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Invariance Entropy for Control Systems on Lie Groups and Homogeneous Spaces

Entropia Invariante para Sistemas de Controle em Grupos de Lie e Espaços Homogêneos

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Abstract

In this Thesis we will analyse the invariance entropy of admissible pairs for control systems on Lie groups and homogeneous spaces. The main goal is to improve the known upper and lower bounds for such entropy and see when it is possible to prove that these bounds coincide, which give us an expression for the entropy. We will show that for induced control-affine systems on the flag manifolds both, the upper and lower bounds are given by the determinant of the unstable part of the system and they differ just on the set where we consider the infimum. For Linear systems on abelian, nilpotents and compact Lie groups we have an expression for the invariance entropy and in the semi-simple case, the upper and lower bounds equality depend on the exponential growth of an associated driftless control-affine system. At the end of the Thesis we introduce a concept of entropy for random control systems and derive general bounds for it.

Resumo

Na presente tese iremos analisar a entropia invariante de pares admissíveis para sistemas de controle sobre grupos de Lie e espaços homogêneos. O objetivo é melhorar os limitantes superiores e inferiores já conhecidos para tal entropia e ver quando é possível mostrar que tais limitantes coincidem, nos dando então uma expressão para ela. Mostraremos que para sistemas afins induzidos em variedades flag os limitantes, tanto superior como inferior, são dados com ínfimo do determinante da parte instável do sistema e diferem apenas em qual conjunto tal ínfimo é considerado. Para sistemas Lineares sobre grupos abelianos, nilpotentes e compactos temos uma expressão para a entropia e no caso semi-simples, a igualdade dos limitantes depende do crescimento exponencial de um sistema de controle afim sem drift associado. No fim da tese é ainda introduzido um conceito de entropia invariante para sistemas aleatórios e limitantes gerais para este são derivados.

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Introduction

Inspired by the work of Nair, Evans, Mareels and Moran [25] about entropy-like notions for measuring the complexity of certain control tasks, and the characterizations of Bowen-Dinaburg of the topological entropy in metric spaces, Colonius and Kawan introduced in [10] the invariance entropy and outer invariance entropy concepts for continuous-time control systems as a measure of how often open-loop control functions have to be updated in order to achieve invariance of a given compact and controlled invariant subset Q of the state space for a fixed set of initial states $K \subset Q$. In subsequent works [19], [18], [20] and [21], Kawan obtained lower and upper bounds for the invariance entropy and in some particular cases an expression was obtained (see for instance Corollary 5.3, Theorem 7.2 and 7.8 of [21]). In [20] is shown that if there is subbundle of constant rank of the tangent space over points in Q where the system is expanding, then the lower bound obtained depends on the infimum sum of the positive Lyapunov exponents of the system and on a quantity that measures how much the solutions of our system starting in K tends to escape the compact controlled invariant set Q. For control-affine system, with assumptions, the upper bound depends also on the sum of the positive Lyapunov exponents but it differs from the lower bound in which set the infimum is considered. Although such bounds goes in the direction of an expression it is hard to say whether the escape entropy vanishes or not.

In order to analyze those bounds and see if one could improve them, at least for some particular cases, we consider a right invariant control-affine system over a semi-simple Lie group G. Associated with it there is a continuous flow, denominated control flow of the system, that acts as a flow of automorphisms on a trivial principal bundle with fiber G. For such settings, the theory developed in [28], [26], [30] and [1] allow us to say explicitly who are the chain control sets and control sets of the induced systems on the flag manifolds of G. It is shown that on every chain control set of every flag manifold the induced control-affine systems are partially hyperbolic sets and there are chain control sets on certain flag manifolds where we actually have hyperbolicity. For such hyperbolic chain control sets it is shown that we can get rid of the escape entropy and we can slightly improve the upper bound showing that lower and upper bounds are almost the same. When hyperbolicity happens in the maximal flag the result is true for all control sets.

Still concerning control-affine system we have a special class of such systems that are the system whose drift generates a flow of automorphisms as introduced in [4] and [5]. For such systems we work with the concept of outer invariance entropy (that is a natural lower bound for the invariance entropy). Its is shown that for some class of groups we have that the outer invariance entropy is given by the sum of the real parts of the eigenvalues of an associated linear derivation what is a natural generalization of the result for linear control systems obtained in [10] for Euclidean spaces. For the semi-simple case what we get is that the outer invariance entropy is bounded below by the same sum of the real parts of the eigenvalues as above and by a negative quantity that depends just on the exponential growth rate of the associated right invariant control-affine system without drift.

At the end a new concept of entropy is defined for continuous-time random control systems and random pairs as a measure for the amount of information necessary to achieve invariance of random weakly invariant compact subsets of the state space. For linear random control systems with compact control range, this entropy is given, in a set of full measure for some invariant measure, by the sum of the real parts of the unstable eigenvalues of the uncontrolled system and if we assume ergodicity such quantity is almost everywhere constant.

Now we briefly sketch the contents of the Thesis:

The first Chapter is divided in two Sections. The first serves as an introduction of the basic control-theoretic notions. We will just consider system given by differential equations, more specifically control-affine systems. Such systems have special properties and one of them is that we can associate to it a continuous flow whose dynamical properties are intrinsically connected with the properties of the solutions of this system. The notion of uniformly hyperbolic sets for control-affine systems is also introduced and it will be central in Chapter 3 and in subsequent chapters.

In the second Section the central notion of invariance entropy for control systems is established and its basic properties are stated, such as the important result about invariance under conjugacy. A related notion, named outer invariance entropy, is also introduced. Such notion is a natural lower bound for the invariance entropy and is, in some respect, better behaved. The last subsection state the bounds obtained for general systems and the concept of escape entropy.

In Chapter 2 the semi-simple theory is considered. The first Section serves to introduce the general notations and results about semi-simple theory. In Section 2.2 is introduced the concept of flag type of a semigroup and the relation between it and the control sets of the semigroup considered. In Section 2.3 a flow on a principal bundle is considered and the flag type of such flow is defined. It is shown that the flag type of a flow is closely related to its finest Morse decomposition on the induced flag bundles. The notion of a vectorial cocycle associated with the flow is also defined and some properties are derived.

In Chapter 3 we consider a right invariant control-affine system on a semi-simple Lie group. The associated control flow acts as a flow of automorphisms on a trivial principal bundle what allow us to apply all the results stated in Chapter 2 to this special case. Such results allow us to characterize all the chain control sets for the induced systems on the flag manifolds and to show that they are partially hyperbolic sets. For some flag manifolds is shown that the maximal control set has escape entropy equals to zero and for some hyperbolic chain control sets that is also true. In particular, when hyperbolicity happens on the maximal flag manifold it is shown that all the control sets on every flag manifolds have escape entropy equals to zero. In the last Section we improve the upper bound over some hyperbolic chain control sets, using the ideas from [21] for projective systems.

In Chapter 4 we consider an admissible pair for a linear system on a Lie group (not necessarily semi-simple) and analyze the outer invariance entropy. It is shown that in many cases a generalization of the result for linear system in $\mathbb{R}^d$ is possible and that is closely related with the geometry of the Lie group considered.

In Chapter 5 we introduce a new concept of invariance entropy by adding a new random component to our system. The addition of such component gives rise to a concept of a family of entropies parametrized by the random component. For the linear case we are able to show that such invariance entropy is given a.e. by the sum of the real part of the unstable eigenvalues of the uncontrolled system and assuming ergodicity is possible to show that this entropy is a.e. constant.

Chapter 1

Control Systems and Invariance Entropy

The aim of this chapter is to introduce the notion of control systems on smooth manifolds given by differential equations and define its invariance entropy.

1.1 Control Systems

Let Ω be a compact convex set of $\mathbb{R}^m$. The set of **admissible control functions** is defined by

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m; u \text{ measurable with } u(t) \in \Omega \text{ a.e.}\}.$$

The set Ω is denominated the **control range** of the system.

The **shift flow** on $\mathcal{U}$ is defined by

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \;\; \theta(t,u) := \theta_t u \text{ with } (\theta_t u)(s) := u(t+s) \text{ for all } t, s \in \mathbb{R}.$$

Let M be a d-dimensional smooth manifold and $F : M \times \mathbb{R}^m \to TM$ be a continuously differentiable function such that for each $u \in \mathbb{R}^m$ we have that $F_u := F(\cdot, u)$ is a $\mathcal{C}^1$-vector field on M.

By a **control system** we understand a family of ordinary differential equations

$$\dot{x}(t) = F(x(t), u(t)), \;\; u \in \mathcal{U} \tag{1.1}$$

on M parametrized by the set of admissible functions $\mathcal{U}$. We call the map F the **right-hand side** of the system.

The assumptions on F implies that there exists, for every initial value $x \in M$ and every control function $u \in \mathcal{U}$, a unique solution $\varphi(\cdot, x, u)$ such that $\varphi(0, x, u) = x$ and for $t > 0$, $\varphi(t, x, u)$ does not depend on the values of u outside of $[0, t)$, that is, if $u_1, u_2 \in \mathcal{U}$ and $u_1(s) = u_2(s)$ for all $s \in [0, t)$, then $\varphi(t, x, u_1) = \varphi(t, x, u_2)$ (see [21]).

We will usually use the notation $\varphi_{t,u}(x)$ instead of $\varphi(t, x, u)$. Since the concept of invariance entropy consider only solutions which stay inside a compact set (or an ε-neighborhood of it) we may assume, without loss of generality, that all solutions are defined on $\mathbb{R}$. Hence, the solutions give rise to a global map

$$\varphi : \mathbb{R} \times \mathcal{U} \times M \to M, \quad (t, u, x) \mapsto \varphi(t, x, u). \tag{1.2}$$

For the control system (1.1) and a state $x \in M$, the sets

$$\mathcal{O}^+_{\leq\tau}(x) := \{y \in M\ ; \exists u \in \mathcal{U},\ t \in [0, \tau];\ y = \varphi(t, x, u)\}$$

and

$$\mathcal{O}^+(x) := \bigcup_{\tau>0} \mathcal{O}^+_{\leq\tau}(x).$$

are called, respectively, **the set of points reachable from x up to time** τ and **the positive orbit of x**. In the same way, the sets

$$\mathcal{O}^-_{\leq\tau}(x) := \{y \in X\ ; \exists u \in \mathcal{U},\ t \in [0, \tau];\ x = \varphi(t, y, u)\}$$

and

$$\mathcal{O}^-(x) := \bigcup_{\tau>0} \mathcal{O}^-_{\leq\tau}(x)$$

are called **the set of points controllable to x within time** τ and **the negative orbit of x**. Moreover, for every $\tau > 0$, **the set of points reachable at time** τ is given by

$$\mathcal{O}_\tau(x) := \{y \in X\ ; \exists u \in \mathcal{U};\ x = \varphi(\tau, y, u)\}.$$

The next Definition is necessary when we are interested in the notion of controllability of control systems.

Definition 1.1.1 *The control system (1.1) is called* **local accessible from** $x \in X$ *if the interior of the sets* $\mathcal{O}^+_{\leq\tau}(x)$ *and* $\mathcal{O}^-_{\leq\tau}(x)$ *are nonempty for every* $\tau > 0$. *It is called* **locally accessible** *if it is locally accessible from every point* $x \in X$.

The following Proposition give us a Lie-algebraic criterion in order to have local accessibility. Such result is known as Krener's criterion and is usually called **the Lie rank condition**.

Proposition 1.1.2 *Consider a control system with right-hand side F and control range Ω and assume that F_u is a $\mathcal{C}^\infty$-vector field for every $u \in \mathbb{R}^m$. Define*

$$\mathcal{F} := \{F_u;\ \ u \in \Omega\} \subset \mathcal{X}^\infty(M).$$

Let $\mathcal{L}(\mathcal{F}) \subset \mathcal{X}^\infty(M)$ be the smallest Lie algebra containing the set $\mathcal{F}$ and $\Delta_{\mathcal{L}(\mathcal{F})}(x) := \{f(x);\ f \in \mathcal{L}(\mathcal{F})\}$ for all $x \in M$. Then if $\Delta_{\mathcal{L}(\mathcal{F})}(x) = T_xM$ for all $x \in M$, the system is locally accessible.

We introduce now a very special class of control systems, the so-called control-affine systems. For such systems we have associated a continuous flow whose properties have intrinsic relations with the solutions of the system. The proofs of the results stated here can be found in [12].

Definition 1.1.3 *Let M be a connected $\mathcal{C}^n$-manifold ($n \geq 3$). A control system given by differential equations is called* **control-affine** *if the right-hand side F has the form*

$$F(x(t), u(t)) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) \quad u \in \mathcal{U} \tag{1.3}$$

with vector fields $f_0, \ldots, f_m \in \mathcal{X}^1(M)$ and control range Ω compact and convex. The vector field f_0 is called the **drift vector field** *and $f_1, \ldots, f_m$ the* **control vector fields** *of the system.*

The set $\mathcal{U}$ of admissible control functions in the case of control-affine systems becomes a compact metrizable space with the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$. A metric compatible with the topology is given by

$$d_{\mathcal{U}}(u_1, u_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\left| \int_{\mathbb{R}} \langle u_1(t) - u_2(t), x_k(t) \rangle \mathrm{d}t \right|}{1 + \left| \int_{\mathbb{R}} \langle u_1(t) - u_2(t), x_k(t) \rangle \mathrm{d}t \right|} \tag{1.4}$$

where $\{x_k\}$ is an arbitrary countable dense subset of $L^1(\mathbb{R}, \mathbb{R}^m)$. Also, in the control-affine case, the control flow

$$\phi : \mathbb{R} \times (\mathcal{U} \times M) \to \mathcal{U} \times M, \quad \phi_t(u, x) = (\theta_t u, \varphi(t, x, u)),$$

defines a continuous dynamical system on $\mathcal{U} \times M$ (see Lemma 4.3.2 of [12]).

The next Definitions introduce the notions of control sets and chain control sets of a control system.

Definition 1.1.4 *A nonempty set $D \subset M$ is called a* **control set** *of the control system (1.1) if*

(i) D is controlled invariant, that is, for every $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}_+, x, u) \subset D$;

(ii) For every $x \in D$ one has $D \subset \operatorname{cl} \mathcal{O}^+(x)$;

(iii) D is maximal with properties (i) and (ii), that is, if $D' \supset D$ satisfies (i) and (ii), then $D' = D$.

We say that a control set D is an **invariant control set** if $\operatorname{cl} D = \operatorname{cl} \mathcal{O}^+(x)$ for all $x \in D$.

Let $x, y \in M$ and $\varepsilon, \tau > 0$. A **controlled (ε, τ)-chain** from x to y is given by $n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $u_0, \ldots, u_{n-1}$ and $\tau_0, \ldots, \tau_{n-1} \geq \tau$ with $x_0 = x$, $x_n = y$ and

$$\varrho(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon \quad \text{for all } i = 0, 1, \ldots, n-1,$$

where ϱ is any metric in M compatible with the given topology.

Definition 1.1.5 *A set $E \subset M$ is called a* **chain control set** *of the system (1.1) if it satisfies the following properties:*

(i) For every $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \in E$;

(ii) For all $x, y \in E$ and $\varepsilon, \tau > 0$, there is a controlled (ε, τ)-chain from x to y (the points in this chain are not necessarily elements of E);

(iii) E is maximal with properties (i) and (ii).

The following properties of chain control sets, whose proofs can be found in [12], Section 4.3, establish a relation between the control-theoretic properties of the control-affine system and the dynamical properties of the associated control flow.

Proposition 1.1.6 *The following assertions hold:*

(i) *Every chain control set E of the control system (1.3) is closed;*

(ii) *Assume that the (1.3) is locally accessible. Then every control set D with nonempty interior is contained in a chain control set E;*

(iii) *Different chain control sets of (1.3) are disjoint;*

(iv) *If M is compact and E is a chain control set of the control system (1.3), then*

$$\mathcal{E} := \{(u,x) \in \mathcal{U} \times M; \quad \varphi(\mathbb{R}, x, u) \subset E\}$$

is a maximal invariant chain transitive set[1] *for the control flow of the control-affine system (1.3). On the other hand, if $\mathcal{E} \subset \mathcal{U} \times M$ is a maximal chain transitive set for the control flow, then the projection of $\mathcal{E}$ to M is a chain control set.*

Next we introduce the notion of hyperbolicity for control-affine systems, notion that we will find when working with induced systems on flag manifolds.

Definition 1.1.7 *Assume that $Q \subset M$ is a compact set which is controlled invariant in forward and in backward time for the control-affine system (1.3), that is, for any $x \in Q$ there exists $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset Q$. Define the* **full time lift of** Q *by*

$$\mathcal{Q} := \{(u,x) \in \mathcal{U} \times M; \quad \varphi(\mathbb{R}, x, u) \subset Q\}.$$

Further assume that for each $(u,x) \in \mathcal{Q}$ the tangent space T_xM can be written as a direct sum

$$T_xM = E^-_{u,x} \oplus E^+_{u,x}$$

of subspaces such that the following statements hold:

1. *For all $t \in \mathbb{R}$ and $(u,x) \in \mathcal{Q}$ we have*

$$(d\varphi_{t,u})_x E^-_{u,x} = E^-_{\phi_t(u,x)} \quad \textit{and} \quad (d\varphi_{t,u})_x E^+_{u,x} = E^+_{\phi_t(u,x)};$$

[1] See Definition on page 32 ahead

2. *There are constants* $c, \mu > 0$ *such that*

$$||(d\varphi_{t,u})_x v|| \leq c^{-1}\mathrm{e}^{-\mu t}||v|| \quad \textit{for all } t \geq 0, (u,x) \in \mathcal{Q}, v \in E^-_{u,x}$$

and

$$||(d\varphi_{t,u})_x v|| \geq c\mathrm{e}^{\mu t}||v|| \quad \textit{for all } t \geq 0, (u,x) \in \mathcal{Q}, v \in E^+_{u,x};$$

Then Q *is called* **uniformly hyperbolic**.

From Lemma 6.4 of [21] the decomposition of the tangent space above vary continuously on (u, x) and when the state space M is compact, the dimension of $E^{\pm}_{u,x}$ are constant on $\mathcal{Q}$.

1.2 Invariance Entropy

This Section gives an introduction to the concepts of invariance entropy for the control system (1.1) and its properties. The proofs can be found mainly in [21].

1.2.1 Definitions and Basic Properties

Invariance entropy is a nonnegative (possibly infinite) quantity which is assigned to a pair (K, Q) of subsets of M, which satisfies the properties described in the following definition.

Definition 1.2.1 *A pair* (K, Q) *of nonempty subsets of* M *is called* **admissible** *for the control system (1.1) if it satisfies the following properties:*

(i) K *is a compact set;*

(ii) For each $x \in K$ *there exists* $u \in \mathcal{U}$ *such that* $\varphi(\mathbb{R}_+, x, u) \in Q$ *(in particular,* $K \subset Q$*).*

Given an admissible pair (K, Q) and $\tau > 0$, a set $\mathcal{S} \subset \mathcal{U}$ is called $(\tau, \mathbf{K}, \mathbf{Q})$-**spanning** if

$$\forall x \in K \ ; \ \exists\, u \in \mathcal{S} \ ; \ \varphi([0,\tau], x, u) \subset Q.$$

By $r_{\mathrm{inv}}(\tau, K, Q)$ we denote the minimal number of elements that such a set can have. If there is no finite set we say that $r_{\mathrm{inv}}(\tau, K, Q) = \infty$. If $K = Q$, we omit the argument K, that is, we write $r_{\text{inv}}(\tau, Q)$ and we speak of (τ, Q)-spanning sets.

Note that the existence of (τ, K, Q)-spanning sets is guaranteed by property (ii); indeed, $\mathcal{U}$ is a (τ, K, Q)-spanning set for every $\tau > 0$. A pair of the form (Q, Q) is admissible if and only if Q is a compact and controlled invariant set.

Definition 1.2.2 *Given an admissible pair (K, Q), we define the* **invariance entropy of $(\mathbf{K}, \mathbf{Q})$** *by*

$$h_{\text{inv}}(K, Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{inv}}(\tau, K, Q).$$

Here, we use the convention that $\log = \log_{\text{e}} = \ln$. If $K = Q$, again we omit the argument K and write $h_{\text{inv}}(Q)$. Moreover, we let $\log \infty := \infty$.

The following Proposition put together the main basic properties of the invariance entropy. Their proofs can be found in [21], Proposition 2.1 to 2.3.

Proposition 1.2.3 *Let (K, Q) be an admissible pair. It holds:*

(i) If $\tau_1 < \tau_2$ then $r_{\text{inv}}(\tau_1, K, Q) \leq r_{\text{inv}}(\tau_2, K, Q)$;

(ii) If $Q \subset P$, then also (K, P) is admissible and $r_{\text{inv}}(\tau, K, Q) \geq r_{\text{inv}}(\tau, K, P)$ for all $\tau > 0$; hence $h_{\text{inv}}(K, Q) \geq h_{\text{inv}}(K, P)$;

(iii) If $L \subset K$ is closed in M, then also (L, Q) is admissible and $r_{\text{inv}}(\tau, L, Q) \leq r_{\text{inv}}(\tau, K, Q)$ for all $\tau > 0$; hence $h_{\text{inv}}(L, Q) \leq h_{\text{inv}}(K, Q)$;

(iv) If Q is open, then $r_{\text{inv}}(\tau, K, Q)$ is finite for all $\tau > 0$.

If Q is compact and controlled invariant we have also:

(v) The number $r_{\text{inv}}(\tau, Q)$ is either finite for all $\tau > 0$ or for none;

(vi) The function $\tau \mapsto \log r_{\text{inv}}(\tau, Q)$, $\mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$, is subadditive and therefore

$$h_{\text{inv}}(Q) = \lim_{\tau \to \infty} \frac{1}{\tau} \log r_{\text{inv}}(\tau, Q) = \inf_{\tau > 0} \frac{1}{\tau} \log r_{\text{inv}}(\tau, Q).$$

Remark 1.2.4 *The preceding proposition implies the equivalence of the following statements:*

- $h_{\text{inv}}(Q)$ *is finite;*
- $r_{\text{inv}}(\tau, Q)$ *is finite for some* τ*;*
- $r_{\text{inv}}(\tau, Q)$ *is finite for all* τ*.*

A trivial example where we have zero entropy is when Q is an invariant set, that is, the solutions starting in Q do not leave Q. Nontrivial examples can be found in [21] page 47.

Another notion of entropy associated with an admissible pair is given by the following Definition.

Definition 1.2.5 *Given an admissible pair* (K, Q) *such that* Q *is closed in* M*, and a metric* ϱ *on* M*, we define the* **outer invariance entropy of** $(\mathbf{K}, \mathbf{Q})$ *by*

$$\begin{aligned} h_{\text{inv,out}}(K, Q; \varrho) &:= \lim_{\varepsilon \searrow 0} h_{\text{inv}}(K, N_\varepsilon(Q)) \\ &= \sup_{\varepsilon > 0} h_{\text{inv}}(K, N_\varepsilon(Q)), \end{aligned}$$

where $N_\varepsilon(Q)$ *denotes the* ε*-neighborhood of* Q*.*

The above Definition is independent of uniformly equivalent metrics (Proposition 2.5 of [21]) and when it is the case, we denote the outer invariance entropy just by $h_{\text{inv,out}}(K, Q)$. This quantity is better behaved than the invariance entropy and they are related by

$$0 \leq h_{\text{inv,out}}(K, Q) \leq h_{\text{inv}}(K, Q) \leq \infty$$

(Proposition 2.4 of [21]). A question that arises is under which conditions we have the equality between them.

The following result (Proposition 3.1 of [10]) shows that in order to calculate the (outer) invariance entropy it is enough to consider steps that are integer multiples.

Proposition 1.2.6 *Let* (K, Q) *be an admissible pair for the control system (1.1). Then for all* $\tau \in \mathbb{R}_+$ *we have*

$$h_{\text{inv}}(K, Q) = \limsup_{\mathbb{N} \ni n \to \infty} \frac{1}{n\tau} \log r_{\text{inv}}(n\tau, K, Q) \tag{1.5}$$

and the same holds for the outer invariance entropy.

An appropriate notion of topological conjugacy for control systems, which preserves the invariance entropy, is given in the next Definition.

Definition 1.2.7 *Consider two control systems $\dot{x}_i(t) = F_i(x_i(t), u_i(t))$ on M_i with solutions $\varphi_i(t_i, x_i, u_i)$ and set of admissible functions given by $\mathcal{U}_i$ corresponding to control ranges Ω_i, $i = 1, 2$. Let $\pi : \mathbb{R}_+ \times M_1 \to M_2$ $(t, x) \mapsto \pi_t(x)$, be a continuous map and $h : \mathcal{U}_1 \to \mathcal{U}_2$ a map such that*

$$\pi_t(\varphi_1(t, x, u)) = \varphi_2(t, \pi_0(x), h(u)) \tag{1.6}$$

holds for all $t \in \mathbb{R}_+$, $x \in M_1$ and $u \in \mathcal{U}_1$. Then:

- *The pair (π, h) is called a* **time-variant semi-conjugacy** *from $\dot{x}_1(t) = F_1(x_1(t), u_1(t))$ to $\dot{x}_2(t) = F_2(x_2(t), u_2(t))$;*
- *If π is independent of $\tau \in \mathbb{R}_+$, we can regard π as a map from M_1 to M_2 and we say that (π, h) is a* **(time-invariant) semi-conjugacy** *from $\dot{x}_1(t) = F_1(x_1(t), u_1(t))$ to $\dot{x}_2(t) = F_2(x_2(t), u_2(t))$;*
- *If the maps $\pi_t : M_1 \to M_2$ are homeomorphisms and $h : \mathcal{U}_1 \to \mathcal{U}_2$ is invertible, we call (π, h) a* **time-variant conjugacy** *from $\dot{x}_1(t) = F_1(x_1(t), u_1(t))$ to $\dot{x}_2(t) = F_2(x_2(t), u_2(t))$.*

The next result (Proposition 2.13 of [21]) give us a relation between the (outer) invariance entropy of conjugated systems.

Proposition 1.2.8 *Let $\dot{x}_i(t) = F_i(x_i(t), u_i(t))$, $i = 1, 2$ be two control systems and let (π, h) be a time-variant semi-conjugacy from $\dot{x}_1(t) = F_1(x_1(t), u_1(t))$ to $\dot{x}_2(t) = F_2(x_2(t), u_2(t))$. Further assume that (K_1, Q_1) is an admissible pair for $\dot{x}_1(t) = F_1(x_1(t), u_1(t))$ and*

$$\pi_t(Q_1) \subset \pi_0(Q_1) \quad \text{for all } t > 0. \tag{1.7}$$

Then $(K_2, Q_2) = (\pi_0(K), \pi_0(Q))$ is an admissible pair for the system $\dot{x}_2(t) = F_2(x_2(t), u_2(t))$ and

$$h_{\text{inv}}(K_1, Q_1) \geq h_{\text{inv}}(K_2, Q_2).$$

Moreover, if Q_1 is compact and the family $\{\pi_t\}_{t \in \mathbb{R}_+}$ is pointwise equicontinuous, then

$$h_{\text{inv,out}}(K_1, Q_1) \geq h_{\text{inv,out}}(K_2, Q_2).$$

A sufficient condition for the existence of a topological conjugacy can also be formulated in terms of the right-hand sides of the systems (see Proposition 2.14 of [21]).

1.2.2 Upper and Lower Bounds for Control-Affine Systems

In this section we will show the known bounds for the invariance entropy, due to Kawan. As before we will be considering control-affine systems over a Riemannian manifold M.

Upper bound

Let M be a d-dimensional Riemannian manifold and let

$$\dot{x}(t) = F(x(t), u(t)) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t))$$

be a control-affine system with control range Ω.

Definition 1.2.9 *We say that the system $\dot{x}(t) = F(x(t), u(t))$ on M is* **strongly accessible** *if for each $x \in M$ there is some $\tau > 0$ such that* $\operatorname{int} \mathcal{O}_\tau(x) \neq \emptyset$

If we denote by $\mathcal{L}_0$ the ideal in $\mathcal{L}(\mathcal{F})$ generated by the vector fields $f_1, \ldots, f_m$ we have that the system is strongly accessible if $\dim \mathcal{L}_0 = d$ (see Proposition 5.6 (vi) in [21]).

For a given $t \in \mathbb{R}$ and $(u, x) \in \mathcal{U} \times M$, the derivative

$$(d\varphi_{t,u})_x : T_xM \to T_{\varphi_{t,u}(x)}M$$

is a linear isomorphism between d-dimensional Euclidean spaces, and hence has well-defined (positive) singular values, which we denote by

$$\sigma_1(t, x, u) \geq \sigma_2(t, x, u) \geq \ldots \geq \sigma_d(t, x, u) > 0.$$

For $0 \leq k \leq d$, the singular value function of order k of $(d\varphi_{t,u})_x$ is denoted by

$$\alpha_k(t, x, u) = \begin{cases} \sigma_1(t, x, u)\sigma_2(t, x, u) \cdots \sigma_k(t, x, u) & \text{for } k > 0 \\ 1 & \text{for } k = 0. \end{cases}$$

We have that for every $k \in \{0, 1, \ldots, d\}$, the function $a^k : \mathbb{R} \times \mathcal{U} \times M$ defined by

$$a_t^k(u, x) := \log \alpha_k(t, x, u)$$

is a subadditive cocycle over the control flow.

When $k = d$, we have the **absolute determinant** of $(d\varphi_{t,u})_x$, that is,

$$\left|\det(d\varphi_{t,u})_x\right| := \sigma_1(t,x,u)\sigma_2(t,x,u)\cdots\sigma_d(t,x,u).$$

For the control function $u \in \mathcal{U}$, the **Lyapunov exponent** at x in the direction $v \in T_xM, v \neq 0_x$, is given by

$$\lambda(u,x;v) := \limsup_{\tau\to\infty} \frac{1}{\tau} \log |(d\varphi_{\tau,u})_x v| \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

We have then the following upper bound for the invariance entropy.

Proposition 1.2.10 *Let D be a control set with nonempty interior of the above system and assume that strongly accessibility holds. Let $(u,x) \in \operatorname{int}\mathcal{U} \times \operatorname{int}D$ such that $\varphi(t,x,u)$ is contained in a compact set of $\operatorname{int}D$ for all $t \geq 0$. Furthermore, assume that there exists $k \in \{0,1,\ldots,d\}$ such that the following are satisfied:*

(i) Every periodic trajectory corresponding to some $(v,y) \in \operatorname{int}\mathcal{U} \times \operatorname{int}D$ has exactly k positive Lyapunov exponents (counted with multiplicity);

(ii) There exists $t_0 \geq 0$ such that $a_t^k(u,x) \geq 0$ for all $t \geq t_0$.

Then for every compact set $K \subset D$ it holds that

$$h_{\text{inv}}(K,Q) \leq \limsup_{\tau\to\infty} \frac{1}{\tau} a_t^k(u,x)$$

The proof of the above Proposition can be found in [19] page 147. Although the condition about strongly accessibility seems to be a little restrictive, for the systems that we will consider it will be equivalent to the rank condition.

Lower Bound and Escape entropy

Consider the control system (1.1) and let (K,Q) be an admissible pair for this system such that Q is compact and controlled invariant. Furthermore, assume that $h_{\text{inv}}(Q) < \infty$.

Following [19] we can associate with the set Q a vector bundle of rank d, called **the extended tangent bundle over $\mathcal{Q}$**, given by:

$$\pi_{\mathcal{Q}} : \bigcup_{(u,x)\in\mathcal{Q}} \{u\} \times T_x M \to \mathcal{Q}, \quad \pi_{\mathcal{Q}}(u,v) = (u, \pi_{TM}(v)), \tag{1.8}$$

where $\pi_{TM} : TM \to M$ is the map sending a tangent vector $v \in T_x M$ to its base point x. The topology considered on $\mathcal{U}$ is the relative topology of $L^\infty(\mathbb{R}, \mathbb{R}^m)$, which turns $\mathcal{Q} \subset \mathcal{U} \times M$ into a metrizable topological space.

By $\pi_{\mathcal{U}} : \mathcal{U} \times M \to \mathcal{U}$ we denote the projection onto the first factor, $\pi_{\mathcal{U}}(u,x) = u$.

We define the **lift of K inside Q** as

$$\mathcal{K} := \{(u,x) \in \mathcal{Q}; \;\; x \in K\}.$$

Moreover, for each $u \in \pi_{\mathcal{U}}\mathcal{K}$ we define the nonempty compact sets

$$K(u,\tau) := \{x \in K; \;\; \varphi_{t,u}(x) \in Q \text{ for all } t \in [0,\tau]\}, \quad \tau > 0.$$

For each $u \in \mathcal{U}$ and $\tau > 0$ the *Bowen-metric* is defined by

$$\varrho_{\tau,u}(x,y) := \max_{t\in[0,\tau]} \varrho(\varphi_{t,u}(x), \varphi_{t,u}(y)).$$

For each $(u,x) \in \mathcal{U} \times M$ and $\tau, \varepsilon > 0$, the *Bowen-ball of order* τ and radius ε centered at $x \in M$, is denoted by

$$B^\tau_\varepsilon(u,x) = \{y \in M; \;\; \varrho_{\tau,u}(x,y) < \varepsilon\}.$$

A set $S \subset M$ is called (u,τ,ε)-separated if for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ one has $\varrho_{\tau,u}(x_1,x_2) \geq \varepsilon$. By $r_{\text{sep}}(u,\tau,\varepsilon,K,Q)$ we denote the maximal cardinality of an (u,τ,ε)-separated subset of $K(u,\tau)$. We say that a set $D \subset M$ (u,τ,ε)-spans another set $E \subset M$ if for every $x \in E$ there is $y \in D$ such that $\varrho_{\tau,u}(x,y) < \varepsilon$. By $r_{\text{span}}(u,\tau,\varepsilon,K,Q)$ we denote the minimal cardinality of a set which (u,τ,ε)-spans $K(u,\tau)$. It is easy to see that a maximal (u,τ,ε)-separated subset S of $K(u,\tau)$ also (u,τ,ε)-spans $K(u,\tau)$ and hence

$$K(u,\tau) \subset \bigcup_{x\in S} B^\tau_\varepsilon(u,x).$$

Definition 1.2.11 *The* **escape entropy of** (K, Q) *is defined as follows:*

$$\bar{r}_{\text{sep}}(\tau, \varepsilon, K, Q) := \sup_{u \in \pi_{\mathcal{U}} \mathcal{K}} r_{\text{sep}}(u, \tau, \varepsilon, K, Q),$$

$$h_{\text{esc}}(\varepsilon, K, Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log \bar{r}_{\text{sep}}(\tau, \varepsilon, K, Q),$$

$$h_{\text{esc}}(K, Q) := \lim_{\varepsilon \searrow 0} h_{\text{esc}}(\varepsilon, K, Q).$$

The next Proposition state the main properties of the escape entropy. It is a collection of the principal results about the escape entropy and the proofs can be found in the Chapter 6 of [21].

Proposition 1.2.12 *The following assertions hold:*

(i) For all $\tau, \varepsilon > 0$ *and* $u \in \mathcal{U}$ *it holds that*

$$r_{\text{span}}(u, \tau, \varepsilon, K, Q) \leq r_{\text{sep}}(u, \tau, \varepsilon, K, Q) \leq r_{\text{span}}(u, \tau, \frac{\varepsilon}{2}, K, Q) < \infty;$$

(ii) It holds that

$$h_{\text{esc}}(K, Q) = \lim_{\varepsilon \searrow 0} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \bar{r}_{\text{span}}(\tau, \varepsilon, K, Q),$$

where $\bar{r}_{\text{span}}(\tau, \varepsilon, K, Q) := \sup_{u \in \pi_{\mathcal{U}} \mathcal{K}} r_{\text{span}}(u, \tau, \varepsilon, K, Q)$*;*

(iii) $h_{\text{esc}}(K, Q) \in [0, \infty)$*;*

(iv) $h_{\text{esc}}(K, Q)$ *is invariant under* $\mathcal{C}^0$*-state equivalence, and hence metric-independent.*

Remark 1.2.13 *In [21] we found two definitions for the escape entropy. The above is just used for uniformly hyperbolic sets. Since we are interested in calculate lower bounds for the invariance entropy on induced flag manifolds where we do have that the chain control sets are uniformly hyperbolic sets, the definition above is the best choice.*

Let us assume that Q is uniformly hyperbolic and consider the subbundle $E^+ \to \mathcal{Q}$ whose fibers are $E^+_{u,x}$. The following result (Theorem 6.2 of [21]) give us a lower bound for the invariance entropy in the case where hyperbolicity holds.

Theorem 1.2.14 *Assume that the vector fields* $f_0, f_1, \ldots, f_m$ *are of class* $\mathcal{C}^2$ *and let* Q *be a uniformly hyperbolic set that satisfies* $h_{\mathrm{inv}}(Q) < \infty$*. Then for each compact set* $K \subset Q$ *of positive volume we have*

$$h_{\mathrm{inv}}(K,Q) \geq \inf_{(u,x)\in\mathcal{Q}} \limsup_{\tau\to\infty} \frac{1}{\tau} \log\left|\det(d\varphi_{\tau,u})_{|E^+_{u,x}}\right| - h_{\mathrm{esc}}(K,Q).$$

The main goal is try to get rid of the escape entropy, in order to get a lower bound just in terms of the positive eigenvalues of $(d\varphi_{t,u})_x$. In this direction we have the following results.

Definition 1.2.15 *Fix a metric* ϱ *of the state space* M *of the system (1.1) and let* $K \subset Q \subset M$ *be nonemtpy sets. Let* $x_1, x_2 \in K$*,* $u \in \mathcal{U}$ *and* $\tau \geq 0$ *with* $\varphi_{t,u}(x_i) \in Q$ *for all* $t \in [0,\tau]$ *and* $i = 1, 2$*. We say that (1.1) restricted to* K *is:*

(i) **uniformly expanding inside** Q *if there are constants* $c, \mu > 0$ *such that*
$$\varrho(\varphi_{\tau,u}(x_1), \varphi_{\tau,u}(x_2)) \geq c\mathrm{e}^{\mu\tau}\varrho(x_1, x_2);$$

(ii) **uniformly contracting inside** Q *if there are constants* $c, \mu > 0$ *such that*
$$\varrho(\varphi_{\tau,u}(x_1), \varphi_{\tau,u}(x_2)) \leq c^{-1}\mathrm{e}^{-\mu\tau}\varrho(x_1, x_2).$$

Remark 1.2.16 *If* $K = Q$ *in the above Definition, we say that the system is* **uniformly contracting or expanding on** Q*.*

Proposition 1.2.17 *Consider the control system (1.1) on* M *and let* (K,Q) *be an admissible pair for it with* Q *compact and controlled invariant. If the system is uniformly contracting or uniformly expanding on* Q*, then* $h_{\mathrm{esc}}(K,Q) = 0$*.*

Proof. The proof for uniformly expanding systems is due to Kawan and can be found in [19] Proposition 7.4.

Let us then assume that the system is uniformly contracting. For $x \in Q$ and $\tau, \varepsilon > 0$, if $y \in B_{c\varepsilon}(x)$ and $t \in [0,\tau]$ we have

$$\varrho(\varphi_{t,u}(x), \varphi_{t,u}(y)) \leq c^{-1}\mathrm{e}^{-\mu t}\varrho(x,y) < \varepsilon$$

what show us that $B_{c\varepsilon}(x) \subset B_\varepsilon^\tau(u,x)$. If we denote by $N = N(\varepsilon)$ the minimal number of $c\varepsilon$-balls necessary to cover K we have that

$$K(u,\tau) \subset \bigcup_{i=1}^{N} B_{c\varepsilon}(x_i) \subset \bigcup_{i=1}^{N} B_\varepsilon^\tau(u,x_i)$$

for some $x_1, \dots x_N \in K$. The set $S = \{x_1, \dots, x_N\}$ is in particular a (u,τ,ε)-spanning set for $K(u,\tau)$ which give us

$$\bar{r}_{\text{span}}(\tau,\varepsilon,K,Q) \leq N(\varepsilon)$$

and consequently

$$h_{\text{esc}}(K,Q) \leq 0.$$

■

Since we do not always have uniformly expanding or contracting on the whole set Q we will look at the projection between manifolds in order to compare their escape entropy. Such idea will be central when we specialize our calculations to the flag manifolds.

Theorem 1.2.18 *Let $\dot{x}_i(t) = F_i(x_i(t), u(t))$, $i = 1,2$ be two control systems with the same set of admissible functions $\mathcal{U}$ and π be a continuous map from M_1 onto M_2 such that $(\pi, \text{id}_{\mathcal{U}})$ is a semi-conjugacy between them.*

Let (K_1, Q_1) be an admissible pair for $\dot{x}_1(t) = F_1(x_1(t), u(t))$, with Q_1 being compact and controlled invariant and consider (K_2, Q_2) be the admissible pair for $\dot{x}_2(t) = F_2(x_2(t), u(t))$ given by the projection of (K_1, Q_1). We have then:

(i) If for any $(u,x) \in \mathcal{K}_2$ there exists $z \in \pi^{-1}(x)$ such that $(u,z) \in \mathcal{K}_1$ then

$$h_{\text{esc}}(K_1,Q_1) \geq h_{\text{esc}}(K_2,Q_2);$$

(ii) If for every $u \in \pi_{\mathcal{U}}\mathcal{K}_1$ and $\tau > 0$ we have that $\varphi_{\tau,u}$ restricted to the subset of the fibers $Q_y := Q_1 \cap \pi^{-1}(y)$ for $y \in K_2$ is uniformly expanding or contracting inside Q, then

$$h_{\text{esc}}(K_1,Q_1) \leq h_{\text{esc}}(K_2,Q_2).$$

Proof. By the semiconjugation property above, we have that

$$\pi_{\mathcal{U}}\mathcal{K}_1 \subset \pi_{\mathcal{U}}\mathcal{K}_2$$

and property (i) implies that $\pi_{\mathcal{U}}\mathcal{K}_1 = \pi_{\mathcal{U}}\mathcal{K}_2$ and

$$\pi(K_1(u,\tau)) = K_2(u,\tau), \qquad \text{for } u \in \pi_{\mathcal{U}}\mathcal{K}_1,\ \tau > 0.$$

Let then $\tau, \varepsilon > 0$ and $u \in \pi_{\mathcal{U}}\mathcal{K}_1$. Since π is uniformly continuous on K_1, there exists $\delta = \delta(\varepsilon)$ such that

$$\varrho_2(\pi(x), \pi(y)) < \varepsilon \quad \text{if} \quad \varrho_1(x,y) < \delta.$$

Using the above and the semiconjugation property, we have that if S is a (u,τ,δ)-spanning set for $K_1(u,\tau)$, its projection $\pi(S)$, is a (u,τ,ε)-spanning set for $\pi(K_1(u,\tau)) = K_2(u,\tau)$ and then

$$r_{\text{span}}(u,\tau,\delta,K_1,Q_1) \geq r_{\text{span}}(u,\tau,\varepsilon,K_2,Q_2).$$

Since $\pi_{\mathcal{U}}\mathcal{K}_1 = \pi_{\mathcal{U}}\mathcal{K}_2$ we have

$$\bar{r}_{\text{span}}(\tau,\delta,K_1,Q_1) \geq \bar{r}_{\text{span}}(\tau,\varepsilon,K_2,Q_2).$$

which implies

$$h_{\text{esc}}(K_1,Q_1) \geq h_{\text{esc}}(K_2,Q_2)$$

and it shows (i).

For item (ii) denote by K_y the intersection $\pi^{-1}(y) \cap K_1$. Consider $\tau, \varepsilon > 0$ and $u \in \pi_{\mathcal{U}}\mathcal{K}_1$ and for each $y \in K_2$ let S_y be an (u,τ,ε)-spanning set for $K_y(u,\tau) = \{z \in K_y; \varphi_{t,u}(z) \in Q_1,\ t \in [0,\tau]\}$ with the minimum number of members. Then

$$U_y = \bigcup_{z \in S_y} B_\varepsilon^\tau(u,z)$$

is an open neighborhood of $K_y(u,\tau)$. Now

$$K_1(u,\tau) \setminus U_y \cap \bigcap_{\gamma>0} \pi^{-1}(\overline{B_\gamma(y)}) = \emptyset.$$

By the finite intersection property for compact sets, there is $W_y = B_\gamma(y)$ for which $U_y \supset \pi^{-1}(\overline{W}_y)$. Let $W_{y_1}, \ldots W_{y_r}$ cover K_2 and let $\delta > 0$ be a Lebesgue number for K_2 for this open cover. Let then S be an (u, τ, δ)-spanning set for $K_2(u, \tau)$ with minimal number of elements. For $x \in S$ let us denote by $y(x)$ the element in $y_1, \dots y_k$ such that

$$B_\delta^\tau(u, x) \subset W_{y(x)}$$

that always exist, since δ is the Lebesgue number of $\left\{W_{y_i}\right\}_{i=1}^r$ and $B_\delta^\tau(u, x) \subset B_\delta(x)$. We claim that

$$K_1(u, \tau) \subset \bigcup_{x \in S} \bigcup_{z \in S_{y(x)}} B_\varepsilon^\tau(u, z).$$

In fact, since $\pi(K_1(u, \tau)) \subset K_2(u, \tau)$ for every $u \in \pi_{\mathcal{U}} \mathcal{K}_1$, we have for $y \in K_1(u, \tau)$ that $\pi(y) \in B_\delta^\tau(u, x)$ for some $x \in S$ and then $y \in \pi^{-1}(\pi(y)) \subset U_{y(x)}$ what give us $y \in B_\varepsilon^\tau(u, z)$ for some $z \in S_{y(x)}$.

Then the set $\cup_{x \in S} S_{y(x)}$ is a (u, τ, ε)-spanning set for $K_1(u, \tau)$ what give us

$$r_{\text{span}}(u, \tau, \varepsilon, K_1, Q_1) \leq r_{\text{span}}(u, \tau, \delta, K_2, Q_2) \cdot \max_{x \in S} r_{\text{span}}(u, \tau, \varepsilon, K_{y(x)}, Q_1).$$

For the uniformly contracting case, an analogous analysis as the one made in Proposition 1.2.17 above allow us to conclude that the number $r_{\text{span}}(u, \tau, \varepsilon, K_y, Q_1)$ is bounded above by a constant that depends just on ε for any $y \in K_2$, what give us

$$r_{\text{span}}(u, \tau, \varepsilon, K_1, Q_1) \leq C_1(\varepsilon) \cdot r_{\text{span}}(u, \tau, \delta, K_2, Q_2)$$

For the uniformly expanding case we have also the same and the proof is analogous of the proof of Proposition 7.4 of [21] as follows: Let $S_y \subset K_y$ be an (u, τ, ε)-separated set for $K_y(u, \tau)$. Let $x_1, x_2 \in S_y$ with $x_1 \neq x_2$ and let $s = s(x_1, x_2) \in [0, \tau]$ such that $\varrho(\varphi_{s,u}(x_1), \varphi_{s,u}(x_2)) = \varrho_{u,\tau}(x_1, x_2)$. Using the cocycle property of φ, we find that

$$\begin{aligned}
&\varrho(\varphi_{\tau,u}(x_1), \varphi_{\tau,u}(x_2)) \\
&\quad = \varrho(\varphi(\tau - s, \varphi(s, x_1, u), \theta_s u), \varphi(\tau - s, \varphi(s, x_2, u), \theta_s u)) \\
&\quad \geq c e^{\mu(\tau - s)} \varrho(\varphi_{s,u}(x_1), \varphi_{s,u}(x_2)) \\
&\quad \geq c e^{\mu(\tau - s)} \geq c\varepsilon.
\end{aligned}$$

Hence, the set $\varphi_{\tau,u}(S_y)$ is a $c\varepsilon$-separated subset of Q_1 with the same cardinality as S_y. By compactness we can cover Q_1 with finitely many balls $B_\eta(x_i)$, $i = 1, \dots n$ of a fixed radius $\eta > 0$ such that

$$\exp^{-1}(B_\eta(x_i)) = B_\eta(0_{x_i})$$

and

$$\varrho(\exp_{x_i}(v), \exp_{x_i}(w)) \leq 2|v - w|, \quad \text{for all } \ v, w \in B_\eta(0_{x_i}) \tag{1.9}$$

that is possible since $(d\exp_x)_{0_x} = \mathrm{id}_{T_xM}$ for all $x \in M$. Then

$$\#\varphi_{\tau,u}(S_y) \leq \sum_{i=1}^{n} \#(\varphi_{\tau,u}(S_y) \cap B_\varepsilon(x_i))$$

$$n \max_{1 \leq i \leq n} \# \exp_{x_i}^{-1}(\varphi_{\tau,u}(S_y) \cap B_\delta(x_i)).$$

Set $N_i = \# \exp_{x_i}^{-1}(\varphi_{\tau,u}(S_y) \cap B_\varepsilon(x_i))$. By (1.9) the set $\exp_{x_i}^{-1}(\varphi_{\tau,u}(S_y) \cap B_\eta(x_i))$ is a $c\varepsilon/2$-separated subset of $B_\eta(0_{x_i})$ and so, $B(x_i, \eta + (c\varepsilon)/4)$ contains N_i disjoint balls of radii $(c\varepsilon)/4$. Letting $d = \dim M_1$, this implies

$$\left(\eta + \frac{c\varepsilon}{4}\right)^d \geq N_i \left(\frac{c\varepsilon}{4}\right)^d \qquad \Rightarrow \qquad N_i \leq \left(\frac{4\eta + c\varepsilon}{c\varepsilon}\right)^d$$

which give us

$$\#S_y = \#\varphi_{\tau,u}(S_y) \leq n \left(\frac{4\eta + c\varepsilon}{c\varepsilon}\right)^d$$

and therefore we obtain for the expanding case, using Proposition 1.2.12,

$$r_{\mathrm{span}}(u, \tau, \varepsilon, K_1, Q_1) \leq C_2(\varepsilon) \cdot r_{\mathrm{span}}(u, \tau, \delta/2, K_2, Q_2)$$

for $C_2(\varepsilon) = n\left(\frac{4\eta + c\varepsilon}{c\varepsilon}\right)^d$. Then, in both cases we have

$$\bar{r}_{\mathrm{span}}(\tau, \varepsilon, K_1, Q_1) \leq C(\varepsilon) \cdot \bar{r}_{\mathrm{span}}(\tau, \delta/2, K_2, Q_2).$$

Applying log, dividing by τ and taking the $\limsup$ give us

$$h_{\mathrm{esc}}(\varepsilon, K_1, Q_1) \leq h_{\mathrm{esc}}(\delta/2, K_2, Q_2) \leq h_{\mathrm{esc}}(K_2, Q_2)$$

and consequently

$$h_{\mathrm{esc}}(K_1, Q_1) \leq h_{\mathrm{esc}}(K_2, Q_2)$$

as desired. ■

Chapter 2

Control Sets and Morse Sets and their Flag type

The aim of this chapter is give some notions of the semigroup theory for semi-simple Lie groups applied to control theory. The notion of flag type of a semigroup and flag type of a flow are introduced and is shown that they are closely connected with the control and Morse sets on the flag manifolds and flag bundles. At the end of the section a vectorial cocycle associated with the Iwasawa decomposition of the group is defined. Such cocycle normally measures exponential growth of the associated flow and it will be important in order to estimate the entropy.

2.1 Semi-simple Theory

We refer to Duistermat-Kolk-Varadarajan [15], Helgason [17], Knapp [22] and Warner [34] for the theory of semi-simple Lie groups and their flag manifolds. In order to set notation let G be a connected noncompact semi-simple Lie group with finite center and Lie algebra $\mathfrak{g}$. Fix a Cartan involution θ of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Associated with the Cartan involution we have the inner product $B_\theta(X, Y) = -\langle X, \theta Y \rangle$, where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form of $\mathfrak{g}$.

For a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ fixed, we denote by Π the set of roots of $\mathfrak{a}$, Π^+ the positive roots corresponding to $\mathfrak{a}$, Σ the set of simple roots in Π^+ and $\Pi^- = -\Pi^+$ the negative roots. For a given root $\alpha \in \Pi$ we denote by $H_\alpha \in \mathfrak{a}$ its coroot so that $B_\theta(H_\alpha, H) = \alpha(H)$

for all $H \in \mathfrak{a}$. The standard Iwasawa decompositions of the Lie algebra $\mathfrak{g}$ associated with this choice of maximal abelian and the Weyl chamber are given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^{\pm}$ where $\mathfrak{n}^{\pm} = \sum_{\alpha \in \Pi^{\pm}} \mathfrak{g}_{\alpha}$ and $\mathfrak{g}_{\alpha}$ is the root space associated to α. As for the global decomposition of the group we write $G = KS$ and $G = KAN^{\pm}$ where K, A and $N^{\pm}$ are the connected subgroups whose Lie algebras are $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}^{\pm}$, respectively.

Let $\mathcal{W}$ be the Weyl group of G. It is constructed either as the subgroup of reflections generated by the roots of $(\mathfrak{g}, \mathfrak{a})$ or as the quotient M^{*}/M where M^{*} and M are respectively the normalizer and the centralizer of $\mathfrak{a}$ in K. There is an unique element $w_0 \in \mathcal{W}$ which take the positive roots Π^{+} to Π^{-}. Such element is called the principal involution of $\mathcal{W}$.

Associated with $\Theta \subset \Sigma$ there are several Lie algebras and groups (cf. [34], Section 1.2.4). We will denote by $\mathfrak{g}(\Theta)$ the semi-simple Lie subalgebra generated by $\mathfrak{g}_{\alpha}$, $\alpha \in \Theta$, and put $\mathfrak{k}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{k}$, $\mathfrak{a}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{a}$, and $\mathfrak{n}^{\pm}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{n}^{\pm}$. The simple roots of $\mathfrak{g}(\Theta)$ are given by Θ, more precisely, by the restriction of the functionals of Θ to $\mathfrak{a}(\Theta)$. The coroots $H_{\alpha}, \alpha \in \Theta$, form a basis for $\mathfrak{a}(\Theta)$. Let $G(\Theta)$ and $K(\Theta)$ be the connected Lie groups with Lie algebras $\mathfrak{g}(\Theta)$ and $\mathfrak{k}(\Theta)$, respectively. Then $G(\Theta)$ is a connected semi-simple Lie group with finite center. Let $A(\Theta) = \exp \mathfrak{a}(\Theta)$, $N^{\pm}(\Theta) = \exp \mathfrak{n}^{\pm}(\Theta)$. We have the Iwasawa decomposition $G(\Theta) = K(\Theta)A(\Theta)N^{\pm}(\Theta)$. Let $\mathfrak{a}_{\Theta} = \{H \in \mathfrak{a};\ \alpha(H) = 0 \text{ for all } \alpha \in \Theta\}$ be the orthogonal complement of $\mathfrak{a}(\Theta)$ in $\mathfrak{a}$ with respect to B_{θ} and put $A_{\Theta} = \exp \mathfrak{a}_{\Theta}$. The subset Θ singles out the subgroup $\mathcal{W}_{\Theta}$ of the Weyl group which acts trivially on $\mathfrak{a}_{\Theta}$. Alternatively $\mathcal{W}_{\Theta}$ can be given as the subgroup generated by the reflections with respect to the roots $\alpha \in \Theta$. The restriction of $w \in \mathcal{W}_{\Theta}$ to $\mathfrak{a}(\Theta)$ furnishes an isomorphism between $\mathcal{W}_{\Theta}$ and the Weyl group $\mathcal{W}(\Theta)$ of $G(\Theta)$.

The standard parabolic subalgebra of type $\Theta \subset \Sigma$ with respect to the chamber $\mathfrak{a}^{+}$ is defined by

$$\mathfrak{p}_{\Theta} = \mathfrak{n}^{-}(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^{+}$$

where $\mathfrak{m}$ is the Lie algebra of M. The corresponding standard parabolic subgroup P_{Θ} is the normalizer of $\mathfrak{p}_{\Theta}$ in G. It has the Langlands decomposition $P_{\Theta} = K_{\Theta}AN^{+}$. The empty set $\Theta = \emptyset$ gives the minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^{+}$ whose minimal parabolic subgroup $P = P_{\emptyset}$ decompose as $P = MAN^{+}$.

We let Z_{Θ} be the centralizer of $\mathfrak{a}_{\Theta}$ in G and $K_{\Theta} = Z_{\Theta} \cap K$. We have that K_{Θ} decomposes as $K_{\Theta} = MK(\Theta)$ and Z_{Θ} decomposes as $Z_{\Theta} = MG(\Theta)A_{\Theta}$

which implies that $Z_\Theta = K_\Theta AN(\Theta)$ is an Iwasawa decomposition of Z_Θ (which is a reductive Lie group). Let $\Theta_1, \Theta_2 \subset \Sigma$, then $\mathfrak{a}_{\Theta_1\cap\Theta_2} = \mathfrak{a}_{\Theta_1} + \mathfrak{a}_{\Theta_2}$. Thus it follows that $Z_{\Theta_1\cap\Theta_2} = Z_{\Theta_1} \cap Z_{\Theta_2}$, $K_{\Theta_1\cap\Theta_2} = K_{\Theta_1} \cap K_{\Theta_2}$ and $P_{\Theta_1\cap\Theta_2} = P_{\Theta_1} \cap P_{\Theta_2}$.

For $H \in \mathfrak{a}$ we denote by Z_H, $\mathcal{W}_H$, etc. The centralizer of H, respectively, in G, $\mathcal{W}$, etc., except when explicitly noted. When $H \in \mathrm{cl}\,\mathfrak{a}^+$ we put

$$\Theta(H) = \{\alpha \in \Sigma;\ \alpha(H) = 0\},$$

and we have $Z_H = Z_{\Theta(H)}$, $K_H = K_{\Theta(H)}$, $N^+(H) = N^+(\Theta(H))$ and $\mathcal{W}_H = \mathcal{W}_{\Theta(H)}$.

Let $\mathfrak{n}_\Theta^\pm = \sum_{\alpha\in\Pi^\pm\setminus\langle\Theta\rangle} \mathfrak{g}_\alpha$ and $N_\Theta^\pm = \exp \mathfrak{n}_\Theta^\pm$ where $\langle\Theta\rangle$ is given by all root in Π that is linear combination of the roots in Θ. Then $N^\pm$ decomposes as $N^\pm = N^\pm(\Theta)N_\Theta^\pm$ where $N^\pm(\Theta)$ normalizes $N_\Theta^\pm$, $N^\pm(\Theta)$ centralizes $N_\Theta^\mp$ and $N^\pm(\Theta) \cap N_\Theta^\pm = 1$. We have that $\mathfrak{g} = \mathfrak{n}_\Theta^- \oplus \mathfrak{p}_\Theta$, $N_\Theta^- \cap P_\Theta = 1$ and P_Θ is the normalizer of $\mathfrak{n}_\Theta^+$ in G. The subgroup P_Θ decomposes as $P_\Theta = Z_\Theta N_\Theta^+$, where Z_Θ normalizes N_Θ^+ and $Z_\Theta \cap N_\Theta^+ = 1$. We write $\mathfrak{p}_\Theta^- = \Theta(\mathfrak{p}_\Theta)$ for the parabolic subalgebra opposed to $\mathfrak{p}_\Theta$. It is conjugated to the parabolic subalgebra $\mathfrak{p}_{\Theta^*}$ where $\Theta^* = -(w_0\Theta)$ is the dual to Θ and w_0 is the principal involution of $\mathcal{W}$. More precisely $\mathfrak{p}_\Theta^- = k\mathfrak{p}_{\Theta^*}$ where $k \in M^*$ is a representative of w_0 in M^*. If P_Θ^- is the parabolic subgroup associated to $\mathfrak{p}_\Theta^-$ then $Z_\Theta = P_\Theta \cap P_\Theta^-$ and $P_\Theta^- = Z_\Theta N_\Theta^-$, where Z_Θ normalizes N_Θ^- and $Z_\Theta \cap N_\Theta^- = 1$.

The flag manifold of type Θ is the orbit $\mathbb{F}_\Theta = \mathrm{Ad}(G)\mathfrak{p}_\Theta$ on the Grassmann manifolds of subspaces of the Lie algebra $\mathfrak{g}$, with base point $b_\Theta = \mathfrak{p}_\Theta$, which identifies with the homogeneous space G/P_Θ. Since the center of G normalizes $\mathfrak{p}_\Theta$, the flag manifolds depends only on the Lie algebra $\mathfrak{g}$ of G. The empty set $\Theta = \emptyset$ gives the maximal flag manifold $\mathbb{F} = \mathbb{F}_\emptyset$ with base point $b_0 = b_\emptyset$. For $\Theta_1 \subset \Theta_2 \subset \Sigma$ there is a G-equivariant projection $\pi_{\Theta_2}^{\Theta_1} : \mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ given by $gb_{\Theta_1} \mapsto gb_{\Theta_2}, g \in G$. When $\Theta_1 = \emptyset$ we denote this fibration just by π_{Θ_2}.

The above subalgebras of $\mathfrak{g}$, which are defined by the choice of the Weyl chamber of $\mathfrak{a}$ and a subset of the associated simple roots, can be defined alternatively by the choice of an element $H \in \mathfrak{a}$ as follows. First note that the eigenspaces of $\mathrm{ad}(H)$ in $\mathfrak{g}$ are the weight spaces $\mathfrak{g}_\alpha$, and the centralizer of $H \in \mathfrak{g}$ is given by $\mathfrak{z}_H = \sum\{\mathfrak{g}_\alpha;\ \alpha(H) = 0\}$ where the sum is taken over $\alpha \in \mathfrak{a}^*$. Now define the negative and positive nilpotent subalgebras of type H given by

$$\mathfrak{n}_H^+ = \sum\{\mathfrak{g}_\alpha;\ \alpha(H) > 0\}, \qquad \mathfrak{n}_H^- = \sum\{\mathfrak{g}_\alpha;\ \alpha(H) < 0\}$$

and the parabolic subalgebra of type H which is given by

$$\mathfrak{p}_H = \sum \{\mathfrak{g}_\alpha;\ \alpha(H) \geq 0\}.$$

Denote by $N_H^\pm = \exp \mathfrak{n}_H^\pm$ and by P_H the normalizer in G of $\mathfrak{p}_H$. Note that $\mathfrak{n}_H^\pm, \mathfrak{p}_H, N_H^\pm$ and P_H are not centralizers of H: These are the only exceptions for the centralizer notation introduced above. We have clearly that

$$\mathfrak{g} = \mathfrak{n}_H^- \oplus \mathfrak{z}_H \oplus \mathfrak{n}_H^+ \qquad \text{and} \qquad \mathfrak{p}_H = \mathfrak{z}_H \oplus \mathfrak{n}_H^+.$$

Define the flag manifold of type H as the orbit on the Grassmann manifolds of subspaces of the Lie algebra $\mathfrak{g}$ given by

$$\mathbb{F}_H = \mathrm{Ad}(G)\mathfrak{p}_H.$$

Now choose a chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ which contains H in its closure, consider the simple roots Σ associated to $\mathfrak{a}^+$ and take $\Theta(H) \subset \Sigma$. Since $\alpha \in \Theta(H)$ if, and only if, $\alpha_{|\mathfrak{a}_{\Theta(H)}} = 0$, we have that

$$\mathfrak{z}_{\Theta(H)} = \mathfrak{z}_H, \qquad \mathfrak{n}_{\Theta(H)}^\pm = \mathfrak{n}_H^\pm, \qquad \mathfrak{p}_{\Theta(H)} = \mathfrak{p}_H.$$

So it follows that

$$\mathbb{F}_H = \mathbb{F}_{\Theta(H)},$$

and that the isotropy of G in $\mathfrak{p}_H$ is

$$P_H = P_{\Theta(H)} = K_{\Theta(H)}AN^+ = K_H AN^+,$$

since $K_{\Theta(H)} = K_H$. We note that we can proceed reciprocaly, that is, if $\mathfrak{a}^+$ and Θ are given, we can choose $H \in \mathrm{cl}\,\mathfrak{a}^+$ such that $\Theta(H) = \Theta$ and describe the objects that depend on $\mathfrak{a}^+$ and Θ by H (clearly such an H is not unique). We remark that the map

$$\mathbb{F}_H \to \mathfrak{s}, \qquad k\mathfrak{p}_H \mapsto \mathrm{Ad}(k)H, \quad \text{for } k \in K,$$

gives an embedding of $\mathbb{F}_H$ in $\mathfrak{s}$ (see Proposition 2.1 of [15]). In fact, the isotropy of K at H is $K_H = K_{\Theta(H)}$ which is, by the above comments, the isotropy of K at $\mathfrak{p}_H$. Define the negative parabolic subalgebra of type H by

$$\mathfrak{p}_H^- = \sum \{\mathfrak{g}_\alpha;\ \alpha(H) \leq 0\}$$

and denote by P_H^- its normalizer in G. Then we have that $P_H^- = P_{\Theta(H)}^-$.

An element $Y = \mathrm{Ad}(g)H$, $H \in \mathrm{cl}\,\mathfrak{a}^+$ is said to be a **split** element. In the same way we call an element of the form ghg^{-1} split if $h \in \mathrm{cl}\,A^+$, where $A^+ = \exp \mathfrak{a}^+$.

An split element $H \in \mathrm{cl}\,\mathfrak{a}^+$ induces a vector field $\tilde{H}$ on a flag manifold $\mathbb{F}_\Theta$ with flow $\exp tH$. This is a gradient vector field with respect to a given Riemannian metric on $\mathbb{F}_\Theta$ (see [15], Section 3). The connected sets of fixed point of this flow are given by

$$\mathrm{fix}_\Theta(H, w) = Z_H w b_\Theta = K_H w b_\Theta,$$

so that they are in bijection with the cosets in $\mathcal{W}_H \setminus \mathcal{W} / \mathcal{W}_\Theta$. In particular, if $H \in \mathfrak{a}^+$ is regular then there are $|\mathcal{W}|/|\mathcal{W}_\Theta|$ isolated singularities. Each w-fixed point connected set has stable manifold given by

$$\mathrm{st}_\Theta(H, w) = N_H^- \mathrm{fix}_\Theta(H, w) = P_H^- w b_\Theta,$$

whose union gives the Bruhat decomposition of $\mathbb{F}_\Theta$:

$$\mathbb{F}_\Theta = \coprod_{\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_\Theta} \mathrm{st}_\Theta(H, w) = \coprod_{\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_\Theta} P_H^- w b_\Theta.$$

The unstable manifold is

$$\mathrm{un}_\Theta(H, w) = N_H^+ \mathrm{fix}_\Theta(H, w) = P_H w b_\Theta.$$

Remark 2.1.1 *For $h \in \mathrm{cl}\,A^+$ we will write $\mathrm{fix}_\Theta(h, w)$ to denote the set of fixed points of $\mathrm{fix}_\Theta(H, w)$ where $h = \exp H$, $H \in \mathrm{cl}\,\mathfrak{a}^+$.*

For each element $w \in \mathcal{W}$ there exist simple roots $\alpha_i \in \Sigma$, $i = 1, \ldots, n$ such that $w = s_1 \cdots s_n$ where s_i are the simple reflection associated to α_i. The **length** $l(w)$ of w is the number of simple roots in a minimal decomposition of w as above. Let $\Pi_w = \Pi^+ \cap w\Pi^-$ be the set of positive roots that are taken to negative ones by w^{-1}. It is a fact that $l(w)$ is equal the cardinality of Π_w.

Also, if $w \in \mathcal{W}$ and $\alpha \in \Sigma$ is a simple root, holds

$$l(ws_\alpha) = l(w) + 1 \quad \text{if and only if ,} \quad w(\alpha) \in \Pi^+$$

and

$$l(ws_\alpha) = l(w) - 1 \quad \text{if and only if ,} \quad w(\alpha) \in \Pi^-$$

For a fixed simple system Σ of the roots, the **Bruhat-Chevalley order** of the Weyl group $\mathcal{W}$ is given as follows. For $w \in \mathcal{W}$ take a minimal decomposition $w = s_1 \cdots s_n$ as product of simple roots. Then $w_1 \leq w$ if and only if there are integers $1 \leq i_1 < \ldots < i_k \leq n$ such that $w_1 = s_{i_1} \cdots s_{i_k}$ is a minimal decomposition of w_1.

In general the order of $\mathcal{W}$ depends on the choice of the simple system of roots Σ, that is, on the set of generators of $\mathcal{W}$. Note however that the order obtained from $-\Sigma$ coincides with the order coming from Σ because both simple systems of roots define the same set of generators.

The Bruhat-Chevalley order are associated with the order of control sets on the flag manifolds (see [29]).

2.2 Control Sets for Semigroups Actions

In this section we will introduce the notion of control set via semigroup theory that is the best approach to work in flag manifolds. The notion of control sets, as defined in Chapter 1, for control systems on flag manifolds coincide with the one here stated and so we will use both approaches.

Let S be a semigroup of diffeomorphisms acting on a Riemannian manifold X. We say that S is accessible in $x \in X$ if $\text{int}(Sx) \neq \emptyset$. If the semigroup S is accessible for every $x \in X$ we say that S is accessible.

Definition 2.2.1 *A subset $D \subset X$ is said to be a* **control set** *for the action of S provided it satisfies*

1. $D \subset \text{cl}(Sx)$ *for every* $x \in D$*;*

2. $\text{int}D \neq \emptyset$*;*

3. *D is maximal with these properties.*

The control sets for S are ordered by putting $D_1 \leq D_2$ if there are $x \in D_1$ and $g \in S$ such that $gx \in D_2$. Equivalently, $D_1 \leq D_2$ if $D_2 \subset \text{cl}(Sx)$ for some, and then for all, $x \in D_1$.

For a given control set D the set $D_0 = \{x \in D; x \in \text{int}(Sx) \cap \text{int}(S^{-1}x)\}$. is called the **set of transitivity** of D or **core** of D. Such set can be empty but when it is not the control set D is said to be **effective**. The invariant control sets are effective, that is, $D_0 \neq \emptyset$ if $SD \subset D$ or $S^{-1}D \subset D$.

We consider now G to be a semi-simple Lie group and $S \subset G$ a semigroup with $\operatorname{int} S \neq \emptyset$. The concept of flag type of a semigroup come from the characterization of the effective control sets in $\mathbb{F}_\Theta$. The demonstration of the results in this section can be found in [31].

Let us consider an Iwasawa decomposition $G = KAN^+$ and let $\mathcal{W}$ the Weyl group of G. In order to describe the effective control sets and define the flag type of S, let us consider the set of the split-regular elements of G that are in the interior of S, that is

$$\mathcal{R}(S) = \{h \in \operatorname{int} S; h = g\bar{h}g^{-1}, \text{ for } g \in G, \bar{h} \in A^+\},$$

where A^+ is a fixed Weyl chamber.

The core of the effective control sets for the action of S on $\mathbb{F}_\Theta$ are given by the fixed points for elements in $\mathcal{R}(S)$ as state the following theorem, whose proof can be found in Section 3 of [31].

Theorem 2.2.2 *For every $w \in \mathcal{W}$ there is a control set $D_\Theta(w) \subset \mathbb{F}_\Theta$ whose core is given by*

$$D_\Theta(w)_0 = \bigcup \{\operatorname{fix}_\Theta(h, w);\ h \in \mathcal{R}(S)\}.$$

Moreover, $D_\Theta^+ = D_\Theta(1)$ is the only control set S-invariant and $D_\Theta^- = D_\Theta(w_0)$ the only S^{-1}-invariant. Also, every effective control set is $D_\Theta(w)$ for some $w \in \mathcal{W}$.

The next theorem allow us to define flag type of a semigroup S. For the proof see [31].

Theorem 2.2.3 *Let $S \subset G$ be a semigroup with nonempty interior. There exists a subset of the simple roots $\Theta(S) \subset \Sigma$ such that the following are equivalents:*

- *(i) $\Theta(S)$ is the smallest subset $\Theta \subset \Sigma$, or the largest flag manifold $\mathbb{F}_\Theta$, such that D_Θ^+ is contained in the open cell of a Bruhat decomposition of $\mathbb{F}_\Theta$, that is, $D_\Theta^+ \subset N_\Theta^- \cdot b_\Theta$ for some Iwasawa decomposition;*
- *(ii) $\Theta(S)$ is the largest subset $\Theta \subset \Sigma$, or the smaller flag manifold $\mathbb{F}_\Theta$ such that $\pi_\Theta^{-1}(D_\Theta^+)$ is the S-invariant control set in $\mathbb{F}$;*

(iii) $\Theta(S)$ *is the only subset* $\Theta \subset \Sigma$*, or the only flag manifold* $\mathbb{F}_\Theta$ *such that* D_Θ^+ *is contained in the open cell in the Bruhat decomposition and* $\pi_\Theta^{-1}(D_\Theta^+)$ *is the* S*-invariant control set in* $\mathbb{F}$*.*

With this we can define the flag type of the semigroup S.

Definition 2.2.4 *The flag type of the semigroup* $S \subset G$ *with nonempty interior is the subset* $\Theta(S) \subset \Sigma$*, or the flag manifold* $\mathbb{F}_\Theta$*, satisfying one of the equivalent conditions of the above theorem.*

This notion of stability allow us to look, structurally, at the flag type of S through its elements. For any $g \in G$ there is an Iwasawa decomposition $G = KAN^+$ such that $g = ehu$ with $e \in K$, $h \in \operatorname{cl} A^+$ and $u \in N^+$ is unipotent, that is, $\operatorname{Ad}(u)$ is an unipotent linear map. Furthermore, the elements e, h and u commutes. Such decomposition is called **Jordan decomposition of** g (see Helgason [17], Chapter IX, Lemma 3.1). The flag type of g is given by

$$\Theta(g) = \{\alpha \in \Sigma;\ \alpha(\log h) = 0\}.$$

The flag type of g says what is the regularity of the vectorial component h in terms of the roots of G. The following theorem relate the flag type of an element in $\operatorname{int} S$ and $\Theta(S)$.

Theorem 2.2.5 *For every* $g \in \operatorname{int} S$ *we have that* $\Theta(g) \subset \Theta(S)$*. Moreover, there is* $g \in \operatorname{int} S$ *with minimal regularity, that is,* $\Theta(g) = \Theta(S)$*. In other words, the flag type of the semigroup* S *is the smallest regularity for the elements in* $\operatorname{int} S$

The fact that $\Theta(g) \subset \Theta(S)$, for $g \in \operatorname{int} S$ implies that on the flags F_Θ, with $\Theta(S) \subset \Theta$, g has at most one fixed point in each control set. We still do not now what happens with the elements in S that are not in the interior. As we will see ahead, for control sets of affine control system we can give a minimal regularity for such elements.

Example 2.2.6 *Let* $G = Sl(n, \mathbb{R})$*. A canonical setting is given by taking* $\mathfrak{a}$ *as the algebra of diagonal matrices with zero trace. The roots are* $\alpha_{ij} = \lambda_i - \lambda_j$ *where* $\lambda_i(H) = a_i$ *if* $H = \operatorname{diag}\{a_1, \ldots, a_n\}$*. A simple system is given by* $\Sigma = \{\alpha_{i,i+1}; i = 1, \ldots, n-1\}$ *and associated to this simple roots we have the positive Weyl chamber* $\mathfrak{a}^+ \subset \mathfrak{a}$ *given by the matrices* $H = \operatorname{diag}\{a_1, \ldots, a_n\}$

with $a_1 > a_2 > \ldots > a_n$. Then, for $h \in \mathrm{cl}\,\mathfrak{a}^+$ we have that $\Theta(h)$ tell us the multiplicity of the eigenvalues of h. The extreme cases are $\Theta(h) = \emptyset$, that is, all the eigenvalues of h are distinct and $\Theta(h) = \Sigma$ which implies that h has just one eigenvalue.

Example 2.2.7 *Let*

$$\dot{g}(t) = f_0(g(t)) + \sum_{i=1}^{m} u_i(t) f_i(g(t))$$

be a right invariant control-affine system on a semi-simple Lie group G, that is, $f_0, f_1, \ldots, f_m$ are right invariant vector fields. The right invariance allows us to induce, on every flag manifold $\mathbb{F}_\Theta$, for $\Theta \subset \Sigma$, control-affine systems

$$\dot{x}_\Theta(t) = \bar{f}_0(x_\Theta(t)) + \sum_{i=1}^{m} u_i(t) \bar{f}_i(x_\Theta(t))$$

where $\bar{f}_i = (\pi_\Theta)_*(f_i)$, $i = 0, 1, \ldots, m$ and $\pi_\Theta : G \to \mathbb{F}_\Theta$ is the canonical projection.

If we assume that the orbit of the control-affine system at $1 \in G$ has nonempty interior, that is $\mathrm{int}\,\mathcal{O}^+(1) \neq \emptyset$, all the effective control sets of the semigroup $S = \mathcal{O}^+(1)$ on $\mathbb{F}_\Theta$ are of the form $D_\Theta(w)$, for some $w \in \mathcal{W}$. By definition of control sets for a semigroup we have that such control sets are also control sets for the induced control-affine system on $\mathbb{F}_\Theta$ with nonempty interior.

Reciprocally, if local accessibility holds for the system on G, all the control sets with nonempty interior for all the induced systems are effective and so they are of the form $D_\Theta(w)$, for some $w \in \mathcal{W}$. In fact, by Proposition 1.23 of [21] we have that $\mathrm{int}D \subset \mathcal{O}^+(x) = Sx$ because we are assuming local accessibility for the system in G. Also by local accessibility we have that $\mathrm{int}S$ is dense in S and consequently we have that $(\mathrm{int}S)D \cap D \neq \emptyset$ and the result follows from Proposition 1.10 of [31].

Remark 2.2.8 *In passing note that we denote both the canonical projections $G \to \mathbb{F}_\Theta$ and $\mathbb{F} \to \mathbb{F}_\Theta$ by π_Θ. The point is that the use of both are normally clear by the context and allow us to avoid extra notation.*

2.3 Morse Decomposition and Flag type of flows

The description of the flag type of semigroups help us to understand the dynamics of flows of automorphisms on G-principal bundles, with G semi-simple. More specifically, let $Q \to X$ be a fiber bundle and G be a semi-simple Lie group such that G acts continuously on the right of Q and its action preserves the fiber of Q. This implies that the fiber of the bundle is homeomorphic to the group G itself.

If $\phi_t : Q \to Q$ is a flow of automorphisms on the G-principal bundle Q, with G semi-simple, we can characterize the Morse components of induced flows on the flag bundles.

Consider $\phi_t : X \to X$ be a flow on a topological space X. For $x \in X$ the **ω-set of** x is given by

$$\omega(x) := \{y \in X; \exists t_n \to +\infty; \phi_{t_n}(x) \to y\}.$$

In the same way, the **ω^*-set of** x is defined as

$$\omega^*(x) := \{y \in X; \exists t_n \to -\infty; \phi_{t_n}(x) \to y\}.$$

Definition 2.3.1 *Let $\phi_t : X \to X$ be a flow on a topological space X. A Morse decomposition of ϕ_t is a finite collection of disjoint subsets $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of X such that:*

(i) each $\mathcal{M}_i$ is compact, isolated [1] *and ϕ_t-invariant;*

(ii) for all $x \in X$ we have $\omega(x), \omega^*(x) \subset \bigcup_i \mathcal{M}_i$;

(iii) suppose there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_l}$ and $x_1, \ldots, x_l \in X \setminus \bigcup_{i=1}^n \mathcal{M}_i$ with $\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, l$; then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

In order to describe the Morse components using the flag type of semigroups, what is done is a description of the chain recurrent components of the flow, that we will define now.

[1] A set $K \subset X$ is isolated if there is a neighborhood N of K such that $K \subset \operatorname{int} N$ and $\phi_t(x) \in N$, for all $t \in \mathbb{R}$ implies $x \in K$

Let $\phi_t : X \to X$ be a flow on a metric space (X, d). For $x, y \in X$ and $\varepsilon, \tau > 0$, a (ε, τ)-chain from x to y is given by $x_0 = x, x_1, \ldots, x_n = y$ and $\tau_0, \ldots, \tau_n \geq \tau$ such that

$$d(\phi_{\tau_i}(x_i), x_{i+1}) < \varepsilon.$$

A subset $Y \subset X$ is said **chain transitive** if for each $x, y \in Y$ and $\varepsilon, \tau > 0$ there is an (ε, τ)-chain from x to y. A point $x \in X$ is **chain recurrent** if $\{x\}$ is chain transitive, that is, if for all $\varepsilon, \tau > 0$, there is a (ε, τ)-chain from x to x. We denote by $\mathcal{R}(\phi)$ the set of the chain recurrent points in X. The relation between the Morse components and chain recurrence is given by the next result.

Proposition 2.3.2 *The flow ϕ_t admits a finest Morse decomposition if, and only if, the chain recurrent set $\mathcal{R}(\phi)$ has finite many connected components. In this case the connected components of $\mathcal{R}(\phi)$ are exactly the Morse components of the finest Morse decomposition.*

Proof. Theorem B.2.26 of [12] ■

Due to this Proposition we can describe the Morse components of the induced flows.

2.3.1 Flag Type of Semigroups of Automophisms

Let $Q \to X$ be a G-principal bundle with G semi-simple and X a compact metric space. Consider the flag bundle associated given by $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta$, that is given by the classes of $(Q \times \mathbb{F}_\Theta)/\sim$ where $(q_1, b_1) \sim (q_2, b_2)$ if and only if, there exists $g \in G$ such that $q_1 = q_2 \cdot g$ and $b_1 = g^{-1} \cdot b_2$. As before, $\mathbb{E}$ denote the associated bundle $Q \times_G \mathbb{F}$.

Let S_Q be a semigroup of local endomorphisms in Q. Such semigroup acts in a standard way in Q, in X and in $\mathbb{E}_\Theta$. Assume that S_Q is accessible and that the action on X is transitive. With this hypothesis we can study the control sets of the action of S_Q in $\mathbb{E}_\Theta$ only by looking its action on the fibers, that is basically just the study of the control sets for the action of a nonempty semigroup of G on the flag manifold $\mathbb{F}_\Theta$.

The general idea is to consider for each $q \in Q$ the semigroup of G given by

$$S^q = \{g \in G;\ \psi(q) = q \cdot g,\ \text{for some } \psi \in S_Q\}.$$

Since S_Q is accessible we have that S^q is open for all $q \in Q$. Moreover, if we denote by $D^q(w)$ the control sets given by the action of S^q in the maximal flag $\mathbb{F}$, we have the following theorem, whose proof can be found in [9].

Theorem 2.3.3 *The control sets for the action of S_Q in $\mathbb{E}$ are given by the sets $\mathbb{D}(w)$, $w \in \mathcal{W}$, which are projected onto X and whose core are given fiberwise by*

$$(\mathbb{D}(w)_0)_q = q \cdot D^q(w)_0, \ q \in Q.$$

Moreover, the flag type of each S^q does not depend on $q \in Q$.

Definition 2.3.4 *The flag type of the semigroup S_Q is the flag type of S^q for $q \in Q$.*

2.3.2 Flag Type for the Morse decomposition of flows

The idea now is to use the above to characterize the chain recurrent components of induced flows on the flag bundles. Let $\phi_t : Q \to Q$ be a flow of automorphisms and consider the local automorphisms of Q that are ε-close to the identity, that is, let $V_\varepsilon = \{\psi \in \mathrm{End}(Q);\ \varrho(\psi(\xi), \xi) < \varepsilon, \text{ for all } \xi \in \mathbb{F}\}$, where the metric in $\mathbb{F}$ is the standard K-invariant Riemann metric on $\mathbb{F}$.

For each $\varepsilon, \tau > 0$ we define the (ε, τ)-shadowing semigroup as

$$S_{\varepsilon,\tau} := \{\psi_s \circ \phi_{\tau_s} \circ \cdots \circ \psi_1 \circ \phi_{\tau_1};\ \tau_i \geq \tau, \psi_i \in V_\varepsilon\}.$$

From this construction it follows that the (ε, τ)-chains in $\mathbb{E}$ coincides with the orbits of the semigroup $S_{\varepsilon,\tau}$. Also, the semigroups $S_{\varepsilon,\tau}$ are accessible so their control sets are characterized by Theorem 2.3.3. If we denote such control sets in $\mathbb{E}_\Theta$ by $\mathbb{D}_\Theta^{\varepsilon,\tau}(w)$, we have that the Morse components of the induced flow in $\mathbb{E}_\Theta$ are given by the intersection of these control sets, as stated in the next theorem.

Theorem 2.3.5 *The finest Morse decomposition of the induced flow $\phi_t : \mathbb{E}_\Theta \to \mathbb{E}_\Theta$ are given by*

$$\mathcal{M}_\Theta(w) = \bigcap_{\varepsilon,\tau} \mathbb{D}_\Theta^{\varepsilon,\tau}(w).$$

Moreover, $\mathcal{M}_\Theta^+ = \mathcal{M}_\Theta(1)$ is the only attractor and $\mathcal{M}_\Theta^- = \mathcal{M}_\Theta(w_0)$ the only repeller.

The proof can also be found in [9], Section 9. We note that for $\varepsilon_1 < \varepsilon_2$ and $\tau_1 > \tau_2$ we have $S_{\varepsilon_1,\tau_1} \subset S_{\varepsilon_2,\tau_2}$ what give us $\Theta(S_{\varepsilon_1,\tau_1}) \subset \Theta(S_{\varepsilon_2,\tau_2})$ and then makes sense the following definition.

Definition 2.3.6 *The flag type of the flow of automorphisms $\phi_t : Q \to Q$ is the subset of simple roots given by*

$$\Theta(\phi) = \bigcap_{\varepsilon,\tau} \Theta(S_{\varepsilon,\tau}).$$

We can also, using $\Theta(\phi)$ and its dual $\Theta(\phi)^*$, give an algebraic description of the Morse components.

Theorem 2.3.7 *Let $\Theta(\phi)$ the flag type of the flow ϕ_t and $\Theta(\phi)^*$ its dual. It holds*

(i) The flow admits only one attractor component in $\mathbb{E}_{\Theta(\phi)}$ and it intersects every fiber in exactly one point. This component is image of a continuous section $\sigma_\phi : X \to \mathbb{E}_{\Theta(\phi)}$. That is,

$$\left(\mathcal{M}^+_{\Theta(\phi)}\right)_x = \sigma_\phi(x).$$

In the same way there is only one repeller component in the dual flag bundle $\mathbb{E}_{\Theta(\phi)^*}$ and it is given as image of a continuous section $\sigma^*_\phi : X \to \mathbb{E}_{\Theta(\phi)^*}$;

(ii) If we consider the equivariant functions[2] $f : Q \to \mathbb{F}_{\Theta(\phi)}$ and $f^* : Q \to \mathbb{F}_{\Theta(\phi)^*}$ associated to σ_ϕ and σ^*_ϕ, respectively, we have that for each $q_0 \in Q$, $f(q)$ and $f^*(q_0)$ are opposed subalgebras and the orbit

$$\{(f(q), f^*(q));\ q \in Q\} = \mathrm{Ad}(G)(f(q_0), f^*(q_0)) \subset \mathbb{F}_{\Theta(\phi)} \times \mathbb{F}_{\Theta(\phi)^*}$$

is open and dense and it is identified with the homogeneous space $\mathrm{Ad}(G)H_\phi = G/Z_\phi$, where $Z_\phi = Z_{H_\phi}$ and H_ϕ is a characteristic element for $\Theta(\phi)$, that is, $\Theta(\phi) = \{\alpha \in \Sigma;\ \alpha(H_\phi) = 0\}$;

[2]That is, for any $g \in G$ and $q \in Q$ we have that $f(q \cdot g) = g^{-1} f(q)$ and the same for f^*.

(iii) The equivariant function $h_\phi : Q \to \mathrm{Ad}(G)H_\phi$, obtained from item (ii) above ($h_\phi(q) \approx (f(q), f^*(q))$), give us an algebraic description of the Morse components of the flow ϕ_t on every flag bundle $\mathbb{E}_\Theta$. The components are given fiberwise by

$$\mathcal{M}_\Theta(w)_x = q \cdot \mathrm{fix}_\Theta(h_\phi(q), w).$$

where $q \in Q$ is any element such that $\pi(q) = x$.

The above theorem says that, fiberwise, the finest Morse decomposition of ϕ_t in $\mathbb{E}_\Theta$ is given by the finest Morse decomposition of the action of some conjugated of H_ϕ on the flag manifold $\mathbb{F}_\Theta$.

By Proposition 5.4 of [30] the Morse components have also the representation

$$\mathcal{M}_\Theta(w) = \{q \cdot wb_\Theta;\ q \in Q_\phi\} = \{r \cdot wb_\Theta;\ r \in R_\phi\},$$

where b_Θ is the origin of $\mathbb{F}_\Theta$, Q_ϕ is a ϕ_t-invariant subbundle of Q with structural group Z_ϕ and R_ϕ a $K_{H_\phi} = K_\phi$-reduction of Q_ϕ. The subbundle Q_ϕ is given by $Q_\phi = h_\phi^{-1}(H_\phi)$.

Example 2.3.8 *Consider the control-affine system given in example (2.2.7). If we consider the trivial principal bundle $Q = \mathcal{U} \times G \to \mathcal{U}$, with right action given by the right translation on G we have that the control flow $\phi_t : Q \to Q$ is a flow of automorphisms. By the above, we have that the Morse sets of the induced control flow on $\mathbb{E}_\Theta = (\mathcal{U} \times G) \times_G \mathbb{F}_\Theta = \mathcal{U} \times \mathbb{F}_\Theta$ is of the form*

$$\mathcal{M}_\Theta(w) = \{q \cdot wb_\Theta;\ q \in Q_\phi\} = \{r \cdot wb_\Theta;\ r \in R_\phi\},$$

where $Q_\phi \subset \mathcal{U} \times G$ is a ϕ_t-invariant subbundle with structural group Z_ϕ and $R_\phi \subset \mathcal{U} \times K$ a K_ϕ-reduction of Q_ϕ. These subbundles are not necessarily trivial, however when it happens we can assume that $\psi_{\tau,u} := \varphi(\tau, 1, u) \in Z_\phi$ for every $\tau > 0$, $u \in \mathcal{U}$ (see [30]).

For every $g \in S = \mathcal{O}^+(1)$, and $\varepsilon, \tau > 0$ there is $q \in Q$ and some potency g^n of g such that $g^n \in S^q_{\varepsilon,\tau}$. Since $S^q_{\varepsilon,\tau}$ is an open semigroup we have that

$$\Theta(g^n) \subset \Theta(S^q_{\varepsilon,\tau}).$$

But the flag type of g^n and of g are the same and the flag type of $S_{\varepsilon,\tau}$ does not depend on $q \in Q$ which give us $\Theta(g) \subset \Theta(S_{\varepsilon,\tau})$ for every $\varepsilon, \tau > 0$ and consequently $\Theta(g) \subset \Theta(\phi)$. That give us a minimal regularity for all the elements in S (closure included). In particular, we have $\Theta(S) \subset \Theta(\phi)$.

2.4 Iwasawa decomposition and $\mathfrak{a}$-cocycle

As above let $Q \to X$ be a G-principal bundle with G semi-simple. Let $G = KAN^+$ be an Iwasawa decomposition. Since G/K is diffeomorphic to the Euclidian space AN^+, the principal bundle Q admits a K-reduction $R \subset Q$ (see [23]). Then we can write an Iwasawa decomposition for Q as $Q = R \cdot AN^+$ and then, every element $q \in Q$ can be written uniquely as $q = r \cdot an$, with $r \in R$, $a \in A$ and $n \in N^+$. Take the canonical projections

$$\mathsf{R} : Q \to R \quad \text{and} \quad \mathsf{A} : Q \to A.$$

The above maps satisfy:

1. If $r \in R$ then $\mathsf{R}(r) = r$ and $\mathsf{A}(r) = 1$;
2. If $q \in Q$ and $g = man \in P = MAN^+$ then $\mathsf{R}(q \cdot g) = \mathsf{R}(q)m$ and $\mathsf{A}(q \cdot g) = \mathsf{A}(q)a$.

Taking the logarithm of A we have the map $\mathsf{a} : Q \to \mathfrak{a}$ given by

$$\mathsf{a}(q) = \log \mathsf{A}(q). \tag{2.1}$$

We will use the same notation for the map of G to $\mathfrak{a}$ that associates for every $g = kan \in KAN^+$ the element $\log a \in \mathfrak{a}$. With these notations, the above property 2. implies that

$$\mathsf{a}(q \cdot g) = \mathsf{a}(q) + \mathsf{a}(g) \text{ for any } q \in Q,\ g \in P.$$

Consider a flow of automorphisms $\phi_t : Q \to Q$. By above we have that $\phi_t^R : R \to R$ defined as $\phi_t^R(r) = \mathsf{R}(\phi_t(r))$ is a flow on R. Moreover, the map

$$\mathsf{a}^\phi : \mathbb{R} \times R \to \mathfrak{a}, \quad \mathsf{a}^\phi(t, r) = \mathsf{a}(\phi_t(r)),$$

is an additive cocycle over ϕ_t^R, that is,

$$\mathsf{a}^\phi(t + s, r) = \mathsf{a}^\phi(t, \phi_s^R(r)) + \mathsf{a}^\phi(s, r).$$

In fact, consider the Iwasawa decomposition $\phi_s(r) = \phi_s^R(r) a_s n_s \in R \cdot AN^+$. Using (2.1) we have that

$$\mathsf{a}^\phi(t + s, r) = \mathsf{a}(\phi_{t+s}(r)) = \mathsf{a}(\phi_t(\phi_s(r)))$$

$$= \mathsf{a}(\phi_t(\phi^R_s(r) \cdot a_s n_s)) = \mathsf{a}(\phi_t(\phi^R_s(r)) \cdot a_s n_s)$$
$$= \mathsf{a}(\phi_t(\phi^R_s(r))) + \log a_s = \mathsf{a}^\phi(t, \phi^R_s(r)) + \mathsf{a}^\phi(s, r).$$

We will usually denote the cocycle a^ϕ just by a. The cocycle a factors to a well defined cocycle on the maximal flag bundle $\mathbb{E} = Q \times_G \mathbb{F} = R \times_K \mathbb{F}$ if we write

$$\mathsf{a}(t, \xi) = \mathsf{a}(t, r)$$

where $\xi = r \cdot b_0$. In fact, if $\xi = r' \cdot b_0$, there exist $m \in M$ such that $r' = r \cdot m$ and then

$$\mathsf{a}(t, r') = \mathsf{a}(t, r \cdot m) = \mathsf{a}(\phi_t(r) \cdot m) = \mathsf{a}(\phi_t(r)) + \mathsf{a}(m) = \mathsf{a}(t, r)$$

since $\mathsf{a}(m) = 0$. We refers to it as the **$\mathfrak{a}$-cocycle of the flow** ϕ_t. For the reverse flow, if we consider

$$\mathsf{a}^*(t, r) = \log \mathsf{A}(\phi_{-t}(r))$$

we also have a well defined cocycle over $\mathbb{E}$ and they are related by (see [1])

$$\mathsf{a}^*(t, \xi) = -\mathsf{a}(t, \phi_{-t}(\xi)).$$

In general, the $\mathfrak{a}$-cocycle does not factor to the partial flag bundles $\mathbb{F}_\Theta$. However, if we compose a with some specific $\beta \in \mathfrak{a}^*$ we still have a cocycle and it factors to the parcial bundle $\mathbb{E}_\Theta$. Such cocycle usually appears when we want to measure the exponential growth rate of the flows.

Example 2.4.1 *The main classical example of cocycles over a partial flag bundle is the one yielding Lyapunov exponents of linear flows on vector bundles. Let $\mathcal{V} \to X$ be a real vector bundle of dimension n, and denote by $B\mathcal{V}$ the bundle of frames of $\mathcal{V}$, which is a principal bundle with structure group $Gl(n, \mathbb{R})$. The elements of $B\mathcal{V}$ are linear isomorphisms $p : \mathbb{R}^n \to \mathcal{V}_x$, and the right action of $Gl(n, \mathbb{R})$ is $(p, g) \mapsto p \circ g$. Endow $\mathcal{V}$ with a Riemannian metric $\langle \cdot, \cdot \rangle$, and let $O\mathcal{V} = \{p : \mathbb{R}^n \to \mathcal{V}_x;\ \ p$ is isometry $\}$ be the orthonormal frame bundle, which is a $O(n)$-reduction of $B\mathcal{V}$.*

An Iwasawa decomposition of $Gl(n, \mathbb{R})$ reads $Gl(n, \mathbb{R}) = O(n)AN$, where A is the subgroup of diagonal matrices with positive entries an N the subgroup of upper triangular matrices with 1's in the diagonal. The subbundle of orthonormal frames together with the Iwasawa decomposition of $Gl(n, \mathbb{R})$ gives rise to the Iwasawa decomposition $B\mathcal{V} = O\mathcal{V} \cdot AN$ of $B\mathcal{V}$. Hence our *vector valued cocycle* $\mathsf{a}(t,\xi)$ *assume values in the space* $\mathfrak{a} = \log A$*, of diagonal matrices. Now a linear flow* Φ_t *on* $\mathcal{V}$ *defines a flow on* $B\mathcal{V}$ *by* $\phi_t(p) = \Phi_t \circ p$*, which is right invariant. Then the Lyapunov exponents of* Φ_t *are given by the asymptotics of the additive cocycle* $\log ||\Phi_t v||$*,* $v \in \mathcal{V}$*. We can read off this cocycle from the* $\mathfrak{a}$*-valued cocycle as follows: Let* e_1 *be the first basic element of* $\mathbb{R}^n$*. Take* $v \in \mathcal{V}_x$ *and let* $r \in O\mathcal{V}_x$ *be such that* $v = r(e_1)$*. Since* $\phi_t(r) = \Phi_t \circ r$ *it follows that* $\Phi_t(v) = \Phi_t \circ r(e_1) = \phi_t(r)(e_1)$*. Now,* $\phi_t(r) = r_t \cdot a_t n_t \in O\mathcal{V} \cdot AN$*, hence* $\Phi_t(v) = r_t(a_t n_t e_1)$*. But* $n_t e_1 = e_1$*. Also,* $r_t \in O\mathcal{V}$ *is an isometry. Therefore*

$$||\Phi_t(v)|| = ||a_t e_1||.$$

That is, $||\Phi_t(v)||$ *is the first eigenvalue of* a_t*. Hence if we let* $\lambda_1 \in \mathfrak{a}^*$ *be given by* $\lambda_1(\mathrm{diag}\{a_1, \ldots, a_n\}) = a_1$ *then*

$$\log ||\Phi_t(v)|| = \lambda_1(\mathsf{a}(t,r)). \tag{2.2}$$

We can write this equality with the cocycle $\mathsf{a}(t,\xi)$ *with* ξ *in the flag bundle instead of* $r \in O\mathcal{V}$*. For this we note that the flag bundle* $E = B\mathcal{V} \times_{Gl(n,\mathbb{R})} \mathbb{F}$ *is the bundle*

$$\mathbb{F}\mathcal{V} = \{rb_0;\ \ r \in O\mathcal{V}\} \qquad b_0 = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset)$$

of complete flags of vector subspaces of $\mathcal{V}$*. By formula (2.2) the Lyapunov ex- ponents of* Φ_t *at* v *have the form*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log ||\Phi_t(v)|| = \lim_{\tau\to\infty} \frac{1}{\tau} \lambda_1(\mathsf{a}(t,\xi)$$

where $\xi \in \mathbb{F}\mathcal{V}$ *is any flag whose one dimensional subspace is spanned by* v*. In other words the Lyapunov exponents of* Φ_t *are determined by the cocycle* $\lambda_1(\mathsf{a}(t,\xi)$*.*

To formalize this factorization we have the following lemma (Lemma 7.1 of [1]).

Lemma 2.4.2 *Let* $\Theta \subset \Sigma$ *and let* V *be an arbitrary vector space. If* $\beta : \mathfrak{a} \to V$ *is a linear map that annihilates on* $\mathfrak{a}(\Theta)$*, then the cocycle* $\mathsf{a}_\beta := \beta \circ \mathsf{a}$ *satisfies* $\mathsf{a}_\beta(t,r) = \mathsf{a}_\beta(t, r\cdot k)$ *for* $k \in K_\Theta$*.*

Using the Lemma above we will show that restricted to the Morse sets we can factor the $\mathfrak{a}$-cocycle. Let $\pi_\Theta : \mathbb{E} \to \mathbb{E}_\Theta$ the canonical projection given by $r \cdot b_0 \mapsto r \cdot b_\Theta$. Since the Morse sets of the induced flow are given by

$$\mathcal{M}_\Theta(w) = R_\phi \cdot wb_\Theta$$

we have that $\pi_\Theta(\mathcal{M}(w)) = \mathcal{M}_\Theta(w)$ and if $r \cdot wb_\Theta = r' \cdot wb_\Theta$, there exist $k \in K_\phi$ such that $r' = r \cdot k$ and $kwb_\Theta = wb_\Theta$ which implies that $k \in K_{\Theta(\phi)\cap w\Theta}$.

Consider then the subsets of the roots given by

$$\Pi^+_{\phi,\Theta,w} := \{\alpha \in \Pi^+ \setminus \langle\Theta(\phi)\rangle;\ w^{-1}\alpha \in \Pi^- \setminus \langle\Theta\rangle\} = \Pi^+ \setminus \langle\Theta(\phi)\rangle \cap w(\Pi^- \setminus \langle\Theta\rangle)$$

and

$$\Pi^-_{\phi,\Theta,w} := \{\alpha \in \Pi^- \setminus \langle\Theta(\phi)\rangle;\ w^{-1}\alpha \in \Pi^- \setminus \langle\Theta\rangle\} = \Pi^- \setminus \langle\Theta(\phi)\rangle \cap w(\Pi^- \setminus \langle\Theta\rangle).$$

and define the functional linear $\chi^+_{\Theta,w}, \chi^-_{\Theta,w} : \mathfrak{a} \to \mathbb{R}$ by

$$\chi^+_{\Theta,w} = \sum_{\alpha\in\Pi^+_{\phi,\Theta,w}} n_\alpha\,\alpha \qquad \chi^-_{\Theta,w} = \sum_{\alpha\in\Pi^-_{\phi,\Theta,w}} n_\alpha\,\alpha$$

where $n_\alpha = \dim \mathfrak{g}_\alpha$. We have then

Lemma 2.4.3 *$\chi^\pm_{\Theta,w}$ annihilates $\mathfrak{a}(\Theta_\phi \cap w\Theta)$.*

Proof. We will show just for $\chi^+_{\Theta,w}$ since for $\chi^-_{\Theta,w}$ is analogous. Consider then $\beta \in \Theta(\phi) \cap w\Theta$ and let

$$r_\beta(\alpha) = \alpha - 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\beta$$

the β-reflection of α. We affirm that $r_\beta(\Pi^+_{\phi,\Theta,w}) = \Pi^+_{\phi,\Theta,w}$. In fact, consider $H \in \mathfrak{a}$ such that $\Theta = \Theta(H)$. Then for each $\alpha \in \Pi^+_{\phi,\Theta,w}$

$$r_\beta(\alpha)(H_\phi) = \alpha(H_\phi) - 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\beta(H_\phi) > 0$$

since $\alpha(H_\phi) > 0$ and $\beta(H_\phi) = 0$. Also

$$w^{-1}r_\beta(\alpha) = w^{-1}\alpha - 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}w^{-1}\beta$$

and then

$$w^{-1}r_\beta(\alpha)(H) = w^{-1}\alpha(H) - 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}w^{-1}\beta(H) < 0$$

since $w^{-1}\alpha(H) < 0$ and $w^{-1}\beta(H) = 0$. Then $r_\beta(\Pi^+_{\phi,\Theta,w}) \subset \Pi^+_{\phi,\Theta,w}$ and since r_β is bijective we have the equality. Also, for any $w \in \mathcal{W}$ we have that $w\mathfrak{g}_\alpha = \mathfrak{g}_{w(\alpha)}$ what give us that $n_\alpha = n_{w(\alpha)}$ and consequently

$$r_\beta(\chi^+_{\Theta,w}) = \sum_{\alpha\in\Pi^+_{\phi,\Theta,w}} n_{r_\beta(\alpha)}\, r_\beta(\alpha) = \sum_{\gamma\in r_\beta(\Pi^+_{\phi,\Theta,w})} n_\gamma\,\gamma = \sum_{\gamma\in\Pi^+_{\phi,\Theta,w}} n_\gamma\,\gamma = \chi^+_{\Theta,w}.$$

But that is true if, and only if, $\chi^+_{\Theta,w}(H_\beta) = \langle\chi^+_{\Theta,w},\beta\rangle = 0$. Since β was arbitrary and

$$\mathfrak{a}(\Theta(\phi)\cap w\Theta) = \mathrm{span}\{H_\beta, \beta\in\Theta(\phi)\cap w\Theta\}$$

we have the desired. ■

By Lemma 2.4.2 above we have

Corollary 2.4.4 *Let us assume that $\langle\Theta(\phi)\rangle \subset w\langle\Theta\rangle$. The maps $\mathsf{a}^\pm_{\Theta,w} : \mathbb{R}\times\mathcal{M}_\Theta(w) \to \mathbb{R}$ given by*

$$\mathsf{a}^\pm_{\Theta,w}(t, r\cdot wb_\Theta) := \chi^\pm_{\Theta,w}(\mathsf{a}(t,r)) \tag{2.3}$$

are well defined cocycles.

Proof. By the above Lemma, we just need to show that for any $w' \in \mathcal{W}_{\Theta(\phi)} \setminus \mathcal{W} \,/\, \mathcal{W}_\Theta$ we have that

$$\mathsf{a}^\pm_{\Theta,w}(t, r\cdot wb_\Theta) = \mathsf{a}^\pm_{\Theta,w'}(t, r\cdot w'b_\Theta).$$

Let then $w_1 \in \mathcal{W}_{\Theta(\phi)}$ and $w' \in \mathcal{W}_\Theta$ such that $w' = w_1ww_2$. Since $w_1\mathfrak{a}(\Theta(\phi)) = \mathfrak{a}(\Theta(\phi))$ and $w_2\mathfrak{a}(\Theta) = \mathfrak{a}(\Theta)$ we have that

$$\mathfrak{a}(\Theta(\phi)\cap w'(\Theta)) = w_1\mathfrak{a}(\Theta(\phi)\cap w(\Theta))$$

and

$$\mathfrak{a}_{\Theta(\phi)\cap w'(\Theta)} = w_1\mathfrak{a}_{\Theta(\phi)\cap w(\Theta)}.$$

Also, since $\chi^{\pm}_{\Theta,w}(\mathfrak{a}(\Theta(\phi)\cap w(\Theta)))=0$ and $\chi^{\pm}_{\Theta,w'}(\mathfrak{a}(\Theta(\phi)\cap w'(\Theta)))=0$ we have that

$$\mathsf{a}^{\pm}_{\Theta,w}(t,r\cdot wb_{\Theta}):=\chi^{\pm}_{\Theta,w}(\mathsf{a}_1(t,r))$$

and

$$\mathsf{a}^{\pm}_{\Theta,w}(t,r\cdot wb_{\Theta}):=\chi^{\pm}_{\Theta,w}(\mathsf{a}'_1(t,r))$$

where $\mathsf{a}_1(t,r))$ and $\mathsf{a}'_1(t,r))$ are the part of $\mathsf{a}(t,r))$ in $\mathfrak{a}_{\Theta(\phi)\cap w(\Theta)}$ and $\mathfrak{a}_{\Theta(\phi)\cap w'(\Theta)}$, respectively. By the above and the unicity of the decompositions we have that $w_1\mathsf{a}_1(t,r))=\mathsf{a}'_1(t,r))$. Then

$$\mathsf{a}^{\pm}_{\Theta,w'}(t,r\cdot w'b_{\Theta}):=\chi^{\pm}_{\Theta,w'}(\mathsf{a}'_1(t,r))=\sum_{\alpha\in\Pi^{\pm}_{\phi,\Theta,w'}} n_{\alpha}\,\alpha(w_1\mathsf{a}_1(t,r))$$

$$=\sum_{\beta\in w_1(\Pi^{\pm}_{\phi,\Theta,w'})} n_{\beta}\,\beta(\mathsf{a}_1(t,r)).$$

But we have also that $w_1\Pi^{\pm}\backslash\langle\Theta(\phi)\rangle=\Pi^{\pm}\backslash\langle\Theta(\phi)\rangle$ and $w_2\Pi^{\pm}\backslash\langle\Theta\rangle=\Pi^{\pm}\backslash\langle\Theta\rangle$ which implies that $w_1(\Pi^{\pm}_{\phi,\Theta,w'})=(\Pi^{\pm}_{\phi,\Theta,w})$ and consequently

$$\sum_{\beta\in w_1(\Pi^{\pm}_{\phi,\Theta,w'})} n_{\beta}\,\beta(\mathsf{a}_1(t,r))=\sum_{\beta\in\Pi^{\pm}_{\phi,\Theta,w}} n_{\beta}\,\beta(\mathsf{a}_1(t,r))=:\mathsf{a}^{\pm}_{\Theta,w}(t,r\cdot wb_{\Theta})$$

showing the result. ■

Such cocycles will be important in the next Chapter in order to estimate the invariance entropy of control-affine systems on flag manifolds.

To conclude the Chapter we will give the description of a construction of a linearization around the attractor component given by the Morse decomposition of the induced flow on $\mathbb{E}_{\Theta(\phi)}$ as given in Section 5 of [30]. The tangent space at the origin of $\mathbb{F}_{\Theta(\phi)}$ identifies with the nilpotent Lie algebra $\mathfrak{n}^-_{\phi}=\mathfrak{n}^-_{\Theta(\phi)}$ and the group Z_{ϕ} normalizes $\mathfrak{n}^-_{\phi}$ which implies that it acts linearly on $\mathfrak{n}^-_{\phi}$ by the adjoint representation. We have then the associated bundle

$$\mathcal{V}_{\phi}=Q_{\phi}\times_{Z_{\phi}}\mathfrak{n}^-_{\phi}\to X.$$

Since the Z_{ϕ}-action is linear we have that the associated bundle $\mathcal{V}_{\phi}\to X$ is a vector bundle and that the flow ϕ_t induces a linear flow Φ_t on $\mathcal{V}_{\phi}$.

Let $b_{\Theta(\phi)}$ be the origin of $\mathbb{F}_{\Theta(\phi)}$ and define the subset

$$\mathbb{B}_\phi = Q_\phi \cdot N_\phi^- b_{\Theta(\phi)}$$

and the mapping $\Psi : \mathcal{V}_\phi \to \mathbb{B}_\phi$

$$\Psi(q \cdot X) = q \cdot (\exp X) b_{\Theta(\phi)}, \ \ q \in Q_\phi, \ X \in \mathfrak{n}_\phi^-.$$

We have then the following Proposition from [30] (Proposition 5.5).

Proposition 2.4.5 *The following statements are true:*

(i) $\mathbb{B}_\phi$ *is an open and dense* ϕ_t*-invariant subset of* $\mathbb{E}_{\Theta(\phi)}$ *which contains the attractor component* $\mathcal{M}^+_{\Theta(\phi)} = \Psi(\mathcal{V}^0_\phi)$*, where* $\mathcal{V}^0_\phi$ *is the zero section of* $\mathcal{V}_\phi$*;*

(ii) ϕ_t *and* Φ_t *are conjugated under* Ψ*, that is,*

$$\phi_t(\Psi(v)) = \Psi(\Phi_t(v)), \quad v \in \mathcal{V}_\phi.$$

There is also a natural metric $(\cdot, \cdot)$ in $\mathcal{V}_\phi \to X$ given by

$$(r \cdot X, r \cdot Y) = B_\theta(X, Y), \quad r \in R_\phi, X, Y \in \mathfrak{n}_\phi^-$$

where B_θ is the inner product in the Lie algebra defined by the Cartan involution θ. That this in fact a metric in the whole $\mathcal{V}_\phi$ follows from the Iwasawa decomposition $Q_\phi = R_\phi \cdot AN^+(\phi)$, since $AN^+(\phi)$ normalizes $\mathfrak{n}_\phi^-$. We have then the following Proposition, that is a slightly modification of Theorem 7.2 of [30].

Proposition 2.4.6 *There exist* $\mu, B \in \mathbb{R}$ *with* $\mu > 0$ *such that*

$$\alpha(\mathsf{a}(\tau, \xi)) \geq \mu\tau + B$$

for $\xi \in \mathcal{M}^+, \alpha \in \Pi^+ \setminus \langle\Theta(\phi)\rangle$ *and* $\tau > 0$.

As a consequence of this Proposition we have that the cocycles on the Morse components can be linearly estimated, that is,

$$\mathsf{a}^+_{\Theta,w}(t, r \cdot wb_\Theta) \geq s^+(\mu t + B) \quad \text{and} \quad \mathsf{a}^-_{\Theta,w}(t, r \cdot wb_\Theta) \leq -s^-(\mu t + B)$$

where $s^\pm = \sum_{\alpha \in \Pi^\pm_{\phi,\Theta,w}} n_\alpha$.

Chapter 3

The Flag Case

In this chapter we will use the control and semigroup theory presented in the previous Chapters in order to get lower and upper bounds for the invariance entropy of an admissible pair (K, Q) of a control-affine system on a flag manifold, induced by a right invariant control-affine system on a semi-simple Lie group G. We will be interested in the case where Q is a chain control set that coincides with the closure of a control set of the induced system, what happens if we have that Q is a hyperbolic chain control set (see [11]).

3.1 Hyperbolic Affine Systems on flag manifolds

From now on we will consider a right-invariant affine control system

$$\dot{g}(t) = f_0(g(t)) + \sum_{i=1}^{m} u_i(t) f_i(g(t)) \tag{3.1}$$

where the state space G is a semi-simple Lie group. The right invariance allows us to induce control-affine systems on the partial flag manifold $\mathbb{F}_\Theta$

$$\dot{x}_\Theta(t) = \bar{f}_0(x_\Theta(t)) + \sum_{i=1}^{m} u_i(t) \bar{f}_i(x_\Theta(t)) \tag{3.2}$$

where $\bar{f}_i = (\pi_\Theta)_* f_i$ with $\pi_\Theta : G \to \mathbb{F}_\Theta$ the canonical projection. Note that we have also the canonical fibration between the flags $\pi^{\Theta_1}_{\Theta_2} : \mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ if

$\Theta_1 \subset \Theta_2$. If $F_{\Theta_i}(x_{\Theta_i}(t), u(t))$ denotes the right hand-side of (3.2) induced on $\mathbb{F}_{\Theta_i}$, $i = 1, 2$ we have

$$(\pi^{\Theta_1}_{\Theta_2})_*\big(F_{\Theta_1}(x_{\Theta_1}(t), u(t))\big) = F_{\Theta_2}(x_{\Theta_2}(t), u(t))$$

and by Definition 1.2.7, the pair $(\pi^{\Theta_1}_{\Theta_2}, \mathrm{id}_{\mathcal{U}})$ is a semi-conjugacy from $F_{\Theta_1}(x_{\Theta_1}(t), u(t))$ to $F_{\Theta_2}(x_{\Theta_2}(t), u(t))$. When $\Theta_1 = \emptyset$ and $\Theta_2 = \Theta$ we will denote the projection also by π_Θ.

If we consider the usual right action on G, we have that the control flow

$$\phi_t : \mathcal{U} \times G \to \mathcal{U} \times G \qquad \phi_t(u, g) = (\theta_t u, \varphi_{t,u}(g))$$

is a right invariant flow on the trivial principal bundle $\mathcal{U} \times G \to \mathcal{U}$. Since $(\mathcal{U} \times G) \times_G \mathbb{F}_\Theta = \mathcal{U} \times \mathbb{F}_\Theta$ we have the induced control flow (that we still denotes by ϕ)

$$\phi_t : \mathcal{U} \times \mathbb{F}_\Theta \to \mathcal{U} \times \mathbb{F}_\Theta \qquad \phi_t(x, u) = (\theta_t u, \varphi(t, x, u)).$$

If we denote by $\psi_{t,u}$ the solution of the control-affine (3.1) at $1 \in G$ we have that the solutions of the induced system (3.2) at $x \in \mathbb{F}_\Theta$ are given by $\varphi(t, x, u) = \psi_{t,u} \cdot x$, that is, the translation of x by $\psi_{t,u}$.

By Theorem 2.3.7 we have a equivariant continuous map $h_\phi : \mathcal{U} \times G \to \mathrm{Ad}(G)H_\phi$, that is ϕ_t-invariant. Associated with h_ϕ we have the block reduction Q_ϕ given by $h_\phi^{-1}(H_\phi) = Q_\phi$.

Consider then the map

$$\mathsf{h} : \mathcal{U} \to \mathrm{Ad}(G)H_\phi, \qquad \mathsf{h}(u) := h_\phi(u, 1).$$

Proposition 3.1.1 *The function* h *defined above has the following properties:*

(i) For every $(u, g) \in Q_\phi$ *we have* $\mathsf{h}(u) = \mathrm{Ad}(g)H_\phi$*;*

(ii) For $u \in \mathcal{U}$, $t \in \mathbb{R}$ *we have* $\mathsf{h}(\theta_t u) = \mathrm{Ad}(\psi_{t,u})\mathsf{h}(u)$

Proof. The equivariance of h_ϕ give us that $h_\phi(u, g) = \mathrm{Ad}(g^{-1})h_\phi(u, 1) = \mathrm{Ad}(g^{-1})\mathsf{h}_\phi(u)$ for $g \in G$, $u \in \mathcal{U}$. Since $Q_\phi = h_\phi^{-1}(H_\phi)$, we have for $(u, g) \in Q_\phi$ that

$$H_\phi = h_\phi(u, g) = \mathrm{Ad}(g^{-1})\mathsf{h}(u)$$

and consequently

$$\mathsf{h}(u) = \mathrm{Ad}(g)H_\phi$$

showing (i).

For (ii), let $u \in \mathcal{U}$ and $t \in \mathbb{R}$, then

$$\mathsf{h}(\theta_t u) = h_\phi(\theta_t u, 1) = h_\phi(\theta_t u, \psi_{t,u}\psi_{t,u}^{-1}) = \mathrm{Ad}(\psi_{t,u})h_\phi(\phi_t(u,1))$$

but since h_ϕ is ϕ_t invariant, we have

$$h_\phi(\phi_t(u,1)) = h_\phi(u,1) = \mathsf{h}(u)$$

and then

$$\mathsf{h}(\theta_t u) = \mathrm{Ad}(\psi_{t,u})\mathsf{h}(u).$$

■

The above properties give us that

$$Q_\phi = \{(u,g) \in \mathcal{U} \times G,\ \mathsf{h}(u) = \mathrm{Ad}(g)H_\phi\}$$

and

$$R_\phi = \{(u,k) \in \mathcal{U} \times K,\ \mathsf{h}(u) = \mathrm{Ad}(k)H_\phi\}.$$

Also, the Morse components of ϕ_t are given fiberwise by

$$\mathcal{M}_\Theta(w)_u = (u,g) \cdot \mathrm{fix}_\Theta(h_\phi(u,g), w).$$

Since $h_\phi(u,g) = \mathrm{Ad}(g^{-1})\mathsf{h}(u)$ and $\mathrm{fix}_\Theta(\mathrm{Ad}(g^{-1})\mathsf{h}(u), w) = g^{-1}\mathrm{fix}_\Theta(\mathsf{h}(u), w)$ we have

$$\mathcal{M}_\Theta(w)_u = \{u\} \times \mathrm{fix}_\Theta(\mathsf{h}(u), w). \tag{3.3}$$

By Proposition 1.1.6 we have that the chain control sets of induced control system on $\mathbb{F}_\Theta$ are given by

$$E_{\Theta,w} = \pi_2(\mathcal{M}_\Theta(w)) = \bigcup_{u \in \mathcal{U}} \mathrm{fix}_\Theta(\mathsf{h}(u), w)$$

where $\pi_2 : \mathcal{U} \times \mathbb{F}_\Theta \to \mathbb{F}_\Theta$ is the projection on the second component. As before, we will denote the chain control sets in $\mathbb{F}$, simple by E_w.

Note also that, since $\Theta(S) \subset \Theta(\phi)$ we have that the effective control sets $D_\Theta(w)$ of the induced system (3.2) on $\mathbb{F}_\Theta$ are contained in the chain control sets $E_{\Theta,w} \subset \mathbb{F}_\Theta$.

We will show that for points $(u,x) \in \mathcal{M}_\Theta(w)$ we have a decomposition of the tangent space of $\mathbb{F}_\Theta$ that is invariant by the control flow.

For $X \in \mathfrak{g}$ we still denote by X the vector field induced on $\mathbb{F}_\Theta$ given by

$$X(x) = \frac{d}{dt}_{|t=0} \mathrm{e}^{tX} \cdot x, \quad x \in \mathbb{F}_\Theta.$$

For a subset $\mathfrak{l} \subset \mathfrak{g}$ we put

$$\mathfrak{l} \cdot x = \{X(x) \in T_x\mathbb{F}_\Theta, \ \ X \in \mathfrak{l}\}$$

and we have that $T_{b_\Theta}\mathbb{F}_\Theta = \mathfrak{n}_\Theta^- \cdot b_\Theta$. We have also that $g \in G$ acts as a diffeomorphism on $\mathbb{F}_\Theta$ and its differential at a point $x \in \mathbb{F}_\Theta$ satisfies

$$dg_x(X(x)) = (\mathrm{Ad}(g)(X))(gx). \tag{3.4}$$

Consider then the subspaces of the tangent space to $x \in \mathrm{fix}_\Theta(\mathsf{h}(u), w)$ given by

$$\mathcal{S}_{\Theta,w}(u,x) := \mathfrak{n}^-_{\mathsf{h}(u)} \cdot x$$

and

$$\mathcal{U}_{\Theta,w}(u,x) := \mathfrak{n}^+_{\mathsf{h}(u)} \cdot x$$

where $\mathfrak{n}^\pm_{\mathsf{h}(u)} = \mathfrak{n}^\pm_{\Theta(\mathsf{h}(u))}$. Since

$$\mathfrak{g} = \mathfrak{n}^-_{\mathsf{h}(u)} \oplus \mathfrak{z}_{\mathsf{h}(u)} \oplus \mathfrak{n}^+_{\mathsf{h}(u)}$$

we have

$$T_x\mathbb{F}_\Theta = \mathcal{S}_{\Theta,w}(u,x) \oplus T_x\mathrm{fix}_\Theta(\mathsf{h}(u), w) \oplus \mathcal{U}_{\Theta,w}(u,x)$$

for all $(u,x) \in \mathcal{M}_\Theta(w)$.

Proposition 3.1.2 *The decomposition*

$$T_x\mathbb{F}_\Theta = \mathcal{S}_{\Theta,w}(u,x) \oplus T_x\mathrm{fix}_\Theta(\mathsf{h}(u), w) \oplus \mathcal{U}_{\Theta,w}(u,x)$$

holds for every $(u,x) \in \mathcal{M}_\Theta(w)$ and the subspaces $\mathcal{S}_{\Theta,w}$ and $\mathcal{U}_{\Theta,w}$ have constant dimensions and are invariant by ϕ.

Proof. For given $(u,x) \in \mathcal{M}_\Theta(w)$ we have that $x = k \cdot wb_\Theta$ with $\mathsf{h}(u) = \mathrm{Ad}(k)H_\phi$ and so, the translation formula (3.4) give us that

$$\mathcal{S}_{\Theta,w}(u,x) = \mathfrak{n}^-_{\mathsf{h}(u)} \cdot x = (dk)_{wb_\Theta}\big(\mathfrak{n}^-_\phi \cdot wb_\Theta\big)$$

and consequently that $\dim \mathcal{S}_{\Theta,w}(u,x) = \dim\Big(\mathfrak{n}^-_\phi \cdot wb_\Theta\Big)$ showing that $\dim \mathcal{S}_{\Theta,w}$ is constant. Analogously for $\mathcal{U}_{\Theta,w}$.

Let us show the invariance of $\mathcal{S}_{\Theta,w}(u,x)$ since for $\mathcal{U}_{\Theta,w}(u,x)$ is analogous. The map $\varphi_{t,u}$ acts in $\mathbb{F}_\Theta$ as the translation by $\psi_{t,u} \in G$. By the translation formula we have then

$$(d\varphi_{t,u})_x\mathcal{S}_{\Theta,w}(u,x) = (d\psi_{t,u})_x\big(\mathfrak{n}^-_{\mathsf{h}(u)} \cdot x\big) = \Big(\mathrm{Ad}(\psi_{t,u})\mathfrak{n}^-_{\mathsf{h}(u)}\Big) \cdot \varphi_{t,u}(x)$$

$$= \mathfrak{n}^-_{\mathrm{Ad}(\psi_{t,u})\mathsf{h}(u)} \cdot \varphi_{t,u}(x).$$

By property (ii) of the Proposition 3.1.1 we have $\mathsf{h}(\theta_t u) = \mathrm{Ad}(\psi_{t,u})\mathsf{h}(u)$ and consequently

$$(d\varphi_{t,u})_x\mathcal{S}_{\Theta,w}(u,x) = \mathcal{S}_{\Theta,w}(\phi_t(u,x)).$$

■

We call the spaces $\mathcal{S}_{\Theta,w}$ and $\mathcal{U}_{\Theta,w}$, respectively, by **stable** and **unstable tangent bundles**. Such names will become clear ahead.

Consider as before the $\mathfrak{a}$-cocycle in $\mathcal{U} \times \mathbb{F}$ over ϕ given by

$$\mathsf{a} : \mathbb{R} \times \mathcal{U} \times \mathbb{F} \to \mathfrak{a} \qquad (t,u,x) \mapsto \mathsf{a}(t,u,x) := \log A(\varphi_{t,u}(x))$$

for $x = k \cdot b_0$.

We have then the following Theorem.

Theorem 3.1.3 *There exist constants $c, \mu > 0$ such that for all $(u,x) \in \mathcal{M}_\Theta(w)$*

$$||(d\varphi_{t,u})_x v|| \leq c^{-1}\mathrm{e}^{-\mu t}||v|| \quad \textit{for all } t \geq 0,\ v \in \mathcal{S}_{\Theta,w}(u,x)$$

and

$$||(d\varphi_{t,u})_x v|| \geq c\mathrm{e}^{\mu t}||v|| \quad \textit{for all } t \geq 0,\ v \in \mathcal{U}_{\Theta,w}(u,x).$$

Proof. Let $k \in K$ such that $x = k \cdot wb_\Theta$ and $\mathsf{h}(u) = \mathrm{Ad}(k)H_\phi$, that is, $(u, k) \in R_\phi$. The ϕ_t-invariance of Q_ϕ, for $t > 0$, give us that

$$\varphi_{t,u}(k) = k_{t,u}a_{t,u}n_{t,u}$$

with $a_{t,u}n_{t,u} \in AN^+(\phi)$ and $k_{t,u} \in K$. Consider in $\mathbb{F}_\Theta$ a K-invariant Riemann metric. By the relations between k and u we have that

$$\mathfrak{n}^-_{\mathsf{h}(u)} \cdot x = (dk)_{wb_\Theta}\left(\mathfrak{n}^-_\phi \cdot wb_\Theta\right)$$

Let $v \in \mathcal{S}_{\Theta,w}(u, x)$ and $\bar{v} \in \mathfrak{n}^-_\phi \cdot wb_\Theta$ such that $v = (dk)_{wb_\Theta}\bar{v}$. Then

$$\begin{aligned} ||(d\varphi_{t,u})_x v|| = ||(d\psi_{t,u})_x v|| = ||(d\psi_{t,u})_x (dk)_{wb_\Theta}\bar{v}|| = ||(d\varphi_{t,u}(k))_{wb_\Theta}\bar{v}|| \\ = ||(dk_{t,u})_{wb_\Theta}(da_{t,u})_{wb_\Theta}(dn_{t,u})_{wb_\Theta}\bar{v}||. \end{aligned} \tag{3.5}$$

Since $N^+(\phi)$ centralizes $\mathfrak{n}^-_\phi$ and the inner product is K-invariant, we have that

$$||(d\varphi_{t,u})_x v|| = ||(da_{t,u})_{wb_\Theta}\bar{v}||.$$

Being $(dk)_{wb_\Theta}$ a isometry we conclude that

$$||(d\varphi_{t,u})_{|\mathcal{S}_{\Theta,w}(u,x)}|| = ||(da_{t,u})_{|\mathfrak{n}^-_\phi \cdot wb_\Theta}|| = ||\mathrm{Ad}(a_{t,u})_{|\mathfrak{n}^-_{\phi,w}}||$$

where $\mathfrak{n}^-_{\phi,w} = \bigoplus_{\alpha\in\Pi^-_{\phi,\Theta,w}} \mathfrak{g}_\alpha$. Now $\mathrm{Ad}(a_{t,u})_{|\mathfrak{n}^-_{\phi,w}}$ is positive definite so that $||\mathrm{Ad}(a_{t,u})_{|\mathfrak{n}^-_{\phi,w}}||$ is equal to its greatest eigenvalue. Since the eigenvalues are $\mathrm{e}^{\alpha(\mathsf{a}(t,u,k\cdot b_0))}$, $\alpha \in \Pi^-_{\phi,\Theta,w}$ we have by Proposition 2.4.6 that

$$||(d\varphi_{t,u})_x v|| \leq c^{-1}\mathrm{e}^{-\mu t}||v||$$

showing the first inequality.

Consider now $v \in \mathcal{U}_{\Theta,w}(u, x)$. We have also that

$$\mathfrak{n}^+_{\mathsf{h}(u)} \cdot x = (dk)_{wb_\Theta}\left(\mathfrak{n}^+_\phi \cdot wb_\Theta\right)$$

and if we define in a similar way as before the vector space

$$\mathfrak{n}^+_{\phi,w} = \bigoplus_{\alpha\in\Pi^+_{\phi,\Theta,w}} \mathfrak{g}_\alpha$$

since for the $t > 0$ we have

$$\varphi_{-t,u}(k) = k^*_{t,u} a^*_{t,u} n^*_{t,u}$$

with $a^*_{t,u} n^*_{t,u} \in AN^-(\phi)$, $k^*_{t,u} \in K$ and $\log a^*_{t,u} = \mathsf{a}^*(t, u, k \cdot b_0)$, we get

$$||(d\varphi_{-t,u})_{|\mathcal{U}_{\Theta,w}(u,x)}|| = ||(da^*_{t,u})_{|\mathfrak{n}^+_\phi \cdot wb_\Theta}|| = ||\mathrm{Ad}(a^*_{t,u})_{|\mathfrak{n}^+_{\phi,w}}||$$

Now $\mathrm{Ad}(a^*_{t,u})_{|\mathfrak{n}^+_{\phi,w}}$ is positive definite so that $||\mathrm{Ad}(a^*_{t,u})_{|\mathfrak{n}^+_{\phi,w}}||$ is equal to its greatest eigenvalue. Since its eigenvalues are given by $\mathrm{e}^{\alpha(\mathsf{a}^*(t,u,k \cdot b_0))}$, $\alpha \in \Pi^+_{\phi,\Theta,w}$ we have, by Proposition 2.4.6 and the fact that $\mathsf{a}^*(t, u, k \cdot b_0) = -\mathsf{a}(t, \phi_{-t}(u, k \cdot b_0))$, that

$$||(d\varphi_{-t,u})_x v|| \leq c^{-1} \mathrm{e}^{-\mu t} ||v||$$

for all $t > 0$ and $(u, x) \in \mathcal{M}_\Theta(w)$ and consequently

$$||(d\varphi_{t,u})_x v|| \geq c \mathrm{e}^{\mu t} ||v||$$

as desired. ■

Remark 3.1.4 *Consider $u \in \mathcal{U}$, $\tau > 0$ and $x \in E_{\Theta,w}$ such that $\varphi_{t,u}(x) \in E_{\Theta,w}$ for all $t \in [0, \tau]$. Since $E_{\Theta,w}$ is a chain control set, there exists $u_1, u_2 \in \mathcal{U}$ such that the function $\bar{u} \in \mathcal{U}$ defined as*

$$\bar{u}(s) = \begin{cases} u_1(s) & \text{if } s < 0 \\ u(s) & \text{if } s \in [0, \tau] \\ u_2(s) & \text{if } s > \tau \end{cases}$$

satisfies $\varphi_{t,\bar{u}}(x) \in E_{\Theta,w}$ or, equivalently, $(\bar{u}, x) \in \mathcal{M}_\Theta(w)$. By Theorem 3.1.3 above, we have for any $v \in \mathcal{S}_{\Theta,w}(\bar{u}, x)$ and $t \geq 0$ that

$$||(d\varphi_{t,\bar{u}})_x v|| \leq c^{-1} \mathrm{e}^{-\mu t} ||v||.$$

Since $\varphi_{t,\bar{u}}$ just depends on $\bar{u}$ restricted to $[0, t)$ we have that

$$||(d\varphi_{t,u})_x v|| \leq c^{-1} \mathrm{e}^{-\mu t} ||v||$$

for any $t \in [0, \tau]$. The same is true for the second inequality in Theorem 3.1.3 and vectors in $\mathcal{U}_{\Theta,w}(\bar{u}, x)$.

Theorem 3.1.3 shows that every chain control set $E_{\Theta,w} \subset \mathbb{F}_\Theta$ are partially hyperbolic. Also, for points $(u,x) \in \mathcal{M}_\Theta(w)$ we have at least $\dim \mathcal{S}_{\Theta,w}(u,x) = \dim\left(\mathfrak{n}_\phi^- \cdot wb_\Theta\right)$ negative Lyapunov exponents and at least $\dim \mathcal{U}_{\Theta,w}(u,x) = \dim\left(\mathfrak{n}_\phi^+ \cdot wb_\Theta\right)$ positive Lyapunov exponents, that is, for each point in $\mathcal{M}_\Theta(w)$ we have a minimal number of positive and negative Lyapunov exponents. A direct consequence of such result is:

Corollary 3.1.5 *Let $\Theta \subset \Sigma$ a subset of the roots and $w \in \mathcal{W}$ such that $\langle\Theta(\phi)\rangle \subset w\langle\Theta\rangle$. Then the chain control sets $E_{\Theta,w}$ of the control-affine system (3.2) on $\mathbb{F}_\Theta$ are uniformly hyperbolic.*

Proof. Follows directly from the fact that the above condition implies that $\mathrm{fix}_\Theta(H_\phi, w) = wb_\Theta$. ■

Remark 3.1.6 *We note also that by [11] if the condition $\langle\Theta(\phi)\rangle \subset w\langle\Theta\rangle$ is satisfied, as in the above Corollary, we have that $\mathrm{cl}D_\Theta(w) = E_{\Theta,w}$, that is, the closure of the control set $D_\Theta(w)$ coincides with the chain control set $E_{\Theta,w}$.*

Remark 3.1.7 *Note that the hyperbolicity condition is independent of the representative $w \in \mathcal{W}$. Indeed, if $w' = w_1ww_2$ with $w_1 \in \mathcal{W}_{\Theta(\phi)}, w_2 \in \mathcal{W}_\Theta$ and $\langle\Theta(\phi)\rangle \subset w\langle\Theta\rangle$, then $\langle\Theta(\phi)\rangle = w_1\langle\Theta(\phi)\rangle \subset w_1w\langle\Theta\rangle = w_1ww_2\langle\Theta\rangle = w'\langle\Theta\rangle$.*

Let $\mathsf{a}^\pm_{\Theta,w}$ be the cocycles over the Morse component $\mathcal{M}_\Theta(w)$ as defined in Chapter 2. The next Theorem relates this cocycles with the map φ.

Theorem 3.1.8 *Assume that $\langle\Theta(\phi)\rangle \subset w\langle\Theta\rangle$. Then, for each $(u,x) \in \mathcal{M}_\Theta(w)$*

$$\left|\det((d\varphi_{t,u})_{|\mathcal{U}_{\Theta,w}(u,x)})\right| = \mathrm{e}^{\mathsf{a}^+_{\Theta,w}(t,u,x)}$$

and

$$\left|\det((d\varphi_{t,u})_{|\mathcal{S}_{\Theta,w}(u,x)})\right| = \mathrm{e}^{\mathsf{a}^-_{\Theta,w}(t,u,x)}$$

for $t \geq 0$.

Proof. Let $k \in K$ such that $x = k \cdot wb_\Theta$. Then

$$(d\varphi_{t,u})_{|\mathcal{U}_{\Theta,w}(u,x)} = (d\varphi_{t,u}(k))_{|\mathfrak{n}_\phi^+ \cdot wb_\Theta}$$

and the Iwasawa decomposition give us $\varphi_{t,u}(k) = k_{t,u}a_{t,u}n_{t,u}$, with $a_{t,u} = \exp \mathsf{a}(t,u,k\cdot b_0)$ and $a_{t,u}n_{t,u} \in AN^+(\phi)$. We have that $N^+(\phi)$ centralizes $\mathfrak{n}_\phi^-$ and then $(dn_{t,u})_{\mathfrak{n}_\phi^-\cdot wb_\Theta} = \mathrm{id}_{\mathfrak{n}_\phi^-\cdot wb_\Theta}$. Also, the fact that K acts on $\mathbb{F}_\Theta$ by isometries give us

$$\left|\det((dk_{t,u})_{|\mathfrak{n}_\phi^+\cdot wb_\Theta})\right| = 1.$$

and consequently

$$\left|\det((d\varphi_{t,u})_{|\mathcal{U}_{\Theta,w}(u,x)})\right| = \left|\det((da_{t,u})_{\mathfrak{n}_\phi^+\cdot wb_\Theta})\right| = \det(\mathrm{Ad}(a_{t,u})_{|\mathfrak{n}_{\phi,w}^+}).$$

But since $a_{t,u} = \exp\big(\mathsf{a}(t,u,k\cdot b_0)\big)$ we have that $\mathrm{Ad}(a_{t,u})_{|\mathfrak{n}_{\phi,w}^+}$ is a diagonal matrix with $\alpha(\mathsf{a}(t,u,k\cdot b_0))$ in the diagonal. Moreover, $\mathfrak{n}_{\phi,w}^+ = \bigoplus_{\alpha\in\Pi_{\phi,\Theta,w}^-}\mathfrak{g}_\alpha$ and using Corollary 2.4.4, we have

$$\det(\mathrm{Ad}(a_{t,u})_{|\mathfrak{n}_{\phi,w}^+}) = \mathrm{e}^{\mathsf{a}_{\Theta,w}^+(t,u,x)}$$

and consequently

$$\left|\det((d\varphi_{t,u})_{|\mathcal{U}_{\Theta,w}(u,x)})\right| = \mathrm{e}^{\mathsf{a}_{\Theta,w}^+(t,u,x)}.$$

In the same way we show that

$$\left|\det((d\varphi_{t,u})_{|\mathcal{S}_{\Theta,w}(u,x)})\right| = \mathrm{e}^{\mathsf{a}_{\Theta,w}^-(t,u,x)}.$$

■

Remark 3.1.9 *We notice that in the proof of the above results we used that $n\cdot wb_\Theta = wb_\Theta$ for $N^\pm(\phi)$. It follows from the fact that we can choose the representative of w in $\mathcal{W}_{\Theta(\phi)}\setminus\mathcal{W}\,/\,\mathcal{W}_\Theta$. Since such choice does not change the set $\mathcal{M}_\Theta(w)$ and the cocycles $\mathsf{a}_{\Theta,w}^\pm(t,u,x)$ we will just assume that we have the adequate representative.*

3.2 Escape Entropy and Lower bounds on Flag manifolds

In this section we will look at the induced control-affine systems on the flag manifolds where we have hyperbolicity on some chain control sets and see the cases where we can get rid of the escape entropy.

By the results in the last section and Theorem 1.2.14 we have the following Corollary for the induced control-affine system (3.2) on $\mathbb{F}_\Theta$.

Corollary 3.2.1 *Consider the induced control-affine system on $\mathbb{F}_\Theta$ and let $w \in \mathcal{W}$ such that $E_{\Theta,w}$ is uniformly hyperbolic. Then, for any compact set $K \subset D_\Theta(w)$ with volume positive it holds that*

$$h_{\text{inv}}(K, E_{\Theta,w}) \geq \inf_{(u,x)\in\mathcal{M}_\Theta(w)} \limsup_{\tau\to\infty} \frac{1}{\tau}\mathsf{a}^+_{\Theta,w}(\tau,u,x) - h_{\text{esc}}(K, E_{\Theta,w}).$$

The idea is then look at the hyperbolic chain control sets and see which conditions we need in order to get rid of the escape entropy.

A first step in this direction is the following Theorem.

Theorem 3.2.2 *Let $\Theta \subset \Sigma$ and assume that $\Theta(\phi) \subset \Theta$. Then, for any compact set $K \subset \text{int}D^+_\Theta$ we have*

$$h_{\text{esc}}(K, D^+_\Theta) = 0$$

where D^+_Θ is the invariant control set in $\mathbb{F}_\Theta$.

Proof. The conditions $\Theta(\phi) \subset \Theta$ implies that for any $(u,x) \in \mathcal{M}^+$ we have $T_x\mathbb{F}_\Theta = \mathcal{S}_{\Theta,1}(u,x)$.

For given $u \in \mathcal{U}$ and $x \in E^+_\Theta$ we have by invariance in positive time of E^+_Θ that $\varphi_{t,u}(x) \in E^+_\Theta$ for any $t \geq 0$. Then, for $x_1, x_2 \in E^+_\Theta$ it holds that

$$\varrho(\varphi_{t,u}(x_1), \varphi_{t,u}(x_2)) \leq \max_{x\in\text{Im}(\gamma)} ||(d\varphi_{t,u})_x||\varrho(x_1,x_2)$$

where $\gamma : [0,1] \to \mathbb{F}_\Theta$ a geodesic connecting x_1 to x_2. For any $x \in K$ there is a convex neighborhood W_x of x contained in $\text{int}D^+_\Theta$, that is, for any two points in W_x there is a geodesic connecting them such that its image is still contained in W_x. Consider then the cover $\{W_x\}_{x\in K}$ of K. Since the set K is compact we have a finite cover $W_1, \ldots, W_n \in \{W_x\}_{x\in K}$ of K. Let $\delta > 0$ be the Lebesgue number of this finite cover. Then, for any $x_1, x_2 \in K$ such that $\varrho(x_1,x_2) < \delta$ there is a geodesic connecting x_1 and x_2 whose image is contained, in particular, in E^+_Θ, which implies by Remark 3.1.4, that $||(d\varphi_{t,u})_x|| \leq c^{-1}\mathrm{e}^{-\mu t}$ for any $t \geq 0$ and $x \in \text{Im}(\gamma)$ and consequently

$$\varrho(\varphi_{t,u}(x_1), \varphi_{t,u}(x_2)) \leq c^{-1}\mathrm{e}^{-\mu t}\varrho(x_1,x_2).$$

Then, for any $u \in \mathcal{U}$, $\tau, \varepsilon > 0$ such that $\varepsilon < \delta$ we have that

$$B_\varepsilon(x) \subset B^\tau_{c^{-1}\varepsilon}(u,x)$$

and then, if $\{B_\varepsilon(x_i)\}_{i=1}^{N(\varepsilon)}$ is a minimal cover of K, $\{B^\tau_{c^{-1}\varepsilon}(u,x_i)\}_{i=1}^{N(\varepsilon)}$ is also a cover of K, which implies that

$$r_{\text{span}}(u,\tau,\varepsilon,K,D^+_\Theta) \leq N(\varepsilon)$$

and consequently

$$h_{\text{esc}}(K,D^+_\Theta) = 0$$

showing the desired. ■

We note that although the invariance entropy in D^+_Θ is trivially zero, the fact that $h_{\text{esc}}(K,D^+_\Theta) = 0$ will help us to estimate the escape entropy on the other control sets and consequently improve the lower bounds for the invariance entropy on them. Also, we do not know if the escape entropy of D^+_Θ id zero without the condition $\Theta(\phi) \subset \Theta$.

The case without multiplicity

Let us assume now that $\Theta(\phi) = \emptyset$, that is, the induced control-affine systems (3.2) are hyperbolic over all chain control set on all flag manifolds. We will show that in this case the escape entropy is always zero what give us a good lower bound for the invariance entropy on every control set. Since all the chain control sets are hyperbolic, we have that all control set $D_\Theta(w)$ satisfies $\text{cl}D_\Theta(w) = E_{w,\Theta}$ for every $w \in \mathcal{W}$ and $\Theta \subset \Sigma$.

Let $\alpha \in \Sigma$ and consider the fibration $\pi_\alpha : \mathbb{F} \to \mathbb{F}_\alpha$. It is well known that the set $N^-_\alpha = \exp\{\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}\}$ is dense on the fiber $\pi_\alpha^{-1}\pi_\alpha(b_0)$, where b_0 is the origin in $\mathbb{F}$.

Since $\Theta(S) \subset \Theta(\phi) = \emptyset$ we have that

$$D(w_1) \leq D(w_2) \qquad \text{iff} \qquad w_1 \geq w_2$$

and consequently the only control sets in $\mathbb{F}$ that project on $D_\alpha(w) \subset \mathbb{F}_\alpha$ are $D(w)$ and $D(ws_\alpha)$ (see Theorem 4.1 of [29]). The following theorem give us a way to compare the escape entropy of the control sets $D(w)$ and $D(ws_\alpha)$.

Theorem 3.2.3 *Let $w \in \mathcal{W}$ and $\alpha \in \Sigma$. For a given compact set $K \subset D(w)$ there exists $K' \subset D(ws_\alpha)$ such that*

1. *If $w(\alpha) < 0$, then*

$$h_{\text{esc}}(K,E_w) \leq h_{\text{esc}}(K',E_{ws_\alpha}).$$

2. If $w(\alpha) > 0$, then

$$h_{\text{esc}}(K, E_w) \geq h_{\text{esc}}(K', E_{ws_\alpha}).$$

Proof. We will just show 1. since 2. is analogous. Then $w(\alpha) < 0$ and we have that $ws_\alpha \leq w$ and consequently $D(w) \leq D(ws_\alpha)$. By the above, such control sets are the only control sets that project onto $D_\alpha(w)$.

Let $K'' = \pi_\alpha(K)$ and $E_{\alpha,w} = \text{cl}D_\alpha(w) = \pi_\alpha(E_{ws_\alpha})$, and consider $K' \subset D(ws_\alpha)$ such that $\pi_\alpha(K') = K''$. Let $(u, x) \in \mathcal{K}''$, where $\mathcal{K}''$ is the lift of K''. The fact that $(u, x) \in \mathcal{K}''$ implies, by the G-invariance of π_α, that for every $z \in \pi_\alpha^{-1}(x)$ we have $\varphi_{t,u}(z) \in \pi_\alpha^{-1}(E_{\alpha,w})$ for $t \geq 0$. Since $\pi_\alpha^{-1}(E_{\alpha,w})$ is compact, $\varphi_{t,u}(z)$ has to converge to a chain control set. Then, if $z \in \pi_\alpha^{-1}(x) \cap E_{ws_\alpha}$ we have, by Theorem 6.3 of [29] and the no-return property[1], that $\varphi_{t,u}(z) \in E_{ws_\alpha}$ for $t \geq 0$. By Theorem 1.2.18 item (i) we have then that

$$h_{\text{esc}}(K', E_{ws_\alpha}) \geq h_{\text{esc}}(K'', E_{\alpha,w})$$

where $K' \subset D(ws_\alpha)$ is a compact set that satisfies $\pi_\alpha(K') = K''$.

Let $(u, x) \in \mathcal{M}(w)$ with $x = k \cdot wb_0$. Since $w(\alpha) < 0$ we have that $wN_\alpha^- \cdot b_0 \subset N^+ \cdot wb_0$ and then $\varphi_{t,u}$ restricted to $kwN_\alpha^- \cdot b_0$ is uniformly expanding because the tangent space to $kwN_\alpha^- \cdot b_0$ is contained in $\mathcal{U}_w(u, x)$. But $kwN_\alpha^- \cdot b_0$ is dense in the fiber $(E_w)_x$ which implies, by the continuity of $\varphi_{t,u}$ and Remark 3.1.4, that uniformly expanding holds in $(E_w)_x$ inside E_w what give us

$$h_{\text{esc}}(K, E_w) \leq h_{\text{esc}}(K'', E_{\alpha,w})$$

by item (ii) of Theorem 1.2.18 and consequently

$$h_{\text{esc}}(K', E_{ws_\alpha}) \geq h_{\text{esc}}(K, E_w)$$

concluding the proof. ■

Remark 3.2.4 *We note that in order to obtain the compact K' we just used that the projection π_α restricted to $D(ws_\alpha)$ is proper and open and the flags are locally compact, that is, we can lift compact sets.*

As a direct corollary we have.

[1] A set Q has the no-return property if given $x \in Q$, $u \in \mathcal{U}$ and $\tau > 0$ such that $\varphi_{\tau,u}(x) \in Q$ it implies that $\varphi_{t,u}(x) \in Q$ for all $t \in [0, \tau]$.

Corollary 3.2.5 *For any control set $D(w) \subset \mathbb{F}$ and any compact set $K \subset \mathrm{int}D(w)$ we have that*

$$h_{\mathrm{esc}}(K, E_w) = 0.$$

Proof. Let us proceed by induction in the length of w. If $l(w) = 1$ we have that $w = s_\alpha$ and since $s_\alpha(\alpha) = -\alpha$, the above Theorem implies that, for any compact set $K \subset \mathrm{int}\, D(s_\alpha)$ there is $K' \subset \mathrm{int}\, D^+ = E^+$ such that

$$h_{\mathrm{esc}}(K, E_{s_\alpha}) \leq h_{\mathrm{esc}}(K', E^+)$$

and Theorem 3.2.2 implies then that $h_{\mathrm{inv}}(K, E_{s_\alpha}) = 0$. Let us assume that $h_{\mathrm{inv}}(K, E_w) = 0$ for every $w \in \mathcal{W}$, $K \in \mathrm{int}\, D(w)$ such that $l(w) < k$ and let w such that $l(w) = k$. Since $k \geq 1$, there exist $\alpha \in \Sigma$ such that $w(\alpha) < 0$ and Theorem 3.2.3 give us that

$$h_{\mathrm{esc}}(K, E_w) \leq h_{\mathrm{esc}}(K', E_{ws_\alpha})$$

for any $K \subset D(w)$ and some compact set $K' \subset D(ws_\alpha)$. Since $w(\alpha) < 0$ we have that $l(ws_\alpha) = l(w) - 1 < k$ and the inductive hypothesis implies $h_{\mathrm{esc}}(K', E_{ws_\alpha}) = 0$ showing the result. ■

That give us also that the control sets on the other flag manifolds have also zero escape entropy.

Corollary 3.2.6 *Let $\Theta \subset \Sigma$ and $w \in \mathcal{W}$. If $K \subset \mathrm{int}D_\Theta(w)$ is a compact set with nonempty interior, then*

$$h_{\mathrm{esc}}(K, E_{\Theta,w}) = 0.$$

Proof. As in the demonstration above, we have that the greatest control set of the induced system on $\mathbb{F}$ that projects onto $D_\Theta(w)$ satisfies condition (i) of Theorem 1.2.18. Using the above Corollary, we have that the escape entropy of all control sets on $\mathbb{F}$ is zero which implies $h_{\mathrm{esc}}(K, E_{\Theta,w}) = 0$ for any compact set $K \subset D_\Theta(w)$. ■

Corollary 3.2.7 *Let $\Theta \subset \Sigma$ and $w \in \mathcal{W}$. If $K \subset D_\Theta(w)$ is a compact set with positive volume, then*

$$h_{\mathrm{inv}}(K, E_w) \geq \inf_{(u,x)\in\mathcal{M}_\Theta(w)} \limsup_{\tau\to\infty} \frac{1}{\tau} \mathsf{a}^+_{\Theta,w}(\tau, u, x).$$

Proof. Its a consequence of the above Corollaries and Proposition 5.1 of [21]. ■

The above Theorems and Corollaries give us a really nice lower bound in the case that the flag type of the control flow has no multiplicity, that is, $\Theta(\phi) = \emptyset$.

The case with multiplicities

The method used above require us to have a fibration between flag manifolds. In this subsection we will try to generalize this in order to show that the escape entropy of the hyperbolic control sets in the flag $\mathbb{F}_{\Theta(\phi)}$ vanishes. In order to do that we will have to assume that there are simple roots in $\Sigma \setminus \Theta(\phi)$ that are orthogonal to $\Theta(\phi)$.

Denote $\Theta(\phi)$ simple by Θ and consider as before the set $N_\alpha^- = \exp\{\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}\}$. Assume that $\alpha \in \Sigma \setminus \Theta$ and $\langle \alpha, \beta \rangle = 0$ for any $\beta \in \Theta$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathfrak{a}^*$ induced by the Cartan-Killing form. Since α has no connection with Θ we have for the projection $\pi_\alpha := \pi^\Theta_{\Theta_\alpha} : \mathbb{F}_\Theta \to \mathbb{F}_{\Theta_\alpha}$ that $N_\alpha^- \cdot b_\Theta$ is dense on the fiber $\pi_\alpha^{-1}\pi_\alpha(b_\Theta)$. We will say that $w \in \mathcal{W}$ is **orthogonal** to Θ if there is a minimal decomposition $w = s_1 \dots s_n$ with $\alpha_i \in \Sigma \setminus \Theta$ and $\langle \alpha_i, \beta \rangle = 0$ for any $i = 1, \dots, n$ and $\beta \in \Theta$.

With that we have a similar result for hyperbolic chain control sets in $\mathbb{F}_\Theta$, as stated in the next Theorem.

Theorem 3.2.8 *Let $w \in \mathcal{W}$ and assume that w is orthogonal to Θ. Then for any compact set $K \subset \mathrm{int} D_\Theta(w)$ we have that*

$$h_{\mathrm{esc}}(K, E_{\Theta,w}) = 0.$$

Proof. Let $w \in \mathcal{W}$ to be orthogonal to Θ and assume that $w(\alpha) < 0$ for $\alpha \in \Sigma \setminus \Theta$ and $\langle \alpha, \beta \rangle = 0$, $\beta \in \Theta$. The fact that $\langle \alpha, \beta \rangle = 0$ for any $\beta \in \Theta$ implies that s_α commutes with any element of $\mathcal{W}_\Theta$ which implies that $D_\Theta(ws_\alpha)$ and $D_\Theta(w)$ are the only control sets that project onto $D_{\Theta_\alpha}(w)$, for $\Theta_\alpha = \Theta \cup \{\alpha\}$ (see Proposition 7.1 of [29]). The proof follows then in the exactly same way as in Theorem 3.2.3. ■

We note that what plays a central role here is that there are just two control sets that projects onto $D_{\Theta_\alpha}(w)$. As in the case without multiplicities we have the same results for hyperbolic chain control sets on smaller flags.

Corollary 3.2.9 *Let $\Theta \in \Sigma$ and $w \in \mathcal{W}$ such that $\Theta(\phi) \subset \Theta$, w is orthogonal to Θ and $w(\Theta) \subset \Pi^+$. Then for any compact set $K \subset \mathrm{int} D_\Theta(w)$ we have*

$$h_{\mathrm{esc}}(K, E_{\Theta,w}) = 0$$

Remark 3.2.10 *Let us comment the conditions on the above Corollary. The first one, that $\Theta(\phi) \subset \Theta$, is just to assure that we have the fibration $\mathbb{F}_{\Theta(\phi)} \to \mathbb{F}_\Theta$; The second one assure hyperbolicity of the chain control sets $E_{\Theta(\phi),w}$ and $E_{\Theta,w}$ on $\mathbb{F}_{\Theta(\phi)}$ and $\mathbb{F}_\Theta$, respectively. The extra condition, that $w(\Theta) \subset \Pi^+$, is to assure that the control set $D_{\Theta(\phi)}(w)$ on $\mathbb{F}_{\Theta(\phi)}$ is the greatest control set of the system that projects onto $D_\Theta(w)$ on $\mathbb{F}_\Theta$ what gives us that condition (i) in Theorem 3.2 is satisfied.*

Concerning the invariance entropy we have then:

Corollary 3.2.11 *Let $\Theta \in \Sigma$ and $w \in \mathcal{W}$ such that $\Theta(\phi) \subset \Theta$, w is orthogonal to Θ and $w(\Theta) \subset \Pi^+$. Then for any compact set $K \subset D_\Theta(w)$ we have*

$$h_{\mathrm{inv}}(K, E_{\Theta,w}) \geq \inf_{(u,x)\in\mathcal{M}_\Theta(w)} \limsup_{\tau\to\infty} \frac{1}{\tau} \mathsf{a}^+_{\Theta,w}(\tau, u, x).$$

The above results show that the geometry of the flag manifolds allows us to get rid of the escape entropy in many cases.

3.3 Upper bound on Hyperbolic Chain Control sets

In this section we will assume that we have $w\langle\Theta\rangle \supset \langle\Theta(\phi)\rangle$, that is, the chain control set $E_{\Theta,w}$ is uniformly hyperbolic. With that we can generalize the results of [21] for induced systems on projective spaces. First of all note that since we are considering the semi-simple case, if we assume local accessibility of the system 3.1 on G we have strongly accessibility. In fact, since the system is right invariant, we just have to assume that $\Delta_{\mathcal{L}(\mathcal{F})}(1) = \mathfrak{g}$. By [33] we have that the ideal generated by $f_1, \ldots, f_m$ has dimension d or $d-1$ where $d = \dim G$. Since the orthogonal complementary of an ideal is also an ideal we must have that this dimension in d, otherwise $\mathfrak{g}$ would have an abelian ideal, what cannot happen in a semi-simple Lie algebra.

Consider the function $\mathsf{h} : \mathcal{U} \to \mathrm{Ad}(G)H_\phi$ defined in the last section. Since we are assuming that $w\langle\Theta\rangle \supset \langle\Theta(\phi)\rangle$ we have that the set $\mathrm{fix}_\Theta(\mathsf{h}(u), w)$ reduces to a point and then we have a well defined function $\sigma_{\Theta,w} : \mathcal{U} \to \mathcal{M}_\Theta(w)$ given by

$$u \mapsto \sigma_{\Theta,w}(u) := (u, \mathrm{fix}_\Theta(\mathsf{h}(u), w)).$$

Lemma 3.3.1 *The function $\sigma_{\Theta,w} : \mathcal{U} \to \mathcal{M}_\Theta(w)$ is a homeomorphism that conjugates the control flow and the shift, that is,*

$$\sigma_{\Theta,w}(\theta_t u) = \phi_t(\sigma_{\Theta,w}(u)).$$

Proof. The conjugation follows directly from the properties of h. If we denote by $\pi_{\Theta,w}$ the restriction to $\mathcal{M}_\Theta(w)$ of the projection on $\mathcal{U}$, we have that $\pi_{\Theta,w} \circ \sigma_{\Theta,w} = \mathrm{id}_\mathcal{U}$ and $\sigma_{\Theta,w} \circ \pi_{\Theta,w} = \mathrm{id}_{\mathcal{M}_\Theta(w)}$. Then, the continuity of $\sigma_{\Theta,w}$ follows from the continuity of $\pi_{\Theta,w}$ and the fact that $\mathcal{M}_\Theta(w)$ is a compact set. ■

We can now slightly improve the upper bound for the invariance entropy of hyperbolic sets as above. The next result was first shown in [21] for projective spaces.

Theorem 3.3.2 *Consider the induced system on $\mathbb{F}_\Theta$ and assume that $w\langle\Theta\rangle \supset \langle\Theta(\phi)\rangle$. Let $D_\Theta(w)$ be the only control set contained in the hyperbolic chain control set $E_{\Theta,w}$. Then, for every compact $K \subset D_\Theta(w)$ we have*

$$h_{\mathrm{inv}}(K, E_{\Theta,w}) \leq \inf_{(u,x)\in\mathcal{M}_\Theta^{\mathrm{Per}}(w)} \limsup_{\tau\to\infty} \frac{1}{\tau} \mathsf{a}^+_{\Theta,w}(\tau, u, x)$$

where the $\mathcal{M}_\Theta^{\mathrm{Per}}(w)$ denotes the subset of $\mathcal{M}_\Theta(w)$ of the periodic points.

Proof. Proposition 1.2.10 assures that

$$h_{\mathrm{inv}}(K, E_{\Theta,w}) \leq \inf_{(u,x)} \limsup_{\tau\to\infty} \frac{1}{\tau} \mathsf{a}^+_{\Theta,w}(\tau, u, x)$$

for periodic points $(u, x) \in \mathcal{M}_\Theta(w)$ such that $(u, x) \in \mathrm{int}\mathcal{U} \times \mathrm{int}D_\Theta(w)$.

As in the Proposition 7.10 of [21], the change of $\mathrm{int}\mathcal{U} \times \mathrm{int}D_\Theta(w)$ to $\mathcal{U} \times \mathrm{int}D_\Theta(w)$ is straightforward.

Consider then an arbitrary periodic point (u,x) with period $\tau>0$. By Proposition 1.6 in [21] we have a sequence of piecewise constant control functions $u_n\in\mathcal{U}$ such that $\psi_{t,u_n}\to\psi_{t,u}$ for $n\to\infty$ uniformly in $t\in[0,\tau]$.

Let us denote by $g=\psi_{\tau,u}$ and similarly $g_n=\psi_{\tau,u_n}$. We have $g,g_n\in S$ and being $\mathrm{int}S$ dense in S we can actually assume that $g_n\in\mathrm{int}S$. Since we can assume that u_n is periodic, we have that $g_nx_n=x_n$, where $x_n=\mathrm{fix}_\Theta(\mathsf{h}(u_n),w))$. Also, as in Lemma 5.2 of [27] we can assume that there is a potency l_n such that $g_n^{l_n}$ is a regular element and consequently $x_n\in D_\Theta(w)_0\subset\mathrm{int}D_\Theta(w)$. The continuity of $\sigma_{\Theta,w}$ implies then that $x_n\to x$.

Since $\mathsf{a}^+_{\Theta,w}$ is continuous we have then

$$\frac{1}{\tau}\mathsf{a}^+_{\Theta,w}(\tau,u_n,x_n)=\frac{1}{\tau}\log A(g_nx_n)\to\frac{1}{\tau}\log A(gx)=\frac{1}{\tau}\mathsf{a}^+_{\Theta,w}(\tau,u,x)$$

and the result follows. ■

The Theorem above together with Theorem 3.2.8 give us the following.

Theorem 3.3.3 *Let (3.2) be the induce control-affine system on $\mathbb{F}_{\Theta(\phi)}$. Let $w\in\mathcal{W}$ orthogonal to $\Theta(\phi)$ and let $K\subset\mathrm{int}\,D_\Theta(w)$ be any compact set with nonempty interior. It holds that*

$$h_{\mathrm{inv}}(K,E_{\Theta,w})\geq\inf_{(u,x)\in\mathcal{M}_\Theta(w)}\limsup_{\tau\to\infty}\frac{1}{\tau}\mathsf{a}^+_{\Theta,w}(\tau,u,x)$$

and

$$h_{\mathrm{inv}}(K,E_{\Theta,w})\leq\inf_{(u,x)\in\mathcal{M}^{\mathrm{Per}}_\Theta(w)}\limsup_{\tau\to\infty}\frac{1}{\tau}\mathsf{a}^+_{\Theta,w}(\tau,u,x).$$

In particular, if $\Theta(\phi)=\emptyset$ the above is true for any control set of the induced system on any flag manifold.

Remark 3.3.4 *We do not know if just under the hypothesis $\langle\Theta(\phi)\rangle\subset w\langle\Theta\rangle$ we have that the Theorem 3.2.8 is still valid. If we had a decomposition of w of the form $w=w_1w_2$ with w_1 orthogonal to $\Theta(\phi)$ and $w_2\in\mathcal{W}_\Theta$, then the result would also be true.*

Remark 3.3.5 *We note, by Lemma 5.5 of [21] and comments before it, that for every periodic point $(u,x)\in\mathcal{M}_\Theta(w)$ with period τ^* and for $w\in\mathcal{W}$ as above that*

$$\limsup_{\tau\to\infty}\frac{1}{\tau}\mathsf{a}^+_{\Theta,w}(\tau,u,x)=\sum_{\alpha\in\Pi^+_{\phi,\Theta,w}}n_\alpha\alpha(\lambda(u,k\cdot b_0))$$

where

$$\lambda(u, k \cdot b_0) = \lim_{\tau \to \infty} \frac{1}{\tau} \mathsf{a}(\tau, u, k \cdot b_0) = \frac{1}{\tau^*} \mathsf{a}(\tau^*, u, k \cdot b_0)$$

and $x = k \cdot w b_\Theta$. *In particular the numbers* $\alpha(\lambda(u, x))$ *for* $\alpha \in \Pi^+_{\phi,\Theta,w}$ *coincides with the positive Lyapunov exponents of the system in* (u, x) *and consequently we have, in the conditions of Theorem 3.3.3, that the invariance entropy* $h_{\mathrm{inv}}(K, E_{\Theta,w})$ *is bounded above by the infimum over the sum of all the positive Lyapunov exponents (counted with multiplicities) for periodic points of the system.*

Chapter 4

Linear Systems on Lie Groups

Our aim in this Chapter is to apply some general results about the outer invariance entropy to a linear system on Lie groups as introduced in [4] and [5]. A linear control system on a Lie group G is defined by

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) f_j(g(t)),$$

where the drift vector field $\mathcal{X}$ is an infinitesimal automorphism, f_j are right invariant vector fields and $u = (u_1, \cdots, u_m)$ belongs to the class of admissible controls functions $\mathcal{U}$.

4.1 Linear Systems

Consider the system

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) f_j(g(t)), \tag{4.1}$$

with the conditions above. Let $(\psi_t)_{t\in\mathbb{R}}$ denote the one parameter group of automorphisms of G generated by $\mathcal{X}$ and by $e \in G$ the identity element of G. For all right invariant vector fields Y, we have

$$[\mathcal{X}, Y](e) = \frac{d}{dt}_{|t=0} (d\psi_{-t})_{\psi_t(e)} Y(\psi_t(e)) = \frac{d}{dt}_{|t=0} (d\psi_{-t})_e Y(e) \tag{4.2}$$

since $\psi_t(e) = e$ for all $t \in \mathbb{R}$. Considering that $\psi_{-t} \circ R_{\psi_t(g)} = R_g \circ \psi_{-t}$, for all $g \in G$, we have that

$$[\mathcal{X}, Y](g) = \frac{d}{dt}_{|t=0} (d\psi_{-t})_{\psi_t(g)} Y(\psi_t(g)) = \frac{d}{dt}_{|t=0} (d\psi_{-t})_{\psi_t(g)} (dR_{\psi_t(g)})_e Y(e) =$$

$$= \frac{d}{dt}_{|t=0} (dR_g)_e (d\psi_{-t})_e Y(e) = (dR_g)_e [\mathcal{X}, Y](e).$$

Then for a given linear vector field, one can associate the derivation $\mathcal{D}$ of $\mathfrak{g}$ defined by

$$\mathcal{D}Y = -[\mathcal{X}, Y], \text{ for all } Y \in \mathfrak{g},$$

that is, $\mathcal{D} = -\text{ad}(\mathcal{X})$. The minus sign in this definition comes from the formula $[Ax, b] = -Ab$ in $\mathbb{R}^n$. It also enable us to avoid a minus sign in the equality

$$\psi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}} Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

stated in the forthcoming proposition.

Proposition 4.1.1 *For all* $t \in \mathbb{R}$

$$(d\psi_t)_e = \mathrm{e}^{t\mathcal{D}}$$

and consequently

$$\psi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}} Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

Proof. Let us first show the equality

$$\frac{d}{dt} (d\psi_t)_e Y(e) = \mathcal{D}(d\psi_t)_e Y(e).$$

This equality has already been shown for $t = 0$ (see equality (4.2) above). In general,

$$\frac{d}{dt} (d\psi_t)_e Y(e) = \frac{d}{ds}_{|s=0} (d\psi_{t+s})_e Y(e) =$$

$$= \frac{d}{ds}_{|s=0} (d\psi_s)_e (d\psi_t)_e Y(e) = \mathcal{D}(d\psi_t)_e Y(e).$$

From the formula above, the first formula of the proposition is immediate. For the second one, note that ψ_t is a Lie group morphism. Therefore

$$\psi_t(\exp Y) = \exp(d\psi_t)_e Y = \exp(\mathrm{e}^{t\mathcal{D}} Y). \tag{4.3}$$

■

We have also the following proposition about the solutions of (4.1).

Proposition 4.1.2 *For a given $u \in \mathcal{U}$, let us denote by $\zeta_{t,u}$ the solution of (4.1) starting at the origin $e \in G$. Then the solutions of (4.1) are given by*

$$\varphi(t, g, u) = \zeta_{t,u}\,\psi_t(g) = L_{\zeta_{t,u}}(\psi_t(g)),$$

for each $g \in G$.

Proof. Let us consider the curve $\alpha(t)$ given by

$$\alpha(t) = \zeta_{t,u}\psi_t(g).$$

We have that $\alpha(0) = g$ and

$$\dot{\alpha}(t) = (dL_{\zeta_{t,u}})_{\psi_t(g)}\frac{d}{dt}\psi_t(g) + (dR_{\psi_t(g)})_{\zeta_{t,u}}\frac{d}{dt}\zeta_{t,u} =$$

$$= (dL_{\zeta_{t,u}})_{\psi_t(g)}\mathcal{X}(\psi_t(g)) + (dR_{\psi_t(g)})_{\zeta_{t,u}}\left\{\mathcal{X}(\zeta_{t,u}) + \sum_{j=1}^{m} u_j(t) f_j(\zeta_{t,u})\right\} =$$

$$= \left\{(dL_{\zeta_{t,u}})_{\psi_t(g)}\mathcal{X}(\psi_t(g)) + (dR_{\psi_t(g)})_{\zeta_{t,u}}\mathcal{X}(\zeta_{t,u})\right\} + \sum_{j=1}^{m} u_j(t) f_j(\alpha(t)).$$

Since ψ_t is a flow of automorphism we have that

$$\mathcal{X}(kh) = \frac{d}{ds}_{|s=0}\psi_s(kh) = (dL_k)_h\mathcal{X}(h) + (dR_h)_k\mathcal{X}(k)$$

what give us, taking $k = \zeta_{t,u}$ and $h = \psi_t(g)$,

$$(dL_{\zeta_{t,u}})_{\psi_t(g)}\mathcal{X}(\psi_t(g)) + (dR_{\psi_t(g)})_{\zeta_{t,u}}\mathcal{X}(\zeta_{t,u}) = \mathcal{X}(\zeta_{t,u}\psi_t(g)) = \mathcal{X}(\alpha(t))$$

and consequently

$$\dot{\alpha}(t) = \mathcal{X}(\alpha(t)) + \sum_{j=1}^{m} f_j(\alpha(t)).$$

By the unicity of the solution, we have the desired. ■

Remark 4.1.3 *The formula for the solution of the linear system (4.1) in the above Proposition corresponds to the variation-of-constants formula in the Euclidean case.*

Remark 4.1.4 *The solution of the system without the drift*

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{4.4}$$

at $e \in G$ *coincides with* $\zeta_{t,u}$. *Since the vector* f_i *are right invariant its solutions at any* $g \in G$ *are given by* $\zeta(t, g, u) = \zeta_{t,u}(g) = \zeta_{t,u} g$.

The idea is to show that the outer invariant entropy for the system (4.1) is given in terms of the positive real parts of the eigenvalues of the derivation $\mathcal{D}$, that generalizes the result for linear control systems in $\mathbb{R}^d$ (Theorem 5.1 of [10]). Before that we will introduce the notion of topological entropy. Let (X, d) be a metric space and $\phi : \mathbb{R} \times X \to X$ be a flow over X. For a given compact set $K \subset X$ and $\varepsilon, \tau > 0$ we say that a set $S_{\text{top}} \subset X$ is (τ, ε)-spanning set for K if, for every $y \in K$ there exist $x \in S_{\text{top}}$ such that

$$d(\phi_t(x), \phi_t(y)) < \varepsilon \quad \text{for all} \quad t \in [0, \tau].$$

If we denote by $r_\tau(\varepsilon, K)$ the minimal cardinallity of a spanning set, the topological entropy of ϕ over K is defined by

$$h_{\text{top}}(\phi, K) := \lim_{\varepsilon \searrow 0} \limsup_{\tau \to \infty} \frac{1}{\tau} r_\tau(\varepsilon, K)$$

and the **topological entropy** of ϕ as

$$h_{\text{top}}(\phi) := \sup_{K \text{compact}} h_{\text{top}}(\phi, K).$$

We should note that the topological entropy does not change for uniformly equivalent metrics. We have then the following Theorems.

Theorem 4.1.5 *Let* (K, Q) *be an admissible pair for the linear system (4.1) on* G *and assume that* Q *is compact. Then, the outer invariant entropy satisfies*

$$h_{\text{inv,out}}(K, Q) \leq \sum_{\lambda_{\mathcal{D}} > 0} \lambda_{\mathcal{D}}$$

where $\lambda_{\mathcal{D}}$ *are the real parts of the eigenvalues of the derivation* $\mathcal{D}$.

Proof. As shown in [21] Proposition 2.5, the invariance entropy does not depends on the metric that we choose. Let then ϱ be a left invariant metric on G. By proposition 4.1.2 we have that two solutions $\varphi_{t,u}(g)$ and $\varphi_{t,u}(g')$, satisfies

$$\varrho(\varphi_{t,u}(g'), \varphi_{t,u}(g)) = \varrho(\zeta_{t,u}\psi_t(g'), \zeta_{t,u}\psi_t(g)) = \varrho(\psi_t(g'), \psi_t(g)).$$

Using such equality consider S_{top} be a minimal (τ, ε)-spanning set for K of the flow ψ_t, that is, for all $g' \in K$ there exists $g \in S_{\text{top}}$ such that

$$\varrho(\psi_t(g), \psi_t(g')) < \varepsilon \quad \text{for all } t \in [0, \tau].$$

Since (K, Q) is admissible and we can assume w.l.o.g. that $S_{\text{top}} \subset K$, there exists for each $g \in S_{\text{top}}$, $u_g \in \mathcal{U}$ such that $\varphi_{t,u_g}(g) \in Q$ for all $t \in \mathbb{R}$. Then for all $g' \in K$, there exists u_g such that

$$\varrho(\varphi_{t,u_g}(g'), \varphi_{t,u_g}(g)) = \varrho(\zeta_{t,u_g}\psi_t(g'), \zeta_{t,u_g}\psi_t(g)) = \varrho(\psi_t(g), \psi_t(g)) < \varepsilon$$

for all $t \in [0, \tau]$, that is, $\varphi_{t,u_g}(g') \in N_\varepsilon(Q)$ showing that $\{u_g;\ g \in S_{\text{top}}\}$ is a (τ, ε)-spanning set for (K, Q) and we conclude that

$$h_{\text{inv,out}}(K, Q) \leq h_{\text{top}}(\psi) = h_{\text{top}}(\psi_1)$$

and by [8] we have that

$$h_{\text{top}}(\psi_1) = \sum_{\alpha;\, |\alpha|>1} \log |\alpha|$$

where α are the eigenvalues of $(d\psi_1)_e$. Since by Proposition (4.1.1) we have that

$$(d\psi_1)_e = \mathrm{e}^{\mathcal{D}},$$

the eigenvalues of $(d\psi_1)_e$ are given by the exponential of the eigenvalues of $\mathcal{D}$ and consequently $|\alpha| = \mathrm{e}^{\lambda_{\mathcal{D}}}$, where $\lambda_{\mathcal{D}}$ is the real part of some eigenvalue of $\mathcal{D}$. Then

$$h_{\text{top}}(\varphi_1) = \sum_{\alpha;\, |\alpha|>1} \log |\alpha| = \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}}$$

as desired. ∎

Remark 4.1.6 *The result proved in Bowen [8] is actually for a right invariant metric on G. But since on a Lie group we always have right and left invariant metrics and right and left Haar measures, Proposition 10 of [8] gives us that the topological entropy of ψ_1 for the left and right invariant metrics are the same.*

As proved in [10], we would also like to show that for linear systems as (4.1), we have actually that the entropy coincides with the positive eigenvalues of $\mathcal{D}$. The next theorem give us then a lower bound in this direction.

Theorem 4.1.7 *Consider the system (4.1) and let (K, Q) be an admissible pair with Q compact. Let d_G be a left invariant Haar measure and assume that $d_G(K) > 0$. Then the estimate*

$$h_{\mathrm{inv,out}}(K, Q) \geq \sum \lambda_{\mathcal{D}}$$

holds, where $\lambda_{\mathcal{D}}$ are the real parts of the eigenvalues of the derivation $\mathcal{D}$.

Proof. By Proposition 1.2.6 we can consider just spanning sets for natural numbers. Then let $n, \varepsilon > 0$ and consider $S = \{u_1, \cdots, u_k\}$ be a (n, ε)-spanning set for (K, Q). Define the sets

$$K_j = \{g \in K; \varphi([0, n], g, u_j) \subset N_\varepsilon(Q)\}, \qquad j = 1, \cdots k.$$

Then by definition of (n, ε)-spanning sets, K_j is a Borel set for each j and also $K = \cup_j K_j$. Also, since the solution map $\varphi_{n,u_j} : G \to G$ is a diffeomorphism, $\varphi_{n,u_j}(K_j)$ is also a Borel set and satisfies

$$d_G(\varphi_{n,u_j}(K_j)) \leq d_G(N_\varepsilon(Q)).$$

Because of the left invariance, we have that

$$d_G(\varphi_{n,u_j}(K_j)) = d_G(\zeta_{n,u_j}\psi_n(K_j)) = d_G(\psi_n(K_j)).$$

Using that $\psi_n \circ L_g = L_{\psi_n(g)} \circ \psi_n$ and that $\det(dL_g)_h = 1$ for each $g, h \in G$, we have that

$$|\det(d\psi_n)_g| = |\det(dL_{\psi_n(g)})_e||\det(d\psi_n)_e||\det(dL_{g^{-1}})_g| = \mathrm{e}^{n \sum \lambda_{\mathcal{D}}}$$

and then

$$d_G(\psi_n(K_j)) = \int_{\psi_n(K_j)} d_G(g) = \int_{K_j} |\det(d\psi_n)_g| d_G(g)$$

$$= \mathrm{e}^{n\sum\lambda_{\mathcal{D}}} \int_{K_j} d_G(g) = \mathrm{e}^{n\sum\lambda_{\mathcal{D}}} d_G(K_j),$$

by left invariance of the Haar measure.

Taking j_0 such that $\max_j d_G(K_j) = d_G(K_{j_0})$ we have

$$d_G(K) \leq \sum_{j=1}^{k} d_G(K_j) \leq k \cdot d_G(K_{j_0}) = k \cdot \frac{d_G(N_\varepsilon(Q))}{\mathrm{e}^{n\sum\lambda_{\mathcal{D}}}}$$

and consequently

$$r_{\mathrm{inv,out}}(n, \varepsilon, K, Q) \geq \mathrm{e}^{n\sum\lambda_{\mathcal{D}}} \cdot \frac{d_G(K)}{d_G(N_\varepsilon(Q))}$$

which implies

$$h_{\mathrm{inv,out}}(K, Q) \geq \sum \lambda_{\mathcal{D}}$$

as desired.

■

We would like to show that as for the linear case in $\mathbb{R}^n$, we have that the entropy coincides with the sum of the positive real part of the eigenvalues of $\mathcal{D}$. As we will see that is closely connected with the geometry of the space that we consider.

4.1.1 Quasi-Invariant Measures

Let G be a locally compact topological group and denote by d_G its left Haar measure. The *right modular function* for G (or more briefly the modular function for G) δ_G is defined by the relations

$$\int_G f(gh^{-1}) d_G(g) = \delta_G(h) \int_G f(g) d_G(g)$$

$$\int_G f(g^{-1}) d_G(g) = \int_G \frac{f(g)}{\delta_G(g)} d_G(g)$$

for all $f \in C_c(G)$. We remind that δ_G is a continuous homomorphism of G into the multiplicative group $\mathbb{R}_+$ of strictly positive real numbers. When G is a Lie group, we have

$$\delta_G(g) = |\det(\mathrm{Ad}(g^{-1}))|.$$

Let H be a closed subgroup of G. The canonical projection of G onto G/H will be denoted by $g \to \pi(g)$ or $g \to \bar{g} = g \cdot H$. Let μ a Radon measure on G/H. The transform μ^g of μ by an element $g \in G$ is the measure

$$f \mapsto \mu(f^{g^{-1}})$$

where $f \in C_c(G/H)$ and

$$f^g(\bar{q}) := f(g^{-1} \cdot \bar{q}) \qquad g, q \in G; \;\; g \cdot \bar{q} = gq \cdot H.$$

Definition 4.1.8 *A positive Radon measure μ on G/H is said to be* **quasi-invariant** *if μ and μ^g are equivalent for all $g \in G$ (i.e. μ and μ^g have the sets of measure zero).*

Although in general positive G-invariant Radon measures do not exist on G/H, nevertheless quasi-invariant measures are always present [13]. In fact if ν is a non-trivial positive Radon measure on G whose null sets are those of the Haar measure, then, putting

$$\mu(f) = \nu(f \circ \pi) \qquad f \in C_c(G/H)$$

one easily checks that μ is quasi-invariant. In turn, the existence of ν follows from the fact that G is countable at infinity.

We shall now give a brief description of the main results on quasi-invariant measures without proofs (for more details see Bourbaki [7]). Let δ_H denote the modular function for H.

In the first place quasi-invariant measures on G/H always exists and any two quasi-invariant measures are equivalent. A way to manufacture quasi-invariant measures is as follows: Let ρ be a strictly positive Borel function on G bounded above and below on compact subsets and verifying for every $h \in H$

$$\rho(gh) = \frac{\delta_H(h)}{\delta_G(h)}\rho(g) \qquad g \in G$$

A function with these properties is called a *rho-function*. Fix now a rho-function ρ. Associated with this ρ is a quasi-invariant measure μ_ρ defined by

$$\int_G f(g)\rho(g)d_G(g) = \int_{G/H} d\mu_\rho(\bar{g}) \int_H f(gh)d_H(h), \qquad f \in C_c(G).$$

Let us recall that the map

$$f \mapsto \bar{f}, \quad \bar{f}(\bar{g}) = \int_H f(gh) d_H(h)$$

is a continuous map of $C_c(G)$ onto $C_c(G/H)$.

For each $f \in C_c(G/H)$ one has the relation

$$\int_{G/H} f(g \cdot \bar{q}) d\mu_\rho(\bar{q}) = \int_{G/H} s_\rho(g^{-1}, \bar{q}) f(\bar{q}) d\mu_\rho(\bar{q}). \tag{4.5}$$

The function s_ρ is obtained in the following way. Let

$$s_\rho(g, q) = \frac{\rho(gq)}{\rho(q)}, \qquad g, q \in G.$$

Then s_ρ passes to a function on $G \times G/H$, which again will be denoted by s_ρ, such that:

(i) $s_\rho(gp, \bar{q}) = s_\rho(p, \bar{q}) s_\rho(g, p \cdot \bar{q}), \qquad g, q, p \in G;$

(ii) $s_\rho(h, \bar{1}) = \delta_H(h)/\delta_G(h), \qquad h \in H;$

(iii) $s_\rho(x, \bar{1})$ is bounded on compact sets as a function of x.

The function s_ρ will be called a multiplier (see Warner [34]).

Remark 4.1.9 *Since $\bar{\mathcal{X}}_K(\bar{g}) > 0$ if, and only if, $\mathcal{X}_{\pi(K)}(\bar{g}) = 1$ we have that for each compact set $K \subset G$, there is $c > 0$ such that*

$$c\,\mu_\rho(\pi(K)) \geq \int_K \rho(g) d_G(g) \tag{4.6}$$

what show us that $\mu_\rho(\pi(K)) > 0$ if $d_G(K) > 0$.

On the other hand every quasi-invariant measure μ gives rise in a canonical manner to a rho-function so that all quasi-invariant measures are obtained in this way.

Remark 4.1.10 *In passing, note that if G is a Lie group, then every positive Radon measure on G/H which, on every local chart of G/H, is equivalent to the Lebesgue measure is, in fact, quasi-invariant.*

The following Theorem assures the existence of rho-functions, and consequently quasi-invariant measures, on Lie groups. Its proof can be found in [34] page 476.

Lemma 4.1.11 (Bruhat) *If G is a Lie group, then there exist C^∞ rho-functions.*

Suppose that G is a Lie group - then, unless specified to the contrary, we shall assume $\rho = \rho_H$ chosen so that $\rho_H \in C^\infty(G)$ and normalized so that $\rho_H(1) = 1$ and hence

$$\rho_H(h) = \frac{\delta_H(h)}{\delta_G(h)}$$

for all $h \in H$. The quasi-invariant measure associated with ρ_H will be denoted by μ_H. Observe that now the multiplier $s_{\rho_H} = s_H$ is in $C^\infty(G \times G/H)$.

Let H be a closed subgroup of G and $d\mu_H(\bar{q})$ a quasi-invariant measure on G/H . Then we shall often write more briefly $d\mu_H(q)$ (similarly in other cases too).

For reference, we shall list here a number of technical lemmas concerning rho-functions and the like.

Let H_1 and H_2 be closed subgroups of our locally compact group G and assume that $H_1 \subset H_2$. Let $d_i = d_{H_i}$ denote the left Haar measure on H_i and δ_i de corresponding modular function ($i = 1, 2$).

Lemma 4.1.12 *Let us suppose that the homogeneous space H_2/H_1 admits a positive H_2-invariant measure ν_2. Let ρ_2 be a rho function on G for the subgroup H_2 (and hence also for the subgroup H_1, δ_1 being equal to δ_2 on H_1). Let also μ_1 and μ_2 be the quasi-invariant measures on G/H_1 and G/H_2 corresponding to ρ_1 and ρ_2, respectively. Then, for a suitable normalization of the Haar measures, we have*

$$\int_{G/H_1} \phi(g)d\mu_1(g) = \int_{G/H_2} d\mu_2(g) \int_{H_2/H_1} \phi(gq)d\nu_2(q), \quad \phi \in C_c(G/H_1).$$

Lemma 4.1.13 *Let us suppose that the homogeneous space G/H_2 admits a positive G-invariant measure ν_2 and let ρ_1 be a rho-function on G for the subgroup H_1 - then $\rho_1|_{H_2}$ is a rho-function on H_2 relative to H_1 (since $\delta_2 = \delta_G$ on H_2). Let μ_1 and μ_2 be quasi-invariant measures on G/H_1 and*

H_2/H_1 corresponding, respectively, to ρ_1 and $\rho_1|_{H_2}$. Then, for a suitable normalization of the Haar measure, we have

$$\int_{G/H_1} \phi(g)d\mu_1(g) = \int_{G/H_2} d\nu_2(g) \int_{H_2/H_1} \frac{\rho_1(gq)}{\rho(q)} \phi(gq)d\mu_2(q), \quad \phi \in C_c(G/H_1).$$

Let us drop the assumption that H_1 is contained in H_2. We shall, however, agree to keep the other conventions which were laid down above. The group $H_2 \times H_1$ operates to the left on G via the prescription

$$x \mapsto (q_2, q_1) \cdot g = q_2 g q_1^{-1}, \qquad g \in G, q_i \in H_i, i = 1, 2.$$

Fix an element $g \in G$ and let Q be the $H_2 \times H_1$-orbit of g, that is, the H_2, H_1 double coset $H_2 g H_1$. The stability subgroup $(H_2 \times H_1)_g$ of $(H_2 \times H_1)$ for the element $g \in Q$ is

$$(H_2 \times H_1)_g = \{(gqg^{-1}, q); q \in H_g\}$$

where $H_g = H_1 \cap (g^{-1} H_2 g)$. Thus there is a natural identification

$$(q_1, q_2)((H_2 \times H_1)_g) \leftrightarrow q_2 g q_1^{-1}$$

between $(H_2 \times H_1)/(H_2 \times H_1)_g$ and Q. We can assume that Q has the topology which renders this identification a homeomorphism.

Let us now suppose that the homogeneous space H_2/gH_1g^{-1} carries a positive H_2-positive invariant measure ν_2. This assumption implies that $\delta_2 = \delta_{gH_1g^{-1}}$ on gH_1g^{-1} and, consequently, if we put

$$\rho(q_2, q_1) = \delta_1(q_1^{-1}), \qquad q_i \in H_i, i = 1, 2,$$

then ρ is a rho-function on $H_2 \times H_1$ relative to $(H_2 \times H_1)_g$. Consider the group $gH_gg^{-1} \times H_1$. Then

$$H_2/gH_1g^{-1} \sim H_2 \times H_1/(gH_1g^{-1}) \times H_1$$

and the image of ν_2 under this identification is an $H_2 \times H_1$-invariant measure on $H_2 \times H_1/(gH_1g^{-1}) \times H_1$ (in particular $\rho|_{gH_1g^{-1} \times H_1}$ is a rho-function for $(H_2 \times H_1)_g$). We may identify $gH_1g^{-1} \times H_1/(H_2 \times H_1)_g$ with H_1:

$$(1, q_1)(H_2 \times H_1)_g \leftrightarrow q_1, \quad q_1 \in H_1.$$

The image, under this identification, of the quasi-invariant measure on

$$gH_g g^{-1} \times H_1/(H_2 \times H_1)_g$$

corresponding to $\rho_{gH_1g^{-1}\times H_1}$ is a right Haar measure on H_1. Let μ denote the quasi invariant measure on $H_2 \times H_1/(H_2 \times H_1)_g \sim Q$ corresponding to ρ. Substituting $H_2 \times H_1$, $gH_g g^{-1} \times H_1$, $(H_2 \times H_1)_g$ for G, H_2 and H_1, respectively, in Lemma 4.1.13 gives us the following result.

Lemma 4.1.14 *Under the above hypothesis, we have, for a suitable normalization of the Haar measures,*

$$\int_Q \phi(q)d\mu(q) = \int_{H_2/gH_g g^{-1}} d\nu_2(q_2) \int_{H_1} \phi(q_2 g q_1) d_1(u_1), \quad \phi \in C_c(Q)$$

4.1.2 Stable and Unstable Lie algebras

Consider now the generalized eigenspaces associated to the derivation $\mathcal{D}$,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; (D-\alpha)^n X = 0 \text{ for some } n \geq 1\}$$

for α an eigenvalue of $\mathcal{D}$. Let as before denote by $\lambda_{\mathcal{D}}$ the real part of the eigenvalues of $\mathcal{D}$. We can decompose $\mathfrak{g}$ by

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^\circ \oplus \mathfrak{g}^-$$

with $\mathfrak{g}^+ = \bigoplus_{\lambda_{\mathcal{D}}>0} \mathfrak{g}_{\lambda_{\mathcal{D}}}$, $\mathfrak{g}^\circ = \bigoplus_{\lambda_{\mathcal{D}}=0} \mathfrak{g}_{\lambda_{\mathcal{D}}}$ and $\mathfrak{g}^- = \bigoplus_{\lambda_{\mathcal{D}}<0} \mathfrak{g}_{\lambda_{\mathcal{D}}}$.

The next Proposition show us that the vector spaces $\mathfrak{g}^\pm$ and $\mathfrak{g}^\circ$ are Lie algebras and that $\mathfrak{g}^\pm$ are actually nilpotent. The proof can be found in [29]

Proposition 4.1.15 *Let $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ a derivation of the Lie algebra $\mathfrak{g}$ of finite dimension over a closed field. Consider the decomposition*

$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the generalized eigenspace associated to the eigenvalue α. Then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

with $\mathfrak{g}_{\alpha+\beta} = 0$ in case $\alpha + \beta$ is not an eigenvalue of $\mathcal{D}$.

Remark 4.1.16 *The Proposition 4.1.15 assumes that the Lie algebra is over a closed field but since a algebra $\mathfrak{g}$ is nilpotent if, and only if, $\mathfrak{g}_{\mathbb{C}}$ is nilpotent we have that $\mathfrak{g}^{\pm}$ and $\mathfrak{g}^{o}$ are Lie algebras with $\mathfrak{g}^{\pm}$ nilpotent.*

Consider for the linear system (4.1), and the associated derivation $\mathcal{D}$, the decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}^{+} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{-}$. We call the $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$, respectively, the **unstable** and **stable Lie algebras associated to the derivation** $\mathcal{D}$.

Let $S_{\mathcal{D}} \subset G$ defined by

$$S_{\mathcal{D}} := \{g \in G; \quad \text{exist a compact set } K \subset G \text{ with } \psi_t(g) \in K \text{ for } t \in \mathbb{R}_{+}\}.$$

We have then.

Proposition 4.1.17 *The set $S_{\mathcal{D}}$ is a closed subgroup of G such that $\mathfrak{s} \supset \mathfrak{g}^{-}$ and $\mathfrak{s} \cap \mathfrak{g}^{+} = \emptyset$, where $\mathfrak{s}$ is its Lie algebra.*

Proof. That $S_{\mathcal{D}}$ is in fact a closed group follows from the fact that ψ_t is an automorphism. For the Lie algebra $\mathfrak{s}$, the afirmations follow from the formula (4.3). ■

Remark 4.1.18 *It is not hard to show, using the $\mathcal{D}$-invariance of the Lie algebras above, that $\mathfrak{s}$ is actually given by the direct sum $\mathfrak{g}^{-} \oplus \ker \mathcal{D}$;*

Consider now the space $\mathfrak{u} \supset \mathfrak{g}^{+}$ such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{s}$. We have that $\mathfrak{g}/\mathfrak{s} \approx \mathfrak{u}$. Define the linear application $\mathcal{D}^{+} : \mathfrak{u} \to \mathfrak{u}$ by $\mathcal{D}^{+}\pi^{*}(X) := \pi^{*}(\mathcal{D}X)$ for $X \in \mathfrak{g}$. That $\mathcal{D}^{+}$ is well defined follows from the $\mathcal{D}$-invariance of $\mathfrak{s}$. Also $\mathcal{D}^{+}$ satisfies $\operatorname{tr} \mathcal{D}^{+} = \operatorname{tr} \mathcal{D}|_{\mathfrak{g}^{+}}$ since $\mathfrak{g}^{+} \subset \mathfrak{u}$ and $\mathfrak{g}^{-} \subset \mathfrak{s}$.

Proposition 4.1.17 together with Theorem 4.1.5 give us the following result.

Corollary 4.1.19 *Consider the linear system (4.1) over G and assume that G is compact. Then, for any admissible pair (K, Q) we have that*

$$h_{\text{inv,out}}(K, Q) = 0.$$

4.2 Linear Systems on Homogeneous Spaces

Let H be a closed subgroup of G. The homogeneous space G/H is the manifold of the left cosets of H and we denote by π the projection onto G/H. For any right invariant vector field $Y \in \mathfrak{g}$ the projection $\pi_* Y$ of Y onto G/H is always well defined and will be referred as an invariant vector field on G/H. It is known that the set $\pi_* := \{\pi_* Y; Y \in \mathfrak{g}\}$ is a Lie algebra and that π_* is a Lie algebra morphism from $\mathfrak{g}$ onto $\pi_* \mathfrak{g}$.

Let $\mathcal{X}$ be a linear vector field on G. We want to assure the existence of a vector field on G/H that is π-related to $\mathcal{X}$. Such a vector field exists if, and only if,

$$\text{for all } x \in G \text{ and } y \in H, \qquad \pi(\psi_t(xy)) = \pi(\psi_t(x)).$$

But $\pi(\psi_t(xy)) = \psi_t(x)\psi_t(y)H$ and the preceding condition is equivalent to

$$\text{for all } y \in H, t \in \mathbb{R} \qquad \psi_t(y) \in H.$$

Thus $\mathcal{X}$ is π-related to a vector field on G/H if, and only if, H is invariant under the flow of $\mathcal{X}$.

Definition 4.2.1 *Let G be a Lie group and $H \subset G$ a closed subgroup. A vector field f on G/H is said to be linear if it is π-related to a linear vector field of G, where π is the canonical projection $\pi : G \to G/H$.*

Consider then $S_{\mathcal{D}}$ as above. Its invariance allow us to consider the quotient space $G/S_{\mathcal{D}}$. Let $\bar{\mathcal{X}}$ be the vector field induced in $G/S_{\mathcal{D}}$ by $\mathcal{X}$.

Considering a geodesic referential we have that $\text{div}\bar{\mathcal{X}}(\bar{g}) = -\text{tr}\,\text{ad}(\bar{\mathcal{X}})_{\bar{g}}$ where $\text{ad}(\bar{\mathcal{X}})_{\bar{g}} : T_{\bar{g}}(G/S_{\mathcal{D}}) \to T_{\bar{g}}(G/S_{\mathcal{D}})$ is the Lie bracket, that is

$$\text{ad}(\bar{\mathcal{X}})_{\bar{g}}(\bar{Y}(\bar{g})) := [\bar{\mathcal{X}}, \bar{Y}](\bar{g}).$$

Also, since $\mathcal{X}$ satisfies

$$[\mathcal{X}, Y](g) = ((dL_g) \circ \text{ad}(\mathcal{X}) \circ (dL_{g^{-1}}))_g(Y(g)) = -((dL_g)_e \circ \mathcal{D} \circ (dL_{g^{-1}})_g(Y(g))$$

we have that $\bar{\mathcal{X}}$ satisfies

$$[\bar{\mathcal{X}}, \bar{Y}](\bar{g}) = -((d\bar{L}_g)_{\bar{e}} \circ \mathcal{D}^+ \circ (d\bar{L}_{g^{-1}})_{\bar{g}}(\bar{Y}(\bar{g})),$$

where $\bar{L}_g(\bar{q}) = g \cdot \bar{q}$ is the translation on $G/S_{\mathcal{D}}$. Consequently

$$\mathrm{div}\bar{\mathcal{X}}(\bar{g}) = \mathrm{tr}((d\bar{L}_g)_{\bar{e}} \circ \mathcal{D}^+ \circ (d\bar{L}_{g^{-1}})_{\bar{g}} = \mathrm{tr}\,\mathcal{D}^+ = \mathrm{tr}\,\mathcal{D}|_{\mathfrak{g}^+} = \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}}$$

Our idea is to project the system (4.1) over the homogeneous space $G/S_{\mathcal{D}}$ and use that the entropy does not increase by semi-conjugation to get a better lower bound.

Consider then on the homogeneous space $G/S_{\mathcal{D}}$ the $\mathcal{C}^\infty$, $\rho = \rho_{\mathcal{D}}$-function and consider the associated multiplier $s_{\mathcal{D}} : G \times G/S_{\mathcal{D}} \to \mathbb{R}_+$. Let also $\zeta_{t,u}(\bar{g}) = \zeta_{t,u} \cdot \bar{g}$ be the solution of the system on $G/S_{\mathcal{D}}$ induced by (4.4).

Proposition 4.2.2 *The map $\sigma = \sigma_{\mathcal{D}} : \mathbb{R} \times \mathcal{U} \times G/S_{\mathcal{D}} \to \mathbb{R}$ defined by*

$$\sigma_t(u, \bar{q}) := \log s_{\mathcal{D}}(\zeta_{t,u}, \bar{q}). \tag{4.7}$$

is a continuous cocycle over the control flow for the system (4.4),

$$\phi_t : \mathcal{U} \times G/S_{\mathcal{D}} \to \mathcal{U} \times G/S_{\mathcal{D}} \qquad (u, \bar{q})) \mapsto (\Theta_t u, \zeta_{t,u}(\bar{q})).$$

Also, in the negative time we have

$$\sigma_{-t}(u, \bar{q}) = -\sigma_t(\phi_{-t}(u, \bar{q})) \quad \textit{for } t > 0.$$

Proof. The continuity follows from the fact that $s_{\mathcal{D}}$ and log are $\mathcal{C}^\infty$ and since the system (4.4) is control-affine, the solutions $\zeta_{t,u}(g)$ are continuous. For the cocycle property, we have the property (iii) of $s_{\mathcal{D}}$, that is,

$$s_{\mathcal{D}}(gp, \bar{q}) = s_{\mathcal{D}}(p, \bar{q}) s_{\mathcal{D}}(g, p \cdot \bar{q}), \quad g, p, q \in G.$$

Then, for $(u, \bar{q}) \in \mathcal{U} \times G/S_{\mathcal{D}}$ and $t, s \in \mathbb{R}$ we have

$$s_{\mathcal{D}}(\zeta_{t+s,u}, \bar{q}) = s_{\mathcal{D}}(\zeta_{t,\Theta_s u}\zeta_{s,u}, \bar{q})$$

$$= s_{\mathcal{D}}(\zeta_{s,u}, \bar{q}) s_{\mathcal{D}}(\zeta_{t,\Theta_s u}, \zeta_{s,u} \cdot \bar{q}) =$$

and then, applying log and using that $\zeta_{s,u}(\bar{q}) = \zeta_{s,u} \cdot \bar{q}$ we get

$$\sigma_{t+s}(u, \bar{q}) = \log s_{\mathcal{D}}(\zeta_{s,u}, \bar{q}) + \log s_{\mathcal{D}}(\zeta_{t,\Theta_s u}, \zeta_{s,u}(\bar{q}))$$

$$=: \sigma_s(u, \bar{q}) + \sigma_t(\phi_s(u, \bar{q})).$$

The last assertion follows also from the property above and the fact that $\zeta_{t,u}^{-1} = \zeta_{-t,\Theta_t u}$. In fact for $(u, \bar{q}) \in \mathcal{U} \times G/S_{\mathcal{D}}$ we have

$$s_{\mathcal{D}}(\zeta_{-t,u}, \bar{q}) = s_{\mathcal{D}}(\zeta_{-t,u}, \zeta_{-t,u}^{-1}\zeta_{-t,u}\bar{q}) =$$

$$s_{\mathcal{D}}(1, \zeta_{-t,u}\bar{q}) \cdot \left[s_{\mathcal{D}}(\zeta_{-t,u}^{-1}, \zeta_{-t,u}\bar{q})\right]^{-1} = \left[s_{\mathcal{D}}(\zeta_{t,\Theta_{-t}u}, \zeta_{-t,u}\bar{q})\right]^{-1}$$

since $s_{\mathcal{D}}(e, \bar{q}) = 1$ for all $\bar{q} \in G/S_{\mathcal{D}}$. Applying log and using the definition we get

$$\sigma_{-t}(u, \bar{q}) = -\sigma_t(\phi_{-t}(u, \bar{q}))$$

concluding the proof. ■

We have then the main result of this section.

Theorem 4.2.3 *Let (K, Q) an admissible pair for the linear system (4.1) and assume that $d_G(K) > 0$, with d_G the left Haar measure. Then the outer invariance entropy satisfies*

$$h_{\text{inv,out}}(K, Q) \geq \sum_{\lambda_{\mathcal{D}} > 0} \lambda_{\mathcal{D}} + g_{\mathcal{D}}(Q)$$

where $\lambda_{\mathcal{D}}$ are the real part of the eigenvalues of the derivation $\mathcal{D}$ associated to the drift $\mathcal{X}$ and

$$g_{\mathcal{D}}(Q) = \sup_{(u,\bar{q}) \in \bar{\mathcal{Q}}} \limsup_{n \to -\infty} \frac{1}{n} \sigma_n(u, \bar{q}).$$

Proof. Consider as before the quotient space $G/S_{\mathcal{D}}$. Since $S_{\mathcal{D}}$ is invariant by the flow of $\mathcal{X}$ we have a π_*-related system to the system (4.1). The solutions of such system are just the projection of the solution of the system (4.1), that is,

$$\bar{\varphi}(t, \pi(g), u) = \pi(\varphi(t, g, u))$$

where φ is the solution of (4.1). But then, for $\bar{q} = q \cdot S_{\mathcal{D}}$

$$\bar{\varphi}(t, \bar{g}, u) = \zeta_{t,u}\psi_t(q)S_{\mathcal{D}} = \zeta_{t,u}\bar{\psi}_t(\bar{q})$$

where $\bar{\psi}_t$ is the curve associated with $\bar{\mathcal{X}} = \pi_*\mathcal{X}$. By the conjugation with π above, the Theorem 3.5 of [10] assures that

$$h_{\text{inv,out}}(K, Q) \geq h_{\text{inv,out}}(\pi K, \pi Q).$$

Denote by $(\bar{K}, \bar{Q})$ the pair $(\pi K, \pi Q)$ and consider the rho-function $\rho_{\mathcal{D}}$ on G and $\mu_{\mathcal{D}}$ the associated measure. Remark 4.1.9, assures that $\mu_{\mathcal{D}}(\pi(K)) > 0$.

Let then $n, \varepsilon > 0$ and consider $S = \{u_1, \cdots, u_k\}$ be a minimal (n, ε)-spanning set for $(\bar{K}, \bar{Q})$. Define the sets

$$K_j = \{\bar{g} \in \bar{K};\ \bar{\varphi}([0,n], \bar{q}, u_j) \subset N_\varepsilon(\bar{Q})\}, \qquad j = 1, \cdots k.$$

Then by definition of (n, ε)-spanning sets, K_j is a Borel set for each j and also $\pi K = \cup_j K_j$. Also, since the solution map $\bar{\varphi}_{n,u_j} : S_{\mathcal{D}} \to S_{\mathcal{D}}$ is a diffeomorphism, $\bar{\varphi}_{n,u_j}(K_j)$ is also a Borel set and satisfies

$$\mu_{\mathcal{D}}(\bar{\varphi}_{n,u_j}(K_j)) \leq \mu_{\mathcal{D}}(N_\varepsilon(\bar{Q})).$$

Since $\mu_{\mathcal{D}}$ is quasi-invariant, we have

$$\mu_{\mathcal{D}}(\bar{\varphi}_{n,u_j}(K_j)) \quad = \quad \int_{\bar{\psi}_n(K_j)} s_{\mathcal{D}}(\zeta_{n,u_j}, \bar{q}) d\mu_{\mathcal{D}}(\bar{q})$$

$$\int_{K_j} s_{\mathcal{D}}(\zeta_{n,u_j}, \bar{\psi}_n(\bar{q})) |\det(d\bar{\psi}_n)_{\bar{q}}| d\mu_{\mathcal{D}}(\bar{q}).$$

By Liouville's trace formula and discussion above, we have that

$$|\det(d\bar{\psi}_n)_{\bar{g}}| = \exp\left(\int_0^n \mathrm{div}\bar{\mathcal{X}}(\psi_s(\bar{q}))\right) ds = \mathrm{e}^{n \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}}}$$

We obtain

$$\mu_{\mathcal{D}}(\bar{\varphi}_{n,u_j}(K_j)) = \mathrm{e}^{n \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}}} \int_{K_j} s_{\mathcal{D}}(\zeta_{n,u_j}, \bar{\psi}_n(\bar{q})) d\mu_{\mathcal{D}}(\bar{q})$$

and using property of $s_{\mathcal{D}}$ we have that

$$s_{\mathcal{D}}(\zeta_{n,u_j}, \bar{\psi}_t(\bar{q})) = \frac{s_{\mathcal{D}}(e, \bar{\varphi}_{n,u_j}(\bar{q}))}{s_{\mathcal{D}}(\zeta_{n,u_j}^{-1}, \bar{\varphi}_n(\bar{q}))} = \left[s_{\mathcal{D}}(\zeta_{n,u_j}^{-1}, \bar{\varphi}_n(\bar{q}))\right]^{-1}.$$

Since $\bar{\varphi}_{t,u_j}(\bar{q}) \in \mathrm{cl}N_\varepsilon(\bar{Q})$, for $t \in [0,n]$ and $u_j \in S \subset \pi_{\mathcal{U}}\bar{\mathcal{Q}}$ the lift of $\bar{Q}$, we have by continuity of $s_{\mathcal{D}}$ that

$$s_{\mathcal{D}}(\zeta_{n,u_j}, \bar{\psi}_t(\bar{g})) \geq \left[\max_{(u,\bar{q})} s_{\mathcal{D}}(\zeta_{n,u}^{-1}, \bar{q})\right]^{-1} \geq \max_{(u,\bar{q})} s_{\mathcal{D}}(\zeta_{n,\Theta_{-n}u}, \zeta_{-n,u}(\bar{q}))$$

where the maximum is taken over the compact set $\pi_{\mathcal{U}}\bar{\mathcal{Q}} \times \mathrm{cl}N_\varepsilon(\bar{Q})$.

We have then

$$\mu_{\mathcal{D}}(\bar{\varphi}_{n,u_j}(K_j)) \geq \mathrm{e}^{n\sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}}}\mathrm{e}^{\max_{(u,\bar{q})}\sigma_n(\phi_{-n}(u,\bar{q}))} \cdot \mu_{\mathcal{D}}(K_j).$$

Taking j_0 such that $\max_j \mu_{\mathcal{D}}(K_j) = \mu_{\mathcal{D}}(K_{j_0})$ we have

$$\mu_{\mathcal{D}}(\bar{K}) \leq \sum_{j=1}^{k} \mu_{\mathcal{D}}(K_j) \leq k\mu_{\mathcal{D}}(K_{j_0})$$

$$\leq k \cdot \frac{\mu_{\mathcal{D}}(N_\varepsilon(\bar{Q}))}{\mathrm{e}^{k\sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}}}\mathrm{e}^{\max_{(u,\bar{q})}\sigma_n(\phi_{-n}(u,\bar{q}))}}$$

and consequently

$$r_{\mathrm{inv}}(n,\varepsilon,\bar{K},\bar{Q}) \geq \mathrm{e}^{n\sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}}}\mathrm{e}^{\max_{(u,\bar{q})}\sigma_n(\phi_{-n}(u,\bar{q}))} \cdot \frac{\mu_{\mathcal{D}}(\bar{K})}{\mu_{\mathcal{D}}(N_\varepsilon(\bar{Q}))}.$$

Taking log and dividing by n we obtain

$$\frac{1}{n}\log r_{\mathrm{inv}}(n,\varepsilon,\bar{K},\bar{Q})$$
$$\geq \sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}} + \max_{(u,\bar{q})}\frac{1}{n}\sigma_n(\phi_{-n}(u,\bar{q})) + \frac{1}{n}\cdot \mathrm{cte}.$$

Since $\sigma_n(\phi_{-n}(u,\bar{q})) = -\sigma_{-n}(u,\bar{q})$ we get

$$\limsup_{n\to-\infty}\max_{(u,\bar{q})}\frac{1}{n}\sigma_n(u,\bar{q}) \geq \sup_{(u,\bar{q})}\limsup_{n\to-\infty}\frac{1}{n}\sigma_n(u,\bar{q})$$

$(u,\bar{q}) \in \pi_{\mathcal{U}}\mathcal{Q} \times \mathrm{cl}N_\varepsilon(\bar{Q})$ and consequently

$$h_{\mathrm{inv,out}}(\bar{K},\bar{Q}) \geq \sum_{\lambda_{\mathcal{D}}>0}\lambda_{\mathcal{D}} + \lim_{\varepsilon\searrow 0}\max_{(u,\bar{q})}\limsup_{n\to-\infty}\frac{1}{n}\sigma_n(u,\bar{q}).$$

But $\pi_{\mathcal{U}}\bar{\mathcal{Q}} \times \bar{Q} \supset \bar{\mathcal{Q}}$ and

$$\lim_{\varepsilon\searrow 0}\max_{(u,\bar{q})\in\pi_{\mathcal{U}}\bar{\mathcal{Q}}\times N_\varepsilon(\bar{Q})}\limsup_{n\to-\infty}\frac{1}{n}\sigma_n(u,\bar{q}) = \max_{(u,\bar{q})\in\pi_{\mathcal{U}}\mathcal{Q}\times\bar{Q}}\limsup_{n\to-\infty}\frac{1}{n}\sigma_n(u,\bar{q})$$

what give us

$$\lim_{\varepsilon\searrow 0}\sup_{(u,\bar{q})\in\pi_{\mathcal{U}}\mathcal{Q}\times N_\varepsilon(\bar{Q})}\limsup_{n\to-\infty}\frac{1}{n}\sigma_n(u,\bar{q}) \geq \sup_{(u,\bar{q})\in\bar{\mathcal{Q}}}\limsup_{n\to-\infty}\frac{1}{n}\sigma_n(u,\bar{q})$$

and implies

$$h_{\text{inv,out}}(\bar{K},\bar{Q}) \geq \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}} + g_{\mathcal{D}}(Q).$$

for

$$g_{\mathcal{D}}(Q) = \sup_{(u,\bar{q})\in\bar{\mathcal{Q}}} \limsup_{n\to-\infty} \frac{1}{n}\sigma_n(u,\bar{q})$$

showing the theorem. ■

Remark 4.2.4 *Using the cocycle property and the fact that the lift $\bar{\mathcal{Q}}$ is invariant by the control flow we can show that*

$$g_{\mathcal{D}}(Q) = \sup_{(u,\bar{q})\in\bar{\mathcal{Q}}} \limsup_{\tau\to-\infty} \frac{1}{\tau}\sigma_\tau(u,\bar{q})$$

for $\tau \in \mathbb{R}$.

We would like very much the quantity $g_{\mathcal{D}}(Q)$ to be equal to zero. A first step in this way, that generalizes the linear case, is the following corollary.

Corollary 4.2.5 *Let (K,Q) an admissible pair for the linear system (4.1) with $d_G(K) > 0$. Let us assume that G is in one of the following categories:*

(i) Abelian;

(ii) Nilpotent;

(iii) Compact.

Then

$$h_{\text{inv,out}}(K,Q) = \sum_{\lambda_{\mathcal{D}}>0} \lambda_{\mathcal{D}}$$

with $\lambda_{\mathcal{D}}$ as above.

Proof. Since for such groups we have $\delta_H = \delta_G \equiv 1$ for all subgroup $H \subset G$ we have that the rho-function associated to the subgroup $S_{\mathcal{D}}$ as above is constant equal to 1 and consequently $g_{\mathcal{D}}(Q) = 0$ what give us the result. ■

Remark 4.2.6 *Note that the above Corollary show us that in some groups, the outer invariance entropy just depends on the geometry of the space.*

4.2.1 The Semi-simple case

We will show that on the semi-simple case we have a relation between $g_{\mathcal{D}}$ and the $\mathfrak{a}$-cocycle. Consider then the trivial principal bundle $Q = \mathcal{U} \times G$. The right invariance of the system (4.4) give us that the control flow associated with this system is a flow of automorphisms and we have then, as before, the $\mathfrak{a}$-cocycle defined on $\mathcal{U} \times \mathbb{F}$.

The map

$$\rho_\Theta = \sum_{\alpha \in \Pi^- \setminus \langle \Theta \rangle} \alpha$$

annihilates $\mathfrak{a}(\Theta)$, we have by Lemma 2.4.2 a well defined cocycle a_Θ on $\mathcal{U} \times \mathbb{F}_\Theta$ given by $\mathsf{a}_\Theta(t, u, \pi_\Theta(q)) = \rho_\Theta(\mathsf{a}(t, u, q))$ and by the relation between a and a^* we have also the cocycle in the negative time $\mathsf{a}^*_\Theta(t, u, \pi_\Theta(q)) = \rho_\Theta(\mathsf{a}^*(t, u, q))$.

Since G is semi-simple, the derivation $\mathcal{D}$ is inner, that is, there is $X \in \mathfrak{g}$ such that $\mathcal{D} = \mathrm{ad}(X)$ and we have that the real parts of the eigenvalues of $\mathcal{D}$ coincides with the eigenvalues of the abelian part of $\mathrm{ad}(X)$. We will assume that $\mathcal{D} = \mathrm{ad}(H)$ with H in the closure a positive Weyl chamber, that is, $H \in \mathrm{cl}\mathfrak{a}^+$. If we consider then $\Theta = \{\alpha \in \Sigma;\ \alpha(H) = 0\}$, we have that the space $G/S_{\mathcal{D}}$ is the flag manifold $\mathbb{F}_\Theta = G/P_\Theta$. Let us denote by $\mathbb{F}_{\mathcal{D}}$ this flag manifold and by $\mathsf{a}_{\mathcal{D}}$, $\mathsf{a}^*_{\mathcal{D}}$ the cocycles associated with the abelian part as above. We have then:

Theorem 4.2.7 *Let (4.1) be a linear system on G, with G semi-simple. For a compact and controlled invariant set $Q \subset G$ we have*

$$g_{\mathcal{D}}(Q) = \sup_{(u, \pi_\Theta(q)) \in \bar{\mathcal{Q}}} \limsup_{\tau \to -\infty} \frac{1}{\tau} \mathsf{a}^*_{\mathcal{D}}(\tau, u, \pi_\Theta(q)).$$

Proof. Let $(u, \pi_\Theta(q)) \in \bar{\mathcal{Q}}$ and $\tau > 0$. Using the property of $s_{\mathcal{D}}$ we have that

$$s_{\mathcal{D}}(\zeta_{-\tau,u}, \pi_\Theta(x)) = \frac{s_{\mathcal{D}}(\zeta_{-\tau,u}(q), b_\Theta)}{s_{\mathcal{D}}(q, b_\Theta)}.$$

Consider the Iwasawa decomposition $\zeta_{-\tau,u}(q) = k^*_{\tau,u} a^*_{\tau,u} n^*_{\tau,u}$. Using again the above property we get

$$s_{\mathcal{D}}(\zeta_{-\tau,u} q, b_\Theta) = s_{\mathcal{D}}(k^*_{\tau,u}, b_\Theta) s_{\mathcal{D}}(a^*_{\tau,u} n^*_{\tau,u}, b_\Theta) = s_{\mathcal{D}}(k{*}_{\tau,u}, b_\Theta) \delta_{\mathcal{D}}(a^*_{\tau,u})$$

because $\delta_N = \delta_G \equiv 1$ and $a^*_{\tau,u} n^*_{\tau,u} b_\Theta = b_\Theta$. Since

$$\delta_{\mathcal{D}}(a^*_{\tau,u}) = \mathrm{e}^{\mathsf{a}^*_{\mathcal{D}}(\tau, u, \pi_\Theta(q))}$$

we have

$$\sigma_{-\tau}(u,\pi_\Theta(q)) = \log\frac{s_{\mathcal{D}}(k^*_{\tau,u},b_\Theta)}{s_{\mathcal{D}}(q,b_\Theta)} + \mathsf{a}^*_{\mathcal{D}}(\tau,u,\pi_\Theta(q)).$$

But $k^*_{\tau,u}$ is in the compact K and then,

$$\limsup_{\tau\to-\infty}\frac{1}{\tau}\sigma_\tau(u,\pi_\Theta(q)) = \limsup_{\tau\to-\infty}\frac{1}{\tau}\mathsf{a}^*_{\mathcal{D}}(\tau,u,\pi_\Theta(q))$$

which implies the Theorem. ■

Remark 4.2.8 *The above Theorem shows that, in order to obtain a formula for the outer invariance entropy in the semi-simple case we have, as one could expect, made assumptions just on the abelian part of the induced control-affine system (4.4).*

Remark 4.2.9 *The existence of controlled invariant sets, or control sets, for linear systems on Lie groups is still an ongoing area. What is expected is the existence of at least one control set around the identity* 1 *of* G *because of the singularity of the drift* $\mathcal{X}$ *but there is still not any work (of my knowledge) about it.*

Chapter 5

Invariance Entropy for Random Control Systems

For continuous-time random control systems, this chapter introduces invariance entropy for random pairs as a measure for the amount of information necessary to achieve invariance of random weakly invariant compact subsets of the state space. For linear random control systems with compact control range, the invariance entropy is given by the sum of the real parts of the unstable eigenvalues of the uncontrolled system if we assume ergodicity.

5.1 Preliminaries

Let $m, d \in \mathbb{N}$, M an open subset of $\mathbb{R}^d$, $(\Omega, \mathcal{F}, (\theta_t)_{t\in\mathbb{R}})$ a measurable dynamical system and $U \subset \mathbb{R}^m$ a compact subset. We define the set of *admissible functions* by

$$\mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m\,;\, u \text{ measurable with } u(t) \in U \text{ a.e.}\}.$$

The *shift flow* is defined by

$$\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U},\ \Theta(t,u) := \Theta_t u \text{ with } \Theta_t u(s) := u(t+s) \text{ for all } t, s \in \mathbb{R}.$$

A *continuous random control system* (RCS) on $M \subset \mathbb{R}^d$ over a metric dynamical system $(\Omega, \mathcal{F}, (\theta_t)_{t\in\mathbb{R}})$ with time $\mathbb{R}$ is a map

$$\varphi : \mathbb{R} \times M \times \Omega \times \mathcal{U} \to M$$

with the following properties:

i) For all $(\omega, u) \in \Omega \times \mathcal{U}$ the map $\varphi(\omega, u) : \mathbb{R} \times M \to M$, given by $\varphi(\omega, u)(t, x) := \varphi(t, x, \omega, u)$, is continuous;

ii) For all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$ the map $\varphi(t, x) : \Omega \times \mathcal{U} \to M$ is measurable;

iii) The map $\varphi(t, \omega, u) : M \to M$ form a cocycle over $\theta \times \Theta$ i.e. they satisfies

$$\varphi(0, \omega, u) = \mathrm{id}_M \qquad \text{for all } \omega \in \Omega \text{ and } u \in \mathcal{U}$$

$$\varphi(t + s, \omega, u) = \varphi(t, \theta_s\omega, \Theta_s u) \circ \varphi(s, \omega, u)$$

where $\varphi(t, \omega, u)x := \varphi(t, x, \omega, u)$.

Let $F : M \times \mathbb{R}^m \times \Omega \to \mathbb{R}^d$ be a Carathéodory application, i.e., continuous on $M \times \mathbb{R}^m$ and measurable on Ω, satisfying $F(x, u, \omega) \in T_x M$ and continuous differentiable in the first argument.

The family

$$\dot{x}(t) = F(x(t), u(t), \theta_t\omega), \qquad u \in \mathcal{U},\ \omega \in \Omega, \tag{5.1}$$

of ordinary differential equations generates a RCS.

For each $x \in M$, $u \in \mathcal{U}$ and $\omega \in \Omega$ the solution of the initial value problem $x(0) = x$ will be denoted by $\varphi(t, x, u, \omega)$. With some assumptions on F we have that the solutions of (5.1) exist. For instance, if we assume that there is an interval I such that for all $x \in M$, $u \in \mathcal{U}$ and $\omega \in \Omega$ the functions $t \in I \mapsto ||D_{\mathrm{x}}F(x, u(t), \theta_t\omega)||$ and $t \in I \mapsto ||F(x, u(t), \theta_t\omega)||$ are locally integrable, then we have that the solutions exist in some interval $J \subset I$, where $J = J(x, u, \omega)$.

The solutions are defined in the sense of Carathéodory, that is, $\varphi(\cdot, x, u, \omega)$ is an absolutely continuous curve which satisfies the corresponding integral equation. Throughout we assume that solutions are defined globally. This assumption is justified by the fact that we consider only trajectories which do not leave a compact subset of the state space M (cf. Sontag [32], Prop. C.3.6]). Thus, we obtain the cocycle property for the solutions φ of (5.1). In this Chapter we will work with RCS that are generated by a family of differential equations.

Definition 5.1.1 (Random Set) *Let $\mathcal{C}(M)$ be the set of all nonvoid compact subsets of M. The compact set-valued map $\mathcal{Q} : \Omega \to \mathcal{C}(M)$, $\omega \mapsto Q_\omega$, is called a random compact set if for each $x \in M$ the map $\omega \mapsto d(x, Q_\omega)$ is measurable, where d is any distance in M.*

Let Q be a compact set. Then we have in a natural way the trivial random compact set defined by $Q_\omega = Q$ for $\omega \in \Omega$. In this situation, we also denote the random control set $\mathcal{Q}$ only by Q. Also, for two compact set-valued maps $\mathcal{K}$ and $\mathcal{Q}$ we say that $\mathcal{K}$ is contained in $\mathcal{Q}$ (and denote in the usual way $\mathcal{K} \subset \mathcal{Q}$) if $K_\omega \subset Q_\omega$ for each $\omega \in \Omega$.

For a given random compact set $\mathcal{Q}$, we say that $\mathcal{Q}$ is *weakly invariant for the* RCS if for each $\omega \in \Omega$ and each $x \in Q_\omega$ there exists $u \in \mathcal{U}$ such that $\varphi(t, x, u, \omega) \in Q_{\theta_t \omega}$ for all $t \geq 0$.

We will also need some notion of continuity for the random compact set. We say that the random compact set is *upper semi-continuous over the flow* θ if for each $t \in \mathbb{R}$, $\omega \in \Omega$

$$\lim_{s \to t} \operatorname{dist}(Q_{\theta_s \omega}, Q_{\theta_t \omega}) = 0,$$

where dist denotes the Hausdorff semi-distance given by

$$\operatorname{dist}(A, B) = \sup_{a \in A} d(a, B)$$

for $d(a, B) = \inf_{b \in B} d(a, b)$. We have the following lemma that will be needed in some proofs below.

Lemma 5.1.2 *Let $\mathcal{Q}$ be an upper semi-continuous random compact set. For given $\omega \in \Omega$, $u \in \mathcal{U}$ we have:*

(i) The function $f_{u,\omega} : \mathbb{R} \times M \to \mathbb{R}_+$, defined by

$$f_{u,\omega}(t, x) := d(\varphi(t, x, u, \omega), Q_{\theta_t \omega}),$$

is lower semi-continuous in the first argument and continuous in the second one. Also, for every compact interval $[0, T]$ we have that

$$\sup_{t \in [0,T]} f_{u,\omega}(t, x) = \sup_{t \in [0,T] \cap \mathbb{Q}} f_{u,\omega}(t, x);$$

(ii) The set $Q_{T,\omega} = \bigcup_{t \in [0,T]} Q_{\theta_t \omega}$ is compact.

Proof. (i) First we need a property of the Hausdorff semi-distance for compact sets: Let B, C be compact sets and a an arbitrary point. Then

$$d(a, B) \leq d(a, C) + \operatorname{dist}(C, B). \tag{5.2}$$

For each $c \in C$ we have that $d(a,B) \le d(a,c) + d(c,B)$. Since C is a compact set there exist $c_0 \in C$ such that $d(a,c_0) = d(a,C)$ and that implies

$$d(a,B) \le d(a,c_0) + d(c_0,B) \le d(a,C) + \max_{c\in C} d(c,B) = d(a,C) + \operatorname{dist}(C,B).$$

For fixed $x \in M$, $u \in \mathcal{U}$ and $\omega \in \Omega$, let $t \in \mathbb{R}$. For a given $\varepsilon > 0$, consider $\delta > 0$ such that

$$d(\varphi(t,x,u,\omega), \varphi(s,x,u,\omega)) < \frac{\varepsilon}{2} \text{ and } \operatorname{dist}(Q_{\theta_s\omega}, Q_{\theta_t\omega}) < \frac{\varepsilon}{2}$$

if $|t-s| < \delta$. That exists because of the continuity of the solution and the upper semi-continuity of the random compact set. Then

$$\begin{aligned} &d(\varphi(t,x,u,\omega), Q_{\theta_t\omega}) \\ &\le d(\varphi(t,x,u,\omega), \varphi(s,x,u,\omega)) + d(\varphi(s,x,u,\omega), Q_{\theta_s\omega}) + \operatorname{dist}(Q_{\theta_s\omega}, Q_{\theta_t\omega}) \\ &< d(\varphi(s,x,u,\omega), Q_{\theta_s\omega}) + \varepsilon \text{ for all } |t-s| < \delta, \end{aligned}$$

where we used (5.2). By the definition of the function f we have

$$f_{u,\omega}(t,x) - \varepsilon < f_{u,\omega}(s,x)$$

showing that f is lower semi-continuous in the first argument.

The continuity in the second argument follows from the continuity of the solution and of the continuity of the metric.

For the second assertion, we just need to show that $\sup_{t\in[0,T]} f_{u,\omega}(t,x) \le \sup_{t\in[0,T]\cap\mathbb{Q}} f_{u,\omega}(t,x)$. Let $t \in [0,T]$. There exists a sequence of rational numbers $t_n \in [0,T]$ that converges to t. Then, for each $\varepsilon > 0$, there exist n_0 such that for all $n \ge n_0$ we have that

$$f_{u,\omega}(t,x) < f_{u,\omega}(t_n,x) + \varepsilon \le \sup_{t\in[0,T]\cap\mathbb{Q}} f_{u,\omega}(t,x) + \varepsilon.$$

Since this relation is valid for all $t \in [0,T]$ we have the desired result.

(ii) Let z_n be a sequence in $Q_{T,\omega}$. Then there is $t_n \in [0,T]$ such that $z_n \in Q_{\theta_{t_n}\omega}$. Since $[0,T]$ compact we can assume that the sequence t_n converges to some $t \in [0,T]$. By the upper semi-continuity of the random compact set, we have that

$$d(z_n, Q_{\theta_t\omega}) \le \operatorname{dist}(Q_{\theta_{t_n}\omega}, Q_{\theta_t\omega}) \to 0$$

as $n \to \infty$. Since $Q_{\theta_t\omega}$ is compact there is $x_n \in Q_{\theta_t\omega}$ such that $d(z_n, Q_{\theta_t\omega}) = d(z_n, x_n)$ and we can also assume that the sequence $x_n \to z \in Q_{\theta_t\omega}$. Consequently $z_n \to z$ showing that $Q_{T,\omega}$ is compact. ∎

5.2 Definition and Elementary Properties

This section will present several definitions for the invariance entropy with relation to a measure. Basic properties are derived.

We call $(\mathcal{K},\mathcal{Q})$ a *random pair* if $\mathcal{Q}$ is a weakly invariant random compact set for (5.1) and $\mathcal{K}$ is a compact set-valued map satisfying $\mathcal{K}\subset\mathcal{Q}$.

Consider then a random pair $(\mathcal{K},\mathcal{Q})$. For given $T,\varepsilon>0$ and $\omega\in\Omega$ we call $\mathcal{S}_\omega\subset\mathcal{U}$ a (T,ε,ω)-*spanning* set for $(\mathcal{K},\mathcal{Q})$ if for every $x\in K_\omega$ there exists $u\in\mathcal{S}_\omega$ with

$$d(\varphi(t,x,u,\omega),Q_{\theta_t\omega})<\varepsilon,\ \text{for all } t\in[0,T],$$

where d is the Euclidean distance. By $r_{\text{inv}}(T,\varepsilon,\omega,\mathcal{K},\mathcal{Q})$ we denote the minimal cardinality of a (T,ε,ω)-spanning set.

Let $0<T_1<T_2$. Since every (T_2,ε,ω)-spanning set is also a (T_1,ε,ω)-spanning, it follows that

$$r_{\text{inv}}(T_1,\varepsilon,\omega,\mathcal{K},\mathcal{Q})\le r_{\text{inv}}(T_2,\varepsilon,\omega,\mathcal{K},\mathcal{Q}). \tag{5.3}$$

Also for every $0<\varepsilon_1<\varepsilon_2$ a (T,ε_1,ω)-spanning set is a (T,ε_2,ω)-spanning set. Then

$$r_{\text{inv}}(T,\varepsilon_1,\omega,\mathcal{K},\mathcal{Q})\ge r_{\text{inv}}(T,\varepsilon_2,\omega,\mathcal{K},\mathcal{Q}). \tag{5.4}$$

Note that a priori the numbers $r_{\text{inv}}(T,\varepsilon,\omega,\mathcal{K},\mathcal{Q})$ are not necessarily finite, but we will show that the assumption that the random compact set $\mathcal{Q}$ is weakly invariant is enough to assure that. For this purpose we define another notion of spanning sets.

We still consider a pair $(\mathcal{K},\mathcal{Q})$ as above and define the *lift over* $\omega\in\Omega$ of $\mathcal{Q}$ by

$$\text{lift}(\mathcal{Q},\omega):=\{(x,u)\in Q_\omega\times\mathcal{U};\varphi(t,x,u,\omega)\in Q_{\theta_t\omega}\text{ for all } t\ge 0\}.$$

For given $\omega\in\Omega$ and $T,\varepsilon>0$ we call a set $\mathcal{S}_w^+\subset\text{lift}(\mathcal{Q},\omega)$ *strongly* (T,ε,ω)-*spanning* for $(\mathcal{K},\mathcal{Q})$ if for every $x\in K_\omega$ there exists $(y,u)\in\mathcal{S}_\omega^+$ with

$$d(\varphi(t,x,u,\omega),\varphi(t,y,u,\omega))<\varepsilon,\ \text{for all } t\in[0,T].$$

We denote by $r_{\text{inv}}^+(T,\varepsilon,\omega,\mathcal{K},\mathcal{Q})$ the minimal cardinality of a strongly (T,ε,ω)-spanning set.

As a function of T,ε, we have that $r_{\text{inv}}^+(T,\varepsilon,\omega,\mathcal{K},\mathcal{Q})$ has the same properties as $r_{\text{inv}}(T,\varepsilon,\omega,\mathcal{K},\mathcal{Q})$.

Proposition 5.2.1 *Let* $(\mathcal{K}, \mathcal{Q})$ *a random pair. Then for all* $\omega \in \Omega$ *and* $T, \varepsilon > 0$

$$r_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq r^{+}_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) < \infty.$$

Proof. Let show first that $r^{+}_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) < \infty$. Consider then $\omega \in \Omega$, $T, \varepsilon > 0$. Since $\mathcal{Q}$ is weakly invariant we have that for each $y \in K_\omega \subset Q_\omega$ there exists $u_y \in \mathcal{U}$ such that $\varphi(t, y, u_y, \omega) \in Q_{\theta_t \omega}$ for all $t \geq 0$. Using the continuity of the solution in (t, y), there exist neighborhoods I_t of t and U_y of y such that if $s \in I_t$ and $x \in U_y$, we have that $d(\varphi(s, y, u_y, \omega), \varphi(s, x, u_y, \omega)) < \varepsilon$. By compactness of $[0, T] \times K_\omega$ we have that there exists a finite set $\{(y_1, u_1), \ldots, (y_n, u_n)\}$ such that, for every $x \in K_\omega$ there is $i \in \{1, \cdots, n\}$ with

$$d(\varphi(t, y_i, u_i, \omega), \varphi(t, x, u_i, \omega)) < \varepsilon \text{ for all } t \in [0, T]$$

and hence $r^{+}_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq n < \infty$ as desired.

For the first inequality let $\omega \in \Omega$, $T, \varepsilon > 0$ and consider a minimal strongly (T, ε, ω)-spanning set $\mathcal{S}^{+}_{\omega} = \{(y_1, u_1), \ldots, (y_n, u_n)\}$ for the random compact set $(\mathcal{K}, \mathcal{Q})$ and define $\mathcal{S}_\omega = \{u_1, \ldots, u_n\}$. We need to show that $\mathcal{S}_\omega$ is a (T, ε, ω)-spanning set for $(\mathcal{K}, \mathcal{Q})$.

Since $\mathcal{S}^{+}_{\omega}$ is a strongly (T, ε, ω)-spanning set for $(\mathcal{K}, \mathcal{Q})$, we have that for each $x \in K_\omega$ there exists $(y_i, u_i) \in \mathcal{S}^{+}_{\omega}$, $i \in \{1, \ldots, n\}$, such that

$$d(\varphi(t, x, u_i, \omega), \varphi(t, y_i, u_i, \omega)) < \varepsilon \text{ for all } t \in [0, T].$$

As $\mathcal{S}^{+}_{\omega} \subset \mathrm{lift}(\mathcal{Q}, \omega)$, we have that $\varphi(t, y_i, \omega, u_i) \in Q_{\theta_t \omega}$ for all $t \geq 0$ and consequently $d(\varphi(t, x, u_i, \omega), Q_{\theta_t \omega}) < \varepsilon$ for all $t \in [0, T]$. Hence the set $\mathcal{S}_\omega$ is (T, ε, ω)-spanning and consequently

$$r_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq r^{+}_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}).$$

■

We define the *invariance entropy* $h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q})$ *of the random pair* $(\mathcal{K}, \mathcal{Q})$ *at* $\omega \in \Omega$, by

$$h_{\mathrm{inv}}(\varepsilon, \omega, \mathcal{K}, \mathcal{Q}) := \limsup_{T \to \infty} \frac{1}{T} \log r_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}),$$

$$h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q}) := \lim_{\varepsilon \searrow 0} h_{\mathrm{inv}}(\varepsilon, \omega, \mathcal{K}, \mathcal{Q}).$$

It follows from (5.4) and Proposition 5.2.1 that the limit

$$\lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon,\omega,\mathcal{K},\mathcal{Q})$$

is well-defined. When the random pair is given by $(\mathcal{Q},\mathcal{Q})$ we denote the entropy just by $h_{\mathrm{inv}}(\omega,\mathcal{Q})$.

In order to compute bounds for $h_{\mathrm{inv}}(\omega,\mathcal{K},\mathcal{Q})$ it will be useful to define another quantity which will be called *strong invariance entropy of the random pair* $(\mathcal{K},\mathcal{Q})$ *at* $\omega\in\Omega$. We define

$$h^{+}_{\mathrm{inv}}(\varepsilon,\omega,\mathcal{K},\mathcal{Q}) := \limsup_{T\to\infty}\frac{1}{T}\log r^{+}_{\mathrm{inv}}(T,\varepsilon,\omega,\mathcal{K},\mathcal{Q}),$$

$$h^{+}_{\mathrm{inv}}(\omega,\mathcal{K},\mathcal{Q}) := \lim_{\varepsilon\searrow 0} h^{+}_{\mathrm{inv}}(\varepsilon,\omega,\mathcal{K},\mathcal{Q}).$$

It follows directly from Proposition 5.2.1 that the two notions of entropy above satisfy $h_{\mathrm{inv}}(\omega,\mathcal{K},\mathcal{Q}) \le h^{+}_{\mathrm{inv}}(\omega,\mathcal{K},\mathcal{Q})$ for all $\omega\in\Omega$. Also, for two random pairs $(\mathcal{K}_1,\mathcal{Q})$ and $(\mathcal{K}_2,\mathcal{Q})$ such that $\mathcal{K}_1\subset\mathcal{K}_2$ we have $h_{\mathrm{inv}}(\omega,\mathcal{K}_1,\mathcal{Q}) \le h_{\mathrm{inv}}(\omega,\mathcal{K}_2,\mathcal{Q})$ for all $\omega\in\Omega$.

The next theorem shows that under some conditions the invariance entropy for random control systems cannot increase under semiconjugation.

Theorem 5.2.2 *Consider two random control systems φ and ψ over open sets $M\subset\mathbb{R}^d$ and $N\subset\mathbb{R}^n$ with control spaces $\mathcal{U}$ and $\mathcal{V}$ corresponding to control ranges U and V, and a measurable dynamical system θ_t on a compact metric space Ω. Let $\pi:\Omega\times M\to N$ be a Carathéodory map and $h:\mathcal{U}\to\mathcal{V}$ be any map with the semiconjugation property*

$$\pi_{\theta_t\omega}(\varphi(t,x,\omega,u)) = \psi(t,\pi_\omega(x),\omega,h(u)) \tag{5.5}$$

for all $x\in\mathbb{R}^d, u\in\mathcal{U}, t\ge 0,\omega\in\Omega$.

Assume also that for each fixed $\omega\in\Omega$, the map

$$(t,x)\in\mathbb{R}\times M\mapsto \pi(\theta_t\omega,x)\in N \tag{5.6}$$

is continuous. Then for a random pair $(\mathcal{K},\mathcal{Q})$ we have:

(i) The pair $(\pi(\mathcal{K}),\pi(\mathcal{Q}))$ is a random pair, where for each $\omega\in\Omega$,

$$\pi(\mathcal{K})_\omega := \pi_\omega(K_\omega) \qquad \textit{and} \qquad \pi(\mathcal{Q})_\omega := \pi_\omega(Q_\omega);$$

(ii) If $\mathcal{Q}$ is upper semi-continuous, then $\pi(\mathcal{Q})$ is upper semi-continuous;

(iii) Let $\omega \in \Omega$ and consider the set

$$M_{T,\omega} := \{(t,x); t \in [0,T] \text{ and } x \in Q_{\theta_t\omega}\}.$$

If for $\omega \in \Omega$ and all $T > 0$ the sets $M_{T,\omega}$ are compact, then

$$h_{\text{inv}}(\omega, \pi(\mathcal{K}), \pi(\mathcal{Q})) \leq h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}).$$

Proof. (i) It is clear that $\pi(\mathcal{Q})$ and $\pi(\mathcal{K})$ are compact set-valued functions and that $\pi(\mathcal{K}) \subset \pi(\mathcal{Q})$. We have to show that $\pi(\mathcal{Q})$ is weakly invariant. Let $\omega \in \Omega$ and $z = \pi_\omega(x) \in \pi_\omega(Q_\omega)$, with $x \in Q_\omega$. Since $\mathcal{Q}$ is weakly invariant, there exists $u \in \mathcal{U}$ such that $\varphi(t,x,\omega,u) \in Q_{\theta_t\omega}$ for all $t \geq 0$. Taking $h(u) \in \mathcal{V}$ and using (5.5) we have

$$\psi(t, \pi_\omega(x), \omega, h(u)) = \pi_{\theta_t\omega}(\varphi(t,x,\omega,u)) \in \pi_{\theta_t\omega}(Q_{\theta_t\omega}) = \pi(Q)_{\theta_t\omega},$$

for all $t \geq 0$ showing the weak invariance.

(ii) Let $\omega \in \Omega$, $\varepsilon > 0$ and $t \in \mathbb{R}$. Since the map $(t,x) \mapsto \pi(\theta_t\omega, x)$ is continuous and $Q_{\theta_t\omega}$ is a compact set, there exist $\delta_1 = \delta_1(t,\varepsilon) > 0$ such that

$$d(\pi_{\theta_s\omega}(x), \pi_{\theta_t\omega}(y)) < \varepsilon, \text{ if } |t-s| < \delta_1 \text{ and } d(x,y) < \delta_1, \tag{5.7}$$

for $x \in Q_{\theta_t\omega}$. Also, since the upper semi-continuity of $\mathcal{Q}$ assures that

$$\lim_{s\to t} \text{dist}(Q_{\theta_s\omega}, Q_{\theta_t\omega}) = 0,$$

there exists $\delta_2 = \delta_2(t,\varepsilon) > 0$ such that

$$\text{dist}(Q_{\theta_s\omega}, Q_{\theta_t\omega}) < \delta_1 \quad \text{if} \quad |t-s| < \delta_2.$$

Then take $\delta = \min\{\delta_1, \delta_2\}$ and consider for each $x \in Q_{\theta_s\omega}$ a point $y_x \in Q_{\theta_t\omega}$ satisfying $d(x, Q_{\theta_t\omega}) = d(x, y_x)$.

Then, for $|t-s| < \delta$, we have that $d(x,y_x) \leq \text{dist}(Q_{\theta_s\omega}, Q_{\theta_t\omega}) < \delta$ implying by (5.7) that

$$d(\pi_{\theta_s\omega}(x), \pi(Q)_{\theta_t\omega}) \leq d(\pi_{\theta_s\omega}(x), \pi_{\theta_t\omega}(y_x)) < \varepsilon$$

and this inequality is valid for all $x \in Q_{\theta_t\omega}$. Then, for each $z \in \pi(Q)_{\theta_s\omega}$, we have that $d(z, \pi(Q)_{\theta_t\omega}) < \varepsilon$ and consequently

$$\operatorname{dist}(\pi(\theta_s\omega, Q_{\theta_s\omega}), \pi(\theta_t\omega, Q_{\theta_t\omega})) < \varepsilon \text{ when } |t-s| < \delta$$

showing that $\pi(\mathcal{Q})$ is upper semi-continuous.

(iii) Take $\omega \in \Omega$ and $T, \varepsilon > 0$. Since $M_{T,\omega}$ is a compact set and the map $(t,x) \mapsto \pi(\theta_t\omega, x)$ is continuous, there exist $\delta > 0$ such that for $(t,x) \in M_{T,\omega}$

$$d(\pi(\theta_s\omega, y), \pi(\theta_t\omega, x)) < \varepsilon \qquad \text{if } |t-s| < \delta \text{ and } d(x,y) < \delta.$$

Consider a minimal (T, δ, ω)-spanning set $\mathcal{S}_\omega \subset \mathcal{U}$ for $(\mathcal{K}, \mathcal{Q})$. This means that for each $x \in K_\omega$, there exists $u \in \mathcal{S}_\omega$ such that

$$d(\varphi(t, x, u, \omega), Q_{\theta_t\omega}) < \delta, \text{ for all } t \in [0, T].$$

Then take $y = \pi_\omega(x) \in \pi(K)_\omega$ and $u \in \mathcal{S}_\omega$ that satisfies the above. Since $Q_{\theta_t\omega}$ is a compact set, there exists for each $t \in [0,T]$ an element $x_t \in Q_{\theta_t\omega}$ such that

$$d(\varphi(t, x, u, \omega), x_t) = d(\varphi(t, x, u, \omega), Q_{\theta_t\omega}) < \delta$$

and consequently

$$d(\psi(t, y, h(u), \omega), \pi(Q)_{\theta_t\omega}) = d(\pi(\theta_t\omega, \varphi(t, x, u, \omega)), \pi(\theta_t\omega, Q_{\theta_t\omega}))$$

$$\leq d(\pi(\theta_t\omega, \varphi(t, x, u, \omega)), \pi(\theta_t\omega, x_t)) < \varepsilon \text{ for all } t \in [0, T].$$

Then $h(\mathcal{S}_\omega) \subset \mathcal{V}$ is a (T, ε, ω)-spanning set for $(\pi(\mathcal{K}), \pi(\mathcal{Q}))$ and that implies $r_{\text{inv}}(T, \varepsilon, \omega, \pi(\mathcal{K}), \pi(\mathcal{Q})) \leq r_{\text{inv}}(T, \delta, \omega, \mathcal{K}, \mathcal{Q})$. Also

$$\limsup_{T\to\infty} \frac{1}{T} \ln r_{\text{inv}}(T, \varepsilon, \omega, (\pi(\mathcal{K}), \pi(\mathcal{Q}))) \leq h_{\text{inv}}(\delta, \omega, \mathcal{K}, \mathcal{Q}) \leq h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}).$$

For $\varepsilon \searrow 0$ we obtain $h_{\text{inv}}(\omega, \pi(\mathcal{K}), \pi(\mathcal{Q})) \leq h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q})$. ■

By Lemma 5.1.2 we notice that the set $M_{T,\omega} \subset [0,T] \times Q_{T,\omega}$ is compact if the random compact set $\mathcal{Q}$ is upper semi-continuous .

5.3 Lower and Upper bounds

In this section we will give lower and upper bounds for the invariance entropy of a random control system. Recall the definition of conditional expectation.

Definition 5.3.1 *Consider a probability space* $(\Omega, \mathcal{F}, \mu)$ *and a sub* σ*-algebra* $\mathcal{G}$ *of* $\mathcal{F}$*. For each real-valued integrable function* g *on* Ω*, the conditional expectation of* g *is defined a.e. as the real-valued function* $\mathbb{E}(g|\mathcal{G}) : \Omega \to \mathbb{R}$ *that satisfies:*

1. $\mathbb{E}(g|\mathcal{G})$ *is* $\mathcal{G}$*-measurable;*
2. *for each* $C \in \mathcal{G}$*,*
$$\int_C g d\mu = \int_C \mathbb{E}(g|\mathcal{G}) d\mu.$$

The conditional expectation always exists when the function g is integrable and it is unique outside of a set of zero measure. The following properties of such functions can be found in [[35] Section 9.7].

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and consider sub σ-algebras $\mathcal{H}$, $\mathcal{G}$ of $\mathcal{F}$. Then the followings assertions holds:

1. If $g \in L^1(\Omega, \mathcal{G}, \mu)$, then $\mathbb{E}(g|\mathcal{G}) = g$;
2. If $g, f \in L^1(\Omega, \mathcal{G}, \mu)$ and $a \in \mathbb{R}$, then $\mathbb{E}(af + g|_{\mathcal{G}}) = a\mathbb{E}(f|\mathcal{G}) + \mathbb{E}(g|\mathcal{G})$;
3. If $g \in L^1(\Omega, \mathcal{G}, \mu)$ and $g \geq 0$, then $\mathbb{E}(g|\mathcal{G}) \geq 0$;
4. (Monotone Convergence): Let g_n nonnegative real valued functions such that $g_n \leq g_{n+1}$ for all $n \in \mathbb{N}$. Then
$$\lim_{n\to\infty} \mathbb{E}(g_n|\mathcal{G}) = \mathbb{E}(\lim_{n\to\infty} g_n|\mathcal{G});$$
5. (Fatou): Let g_n nonnegative real valued functions. Then
$$\mathbb{E}(\liminf_{n\to\infty} g_n|\mathcal{G}) \leq \liminf_{n\to\infty} \mathbb{E}(g_n|\mathcal{G});$$
6. (Dominated Convergence): Let g_n real-valued functions and $f \in L^1(\Omega, \mathcal{F}, \mu)$ such that for all $n \in \mathbb{N}$ $|g_n| \leq f$ and $g = \lim_{n\to\infty} g_n$. Then
$$\lim_{n\to\infty} \mathbb{E}(g_n|\mathcal{G}) = \mathbb{E}(g|\mathcal{G}).$$

7. (Jensen inequality): Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. For a real-valued function g, if $\varphi \circ g \in L^1(\Omega, \mathcal{F}, \mu)$, then

$$\varphi(\mathbb{E}(g|\mathcal{G})) \leq \mathbb{E}(\varphi \circ g|\mathcal{G});$$

8. If $\mathcal{H}$ is a subalgebra of $\mathcal{G}$, then

$$\mathbb{E}(g|\mathcal{H}) = \mathbb{E}(\mathbb{E}(g|\mathcal{G})|\mathcal{H})$$

for each real valued function g;

9. If $g \in L^1(\Omega, \mathcal{F}, \mu)$ and f is a bounded real-valued function that is $\mathcal{G}$-measurable, then

$$\mathbb{E}(fg|\mathcal{G}) = f\mathbb{E}(g|\mathcal{G}).$$

We will also need the following lemma.

Lemma 5.3.2 *Let $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a Carathéodory map and $\mathcal{Q}$ a compact set-valued map. Then:*

(i) For a given $\varepsilon > 0$, the set-valued map $\mathcal{Q}_\varepsilon : \Omega \to \mathcal{C}(M)$ defined by $\omega \mapsto \mathrm{cl}(N_\varepsilon(Q_\omega))$, is measurable.

(ii) The maps $m^f_{\mathcal{Q}}, M^f_{\mathcal{Q}} : \Omega \to \mathbb{R}$ defined by

$$m^f_{\mathcal{Q}}(\omega) := \min_{x \in Q_\omega} f(x,\omega) \quad \text{and} \quad M^f_{\mathcal{Q}}(\omega) := \max_{x \in Q_\omega} f(x,\omega)$$

are measurable.

(iii) For each $\omega \in \Omega$ we have $m^f_{\mathcal{Q}_\varepsilon}(\omega) \to m^f_{\mathcal{Q}}(\omega)$ and $M^f_{\mathcal{Q}_\varepsilon}(\omega) \to M^f_{\mathcal{Q}}(\omega)$ for $\varepsilon \searrow 0$.

Proof. The assertions (i) and (ii) are proven in [3]

Let us show (iii). Since $Q_\omega \subset \mathrm{cl}(N_\varepsilon(Q_\omega))$ for each $\varepsilon > 0$ we have $m^f_{\mathcal{Q}}(\omega) \geq m^f_{\mathcal{Q}_\varepsilon}(\omega)$ and $M^f_{\mathcal{Q}}(\omega) \leq M^f_{\mathcal{Q}_\varepsilon}(\omega)$ for all $\omega \in \Omega$. We will show that for each $\delta > 0$ given, there exists $\varepsilon_0 > 0$ such that $m^f_{\mathcal{Q}}(\omega) - m^f_{\mathcal{Q}_\varepsilon}(\omega) < \delta$ and $M^f_{\mathcal{Q}_\varepsilon}(\omega) - M^f_{\mathcal{Q}}(\omega) < \delta$ if $\varepsilon < \varepsilon_0$.

Since for a fixed ω the function $x \mapsto f(x,\omega)$ is uniformly continuous over Q_ω we have that for each $\delta > 0$ there exists $\varepsilon_0 > 0$ such that $|f(x,\omega) -$

$f(y,\omega)| < \delta$ if $||x-y|| < \varepsilon_0$ for $x \in Q_\omega$. Then each point $y \in N_\varepsilon(Q_\omega)$ satisfies

$$f(y,\omega) = f(x,\omega) + f(y,\omega) - f(x,\omega) \geq m^f_{\mathcal{Q}}(\omega) - \delta$$

and

$$f(y,\omega) = f(x,\omega) + f(y,\omega) - f(x,\omega) \leq M^f_{\mathcal{Q}}(\omega) + \delta,$$

where x is a point in Q_ω such that $||x-y|| < \varepsilon_0$. Consequently $0 \leq m^f_{\mathcal{Q}}(\omega) - m^f_{\mathcal{Q}_\varepsilon}(\omega) \leq \delta$ and $M^f_{\mathcal{Q}_\varepsilon}(\omega) - M^f_{\mathcal{Q}}(\omega) \leq \delta$ as desired. ■

Let $(\Omega, \mathcal{F}, (\theta_t)_{t\in\mathbb{R}})$ be a measurable dynamical system. The *(Birkhoff-Khintchin) ergodic theorem* states that for any θ-invariant measure μ on $(\Omega, \mathcal{F})$ and any $f \in L^1(\Omega, \mathcal{F}, \mu)$ the limits

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\theta_s\omega)\mathrm{ds} = \lim_{t\to\infty} \frac{1}{t} \int_{-t}^0 f(\theta_s\omega)\mathrm{ds} = \bar{f}(\omega)$$

exist for all ω in a set Ω_f of full measure and also that the function $\bar{f}$ (defined outside Ω_f by $\bar{f}(\omega) = 0$) is a version of $\mathbb{E}(f|\mathcal{I})$, where $\mathcal{I} \subset \mathcal{F}$ is the sub σ-algebra of the measurable invariant sets of $((\theta_t)_{t\in\mathbb{R}})$. If the measure μ is ergodic the above limit is μ-a.e. constant.

With these prerequisites we can now give lower and upper bounds for the invariance entropy.

Theorem 5.3.3 *Consider the RCS (5.1) and let $(\mathcal{K}, \mathcal{Q})$ be a random pair with the additional assumptions that $\mathcal{K}$ has nonempty interior and $\mathcal{Q}$ is upper semi-continuous. Let μ be a θ-invariant measure over Ω and assume that for each $(x,u) \in M \times U$ we have that $\|\frac{\partial F}{\partial x}(x,u,\cdot)\| \in L^1(\Omega, \mathcal{F}, \mu)$. Then for each $\omega \in \Omega$ the following estimate holds:*

$$h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq \mathbb{E}(m^f_{\mathcal{Q}}|_{\mathcal{I}})(\omega) - \lambda(\omega, \mathcal{Q}), \tag{5.8}$$

where f is the Carathéodory function defined as

$$f(x,\omega) := \min_{u\in U} \mathrm{div_x} F(x,u,\omega)$$

and $\lambda(\omega, \mathcal{Q}) := \lim_{\varepsilon\searrow 0} \liminf_{T\to\infty} \frac{1}{T} \ln \lambda^d(N_\varepsilon(Q_{\theta_T\omega}))$.

Proof. Define the integrable functions $m^f_{\mathcal{Q}_\varepsilon}, m^f_{\mathcal{Q}} : \Omega \to \mathbb{R}$ as in Lemma 5.3.2. Let $\omega \in \Omega$ and $T, \varepsilon > 0$ and consider a minimal (T, ε, ω)-spanning set $\mathcal{S}_\omega = \{u_1, \ldots, u_n\}$ for $(\mathcal{K}, \mathcal{Q})$. Define the following sets:

$$K_{\omega,j} = \{x \in K_\omega; d(\varphi(t, x, \omega, u_j), Q_{\theta_t\omega}) < \varepsilon, t \in [0, T]\}, \quad j = 1, 2, \ldots, n.$$

The set $K_{\omega,j}$ is a Borel set. In fact, since by Lemma 5.1.2

$$K_{\omega,j} = \{x \in K_\omega; g_j(x) < \varepsilon\} \text{ where } g_j(x) = \sup_{t\in[0,T]\cap\mathbb{Q}} f_{u_j,\omega}(t, x)$$

for the lower semi-continuous in the first arguments functions $f_{u_j,\omega}(t, x)$, we have that $K_{\omega,j}$ is a Borel set.

As $\varphi(T, K_{\omega,j}, \omega, u_j) \subset N_\varepsilon(Q_{\theta_T\omega})$ for $j = 1, \ldots, n$, we obtain in particular

$$\lambda^d(\varphi(T, K_{\omega,j}, \omega, u_j)) \leq \lambda^d(N_\varepsilon(Q_{\theta_T\omega})), \text{ for } j = 1, \ldots, n,$$

where λ^d is the Lebesgue measure on $\mathbb{R}^d$.

Moreover, by the transformation theorem and Liouville's trace formula we get for all $j = 1, \ldots, n$

$$\lambda^d(\varphi(T, K_{\omega,j}, \omega, u_j)) = \int_{K_{\omega,j}} \left|\det\frac{\partial\varphi}{\partial x}(T, x, \omega, u_j)\right| \mathrm{dx}$$

$$\geq \lambda^d(K_{\omega,j}) \cdot \inf_{\substack{(x,u)\in K_\omega\times\mathcal{U} \\ \varphi(t,x,\omega,u)\in N_\varepsilon(Q_{\theta_t\omega}) \\ \forall t\in[0,T]}} \left|\det\frac{\partial\varphi}{\partial x}(T, x, \omega, u_j)\right|$$

$$= \lambda^d(K_{\omega,j}) \cdot \inf_{\substack{(x,u)\in K_\omega\times\mathcal{U} \\ \varphi(t,x,\omega,u)\in N_\varepsilon(Q_{\theta_t\omega}) \\ \forall t\in[0,T]}} \exp\int_0^T \mathrm{div_x} F(\varphi(s, x, \omega, u), u(s), \theta_s\omega)\mathrm{ds}$$

$$\geq \lambda^d(K_{\omega,j}) \cdot \exp\int_0^T \min_{(x,u)\in\mathrm{cl}(N_\varepsilon(Q_{\theta_s\omega}))\times U} \mathrm{div_x} F(x, u, \theta_s\omega)\mathrm{ds}$$

$$= \lambda^d(K_{\omega,j}) \cdot \exp\int_0^T m^f_{\mathcal{Q}_\varepsilon}(\theta_s\omega)\mathrm{ds}.$$

Take $j_0 \in \{1, \dots, n\}$ such that $\lambda^d(K_{\omega, j_0}) = \max_{1 \le j \le n} \lambda^d(K_{\omega, j})$. This implies

$$\lambda^d(K_\omega) \le \sum_{j=1}^{n} \lambda^d(K_{\omega, j}) \le n \cdot \lambda^d(K_{\omega, j_0})$$

$$\le n \cdot \frac{\lambda^d(\varphi(T, K_{\omega,j}, \omega, u_j))}{\exp \int_0^T m^f_{\mathcal{Q}_\varepsilon}(\theta_s \omega) \mathrm{d}s} \le n \cdot \frac{\lambda^d(N_\varepsilon(Q_{\theta_T \omega}))}{\exp \int_0^T m^f_{\mathcal{Q}_\varepsilon}(\theta_s \omega) \mathrm{d}s}.$$

Consequently, with $n = r_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q})$ we get that

$$r_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \ge \frac{\lambda^d(K_\omega)}{\lambda^d(N_\varepsilon(Q_{\theta_T \omega}))} \exp \int_0^T m^f_{\mathcal{Q}_\varepsilon}(\theta_s \omega) \mathrm{d}s,$$

and then

$$\limsup_{T \to \infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \ge -\liminf_{T \to \infty} \frac{1}{T} \ln \Big(\lambda^d(N_\varepsilon(Q_{\theta_T \omega})) \Big) +$$

$$+ \limsup_{T \to \infty} \frac{1}{T} \int_0^T m^f_{\mathcal{Q}_\varepsilon}(\theta_s \omega) \mathrm{d}s.$$

By the assumptions the functions $m^f_{\mathcal{Q}_\varepsilon} \in L^1(\Omega, \mathcal{F}, \mu)$, and hence the Birkhof-Chintchin ergodic theorem implies that the second limit on the right side of the equation above exists and is equal to

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T m^f_{\mathcal{Q}_\varepsilon}(\theta_s \omega) \mathrm{d}s = \mathbb{E}(m^f_{\mathcal{Q}_\varepsilon} | \mathcal{I})(\omega)$$

for almost every $\omega \in \Omega$. By Lemma 5.3.2 the functions $m^f_{\mathcal{Q}_\varepsilon}$ converges to $m^f_{\mathcal{Q}}$ and consequently the conditional expectation $\mathbb{E}(m^f_{\mathcal{Q}_\varepsilon} | \mathcal{I})$ converges to $\mathbb{E}(m^f_{\mathcal{Q}} | \mathcal{I})$ as ε goes to 0. Therefore, we have

$$h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \ge \mathbb{E}(m^f_{\mathcal{Q}} | \mathcal{I})(\omega) - \lambda(\omega, \mathcal{Q})$$

as desired. ■

In the case where the Lebesgue measure of the family $\mathcal{Q}$ is bounded, it is easy to see that $\lambda(\omega, \mathcal{Q}) = 0$ and then we have the following corollary.

Corollary 5.3.4 *Under the assumptions of Theorem 5.3.3 assume additionally that there is $M > 0$ such that $\lambda^d(Q_\omega) \leq M$ for all $\omega \in \Omega$ and that the measure μ is ergodic. Then we have*

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq \int_\Omega m^f_{\mathcal{Q}} \, \mathrm{d}\mu$$

with $f(x, \omega) = \min_{u \in U} \operatorname{div}_x F(x, u, \omega)$.

The next theorem provides an upper bound for the strong invariance entropy of a random pair and hence for the invariance entropy of such pair. For the proof recall the definition of fractal dimension: Let $Z \subset X$ be a totally bounded subset of a metric space (X, d) and let $b(\varepsilon, Z)$ be the minimal cardinality of a cover of Z by ε-balls. Then the **fractal dimension** of Z is defined by

$$\dim_F(Z) := \limsup \frac{\ln b(\varepsilon, Z)}{\ln(1/\varepsilon)} \in \mathbb{R} \cup \{\infty\}.$$

The fractal dimension depends on the metric and is not a topological invariant. But for a relatively compact open subset of a differentiable manifold it equals the topological dimension.

Theorem 5.3.5 *Consider the control system (5.1) and let $(\mathcal{K}, \mathcal{Q})$ a random pair. Suppose that we have a θ-invariant measure μ over Ω such that $\|\frac{\partial F}{\partial x}(x, u, \cdot)\| \in L^1(\Omega, \mathcal{F}, \mu)$. Then the Carathéodory function $L : M \times \Omega \to \mathbb{R}$ defined by*

$$L(x, \omega) := \max_{u \in U} \left\| \frac{\partial F}{\partial x}(x, u, \omega) \right\|,$$

satisfies

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \leq \mathbb{E}(M^L_{\mathcal{Q}} | \mathcal{I})(\omega) \cdot \dim_F(K_\omega)$$

μ-almost every point.

Proof. Let $T, \varepsilon > 0$ be given. Assume that $\varepsilon > 0$ is sufficient small such that $N_{2\varepsilon}(Q_\omega) \subset M$ for all $\omega \in \Omega$. Let us fix $\omega \in \Omega$. By [[14], corollary 11.26] we have assured the existence of a C^∞-function $g_{\varepsilon,\omega} : \mathbb{R}^d \to [0, 1]$ such that $\operatorname{supp}(g_{\varepsilon,\omega}) \subset N_{2\varepsilon}(Q_\omega)$ and $g_{\varepsilon,\omega} = 1$ on $\operatorname{cl}(N_\varepsilon(Q_\omega))$. In fact such function is defined by

$$g_{\varepsilon,\omega}(x) = \phi_{\varepsilon/3} * 1_{V^\omega_{\varepsilon/3}}(x) = \int_{\mathbb{R}^d} \phi_{\varepsilon/3}(x-y)(y) 1_{V^\omega_{\varepsilon/3}}(x),$$

where $\phi_t(x) := t^{-n}\varphi(x/t)$ for a C^∞-function ϕ as in the lemma 11.23 of [14], and

$$V^\omega_{\varepsilon/3} := \left\{ x \in \mathbb{R}^d; d(x, \mathrm{cl}(N_\varepsilon(Q_\omega))) < \frac{\varepsilon}{3} \right\}.$$

Since $N_\varepsilon(\mathcal{Q})$ is a random set, the function $1_{V^\omega_{\varepsilon/3}}$ is measurable and consequently the function $g_\varepsilon(\omega, x) := g_{\varepsilon,\omega}(x)$ is a Carathéodory function. We define $\tilde{F}(x,u,\omega) := g_\varepsilon(\omega,x)F(x,u,\omega)$, $\tilde{F} : \mathbb{R}^d \times \mathbb{R}^m \times \Omega \to \mathbb{R}^d$. Then $\tilde{F}$ is continuous and continuously differentiable with respect to the first argument and is measurable with respect to Ω. We consider then the random control system

$$\dot{x}(t) = \tilde{F}(x(t), u(t), \theta_t\omega), \qquad u \in \mathcal{U}. \tag{5.9}$$

We denote its solutions by $\psi(t, y, u, \omega)$. Note that for each $t \in [0, T]$, we have that if $\varphi(t, x, u, \omega) \subset N_\varepsilon(Q_{\theta_t\omega})$ then the solutions of (5.1) and (5.9) coincide.

Define the integrable functions $M^L_{\mathcal{Q}_{2\varepsilon}}$ as in Lemma 5.3.2. Then we have that $M^L_{\mathcal{Q}_{2\varepsilon}} \to M^L_{\mathcal{Q}}$ pointwise and $0 \le M^L_{\mathcal{Q}}(\omega) \le M^L_{\mathcal{Q}_{2\varepsilon}}(\omega)$.

For each ω, we have that $M^L_{\mathcal{Q}_{2\varepsilon}}(\omega)$ is a global Lipschitz constant for the first variable, that is,

$$||\tilde{F}(x_1, u, \omega) - \tilde{F}(x_2, u, \omega)|| \le M^L_{\mathcal{Q}_{2\varepsilon}}(\omega)||x_1 - x_2||$$

for all $x_1, x_2 \in \mathbb{R}^d$, $u \in U$ and $\omega \in \Omega$.

Note that $(\mathcal{K}, \mathcal{Q})$ is also a random pair with respect to the system (5.9) and the lift $\mathrm{lift}(\mathcal{Q}, \omega)$ is the same for the systems (5.9) and (5.1). Also the strongly (T, ε, ω)-spanning sets of system (5.9) coincides with those of system (5.1).

Now let $\mathcal{S}^+_\omega = \{(y_1, u_1), \ldots, (y_n, u_n)\} \subset \mathrm{lift}(\mathcal{Q}, \omega)$ be a minimal strongly (T, ε, ω)-spanning set for $\mathcal{Q}$ and define the sets:

$$N^i_\omega = \{x \in \mathbb{R}^d;\ d_{T,u_i,\omega}(x, y_i) < \varepsilon\}$$

where

$$d_{T,u_i,\omega}(x, y_i) = \max_{t\in[0,T]} ||\psi(t, x, u_i, \omega) - \psi(t, y_i, u_i, \omega)||.$$

Notice that it does not matter whether we consider trajectories of system (5.1) or of system (5.9), since the trajectory $\varphi(t, y_i, u_i, \omega)$ is contained in $Q_{\theta_t\omega}$

for each $t \geq 0$ and $\varphi(t, x, u_i, \omega)$ is ε-close to it for all $t \in [0, T]$. By definition of strongly spanning sets, K_ω is contained in $\bigcup_{i=1}^n N_\omega^i$. Let $x \in \mathbb{R}^d$ be a point such that for some $i \in \{1, \ldots, n\}$ we have

$$||x - y_i|| < \mathrm{e}^{-\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}}\varepsilon.$$

It follows that

$$||\psi(t, x, u_i, \omega) - \psi(t, y_i, u_i, \omega)|| \leq$$
$$\leq ||x - y_i|| + \int_0^t M^L_{\mathcal{Q}_{2\varepsilon}}(\omega)||\psi(s, x, u_i, \omega) - \psi(s, y_i, u_i, \omega)||\mathrm{ds}$$

for all $t \geq 0$ and by Gronwall's Lemma

$$||\psi(t, x, u_i, \omega) - \psi(t, y_i, u_i, \omega)|| \leq ||x - y_i||e^{\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}} < \varepsilon \tag{5.10}$$

for all $t \in [0, T]$.

The equation (5.10) is also true for φ instead of ψ. It follows that $x \in N_\omega^i$ and thus N_ω^i contains the ball $B_{c(T,\varepsilon)}(y_i)$, where $c(T, \varepsilon) := \mathrm{e}^{-\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}}\varepsilon$.

Suppose now that there exists a cover of K_ω consisting of $c(T, \varepsilon)$-balls centered at points $x_1, \ldots, x_N \in K_\omega$ such that $N < n$. Assign to each x_i as above a control function v_i such that $(x_i, v_i) \in \mathrm{lift}(\mathcal{Q}, \omega)$. Then the ball $B_{c(T,\varepsilon)}(x_i)$ is contained in the set

$$V_\omega^i = \{x \in \mathbb{R}^d; d_{T,v_i,\omega}(x, y_i) < \varepsilon\}, \qquad i = 1, \ldots, N.$$

Thus the set $\{(x_1, v_1), \ldots, (x_N, v_N)\}$ is also strongly (T, ε, ω)-spanning, which contradicts the minimality of S_ω^+. It follows that

$$r^+_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq b(c(T, \varepsilon), K_\omega).$$

We have that

$$\ln c(T, \varepsilon)^{-1} = \int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds} - \ln(\varepsilon)$$

and

$$\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds} = \ln(c(T, \varepsilon)^{-1}) + \ln(\varepsilon) = \ln c(T, \varepsilon)^{-1}\left(1 + \frac{\ln(\varepsilon)}{\ln c(T, \varepsilon)^{-1}}\right).$$

If the integral $\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}$ is bounded, that is, there is $M > 0$ such that $\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds} \leq M$ for all $T \geq 0$, we have that

$$r^+(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq b(c(T, \varepsilon), K_\omega) \leq b(e^{-M}\varepsilon, K_\omega)$$

and consequently $h^+_{\mathrm{inv}}(w, \mathcal{K}, \mathcal{Q}) = 0$. Also, by the (Birkhoff-Khintchin) ergodic theorem, we have

$$\mathbb{E}(M^L_{\mathcal{Q}}|\mathcal{I})(w) = \lim_{t\to\infty} \frac{1}{t}\int_0^t M^L_{\mathcal{Q}}(\theta_s\omega)\mathrm{ds}$$

and the fact that the integral above is bounded implies that $\mathbb{E}(M^L_{\mathcal{Q}}|\mathcal{I})(w) = 0$

Hence we can assume then that $\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds} \to \infty$ as $T \to \infty$ and consequently $c(T, \varepsilon) \to 0$ as $T \to \infty$.

Then

$$h^+_{\mathrm{inv}}(\varepsilon, \omega, \mathcal{K}, \mathcal{Q}) = \limsup_{T\to\infty} \frac{1}{T} \ln r^+_{\mathrm{inv}}(T, \varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq$$

$$\limsup_{T\to\infty} \frac{1}{T} \ln b(c(T, \varepsilon), K_\omega) = \limsup_{T\to\infty} \frac{1}{T}\left(\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}\cdot \frac{\ln b(c(T, \varepsilon), K_\omega)}{\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}}\right).$$

Since for each T,

$$\frac{1}{T}\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds} \qquad \text{and} \qquad \frac{\ln b(c(T, \varepsilon), K_\omega)}{\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}}$$

are nonnegative, we have that

$$h^+_{\mathrm{inv}}(\varepsilon, \omega, \mathcal{K}, \mathcal{Q}) \leq \limsup_{T\to\infty} \frac{1}{T}\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds} \cdot \limsup_{T\to\infty} \frac{\ln b(c(T, \varepsilon), K_\omega)}{\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}},$$

and as we are assuming that the measure μ is θ-invariant, the first lim sup on the right-hand side of the inequality above is, μ-a.e., a limit and since $M^L_{\mathcal{Q}_{2\varepsilon}} \in L^1(\Omega, \mathcal{F}, \mu)$ it is equal to $\mathbb{E}(M^L_{\mathcal{Q}_{2\varepsilon}}|\mathcal{I})(\omega)$. For the second lim sup, we note that

$$\frac{\ln b(c(T, \varepsilon), K_\omega)}{\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}} = \frac{\ln b(c(T, \varepsilon), K_\omega)}{\ln(c(T, \varepsilon)^{-1})\left(1 + \frac{\ln(\varepsilon)}{(\ln c(T,\varepsilon)^{-1})}\right)}$$

and consequently

$$\limsup_{T\to\infty} \frac{\ln b(c(T,\varepsilon),K_\omega)}{\int_0^T M^L_{\mathcal{Q}_{2\varepsilon}}(\theta_s\omega)\mathrm{ds}} = \dim_F(K_\omega).$$

Then we have that

$$h^+_{\text{inv}}(\varepsilon,\omega,\mathcal{K},\mathcal{Q}) \le \mathbb{E}(M^L_{\mathcal{Q}_{2\varepsilon}}|\mathcal{I})(\omega)\cdot\dim_F(K_\omega)$$

and since $M^L_{\mathcal{Q}_{2\varepsilon}} \to M^L_{\mathcal{Q}}$ and $M^L_{\mathcal{Q}_{2\varepsilon}} \le M^L_{\mathcal{Q}_{2\varepsilon'}}$ if $\varepsilon\le\varepsilon'$ we have that for $\varepsilon \searrow 0$

$$h^+_{\text{inv}}(\omega,\mathcal{K},\mathcal{Q}) = \lim_{\varepsilon\searrow 0} h^+(\varepsilon,\omega,\mathcal{K},\mathcal{Q}) \le \lim_{\varepsilon\searrow 0}\mathbb{E}(M^L_{\mathcal{Q}_{2\varepsilon}}|\mathcal{I})(\omega)\cdot\dim_F(K_\omega) =$$

$$= \mathbb{E}(\lim_{\varepsilon\searrow 0} M^L_{\mathcal{Q}_{2\varepsilon}}|\mathcal{I})(\omega)\cdot\dim_F(K_\omega) = \mathbb{E}(M^L_{\mathcal{Q}}|\mathcal{I})(\omega)\cdot\dim_F(K_\omega).$$

■

Notice that since for a compact set $K\subset\mathbb{R}^d$ we have $\dim_F K \le d$, we also have that $h^+_{\text{inv}}(\omega,\mathcal{K},\mathcal{Q}) \le \mathbb{E}(d\cdot M^L_{\mathcal{Q}}|\mathcal{I})(\omega)$. As for the lower bound, if we assume that the measure μ is ergodic we have the following result.

Corollary 5.3.6 *Assume in addition to the conditions of Theorem 5.3.5 that the measure μ is ergodic. Then*

$$h^+_{\text{inv}}(\omega,\mathcal{K},\mathcal{Q}) \le d\cdot\int_\Omega M^L_{\mathcal{Q}}\,\mathrm{d}\mu$$

with L defined as above.

Example 5.3.7 *With the theorems above we are able to compute the invariance entropy of a random pair $(\mathcal{K},\mathcal{Q})$ for a one-dimensional linear control system given by*

$$\dot{x}(t) = a(\theta_t\omega)x(t) + u(t) := F(x(t),u(t),\theta_t\omega), \qquad u\in\mathcal{U},\omega\in\Omega,$$

with a nonnegative integrable function $a:\Omega\to\mathbb{R}$. For this system, if the random pair $(\mathcal{K},\mathcal{Q})$ is such that $\mathcal{K}$ has nonvoid interior and $\mathcal{Q}$ is upper semi-continuous and has bounded Lebesgue measure, Theorems 5.3.3 and 5.3.5 yield

$$h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \in \Big[\mathbb{E}(f_1|\mathcal{I})(\omega),\ \mathbb{E}(f_2|\mathcal{I})(\omega)\Big],$$

where the functions f_1 and f_2 are given by

$$f_1(\omega) = \min_{(x,u)\in Q_\omega \times U} \frac{\partial F}{\partial x}(x, u, \omega) \qquad \text{and} \qquad f_2(\omega) = \max_{(x,u)\in Q_\omega \times U} \left|\frac{\partial F}{\partial x}(x, u, \omega)\right|,$$

respectively. Since $\frac{\partial F}{\partial x}(x, u, \omega) = a(\omega) \geq 0$ we obtain $h_{\mathrm{inv}}(\omega, \mathcal{Q}) = \mathbb{E}(a|\mathcal{I})(\omega)$. Also, if the measure μ is ergodic, then

$$h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q}) = \int_\Omega a \,\mathrm{d}\mu.$$

5.4 The Linear case

Now we consider a system of the form

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)u(t) \tag{5.11}$$

with $A : \Omega \to \mathbb{R}^{d\times d}$ and $B : \Omega \to \mathbb{R}^{d\times m}$ integrable. The solutions in such a case are given by

$$\varphi(t, x, \omega, u) = \varphi_L(t, \omega)x + \int_0^t \varphi_L(t - s, \theta_s\omega)B(\theta_s\omega)u(s)\mathrm{ds}, \tag{5.12}$$

where φ_L is the solution for the associated random dynamical system determined by A.

We have the following theorem, called Oseledet's Multiplicative Theorem, that helps us in the linear case.

Theorem 5.4.1 *Consider a random dynamical system $\Phi = (\theta, \varphi_L) : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d$ and assume*

$$\sup_{0\leq t\leq 1} \log^+ ||\varphi_L(t, \omega)^{\pm 1}|| \in L^1(\Omega, \mathcal{F}, \mu),$$

where $||\cdot||$ is any norm in $Gl(d, \mathbb{R})$ and $\log^+$ denotes the positive part of $\log$.

Then there exists a θ-invariant set $\Omega_0 \subset \Omega$ of full μ-measure such that for each $\omega \in \Omega_0$ there exists a splitting $\mathbb{R}^d = \bigoplus_{j=1}^{l(\omega)} L_j(\omega)$ of $\mathbb{R}^d$ into linear subspaces with the following properties:

(i) The number of subspaces is θ-invariant, i.e., $l(\theta_t\omega) = l(\omega)$ for all $t \in \mathbb{R}$ and the dimensions of the subspaces are θ-invariant, i.e., $\dim L_j(\theta_t\omega) = \dim L_j(\omega) := d_j(\omega)$.

(ii) The subspaces are invariant by the flow Φ, i.e.,

$$\varphi_L(t,\omega)L_j(\omega) = L_j(\theta_t\omega).$$

(iii)There exist finitely many numbers $\lambda_1(\omega) < \cdots < \lambda_{l(\omega)}(\omega)$ in $\mathbb{R}$, such that for each $x \in \mathbb{R}^d \setminus \{0\}$ the Lyapunov exponent $\lambda(x,\omega)$ exists as a limit and

$$\lambda(x,\omega) = \lim_{t\to\pm\infty} \frac{1}{t} \ln ||\varphi_L(t,\omega)x|| = \lambda_j(\omega)$$

if, and only if, $x \in L_j(\omega)$.

(iv) The following maps are measurable: $l : \Omega \to \{1,\ldots,d\}$ with the discrete σ-algebra, and for each $j = 1,\ldots,l(\omega)$ the maps $L_j : \Omega \to Gr_{d_j}(\mathbb{R}^d)$, with the Borel σ-algebra, $d_j : \Omega \to \{1,\ldots,d\}$ with the discrete σ-algebra and $\lambda_j : \Omega \to \mathbb{R}$ with the Borel σ-algebra.

(v) If the base flow θ is ergodic, then the maps l, d_j and λ_j are constant on Ω_0 and we usually denote them without the variable ω.

The linear random equation

$$\dot{x} = A(\theta_t\omega)x$$

gives rises to a RDS given by (θ,φ_L). In general the Lyapunov exponents for such a system are difficult to compute explicitly but the average can sometimes be computed explicitly. In the ergodic case, the average Lyapunov exponent $\bar{\lambda} := \frac{1}{d}\sum d_j\lambda_j$ is given by $\frac{1}{d}\mathrm{tr}\mathbb{E}(A|\mathcal{I})$.

For each $\omega \in \Omega$ let us consider the spaces

$$L^+(\omega) := \bigoplus_j \{L_j(w);\ \lambda(w) > 0\} \quad \text{and} \quad L_0^-(\omega) := \bigoplus_j \{L_j(w);\ \lambda(w) \leq 0\}.$$

The above theorem assures then that for each ω in a set of full measure, $\mathbb{R}^d = L^+(\omega) \oplus L_0^-(\omega)$. Define $\pi^+ : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ as $(\omega,x) \mapsto x_\omega^+ \in L^+(\omega)$. Also, property (ii) assures that

$$\pi^+_{\theta_t\omega} \circ \varphi_L(t,\omega) = \varphi_L(t,\omega) \circ \pi^+_\omega$$

and consequently $(t,x) \mapsto \pi^+_{\theta_t\omega}(x) = \pi^+(\theta_t\omega, x)$ is continuous for each ω in a set of full measure.

Pick measurable unit vectors (in the standard Euclidean norm) in such a way that the $v_1(\omega), \ldots, v_{d_1(\omega)}(\omega)$ are orthogonal and taken from $L_1(\omega)$, $v_{d_1(\omega)+1}(\omega), \ldots, v_{d_2(\omega)}(\omega)$ are orthogonal and taken from $L_2(\omega)$ and so on. The existence of such measurable is guaranteed in [Corollary 4.3.12 [2]]. For $\omega \in \Omega_0$ fixed, consider then map $T(\omega) : \mathbb{R}^{d(\omega)} \to L^+(\omega)$ that associates the standard basis $\{e_i\}$ with the basis of $L^+(\omega)$ given by the $v_i(\omega)$ above.

Since $d(\theta_t\omega) = d(\omega)$ for all $t \in \mathbb{R}$ we can define the linear random dynamical system $\varphi^+_L : \mathbb{R} \times \mathbb{R}^{d(\omega)} \times \Omega_\omega \times \to \mathbb{R}^{d(\omega)}$ by

$$\varphi^+_L(t, \bar{\omega})x = T(\theta_t\bar{\omega})^{-1}\varphi_L(t, \bar{\omega})T(\bar{\omega})$$

where $\Omega_\omega := \{\theta_t\omega,\, t \in \mathbb{R}\}$. It is not hard to show that the Lyapunov coefficients of φ^+ are just the positive ones associated to φ.

Using the expression (5.12), we have that

$$\varphi^+(t, P^+(\bar{\omega})x, \bar{\omega}, u) = P^+(\theta_t\bar{\omega})\varphi(t, x, \bar{\omega}, u)$$

is a linear RCS, where $P^+(\bar{\omega}) := T^{-1}(\bar{\omega}) \circ \pi^+_{\bar{\omega}}$ and we are considering φ just over Ω_ω (that is not a problem, because our entropy just depends on Ω_ω).

Since the projection π^+ together with φ^+ and φ clearly satisfies the hypothesis of Theorem 5.2.2, we have that for a random dynamical pair $(\mathcal{K}, \mathcal{Q})$, that

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq h_{\text{inv}}(\omega, \pi^+(\mathcal{K}), \pi^+(\mathcal{Q})).$$

Corollary 5.4.2 *Consider a linear RCS given by (5.11) and let $(\mathcal{K}, \mathcal{Q})$ be a random pair. Assume that $\mathcal{K}$ has nonvoid interior and that $\mathcal{Q}$ is upper semi-continuous and has Lebesgue measure bounded. Suppose also that there is a θ-invariant measure over Ω such that $A \in L^1(\Omega, \mathcal{F}, \mu)$. Then there is a full measure set Ω_0 such that for each $\omega \in \Omega_0$*

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq \sum_{\lambda_i(\omega) > 0} d_i(\omega)\lambda_i(\omega).$$

Proof. Since $A \in L^1(\Omega, \mathcal{F}, \mu)$ we have a full measure set Ω_0 such that the properties in Theorem 5.4.1 are satisfied for the random dynamical (θ, φ) generated by A. If we define as above the RCS φ^+ we have that

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq h_{\text{inv}}(\omega, \pi^+(\mathcal{K}), \pi^+(\mathcal{Q}))$$

and we just need to estimate a lower bound for the entropy of φ^+. But from the demonstration of the Theorem 5.3.3 it is not hard to show that in the linear case we have that

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq \limsup_{T\to\infty} \frac{1}{T} \log|\det \varphi_L(t,\omega)| - \lambda(\omega, \mathcal{Q})$$

and in particular

$$h_{\text{inv}}(\omega, \mathcal{K}, \mathcal{Q}) \geq h_{\text{inv}}(\omega, \pi^+(\mathcal{K}), \pi^+(\mathcal{Q})) \geq \limsup_{T\to\infty} \frac{1}{T} \log|\det \varphi_L^+(t,\omega)|$$

because we are assuming that $\mathcal{Q}$ has Lebesgue measure bounded. The result follows then if we use the Theorem of Furstenberg-Kesten, that assures that for every $\omega \in \Omega$, we have that

$$\lim_{t\to\infty} \frac{1}{t} \log|\det \varphi_L^+(t,\omega)| = \sum_{\lambda_i(\omega)>0} d_i(\omega)\lambda_i(\omega).$$

■

For the upper bound we need the notion of topological entropy for a linear random dynamical system (LRDS). Notions of entropy for random dynamical systems and nonautonomous dynamical systems have appeared in the papers of Bogenschütz [6], Froyland and Stancevic [16] and Kolyada and Snoha [24].

Let $\mathcal{Q}$ be a random compact set in $\mathbb{R}^d$. For $\varepsilon, T > 0$ and $\omega \in \Omega$ we call the set $R \subset \mathbb{R}^d$ a (T, ε)-spanning set for the compact set Q_ω if for every $x \in Q_\omega$ there exists $y \in R$ such that

$$d(\varphi_L(t,x,\omega), \varphi_L(t,y,\omega)) < \varepsilon \text{ for all } t \in [0,T].$$

Denote by $r_\omega(T,\varepsilon)$ the minimal cardinality of a (T,ε)-spanning set and define the topological entropy of Q_ω for an LRDS by the number

$$h(Q_\omega) := \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \ln r_\omega(T,\varepsilon).$$

If $(\mathcal{K}, \mathcal{Q})$ is a random pair for the LRCS in (5.11) and $\mathcal{Q}$ is upper semicontinuous it is not hard to show that

$$h^+_{\text{inv}}(\omega, \mathcal{Q}) \leq h(Q_\omega).$$

In fact, since the solutions of the LRDS are given by

$$\varphi(t,x,\omega,u)=\varphi_L(t,\omega)x+\int_0^t \varphi_L(t-s,\omega)B(\theta_s\omega)u(s)\text{ds},$$

we have that for given $x,y\in Q_\omega$

$$||\varphi(t,x,\omega,u)-\varphi(t,y,\omega,u)||=||\varphi_L(t,\omega)x-\varphi_L(t,\omega)y||.$$

If we consider a (T,ε)-spanning set $R\subset\mathbb{R}^d$ for Q_ω, we just have to associate, to each $x\in R$ a control function $u\in\mathcal{U}$ such that $\varphi(t,x,u,\omega)\in Q_{\theta_t\omega}$ and we have $r^+_{\text{inv}}(T,\varepsilon,\omega,\mathcal{Q})\le r_\omega(T,\varepsilon)$ and consequently $h^+_{\text{inv}}(\omega,\mathcal{Q})\le h(Q_\omega)$.

We have two important properties for the topological entropy given by the following lemmas.

Lemma 5.4.3 *Consider a LRCS as above and suppose that for each $\omega\in\Omega$ there is an invariant decomposition $\mathbb{R}^d=\mathcal{W}_{1,\omega}\oplus\mathcal{W}_{2,\omega}$ in the sense that $\varphi_L(t,\omega)\mathcal{W}_{i,\omega}=\mathcal{W}_{i,\theta_t\omega}$, $i=1,2$. Denote the corresponding projections by $\pi_{i,\omega}:\mathbb{R}^d\to\mathcal{W}_{i,\omega}\subset\mathbb{R}^d$. Then for every random compact set $\mathcal{Q}$ the topological entropy satisfies*

$$h(Q_\omega)\le h(\pi_{1,\omega}(Q_\omega))+h(\pi_{2,\omega}(Q_\omega)).$$

Proof. For $\omega\in\Omega$ and $T,\varepsilon>0$ let $R_i\subset\mathbb{R}^d$ be minimal (T,ε)-spanning sets for $\pi_{i,\omega}(Q_\omega)$ with cardinalities $r^i_\omega(T,\varepsilon)$, i=1, 2, respectively. Then consider $R=R_1\oplus R_2$. For a given $x\in Q_\omega$ we have that $x=\pi_{1,\omega}(x)+\pi_{2,\omega}(x)$ and for $\pi_{i,\omega}(x)$ there exists $y_i\in R_i$ such that $d(\varphi_L(t,\pi_{i,\omega}(x),\omega),\varphi_L(t,y_i,\omega))<\varepsilon$ for $t\in[0,T]$. Since the distance here is the usual one given by a norm in $\mathbb{R}^d$ we have that for $y=y_1+y_2$

$$\begin{aligned}d(\varphi_L(t,x,\omega),\varphi_L(t,y,\omega))&=||\varphi_L(t,\pi_{1,\omega}(x)+\pi_{2,\omega}(x),\omega)-\varphi_L(t,y_1+y_2,\omega)||\\ &\le||\varphi_L(t,\pi_{1,\omega}(x),\omega)-\varphi_L(t,y_1,\omega)||+||\varphi_L(t,\pi_{2,\omega}(x),\omega)-\varphi_L(t,y_2,\omega)||\\ &=d(\varphi_L(t,\pi_{1,\omega}(x),\omega),\varphi_L(t,y_1,\omega))+d(\varphi_L(t,\pi_{2,\omega}(x),\omega),\varphi_L(t,y_2,\omega))<2\varepsilon,\end{aligned}$$

for all $t\in[0,T]$. This shows that the set R is a $(T,2\varepsilon)$-spanning set for Q_ω and so

$$r_\omega(T,2\varepsilon)\le r^1_\omega(T,\varepsilon)\cdot r^2_\omega(T,\varepsilon).$$

Also,

$$\limsup_{T\to\infty}\frac{1}{T}\ln r_\omega(T,2\varepsilon)\le\limsup_{T\to\infty}\frac{1}{T}\ln r^1_\omega(T,\varepsilon)+\limsup_{T\to\infty}\frac{1}{T}\ln r^2_\omega(T,\varepsilon).$$

Taking $\varepsilon \searrow 0$ we obtain

$$h(Q_\omega) \leq h(\pi_{1,\omega}(Q_\omega)) + h(\pi_{2,\omega}(Q_\omega))$$

as desired. ■

By induction we can show that the result above is valid for each finite sum.

Lemma 5.4.4 *Consider a LRDS as before and let μ be a θ-invariant measure on Ω. With the notation of Theorem 5.4.1 for a LRDS, there exists Ω_0 such that for each $\omega \in \Omega_0$, $\mathbb{R}^d = \oplus_{j=1}^{l} L_j(\omega)$. Then for a random compact set $\mathcal{Q}$ we have for $\omega \in \Omega_0$ that the topological entropy for $\pi_{i,\omega}(Q_\omega)$ satisfies*

$$h(\pi_{i,\omega}(Q_\omega)) \leq d_i(\omega) \cdot \lambda_i(\omega) \qquad \text{if } \lambda_i(\omega) > 0,$$

and

$$h(\pi_{i,\omega}(Q_\omega)) = 0 \qquad \text{if } \lambda_i(\omega) \leq 0,$$

where $d_i(\omega)$ is the dimension of the spaces $L_i(\omega)$ and $\pi_{i,\omega} : \mathbb{R}^d \to L_i(\omega) \subset \mathbb{R}^d$ are the projections $x \mapsto \pi_i(\omega, x) = x_{i,\omega}$ given by the Theorem 5.4.1.

Proof. Let $\omega \in \Omega_0$ fixed and assume, for simplicity of notation, that $Q_\omega \subset L_i(\omega)$ for some i. Suppose first that $\lambda_i(\omega) < 0$ (for the case $\lambda_i(\omega) = 0$ we will show that the first inequality holds and since $h(Q_\omega) \geq 0$ we have the result). If $\lambda_i(\omega) < 0$ there is, for each $x \in L_i(\omega)$, a $T_x > 0$ such that $||\varphi_L(T, \omega)x|| < 1$ for all $T > T_x$. Then, since φ_L is uniformly continuous on the compact set $B_\omega^i = \{x \in V_\omega^i; ||x|| \leq 1\}$, there is a $T > 0$ such that $||\varphi_L(t, \omega)x|| < 1$ for all $x \in B_\omega^i$ and $t \geq T$. Consequently for all $t \geq T$, $||\varphi_L(t, \omega)_{|L_i(\omega)}|| \leq 1$ and then the cardinality of any (S, ε)-spanning set for Q_ω, for $S \geq T$ satisfies $r_\omega(S, \varepsilon) \leq r_\omega(T, \varepsilon)$, showing that $h(Q_\omega) = 0$.

Suppose now $\lambda_i(\omega) \geq 0$. Since Q_ω is a compact set, there exists $N(\omega) \in \mathbb{N}$ such that

$$Q_\omega \subset [-N(\omega), N(\omega)]^{d_i(\omega)}.$$

.

For $\delta > 0$ and $M := \lceil \frac{1}{\delta} \rceil$ every point in $[-N(\omega), N(\omega)]$ has distance less than $\frac{1}{M} \leq \delta$ to one of the $2MN(\omega) + 1$ points in

$$S(\omega) = \left\{ x_i = \frac{i}{M}; i = -MN(\omega), \ldots, MN(\omega) \right\}.$$

Then in the max-norm, every point in $Q_\omega \subset [-N(\omega), N(\omega)]^{d_i(\omega)}$ has distance less than $\frac{1}{M} \leq \delta$ to one of the $(2MN(\omega)+1)^{d_i(\omega)}$ points in the product $S(\omega)^{d_i(\omega)}$. Denote by $\varphi_L^i(t,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ the linear map given by $\varphi_L^i(t,\omega) = \varphi_L(t,\omega)_{|L_i(\omega)}$. We have

$$||\varphi_L(t,x,\omega) - \varphi_L(t,y,\omega)|| \leq ||\varphi_L^i(t,\omega)||||x-y||$$

and then the set $S(\omega)^{d_i(\omega)}$ is a $(T, \delta \max_{t\in[0,T]} ||\varphi_L^i(t,\omega)||, \omega)$-spanning set of cardinality

$$(2MN(\omega)+1)^{d_i(\omega)} \leq M^{d_i(\omega)}(2N(\omega)+1)^{d_i(\omega)}$$

$$= \left\lceil \frac{1}{\delta} \right\rceil^{d_i(\omega)} (2N(\omega)+1)^{d_i(\omega)} \leq \left(\frac{1}{\delta}+1\right)^{d_i(\omega)} (2N(\omega)+1)^{d_i(\omega)}.$$

Thus for $\varepsilon > 0$ and $\delta := \varepsilon\big[\max_{t\in[0,T]} ||\varphi_L^i(t,\omega)||\big]^{-1}$ we find that the minimal cardinality of a (T,ε)-spanning set for Q_ω satisfies

$$r_\omega(T,\varepsilon) \leq \left[\varepsilon^{-1} \max_{t\in[0,T]} ||\varphi_L^i(t,\omega)|| + 1\right]^{d_i(\omega)} (2N(\omega)+1)^{d_i(\omega)}$$

$$= \left[\max_{t\in[0,T]} ||\varphi_L^i(t,\omega)|| + \varepsilon\right]^{d_i(\omega)} \varepsilon^{-d_i(\omega)}(2N(\omega)+1)^{d_i(\omega)}.$$

Let

$$\limsup_{T\to\infty} \frac{1}{T} \ln r_\omega(T,\varepsilon) = \lim_{T_j\to\infty} \frac{1}{T_j} \ln r_\omega(T_j,\varepsilon).$$

There are $\tau_j \in [0,T_j]$ with $||\varphi_L^i(\tau_j,\omega)|| = \max_{t\in[0,T_j]} ||\varphi_L^i(t,\omega)||$. If τ_j and hence $||\varphi_L^i(\tau_j,\omega)||$ remains bounded for $j\to\infty$, it is easy to see that $h(Q_\omega) = 0$. Hence we may assume that there is a subsequence of (τ_j), that we again denote by (τ_j), with $\tau_j \to \infty$. It follows that

$$\limsup_{T\to\infty} \frac{1}{T} \ln r_\omega(T,\varepsilon) = \lim_{T_j\to\infty} \frac{1}{T_j} \ln r_\omega(T_j,\varepsilon)$$

$$\leq \lim_{j\to\infty} \frac{1}{\tau_j} \ln r_\omega(T_j,\varepsilon)$$

$$\leq \lim_{j\to\infty} \frac{1}{\tau_j} \big[\ln(||\varphi_L^i(\tau_j,\omega)|| + \varepsilon)^{d_i(\omega)} - d_i(\omega)\ln\varepsilon + d_i(\omega)\ln(2N(\omega)+1)\big]$$

$$= d_i(\omega) \lim_{j\to\infty} \frac{1}{\tau_j} \ln ||\varphi_L(\tau_j,\omega)|| \leq d_i(\omega) \limsup_{T\to\infty} \frac{1}{T} \ln ||\varphi_L^i(T,\omega)||.$$

We will show that

$$\limsup_{T\to\infty} \frac{1}{T} \ln ||\varphi_L^i(T,\omega)|| \leq \lambda_i(\omega),$$

and then for $\varepsilon \searrow 0$ we have that

$$h(Q_\omega) \leq d_i(\omega) \cdot \lambda_i(\omega).$$

Consider a sequence $T_n \to \infty$ that satisfies

$$\limsup_{T\to\infty} \frac{1}{T} \ln ||\varphi_L^i(T,\omega)|| = \lim_{n\to\infty} \frac{1}{T_n} \ln ||\varphi_L^i(T_n,\omega)||.$$

We need to show that for each $\varepsilon > 0$, there is n_0 such that

$$\frac{1}{T_n} \ln ||\varphi_L^i(T_n,\omega)|| - \lambda_i(\omega) < \varepsilon \qquad \forall n \geq n_0.$$

But for each $x \in L_i(\omega)$, there exists n_x such that for $n \geq n_x$ we have

$$0 \leq \frac{1}{T_n} \ln ||\varphi_L^i(T_n,\omega)x|| - \lambda_i(\omega) < \varepsilon \tag{5.13}$$

and by continuity that is actually valid in a small neighbourhood of x. Then, over the compact set $D^i = \{x \in L_i(\omega); ||x|| \leq 1\}$ there is n_0 such that for all $n \geq n_0$ and all $x \in D^i$ we have that (5.13) holds. Consequently, taking the supremum over D^i, we have

$$\sup_{x\in D^i} \left\{ \frac{1}{T_n} \ln ||\varphi_L^i(T_n,\omega)x|| - \lambda_i(\omega) \right\} < \varepsilon$$

and since

$$\sup_{x\in D^i} \left\{ \frac{1}{T_n} \ln ||\varphi_L^i(T_n,\omega)x|| - \lambda_i(\omega) \right\} = \frac{1}{T_n} \ln \left\{ \sup_{x\in D^i} ||\varphi_L^i(T_n,\omega)x|| \right\} - \lambda_i(\omega),$$

we have the result. ■

We can give then an upper bound for the invariance entropy for a LRCS given by (5.11).

Theorem 5.4.5 *Let $\mathcal{Q}$ be a random compact set for the LRCS given by (5.11) and let μ an θ-invariant measure in Ω. Then the invariance entropy at satisfies*

$$h_{\mathrm{inv}}(\omega, \mathcal{Q}) \leq \sum_{\lambda_i(\omega)>0} d_i(\omega)\lambda_i(\omega),$$

ω in a set of full measure.

Proof. We have already shown that $h_{\mathrm{inv}}(\omega, \mathcal{Q}) \leq h^+_{\mathrm{inv}}(\omega, \mathcal{Q}) \leq h(Q_\omega)$. If we consider the random compact sets $\mathcal{Q}_i = \pi_{i,\omega}(\mathcal{Q})$ defined as before, we get by Lemma 5.4.3 that $h(Q_\omega) \leq \sum_{i=1}^{r} h(Q_{i,\omega})$, where $Q_{i,\omega} = \pi_{i,\omega}(Q_\omega)$. Also, by Lemma 5.4.4 we know that

$$h(Q_{i,\omega}) \begin{cases} \leq d_i(\omega) \cdot \lambda_i(\omega) & \text{if } \lambda_i(\omega) > 0, \\ \quad = 0 & \text{if } \lambda_i(\omega) \leq 0, \end{cases}$$

where $d_i(\omega)$ is the dimension of $L_i(\omega)$. If we put that everything together we get that $h(Q_\omega) \leq \sum_{\lambda_i(\omega)>0} d_i(\omega)\lambda_i(\omega)$ as desired. ■

Corollary 5.4.6 *Let $(\mathcal{K}, \mathcal{Q})$ be a random pair for the system (5.11) and let μ a θ-invariant measure. Assume that $\mathcal{K}$ has nonvoid interior and $\mathcal{Q}$ is upper semi-continuous and has bounded Lebesgue measure. Then there is a set of full measure Ω_0 of Ω such that*

$$h_{\mathrm{inv}}(\omega, \mathcal{K}, \mathcal{Q}) = \sum_{\lambda_i(\omega)>0} d_i(\omega)\lambda_i(\omega)$$

for every $\omega \in \Omega_0$. In particular if μ is ergodic, the invariance entropy is constant μ a.e.

We notice that the above formula says us that in the linear case, the invariance entropy of a random pair, under some assumptions, is measurable, invariant by the flow θ and constant in the ergodic case. A question that arises is: Are that true also for the general case?

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