.
- Finally we use that $\mathcal{E}$ is torsion free if and only if it is a locally 1^{st} -syzygy sheaf, and $\mathcal{E}$ is reflexive if and only if it is a locally 2^{nd} -syzygy sheaf. [16, Okonek, Schneider & Spindler, p.148-149] $\square$

2.2 Horrocks monads

In this section X is a projective variety over k with a given very ample invertible sheaf $\mathcal{O}_X(1)$. From this section and until the end of the dissertation a monad will **always**

have cohomology $\mathcal{E}$ with $\text{Sing}(\mathcal{E}) = \emptyset$ (thus it is always locally free).

It is interesting to notice that if $\mathcal{E}$ is locally free then α^* is surjective, so a monad

$$\mathcal{V}_\bullet : \mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1 \xrightarrow{\beta} \mathcal{V}_2 \quad (2.27)$$

gives another monad by dualizing $\mathcal{V}_\bullet$:

$$\mathcal{V}_\bullet^* : \mathcal{V}_2^* \xrightarrow{\beta^*} \mathcal{V}_1^* \xrightarrow{\alpha^*} \mathcal{V}_0^* \quad (2.28)$$

Denote $H_*^1(\mathcal{F}) = \bigoplus_{k \in \mathbb{Z}} H^1(\mathcal{F}(k))$. We will now define a Horrocks monad as follows

Definition 2.2.1. [14, Jardim & Martins] A monad is said to be Horrocks if

- (i) $\mathcal{V}_0 = \bigoplus_{i=1}^r \omega_X(k_i)$ for $k_i \in \mathbb{Z}$
- (ii) $\mathcal{V}_2 = \bigoplus_{j=1}^s \mathcal{O}_X(l_j)$ for $l_j \in \mathbb{Z}$
- (iii) $H_*^1(\mathcal{V}_1) = H_*^{n-1}(\mathcal{V}_1) = 0$

We will now define an ACM variety

Definition 2.2.2. [?, Eisenbud] A projective variety $X \hookrightarrow \mathbb{P}^r$ of pure dimension n is arithmetically Cohen-Macaulay (ACM) if its homogeneous coordinate ring $S(X) = H_*^0(\mathcal{O}_X)$ is a Cohen-Macaulay ring

This is equivalent to saying that $H_*^1(\mathbb{P}^r, \mathcal{I}_X) = 0$ and $H_*^p(\mathcal{O}_X) = 0$ for $1 \leq p \leq n-1$ where $\mathcal{I}_X$ is the saturated ideal sheaf of X .

Theorem 2.2.3. [14, Jardim & Martins, Theorem 2.3] Let X be an ACM variety of dimension $n \geq 3$ and let $\mathcal{E}$ be a locally free sheaf on X . There exists a bijective correspondence between collections $h_1, \dots, h_r, g_1, \dots, g_s$ with $h_i \in H^1(\mathcal{E}^* \otimes \omega_X(k_i))$ and $g_j \in H^1(\mathcal{E}(-l_j))$, for integers k_i and l_j , and isomorphism classes of monads of the form

$$\mathcal{V}_\bullet : \bigoplus_{i=1}^r \omega_X(k_i) \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \bigoplus_{j=1}^s \mathcal{O}_X(l_j) \quad (2.29)$$

whose cohomology is isomorphic to $\mathcal{E}$. This correspondence is such that $\mathcal{V}_\bullet$ is Horrocks if and only if the g_j generate $H_*^1(\mathcal{E})$ and the h_i generate $H_*^1(\mathcal{E}^* \otimes \omega_X)$ as $S(X)$ -modules.

Proof. Let $\sum_i h_i \in \bigoplus_i H^1(\mathcal{E}^* \otimes \omega_X(k_i)) \cong H^1(\mathcal{E}^* \otimes (\bigoplus_i \omega_X(k_i))) \cong \text{Ext}^1(\mathcal{O}_X, \mathcal{E}^* \otimes \bigoplus_i \omega_X(k_i)) \cong \text{Ext}^1(\mathcal{E}, \bigoplus_i \omega_X(h_i))$. This element defines an extension

$$0 \rightarrow \bigoplus_{i=1}^r \omega_X(k_i) \xrightarrow{\alpha} \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0 \quad (2.30)$$

In particular we have, for every p , the following exact sequence in cohomology

$$\bigoplus_{i=1}^r H^p(\omega_X(k_i)) \rightarrow H^p(\mathcal{K}) \rightarrow H^p(\mathcal{E}) \rightarrow \bigoplus_{i=1}^r H^{p+1}(\omega_X(k_i)) \quad (2.31)$$

But $0 \stackrel{ACM}{=} H^{n-p}(\mathcal{O}_X(-k_i)) \cong H^p(\omega_X(k_i))$ hence $H^p(\mathcal{K}(m)) \cong H^p(\mathcal{E}(m))$ for $1 \leq p \leq n-2$ and any m .

Let $g'_j \in H^1(\mathcal{K}(-l_j))$ be the image through the isomorphism above for every g_j . We have the isomorphisms $\bigoplus_j H^1(\mathcal{K}(-l_j)) \cong H^1(\bigoplus_j \mathcal{K}(-l_j)) \cong Ext^1(\mathcal{O}_X, \bigoplus_j \mathcal{K}(-l_j)) \cong Ext^1(\bigoplus_j \mathcal{O}_X(l_j), \mathcal{K})$, so $\sum_j g'_j \in \bigoplus_j H^1(\mathcal{K}(-l_j))$ defines an extension :

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \xrightarrow{\beta} \bigoplus_{j=1}^s \mathcal{O}_X(l_j) \rightarrow 0 \quad (2.32)$$

Putting (2.30) and (2.32) together gives the monad :

$$\bigoplus_{i=1}^r \omega_X(k_i) \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \bigoplus_{j=1}^s \mathcal{O}_X(l_j) \quad (2.33)$$

Since $\mathcal{K} = \ker \beta$ and $\text{coker } \alpha = \mathcal{E}$ we have $\mathcal{E}$ as the cohomology of (2.33). Now, conversely if we have an isomorphism class as in (2.29), then there are two exact sequences

$$0 \rightarrow \bigoplus_{i=1}^r \omega_X(k_i) \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0 \quad (2.34)$$

and

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \bigoplus_{j=1}^s \mathcal{O}_X(l_j) \rightarrow 0 \quad (2.35)$$

with $\mathcal{K} = \ker \beta$. The first one corresponds to an element $h \in Ext^1(\mathcal{E}, \bigoplus_i \omega_X(k_i)) \cong Ext^1(\mathcal{O}_X, \mathcal{E}^* \otimes \bigoplus_i \omega_X(k_i)) \cong H^1(\mathcal{E}^* \otimes \bigoplus_i \omega_X(k_i))$, while the second sequence corresponds to an element $g \in Ext^1(\bigoplus_{j=1}^s \mathcal{O}_X(l_j), \mathcal{K}) \cong \bigoplus_j H^1(\mathcal{K}(-l_j)) \cong \bigoplus_j H^1(\mathcal{E}(-l_j))$. Set $h = \sum_i h_i$ and $g = \sum_j g_j$, we still need to prove the last statement.

First notice that conditions (i) and (ii) of (2.2.1) are satisfied automatically. Since X is ACM, $H_*^1(\bigoplus_j \mathcal{O}_X(l_j)) = 0$. But then using the exact sequence (2.35) we have the following exact sequence :

$$H^0(\bigoplus_j \mathcal{O}_X(l_j + k)) \xrightarrow{\delta_k} H^1(\mathcal{K}(k)) \twoheadrightarrow H^1(\mathcal{F}(k)) \rightarrow 0 \quad (2.36)$$

Hence δ_k is surjective for every k if and only if $H_*^1(\mathcal{F}) = 0$. Those morphisms are defined by construction taking $\sum f_j$ to $\sum f_j g'_j$, each $f_j \in \mathcal{O}_X(l_j + k)$ has degree $l_j + k$, this is because given an element $\xi \in Ext^1(\mathcal{O}_X, \mathcal{K})$ corresponding to an extension

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X \rightarrow 0 \quad (2.37)$$

And an element $g \in H^1(\mathcal{K}) \cong \text{Ext}^1(\mathcal{O}_X, \mathcal{K})$. Then the morphism $H^0(\mathcal{O}_X) \rightarrow H^1(\mathcal{K})$ has to send 1 to g .

Surjectivity of each δ_k is equivalent to saying that g'_j generates $H^1_*(\mathcal{K})$ as $H^0_*(\mathcal{O}_X)$ -module. Finally, using Serre duality $H^{n-1}(\mathcal{F}(k)) \cong \text{Ext}^1(\mathcal{F}(k), \omega_X)^* \cong \text{Ext}^1(\mathcal{O}_X, \mathcal{F}^* \otimes \omega_X(-k))^* \cong H^1(\mathcal{F}^* \otimes \omega_X(-k))^*$ hence $H^{n-1}_*(\mathcal{F}) = 0$ if and only if $H^1_*(\mathcal{F}^* \otimes \omega_X) = 0$. Since $\mathcal{E}$ is locally free the dual of $\mathcal{V}_\bullet$ is a monad, tensoring it by ω_X gives

$$\mathcal{V}_\bullet^* : \bigoplus_{j=1}^s \omega_X(-l_j) \rightarrow \mathcal{F}^* \otimes \omega_X \rightarrow \bigoplus_{i=1}^r \mathcal{O}_X(-k_i) \quad (2.38)$$

Which gives

$$H^0\left(\bigoplus_{i=1}^r \mathcal{O}_X(-k_i + k)\right) \xrightarrow{\delta_k} H^1\left(\bigoplus_{j=1}^s \omega_X(-l_j + k)\right) \rightarrow H^1(\mathcal{F}^* \otimes \omega_X(k)) \rightarrow 0 \quad (2.39)$$

The same argument shows that $H^{n-1}_*(\mathcal{F}) = 0$ if and only if h_i generate $H^1_*(\mathcal{E}^* \otimes \omega_X)$ as $H^0_*(\mathcal{O}_X)$ -module. $\square$

The theorem leads to an interesting result not only concerning the case $X = \mathbb{P}^n$, but this time considering X as an ACM variety of dimension ≥ 3 .

Corollary 2.2.4. [14, Jardim & Martins] *Every locally free sheaf $\mathcal{E}$ on an ACM variety of dimension $n \geq 3$ is the cohomology of a monad*

$$\mathcal{V}_\bullet : \bigoplus_{i=1}^r \omega_X(k_i) \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \bigoplus_{j=1}^s \mathcal{O}_X(l_j) \quad (2.40)$$

such that

$$(i) \ H^1_*(\mathcal{F}) = H^{n-1}_*(\mathcal{F}) = 0$$

$$(ii) \text{ for } n \geq 4, \ H^p_*(\mathcal{F}(k)) \cong H^p_*(\mathcal{E}(k)) \text{ for } 2 \leq p \leq n-2$$

Proof. $\mathcal{E}$ is locally free so $H^1_*(\mathcal{E})$ and $H^1_*(\mathcal{E}^* \otimes \omega_X)$ are finitely generated as $H^0_*(\mathcal{O}_X)$ -modules. Then we can pick any set of generators and apply last theorem to generate the desired monad. $\square$

Since every Horrocks monad is a complex, and every morphism of Horrocks monads is a morphism of complexes it is immediate to see that Horrocks monads on X form a full subcategory $\mathcal{H}(X)$ of $Ch_\bullet(X)$ the abelian category of complexes of sheaves on X . Since $\mathcal{H}(X)$ is a full subcategory of an abelian category, we also have that $\mathcal{H}(X)$ is additive.

Remark 2.2.5. Let $\mathcal{V}_\bullet$ and $\mathcal{W}_\bullet$ be two monads and $\phi_\bullet : \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ be a morphism of monads. Then $\phi_\bullet$ defines a morphism of sheaves between the cohomologies of $\mathcal{V}_\bullet$ and $\mathcal{W}_\bullet$. Moreover, this defines a functor $\mathbf{C} : \mathcal{H}(X) \rightarrow \mathcal{V}(X)$ from the category of Horrocks monads to the category of locally free sheaves on X .

Proof. Let $\mathcal{E}$ and $\mathcal{F}$ be the cohomology of $V_\bullet$ and $W_\bullet$ respectively. Considering both monads as complexes over $\mathcal{V}(X)$ we obtain a morphism which we call $\mathbf{C}(\phi) : \mathcal{E} \rightarrow \mathcal{F}$. Any composition of morphisms of complexes $\phi_\bullet \circ \psi_\bullet$ goes to $\mathbf{C}(\phi_\bullet \circ \psi_\bullet) = \mathbf{C}(\phi) \circ \mathbf{C}(\psi)$ since everything is done in $Ch_\bullet(X)$. $\square$

Theorem (2.2.3) states that $\mathbf{C}$ is surjective on objects, but more can be asked of it.

Theorem 2.2.6. [14, Jardim & Martins, Theorem 2.5] *Let X be an ACM projective variety and*

$$\mathcal{M}_\bullet : \mathcal{M}_0 \xrightarrow{\alpha} \mathcal{M}_1 \xrightarrow{\beta} \mathcal{M}_2 \quad (2.41)$$

$$\mathcal{N}_\bullet : \mathcal{N}_0 \xrightarrow{\alpha'} \mathcal{N}_1 \xrightarrow{\beta'} \mathcal{N}_2 \quad (2.42)$$

be Horrocks monads. Consider the morphism

$$\begin{aligned} \rho : \operatorname{Hom}(\mathcal{M}_1, \mathcal{N}_0) \oplus \operatorname{Hom}(\mathcal{M}_2, \mathcal{N}_1) &\rightarrow \operatorname{Hom}(\mathcal{M}_\bullet, \mathcal{N}_\bullet) \\ (\psi_1, \psi_2) &\mapsto (\psi_1\alpha, \alpha'\psi_1 + \psi_2\beta, \beta'\psi_2) \end{aligned}$$

Then the sequence

$$0 \rightarrow \operatorname{Im}(\rho) \xrightarrow{i} \operatorname{Hom}(\mathcal{M}_\bullet, \mathcal{N}_\bullet) \xrightarrow{\pi} \operatorname{Hom}(\mathbf{C}(\mathcal{M}_\bullet), \mathbf{C}(\mathcal{N}_\bullet)) \rightarrow 0 \quad (2.43)$$

is exact, where i is the inclusion and π the natural morphism.

Proof. First we need to prove that ρ is well defined. Denote $\gamma_0 = \psi_1\alpha$, $\gamma_1 = \alpha'\psi_1 + \psi_2\beta$ and $\gamma_2 = \beta'\psi_2$.

$$\begin{array}{ccccc} \mathcal{M}_0 & \xrightarrow{\alpha} & \mathcal{M}_1 & \xrightarrow{\beta} & \mathcal{M}_2 \\ \downarrow \gamma_0 & & \downarrow \gamma_1 & & \downarrow \gamma_2 \\ \mathcal{N}_0 & \xrightarrow{\alpha'} & \mathcal{N}_1 & \xrightarrow{\beta'} & \mathcal{N}_2 \end{array}$$

But then $\gamma_1\alpha = \alpha'\psi_1\alpha + \psi_2\beta\alpha \stackrel{\beta\alpha=0}{=} \alpha'\psi_1\alpha = \alpha'\gamma_0$ and $\beta'\gamma_1 = \beta'\alpha'\psi_1 + \beta'\psi_2\beta \stackrel{\beta'\alpha'=0}{=} \beta'\psi_2\beta = \gamma_2\beta$, hence ρ is well defined.

The map i is injective by definition of image. To show $\pi i = 0$ we can apply diagram chasing because the category of sheaves on an abelian category is an abelian category: Let $x + \operatorname{Im}\alpha \in \mathcal{E} = \ker\beta / \operatorname{Im}\alpha$. If $\gamma = (\gamma_0, \gamma_1, \gamma_2) = \rho(\psi_1, \psi_2)$ then $H(\gamma)(x + \operatorname{Im}\alpha) = \gamma_1(x) + \operatorname{Im}\alpha' = \alpha'\psi_1(x) + \psi_2\beta(x) + \operatorname{Im}\alpha' = 0$ because $\alpha'(\psi_1(x)) \in \operatorname{Im}\alpha'$ and $x \in \ker\beta$ hence $\beta(x) = 0$.

To establish the surjectivity of π , consider the exact sequences :

$$\begin{aligned} (i) \quad & 0 \rightarrow \ker \beta \xrightarrow{j} \mathcal{M}_1 \xrightarrow{\beta} \mathcal{M}_2 \rightarrow 0 \\ (ii) \quad & 0 \rightarrow \ker \beta' \xrightarrow{j'} \mathcal{N}_1 \xrightarrow{\beta'} \mathcal{N}_2 \rightarrow 0 \\ (iii) \quad & 0 \rightarrow \mathcal{M}_0 \xrightarrow{\alpha} \ker \beta \xrightarrow{p} \mathcal{E} \rightarrow 0 \\ (iv) \quad & 0 \rightarrow \mathcal{N}_0 \xrightarrow{\alpha'} \ker \beta' \xrightarrow{p'} \mathcal{F} \rightarrow 0 \end{aligned}$$

Let $\phi : \mathcal{E} \rightarrow \mathcal{F}$, define $\tilde{\phi} = \phi \circ p : \ker \beta \rightarrow \mathcal{F}$. Applying the functor $\text{Hom}(\ker \beta, -)$ to the sequence (iv) results

$$\text{Hom}(\ker \beta, \ker \beta') \rightarrow \text{Hom}(\ker \beta, \mathcal{F}) \rightarrow \text{Ext}^1(\ker \beta, \mathcal{N}_0) \quad (2.44)$$

Now, $\text{Ext}^1(\ker \beta, \mathcal{N}_0) = 0$, indeed applying $\text{Hom}(-, \mathcal{N}_0)$ to the sequence (i) we get

$$\text{Ext}^1(\mathcal{M}_1, \mathcal{N}_0) \rightarrow \text{Ext}^1(\ker \beta, \mathcal{N}_0) \rightarrow \text{Ext}^2(\mathcal{M}_2, \mathcal{N}_0) \quad (2.45)$$

But $\text{Ext}^2(\mathcal{M}_2, \mathcal{N}_0) \cong \text{Ext}^2(\mathcal{O}_X, \mathcal{M}_2^* \otimes \mathcal{N}_0) \cong H^2(\mathcal{M}_2^* \otimes \mathcal{N}_0) = 0$ because $\mathcal{M}_2$ is a sum of line bundles, $\mathcal{N}_0$ is a sum of twists of the canonical bundle, so we would have a sum of terms $H^2(\omega_X(k_j))$ which are zero since X is ACM. The same goes for $\text{Ext}^1(\mathcal{M}_1, \mathcal{N}_0) \cong H^1(\mathcal{M}_1^* \otimes \mathcal{N}_0) = 0$ by definition of a Horrocks monad. But then $\text{Ext}^1(\ker \beta, \mathcal{N}_0) = 0$ and hence $\text{Hom}(\ker \beta, \ker \beta') \rightarrow \text{Hom}(\ker \beta, \mathcal{F})$ is surjective so there is a map $\Phi \in \text{Hom}(\ker \beta, \ker \beta')$ with $\tilde{\phi} = p' \circ \Phi$, in other words the right square in the diagram below is commutative :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}_0 & \xrightarrow{\alpha} & \ker \beta & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & & & \downarrow \Phi & \searrow \tilde{\phi} & \downarrow \phi \\ 0 & \longrightarrow & \mathcal{N}_0 & \xrightarrow{\alpha'} & \ker \beta' & \longrightarrow & \mathcal{F} \longrightarrow 0 \end{array}$$

Applying $\text{Hom}(\mathcal{M}_0, -)$ to the sequence (iv) results in

$$0 \rightarrow \text{Hom}(\mathcal{M}_0, \mathcal{N}_0) \rightarrow \text{Hom}(\mathcal{M}_0, \ker \beta') \rightarrow \text{Hom}(\mathcal{M}_0, \mathcal{F}) \rightarrow \text{Ext}^1(\mathcal{M}_0, \mathcal{N}_0) = 0 \quad (2.46)$$

exact since since X is ACM. But $p' \circ \Phi \circ \alpha = \tilde{\phi} \circ \alpha = \phi \circ p \circ \alpha = 0$ because $p \circ \alpha = 0$ so $\Phi \circ \alpha$ is in the kernel of the second morphism. Therefore there exists a map $g \in \text{Hom}(\mathcal{M}_0, \mathcal{N}_0)$ such that $\alpha' g = \Phi \alpha$ hence making the following diagram commutative.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}_0 & \xrightarrow{\alpha} & \ker \beta & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow g & & \downarrow \Phi & \searrow \tilde{\phi} & \downarrow \phi \\ 0 & \longrightarrow & \mathcal{N}_0 & \xrightarrow{\alpha'} & \ker \beta' & \longrightarrow & \mathcal{F} \longrightarrow 0 \end{array}$$

The composition of Φ with the first map in (ii) gives a map $\tilde{\Phi} : \ker \beta \rightarrow \mathcal{N}_1$. Applying the functor $\text{Hom}(-, \mathcal{N}_1)$ to the sequence (i) yields

$$\text{Hom}(\mathcal{M}_1, \mathcal{N}_1) \rightarrow \text{Hom}(\ker \beta, \mathcal{N}_1) \rightarrow \text{Ext}^1(\mathcal{M}_2, \mathcal{N}_1) = 0 \quad (2.47)$$

So there is a $h : \mathcal{M}_1 \rightarrow \mathcal{N}_1$ such that the left square in the diagram below commutes :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker \beta & \longrightarrow & \mathcal{M}_1 & \xrightarrow{\beta} & \mathcal{M}_2 \longrightarrow 0 \\
& & \downarrow \Phi & \searrow \tilde{\Phi} & \downarrow h & & \\
0 & \longrightarrow & \ker \beta' & \longrightarrow & \mathcal{N}'_1 & \xrightarrow{\beta'} & \mathcal{N}_2 \longrightarrow 0
\end{array}$$

Applying the functor $\text{Hom}(-, \mathcal{N}_2)$ to (i)

$$0 \rightarrow \text{Hom}(\mathcal{M}_2, \mathcal{N}_2) \rightarrow \text{Hom}(\mathcal{M}_1, \mathcal{N}_2) \rightarrow \text{Hom}(\ker \beta, \mathcal{N}_2) \rightarrow \text{Ext}^1(\mathcal{M}_2, \mathcal{N}_2) = 0 \quad (2.48)$$

Using the same argument as before we have that $\beta' \circ h$ is in the kernel of the second map, so there is a map $l : \mathcal{M}_2 \rightarrow \mathcal{N}_2$ with $\beta' \circ h = l \circ \beta$. Thus making the diagram commute

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker \beta & \longrightarrow & \mathcal{M}_1 & \xrightarrow{\beta} & \mathcal{M}_2 \longrightarrow 0 \\
& & \downarrow \Phi & \searrow \tilde{\Phi} & \downarrow h & & \downarrow l \\
0 & \longrightarrow & \ker \beta' & \longrightarrow & \mathcal{N}'_1 & \xrightarrow{\beta'} & \mathcal{N}_2 \longrightarrow 0
\end{array}$$

By construction (g, h, l) is a morphism of complexes from $\mathcal{M}_\bullet$ to $\mathcal{N}_\bullet$.

Notice that $\pi(g, h, l) = \phi : \text{if } x \in \mathcal{E} \text{ then } \pi(g, h, l)(x + \text{Im} \alpha) = h(x) + \text{Im} \alpha' = p' \circ h(x) = p' \circ h \circ j(x) = p' \circ \tilde{\Phi}(x) = p' \circ j' \circ \Phi(x) = p' \circ \Phi(x) = \phi \circ p(x) = \phi(x + \text{Im} \alpha)$, in which we are considering $j(x) = x$ and $j'(y) = y$ since j, j' are inclusions.

Therefore π is surjective. To prove exactness suppose $(g, h, l) \in \ker \pi$, then we get the following diagram :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}_0 & \xrightarrow{\alpha} & \ker \beta & \xrightarrow{p} & \mathcal{E} \longrightarrow 0 \\
& & \downarrow g & & \downarrow h' & & \downarrow 0 \\
0 & \longrightarrow & \mathcal{N}_0 & \xrightarrow{\alpha'} & \ker \beta' & \xrightarrow{p'} & \mathcal{F} \longrightarrow 0
\end{array}$$

where we consider the restriction of h as $h' : \ker \beta \rightarrow \ker \beta'$. Applying the functor $\text{Hom}(\ker \beta, -)$ on the sequence (iv) yields

$$0 \rightarrow \text{Hom}(\ker \beta, \mathcal{N}_0) \rightarrow \text{Hom}(\ker \beta, \ker \beta') \rightarrow \text{Hom}(\ker \beta, \mathcal{F}) \rightarrow \text{Ext}^1(\ker \beta, \mathcal{N}_0) = 0 \quad (2.49)$$

But the diagram above states that $p' \circ h' = 0$ so $h' \in \ker(\text{Hom}(\ker \beta, \mathcal{N}_0) \rightarrow \text{Hom}(\ker \beta, \ker \beta'))$ hence there exists a map $\psi_1 : \ker \beta \rightarrow \mathcal{N}_0$ with $\alpha' \psi_1 = h'$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{M}_0 & \xrightarrow{\alpha} & \ker \beta & \xrightarrow{p} & \mathcal{E} \longrightarrow 0 \\
& & \downarrow g & \nwarrow \psi_1 & \downarrow h' & & \downarrow 0 \\
0 & \longrightarrow & \mathcal{N}_0 & \xrightarrow{\alpha'} & \ker \beta' & \xrightarrow{p'} & \mathcal{F} \longrightarrow 0
\end{array}$$

Applying the functor $\text{Hom}(-, \mathcal{N}_0)$ to (i)

$$0 \rightarrow \text{Hom}(\mathcal{M}_2, \mathcal{N}_0) \rightarrow \text{Hom}(\mathcal{M}_1, \mathcal{N}_0) \rightarrow \text{Hom}(\ker \beta, \mathcal{N}_0) \rightarrow \text{Ext}^1(\mathcal{M}_2, \mathcal{N}_0) = 0 \quad (2.50)$$

Then there is a map $\Psi_1 : \mathcal{M}_1 \rightarrow \mathcal{N}_0$ extending ψ_1 .

Since $Im\alpha \subset ker\beta$, $\Psi \circ \alpha = \psi \circ \alpha$, by the last diagram $h' \circ \alpha = \alpha' \circ g$ hence $\alpha' \circ \psi_1 \circ \alpha = h' \circ \alpha = \alpha' \circ g$ which by injectivity implies $\psi_1 \circ \alpha = g$ hence $\Psi_1 \circ \alpha = g$ (so g is the image of ρ on the first coordinate).

Consider the map $H = h - \alpha' \circ \Psi_1 : \mathcal{M}_1 \rightarrow \mathcal{N}_1$, the restriction of H on $ker\beta$ is $h' - \alpha' \circ \psi_1 = 0$ so H lies in the kernel of the second map in the exact sequence

$$Hom(\mathcal{M}_2, \mathcal{N}_1) \rightarrow Hom(\mathcal{M}_1, \mathcal{N}_1) \rightarrow Hom(ker\beta, \mathcal{N}_1) \quad (2.51)$$

obtained from (i). But then there is $\Psi_2 : \mathcal{M}_2 \rightarrow \mathcal{N}_1$ such that $H = \Psi_2 \circ \beta$ hence $h = \Psi_2\beta + \alpha'\Psi_1$ (so h is the image of ρ on the second coordinate).

Finally, $l\beta = \beta'h = \beta'\Psi_2\beta + \beta'\alpha'\Psi_1 = \beta'\Psi_2\beta$ where the first equality comes from the fact that (g, h, l) is a morphism of complexes, and the third is due to $\beta'\alpha' = 0$. But this implies $l = \beta'\Psi_2$ because β is surjective (so l is the image of ρ on the third coordinate).

Those three equalities prove $(g, h, l) = (\Psi_1\alpha, \alpha'\Psi_1 + \Psi_2\beta, \beta'\Psi_2) \in Im \rho$ so the sequence is exact. $\square$

Theorem 2.2.7. [14, Jardim & Martins, Theorem 2.6] *Let X be an ACM variety of dimension $n \geq 3$. The functor $\mathbf{C}$ is additive, essentially surjective, full and exact.*

Proof. It is essentially surjective by Theorem (2.2.3), full by Theorem (2.2.6) and additive because $\pi(f + g) = \pi(f) + \pi(g)$ (the same π in the last theorem).

Consider the following exact sequence of monads:

$$\begin{array}{ccccc}
 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{L}_0 & \xrightarrow{\alpha_L} & \mathcal{L}_1 & \xrightarrow{\beta_L} & \mathcal{L}_2 \\
 \downarrow g_1 & & \downarrow h_1 & & \downarrow l_1 \\
 \mathcal{M}_0 & \xrightarrow{\alpha_M} & \mathcal{M}_1 & \xrightarrow{\beta_M} & \mathcal{M}_2 \\
 \downarrow g_2 & & \downarrow h_2 & & \downarrow l_2 \\
 \mathcal{N}_0 & \xrightarrow{\alpha_N} & \mathcal{N}_1 & \xrightarrow{\beta_N} & \mathcal{N}_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

with $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ the cohomology of the first, second and third rows. Let $\bar{h}_1 : \mathcal{E} \rightarrow \mathcal{F}$ and $\bar{h}_2 : \mathcal{F} \rightarrow \mathcal{G}$ be the induced maps. We have $\bar{h}_2\bar{h}_1 = 0$ because $\mathbf{C}(\beta\alpha) = \mathbf{C}(0) = 0$.

Using the same argument from the proof of Theorem (2.2.6), we can verify the injectivity and surjectivity of $\bar{h}_1, \bar{h}_2$ respectively using diagram chasing : Let $\bar{x} = x + im\alpha_L \in \mathcal{E}$ and suppose $\bar{h}_1(\bar{x}) = h_1(x) + im\alpha_M = 0$. Then $h_1(x) \in im\alpha_M$ implies the existence of an unique $y \in \mathcal{M}_0$ with $\alpha_M(y) = x$ because α_M is injective. Using injectivity of g_1 gives another unique $z \in \mathcal{L}_0$ with $g_1(z) = y$. Since the diagram is commutative and α_L is injective we

have $\alpha_L(z) = x$ hence $\bar{x} = 0$.

Let $\bar{y} = y + im\alpha_N \in \mathcal{F}$. Being the map h_2 surjective, there is an element $x \in \mathcal{M}_1$ with $h_2(x) = y \in \ker\beta_N$. Since every square is commutative, $l_2 \circ \beta_M(x) = 0$ so $\beta_M(x) \in \ker l_2$, but then $\beta_M(x) \in im l_1$ and thus there is a unique $z \in \mathcal{L}_2$ with $l_1(z) = \beta_M(x)$. Since β_L is surjective there is a (non unique) $z' \in \mathcal{L}_1$ such that $\beta_L(z') = z$ which implies $\beta_M \circ h_1(z') = \beta_M(x)$.

Now the fact that α_L is a monomorphism implies there is a unique z'' with $\alpha_L(z'') = z'$ hence $\alpha_M \circ g_1(z'') = h_1(z')$. But then $h_1(z') \in \ker\beta_M$ implies that $0 = l_1 \circ \beta_L(z') = l_1(z)$ which implies $0 = l_1(z) = \beta_M(x)$ hence $x \in \ker\beta_M$ so the element $\bar{x} = x + im\alpha_M$ is well defined and $\bar{h}_2(\bar{x}) = \bar{y}$.

Finally if $\bar{x} \in \ker\bar{h}_2$ we have $h_2(x) \in im\alpha_N$ so there is an element $y \in \mathcal{N}_0$ with $\alpha_N(y) = h_2(x)$. But since g_2 is an epimorphism so there exists $z \in \mathcal{M}_0$ such that $g_2(z) = y$ and $h_2 \circ \alpha_M(z) = \alpha_N \circ g_2(z) = \alpha_N(y) = h_2(x)$. So $h_2(\alpha_M(z) - x) = 0$.

It is known [15, MacLane, Theorem 3 (vi), p.201] that implies $\alpha_M(z) - x \in im h_1$, which implies the existence of an element $z' \in \mathcal{L}_1$ with $h_1(z') = \alpha_M(z) - x$. Then $l_1 \circ \beta_L(z') = \beta_M(\alpha_M(z) - x) = 0$ because $\beta_M \circ \alpha_M = 0$ and $x \in \ker\beta_M$. So $\beta_L(z') \in \ker l_1$ implies $\beta_L(z') = 0$ hence $z' \in \ker\beta_L$ and finally $\bar{z}' = z' + im\alpha_L$ is well defined with $h_1(\bar{z}') = \bar{x}$ as desired. $\square$

Corollary 2.2.8. *If X is an ACM variety of dimension $n \geq 3$ then the category $\mathcal{V}(X)$ of locally free sheaves on X is equivalent to a quotient of the category $\mathcal{H}(X)$ of Horrocks monads on X .*

With Theorem (2.2.3) in hands we are ready to begin the study of the classification of vector bundles.

3 Toward the classification

In this chapter we will introduce the spectrum of a reflexive sheaf and the Hartshorne-Serre correspondence. Then we will give a few families of curves which we can use the correspondence to obtain families of vector bundles of rank 2 with $c_1 = -1$, each vector bundle has an spectrum and a monad associated.

3.1 Spectrum and its properties

In [3, Barth & Elenewajg] Barth and Elenewajg introduced the concept of the spectrum of a stable vector bundle $\mathcal{E}$ of rank 2 on $\mathbb{P}^3$ with $c_1 = 0$, we recall their construction. Let L be a general line on $\mathbb{P}^3$, $p : X \rightarrow \mathbb{P}^3$ the blowing-up of $\mathbb{P}^3$ along L and $q : X \rightarrow \mathbb{P}^1$ the morphism determined by the pencil of planes through L . The sheaf $\mathcal{H} := R^1 q_* p^* \mathcal{E}(-1)$ is locally free of rank c_2 on $\mathbb{P}^1$ [3, Barth & Elenewajg, Proposition 2.2.1 p.9]. Grothendieck's theorem states that $\mathcal{H}$ is a finite sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(k_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(k_{c_2})$ for suitable integers k_i . The sequence $\xi = \{k_i\}_{i=1, \dots, c_2}$ is called the **spectrum of $\mathcal{E}$** .

In [10, Hartshorne, Theorem 7.1 p.151] Hartshorne generalized the concept of spectrum for any reflexive sheaf $\mathcal{F}$ of rank 2 with $c_1 = 0$ or -1 and $H^0(\mathcal{F}(-1)) = 0$ on $\mathbb{P}^3$. In this chapter we won't be interested in the case $c_1 = 0$ since it has already been studied in [11, Hartshorne & Rao], but almost every result in this chapter is based on [11, Hartshorne & Rao]. We start with some fundamental facts and definitions from Hartshorne's paper, starting from an equivalent definition for spectrum.

Theorem 3.1.1. [10, Hartshorne, Theorem 7.1] *Let $\mathcal{F}$ be a rank 2 reflexive on X with $c_1 = 0$ or -1 . Assume $H^0(\mathcal{F}(-1)) = 0$. Then there is an unique sequence of integers (up to ordering) $\chi = \{k_i\}$, $i = 1, \dots, c_2$ satisfying the following properties:*

- (1) $h^1(\mathbb{P}^3, \mathcal{F}(l)) = h^0(\mathbb{P}^1, \mathcal{H}(l+1))$ for $l \leq -1$
- (2) $h^2(\mathbb{P}^3, \mathcal{F}(l)) = h^1(\mathbb{P}^1, \mathcal{H}(l+1))$ for $l \geq -3$ if $c_1 = 0$ or $l \geq -2$ if $c_1 = -1$

Where $\mathcal{H} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(k_i)$. This set of integers is called the spectrum of $\mathcal{F}$.

In his proof Hartshorne gives a method on how to compute χ . Define $n_l = h^1(\mathcal{F}(-l)) - h^1(\mathcal{F}(-l-1))$ for $l \geq 1$, then $n_l = \#\{k_j \in \chi | k_j \geq l-1\}$, which results $n_l - n_{l-1} = \#\{k_j = l-1\}$. In other words

$$\#\{k_j = l\} = h^1(\mathcal{F}(-l-1)) - 2h^1(\mathcal{F}(-l-2)) + h^1(\mathcal{F}(-l-3)), \quad l \geq 0 \quad (3.1)$$

We present some results which will help us in the computation of the spectrum

Proposition 3.1.2. [10, Hartshorne, Proposition 7.2] If $\mathcal{F}$ is locally free, then

$$\{-k_i\} = \{k_i\} \text{ if } c_1 = 0 \quad (3.2)$$

$$\{-k_i\} = \{k_i + 1\} \text{ if } c_1 = -1 \quad (3.3)$$

Theorem 3.1.3. [10, Hartshorne, Theorem 7.5] Let $\mathcal{F}$ be as in the previous theorem and $\chi = \{k_j\}$ its spectrum.

(a) Assume $H^0(\mathcal{F}(-1)) = 0$

1) If there is a $k \geq 1$ in χ , then $1, 2, \dots, k$ also occur in χ

2) If there is a $k \leq -2$ then $-1, -2, \dots, k$ also occur if $c_1 = 0$ and $-2, -3, \dots, k$ also occur if $c_1 = -1$

(b) Assume $\mathcal{F}$ is stable.

1) If there is a $k \geq 1$ then $0, 1, \dots, k$ also occur.

2) If there is a $k \leq -1$ then $-1, -2, \dots, k$ also occur. Furthermore, if $c_1 = 0$, then either 0 also occurs, or -1 occurs at least twice.

Proposition 3.1.4. [10, Hartshorne, Proposition 5.1] Let $\mathcal{F}$ be a semistable rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1 = 0$ or -1 . Let $k = \max\{-k_i\}$ as k_i runs over the spectrum of $\mathcal{F}$. Assume there is a r_0 with $-k < r_0 \leq -2$ which occurs only once in the spectrum. Then each k_i with $-k \leq k_i \leq r_0$ occurs exactly once in the spectrum

Now, let $\mathcal{F}$ be a rank 2 locally free sheaf with $c_1 = -1$, $H^0(\mathcal{F}(-1)) = 0$ and χ its spectrum. If $k_j \in \chi$, by symmetry $-k_j - 1$ occurs in χ .

Define $p_j = \{-j - 1, j\}$ for each integer. We will say that p_j occurs in χ if $-j - 1$ and j are in χ . Note that $p_j = p_{-j-1}$ so we will only deal with $j \geq 0$.

For example, $p_0 p_1^2$ denotes the spectrum $\{-2^2, -1, 0, 1^2\}$.

By Theorem (3.1.3), if p_j occurs in χ and $j \geq 1$ then every p_l with $1 \leq l \leq j$ also occurs, if $\mathcal{F}$ is stable then every p_l with $0 \leq l \leq j$ occurs.

With this in mind, given an integer c_2 we can write a list of possible spectra a stable bundle satisfying the hypothesis of (3.1.1) can have:

c_2	χ
1	$\emptyset$
2	p_0
3	$\emptyset$
4	$p_0^2, p_0 p_1$
5	$\emptyset$
6	$p_0^3, p_1 p_0^2, p_1^2 p_0, p_2 p_1 p_0$
7	$\emptyset$
8	$p_0^4, p_1 p_0^3, p_1^2 p_0^2, p_1^3 p_0, p_2 p_1^2 p_0, p_2 p_1 p_0^2, p_3 p_2 p_1 p_0$

Notice that we don't have $p_2^2 p_1 p_0$ in the line $c_2 = 8$, because Proposition (3.1.4) tells us this case cannot happen. To see this notice that $\max\{-k_i\}_{i=1,\dots,8} = 3$ and take $r_0 = -2$, we have $-3 = \max\{k_i\}_{i=1,\dots,8} < r_0 = -2$ and -2 occurs only once in the spectrum, so Proposition (3.1.4) states that any k_i such that $-3 = -k \leq k_i \leq -2$ must occur only once in the spectrum, this includes -3 . But -3 appears twice in $\chi = p_2^2 p_1 p_0$ so χ can't happen. Since any integer in χ must have another different integer associated with it we see there is no rank 2 locally free sheaf with $c_1 = -1$ satisfying $H^0(\mathcal{F}(-1)) = 0$ with odd c_2 .

It is interesting to notice this is not the case for $c_1 = 0$. In this case 0 is not associated with any other integer, so for instance it is possible to have a spectrum $\chi = \{-1, 0^3, 1\}$.

A general formula for any c_2 is given by $\prod_{i=1}^{\frac{c_2}{2}} p_i^{k_i}$ with:

- (i) $\sum_{i=1}^{\frac{c_2}{2}} k_i = c_2$
- (ii) If $p_k \in \chi$ then $p_j \in \chi$ for every $0 \leq j \leq k$
- (iii) If $p_k \in \chi$ is such that $k = \max\{l \mid p_l \in \chi\}$ and p_j occurs only once with $p_1 \leq p_j < p_k$ then any $p_{j'}$ with $p_j \leq p_{j'} \leq p_k$ occurs only once

Theorem (2.2.3) tells us that every locally free sheaf of rank 2 in $\mathbb{P}^3$ is the cohomology of a monad of the form

$$\mathcal{V}_0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_2 \quad (3.4)$$

where $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$ are sums of line bundles. Such monad can be chosen to be minimal in the sense that it is built from a minimal resolution of the first cohomology module [18, Rao] and [5, Decker].

In [5, Decker, Proposition 1], Decker shows that $\mathcal{E}$ is the cohomology of a monad of the form

$$\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(-c_i - 1) \rightarrow \bigoplus_{i=1}^{r+1} (\mathcal{O}_{\mathbb{P}^3}(-b_i - 1) \oplus \mathcal{O}_{\mathbb{P}^3}(b_i)) \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(c_i) \quad (3.5)$$

Note that (3.5) is Horrocks, hence by Theorem (2.2.3) r coincides with the number of generators of $H_*^1(\mathcal{E})$ as a graded module over the ring of polynomials in four variables, while $-c_i$ are the degrees of these generators.

3.2 The Hartshorne-Serre correspondence

In [7, Hartshorne], Hartshorne, following the ideas of previous works of Horrocks [12, Horrocks, 1968] and Serre [22, Serre, 1960] proved that a nonsingular subvariety Y of codimension 2 in $\mathbb{P}^n$ with $n \geq 3$ occurs as the zero-set of a section of a bundle $\mathcal{E}$ of rank 2 on $\mathbb{P}^n$ if and only if its canonical sheaf ω_Y is a multiple of the hyperplane sheaf $\mathcal{O}_Y(1)$ (if this last condition is satisfied we say Y is **subcanonical**).

In [10, Hartshorne] Hartshorne provided a version of the correspondence between rank 2 reflexive sheaves on $\mathbb{P}^3$ and arbitrary curves in $\mathbb{P}^3$.

Recall that a projective variety $Y \rightarrow \mathbb{P}^n$ of codimension r is a local complete intersection if the ideal sheaf can be locally generated by r elements at every point.

Theorem 3.2.1. [9, Hartshorne, Theorem 1.1] *Fix a line bundle $\mathcal{L}$ on $\mathbb{P}^3$, there is a bijective correspondence between*

- (i) *the set of triples $(\mathcal{E}, s, \phi)$ modulo equivalence relation $\sim$, where $\mathcal{E}$ is a vector bundle of rank 2 on $\mathbb{P}^3$, $s \in H^0(\mathcal{E})$ is a global section whose scheme of zeros Y has codimension 2, $\phi : \wedge^2 \mathcal{E} \rightarrow \mathcal{L}$ is an isomorphism and $(\mathcal{E}, s, \phi) \sim (\mathcal{E}', s', \phi')$ if and only if there is an isomorphism $\psi : \mathcal{E} \rightarrow \mathcal{E}'$ and an element $\lambda \in k - \{0\}$ such that $s' = \lambda\psi(s)$ and $\phi' = \lambda^2 \phi \circ (\wedge^2 \psi)^{-1}$.*
- (ii) *the set of pairs (Y, ξ) where Y is a locally complete intersection curve in $\mathbb{P}^3$ and $\xi : \mathcal{L} \otimes \omega_{\mathbb{P}^3} \otimes \mathcal{O}_Y \rightarrow \omega_Y$ is an isomorphism.*

If Y is a subcanonical curve with $\omega_Y \cong \mathcal{O}_Y(a)$ we can take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(a+4)$, thus $\omega_Y \cong \mathcal{L} \otimes \omega_{\mathbb{P}^3} \otimes \mathcal{O}_Y$. The theorem then gives a bundle $\mathcal{E}$, a section s and an isomorphism $\det(\mathcal{E}) \cong \mathcal{L}$. Since $c_1(\det(\mathcal{E})) = c_1(\mathcal{E})$ [8, Hartshorne, p.430] we get $c_1(\mathcal{E}) = a+4$. We also have $c_2(\mathcal{E}) = [Y] = \deg(Y)[H]^2$ [8, Hartshorne, C6 p.431] so the second Chern class corresponds to the degree of Y . From the proof of theorem 3.2.1 we get an exact sequence:

$$0 \rightarrow \mathcal{L}^* \rightarrow \mathcal{E}^* \rightarrow \mathcal{I}_Y \rightarrow 0 \quad (3.6)$$

Tensor it by $\mathcal{O}_{\mathbb{P}^3}(a+4)$ and define $\mathcal{F} = \mathcal{E}^*(a+4)$.

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{F} \rightarrow \mathcal{I}_Y(a+4) \rightarrow 0 \quad (3.7)$$

The Chern classes of $\mathcal{F}$ are $c_1(\mathcal{F}) = -(a+4) + 2(a+4) = c_1(\mathcal{E})$ and $c_2(\mathcal{F}) = c_2(\mathcal{E}) - (a+4)c_1(\mathcal{E}) + (a+4)^2 = c_2(\mathcal{E})$.

So from now on the **vector bundle $\mathcal{E}$ corresponding to a subcanonical curve Y** will be the vector bundle satisfying sequence (3.7).

Now, let $\mathcal{T}$ be the tangent bundle of $\mathbb{P}^3$. From the sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^4 \rightarrow \mathcal{T} \rightarrow 0$ we obtain $td(\mathcal{T}) = td(\mathcal{O}_{\mathbb{P}^3}(1)^4)$ where td is the Todd class, so $td(\mathcal{T}) = 1 + 2H + \frac{11}{6}H^2 + H^3$. Let $\mathcal{F}$ be a rank 2 bundle on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{F}) = c_1$ and $c_2(\mathcal{F}) = c_2$. The Chern character $ch(\mathcal{F})$ is $2 + c_1H + \frac{c_1^2 - 2c_2}{2} + \frac{c_1^3 - 3c_1c_2}{6}$. Multiplying $td(\mathcal{T})$ with $ch(\mathcal{F})$ and summing the degree 3 terms gives $2 + \frac{11c_1}{6} + (c_1^2 - 2c_2) + \frac{c_1^3 - 3c_1c_2}{6}$. Rearranging terms and using the Riemann-Roch theorem gives

$$\chi^e(\mathcal{F}) = \binom{c_1 + 3}{3} - 2c_2 - \frac{c_1c_2}{2} + 1 \quad (3.8)$$

We will give a few examples of bundles corresponding to curves

Example 3.2.2 (Disjoint union of conics). Let $Y = \bigsqcup_{i=1}^r C_i$ be a disjoint union of r conics in $\mathbb{P}^3$ and i the inclusion morphism. We have $\deg(Y) = 2r$ and $\omega_Y \cong \mathcal{O}_Y(-1)$. Let $\mathcal{E}'$ be the

vector bundle corresponding to Y . We have an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{E}' \rightarrow \mathcal{I}_Y(3) \rightarrow 0$ with $c_1(\mathcal{E}') = 3$ and $c_2(\mathcal{E}') = 2r$. Let $\mathcal{E} = \mathcal{E}'(-2)$, then $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2r - 2$. The sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Y(1) \rightarrow 0 \quad (3.9)$$

shows that if $r \geq 2$ then $H^0(\mathcal{E}) = 0$ since Y is not contained in any hyperplane (this implies $H^0(\mathcal{I}_Y(1)) = 0$). Since $\mathcal{E}$ is normalized it means $\mathcal{E}$ is stable. Notice $\mathcal{E}^* \cong \mathcal{E}(1)$ because $c_1(\mathcal{E}) = -1$.

Now, since $H^0(\mathcal{E}(k)) = 0$ for every $k \leq 0$ using Serre duality we have $H^3(\mathcal{E}(-k-3)) = 0$ for $k \leq 0$. In other words $H^3(\mathcal{E}(k)) = 0$ for $k \geq -3$.

From sequence (3.9) twisted by $k \in \mathbb{Z}$ we get $H^1(\mathcal{E}(k)) \cong H^1(\mathcal{I}_Y(k+1))$. From

$$0 \rightarrow \mathcal{I}_Y(k) \rightarrow \mathcal{O}_{\mathbb{P}^n}(k) \rightarrow \mathcal{O}_Y(k) \rightarrow 0$$

since $H^0(\mathcal{O}_Y(k)) = \bigoplus_{i=1}^r H^0(\mathcal{O}_{C_i}(k)) = 0$ for $k \leq -1$ we get $H^1(\mathcal{I}_Y(k)) = 0$ for $k \leq -1$, hence $H^1(\mathcal{E}(k)) = 0$ for $k \leq -2$. By Serre duality $H^1(\mathcal{E}(k)) \cong H^2(\mathcal{E}(-k-3))^*$ so $H^2(\mathcal{E}(k)) = 0$ for $k \geq -1$.

We wish to compute the spectrum of $\mathcal{E}$. Notice that the number of p_i in the spectrum where $i \geq 1$ is zero since $\#p_i = h^1(\mathcal{E}(-1-i)) - 2h^1(\mathcal{E}(-2-i)) + h^1(\mathcal{E}(-3-i))$ for $i \geq 1$. So $\chi(\mathcal{E})$ can only have p_0 's, which implies $\chi(\mathcal{E}) = p_0^{\frac{2r-2}{2}} = p_0^{r-1}$.

We could also use formula (3.1): Since $H^i(\mathcal{E}(-1)) = 0$ for $i \neq 1$ we can use equation (3.8) to compute $h^1(\mathcal{E}(-1))$. Since $c_1(\mathcal{E}(-1)) = -1 - 2 = -3$ and $c_2(\mathcal{E}(-1)) = 2r - 2 + (-1)(-1) + 1 = 2r$, using formula (3.8) we get $h^1(\mathcal{E}(-1)) = r - 1$, hence $\#p_0 = r - 1$, in other words $\chi(\mathcal{E}) = p_0^{r-1}$.

Example 3.2.3. Let $m \geq 2$ be an integer, Y_1 a plane curve of degree $2m - 2$ and Y_2 a complete intersection of surfaces of degree m and $m - 1$. Let $Y = Y_1 \sqcup Y_2$. Since $\omega_{Y_i} \cong \mathcal{O}_{Y_i}(2m-5)$ we have $\omega_Y \cong \mathcal{O}_Y(2m-5)$ with $\deg(Y) = m(m-1) + 2m - 2 = m^2 + m - 2$. The correspondence gives a vector bundle $\mathcal{E}'$ satisfying

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E}' \rightarrow \mathcal{I}_Y(2m-1) \rightarrow 0 \quad (3.10)$$

We wish to normalize $\mathcal{E}'$, so consider $\mathcal{E} = \mathcal{E}'(-m)$. Then $c_1(\mathcal{E}) = 2m - 1 + 2(-m) = -1$ and $c_2(\mathcal{E}) = \deg(Y) + (2m-1)(-m) + m^2 = 2m - 2$. So the new exact sequence is

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-m) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Y(m-1) \rightarrow 0 \quad (3.11)$$

$\mathcal{E}$ is stable because Y is not contained in any surface of degree $m-1$. Y has two components so from the exact sequence $0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}) \rightarrow H^0(\mathcal{O}_Y) \rightarrow H^1(\mathcal{I}_Y) \rightarrow 0$ we get $h^1(\mathcal{I}_Y) = 1$, which implies $H^1(\mathcal{E}(-m+1)) \neq 0$. But $c_2 = 2m - 2$ so $H^1(\mathcal{E}(-\frac{c_2}{2})) \neq 0$ which implies $\chi(\mathcal{E}) = p_0 p_1 \dots p_{\frac{c_2-2}{2}}$.

Example 3.2.4. Let Y_1 be the disjoint union of two plane curves of degree 4 and Y_2 be a curve from the intersection of two surfaces of degree 3 and 2. We have $\omega_{Y_i} = \mathcal{O}_{Y_i}(1)$. Let $\mathcal{E}'_i$ be the vector bundle corresponding to Y_i .

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E}'_i \rightarrow \mathcal{I}_{Y_i}(5) \rightarrow 0 \quad (3.12)$$

$c_1(\mathcal{E}'_i) = 5$ and $c_2(\mathcal{E}'_i) = \deg(Y_i)$. Set $\mathcal{E}_i = \mathcal{E}'_i(-3)$ so that $c_1(\mathcal{E}_i) = -1$, $c_2(\mathcal{E}_1) = 2$ and $c_2(\mathcal{E}_2) = 0$.

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-3) \rightarrow \mathcal{E}_i \rightarrow \mathcal{I}_{Y_i}(2) \rightarrow 0 \quad (3.13)$$

It only makes sense to talk about the spectrum of $\mathcal{E}_1$ since the spectrum is defined only if $c_2 > 0$:

The number of ones in the spectrum is given by $\#1 = h^1(\mathcal{E}_1(-2)) - 2h^1(\mathcal{E}_1(-3)) + h^1(\mathcal{E}_1(-4))$. From (3.13) we have $H^1(\mathcal{E}_1(k)) \cong H^1(\mathcal{I}_{Y_1}(k+2))$. We already know that $H^1(\mathcal{I}_{Y_1}(k)) = 0$ for $k < 0$ since Y_1 is a disjoint union of complete intersections, indeed if Z is a complete intersection curve of surfaces of degree d_1 and d_2 then using the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1 - d_2) \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_1) \rightarrow \mathcal{I}_Z \rightarrow 0$$

we conclude $H^1(\mathcal{I}_Z(k)) = 0$ for $k < 0$. Looking at the exact sequence

$$0 \rightarrow \mathcal{I}_{Y_1} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{Y_1} \rightarrow 0$$

and noticing that $H^0(\mathcal{I}_{Y_1}) = 0$ and $h^0(\mathcal{O}_{Y_1})$ is the number of connected components, we get $h^1(\mathcal{I}_{Y_1}) = 1$. So the spectrum of Y_1 is $\chi(\mathcal{E}_1) = p_1$. In particular this implies that $\mathcal{E}_1$ is not stable (Theorem 3.1.3), hence Y_1 is contained in a surface of degree 2.

Now consider $Y = Y_1 \sqcup Y_2$. Let $\mathcal{E}$ be the normalized vector bundle corresponding to Y . Y is not contained in any surface of degree 2 so $\mathcal{E}$ is stable. We have $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 10$. Using the same argument as before $\#1 = 2$ so p_1^2 appears in $\chi(\mathcal{E})$. We can only complete the spectrum with p_0 because $h^1(\mathcal{E}(k)) = 0$ for $k \leq -3$ so $\chi(\mathcal{E}) = p_0^3 p_1^2$. We could keep on adding those curves in order to generate other stable vector bundles with only p_1 and p_0 .

We now have three families of curves, each generating a family of vector bundles with a specific spectrum, it is natural to ask what kind of monad we can get from each family. We would like to use Theorem (2.2.3), but first we need to find the amount of minimal generators of $H_*^1(\mathcal{E})$ and their respective degrees.

Let $\mathcal{E}$ be a rank 2 stable vector bundle on $\mathbb{P}^3$ with $c_1 = -1$ and call $M := H_*^1(\mathcal{E})$ the first cohomology module over $S = k[x_0, x_1, x_2, x_3]$ with M_j being the degree j th submodule of M . Given $l \in \mathbb{Z}$ define $\rho(l)$ to be amount of minimal generators for M in degree l and let $m_j := h^1(\mathcal{E}(j)) = \dim(M_j)$. It has a spectrum of the form $\chi(\mathcal{E}) = \{(-k-1)^{s(k)}, \dots, -1^{s(0)}, 0^{s(0)}, \dots, k^{s(k)}\}$, since $s(k+1) = 0$, $s(k) \neq 0$ and $s(l) = h^1(\mathcal{E}(-l-1)) - 2h^1(\mathcal{E}(-l-2)) + h^1(\mathcal{E}(-l-3))$ we have that M_{-k-1} is the lowest degree part of M

such that it is nonzero. In particular this implies $\rho(-k-1) = m_{-k-1}$, using the fact that $m_l = \#\{k_j \in \chi(\mathcal{E}) | k_j \geq l-1\}$ gives:

$$\rho(-k-1) = s(k) \quad (3.14)$$

In other words, if we know how many times the greatest integer k appears in $\chi(\mathcal{E})$, then we know the amount of minimal generators in degree $-k-1$ for M , which gives a summand of our monad. Following the notation of equation (3.5) the number $k+1$ will be the largest number between the c_i 's.

We still need to find the other $c_i < k+1$, the following proposition given by Hartshorne is useful

Proposition 3.2.5. [11, Hartshorne & Rao, p.796] For $0 \leq i < k$

$$s(i) - 2\sum_{j \geq i+1} s(j) \leq \rho(-i-1) \leq s(i) - 1 \quad (3.15)$$

Proof. Since we are in the case $c_1 = -1$, the restriction of $\mathcal{E}$ to the general plane is stable ([2, Barth, Theorem 3]). So the map $M_{-l-1} \rightarrow M_{-l}$ is injective for every $l \geq 0$. Define N_{-l} by the exact sequence $0 \rightarrow M_{-l-1} \rightarrow M_{-l} \rightarrow N_{-l} \rightarrow 0$ and call n_{-l} the dimension of N_{-l} . Let $x \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, we have a natural map $N_{-l-1} \xrightarrow{x} N_{-l}$, consider then the map $N_{-l-1} \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow N_{-l}$. From the proof of [10, Hartshorne, Theorem 5.3] we have that this map has image of dimension $> n_{-l-1}$. Let $N = \bigoplus_{k \in \mathbb{Z}} N_k$, then $\rho(-l)$ is also the number of generators for N in degree $-l$ which results for $l \geq 1$

$$\rho(-l) = n_{-l} - n_{-l-1} - 1 \quad (3.16)$$

But $n_{-l} - n_{-l-1} = s(l-1)$, hence

$$\rho(-l) \leq s(l-1) - 1 \quad (3.17)$$

Notice that $N_{-l-1} \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow N_{-l}$ has image $\leq 3n_{-l-1}$ since $h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$. Hence N_{-l} must contain at least $n_{-l} - 3n_{-l-1}$ minimal generators for N , equation (3.1) gives the inequality. $\square$

We have seen that the a monad for $\mathcal{E}$ must be of the form

$$\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(-c_i - 1) \rightarrow \bigoplus_{i=1}^{r+1} (\mathcal{O}_{\mathbb{P}^3}(-b_i - 1) \oplus \mathcal{O}_{\mathbb{P}^3}(b_i)) \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(c_i) \quad (3.18)$$

For the sake of simplicity denote it by

$$\mathcal{V}_0^*(-1) \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_0 \quad (3.19)$$

Next proposition gives us a tool to limit the negative degrees appearing in the summands of $\mathcal{V}_0$:

Proposition 3.2.6. *Suppose $\mathcal{V}_0$ has r summands with degrees $\leq l$. Then $\mathcal{V}_1$ must contain at least $r + 3$ summands with degrees $\geq -l$*

Proof. $\mathcal{V}_0^*(-1)$ has r summands with degrees $\geq -l - 1$, and since the monad is minimal these must embed into the summand of $\mathcal{V}_1$ consisting of terms with degrees $\geq -l$. The quotient of this embedding has rank 3 or more $\square$

With equation (3.14) we can construct a monad for the first three examples:

Example 3.2.7. (Disjoint union of conics) Let Y and $\mathcal{E}$ be the curve and the vector bundle of example (3.2.2). We already know that $\chi(\mathcal{E}) = p_0^{r-1}$ so using the notation from equation (3.14) we have $k = 0$ and $s(k) = r - 1$. Then $\rho(-1) = r - 1$ and hence we know that there is a $c_i = 1$ which appears $r - 1$ times in $\mathcal{V}_0$. Since 0 was the largest integer in the spectrum, there can be no integer greater than 1 in $\mathcal{V}_0$, so we can only complete it with $\mathcal{O}_{\mathbb{P}^3}$'s. But if we had any summand $\mathcal{O}_{\mathbb{P}^3}$ in $\mathcal{V}_0$ the monad wouldn't be minimal, hence we have an unique monad

$$\mathcal{O}_{\mathbb{P}^3}(-2)^{r-1} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^r \oplus \mathcal{O}_{\mathbb{P}^3}^r \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{r-1} \quad (3.20)$$

Example 3.2.8. Let $\mathcal{E}$ be the vector bundle of example 3.2.3. We already have $\chi(\mathcal{E}) = p_0 p_1 \dots p_{m-2}$ where $c_2 = 2m - 2$ and $m \geq 2$. Then $k = m - 2$ and $\rho(-m + 1) = s(k) = 1$, so $\mathcal{O}_{\mathbb{P}^3}(m - 1)$ appears only once in $\mathcal{V}_0$. Using proposition (3.2.5) we get that $\mathcal{V}_0 = \mathcal{O}_{\mathbb{P}^3}(m - 1)$.

$$\mathcal{O}_{\mathbb{P}^3}(-m) \rightarrow \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(-a - 1) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(-b - 1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(m - 1) \quad (3.21)$$

We can compute a and b using the Chern polynomial c_t : If $\mathcal{V}_0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is a monad, using the associated exact sequences

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_2 \rightarrow 0 \quad (3.22)$$

and

$$0 \rightarrow \mathcal{V}_0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0 \quad (3.23)$$

and using the fact that c_t is multiplicative, we get $c_t(\mathcal{E}) = \frac{c_t(\mathcal{V}_1)}{c_t(\mathcal{V}_0)c_t(\mathcal{V}_2)}$, we can then compare each coefficient since $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 2m - 2$. In our case the denominator will be $1 - t - m(m - 1)t^2$, and its inverse is $1 + t + (1 + m(m - 1))t^2 + (1 + 2m(m - 1))t^3$. Multiplying this last polynomial with the numerator gives the polynomial $1 - t + (m^2 - m - b^2 - a - a^2)t^2$, using again that $c_2 = 2m - 2$ we are left with a diophantine equation

$$(m - 1)(m - 2) = b + b^2 + a + a^2 \quad (3.24)$$

We already know the case $m = 2$, so let $m = 3$ ($c_2 = 4$)

$$2 = b + b^2 + a + a^2 \quad (3.25)$$

has 5 solutions, all of them give the same monad

$$\mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2) \quad (3.26)$$

The case $m = 4$ ($c_2 = 6$) gives

$$\mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(3) \quad (3.27)$$

which suggests that a general solution is given by

$$\mathcal{O}_{\mathbb{P}^3}(-m) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-m+1) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(m-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(m-1) \quad (3.28)$$

In order to verify it just take $a = -1$ and $b = -m + 1$.

Example 3.2.9. Let $\mathcal{E}$ be the vector bundle from example (3.2.4). $\chi(\mathcal{E}) = p_0^3 p_1^2$, $c_2(\mathcal{E}) = 10$ and $\rho(-2) = s(1) = 2$. Using proposition (3.2.5) we get $\rho(-1) \leq 2$, so we need to analyze three different cases.

Case 1: if $\rho(-1) = 0$ the monad will be

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(-a-1) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(-b-1) \oplus \mathcal{O}_{\mathbb{P}^3}(c) \oplus \mathcal{O}_{\mathbb{P}^3}(-c-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2)^2 \quad (3.29)$$

The same computation gives the diophantine equation

$$a + a^2 + b + b^2 + c + c^2 = 2 \quad (3.30)$$

Which has 24 solutions, all of them giving the same monad

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}^2 \oplus \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2)^2 \quad (3.31)$$

Case 2: if $\rho(-1) = 1$ we have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^3}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(-a-1) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(-b-1) \oplus \mathcal{O}_{\mathbb{P}^3}(c) \oplus \mathcal{O}_{\mathbb{P}^3}(-c-1) \\ \oplus \mathcal{O}_{\mathbb{P}^3}(d) \oplus \mathcal{O}_{\mathbb{P}^3}(-d-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)^2 \end{aligned} \quad (3.32)$$

which gives the diophantine equation

$$a + a^2 + b + b^2 + c + c^2 + d + d^2 = 4 \quad (3.33)$$

it has 96 solutions giving the monad

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}^2 \oplus \mathcal{O}_{\mathbb{P}^3}(1)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)^2 \quad (3.34)$$

Case 3: if $\rho(-1) = 2$ we have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^3}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(-a-1) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(-b-1) \oplus \mathcal{O}_{\mathbb{P}^3}(c) \oplus \mathcal{O}_{\mathbb{P}^3}(-c-1) \\ \oplus \mathcal{O}_{\mathbb{P}^3}(d) \oplus \mathcal{O}_{\mathbb{P}^3}(-d-1) \oplus \mathcal{O}_{\mathbb{P}^3}(e) \oplus \mathcal{O}_{\mathbb{P}^3}(-e-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(2)^2 \end{aligned} \quad (3.35)$$

gives the diophantine equation

$$a + a^2 + b + b^2 + c + c^2 + d + d^2 + e + e^2 = 6 \quad (3.36)$$

there are two monads satisfying the equation:

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^4 \oplus \mathcal{O}_{\mathbb{P}^3}^4 \oplus \mathcal{O}_{\mathbb{P}^3}(2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(2)^2 \quad (3.37)$$

which cannot occur, since we are assuming the monad to be minimal, and

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}^2 \oplus \mathcal{O}_{\mathbb{P}^3}(1)^3 \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(2)^2 \quad (3.38)$$

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