Universidade Estadual de Campinas

INSTITUTO DE MATEMÁTICA, ESTATíSTICA E COMPUTAÇÃO CIENTíFICA
DEPARTAMENTO DE MATEMÁTICA APLICADA

As Equações de Movimento de Fluidos Viscosos Incompressíveis com Fenômenos de Difusão

Autor: Pedro Danizete Damázio

Orientador: Prof. Dr. Marko Antonio Rojas Medar

Campinas, São Paulo
Maio de 2003

AS EQUAÇÕES DE MOVIMENTO DE FLUIDOS VISCOSOS
INCOMPRESSíVEIS COM FENÔMENOS DE DIFUSÃO

Este exemplar corresponde à redação final da tese devidamente corrigida e defendida por *Pedro Danizete Damázio* e aprovada pela comissão julgadora.

Campinas, 05 de Maio de 2003.

Prof. Dr. Marko Antonio Rojas Medar
Orientador

Banca Examinadora
1. Prof. Dr. Marko Antonio Rojas Medar
2. Prof. Dr. Luiz Adauto da Justa Medeiros
3. Prof. Dr. José Luiz Boldrini
4. Prof. Dr. João Batista de Mendonça Xavier
5. Prof. Dr. Ma To Fu

Tese apresentada ao Instituto de Matemática Estatística e Computação Científica,UNICAMP, como requesito parcial para obtenção do título de **Doutor em Matemática Aplicada.**

Conteúdo

Capítulo 1

Resultados Preliminares

1.1 Introdução

Neste trabalho faremos um estudo das equações de movimento de fluidos viscosos incompressíveis com fenômenos de difusão, no que diz respeito à existência, unicidade e regularidade de soluções.

Para situar o problema, faremos inicialmente, uma ligeira discussão sobre a sua formulação. Consideramos o escoamento de um fluido numa região $\Omega \subset I\!R^n$ ($n = 2$ ou 3), consistindo de dois componentes com densidades características ρ_1 e ρ_2 (supostas constantes positivas) e velocidades características $\mathbf{v}_1 = \mathbf{v}_1(x,t)$ e $\mathbf{v}_2 = \mathbf{v}_2(x,t)$, respectivamente. Admita que as concentrações de massa e de volume do primeiro fluido sejam dadas por $a(x,t)$ e $b(x,t)$, respectivamente. Definimos a densidade média pela expressão $\rho(x,t) = b(x,t)\rho_1 + (1 - b(x,t))\rho_2$, a velocidade média de massa da mistura por $\mathbf{v}(x,t) = a(x,t)\mathbf{v}_1(x,t) + (1 - a(x,t))\mathbf{v}_2(x,t)$ e a velocidade média de volume da mistura por $\mathbf{u}(x,t) = b(x,t)\mathbf{v}_1(x,t) + (1 - b(x,t))\mathbf{v}_2(x,t)$.

As equações fundamentais do movimento do fluido, obtidas fazendo uso das leis de conservação de massa e de momento linear (ver, por exemplo, Lukaszewicz, [26], pág. 15 e Guillén [17], pág. 97) na região $\Omega \times [0,T]$ são dadas por:

$$\begin{cases} \rho\mathbf{v}_t + (\rho\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} - (\mu + \mu^{'})\nabla(\nabla\cdot\mathbf{v}) = \rho F - \nabla\tilde{p}, \\ \rho_t + \nabla\cdot(\rho\mathbf{v}) = 0, \\ \operatorname{div}\mathbf{u} = 0, \end{cases}$$

onde μ e $\mu^{'}$ são os coeficientes de viscosidade dados, supostos constantes e satisfazendo $\mu > 0$ e $3\mu^{'} + 2\mu \geq 0$, F é a função de densidade de forças externas e $\tilde{p}$ a pressão do fluido. Admitindo que o processo de difusão de massa obedece a lei de Fick (que relaciona as velocidades locais do fluido com a velocidade de massa e dada pela expressão

$$\mathbf{v} = \mathbf{u} - \frac{\lambda}{\rho}\nabla\rho$$

onde $\lambda > 0$, suposto constante, é o coeficiente de difusão molecular), podemos reescrever o sistema acima inteiramente em função da velocidade de massa ou da velocidade de volume. Uma vez que a condição de incompressibilidade $\operatorname{div}\mathbf{u} = 0$ é de grande importância teórica, reescrevemos as equações em função da velocidade de volume $\mathbf{u}$ obtendo assim, o seguinte sistema:

$$\rho\mathbf{u}_t + (\rho\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} - \lambda(\mathbf{u}\cdot\nabla)\nabla\rho - \lambda(\nabla\rho\cdot\nabla)\mathbf{u}$$

$$-\lambda^2\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho + \lambda^2\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho - \lambda^2\frac{1}{\rho}\Delta\rho\nabla\rho + \nabla p = \rho F, \qquad (1.1)$$

$$\rho_t - \lambda\Delta\rho + \mathbf{u}\cdot\nabla\rho = 0, \tag{1.2}$$

$$\text{div } \mathbf{u} = 0, \tag{1.3}$$

onde $p = \tilde{p} + \lambda\mathbf{u}\cdot\nabla\rho - \lambda^2\Delta\rho + \lambda(2\mu + \mu^{'})\Delta\ln\rho$ é a pressão modificada.

Ao sistema de equações acima agregamos as seguintes condições iniciais e de fronteira:

$$\mathbf{u}|_{\Gamma\times(0,T)} = 0, \quad \mathbf{u}(0) = \mathbf{u}_\circ(x), \tag{1.4}$$

$$\left.\frac{\partial\rho}{\partial\mathbf{n}}\right|_{\Gamma\times(0,T)} = 0, \quad \rho(0) = \rho_\circ(x), \tag{1.5}$$

onde Γ é a fronteira de Ω e $\mathbf{n}$ é a normal unitária a Γ. Notamos que as condições de fronteira nos dizem que as partículas do fluido aderem a Γ e que não há fluxo de massa através de Γ. Neste trabalho, estaremos considerando que a densidade inicial $\rho_\circ$ satisfaz $0 < \alpha \leq \rho_\circ \leq \beta$.

Observamos inicialmente, que tal modelo inclui, com caso particular, as conhecidas equações de Navier-Stokes (caso em que $\lambda = 0$ e a densidade ρ é constante).

Com relação ao estudo de tal sistema, alguns resultados já são conhecidos.

Em [21], Kazhikov, Smagulov impõem que o coeficiente de difusão λ é suficientemente pequeno (de modo que se podem ser desconsiderados os termos em λ^2) e que a densidade inicial é estritamente positiva e limitada; sob tais hipóteses, provam a existência de soluções fracas e unicidade de soluções fortes nos casos $n = 2$ e $n = 3$, sendo que no último caso a solução está definida num intervalo $(0, T_0)$ suficientemente pequeno.

Em seu trabalho [32], Salvi prova (via aproximações semi-Galerkin espectral) a existência de soluções fracas em domínios cilíndricos e não-cilíndricos de $I\!R^n$ (n arbitrário) para o modelo simplificado, ou seja, aquele que não apresenta os termos que envolvem λ^2; são feitas as mesmas imposições de Kazhikhov-Smagulov sobre os coeficientes de viscosidade μ e de difusão λ e sobre a densidade inicial. (Aqui, por aproximações semi-Galerkin espectral, entendemos aproximações finito-dimensional da velocidade e infinito-dimensional da densidade.) Basicamente, os dois trabalhos se distingüem pelo fato de no primeiro serem obtidas estimativas a priori para um tipo de derivada fracionária da velocidade (resultado que não pode ser estendido para $n > 4$) e no segundo, os resultados de compacidade serem obtidos a partir de estimativas a priori para a derivada temporal do termo $\rho\mathbf{u}$ (como no caso das equações de Navier-Stokes). A questão da unicidade e da regularidade não é considerada, uma vez que a unicidade de soluções fracas é um problema em aberto mesmo no caso bidimensional.

Em seu outro artigo [33], Salvi trabalha com o mesmo modelo e as mesmas hipóteses para mostrar a existência de soluções fracas para o problema formulado em termos de um sistema de desigualdades variacionais.

Também trabalhando com sistema de desigualdades diferenciais em [29] e [28], Prouse prova a existência e unicidade de soluções para o modelo de Graffi (aquele em que não aparecem os termos em λ e λ^2) associado ao movimento de uma mistura de dois fluidos viscosos incompressíveis. Interessantes considerações relativas à consistência física para o modelo de Graffi são feitas, sendo que tais considerações permanecem válidas para o modelo de difusão completo.

O caso em que $\Omega = I\!R^3$ e o modelo é o simplificado, é tratado por Secchi, em [36]. Sem qualquer restrição aos coeficientes de viscosidade μ e de difusão λ, o autor prova a existência (local no tempo) e a unicidade de solução, fazendo uso de argumentos de ponto fixo.

Posteriormente, em [35], Secchi considera o modelo completo e prova a existência de uma única solução global no caso bidimensional, admitindo que λ/μ é suficientemente pequeno. Também analisa o comportamento da solução quando $\lambda \to 0$, mostrando que existe uma subseqüência de soluções do problema inicial convergindo para a solução das correspondentes equações de Navier-Stokes não-homogêneas. No caso $n = 2$ a convergência se dá em todo intervalo finito $[0, T]$ e no caso $n = 3$ a convergência ocorre num pequeno intervalo de tempo, independente de λ.

Fazendo linearização das equações de momento e de difusão e utilizando-se de argumentos de pontos fixos e sem nenhuma imposição adicional aos coeficientes de viscosidade e de difusão , da Veiga [5] trabalha com o modelo completo e apresenta resultados de existência e unicidade de solução forte local para dados iniciais arbitrários e global para dados iniciais e força externa pequenos. Além disso, trata-se também do comportamento assintótico da solução, mostrando-se que se a força externa é identicamente nula então a solução decai exponencialmente para a solução de equilíbrio, conforme $t \to +\infty$. Para outras questões relacionadas ao comportamento assintótico e à periodicidade de soluções, podem ser consultados [4] e [6] do mesmo autor.

Passamos agora, a detalhar a metodologia adotada neste trabalho. No Capítulo 2, fazemos uso das aproximações semi-Galerkin espectral para provarmos a existência de solução forte local para o modelo completo, sem nenhuma condição adicional sobre os coeficientes de viscosidade e de difusão.

Este procedimento tem a vantagem (em relação, por exemplo, ao método de pontos fixos) de nos fornecer, então, um método construtivo para a obtenção da solução. A prova da unicidade é conseguida fazendo-se uso da Desigualdade de Gronwall. Convém salientar que, procedendo de maneira análoga ao caso das equações de Navier-Stokes e impondo pequenez dos dados iniciais é possível obter soluções fortes globais no tempo.

No Capítulo 3, investigamos as taxas de convergência nas normas de L^2 e de H^1, das aproximações acima mencionadas para a solução do problema. Mostramos que é possível obter estimativas de erro uniformes e também, estimativas melhoradas na norma de L^2.

No Capítulo 4 deste trabalho, utilizamos um método iterativo para chegar à solução do modelo simplificado. Num primeiro momento, mostramos que as seqüências de aproximações da velocidade e da densidade são uniformemente limitadas em espaços de Banach adequados e, em seguida, que tais seqüências são de Cauchy nestes mesmos espaços; ao mostrar este último fato, automaticamente já obtemos as taxas de convergência das aproximações para a solução do problema. Convém observar que o método iterativo proposto neste capítulo revelou-se inadequado para o tratamento do problema completo pois com tal procedimento, não conseguimos obter estimativas a priori de ordens mais altas para as aproximações da densidade. Acreditamos que adotando um outro tipo de linearização e trabalhando simultaneamente com as equações de momento e de difusão, tal dificuldade pode, provavelmente, ser superada.

1.2 Preliminares

Seja $\Omega \subset I\!\!R^n$ um domínio limitado. No que segue, denotaremos por $C^\circ(\Omega)$ (ou simplesmente, $C(\Omega)$) o espaço das funções contínuas definidas em Ω e

$$C^m(\Omega) = \{u \in C^\circ(\Omega); \partial^k u \in C^\circ(\Omega) \quad \forall |k| \leq m\},$$

onde $k = (k_1, ..., k_n) \in N^n$, $|k| = k_1 + ... + k_n$ e $\partial^k = \dfrac{\partial^{|k|}}{\partial x_1^{k_1} ... \partial x_n^{k_n}}$.

Como as funções de $C^m(\Omega)$ não são necessariamente limitadas (já que Ω é aberto), introduzimos o espaço

$$C^m(\overline{\Omega}) = \{u \in C^m(\Omega); \partial^k u \text{ é limitada e uniformemente contínua em } \Omega, \quad \forall 0 \leq |k| \leq m\}.$$

E também,

$$C^{m,1}(\overline{\Omega}) = \{u \in C^m(\overline{\Omega}); \partial^k u \text{ é Lipschitz-contínua em } \Omega, \forall 0 \leq |k| \leq m\}.$$

Antes de introduzirmos os espaços de Sobolev, convém notar que as propriedades de tais espaços sobre um domínio Ω, exige certa regularidade sobre a fronteira Γ. A seguinte definição, retirada de Girault-Raviart [16], nos dará o sentido exato de tal regularidade.

Definição 1.1 Seja Ω um subconjunto aberto de $I\!\!R^n$. Dizemos que Γ é contínua (resp. Lipschitz-contínua, de classe C^m, de classe $C^{m,1}$, para algum inteiro $m > 0$) se para cada $x \in \Gamma$ existe uma vizinhança $\mathcal{O}$ de x em $I\!\!R^n$ e novas coordenadas ortogonais $y = (y', y_n)$ onde $y' = (y_1, ..., y_{n-1})$, tais que:

i) $\mathcal{O}$ é um hipercubo nas novas coordenadas:

$$\mathcal{O} = \{y; -a_j < y_j < a_j, \quad 1 \leq j \leq n\};$$

ii) existe uma função ϕ contínua (resp. Lipschitz-contínua, C^m, $C^{m,1}$) definida em

$$\mathcal{O}' = \{y'; -a_j < y_j < a_j, \quad 1 \leq j \leq n-1\}$$

satisfazendo

$$|\phi(y')| \leq a_n/2 \quad \forall y' \in \mathcal{O}',$$

$$\Omega \cap \mathcal{O} = \{y; y_n < \phi(y')\}, \quad \Gamma \cap \mathcal{O} = \{y; y_n = \phi(y')\} = \{y; y = (y', \phi(y'))\}.$$

Basicamente, esta definição nos diz que, localmente, Ω está "abaixo" do gráfico de alguma função ϕ, Γ é representada pelo gráfico de ϕ e a regularidade de Γ é determinada pela regularidade de ϕ. Com esta definição, temos que um domínio com fronteira contínua está sempre de um mesmo lado de Γ.

Diremos que o aberto Ω é Lipschitz-contínuo se Γ é Lipschitz-contínua. Vale notar que se Γ é Lipschitz-contínua então Γ admite um vetor normal unitário $\mathbf{n}$ em quase todo ponto $x \in \Gamma$; além disso, se $m \geq 1$ e Γ é de classe C^m então Γ admite vetores normais que pertencem a $(C^{m-1}(\Gamma))^n$.

Consideremos os usuais espaços de Sobolev assim definidos:

$$W^{m,q}(D) = \{f \in L^q(D) \mid |\partial^k f|_{L^q(D)} < +\infty, |k| \leq m\},$$

onde $m = 0, 1, 2, ...$, $1 \leq q \leq +\infty$, $D = \Omega$ ou $D = (0,T) \times \Omega$, $0 < T < \infty$, com a norma usual de $L^q(D)$. Prova-se que $W^{m,q}(D)$ munido da norma

$$|f|_{W^{m,q}(D)} = \Big(\sum_{|k| \leq m} \int_D |\partial^k f(x)|^q dx \Big)^{1/q} \qquad \text{se } q < \infty$$

ou

$$|f|_{W^{m,\infty}(D)} = \max_{|k| \leq m} \Big(\sup_{x \in D} \text{ess} |\partial^k f(x)| \Big), \qquad \text{se } q = \infty$$

é um espaço de Banach. Para estas e outras propriedades dos espaços $W^{m,q}(D)$ indicamos Adams, [1]. No caso especial em que $q = 2$, usaremos a notação $H^m(D)$ para o espaço $W^{m,2}(D)$, $H^m_\circ(D)$ para o fecho de $C^\infty_\circ(D)$ na norma de $H^m(D)$ (onde por $C^\infty_\circ(D)$ entendemos o espaço das funções de classe C^∞ cuja restrição (no sentido de traços) à fronteira ∂D é nula) e $H^{-m}(D)$ para o dual de $H^m_\circ(D)$. Além destes, definimos os seguintes espaços funcionais:

$$H^k_{N}(\Omega) = \{\rho \in H^k(\Omega) \mid \frac{\partial \rho}{\partial \mathbf{n}} = 0 \text{ sobre } \Gamma, \quad \int_\Omega \rho(x) dx = \int_\Omega \rho_\circ(x) dx\}, k \geq 2.$$

e

$$H^k_{N,0}(\Omega) = \{\rho \in H^k(\Omega) \mid \frac{\partial \rho}{\partial \mathbf{n}} = 0 \text{ sobre } \Gamma, \quad \int_\Omega \rho(x) dx = 0\}, k \geq 2.$$

Com relação a tais espaços, devemos observar que $H^k_{N}(\Omega)$ é subespaço afim de $H^k(\Omega)$ e $H^k_{N,0}(\Omega)$ é subespaço fechado de $H^k(\Omega)$; além disso, as normas $|\rho|_{H^2}$ e $|\Delta\rho|$ são equivalentes em $H^2_{N,0}(\Omega)$ e $|\rho|_{H^3(\Omega)}$ e $|\nabla\Delta\rho|$ são equivalentes em $H^3_{N,0}(\Omega)$, conforme Kamenetski, [15].

Vale notar que $H^2_{N}(\Omega) = C_\circ + H^2_{N,0}(\Omega)$, onde $C_\circ = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \rho_\circ(x) dx$.

Apresentamos na seqüência o fundamental teorema de imersões de Sobolev que essencialmente, relaciona diferentes espaços de Sobolev e espaços de funções suaves.

Teorema 1.1 *Seja Ω um subconjunto de $I\!R^n$ com fronteira Γ Lipschitz-contínua e $q^* \in I\!R$ com $1 \leq q^* < \infty$ e m e $j \in N$ com $j \leq m$. As seguintes imersões ocorrem algébrica e topologicamente:*

$$W^{m,q^*}(\Omega) \quad \subset \quad \begin{cases} W^{j,q}(\Omega) & \text{se } 1/q = 1/q^* - (m-j)/n > 0, \\ W^{j,q}_{loc}(\Omega) & \forall q \in [0,\infty) \text{ se } 1/q^* = (m-j)/n, \\ C^j(\Omega) & \text{se } 1/q^* < (m-j)/n. \end{cases}$$

Além disso, se Ω é limitado, a última inclusão ocorre em $C^j(\overline{\Omega})$ e a imersão de $W^{m,q^*}(\Omega)$ em $W^{j,q}(\Omega)$ é compacta para todo real q satisfazendo:

$$\begin{cases} 1 \leq q < nq^*/(n - (m-j)q^*) \text{ se } n > (m-j)q^*, \\ \text{ou} \\ 1 \leq q < \infty \text{ se } n = (m-j)q^*. \end{cases}$$

Merece destaque o caso especial em que Ω é limitado e $n = 3$, $q^* = 2$, $m = 1$ e $j = 0$; neste caso, pelo teorema acima teremos $H^1(\Omega) \subset L^6(\Omega)$ e como $L^6(\Omega) \subset L^q(\Omega)$ se $1 \leq q \leq 6$, então teremos $H^1(\Omega) \subset L^q(\Omega)$, com imersão contínua para $1 \leq q \leq 6$ e com imersão compacta para $1 \leq q < 6$.

Também de uso freqüente neste trabalho são os resultados relacionados à interpolação nos espaços de Sobolev. Iniciamos com um resultado de interpolação nos espaços $L^q(\Omega)$.

Lema 1.2 (*Desigualdade de Interpolação*). *Se $f \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega)$ com $1 \leq q_1 \leq q_2 \leq \infty$, então $f \in L^r(\Omega)$ para todo $q_1 \leq r \leq q_2$ e*

$$|f|_{L^r(\Omega)} \leq |f|^k_{L^{q_1}(\Omega)} |f|^{1-k}_{L^{q_2}(\Omega)}$$

onde $\dfrac{1}{r} = \dfrac{k}{q_1} + \dfrac{1-k}{q_2}$ *com* $0 \leq k \leq 1$.

Observamos que, usando a imersão $H^1(\Omega) \subset L^q(\Omega)$, para $1 \leq q \leq 6$, obteremos algumas situações que serão freqüentemente usadas, dentre elas, podemos destacar:

$$|u|_{L^3(\Omega)} \leq |u|^{1/2}_{L^2(\Omega)} |u|^{1/2}_{L^6(\Omega)} \leq |u|^{1/2}_{L^2(\Omega)} |u|^{1/2}_{H^1(\Omega)};$$

$$|u|_{L^4(\Omega)} \leq |u|^{1/2}_{L^3(\Omega)} |u|^{1/2}_{L^6(\Omega)} \leq |u|^{1/2}_{L^3(\Omega)} |u|^{1/2}_{H^1(\Omega)};$$

$$|u|_{L^4(\Omega)} \leq |u|^{1/4}_{L^2(\Omega)} |u|^{3/4}_{L^6(\Omega)} \leq |u|^{1/4}_{L^2(\Omega)} |u|^{3/4}_{H^1(\Omega)};$$

$$|u|_{L^6(\Omega)} \leq |u|^{1/2}_{L^3(\Omega)} |u|^{1/2}_{L^\infty(\Omega)}.$$

Destacamos agora, um resultado (devido a Gagliardo-Nirenberg, [27]) do qual podemos deduzir outras desigualdades de interpolação que também nos serão úteis. Antes porém, lembremos que r é dito ser a *média harmônica* de q_1 e q_2 se $\frac{1}{r} = \frac{1}{2}(\frac{1}{q_1} + \frac{1}{q_2})$.

Lema 1.3 *Sejam Ω um subconjunto aberto de $I\!R^n$ com Γ Lipschitz-contínua, $u \in W^{2,q_1}(\Omega) \cap L^{q_2}(\Omega)$ com $1 \leq q_1, q_2 \leq \infty$ e r a média harmônica de q_1 e q_2. Então $u \in W^{1,r}(\Omega)$ e*

$$|Du|_{L^r(\Omega)} \leq C |u|^{1/2}_{W^{2,q_1}(\Omega)} |u|^{1/2}_{L^{q_2}(\Omega)}.$$

Como o caso $q_1 = 2$ será o mais usado, convém verificarmos esta desigualdade para alguns valores de r e q_2. Teremos:

$$\begin{array}{lcl} r = 2,\ q_2 = 2 & : & |Du|^2_{L^2(\Omega)} \leq C |u|_{H^2(\Omega)} |u|_{L^2(\Omega)}, \\ r = 3,\ q_2 = 6 & : & |Du|^2_{L^3(\Omega)} \leq C |u|_{H^2(\Omega)} |u|_{L^6(\Omega)}, \\ r = 4,\ q_2 = \infty & : & |Du|^2_{L^4(\Omega)} \leq C |u|_{H^2(\Omega)} |u|_{L^\infty(\Omega)}. \end{array}$$

Sendo B um espaço de Banach, denotaremos por $L^q(0,T;B)$ o espaço de Banach das (classes de) funções definidas no intervalo $(0,T)$ com valores em B, fortemente mensuráveis e que são L^q-integráveis no sentido de Bochner, munido de sua norma natural. Vale notar que se $f \in L^q(0,T;B)$ então a ela está associada uma distribuição ainda denotada por f, de modo que podemos falar em $\frac{\partial f}{\partial t}$ (no sentido de distribuição). O próximo resultado estabelece um critério de continuidade para funções de $L^q(0,T;B)$.

Lema 1.4 *Se* $f \in L^q(0,T;B)$ *e* $\partial f/\partial t \in L^q(0,T;B)$, *para* $1 \leq q \leq \infty$, *então existe* $f^* \in C([0,T];B)$ *tal que* $f = f^*$ *q.t.p. em* $[0,T]$.

Consideremos agora, três espaços de Banach B_0, B e B_1, com $B_0 \subset B \subset B_1$, B_0 e B_1 reflexivos, sendo que a imersão $B_0 \subset B$ é compacta. Definamos

$$\mathcal{W}_{q_0,q_1} = \{f | f \in L^{q_0}(0,T;B_0), \partial f/\partial t \in L^{q_1}(0,T;B_1)\}$$

onde $1 < q_0, q_1 < \infty$. É de fácil verificação que munido da norma

$$|f|_{L^{q_0}(0,T;B_0)} + |\partial f/\partial t|_{L^{q_1}(0,T;B_1)},$$

o espaço $\mathcal{W}_{q_0,q_1}$ é um espaço de Banach. Nessas condições, temos o seguinte resultado (também conhecido como *Lema de Aubin*):

Lema 1.5 *A imersão do espaço* $\mathcal{W}_{q_0,q_1}$ *em* $L^{q_0}(0,T;B)$ *é compacta.*

Estes dois últimos lemas podem ser encontrados em Lions, [25]. É importante salientar que o lema anterior nos diz que os conjuntos limitados de $\mathcal{W}_{q_0,q_1}$ são relativamente compactos em $L^{q_0}(0,T;B_0)$. Sua principal utilidade neste trabalho será, então, a de garantir que seqüências limitadas em espaços $\mathcal{W}_{q_0,q_1}$ possuam subseqüências convergindo fortemente em espaços $L^{q_0}(0,T;B)$.

O resultado seguinte, cuja demonstração pode ser encontrada em J. Simon, [38], generaliza o anterior e estabelece outras imersões compactas. Vale notar que o caso em $q = 1$ já havia sido provado em Temam, [39].

Lema 1.6 *Sejam* $B_0 \subset B \subset B_1$ *espaços de Banach, sendo que a imersão* $B_0 \subset B$ *é compacta. Então, são também compactas as seguintes imersões:*

$$\begin{array}{ll} \text{(i)} & L^q(0,T;B_0) \cap \{\phi : \frac{\partial \phi}{\partial t} \in L^1(0,T;B_1)\} \to L^q(0,T;B) \textit{ se } 1 \leq q \leq \infty, \\ \text{(ii)} & L^\infty(0,T;B_0) \cap \{\phi : \frac{\partial \phi}{\partial t} \in L^r(0,T;B_1)\} \to C(0,T;B) \textit{ se } 1 < r \leq \infty. \end{array}$$

Apresentamos a seguir, uma versão do *Teorema de dualidade de De Rham* para distribuições (ver a referência J. Simom, [37]).

Lema 1.7 *Seja* $h \in \mathcal{D}'(0,T;H^{-1}(\Omega))$ *uma distribuição tal que* $(h,\varphi)_\Omega = 0$ *para toda* $\varphi \in C_0^\infty(\Omega)$, *com* $\operatorname{div} \varphi = 0$. *Então existe* $g \in D'(0,T;L^2(\Omega))$ *tal que* $h = \nabla g$.

Colocamos em seguida, um resultado essencial sobre desigualdades diferenciais que usaremos mais adiante para garantir a existência de um intervalo $[0,T]$ onde deverão estar definidas todas as soluções aproximadas do problema inicial. (Ver, por exemplo, J. Simon, [37].)

Lema 1.8 *Sejam* $g \in W^{1,1}(0,T)$ *e* $h \in L^1(0,T)$ *satisfazendo*

$$\frac{dg}{dt} \leq F(g) + h \quad \text{em } [0,T], \quad g(0) \leq g_0$$

onde $F : I\!R \to I\!R$ *é limitada em conjuntos limitados. Então para todo* $\varepsilon > 0$, *existe* $T_\varepsilon > 0$ *independente de* g *tal que*

$$g(t) \leq g_0 + \varepsilon \quad \forall t \leq T_\varepsilon.$$

Introduziremos agora, espaços funcionais que aparecem em formulações equivalentes àquela feita para o problema inicial. Convém notar que, por uma questão de simplicidade de notação, utilizaremos sempre $W^{m,q}(\Omega)$ para representar o cartesiano $(W^{m,q}(\Omega))^n$, qualquer que seja n. Primeiramente definimos:

$$\begin{aligned}
&\mathcal{V} = \{\mathbf{v} \in (C_\circ^\infty(\Omega))^n |\ \text{div } \mathbf{v} = 0\},\\
&H = \text{fecho de } \mathcal{V} \text{ na norma de } L^2(\Omega),\\
&V = \text{fecho de } \mathcal{V} \text{ na norma de } H^1(\Omega).
\end{aligned}$$

Admitindo-se que Ω é limitado e Γ é Lipschtiz-contínua, é possível mostrar que

$$V = \{\mathbf{v} \in H_\circ^1(\Omega) |\ \text{div } \mathbf{v} = 0\};$$

além disso, colocando $|\mathbf{u}|_V = (\nabla \mathbf{u}, \nabla \mathbf{u})^{1/2}$ e utilizando a *desigualdade de Poincaré* , temos que a semi-norma $|\cdot|_V = (\cdot,\cdot)^{1/2}$ é uma norma em $H_\circ^1(\Omega)$ (e, em particular, em V) equivalente à norma de $H^1(\Omega)$.

Por outro lado, definindo o espaço $G = \{\varphi | \varphi = \nabla p, p \in H^1(\Omega)\}$, temos que os espaços H e G são mutuamente ortogonais com relação ao produto interno usual de $L^2(\Omega)$ e então, a decomposição de Helmholtz nos fornece o fato de $L^2(\Omega) = H \oplus G$. Além disso, é possível caracterizar o subespaço H de $L^2(\Omega)$ colocando

$$H = \{\mathbf{v} \in L^2(\Omega); \text{div } \mathbf{v} = 0, \ \ \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0\}$$

onde $\mathbf{v} \cdot \mathbf{n}|_\Gamma$ representa a *componente normal de* $\mathbf{v}$ *sobre* Γ. Para a demonstração de tais fatos, sugerimos Constantin-Foias, [13].

Neste trabalho, denotaremos por P o operador projeção ortogonal de $L^2(\Omega)$ sobre o subespaço H. Utilizando-nos da projeção P, teremos então, definido o operador de Stokes $A : dom(A) \to H$, dado por $A = -P\Delta$ e cujo domínio $dom(A)$ é o espaço $H^2(\Omega) \cap V$. É possível mostrar (ver, por exemplo, Constantin-Foias, [13]) que A é um operador auto-adjunto definido positivo caracterizado por

$$(A\mathbf{w}, \mathbf{v}) = (\nabla \mathbf{w}, \nabla \mathbf{v}) \ \forall\ \mathbf{w} \in dom(A), \ \forall\ \mathbf{v} \in V,$$

onde o símbolo $(\cdot,\cdot)$ denota o produto interno de $L^2(\Omega)$.

Observamos que para garantir as propriedades de regularidade do operador A, em geral, admite-se que a fronteira Γ do domínio Ω seja de classe C^3 com o intuito de utilizar-se os resultados de

Cattabriga [11]; em vez destes, usaremos resultados mais fortes devidos a Amrouche-Girault [2], os quais asseguram, quando Γ é de classe $C^{1,1}$, que se $A\mathbf{u} \in L^2(\Omega)$ então $\mathbf{u} \in H^2(\Omega)$ e que $|\mathbf{u}|_{H^2(\Omega)}$ e $|A\mathbf{u}|$ são normas equivalentes.

Denotaremos respectivamente por φ_k e por λ_k ($k \in I\!N$) a k-ésima autofunção e o k-ésimo autovalor do operador de Stokes definido sobre $V \cap H^2(\Omega)$. Prova-se que o conjunto de funções $\{\varphi_k\}_{k\in I\!N}$ é um conjunto ortogonal completo nos espaços H, V e $V \cap H^2(\Omega)$ com respeito ao seus produtos internos usuais $(\mathbf{u}, \mathbf{v})$, $(\nabla\mathbf{u}, \nabla\mathbf{v})$ e $(A\mathbf{u}, A\mathbf{v})$, respectivamente.

Denotaremos por V_k o espaço gerado pelas k primeiras autofunções do operador de Stokes A, ou seja, $V_k = [\varphi_1, ..., \varphi_k]$ e por P_k a projeção ortogonal de $L^2(\Omega)$ sobre V_k.

Apresentamos em seguida, um resultado que ser-nos-á de grande utilidade na obtenção das taxas de convergência das aproximações semi-Galerkin espectral e cuja demonstração pode ser encontrada na referência Rautmann [30].

Lema 1.9 *Se* $\mathbf{v} \in V$ *então*

$$|\mathbf{v} - P_k\mathbf{v}|^2 \leq \frac{1}{\lambda_{k+1}}|\nabla\mathbf{v}|^2.$$

Se $\mathbf{v} \in V \cap H^2(\Omega)$ *então*

$$|\nabla\mathbf{v} - \nabla P_k\mathbf{v}|^2 \leq \frac{1}{\lambda_{k+1}}|A\mathbf{v}|^2 \text{ e } |\mathbf{v} - P_k\mathbf{v}|^2 \leq \frac{1}{\lambda_{k+1}^2}|A\mathbf{v}|^2.$$

Destinamos esta última parte do capítulo à apresentação de algumas desigualdades clássicas que serão de uso freqüente.

Lema 1.10 (*Desigualdade de Gronwall*) *Sejam* $\varphi \in L^\infty(0,T)$, $\varphi(t) \geq 0$ *q.t.p. em* [0,T] *e* $b \in L^1(0,T)$, $b(t) \geq 0$ *q.t.p. em* [0,T] *tais que*

$$\varphi(t) \leq C + \int_0^t b(s)\varphi(s)ds,$$

para quase todo $t \in [0,T]$. *Então*

$$\varphi(t) \leq C \exp \int_0^t b(s)ds,$$

para quase todo $t \in [0,T]$.

Notamos que se a constante $C = 0$ então deverá ser $\varphi(t) = 0$, para quase todo $t \in [0,T]$. (Para a demonstração deste resultado, consultar Lions, [25].) Temos também a seguinte versão para a desigualdade acima (conforme Rautmann, [30]):

Lema 1.11 (*Desigualdade de Gronwall Generalizada*). *Sejam* φ *e* ψ *funções contínuas não-negativas em* $[0,T]$, $a(t)$ *uma função absolutamente contínua com* $a'(t) \geq 0$ *e* $b(t) \geq 0$ *integrável em* $[0,T]$, *tais que*

$$\varphi(t) + \int_0^t \psi(s)ds \leq a(t) + \int_0^t b(s)\varphi(s)ds, \qquad \forall\, 0 \leq t \leq T.$$

Então

$$\varphi(t) + \int_0^t \psi(s)ds \leq a(t) \cdot (1 + \int_0^t b(s)ds) \cdot e^{\int_0^t b(s)ds},$$

para todo $t \in [0, T]$.

Também de uso freqüente são as desigualdades fornecidas pelo teorema que se segue, cuja demonstração pode ser encontrada em [9].

Sejam q e $q^{'}$ *expoentes conjugados,* ou seja, $\frac{1}{q} + \frac{1}{q^{'}} = 1$, para $1 < q < \infty$, $q^{'} = \infty$ se $q = 1$ e $q^{'} = 1$ se $q = \infty$.

Teorema 1.12

i) (*Desigualdade de Young*). *Se q e $q^{'}$ são expoentes conjugados, com* $1 < q < \infty$ *e a e b são números reais positivos então*

$$ab \leq \frac{1}{q}a^q + \frac{1}{q^{'}}b^{q^{'}}.$$

(Observamos que é possível obter uma forma mais geral para a desigualdade de Young. Para isto, podemos escrever

$$ab = ((q\varepsilon)^{1/q}a)(\frac{1}{(q\varepsilon)^{1/q}}b)$$

e então, usando a desigualdade acima e lembrando que $q > 1$ teremos, após algumas majorações convenientes:

$$ab \leq \varepsilon a^q + C_\varepsilon b^{q^{'}},$$

com $C_\varepsilon = \varepsilon^{-\frac{1}{q-1}}$.)

ii) (*Desigualdade de Hölder*). *Se* q_1 *e* q_2 *são expoentes conjugados com* $1 \leq q_1 \leq \infty$, $f_1 \in L^{q_1}(\Omega)$ *e* $f_2 \in L^{q_2}(\Omega)$, *então* $f_1 \cdot f_2 \in L^1(\Omega)$ *e*

$$\int_\Omega |f_1 \cdot f_2| \leq |f_1|_{L^{q_1}(\Omega)} |f_2|_{L^{q_2}(\Omega)}.$$

(Usaremos também, com bastante freqüência, a desigualdade de Hölder generalizada: *Se* $f_1, f_2, ... f_m$ *são funções tais que* $f_i \in L^{q_i}(\Omega)$, *para* $1 \leq i \leq m$, *com* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + ... + \frac{1}{q_m} \leq 1$, *então o produto* $f = f_1 f_2 ... f_m$ *pertence a* $L^q(\Omega)$ *e*

$$|f|_{L^q(\Omega)} \leq |f_1|_{L^{q_1}(\Omega)} |f_2|_{L^{q_2}(\Omega)} \cdots |f_m|_{L^{q_m}(\Omega)}.)$$

Finalmente, gostaríamos de observar que, com relação à notação, por simplicidade usaremos sempre o símbolo $|\cdot|_X$ para significar a norma em qualquer espaço funcional X. Nos casos em que $X = L^q(\Omega)$ e $X = H^m(\Omega)$ colocaremos simplesmente $|\cdot|_q$ e $|\cdot|_{H^m}$ para designar suas normas; também, sempre omitiremos o índice quando estiver claro no contexto à norma de que espaço X nos referimos.

Capítulo 2

Soluções Fortes para o Problema de Transporte com Difusão

2.1 Introdução

Este capítulo é inteiramente destinado à investigação da existência de soluções fortes locais para o modelo completo descrito no capítulo anterior; também são analisadas questões relacionadas à unicidade bem como à regularidade de eventuais soluções. Daremos a seguir, uma idéia do procedimento a ser seguido para chegar a tais resultados.

Inicialmente, consideraremos uma formulação equivalente àquela do problema inicial. Em seguida, definiremos o problema aproximado de ordem k e trabalhando com a seqüência de soluções aproximadas, procuraremos estabelecer estimativas a priori que sejam independentes do nível de aproximação.

Em posse de tais limitações uniformes, faremos uso de resultados de compacidade (notadamente do *Lema de Aubin*) para garantir a existência de uma subseqüência de aproximações convergindo fortemente em espaços convenientes. Veremos então, que o elemento limite é solução do problema aqui proposto (equivalente ao inicial). Vale notar que as estimativas obtidas para as soluções aproximadas permanecem válidas para o elemento limite.

Utilizando argumentos de dualidade, recuperamos a pressão, garantindo assim, a existência de solução do problema original.

O próximo passo será mostrar então, que a solução assume o dado inicial continuamente. A parte final do capítulo é destinada a mostrar a unicidade da velocidade solução e da densidade solução; a unicidade da pressão solução é garantida, a menos de constantes aditivas.

2.2 Existência, unicidade e regularidade de soluções

Passemos assim, à formulação do "novo" problema. Usando o operador projeção P, vemos que o problema (1.1)-(1.5) pode ser reescrito na seguinte forma:

$$\left.\begin{array}{l} P(\rho\mathbf{u}_t + (\rho\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda[(\mathbf{u}\cdot\nabla)\nabla\rho + ((\nabla\rho\cdot\nabla)\mathbf{u}] - \rho F) + \mu A\mathbf{u} \\ +\lambda^2 P(\dfrac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho - \dfrac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \dfrac{1}{\rho}\Delta\rho\nabla\rho) = 0 \\ \dfrac{\partial\rho}{\partial t} + \mathbf{u}.\nabla\rho - \lambda\Delta\rho = 0 \\ \mathbf{u}(0) = \mathbf{u}_0,\ \rho(0) = \rho_0,\ \mathbf{u}\Big|_{\Gamma\times(0,T)} = 0 \text{ e } \dfrac{\partial\rho}{\partial\mathbf{n}}\Big|_{\Gamma\times(0,T)} = 0. \end{array}\right\} \tag{2.6}$$

De forma inteiramente análoga àquela feita para o caso das equações de Navier-Stokes (ver Girault-Raviart [16]), mostra-se que a formulação acima é equivalente à seguinte na forma fraca:

$$\left.\begin{array}{l}
(\rho\mathbf{u}_t, v) + ((\rho\mathbf{u}\cdot\nabla)\mathbf{u}, v) + \mu(A\mathbf{u}, v) - \lambda((\mathbf{u}\cdot\nabla)\nabla\rho, v) \\
-\lambda((\nabla\rho\cdot\nabla)\mathbf{u}, v) + \lambda^2(\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho, v) - \lambda^2(\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho, v) \\
+\lambda^2(\frac{1}{\rho}\Delta\rho\nabla\rho, v) = (\rho F, v) \ \ \forall\, v \in V \\
\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho - \lambda\Delta\rho = 0 \qquad \text{para} \ \ 0 < t < T \\
\mathbf{u}(0) = \mathbf{u}_0, \ \rho(0) = \rho_0, \ \mathbf{u}\Big|_{\Gamma\times(0,T)} = 0 \text{ e } \frac{\partial\rho}{\partial\mathbf{n}}\Big|_{\Gamma\times(0,T)} = 0.
\end{array}\right\} \tag{2.7}$$

Definimos para cada $k \in I\!N$ as aproximações semi-Galerkin espectral de $(\mathbf{u}, \rho)$ como sendo a solução $(\mathbf{u}^k, \rho^k) \in C^1([0, T^k]; H^2(\Omega) \cap V) \times C^2(\overline{\Omega} \times [0, T^k])$ do problema:

$$\left.\begin{array}{l}
(\rho^k\mathbf{u}^k_t, v) + ((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k, v) + \mu(A\mathbf{u}^k, v) - \lambda((\mathbf{u}^k\cdot\nabla)\nabla\rho^k, v) \\
-\lambda((\nabla\rho^k\cdot\nabla)\mathbf{u}^k, v) + \lambda^2(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k, v) \\
-\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k, v) \\
+\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k, v) = (\rho^k F, v) \ \ \forall\, v \in V_k \\
\frac{\partial\rho^k}{\partial t} + \mathbf{u}^k\cdot\nabla\rho^k - \lambda\Delta\rho^k = 0 \qquad \forall \ (x,t) \in \Omega\times(0, T^k) \\
\frac{\partial\rho^k}{\partial\mathbf{n}} = 0, \quad \forall x \in \Gamma \\
\mathbf{u}^k(x, 0) = P_k\mathbf{u}_0, \ \rho^k(0, x) = \rho_0(x), \quad \forall \ x \in \Omega.
\end{array}\right\} \tag{2.8}$$

Observamos que a expressão *"aproximações semi-Galerkin espectral"* se deve ao fato de estarmos fazendo aproximações finito-dimensional para a velocidade $\mathbf{u}$ e infinito-dimensional para a densidade ρ. Além disso, referir-nos-emos a (2.8) como sendo o problema aproximado.

Vale lembrar que para todo $k \in I\!N$ o sistema (de EDO's) acima admite uma única solução $(\mathbf{u}^k, \rho^k)$ definida em $[0, T^k]$, com $0 < T^k \leq T$ (conforme *teorema de Carathéodory*; ver, por exemplo, Coddington-Levinson, [12]). No entanto, as estimativas que obteremos (e que serão independentes do nível de aproximação k) nos permitirão mostrar que existe T^*, com $0 < T^* \leq T^k$ para todo $k \geq 1$, tal que todas as soluções u^k estarão definidas no intervalo $[0, T^*]$.

Nosso principal objetivo nesta seção, é mostrar que as aproximações $(\mathbf{u}^k, \rho^k)$ convergem num sentido apropriado para a solução $(\mathbf{u}, \rho)$ do problema (2.7), conforme $k \to \infty$. Temos então, o seguinte:

Teorema 2.1. *Sejam $\mathbf{u}_0 \in D(A)$, $\rho_0 \in H^3_N$, com $0 < \alpha \leq \rho_0 \leq \beta$, $F \in L^2(0, T; H^1(\Omega))$ e $F_t \in L^2(0, T; L^2(\Omega))$. Então para todo $T > 0$, existem $T^* \in]0, T]$, e um par de funções $(\mathbf{u}, \rho)$ com*

$\rho \in L^\infty(0,T^*;H^3_N) \cap L^2(0,T^*;H^4_N) \cap C([0,T^*];H^2_N)$ e $\mathbf{u} \in L^\infty(0,T^*;D(A)) \cap L^2(0,T^*;W^{1,\infty}(\Omega)) \cap C([0,T^*];V)$ tais que $(\mathbf{u},\rho)$ é a única solução do problema (2.6) em $[0,T^*]\times\Omega$.

Além disso, as aproximações $\mathbf{u}^k, \rho^k$ satisfazem as seguintes estimativas:

$$
\left.\begin{array}{l}
\alpha \le \rho^k \le \beta; \qquad |\nabla \mathbf{u}^k(t)| \le F_1(t); \qquad |A\mathbf{u}^k(t)| \le F_2(t); \\[2ex]
|\mathbf{u}^k_t(t)|^2 + \displaystyle\int_0^t |\nabla \mathbf{u}^k_t(s)|^2 ds \le F_3(t); \\[2ex]
\displaystyle\int_0^t |\nabla \mathbf{u}^k(s)|^2_\infty ds \le F_4(t); \qquad \int_0^t |\nabla \rho^k(t)|^2_\infty \le F_5(t); \\[2ex]
|\nabla \Delta \rho^k(t)|^2 + \displaystyle\int_0^t |\rho^k(s)|^2_{H^4_N} ds \le F_6(t), \\[2ex]
|\nabla \rho^k_t(t)|^2 + \displaystyle\int_0^t |\rho^k_t(s)|^2_{H^2_N} ds \le F_7(t), \\[2ex]
\sigma(t)|\nabla \mathbf{u}^k_t(t)|^2 + \displaystyle\int_0^t \sigma(s)|\nabla \mathbf{u}^k_{tt}(s)|^2 ds \le F_8(t),
\end{array}\right\} \tag{2.9}
$$

onde $\sigma(t) = min\{t,1\}$. Estimativas análogas são verificadas pela solução $(\mathbf{u},\rho)$.

As funções do lado direito das estimativas acima **dependem** *do argumento t, de T^*, de Γ e dos dados iniciais do problema. No intervalo em questão tais funções são contínuas na variável t.*

Demonstração. Será feita em quatro estágios:

Parte 1: Estimativas a priori.

Consideremos a segunda equação de (2.8); pelo princípio do máximo para a solução de equações parabólicas, temos que para todo $k \in I\!N$, vale $\alpha \le \rho^k \le \beta$. Em seguida, tomando o produto interno com ρ^k e integrando sobre Ω temos:

$$\frac{d}{dt}|\rho^k|^2 + 2\lambda|\nabla\rho^k|^2 = 0,$$

uma vez que $(\mathbf{u}^k \cdot \nabla\rho^k, \rho^k) = -(\text{div } \mathbf{u}^k, \frac{1}{2}|\rho^k|^2) = 0$, $\mathbf{u}^k\Big|_\Gamma = 0$ e $\dfrac{\partial \rho^k}{\partial \mathbf{n}}\Big|_\Gamma = 0$.

Agora, integrando entre 0 e t obtemos:

$$|\rho^k(t)|^2 + 2\lambda \int_0^t |\nabla\rho^k|^2 ds \le |\rho^k_0|^2.$$

Concluímos então que

$$\rho^{\mathbf{k}} \in \mathbf{L}^\infty(\mathbf{0},\mathbf{T}^*;\mathbf{L}^2(\boldsymbol{\Omega})) \cap \mathbf{L}^2(\mathbf{0},\mathbf{T}^*;\mathbf{H}^1).$$

Fazendo $v = \mathbf{u}^k$ na primeira equação de (2.8) teremos:

$$\frac{1}{2}\frac{d}{dt}|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + \mu|\nabla\mathbf{u}^k|^2 = \frac{1}{2}(\rho_t^k\mathbf{u}^k, \mathbf{u}^k) - (\rho^k\mathbf{u}^k\cdot\nabla\mathbf{u}^k, \mathbf{u}^k)$$

$$+\lambda((\nabla\rho^k\cdot\nabla)\mathbf{u}^k, \mathbf{u}^k) + \lambda((\mathbf{u}^k\cdot\nabla)\nabla\rho^k, \mathbf{u}^k) - \lambda^2(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k, \mathbf{u}^k)$$

$$+\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k, \mathbf{u}^k) - \lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k, \mathbf{u}^k) + (\rho^k F, \mathbf{u}^k).$$

O próximo passo é estimar os termos do lado direito da igualdade acima. Observamos que, assim como é usualmente feito para se obter estimativas a priori, procederemos utilizando-nos das desigualdades de Hölder, de Young, de interpolação, das imersões clássicas de Sobolev, do lema de Gronwall, etc. Assim sendo, teremos:

$$\begin{aligned}
|\frac{1}{2}(\rho_t^k\mathbf{u}^k, \mathbf{u}^k)| &\leq \frac{\lambda}{2}|(\Delta\rho^k\mathbf{u}^k, \mathbf{u}^k)| + \frac{1}{2}|((\mathbf{u}^k\cdot\nabla\rho^k)\mathbf{u}^k, \mathbf{u}^k)| \\
&\leq \frac{\lambda}{2}|\Delta\rho^k||\mathbf{u}^k|_4^2 + \frac{1}{2}|\mathbf{u}^k|_4^2|\nabla\rho^k|_6|\mathbf{u}^k|_3 \\
&\leq C|\Delta\rho^k||\nabla\mathbf{u}^k|^2 + C|\nabla\mathbf{u}^k|^3|\Delta\rho^k| \\
&\leq C|\Delta\rho^k|^2 + C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6;
\end{aligned}$$

$$\begin{aligned}
|(\rho^k\mathbf{u}^k\cdot\nabla\mathbf{u}^k, \mathbf{u}^k)| &\leq |\rho^k|_\infty|\mathbf{u}^k|_6|\nabla\mathbf{u}^k|_3|\mathbf{u}^k| \\
&\leq C|\nabla\mathbf{u}^k||A\mathbf{u}^k|^{\frac{1}{2}}|\nabla\mathbf{u}^k|^{\frac{1}{2}}|\rho^{k\,\frac{1}{2}}\mathbf{u}^k| \\
&\leq C|\nabla\mathbf{u}^k|^3|A\mathbf{u}^k| + C|\rho^{k\,\frac{1}{2}}\mathbf{u}^k|^2 \\
&\leq C|\nabla\mathbf{u}^k|^6 + \delta|A\mathbf{u}^k|^2 + C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2;
\end{aligned}$$

$$\begin{aligned}
|\lambda((\nabla\rho^k\cdot\nabla)\mathbf{u}^k, \mathbf{u}^k)| &\leq \lambda|\nabla\rho^k|_6|\nabla\mathbf{u}^k|_3|\mathbf{u}^k| \\
&\leq C|\Delta\rho^k||A\mathbf{u}^k|^{\frac{1}{2}}|\nabla\mathbf{u}^k|^{\frac{1}{2}}|\mathbf{u}^k| \\
&\leq C|\Delta\rho^k|^2|A\mathbf{u}^k||\nabla\mathbf{u}^k| + C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2 \\
&\leq C|\Delta\rho^k|^4|\nabla\mathbf{u}^k|^2 + \delta|A\mathbf{u}^k|^2 + C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2 \\
&\leq C|\Delta\rho^k|^8 + C|\nabla\mathbf{u}^k|^4 + \delta|A\mathbf{u}^k|^2 + C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2;
\end{aligned}$$

$$\begin{aligned}
|\lambda((\mathbf{u}^k\cdot\nabla)\nabla\rho^k, \mathbf{u}^k)| &= \lambda|-((\mathbf{u}^k\cdot\nabla)\mathbf{u}^k, \nabla\rho^k)| \\
&\leq C|\mathbf{u}^k||\nabla\mathbf{u}^k|_3|\nabla\rho^k|_6 \\
&\leq C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + \delta|A\mathbf{u}^k|^2 + C|\nabla\mathbf{u}^k|^4 + C|\Delta\rho^k|^8;
\end{aligned}$$

$$|(\rho^k F, \mathbf{u}^k)| \leq \beta|F||\mathbf{u}^k| \leq C|F|^2 + C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2.$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\mathbf{u}^k)| &\leq C|\nabla\rho^k|_4|\nabla^2\rho^k||\mathbf{u}^k|_4 \\
&\leq C|\Delta\rho^k|^2|\nabla\mathbf{u}^k| \\
&\leq C|\Delta\rho^k|^4+\frac{\mu}{4}|\nabla\mathbf{u}^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}^k)| &\leq C|\nabla\rho^k|_6^3|\mathbf{u}^k| \\
&\leq C|\Delta\rho^k|^3|\mathbf{u}^k| \\
&\leq C|\Delta\rho^k|^6+C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k,\mathbf{u}^k)| &\leq C|\Delta\rho^k||\nabla\rho^k|_4|\mathbf{u}^k|_4 \\
&\leq C|\Delta\rho^k|^2|\nabla\mathbf{u}^k| \\
&\leq C|\Delta\rho^k|^4+\frac{\mu}{4}|\nabla\mathbf{u}^k|^2.
\end{aligned}
$$

Como conseqüência das estimativas acima temos:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2+\frac{\mu}{2}|\nabla\mathbf{u}^k|^2\leq C|\nabla\mathbf{u}^k|^4+C|\nabla\mathbf{u}^k|^6+3\delta|A\mathbf{u}^k|^2 \\
&+C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2+C|\Delta\rho^k|^2+C|\Delta\rho^k|^4+C|\Delta\rho^k|^6+C|\Delta\rho^k|^8+C|F|^2.
\end{aligned}
\tag{2.10}
$$

Agora, fazendo $v=\mathbf{u}_t^k$ na primeira equação de (2.8) obtemos, após uma integração por partes em Ω:

$$
\begin{aligned}
&\tfrac{\mu}{2}\tfrac{d}{dt}|\nabla\mathbf{u}^k|^2+|(\rho^k)^{1/2}\mathbf{u}_t^k|^2\leq|(\rho^kF,\mathbf{u}_t^k)|+|((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)| \\
&+\lambda|((\nabla\rho^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)|+\lambda|((\mathbf{u}^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)|+\lambda^2|(\tfrac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)| \\
&+\lambda^2|(\tfrac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}_t^k)|+\lambda^2|(\tfrac{1}{\rho^k}\Delta\rho^k\nabla\rho^k,\mathbf{u}_t^k)|.
\end{aligned}
$$

Os termos do lado direito da expressão acima são estimados como segue:

$$
\begin{aligned}
|(\rho^kF,\mathbf{u}_t^k)| &\leq |\rho^k|_\infty|F||\mathbf{u}_t^k| \\
&\leq C|F|^2+\varepsilon|\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)| &\leq |\rho^k|_\infty|\mathbf{u}^k|_6|\nabla\mathbf{u}^k|_3|\mathbf{u}_t^k| \\
&\leq C_\varepsilon|\nabla\mathbf{u}^k|^2|\nabla\mathbf{u}^k|_3^2+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C_{\varepsilon,\delta}|\nabla\mathbf{u}^k|^6+\delta|A\mathbf{u}^k|^2+\varepsilon|\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\nabla\rho^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)| &\leq \lambda|\nabla\rho^k|_6|\nabla\mathbf{u}^k|_3|\mathbf{u}_t^k| \\
&\leq C_\varepsilon|\Delta\rho^k|^2|\nabla\mathbf{u}^k|_3^2+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C_\varepsilon|\Delta\rho^k|^2|\nabla\mathbf{u}^k||A\mathbf{u}^k|+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C_{\varepsilon,\delta}|\Delta\rho^k|^4|\nabla\mathbf{u}^k|^2+\delta|A\mathbf{u}^k|^2+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C|\Delta\rho^k|^8+C|\nabla\mathbf{u}^k|^4+\delta|A\mathbf{u}^k|^2+\varepsilon|\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$\begin{aligned}
|\lambda((\mathbf{u}^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)| &\leq \lambda|\mathbf{u}^k|_6|\Delta\rho^k|_3|\mathbf{u}_t^k| \\
&\leq C_\varepsilon|\nabla\mathbf{u}^k|^2|\Delta\rho^k|_3^2+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C_\varepsilon|\nabla\mathbf{u}^k|^2|\rho^k|_\infty^{\frac{2}{3}}|\nabla\Delta\rho^k|^{\frac{4}{3}}+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C|\nabla\mathbf{u}^k|^6+\gamma|\nabla\Delta\rho^k|^2+\varepsilon|\mathbf{u}_t^k|^2;
\end{aligned}$$

$$\begin{aligned}
|\lambda^2(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho^k|_6|\nabla^2\rho^k|_3|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho^k||\rho^k|_\infty^{\frac{1}{3}}|\nabla\Delta\rho^k|^{\frac{2}{3}}|\mathbf{u}_t^k| \\
&\leq C_\varepsilon|\Delta\rho^k|^2|\nabla\Delta\rho^k|^{\frac{4}{3}}+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C_{\varepsilon,\gamma}|\Delta\rho^k|^6+\gamma|\nabla\Delta\rho^k|^2+\varepsilon|\mathbf{u}_t^k|^2;
\end{aligned}$$

$$\begin{aligned}
|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho^k|_6^3|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho^k|^3|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho^k|^6+\varepsilon|\mathbf{u}_t^k|^2;
\end{aligned}$$

$$\begin{aligned}
|\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\Delta\rho^k|_3|\nabla\rho^k|_6|\mathbf{u}_t^k| \\
&\leq C|\rho^k|_\infty^{\frac{1}{3}}|\nabla\Delta\rho^k|^{\frac{2}{3}}|\Delta\rho^k||\mathbf{u}_t^k| \\
&\leq C_\varepsilon|\Delta\rho^k|^2|\nabla\Delta\rho^k|^{\frac{4}{3}}+\varepsilon|\mathbf{u}_t^k|^2 \\
&\leq C_{\varepsilon,\gamma}|\Delta\rho^k|^6+\gamma|\nabla\Delta\rho^k|^2+\varepsilon|\mathbf{u}_t^k|^2.
\end{aligned}$$

Usando as estimativas acima com $\varepsilon=\dfrac{\alpha}{14}$ e lembrando que $0<\alpha\leq\rho^k(x,t)\leq\beta$, temos:

$$\begin{aligned}
\frac{\mu}{2}\frac{d}{dt}|\nabla\mathbf{u}^k|^2+\frac{\alpha}{2}|\mathbf{u}_t^k|^2 &\leq C|F|^2+C|\nabla\mathbf{u}^k|^4+C|\nabla\mathbf{u}^k|^6+C|\Delta\rho^k|^6 \\
&+ 2\delta|A\mathbf{u}^k|^2+C|\Delta\rho^k|^8+3\gamma|\nabla\Delta\rho^k|^2.
\end{aligned}\tag{2.11}$$

Consideremos agora a segunda equação de (2.8); aplicando o operador Δ na mesma e tomando o produto interno de $L^2(\Omega)$ com $\Delta\rho^k$ obtemos:

$$\frac{1}{2}\frac{d}{dt}|\Delta\rho^k|^2+(\nabla(\lambda\Delta\rho^k-\mathbf{u}^k\cdot\nabla\rho^k),\nabla\Delta\rho^k)=0,$$

uma vez que $\dfrac{\partial}{\partial\mathbf{n}}(\lambda\Delta\rho^k-\mathbf{u}^k\cdot\nabla\rho^k)=\dfrac{\partial}{\partial\mathbf{n}}\Big(\dfrac{\partial\rho^k}{\partial t}\Big)=\dfrac{\partial}{\partial t}\Big(\dfrac{\partial\rho^k}{\partial\mathbf{n}}\Big)=0$ sobre Γ; ou seja,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\Delta\rho^k|^2+\lambda|\nabla\Delta\rho^k|^2 &\leq |(\nabla(\mathbf{u}^k\cdot\nabla\rho^k),\nabla\Delta\rho^k)| \\
&\leq |\nabla(\mathbf{u}^k\cdot\nabla\rho^k)||\nabla\Delta\rho^k| \\
&\leq C_\gamma|\nabla(\mathbf{u}^k\cdot\nabla\rho^k)|^2+\gamma|\nabla\Delta\rho^k|^2,
\end{aligned}$$

onde γ é uma constante positiva arbitrária, a ser escolhida posteriormente.

Mas,

$$\begin{aligned}
|\nabla(\mathbf{u}^k\cdot\nabla\rho^k)|^2 &\leq C|\nabla\mathbf{u}^k\cdot\nabla\rho^k|^2+C|\mathbf{u}^k\cdot\nabla(\nabla\rho^k)|^2\\
&\leq C|\nabla\mathbf{u}^k|_3^2|\nabla\rho^k|_6^2+C|\mathbf{u}^k|_6^2|\nabla(\nabla\rho^k)|_3^2\\
&\leq C|\nabla\mathbf{u}^k||A\mathbf{u}^k||\Delta\rho^k|^2+C|\nabla\mathbf{u}^k|^2|\rho^k|_\infty^{\frac{2}{3}}|\nabla\Delta\rho^k|^{\frac{4}{3}}\\
&\leq C|\nabla\mathbf{u}^k|^2|\Delta\rho^k|^4+\delta|A\mathbf{u}^k|^2+C|\nabla\mathbf{u}^k|^6+\gamma|\nabla\Delta\rho^k|^2\\
&\leq C|\nabla\mathbf{u}^k|^4+C|\Delta\rho^k|^8+C|\nabla\mathbf{u}^k|^6+\delta|A\mathbf{u}^k|^2+\gamma|\nabla\Delta\rho^k|^2.
\end{aligned}$$

A última desigualdade torna-se então na seguinte:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\Delta\rho^k|^2+(\lambda-2\gamma)|\nabla\Delta\rho^k|^2 &\leq \delta|A\mathbf{u}^k|^2+C|\nabla\mathbf{u}^k|^4\\
&+ C|\nabla\mathbf{u}^k|^6+C|\Delta\rho^k|^8.
\end{aligned}\tag{2.12}$$

Somando-se as desigualdades (2.10), (2.11) e (2.12) resulta que (depois de escolhermos $\gamma=\dfrac{\lambda}{10}$):

$$\begin{gathered}
\frac{1}{2}\frac{d}{dt}\Big\{|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2+\mu|\nabla\mathbf{u}^k|^2+|\Delta\rho^k|^2\Big\}+\frac{\mu}{2}|\nabla\mathbf{u}^k|^2+\frac{\alpha}{2}|\mathbf{u}_t^k|^2\\
+\frac{\lambda}{2}|\nabla\Delta\rho^k|^2\leq 6\delta|A\mathbf{u}^k|^2+C|\Delta\rho^k|^2+C|\Delta\rho^k|^4+C|\Delta\rho^k|^6+C|\Delta\rho^k|^8\\
+C|\nabla\mathbf{u}^k|^4+C|\nabla\mathbf{u}^k|^6+C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2+C|F|^2.
\end{gathered}\tag{2.13}$$

Além disso, fazendo $v=A\mathbf{u}^k$ na primeira equação de (2.8) teremos:

$$\begin{gathered}
\mu|A\mathbf{u}^k|^2\leq|(\rho^kF,A\mathbf{u}^k)|+|(\rho^k\mathbf{u}_t^k,A\mathbf{u}^k)|+|((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,A\mathbf{u}^k)|\\
+\lambda|((\nabla\rho^k\cdot\nabla)\mathbf{u}^k,A\mathbf{u}^k)|+\lambda|((\mathbf{u}^k\cdot\nabla)\nabla\rho^k,A\mathbf{u}^k)|\\
+\lambda^2|(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,A\mathbf{u}^k)|+\lambda^2|(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,A\mathbf{u}^k)|\\
+\lambda^2|(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k,A\mathbf{u}^k)|.
\end{gathered}$$

Estimamos os termos à direita na expressão acima como segue:

$$|(\rho^k\mathbf{u}_t^k,A\mathbf{u}^k)|\leq\beta|\mathbf{u_t}^k||A\mathbf{u}^k|\leq\frac{\beta^2}{2\mu}|\mathbf{u}_t^k|^2+\frac{\mu}{2}|A\mathbf{u}^k|^2;$$

$$\begin{aligned}
|((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,A\mathbf{u}^k)| &\leq \beta|\mathbf{u}^k|_6|\nabla\mathbf{u}^k|_3|A\mathbf{u}^k|\\
&\leq C|\nabla\mathbf{u}^k||A\mathbf{u}^k|^{\frac{1}{2}}|\nabla\mathbf{u}^k|^{\frac{1}{2}}|A\mathbf{u}^k|\\
&= C|\nabla\mathbf{u}^k|^{\frac{3}{2}}|A\mathbf{u}^k|^{\frac{3}{2}}\leq\theta|A\mathbf{u}^k|^2+C|\nabla\mathbf{u}^k|^6;
\end{aligned}$$

$$\begin{aligned}
\lambda|((\mathbf{u}^k\cdot\nabla)\nabla\rho^k,A\mathbf{u}^k)| &\leq \lambda|\mathbf{u}^k|_6|\nabla^2\rho^k|_3|A\mathbf{u}^k|\\
&\leq C|\nabla\mathbf{u}^k||\rho^k|_\infty^{\frac{1}{3}}|\nabla\Delta\rho^k|^{\frac{2}{3}}|A\mathbf{u}^k|\\
&\leq \theta|A\mathbf{u}^k|^2+C|\nabla\mathbf{u}^k|^2|\nabla\Delta\rho^k|^{\frac{4}{3}}\\
&\leq \theta|A\mathbf{u}^k|^2+C|\nabla\mathbf{u}^k|^6+\eta|\nabla\Delta\rho^k|^2;
\end{aligned}$$

$$
\begin{aligned}
\lambda|((\nabla\rho^k\cdot\nabla)\mathbf{u}^k, A\mathbf{u}^k)| &\leq \lambda|\nabla\rho^k|_6|\nabla\mathbf{u}^k|_3|A\mathbf{u}^k|\\
&\leq C|\Delta\rho^k||\nabla\mathbf{u}^k|^{\frac{1}{2}}|A\mathbf{u}^k|^{\frac{3}{2}}\\
&\leq \theta|A\mathbf{u}^k|^2 + C|\Delta\rho^k|^4|\nabla\mathbf{u}^k|^2\\
&\leq \theta|A\mathbf{u}^k|^2 + C|\Delta\rho^k|^8 + C|\nabla\mathbf{u}^k|^4;
\end{aligned}
$$

$$
|(\rho^k F, A\mathbf{u}^k)| \leq \beta|F||A\mathbf{u}^k| \leq C|F|^2 + \theta|A\mathbf{u}^k|^2.
$$

$$
\begin{aligned}
\lambda^2|(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k, A\mathbf{u}^k)| &\leq C|\nabla\rho^k|_6|\nabla^2\rho^k|_3|A\mathbf{u}^k|\\
&\leq C|\Delta\rho^k||\rho^k|_\infty^{\frac{1}{3}}|\nabla\Delta\rho^k|^{\frac{2}{3}}|A\mathbf{u}^k|\\
&\leq C_\theta|\Delta\rho^k|^2|\nabla\Delta\rho^k|^{\frac{4}{3}} + \theta|A\mathbf{u}^k|^2\\
&\leq C_{\theta,\eta}|\Delta\rho^k|^6 + \eta|\nabla\Delta\rho^k|^2 + \theta|A\mathbf{u}^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
\lambda^2|(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k, A\mathbf{u}^k)| &\leq C|\nabla\rho^k|_6^3|A\mathbf{u}^k|\\
&\leq C|\Delta\rho^k|^3|A\mathbf{u}^k|\\
&\leq C_\theta|\Delta\rho^k|^6 + \theta|A\mathbf{u}^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
\lambda^2|(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k, A\mathbf{u}^k)| &\leq C|\Delta\rho^k|_3|\nabla\rho^k|_6|A\mathbf{u}^k|\\
&\leq C|\rho^k|_\infty^{\frac{1}{3}}|\nabla\Delta\rho^k|^{\frac{2}{3}}|\Delta\rho^k||A\mathbf{u}^k|\\
&\leq C_\theta|\Delta\rho^k|^2|\nabla\Delta\rho^k|^{\frac{4}{3}} + \theta|A\mathbf{u}^k|^2\\
&\leq C_{\theta,\eta}|\Delta\rho^k|^6 + \eta|\nabla\Delta\rho^k|^2 + \theta|A\mathbf{u}^k|^2.
\end{aligned}
$$

Então, tomando $\theta = \dfrac{\mu}{28}$ obtemos a seguinte desigualdade:

$$
\begin{aligned}
&\frac{\mu}{4}|A\mathbf{u}^k|^2 \leq \frac{\beta^2}{2\mu}|\mathbf{u}_t^k|^2 + C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6\\
&+C|\Delta\rho^k|^6 + C|\Delta\rho|^8 + 3\eta|\nabla\Delta\rho|^2 + C|F|^2.
\end{aligned}
$$

Multiplicando a desigualdade acima por $\dfrac{\alpha\mu}{2\beta^2}$ e somando-a com (2.13) teremos:

$$
\frac{1}{2}\frac{d}{dt}(|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + \mu|\nabla\mathbf{u}^k|^2 + |\Delta\rho^k|^2) + \frac{\mu}{2}|\nabla\mathbf{u}^k|^2 + \frac{\alpha}{4}|\mathbf{u}_t^k|^2
$$

$$
+(\frac{\lambda}{2} - 3\eta\frac{\alpha\mu}{2\beta^2})|\nabla\Delta\rho^k|^2 + (\frac{\alpha\mu^2}{8\beta^2} - 6\delta)|A\mathbf{u}^k|^2 \leq C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6
$$

$$+C|\Delta\rho^k|^2+C|\Delta\rho^k|^4+C|\Delta\rho^k|^6+C|\Delta\rho^k|^8+C|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2+C|F|^2.$$

Finalmente, tomamos $\eta=\dfrac{\lambda\beta^2}{6\alpha\mu}$ e $\delta=\dfrac{\alpha\mu^2}{96\beta^2}$; assim, obtemos:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2+\mu|\nabla\mathbf{u}^k|^2+|\Delta\rho^k|^2)+\frac{\mu}{2}|\nabla\mathbf{u}^k|^2+\frac{\alpha}{4}|\mathbf{u}^k_t|^2+\frac{\alpha\mu^2}{16\beta^2}|A\mathbf{u}^k|^2\\
&+\frac{\lambda}{4}|\nabla\Delta\rho^k|^2\leq C\{|F|^2+|\nabla\mathbf{u}^k|^4+|\nabla\mathbf{u}^k|^6+|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k|^2\\
&+|\Delta\rho^k|^2+|\Delta\rho^k|^4+|\Delta\rho^k|^6+|\Delta\rho|^8\}.
\end{aligned}\tag{2.14}$$

Definindo

$$G_k(t)=|(\rho^k(t))^{\frac{1}{2}}\mathbf{u}^k(t)|^2+\mu|\nabla\mathbf{u}^k(t)|^2+|\Delta\rho^k(t)|^2,$$

podemos ver pela desigualdade acima que:

$$\frac{d}{dt}G_k(t)\quad\leq\quad C|F|^2+CG_k(t)+CG_k^2(t)+CG_k^3(t)+CG_k^4(t)$$

Assim, ficamos com o seguinte sistema:

$$\begin{cases}G_k'(t)\leq C|F|^2+CG_k(t)+CG_k^2(t)+CG_k^3(t)+CG_k^4(t)\\ G_k(0)=|\rho_\circ^{\frac{1}{2}}u_\circ|+\mu|\nabla\mathbf{u}_\circ|^2+|\Delta\rho_\circ|^2.\end{cases}$$

Usando o Lema 1.8 de desigualdades diferenciais, temos que:

$$G_k(t)\leq\varphi(t),$$

para todo t no intervalo maximal de existência de φ, onde

$$\begin{cases}\varphi'(t)&=&C|F|^2+C\varphi+C\varphi^2+C\varphi^3+C\varphi^4\\ \varphi(0)&=&G_k(0).\end{cases}\tag{2.15}$$

Portanto, existe T^*, com $0<T^*\leq T$ tal que, para alguma constante $M>0$, teremos:

$$G_k(t)\leq M,\qquad\forall\ t\ \in\ [0,T^*].$$

Voltando à expressão (2.14) e integrando de 0 até t, com $t\in[0,T^*]$, teremos:

$$\begin{aligned}
&|(\rho^k)^{\frac{1}{2}}(t)\mathbf{u}^k(t)|^2+\mu|\nabla\mathbf{u}^k(t)|^2+|\Delta\rho^k(t)|^2+\mu\int_0^t|\nabla\mathbf{u}^k(s)|^2\\
+\ &\frac{\alpha}{2}\int_0^t|\mathbf{u}^k_t(s)|^2ds+\frac{\alpha\mu^2}{8\beta^2}\int_0^t|A\mathbf{u}^k(s)|^2ds+\frac{\lambda}{2}\int_0^t|\nabla\Delta\rho^k(s)|^2ds
\end{aligned}$$

$$
\begin{aligned}
&\leq && |\rho_\circ^{\frac{1}{2}}\mathbf{u}_\circ|^2 + \mu|\nabla\mathbf{u}_\circ|^2 + |\Delta\rho_\circ|^2 + C\int_0^t |F(s)|^2 ds \\
&+ && C\int_0^t \{|(\rho^k(s))^{\frac{1}{2}}\mathbf{u}^k(s)|^2 + |\nabla\mathbf{u}^k(s)|^4 + |\nabla\mathbf{u}^k(s)|^6 \\
&+ && |\Delta\rho^k|^2 + |\Delta\rho^k(s)|^4 + |\Delta\rho^k|^6 + |\Delta\rho^k(s)|^8\} ds \\
&\leq && |\rho_\circ^{\frac{1}{2}}\mathbf{u}_\circ| + \mu|\nabla\mathbf{u}_\circ|^2 + |\Delta\rho_\circ|^2 + C|F|^2_{L^2(0,T^*;L^2(\Omega))} + CT^*.
\end{aligned}
$$

De onde podemos concluir que

$$
\begin{aligned}
&\mathbf{u^k} \in \mathbf{L^\infty(0, T^*; H) \cap L^2(0, T^*; V)}, \\
&\mathbf{u^k} \in \mathbf{L^\infty(0, T^*; V) \cap L^2(0, T^*; D(A))}, \\
&\mathbf{u^k_t} \in \mathbf{L^2(0, T^*; H)}, \\
&\rho^{\mathbf{k}} \in \mathbf{L^\infty(0, T^*; H^2_N) \cap L^2(0, T^*; H^3_N)},
\end{aligned}
$$

uniformemente em k.

Novamente usando a segunda equação de (2.8) vemos que:

$$|\rho_t^k| \leq |\mathbf{u}^k \cdot \nabla\rho^k| + \lambda|\Delta\rho^k|;$$

mas,

$$|\mathbf{u}^k \cdot \nabla\rho^k| \leq |\mathbf{u}^k|_4 |\nabla\rho^k|_4 \leq C|\nabla\mathbf{u}^k||\Delta\rho^k|.$$

Assim sendo,

$$|\rho_t^k|_{L^\infty(0,T^*,L^2(\Omega))} \leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)} \cdot (1 + |\mathbf{u}^k|_{L^\infty(0,T^*;V)}) \leq C$$

Isto nos diz que

$$\rho^{\mathbf{k}}_{\mathbf{t}} \in \mathbf{L^\infty(0, T^*; L^2(\Omega))},$$

uniformemente em k.

Agora, aplicando o operador ∇ na segunda equação de (2.8) obteremos:

$$\nabla\rho_t^k = \lambda\nabla\Delta\rho^k - \nabla(\mathbf{u}^k \cdot \nabla\rho^k).$$

Então,

$$
\begin{aligned}
|\nabla\rho_t^k|^2 &\leq C|\nabla\Delta\rho^k|^2 + C|\nabla\mathbf{u}^k \cdot \nabla\rho^k|^2 + C|\mathbf{u}^k \cdot \nabla^2\rho^k|^2 \\
&\leq C|\nabla\Delta\rho^k|^2 + C|\nabla\mathbf{u}^k|^2|\nabla\rho^k|^2_\infty + C|\mathbf{u}^k|^2_\infty|\Delta\rho^k|^2 \\
&\leq C|\rho^k|_{H^3_N} + C|A\mathbf{u}^k|^2.
\end{aligned}
$$

Agora, integrando de 0 até t teremos que:

$$\int_0^t |\nabla\rho_t^k(s)|^2 ds \leq C\int_0^t |\rho^k(s)|^2_{H^3_N} ds + C\int_0^t |A\mathbf{u}^k(s)|^2 ds \leq C.$$

e, portanto, podemos concluir que

$$\rho_{\mathbf{t}}^{\mathbf{k}} \in \mathbf{L}^{\infty}(\mathbf{0}, \mathbf{T}^*; \mathbf{L^2}(\boldsymbol{\Omega})) \cap \mathbf{L^2}(\mathbf{0}, \mathbf{T}^*; \mathbf{H^1}(\boldsymbol{\Omega}))$$

uniformemente em k.

Usando a primeira equação de (2.8) e lembrando que estamos trabalhando com a base espectral, vemos que

$$\begin{aligned} \mu A\mathbf{u}^k = P_k\{&-\rho^k\mathbf{u}_t^k - (\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k + \lambda[(\mathbf{u}^k\cdot\nabla)\nabla\rho^k + (\nabla\rho^k\cdot\nabla)\mathbf{u}^k] \\ &+\rho^k F - \lambda^2[\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k - \frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k + \frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k]\}. \end{aligned} \tag{2.16}$$

Tomando a norma L^2 na equação acima obtemos as seguintes estimativas para os termos que se encontram do lado direito da expressão acima:

$$|\rho^k\mathbf{u}_t^k| \leq \beta|\mathbf{u}_t^k|;$$

$$\begin{aligned} |(\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k| &\leq \beta|\mathbf{u}^k|_\infty|\nabla\mathbf{u}^k| \leq |\nabla\mathbf{u}^k|^{\frac{1}{2}}|A\mathbf{u}^k|^{\frac{1}{2}}|\nabla\mathbf{u}^k| \\ &\leq C|\nabla\mathbf{u}^k|^3 + \varepsilon|A\mathbf{u}^k|; \end{aligned}$$

$$\begin{aligned} |\lambda(\mathbf{u}^k\cdot\nabla)\nabla\rho^k| &\leq C|\mathbf{u}^k|_\infty|\Delta\rho^k \leq C|\nabla\mathbf{u}^k|^{\frac{1}{2}}|A\mathbf{u}^k|^{\frac{1}{2}}|\Delta\rho^k| \\ &\leq C|\Delta\rho^k|^2|\nabla\mathbf{u}^k| + \varepsilon|A\mathbf{u}^k|; \end{aligned}$$

$$|\lambda(\nabla\rho^k\cdot\nabla)\mathbf{u}^k| \leq C|\nabla\rho^k|_6|\nabla\mathbf{u}^k|_3 \leq C|\Delta\rho^k|^2|\nabla\mathbf{u}^k| + \varepsilon|A\mathbf{u}^k|;$$

$$|\rho^k F| \leq \beta|F|;$$

$$\begin{aligned} |\frac{\lambda^2}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k| &\leq C|\nabla\rho^k|_6|\nabla^2\rho^k|_3 \\ &\leq C|\Delta\rho^k||\nabla\Delta\rho^k| \leq C|\nabla\Delta\rho^k|; \end{aligned}$$

$$|\frac{\lambda^2}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k| \leq C|\nabla\rho^k|_6^3 \leq C|\Delta\rho^k|^3;$$

$$\begin{aligned} |\frac{\lambda^2}{\rho^k}\Delta\rho^k\nabla\rho^k| &\leq C|\Delta\rho^k|_3|\nabla\rho^k|_6 \\ &\leq C|\Delta\rho^k||\nabla\Delta\rho^k| \leq C|\nabla\Delta\rho^k|. \end{aligned}$$

Das estimativas acima podemos ver que (após escolher ε conveniente):

$$|A\mathbf{u}^k|^2 \leq C + C|F|^2 + C|\mathbf{u}_t^k|^2 + C|\nabla\Delta\rho^k|^2 \leq C + C|\mathbf{u}_t^k|^2 + C|\nabla\Delta\rho^k|^2, \tag{2.17}$$

já que, pelo Lema 1.4, tem-se $F \in C([0,T^*];L^2(\Omega))$.

Calculando a derivada com relação à variável t da segunda equação do sistema (2.8) teremos:

$$\rho^k_{tt} - \lambda\Delta\rho^k_t = -\mathbf{u}^k_t \cdot \nabla\rho^k - \mathbf{u}^k \cdot \nabla\rho^k_t,$$

e aplicando, em seguida, o operador ∇ na equação acima ficaremos com a expressão:

$$\nabla\rho^k_{tt} - \lambda\nabla\Delta\rho^k_t = -\nabla\mathbf{u}^k_t \cdot \nabla\rho^k - \mathbf{u}^k_t \cdot \nabla^2\rho^k - \nabla\mathbf{u}^k \cdot \nabla\rho^k_t - \mathbf{u}^k \cdot \nabla^2\rho^k_t.$$

Tomando o produto interno de $L^2(\Omega)$ dos termos da equação acima com o termo $\nabla\rho^k_t$ obteremos, após uma integração por partes, a seguinte desigualdade diferencial:

$$\begin{array}{l}\dfrac{1}{2}\dfrac{d}{dt}|\nabla\rho^k_t|^2 + \lambda|\Delta\rho^k_t|^2 \leq |(\nabla\mathbf{u}^k_t \cdot \nabla\rho^k, \nabla\rho^k_t)| \\ +|(\mathbf{u}^k_t \cdot \nabla^2\rho^k, \nabla\rho^k_t)| + |(\nabla\mathbf{u}^k \cdot \nabla\rho^k_t, \nabla\rho^k_t)| + |(\mathbf{u}^k \cdot \nabla^2\rho^k_t, \nabla\rho^k_t)|.\end{array}$$

Os termos à direita na expressão acima podem ser assim estimados:

$$\begin{array}{rcl}|(\nabla\mathbf{u}^k_t \cdot \nabla\rho^k, \nabla\rho^k_t)| & \leq & |\nabla\mathbf{u}^k_t||\nabla\rho^k|_\infty|\nabla\rho^k_t| \\ & \leq & |\nabla\mathbf{u}^k_t||\nabla\Delta\rho^k||\nabla\rho^k_t| \\ & \leq & \gamma|\nabla\mathbf{u}^k_t|^2 + C|\nabla\Delta\rho^k|^2|\nabla\rho^k_t|^2;\end{array}$$

$$\begin{array}{rcl}|(\mathbf{u}^k_t \cdot \nabla^2\rho^k, \nabla\rho^k_t)| & \leq & |\mathbf{u}^k_t|_4|\nabla^2\rho^k|_4|\nabla\rho^k_t| \\ & \leq & C|\nabla\mathbf{u}^k_t||\nabla\Delta\rho^k||\nabla\rho^k_t| \\ & \leq & \gamma|\nabla\mathbf{u}^k_t|^2 + C|\nabla\Delta\rho^k|^2|\nabla\rho^k_t|^2;\end{array}$$

$$\begin{array}{rcl}|(\nabla\mathbf{u}^k \cdot \nabla\rho^k_t, \nabla\rho^k_t)| & \leq & |\nabla\mathbf{u}^k|_4|\nabla\rho^k_t|_4|\nabla\rho^k_t| \\ & \leq & C|A\mathbf{u}^k||\Delta\rho^k_t||\nabla\rho^k_t| \\ & \leq & \delta|\Delta\rho^k_t|^2 + C|A\mathbf{u}^k|^2|\nabla\rho^k_t|^2;\end{array}$$

$$\begin{array}{rcl}|(\mathbf{u}^k \cdot \nabla^2\rho^k_t, \rho^k_t)| & \leq & C|\mathbf{u}^k|_\infty|\Delta\rho^k_t||\nabla\rho^k_t| \\ & \leq & \delta|\Delta\rho^k_t|^2 + C|A\mathbf{u}^k|^2|\nabla\rho^k_t|^2.\end{array}$$

Assim, a desigualdade diferencial acima nos leva à seguinte:

$$\begin{array}{l}\dfrac{1}{2}\dfrac{d}{dt}|\nabla\rho^k_t|^2 + \lambda|\Delta\rho^k_t|^2 \leq 2\gamma|\nabla\mathbf{u}^k_t|^2 \\ +2\delta|\Delta\rho^k_t|^2 + C(|A\mathbf{u}^k|^2 + |\nabla\Delta\rho^k|^2)|\nabla\rho^k_t|^2.\end{array} \tag{2.18}$$

Agora, calculando a derivada da equação (2.16) com relação a t e tomando o produto interno com $\mathbf{u}^k_t$ obteremos a seguinte identidade:

$$\begin{array}{rcl}\dfrac{1}{2}\dfrac{d}{dt}|(\rho^k)^{\frac{1}{2}}\mathbf{u}^k_t|^2 + \mu|\nabla\mathbf{u}^k_t|^2 & = & -\dfrac{1}{2}(\rho^k_t\mathbf{u}^k_t, \mathbf{u}^k_t) - (\rho^k_t(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_t) \\ & - & (\rho^k(\mathbf{u}^k_t \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_t) - (\rho^k(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k_t, \mathbf{u}^k_t) + (\rho^k_t F, \mathbf{u}^k_t) + (\rho^k F_t, \mathbf{u}^k_t) \\ & + & \lambda\{((\nabla\rho^k_t \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_t) + \lambda((\nabla\rho^k \cdot \nabla)\mathbf{u}^k_t, \mathbf{u}^k_t) + ((\mathbf{u}^k_t \cdot \nabla)\nabla\rho^k, \mathbf{u}^k_t)\}\end{array}$$

$$
\begin{aligned}
&+ \quad \lambda((\mathbf{u}^k \cdot \nabla)\nabla\rho_t^k, \mathbf{u}_t^k) + \lambda^2(\frac{1}{(\rho^k)^2}\rho_t^k(\nabla\rho^k \cdot \nabla)\nabla\rho^k, \mathbf{u}_t^k) \\
&- \quad \lambda^2\{(\frac{1}{\rho^k}(\nabla\rho_t^k \cdot \nabla)\nabla\rho^k, \mathbf{u}_t^k) - (\frac{1}{\rho^k}(\nabla\rho^k \cdot \nabla)\nabla\rho_t^k, \mathbf{u}_t^k)\} \\
&- \quad \lambda^2\{(\frac{2\rho_t^k}{(\rho^k)^3}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho^k, \mathbf{u}_t^k) + (\frac{1}{(\rho^k)^2}(\nabla\rho_t^k \cdot \nabla\rho^k)\nabla\rho^k, \mathbf{u}_t^k)\} \\
&+ \quad \lambda^2\{(\frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho_t^k)\nabla\rho^k, \mathbf{u}_t^k) + (\frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho_t^k, \mathbf{u}_t^k)\} \\
&+ \quad \lambda^2\{(\frac{\rho_t^k}{(\rho^k)^2}\Delta\rho^k\nabla\rho^k, \mathbf{u}_t^k) - (\frac{1}{\rho^k}\Delta\rho_t^k\nabla\rho^k, \mathbf{u}_t^k) - (\frac{1}{\rho^k}\Delta\rho^k\nabla\rho_t^k, \mathbf{u}_t^k)\}.
\end{aligned}
$$

Estimamos os termos à direita na expressão acima como segue:

$$
\begin{aligned}
|\frac{1}{2}(\rho_t^k\mathbf{u}_t^k, \mathbf{u}_t^k)| &\leq C|\rho_t^k|_4|\mathbf{u}_t^k||\mathbf{u}_t^k|_4 \\
&\leq C|\rho_t^k|_{H^1}^2|\mathbf{u}_t^k|^2 + \gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\rho_t^k(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}_t^k)| &\leq |\rho_t^k|_\infty|\mathbf{u}^k|_4|\nabla\mathbf{u}^k|_4|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho_t^k||\nabla\mathbf{u}^k||A\mathbf{u}^k||\mathbf{u}_t^k| \\
&\leq C|\nabla\mathbf{u}^k|^2|A\mathbf{u}^k|^2|\mathbf{u}_t^k|^2 + \delta|\Delta\rho_t^k|^2
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^k\mathbf{u}_t^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}_t^k)| &\leq \beta|\mathbf{u}_t^k|_3|\nabla\mathbf{u}^k||\mathbf{u}_t^k|_6 \\
&\leq \beta|\mathbf{u}_t^k|^{\frac{1}{2}}|\nabla\mathbf{u}_t^k|^{\frac{3}{2}}|\nabla\mathbf{u}^k| \\
&\leq C|\nabla\mathbf{u}^k|^4|\mathbf{u}_t^k|^2 + \gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^k\mathbf{u}^k \cdot \nabla)\mathbf{u}_t^k, \mathbf{u}_t^k)| &\leq \beta|\mathbf{u}^k|_\infty|\nabla\mathbf{u}_t^k||\mathbf{u}^k| \\
&\leq C|A\mathbf{u}^k|^2|\mathbf{u}_t^k|^2 + \gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\rho_t^k F, \mathbf{u}_t^k)| &\leq |F|_3|\rho_t^k||\mathbf{u}_t^k|_6 \\
&\leq C|F|_{H^1}^2|\rho_t^k|^2 + \gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
|(\rho^k F_t, \mathbf{u}_t^k)| \leq \beta|F_t||\mathbf{u}_t^k| \leq C|F_t|^2 + C|\mathbf{u}_t^k|^2;
$$

$$
\begin{aligned}
|\lambda((\nabla\rho_t^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}_t^k)| &\leq C|\nabla\rho_t^k||\nabla\mathbf{u}^k|_4|\mathbf{u}_t^k|_4 \\
&\leq C|\nabla\rho_t^k||A\mathbf{u}^k||\nabla\mathbf{u}_t^k| \\
&\leq C|A\mathbf{u}^k|^2|\nabla\rho_t^k|^2 + \gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\nabla\rho^k\cdot\nabla)\mathbf{u}_t^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho^k|_\infty|\nabla\mathbf{u}_t^k||\mathbf{u}_t^k| \\
&\leq C|\nabla\Delta\rho^k|^2|\mathbf{u}_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\mathbf{u}_t^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\mathbf{u}_t^k||\Delta\rho^k|_4|\mathbf{u}_t^k|_4 \\
&\leq C|\mathbf{u}_t^k||\nabla\Delta\rho^k||\nabla\mathbf{u}_t^k| \\
&\leq C|\nabla\Delta\rho^k|^2|\mathbf{u}_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\mathbf{u}^k\cdot\nabla)\nabla\rho_t^k,\mathbf{u}_t^k)| &\leq C|\mathbf{u}^k|_\infty|\Delta\rho_t^k||\mathbf{u}_t^k| \\
&\leq C|A\mathbf{u}^k|^2|\mathbf{u}_t^k|^2+\delta|\Delta\rho_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{\rho_t^k}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\rho_t^k|_\infty|\nabla\rho^k|_4|\nabla^2\rho^k|_4|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho_t^k||\Delta\rho^k||\nabla\Delta\rho^k||\mathbf{u}_t^k| \\
&\leq C|\Delta\rho^k|^2|\nabla\Delta\rho^k|^2|\mathbf{u}_t^k|^2+\delta|\Delta\rho_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{1}{\rho^k}(\nabla\rho_t^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho_t^k||\nabla^2\rho^k|_4|\mathbf{u}_t^k|_4 \\
&\leq C|\nabla\rho_t^k||\nabla\Delta\rho^k||\nabla\mathbf{u}_t^k| \\
&\leq C|\nabla\Delta\rho^k|^2|\nabla\rho_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho_t^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho^k|_\infty|\nabla^2\rho_t^k||\mathbf{u}_t^k| \\
&\leq C|\nabla\Delta\rho^k||\Delta\rho_t^k||\mathbf{u}_t^k| \\
&\leq C|\nabla\Delta\rho^k|^2|\mathbf{u}_t^k|^2+\delta|\Delta\rho_t^k|^2
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{2\rho_t^k}{(\rho^k)^3}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\rho_t^k|_\infty|\nabla\rho^k|_6^3|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho_t^k||\Delta\rho^k|^3|\mathbf{u}_t^k| \\
&\leq C|\Delta\rho^k|^6|\mathbf{u}_t^k|^2+\delta|\Delta\rho_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{1}{(\rho^k)^2}(\nabla\rho_t^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho_t^k||\nabla\rho^k|_6^2|\mathbf{u}_t^k|_6 \\
&\leq C|\nabla\rho_t^k||\Delta\rho^k|^2|\nabla\mathbf{u}_t^k| \\
&\leq C_\varepsilon|\Delta\rho^k|^4|\nabla\rho_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2;
\end{aligned}
$$

$$\begin{aligned}|(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho_t^k)\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\nabla\rho_t^k||\nabla\rho^k|_6^2|\mathbf{u}_t^k|_6\\ &\leq C_\varepsilon|\Delta\rho^k|^4|\nabla\rho_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2;\end{aligned}$$

e, da mesma forma,

$$|(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho_t^k,\mathbf{u}_t^k)| \leq C_\varepsilon|\Delta\rho^k|^4|\nabla\rho_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2;$$

$$\begin{aligned}|(\frac{\rho_t^k}{(\rho^k)^2}\Delta\rho^k\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\rho_t^k|_\infty|\Delta\rho^k|_4|\nabla\rho^k|_4|\mathbf{u}_t^k|\\ &\leq C|\Delta\rho_t^k||\nabla\Delta\rho^k||\Delta\rho^k||\mathbf{u}_t^k|\\ &\leq C|\Delta\rho^k|^2|\nabla\Delta\rho^k|^2|\mathbf{u}_t^k|^2+\delta|\Delta\rho_t^k|^2;\end{aligned}$$

$$\begin{aligned}|(\frac{1}{\rho^k}\Delta\rho_t^k\nabla\rho^k,\mathbf{u}_t^k)| &\leq C|\Delta\rho_t^k||\nabla\rho^k|_\infty|\mathbf{u}_t^k|\\ &\leq C|\Delta\rho_t^k||\nabla\Delta\rho^k||\mathbf{u}_t^k|\\ &\leq C|\nabla\Delta\rho^k|^2|\mathbf{u}_t^k|^2+\delta|\Delta\rho_t^k|^2;\end{aligned}$$

$$\begin{aligned}|(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho_t^k,\mathbf{u}_t^k)| &\leq C|\Delta\rho^k|_6|\nabla\rho_t^k||\mathbf{u}_t^k|_3\\ &\leq C_\varepsilon|\nabla\Delta\rho^k|^2|\nabla\rho_t^k|^2+\gamma|\nabla\mathbf{u}_t^k|^2.\end{aligned}$$

As estimativas acima obtidas nos permitem chegar à seguinte desigualdade diferencial (fazendo uso do fato de serem $\nabla\mathbf{u}^k$ e $\Delta\rho^k$ uniformemente limitados):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+\mu|\nabla\mathbf{u}_t^k|^2 &\leq C(|F|_{H^1}^2|\rho_t^k|^2+|F_t|^2)+C(|\rho_t^k|_{H^1}^2+|A\mathbf{u}^k|^2+|\nabla\Delta\rho^k|^2+1)|\mathbf{u}_t^k|^2\\ &+ C(|A\mathbf{u}^k|^2+|\nabla\Delta\rho^k|^2)|\nabla\rho_t^k|^2+12\gamma|\nabla\mathbf{u}_t^k|^2+7\delta|\Delta\rho_t^k|^2.\end{aligned}$$

Somando a desigualdade acima àquela dada por (2.18) teremos:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla\rho_t^k|^2)+\mu|\nabla\mathbf{u}_t^k|^2+\lambda|\Delta\rho_t^k|^2\\ \leq\ & C(|F|_{H^1}^2|\rho_t^k|^2+|F_t|^2)+C(|\rho_t^k|_{H^1}^2+|A\mathbf{u}^k|^2+|\nabla\Delta\rho^k|^2+1)|\mathbf{u}_t^k|^2\\ +\ & C(|A\mathbf{u}^k|^2+|\nabla\Delta\rho^k|^2+1)|\nabla\rho_t^k|^2+14\gamma|\nabla\mathbf{u}_t^k|^2+9\delta|\Delta\rho_t^k|^2.\end{aligned}$$

Escolhendo $\gamma=\dfrac{\mu}{28}$ e $\delta=\dfrac{\lambda}{18}$, a desigualdade acima pode ser reescrita na forma:

$$\frac{1}{2}\frac{d}{dt}(|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla\rho_t^k|^2)+\frac{\mu}{2}|\nabla\mathbf{u}_t^k|^2+\frac{\lambda}{2}|\Delta\rho_t^k|^2\leq C(|F|_{H^1}^2|\rho_t^k|^2+|F_t|^2)$$

$$+C(|\rho_t^k|^2_{_{H^1}}+|A\mathbf{u}^k|^2+|\nabla\Delta\rho^k|^2+1)(|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla\rho_t^k|^2).$$

$$=C\varphi(t)+C\psi(t)(|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla\rho_t^k|^2),$$

com $\varphi,\ \psi\in L^1(0,T^*)$.

Multiplicando a desigualdade acima por 2 e integrando-a de 0 até t obtemos:

$$|(\rho^k(t))^{\frac{1}{2}}\mathbf{u}_t^k(t)|^2+|\nabla\rho_t^k(t)|^2+\mu\int_0^t|\nabla\mathbf{u}_t^k(s)|^2ds+\lambda\int_0^t|\Delta\rho_t^k(s)|^2ds$$

$$\leq|\rho_\circ^{\frac{1}{2}}\mathbf{u}_t^k(0)|^2+|\nabla\rho_t^k(0)|^2+C\int_0^t\varphi(s)ds+C\int_0^t\psi(s)(|(\rho^k(s))^{\frac{1}{2}}\mathbf{u}_t^k(s)|^2+|\nabla\rho_t^k(s)|^2ds.$$

Usando o lema de Gronwall e o fato de $0<\alpha\leq\rho^k\leq\beta$ obtemos:

$$\alpha|\mathbf{u}_t^k(t)|^2+|\nabla\rho_t^k(t)|^2+\mu\int_0^t|\nabla\mathbf{u}_t^k(s)|^2ds+\lambda\int_0^t|\Delta\rho_t^k(s)|^2ds$$

$$\leq C\bigg(|\rho_\circ^{\frac{1}{2}}\mathbf{u}_t^k(0)|^2+|\nabla\rho_t^k(0)|^2+\int_0^t\varphi(s)ds\bigg)\cdot\ \exp\ (C\int_0^t\psi(s)ds)<+\infty.$$

Mostremos em seguida, que $|\mathbf{u}_t^k(0)|$ e $|\nabla\rho_t^k(0)|$ são limitados uniformemente em k. De fato, usando a primeira equação de (2.8) com o multiplicador $v=\mathbf{u}_t^k$ teremos:

$$|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2=(-(\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k-\mu A\mathbf{u}^k+\lambda(\mathbf{u}^k\cdot\nabla)\nabla\rho^k+\lambda((\nabla\rho^k\cdot\nabla)\mathbf{u}^k$$

$$-\lambda^2\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k+\lambda^2\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k-\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k+\rho^kF,\mathbf{u}_t^k)$$

$$=(\Phi^k,\mathbf{u}_t^k).$$

Usando o fato de $|(\rho^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2\geq\alpha|\mathbf{u}_t^k|^2$ teremos

$$\alpha|\mathbf{u}_t^k|^2\leq|(\Phi^k,\mathbf{u}_t^k)|\Longrightarrow|\mathbf{u}_t^k|^2\leq\frac{1}{\alpha}|(\Phi^k,\mathbf{u}_t^k)|\leq C|\Phi^k|^2+\frac{1}{2}|\mathbf{u}_t^k|^2\Longrightarrow|\mathbf{u}_t^k|^2\leq C|\Phi^k|^2.$$

Em particular, para $t=0$ teremos:

$$|\mathbf{u}_t^k(0)|^2\leq C|\Phi^k(0)|^2.$$

Mas,

$$|\Phi^k(0)|^2\leq C(|(\rho^k(0)\mathbf{u}^k(0)\cdot\nabla)\mathbf{u}^k(0)|^2+|A\mathbf{u}^k(0)|^2+|(\mathbf{u}^k(0)\cdot\nabla)\nabla\rho^k(0)|^2$$

$$+|(\nabla\rho^k(0)\cdot\nabla)\mathbf{u}^k(0)|^2+|\frac{1}{\rho^k(0))}(\nabla\rho^k(0))\cdot\nabla)\nabla\rho^k(0)|^2$$

$$+|\frac{1}{(\rho^k(0))^2}(\nabla\rho^k(0)\cdot\nabla\rho^k(0))\nabla\rho^k(0)|^2+|\frac{1}{\rho^k(0)}\Delta\rho^k(0)\nabla\rho^k(0)|^2+|\rho^k(0)F(0)|^2).$$

$$\leq C(\beta^2|\nabla \mathbf{u}_\circ|^2|A\mathbf{u}_\circ|^2 + |A\mathbf{u}_\circ|^2 + |\Delta\rho_\circ|^2|A\mathbf{u}_\circ|^2$$

$$+\frac{1}{\alpha^4}|\Delta\rho_\circ|^6 + \frac{1}{\alpha^2}|\Delta\rho_\circ|^2|\nabla\Delta\rho_\circ|^2 + \beta^2|F(0)|^2)$$

$$\Longrightarrow |\Phi^k(0)|^2 \leq C$$

e, portanto, $|\mathbf{u}_t^k(0)|$ é limitado uniformemente em k. Além disso,

$$|\nabla\rho_t^k(0)| \leq |\nabla\mathbf{u}^k(0)\cdot\nabla\rho^k(0)| + |\mathbf{u}^k(0)\cdot\nabla^2\rho^k(0)| + \lambda|\nabla\Delta\rho^k(0)|$$

$$\leq C(|A\mathbf{u}_\circ||\Delta\rho_\circ| + |\nabla\Delta\rho_\circ|) \leq C,$$

e assim, $|\nabla\rho_t^k(0)|$ é também limitado uniformemente em k. Concluímos então que

$$\mathbf{u_t^k} \in \mathbf{L^\infty(0,T^*;H)\cap L^2(0,T^*;V)}$$
$$\rho_\mathbf{t}^\mathbf{k} \in \mathbf{L^\infty(0,T^*;H^1)\cap L^2(0,T^*;H^2_N)}$$

uniformemente em k e devido a (2.17) conseguimos:

$$\mathbf{u^k} \in \mathbf{L^\infty(0,T^*;D(A))},$$

uniformemente em k. Além disso,

$$\lambda|\nabla\Delta\rho^k| = |\nabla\rho_t^k + \nabla\mathbf{u}^k\cdot\nabla\rho^k + \mathbf{u}^k\cdot\nabla^2\rho^k|$$

$$\leq C(|\nabla\rho_t^k| + |\nabla\mathbf{u}^k|_4|\nabla\rho^k|_4 + |\mathbf{u}^k|_\infty|\nabla^2\rho^k|)$$
$$\leq C(|\nabla\rho_t^k| + |A\mathbf{u}^k||\Delta\rho^k|) \leq C;$$

e também,

$$\lambda|\Delta^2\rho^k| = |\Delta\rho_t^k + \Delta\mathbf{u}^k\cdot\nabla\rho^k + \mathbf{u}\cdot\Delta\nabla\rho^k|$$
$$\leq C(|\Delta\rho_t^k| + |\Delta\mathbf{u}^k||\nabla\rho^k|_\infty + |\mathbf{u}^k|_\infty|\nabla\Delta\rho^k|)$$
$$\leq C(|\Delta\rho_t^k| + |A\mathbf{u}^k||\rho^k|_{H^3_N}),$$

e, portanto,

$$\lambda\int_0^t |\Delta^2\rho^k|^2 ds \leq C\int_0^t (|\Delta\rho_t^k|^2 + |A\mathbf{u}^k|^2|\rho^k|^2_{H^3_N})ds \leq C.$$

Concluímos que

$$\rho^\mathbf{k} \in \mathbf{L^\infty(0,T^*;H^3_N)\cap L^2(0,T^*;H^4_N)}$$

Agora, podemos obter estimativas melhores para as aproximações.

Podemos ver pela primeira equação de (2.8) que:

$$
\begin{aligned}
\mu A\mathbf{u}^k &= -P_k[\rho^k\mathbf{u}_t^k + (\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k - \rho^k F - \lambda(\mathbf{u}^k\cdot\nabla)\nabla\rho^k \\
&\quad -\lambda(\nabla\rho^k\cdot\nabla)\mathbf{u}^k)] - \lambda^2 P_k[\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k \\
&\quad -\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k + \frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k].
\end{aligned} \tag{2.19}
$$

Assim, temos as seguintes estimativas na norma $L^6(\Omega)$ para os termos que se encontram do lado direito da expressão acima:

$$|P_k(\rho^k\mathbf{u}_t^k)|_6 \leq C|\rho^k|_\infty|\mathbf{u}_t^k|_6 \leq C|\nabla\mathbf{u}_t^k|;$$

$$|P_k((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k)|_6 \leq C|\rho^k|_\infty|\mathbf{u}^k|_\infty|\nabla\mathbf{u}^k|_6 \leq C|A\mathbf{u}^k|^2;$$

$$|P_k(\rho^k F)|_6 \leq C|\rho^k|_\infty|F|_6 \leq C|F|_{H^1};$$

$$|\lambda P_k((\mathbf{u}\cdot\nabla)\nabla\rho^k)|_6 \leq C|\mathbf{u}^k|_\infty|\nabla^2\rho^k|_6 \leq C|A\mathbf{u}^k||\nabla\Delta\rho^k|;$$

$$
\begin{aligned}
|\lambda P_k((\nabla\rho^k\cdot\nabla)\mathbf{u}^k)|_6 &\leq C|\nabla\rho^k|_\infty|\nabla\mathbf{u}^k|_6 \\
&\leq C|\nabla\Delta\rho^k||A\mathbf{u}^k|;
\end{aligned}
$$

$$
\begin{aligned}
|P_k(\frac{\lambda^2}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k)|_6 &\leq C|\nabla\rho^k|_\infty|\nabla^2\rho^k|_6 \\
&\leq C|\nabla\Delta\rho^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|P_k(\frac{\lambda^2}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k)|_6 &\leq C|\nabla\rho^k|_\infty^2|\nabla\rho^k|_6 \\
&\leq C|\nabla\Delta\rho^k|^3;
\end{aligned}
$$

$$
\begin{aligned}
|P_k(\frac{\lambda^2}{\rho^k}\Delta\rho^k\nabla\rho^k)|_6 &\leq C|\nabla\rho^k|_\infty|\Delta\rho^k|_6 \\
&\leq C|\nabla\Delta\rho^k|^2.
\end{aligned}
$$

Logo, usando as estimativas acima, teremos:

$$\int_0^t |A\mathbf{u}^k(s)|_6^2 ds \leq C.$$

Agora, usando os resultados de Amrouche-Girault [2] teremos:

$$\int_0^t |\mathbf{u}^k(s)|^2_{W^{2,6}} ds \leq C.$$

Então, usamos a imersão de Sobolev de $W^{2,6}(\Omega) \subset W^{1,\infty}(\Omega)$ para obter:

$$\int_0^t |\nabla \mathbf{u}^k(s)|^2_{\infty} ds \leq C.$$

Portanto, temos que

$$\mathbf{u^k} \in \mathbf{L^2(0, T^*; W^{1,\infty}(\Omega))},$$

uniformemente em k.

Podemos ainda, conseguir estimativas uniformes em k para a derivada segunda $\mathbf{u}^k_{tt}$; para isto começando derivando a equação com respeito à variável t e em seguida, considerando $v = \mathbf{u}^k_{tt}$. Após uma integração por partes teremos:

$$|(\rho^k)^{1/2}\mathbf{u}^k_{tt}|^2 + \frac{\mu}{2}\frac{d}{dt}|\nabla \mathbf{u}^k_t|^2 = -(\rho^k_t \mathbf{u}^k_t, \mathbf{u}^k_{tt})$$

$$-((\rho^k_t \mathbf{u}^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_{tt}) - ((\rho^k \mathbf{u}^k_t \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_{tt}) - ((\rho^k \mathbf{u}^k \cdot \nabla)\mathbf{u}^k_t, \mathbf{u}^k_{tt})$$
$$+\lambda((\mathbf{u}^k_t \cdot \nabla)\nabla\rho^k, \mathbf{u}^k_{tt}) + \lambda((\mathbf{u}^k \cdot \nabla)\nabla\rho^k_t, \mathbf{u}^k_{tt})$$
$$+\lambda((\nabla\rho^k_t \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_{tt}) + \lambda((\nabla\rho^k \cdot \nabla)\mathbf{u}^k_t, \mathbf{u}^k_{tt})$$
$$+\lambda^2 \frac{\rho^k_t}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla)\nabla\rho^k, \mathbf{u}^k_{tt}) - \lambda^2(\frac{1}{\rho^k}(\nabla\rho^k_t \cdot \nabla)\nabla\rho^k, \mathbf{u}^k_{tt})$$
$$-\lambda^2(\frac{1}{\rho^k}(\nabla\rho^k \cdot \nabla)\nabla\rho^k_t, \mathbf{u}^k_{tt}) - 2\lambda^2(\frac{\rho^k_t}{(\rho^k)^4}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho^k, \mathbf{u}^k_{tt})$$
$$+\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k_t \cdot \nabla\rho^k)\nabla\rho^k, \mathbf{u}^k_{tt}) + \lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho^k_t)\nabla\rho^k, \mathbf{u}^k_{tt})$$
$$+\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho^k_t, \mathbf{u}^k_{tt}) + \lambda^2(\frac{\rho^k_t}{(\rho^k)^2}\Delta\rho^k\nabla\rho^k, \mathbf{u}^k_{tt})$$
$$-\lambda^2(\frac{1}{\rho^k}\Delta\rho^k_t\nabla\rho^k, \mathbf{u}^k_{tt}) - \lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k_t, \mathbf{u}^k_{tt})$$
$$+(\rho^k_t F, \mathbf{u}^k_{tt}) + (\rho^k F_t, \mathbf{u}^k_{tt})$$

Podemos estimar os termos à direita da seguinte forma:

$$|(\rho^k_t \mathbf{u}^k_t, \mathbf{u}^k_{tt})| \leq |\rho^k_t|_4 |\mathbf{u}^k_t|_4 |\mathbf{u}^k_{tt}| \leq C_\varepsilon |\nabla \mathbf{u}^k_t|^2 + \varepsilon |\mathbf{u}^k_{tt}|^2;$$

$$|((\rho^k_t \mathbf{u}^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k_{tt})| \leq |\rho^k_t|_4 |\mathbf{u}^k|_\infty |\nabla \mathbf{u}^k|_4 |\mathbf{u}^k_{tt}| \leq C_\varepsilon |\rho^k_t|^2_{H^1} |A\mathbf{u}^k|^4 + \varepsilon |\mathbf{u}^k_{tt}|^2;$$

$$|(\rho^k\mathbf{u}_t^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_{tt}^k)|\le|\rho^k|_\infty|\mathbf{u}_t^k|_4|\nabla\mathbf{u}^k|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon|\nabla\mathbf{u}_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}_t^k,\mathbf{u}_{tt}^k)|\le|\rho^k|_\infty|\mathbf{u}^k|_\infty|\nabla\mathbf{u}_t^k||\mathbf{u}_{tt}^k|\le C_\varepsilon|\nabla\mathbf{u}_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|((\mathbf{u}_t^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_{tt}^k)|\le|\mathbf{u}_t^k|_4|\nabla^2\rho^k|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon|\nabla\mathbf{u}_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|((\mathbf{u}^k\cdot\nabla)\nabla\rho_t^k,\mathbf{u}_{tt}^k)|\le|\mathbf{u}^k|_\infty|\nabla^2\rho_t^k||\mathbf{u}_{tt}^k|\le C_\varepsilon|\Delta\rho_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|((\nabla\rho_t^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_{tt}^k)|\le|\nabla\rho_t^k|_4|\nabla\mathbf{u}^k|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon|\Delta\rho_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|((\nabla\rho^k\cdot\nabla)\mathbf{u}_t^k,\mathbf{u}_{tt}^k)|\le|\nabla\rho^k|_\infty|\nabla\mathbf{u}_t^k||\mathbf{u}_{tt}^k|\le C_\varepsilon|\nabla\mathbf{u}_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{\rho_t^k}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\rho_t^k|_4|\nabla\rho^k|_\infty|\nabla^2\rho^k|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{\rho^k}(\nabla\rho_t^k\cdot\nabla)\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\nabla\rho_t^k|_4|\nabla^2\rho^k|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon|\Delta\rho_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho_t^k,\mathbf{u}_{tt}^k)|\le C|\nabla\rho^k|_\infty|\nabla^2\rho_t^k||\mathbf{u}_{tt}^k|\le C_\varepsilon|\Delta\rho_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{\rho_t^k}{(\rho^k)^4}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\rho_t^k||\nabla\rho^k|_\infty^3|\mathbf{u}_{tt}^k|\le C_\varepsilon+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{(\rho^k)^2}(\nabla\rho_t^k\cdot\nabla\rho^k)\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\nabla\rho_t^k||\nabla\rho^k|_\infty^2|\mathbf{u}_{tt}^k|\le C_\varepsilon+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho_t^k)\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\nabla\rho^k|_\infty^2|\nabla\rho_t^k||\mathbf{u}_{tt}^k|\le C_\varepsilon+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho_t^k,\mathbf{u}_{tt}^k)|\le C|\nabla\rho^k|_\infty^2|\nabla\rho_t^k||\mathbf{u}_{tt}^k|\le C_\varepsilon+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{\rho_t^k}{(\rho^k)^2}\Delta\rho^k\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\rho_t^k|_4|\Delta\rho^k|_4|\nabla\rho^k|_\infty|\mathbf{u}_{tt}^k|\le C_\varepsilon+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{\rho^k}\Delta\rho_t^k\nabla\rho^k,\mathbf{u}_{tt}^k)|\le C|\Delta\rho_t^k||\nabla\rho^k|_\infty|\mathbf{u}_{tt}^k|\le C_\varepsilon|\Delta\rho_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho_t^k,\mathbf{u}_{tt}^k)|\le|\Delta\rho^k|_4|\nabla\rho_t^k|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon|\Delta\rho_t^k|^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\rho_t^kF,\mathbf{u}_{tt}^k)|\le|\rho_t^k|_4|F|_4|\mathbf{u}_{tt}^k|\le C_\varepsilon|F|_{H^1}^2+\varepsilon|\mathbf{u}_{tt}^k|^2;$$
$$|(\rho^kF_t,\mathbf{u}_{tt}^k)|\le C|F_t||\mathbf{u}_{tt}^k|\le C_\varepsilon|F_t|^2+\varepsilon|\mathbf{u}_{tt}^k|^2.$$

Usando as estimativas acima obtidas, ficamos com a seguinte expressão:

$$|(\rho^k)^{1/2}\mathbf{u}_{tt}^k|^2+\frac{\mu}{2}\frac{d}{dt}|\nabla\mathbf{u}_t^k|^2$$
$$\le C(|\nabla\mathbf{u}_t^k|^2+|\Delta\rho_t^k|^2+|F|_{H^1}^2+|F_t|^2+1)+20\varepsilon|\mathbf{u}_{tt}^k|^2.$$

Lembrando que $0 < \alpha \leq \rho^k$, para todo $k \geq 1$ e escolhendo $\varepsilon = \frac{\alpha}{40}$, vemos que a desigualdade acima fica na seguinte forma:

$$|\mathbf{u}_{tt}^k|^2 + \frac{d}{dt}|\nabla \mathbf{u}_t^k|^2 \leq C(|\nabla \mathbf{u}_t^k|^2 + |\Delta \rho_t^k|^2 + |F|_{H^1}^2 + |F_t|^2 + 1).$$

Assim, multiplicando a desigualdade acima por $\sigma(t)$, onde $\sigma(t) = min\{t, 1\}$, e integrando-a entre ε e t, com $0 < \varepsilon < t$ teremos:

$$\int_\varepsilon^t \sigma(s)|\mathbf{u}_{tt}^k(s)|^2 ds + \int_\varepsilon^t \sigma(s)\frac{d}{dt}|\nabla \mathbf{u}_t^k(s)|^2 ds$$

$$\leq \int_\varepsilon^t C(|\nabla \mathbf{u}_t^k(s)|^2 + |\Delta \rho_t^k(s)|^2 + |F(s)|_{H^1}^2 + |F_t(s)|^2 + 1)ds. \tag{2.20}$$

Mas,

$$\int_\varepsilon^t \sigma(s)\frac{d}{dt}|\nabla \mathbf{u}_t^k(s)|^2 ds = \sigma(t)|\nabla \mathbf{u}_t^k(t)|^2 - \sigma(\varepsilon)|\nabla \mathbf{u}_t^k(\varepsilon)|^2 - \int_\varepsilon^t \sigma^{'}(s)|\nabla \mathbf{u}_t^k(s)|^2 ds.$$

Como $\sigma^{'}(t) \leq 1$ q.t.p. em $[0, t]$ então, usando este fato em (2.20) teremos:

$$\sigma(t)|\nabla \mathbf{u}_t^k(t)|^2 + \int_\varepsilon^t \sigma(s)|\mathbf{u}_{tt}^k(s)|^2 ds$$

$$\leq \sigma(\varepsilon)|\nabla \mathbf{u}_t^k(\varepsilon)|^2 + \int_\varepsilon^t C(|\nabla \mathbf{u}_t^k(s)|^2 + |\Delta \rho_t^k(s)|^2 + |F(s)|_{H^1}^2 + |F_t(s)|^2 + 1)ds.$$

Escolhemos então, uma seqüência (ε_n) tal que $0 < \varepsilon_n < 1$ para todo $n \geq 1$, $\varepsilon_n \to 0$ e, além disso, $\sigma(\varepsilon_n)|\nabla \mathbf{u}_t^k(\varepsilon_n)|^2 \to 0$ (o que é sempre possível já que $\nabla \mathbf{u}_t^k \in L^2(0, t)$). Fazendo o limite quando $\varepsilon_n \to 0$ teremos:

$$\sigma(t)|\nabla \mathbf{u}_t^k(t)|^2 + \int_0^t \sigma(s)|\mathbf{u}_{tt}^k(s)|^2 ds$$

$$\leq \int_0^t C(|\nabla \mathbf{u}_t^k(s)|^2 + |\Delta \rho_t^k(s)|^2 + |F(s)|_{H^1}^2 + |F_t(s)|^2 + 1)ds \leq C,$$

Logo, podemos concluir que para todo $\delta > 0$, teremos

$$\mathbf{u}_t^k \in \mathbf{L}^\infty(\delta, \mathbf{T}^*; \mathbf{V}) \qquad \text{e} \qquad \mathbf{u}_{tt}^k \in \mathbf{L}^\mathbf{2}(\delta, \mathbf{T}^*; \mathbf{H}),$$

uniformemente em k.

Todas as limitações uniformes anteriores implicam as seguintes convergências (passando a subseqüências, se necessário):

$$(1) \quad \mathbf{u}^k \rightharpoonup \mathbf{u} \text{ fraco - } * \text{ em } \mathbf{L}^\infty(\mathbf{0}, \mathbf{T}^*; \mathbf{V});$$
$$(2) \quad \mathbf{u}^k \rightharpoonup \mathbf{u} \text{ fraco - } * \text{ em } \mathbf{L}^\infty(\mathbf{0}, \mathbf{T}^*; \mathbf{D}(\mathbf{A}));$$

(3) $\mathbf{u}^k \rightharpoonup \mathbf{u}$ fraco em $\mathbf{L^2(0,T^*;D(A))}$;
(4) $\mathbf{u}^k \rightharpoonup \mathbf{u}$ fraco em $\mathbf{L^2(0,T^*;W^{1,\infty}(\Omega))}$;
(5) $\mathbf{u}_t^k \rightharpoonup \mathbf{u}_t$ fraco - $*$ em $\mathbf{L^\infty(0,T^*;H)}$;
(6) $\mathbf{u}_t^k \rightharpoonup \mathbf{u}_t$ fraco em $\mathbf{L^2(0,T^*;V)}$;
(7) $\mathbf{u}_t^k \rightharpoonup \mathbf{u}_t$ fraco - $*$ em $\mathbf{L^\infty}(\delta,\mathbf{T^*;V})$;
(8) $\mathbf{u}_{tt}^k \rightharpoonup \mathbf{u}_{tt}$ fraco em $\mathbf{L^2}(\delta,\mathbf{T^*;H})$;
(9) $\rho^k \rightharpoonup \rho$ fraco - $*$ em $\mathbf{L^\infty(0,T^*;H^3_N)}$;
(10) $\rho^k \rightharpoonup \rho$ fraco em $\mathbf{L^2(0,T^*;H^4_N)}$;
(11) $\mathbf{u}^k \longrightarrow \mathbf{u}$ forte em $\mathbf{L^2(0,T^*;V)}$;
(12) $\rho^k \longrightarrow \rho$ forte em $\mathbf{L^q(\Omega)}$, $\quad \forall q \geq 1$;
(13) $\rho^k \longrightarrow \rho$ forte em $\mathbf{L^2(0,T^*;H^2_N)}$;
(14) $\rho_t^k \rightharpoonup \rho_t$ fraco - $*$ em $\mathbf{L^\infty(0,T^*;H^1)}$;
(15) $\rho_t^k \rightharpoonup \rho_t$ fraco em $\mathbf{L^2(0,T^*;H^2_N)}$;
(16) $\rho_t^k \rightharpoonup \rho_t$ fraco em $\mathbf{L^2(0,T^*;H^1)}$;
(17) $\rho_t^k \rightharpoonup \rho_t$ fraco em $\mathbf{L^2(Q_{T^*})}$.

Observamos que as convergências em (11) e (13) foram obtidas usando o Lema 1.5 (*Aubin*).

Parte 2: passagem ao limite.

Para passar ao limite definimos

$$\phi^m = \sum_{i=1}^{m} c_{im}(t)\varphi_i(x), \tag{2.21}$$

onde $\varphi_i(x)$ é a i-ésima autofunção do operador de Stokes.

Para verificar as convergências abaixo tomamos $k > m$:

$$(i) \qquad \int_0^{T^*} (\rho^k \mathbf{u}_t^k, \phi^m)dt \longrightarrow \int_0^{T^*} (\rho \mathbf{u}_t, \phi^m)dt;$$

$$(ii) \qquad \int_0^{T^*} ((\rho^k \mathbf{u}^k \cdot \nabla)\mathbf{u}^k, \phi^m)dt \longrightarrow \int_0^{T^*} ((\rho u \cdot \nabla)\mathbf{u}, \phi^m)dt;$$

$$(iii) \qquad \int_0^{T^*} (A\mathbf{u}^k, \phi^m)dt \longrightarrow \int_0^{T^*} (A\mathbf{u}, \phi^m)dt;$$

$$(iv) \qquad \int_0^{T^*} ((\nabla\rho^k \cdot \nabla)\mathbf{u}^k, \phi^m)dt \longrightarrow \int_0^{T^*} ((\nabla\rho \cdot \nabla)\mathbf{u}, \phi^m)dt;$$

$$(v) \qquad \int_0^{T^*}((\mathbf{u}^k\cdot\nabla)\nabla\rho^k,\phi^m)dt \longrightarrow \int_0^{T^*}((\mathbf{u}\cdot\nabla)\nabla\rho,\phi^m)dt;$$

$$(vi) \qquad \int_0^{T^*}(\rho^k F,\phi^m)dt \longrightarrow \int_0^{T^*}(\rho F,\phi^m)dt.$$

$$(vii) \qquad \int_0^{T^*}(\frac{\lambda^2}{\rho^k}(\nabla\rho^k\cdot)\nabla\rho^k,\phi^m)dt \longrightarrow \int_0^{T^*}(\frac{\lambda^2}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho,\phi^m)dt$$

$$(viii) \qquad \int_0^{T^*}(\frac{\lambda^2}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k,\phi^m)dt \longrightarrow \int_0^{T^*}(\frac{\lambda^2}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho,\phi^m)dt$$

$$(ix) \qquad \int_0^{T^*}(\frac{\lambda^2}{\rho^k}\Delta\rho^k\nabla\rho^k,\phi^m)dt \longrightarrow \int_0^{T^*}(\frac{\lambda^2}{\rho}\Delta\rho\nabla\rho,\phi^m)dt$$

Mostremos a convergência (i).

$$\Big|\int_0^{T^*}\int_\Omega \rho^k\mathbf{u}_t^k\phi^m dxdt - \int_0^{T^*}\int_\Omega \rho\mathbf{u}_t\phi^m dxdt\Big|$$

$$\leq \int_0^{T^*}\int_\Omega |(\rho^k-\rho)\mathbf{u}_t^k\phi^m|dxdt + |\int_0^{T^*}\int_\Omega \rho(\mathbf{u}_t^k-\mathbf{u}_t)\phi^m dxdt|;$$

Porque $\rho\phi^m \in L^2(0,T^*;L^2(\Omega))$ e $\mathbf{u}_t^k \rightharpoonup \mathbf{u}_t$ fraco em $L^2(0,T^*;H)$ temos que o segundo termo da desigualdade acima converge a zero quando $k \longrightarrow \infty$.
Para o primeiro termo temos:

$$\int_0^{T^*}\int_\Omega |(\rho^k-\rho)\mathbf{u}_t^k\phi^m|dxdt \leq \int_0^{T^*}|\rho^k-\rho|_\infty|\mathbf{u}_t^k||\phi^m|dt$$

$$\leq |\phi^m|_{L^\infty(0,T^*;L^2(\Omega))}\int_0^{T^*}|\rho^k-\rho|_{H^2_N}|\mathbf{u}_t^k|dt$$

$$\leq C(\int_0^{T^*}|\rho^k-\rho|^2_{H^2_N}dt)^{\frac{1}{2}}\cdot(\int_0^{T^*}|\mathbf{u}_t^k|^2dt)^{\frac{1}{2}}$$

$$\leq C(\int_0^{T^*}|\rho^k-\rho|^2_{H^2_N}dt)^{\frac{1}{2}},$$

o qual converge a zero devido à convergência (13).

Para provar *(ii)* tomamos:

$$\Big|\int_0^{T^*}\int_\Omega(\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\phi^m dxdt-\int_0^{T^*}\int_\Omega(\rho u\cdot\nabla)\mathbf{u}\phi^m dxdt\Big|$$

$$\leq\int_0^{T^*}\int_\Omega\{|((\rho^k-\rho)\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\phi^m|+|(\rho(\mathbf{u}^k-\mathbf{u})\cdot\nabla)\mathbf{u}^k\phi^m|\}dxdt$$

$$+\int_0^{T^*}\int_\Omega|(\rho\mathbf{u}\cdot\nabla)(\mathbf{u}^k-\mathbf{u})\phi^m|dxdt$$

$$\leq\int_0^{T^*}|\rho^k-\rho|_\infty|\mathbf{u}^k|_\infty|\nabla\mathbf{u}^k||\phi^m|dt$$

$$+\int_0^{T^*}|\rho|_\infty|\mathbf{u}^k-\mathbf{u}|_{{}_4}|\nabla\mathbf{u}^k||\phi^m|_{{}_4}dt$$

$$+\int_0^{T^*}|\rho|_\infty|\mathbf{u}^k|_{{}_4}|\nabla(\mathbf{u}^k-\mathbf{u})||\phi^m|_{{}_4}dt$$

$$\leq C|\mathbf{u}^k|_{{}_{L^\infty(0,T^*;V)}}|\phi^m|_{{}_{L^\infty(0,T^*;L^2(\Omega))}}(\int_0^{T^*}|\rho^k-\rho|^2_{{}_{H^2_N}}dt)^{\frac{1}{2}}\cdot(\int_0^{T^*}|A\mathbf{u}^k|^2dt)^{\frac{1}{2}}$$

$$+C|\rho|_{{}_{L^\infty(Q_{T^*})}}|\mathbf{u}^k|_{{}_{L^\infty(0,T^*;V)}}(\int_0^{T^*}|\mathbf{u}^k-\mathbf{u}|^2_V dt)^{\frac{1}{2}}\cdot(\int_0^{T^*}|\phi^m|^2_V dt)^{\frac{1}{2}}$$

$$+C|\rho|_{{}_{L^\infty(Q_{T^*})}}|\mathbf{u}^k|_{{}_{L^\infty(0,T^*;V)}}(\int_0^{T^*}|\mathbf{u}^k-\mathbf{u}|^2_V dt)^{\frac{1}{2}}\cdot(\int_0^{T^*}|\phi^m|^2_V dt)^{\frac{1}{2}}$$

$$\leq C_1|\rho^k-\rho|_{{}_{L^2(0,T^*;H^2_N)}}+C_2|\mathbf{u}^k-\mathbf{u}|_{{}_{L^2(0,T^*;V)}};$$

usando as convergências (11) e (13) podemos inferir que a última expressão tende a zero quando $k\longrightarrow\infty$.

O limite em *(iii)* é provado como segue:

$$\mu\Big|\int_0^{T^*}\int_\Omega(A(\mathbf{u}^k-\mathbf{u}))\phi^m dxdt\Big|\leq\mu\int_0^{T^*}\int_\Omega|\nabla(\mathbf{u}^k-\mathbf{u})\nabla\phi^m|dxdt$$

$$\leq \mu(\int_0^{T^*} |\mathbf{u}^k - \mathbf{u}|_V^2 dt)^{\frac{1}{2}} (\int_0^{T^*} |\phi^m|_V^2 dt)^{\frac{1}{2}};$$

tal termo tende a zero devido à convergência forte em (11).

Em *(iv)* temos:

$$\lambda \Big| \int_0^{T^*} \int_\Omega ((\nabla\rho^k \cdot \nabla)\mathbf{u}^k - (\nabla\rho \cdot \nabla)\mathbf{u})\phi^m dxdt \Big|$$

$$\leq \lambda \int_0^{T^*} \int_\Omega \{|(\nabla\rho^k \cdot \nabla)(\mathbf{u}^k - \mathbf{u})\phi^m| dxdt + |(\nabla(\rho^k - \rho) \cdot \nabla)\mathbf{u}\phi^m|\} dxdt$$

$$\leq \lambda \int_0^{T^*} |\nabla\rho^k|_4 |\nabla(\mathbf{u}^k - \mathbf{u})| |\phi^m|_4 dt$$

$$+\lambda \int_0^{T^*} |\nabla(\rho^k - \rho)|_4 |\nabla\mathbf{u}| |\phi^m|_4 dt$$

$$\leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)} |\phi^m|_{L^2(0,T^*;V)} |\mathbf{u}^k - \mathbf{u}|_{L^2(0,T^*;V)}$$

$$+C|\mathbf{u}|_{L^\infty(0,T^*;V)} |\phi^m|_{L^2(0,T^*;V)} |\rho^k - \rho|_{L^2(0,T^*;H^2_N)}$$

$$\leq C|\mathbf{u}^k - \mathbf{u}|_{L^2(0,T^*;V)} + C|\rho^k - \rho|_{L^2(0,T^*;H^2_N)};$$

usando as convergências em (11) e (13) temos que o limite é zero.

Analogamente, para *(v)* temos:

$$\lambda \Big| \int_0^{T^*} \int_\Omega ((\mathbf{u}^k \cdot \nabla)\nabla\rho^k - (\mathbf{u} \cdot \nabla)\nabla\rho)\phi^m dxdt \Big|$$

$$\leq \lambda \int_0^{T^*} \int_\Omega \{|((\mathbf{u}^k - \mathbf{u}) \cdot \nabla)\nabla\rho^k\phi^m| + |(\mathbf{u} \cdot \nabla)\nabla(\rho^k - \rho)\phi^m|\} dxdt$$

$$\leq C \int_0^{T^*} \{|\mathbf{u}^k - \mathbf{u}|_4 |\rho^k|_{H^2_N} |\phi^m|_4 + |\mathbf{u}|_4 |\rho^k - \rho|_{H^2_N} |\phi^m|_4\} dt$$

$$\leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)}|\mathbf{u}^k-\mathbf{u}|_{L^2(0,T^*;V)}|\phi^m|_{L^2(0,T^*;V)}$$

$$+C|\mathbf{u}|_{L^\infty(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}|\phi^m|_{L^2(0,T^*;V)}$$

$$\leq C|\mathbf{u}^k-\mathbf{u}|_{L^2(0,T^*;V)}+C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)};$$

e considerando as convergências fortes em (11) e (13) temos que o limite é zero.

Em *(vi)* temos:

$$\Big|\int_0^{T^*}\int_\Omega(\rho^kF-\rho F)\phi^m dxdt\Big|\leq\int_0^{T^*}|\rho^k-\rho|_4|F|_4|\phi^m|dt$$

$$\leq C|\rho^k-\rho|_{L^2(0,T^*;H^1)}|F|_{L^2(0,T^*;H^1)}|\phi^m|_{L^\infty(0,T^*;H)}$$

$$\leq C|\rho^k-\rho|_{L^2(0,T^*;H^2)}$$

e usando a convergência em (13) temos que o último termo tende a zero quando $k\longrightarrow\infty$.

Para provarmos a convergência (vii), notemos primeiramente que:

$$(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k-\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho,\phi^m)=(\frac{\rho-\rho^k}{\rho^k\rho}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\phi^m)$$

$$+(\frac{1}{\rho}(\nabla(\rho^k-\rho)\cdot\nabla)\nabla\rho^k,\phi^m)+(\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla(\rho^k-\rho),\phi^m).$$

Analisemos, então a convergência dos termos à direita na expressão acima.

$$\begin{aligned}&\int_0^{T^*}\Big|(\frac{\rho-\rho^k}{\rho^k\rho}(\nabla\rho^k\cdot\nabla)\nabla\rho^k,\phi^m)\Big|dt\\&\leq C\int_0^{T^*}|\rho-\rho^k|_6|\nabla\rho^k|_6|\nabla^2\rho^k||\phi^m|_6dt\\&\leq C\int_0^{T^*}|\nabla(\rho-\rho^k)||\Delta\rho^k|^2|\nabla\phi^m|dt\\&\leq C\int_0^{T^*}|\nabla(\rho-\rho^k)|^2dt;\end{aligned}$$

tal termo tende a zero, quando $k \longrightarrow \infty$ devido à convergência (13).
Para o próximo termo temos:

$$\begin{aligned}
&\int_0^{T^*} \Big|(\frac{1}{\rho}(\nabla(\rho^k - \rho)\cdot\nabla)\nabla\rho^k, \phi^m)\Big| dt \\
&\leq C\int_0^{T^*} |\nabla(\rho^k-\rho)|_4 |\nabla^2\rho^k||\phi^m|_4 dt \\
&\leq C\int_0^{T^*} |\Delta(\rho^k-\rho)||\Delta\rho^k||\nabla\phi^m| dt \\
&\leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)} |\phi^m|_{L^2(0,T^*;V)} |\rho^k-\rho|_{L^2(0,T^*;H^2_N)} \\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)},
\end{aligned}$$

o qual vai a zero quando $k \longrightarrow \infty$ conforme a convergência (13).
Além disso, temos que:

$$\begin{aligned}
&\int_0^{T^*} \Big|(\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla(\rho^k - \rho), \phi^m)\Big| dt \\
&\leq C\int_0^{T^*} |\nabla\rho|_4 |\nabla^2(\rho^k-\rho)||\phi^m|_4 dt \\
&\leq C\int_0^{T^*} |\Delta\rho||\Delta(\rho^k-\rho)||\nabla\phi^m| dt \\
&\leq C|\rho|_{L^\infty(0,T^*;H^2_N)} |\phi^m|_{L^2(0,T^*;V)} |\rho^k-\rho|_{L^2(0,T^*;H^2_N)} \\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)},
\end{aligned}$$

sendo que este último converge a zero quando $k \longrightarrow \infty$ pela convergência (13).
Para a convergência em $(viii)$, notemos que:

$$(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k - \frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho, \phi^m) = (\frac{(\rho^k)^2-\rho^2}{(\rho^k\rho)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k, \phi^m)$$

$$+(\frac{1}{\rho^2}(\nabla(\rho^k-\rho)\cdot\nabla\rho^k)\nabla\rho^k, \phi^m) + (\frac{1}{\rho^2}(\nabla\rho\cdot\nabla(\rho^k-\rho)\nabla\rho^k, \phi^m)$$

$$+(\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla(\rho^k-\rho), \phi^m).$$

Assim sendo, mostremos para o primeiro termo da direita na expressão acima.

$$\begin{aligned}
&\int_0^{T^*} \Big|(\frac{(\rho^k)^2-\rho^2}{(\rho^k\rho)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k, \phi^m)\Big| dt \\
&\leq C\int_0^{T^*} |\rho^k-\rho|_4 |\nabla\rho^k|_6^3 |\phi^m|_4 dt \\
&\leq C\int_0^{T^*} |\rho^k-\rho|_{H^1} |\Delta\rho^k|^3 |\nabla\phi^m| dt \\
&\leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)} |\phi^m|_{L^2(0,T^*;V)} |\rho^k-\rho|_{L^2(0,T^*;H^1)} \\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^1)}.
\end{aligned}$$

O termo acima vai a zero devido à convergência (13).

Para o segundo termo temos:

$$\begin{aligned}
&\int_0^{T^*} \Big|(\frac{1}{\rho^2}(\nabla(\rho^k-\rho)\nabla\rho^k)\nabla\rho^k, \phi^m)\Big| dt\\
&\leq C\int_0^{T^*} |\nabla(\rho^k-\rho)||\nabla\rho^k|_6^2|\phi^m|_6 dt\\
&\leq C\int_0^{T^*} |\nabla(\rho^k-\rho)||\Delta\rho^k|^2|\nabla\phi^m| dt\\
&\leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)}|\phi^m|_{L^2(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^1)}\\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^1)},
\end{aligned}$$

e tal termo converge a zero quando $k \longrightarrow \infty$ devido à (13).

Para o terceiro termo temos:

$$\begin{aligned}
&\int_0^{T^*} \Big|(\frac{1}{\rho^2}(\nabla\rho\cdot\nabla(\rho^k-\rho))\nabla\rho^k, \phi^m)\Big| dt\\
&\leq C\int^{T^*} |\nabla\rho|_6|\nabla(\rho^k-\rho)||\nabla\rho^k|_6|\phi^m|_6 dt\\
&\leq C\int_0^{T^*} |\Delta\rho||\nabla(\rho^k-\rho)||\Delta\rho^k||\nabla\phi^m| dt\\
&\leq C|\rho|_{L^\infty(0,T^*;H^2_N)}|\rho^k|_{L^\infty(0,T^*;H^2_N)}|\phi^m|_{L^2(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^1)}\\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^1).}
\end{aligned}$$

Assim, este termo também tende a zero quando $k \longrightarrow \infty$, usando a convergência (13).

Para o último termo de $(viii)$, temos:

$$\begin{aligned}
&\int_0^{T^*} \Big|(\frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla(\rho^k-\rho), \phi^m)\Big| dt\\
&\leq C\int_0^{T^*} |\nabla\rho|_6^2|\nabla(\rho^k-\rho)||\phi^m|_6 dt\\
&\leq C\int_0^{T^*} |\Delta\rho|^2|\nabla(\rho^k-\rho)||\nabla\phi^m| dt\\
&\leq C|\rho|^2_{L^\infty(0,T^*;H^2_N)}|\phi^m|_{L^2(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^1)}\\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^1)}
\end{aligned}$$

e, usando a convergência (13) vemos que o termo acima tende a zero quando $k \longrightarrow \infty$; assim sendo, temos mostrada a convergência em $(viii)$.

Para (ix), vemos que:

$$(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k - \frac{1}{\rho}\Delta\rho\nabla\rho, \phi^m) = (\frac{\rho^k-\rho}{\rho^k\rho}\Delta\rho^k\nabla\rho^k, \phi^m)$$

$$+(\frac{1}{\rho}\Delta(\rho^k-\rho)\nabla\rho^k,\phi^m)+(\frac{1}{\rho}\Delta\rho\nabla(\rho^k-\rho),\phi^m).$$

Analisemos então, cada uma destas convergências.

$$\begin{aligned}
&\int_0^{T^*}\Big|(\frac{\rho^k-\rho}{\rho^k\rho}\Delta\rho^k\nabla\rho^k,\phi^m)\Big|dt\\
&\leq C\int_0^{T^*}|\rho^k-\rho|_6|\Delta\rho^k||\nabla\rho^k|_6|\phi^m|_6dt\\
&\leq C\int_0^{T^*}|\rho^k-\rho|_{H^1}|\Delta\rho^k|^2|\nabla\phi^m|dt\\
&\leq C|\rho^k|^2_{L^\infty(0,T^*;H^2_N)}|\phi^m|_{L^2(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^1)}\\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^1)};
\end{aligned}$$

$$\begin{aligned}
&\int_0^{T^*}\Big|(\frac{1}{\rho}\Delta(\rho^k-\rho)\nabla\rho^k,\phi^m)\Big|dt\\
&\leq C\int_0^{T^*}|\Delta(\rho^k-\rho)||\nabla\rho^k|_4|\phi^m|_4dt\\
&\leq C\int_0^{T^*}|\Delta(\rho^k-\rho)||\Delta\rho^k||\nabla\phi^m|dt\\
&\leq C|\rho^k|_{L^\infty(0,T^*;H^2_N)}|\phi^m|_{L^2(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}\\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)};
\end{aligned}$$

$$\begin{aligned}
&\int_0^{T^*}\Big|(\frac{1}{\rho}\Delta\rho\nabla(\rho^k-\rho),\phi^m)\Big|dt\\
&\leq C\int_0^{T^*}|\Delta\rho||\nabla(\rho^k-\rho)|_4|\phi^m|_4dt\\
&\leq C\int_0^{T^*}|\Delta\rho||\Delta(\rho^k-\rho)||\nabla\phi^m|dt\\
&\leq C|\rho|_{L^\infty(0,T^*;H^2_N)}|\rho^k-\rho|_{L^2(0,T^*;H^2)}|\phi^m|_{L^2(0,T^*;V)}\\
&\leq C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}
\end{aligned}$$

e, portanto, usando a convergência (13) segue o resultado.

Assim, passando o limite obtemos:

$$\begin{aligned}
&\int_0^{T^*}(\rho\mathbf{u}_t+(\rho\mathbf{u}\cdot\nabla)\mathbf{u}+\mu\Delta\mathbf{u}-\lambda(\nabla\rho\cdot\nabla)\mathbf{u}-\lambda(\mathbf{u}\cdot\nabla)\nabla\rho\\
&+\frac{\lambda^2}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho-\frac{\lambda^2}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho+\frac{\lambda^2}{\rho}\Delta\rho\nabla\rho-\rho F,\phi^m)dt=0,
\end{aligned} \tag{2.22}$$

para toda ϕ^m dada por (2.21).

Além disso, temos:

$$|(\rho\mathbf{u}\cdot\nabla)\mathbf{u}|^2_{L^2(0,T^*;L^2(\Omega))} = \int_0^{T^*}\int_\Omega |(\rho\mathbf{u}\cdot\nabla)\mathbf{u}|^2 dxdt \leq \int_0^{T^*} |\rho|^2_{L^\infty}|\mathbf{u}|^2_4|\nabla\mathbf{u}|^2_4 dt$$

$$\leq C|\rho|^2_{L^\infty(Q_{T^*})}|\mathbf{u}|^2_{L^\infty(0,T^*;V)}|\mathbf{u}|^2_{L^2(0,T^*;D(A))} < +\infty;$$

$$|(\nabla\rho\cdot\nabla)\mathbf{u}|^2_{L^2(0,T^*;L^2(\Omega))} = \int_0^{T^*}\int_\Omega |(\nabla\rho\cdot\nabla)\mathbf{u}|^2 dxdt \leq \int_0^{T^*} |\nabla\rho|^2_4|\nabla\mathbf{u}|^2_4 dt$$

$$\leq C|\rho|^2_{L^\infty(0,T^*;H^2_N)}\cdot|\mathbf{u}|^2_{L^2(0,T^*;D(A))} < +\infty;$$

$$|(\mathbf{u}\cdot\nabla)\nabla\rho|^2_{L^2(0,T^*;L^2(\Omega))} = \int_0^{T^*}\int_\Omega |(\mathbf{u}\cdot\nabla)\nabla\rho|^2 dxdt \leq \int_0^{T^*} |\mathbf{u}|^2_4|\nabla^2\rho|^2_4 dt$$

$$\leq C|\mathbf{u}|^2_{L^\infty(),T^*;V)}|\rho|^2_{L^2(0,T^*;H^3_N)} < +\infty;$$

$$|\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho|^2_{L^2(0,T^*;L^2(\Omega))} = \int_0^{T^*}\int_\Omega |\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho|^2 dxdt \leq \int_0^{T^*} |\frac{1}{\rho}|^2_\infty|\nabla\rho|^2_4|\nabla^2\rho|^2_4 dt$$

$$\leq C|\rho|^2_{L^\infty(0,T^*;H^2_N)}|\rho|^2_{L^2(0,T^*;H^3_N)} < +\infty;$$

$$|\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho|^2_{L^2(0,T^*;L^2(\Omega))} = \int_0^{T^*}\int_\Omega |\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho|^2 dxdt \leq \int_0^{T^*} |\frac{1}{\rho^2}|^2_\infty|\nabla\rho|^6_3 dt$$

$$\leq C|\rho|^6_{L^\infty(0,T^*;H^2_N)} < +\infty;$$

$$|\frac{1}{\rho}\Delta\rho\nabla\rho|^2_{L^2(0,T^*;L^2(\Omega))} = \int_0^{T^*}\int_\Omega |\frac{1}{\rho}\Delta\rho\nabla\rho|^2 dxdt \leq \int_0^{T^*} |\frac{1}{\rho}|^2_\infty|\Delta\rho|^2_4|\nabla\rho|^2_4 dt$$

$$\leq C|\rho|^2_{L^\infty(0,T^*;H^2_N)}|\rho|^2_{L^2(0,T^*;H^3_N)} < +\infty.$$

Mas isto significa

$$Lu = \rho \mathbf{u}_t + (\rho \mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} - \lambda(\nabla\rho \cdot \nabla)\mathbf{u} - \lambda(\mathbf{u} \cdot \nabla)\nabla\rho + \frac{\lambda^2}{\rho}(\nabla\rho \cdot \nabla)\nabla\rho$$

$$-\frac{\lambda^2}{\rho^2}(\nabla\rho \cdot \nabla\rho)\nabla\rho + \frac{\lambda^2}{\rho}\Delta\rho\nabla\rho - \rho F \quad \in \quad L^2(0, T^*; L^2(\Omega)).$$

Pelo fato das funções ϕ^m serem densas em $L^2(0, T^*; H)$ temos que (2.22) também é válida para toda $\phi \in L^2(0, T^*; H)$, e assim, $L\mathbf{u} \in L^2(0, T^*; H)^{\perp}$.

Portanto, pelo Lema 1.7 (*De Rham*), existe $p \in L^2(0, T^*; H^1(\Omega))$ tal que

$$\begin{aligned} \rho \mathbf{u}_t + (\rho \mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} - \lambda(\nabla\rho \cdot \nabla)\mathbf{u} - \lambda(\mathbf{u} \cdot \nabla)\nabla\rho + \frac{\lambda^2}{\rho}(\nabla\rho \cdot \nabla)\nabla\rho \\ -\frac{\lambda^2}{\rho^2}(\nabla\rho \cdot \nabla\rho)\nabla\rho + \frac{\lambda^2}{\rho}\Delta\rho\nabla\rho - \rho F = \nabla p. \end{aligned} \tag{2.23}$$

Agora, tomando o produto interno de $\varphi \in L^2(Q_{T^*})$, onde $Q_{T^*} = \Omega \times [0, T^*]$, com a segunda equação de (2.8) obteremos:

$$\int_0^{T^*} \int_\Omega [\rho_t^k + \mathbf{u}^k . \nabla\rho^k - \lambda\Delta\rho^k]\varphi dxdt = 0.$$

Vamos mostrar em seguida que a integral acima converge para

$$\int_0^{T^*} \int_\Omega [\rho_t + \mathbf{u} \cdot \nabla\rho - \lambda\Delta\rho]\varphi dxdt.$$

Dedido à convergência fraca de ρ_t^k para ρ_t em $L^2(Q_{T^*})$ temos que:

$$\int_0^{T} \int_\Omega (\rho_t^k - \rho_t)\varphi dxdt \longrightarrow 0.$$

Temos também que:

$$\Big| \int_0^{T^*} \int_\Omega \{\mathbf{u}^k \cdot \nabla\rho^k - \mathbf{u} \cdot \nabla\rho\}\varphi dxdt \Big| \leq \int_0^{T^*} \{|\mathbf{u}^k - \mathbf{u}|_4 |\nabla\rho^k|_4$$

$$+|\mathbf{u}|_4 |\nabla(\rho^k - \rho)|_4\}|\varphi| dt$$

$$\leq C \int_0^{T^*} \{|\mathbf{u}^k - \mathbf{u}|_V |\rho^k|_{H_N^2} + |\mathbf{u}|_V |(\rho^k - \rho)|_{H_N^2}\}|\varphi| dt$$

$$\leq C\{|\rho^k|_{L^\infty(0,T^*;H^2_N)}|\mathbf{u}^k-\mathbf{u}|_{L^2(0,T^*;V)}|\varphi|_{L^2(Q_{T^*})}$$

$$+|\mathbf{u}|_{L^\infty(0,T^*;V)}|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}\}|\varphi|_{L^2(Q_{T^*})}$$

$$\leq C|\mathbf{u}^k-\mathbf{u}|_{L^2(0,T^*;V)}+C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}\longrightarrow 0.$$

Finalmente, temos que:

$$\lambda\Big|\int_0^{T^*}\int_\Omega \Delta(\rho^k-\rho)\varphi dxdt\Big|\leq \lambda|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}|\varphi|_{L^2(Q_{T^*})}$$

$$\leq C|\rho^k-\rho|_{L^2(0,T^*;H^2_N)}\longrightarrow 0,$$

para toda $\varphi\in L^2(Q_{T^*})$; usando o Lema de Du Bois Raymond, [3], pág. 108, concluímos que:

$$\rho_t+\mathbf{u}\cdot\nabla\rho-\lambda\Delta\rho=0$$

q.t.p. em Q_{T^*}.

Além disso, temos:

$$|\rho_t|\leq|\mathbf{u}\cdot\nabla\rho|+\lambda|\Delta\rho|\leq C|\nabla\mathbf{u}||\nabla^2\rho|+\lambda|\Delta\rho|,$$

o que implica

$$|\rho_t|_{L^\infty(0,T^*;L^2(\Omega))}\leq C.$$

Portanto,

$$\rho_t+\mathbf{u}\cdot\nabla\rho-\lambda\Delta\rho=0 \ \text{ em } \ L^\infty(0,T^*;L^2(\Omega)).$$

Parte 3: continuidade das soluções.

Observamos inicialmente que, como $\mathbf{u}\in L^2(0,T^*;D(A))$ e $\mathbf{u}_t\in L^2(0,T^*;H)$ então pelo Lema 1.4, temos que $\mathbf{u}\in C([0,T^*];V)$, ou seja, existe $\tilde{\mathbf{u}}$ contínua de $[0,T^*]$ em V tal que $\mathbf{u}=\tilde{\mathbf{u}}$, q.t.p. em $[0,T^*]$.

No que segue, mostraremos que a solução $\mathbf{u}$ obtida pelo método de Galerkin assume seu dado inicial continuamente em V e $D(A)$, respectivamente.

Notamos inicialmente que, para todo $t\in[0,T^*]$ e todo $k\geq 1$, temos:

$$\nabla \mathbf{u}^k(t) - \nabla \mathbf{u}^k(0) = \int_0^t \nabla \mathbf{u}_t^k(s)ds$$

e, portanto,

$$|\nabla \mathbf{u}^k(t) - \nabla \mathbf{u}^k(0)| \leq \int_0^t |\nabla \mathbf{u}_t^k(s)|ds \leq (\int_0^t |\nabla \mathbf{u}_t^k(s)|^2 ds)^{\frac{1}{2}} \cdot t^{\frac{1}{2}} \leq Ct^{\frac{1}{2}}. \tag{2.24}$$

Como $\{\mathbf{u}^k\}$ é limitada em $L^\infty(0,T^*;D(A))$ e $\{\mathbf{u}_t^k\}$ é limitada em $L^2(0,T^*;V)$, o Lema 1.5 (*Aubin-Lions*) nos garante que $\mathbf{u}^k \to u$ fortemente em $C^0([0,T^*];V)$; assim, fazendo o limite quando $k \to +\infty$ em (2.24) teremos:

$$|\nabla \mathbf{u}(t) - \nabla \mathbf{u}(0)| \leq Ct^{\frac{1}{2}}. \qquad \forall t \in [0,T^*].$$

Logo,

$$\lim_{t\to 0^+} |\mathbf{u}(t) - \mathbf{u}(0)|_V = \lim_{t\to 0^+} |\nabla \mathbf{u}(t) - \nabla \mathbf{u}(0)| = 0.$$

Mostremos agora, que $\lim\limits_{t\to 0^+} |A\mathbf{u}(t) - A\mathbf{u}_\circ| = 0$. Notemos que, como $\mathbf{u}(t) \to \mathbf{u}_\circ$ fortemente em V e como $|A\mathbf{u}|$ é limitada em $[0,T^*]$ então $A\mathbf{u}(t) \to A\mathbf{u}_\circ$ fracamente em $L^2(\Omega)$ e, portanto, $\liminf\limits_{t\to 0^+} |A\mathbf{u}(t)| \geq |A\mathbf{u}_\circ|$; somente nos falta agora, mostrar que $\limsup\limits_{t\to 0^+} |A\mathbf{u}(t)| \leq |A\mathbf{u}_\circ|$. Para tal, começamos tomando $v = A\mathbf{u}_t^k$ na primeira equação de (2.8) e em seguida, fazemos uma integração na variável t para obter:

$$|A\mathbf{u}^k(t)|^2 - |A\mathbf{u}^k(0)|^2$$

$$= \frac{2}{\mu}\int_0^t (-\rho^k \mathbf{u}_t^k, A\mathbf{u}_t^k)ds - \frac{2}{\mu}\int_0^t ((\rho^k \mathbf{u}^k \cdot \nabla)\mathbf{u}^k + \lambda(\mathbf{u}^k \cdot \nabla)\nabla\rho^k + \lambda(\nabla\rho^k \cdot \nabla)\mathbf{u}^k$$

$$-\lambda^2 \frac{1}{\rho^k}(\nabla\rho^k \cdot \nabla)\nabla\rho^k + \lambda^2 \frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho^k - \lambda^2 \frac{1}{\rho^k}\Delta\rho^k \nabla\rho^k - \rho^k F, A\mathbf{u}_t^k)ds$$

$$= \frac{2}{\mu}\int_0^t (-\rho^k \mathbf{u}_t^k, A\mathbf{u}_t^k)ds + \frac{2}{\mu}\int_0^t (h^k, A\mathbf{u}_t^k)ds$$

$$= \frac{2}{\mu}\int_0^t (-\rho^k \mathbf{u}_t^k, A\mathbf{u}_t^k)ds + \frac{2}{\mu}\Big[(h^k(t), A\mathbf{u}^k(t)) - (h^k(0), A\mathbf{u}^k(0)) - \int_0^t (h_t^k, A\mathbf{u}^k)ds\Big]. \tag{2.25}$$

Notemos que

$$h_t^k = -(\rho_t^k \mathbf{u}^k \cdot \nabla)\mathbf{u}^k - (\rho^k \mathbf{u}_t^k \cdot \nabla)\mathbf{u}^k - (\rho^k \mathbf{u}^k \cdot \nabla)\mathbf{u}_t^k$$

$$+\lambda(\mathbf{u}_t^k \cdot \nabla)\nabla\rho^k + \lambda(\mathbf{u}^k \cdot \nabla)\nabla\rho_t^k + \lambda(\nabla\rho_t^k \cdot \nabla)\mathbf{u}^k + \lambda(\nabla\rho^k \cdot \nabla)\mathbf{u}_t^k$$

$$+\lambda^2 \frac{\rho_t^k}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla)\nabla\rho^k - \lambda^2 \frac{1}{\rho^k}(\nabla\rho_t^k \cdot \nabla)\nabla\rho^k - \lambda^2 \frac{1}{\rho^k}(\nabla\rho^k \cdot \nabla)\nabla\rho_t^k$$

$$-\lambda^2 \frac{2\rho_t^k}{(\rho^k)^3}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho^k + \lambda^2 \frac{1}{(\rho^k)^2}(\nabla\rho_t^k \cdot \nabla\rho^k)\nabla\rho^k + \lambda^2 \frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho_t^k)\nabla\rho^k$$

$$+\lambda^2 \frac{1}{(\rho^k)^2}(\nabla\rho^k \cdot \nabla\rho^k)\nabla\rho_t^k + \lambda^2 \frac{\rho_t^k}{(\rho^k)^3}\Delta\rho^k\nabla\rho^k - \lambda^2 \frac{1}{\rho^k}\Delta\rho_t^k\nabla\rho^k$$

$$-\lambda^2 \frac{1}{\rho^k}\Delta\rho^k\nabla\rho_t^k - \rho_t^k F - \rho^k F_t$$

Agora, vamos estimar os termos que se encontram à direita na igualdade acima.

$$|(\rho_t^k \mathbf{u}^k \cdot \nabla)\mathbf{u}^k| \leq |\rho_t^k|_4 |\mathbf{u}^k|_\infty |\nabla\mathbf{u}^k|_4 \leq C|\rho_t^k|_{H^1}|A\mathbf{u}^k|^2 \leq C;$$

$$|(\rho^k \mathbf{u}_t^k \cdot \nabla)\mathbf{u}^k| \leq |\rho^k|_\infty |\mathbf{u}_t^k|_4 |\nabla\mathbf{u}^k|_4 \leq C|\nabla\mathbf{u}_t^k||A\mathbf{u}^k| \leq C|\nabla\mathbf{u}_t^k|;$$

$$|(\rho^k \mathbf{u}^k \cdot \nabla)\mathbf{u}_t^k| \leq |\rho^k|_\infty |\mathbf{u}^k|_\infty |\nabla\mathbf{u}_t^k| \leq C|A\mathbf{u}^k||\nabla\mathbf{u}_t^k| \leq C|\nabla\mathbf{u}_t^k|;$$

$$|(\mathbf{u}_t^k \cdot \nabla)\nabla\rho^k| \leq |\mathbf{u}_t^k|_4 |\nabla^2\rho^k|_4 \leq C|\nabla\mathbf{u}_t^k||\nabla\Delta\rho^k| \leq C|\nabla\mathbf{u}_t^k|;$$

$$|(\mathbf{u}^k \cdot \nabla)\nabla\rho_t^k| \leq |\mathbf{u}^k|_\infty |\nabla^2\rho_t^k| \leq C|A\mathbf{u}^k||\Delta\rho_t^k| \leq C|\Delta\rho_t^k|;$$

$$|(\nabla\rho_t^k \cdot \nabla)\mathbf{u}^k| \leq |\nabla\rho_t^k|_4 |\nabla\mathbf{u}^k|_4 \leq C|\Delta\rho_t^k||A\mathbf{u}^k| \leq C|\Delta\rho_t^k|;$$

$$|(\nabla\rho^k \cdot \nabla)\mathbf{u}_t^k| \leq |\nabla\rho^k|_\infty |\nabla\mathbf{u}_t^k| \leq C|\nabla\Delta\rho^k||\nabla\mathbf{u}_t^k| \leq C|\nabla\mathbf{u}_t^k|;$$

$$|\frac{\rho_t^k}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla)\nabla\rho^k| \le C|\rho_t^k|_4|\nabla\rho^k|_\infty|\nabla^2\rho^k|_4 \le C|\rho_t^k|_{_{H^1}}|\nabla\Delta\rho^k|^2 \le C;$$

$$|\frac{1}{\rho^k}(\nabla\rho_t^k\cdot\nabla)\nabla\rho^k| \le C|\nabla\rho_t^k|_4|\nabla^2\rho^k|_4 \le C|\Delta\rho_t^k||\nabla\Delta\rho^k| \le C|\Delta\rho_t^k|;$$

$$|\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho_t^k| \le C|\nabla\rho^k|_\infty|\Delta\rho_t^k| \le C|\nabla\Delta\rho^k||\Delta\rho_t^k| \le C|\Delta\rho_t^k|;$$

$$|\frac{2\rho_t^k}{(\rho^k)^3}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k| \le C|\rho_t^k||\nabla\rho^k|_\infty^3 \le C|\rho_t^k||\nabla\Delta\rho^k|^3 \le C;$$

$$|\frac{1}{(\rho^k)^2}(\nabla\rho_t^k\cdot\nabla\rho^k)\nabla\rho^k| \le C|\nabla\rho_t^k||\nabla\rho^k|_\infty^2 \le C|\nabla\rho_t^k||\nabla\Delta\rho^k|^2 \le C;$$

$$|\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho_t^k)\nabla\rho^k| \le C|\nabla\rho^k|_\infty^2|\nabla\rho_t^k| \le C|\nabla\Delta\rho^k|^2|\nabla\rho_t^k| \le C;$$

$$|\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho_t^k| \le C|\nabla\rho^k|_\infty^2|\nabla\rho_t^k| \le C|\nabla\Delta\rho^k|^2|\nabla\rho_t^k| \le C;$$

$$|\frac{\rho_t^k}{(\rho^k)^3}\Delta\rho^k\nabla\rho^k| \le C|\rho_t^k|_4|\Delta\rho^k|_4|\nabla\rho^k|_\infty \le C|\rho_t^k|_{_{H^1}}|\nabla\Delta\rho^k|^2 \le C;$$

$$|\frac{1}{\rho^k}\Delta\rho_t^k\nabla\rho^k| \le C|\Delta\rho_t^k||\nabla\rho^k|_\infty \le C|\Delta\rho_t^k||\nabla\Delta\rho^k| \le C|\Delta\rho_t^k|;$$

$$|\frac{1}{\rho^k}\Delta\rho^k\nabla\rho_t^k| \le C|\Delta\rho^k|_4|\nabla\rho_t^k|_4 \le C|\nabla\Delta\rho^k||\Delta\rho_t^k| \le C|\Delta\rho_t^k|;$$

$$|\rho_t^kF| \le |\rho_t^k|_4|F|_4 \le C|\rho_t^k|_{_{H^1}}|F|_{_{H^1}} \le C|F|_{_{H^1}};$$

$$|\rho^kF_t| \le C|F_t|.$$

Com estas estimativas vemos que

$$|h_t^k| \leq C(|\nabla \mathbf{u}_t^k| + |\Delta \rho_t^k| + |F|_{_{H^1}} + |F_t| + 1).$$

e assim,

$$\frac{2}{\mu}\int_0^t |(h_t^k, A\mathbf{u}^k)|ds \leq C|A\mathbf{u}^k|_{_{L^\infty(0,T^*)}}(\int_0^t |h_t^k|^2 ds)^{\frac{1}{2}} \cdot t^{\frac{1}{2}} \leq C \cdot t^{\frac{1}{2}}. \tag{2.26}$$

Além disso, vemos que:

$$\frac{2}{\mu}\int_0^t |(\rho^k \mathbf{u}_t^k, A\mathbf{u}_t^k)|ds = \frac{2}{\mu}\int_0^t |(\nabla \rho^k \mathbf{u}_t^k + \rho^k \nabla \mathbf{u}_t^k, \nabla \mathbf{u}_t^k)|ds$$

$$\leq \frac{2}{\mu}\int_0^t |(\nabla \rho^k \mathbf{u}_t^k, \nabla \mathbf{u}_t^k)|ds + \frac{2}{\mu}\int_0^t |(\rho^k \nabla \mathbf{u}_t^k, \nabla \mathbf{u}_t^k)|ds$$

$$\leq \frac{2}{\mu}\int_0^t |\nabla \rho^k|_\infty |\mathbf{u}_t^k||\nabla \mathbf{u}_t^k|ds + \frac{2}{\mu}\int_0^t |\rho^k|_\infty |\nabla \mathbf{u}_t^k|^2 ds$$

$$\leq C \cdot t^{\frac{1}{2}} + C\int_0^t |\nabla \mathbf{u}_t^k|^2 ds. \tag{2.27}$$

Assim, voltando à expressão (2.25) e usando as desigualdades (2.26) e (2.27) teremos:

$$|A\mathbf{u}^k(t)|^2 \leq |A\mathbf{u}^k(0)|^2 + C \cdot t^{\frac{1}{2}} + C\int_0^t |\nabla \mathbf{u}_t^k|^2 ds$$

$$+\frac{2}{\mu}|(h^k(t), A\mathbf{u}^k(t)) - (h^k(0), A\mathbf{u}^k(0))| \tag{2.28}$$

Definindo

$$h(t) = (-(\rho \mathbf{u} \cdot \nabla)\mathbf{u} + \lambda(\mathbf{u} \cdot \nabla)\nabla \rho + \lambda(\nabla \rho \cdot \nabla)\mathbf{u} - \lambda^2 \frac{1}{\rho}(\nabla \rho \cdot \nabla)\nabla \rho$$

$$+\lambda^2 \frac{1}{(\rho)^2}(\nabla \rho \cdot \nabla \rho)\nabla \rho - \lambda^2 \frac{1}{\rho}\Delta \rho \nabla \rho - \rho F)(t)$$

para todo $t \in [0, T^*]$, teremos que $h^k(t) \to h(t)$ fracamente em $L^2(\Omega)$, para cada t fixado e fazendo $k \to +\infty$, obteremos:

$$\liminf_{k \to +\infty} |A\mathbf{u}^k(t)|^2 \leq \liminf_{k \to +\infty} |A\mathbf{u}^k(0)|^2 + C \cdot t^{\frac{1}{2}} + C \liminf_{k \to +\infty} \int_0^t |\nabla \mathbf{u}_t^k|^2 ds$$

$$+\frac{2}{\mu}\liminf_{k\to+\infty}|(h^k(t),A\mathbf{u}^k(t))-(h^k(0),A\mathbf{u}^k(0))|$$

$$\Longrightarrow |A\mathbf{u}(t)|^2\leq |A\mathbf{u}(0)|^2+C\cdot t^{\frac{1}{2}}+C\int_0^t|\nabla\mathbf{u}_t|^2ds+\frac{2}{\mu}|(h(t),A\mathbf{u}(t))-(h(0),A\mathbf{u}(0))|.$$

Em seguida, considerando que $h(t)\to h(0)$ fortemente em $L^2(\Omega)$ e $A\mathbf{u}(t)\to A\mathbf{u}_\circ$ fracamente em $L^2(\Omega)$, concluiremos que

$$\limsup_{t\to0^+}|A\mathbf{u}(t)|\leq |A\mathbf{u}_\circ|.$$

Portanto, sabendo que $A\mathbf{u}(t)\to A\mathbf{u}_\circ$ fracamente em $L^2(\Omega)$, que $\lim\limits_{t\to0^+}|A\mathbf{u}(t)|=|A\mathbf{u}_\circ|$ e que o espaço $L^2(\Omega)$ é uniformemente convexo, podemos inferir que

$$\lim_{t\to0^+}|A\mathbf{u}(t)-A\mathbf{u}_\circ|=0,$$

ou seja, a velocidade solução $\mathbf{u}$ assume seu dado inicial continuamente em $\mathbf{D}(\mathbf{A})$. De maneira inteiramente análoga, mostra-se que a densidade solução ρ assume também seu dado inicial continuamente em $\mathbf{H^3_N}$.

Parte 4: unicidade de solução.

Sejam $(\mathbf{u},\rho)$ e $(\mathbf{u}^1,\rho^1)$ duas soluções do problema inicial (2.7); definamos $z=\rho-\rho^1$ e $\mathbf{w}=\mathbf{u}-\mathbf{u}^1$.

Usando a segunda equação de (2.7), temos:

$$(z_t+\mathbf{u}\cdot\nabla z+\mathbf{w}\cdot\nabla\rho^1-\lambda\Delta z,\psi)=0,$$

para toda $\psi\in L^2(Q_{T^*})$.

Em particular, fazendo $\psi=z$ teremos:

$$\frac{1}{2}\frac{d}{dt}|z|^2=\lambda(\Delta z,z)-(\mathbf{w}\cdot\nabla\rho^1,z)$$

já que $(\mathbf{u}\cdot\nabla z,z)=0$. Os dois termos à direita na expressão acima podem ser assim estimados:

$$|(\Delta z,z)|\leq C|z|^2+\delta|\Delta z|^2;$$

$$|(\mathbf{w}\cdot\nabla\rho^1,z)|\leq|\mathbf{w}||\nabla\rho^1|_\infty|z|\leq C|\mathbf{w}|^2+C|z|^2.$$

Assim,

$$\frac{1}{2}\frac{d}{dt}|z|^2 \leq C|\mathbf{w}|^2 + C|z|^2 + \delta|\Delta z|^2 \tag{2.29}$$

Por outro lado, fazendo $\psi = -\Delta z$ obtemos:

$$-(z_t, \Delta z) + \lambda(\Delta z, \Delta z) - (\mathbf{u} \cdot \nabla z, \Delta z) - (\mathbf{w} \cdot \nabla \rho^1, \Delta z) = 0.$$

E isto implica

$$\frac{1}{2}\frac{d}{dt}|\nabla z|^2 + \lambda|\Delta z|^2 = (\mathbf{u} \cdot \nabla z, \Delta z) + (\mathbf{w} \cdot \nabla \rho^1, \Delta z)$$

$$\leq |\mathbf{u}|_\infty |\nabla z||\Delta z| + |\mathbf{w}||\nabla \rho^1|_\infty |\Delta z|$$

$$\leq 2\varepsilon|\Delta z|^2 + C|A\mathbf{u}|^2|\nabla z|^2 + C|\rho^1|^2_{H^3_N}|\mathbf{w}|^2$$

Depois de escolher $\varepsilon = \dfrac{\lambda}{4}$ e levando em conta as limitações de $|A\mathbf{u}|$ e $|\rho^1|_{H^3}$, a desigualdade acima toma a seguinte forma:

$$\frac{1}{2}\frac{d}{dt}|\nabla z|^2 + \frac{\lambda}{2}|\Delta z|^2 \leq C|z|^2_{H^1} + C|\mathbf{w}|^2. \tag{2.30}$$

Consideramos então a primeira equação de (2.7) aplicada em $(\mathbf{u}, \rho)$ e $(\mathbf{u}^1, \rho^1)$.

Fazendo a diferença entre elas e tomando o produto interno com $v = \mathbf{w}$ obtemos:

$$(\rho \mathbf{w}_t, \mathbf{w}) + \mu(A\mathbf{w}, \mathbf{w}) = -(z\mathbf{u}^1_t, \mathbf{w}) - ((z\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{w})$$

$$-((\rho^1 \mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{w}) - ((\rho^1 \mathbf{u}^1 \cdot \nabla)\mathbf{w}, \mathbf{w}) + \lambda((\mathbf{u} \cdot \nabla)\nabla z, \mathbf{w}) + \lambda((\mathbf{w} \cdot \nabla)\nabla \rho^1, \mathbf{w})$$

$$+\lambda((\nabla z \cdot \nabla)\mathbf{u}, \mathbf{w}) + \lambda((\nabla \rho^1 \cdot \nabla)\mathbf{w}, \mathbf{w}) + (zF, \mathbf{w})$$

$$-\lambda^2\{(-\frac{z}{\rho\rho^1}(\nabla\rho \cdot \nabla)\nabla\rho + \frac{1}{\rho^1}(\nabla z \cdot \nabla)\nabla\rho + \frac{1}{\rho^1}(\nabla\rho^1 \cdot \nabla)\nabla z$$

$$+\frac{z(\rho^1 + \rho)}{(\rho\rho^1)^2}(\nabla\rho \cdot \nabla\rho)\nabla\rho - \frac{1}{(\rho^1)^2}(\nabla z \cdot \nabla\rho)\nabla\rho - \frac{1}{(\rho^1)^2}(\nabla\rho^1 \cdot \nabla z)\nabla\rho$$

$$-\frac{1}{(\rho^1)^2}(\nabla\rho^1 \cdot \nabla\rho^1)\nabla z - \frac{z}{\rho\rho^1}\Delta\rho\nabla\rho + \frac{1}{\rho^1}\Delta z\nabla\rho + \frac{1}{\rho^1}\Delta\rho^1\nabla z, \mathbf{w})\}.$$

Mas,

$$
\begin{aligned}
(\rho \mathbf{w}_t, \mathbf{w}) &= \frac{1}{2}\frac{d}{dt}(\rho \mathbf{w}, \mathbf{w}) - \frac{1}{2}(\rho_t \mathbf{w}, \mathbf{w}) \\
&= \frac{1}{2}\frac{d}{dt}(\rho \mathbf{w}, \mathbf{w}) + \frac{1}{2}((\mathbf{u}\cdot\nabla\rho)\mathbf{w}, \mathbf{w}) - \frac{\lambda}{2}(\Delta\rho \mathbf{w}, \mathbf{w});
\end{aligned}
$$

logo,

$$\frac{1}{2}\frac{d}{dt}(\rho \mathbf{w}, \mathbf{w}) + \mu|\nabla \mathbf{w}|^2 = -\frac{1}{2}((\mathbf{u}\cdot\nabla\rho)\mathbf{w}, \mathbf{w}) + \frac{\lambda}{2}(\Delta\rho \mathbf{w}, \mathbf{w})$$

$$+(zF, \mathbf{w}) - (z\mathbf{u}^1_t, \mathbf{w}) - ((z\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{w}) - ((\rho^1\mathbf{w}\cdot\nabla)\mathbf{u}, \mathbf{w}) - ((\rho^1\mathbf{u}^1\cdot\nabla)\mathbf{w}, \mathbf{w})$$

$$+\lambda\{((\mathbf{u}\cdot\nabla)\nabla z, \mathbf{w}) + ((\mathbf{w}\cdot\nabla)\nabla\rho^1, \mathbf{w}) + ((\nabla z\cdot\nabla)\mathbf{u}, \mathbf{w}) + ((\nabla\rho^1\cdot\nabla)\mathbf{w}, \mathbf{w})\}$$

$$-\lambda^2\{(-\frac{z}{\rho\rho^1}(\nabla\rho\cdot\nabla)\nabla\rho + \frac{1}{\rho^1}(\nabla z\cdot\nabla)\nabla\rho + \frac{1}{\rho^1}(\nabla\rho^1\cdot\nabla)\nabla z$$

$$+\frac{z(\rho^1+\rho)}{(\rho\rho^1)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho - \frac{1}{(\rho^1)^2}(\nabla z\cdot\nabla\rho)\nabla\rho - \frac{1}{(\rho^1)^2}(\nabla\rho^1\cdot\nabla z)\nabla\rho$$

$$-\frac{1}{(\rho^1)^2}(\nabla\rho^1\cdot\nabla\rho^1)\nabla z - \frac{z}{\rho\rho^1}\Delta\rho\nabla\rho + \frac{1}{\rho^1}\Delta z\nabla\rho + \frac{1}{\rho^1}\Delta\rho^1\nabla z, \mathbf{w})\}.$$

Estimamos os termos do lado direito como segue:

$$\frac{1}{2}|((\mathbf{u}\cdot\nabla\rho)\mathbf{w}, \mathbf{w})| \leq C|\mathbf{u}|_\infty|\nabla\rho|_\infty|\mathbf{w}|^2 \leq C|A\mathbf{u}||\rho|_{H^3_N}|\mathbf{w}|^2;$$

$$
\begin{aligned}
\frac{\lambda}{2}|(\Delta\rho \mathbf{w}, \mathbf{w})| &= \frac{\lambda}{2}|(\nabla\rho, \nabla(\mathbf{w}^2))| \\
&\leq C|\nabla\rho||\mathbf{w}||\nabla\mathbf{w}| \\
&\leq \varepsilon|\nabla\mathbf{w}|^2 + C|\nabla\rho|^2|\mathbf{w}|^2;
\end{aligned}
$$

$$|(z\mathbf{u}^1_t, \mathbf{w})| \leq |z|_\infty|\mathbf{u}^1_t||\mathbf{w}| \leq \delta|\Delta z|^2 + C|\mathbf{u}^1_t|^2|\mathbf{w}|^2;$$

$$|((z\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{w})| \leq C|z|_\infty|\mathbf{u}|_\infty|\nabla\mathbf{u}||\mathbf{w}|$$

$$\leq \delta|\Delta z|^2 + C|A\mathbf{u}|^2|\nabla\mathbf{u}|^2|\mathbf{w}|^2;$$

$$|((\rho^1\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{w})| \leq |\rho^1|_\infty|\nabla\mathbf{w}||A\mathbf{u}||\mathbf{w}| \leq \varepsilon|\nabla\mathbf{w}|^2 + C|A\mathbf{u}|^2|\mathbf{w}|^2;$$

$$|((\rho^1\mathbf{u}^1\cdot\nabla)\mathbf{w},\mathbf{w})| \leq |\rho^1|_\infty|A\mathbf{u}^1||\nabla\mathbf{w}||\mathbf{w}| \leq \varepsilon|\nabla\mathbf{w}|^2 + C|A\mathbf{u}^1|^2|\mathbf{w}|^2;$$

$$|\lambda((\mathbf{u}\cdot\nabla)\nabla z,\mathbf{w})| \leq C|A\mathbf{u}||\Delta z||\mathbf{w}| \leq \delta|\Delta z|^2 \ + C|A\mathbf{u}|^2|\mathbf{w}|^2;$$

$$|\lambda((\mathbf{w}\cdot\nabla)\nabla\rho^1,\mathbf{w})| \leq \lambda|\nabla\mathbf{w}||\rho^1|_{H^3_N}|\mathbf{w}| \leq \varepsilon|\nabla\mathbf{w}|^2 + C|\rho^1|^2_{H^3_N}|\mathbf{w}|^2;$$

$$|\lambda((\nabla z\cdot\nabla)\mathbf{u},\mathbf{w})| \leq C|\Delta z||A\mathbf{u}||\mathbf{w}| \leq \delta|\Delta z|^2 + C|A\mathbf{u}|^2|\mathbf{w}|^2;$$

$$|\lambda((\nabla\rho^1\cdot\nabla)\mathbf{w},\mathbf{w})| \leq C|\rho^1|_{H^3_N}|\nabla\mathbf{w}||\mathbf{w}| \leq \varepsilon|\nabla\mathbf{w}|^2 + C|\rho^1|^2_{H^3_N}|\mathbf{w}|^2;$$

$$|(zF,\mathbf{w})| \leq C|z|_{H^1}|F||\nabla\mathbf{w}| \leq \varepsilon|\nabla\mathbf{w}|^2 + C|F|^2|z|^2_{H^1}.$$

$$\begin{aligned}|\lambda^2(\frac{z}{\rho\rho^1}(\nabla\rho\cdot\nabla)\nabla\rho,\mathbf{w})| &\leq C|z|_6|\nabla\rho|_6|\nabla^2\rho||\mathbf{w}|_6\\ &\leq C|z|_{H^1}|\Delta\rho|^2|\nabla\mathbf{w}|\\ &\leq C_\varepsilon|\Delta\rho|^4|z|^2_{H^1} + \varepsilon|\nabla\mathbf{w}|^2;\end{aligned}$$

$$\begin{aligned}|\lambda^2(\frac{1}{\rho^1}(\nabla z\cdot\nabla)\nabla\rho,\mathbf{w})| &\leq C|\nabla z||\nabla^2\rho|_4|\mathbf{w}|_4\\ &\leq C|\nabla z||\nabla\Delta\rho||\nabla\mathbf{w}|\\ &\leq C_\varepsilon|\nabla\Delta\rho|^2|\nabla z|^2 + \varepsilon|\nabla\mathbf{w}|^2;\end{aligned}$$

$$\begin{aligned}&|\lambda^2(\frac{1}{\rho^1}(\nabla\rho^1\cdot\nabla)\nabla z,\mathbf{w})| = \Big|\sum_{i,j}\int_\Omega\frac{1}{\rho^1}\frac{\partial\rho^1}{\partial x_i}\frac{\partial}{\partial x_i}(\frac{\partial z}{\partial x_j})\cdot\mathbf{w}_j\Big|\\ &= \Big|-\sum_{i,j}\int_\Omega\frac{\partial z}{\partial x_i}\cdot\frac{\partial}{\partial x_j}(\frac{1}{\rho^1}\frac{\partial\rho^1}{\partial x_i}\mathbf{w}_j) + \sum_{i,j}\int_\Gamma\frac{1}{\rho^1}\frac{\partial\rho^1}{\partial x_i}\mathbf{w}_j\frac{\partial z}{\partial x_i}\cdot\mathbf{n}_i\Big|\\ &= \Big|-\sum_{i,j}\int_\Omega\frac{\partial z}{\partial x_i}\Big[\frac{1}{(\rho^1)^2}\frac{\partial\rho^1}{\partial x_j}\frac{\partial\rho^1}{\partial x_i}\mathbf{w}_j + \frac{1}{\rho^1}\frac{\partial^2\rho^1}{\partial x_j\partial x_i}\mathbf{w}_j + \frac{1}{\rho^1}\frac{\partial\rho^1}{\partial x_i}\frac{\partial\mathbf{w}_j}{\partial x_j}\Big]\Big|\\ &\leq |-\sum_{i,j}\int_\Omega\frac{\partial z}{\partial x_i}\frac{1}{(\rho^1)^2}\frac{\partial\rho^1}{\partial x_j}\frac{\partial\rho^1}{\partial x_i}\mathbf{w}_j| + |\sum_{i,j}\int_\Omega\frac{\partial z}{\partial x_i}\frac{1}{\rho^1}\frac{\partial^2\rho^1}{\partial x_j\partial x_i}\mathbf{w}_j|\\ &\leq C|\nabla z||\nabla\rho^1|^2_6|\mathbf{w}|_6 + C|\nabla z||\Delta\rho^1|_4|\mathbf{w}|_4\\ &\leq C|\nabla z||\Delta\rho^1|^2|\nabla\mathbf{w}| + C|\nabla z||\nabla\Delta\rho^1||\nabla\mathbf{w}|\\ &\leq C_\varepsilon(|\nabla\Delta\rho^1|^2 + |\Delta\rho^1|^4)|\nabla z|^2 + 2\varepsilon|\nabla\mathbf{w}|^2;\end{aligned}$$

$$
\begin{aligned}
|\lambda^2(\frac{z(\rho^1+\rho)}{(\rho\rho^1)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho,\mathbf{w})| &\le C|z|_4|\nabla\rho|_6^3|\mathbf{w}|_4\\
&\le C|z|_{_{H^1}}|\Delta\rho|^3|\nabla\mathbf{w}|\\
&\le C_\varepsilon|\Delta\rho|^6|z|^2_{_{H^1}}+\varepsilon|\nabla\mathbf{w}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{\lambda^2}{(\rho^1)^2}(\nabla z\cdot\nabla\rho)\nabla\rho,\mathbf{w})| &\le C\frac{1}{|\rho^1|^2_\infty}|\nabla z||\nabla\rho|_6^2|\mathbf{w}|_6\\
&\le C|\nabla z||\Delta\rho|^2|\nabla\mathbf{w}|\\
&\le C_\varepsilon|\Delta\rho|^4|\nabla z|^2+\varepsilon|\nabla\mathbf{w}|^;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{\lambda^2}{(\rho^1)^2}(\nabla\rho^1\cdot\nabla z)\nabla\rho,\mathbf{w})| &\le C\frac{1}{|\rho^1|^2_\infty}|\nabla\rho^1|_6|\nabla z||\nabla\rho|_6|\mathbf{w}|_6\\
&\le C|\Delta\rho^1||\nabla z||\Delta\rho||\nabla\mathbf{w}|\\
&\le C_\varepsilon|\Delta\rho^1|^2|\Delta\rho|^2|\nabla z|^2+\varepsilon|\nabla\mathbf{w}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{\lambda^2}{(\rho^1)^2}(\nabla\rho^1\cdot\nabla\rho^1)\nabla z,\mathbf{w})| &\le C\frac{1}{|\rho^1|^2_\infty}|\nabla\rho^1|_6^2|\nabla z||\mathbf{w}|_6\\
&\le C|\Delta\rho^1|^2|\nabla z||\nabla\mathbf{w}|\\
&\le C_\varepsilon|\Delta\rho^1|^4|\nabla z|^2+\varepsilon|\nabla\mathbf{w}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{z}{\rho\rho^1}\Delta\rho\nabla\rho,\mathbf{w})| &\le \frac{C}{|\rho\rho^1|_\infty}|z|_6|\Delta\rho||\nabla\rho|_6|\mathbf{w}|_6\\
&\le C|z|_{_{H^1}}|\Delta\rho|^2|\nabla\mathbf{w}|\\
&\le C_\varepsilon|\Delta\rho|^4|z|^2_{_{H^1}}+\varepsilon|\nabla\mathbf{w}|^2;
\end{aligned}
$$

$$
\begin{aligned}
&|(\frac{1}{\rho^1}\Delta z\nabla\rho,\mathbf{w})| = \Big|\sum_{i,j}\int_\Omega\frac{\partial^2 z}{\partial x_i^2}\frac{1}{\rho^1}\frac{\partial\rho}{\partial x_j}\mathbf{w}_j\Big|\\
&= \Big|-\sum_{i,j}\int_\Omega\frac{\partial z}{\partial x_i}\frac{\partial}{\partial x_i}(\frac{1}{\rho^1}\frac{\partial\rho}{\partial x_j}\mathbf{w}_j)+\sum_{i,j}\int_\Gamma(\frac{1}{\rho^1}\frac{\partial\rho}{\partial x_j}\mathbf{w}_j)\frac{\partial z}{\partial x_i}\cdot\mathbf{n}_i\Big|\\
&\Big|-\sum_{i,j}\int_\Omega\frac{\partial z}{\partial x_i}[\frac{1}{(\rho^1)^2}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}\mathbf{w}_j+\frac{1}{\rho^1}\frac{\partial^2\rho}{\partial x_i\partial x_j}\mathbf{w}_j+\frac{1}{\rho^1}\frac{\partial\rho}{\partial x_j}\frac{\partial\mathbf{w}_j}{\partial x_i}]\Big|\\
&\le \frac{C}{|\rho^1|^2_\infty}|\nabla z||\nabla\rho|_6^2|\mathbf{w}|_6+\frac{C}{|\rho^1|_\infty}|\nabla z||\Delta^2\rho|_4|\mathbf{w}|_4+\frac{C}{|\rho^1|_\infty}|\nabla z||\nabla\rho|_\infty|\nabla\mathbf{w}|\\
&\le C|\nabla z||\Delta\rho|^2|\nabla\mathbf{w}|+C|\nabla z||\nabla\Delta\rho||\nabla\mathbf{w}|\\
&\le C_\varepsilon(|\Delta\rho|^4+|\nabla\Delta\rho|^2)|\nabla z|^2+2\varepsilon|\nabla\mathbf{w}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\frac{1}{\rho^1}\Delta\rho^1\nabla z, \mathbf{w})| &\leq \frac{C}{|\rho^1|_\infty}|\Delta\rho^1|_4|\nabla z||\mathbf{w}|_4 \\
&\leq C|\nabla\Delta\rho^1||\nabla z||\nabla\mathbf{w}| \\
&\leq C_\varepsilon|\nabla\Delta\rho^1|^2|\nabla z|^2 + \varepsilon|\nabla\mathbf{w}|^2.
\end{aligned}
$$

Então,

$$
\frac{1}{2}\frac{d}{dt}(\rho\mathbf{w}, \mathbf{w}) + (\mu - 19\varepsilon)|\nabla\mathbf{w}|^2 \leq C_{\varepsilon,\delta}\{|A\mathbf{u}||\rho|_{H^3_N} + |\nabla\rho|^2 + |\mathbf{u}^1_t|^2
$$

$$
+|A\mathbf{u}|^2|\nabla\mathbf{u}|^2 + |A\mathbf{u}|^2 + |A\mathbf{u}^1|^2 + |\rho^1|^2_{H^3_N}\}|\mathbf{w}|^2
$$

$$
+C_\varepsilon\{|F|^2 + |\Delta\rho|^4 + |\Delta\rho|^6 + |\rho|^2_{H^3_N} + |\rho^1|^2_{H^3_N}
$$

$$
+|\Delta\rho^1|^4 + |\Delta\rho^1|^2|\Delta\rho|^2\}|z|^2_{H^1} + 4\delta|\Delta z|^2,
$$

e usando as estimativas obtidas na parte 1, vemos que a desigualdade acima fica na forma:

$$
\frac{1}{2}\frac{d}{dt}(\rho\mathbf{w}, \mathbf{w}) + (\mu - 19\varepsilon)|\nabla\mathbf{w}|^2 \leq C|\mathbf{w}|^2 + C|z|^2_{H^1} + 4\delta|\Delta z|^2 \tag{2.31}
$$

Somando-se as desigualdades (2.29),
(2.30) e (2.31) teremos:

$$
\frac{1}{2}\frac{d}{dt}\{(\rho\mathbf{w}, \mathbf{w}) + |z|^2_{H^1}\} + (\mu - 19\varepsilon)|\nabla\mathbf{w}|^2 + (\frac{\lambda}{2} - 5\delta)|\Delta z|^2
$$

$$
\leq C|\mathbf{w}|^2 + C|z|^2_{H^1}.
$$

Escolhendo $\varepsilon = \dfrac{\mu}{38}$ e $\delta = \dfrac{\lambda}{20}$ segue-se que:

$$
\frac{1}{2}\frac{d}{dt}\{(\rho\mathbf{w}, \mathbf{w}) + |z|^2_{H^1}\} + \frac{\mu}{2}|\nabla\mathbf{w}|^2 + \frac{\lambda}{4}|\Delta z|^2
$$

$$
\leq C|\mathbf{w}|^2 + C|z|^2_{H^1},
$$

Como $|\mathbf{w}|^2 \leq \dfrac{1}{\alpha}(\rho\mathbf{w}, \mathbf{w})$, temos:

$$\frac{d}{dt}\{(\rho\mathbf{w},\mathbf{w}) + |z|^2_{H^1}\} \leq C((\rho\mathbf{w},\mathbf{w}) + |z|^2_{H^1}),$$

e integrando de 0 a t, com $t \in [0, T^*]$:

$$(\rho\mathbf{w},\mathbf{w}) + |z|^2_{H^1} \leq (\rho_\circ\mathbf{w}(0),\mathbf{w}(0)) + |z(0)|^2_{H^1}$$

$$+C\int_0^t ((\rho(s)\mathbf{w}(s),\mathbf{w}(s)) + |z(s)|^2_{H^1})ds.$$

Agora, usando o lema de Gronwall obtemos:

$$\begin{aligned}(\rho\mathbf{w},\mathbf{w}) + |z|^2_{H^1} &\leq [(\rho_\circ\mathbf{w}(0),\mathbf{w}(0)) + |z(0)|^2_{H^1}] \cdot e^{C\int_0^{T^*} ds} \\ &\leq C[(\rho_\circ\mathbf{w}(0),\mathbf{w}(0)) + |z(0)|^2_{H^1}].\end{aligned}$$

Porque $\mathbf{w}(0) = 0$, $|z(0)|^2_{H^1} = 0$ concluímos que $(\rho\mathbf{w},\mathbf{w}) = 0$ e $|z|^2_{H^1} = 0$ (z é constante); além disso, $z(0) = 0$ e $\rho > 0$ implicam $\mathbf{w} = 0$ e $z = 0$.

Observação. Convém notar que H. Beirão da Veiga provou em [4] a regularidade da solução do problema (2.6) (com termos envolvendo λ^2) usando linearização e argumentos de ponto fixo; além disso, assume-se que $\mathbf{u}_0 \in V$, o que naturalmente fornece estimativas mais fracas que aquelas aqui obtidas.

Capítulo 3

Taxas de Convergência para as Aproximações Semi-Galerkin

3.1 Introdução

Este capítulo é dedicado inteiramente à investigação das taxas de convergência nas normas de L^2 e de H^1, das aproximações semi-Galerkin espectral para a solução do problema completo. Mostramos que é possível obter estimativas de erro uniformes e também, estimativas melhoradas na norma de L^2.

3.2 Estimativas de erro otimais na norma H^1 para a velocidade

Nesta seção provaremos algumas estimativas de erro otimais locais no tempo para a velocidade na norma H^1 e para a densidade na norma H^2. Aqui, por otimalidade entendemos que as soluções aproximadas convergem para a solução do problema (2.2) na melhor taxa possível, medida por potências do inverso do autovalor λ_{k+1} do operador de Stokes, considerando-se as aproximações no subespaço V_k. Começaremos trabalhando como em Heywood [18] no caso das equações de Navier-Stokes clássicas. Para tal, definimos:

Definição 3.1 Sejam $(\mathbf{u}, \rho)$ a solução forte do problema (2.2) dada pelo Teorema 2.1 e $(\mathbf{u}^k, \rho^k)$ a solução do problema aproximado de nível k. Definimos então:

$$\begin{array}{ll} i) & \mathbf{v}^k = P_k \mathbf{u} \\ ii) & \theta^k = \mathbf{v}^k - \mathbf{u}^k \\ iii) & E^k = \mathbf{u} - \mathbf{v}^k \\ iv) & \pi^k = \rho - \rho^k \end{array} \tag{3.32}$$

No que se segue denotaremos por $\bar{c} > 0$ uma constante positiva independente de k que pode depender das funções F_i dadas no Teorema 2.1.

Para as variáveis acima definidas vale o seguinte resultado:

Lema 3.1

$$|\theta^k(t)|^2 + |\pi^k(t)|^2 + C_0 \int_0^t (|\nabla \theta^k(s)|^2 + |\nabla \pi^k(s)|^2) ds$$

$$\leq \frac{\overline{c}}{\lambda_{k+1}^2} + \frac{\overline{c}}{\lambda_{k+1}}.$$

Demonstração. Observamos que $\mathbf{v}^k = P_k\mathbf{u}$ satisfaz:

$$\begin{aligned} & P_k(\rho\mathbf{u}_t + \rho\mathbf{u}.\nabla\mathbf{u} - \rho F) - \lambda P_k((\mathbf{u}.\nabla)\nabla\rho + (\nabla\rho.\nabla)\mathbf{u}) + \mu A\mathbf{v}^k \\ & \quad +\lambda^2 P_k(\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \frac{1}{\rho}\Delta\rho\nabla\rho) = 0. \end{aligned} \tag{3.33}$$

Subtraindo de (3.33) a primeira equação em (2.3) obtemos:

$$\begin{aligned} & P_k[\pi^k(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - F) + \rho^k E_t^k + (\rho^k\theta^k\cdot\nabla)\mathbf{u} \\ & +(\rho^k E^k\cdot\nabla)\mathbf{u} + (\rho^k\mathbf{u}^k\cdot\nabla)\theta^k + (\rho^k\mathbf{u}^k\cdot\nabla)E^k] + P_k(\rho^k\theta_t^k) + \mu A\theta^k \\ & -\lambda P_k[(\theta^k\cdot\nabla)\nabla\rho + (E^k\cdot\nabla)\nabla\rho + (\mathbf{u}^k\cdot\nabla)\nabla\pi^k \\ & +(\nabla\pi^k\cdot\nabla)\mathbf{u} + (\nabla\rho^k\cdot\nabla)\theta^k + (\nabla\rho^k\cdot\nabla)E^k] \\ & -\lambda^2 P_k[-\frac{\pi^k}{\rho\rho^k}(\nabla\rho\cdot\nabla)\nabla\rho + \frac{1}{\rho^k}(\nabla\pi^k\cdot\nabla)\nabla\rho + \frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\pi^k] \\ & -\lambda^2 P_k[\pi^k\frac{\rho+\rho^k}{(\rho\rho^k)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho - \frac{1}{(\rho^k)^2}(\nabla\pi^k\cdot\nabla\rho)\nabla\rho] \\ & -\lambda^2 P_k[-\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\pi^k)\nabla\rho - \frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\pi^k] \\ & -\lambda^2 P_k[\frac{\pi^k}{\rho\rho^k}\Delta\rho\nabla\rho + \frac{1}{\rho^k}\Delta\pi^k\nabla\rho + \frac{1}{\rho^k}\Delta\rho^k\nabla\pi^k] = 0. \end{aligned} \tag{3.34}$$

Tomando o produto interno em $L^2(\Omega)$ da igualdade acima com $\nu\theta^k$ ($\nu > 0$, a ser escolhido posteriormente), obtemos:

$$\begin{aligned} & (\rho^k\theta_t^k, \nu\theta^k) + \mu\nu|\nabla\theta^k|^2 \\ = \;& -(\pi^k(\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} - F), \nu\theta^k) - (\rho^k E_t^k, \nu\theta^k) \\ & -(\rho^k\theta^k.\nabla\mathbf{u}, \nu\theta^k) - (\rho^k E^k.\nabla\mathbf{u}, \nu\theta^k) \\ & -(\rho^k\mathbf{u}^k.\nabla\theta^k, \nu\theta^k) - (\rho^k\mathbf{u}^k.\nabla E^k, \nu\theta^k) \\ & +\lambda((\theta^k.\nabla)\nabla\rho, \nu\theta^k) + \lambda((E^k.\nabla)\nabla\rho, \nu\theta^k) \\ & +\lambda((\mathbf{u}^k.\nabla)\nabla\pi^k, \nu\theta^k) + \lambda((\nabla\pi^k.\nabla)\mathbf{u}, \nu\theta^k) \\ & +\lambda((\nabla\rho^k.\nabla)\theta^k, \nu\theta^k) + \lambda((\nabla\rho^k.\nabla)E^k, \nu\theta^k) \\ & +(\lambda^2[-\frac{\pi^k}{\rho\rho^k}(\nabla\rho\cdot\nabla)\nabla\rho + \frac{1}{\rho^k}(\nabla\pi^k\cdot\nabla)\nabla\rho + \frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\pi^k], \nu\theta^k) \\ & +(\lambda^2[\pi^k\frac{\rho+\rho^k}{(\rho\rho^k)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho - \frac{1}{(\rho^k)^2}(\nabla\pi^k\cdot\nabla\rho)\nabla\rho], \nu\theta^k) \end{aligned} \tag{3.35}$$

$$
\begin{aligned}
&+(\lambda^2[-\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\pi^k)\nabla\rho-\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\pi^k],\nu\theta^k)\\
&+(\lambda^2[\frac{\pi^k}{\rho\rho^k}\Delta\rho\nabla\rho+\frac{1}{\rho^k}\Delta\pi^k\nabla\rho+\frac{1}{\rho^k}\Delta\rho^k\nabla\pi^k],\nu\theta^k).
\end{aligned}
$$

Observamos que:

$$
\begin{aligned}
(\rho^k\theta_t^k,\nu\theta^k) &= \frac{\nu}{2}\frac{d}{dt}|(\rho^k)^{1/2}\theta^k|^2-\frac{\nu}{2}(\rho_t^k\theta^k,\theta^k);\\
\left|\frac{\nu}{2}(\rho_t^k\theta^k,\theta^k)\right| &= |\frac{\nu}{2}\lambda(\Delta\rho^k\theta^k,\theta^k)-\frac{\nu}{2}(\mathbf{u}^k\cdot\nabla\rho^k\theta^k,\theta^k)\\
&\leq c_\varepsilon|\Delta\rho^k||\theta^k|_6|\theta^k|_3+C|A\mathbf{u}^k||\nabla\rho^k|_6|\theta^k|_3|\theta^k|\\
&\leq C|\theta^k|^2+2\varepsilon|\nabla\theta^k|^2;\\
|(\rho^kE_t^k,\nu\theta^k)| &\leq \frac{1}{2\nu}|\rho^k|_\infty^2|E_t^k|^2+\frac{\nu}{2}|\theta^k|^2.
\end{aligned}
$$

Além disso, usando as desigualdades de Hölder e de Young , obtemos para todo $\varepsilon>0$,

$$
\begin{aligned}
|((\rho^k\theta^k\cdot\nabla)\mathbf{u},\nu\theta^k)| &\leq \nu|\rho^k|_\infty|\theta^k|\;|\nabla\mathbf{u}|_4|\theta^k|_4\\
&\leq c_\varepsilon|\rho^k|_\infty^2|A\mathbf{u}|^2\;|\theta^k|^2+\varepsilon|\nabla\theta^k|^2,
\end{aligned}
$$

graças à imersão de Sobolev $H^1(\Omega)\hookrightarrow L^4(\Omega)$, e à equivalência das normas $|\mathbf{u}|_{H^2}$ e $|A\mathbf{u}|$.

Analogamente, obtemos:

$$
|((\rho^kE^k\cdot\nabla)\mathbf{u},\nu\theta^k)|\leq C_\varepsilon|\rho^k|_\infty^2|A\mathbf{u}|^2|E^k|^2+\varepsilon|\nabla\theta^k|^2.
$$

Usando novamente as desigualdades de Hölder e de Young juntamente com a imersão de Sobolev $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, obtemos:

$$
\begin{aligned}
|(\rho^k\mathbf{u}^k.\nabla\theta^k,\nu\theta^k)| &\leq \nu|\rho^k|_\infty|\mathbf{u}^k|_\infty|\nabla\theta^k|\;|\theta^k|\\
&\leq c_\varepsilon|\rho^k|_\infty^2|A\mathbf{u}^k|^2\;|\theta^k|^2+\varepsilon|\nabla\theta^k|^2.
\end{aligned}
$$

Observamos, ainda, que:

$$
\begin{aligned}
((\rho^k\mathbf{u}^k\cdot\nabla)E^k,\nu\theta^k)| &= \nu\left|\sum_{i,j}\int_\Omega\rho^k\mathbf{u}_j^k\frac{\partial E_i^k}{\partial x_j}\theta_i^k\right|\\
&= \nu\left|\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_j}(\rho^k\mathbf{u}_j^k\theta_i^k)E_i^k\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \nu\left|\sum_{i,j}\int_\Omega \frac{\partial\rho^k}{\partial x_j}\mathbf{u}_j^k\theta_i^k E_i^k\right| + \nu\left|\sum_{i,j}\int_\Omega \rho^k\mathbf{u}_j^k\frac{\partial\theta_i^k}{\partial x_j}E_i^k\right| \\
&\leq \nu|\nabla\rho^k|_\infty|\mathbf{u}^k|_4|E^k||\theta^k|_4 \\
&\quad +\nu|\rho^k|_\infty|\mathbf{u}^k|_\infty|\nabla\theta^k|\ |E^k| \\
&\leq c_\varepsilon|\nabla\rho^k|_\infty^2|\nabla\mathbf{u}^k|^2|E^k|^2 + c_\varepsilon|\rho^k|_\infty^2|A\mathbf{u}^k|^2|E^k|^2 \\
&\quad +2\varepsilon|\nabla\theta^k|^2,
\end{aligned}
$$

$$
\begin{aligned}
&|(\pi^k(\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} - F), \nu\theta^k)| \\
&\leq \nu|\pi^k||\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} - F|_4|\theta^k|_4 \\
&\leq c_\varepsilon|\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} - F|_4^2|\pi^k|^2 + \varepsilon|\nabla\theta^k|^2.
\end{aligned}
$$

Os termos que envolvem λ são estimados como segue:

$$
\begin{aligned}
|\lambda((\theta^k\cdot\nabla)\nabla\rho, \nu\theta^k)| &= \nu\left|\lambda\sum_{i,j}\int_\Omega \theta_j^k\frac{\partial}{\partial x_j}\frac{\partial\rho}{\partial x_i}\theta_i^k\right| \\
&= \nu\left|\lambda\sum_{i,j}\int_\Omega \frac{\partial}{\partial x_j}(\theta_j^k\theta_i^k)\frac{\partial\rho}{\partial x_i}\right| \\
&= \nu\left|\lambda\sum_{i,j}\int_\Omega \theta_j^k\frac{\partial\theta_i^k}{\partial x_j}\frac{\partial\rho}{\partial x_i}\right| \\
&\leq \nu\lambda|\nabla\rho|_\infty|\nabla\theta^k|\ |\theta^k| \\
&\leq c_\varepsilon|\nabla\rho|_\infty^2|\theta^k|^2 + \varepsilon|\nabla\theta^k|^2.
\end{aligned}
$$

Analogamente, temos:

$$
|\lambda((E^k\cdot\nabla)\nabla\rho, \nu\theta^k)| \leq c_\varepsilon|\nabla\rho|_\infty^2|E^k|^2 + \varepsilon|\nabla\theta^k|^2.
$$

E também:

$$
\begin{aligned}
&|\lambda((\mathbf{u}^k\cdot\nabla)\nabla\pi^k, \nu\theta^k)| = \left|\nu\lambda\sum_{i,j}\int_\Omega \mathbf{u}_j^k\frac{\partial}{\partial x_j}\frac{\partial\pi^k}{\partial x_i}\theta_i^k\right| \\
&= \left|\nu\lambda\sum_{i,j}\int_\Omega \frac{\partial}{\partial x_i}(\mathbf{u}_j^k\theta_i^k)\frac{\partial\pi^k}{\partial x_j}\right| \\
&= \left|\nu\lambda\sum_{i,j}\int_\Omega \frac{\partial\mathbf{u}_j^k}{\partial x_i}\theta_i^k\frac{\partial\pi^k}{\partial x_j}\right| \\
&\leq \nu\lambda|\nabla\mathbf{u}^k|_\infty|\nabla\pi^k|\ |\theta^k| \\
&\leq c_\delta|\nabla\mathbf{u}^k|_\infty^2|\theta^k|^2 + \delta|\nabla\pi^k|^2,
\end{aligned}
$$

onde $\delta > 0$.

Além disso, temos que:

$$
\begin{array}{rcl}
|\lambda((\nabla\pi^k\cdot\nabla)\mathbf{u},\nu\theta^k)| & \leq & \nu\lambda|\nabla\pi^k|\;|\nabla\mathbf{u}|_\infty|\theta^k| \\
& \leq & c_\delta|\nabla\mathbf{u}|^2_{L^\infty}|\theta^k|^2+\delta|\nabla\pi^k|^2; \\
|\lambda((\nabla\rho^k\cdot\nabla)\theta^k,\nu\theta^k)| & \leq & \nu\lambda|\nabla\rho^k|_\infty\;|\nabla\theta^k|\;|\theta^k| \\
& \leq & c_\varepsilon|\nabla\rho^k|^2_\infty|\theta^k|^2+\varepsilon|\nabla\theta^k|^2;
\end{array}
$$

$$
\begin{aligned}
&|\lambda((\nabla\rho^k\cdot\nabla)E^k,\nu\theta^k)| \\
&\quad= \left|\nu\lambda\sum_{i,j}\int_\Omega\frac{\partial\rho^k}{\partial x_j}\frac{\partial E_i^k}{\partial x_j}\theta_i^k\right| \\
&\quad= \left|-\nu\lambda\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_j}\left(\frac{\partial\rho^k}{\partial x_j}\theta_i^k\right)E_i^k\right| \\
&\quad\leq \left|\nu\lambda\sum_{i,j}\int_\Omega\frac{\partial^2\rho^k}{\partial x_i^2}\theta_i^kE_i^k\right|+\left|\nu\lambda\sum_{i,j}\int_\Omega\frac{\partial\rho^k}{\partial x_j}\frac{\partial\theta_i^k}{\partial x_j}E_i^k\right| \\
&\quad\leq \nu\lambda|\Delta\rho^k|_4|\theta^k|_4|E^k|+\nu\lambda|\nabla\rho^k|_\infty|\nabla\theta^k|\;|E^k| \\
&\quad\leq c_\varepsilon|\Delta\rho^k|_4^2|E^k|^2+c_\varepsilon|\nabla\rho^k|^2_\infty|E^k|^2+2\varepsilon|\nabla\theta^k|^2;
\end{aligned}
$$

$$
\begin{array}{rcl}
|\lambda^2(\dfrac{\pi^k}{\rho\rho^k}(\nabla\rho\cdot\nabla)\nabla\rho,\nu\theta^k)| & \leq & C\nu\lambda^2|\pi^k||\nabla\rho|_6|\nabla^2\rho|_6|\theta^k|_6 \\
& \leq & c|\pi^k||\Delta\rho||\nabla\Delta\rho||\nabla\theta^k| \\
& \leq & c_\varepsilon|\Delta\rho|^2|\nabla\Delta\rho|^2|\pi^k|^2+\varepsilon|\nabla\theta^k|^2;
\end{array}
$$

$$
\begin{array}{rcl}
|\lambda^2(\dfrac{1}{\rho^k}(\nabla\pi^k\cdot\nabla)\nabla\rho^k,\nu\theta^k)| & \leq & C\nu\lambda^2|\nabla\pi^k||\nabla^2\rho^k|_3|\theta^k|_6 \\
& \leq & C|\nabla\pi^k||\Delta\rho^k|^{\frac12}|\nabla\Delta\rho^k|^{\frac12}|\nabla\theta^k| \\
& \leq & \nu c_\delta|\Delta\rho^k||\nabla\Delta\rho^k||\nabla\theta^k|^2+\delta|\nabla\pi^k|^2;
\end{array}
$$

$$
\begin{aligned}
&|\lambda^2\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\pi^k,\nu\theta^k)| = \nu\lambda^2\Big|\sum_{i,j}\int_\Omega\frac{1}{\rho^k}\frac{\partial\rho^k}{\partial x_i}\frac{\partial}{\partial x_i}\frac{\partial\pi^k}{\partial x_j}\theta_j^k\Big| \\
&= \nu\lambda^2\Big|-\sum_{i,j}\int_\Omega\frac{\partial\pi^k}{\partial x_i}\frac{\partial}{\partial x_j}\Big(\frac{1}{\rho^k}\frac{\partial\rho^k}{\partial x_i}\theta_j^k\Big)+\sum_{i,j}\int_\Omega\Big(\frac{1}{\rho^k}\frac{\partial\rho^k}{\partial x_i}\theta_j^k\Big)\frac{\partial\pi^k}{\partial\mathbf{n}}\Big|
\end{aligned}
$$

$$\leq \nu\lambda^2\Big[\Big|\sum_{i,j}\int_\Omega \frac{\partial \pi^k}{\partial x_i}\frac{1}{(\rho^k)^2}\frac{\partial \rho^k}{\partial x_j}\frac{\partial \rho^k}{\partial x_i}\theta_j^k\Big| + \Big|\sum_{i,j}\int_\Omega \frac{\partial \pi^k}{\partial x_i}\frac{1}{\rho^k}\frac{\partial^2 \rho^k}{\partial x_j \partial x_i}\theta_j^k\Big|\Big]$$
$$\leq |\nabla \pi^k||\nabla \rho^k|_\infty^2|\theta^k| + c|\nabla \pi^k||\Delta \rho^k|_3|\theta^k|_6$$
$$\leq c_\delta|\nabla \rho^k|_\infty^4|\theta^k|^2 + \nu c_\delta|\nabla \Delta \rho^k|^2|\nabla \theta^k|^2 + 2\delta|\nabla \pi^k|^2;$$

$$\begin{aligned}|\lambda^2(\pi^k\frac{\rho+\rho^k}{(\rho\rho^k)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho,\nu\theta^k)| &\leq \nu\lambda^2|\pi^k||\nabla\rho|_\infty|\nabla\rho|_6^2|\theta^k|_6\\ &\leq c|\pi^k||\nabla\rho|_\infty|\Delta\rho|^2|\nabla\theta^k|\\ &\leq c_\varepsilon|\nabla\rho|_\infty^2|\Delta\rho|^4|\pi^k|^2 + \varepsilon|\nabla\theta^k|^2;\end{aligned}$$

$$\begin{aligned}|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\pi^k\cdot\nabla\rho)\nabla\rho,\nu\theta^k)| &\leq \nu\lambda^2|\nabla\pi^k||\nabla\rho|_\infty^2|\theta^k|\\ &\leq c_\delta|\nabla\rho|_\infty^4|\theta^k|^2 + \delta|\nabla\pi^k|^2;\end{aligned}$$

$$\begin{aligned}|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\pi^k)\nabla\rho,\nu\theta^k)| &\leq \nu\lambda^2|\nabla\rho^k|_\infty|\nabla\pi^k||\nabla\rho|_\infty|\theta^k|\\ &\leq c_\delta|\nabla\rho^k|_\infty^2|\nabla\rho|_\infty^2|\theta^k|^2 + \delta|\nabla\pi^k|^2;\\ |\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\pi^k,\nu\theta^k)| &\leq \nu\lambda^2|\nabla\rho^k|_\infty^2|\nabla\pi^k||\theta^k|\\ &\leq c_\delta|\nabla\rho^k|_\infty^4|\theta^k|^2 + \delta|\nabla\pi^k|^2;\end{aligned}$$

$$\begin{aligned}|\lambda^2(\frac{\pi^k}{\rho\rho^k}\Delta\rho\nabla\rho,\nu\theta^k)| &\leq \nu\lambda^2|\pi^k||\Delta\rho|_3|\nabla\rho|_6|\theta^k|_6\\ &\leq c|\pi^k||\Delta\rho|^{\frac{3}{2}}|\nabla\Delta\rho|^{\frac{1}{2}}|\nabla\theta^k|\\ &\leq c_\varepsilon|\Delta\rho|^3|\nabla\Delta\rho||\pi^k|^2 + \varepsilon|\nabla\theta^k|^2;\end{aligned}$$

$$|\lambda^2(\frac{1}{\rho^k}\Delta\pi^k\nabla\rho,\nu\theta^k)| = \nu\lambda^2\Big|\sum_{i,j}\int_\Omega \frac{1}{\rho^k}\frac{\partial^2\pi^k}{\partial^2 x_j^2}\frac{\partial\rho}{\partial x_i}\theta_i^k\Big|$$
$$= \nu\lambda^2\Big| -\sum_{i,j}\int_\Omega \frac{\partial\pi^k}{\partial x_j}\frac{\partial}{\partial x_j}(\frac{1}{\rho^k}\frac{\partial\rho}{\partial x_i}\theta_i^k) + \sum_{i,j}\int_\Gamma(\frac{1}{\rho^k}\frac{\partial\rho}{\partial x_i}\theta_i^k)\frac{\partial\pi^k}{\partial\mathbf{n}}\Big|$$
$$\leq \nu\lambda^2\Big[\Big|\sum_{i,j}\int_\Omega \frac{\partial\pi^k}{\partial x_j}\frac{1}{(\rho^k)^2}\frac{\partial\rho^k}{\partial x_j}\frac{\partial\rho}{\partial x_i}\theta_i^k\Big| + \Big|\sum_{i,j}\int_\Omega \frac{\partial\pi^k}{\partial x_j}\frac{1}{\rho^k}\frac{\partial^2\rho}{\partial x_j\partial x_i}\theta_i^k\Big|$$
$$+\Big|\sum_{i,j}\int_\Omega \frac{\partial\pi^k}{\partial x_j}\frac{1}{\rho^k}\frac{\partial\rho}{\partial x_i}\frac{\partial\theta_i^k}{\partial x_j}\Big|\Big]$$

$$
\begin{aligned}
&\le c|\nabla\pi^k||\nabla\rho^k|_6|\nabla\rho|_6|\theta^k|_6 + c|\nabla\pi^k||\Delta\rho|_3|\theta^k|_6 + |\nabla\pi^k||\nabla\rho|_\infty|\nabla\theta^k| \\
&\le c|\nabla\pi^k||\Delta\rho^k||\Delta\rho||\nabla\theta^k| + c|\nabla\pi^k||\Delta\rho|^{\frac{1}{2}}|\nabla\Delta\rho|^{\frac{1}{2}}|\nabla\theta^k| + c|\nabla\pi^k||\nabla\rho|_\infty|\nabla\theta^k| \\
&\le \nu c_\delta(|\Delta\rho^k|^2|\Delta\rho|^2 + |\Delta\rho||\nabla\Delta\rho| + |\nabla\rho|_\infty^2)|\nabla\theta^k|^2 + 3\delta|\nabla\pi^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\pi^k, \nu\theta^k)| &\le \nu\lambda^2|\Delta\rho^k|_3|\nabla\pi^k||\theta^k|_6 \\
&\le c|\Delta\rho^k|^{\frac{1}{2}}|\nabla\Delta\rho^k|^{\frac{1}{2}}|\nabla\pi^k||\nabla\theta^k| \\
&\le \nu c_\delta|\Delta\rho^k||\nabla\Delta\rho^k||\nabla\theta^k|^2 + \delta|\nabla\pi^k|^2.
\end{aligned}
$$

Consideramos em seguida, a equação de π^k; temos então:

$$
\pi_t^k + \theta^k.\nabla\rho + E^k.\nabla\rho + \mathbf{u}^k.\nabla\pi^k - \lambda\Delta\pi^k = 0. \tag{3.36}
$$

Tomando o produto interno em $L^2(\Omega)$ da identidade acima com π^k, obtemos:

$$
\frac{d}{dt}|\pi^k|^2 + \lambda|\nabla\pi^k|^2 = -(\theta^k.\nabla\rho, \pi^k) - (E^k.\nabla\rho, \pi^k), \tag{3.37}
$$

já que

$$
\int_\Omega \mathbf{u}^k.\nabla\pi^k\pi^k = \int_\Omega \mathbf{u}^k.\nabla\frac{|\pi^k|}{2} = -\int_\Omega \text{ div } \mathbf{u}^k\frac{|\pi^k|}{2}^2 = 0.
$$

Estimamos os termos que se encontram do lado direito da igualdade (3.37) como segue:

$$
\begin{aligned}
|(\theta^k\cdot\nabla\rho, \pi^k)| &\le |\theta^k|\ |\nabla\rho|_\infty|\pi^k| \\
&\le \frac{1}{2}|\nabla\rho|_\infty^2|\theta^k|^2 + \frac{1}{2}|\pi^k|^2, \\
|(E^k\cdot\nabla\rho, \pi^k)| &\le \frac{1}{2}|\nabla\rho|_\infty^2|E^k|^2 + \frac{1}{2}|\pi^k|^2.
\end{aligned}
$$

Assim, as identidades (3.35) e (3.37) e as estimativas acima nos fornecem a seguinte desigualdade diferencial:

$$
\frac{1}{2}\frac{d}{dt}(|\pi^k|^2 + |(\rho^k)^{1/2}\theta^k|^2) + \mu|\nabla\theta^k|^2 + \lambda|\nabla\pi^k|^2
$$

$$
\begin{aligned}
\le\ & \frac{1}{2\nu}|\rho^k|_\infty^2|E_t^k|^2 + \frac{1}{2}|\theta^k|^2(\nu + |\rho_t^k|_\infty + |\nabla\rho^k|_\infty^2) \\
&+c_\varepsilon|\theta^k|^2[|\rho^k|_\infty^2|A\mathbf{u}|^2 + |\rho^k|_\infty^2|A\mathbf{u}^k|^2 + \lambda^2|\nabla\rho|_\infty^2 \\
&+\lambda^2|\nabla\rho^k|_\infty^2] + c_\delta|\theta^k|^2[\lambda^2|\nabla\mathbf{u}^k|_\infty^2 + \lambda^2|\nabla\mathbf{u}|_\infty^2 + |\nabla\rho^k|_\infty^4 + |\nabla\rho^k|_\infty^2|\nabla\rho|_\infty^2]
\end{aligned}
$$

$$
\begin{aligned}
&+c_\varepsilon|E^k|^2[|\nabla\rho^k|_\infty^2|\nabla\mathbf{u}^k|^2+|\rho^k|_\infty^2|A\mathbf{u}^k|^2+|\rho^k|_\infty^2|A\mathbf{u}|^2\\
&+\lambda^2|\Delta\rho^k|_4^2+\lambda^2|\nabla\rho^k|_\infty^2+\lambda^2|\nabla\rho|_\infty^2]\\
&+c_\varepsilon|\pi^k|^2(\ |\mathbf{u}_t|_4^2+|\mathbf{u}.\nabla\mathbf{u}|_4^2+|F|_4^2+|\Delta\rho|^2|\nabla\Delta\rho|^2+|\nabla\rho|_\infty^2|\Delta\rho|^4\\
&+|\Delta\rho|^3|\nabla\Delta\rho|)\\
&+\nu c_\delta|\nabla\theta^k|^2(|\Delta\rho^k||\nabla\Delta\rho^k|+|\nabla\Delta\rho^k|^2+|\Delta\rho^k|^2|\Delta\rho|^2+|\Delta\rho||\nabla\Delta\rho|+|\nabla\rho|_\infty^2)\\
&+13\varepsilon|\nabla\theta^k|^2+12\delta|\nabla\pi^k|^2+\frac{1}{2}|\nabla\rho|_\infty^2|E^k|^2+|\pi^k|^2\\
&=\frac{1}{2}|\rho^k|_\infty^2|E_t^k|^2+c|E^k|^2\varphi_\circ(t)+c|\theta^k|^2\varphi_1(t)+c|\pi^k|^2\varphi_2(t)\\
&+\nu c_\delta|\nabla\theta^k|^2\varphi_3(t)+13\varepsilon|\nabla\theta^k|^2+12\delta|\nabla\pi^k|^2.
\end{aligned}
$$

Como $\varphi_\circ(t)$ e $\varphi_3(t)$ são uniformemente limitadas em t, temos que existe $M>0$ tal que $|\varphi_\circ(t)|<M$ e $\varphi_3(t)<M$ para todo t. Assim sendo, escolhemos $\nu=\dfrac{\varepsilon}{c_\delta M}$ e a desigualdade acima fica da seguinte forma:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\pi^k|^2+|(\rho^k)^{1/2}\theta^k|^2)+\mu|\nabla\theta^k|^2+\lambda|\nabla\pi^k|^2\\
&\le c|E^k|^2+c|E_t^k|^2+c|\theta^k|^2\varphi_1(t)+c|\pi^k|^2\varphi_2(t)+14\varepsilon|\nabla\theta^k|^2+12\delta|\nabla\pi^k|^2.
\end{aligned}
$$

Escolhendo $\varepsilon=\dfrac{\mu}{28},\quad \delta=\dfrac{\lambda}{24}$, a desigualdade diferencial acima acarreta a seguinte desigualdade integral:

$$
\begin{aligned}
&\frac{1}{2}|\pi^k(t)|^2+\frac{1}{2}|(\rho^k(t))^{1/2}\theta^k(t)|^2+\ \frac{\mu}{2}\int_0^t|\nabla\theta^k(s)|^2ds+\frac{\lambda}{2}\int_0^t|\nabla\pi^k(s)|^2ds\\
&\le\frac{1}{2}|\pi^k(0)|^2+\frac{1}{2}|(\rho^k(0))^{1/2}\theta^k(0)|^2+c\int_0^t|E_t^k(s)|^2ds+c\int_0^t|E^k(s)|^2ds\\
&+c\int_0^t(\varphi_1(s)+\varphi_2(s))[\alpha|\theta^k|^2+|\pi^k(s)|^2]ds
\end{aligned}
$$

graças às estimativas dadas no Teorema 2.1.

Por outro lado, usando o Lema 1.9, o fato de $|\rho^k(t))^{1/2}\theta^k(t)|^2\ge\alpha|\theta^k(t)|^2$ e $|\pi^k(0)|=|\theta^k(0)|=0$, obtemos:

$$
\begin{aligned}
&|\pi^k(t)|^2+\alpha|\theta^k(t)|^2+\mu\int_0^t|\nabla\theta^k(s)|^2ds+\lambda\int_0^t|\nabla\pi^k(s)|^2ds\\
&\le\frac{ct}{\lambda_{k+1}^2}+\frac{c}{\lambda_{k+1}}\int_0^t|\nabla\mathbf{u}_t(s)|^2ds+c\int_0^t(\varphi_1(s)+\varphi_2(s))[\alpha|\theta^k|^2+|\pi^k(s)|^2]ds
\end{aligned}
$$

Aplicando então, a desigualdade de Gronwall, obtemos:

$$
\begin{aligned}
&|\pi^k(t)|^2 + \alpha|\theta^k(t)|^2 + \mu \int_0^t |\nabla\theta^k(s)|^2 ds + \lambda \int_0^t |\nabla\pi^k(s)|^2 ds \\
&\leq \left| \frac{ct}{\lambda_{k+1}^2} + \frac{c}{\lambda_{k+1}} \int_0^t |\nabla \mathbf{u}_t(s)|^2 ds \right| \ \exp\ \left[c\int_0^t (\varphi_1(s) + \varphi_2(s)) ds \right] \\
&\leq \overline{c} \left| \frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}} \right|
\end{aligned}
$$

para todo $t \in [0, T]$, graças às estimativas fornecidas pelo Teorema 2.1. Isto prova o resultado estabelecido. $\blacksquare$

Observação 3.1 O termo $\displaystyle\int_0^t |E_t^k(s)|^2 ds$ poderia ser estimado com taxa otimal se $\displaystyle\int_0^T |A\mathbf{u}_t(s)|^2 ds \leq c$. Na verdade, neste caso, teríamos

$$
\int_0^t |E_t^k(s)|^2 ds \leq \frac{c}{\lambda_{k+1}^2} \int_0^t |A\mathbf{u}_t(s)|^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}^2},
$$

usando o Lema 1.9, e o resto da análise poderia ser feita exatamente como antes e o resultado final seria uma estimativa da ordem de $\dfrac{1}{\lambda_{k+1}^2}$ no Lema 3.1.

Entretanto, assim como foi destacado por Heywood e Rannacher [20], mesmo no caso das equações clássicas de Navier-Stokes (caso onde tem-se ρ constante) uma condição do tipo $\displaystyle\int_0^T |A\mathbf{u}_t(s)|^2 ds \leq c$ (que implicaria $\displaystyle\int_0^T |\mathbf{u}_t(s)|_{H^2}^2 ds \leq c$) exigiria que a condição inicial satisfizesse uma condição de compatibilidade não local (ver Corolário 2.1 e condição (1.5) em [20]), o que não pode ser esperado a menos que a condição inicial seja bastante especial.

Na próxima seção daremos estimativas melhores que aquelas acima apresentadas (ver Teorema 3.3).

O resultado abaixo nos dá uma estimativa de erro na norma L^2 para a velocidade e a densidade.

Teorema 3.1 *Suponhamos que as hipóteses do Teorema 2.1 sejam satisfeitas. Então, as aproximações* $\mathbf{u}^k, \rho^k$ *satisfazem*

$$
|\mathbf{u}(t) - \mathbf{u}^k(t)|^2 + |\rho(t) - \rho^k(t)|^2 \leq \overline{c}\left[\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}} \right]
$$

para todo $t \in [0, T]$.

Demonstração. Pela definição (3.32)

$$
\mathbf{u} - \mathbf{u}^k = E^k + \theta^k.
$$

Então, usando os Lemas 1.9 e 3.1, obtemos:

$$|\mathbf{u}(t)-\mathbf{u}^k(t)|^2 \leq |E^k(t)|^2+|\theta^k(t)|^2 \leq \overline{c}\Big[\frac{1}{\lambda_{k+1}^2}+\frac{1}{\lambda_{k+1}}\Big]. \qquad \blacksquare$$

O próximo passo é obter estimativas de ordem mais altas.

Lema 3.2

$$\mu|\nabla\theta^k(t)|^2+\lambda|\nabla\pi^k(t)|^2+c\int_0^t[|\pi_t^k(s)|^2+|\theta_t^k(s)|^2]ds \leq \overline{c}\left[\frac{1}{\lambda_{k+1}^2}+\frac{1}{\lambda_{k+1}}\right].$$

Demonstração. Tomando o produto interno em $L^2(\Omega)$ da identidade (3.34) com $v=\delta\theta_t^k$, com $\delta>0$ (a ser determinado posteriormente), obtemos:

$$\begin{aligned}
&\frac{\delta\mu}{2}\frac{d}{dt}|\nabla\theta^k|^2+\delta|(\rho^k)^{1/2}\theta_t^k|^2 \\
&= -(\rho^k E_t^k,\delta\theta_t^k)-(\pi^k(\mathbf{u}_t+\mathbf{u}.\nabla\mathbf{u}-F),\delta\theta_t^k)-((\rho^k\theta^k\cdot\nabla)\mathbf{u},\delta\theta_t^k) \\
&\quad -((\rho^k E^k\cdot\nabla)\mathbf{u},\delta\theta_t^k)-((\rho^k\mathbf{u}^k\cdot\nabla)\theta^k,\delta\theta_t^k)-((\rho^k\mathbf{u}^k\cdot\nabla)E^k,\delta\theta_t^k) \\
&\quad +\lambda((\theta^k\cdot\nabla)\nabla\rho,\delta\theta_t^k)+\lambda((E^k\cdot\nabla)\nabla\rho,\delta\theta_t^k)+\lambda((\mathbf{u}^k\cdot\nabla)\nabla\pi^k,\delta\theta_t^k) \\
&\quad +\lambda((\nabla\pi^k\cdot\nabla)\mathbf{u},\delta\theta_t^k)+\lambda((\nabla\rho^k\cdot\nabla)\theta^k,\delta\theta_t^k)+\lambda((\nabla\rho^k\cdot\nabla)E^k,\delta\theta_t^k) \\
&\quad +\lambda^2(-\frac{\pi^k}{\rho\rho^k}(\nabla\rho\cdot\nabla)\nabla\rho,\delta\theta_t^k)+\lambda^2(\frac{1}{\rho^k}(\nabla\pi^k\cdot\nabla)\nabla\rho,\delta\theta_t^k) \\
&\quad +\lambda^2(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\pi^k,\delta\theta_t^k)+\lambda^2(\pi^k\frac{\rho+\rho^k}{(\rho\rho^k)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho,\delta\theta_t^k) \\
&\quad -\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\pi^k\cdot\nabla\rho)\nabla\rho,\delta\theta_t^k)-\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\pi^k)\nabla\rho,\delta\theta_t^k) \\
&\quad -\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\pi^k,\delta\theta_t^k)-\lambda^2(\frac{\pi^k}{\rho\rho^k}\Delta\rho\nabla\rho,\delta\theta_t^k) \\
&\quad +\lambda^2(\frac{1}{\rho^k}\Delta\pi^k\nabla\rho,\delta\theta_t^k)+\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\pi^k,\delta\theta_t^k).
\end{aligned}$$

Passemos agora, às estimativas dos termos que se encontram do lado direito da expressão acima.

$$\begin{aligned}
|(\pi^k\mathbf{u}_t,\delta\theta_t^k)| &\leq c\delta|\pi^k|_4|\mathbf{u}_t|_4|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\nabla\mathbf{u}_t|^2|\pi^k|_{H^1}^2+\varepsilon|\theta_t^k|^2;
\end{aligned}$$

$$\begin{aligned}
|(\pi^k\mathbf{u}\cdot\nabla\mathbf{u},\delta\theta_t^k)| &\leq c\delta|\pi^k|_4|\mathbf{u}|_\infty|\nabla\mathbf{u}|_4|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|A\mathbf{u}|^4|\pi^k|_{H^1}^2+\varepsilon|\theta_t^k|^2;
\end{aligned}$$

$$
\begin{aligned}
|(\pi^k F, \delta\theta_t^k)| &\leq c\delta|\pi^k|_4|F|_4|\theta_t^k| \\
&\leq c|\pi^k|_{_{H^1}}|F|_{_{H^1}}|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|F|^2_{_{H^1}}|\pi^k|^2_{_{H^1}} + \varepsilon|\theta_t^k|^2;
\end{aligned}
$$

Usando as desigualdades de Cauchy-Schwarz e de Young, obtemos para todo $\varepsilon > 0$,

$$
|(\rho^k E_t^k, \delta\theta_t^k)| \leq c_{\varepsilon,\delta}|\rho^k|^2_\infty |E_t^k|^2 + \varepsilon|\theta_t^k|^2.
$$

As desigualdades de Hölder e de Young e a imersão de Sobolev $H^1(\Omega) \hookrightarrow L^4(\Omega)$ implicam

$$
\begin{aligned}
|((\rho^k\theta^k \cdot \nabla)\mathbf{u}, \delta\theta_t^k)| &\leq \delta|\rho^k|_\infty||\theta^k|_4||\nabla\mathbf{u}|_4||\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\rho^k|^2_\infty|A\mathbf{u}|^2|\nabla\theta^k|^2 + \varepsilon|\theta_t^k|^2.
\end{aligned}
$$

Analogamente, também temos:

$$
|((\rho^k E^k \cdot \nabla)\mathbf{u}, \delta\theta_t^k)| \leq c_{\varepsilon,\delta}|\rho^k|^2_\infty|A\mathbf{u}|^2|\nabla E^k|^2 + \varepsilon|\theta_t^k|^2.
$$

Além disso,

$$
\begin{aligned}
|((\rho^k\mathbf{u}^k \cdot \nabla)\theta^k, \delta\theta_t^k)| &\leq \delta|\rho^k|_\infty|\mathbf{u}^k|_\infty|\nabla\theta^k|\ |\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\rho^k|^2_\infty|A\mathbf{u}^k|^2|\nabla\theta^k|^2 + \varepsilon|\theta_t^k|^2
\end{aligned}
$$

devido à imersão de Sobolev $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

Similarmente,

$$
|((\rho^k\mathbf{u}^k \cdot \nabla)E^k, \delta\theta_t^k)| \leq c_{\varepsilon,\delta}|\rho^k|^2_\infty|A\mathbf{u}^k|^2|\nabla E^k|^2 + \varepsilon|\theta_t^k|^2.
$$

As desig**u**aldades de Hölder e de Young juntas implicam:

$$
\begin{aligned}
|(\lambda((\nabla\rho^k.\nabla)\theta^k, \delta\theta_t^k)| &\leq c_{\varepsilon,\delta}\lambda^2|\nabla\rho^k|^2_\infty|\nabla\theta^k|^2 + \varepsilon|\theta_t^k|^2; \\
|(\lambda((\nabla\rho^k.\nabla)E^k, \delta\theta_t^k)| &\leq c_{\varepsilon,\delta}\lambda^2|\nabla\rho^k|^2_\infty|\nabla E^k|^2 + \varepsilon|\theta_t^k|^2; \\
|(\lambda((\theta^k.\nabla)\nabla\rho, \delta\theta_t^k)| &\leq c_{\varepsilon,\delta}\lambda^2|\nabla\theta^k|^2|\Delta\rho|^2_4 + \varepsilon|\theta_t^k|^2; \\
|(\lambda((E^k.\nabla)\nabla\rho, \delta\theta_t^k)| &\leq c_{\varepsilon,\delta}\lambda^2|\Delta\rho|^2_4|\nabla E^k|^2 + \varepsilon|\theta_t^k|^2; \\
|(\lambda((\nabla\pi^k.\nabla)\mathbf{u}, \delta\theta_t^k)| &\leq c_{\varepsilon,\delta}\lambda^2|\nabla\pi^k|^2|\nabla\mathbf{u}|^2_\infty + \varepsilon|\theta_t^k|^2;
\end{aligned}
$$

e, também,

$$
\begin{aligned}
&|(\lambda((\mathbf{u}^k.\nabla)\nabla\pi^k, \delta\theta_t^k)| \\
&\quad = |\delta\lambda\sum_j\int_\Omega \mathbf{u}_j^k\frac{\partial}{\partial x_j}\nabla\pi^k(\theta_t^k)_i| \\
&\quad = |\delta\lambda\sum_{i,j}\int_\Omega \mathbf{u}_j^k\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\pi^k(\theta_t^k)_i| \\
&\quad = |-\delta\lambda\sum_{i,j}\int_\Omega \frac{\partial}{\partial x_i}\mathbf{u}_j^k\frac{\partial}{\partial x_j}\pi^k(\theta_t^k)_i| \\
&\quad \leq c_{\varepsilon,\delta}\lambda^2|\nabla\mathbf{u}^k|^2_\infty|\nabla\pi^k|^2 + \varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{\pi^k}{\rho\rho^k}(\nabla\rho\cdot\nabla)\nabla\rho,\delta\theta_t^k)| &\leq c\delta|\pi^k|_4|\nabla\rho|_\infty|\nabla^2\rho|_4|\theta_t^k| \\
&\leq c\delta|\pi^k|_{_{H^1}}|\nabla\Delta\rho|^2|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\nabla\Delta\rho|^4|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{\rho^k}(\nabla\pi^k\cdot\nabla)\nabla\rho,\delta\theta_t^k)| &\leq c\delta|\nabla\pi^k||\nabla^2\rho|_\infty|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\rho|^2_{_{H^4_N}}|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
&|\lambda^2(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\pi^k,\delta\theta_t^k)| = \Big|\delta\lambda^2\sum_{i,j}\int_\Omega\frac{1}{\rho^k}\frac{\partial\rho^k}{\partial x_j}\frac{\partial}{\partial x_j}\Big(\frac{\partial\pi^k}{\partial x_i}\Big)(\theta_t^k)_i\Big| \\
&= \Big|-\delta\lambda^2\sum_{i,j}\int_\Omega\frac{\partial\pi^k}{\partial x_j}\frac{\partial}{\partial x_i}\Big(\frac{1}{\rho^k}\frac{\partial\rho^k}{\partial x_j}(\theta_t^k)_i\Big)+\delta\lambda^2\sum_{i,j}\int_\Gamma\frac{1}{\rho^k}\frac{\partial\rho^k}{\partial x_j}(\theta_t^k)_i\frac{\partial\pi^k}{\partial\mathbf{n}}\Big| \\
&= \Big|-\delta\lambda^2\sum_{i,j}\int_\Omega\frac{\partial\pi^k}{\partial x_j}\frac{1}{(\rho^k)^2}\frac{\partial\rho^k}{\partial x_i}\frac{\partial\rho^k}{\partial x_j}(\theta_t^k)_i-\delta\lambda^2\sum_{i,j}\int_\Omega\frac{\partial\pi^k}{\partial x_j}\frac{1}{\rho^k}\frac{\partial^2\rho^k}{\partial x_i\partial x_j}(\theta_t^k)_i\Big| \\
&\leq c|\nabla\pi^k||\nabla\rho^k|^2_\infty|\theta_t^k|+c|\nabla\pi^k||\Delta\rho^k|_\infty|\theta_t^k| \\
&\leq c_{\varepsilon,\delta,\lambda}[|\nabla\Delta\rho^k|^4+|\rho^k|^2_{_{H^4_N}}]|\pi^k|^2_{_{H^1}}+2\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\pi^k\frac{\rho+\rho^k}{(\rho\rho^k)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho,\delta\theta_t^k)| &\leq c\delta|\pi^k|_4|\nabla\rho|^2_\infty|\nabla\rho|_4|\theta_t^k| \\
&\leq c\delta|\pi^k|_{_{H^1}}|\nabla\Delta\rho|^2|\Delta\rho||\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\Delta\rho|^2|\nabla\Delta\rho|^4|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\pi^k\cdot\nabla\rho)\nabla\rho,\delta\theta_t^k)| &\leq c\delta|\nabla\pi^k||\nabla\rho|^2_\infty|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\nabla\Delta\rho|^4|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\pi^k)\nabla\rho,\delta\theta_t^k)| &\leq c\delta|\nabla\rho^k|_\infty|\nabla\pi^k||\nabla\rho|_\infty|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\nabla\Delta\rho^k|^2|\nabla\Delta\rho|^2|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\pi^k,\delta\theta_t^k)| &\leq c\delta|\nabla\rho^k|^2_\infty|\nabla\pi^k||\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\nabla\Delta\rho^k|^4|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda^2(\frac{\pi^k}{\rho\rho^k}\Delta\rho\nabla\rho,\delta\theta_t^k)| &\leq c\delta|\pi^k|_4|\Delta\rho|_4|\nabla\rho|_\infty|\theta_t^k| \\
&\leq c\delta|\pi^k|_{_{H^1}}|\nabla\Delta\rho|^2|\theta_t^k| \\
&\leq c_{\varepsilon,\delta}|\nabla\Delta\rho|^4|\pi^k|^2_{_{H^1}}+\varepsilon|\theta_t^k|^2;
\end{aligned}
$$

Observemos que usando a equação (3.36) podemos escrever

$$|\lambda^2(\frac{1}{\rho^k}\Delta\pi^k\nabla\rho,\delta\theta_t^k)| = |\lambda(\frac{1}{\rho^k}(\pi_t^k + (\theta^k + E^k)\cdot\nabla\rho + \mathbf{u}^k\cdot\nabla\pi^k)\nabla\rho,\delta\theta_t^k)|.$$

Assim, podemos estimar tal termo da forma seguinte:

$$|(\frac{\lambda}{\rho^k}\pi_t^k\cdot\nabla\rho,\delta\theta_t^k)| \leq C\delta|\pi_t^k||\nabla\rho|_\infty|\theta_t^k| \leq C_\varepsilon\delta^2|\pi_t^k|^2 + \varepsilon|\theta_t^k|^2;$$

$$|(\frac{\lambda}{\rho^k}(\theta^k\cdot\nabla\rho)\nabla\rho,\delta\theta_t^k)| \leq C\delta|\theta^k|_4|\nabla\rho|_4|\nabla\rho|_\infty|\theta_t^k| \leq C_{\varepsilon,\delta}|\Delta\rho|^2|\nabla\Delta\rho|^2|\nabla\theta^k|^2 + \varepsilon|\theta_t^k|^2;$$

$$|(\frac{\lambda}{\rho^k}(E^k\cdot\nabla\rho)\nabla\rho,\delta\theta_t^k)| \leq C_{\varepsilon,\delta}|\Delta\rho|^2|\nabla\Delta\rho|^2|\nabla E^k|^2 + \varepsilon|\theta_t^k|^2;$$

$$\begin{aligned} |(\frac{\lambda}{\rho^k}(\mathbf{u}^k\cdot\nabla\pi^k)\nabla\rho,\delta\theta_t^k)| &\leq C\delta|\mathbf{u}^k|_\infty|\nabla\pi^k||\nabla\rho|_\infty|\theta_t^k| \\ &\leq C_{\varepsilon,\delta}|A\mathbf{u}^k|^2|\nabla\Delta\rho|^2|\nabla\pi^k|^2 + \varepsilon|\theta_t^k;| \end{aligned}$$

$$\begin{aligned} |\lambda^2(\frac{1}{\rho^k}\Delta\rho^k\nabla\pi^k,\delta\theta_t^k)| &\leq c\delta|\Delta\rho^k|_\infty|\nabla\pi^k||\theta_t^k| \\ &\leq C_{\varepsilon,\delta}|\rho^k|^2_{H^4_N}|\pi^k|^2_{H^1} + \varepsilon|\theta_t^k|^2. \end{aligned}$$

Assim, usando as estimativas acima (e lembrando que $|(\rho^k)^{\frac{1}{2}}\theta_t^k|^2 \geq \alpha|\theta_t^k|^2$) teremos:

$$\begin{aligned} \frac{\delta\mu}{2}\frac{d}{dt}|\nabla\theta^k|^2 + \delta\alpha|\theta_t^k|^2 &\leq c_\varepsilon|\rho^k|^2_\infty|E_t^k|^2 + c_\varepsilon|\nabla E^k|^2\cdot\varphi_1(t) + \frac{C}{4\varepsilon}\delta^2|\pi_t^k|^2 \\ &+ c_\varepsilon|\nabla\theta^k|^2\cdot\varphi_1(t) + c_\varepsilon|\pi^k|^2_{H^1}\cdot\varphi_2(t) + 28\varepsilon|\theta_t^k|^2, \end{aligned} \tag{3.38}$$

sendo que

$$\begin{aligned} \varphi_1(t) &= |\rho^k|^2_\infty|A\mathbf{u}|^2 + |\rho^k|^2_\infty|A\mathbf{u}^k|^2 + |\nabla\rho^k|^2_\infty + |\nabla\Delta\rho^k|^2 + |\Delta\rho|^2|\nabla\Delta\rho|^2; \\ \varphi_2(t) &= |\nabla\mathbf{u}_t|^2 + |A\mathbf{u}|^4 + |F|^2_{H^1} + |\nabla\mathbf{u}|^2_\infty + |\nabla\mathbf{u}^k|^2_\infty + |\nabla\Delta\rho|^4 + |\rho|^2_{H^4_N} \\ &\quad + |\nabla\Delta\rho^k|^4 + |\rho^k|^2_{H^4_N} + |\Delta\rho|^2|\nabla\Delta\rho|^2 + |\nabla\Delta\rho^k|^2|\nabla\Delta\rho|^2 \\ &\quad + |A\mathbf{u}^k|^2|\nabla\Delta\rho|^2 + |A\mathbf{u}^k|^2 + 1, \end{aligned}$$

com $\varphi_1 \in L^\infty[0,T^*]$ e $\varphi_2 \in L^1[0,T^*]$. Além disso, usando a equação (3.36) vemos que

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\pi^k|^2 + \lambda|\nabla\pi^k|^2 &= -(E^k\cdot\nabla\rho,\pi^k) - (\theta^k\cdot\nabla\rho,\pi^k) \\ &\leq C|E^k|^2|\nabla\rho|^2_\infty + C|\theta^k|^2|\nabla\rho|^2_\infty + C|\pi^k|^2_{H^1}. \end{aligned} \tag{3.39}$$

Por outro lado, multiplicando a igualdade (3.36) por π_t^k e integrando sobre Ω temos, para todo $\gamma > 0$,

$$\begin{aligned} &|\pi_t^k|^2 + \frac{\lambda}{2}\frac{d}{dt}|\nabla\pi^k|^2 \\ &= (\theta^k.\nabla\rho, \pi_t^k) + (E^k.\nabla\rho, \pi_t^k) + (\mathbf{u}^k.\nabla\pi^k, \pi_t^k) \\ &\leq c_\gamma|\nabla\rho|_\infty^2|\theta^k|^2 + c_\gamma|\nabla\rho|_\infty^2|E^k|^2 \\ &+c_\gamma|A\mathbf{u}^k|^2|\nabla\pi^k|^2 + 3\gamma|\pi_t^k|^2. \end{aligned} \tag{3.40}$$

Somando-se as desigualdade (3.38), (3.39) e (3.40) e escolhendo $\varepsilon = \frac{\alpha}{28n}$ onde $\frac{1}{n} < \frac{3\alpha}{28C}$, $\gamma = \frac{1}{12}$ e $\delta > 0$ tal que $\frac{1}{n} < \delta < (\frac{3\varepsilon}{C})^{\frac{1}{2}}$ ficaremos com a seguinte desigualdade:

$$\begin{aligned} &\frac{d}{dt}\{|\nabla\theta^k|^2 + |\pi^k|^2_{_{H^1}}\} + C|\theta_t^k|^2 + C|\pi_t^k|^2 \\ &\quad\leq c|E_t^k|^2 + c|\theta^k|^2 + c|E^k|^2 + c|\nabla E^k|^2\varphi_1(t) \\ &\quad+c(\varphi_1 + \varphi_2)(|\nabla\theta^k|^2 + |\pi^k|^2_{_{H^1}}). \end{aligned}$$

Integrando a desigualdade acima de 0 até t, obtemos:

$$\begin{aligned} &|\nabla\theta^k(t)|^2 + |\pi^k(t)|^2_{_{H^1}} + \int_0^t |\theta_t^k(s)|^2 ds + \int_0^t |\pi_t^k(s)|^2 ds \\ &\quad\leq c\int_0^t |E_t^k(s)|^2 ds + c\int_0^t (|\theta^k(s)|^2 + |E^k(s)|^2) ds + c\int_0^t |\nabla E^k(s)|^2\varphi_1(s) ds \\ &\quad+c\int_0^t (\varphi_1(s) + \varphi_2(s))(|\nabla\theta^k(s)|^2 + |\pi^k(s)|^2_{_{H^1}}) ds \\ &\quad\leq \frac{c}{\lambda_{k+1}} + c\int_0^t (\varphi_1(s) + \varphi_2(s))(|\nabla\theta^k(s)|^2 + |\pi^k(s)|^2_{_{H^1}}) ds, \end{aligned}$$

já que $|\nabla\theta^k(0)|^2 = |\pi^k(0)|^2_{_{H^1}} = 0$ e $\varphi_1 \in L^\infty(0,T)$; além disso, usando o Teorema 2.1, vemos que as estimativas ali dadas implicam para todo $t \in [0,T]$,

$$\int_0^t |\nabla\mathbf{u}_t(s)|^2 ds \leq c, \qquad \int_0^t \varphi_2(s) ds \leq c, \qquad \int_0^t [\varphi_1(s) + \varphi_2(s)] ds \leq c.$$

Consequentemente, aplicando a desigualdade de Gronwall conseguimos o resultado. ■

Corolário 3.1

$$\int_0^t |\Delta\pi^k(s)|^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}}$$

para todo $t \in [0,T]$.

Demonstração. De fato, a igualdade (3.36) implica:

$$\lambda\Delta\pi^k = \pi_t^k + \theta^k.\nabla\rho + E^k.\nabla\rho + \mathbf{u}^k.\nabla\pi^k.$$

Consequentemente,

$$\begin{aligned}\lambda|\Delta\pi^k|^2 \ &\leq\ c(|\pi_t^k|^2 + |\nabla\rho|_\infty^2|\theta^k|^2 + \\ &\qquad |\nabla\rho|_\infty^2|E^k|^2 + |A\mathbf{u}^k|^2|\nabla\pi^k|^2).\end{aligned}$$

Assim, integrando na variável temporal de 0 até t, e usando os Lemas 1.9 e 3.2, obtemos o resultado desejado. ■

Trabalhando de maneira análoga à prova do Teorema 3.1 obtemos a seguinte estimativa de erro otimal na norma H^1 para a velocidade.

Teorema 3.2 *Suponhamos que as hipóteses do Teorema 2.1 sejam satisfeitas. Então, as aproximações* $\mathbf{u}^k$ *satisfazem:*

$$|\nabla\mathbf{u}(t) - \nabla\mathbf{u}^k(t)|^2 + \int_0^t |\mathbf{u}_t(s) - \mathbf{u}_t^k(s)|^2 ds \leq \frac{\bar{c}}{\lambda_{k+1}}$$

para todo $t \in [0, T]$. ■

Corolário 3.2

$$\int_0^t |A\mathbf{u}(s) - A\mathbf{u}^k(s)|^2 ds \leq \frac{\bar{c}}{\lambda_{k+1}}$$

para todo $t \in [0, T]$.

Demonstração. Inicialmente, consideramos as duas seguintes equações:

$$\begin{aligned}A\mathbf{u} \ &=\ P[-\rho\mathbf{u}_t - \rho\mathbf{u}\cdot\nabla\mathbf{u} + \rho F + \lambda(\mathbf{u}\cdot\nabla)\nabla\rho + \lambda(\nabla\rho\cdot\nabla)\mathbf{u} \\ &\qquad + \frac{\lambda^2}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho - \frac{\lambda^2}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \frac{\lambda^2}{\rho}\Delta\rho\nabla\rho];\end{aligned}$$

$$\begin{aligned}A\mathbf{u}^k \ &=\ P_k[-\rho^k\mathbf{u}_t^k - \rho^k\mathbf{u}^k\cdot\nabla\mathbf{u}^k + \rho^k F + \lambda(\mathbf{u}^k\cdot\nabla)\nabla\rho^k + \lambda(\nabla\rho^k\cdot\nabla)\mathbf{u}^k \\ &\qquad + \frac{\lambda^2}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k - \frac{\lambda^2}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k + \frac{\lambda^2}{\rho^k}\Delta\rho^k\nabla\rho^k];\end{aligned}$$

e fazemos a diferença entre elas. (Lembremos que $|P_k\mathbf{v}|^2 \leq |\mathbf{v}|^2$ para todo $\mathbf{v} \in L^2(\Omega)$ pois o operador projeção P_k é contínuo e também, usando a continuidade de P e o Lema 1.9 (*Rautmann*), temos que $|(P - P_k)\mathbf{v}|^2 \leq \frac{C}{\lambda_{k+1}}|\mathbf{v}|^2_{H^1}$ para todo $\mathbf{v} \in H^1(\Omega)$ conforme [31].) Em seguida, tomamos a norma dos termos resultantes, os quais estimamos do modo como segue:

$$
\begin{aligned}
& |P(\rho\mathbf{u}_t) - P_k(\rho^k\mathbf{u}_t^k)|^2 \\
\leq\ & c|(P-P_k)\rho\mathbf{u}_t|^2 + c|P_k(\rho-\rho^k)\mathbf{u}_t|^2 + c|P_k[\rho^k(\mathbf{u}_t - \mathbf{u}_t^k)]|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|\rho\mathbf{u}_t|_{_{H^1}}^2 + c|\pi^k\mathbf{u}_t|^2 + c|(E_t^k + \theta_t^k)|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}[|\rho|_\infty^2| + |\nabla\rho|_\infty^2]|\nabla\mathbf{u}_t|^2 + c|\pi^k|_{_{H^1}}^2|\nabla\mathbf{u}_t|^2 + c|\theta_t^k|^2 \\
\leq\ & \frac{\overline{c}}{\lambda_{k+1}}|\nabla\mathbf{u}_t|^2 + c|\theta_t^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
|P(\rho F) - P_k(\rho^k F)|^2 &\leq c|(P-P_k)\rho F|^2 + c|P_k(\rho-\rho^k)F|^2 \\
&\leq \frac{c}{\lambda_{k+1}}|\rho F|_{_{H^1}}^2 + c|\pi^k F|^2 \\
&\leq \frac{c}{\lambda_{k+1}}[|\rho|_\infty^2 + |\nabla\rho|_\infty^2]|F|_{_{H^1}}^2 + c|\pi^k|_{_{H^1}}^2|F|_{_{H^1}}^2 \\
&\leq \frac{\overline{c}}{\lambda_{k+1}}|F|_{_{H^1}}^2;
\end{aligned}
$$

$$
\begin{aligned}
& |P((\rho\mathbf{u}\cdot\nabla)\mathbf{u}) - P_k((\rho^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k)|^2 \\
\leq\ & c|(P-P_k)(\rho\mathbf{u}\cdot\nabla)\mathbf{u}|^2 + c|P_k(\rho(\mathbf{u}-\mathbf{u}^k)\cdot\nabla)\mathbf{u}|^2 \\
+\ & c|P_k((\rho-\rho^k)\mathbf{u}^k\cdot\nabla)\mathbf{u}|^2 + c|P_k((\rho^k\mathbf{u}^k\cdot\nabla)(E^k+\theta^k))|^2 \\
\leq\ & \frac{\overline{c}}{\lambda_{k+1}}|(\rho\mathbf{u}\cdot\nabla)\mathbf{u}|_{_{H^1}}^2 + c|((E^k+\theta^k)\cdot\nabla)\mathbf{u}|^2 \\
+\ & c|(\pi^k\mathbf{u}^k\cdot\nabla)\mathbf{u}|^2 + |(\mathbf{u}^k\cdot\nabla)(E^k+\theta^k)|^2 \\
\leq\ & \frac{\overline{c}}{\lambda_{k+1}}[|A\mathbf{u}|^2 + |A\mathbf{u}|^4] + c|\nabla E^k + \nabla\theta^k|^2|A\mathbf{u}|^2 \\
+\ & c|\pi^k|_{_{H^1}}^2|A\mathbf{u}|^2|A\mathbf{u}^k|^2 + c|A\mathbf{u}^k|^2|\nabla E^k + \nabla\theta^k|^2 \\
\leq\ & \frac{\overline{c}}{\lambda_{k+1}};
\end{aligned}
$$

$$
\begin{aligned}
& |P((\mathbf{u}\cdot\nabla)\nabla\rho) - P_k((\mathbf{u}^k\cdot\nabla)\nabla\rho^k)|^2 \\
\leq\ & c|(P-P_k)((\mathbf{u}\cdot\nabla)\nabla\rho)|^2 + c|P_k((\mathbf{u}\cdot\nabla)\nabla(\rho-\rho^k))|^2 \\
+\ & c|P_k(((\mathbf{u}-\mathbf{u}^k)\cdot\nabla)\nabla\rho^k)|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|(\mathbf{u}\cdot\nabla)\nabla\rho|_{_{H^1}}^2 + c|(\mathbf{u}\cdot\nabla)\nabla\pi^k|^2 \\
+\ & c|((E^k+\theta^k)\cdot\nabla)\nabla\rho^k|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|A\mathbf{u}|^2|\nabla\Delta\rho|^2 + c|A\mathbf{u}|^2|\Delta\pi^k|^2 + |\nabla E^k + \nabla\theta^k|^2|\nabla\Delta\rho^k|^2 \\
\leq\ & \frac{\overline{c}}{\lambda_{k+1}} + c|\Delta\pi^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
& |P(\nabla\rho\cdot\nabla)\mathbf{u}) - P_k((\nabla\rho^k\cdot\nabla)\mathbf{u}^k)|^2 \\
\leq\ & c|(P-P_k)((\nabla\rho\cdot\nabla)\mathbf{u}|^2 + c|P_k((\nabla(\rho-\rho^k)\cdot\nabla)\mathbf{u})|^2 \\
+\ & c|P_k((\nabla\rho\cdot\nabla)(\mathbf{u}-\mathbf{u}^k))|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|(\nabla\rho\cdot\nabla)\mathbf{u}|^2_{_{H^1}} + c|(\nabla\pi^k\cdot\nabla)\mathbf{u}|^2 + c|(\nabla\rho^k\cdot\nabla)(E^k+\theta^k)|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}[|\nabla\Delta\rho|^2|A\mathbf{u}|^2 + |\nabla\mathbf{u}|^2_\infty] + c|\nabla\rho^k|^2_\infty|\nabla E^k|^2 + c|\nabla\rho^k|^2_\infty|\nabla\theta^k|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}(1+|\nabla\mathbf{u}|^2_\infty);
\end{aligned}
$$

$$
\begin{aligned}
& \lambda^4|P(\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho) - P_k(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\rho^k)|^2 \\
\leq\ & c|(P-P_k)(\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho)|^2 + c|P_k(\frac{\pi^k}{\rho\rho^k}(\nabla\rho\cdot\nabla)\nabla\rho)|^2 \\
+\ & c|P_k(\frac{1}{\rho^k}(\nabla\pi^k\cdot\nabla)\nabla\rho)|^2 + c|P_k(\frac{1}{\rho^k}(\nabla\rho^k\cdot\nabla)\nabla\pi^k)|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|\frac{1}{\rho}(\nabla\rho\cdot\nabla)\nabla\rho|^2_{_{H^1}} + c|\pi^k|^2_4|\nabla\rho|^2_\infty|\nabla^2\rho|^2_4 \\
+\ & c|\nabla\pi^k|^2_4|\nabla^2\rho|^2_4 + c|\nabla\rho^k|^2_\infty|\nabla^2\pi^k|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|\nabla\Delta\rho|^2 + c|\pi^k|^2_{_{H^1}}|\nabla\Delta\rho|^4 \\
+\ & c|\Delta\pi^k|^2(|\nabla\Delta\rho|^2 + |\nabla\Delta\rho^k|^2) \\
\leq\ & \frac{c}{\lambda_{k+1}} + c|\Delta\pi^k|^2;
\end{aligned}
$$

$$
\begin{aligned}
& \lambda^4|P(\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho) - P_k(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\rho^k)|^2 \\
\leq\ & c|(P-P_k)(\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho)|^2 + c|P_k(\frac{\pi^k(\rho^k+\rho)}{(\rho\rho^k)^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho)|^2 \\
+\ & c|P_k(\frac{1}{(\rho^k)^2}(\nabla\pi^k\cdot\nabla\rho)\nabla\rho)|^2 + c|P_k(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\pi^k)\nabla\rho)|^2 \\
+\ & c|P_k(\frac{1}{(\rho^k)^2}(\nabla\rho^k\cdot\nabla\rho^k)\nabla\pi^k)|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|\frac{1}{\rho^2}(\nabla\rho\cdot\nabla\rho)\nabla\rho|^2_{_{H^1}} + c|\pi^k|^2|\nabla\rho|^6_\infty + c|\nabla\pi^k|^2|\nabla\rho|^4_\infty \\
+\ & c|\nabla\rho^k|^2_\infty|\nabla\pi^k|^2|\nabla\rho|^2_\infty + c|\nabla\rho^k|^4_\infty|\nabla\pi^k|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}} + c|\pi^k|^2_{_{H^1}}|\nabla\Delta\rho|^6 + c|\pi^k|^2_{_{H^1}}|\nabla\Delta\rho|^4 \\
+\ & c|\nabla\Delta\rho^k|^2|\pi^k|^2_{_{H^1}}|\nabla\Delta\rho|^2 + c|\nabla\Delta\rho^k|^4|\pi^k|^2_{_{H^1}} \\
\leq\ & \frac{c}{\lambda_{k+1}};
\end{aligned}
$$

$$
\begin{aligned}
& \lambda^4|P(\frac{1}{\rho}\Delta\rho\nabla\rho) - P_k(\frac{1}{\rho^k}\Delta\rho^k\nabla\rho^k)| \\
\leq\ & c|(P-P_k)(\frac{1}{\rho}\Delta\rho\nabla\rho)|^2 + c|P_k(\frac{\pi^k}{\rho\rho^k}\Delta\rho^k\nabla\rho)|^2 \\
+\ & c|P_k(\frac{1}{\rho^k}\Delta\pi^k\nabla\rho)|^2 + c|P_k(\frac{1}{\rho^k}\Delta\rho^k\nabla\pi^k)|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}}|\frac{1}{\rho}\Delta\rho\nabla\rho|^2_{H^1} + c|\pi^k|_3^2|\Delta\rho^k|_6^2|\nabla\rho|_\infty^2 \\
+\ & c|\Delta\pi^k|^2|\nabla\rho|^2_\infty + c|\Delta\rho^k|_4^2|\nabla\pi^k|^2 \\
\leq\ & \frac{c}{\lambda_{k+1}} + c|\pi^k|^2_{H^1}|\nabla\Delta\rho^k|^2|\nabla\Delta\rho|^2 \\
+\ & c|\Delta\pi^k|^2|\nabla\Delta\rho|^2 + c|\nabla\Delta\rho^k|^2|\pi^k|^2_{H^1} \\
\leq\ & \frac{c}{\lambda_{k+1}} + c|\Delta\pi^k|^2;
\end{aligned}
$$

Assim,

$$\mu|A\mathbf{u} - A\mathbf{u}^k|^2 \leq \frac{c}{\lambda_{k+1}}(|\nabla\mathbf{u}_t|^2 + |\nabla\mathbf{u}|^2_\infty + |F|^2_{H^1} + 1) + c|\theta_t^k|^2 + c|\Delta\pi^k|^2,$$

e, portanto, integrando entre 0 e t e usando as estimativas acima obtemos:

$$\int_0^t |A\mathbf{u}(s) - A\mathbf{u}^k(s)|^2 ds \leq \frac{\bar{c}}{\lambda_{k+1}}$$

para todo $t \in [0, T]$. ■

3.3 Estimativas melhoradas na norma L^2

Como pudemos notar, as estimativas fornecidas pelo Teorema 3.1 não são otimais, já que esperava-se obter uma taxa de convergência da ordem de $1/\lambda_{k+1}^2$ em vez de $1/\lambda_{k+1}$. Apesar de não estarmos em condições de obter a taxa esperada, mostraremos nesta seção que é possível melhorar as taxas obtidas na seção anterior e o faremos usando um argumento de *bootstrap*.

A questão sobre a possibilidade de obter-se essa taxa otimal de convergência na norma $L^2(\Omega)$ já foi colocada anteriormente por Rautmann em seu trabalho [30], para o caso das equações de Navier-Stokes clássicas (ou seja, quando a densidade é constante); Boldrini e Rojas-Medar responderam positivamente a esta questão no trabalho [31].

Teorema 3.3

$$|\mathbf{u}(t) - \mathbf{u}^k(t)|^2 + |\rho(t) - \rho^k(t)|^2 + \int_0^t (|\nabla\mathbf{u}(s) - \nabla\mathbf{u}^k(s)|^2 + |\nabla\rho(s) - \nabla\rho^k(s)|^2) ds$$

$$\leq \frac{\bar{c}}{\lambda_{k+1}^{3/2}}$$

para todo $t \in [0,T]$.

Demonstração. Observamos que a taxa otimal de convergência não é obtida no Teorema 3.1 devido ao seguinte termo:

$$|\int_0^t (\rho^k E_t^k, \theta^k) ds|.$$

Assim sendo, usaremos o Lema 3.2 e poderemos estimar o referido termo de maneira alternativa. Fazendo uma integração por partes com relação a t e lembrando que $\theta^k(0) = 0$, temos então:

$$\begin{aligned}
|\int_0^t (\rho^k E_t^k, \theta^k) ds &= |-\int_0^t (\rho_t^k E^k, \theta^k) ds - \int_0^t (\rho^k E^k, \theta_t^k) ds + (\rho(t) E^k(t), \theta^k(t)| \\
&\leq \int_0^t |(\rho_t^k E^k, \theta^k)| ds + \int_0^t |(\rho^k E^k, \theta_t^k)| ds + |(\rho(t) E^k(t), \theta^k(t)| \\
&\leq \int_0^t |\rho_t^k|_4 |E^k|_4 |\theta^k| ds + \int_0^t |\rho^k|_\infty |E^k| |\theta_t^k| ds + |\rho|_\infty |E^k| |\theta^k| \\
&\leq \frac{c}{\lambda_{k+1}^{3/2}} + \frac{c}{\lambda_{k+1}} \Big[\int_0^t |\theta_t^k|^2 ds\Big]^{1/2} \\
&\leq \frac{\bar{c}}{\lambda_{k+1}^{3/2}}. \qquad \blacksquare
\end{aligned}$$

Como conseqüência do resultado acima, podemos melhorar também a taxa de convergência para a densidade, obtendo assim:

Teorema 3.4

$$|\nabla(\rho - \rho^k)(s)|^2 + \int_0^t |\rho_t(s) - \rho_t^k(s)|^2 ds \leq \frac{\bar{c}}{\lambda_{k+1}^{3/2}}$$

para todo $t \in [0,T]$.

Demonstração. Começamos multiplicando a equação (3.36) por π_t^k e fazendo uma integração por partes; deste modo, obtemos:

$$|\pi_t^k|^2 + \lambda \frac{d}{dt} |\nabla \pi^k|^2 = -(\mathbf{u}^k \nabla \pi^k, \pi_t^k) - ((\mathbf{u} - \mathbf{u}^k) \nabla \rho, \pi_t^k).$$

Os dois termos que se encontram do lado direito da igualdade acima podem ser estimados da seguinte forma:

$$\begin{aligned}
|((\mathbf{u} - \mathbf{u}^k) \nabla \rho, \pi_t^k)| &\leq \varepsilon |\pi_t^k|^2 + c |\mathbf{u} - \mathbf{u}^k|^2 |\nabla \rho|_\infty^2 \\
&\leq \varepsilon |\pi_t^k|^2 + \frac{c}{\lambda_{k+1}^{3/2}} |\nabla \Delta \rho|^2;
\end{aligned}$$

$$|(\mathbf{u}^k\nabla\pi^k,\pi_t^k)| \quad \leq \quad \varepsilon|\pi_t^k|^2 + c|A\mathbf{u}^k|^2|\nabla\pi^k|^2.$$

Ficamos, então, com a seguinte inequação diferencial:

$$\frac{d}{dt}|\nabla\pi^k|^2 + c|\pi_t^k|^2 \quad \leq \quad \frac{c}{\lambda_{k+1}^{3/2}}|\nabla\Delta\rho|^2 + c|A\mathbf{u}^k|^2|\nabla\pi^k|^2.$$

Integrando a inequação acima entre 0 e t obtemos:

$$|\nabla\pi^k(t)|^2 + c\int_0^t |\pi_t^k|^2 ds \leq \frac{c}{\lambda_{k+1}^{3/2}} + c\int_0^t |A\mathbf{u}^k|^2|\nabla\pi^k|^2 ds,$$

uma vez que $|\nabla\pi^k(0)| = 0$.

Consequentemente, chegamos a

$$|\nabla\pi^k(t)|^2 + c\int_0^t |\pi_t^k|^2 ds \leq \frac{c}{\lambda_{k+1}^{3/2}} \cdot \exp c\int_0^t |A\mathbf{u}^k|^2 ds. \qquad \blacksquare$$

Corolário 3.3

$$\int_0^t |\Delta\pi^k|^2 ds \leq \frac{\bar{c}}{\lambda_{k+1}^{3/2}},$$

para todo $t \in [0, T]$.

Demonstração. Podemos ver pela equação (3.36) que:

$$\lambda\Delta\pi^k = \pi_t^k + (\mathbf{u} - \mathbf{u}^k)\nabla\rho + \mathbf{u}^k\nabla\pi^k.$$

Logo,

$$\begin{aligned} \lambda|\Delta\pi^k|^2 \quad &\leq \quad c|\pi_t^k|^2 + c|\mathbf{u} - \mathbf{u}^k|_4^2|\nabla\rho|_4^2 + c|\mathbf{u}^k|_\infty^2|\nabla\pi^k|^2 \\ &\leq \quad c|\pi_t^k|^2 + c|\nabla(\mathbf{u} - \mathbf{u}^k)|^2|\Delta\rho|^2 + c|A\mathbf{u}^k|^2|\nabla\pi^k|^2, \end{aligned}$$

e, portanto, o resultado é conseguido fazendo simplesmente uma integração:

$$\int_0^t |\Delta\pi^k|^2 ds \leq c\int_0^t [|\pi_t^k|^2 + |\nabla(\mathbf{u} - \mathbf{u}^k)|^2 + |\nabla\pi^k|^2] ds \leq \frac{\bar{c}}{\lambda_{k+1}^{3/2}}. \qquad \blacksquare$$

Proposição 3.1

$$|\pi_t^k(t)|^2 + \int_0^t |\nabla\pi_t^k(s)|^2 ds \leq \frac{\bar{c}}{\lambda_{k+1}} \qquad e \qquad |\Delta\pi^k(t)|^2 \leq \frac{\bar{c}}{\lambda_{k+1}},$$

para todo $t \in [0,T]$.

Demonstração. Derivando a equação (3.36) com relação à variável t e tomando o produto interno em $L^2(\Omega)$ com π_t^k, obtemos:

$$\frac{1}{2}\frac{d}{dt}|\pi_t^k|^2 + \lambda|\nabla\pi_t^k|^2 = (\theta_t^k \cdot \nabla\rho, \pi_t^k) + (\theta^k \cdot \nabla\rho_t, \pi_t^k)$$
$$+(E_t^k \cdot \nabla\rho, \pi_t^k) + (E^k \cdot \nabla\rho_t, \pi_t^k) + (\mathbf{u}_t^k \cdot \nabla\pi^k, \pi_t^k).$$

Então, usando as desigualdades de Hölder e de Young, estimamos os termos que se encontram no lado direito da expressão acima e como resultado obtemos a seguinte desigualdade diferencial:

$$\frac{d}{dt}|\pi_t^k|^2 + |\nabla\pi_t^k|^2 \leq c|\theta_t^k|^2 + \frac{c}{\lambda_{k+1}}|\Delta\rho_t|^2 + \frac{c}{\lambda_{k+1}^{\frac{3}{4}}}|\nabla\mathbf{u}_t^k||\Delta\pi^k| + c|\pi_t^k|^2 + \frac{c}{\lambda_{k+1}}.$$

Logo, integrando a expressão acima entre 0 e t e observando que

$$\begin{aligned} |\pi_t^k(0)| &\leq \lambda|\Delta(\rho-\rho^k)(0)| + |(\mathbf{u}-\mathbf{u}^k)(0)|_3|\nabla\rho|_6 + |\mathbf{u}^k|_\infty|\nabla(\rho-\rho^k)(0)| \\ &\leq C|\nabla(\mathbf{u}-\mathbf{u}^k)(0)| \leq c|\nabla\theta^k(0)| \leq \frac{c}{\lambda_{k+1}}, \end{aligned}$$

obtemos:

$$|\pi_t^k|^2 + \int_0^t |\nabla\pi_t^k|^2 ds \leq c\int_0^t |\theta_t^k|^2 ds + \frac{c}{\lambda_{k+1}}\int_0^t |\Delta\rho_t|^2 ds$$
$$+\frac{c}{\lambda_{k+1}^{3/4}}\int_0^t |\nabla\mathbf{u}_t^k||\Delta\pi^k| ds + c\int_0^t |\pi_t^k|^2 ds + \frac{c}{\lambda_{k+1}}.$$
$$\leq \frac{c}{\lambda_{k+1}} + \frac{c}{\lambda_{k+1}^{3/4}}\Big(\int_0^t |\nabla\mathbf{u}_t^k|^2 ds\Big)^{\frac{1}{2}}\Big(\int_0^t |\Delta\pi^k|^2 ds\Big)^{\frac{1}{2}} + c\int_0^t |\pi_t^k|^2 ds$$
$$\leq \frac{c}{\lambda_{k+1}} + c\int_0^t |\pi_t^k|^2 ds,$$

Usando o Lema 3.2 obtemos o resultado. A segunda estimativa segue diretamente da estimativa anterior e da equação (3.36). ∎

Capítulo 4

Método Iterativo para o Modelo de Difusão Simplificado

4.1 Introdução

Neste capítulo utilizaremos um processo iterativo para determinar soluções para o modelo simplificado, isto é, aquele em cuja equação de conservação de momento não aparecem os termos com coeficientes λ^2. Tal método, proposto por Zarubin [40], propõe determinar, num primeiro momento, estimativas *a priori* para a seqüência de aproximações $(\mathbf{u}^n, \rho^n)$ que sejam independentes do nível n de aproximação; em seguida, mostrar que tal seqüência é de Cauchy em espaços de Banach adequados e finalmente passar ao limite na seqüência de problemas aproximados, mostrando que o limite $(\mathbf{u}, \rho)$ da referida seqüência é solução do problema simplificado proposto.

Primeiramente, cabe esclarecer que entenderemos como o problema simplificado, aquele onde, uma vez dado $T > 0$ procura-se determinar um intervalo $[0, T^*]$, com $0 < T^* \leq T$, e uma tripla $(\mathbf{u}, \rho, p)$, solução do sistema de equações com condições iniciais e de fronteira:

$$\begin{cases} \rho\mathbf{u}_t - \mu\Delta\mathbf{u} + (\rho\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda(\mathbf{u}\cdot\nabla)\nabla\rho - \lambda(\nabla\rho\cdot\nabla)\mathbf{u} + \nabla p = \rho F \text{ em } Q \\ \text{div }\mathbf{u} = 0, \qquad \mathbf{u}|_{\Gamma\times(0,T^*)} = 0, \qquad \mathbf{u}(0) = \mathbf{u}^\circ, \\ \rho_t - \lambda\Delta\rho + \mathbf{u}\cdot\nabla\rho = 0, \text{ em } Q, \\ \left.\dfrac{\partial\rho}{\partial\mathbf{n}}\right|_{\Gamma\times(0,T^*)} = 0, \qquad \rho(0) = \rho^\circ, \end{cases} \tag{4.41}$$

onde consideramos $Q = \Omega\times(0, T^*)$, os dados iniciais $\mathbf{u}^\circ(x,t) = \mathbf{u}^\circ(x) \in D(A)$ e $\rho^\circ(x,t) = \rho^\circ(x) \in H^3_N$, com $0 < \alpha \leq \rho^\circ \leq \beta$ e a força $F \in L^2(0,T;H^1(\Omega))$, com $F_t \in L^2(0,T;L^2(\Omega))$.

Assim sendo, colocaremos o problema aproximado da seguinte forma: se $\mathbf{u}^n$ e ρ^n são conhecidas definimos $\mathbf{u}^{n+1}$ e ρ^{n+1} como solução no intervalo $[0, T^*]$ ($T^* > 0$ a ser determinado posteriormente) do seguinte sistema de equações lineares:

$$\begin{aligned} P(\rho^n\mathbf{u}_t^{n+1}) + \mu A\mathbf{u}^{n+1} = P[&-(\rho^n\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + \rho^n F \\ &+\lambda(\mathbf{u}^n\cdot\nabla)\nabla\rho^n + \lambda(\nabla\rho^n\cdot\nabla)\mathbf{u}^n], \end{aligned} \tag{4.42}$$

$$\text{div }\mathbf{u}^{n+1} = 0, \qquad \mathbf{u}^{n+1}|_{\Gamma\times(0,T^*)} = 0, \qquad \mathbf{u}^{n+1}(0) = \mathbf{u}^\circ, \tag{4.43}$$

$$\rho_t^{n+1} - \lambda\Delta\rho^{n+1} + \mathbf{u}^n\cdot\nabla\rho^{n+1} = 0, \tag{4.44}$$

$$\left.\frac{\partial\rho^{n+1}}{\partial\mathbf{n}}\right|_{\Gamma\times(0,T^*)} = 0, \qquad \rho^{n+1}(0) = \rho^\circ. \tag{4.45}$$

Observamos que o sistema acima admite solução para cada n e, para conhecer tal solução, poderíamos usar, por exemplo, o método de Semi-Galerkin espectral assim como foi feito no capítulo 2.

4.2 Estimativas a Priori para as Aproximações da Densidade e da Velocidade

Neste seção estamos interessados em obter estimativas uniformes em n. Para tal, lembramos que pelo princípio do máximo para soluções de equações parabólicas, teremos $\min \rho^\circ \leq \rho^n \leq \max \rho^\circ$, para todo n, ou seja, $0 < \alpha \leq \rho^n \leq \beta$; em seguida, multiplicamos a equação da densidade (4.44) por ρ^{n+1} e tomamos o produto interno de $L^2(\Omega)$. Fazendo uma integração por partes obtemos a equação

$$\frac{1}{2}\frac{d}{dt}|\rho^{n+1}|^2 + \lambda|\nabla\rho^{n+1}|^2 = 0,$$

já que $(\mathbf{u}^n \cdot \nabla\rho^{n+1}, \rho^{n+1}) = -\frac{1}{2}(\text{ div } \mathbf{u}^n, |\rho^{n+1}|^2) = 0$ e $\frac{\partial\rho^{n+1}}{\partial \mathbf{n}}(x,t) = 0$ em Γ. Integrando entre 0 e t obtemos:

$$|\rho^{n+1}|^2 + \int_0^t |\nabla\rho^{n+1}|^2 ds \leq C|\rho^{n+1}(0)|^2 \leq C|\rho^\circ|^2,$$

ou seja,

$$\rho^{\mathbf{n+1}} \in \mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{L^2}(\boldsymbol{\Omega})) \cap \mathbf{L^2}(\mathbf{0},\mathbf{T};\mathbf{H^1}(\boldsymbol{\Omega})).$$

Multiplicando novamente a equação da densidade (4.44) por $-\Delta\rho^{n+1}$ e fazendo uma integração por partes temos:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\nabla\rho^{n+1}|^2 + \lambda|\Delta\rho^{n+1}|^2 &\leq |(\mathbf{u}^n \cdot \nabla\rho^{n+1}, \Delta\rho^{n+1})| \\
&\leq |\mathbf{u}^n|_6|\nabla\rho^{n+1}|_3|\Delta\rho^{n+1}| \\
&\leq C|\nabla\mathbf{u}^n||\nabla\rho^{n+1}|^{\frac{1}{2}}|\nabla\rho^{n+1}|_{H^1}^{\frac{1}{2}}|\Delta\rho^{n+1}| \\
&\leq C|\nabla\mathbf{u}^n|^2|\nabla\rho^{n+1}||\Delta\rho^{n+1}| + \frac{\lambda}{4}|\Delta\rho^{n+1}|^2 \\
&\leq C|\nabla\mathbf{u}^n|^4|\nabla\rho^{n+1}|^2 + \frac{\lambda}{2}|\Delta\rho^{n+1}|^2,
\end{aligned}$$

ou seja,

$$\frac{1}{2}\frac{d}{dt}|\nabla\rho^{n+1}|^2 + \frac{\lambda}{2}|\Delta\rho^{n+1}|^2 \leq C|\nabla\mathbf{u}^n|^4|\nabla\rho^{n+1}|^2.$$

Integrando a inequação acima entre 0 e t temos

$$|\nabla\rho^{n+1}|^2 + \int_0^t |\Delta\rho^{n+1}|^2 ds \leq C|\nabla\rho^\circ|^2 + C\int_0^t |\nabla\mathbf{u}^n|^4|\nabla\rho^{n+1}|^2 ds.$$

Aplicando o lema de Gronwall à inequação acima temos:

$$|\nabla\rho^{n+1}|^2 + \int_0^t |\Delta\rho^{n+1}|^2 ds \leq C|\nabla\rho^\circ|^2 exp(C\int_0^t |\nabla\mathbf{u}^n|^4 ds) \tag{4.46}$$

e, admitindo-se que $|\nabla\mathbf{u}^n|$ é limitado uniformemente em n, temos:

$$|\nabla\rho^{n+1}|^2 + \int_0^t |\Delta\rho^{n+1}|^2 ds \leq C_0$$

e, portanto,

$$\rho^{\mathbf{n+1}} \in \mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H^1}(\boldsymbol{\Omega})) \cap \mathbf{L^2}(\mathbf{0},\mathbf{T};\mathbf{H^2_N}(\boldsymbol{\Omega})).$$

Aplicamos, em seguida, o operador Δ à equação (4.44) e multiplicamos a mesma por $\Delta\rho^{n+1}$. Fazendo uma integração por partes obtemos:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Delta\rho^{n+1}|^2 &+ (\nabla(\lambda\Delta\rho^{n+1} - \mathbf{u}^n\cdot\nabla\rho^{n+1}), \nabla\Delta\rho^{n+1})\\ &= \int_\Gamma \Delta\rho^{n+1}\frac{\partial}{\partial\mathbf{n}}(\lambda\Delta\rho^{n+1} - \mathbf{u}^n\cdot\nabla\rho^{n+1})\\ &= \int_\Gamma \Delta\rho^{n+1}\frac{\partial\rho_t^{n+1}}{\partial\mathbf{n}} = 0.\end{aligned}$$

Assim,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\Delta\rho^{n+1}|^2 + \lambda|\nabla\Delta\rho^{n+1}|^2 \leq |\nabla(\mathbf{u}^n\cdot\nabla\rho^{n+1})||\nabla\Delta\rho^{n+1}|\\ &\leq C|\nabla\mathbf{u}^n|_4|\nabla\rho^{n+1}|_4|\nabla\Delta\rho^{n+1}| + C|\mathbf{u}^n|_6|\nabla^2\rho^{n+1}|_3|\nabla\Delta\rho^{n+1}|\\ &\leq C|A\mathbf{u}^n|^2|\Delta\rho^{n+1}|^2 + C|\nabla\mathbf{u}^n|^2|\Delta\rho^{n+1}|^2 + \frac{\lambda}{2}|\nabla\Delta\rho^{n+1}|^2.\end{aligned}$$

Integrando entre 0 e t, obtemos:

$$\begin{aligned}&|\Delta\rho^{n+1}|^2 + \int_0^t |\nabla\Delta\rho^{n+1}|^2 ds\\ &\leq C|\Delta\rho^\circ|^2 + C\int_0^t (|A\mathbf{u}^n|^2 + |\nabla\mathbf{u}^n|^2)|\Delta\rho^{n+1}|^2 ds.\end{aligned}$$

Aplicando o lema de Gronwall à inequação diferencial acima conseguimos:

$$|\Delta\rho^{n+1}|^2 + \int_0^t |\nabla\Delta\rho^{n+1}|^2 ds$$

$$\leq C|\Delta\rho^\circ|^2 \cdot exp(C\int_0^t |A\mathbf{u}^n|^2 + |\nabla\mathbf{u}^n|^2 ds) \leq C_1, \tag{4.47}$$

admitindo-se que $|A\mathbf{u}^n|$ e $|\nabla\mathbf{u}^n|$ são limitados uniformemente em n (conforme estimativas a seguir). Logo,

$$\rho^{\mathbf{n+1}} \in \mathbf{L}^{\infty}(\mathbf{0},\mathbf{T};\mathbf{H^2_N}(\boldsymbol{\Omega})) \cap \mathbf{L^2}(\mathbf{0},\mathbf{T};\mathbf{H^3_N}(\boldsymbol{\Omega})).$$

Consideramos em seguida, a equação (4.42) tomando o produto interno de $L^2(\Omega)$ com $\mathbf{u}_t^{n+1}$. Após uma integração por partes, obtemos:

$$\frac{\mu}{2}\frac{d}{dt}|\nabla \mathbf{u}^{n+1}|^2 + |(\rho^n)^{\frac{1}{2}}\mathbf{u}_t^{n+1}|^2 = ((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}_t^{n+1})$$

$$+(\rho^n F, \mathbf{u}_t^{n+1}) + \lambda((\nabla\rho^n \cdot \nabla)\mathbf{u}^n, \mathbf{u}_t^{n+1}) + \lambda((\mathbf{u}^n \cdot \nabla)\nabla\rho^n, \mathbf{u}_t^{n+1})$$

Os termos à direita na expressão acima são estimados como segue:

$$\begin{aligned}
|((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}_t^{n+1})| &\leq \beta|\mathbf{u}^n|_6|\nabla\mathbf{u}^{n+1}|_3|\mathbf{u}_t^{n+1}| \\
&\leq C|\nabla\mathbf{u}^n||\nabla\mathbf{u}^{n+1}|^{\frac{1}{2}}|A\mathbf{u}^{n+1}|^{\frac{1}{2}}|\mathbf{u}_t^{n+1}| \\
&\leq C_\delta|\nabla\mathbf{u}^n|^2|\nabla\mathbf{u}^{n+1}||A\mathbf{u}^{n+1}| + \delta|\mathbf{u}_t^{n+1}|^2 \\
&\leq C_\delta|\nabla\mathbf{u}^n|^2|\nabla\mathbf{u}^{n+1}|_3|A\mathbf{u}^{n+1}| + \delta|\mathbf{u}_t^{n+1}|^2 \\
&\leq C_\delta|\nabla\mathbf{u}|^2|\nabla\mathbf{u}^{n+1}|^{\frac{1}{2}}|A\mathbf{u}^{n+1}|^{\frac{3}{2}} + \delta|\mathbf{u}_t^{n+1}|^2 \\
&\leq C_{\delta,\gamma}|\nabla\mathbf{u}^n|^8|\nabla\mathbf{u}^{n+1}|^2 + \gamma|A\mathbf{u}^{n+1}|^2 + \delta|\mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$|(\rho^n F, \mathbf{u}_t^{n+1})| \leq C|F|^2 + \delta|\mathbf{u}_t^{n+1}|^2;$$

$$\begin{aligned}
|((\nabla\rho^n \cdot \nabla)\mathbf{u}^n, \mathbf{u}_t^{n+1})| &\leq C|\nabla\rho^n|_6|\nabla\mathbf{u}^n|_3|\mathbf{u}_t^{n+1}| \\
&\leq C|\Delta\rho^n||\nabla\mathbf{u}^n|^{\frac{1}{2}}|A\mathbf{u}^n|^{\frac{1}{2}}|\mathbf{u}_t^{n+1}| \\
&\leq C_\delta|\Delta\rho^n|^2|\nabla\mathbf{u}^n||A\mathbf{u}^n| + \delta|\mathbf{u}_t^{n+1}|^2
\end{aligned}$$

$$\begin{aligned}
|((\mathbf{u}^n \cdot \nabla)\nabla\rho^n, \mathbf{u}_t^{n+1})| &\leq C|\mathbf{u}^n|_6|\nabla^2\rho^n|_3|\mathbf{u}_t^{n+1}| \\
&\leq C|\nabla\mathbf{u}^n||\Delta\rho^n|^{\frac{1}{2}}|\nabla\Delta\rho^n|^{\frac{1}{2}}|\mathbf{u}_t^{n+1}| \\
&\leq C_\delta|\nabla\mathbf{u}^n|^2|\Delta\rho^n||\nabla\Delta\rho^n| + \delta|\mathbf{u}_t^{n+1}|^2
\end{aligned}$$

Usando as estimativas acima obtidas e escolhendo $\delta = \dfrac{\alpha}{8}$ chegamos à seguinte inequação diferencial:

$$\begin{aligned}
\frac{\mu}{2}\frac{d}{dt}|\nabla\mathbf{u}^{n+1}|^2 + \frac{\alpha}{2}|\mathbf{u}_t^{n+1}|^2 &\leq C|\nabla\mathbf{u}^n|^8|\nabla\mathbf{u}^{n+1}|^2 + \gamma|A\mathbf{u}^{n+1}|^2 \\
+C|F|^2 + C|\Delta\rho^n|^2|\nabla\mathbf{u}^n||A\mathbf{u}^n| &+ C|\nabla\mathbf{u}^n|^2|\Delta\rho^n||\nabla\Delta\rho^n|.
\end{aligned} \tag{4.48}$$

Além disso, temos as seguintes estimativas:

$$|(\rho^n \mathbf{u}_t^{n+1}, A\mathbf{u}^{n+1})| \leq C_\varepsilon\beta^2|\mathbf{u}_t^{n+1}|^2 + \varepsilon|A\mathbf{u}^{n+1}|^2;$$

$$\begin{aligned}
|((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, A\mathbf{u}^{n+1})| &\leq C|\rho^n|_\infty|\mathbf{u}^n|_\infty|\nabla\mathbf{u}^{n+1}||A\mathbf{u}^{n+1}| \\
&\leq C_\varepsilon|\nabla\mathbf{u}^n||A\mathbf{u}^n||\nabla\mathbf{u}^{n+1}|^2 + \varepsilon|A\mathbf{u}^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\nabla\rho^n\cdot\nabla)\mathbf{u}^n, A\mathbf{u}^{n+1})| &\leq C|\nabla\rho^n|_6|\nabla\mathbf{u}^n|_3|A\mathbf{u}^{n+1}|\\
&\leq C|\Delta\rho^n||\nabla\mathbf{u}^n|^{\frac{1}{2}}|A\mathbf{u}^n|^{\frac{1}{2}}|A\mathbf{u}^{n+1}|\\
&\leq C_{\varepsilon}|\Delta\rho^n|^2|\nabla\mathbf{u}^n||A\mathbf{u}^n|+\varepsilon|A\mathbf{u}^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\mathbf{u}^n\cdot\nabla)\nabla\rho^n, A\mathbf{u}^{n+1})| &\leq C|\mathbf{u}^n|_6|\nabla^2\rho^n|_3|A\mathbf{u}^{n+1}|\\
&\leq C|\nabla\mathbf{u}^n||\Delta\rho^n|^{\frac{1}{2}}|\nabla\Delta\rho^n|^{\frac{1}{2}}|A\mathbf{u}^{n+1}|\\
&\leq C_{\varepsilon}|\nabla\mathbf{u}^n|^2|\Delta\rho^n||\nabla\Delta\rho^n|+\varepsilon|A\mathbf{u}^{n+1}|^2;
\end{aligned}$$

$$|(\rho^n F, A\mathbf{u}^{n+1})| \leq C_{\varepsilon}|F|^2+\varepsilon|A\mathbf{u}^{n+1}|^2.$$

Assim, escolhendo $\varepsilon = \dfrac{\mu}{12}$ temos a seguinte estimativa:

$$\frac{\mu}{2}|A\mathbf{u}^{n+1}|^2 \leq C_{\varepsilon}\beta^2|\mathbf{u}_t^{n+1}|^2 + C|\nabla\mathbf{u}^n||A\mathbf{u}^n||\nabla\mathbf{u}^{n+1}|^2$$

$$+C|\Delta\rho^n|^2|\nabla\mathbf{u}^n||A\mathbf{u}^n| + C|\nabla\mathbf{u}^n|^2|\Delta\rho^n||\nabla\Delta\rho^n| + C|F|^2.$$

Multiplicando a inequação acima por $\dfrac{\alpha}{4C_{\varepsilon}\beta^2}$, somando-a à inequação (4.48) e escolhendo a constante $\gamma = \dfrac{\alpha\mu}{16C_{\varepsilon}\beta^2}$ temos:

$$\frac{\mu}{2}\frac{d}{dt}|\nabla\mathbf{u}^{n+1}|^2 + \frac{\alpha}{4}|\mathbf{u}_t^{n+1}|^2 + \frac{\alpha\mu}{16C_{\varepsilon}\beta^2}|A\mathbf{u}^{n+1}|^2$$

$$\leq C(|\nabla\mathbf{u}^n|^8 + |\nabla\mathbf{u}^n||A\mathbf{u}^n|)|\nabla\mathbf{u}^{n+1}|^2$$

$$+C(|F|^2 + |\Delta\rho^n|^2|\nabla\mathbf{u}^n||A\mathbf{u}^n| + |\nabla\mathbf{u}^n|^2|\Delta\rho^n||\nabla\Delta\rho^n|).$$

Logo, integrando entre 0 e t, com $t \leq T$, temos:

$$|\nabla\mathbf{u}^{n+1}|^2 + \int_0^t |\mathbf{u}_t^{n+1}|^2 ds + \int_0^t |A\mathbf{u}^{n+1}|^2 ds$$

$$\leq |\nabla\mathbf{u}^{\circ}|^2 + C\int_0^t (|F|^2 + |\Delta\rho^n|^2|\nabla\mathbf{u}^n||A\mathbf{u}^n| + |\nabla\mathbf{u}^n|^2|\Delta\rho^n||\nabla\Delta\rho^n|)ds$$

$$+C\int_0^t (|\nabla\mathbf{u}^n|^8 + |\nabla\mathbf{u}^n||A\mathbf{u}^n|)|\nabla\mathbf{u}^{n+1}|^2 ds,$$

onde C é uma constante independente de n. Aplicando o lema de Gronwall à desigualdade acima obtemos:

$$|\nabla\mathbf{u}^{n+1}|^2 + \int_0^t |\mathbf{u}_t^{n+1}|^2 ds + \int_0^t |A\mathbf{u}^{n+1}|^2 ds$$

$$\leq \Big[|\nabla \mathbf{u}^\circ|^2 + C\Big(\int_0^t (|F|^2 ds + |\Delta\rho^n|^2 |\nabla \mathbf{u}^n||A\mathbf{u}^n| + |\nabla \mathbf{u}^n|^2 |\Delta\rho^n||\nabla\Delta\rho^n|)ds\Big)\Big] \tag{4.49}$$
$$\cdot\Big(1 + C\int_0^t (|\nabla \mathbf{u}^n|^8 + |\nabla \mathbf{u}^n||A\mathbf{u}^n|)ds\Big)\cdot e^{C\int_0^t (|\nabla \mathbf{u}^n|^8 + |\nabla \mathbf{u}^n||A\mathbf{u}^n|)ds}.$$

Nosso próximo passo é obter uma limitação para as normas de $\mathbf{u}^{n+1}$ e de ρ^{n+1} que seja independente de n. Observamos inicialmente que pelo processo iterativo adotado deveríamos trabalhar simultaneamente com as equações (4.49), (4.46) e (4.47) nesta mesma ordem. Entretanto, como as estimativas dadas pelas desigualdades (4.46) e (4.47) dependem unicamente de $|\nabla \mathbf{u}^n|$ e de $\int_0^t |A\mathbf{u}^n|^2 ds$, faremos aqui apenas as iterações necessárias para obter a limitação (uniforme em n) destes termos, entendendo que estas implicam a limitação uniforme de $|\nabla\rho^n|$ e $|\Delta\rho^n|$, conforme indicam (4.46) e (4.47).

Analisemos, inicialmente, os casos $n = 0, 1, 2$.

Caso $n = 0$

Neste caso, a expressão (4.49) fica:

$$|\nabla \mathbf{u}^1|^2 + \int_0^t |\mathbf{u}_t^1|^2 ds + \int_0^t |A\mathbf{u}^1|^2 ds$$
$$\leq \Big[|\nabla \mathbf{u}^\circ|^2 + C\Big(\int_0^t (|F|^2 ds + |\Delta\rho^\circ|^2 |\nabla \mathbf{u}^\circ||A\mathbf{u}^\circ| + |\nabla \mathbf{u}^\circ|^2 |\Delta\rho^\circ||\nabla\Delta\rho^\circ|)ds\Big)\Big]$$
$$\cdot\Big(1 + C\int_0^t (|\nabla \mathbf{u}^\circ|^8 + |\nabla \mathbf{u}^\circ||A\mathbf{u}^\circ|)ds\Big)\cdot e^{C\int_0^t (|\nabla \mathbf{u}^\circ|^8 + |\nabla \mathbf{u}^\circ||A\mathbf{u}^\circ|)ds}.$$

Em seguida, escolhemos $C_3 > 1$ tal que

$$|\nabla \mathbf{u}^\circ|^2 \leq \frac{C_3}{4}, \quad C\int_0^T |F|^2 ds \leq \frac{C_3}{4}, \quad C|\nabla \mathbf{u}^\circ||\Delta\rho^\circ||A\mathbf{u}^\circ|T \leq \frac{C_3}{4},$$

$$C|\nabla \mathbf{u}^\circ|^2|\Delta\rho^\circ||\nabla\Delta\rho^\circ|T \leq \frac{C_3}{4}, \quad (1 + C\int_0^t (|\nabla \mathbf{u}^\circ|^8 + |\nabla \mathbf{u}^\circ||A\mathbf{u}^\circ|)ds) \leq C_3^{1/2}$$

e $e^{(C|\nabla \mathbf{u}^\circ|^8 T + C|\nabla \mathbf{u}^\circ||A\mathbf{u}^\circ|T)} \leq C_3^{1/2}$.

Assim,

$$|\nabla \mathbf{u}^1|^2 + \int_0^t |\mathbf{u}_t^1|^2 ds + \int_0^t |A\mathbf{u}^1|^2 ds \leq C_3^2,$$

para todo t, com $0 \leq t \leq T$.

Caso n=1

$$|\nabla \mathbf{u}^2|^2 + \int_0^t |\mathbf{u}_t^2|^2 ds + \int_0^t |A\mathbf{u}^2|^2 ds$$

$$
\begin{aligned}
&\leq \Big[|\nabla \mathbf{u}^\circ|^2 + C\int_0^t |F|^2 ds + C\int_0^t (|\nabla \mathbf{u}^1||\Delta\rho^1|^2|A\mathbf{u}^1| + |\nabla \mathbf{u}^1|^2|\Delta\rho^1||\nabla\Delta\rho^1|)ds\Big] \\
&\quad \cdot\Big(1 + C\int_0^t |\nabla \mathbf{u}^1|^8 + |\nabla \mathbf{u}^1||A\mathbf{u}^1|ds\Big)\cdot e^{(C\int_0^t |\nabla \mathbf{u}^1|^8 + |\nabla \mathbf{u}^1||A\mathbf{u}^1|ds)} \\
&\leq \Big[\frac{C_3}{4} + \frac{C_3}{4} + CC_3C_1(\int_0^t |A\mathbf{u}^1|^2 ds)^{\frac12}T^{\frac12} + CC_3^2C_1^{\frac12}(\int_0^t |\nabla\Delta\rho^1|^2 ds)^{\frac12}T^{\frac12}\Big] \\
&\quad \cdot\Big(1 + CC_3^8T + CC_3(\int_0^t |A\mathbf{u}^1|^2 ds)^{\frac12}T^{\frac12}\Big)\cdot e^{(CC_3^8T + CC_3(\int_0^t |A\mathbf{u}^1|^2 ds)^{\frac12}T^{\frac12})} \\
&\leq \Big[\frac{C_3}{4} + \frac{C_3}{4} + CC_3^2C_1T^{\frac12} + CC_3^2C_1T^{\frac12}\Big]\cdot\Big(1 + CC_3^8T + CC_3^2T^{\frac12}\Big)\cdot e^{(CC_3^8T + CC_3^2T^{\frac12})}.
\end{aligned}
$$

Escolhemos, então, $T_1 \leq T$ tal que

$$
2CC_3^2C_1T_1^{\frac12} \leq \frac{C_3}{2}, \qquad 1 + CC_3^8T_1 + CC_3^2T_1^{\frac12} \leq C_3^{\frac12} \qquad \text{e} \qquad e^{(CC_3^8T_1 + CC_3T_1^{\frac12})} \leq C_3^{\frac12}.
$$

e, assim,

$$
|\nabla \mathbf{u}^2|^2 + \int_0^t |\mathbf{u}_t^2|^2 ds + \int_0^t |A\mathbf{u}^2|^2 ds \leq C_3^2,
$$

para todo t, com $0 \leq t \leq T_1$.

Caso n=2

$$
|\nabla \mathbf{u}^3|^2 + \int_0^t |\mathbf{u}_t^3|^2 ds + \int_0^t |A\mathbf{u}^3|^2 ds
$$

$$
\begin{aligned}
&\leq \Big[|\nabla \mathbf{u}^\circ|^2 + C\int_0^t |F|^2 ds + C\int_0^t (|\nabla \mathbf{u}^2||\Delta\rho^2|^2|A\mathbf{u}^2| + |\nabla \mathbf{u}^2|^2|\Delta\rho^2||\nabla\Delta\rho^2|)ds\Big] \\
&\quad \cdot\Big(1 + C\int_0^t |\nabla \mathbf{u}^2|^8 + |\nabla \mathbf{u}^2||A\mathbf{u}^2|ds\Big)\cdot e^{(C\int_0^t |\nabla \mathbf{u}^2|^8 + |\nabla \mathbf{u}^2||A\mathbf{u}^2|ds)} \\
&\leq \Big[\frac{C_3}{4} + \frac{C_3}{4} + CC_3C_1(\int_0^t |A\mathbf{u}^2|^2 ds)^{\frac12}T_1^{\frac12} + CC_3^2C_1^{\frac12}(\int_0^t |\nabla\Delta\rho^2|^2 ds)^{\frac12}T_1^{\frac12}\Big] \\
&\quad \cdot\Big(1 + CC_3^8T_1 + CC_3(\int_0^t |A\mathbf{u}^2|^2 ds)^{\frac12}T_1^{\frac12}\Big)\cdot e^{(CC_3^8T_1 + CC_3(\int_0^t |A\mathbf{u}^2|^2 ds)^{\frac12}T_1^{\frac12})} \\
&\leq \Big[\frac{C_3}{4} + \frac{C_3}{4} + CC_3^2C_1T_1^{\frac12} + CC_3^2C_1T_1^{\frac12}\Big]\cdot\Big(1 + CC_3^8T_1 + CC_3^2T_1^{\frac12}\Big)\cdot e^{(CC_3^8T_1 + CC_3^2T_1^{\frac12})}.
\end{aligned}
$$

Como no passo anterior escolhemos $T_1 \leq T$ de modo que

$$2CC_3^2C_1T_1^{\frac{1}{2}} \leq \frac{C_3}{2}, \qquad 1 + CC_3^8T_1 + CC_3^2T_1^{\frac{1}{2}} \leq C_3^{\frac{1}{2}} \qquad \text{e} \qquad e^{(CC_3^8T_1 + CC_3T_1^{\frac{1}{2}})} \leq C_3^{\frac{1}{2}},$$

então teremos também

$$|\nabla \mathbf{u}^3|^2 + \int_0^t |\mathbf{u}_t^3|^2 ds + \int_0^t |A\mathbf{u}^3|^2 ds \leq C_3^2,$$

para todo t, com $0 \leq t \leq T_1$. Assim, em todas as iterações seguintes podemos sempre considerar o mesmo intervalo $[0, T_1]$. Provemos então que a estimativa acima é válida também para todo $n \geq 4$. (Obviamente, estamos admitindo nesta etapa que as estimativas para ρ^n dadas por (4.46) e (4.47) também são válidas para n=0,1,2.)

Suponhamos que

$$|\nabla \mathbf{u}^n|^2 + \int_0^t |\mathbf{u}_t^n|^2 ds + \int_0^t |A\mathbf{u}^n|^2 ds \leq C_3^2$$

e que

$$|\Delta \rho^n|^2 + \int_0^t |\nabla \Delta \rho^n|^2 ds \leq C_1$$

para todo t, com $0 \leq t \leq T_1$.

Logo,

$$\begin{aligned}
&|\nabla \mathbf{u}^{n+1}|^2 + \int_0^t |\mathbf{u}_t^{n+1}|^2 + \int_0^t |A\mathbf{u}^{n+1}|^2 ds \\
&\leq \left[|\nabla \mathbf{u}^\circ|^2 + C \int_0^t (|F|^2 + |\nabla \mathbf{u}^n||\Delta \rho^n|^2 |A\mathbf{u}^n| + C|\nabla \mathbf{u}^n|^2 |\Delta \rho^n||\nabla \Delta \rho^n| ds) \right] \\
&\cdot \left(1 + \int_0^t C(|\nabla \mathbf{u}^n|^8 + |\nabla \mathbf{u}^n||A\mathbf{u}^n|) ds \right) \cdot e^{\int_0^t C(|\nabla \mathbf{u}^n|^8 + |\nabla \mathbf{u}^n||A\mathbf{u}^n|) ds} \\
&\leq \left[\frac{C_3}{4} + \frac{C_3}{4} + CC_3C_1 (\int_0^t |A\mathbf{u}^n|^2 ds)^{\frac{1}{2}} T_1^{\frac{1}{2}} + CC_3^2 C_1^{\frac{1}{2}} (\int_0^t |\nabla \Delta \rho^n|^2 ds)^{\frac{1}{2}} T_1^{\frac{1}{2}} \right] \\
&\cdot \left(1 + CC_3^8 T_1 + CC_3 (\int_0^t |A\mathbf{u}^n|^2 ds)^{\frac{1}{2}} T_1^{\frac{1}{2}} \right) \cdot e^{(CC_3^8T_1 + CC_3(\int_0^t |A\mathbf{u}^n|^2 ds)^{\frac{1}{2}} T_1^{\frac{1}{2}})} \\
&\leq \left[\frac{C_3}{4} + \frac{C_3}{4} + 2CC_3^2C_1T_1^{\frac{1}{2}} \right] \cdot \left(1 + CC_3^8T_1 + CC_3^2T_1^{\frac{1}{2}} \right) \cdot e^{(CC_3^8T_1 + CC_3^2T_1^{\frac{1}{2}})} \\
&\leq (\frac{C_3}{4} + \frac{C_3}{4} + \frac{C_3}{2}) \cdot C_3^{\frac{1}{2}} \cdot C_3^{\frac{1}{2}} = C_3^2
\end{aligned}$$

para todo t, com $0 \leq t \leq T_1$. Portanto, para todo n,

$$|\nabla \mathbf{u}^{n+1}|^2 + \int_0^t |\mathbf{u}_t^{n+1}|^2 ds + \int_0^t |A\mathbf{u}^{n+1}|^2 ds \leq C_3^2,$$

para todo t, com $0 \leq t \leq T_1$.

A estimativa uniforme acima obtida nos permitirá obter estimativas melhores para as aproximações ρ^n. Usando a equação (4.44) é fácil ver que:

$$|\rho_t^n|^2 \leq C|\Delta\rho^n|^2 + C|\nabla\mathbf{u}^n|^2|\Delta\rho^n|^2 \leq C_7 \tag{4.50}$$

Derivando a equação (4.44) com relação à variável t, aplicando o operador ∇ e, em seguida, tomando o produto interno com $\nabla\rho_t^n$ obtemos (após uma integração por partes):

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\nabla\rho_t^{n+1}|^2 + \lambda|\Delta\rho_t^{n+1}|^2 &= ((\mathbf{u}^n\cdot\nabla\rho^{n+1})_t, \Delta\rho_t^{n+1}) \\
&\leq |(\mathbf{u}^n\cdot\nabla\rho^{n+1})_t||\Delta\rho_t^{n+1}| \\
&\leq \frac{\lambda}{4}|\Delta\rho_t^{n+1}|^2 + C|(\mathbf{u}^n\cdot\nabla\rho^{n+1})_t|^2 \\
&\leq \frac{\lambda}{2}|\Delta\rho_t^{n+1}|^2 + C|\nabla\mathbf{u}_t^n|^2|\Delta\rho^{n+1}|^2 + C|\nabla\mathbf{u}^n|^4|\nabla\rho_t^{n+1}|^2.
\end{aligned}$$

Logo,

$$|\nabla\rho_t^{n+1}|^2 + \int_0^t |\Delta\rho_t^{n+1}|^2 ds \leq |\nabla\rho_t^{n+1}(0)|^2 + C\int_0^t |\nabla\mathbf{u}_t^n|^2 ds + \int_0^t C|\nabla\mathbf{u}^n|^4|\nabla\rho_t^{n+1}|^2 ds.$$

Temos também que:

$$|\nabla\rho_t^{n+1}(0)|^2 \leq C|\nabla\Delta\rho^{n+1}(0)|^2 + C|\nabla\mathbf{u}^n(0)|^2|\nabla\Delta\rho^{n+1}(0)|^2 \tag{4.51}$$

$$\leq C|\nabla\Delta\rho^\circ|^2(1 + C_3^2) \leq C.$$

Usando esta informação na desigualdade anterior teremos:

$$|\nabla\rho_t^{n+1}|^2 + \int_0^t |\Delta\rho_t^{n+1}|^2 ds \leq C + C\int_0^t |\nabla\mathbf{u}_t^n|^2 ds + \int_0^t C|\nabla\mathbf{u}^n|^4|\nabla\rho_t^{n+1}|^2 ds.$$

Em seguida, aplicamos o lema de Gronwal na desigualdade acima e então:

$$|\nabla\rho_t^{n+1}|^2 + \int_0^t |\Delta\rho_t^{n+1}|^2 ds$$

$$\leq \left(C + C\int_0^t |\nabla\mathbf{u}_t^n|^2 ds\right)\cdot\left(1 + \int_0^t C|\nabla\mathbf{u}^n|^4 ds\right)\cdot e^{\int_0^t C|\nabla\mathbf{u}^n|^4 ds}.$$

Logo,

$$|\nabla\rho_t^{n+1}|^2 + \int_0^t |\Delta\rho_t^{n+1}|^2 ds \leq C_2, \tag{4.52}$$

já que, pelo passo anterior $\int_0^t |\nabla\mathbf{u}_t^n|^2 ds \leq C_4^2$ e pela última estimativa para $\mathbf{u}^n$ tínhamos $|\nabla\mathbf{u}^n| \leq C_3$.

Por outro lado,

$$\begin{aligned}
|\nabla\Delta\rho^{n+1}(t)|^2 &\leq C(|\nabla\rho_t^{n+1}(t)|^2+|\nabla\mathbf{u}^n(t)\nabla\rho^{n+1}(t)|^2+|\mathbf{u}^n(t)\nabla^2\rho^{n+1}(t)|^2)\\
&\leq CC_2+C|\nabla\mathbf{u}^n(t)|_3^2|\nabla\rho^{n+1}(t)|_6^2+C|\mathbf{u}^n(t)|_\infty^2|\Delta\rho^{n+1}(t)|^2\\
&\leq CC_2+C|A\mathbf{u}^n(t)|^2|\Delta\rho^{n+1}(t)|^2\\
&\leq CC_2+CC_8C_1,
\end{aligned}$$

onde estamos admitindo já provada a estimativa $|A\mathbf{u}^n(t)|^2\leq C_8$. Portanto,

$$|\nabla\Delta\rho^{n+1}(t)|^2\leq C_5. \tag{4.53}$$

Além disso, ainda admitindo-se que o termo $|A\mathbf{u}^n|$ é limitado uniformemente em n (conforme veremos mais adiante), é possível obter também uma estimativa de ordem mais alta para a densidade. Aplicando o operador Δ à equação (4.44) e em seguida, calculando a norma $L^2(\Omega)$ da expressão obtida, teremos:

$$|\Delta^2\rho^n|^2\leq C|\Delta\rho_t^n|^2+C|A\mathbf{u}^{n-1}|^2|\nabla\Delta\rho^n|^2$$

e integrando entre 0 e t para $0<t<T$, teremos:

$$\int_0^t|\Delta^2\rho^n|^2ds\leq C\int_0^t(|\Delta\rho_t^n|^2+|A\mathbf{u}^{n-1}|^2|\nabla\Delta\rho^n|^2)ds\leq CC_2+CC_8C_5\leq C_6.$$

Assim, esta última estimativa e aquela dada pela expressão (4.53) nos permitem concluir que

$$\rho^{\mathbf{n}}\in\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{3}}_{\mathbf{N}}(\boldsymbol{\Omega}))\cap\mathbf{L}^{\mathbf{2}}(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{4}}_{\mathbf{N}}(\boldsymbol{\Omega})),$$

uniformemente em n.

Derivando a equação (4.42) com relação à variável t e tomando o produto interno de $L^2(\Omega)$ com o multiplicador $\mathbf{u}_t^{n+1}$, temos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|(\rho^n)^{\frac{1}{2}}\mathbf{u}_t^{n+1}|^2+\mu(A\mathbf{u}_t^{n+1},\mathbf{u}_t^{n+1})=-\frac{1}{2}(\rho_t^n\mathbf{u}_t^{n+1},\mathbf{u}_t^{n+1})\\
&-((\rho_t^n\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{u}_t^{n+1})-((\rho^n\mathbf{u}_t^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{u}_t^{n+1})-((\rho^n\mathbf{u}^n\cdot\nabla)\mathbf{u}_t^{n+1},\mathbf{u}_t^{n+1})\\
&+(\rho_t^nF,\mathbf{u}_t^{n+1})+(\rho^nF_t,\mathbf{u}_t^{n+1})+\lambda((\nabla\rho_t^n\cdot\nabla)\mathbf{u}^n,\mathbf{u}_t^{n+1})\\
&+\lambda((\nabla\rho^n\cdot\nabla)\mathbf{u}_t^n,\mathbf{u}_t^{n+1})+\lambda((\mathbf{u}_t^n\cdot\nabla)\nabla\rho^n,\mathbf{u}_t^{n+1},\mathbf{u}_t^{n+1})+\lambda((\mathbf{u}^n\cdot\nabla)\nabla\rho_t^n,\mathbf{u}_t^{n+1}).
\end{aligned}$$

Usando o fato de $\mathbf{u}_t^{n+1}\in V$ temos que

$$\mu(A\mathbf{u}_t^{n+1},\mathbf{u}_t^{n+1})=\mu|\nabla\mathbf{u}_t^{n+1}|^2.$$

Os termos à direita na expressão acima são estimados como segue:

$$\begin{aligned}
|\frac{1}{2}(\rho_t^n \mathbf{u}_t^{n+1}, \mathbf{u}_t^{n+1})| &\leq C|\rho_t^n|_3 |\mathbf{u}_t^{n+1}||\mathbf{u}_t^{n+1}|_6 \\
&\leq C|\rho_t^n|_{H^1} |\mathbf{u}_t^{n+1}||\nabla \mathbf{u}_t^{n+1}| \\
&\leq C_\varepsilon |\rho_t^n|_{H^1}^2 |(\rho^n)^{\frac{1}{2}} \mathbf{u}_t^{n+1}|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho_t^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}_t^{n+1})| &\leq C|\rho_t^n|_6 |\mathbf{u}^n|_\infty |\nabla \mathbf{u}^{n+1}||\mathbf{u}_t^{n+1}|_3 \\
&\leq C|\rho_t^n|_{H^1} |\nabla \mathbf{u}^n|^{\frac{1}{2}} |A\mathbf{u}^n|^{\frac{1}{2}} |\nabla \mathbf{u}^{n+1}||\nabla \mathbf{u}_t^{n+1}| \\
&\leq C_\varepsilon |\rho_t^n|_{H^1}^2 |\nabla \mathbf{u}^n||A\mathbf{u}^n||\nabla \mathbf{u}^{n+1}|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^n \mathbf{u}_t^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}_t^{n+1})| &\leq C|\rho^n|_\infty |\mathbf{u}_t^n|_3 |\nabla \mathbf{u}^{n+1}||\mathbf{u}_t^{n+1}|_6 \\
&\leq C|\mathbf{u}_t^n|^{\frac{1}{2}} |\nabla \mathbf{u}_t^n|^{\frac{1}{2}} |\nabla \mathbf{u}^{n+1}||\nabla \mathbf{u}_t^{n+1}| \\
&\leq C_\varepsilon |\mathbf{u}_t^n||\nabla \mathbf{u}_t^n||\nabla \mathbf{u}^{n+1}|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}_t^{n+1}, \mathbf{u}_t^{n+1})| &\leq C|\rho^n|_\infty |\mathbf{u}^n|_6 |\nabla \mathbf{u}_t^{n+1}||\mathbf{u}_t^{n+1}|_3 \\
&\leq C|\nabla \mathbf{u}^n||\nabla \mathbf{u}_t^{n+1}|^{\frac{3}{2}} |\mathbf{u}_t^{n+1}|^{\frac{1}{2}} \\
&\leq C_\varepsilon |\nabla \mathbf{u}^n|^4 |(\rho^n)^{\frac{1}{2}} \mathbf{u}_t^{n+1}|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\rho_t^n F, \mathbf{u}_t^{n+1})| &\leq C|\rho^n|_3 |F||\mathbf{u}_t^{n+1}|_6 \leq C|\rho_t^n|_{H^1} |F||\nabla \mathbf{u}_t^{n+1}| \\
&\leq C_\varepsilon |\rho_t^n|_{H^1}^2 |F|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$|(\rho^n F_t, \mathbf{u}_t^{n+1})| \leq C|\rho^n|_\infty |F_t||\mathbf{u}_t^{n+1}| \leq C_\varepsilon |F_t|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|;$$

$$\begin{aligned}
|\lambda((\nabla \rho_t^n \cdot \nabla)\mathbf{u}^n, \mathbf{u}_t^{n+1})| &\leq C|\nabla \rho_t^n||\nabla \mathbf{u}^n|_3 |\mathbf{u}_t^{n+1}|_6 \\
&\leq C|\nabla \rho_t^n||\nabla \mathbf{u}^n|^{\frac{1}{2}} |A\mathbf{u}^n|^{\frac{1}{2}} |\nabla \mathbf{u}_t^{n+1}| \\
&\leq C_\varepsilon |\nabla \mathbf{u}^n||A\mathbf{u}^n||\nabla \rho_t^n|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$\begin{aligned}
&|\lambda((\nabla \rho^n \cdot \nabla)\mathbf{u}_t^n, \mathbf{u}_t^{n+1})| = \Big|\lambda \sum_{i,j} \int_\Omega \frac{\partial \rho^n}{\partial x_i} \frac{\partial}{\partial x_i} (\mathbf{u}_t^n)_j (\mathbf{u}_t^{n+1})_j \Big| \\
&\leq \Big| -\lambda \sum_{i,j} \int_\Omega (\mathbf{u}_t^n)_j \frac{\partial}{\partial x_i} (\frac{\partial \rho^n}{\partial x_i} (\mathbf{u}_t^{n+1})_j) \Big| + \Big| \lambda \sum_{i,j} \int_\Gamma \frac{\partial \rho^n}{\partial x_i} (\mathbf{u}_t^{n+1})_j (\mathbf{u}_t^n)_j \cdot \nu_j \Big| \\
&\leq \Big| \lambda \sum_{i,j} \int_\Omega (\mathbf{u}_t^n)_j \frac{\partial^2 \rho^n}{\partial x_i^2} (\mathbf{u}_t^{n+1})_j \Big| + \Big| \lambda \sum_{i,j} \int_\Omega (\mathbf{u}_t^n)_j \frac{\partial \rho^n}{\partial x_i} \frac{\partial}{\partial x_i} (\mathbf{u}_t^{n+1})_j \Big| \\
&\leq C|\mathbf{u}_t^n|_3 |\Delta \rho^n||\mathbf{u}_t^{n+1}|_6 + C|\mathbf{u}_t^n|_3 |\nabla \rho^n|_6 |\nabla \mathbf{u}_t^{n+1}| \\
&\leq C|\mathbf{u}_t^n|^{\frac{1}{2}} |\nabla \mathbf{u}_t^n|^{\frac{1}{2}} |\Delta \rho^n||\nabla \mathbf{u}_t^{n+1}| \\
&\leq C_\varepsilon |\mathbf{u}_t^n||\nabla \mathbf{u}_t^n||\Delta \rho^n|^2 + \varepsilon |\nabla \mathbf{u}_t^{n+1}|^2;
\end{aligned}$$

$$
\begin{aligned}
|\lambda((\mathbf{u}_t^n\cdot\nabla)\nabla\rho^n,\mathbf{u}_t^{n+1})| &\le C|\mathbf{u}_t^n|_3|\nabla^2\rho^n||\mathbf{u}_t^{n+1}|_6\\
&\le C|\mathbf{u}_t^n|^{\frac12}|\nabla\mathbf{u}_t^n|^{\frac12}|\Delta\rho^n||\nabla\mathbf{u}_t^{n+1}|\\
&\le C_\varepsilon|\Delta\rho^n|^2|\mathbf{u}_t^n||\nabla\mathbf{u}_t^n|+\varepsilon|\nabla\mathbf{u}_t^{n+1}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\mathbf{u}^n\cdot\nabla)\nabla\rho_t^n,\mathbf{u}_t^{n+1})| &= |-\lambda((\mathbf{u}^n\cdot\nabla)\mathbf{u}_t^{n+1},\nabla\rho_t^n)|\le C|\mathbf{u}^n|_6|\nabla\mathbf{u}_t^{n+1}||\nabla\rho_t^n|_3\\
&\le C|\nabla\mathbf{u}^n||\nabla\mathbf{u}_t^{n+1}||\nabla\rho_t^n|^{\frac12}|\nabla\rho_t^n|^{\frac12}_{H^1}\\
&\le C_\varepsilon|\nabla\mathbf{u}^n|^2|\nabla\rho_t^n||\Delta\rho_t^n|+\varepsilon|\nabla\mathbf{u}_t^{n+1}|^2
\end{aligned}
$$

Com as estimativas anteriores e tomando $\varepsilon=\dfrac{\mu}{20}$ chegamos à seguinte inequação:

$$
\begin{aligned}
\frac{d}{dt}|(\rho^n)^{\frac12}\mathbf{u}_t^{n+1}|^2+|\nabla\mathbf{u}_t^{n+1}|^2 &\le C[|\rho_t^n|^2_{H^1}+|\nabla\mathbf{u}^n|^4]|(\rho^n)^{\frac12}\mathbf{u}_t^{n+1}|^2\\
&+C[|\nabla\mathbf{u}^{n+1}|^2+|\Delta\rho^n|^2]|\mathbf{u}_t^n||\nabla\mathbf{u}_t^n|+C[|\nabla\rho_t^n||\nabla\mathbf{u}^{n+1}|^2\\
&+C|\nabla\rho_t^n|^2]|\nabla\mathbf{u}^n||A\mathbf{u}^n|\\
&+C|\nabla\mathbf{u}^n|^2|\nabla\rho_t^n||\Delta\rho_t^n|+C|\rho_t^n|^2_{H^1}|F|^2+C|F_t|^2.
\end{aligned}
$$

Integrando a inequação acima entre 0 e t, temos:

$$
|(\rho^n)^{\frac12}\mathbf{u}_t^{n+1}|^2+\int_0^t|\nabla\mathbf{u}_t^{n+1}|^2ds\le C|(\rho^n(0))^{\frac12}\mathbf{u}_t^{n+1}(0)|^2
$$

$$
+C\int_0^t[|\rho_t^n|^2_{H^1}+|\nabla\mathbf{u}^n|^4]|(\rho^n)^{\frac12}\mathbf{u}_t^{n+1}|^2ds
$$

$$
+C\int_0^t[|\nabla\mathbf{u}^{n+1}|^2+|\Delta\rho^n|^2]|\mathbf{u}_t^n||\nabla\mathbf{u}_t^n|ds+C\int_0^t[|\nabla\rho_t^n|^2|\nabla\mathbf{u}^{n+1}|^2+|\nabla\rho_t^n|^2]|\nabla\mathbf{u}^n||A\mathbf{u}^n|ds
$$

$$
+C\int_0^t|\nabla\mathbf{u}^n|^2|\nabla\rho_t^n||\Delta\rho_t^n|ds+C\int_0^t|\rho_t^n|^2_{H^1}|F|^2ds+C\int_0^t|F_t|^2ds,
$$

e, usando o lema de Gronwall:

$$
|(\rho^n)^{\frac12}\mathbf{u}_t^{n+1}|^2+\int_0^t|\nabla\mathbf{u}_t^{n+1}|^2ds\le\Big[|(\rho^\circ)^{\frac12}\mathbf{u}_t^{n+1}(0)|^2
$$

$$
+C\int_0^t(|\nabla\mathbf{u}^{n+1}|^2+|\Delta\rho^n|^2)|\mathbf{u}_t^n||\nabla\mathbf{u}_t^n|ds+C\int_0^t(|\nabla\rho_t^n|^2|\nabla\mathbf{u}^{n+1}|^2+|\nabla\rho_t^n|^2)|\nabla\mathbf{u}^n||A\mathbf{u}^n|ds
$$

$$
+C\int_0^t|\nabla\mathbf{u}^n|^2|\nabla\rho_t^n||\Delta\rho_t^n|ds+C\int_0^t|\rho_t^n|^2_{H^1}|F|^2ds+C\int_0^t|F_t|^2ds\Big]
$$

$$
\cdot\Big[1+C\int_0^t(|\nabla\rho_t^n|^2+|\nabla\mathbf{u}^n|^4ds\Big]\cdot e^{\left(C\int_0^t(|\nabla\rho_t^n|^2+|\nabla\mathbf{u}^n|^4)ds\right)}.
$$

Usando as estimativas já provadas (4.47), (4.49), (4.50) e (4.52) e lembrando que $0 < \alpha \leq \rho^n$, $\rho^\circ \leq \beta$, teremos:

$$|\mathbf{u}_t^{n+1}|^2 + \int_0^t |\nabla \mathbf{u}_t^{n+1}|^2 ds \leq \Big[C|\mathbf{u}_t^{n+1}(0)|^2$$

$$+C(C_3^2 + C_1)\int_0^t |\mathbf{u}_t^n||\nabla \mathbf{u}_t^n| ds + C(C_3^2+1)C_3 \int_0^t |\nabla \rho_t^n|^2 |A\mathbf{u}^n| ds$$

$$+CC_3^2 \int_0^t |\nabla \rho_t^n||\Delta \rho_t^n| ds + C\int_0^t |\rho_t^n|^2_{H^1}|F|^2 ds + C\int_0^t |F_t|^2 ds \Big]$$

$$\cdot \Big[1 + CC_3^4 \cdot t + C\int_0^t |\nabla \rho_t^n|^2 ds\Big] \cdot e^{(CC_3^4 \cdot t + C\int_0^t |\nabla \rho_t^n|^2 ds)},$$

para todo $t \in [0, T_1]$.

Observamos que na desigualdade acima usamos o fato de ser $|\mathbf{u}_t^n(0)|$ limitado uniformemente em n. Podemos verificar a validade de tal fato usando a equação (4.42) aplicada em $t = 0$ e tomando o produto interno da mesma com o termo $\mathbf{u}_t^n(0)$, conforme mostramos a seguir.

$$|(\rho^n(0))^{1/2}\mathbf{u}_t^{n+1}(0)|^2 = -\frac{\mu}{2}(A\mathbf{u}^{n+1}(0), \mathbf{u}_t^{n+1}(0)) - ((\rho^n(0)\mathbf{u}^n(0)\cdot\nabla)\mathbf{u}^{n+1}(0), \mathbf{u}_t^{n+1}(0))$$

$$-(\rho^n(0)F(0), \mathbf{u}_t^{n+1}(0)) - \lambda(\mathbf{u}^n(0)\cdot\nabla)\nabla\rho^n(0), \mathbf{u}_t^{n+1})(0) - \lambda((\nabla\rho^n(0)\cdot\nabla)\mathbf{u}^n(0), \mathbf{u}_t^{n+1}(0));$$

logo,

$$\alpha|\mathbf{u}_t^{n+1}(0)|^2 \leq \frac{\mu}{2}|(A\mathbf{u}^{n+1}(0), \mathbf{u}_t^{n+1}(0))| + |((\rho^n(0)\mathbf{u}^n(0)\cdot\nabla)\mathbf{u}^{n+1}(0), \mathbf{u}_t^{n+1}(0))|$$

$$+|(\rho^n(0)F(0), \mathbf{u}_t^{n+1}(0))| + \lambda|(\mathbf{u}^n(0)\cdot\nabla)\nabla\rho^n(0), \mathbf{u}_t^{n+1}(0))|$$

$$+\lambda|((\nabla\rho^n(0)\cdot\nabla)\mathbf{u}^n(0), \mathbf{u}_t^{n+1}(0))|$$

$$\leq C\Big[|A\mathbf{u}^{n+1}(0)| + |\rho^n(0)|_\infty|\mathbf{u}^n(0)|_4|\nabla\mathbf{u}^{n+1}(0)|_4 + |\rho^n(0)|_\infty|F(0)| + |\mathbf{u}^n(0)|_4|\nabla^2\rho^n(0)|_4$$

$$+|\nabla\rho^n(0)|_4|\nabla\mathbf{u}^n(0)|_4\Big] \cdot |\mathbf{u}_t^{n+1}(0)|$$

$$\leq C\Big[|A\mathbf{u}^\circ| + |\nabla\mathbf{u}^\circ||A\mathbf{u}^\circ| + |F(0)| + |\nabla\mathbf{u}^\circ||\nabla\Delta\rho^\circ| + |\Delta\rho^\circ||A\mathbf{u}^\circ|\Big] \cdot |\mathbf{u}_t^{n+1}(0)|$$

$$\leq C_\alpha\Big[|A\mathbf{u}^\circ| + |\nabla\mathbf{u}^\circ||A\mathbf{u}^\circ| + |F(0)| + |\nabla\mathbf{u}^\circ||\nabla\Delta\rho^\circ| + |\Delta\rho^\circ||A\mathbf{u}^\circ|\Big]^2 + \frac{\alpha}{2}|\mathbf{u}_t^{n+1}(0)|^2$$

$$\Longrightarrow |\mathbf{u}_t^{n+1}(0)|^2 \leq C_\alpha\Big[|A\mathbf{u}^\circ| + |\nabla\mathbf{u}^\circ||A\mathbf{u}^\circ| + |F(0)| + |\nabla\mathbf{u}^\circ||\nabla\Delta\rho^\circ| + |\Delta\rho^\circ||A\mathbf{u}^\circ|\Big]^2 \leq C.$$

Em seguida, analisamos os casos para $n = 0, 1, 2$.

Caso $n = 0$

Neste caso a expressão acima toma a seguinte forma:

$$|\mathbf{u}_t^1|^2 + \int_0^t |\nabla \mathbf{u}_t^1|^2 ds \leq \Big[C|\mathbf{u}_t^1(0)|^2$$

$$+C(C_3^2 + C_1)\int_0^t |\mathbf{u}_t^\circ||\nabla \mathbf{u}_t^\circ| ds + C(C_3^2+1)C_3 \int_0^t |\nabla \rho_t^\circ|^2 |A\mathbf{u}^\circ| ds$$

$$+CC_3^2 \int_0^t |\nabla \rho_t^\circ||\Delta \rho_t^\circ| ds + C\int_0^t |\rho_t^\circ|^2_{H^1} |F|^2 ds + C\int_0^t |F_t|^2 ds \Big]$$

$$\cdot \Big[1 + CC_3^4 \cdot t + C\int_0^t |\nabla \rho_t^\circ|^2 ds \Big] \cdot e^{(CC_3^4 \cdot t + C\int_0^t |\nabla \rho_t^\circ|^2 ds)}.$$

Escolhemos em seguida $C_4 > 1$ tal que

$$C|\mathbf{u}_t^1(0)|^2 \leq \frac{C_4}{6}, \quad C(C_3^2 + C_1)\int_0^{T_1} |\mathbf{u}_t^\circ||\nabla \mathbf{u}_t^\circ| ds \leq \frac{C_4}{6},$$

$$C(C_3^2+1)C_3 \int_0^{T_1} |\nabla \rho_t^\circ|^2 |A\mathbf{u}^\circ| ds \leq \frac{C_4}{6}, \quad CC_3^2 \int_0^{T_1} |\nabla \rho_t^\circ||\Delta \rho_t^\circ| ds \leq \frac{C_4}{6},$$

$$C\int_0^{T_1} |\rho_t^\circ|^2_{H^1} |F|^2 ds \leq \frac{C_4}{6}, \quad C\int_0^{T_1} |F_t|^2 ds \leq \frac{C_4}{6},$$

$$(1 + CC_3^4 \cdot T_1 + C\int_0^{T_1} |\nabla \rho_t^\circ|^2 ds) \leq C_4^{\frac{1}{2}} \quad \text{e} \quad e^{(CC_3^4 \cdot T_1 + C\int_0^{T_1} |\nabla \rho_t^\circ|^2 ds)} \leq C_4^{\frac{1}{2}}$$

para todo $t \in [0, T_1]$. Com esta escolha de C_4 teremos:

$$|\mathbf{u}_t^1|^2 + \int_0^t |\nabla \mathbf{u}_t^1|^2 ds \leq C_4^2,$$

para todo $t \in [0, T_1]$. Para a densidade teremos

$$|\nabla \rho_t^1|^2 + \int_0^t |\Delta \rho_t^1|^2 ds$$

$$\leq \Big(C + C|\nabla \mathbf{u}_t^\circ|^2 T_1 \Big) \cdot \Big(1 + C|\nabla \mathbf{u}^\circ|^4 T_1 \Big) \cdot e^{C|\nabla \mathbf{u}^\circ|^4 T_1}.$$

Escolhemos então $C_2 > 1$ tal que

$$(C + C|\nabla \mathbf{u}_t^\circ|^2 T_1) \leq C_2^{\frac{1}{3}}, \quad (1 + C|\nabla \mathbf{u}^\circ|^4 T_1) \leq C_2^{\frac{1}{3}}, \quad e^{C|\nabla \mathbf{u}^\circ|^4 T_1} \leq C_2^{\frac{1}{3}}$$

$$(C + CC_4^2) \leq C_2^{\frac{1}{3}}), \quad (1 + CC_3^4 T_1) \leq C_2^{\frac{1}{3}} \quad \text{e} \quad e^{CC_3^4 T_1} \leq C_2^{\frac{1}{3}}$$

(sendo que estas três últimas condições sobre C_2 serão úteis apenas a partir da próxima iteração) e então teremos que

$$|\nabla \rho_t^1|^2 + \int_0^t |\Delta \rho_t^1|^2 ds \leq C_2,$$

para todo $t \in [0, T_1]$. Analisemos agora o

Caso $n = 1$

$$|\mathbf{u}_t^2|^2 + \int_0^t |\nabla \mathbf{u}_t^2|^2 ds \leq \Big[C|\mathbf{u}_t^2(0)|^2$$

$$+C(C_3^2 + C_1) \int_0^t |\mathbf{u}_t^1||\nabla \mathbf{u}_t^1| ds + C(C_3^2 + 1)C_3 \int_0^t |\nabla \rho_t^1|^2 |A\mathbf{u}^1| ds$$

$$+CC_3^2 \int_0^t |\nabla \rho_t^1||\Delta \rho_t^1| ds + C \int_0^t |\rho_t^1|^2_{H^1} |F|^2 ds + C \int_0^t |F_t|^2 ds \Big]$$

$$\cdot \Big[1 + CC_3^4 \cdot t + C \int_0^t |\nabla \rho_t^1|^2 ds \Big] \cdot e^{(CC_3^4 \cdot t + C \int_0^t |\nabla \rho_t^1|^2 ds)}$$

$$\leq \Big[C|\mathbf{u}_t^2(0)|^2 + C(C_3^2 + C_1)C_4 (\int_0^t |\nabla \mathbf{u}_t^1|^2 ds)^{\frac{1}{2}} \cdot t^{\frac{1}{2}} + C(C_3^2 + 1)C_3 C_2 C_8^{\frac{1}{2}} \cdot t$$

$$+CC_3^2 C_2^{\frac{1}{2}} (\int_0^t |\Delta \rho_t^1|^2 ds)^{\frac{1}{2}} \cdot t^{\frac{1}{2}} + C(C_2 + C_7) \int_0^t |F|^2 ds + C \int_0^t |F_t|^2 ds \Big]$$

$$\cdot \Big[1 + CC_3^4 \cdot t + CC_2 \cdot t \Big] \cdot e^{(CC_3^4 \cdot t + CC_2 \cdot t)}$$

$$\leq \Big[C|\mathbf{u}_t^2(0)|^2 + C(C_3^2 + C_1)C_4^2 \cdot t^{\frac{1}{2}} + C(C_3^2 + 1)C_3 C_2 C_8^{\frac{1}{2}} \cdot t$$

$$+CC_3^2 C_2 \cdot t^{\frac{1}{2}} + C(C_2 + C_7) \int_0^t |F|^2 ds + C \int_0^t |F_t|^2 ds \Big]$$

$$\cdot \Big[1 + CC_3^4 \cdot t + CC_2 \cdot t \Big] \cdot e^{(CC_3^4 \cdot t + CC_2 \cdot t)}.$$

(Observamos que nesta passagem foi usado $|A\mathbf{u}^n|^2 \leq C_8$ e que será provado posteriormente.) Escolhemos agora $T_2 \leq T_1$ tal que

$$C(C_3^2 + C_1)C_4^2 \cdot T_2^{\frac{1}{2}} \leq \frac{C_4}{6}, \quad C(C_3^2 + 1)C_3 C_2 C_8^{\frac{1}{2}} \cdot T_2 \leq \frac{C_4}{6}, \quad CC_3^2 C_2 \cdot T_2^{\frac{1}{2}} \leq \frac{C_4}{6},$$

$$C(C_2 + C_7) \int_0^{T_2} |F|^2 ds \leq \frac{C_4}{6}, \quad C \int_0^{T_2} |F_t|^2 ds \leq \frac{C_4}{6},$$

$$[1 + CC_3^4 \cdot T_2 + CC_2 \cdot T_2] \leq \frac{C_4^{\frac{1}{2}}}{6} \quad \text{e} \quad e^{(CC_3^4 \cdot T_2 + CC_2 \cdot T_2)} \leq \frac{C_4^{\frac{1}{2}}}{6}.$$

Com essa escolha de T_2 (e lembrando que estamos admitindo que $C|\mathbf{u}_t^n(0)|^2 \leq \frac{C_4}{6}$) obteremos:

$$|\mathbf{u}_t^2|^2 + \int_0^t |\nabla \mathbf{u}_t^2|^2 ds \leq C_4^2,$$

para todo $t \in [0, T_2]$.

Assim como no passo anterior, teremos para a densidade

$$|\nabla\rho_t^2|^2+\int_0^t|\Delta\rho_t^2|^2ds$$

$$\leq\Big(C+C\int_0^t|\nabla\mathbf{u}_t^1|^2ds\Big)\cdot\Big(1+C|\nabla\mathbf{u}^1|^4\cdot T_2\Big)\cdot e^{C|\nabla\mathbf{u}^1|^4\cdot T_2}$$
$$\leq\Big(C+CC_4^2\Big)\cdot\Big(1+CC_3^4\cdot T_2\Big)\cdot e^{CC_3^4\cdot T_2}$$

Lembrando que, pela escolha de C_2 temos

$$(C+CC_4^2)\leq C_2^{\frac{1}{3}},\quad (1+CC_3^4T_2)\leq(1+CC_3^4T_1)\leq C_2^{\frac{1}{3}}$$

$$\text{e}\ \ e^{CC_3^4T_1}\leq e^{CC_3^4T_2}\leq C_2^{\frac{1}{3}},$$

conseguimos

$$|\nabla\rho_t^2|^2+\int_0^t|\Delta\rho_t^2|^2ds\leq C_2,$$

para todo $t\in[0,T_2]$. Veremos que a partir da próxima iteração ($n=2$) todas as estimativas são válidas no mesmo intervalo $[0,T_2]$.

Caso $n=2$

Para $n=2$ teremos:

$$|\mathbf{u}_t^3|^2+\int_0^t|\nabla\mathbf{u}_t^3|^2ds\leq\Big[C|\mathbf{u}_t^3(0)|^2$$

$$+C(C_3^2+C_1)\int_0^t|\mathbf{u}_t^2||\nabla\mathbf{u}_t^2|ds+C(C_3^2+1)C_3\int_0^t|\nabla\rho_t^2|^2|A\mathbf{u}^2|ds$$
$$+CC_3^2\int_0^t|\nabla\rho_t^2||\Delta\rho_t^2|ds+C\int_0^t|\rho_t^2|_{H^1}^2|F|^2ds+C\int_0^t|F_t|^2ds\Big]$$
$$\cdot\Big[1+CC_3^4\cdot t+C\int_0^t|\nabla\rho_t^2|^2ds\Big]\cdot e^{(CC_3^4\cdot t+C\int_0^t|\nabla\rho_t^2|^2ds)}$$
$$\leq\Big[C|\mathbf{u}_t^3(0)|^2+C(C_3^2+C_1)C_4(\int_0^t|\nabla\mathbf{u}_t^2|^2ds)^{\frac{1}{2}}\cdot t^{\frac{1}{2}}+C(C_3^2+1)C_3C_2C_8^{\frac{1}{2}}\cdot t$$
$$+CC_3^2C_2^{\frac{1}{2}}(\int_0^t|\Delta\rho_t^2|^2ds)^{\frac{1}{2}}\cdot t^{\frac{1}{2}}+C(C_2+C_7)\int_0^t|F|^2ds+C\int_0^t|F_t|^2ds\Big]$$
$$\cdot\Big[1+CC_3^4\cdot t+CC_2\cdot t\Big]\cdot e^{(CC_3^4\cdot t+CC_2\cdot t)}$$
$$\leq\Big[C|\mathbf{u}_t^3(0)|^2+C(C_3^2+C_1)C_4^2\cdot t^{\frac{1}{2}}+C(C_3^2+1)C_3C_2C_8^{\frac{1}{2}}\cdot t$$
$$+CC_3^2C_2\cdot t^{\frac{1}{2}}+C(C_2+C_7)\int_0^t|F|^2ds+C\int_0^t|F_t|^2ds\Big]$$

$$\cdot\Big[1 + CC_3^4 \cdot t + CC_2 \cdot t\Big] \cdot e^{(CC_3^4 \cdot t + CC_2 \cdot t)}.$$

$$\leq \Big[C|\mathbf{u}_t^3(0)|^2 + C(C_3^2 + C_1)C_4^2 \cdot T_2^{\frac{1}{2}} + C(C_3^2 + 1)C_3 C_2 C_8^{\frac{1}{2}} \cdot T_2$$

$$+ CC_3^2 C_2 \cdot T_2^{\frac{1}{2}} + C(C_2 + C_7)\int_0^{T_2} |F|^2 ds + C\int_0^{T_2} |F_t|^2 ds\Big]$$

$$\cdot\Big[1 + CC_3^4 \cdot T_2 + CC_2 \cdot T_2\Big] \cdot e^{(CC_3^4 \cdot T_2 + CC_2 \cdot T_2)}.$$

Pela escolha feita de T_2 vemos que

$$|\mathbf{u}_t^3|^2 + \int_0^t |\nabla \mathbf{u}_t^3|^2 ds \leq C_4^2,$$

para todo $t \in [0, T_2]$. E para a densidade teremos

$$|\nabla \rho_t^3|^2 + \int_0^t |\Delta \rho_t^3|^2 ds$$

$$\leq \Big(C + C\int_0^t |\nabla \mathbf{u}_t^2|^2 ds\Big) \cdot \Big(1 + C|\nabla \mathbf{u}^2|^4 \cdot T_2\Big) \cdot e^{C|\nabla \mathbf{u}^2|^4 \cdot T_2}$$

$$\leq \Big(C + CC_4^2\Big) \cdot \Big(1 + CC_3^4 \cdot T_2\Big) \cdot e^{CC_3^4 \cdot T_2}$$

$$\leq C_2,$$

para todo $t \in [0, T_2]$.

A seguir, provaremos por indução matemática, que estas estimativas também são válidas para $n \geq 3$; para tal suponhamos que

$$|\mathbf{u}_t^n|^2 + \int_0^t |\nabla \mathbf{u}_t^n|^2 ds \leq C_4^2,$$

e

$$|\nabla \rho_t^n|^2 + \int_0^t |\Delta \rho_t^n|^2 ds \leq C_2,$$

para todo $t \in [0, T_2]$. Assim,

$$|\mathbf{u}_t^{n+1}|^2 + \int_0^t |\nabla \mathbf{u}_t^n|^2 ds \leq \Big[C|\mathbf{u}_t^{n+1}(0)|^2$$

$$+ C(C_3^2 + C_1)\int_0^t |\mathbf{u}_t^n||\nabla \mathbf{u}_t^n| ds + C(C_3^2 + 1)C_3 \int_0^t |\nabla \rho_t^n|^2 |A\mathbf{u}^n| ds$$

$$+ CC_3^2 \int_0^t |\nabla \rho_t^n||\Delta \rho_t^n| ds + C\int_0^t |\rho_t^n|^2_{H^1} |F|^2 ds + C\int_0^t |F_t|^2 ds\Big]$$

$$\cdot\Big[1 + CC_3^4 \cdot t + C\int_0^t |\nabla \rho_t^n|^2 ds\Big] \cdot e^{(CC_3^4 \cdot t + C\int_0^t |\nabla \rho_t^n|^2 ds)}$$

$$\text{(hipótese de indução)} \leq \Big[C|\mathbf{u}_t^{n+1}(0)|^2 + C(C_3^2 + C_1)C_4^2 \cdot T_2^{\frac{1}{2}} + C(C_3^2+1)C_3C_2C_7^{\frac{1}{2}} \cdot T_2$$

$$+CC_3^2C_2 \cdot T_2^{\frac{1}{2}} + C(C_2 + C_7)\int_0^{T_2} |F|^2 ds + C\int_0^{T_2} |F_t|^2 ds\Big]$$

$$\cdot\Big[1 + CC_3^4 \cdot T_2 + CC_2 \cdot T_2\Big] \cdot e^{(CC_3^4 \cdot t + CC_2 \cdot T_2)}$$

Usando a limitação uniforme de $|\mathbf{u}_t^n(0)|$ e a escolha de T_2 chegamos à estimativa:

$$|\mathbf{u}_t^{n+1}|^2 + \int_0^t |\nabla \mathbf{u}_t^n|^2 ds \leq C_4^2,$$

para todo $t \in [0, T_2]$. Além disso, temos:

$$|\nabla \rho_t^{n+1}|^2 + \int_0^t |\Delta \rho_t^{n+1}|^2 ds$$

$$\leq \left(C + C\int_0^t |\nabla \mathbf{u}_t^n|^2 ds\right) \cdot \left(1 + \int_0^t C|\nabla \mathbf{u}^n|^4 ds\right) \cdot e^{\int_0^t C|\nabla \mathbf{u}^n|^4 ds}$$

$$\text{(hipótese de indução)} \leq \left(C + CC_4^2\right) \cdot \left(1 + CC_3^4 \cdot T_2\right) \cdot e^{CC_3^4 \cdot T_2}.$$

Pela escolha de C_2 e de T_2, concluímos que

$$|\nabla \rho_t^{n+1}|^2 + \int_0^t |\Delta \rho_t^{n+1}|^2 ds \leq C_2,$$

para todo $t \in [0, T_2]$.

Com os resultados obtidos até agora, podemos estimar o termo $|A\mathbf{u}^{n+1}|$. Pela equação (4.42), vemos que:

$$\mu A\mathbf{u}^{n+1} = P[-\rho^n \mathbf{u}_t^{n+1} - (\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \rho^n F$$
$$+\lambda((\nabla \rho^n \cdot \nabla)\mathbf{u}^n) + \lambda(\mathbf{u}^n \cdot \nabla)\nabla \rho^n]. \tag{4.54}$$

Assim,

$$\mu|A\mathbf{u}^{n+1}|^2 \leq C|\rho^n \mathbf{u}_t^{n+1}|^2 + C|(\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}|^2$$

$$+C|\rho^n F|^2 + C|(\nabla \rho^n \cdot \nabla)\mathbf{u}^n|^2 + C|(\mathbf{u}^n \cdot \nabla)\nabla \rho^n|^2.$$

Os termos à direita na desigualdade acima são estimados da seguinte maneira:

$$\begin{aligned} |(\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}|^2 &\leq C|\rho^n|_\infty^2 |\mathbf{u}^n|_4^2 |\nabla \mathbf{u}^{n+1}|_4^2 \\ &\leq C|\nabla \mathbf{u}^n|^2 |\nabla \mathbf{u}^{n+1}||A\mathbf{u}^{n+1}| \\ &\leq C|\nabla \mathbf{u}^n|^4 |\nabla \mathbf{u}^{n+1}|^2 + \frac{\mu}{2}|A\mathbf{u}^{n+1}|^2; \end{aligned}$$

$$|\rho^n F|^2 \leq C|\rho^n|_\infty^2 |F|^2 \leq C|F|^2;$$

$$|(\nabla\rho^n \cdot \nabla)\mathbf{u}^n|^2 \leq C|\nabla\rho^n|_\infty^2 |\nabla\mathbf{u}^n|^2 \leq C|\nabla\Delta\rho^n|^2 |\nabla\mathbf{u}^n|^2;$$

$$|(\mathbf{u}^n \cdot \nabla)\nabla\rho^n|^2 \leq C|\mathbf{u}^n|_3^2 |\nabla^2\rho^n|_6^2 \leq C|\nabla\mathbf{u}^n|^2 |\nabla\Delta\rho^n|^2.$$

Logo,

$$\begin{aligned}
|A\mathbf{u}^{n+1}|^2 &\leq C|\mathbf{u}_t^{n+1}|^2 + C|\nabla\mathbf{u}^n|^4 |\nabla\mathbf{u}^{n+1}|^2 + C|F|^2 + C|\nabla\Delta\rho^n|^2 |\nabla\mathbf{u}^n|^2 \\
&\leq CC_4^2 + CC_3^6 + CC_5C_3^2 + C|F|^2 \\
&\leq C_8,
\end{aligned}$$

já que, pelo Lema 1.4, $F \in C([0,T]; L^2(\Omega))$.

Assim, temos que

$$\mathbf{u^n} \in \mathbf{L}^\infty(\mathbf{0}, \mathbf{T_2}; \mathbf{D}(\mathbf{A})),$$

uniformemente em n. Por outro lado, temos:

$$|\rho^n \mathbf{u}_t^{n+1}|_6 \leq C|\rho^n|_\infty |\mathbf{u}_t^{n+1}|_6 \leq C|\nabla\mathbf{u}_t^{n+1}|;$$

$$|(\rho^n\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}|_6 \leq C|\rho^n|_\infty |\mathbf{u}^n|_\infty |\nabla\mathbf{u}^{n+1}|_6 \leq C|A\mathbf{u}^n||A\mathbf{u}^{n+1}|;$$

$$|\rho^n F|_6 \leq C|\rho^n|_\infty |F|_6 \leq C|F|_{H^1};$$

$$|(\nabla\rho^n \cdot \nabla)\mathbf{u}^n|_6 \leq C|\nabla\rho^n|_\infty |\nabla\mathbf{u}^n|_6 \leq C|\nabla\Delta\rho^n||A\mathbf{u}^n|;$$

$$|(\mathbf{u}^n \cdot \nabla)\nabla\rho^n|_6 \leq C|\mathbf{u}^n|_\infty |\nabla^2\rho^n|_6 \leq C|\nabla\Delta\rho^n||A\mathbf{u}^n|.$$

Assim,

$$\begin{aligned}
|A\mathbf{u}^{n+1}|_6^2 &\leq C|\nabla\mathbf{u}_t^{n+1}|^2 + C|A\mathbf{u}^n|^2 |A\mathbf{u}^{n+1}|^2 + C|F|_{H^1}^2 + C|\nabla\Delta\rho^n|^2 |A\mathbf{u}^n|^2 \\
&\leq C|\nabla\mathbf{u}_t^{n+1}|^2 + C|F|_{H^1}^2 + C.
\end{aligned}$$

Segue-se que

$$\int_0^t |A\mathbf{u}^{n+1}|_6^2 ds \leq C\int_0^t (|\nabla\mathbf{u}_t^{n+1}|^2 + |F|_{H^1}^2 + 1) ds \leq C.$$

Por resultados de Amrouche-Girault [2] temos

$$\int_0^t |u|_{W^{2,6}}^2 ds \leq C,$$

e usando a imersão de Sobolev $W^{2,6}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, temos

$$\int_0^t |\nabla \mathbf{u}^{n+1}|_\infty^2 ds \leq C_9,$$

de onde se conclui que

$$\nabla \mathbf{u}^{\mathbf{n+1}} \in \mathbf{L}^{\mathbf{2}}(\mathbf{0}, \mathbf{T_2}; \mathbf{L}^{\infty}(\mathbf{\Omega})),$$

uniformemente em n.

Mostraremos em seguida que a velocidade aproximada $\mathbf{u}^n$ é uma função de $H^3(\Omega) \cap V$. Se definirmos, para cada n,

$$g^{n+1} = \rho^n \mathbf{u}_t^{n+1} + (\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} - \rho^n F - \lambda(\nabla \rho^n \cdot \nabla)\mathbf{u}^n - \lambda(\mathbf{u}^n \cdot \nabla)\nabla \rho^n,$$

então as estimativas anteriores nos permitirão concluir que $g^{n+1} \in L^2(0, T^*; H^1(\Omega))$. De fato,

$$|\partial_{x_i}(\rho^n \mathbf{u}_t^{n+1})| \leq |\partial_{x_i}\rho^n \cdot \mathbf{u}_t^{n+1}| + |\rho^n \cdot \partial_{x_i}\mathbf{u}_t^{n+1}| \leq |\nabla \rho^n|_4 |\mathbf{u}_t^{n+1}|_4 + |\rho^n|_\infty |\nabla \mathbf{u}_t^{n+1}|$$

$$\leq |\Delta \rho^n||\nabla \mathbf{u}_t^{n+1}| + \beta|\nabla \mathbf{u}_t^{n+1}| \leq C|\nabla \mathbf{u}_t^{n+1}|;$$

$$|\partial_{x_i}((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1})| \leq |\nabla \rho^n|_4 |\mathbf{u}^n|_\infty |\nabla \mathbf{u}^{n+1}|_4 + |\rho^n|_\infty |\nabla \mathbf{u}^n|_4 |\nabla \mathbf{u}^{n+1}|_4$$

$$+|\rho|_\infty^n |\mathbf{u}^n|_\infty |\nabla^2 \mathbf{u}^{n+1}| \leq C|A\mathbf{u}^{n+1}|^2 \leq C;$$

$$|\partial_{x_i}(\rho^n F)| \leq |\nabla \rho^n|_4 |F|_4 + |\rho^n|_\infty |\nabla F| \leq C|F|_{H^1};$$

$$|\partial_{x_i}((\nabla \rho^n \cdot \nabla)\mathbf{u}^n)| \leq |\nabla^2 \rho^n|_4 |\nabla \mathbf{u}^n|_4 + |\nabla \rho^n|_\infty |\nabla^2 \mathbf{u}^n| \leq |\nabla \Delta \rho^n||A\mathbf{u}^n| \leq C;$$

$$|\partial_{x_i}((\mathbf{u}^n \cdot \nabla)\nabla \rho^n)| \leq |\nabla \mathbf{u}^n|_4 |\nabla \rho^n|_4 + |\mathbf{u}^n|_\infty |\nabla \Delta \rho^n| \leq |A\mathbf{u}^n||\Delta \rho^n| + |A\mathbf{u}^n||\nabla \Delta \rho^n| \leq C.$$

Logo, com estas últimas estimativas vemos que $|\partial_{x_i} g|^2 \leq C|\nabla \mathbf{u}_t^{n+1}|^2 + C|F|_{H^1}^2$ e concluímos que $g \in L^2(0, T^*; H^1(\Omega))$. Assim, se $\varphi \in D(\Omega)$ podemos multiplicar a equação (4.54) por $\partial_{x_i}\varphi$ e integrar sobre Ω; após uma integração por partes, teremos:

$$(\nabla \partial_{x_i} \mathbf{u}^{n+1}, \nabla \varphi) = -(\partial_{x_i} g^{n+1}, \varphi)$$

e, utilizando os resultados sobre o operador de Stokes de *Amrouche-Girault* [2] teremos

$$\mathbf{u}^{\mathbf{n+1}} \in \mathbf{L}^{\mathbf{2}}(\mathbf{0}, \mathbf{T}^*; \mathbf{H}^{\mathbf{3}}(\mathbf{\Omega}) \cap \mathbf{V}).$$

uniformemente em n. Além disso, para todo n, temos

$$\int_0^t |\mathbf{u}^n|^2_{H^3} ds \leq C \int_0^t |g^n|^2_{H^1} ds \leq C_{10}$$

Passemos agora a buscar uma estimativa uniforme em n para $\mathbf{u}^n_{tt}$. Começamos por derivar a equação (4.42) com relação à variável t e tomar o produto interno de $L^2(\Omega)$ com o termo $\mathbf{u}^{n+1}_{tt}$ para obter:

$$|(\rho^n)^{\frac{1}{2}} \mathbf{u}^{n+1}_{tt}|^2 + \frac{\mu}{2} \frac{d}{dt} |\nabla \mathbf{u}^{n+1}_t|^2 = -(\rho^n_t \mathbf{u}^{n+1}_t, \mathbf{u}^{n+1}_{tt})$$

$$-((\rho^n_t \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}^{n+1}_{tt}) - ((\rho^n \mathbf{u}^n_t \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}^{n+1}_{tt}) - ((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}_t, \mathbf{u}^{n+1}_{tt})$$
$$+(\rho^n_t F, \mathbf{u}^{n+1}_{tt}) + (\rho^n F_t, \mathbf{u}^{n+1}_{tt}) + \lambda((\nabla \rho^n_t \cdot \nabla)\mathbf{u}^n, \mathbf{u}^{n+1}_{tt})$$
$$+\lambda((\nabla \rho^n \cdot \nabla)\mathbf{u}^n_t, \mathbf{u}^{n+1}_{tt}) + \lambda((\mathbf{u}^n_t \cdot \nabla)\nabla \rho^n, \mathbf{u}^{n+1}_t, \mathbf{u}^{n+1}_{tt}) + \lambda((\mathbf{u}^n \cdot \nabla)\nabla \rho^n_t, \mathbf{u}^{n+1}_{tt}).$$

Passamos agora a estimar os termos que encontram-se do lado direito da igualdade acima.

$$\begin{aligned}
|(\rho^n_t \mathbf{u}^{n+1}_t, \mathbf{u}^{n+1}_{tt})| &\leq |\rho^n_t|_4 |\mathbf{u}^{n+1}_t|_4 |\mathbf{u}^{n+1}_{tt}| \\
&\leq C|\rho^n_t|_{H^1} |\nabla \mathbf{u}^{n+1}_t||\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon |\nabla \mathbf{u}^{n+1}_t|^2 + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^n_t \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}^{n+1}_{tt})| &\leq |\rho^n_t|_4 |\mathbf{u}^n|_\infty |\nabla \mathbf{u}^{n+1}|_4 |\mathbf{u}^{n+1}_{tt}| \\
&\leq |\rho^n_t|_{H^1} |A\mathbf{u}^n||A\mathbf{u}^{n+1}||\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^n \mathbf{u}^n_t \cdot \nabla)\mathbf{u}^{n+1}, \mathbf{u}^{n+1}_{tt})| &\leq |\rho^n|_\infty |\mathbf{u}^n_t|_4 |\nabla \mathbf{u}^{n+1}|_4 |\mathbf{u}^{n+1}_{tt}| \\
&\leq C|\nabla \mathbf{u}^n_t||A\mathbf{u}^{n+1}||\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon |\nabla \mathbf{u}^n_t|^2 + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}_t, \mathbf{u}^{n+1}_{tt})| &\leq |\rho^n|_\infty |\mathbf{u}^n|_\infty |\nabla \mathbf{u}^{n+1}_t||\mathbf{u}^{n+1}_{tt}| \\
&\leq C|A\mathbf{u}^n||\nabla \mathbf{u}^{n+1}_t||\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon |\nabla \mathbf{u}^{n+1}_t|^2 + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\rho^n_t F, \mathbf{u}^{n+1}_{tt})| &\leq |\rho^n_t|_4 |F|_4 |\mathbf{u}^{n+1}_{tt}| \\
&\leq |\rho^n_t|_{H^1} |F|_{H^1} |\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon |F|^2_{H^1} + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\rho^n F_t, \mathbf{u}^{n+1}_{tt})| &\leq |\rho^n|_\infty |F_t||\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon |F_t|^2 + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\nabla \rho^n_t \cdot \nabla)\mathbf{u}^n, \mathbf{u}^{n+1}_{tt})| &\leq |\nabla \rho^n_t|_4 |\nabla \mathbf{u}^n|_4 |\mathbf{u}^{n+1}_{tt}| \\
&\leq |\Delta \rho^n_t||A\mathbf{u}^n||\mathbf{u}^{n+1}_{tt}| \\
&\leq C_\varepsilon |\Delta \rho^n_t|^2 + \varepsilon |\mathbf{u}^{n+1}_{tt}|^2;
\end{aligned}$$

$$
\begin{aligned}
|((\nabla\rho^n\cdot\nabla)\mathbf{u}_t^n,\mathbf{u}_{tt}^{n+1})| &\le |\nabla\rho^n|_\infty|\nabla\mathbf{u}_t^n||\mathbf{u}_{tt}^{n+1}|\\
&\le |\nabla\Delta\rho^n||\nabla\mathbf{u}_t^n||\mathbf{u}_{tt}^{n+1}|\\
&\le C_\varepsilon|\nabla\mathbf{u}_t^n|^2+\varepsilon|\mathbf{u}_{tt}^{n+1}|^2;\\
|((\mathbf{u}_t^n\cdot\nabla)\nabla\rho^n,\mathbf{u}_{tt}^{n+1})| &\le |\mathbf{u}_t^n|_4|\nabla^2\rho^n|_4|\mathbf{u}_{tt}^{n+1}|\\
&\le |\nabla\mathbf{u}_t^n||\nabla\Delta\rho^n||\mathbf{u}_{tt}^{n+1}|\\
&\le C_\varepsilon|\nabla\mathbf{u}_t^n|^2+\varepsilon|\mathbf{u}_{tt}^{n+1}|^2;\\
|((\mathbf{u}^n\cdot\nabla)\nabla\rho_t^n,\mathbf{u}_{tt}^{n+1})| &\le |\mathbf{u}^n|_\infty|\nabla^2\rho_t^n||\mathbf{u}_{tt}^{n+1}|\\
&\le |A\mathbf{u}^n||\Delta\rho_t^n||\mathbf{u}_{tt}^{n+1}|\\
&\le C_\varepsilon|\Delta\rho_t^n|^2+\varepsilon|\mathbf{u}_{tt}^{n+1}|^2.
\end{aligned}
$$

Escolhendo $\varepsilon=\frac{\alpha}{20}$, $\delta=min\{\frac{\alpha}{2},\frac{\mu}{2}\}$ e utilizando as estimativas acima obtidas teremos:

$$
|\mathbf{u}_{tt}^{n+1}|^2+\frac{d}{dt}|\nabla\mathbf{u}_t^{n+1}|^2
$$

$$
\le C_\delta\Big[|\nabla\mathbf{u}_t^{n+1}|^2+|\nabla\mathbf{u}_t^n|^2+|\Delta\rho_t^n|^2+|F|^2_{_{H^1}}+|F_t|^2+1\Big].
$$

Em seguida, multiplicando a desigualdade acima pela função $\sigma(t)=min\{1,t\}$ e integrando-a em (γ,t), onde $\gamma>0$, teremos:

$$
\int_\gamma^t\sigma(s)\frac{d}{ds}|\nabla\mathbf{u}_t^{n+1}(s)|^2ds+\int_\gamma^t\sigma(s)|\mathbf{u}_{tt}^{n+1}(s)|^2ds
$$

$$
\le C\Big[\int_\gamma^t(|\nabla\mathbf{u}_t^{n+1}(s)|^2+|\nabla\mathbf{u}_t^n(s)|^2+|\Delta\rho_t^n(s)|^2+|F(s)|^2_{_{H^1}}+|F_t(s)|^2+1)ds\Big].
$$

Lembrando que

$$
\int_0^t|\nabla\mathbf{u}_t^n(s)|^2ds\le C_4^2\quad\text{e}\quad\int_0^t|\Delta\rho_t^n(s)|^2ds\le C_2,
$$

uniformemente em n,

$$
\int_0^t|F(s)|^2_{_{H^1}}ds<+\infty\quad\text{e}\quad\int_0^t|F_t(s)|^2ds<+\infty,
$$

então a desigualdade acima pode ser reescrita na seguinte forma:

$$
\int_\gamma^t\sigma(s)\frac{d}{ds}|\nabla\mathbf{u}_t^{n+1}(s)|^2ds+\int_\gamma^t\sigma(s)|\mathbf{u}_{tt}^{n+1}(s)|^2ds\le C.
$$

Notamos ainda, que

$$
\int_\gamma^t\sigma(s)\frac{d}{ds}|\nabla\mathbf{u}_t^{n+1}(s)|^2ds=\sigma(t)|\nabla\mathbf{u}_t^{n+1}(t)|^2
$$

$$
-\sigma(\gamma)|\nabla\mathbf{u}_t^{n+1}(\gamma)|^2-\int_\gamma^t\sigma'(s)|\nabla\mathbf{u}_t^{n+1}(s)|^2ds.
$$

Sendo $|\nabla \mathbf{u}_t^{n+1}(t)|^2 \in L^1(0,T_2)$, podemos escolher uma seqüência (γ_k), com $\gamma_k > 0$ e $\gamma_k \to 0$, tal que $\sigma(\gamma_k)|\nabla \mathbf{u}_t^{n+1}(\gamma_k)|^2 \to 0$ quando $\gamma_k \to 0$. Fazendo então, $\gamma_k \to 0$ teremos:

$$\sigma(t)|\nabla \mathbf{u}_t^{n+1}(t)|^2 + \int_0^t \sigma(s)|\mathbf{u}_{tt}^{n+1}(s)|^2 ds$$

$$\leq C + \int_0^t \sigma'(s)|\nabla \mathbf{u}_t^{n+1}(s)|^2 ds \leq C + \int_0^t |\nabla \mathbf{u}_t^{n+1}(s)|^2 ds \leq C_{11},$$

uma vez que $\sigma'(t) \leq 1$ q.t.p. em $[0,T_2]$.

Assim, para todo $\gamma > 0$ temos

$$\mathbf{u}_t^n \in \mathbf{L}^\infty(\gamma, \mathbf{T_2}; \mathbf{V}) \qquad \text{e} \qquad \mathbf{u}_{tt}^n \in \mathbf{L^2}(\gamma, \mathbf{T_2}; \mathbf{H}),$$

uniformemente em n.

Podemos, então, destacar os principais resultados obtidos nessa seção sob a forma do seguinte

Teorema 4.1 *Seja* $(\mathbf{u}^n, \rho^n)$ *a solução do problema* (4.42)-(4.45) *de nível* n *no intervalo* $[0,T_2]$. *São verificadas as seguintes estimativas*:

$$\begin{aligned}
&|\nabla \mathbf{u}^n(t)|^2 + \int_0^t |\mathbf{u}_t^n(s)|^2 ds + \int_0^t |A\mathbf{u}^n(s)|^2 ds \leq C_3^2,\\
&|\rho_t^n(t)|^2 \leq C_7,\\
&|\nabla \rho_t^{n+1}(t)|^2 + \int_0^t |\Delta \rho_t^{n+1}(s)|^2 ds \leq C_2,\\
&|\nabla \Delta \rho^n(t)|^2 \leq C_5,\\
&\int_0^t |\Delta^2 \rho^n(s)|^2 ds \leq C_6,\\
&|\mathbf{u}_t^n(t)|^2 + \int_0^t |\nabla \mathbf{u}_t^n(s)|^2 ds \leq C_4^2,\\
&|A\mathbf{u}^n(t)|^2 \leq C_8,\\
&\int_0^t |\nabla \mathbf{u}^n(s)|_\infty^2 ds \leq C_9,\\
&\int_0^t |\mathbf{u}^n(s)|_{H^3}^2 ds \leq C_{10}\\
&\sigma(t)|\nabla \mathbf{u}_t^n(t)|^2 + \int_0^t \sigma(s)|\mathbf{u}_{tt}^n(s)|^2 ds \leq C_{11},
\end{aligned}$$

para todo $n \geq 1$ *e todo* $t \in [0,T_2]$.

4.3 Taxas de convergência na norma L^2

Destinamos esta seção à obtenção das primeiras taxas de convergência da seqüência de aproximações $(\mathbf{u}^n, \rho^n)$ para a solução $(\mathbf{u}, \rho)$ do problema simplificado. Mais precisamente, mostraremos que $\{\mathbf{u}^n\}$ é seqüência de Cauchy em $L^\infty(0,T_2;H) \cap L^2(0,T_2;V)$ e $\{\rho^n\}$ é seqüência de Cauchy em $L^\infty(0,T_2;L^2(\Omega)) \cap L^2(0,T_2;H^1(\Omega))$, conforme indica a seguinte

Proposição 4.2 *Seja* $\{(\mathbf{u}^n, \rho^n)\}$ *a seqüência de aproximações do problema simplificado. Então existem* $\mathbf{u} \in L^\infty(0, T_2; H) \cap L^2(0, T_2; V)$ *e* $\rho \in L^\infty(0, T_2; L^2(\Omega)) \cap L^2(0, T_2; H^1(\Omega))$ *tais que:*

$$|\mathbf{u}^n(t) - u(t)|^2 + |\rho^n(t) - \rho(t)|^2 \leq M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}} \tag{4.55}$$

$$\int_0^t |\nabla \mathbf{u}^n(s) - u(s)|^2 ds + \int_0^t |\nabla \rho^n(s) - \nabla \rho(s)|^2 ds \leq M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}}, \tag{4.56}$$

para todo $n \geq 1$ *e todo* $t \in [0, T_2]$.

Demonstração. Sejam $\mathbf{u}^{n,s} = \mathbf{u}^{n+s} - \mathbf{u}^n$ e $\rho^{n,s} = \rho^{n+s} - \rho^n$, para todos n, $s \geq 1$. Notamos, inicialmente, que

$$\begin{gathered} P(\rho^{n-1}\mathbf{u}_t^{n,s}) + \mu A\mathbf{u}^{n,s} = -P[\rho^{n-1,s}\mathbf{u}_t^{n+s} + (\rho^{n-1+s}\mathbf{u}^{n-1,s} \cdot \nabla)\mathbf{u}^{n+s} \\ +(\rho^{n-1+s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n,s} + (\rho^{n-1,s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^n - \rho^{n-1,s}F \\ -\lambda(\mathbf{u}^{n-1,s} \cdot \nabla)\nabla\rho^{n-1+s} - \lambda(\mathbf{u}^{n-1} \cdot \nabla)\nabla\rho^{n-1,s} \\ -\lambda(\nabla\rho^{n-1,s} \cdot \nabla)\mathbf{u}^{n-1+s} - \lambda(\nabla\rho^{n-1} \cdot \nabla)\mathbf{u}^{n-1,s}. \end{gathered} \tag{4.57}$$

Tomando o produto interno de $L^2(\Omega)$ na igualdade acima com $\mathbf{u}^{n,s}$ teremos:

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}|(\rho^{n-1})^{\frac{1}{2}}\mathbf{u}^{n,s}|^2 + \mu|\nabla\mathbf{u}^{n,s}|^2 = (\frac{1}{2}\rho_t^{n-1}\mathbf{u}^{n,s} - \rho^{n-1,s}\mathbf{u}_t^{n+s} \\ -(\rho^{n-1+s}\mathbf{u}^{n-1,s} \cdot \nabla)\mathbf{u}^{n+s} - (\rho^{n-1+s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n,s} - (\rho^{n-1,s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^n \\ +\rho^{n-1,s}F + \lambda(\mathbf{u}^{n-1,s} \cdot \nabla)\nabla\rho^{n-1+s} + \lambda(\mathbf{u}^{n-1} \cdot \nabla)\nabla\rho^{n-1,s} \\ +\lambda(\nabla\rho^{n-1,s} \cdot \nabla)\mathbf{u}^{n-1+s} + \lambda(\nabla\rho^{n-1} \cdot \nabla)\mathbf{u}^{n-1,s}, \mathbf{u}^{n,s}). \end{gathered}$$

O próximo passo é, então, estimar os termos que se encontram do lado direito da igualdade acima. (Observamos que, em algumas situações, se faz necessária uma integração por partes antes de majorar o termo em questão. Como o termo $\mathbf{u}^{n,s}$ se anula em Γ, omitimos as integrais de fronteira que possuem o referido termo em seu integrando e consideramos apenas as integrais no interior da região Ω.)

$$\begin{aligned} |(\rho_t^{n-1}\mathbf{u}^{n,s}, \mathbf{u}^{n,s})| &\leq C|\rho_t^{n-1}|_3|\mathbf{u}^{n,s}||\mathbf{u}^{n,s}|_6 \\ &\leq C_\varepsilon|\rho_t^{n-1}||\rho_t^{n-1}|_{H^1}|\mathbf{u}^{n,s}|^2 + \varepsilon|\nabla\mathbf{u}^{n,s}|^2; \end{aligned}$$

$$\begin{aligned} |(\rho^{n-1,s}\mathbf{u}_t^{n+s}, \mathbf{u}^{n,s})| &\leq C|\rho^{n-1,s}||\mathbf{u}_t^{n+s}|_3|\mathbf{u}^{n,s}|_6 \\ &\leq C|\mathbf{u}_t^{n+s}|^{\frac{1}{2}}|\nabla\mathbf{u}_t^{n+s}|^{\frac{1}{2}}|\rho^{n-1,s}||\nabla\mathbf{u}^{n,s}| \\ &\leq C_\varepsilon|\mathbf{u}_t^{n+s}||\nabla\mathbf{u}_t^{n+s}||\rho^{n-1,s}|^2 + \varepsilon|\nabla\mathbf{u}^{n,s}|^2; \end{aligned}$$

$$
\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}^{n,s}| &\le C|\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1,s}||\nabla\mathbf{u}^{n+s}|_3|\mathbf{u}^{n,s}|_6\\
&\le C|A\mathbf{u}^{n+s}||\mathbf{u}^{n-1,s}||\nabla\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|A\mathbf{u}^{n+s}|^2|\mathbf{u}^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s},\mathbf{u}^{n,s})| &\le C|\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^{n,s}||\mathbf{u}^{n,s}|\\
&\le c_\varepsilon|A\mathbf{u}^{n-1}|^2|\mathbf{u}^{n,s}|^2+\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n},\mathbf{u}^{n,s})| &\le C|\rho^{n-1,s}||\mathbf{u}^{n-1}|_6|\nabla\mathbf{u}^{n}|_6|\mathbf{u}^{n,s}|_6\\
&\le C|\nabla\mathbf{u}^{n-1}||A\mathbf{u}^{n}||\rho^{n-1,s}||\nabla\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}^{n-1}|^2|A\mathbf{u}^{n}|^2|\rho^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\rho^{n-1,s}F,\mathbf{u}^{n,s})| &\le C|\rho^{n-1,s}||F|_3|\mathbf{u}^{n,s}|_6\\
&\le C_\varepsilon|F||F|_{H^1}|\rho^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s},\mathbf{u}^{n,s})| &\le C|\mathbf{u}^{n-1,s}||\nabla^2\rho^{n-1+s}|_3|\mathbf{u}^{n,s}|_6\\
&\le C_\varepsilon|\nabla\Delta\rho^{n-1+s}|^2|\mathbf{u}^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho^{n-1,s},\mathbf{u}^{n,s})| &= \Big|\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_j}(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_i}(\mathbf{u}^{n,s})_j\rho^{n-1,s}\Big|\\
&\le C|\nabla\mathbf{u}^{n-1}|_\infty|\rho^{n-1,s}||\nabla\mathbf{u}^{n,s}|\\
&\le C|A\mathbf{u}^{n-1}|^{\frac12}|A\nabla\mathbf{u}^{n-1}|^{\frac12}|\rho^{n-1,s}||\nabla\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|A\mathbf{u}^{n-1}||\mathbf{u}^{n-1}|_{H^3}|\rho^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
&|((\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s},\mathbf{u}^{n,s})| = \Big|-\sum_{i,j}\int_\Omega\rho^{n-1,s}\frac{\partial}{\partial x_i}\Big(\frac{\partial\mathbf{u}_j^{n-1+s}}{\partial x_i}\mathbf{u}_j^{n,s}\Big)\Big|\\
&= \Big|-\sum_{i,j}\int_\Omega\rho^{n-1,s}\frac{\partial^2\mathbf{u}_j^{n-1+s}}{\partial x_i^2}\mathbf{u}_j^{n,s}-\sum_{i,j}\int_\Omega\rho^{n-1,s}\frac{\partial\mathbf{u}_j^{n-1+s}}{\partial x_i}\frac{\partial\mathbf{u}_j^{n,s}}{\partial x_i}\Big|\\
&\le C|\rho^{n-1,s}||A\mathbf{u}^{n-1+s}|_3|\mathbf{u}_j^{n,s}|_6+C|\rho^{n-1,s}||\nabla\mathbf{u}^{n-1+s}|_\infty|\nabla\mathbf{u}^{n,s}|\\
&\le C|\rho^{n-1,s}||A\mathbf{u}^{n-1+s}|^{\frac12}|\nabla A\mathbf{u}^{n-1+s}|^{\frac12}|\nabla\mathbf{u}^{n,s}|\\
&+C|\rho^{n-1,s}||A\mathbf{u}^{n-1+s}|^{\frac12}|\nabla A\mathbf{u}^{n-1+s}|^{\frac12}|\nabla\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|A\mathbf{u}^{n-1+s}||\mathbf{u}^{n-1+s}|_{H^3}|\rho^{n-1,s}|^2+2\varepsilon|\nabla\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
&|((\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},\mathbf{u}^{n,s})| = \Big|\sum_{i,j}\int_\Omega \frac{\partial\rho}{\partial x_i}^{n-1}\frac{\partial \mathbf{u}_j^{n-1,s}}{\partial x_i}\mathbf{u}_j^{n,s}\Big| \\
&= \Big| -\sum_{i,j}\int_\Omega \mathbf{u}_j^{n-1,s}\frac{\partial}{\partial x_i}(\frac{\partial\rho}{\partial x_i}^{n-1}\mathbf{u}_j^{n,s}) + \sum_{i,j}\int_\Gamma \frac{\partial\rho}{\partial x_i}^{n-1}\mathbf{u}_j^{n,s}\mathbf{u}_j^{n-1,s}\cdot\mathbf{n_j}\Big| \\
&\leq \Big| -\sum_{i,j}\int_\Omega \mathbf{u}_j^{n-1,s}\frac{\partial^2\rho}{\partial x_i^2}^{n-1}\mathbf{u}_j^{n,s}\Big| + \Big| -\sum_{i,j}\int_\Omega \mathbf{u}_j^{n-1,s}\frac{\partial\rho}{\partial x_i}^{n-1}\frac{\partial \mathbf{u}_j^{n,s}}{\partial x_i}\Big| \\
&\leq C|\mathbf{u}^{n-1,s}||\Delta\rho^{n-1}|_3|\mathbf{u}^{n,s}|_6 + C|\mathbf{u}^{n-1,s}||\nabla\rho^{n-1}|_\infty|\nabla\mathbf{u}^{n,s}| \\
&\leq C|\nabla\Delta\rho^{n-1}||\mathbf{u}^{n-1,s}||\nabla\mathbf{u}^{n,s}| + C|\nabla\Delta\rho^{n-1}||\mathbf{u}^{n-1,s}||\nabla\mathbf{u}^{n,s}| \\
&\leq C_\varepsilon|\nabla\Delta\rho^{n-1}|^2|\mathbf{u}^{n-1,s}|^2 + 2\varepsilon|\nabla\mathbf{u}^{n,s}|^2.
\end{aligned}
$$

Assim, escolhendo $\varepsilon = \dfrac{\mu}{24}$, as estimativas acima implicam:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|(\rho^{n-1})^{\frac{1}{2}}\mathbf{u}^{n,s}|^2 + \frac{\mu}{2}|\nabla\mathbf{u}^{n,s}|^2 &\leq C(|\rho_t^{n-1}||\rho_t^{n-1}|_{H^1} + |A\mathbf{u}^{n-1}|^2)|\mathbf{u}^{n,s}|^2 \\
&+C(|A\mathbf{u}^{n+s}|^2 + |\nabla\Delta\rho^{n-1+s}|^2 + |\nabla\Delta\rho^{n-1}|^2)|\mathbf{u}^{n-1,s}|^2 \\
&+C(|\mathbf{u}_t^{n+s}||\nabla\mathbf{u}_t^{n+s}| + |\nabla\mathbf{u}^{n-1}|^2|A\mathbf{u}^n|^2 \\
&+|F||F|_{H^1} + |A\mathbf{u}^{n-1}||\mathbf{u}^{n-1}|_{H^3} + |A\mathbf{u}^{n-1+s}||\mathbf{u}^{n-1+s}|_{H^3})|\rho^{n-1,s}|^2 \qquad (4.58) \\
&\leq C|\mathbf{u}^{n,s}|^2 + C|\mathbf{u}^{n-1,s}|^2 + C(|\mathbf{u}_t^{n+s}||\nabla\mathbf{u}_t^{n+s}||\nabla\mathbf{u}^{n-1}|^2|A\mathbf{u}^n|^2 \\
&+|F||F|_{H^1} + |A\mathbf{u}^{n-1}||\mathbf{u}^{n-1}|_{H^3} + |A\mathbf{u}^{n-1+s}||\mathbf{u}^{n-1+s}|_{H^3})|\rho^{n-1,s}|^2
\end{aligned}
$$

Por outro lado, temos que:

$$
\rho_t^{n,s} - \lambda\Delta\rho^{n,s} = -\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s} - \mathbf{u}^{n-1}\cdot\nabla\rho^{n,s}
$$

Tomando o produto interno de $L^2(\Omega)$ com $\rho^{n,s}$ temos:

$$
\frac{1}{2}\frac{d}{dt}|\rho^{n,s}|^2 + \lambda|\nabla\rho^{n,s}|^2 = -(\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s},\rho^{n,s}) - (\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s},\rho^{n,s}),
$$

Estimamos os termos à direita como segue abaixo.

$$
\begin{aligned}
|(\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s},\rho^{n,s})| &\leq C|\mathbf{u}^{n-1,s}||\nabla\rho^{n+s}|_\infty|\rho^{n,s}| \\
&\leq C|\mathbf{u}^{n-1,s}||\nabla\Delta\rho^{n+s}||\rho^{n,s}| \\
&\leq C|\nabla\Delta\rho^{n+s}|^2|\rho^{n,s}|^2 + C|\mathbf{u}^{n-1,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s},\rho^{n,s})| &\leq C|\mathbf{u}^{n-1}|_\infty|\nabla\rho^{n,s}||\rho^{n,s}|\\
&\leq C|A\mathbf{u}^{n-1}||\rho^{n,s}||\nabla\rho^{n,s}|\\
&\leq C_\lambda|A\mathbf{u}^{n-1}|^2|\rho^{n,s}|^2+\frac{\lambda}{2}|\nabla\rho^{n,s}|^2.
\end{aligned}
$$

Assim,

$$
\begin{aligned}
\frac{d}{dt}|\rho^{n,s}|^2+\lambda|\nabla\rho^{n,s}|^2 &\leq C(|\nabla\Delta\rho^{n+s}|^2+|A\mathbf{u}^{n-1}|^2)|\rho^{n,s}|^2+C|\mathbf{u}^{n-1,s}|^2\\
&\leq C|\rho^{n,s}|^2+C|\mathbf{u}^{n-1,s}|^2.
\end{aligned}
\tag{4.59}
$$

Somando-se as inequações (4.58) e (4.59) resulta:

$$
\frac{d}{dt}(|(\rho^{n-1})^{\frac{1}{2}}\mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2)+(|\nabla\mathbf{u}^{n,s}|^2+|\nabla\rho^{n,s}|^2)\leq C(|\mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2)
$$

$$
+C|\mathbf{u}^{n-1,s}|^2+C(|\mathbf{u}_t^{n+s}||\nabla\mathbf{u}_t^{n+s}|+|\nabla\mathbf{u}^{n-1}|^2|A\mathbf{u}^n|^2+|F||F|_{H^1}
$$

$$
+|A\mathbf{u}^{n-1}||\mathbf{u}^{n-1}|_{H^3}+|A\mathbf{u}^{n-1+s}||\mathbf{u}^{n-1+s}|_{H^3})|\rho^{n-1,s}|^2.
$$

Integrando entre 0 e t, com $0\leq t\leq T_2$ e lembrando que $|(\rho^{n-1}(0))^{\frac{1}{2}}\mathbf{u}^{n,s}(0)|=0$ e $|\rho^{n,s}(0)|=0$, teremos:

$$
|\mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2+\int_0^t|\nabla\mathbf{u}^{n,s}|^2d\tau+\int_0^t|\nabla\rho^{n,s}|^2d\tau\leq C\int_0^t(|\mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2)d\tau
$$

$$
+C\int_0^t|\mathbf{u}^{n-1,s}|^2d\tau+C\int_0^t(|\nabla\mathbf{u}_t^{n+s}|+|F|_{H^1})|\rho^{n-1,s}|^2d\tau
$$

$$
+C\int_0^t(|A\mathbf{u}^{n-1}||\mathbf{u}^{n-1}|_{H^3}+|A\mathbf{u}^{n-1+s}||\mathbf{u}^{n-1+s}|_{H^3}+1)|\rho^{n-1,s}|^2d\tau.
$$

Usando o lema de Gronwall, temos:

$$
|\mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2+\int_0^t|\nabla\mathbf{u}^{n,s}|^2d\tau+\int_0^t|\nabla\rho^{n,s}|^2d\tau\leq\Big[C\int_0^t|\mathbf{u}^{n-1,s}|^2d\tau
$$

$$
+C\int_0^t(|\nabla\mathbf{u}_t^{n+s}|+|F|_{H^1}+|\mathbf{u}^{n-1}|_{H^3}+|\mathbf{u}^{n-1+s}|_{H^3}+1)|\rho^{n-1,s}|^2d\tau\Big]\cdot e^{(C\int_0^t d\tau)}
$$

$$
\leq C\int_0^t(|\nabla\mathbf{u}_t^{n+s}|+|F|_{H^1}+|\mathbf{u}^{n-1}|_{H^3}+|\mathbf{u}^{n-1+s}|_{H^3}+1)(|\mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2)d\tau \tag{4.60}
$$

$$\leq C\Big[\int_0^t(|\nabla \mathbf{u}_t^{n+s}|^2+|F|^2_{H^1}+|\mathbf{u}^{n-1}|^2_{H^3}+|\mathbf{u}^{n-1+s}|^2_{H^3}+1)d\tau\Big]^{\frac{1}{2}}$$

$$\cdot\Big[\int_0^t(|\mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2)^2d\tau\Big]^{\frac{1}{2}}$$

$$\leq C\Big[\int_0^t(|\mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2)^2d\tau\Big]^{\frac{1}{2}}.$$

Seja

$$\varphi_{n,s}(t)=|\mathbf{u}^{n,s}(t)|^2+|\rho^{n,s}(t)|^2, \tag{4.61}$$

para todo $t\in[0,T_2]$. Pela desigualdade acima podemos ver que

$$\begin{aligned}\varphi^2_{n,s}(t) &\leq C\int_0^t \varphi^2_{n-1,s}(\tau)d\tau\\ &\leq C^nM^2\int_0^t\frac{\tau^{n-1}}{(n-1)!}d\tau\\ &= C^nM^2\frac{t^n}{n!},\end{aligned}$$

onde $M=M(\mathbf{u}^\circ,\rho^\circ,C_3)$.

Portanto,

$$\varphi^2_{n,s}(t)\leq M^2\frac{(CT_2)^n}{n!},$$

para todo n e todo s.

Notamos que $\varphi^2_{n,s}(t)$ é majorada pelo n-ésimo termo da série $\sum_{k=0}^\infty M^2\frac{(CT_2)^k}{k!}=M^2e^{CT_2}$; logo, $\varphi_{n,s}(t)\longrightarrow 0$ quando $n\longrightarrow\infty$ (e s fixado); assim,

$$\varphi_{n,s}(t)\leq M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}}, \tag{4.62}$$

para todos n, $s\in N$ e todo t, com $0\leq t\leq T_2$.

Voltando à desigualdade (4.60), vemos que

$$\Big(\int_0^t|\nabla\mathbf{u}^{n,s}|^2d\tau+\int_0^t|\nabla\rho^{n,s}|^2d\tau\Big)^2\leq C\Big[\int_0^t(|\mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2)^2d\tau\Big]$$

.

$$=C\int_0^t\varphi^2_{n-1,s}(\tau)d\tau\leq C\int_0^tM^2\frac{(C\tau)^{n-1}}{(n-1)!}d\tau=M^2\frac{(Ct)^n}{n!}\leq M^2\frac{(CT_2)^n}{n!};$$

portanto,

$$\int_0^t|\nabla\mathbf{u}^{n,s}|^2d\tau+\int_0^t|\nabla\rho^{n,s}|^2d\tau\leq\Big[M^2\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}}, \tag{4.63}$$

para todos n, $s \in N$ e todo $t \in [0, T_2]$. Assim, usando as desigualdades (4.62) e (4.63) concluímos que $\{\mathbf{u}^n\}$ é uma seqüência de Cauchy nos espaços de Banach $L^\infty(0, T_2; H)$ e $L^2(0, T_2; V)$ e que $\{\rho^n\}$ é uma seqüência de Cauchy nos espaços de Banach $L^\infty(0, T_2; L^2(\Omega))$ e $L^2(0, T_2; H^1(\Omega))$. Logo, existem $\mathbf{u} \in L^\infty(0, T_2; H) \cap L^2(0, T_2; V)$ e $\rho \in L^\infty(0, T_2; L^2(\Omega)) \cap L^2(0, T_2; H^1(\Omega))$ tais que

$$\lim_{n \to +\infty} \mathbf{u}^n = \mathbf{u} \quad \text{em} \quad L^\infty(0, T_2; H) \cap L^2(0, T_2; V)$$
$$\lim_{n \to +\infty} \rho^n = \rho \quad \text{em} \quad L^\infty(0, T_2; L^2(\Omega)) \cap L^2(0, T_2; H^1(\Omega))$$

Finalmente, fazendo $s \to +\infty$ em (4.62) e (4.63) obtemos as taxas (4.55) e (4.56) do enunciado.
$\blacksquare$

4.4 Taxas de Convergência na norma H^1

Nesta seção, trabalharemos na direção de obtermos taxas de convergência paras as seqüências de aproximações $\{\mathbf{u}^n\}$ e $\{\rho^n\}$ na norma de H^1 bem como taxas de convergência para suas derivadas temporais. A seguir, temos o principal resultado desta seção.

Proposição 4.3 *Sejam $\{(\mathbf{u}^n, \rho^n)\}$ e $(\mathbf{u}, \rho)$ nas condições da Proposição 4.2. Então,*

$$|\nabla \mathbf{u}^n(t) - \nabla \mathbf{u}(t)|^2 + |\rho^n(t) - \rho(t)|^2_{H^1} \leq M \Big[\frac{(CT_2)^n}{n!} \Big]^{\frac{1}{2}}, \tag{4.64}$$

$$\int_0^t |\mathbf{u}_t^n(\tau) - \mathbf{u}_t(\tau)|^2 d\tau + \int_0^t |A\mathbf{u}^n(\tau) - A\mathbf{u}(\tau)|^2 d\tau \leq M \Big[\frac{(CT_2)^n}{n!} \Big]^{\frac{1}{2}}, \tag{4.65}$$

$$|\Delta\rho^n(t) - \Delta\rho(t)|^2 + \int_0^t |\nabla\Delta\rho^n(\tau) - \nabla\Delta\rho(\tau)|^2 d\tau \leq K_1 \Big[\frac{(CT_2)^{n-1}}{(n-1)!} \Big]^{\frac{1}{4}}, \tag{4.66}$$

$$|\rho_t^n(t) - \rho_t(t)|^2 \leq K_2 \Big[\frac{(CT_2)^{n-1}}{(n-1)!} \Big]^{\frac{1}{4}}, \tag{4.67}$$

$$\int_0^t |\nabla\rho_t^n(\tau) - \nabla\rho_t(\tau)|^2 d\tau \leq K_3 \Big[\frac{(CT_2)^{n-1}}{(n-1)!} \Big]^{\frac{1}{4}}, \tag{4.68}$$

para todo $n \geq 1$ e todo $t \in [0, T_2]$.

Demonstração. Começamos tomando o produto interno de $L^2(\Omega)$ de (4.57) com o termo $\mathbf{u}_t^{n,s}$. Após uma integração por partes, obteremos:

$$\frac{\mu}{2}\frac{d}{dt}|\nabla \mathbf{u}^{n,s}|^2 + |(\rho^{n-1})^{\frac{1}{2}} \mathbf{u}_t^{n,s}|^2 = (-\rho^{n-1,s}\mathbf{u}_t^{n+s}$$

$$-(\rho^{n-1+s}\mathbf{u}^{n-1,s} \cdot \nabla)\mathbf{u}^{n+s} - (\rho^{n-1+s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n,s} - (\rho^{n-1,s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^n$$

$$+\rho^{n-1,s}F + \lambda(\mathbf{u}^{n-1,s} \cdot \nabla)\nabla\rho^{n-1+s} + \lambda(\mathbf{u}^{n-1} \cdot \nabla)\nabla\rho^{n-1,s}$$

$$+\lambda(\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s}+\lambda(\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},\mathbf{u}_t^{n,s}).$$

Passamos agora a estimar os termos que se encontram do lado direito da igualdade acima; lembramos novamente que nos casos onde for efetuada uma integração por partes cujo integrando na integral de fronteira seja nulo, omitiremos a referida integral.

$$\begin{aligned}
|(\rho^{n-1,s}\mathbf{u}_t^{n+s},\mathbf{u}_t^{n,s})| &\leq C|\rho^{n-1,s}|_6|\mathbf{u}_t^{n+s}|_3|\mathbf{u}_t^{n,s}|\\
&\leq C|\rho^{n-1,s}|_{H^1}|\mathbf{u}_t^{n+s}|^{\frac{1}{2}}|\nabla\mathbf{u}_t^{n+s}|^{\frac{1}{2}}|\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\nabla\mathbf{u}_t^{n+s}||\rho^{n-1,s}|^2_{H^1}+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{n,s}| &\leq C|\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1,s}|_4|\nabla\mathbf{u}^{n+s}|_4|\mathbf{u}_t^{n,s}|\\
&\leq C|\Delta\rho^{n-1+s}||A\mathbf{u}^{n+s}||\nabla\mathbf{u}^{n-1,s}||\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s},\mathbf{u}_t^{n,s})| &\leq C|\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^{n,s}||\mathbf{u}_t^{n,s}|\\
&\leq |\Delta\rho^{n-1+s}||A\mathbf{u}^{n-1}||\nabla\mathbf{u}^{n,s}||\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n,s}|^2+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n},\mathbf{u}_t^{n,s})| &\leq C|\rho^{n-1,s}|_4|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^{n}|_4|\mathbf{u}_t^{n,s}|\\
&\leq C|\rho^{n-1,s}|_{H^1}|A\mathbf{u}^{n-1}||A\mathbf{u}^{n}||\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\rho^{n-1,s}|^2_{H^1}+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\rho^{n-1,s}F,\mathbf{u}_t^{n,s})| &\leq C|\rho^{n-1,s}|_6|F|_3|\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|F|_3^2|\rho^{n-1,s}|^2_{H^1}+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s},\mathbf{u}_t^{n,s})| &\leq C|\mathbf{u}^{n-1,s}|_4|\nabla^2\rho^{n-1+s}|_4|\mathbf{u}_t^{n,s}|\\
&\leq C|\nabla\Delta\rho^{n-1+s}||\nabla\mathbf{u}^{n-1,s}||\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho^{n-1,s},\mathbf{u}_t^{n,s})| &= \Big|\sum_{i,j}\int_\Omega(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_i}\Big(\frac{\partial\rho^{n-1,s}}{\partial x_j}\Big)(\mathbf{u}_t^{n,s})_j\Big|\\
&= \Big|\sum_{i,j}\int_\Omega(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_j}\Big(\frac{\partial\rho^{n-1,s}}{\partial x_i}\Big)(\mathbf{u}_t^{n,s})_j\Big|\\
&= \Big|-\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_j}\Big((\mathbf{u}^{n-1})_i(\mathbf{u}_t^{n,s})_j\Big)\frac{\partial\rho^{n-1,s}}{\partial x_i}\Big|\\
&= \Big|-\sum_{i,j}\int_\Omega\Big(\frac{\partial}{\partial x_j}(\mathbf{u}^{n-1})_i\Big)(\mathbf{u}_t^{n,s})_j\frac{\partial\rho^{n-1,s}}{\partial x_i}\Big|\\
&\leq C|\nabla\mathbf{u}^{n-1}|_\infty|\nabla\rho^{n-1,s}||\mathbf{u}_t^{n,s}|\\
&\leq C|A\mathbf{u}^{n-1}|^{\frac{1}{2}}|A\nabla\mathbf{u}^{n-1}|^{\frac{1}{2}}|\nabla\rho^{n-1,s}||\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\mathbf{u}^{n-1}|_{H^3}|\rho^{n-1,s}|^2_{H^1}+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$
\begin{array}{rcl}
|((\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s},\mathbf{u}_t^{n,s})| & \leq & C|\nabla\rho^{n-1,s}||\nabla\mathbf{u}^{n-1+s}|_\infty|\mathbf{u}_t^{n,s}| \\
& \leq & C|\rho^{n-1,s}|_{_{H^1}}|A\mathbf{u}^{n-1+s}|^{\frac{1}{2}}|A\nabla\mathbf{u}^{n-1+s}|^{\frac{1}{2}}|\mathbf{u}_t^{n,s}| \\
& \leq & C_\varepsilon|\mathbf{u}^{n-1+s}|_{_{H^3}}|\rho^{n-1,s}|^2_{_{H^1}}+\varepsilon|\mathbf{u}_t^{n,s}|^2;
\end{array}
$$

$$
\begin{array}{rcl}
|((\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},\mathbf{u}_t^{n,s})| & \leq & C|\nabla\rho^{n-1}|_\infty|\nabla\mathbf{u}^{n-1,s}||\mathbf{u}_t^{n,s}| \\
& \leq & C|\nabla\Delta\rho^{n-1}||\nabla\mathbf{u}^{n-1,s}||\mathbf{u}_t^{n,s}| \\
& \leq & C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|\mathbf{u}_t^{n,s}|^2.
\end{array}
$$

Usando estas estimativas e escolhendo $\varepsilon=\frac{1}{18}$ (e lembrando que $0<\alpha\leq\rho^n$) teremos a seguinte desigualdade :

$$
\frac{\mu}{2}\frac{d}{dt}|\nabla\mathbf{u}^{n,s}|^2+|(\rho^{n-1})^{\frac{1}{2}}\mathbf{u}_t^{n,s}|^2\leq C|\nabla\mathbf{u}^{n,s}|^2+C|\nabla\mathbf{u}^{n-1,s}|^2
$$

$$
+C\Big(|\nabla\mathbf{u}_t^{n+s}|+|F|_3^2+|\mathbf{u}^{n-1}|_{_{H^3}}+|\mathbf{u}^{n-1+s}|_{_{H^3}}+1\Big)|\rho^{n-1,s}|^2_{_{H^1}}. \tag{4.69}
$$

Tomando o produto interno de $L^2(\Omega)$ na igualdade (4.57) com o termo $\theta A\mathbf{u}^{n,s}$, com $\theta>0$ (a ser escolhido posteriormente), teremos:

$$
\begin{gathered}
\theta\mu|A\mathbf{u}^{n,s}|^2=(\rho^{n-1}\mathbf{u}_t^{n,s}-\rho^{n-1,s}\mathbf{u}_t^{n+s},\theta A\mathbf{u}^{n,s}) \\
(-(\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s}-(\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s}-(\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n},\theta A\mathbf{u}^{n,s}) \\
(+\rho^{n-1,s}F+\lambda(\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s}+\lambda(\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho^{n-1,s},\theta A\mathbf{u}^{n,s}) \\
(+\lambda(\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s}+\lambda(\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},\theta A\mathbf{u}^{n,s}).
\end{gathered}
$$

Passemos agora, às estimativas dos termos que estão do lado direito da expressão acima.

$$
\begin{array}{rcl}
|(\rho^{n-1}\mathbf{u}_t^{n,s},\theta A\mathbf{u}^{n,s})| & \leq & C\theta|\rho^{n-1}|_\infty|\mathbf{u}_t^{n,s}||A\mathbf{u}^{n,s}| \\
& \leq & C\theta^2|\mathbf{u}_t^{n,s}|^2+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{array}
$$

$$
\begin{array}{rcl}
|(\rho^{n-1,s}\mathbf{u}_t^{n+s},\theta A\mathbf{u}^{n,s})| & \leq & C\theta|\rho^{n-1,s}|_6|\mathbf{u}_t^{n+s}|_3|A\mathbf{u}^{n,s}| \\
& \leq & C|\rho^{n-1,s}|_{_{H^1}}|\mathbf{u}_t^{n+s}|^{\frac{1}{2}}|\nabla\mathbf{u}_t^{n+s}|^{\frac{1}{2}}|A\mathbf{u}^{n,s}| \\
& \leq & C_\varepsilon|\nabla\mathbf{u}_t^{n+s}||\rho^{n-1,s}|^2_{_{H^1}}+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{array}
$$

$$
\begin{array}{rcl}
|((\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s},\theta A\mathbf{u}^{n,s})| & \leq & C\theta|\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1,s}|_4|\nabla\mathbf{u}^{n+s}|_4|A\mathbf{u}^{n,s}| \\
& \leq & C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{array}
$$

$$
\begin{array}{rcl}
|((\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s},\theta A\mathbf{u}^{n,s})| & \leq & C\theta|\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^{n,s}||A\mathbf{u}^{n,s}| \\
& \leq & C_\varepsilon|\nabla\mathbf{u}^{n,s}|^2+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{array}
$$

$$
\begin{aligned}
|((\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^n,\theta A\mathbf{u}^{n,s})| &\le C\theta|\rho^{n-1,s}|_4|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^n|_4|A\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|\rho^{n-1,s}|^2_{H^1}+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\rho^{n-1,s}F,\theta A\mathbf{u}^{n,s})| &\le C\theta|\rho^{n-1,s}|_6|F|_3|A\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|F|_3^2|\rho^{n-1,s}|^2_{H^1}+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s},\theta A\mathbf{u}^{n,s})| &\le C\theta|\mathbf{u}^{n-1,s}|_4|\nabla^2\rho^{n-1+s}|_4|A\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho^{n-1,s},\theta A\mathbf{u}^{n,s})| &= \Big|\theta\sum_{i,j}\int_\Omega(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_i}\Big(\frac{\partial\rho^{n-1,s}}{\partial x_j}\Big)(A\mathbf{u}^{n,s})_j\Big|\\
&= \Big|\theta\sum_{i,j}\int_\Omega(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_j}\Big(\frac{\partial\rho^{n-1,s}}{\partial x_i}\Big)(A\mathbf{u}^{n,s})_j\Big|\\
&= \Big|-\theta\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_j}\Big((\mathbf{u}^{n-1})_i(A\mathbf{u}^{n,s})_j\Big)\frac{\partial\rho^{n-1,s}}{\partial x_i}\Big|\\
&= \Big|-\theta\sum_{i,j}\int_\Omega\Big(\frac{\partial}{\partial x_j}(\mathbf{u}^{n-1})_i\Big)(A\mathbf{u}^{n,s})_j\frac{\partial\rho^{n-1,s}}{\partial x_i}\Big|\\
&\le C\theta|\nabla\mathbf{u}^{n-1}|_\infty|\nabla\rho^{n-1,s}||A\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|\mathbf{u}^{n-1}|_{H^3}|\rho^{n-1,s}|^2_{H^1}+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s},\theta A\mathbf{u}^{n,s})| &\le C\theta|\nabla\rho^{n-1,s}||\nabla\mathbf{u}^{n-1+s}|_\infty|A\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|\mathbf{u}^{n-1+s}|_{H^3}|\rho^{n-1,s}|^2_{H^1}+\varepsilon|A\mathbf{u}^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},\theta A\mathbf{u}^{n,s})| &\le C\theta|\nabla\rho^{n-1}|_\infty|\nabla\mathbf{u}^{n-1,s}||A\mathbf{u}^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|A\mathbf{u}^{n,s}|^2.
\end{aligned}
$$

Assim, estas estimativas nos permitem concluir que:

$$
(\theta\mu-10\varepsilon)|A\mathbf{u}^{n,s}|^2\le C\theta^2|\mathbf{u}_t^{n,s}|^2+C|\nabla\mathbf{u}^{n,s}|^2+C|\nabla\mathbf{u}^{n-1,s}|^2
$$

$$
+C\Big(|\nabla\mathbf{u}_t^{n+s}|+|F|_3^2+|\mathbf{u}^{n-1}|_{H^3}+|\mathbf{u}^{n-1+s}|_{H^3}+1\Big)\cdot|\rho^{n-1,s}|^2_{H^1}. \tag{4.70}
$$

Por outro lado, vemos que

$$
\rho_t^{n,s}=\lambda\Delta\rho^{n,s}-\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s}-\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s}. \tag{4.71}
$$

Então, tomando o produto interno de $L^2(\Omega)$ da expressão acima com o termo $\rho^{n,s}$ teremos:

$$
\frac{1}{2}\frac{d}{dt}|\rho^{n,s}|^2=\lambda(\Delta\rho^{n,s},\rho^{n,s})-(\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s},\rho^{n,s})-(\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s},\rho^{n,s}).
$$

$$\leq C|\nabla\rho^{n,s}|^2 + C|\nabla\rho^{n+s}|^2|\nabla\mathbf{u}^{n-1,s}|^2 + C|\rho^{n,s}|^2_{H^1} + C|\nabla\mathbf{u}^{n-1}|\rho^{n,s}|^2_{H^1}$$

$$\leq C|\nabla\mathbf{u}^{n-1,s}|^2 + C|\rho^{n,s}|^2_{H^1};$$

ou seja,

$$\frac{1}{2}\frac{d}{dt}|\rho^{n,s}|^2 \leq C|\nabla\mathbf{u}^{n-1,s}|^2 + C|\rho^{n,s}|^2_{H^1}. \tag{4.72}$$

Além disso,

$$\frac{\lambda}{2}\frac{d}{dt}|\nabla\rho^{n,s}|^2 + |\rho_t^{n,s}|^2 \leq |(\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s},\rho_t^{n,s})| + |(\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s},\rho_t^{n,s})|$$

$$\leq C|\mathbf{u}^{n-1,s}|_4|\nabla\rho^{n+s}|_4|\rho_t^{n,s}| + C|\mathbf{u}^{n-1}|_\infty|\nabla\rho^{n,s}||\rho_t^{n,s}|$$

$$\leq C|\Delta\rho^{n+s}|^2|\nabla\mathbf{u}^{n-1,s}|^2 + C|A\mathbf{u}^{n-1}|^2|\nabla\rho^{n,s}|^2 + \frac{1}{2}|\rho_t^{n,s}|^2;$$

ou seja,

$$\frac{\lambda}{2}\frac{d}{dt}|\nabla\rho^{n,s}|^2 + \frac{1}{2}|\rho_t^{n,s}|^2 \leq C|\nabla\mathbf{u}^{n-1,s}|^2 + C|\nabla\rho^{n,s}|^2. \tag{4.73}$$

Portanto, somando-se as desigualdades (4.69), (4.70), (4.72) e (4.73) teremos

$$\frac{1}{2}\frac{d}{dt}\Big(\mu|\nabla\mathbf{u}^{n,s}|^2 + |\rho^{n,s}|^2 + \lambda|\nabla\rho^{n,s}|^2\Big) + \alpha|u_t^{n,s}|^2 + \frac{1}{2}|\rho_t^{n,s}|^2$$

$$+(\theta\mu - 10\varepsilon)|A\mathbf{u}^{n,s}|^2 \leq C|\nabla\mathbf{u}^{n,s}|^2 + C|\rho^{n,s}|^2_{H^1} + C\theta^2|\mathbf{u}_t^{n,s}|^2$$

$$+C|\nabla\mathbf{u}^{n-1,s}|^2 + C\Big(|\nabla\mathbf{u}_t^{n+s}| + |F|_3^2 + |\mathbf{u}^{n-1}|_{H^3} + |\mathbf{u}^{n-1+s}|_{H^3} + 1\Big)\cdot|\rho^{n-1,s}|^2_{H^1}$$

$$\leq C\Big(|\nabla\mathbf{u}^{n,s}|^2 + |\rho^{n,s}|^2_{H^1}\Big) + C\theta^2|\mathbf{u}_t^{n,s}|^2$$

$$+C\Big(|\nabla\mathbf{u}_t^{n+s}| + |\mathbf{u}^{n-1}|_{H^3} + |\mathbf{u}^{n-1+s}|_{H^3} + 1\Big)\cdot\Big(|\nabla\mathbf{u}^{n-1,s}|^2 + |\rho^{n-1,s}|^2_{H^1}\Big),$$

onde estamos usando o fato de $F \in C([0,T_2];L^3(\Omega))$. Portanto,

$$\frac{1}{2}\Big(\mu|\nabla\mathbf{u}^{n,s}|^2 + |\rho^{n,s}|^2 + \lambda|\nabla\rho^{n,s}|^2\Big) + (\alpha - C\theta^2)\int_0^t|\mathbf{u}_t^{n,s}|^2ds + \frac{1}{2}\int_0^t|\rho_t^{n,s}|^2ds$$

$$+(\theta\mu - 10\varepsilon)\int_0^t|A\mathbf{u}^{n,s}|^2ds \leq C\Big(|\nabla\mathbf{u}^{n,s}|^2 + |\rho^{n,s}|^2_{H^1}\Big)$$

$$+C\Big(|\nabla \mathbf{u}_t^{n+s}|+|\mathbf{u}^{n-1}|_{_{H^3}}+|\mathbf{u}^{n-1+s}|_{_{H^3}}+1\Big)\cdot\Big(|\nabla \mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2_{_{H^1}}\Big),$$

uma vez que $|\nabla \mathbf{u}^{n,s}(0)|=|\rho^{n,s}(0)|^2=|\nabla\rho^{n,s}(0)|=0$. Em seguida, escolhendo $\theta=(\frac{\alpha}{4C})^{\frac{1}{2}}$ e $\varepsilon=\frac{\mu}{20}((\frac{\alpha}{C})^{\frac{1}{2}}-\frac{1}{2})$, teremos $\frac{\alpha}{2}-C\theta^2=\frac{\alpha}{4}$ e $\mu\theta-10\varepsilon=\frac{\mu}{4}$; daí, denotando $\delta=min\{\frac{1}{2},\,\frac{\alpha}{4},\,\frac{\mu}{4},\,\frac{\lambda}{2}\}$ teremos:

$$|\nabla \mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2_{_{H^1}}+\int_0^t|\mathbf{u}_t^{n,s}|^2ds+\int_0^t|\rho_t^{n,s}|^2ds+\int_0^t|A\mathbf{u}^{n,s}|^2ds$$

$$\leq\frac{1}{\delta}\Big[C\int_0^t\Big(|\nabla \mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2_{_{H^1}}\Big)ds$$

$$+C\int_0^t\Big(|\nabla \mathbf{u}_t^{n+s}|+|\mathbf{u}^{n-1}|_{_{H^3}}+|\mathbf{u}^{n-1+s}|_{_{H^3}}+1\Big)\cdot\Big(|\nabla \mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2_{_{H^1}}\Big)ds\Big]$$

$$\leq C\int_0^t\Big(|\nabla \mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2_{_{H^1}}\Big)ds$$

$$+C\int_0^t\Big(|\nabla \mathbf{u}_t^{n+s}|+|\mathbf{u}^{n-1}|_{_{H^3}}+|\mathbf{u}^{n-1+s}|_{_{H^3}}+1\Big)\cdot\Big(|\nabla \mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2_{_{H^1}}\Big)ds.$$

Usando o Lema de Gronwal com

$\varphi(t)=|\nabla \mathbf{u}^{n,s}(t)|^2+|\rho^{n,s}(t)|^2,$

$\varphi^*(t)=|\mathbf{u}^{n,s}(t)|^2+|\rho^{n,s}(t)|^2_{_{H^1}}+|A\mathbf{u}^{n,s}(t)|,$

$a(t)=\displaystyle\int_0^t C\Big(|\nabla \mathbf{u}_t^{n+s}(\tau)|+|\mathbf{u}^{n-1}(\tau)|_{_{H^3}}+|\mathbf{u}^{n-1+s}(\tau)|_{_{H^3}}+1\Big)\cdot\Big(|\nabla \mathbf{u}^{n-1,s}(\tau)|^2+|\rho^{n-1,s}(\tau)|^2_{_{H^1}}\Big)d\tau,$

$b(t)=C,$

teremos

$$|\nabla \mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2_{_{H^1}}+\int_0^t|\mathbf{u}_t^{n,s}|^2ds+\int_0^t|\rho_t^{n,s}|^2ds+\int_0^t|A\mathbf{u}^{n,s}|^2ds$$

$$\leq C\int_0^t\Big(|\nabla \mathbf{u}_t^{n+s}|+|\mathbf{u}^{n-1}|_{_{H^3}}+|\mathbf{u}^{n-1+s}|_{_{H^3}}+1\Big)\cdot\Big(|\nabla \mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2_{_{H^1}}\Big)ds.$$

Ainda, pela desigualdade de Cauchy-Schwarz, a desigualdade acima assume a seguinte forma:

$$|\nabla \mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2_{_{H^1}}+\int_0^t|\mathbf{u}_t^{n,s}|^2ds+\int_0^t|\rho_t^{n,s}|^2ds+\int_0^t|A\mathbf{u}^{n,s}|^2ds$$

$$\leq C\Big[\int_0^t(|\nabla \mathbf{u}_t^{n+s}|^2+|\mathbf{u}^{n-1}|^2_{_{H^3}}+|\mathbf{u}^{n-1+s}|^2_{_{H^3}}+1)d\tau\Big]^{\frac{1}{2}}\cdot\Big[\int_0^t(|\nabla \mathbf{u}^{n-1,s}|^2+|\rho^{n-1,s}|^2_{_{H^1}})^2d\tau\Big]^{\frac{1}{2}}.$$

Finalmente, usando a limitação uniforme de $\mathbf{u}^n$ em $L^2(0,T_2;H^3(\Omega))$ e de $\mathbf{u}_t^n$ em $L^2(0,T_2;V)$, concluímos que:

$$|\nabla \mathbf{u}^{n,s}|^2+|\rho^{n,s}|^2_{_{H^1}}+\int_0^t|\mathbf{u}_t^{n,s}|^2ds+\int_0^t|\rho_t^{n,s}|^2ds+\int_0^t|A\mathbf{u}^{n,s}|^2ds$$

$$\leq C\Big[\int_0^t (|\nabla \mathbf{u}^{n-1,s}|^2 + |\rho^{n-1,s}|^2_{H^1})^2 d\tau\Big]^{\frac{1}{2}}. \tag{4.74}$$

Assim, definindo $\psi_{n,s}(t) = |\nabla \mathbf{u}^{n,s}(t)|^2 + |\rho^{n,s}(t)|^2_{H^1}$, para todo $t \in [0, T_2]$, a desigualdade acima mostra-nos que

$$\psi^2_{n,s}(t) \leq C\int_0^t \psi^2_{n-1,s}(\tau)d\tau.$$

Trabalhando com esta desigualdade como anteriormente feito para $\varphi_{n,s}(t)$ dada por (4.61) teremos:

$$\begin{aligned}
\psi^2_{n,s}(t) &\leq C\int_0^t \psi^2_{n-1,s}(\tau)d\tau \\
&\leq C^n M^2 \int_0^t \frac{\tau^{n-1}}{(n-1)!}d\tau \\
&= C^n M^2 \frac{t^n}{n!} \leq M^2 \frac{(CT_2)^n}{n!},
\end{aligned}$$

onde $M = M(\mathbf{u}^\circ, \rho^\circ, C_3, C_0)$; assim sendo, teremos que

$$\psi_{n,s}(t) \leq M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}},$$

para todos n, $s \in N$ (s fixado), e todo $t \in [0, T_2]$. Usando a definição de ψ, vemos que

$$|\nabla \mathbf{u}^{n,s}(t)| \leq M^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}}, \quad |\rho^{n,s}(t)|_{H^1} \leq M^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}}, \tag{4.75}$$

para todos n, $s \in N$ (s fixado), e todo $t \in [0, T_2]$. Além destas, obteremos, usando a desigualdade (4.74), taxas de convergência para outros termos conforme abaixo:

$$\int_0^t |\mathbf{u}^{n,s}_t|^2 d\tau + \int_0^t |\rho^{n,s}_t|^2 d\tau + \int_0^t |A\mathbf{u}^{n,s}|^2 d\tau \leq \Big[C\int_0^t \psi^2_{n-1,s}(\tau)d\tau\Big]^{\frac{1}{2}} \leq M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}},$$

e assim,

$$\Big(\int_0^t |\mathbf{u}^{n,s}_t|^2 d\tau\Big)^{\frac{1}{2}} \leq M^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}}, \quad \Big(\int_0^t |A\mathbf{u}^{n,s}|^2 d\tau\Big)^{\frac{1}{2}} \leq M^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}},$$

$$\text{e } \Big(\int_0^t |\rho^{n,s}_t|^2 d\tau\Big)^{\frac{1}{2}} \leq M^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}}, \tag{4.76}$$

para todos n, $s \in N$ (s fixado), e todo $t \in [0, T_2]$. Agora, usando a igualdade (4.71) podemos também estabelecer uma taxa de convergência para $\Delta\rho^n$; de fato, temos que:

$$\lambda|\Delta\rho^{n,s}|^2 \leq C|\rho^{n,s}_t|^2 + C|\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s}|^2 + C|\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s}|^2$$

$$\leq C|\rho^{n,s}_t|^2 + C|\mathbf{u}^{n-1,s}|^2 + C|\nabla\rho^{n,s}|^2$$

$$\Longrightarrow \Big[\int_0^t |\Delta\rho^{n,s}|^2 d\tau\Big]^{\frac{1}{2}} \leq C\Big[\int_0^t |\rho^{n,s}_t|^2 d\tau\Big]^{\frac{1}{2}}$$

$$+C\Big[\int_0^t |\mathbf{u}^{n-1,s}|^2 d\tau\Big]^{\frac{1}{2}} + C\Big[\int_0^t |\nabla\rho^{n,s}|^2 d\tau\Big]^{\frac{1}{2}} \tag{4.77}$$

Usando (4.61) e (4.62) vemos que $|\mathbf{u}^{n,s}(t)|^2 \leq M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}}$. Assim,

$$C\Big[\int_0^t |\mathbf{u}^{n-1,s}(\tau)|^2 d\tau\Big]^{\frac{1}{2}} \leq C\Big[\int_0^t M\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}} d\tau\Big]^{\frac{1}{2}}$$

$$= CM^{\frac{1}{2}}\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}} T_2^{\frac{1}{2}} = M^*\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}};$$

e também, admitindo $CT_2 < n$ e usando (4.75) e (4.76) teremos:

$$(i)\quad \Big(\int_0^t |\rho_t^{n,s}|^2 d\tau\Big)^{\frac{1}{2}} \leq M^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}} = M^{\frac{1}{2}}\Big[\frac{CT_2}{n}\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}} < M^{\frac{1}{2}}\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}$$

$$(ii)\Big(|\rho^{n,s}(\tau)|^2_{H^1} d\tau\Big)^{\frac{1}{2}} \leq C\Big[\int_0^t M\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}} d\tau\Big]^{\frac{1}{2}} \leq CM^{\frac{1}{2}}\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{4}} T_2^{\frac{1}{2}}$$

$$= CM^{\frac{1}{2}}\Big[\frac{CT_2}{n}\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}} T_2^{\frac{1}{2}} = M^{**}\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}$$

Voltando à (4.77) e colocando $K = K(C, M^*, M^{**})$, teremos

$$\Big[\int_0^t |\Delta\rho^{n,s}|^2 d\tau\Big]^{\frac{1}{2}} \leq K\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}, \tag{4.78}$$

para todos n, $s \in N$ com $CT_2 < n$ e s fixado, e para todo $t \in [0, T_2]$.

Além desta última, conseguimos também uma taxa de convergência para ρ^n em $L^\infty(0, T_2; H^2_N)$. De fato, considerando a equação (4.71), aplicando o operador ∇ na mesma e, em seguida, fazendo o produto interno de $L^2(\Omega)$ da equação obtida com o termo $-\nabla\Delta\rho^{n,s}$, teremos, após uma integração por partes:

$$\frac{1}{2}\frac{d}{dt}|\Delta\rho^{n,s}|^2 + \lambda|\nabla\Delta\rho^{n,s}|^2 \leq |(\nabla\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s}, \nabla\Delta\rho^{n,s})$$

$$+(\mathbf{u}^{n-1,s}\cdot\nabla^2\rho^{n+s}, \nabla\Delta\rho^{n,s}) + (\nabla\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s}, \nabla\Delta\rho^{n,s}) + (\mathbf{u}^{n-1}\cdot\nabla^2\rho^{n,s}, \nabla\Delta\rho^{n,s})|.$$

Então, estimamos os termos à direita na expressão acima do seguinte modo:

$$\begin{aligned}
|(\nabla\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s}, \nabla\Delta\rho^{n,s})| &\leq C|\nabla\mathbf{u}^{n-1,s}||\nabla\rho^{n+s}|_\infty|\nabla\Delta\rho^{n,s}| \\
&\leq C|\nabla\mathbf{u}^{n-1,s}||\nabla\Delta\rho^{n+s}||\nabla\Delta\rho^{n,s}| \\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2 + \varepsilon|\nabla\Delta\rho^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\mathbf{u}^{n-1,s}\cdot\nabla^2\rho^{n+s}, \nabla\Delta\rho^{n,s})| &\leq C|\mathbf{u}^{n-1,s}|_4|\nabla^2\rho^{n+s}|_4|\nabla\Delta\rho^{n,s}| \\
&\leq C|\nabla\mathbf{u}^{n-1,s}||\nabla\Delta\rho^{n+s}||\nabla\Delta\rho^{n,s}| \\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2 + \varepsilon|\nabla\Delta\rho^{n,s}|^2;
\end{aligned}$$

$$
\begin{aligned}
|(\nabla \mathbf{u}^{n-1}\cdot\nabla\rho^{n,s},\nabla\Delta\rho^{n,s})| &\leq C|\nabla\mathbf{u}^{n-1}|_4|\nabla\rho^{n,s}|_4|\nabla\Delta\rho^{n,s}|\\
&\leq C|A\mathbf{u}^{n-1}||\Delta\rho^{n,s}||\nabla\Delta\rho^{n,s}|\\
&\leq C_\varepsilon|\Delta\rho^{n,s}|^2+\varepsilon|\nabla\Delta\rho^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}^{n-1}\cdot\nabla^2\rho^{n,s},\nabla\Delta\rho^{n,s})| &\leq C|\mathbf{u}^{n-1}|_\infty|\nabla^2\rho^{n,s}||\nabla\Delta\rho^{n,s}|\\
&\leq C|A\mathbf{u}^{n-1}||\Delta\rho^{n,s}||\nabla\Delta\rho^{n,s}|\\
&\leq C_\varepsilon|\Delta\rho^{n,s}|^2+\varepsilon|\nabla\Delta\rho^{n,s}|^2;
\end{aligned}
$$

Logo, escolhendo $\varepsilon = \frac{\lambda}{8}$ teremos:

$$
\frac{1}{2}\frac{d}{dt}|\Delta\rho^{n,s}|^2+\frac{\lambda}{2}|\nabla\Delta\rho^{n,s}|^2 \leq C|\nabla\mathbf{u}^{n-1,s}|^2+C|\Delta\rho^{n,s}|^2
$$

$$
\Longrightarrow |\Delta\rho^{n,s}|^2+\int_0^t|\nabla\Delta\rho^{n,s}|^2d\tau \leq C\int_0^t|\nabla\mathbf{u}^{n-1,s}|^2d\tau+C\int_0^t|\Delta\rho^{n,s}|^2d\tau,
$$

$$
\Longrightarrow |\Delta\rho^{n,s}|^2+\int_0^t|\nabla\Delta\rho^{n,s}|^2d\tau \leq K_1\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}} \tag{4.79}
$$

onde $K_1 = K_1(C, M^*, M^{**})$, $n,\ s \in N$ (s fixado) e $t \in [0, T_2]$.

Usando ainda (4.71), podemos ver que:

$$
\rho_t^{n,s} = \lambda\Delta\rho^{n,s}-\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s}-\mathbf{u}^{n-1}\cdot\nabla\rho^{n,s}
$$

e

$$
\nabla\rho_t^{n,s} = \lambda\nabla\Delta\rho^{n,s}-\nabla\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s}
$$

$$
-\mathbf{u}^{n-1,s}\cdot\nabla^2\rho^{n+s}-\nabla\mathbf{u}^{n-1}\cdot\rho^{n,s}-\mathbf{u}^{n-1}\cdot\nabla^2\rho^{n,s};
$$

donde, usando as últimas estimativas teremos:

$$
|\rho_t^{n,s}|^2 \leq C|\Delta\rho^{n,s}|^2+C|\mathbf{u}^{n-1,s}|^2+|\nabla\rho^{n,s}|^2
$$

$$
\Longrightarrow |\rho_t^{n,s}|^2 \leq K_2\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}, \tag{4.80}
$$

e também,

$$
\int_0^t|\nabla\rho_t^{n,s}|^2d\tau \leq C\int_0^t|\nabla\Delta\rho^{n,s}|^2d\tau+C\int_0^t|\nabla\mathbf{u}^{n-1,s}|^2d\tau
$$

$$
+C\int_0^t|\mathbf{u}^{n-1,s}|^2d\tau+C\int_0^t|\nabla\rho^{n,s}|^2d\tau+C\int_0^t|\Delta\rho^{n,s}|^2d\tau
$$

$$
\Longrightarrow \int_0^t|\nabla\rho_t^{n,s}|^2d\tau \leq K_3\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}, \tag{4.81}
$$

onde $K_2 = K_2(C, M^*, M^{**})$, $K_3 = K_3(C, M^*, M^{**})$, $n,\ s \in N$ (s fixado) e $t \in [0, T_2]$.

Assim, usando as desigualdades (4.75), (4.76), (4.78), (4.79), (4.80) e (4.81) concluímos que $\{\mathbf{u}^n\}$ é seqüência de Cauchy em $L^\infty(0,T_2;V)$ e $L^2(0,T_2;D(A))$, $\{\mathbf{u}_t^n\}$ é seqüência de Cauchy em $L^2(0,T_2;H)$ (*), $\{\rho^n\}$ é seqüência de Cauchy em $L^2(0,T_2;H^2_{_N})$, $L^\infty(0,T_2;H^2_{_N})$ e $L^2(0,T_2;H^3_{_N})$ e $\{\rho_t^n\}$ é seqüência de Cauchy em $L^2(0,T_2;L^2(\Omega))$, $L^\infty(0,T_2;L^2(\Omega))$ e $L^2(0,T_2;H^1_{_N})$ (*). Como todos estes espaços mencionados são espaços de Banach concluímos que existem $\mathbf{u}\in L^\infty(0,T_2;V)\cap L^2(0,T_2;D(A))$ e $\rho\in L^\infty(0,T_2;H^2_{_N})\cap L^2(0,T_2;H^3_{_N})$ tais que

$$\lim_{n\to+\infty}\mathbf{u}^n=\mathbf{u} \text{ em } L^\infty(0,T_2;V)\cap L^2(0,T_2;D(A)),$$

$$\lim_{n\to+\infty}\mathbf{u}_t^n=\mathbf{u}_t \text{ em } L^2(0,T_2;H),$$

$$\lim_{n\to+\infty}\rho^n=\rho \text{ em } L^\infty(0,T_2;H^2_{_N})\cap L^2(0,T_2;H^3_{_N}),$$

$$\lim_{n\to+\infty}\rho_t^n=\rho_t \text{ em } L^\infty(0,T_2;L^2(\Omega))\cap L^2(0,T_2;H^1_{_N}),$$

sendo que tais convergências são todas fortes.

Finalmente, fazendo $s\to+\infty$ em (4.74), (4.79), (4.80) e (4.81) obteremos as taxas (4.64), (4.65), (4.66), (4.67) e (4.68). ■

Convém observar que, por (*), inicialmente podemos garantir que existem $\alpha\in L^2(0,T_2;H)$ e $\beta\in L^2(0,T_2;L^2(\Omega))$ tais que $\lim_{n\to+\infty}\mathbf{u}_t^n=\alpha$ e $\lim_{n\to+\infty}\rho_t^n=\beta$; daí, como convergência forte implica convergência no sentido de distribuições, usando a unicidade do limite no sentido de distribuições concluímos que $\alpha=\mathbf{u}_t$ e $\beta=\rho_t$.

Procedendo exatamente como no Capítulo 2, prova-se que $(\mathbf{u},\rho)$ é a única solução do sistema (4.42)-(4.45).

4.5 Taxa de Convergência para Derivadas Temporais

Nesta etapa deste trabalho estabelecemos as últimas taxas de convergência para as aproximações $\mathbf{u}^n$ e ρ^n da velocidade e da densidade, respectivamente. As principais taxas obtidas estão relacionadas na seguinte

Proposição 4.4 *Sejam $(\mathbf{u}^n,\rho^n)$ e $(\mathbf{u},\rho)$ nas condições da Proposição 4.3. Então,*

$$\sigma(t)|\mathbf{u}_t^n(t)-\mathbf{u}_t(t)|^2+\sigma(t)|\rho_t^n(t)-\rho_t(t)|^2_{_{H^1}}\le C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac12}, \tag{4.82}$$

$$\int_0^t\sigma(\tau)|\nabla\mathbf{u}_t^n(\tau)-\nabla\mathbf{u}_t(\tau)|^2d\tau\le C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac12}, \tag{4.83}$$

$$\int_0^t\sigma(\tau)|\Delta\rho_t^n(\tau)-\Delta\rho_t(\tau)|^2d\tau\le C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac12}, \tag{4.84}$$

$$\sigma(t)|A\mathbf{u}^n(t) - A\mathbf{u}(t)|^2 \leq C^{**}\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}, \tag{4.85}$$

$$\sigma(t)|\nabla\Delta\rho^n(t) - \nabla\Delta\rho(t)|^2 \leq K_4\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}, \tag{4.86}$$

$$\int_0^t \sigma(\tau)|\Delta^2\rho^n(\tau) - \Delta^2\rho(\tau)|^2 d\tau \leq K_5\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}, \tag{4.87}$$

para todo $n \geq 1$ *e todo* $t \in [0, T_2]$.

Demonstração. Derivando a equação (4.57) com relação à variável temporal e tomando o produto interno de $L^2(\Omega)$ na igualdade acima com $\mathbf{u}_t^{n,s}$ teremos:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|(\rho^{n-1})^{1/2}\mathbf{u}_t^{n,s}|^2 + \mu|\nabla\mathbf{u}_t^{n,s}|^2 = -\frac{1}{2}(\rho_t^{n-1}\mathbf{u}_t^{n,s}, \mathbf{u}_t^{n,s}) - (\rho_t^{n-1,s}\mathbf{u}_t^{n+s} + \rho^{n-1,s}\mathbf{u}_{tt}^{n+s}\\
+(\rho_t^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s} + (\rho^{n-1+s}\mathbf{u}_t^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s} + (\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}_t^{n+s}\\
+(\rho_t^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s} + (\rho^{n-1+s}\mathbf{u}_t^{n-1}\cdot\nabla)\mathbf{u}^{n,s} + (\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}_t^{n,s}\\
+(\rho_t^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n} + (\rho^{n-1,s}\mathbf{u}_t^{n-1}\cdot\nabla)\mathbf{u}^{n} + (\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}_t^{n}\\
-\rho_t^{n-1,s}F - \rho^{n-1,s}F_t - \lambda(\mathbf{u}_t^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s} - \lambda(\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho_t^{n-1+s}\\
-\lambda(\mathbf{u}_t^{n-1}\cdot\nabla)\nabla\rho^{n-1,s} - \lambda(\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho_t^{n-1,s} - \lambda(\nabla\rho_t^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s}\\
-\lambda(\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}_t^{n-1+s} - \lambda(\nabla\rho_t^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s} - \lambda(\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}_t^{n-1,s}, \mathbf{u}_t^{n,s}).
\end{aligned}$$

Em seguida, estimamos os termos que se encontram do lado direito como segue.

$$\begin{aligned}
|(\rho_t^{n-1}\mathbf{u}_t^{n,s}, \mathbf{u}_t^{n,s})| &\leq |\rho_t^{n-1}|_4|\mathbf{u}_t^{n,s}|_4|\mathbf{u}_t^{n,s}|\\
&\leq C|\rho_t^{n-1}|_{_{H^1}}|\nabla\mathbf{u}_t^{n,s}||\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\mathbf{u}_t^{n,s}|^2 + \varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\rho_t^{n-1,s}\mathbf{u}_t^{n+s}, \mathbf{u}_t^{n,s})| &\leq |\rho_t^{n-1,s}|_4|\mathbf{u}_t^{n+s}||\mathbf{u}_t^{n,s}|_4\\
&\leq C|\rho_t^{n-1,s}|_{_{H^1}}|\mathbf{u}_t^{n+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\rho_t^{n-1,s}|_{_{H^1}}^2 + \varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|(\rho^{n-1,s}\mathbf{u}_{tt}^{n+s}, \mathbf{u}_t^{n,s})| &\leq |\rho^{n-1,s}|_4|\mathbf{u}_{tt}^{n+s}||\mathbf{u}_t^{n,s}|_4\\
&\leq C|\rho^{n-1,s}|_{_{H^1}}|\mathbf{u}_{tt}^{n+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\mathbf{u}_{tt}^{n+s}|^2|\rho^{n-1,s}|_{_{H^1}}^2 + \varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$\begin{aligned}
|((\rho_t^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s}, \mathbf{u}_t^{n,s})| &\leq |\rho_t^{n-1+s}|_6|\mathbf{u}^{n-1,s}||\nabla\mathbf{u}^{n+s}|_6|\mathbf{u}_t^{n,s}|_6\\
&\leq C|\rho_t^{n-1+s}|_{_{H^1}}|\mathbf{u}^{n-1,s}||A\mathbf{u}^{n+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\leq C_\varepsilon|\rho_t^{n-1+s}|_{_{H^1}}^2|\mathbf{u}^{n-1,s}|^2 + \varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}$$

$$
\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}_t^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{n,s})| &\le |\rho^{n-1+s}|_\infty|\mathbf{u}_t^{n-1,s}||\nabla\mathbf{u}^{n+s}|_4|\mathbf{u}_t^{n,s}|_4\\
&\le C|\mathbf{u}_t^{n-1,s}||A\mathbf{u}^{n+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\mathbf{u}_t^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}_t^{n+s},\mathbf{u}_t^{n,s})| &\le |\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1,s}|_4|\nabla\mathbf{u}_t^{n+s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}^{n-1,s}||\nabla\mathbf{u}_t^{n+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}_t^{n+s}|^2|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho_t^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s},\mathbf{u}_t^{n,s})| &\le |\rho_t^{n-1+s}|_4|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^{n,s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\rho_t^{n-1+s}|_{H^1}|A\mathbf{u}^{n-1}||\nabla\mathbf{u}^{n,s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}^{n,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}_t^{n-1}\cdot\nabla)\mathbf{u}^{n,s},\mathbf{u}_t^{n,s})| &\le |\rho^{n-1+s}|_\infty|\mathbf{u}_t^{n-1}|_4|\nabla\mathbf{u}^{n,s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}_t^{n-1}||\nabla\mathbf{u}^{n,s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}_t^{n-1}|^2|\nabla\mathbf{u}^{n,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}_t^{n,s},\mathbf{u}_t^{n,s})| &\le |\rho^{n-1+s}|_\infty|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}_t^{n,s}||\mathbf{u}_t^{n,s}|\\
&\le C|A\mathbf{u}^{n-1}||\nabla\mathbf{u}_t^{n,s}||\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\mathbf{u}_t^{n,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho_t^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n},\mathbf{u}_t^{n,s})| &\le |\rho_t^{n-1,s}|_4|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}^{n}|_4|\mathbf{u}_t^{n,s}|\\
&\le C|\rho_t^{n-1,s}|_{H^1}|A\mathbf{u}^{n-1}||A\mathbf{u}^{n}||\mathbf{u}_t^{n,s}|\\
&\le C|\rho_t^{n-1,s}|_{H^1}^2+C|\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1,s}\mathbf{u}_t^{n-1}\cdot\nabla)\mathbf{u}^{n},\mathbf{u}_t^{n,s})| &\le |\rho^{n-1,s}|_6|\mathbf{u}_t^{n-1}||\nabla\mathbf{u}^{n}|_6|\mathbf{u}_t^{n,s}|_6\\
&\le C|\rho^{n-1,s}|_{H^1}|\mathbf{u}_t^{n-1}||A\mathbf{u}^{n}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\rho^{n-1,s}|_{H^1}^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}_t^{n},\mathbf{u}_t^{n,s})| &\le |\rho^{n-1,s}|_4|\mathbf{u}^{n-1}|_\infty|\nabla\mathbf{u}_t^{n}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\rho^{n-1,s}|_{H^1}|A\mathbf{u}^{n-1}||\nabla\mathbf{u}_t^{n}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\rho^{n-1,s}|_{H^1}^2|\nabla\mathbf{u}_t^{n}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\rho_t^{n-1,s}F,\mathbf{u}_t^{n,s})| &\le |\rho_t^{n-1,s}|_4|F||\mathbf{u}_t^{n,s}|_4\\
&\le C|\rho_t^{n-1,s}|_{H^1}|F||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\rho_t^{n-1,s}|_{H^1}^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\rho^{n-1,s}F_t,\mathbf{u}_t^{n,s})| &\le |\rho^{n-1,s}|_4|F_t||\mathbf{u}_t^{n,s}|_4\\
&\le C|\rho^{n-1,s}|_{H^1}|F_t||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\rho^{n-1,s}|_{H^1}^2|F_t|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\mathbf{u}_t^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s},\mathbf{u}_t^{n,s})| &\le |\mathbf{u}_t^{n-1,s}||\nabla^2\rho^{n-1+s}|_4|\mathbf{u}_t^{n,s}|_4\\
&\le C|\mathbf{u}_t^{n-1,s}||\nabla\Delta\rho^{n-1+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\mathbf{u}_t^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|((\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho_t^{n-1+s},\mathbf{u}_t^{n,s})| &\le |\mathbf{u}^{n-1,s}|_4|\nabla^2\rho_t^{n-1+s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}^{n-1,s}||\Delta\rho_t^{n-1+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}^{n-1,s}|^2|\Delta\rho_t^{n-1+s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;\\
|((\mathbf{u}_t^{n-1}\cdot\nabla)\nabla\rho^{n-1,s},\mathbf{u}_t^{n,s})| &\le |\mathbf{u}_t^{n-1}|_4|\nabla^2\rho^{n-1,s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}_t^{n-1}||\Delta\rho^{n-1,s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\mathbf{u}_t^{n-1}|^2|\Delta\rho^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;\\
|((\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho_t^{n-1,s},\mathbf{u}_t^{n,s})| &= \Big|\sum_{i,j}\int_\Omega(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_i}\Big(\frac{\partial}{\partial x_j}(\rho_t^{n-1,s})\Big)(\mathbf{u}_t^{n,s})_j\Big|\\
&= \Big|\sum_{i,j}\int_\Omega(\mathbf{u}^{n-1})_i\frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}(\rho_t^{n-1,s})\Big)(\mathbf{u}_t^{n,s})_j\Big|\\
&= \Big|\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_i}(\rho_t^{n-1,s})\frac{\partial}{\partial x_j}\Big((\mathbf{u}^{n-1})_i(\mathbf{u}_t^{n,s})_j\Big)\Big|\\
&= \Big|\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_i}(\rho_t^{n-1,s})\frac{\partial}{\partial x_j}((\mathbf{u}^{n-1})_i)(\mathbf{u}_t^{n,s})_j\Big|\\
&\le |\nabla\rho_t^{n-1,s}||\nabla\mathbf{u}^{n-1}|_4|\mathbf{u}_t^{n,s}|_4\\
&\le C|\nabla\rho_t^{n-1,s}||A\mathbf{u}^{n-1}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\rho_t^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;\\
|((\nabla\rho_t^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s},\mathbf{u}_t^{n,s})| &\le |\nabla\rho_t^{n-1,s}||\nabla\mathbf{u}^{n-1+s}|_4|\mathbf{u}_t^{n,s}|_4\\
&\le C|\nabla\rho_t^{n-1,s}||A\mathbf{u}^{n-1+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\nabla\rho_t^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;\\
|((\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}_t^{n-1+s},\mathbf{u}_t^{n,s})| &\le |\nabla\rho^{n-1,s}|_4|\nabla\mathbf{u}_t^{n-1+s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\Delta\rho^{n-1,s}||\nabla\mathbf{u}_t^{n-1+s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\Delta\rho^{n-1,s}|^2|\nabla\mathbf{u}_t^{n-1+s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;\\
|((\nabla\rho_t^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},\mathbf{u}_t^{n,s})| &\le |\nabla\rho_t^{n-1}|_4|\nabla\mathbf{u}^{n-1,s}||\mathbf{u}_t^{n,s}|_4\\
&\le C|\Delta\rho_t^{n-1}||\nabla\mathbf{u}^{n-1,s}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\Delta\rho_t^{n-1}|^2|\nabla\mathbf{u}^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;\\
|((\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}_t^{n-1,s},\mathbf{u}_t^{n,s})| &= \Big|\sum_{i,j}\int_\Omega\frac{\partial}{\partial x_i}(\rho^{n-1})\frac{\partial}{\partial x_i}((\mathbf{u}_t^{n-1,s})_j)(\mathbf{u}_t^{n,s})_j\Big|\\
&= \Big|\sum_{i,j}\int_\Omega(\mathbf{u}_t^{n-1,s})_j\frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_i}(\rho^{n-1})\cdot(\mathbf{u}_t^{n,s})_j)\Big|\\
&\le \Big|\sum_{i,j}\int_\Omega(\mathbf{u}_t^{n-1,s})_j\frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_i}(\rho^{n-1}))\cdot(\mathbf{u}_t^{n,s})_j\Big|\\
&+ \Big|\sum_{i,j}\int_\Omega(\mathbf{u}_t^{n-1,s})_j\frac{\partial}{\partial x_i}(\rho^{n-1})\cdot\frac{\partial}{\partial x_i}((\mathbf{u}_t^{n,s})_j)\Big|\\
&\le |\mathbf{u}_t^{n-1,s}||\Delta\rho^{n-1}|_4|\mathbf{u}_t^{n,s}|_4\\
&+ |\mathbf{u}_t^{n-1,s}||\nabla\rho^{n-1}|_\infty|\nabla\mathbf{u}_t^{n,s}|\\
&\le C|\mathbf{u}_t^{n-1,s}||\nabla\Delta\rho^{n-1}||\nabla\mathbf{u}_t^{n,s}|\\
&\le C_\varepsilon|\mathbf{u}_t^{n-1,s}|^2+\varepsilon|\nabla\mathbf{u}_t^{n,s}|^2;
\end{aligned}
$$

Estas estimativas implicam (escolhendo $\varepsilon = \frac{\mu}{42}$):

$$\frac{1}{2}\frac{d}{dt}|(\rho^{n-1})^{1/2}\mathbf{u}_t^{n,s}|^2 + \frac{\mu}{2}|\nabla \mathbf{u}_t^{n,s}|^2 \leq C|\mathbf{u}_t^{n,s}|^2 + C|\rho_t^{n-1,s}|^2_{_{H^1}} + C|\mathbf{u}_t^{n-1,s}|^2$$
$$+C|\mathbf{u}^{n-1,s}|^2|\rho_t^{n-1+s}|^2_{_{H^1}} + C|\nabla \mathbf{u}^{n-1,s}|^2(|\nabla \mathbf{u}_t^{n+s}|^2 + |\Delta\rho_t^{n-1+s}|^2 + |\Delta\rho_t^{n-1}|^2)$$
$$+C|\nabla \mathbf{u}^{n,s}|^2(1+|\nabla \mathbf{u}_t^{n-1}|^2) + C|\rho^{n-1,s}|^2_{_{H^1}}(1+|\mathbf{u}_{tt}^{n+s}|^2 + |\nabla \mathbf{u}_t^{n}|^2 + |F_t|^2) \tag{4.88}$$
$$+C|\Delta\rho^{n-1,s}|^2(|\nabla \mathbf{u}_t^{n-1+s}|^2 + |\nabla \mathbf{u}_t^{n-1}|^2).$$

Temos que

$$\rho_t^{n,s} - \lambda\Delta\rho^{n,s} = -\mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s} - \mathbf{u}^{n-1}\cdot\nabla\rho^{n,s}.$$

Assim, derivando a igualdade acima com respeito à variável t e, em seguida, tomando o produto interno com $\rho_t^{n,s}$ teremos:

$$\frac{1}{2}\frac{d}{dt}|\rho_t^{n,s}|^2 + \lambda|\nabla\rho_t^{n,s}|^2 = -(\mathbf{u}_t^{n-1,s}\cdot\nabla\rho^{n+s}, \rho_t^{n,s})$$
$$-(\mathbf{u}^{n-1,s}\cdot\nabla\rho_t^{n+s}, \rho_t^{n,s}) - (\mathbf{u}_t^{n-1}\cdot\nabla\rho^{n,s}, \rho_t^{n,s}) - (\mathbf{u}^{n-1}\cdot\nabla\rho_t^{n,s}, \rho_t^{n,s})$$
$$\leq |\mathbf{u}_t^{n-1,s}||\nabla\rho^{n+s}|_\infty|\rho_t^{n,s}| + |\mathbf{u}^{n-1,s}|_4|\nabla\rho_t^{n+s}||\rho_t^{n,s}|_4$$
$$+|\mathbf{u}_t^{n-1}||\nabla\rho^{n,s}||\rho_t^{n,s}|_\infty + |\mathbf{u}^{n-1}|_\infty|\nabla\rho_t^{n,s}||\rho_t^{n,s}|$$
$$\leq C|\mathbf{u}_t^{n-1,s}|^2 + C|\nabla \mathbf{u}^{n-1,s}|^2 + C|\rho_t^{n,s}|^2_{_{H^1}}$$
$$+C_\delta|\nabla\rho^{n,s}|^2 + \delta|\Delta\rho_t^{n,s}|^2 + \frac{\lambda}{2}|\nabla\rho_t^{n,s}|^2.$$

Portanto,

$$\frac{1}{2}\frac{d}{dt}|\rho_t^{n,s}|^2 + \frac{\lambda}{2}|\nabla\rho_t^{n,s}|^2 \leq C|\mathbf{u}_t^{n-1,s}|^2$$
$$+C|\nabla \mathbf{u}^{n-1,s}|^2 + C|\rho_t^{n,s}|^2_{_{H^1}} + C_\delta|\nabla\rho^{n,s}|^2 + \delta|\Delta\rho_t^{n,s}|^2. \tag{4.89}$$

Além disso, temos que:

$$\frac{1}{2}\frac{d}{dt}|\nabla\rho_t^{n,s}|^2 + \lambda|\Delta\rho_t^{n,s}|^2 \leq |(\nabla \mathbf{u}_t^{n-1,s}\cdot\nabla\rho^{n+s}, \nabla\rho_t^{n,s})$$
$$+(\mathbf{u}_t^{n-1,s}\cdot\nabla^2\rho^{n+s} + \nabla \mathbf{u}^{n-1,s}\cdot\nabla\rho_t^{n+s} + \mathbf{u}^{n-1,s}\cdot\nabla^2\rho_t^{n+s}, \nabla\rho_t^{n,s})$$
$$+(\nabla \mathbf{u}_t^{n-1}\cdot\nabla\rho^{n,s} + \mathbf{u}_t^{n-1}\cdot\nabla^2\rho^{n,s} + \nabla \mathbf{u}^{n-1}\cdot\nabla\rho_t^{n,s} + \mathbf{u}^{n-1}\cdot\nabla^2\rho_t^{n,s}, \nabla\rho_t^{n,s})|.$$

Passemos então, a estimar os termos que se encontram no lado direito da desigualdade acima.

$$
\begin{aligned}
&|(\nabla \mathbf{u}_t^{n-1,s}\cdot\nabla\rho^{n+s},\nabla\rho_t^{n,s})|\\
&=\Big|\sum_{i,j}\int_\Omega \frac{\partial}{\partial x_j}(\mathbf{u}_t^{n-1,s})_i\frac{\partial\rho^{n+s}}{\partial x_j}\frac{\partial\rho_t^{n,s}}{\partial x_i}\Big|\\
&=\Big|\sum_{i,j}\int_\Omega (\mathbf{u}_t^{n-1,s})_i\frac{\partial}{\partial x_j}\Big(\frac{\partial\rho^{n+s}}{\partial x_j}\frac{\partial\rho_t^{n,s}}{\partial x_i}\Big)\Big|\\
&\le\Big|\sum_{i,j}\int_\Omega (\mathbf{u}_t^{n-1,s})_i\frac{\partial^2\rho^{n+s}}{\partial x_j^2}\frac{\partial\rho_t^{n,s}}{\partial x_i}\Big|+\Big|\sum_{i,j}\int_\Omega (\mathbf{u}_t^{n-1,s})_i\frac{\partial\rho^{n+s}}{\partial x_j}\frac{\partial^2\rho_t^{n,s}}{\partial x_j x_i}\Big|\\
&\le C|\mathbf{u}_t^{n-1,s}||\Delta\rho^{n+s}|_4|\nabla\rho_t^{n,s}|_4+C|\mathbf{u}_t^{n-1,s}||\nabla\rho^{n+s}|_\infty|\Delta\rho_t^{n,s}|\\
&\le C|\mathbf{u}_t^{n-1,s}||\nabla\Delta\rho^{n+s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\mathbf{u}_t^{n-1,s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}_t^{n-1,s}\cdot\nabla^2\rho^{n+s},\nabla\rho_t^{n,s})| &\le |\mathbf{u}_t^{n-1,s}||\nabla^2\rho^{n+s}|_4|\nabla\rho_t^{n,s}|_4\\
&\le C|\mathbf{u}_t^{n-1,s}||\nabla\Delta\rho^{n+s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\mathbf{u}_t^{n-1,s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\nabla\mathbf{u}^{n-1,s}\cdot\nabla\rho_t^{n+s},\nabla\rho_t^{n,s})| &\le |\nabla\mathbf{u}^{n-1,s}||\nabla\rho_t^{n+s}|_4|\nabla\rho_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}^{n-1,s}||\Delta\rho_t^{n+s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\nabla\mathbf{u}^{n-1,s}|^2|\Delta\rho_t^{n+s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}^{n-1,s}\cdot\nabla^2\rho_t^{n+s},\nabla\rho_t^{n,s})| &\le |\mathbf{u}^{n-1,s}|_4|\nabla^2\rho_t^{n+s}||\nabla\rho_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}^{n-1,s}||\Delta\rho_t^{n+s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\nabla\mathbf{u}^{n-1,s}|^2|\Delta\rho_t^{n+s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\nabla\mathbf{u}_t^{n-1}\cdot\nabla\rho^{n,s},\nabla\rho_t^{n,s})| &\le |\nabla\mathbf{u}_t^{n-1}||\nabla\rho^{n,s}|_4|\nabla\rho_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}_t^{n-1}||\Delta\rho^{n,s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\nabla\mathbf{u}_t^{n-1}|^2|\Delta\rho^{n,s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}_t^{n-1}\cdot\nabla^2\rho^{n,s},\nabla\rho_t^{n,s})| &\le |\mathbf{u}_t^{n-1}|_4|\nabla^2\rho^{n,s}||\nabla\rho_t^{n,s}|_4\\
&\le C|\nabla\mathbf{u}_t^{n-1}||\Delta\rho^{n,s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\nabla\mathbf{u}_t^{n-1}|^2|\Delta\rho^{n,s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\nabla\mathbf{u}^{n-1}\cdot\nabla\rho_t^{n,s},\nabla\rho_t^{n,s})| &\le |\nabla\mathbf{u}^{n-1}|_4|\nabla\rho_t^{n,s}||\nabla\rho_t^{n,s}|_4\\
&\le C|A\mathbf{u}^{n-1}||\nabla\rho_t^{n,s}||\Delta\rho_t^{n,s}|\\
&\le C_\delta|\nabla\rho_t^{n,s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

$$
\begin{aligned}
|(\mathbf{u}^{n-1}\cdot\nabla^2\rho_t^{n,s},\nabla\rho_t^{n,s})| &\le |\mathbf{u}^{n-1}|_\infty|\nabla^2\rho_t^{n,s}||\nabla\rho_t^{n,s}|\\
&\le C|A\mathbf{u}^{n-1}||\Delta\rho_t^{n,s}||\nabla\rho_t^{n,s}|\\
&\le C_\delta|\nabla\rho_t^{n,s}|^2+\delta|\Delta\rho_t^{n,s}|^2;
\end{aligned}
$$

Usando estas estimativas teremos

$$\frac{1}{2}\frac{d}{dt}|\nabla\rho_t^{n,s}|^2+(\lambda-8\delta)|\Delta\rho_t^{n,s}|^2\leq C|\mathbf{u}_t^{n-1,s}|^2$$

$$+C|\nabla\mathbf{u}^{n-1,s}|^2|\Delta\rho_t^{n+s}|^2+C|\nabla\mathbf{u}_t^{n-1}|^2|\Delta\rho^{n,s}|^2+C|\nabla\rho_t^{n,s}|^2.$$

Assim, usando a desigualdade acima, a desigualdade (4.89) e tomando $\delta=\frac{\lambda}{18}$ ficaremos com a seguinte desigualdade:

$$\frac{1}{2}\frac{d}{dt}|\rho_t^{n,s}|^2_{H^1}+\frac{\lambda}{2}|\Delta\rho_t^{n,s}|^2\leq C|\mathbf{u}_t^{n-1,s}|^2$$

$$+C|\nabla\mathbf{u}^{n-1,s}|^2(1+|\Delta\rho_t^{n+s}|^2)+C|\rho_t^{n,s}|^2_{H^1}+C|\nabla\mathbf{u}_t^{n-1}|^2|\Delta\rho^{n,s}|^2+C|\nabla\rho^{n,s}|^2.$$

Somando-se a desigualdade (4.88) a esta última ficaremos com

$$\frac{1}{2}\frac{d}{dt}(|(\rho^{n-1})^{1/2}\mathbf{u}_t^{n,s}|^2+|\rho_t^{n,s}|^2_{H^1})+\frac{\mu}{2}|\nabla\mathbf{u}_t^{n,s}|^2+\frac{\lambda}{2}|\Delta\rho_t^{n,s}|^2$$

$$\leq C|\mathbf{u}_t^{n,s}|^2+C|\rho_t^{n,s}|^2_{H^1}+C|\mathbf{u}_t^{n-1,s}|^2+C|\rho_t^{n-1,s}|^2_{H^1}+C|\mathbf{u}^{n-1,s}|^2$$

$$+C|\nabla\mathbf{u}^{n,s}|^2(1+|\nabla\mathbf{u}_t^{n-1}|^2)+C|\nabla\rho^{n,s}|^2+C|\Delta\rho^{n,s}|^2|\nabla\mathbf{u}_t^{n-1}|^2$$

$$+C|\nabla\mathbf{u}^{n-1,s}|^2(|\nabla\mathbf{u}_t^{n+s}|^2+|\Delta\rho_t^{n+s}|^2+|\Delta\rho_t^{n-1+s}|^2+|\Delta\rho_t^{n-1}|^2+1)$$

$$+C|\rho^{n-1,s}|^2_{H^1}(1+|\mathbf{u}_{tt}^{n+s}|^2+|\nabla\mathbf{u}_t^{n}|^2+|F_t|^2)+C|\Delta\rho^{n-1,s}|^2(|\nabla\mathbf{u}_t^{n-1+s}|^2+|\nabla\mathbf{u}_t^{n-1}|^2).$$

Multiplicando a desigualdade acima pela função $\sigma(t)=min\{t,1\}$, para $0\leq t\leq T_2$ teremos:

$$\frac{1}{2}\sigma(t)\frac{d}{dt}(|(\rho^{n-1})^{1/2}\mathbf{u}_t^{n,s}|^2+|\rho_t^{n,s}|^2_{H^1})+\frac{\mu}{2}\sigma(t)|\nabla\mathbf{u}_t^{n,s}|^2+\frac{\lambda}{2}\sigma(t)|\Delta\rho_t^{n,s}|^2$$

$$\leq C|\mathbf{u}_t^{n,s}|^2+C|\rho_t^{n,s}|^2_{H^1}+C|\mathbf{u}_t^{n-1,s}|^2+C|\rho_t^{n-1,s}|^2_{H^1}+C|\mathbf{u}^{n-1,s}|^2$$

$$+C|\nabla\mathbf{u}^{n,s}|^2(1+|\nabla\mathbf{u}_t^{n-1}|^2)+C|\nabla\rho^{n,s}|^2+C|\Delta\rho^{n,s}|^2|\nabla\mathbf{u}_t^{n-1}|^2$$

$$+C|\nabla\mathbf{u}^{n-1,s}|^2(|\nabla\mathbf{u}_t^{n+s}|^2+|\Delta\rho_t^{n+s}|^2+|\Delta\rho_t^{n-1+s}|^2+|\Delta\rho_t^{n-1}|^2+1)$$

$$+C|\rho^{n-1,s}|^2_{H^1}(1+|\nabla\mathbf{u}_t^{n}|^2+|F_t|^2)+C|\rho^{n-1,s}|^2_{H^1}\sigma(t)|\mathbf{u}_{tt}^{n+s}|^2$$

$$+C|\Delta\rho^{n-1,s}|^2(|\nabla\mathbf{u}_t^{n-1+s}|^2+|\nabla\mathbf{u}_t^{n-1}|^2).$$

Integrando a desigualdade acima em $[\varepsilon_k,t]$, com $0<\varepsilon_k<t$, teremos como resultado:

$$\frac{1}{2}\sigma(t)(|(\rho^{n-1})^{1/2}(t)\mathbf{u}_t^{n,s}(t)|^2+|\rho_t^{n,s}(t)|^2_{H^1})+\frac{\mu}{2}\int_{\varepsilon_k}^t\sigma(\tau)|\nabla\mathbf{u}_t^{n,s}(\tau)|^2d\tau$$

$$+\frac{\lambda}{2}\int_{\varepsilon_k}^t\sigma(\tau)|\Delta\rho_t^{n,s}(\tau)|^2d\tau\leq\frac{1}{2}\sigma(\varepsilon_k)(|(\rho^{n-1})^{1/2}(\varepsilon_k)\mathbf{u}_t^{n,s}(\varepsilon_k)|^2+|\rho_t^{n,s}(\varepsilon_k)|^2_{H^1})$$

$$+\frac{1}{2}\int_{\varepsilon_k}^t\sigma^{'}(\tau)(|(\rho^{n-1})^{1/2}(\tau)\mathbf{u}_t^{n,s}(\tau)|^2+|\rho_t^{n,s}(\tau)|^2_{H^1})d\tau$$

$$+C\int_{\varepsilon_k}^t |\mathbf{u}_t^{n,s}(\tau)|^2 d\tau + C\int_{\varepsilon_k}^t |\rho_t^{n,s}(\tau)|_{H^1}^2 d\tau$$

$$+C\int_{\varepsilon_k}^t |\mathbf{u}_t^{n-1,s}(\tau)|^2 d\tau + C\int_{\varepsilon_k}^t |\rho_t^{n-1,s}(\tau)|_{H^1}^2 d\tau + C\int_{\varepsilon_k}^t |\mathbf{u}^{n-1,s}(\tau)|^2 d\tau$$

$$+C\int_{\varepsilon_k}^t |\nabla \mathbf{u}^{n,s}(\tau)|^2 (1+|\nabla \mathbf{u}_t^{n-1}(\tau)|^2) d\tau + C\int_{\varepsilon_k}^t |\nabla \rho^{n,s}(\tau)|^2 d\tau$$

$$+C\int_{\varepsilon_k}^t |\Delta \rho^{n,s}(\tau)|^2 |\nabla \mathbf{u}_t^{n-1}(\tau)|^2 d\tau + C\int_{\varepsilon_k}^t |\nabla \mathbf{u}^{n-1,s}(\tau)|^2 (|\nabla \mathbf{u}_t^{n+s}(\tau)|^2$$

$$+|\Delta \rho_t^{n+s}(\tau)|^2 + |\Delta \rho_t^{n-1+s}(\tau)|^2 + |\Delta \rho_t^{n-1}(\tau)|^2 + 1) d\tau$$

$$+C\int_{\varepsilon_k}^t |\rho^{n-1,s}(\tau)|_{H^1}^2 (1+|\nabla \mathbf{u}_t^{n}(\tau)|^2 + |F_t(\tau)|^2) d\tau + C\int_{\varepsilon_k}^t |\rho^{n-1,s}(\tau)|_{H^1}^2 \sigma(\tau) |\mathbf{u}_{tt}^{n+s}(\tau)|^2 d\tau$$

$$+C\int_{\varepsilon_k}^t |\Delta \rho^{n-1,s}(\tau)|^2 (|\nabla \mathbf{u}_t^{n-1+s}(\tau)|^2 + |\nabla \mathbf{u}_t^{n-1}(\tau)|^2) d\tau.$$

Podemos escolher uma seqüência (ε_k), com $0 < \varepsilon_k < t$, tal que $\varepsilon_k \to 0$, quando $k \to +\infty$ e também $\sigma(\varepsilon_k)(|(\rho^{n-1})^{1/2}(\varepsilon_k)\mathbf{u}_t^{n,s}(\varepsilon_k)|^2 + |\rho_t^{n,s}(\varepsilon_k)|_{H^1}^2) \to 0$, quando $k \to +\infty$; passando ao limite quando $k \to +\infty$ e lembrando que $\sigma'(t) \leq 1$ q.t.p. em $[0, T_2]$, teremos :

$$\sigma(t)(|\mathbf{u}_t^{n,s}(t)|^2 + |\rho_t^{n,s}(t)|_{H^1}^2) + \int_0^t \sigma(\tau)|\nabla \mathbf{u}_t^{n,s}(\tau)|^2 d\tau + \int_0^t \sigma(\tau)|\Delta \rho_t^{n,s}(\tau)|^2 d\tau$$

$$\leq C\int_0^t |\mathbf{u}_t^{n,s}(\tau)|^2 d\tau + C\int_0^t |\rho_t^{n,s}(\tau)|_{H^1}^2 d\tau$$

$$+C\int_0^t |\mathbf{u}_t^{n-1,s}(\tau)|^2 d\tau + C\int_0^t |\rho_t^{n-1,s}(\tau)|_{H^1}^2 d\tau + C\int_0^t |\mathbf{u}^{n-1,s}(\tau)|^2 d\tau$$

$$+C\int_0^t |\nabla \mathbf{u}^{n,s}(\tau)|^2 (1+|\nabla \mathbf{u}_t^{n-1}(\tau)|^2) d\tau + C\int_0^t |\nabla \rho^{n,s}(\tau)|^2 d\tau$$

$$+C\int_0^t |\Delta \rho^{n,s}(\tau)|^2 |\nabla \mathbf{u}_t^{n-1}(\tau)|^2 d\tau + C\int_0^t |\nabla \mathbf{u}^{n-1,s}(\tau)|^2 (|\nabla \mathbf{u}_t^{n+s}(\tau)|^2$$

$$+|\Delta \rho_t^{n+s}(\tau)|^2 + |\Delta \rho_t^{n-1+s}(\tau)|^2 + |\Delta \rho_t^{n-1}(\tau)|^2 + 1) d\tau$$

$$+C\int_0^t |\rho^{n-1,s}(\tau)|_{H^1}^2 (1+|\nabla \mathbf{u}_t^{n}(\tau)|^2 + |F_t(\tau)|^2) d\tau + C\int_0^t |\rho^{n-1,s}(\tau)|_{H^1}^2 \sigma(\tau) |\mathbf{u}_{tt}^{n+s}(\tau)|^2 d\tau$$

$$+C\int_0^t |\Delta \rho^{n-1,s}(\tau)|^2 (|\nabla \mathbf{u}_t^{n-1+s}(\tau)|^2 + |\nabla \mathbf{u}_t^{n-1}(\tau)|^2) d\tau$$

$$(*) \qquad \leq CM\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}} + CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}} + CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}} \int_0^t d\tau$$

$$+CM\Big[\frac{(CT_2)^n}{n!}\Big]^{\frac{1}{2}} \int_0^t (1+|\nabla \mathbf{u}_t^{n-1}(\tau)|^2) d\tau + CK_1\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}} \int_0^t |\nabla \mathbf{u}_t^{n-1}(\tau)|^2 d\tau$$

$$+CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}}\int_0^t(|\nabla\mathbf{u}_t^{n+s}(\tau)|^2+|\Delta\rho_t^{n+s}(\tau)|^2+|\Delta\rho_t^{n-1+s}(\tau)|^2+|\Delta\rho_t^{n-1}(\tau)|^2+1)d\tau$$

$$+CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}}\int_0^t(1+|\nabla\mathbf{u}_t^{n}(\tau)|^2+|F_t(\tau)|^2+\sigma(\tau)|\mathbf{u}_{tt}^{n+s}(\tau)|^2)d\tau$$

$$+CK_1\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{4}}\int_0^t(|\nabla\mathbf{u}_t^{n-1+s}(\tau)|^2+|\nabla\mathbf{u}_t^{n-1}(\tau)|^2)d\tau.$$

$$\leq CM\Big[\frac{(CT_2)^{n}}{n!}\Big]^{\frac{1}{2}}+CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}}+CK_1\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}+CK_1\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{4}}$$

$$\leq CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}}+CK_1\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{4}}\leq C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}, \tag{4.90}$$

sendo que, em (*), foram usadas as desigualdades (4.62), (4.75), (4.76), (4.79) e (4.80).

Lembremos que

$$\mu A\mathbf{u}^{n,s}=-P(\rho^{n-1}\mathbf{u}_t^{n,s})-P[\rho^{n-1,s}\mathbf{u}_t^{n+s}+(\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s}$$
$$+(\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s}+(\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n}-\rho^{n-1,s}F$$
$$-\lambda(\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s}-\lambda(\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho^{n-1,s}$$
$$-\lambda(\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s}-\lambda(\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s},$$

e que, assim , teremos:

$$|A\mathbf{u}^{n,s}|^2\leq C\Big[|\rho^{n-1}\mathbf{u}_t^{n,s}|^2+|\rho^{n-1,s}\mathbf{u}_t^{n+s}|^2+|(\rho^{n-1+s}\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{n+s}|^2$$
$$+|(\rho^{n-1+s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n,s}|^2+|(\rho^{n-1,s}\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{n}|^2+|\rho^{n-1,s}F|^2$$
$$|(\mathbf{u}^{n-1,s}\cdot\nabla)\nabla\rho^{n-1+s}|^2+|(\mathbf{u}^{n-1}\cdot\nabla)\nabla\rho^{n-1,s}|^2$$
$$+|(\nabla\rho^{n-1,s}\cdot\nabla)\mathbf{u}^{n-1+s}|^2+|(\nabla\rho^{n-1}\cdot\nabla)\mathbf{u}^{n-1,s}|^2\Big]$$

$$\leq C\Big[|\mathbf{u}_t^{n,s}|^2+|\rho^{n-1,s}|^2_{H^1}|\nabla\mathbf{u}_t^{n+s}|^2+|\nabla\mathbf{u}^{n-1,s}|^2|A\mathbf{u}^{n+s}|^2+|A\mathbf{u}^{n-1}|^2|\nabla\mathbf{u}^{n,s}|^2$$
$$+|\rho^{n-1,s}|^2_{H^1}|A\mathbf{u}^{n-1}|^2|A\mathbf{u}^{n}|^2+|\rho^{n-1,s}|^2_{H^1}|F|^2_3+|\nabla\mathbf{u}^{n-1,s}|^2|\nabla\Delta\rho^{n-1+s}|^2$$
$$+|A\mathbf{u}^{n-1}|^2|\Delta\rho^{n-1,s}|^2+|\Delta\rho^{n-1,s}|^2|A\mathbf{u}^{n-1+s}|^2+|\nabla\Delta\rho^{n-1}|^2|\nabla\mathbf{u}^{n-1,s}|^2\Big]$$

$$\leq C\Big[|\mathbf{u}_t^{n,s}|^2+|\rho^{n-1,s}|^2_{H^1}(1+|\nabla\mathbf{u}_t^{n+s}|^2)+|\nabla\mathbf{u}^{n-1,s}|^2+|\nabla\mathbf{u}^{n,s}|^2+|\Delta\rho^{n-1,s}|^2\Big];$$

e então, multiplicando a desigualdade acima pela função $\sigma(t)$ teremos:

$$\sigma(t)|A\mathbf{u}^{n,s}|^2 \leq C\Big[\sigma(t)|\mathbf{u}_t^{n,s}|^2 + |\rho^{n-1,s}|^2_{_{H^1}}(1+\sigma(t)|\nabla \mathbf{u}_t^{n+s}|^2)$$

$$+|\nabla \mathbf{u}^{n-1,s}|^2 + |\nabla \mathbf{u}^{n,s}|^2 + |\Delta\rho^{n-1,s}|^2\Big]$$

$$\leq C\Big[C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} + M\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}} + K_1\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{4}}\Big]$$

$$\leq C^{**}\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}. \tag{4.91}$$

Podemos agora, conseguir taxas de convergência de ordem mais alta para as aproximações da densidade. Mais precisamente, considerando a igualdade

$$\lambda\Delta\rho^{n,s} = \rho_t^{n,s} + \mathbf{u}^{n-1,s}\cdot\nabla\rho^{n+s} + \mathbf{u}^{n-1}\cdot\nabla\rho^{n,s},$$

deduzimos que

$$|\nabla\Delta\rho^{n,s}|^2 \leq C\Big(|\nabla\rho_t^{n,s}|^2 + |A\mathbf{u}^{n-1,s}|^2 + |\Delta\rho^{n,s}|^2\Big),$$

e daí, multiplicando esta desigualdade por $\sigma(t)$ ficaremos com

$$\sigma(t)|\nabla\Delta\rho^{n,s}(t)|^2 \leq C\Big(\sigma(t)|\rho_t^{n,s}(t)|^2_{_{H^1}} + \sigma(t)|A\mathbf{u}^{n-1,s}(t)|^2 + |\Delta\rho^{n,s}|^2\Big)$$

$$\leq C\Big[C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} + C^{**}\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} + K_1\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{4}}\Big]$$

$$\leq K_4\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}. \tag{4.92}$$

Além desta última, conseguimos também :

$$|\Delta^2\rho^{n,s}|^2 \leq C\Big(|\Delta\rho_t^{n,s}|^2 + |A\mathbf{u}^{n-1,s}|^2 + |\nabla\Delta\rho^{n,s}|^2\Big)$$

$$\Longrightarrow \int_0^t \sigma(t)|\Delta^2\rho^{n,s}|^2 \leq C\Big(\int_0^t \sigma(t)|\Delta\rho_t^{n,s}|^2 + \int_0^t \sigma(t)|A\mathbf{u}^{n-1,s}|^2 + \int_0^t \sigma(t)|\nabla\Delta\rho^{n,s}|^2\Big)$$

$$\leq C\Big[C^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} + C^{**}\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} + K_4\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}\Big]$$

$$\leq K_5\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} \tag{4.93}$$

Usando as taxas dadas em (4.90), (4.91), (4.92) e (4.93) podemos afirmar que para todo $\gamma > 0$, $\{\mathbf{u}^n\}$ é seqüência de Cauchy em $L^\infty(\gamma, T_2; D(A))$, $\{\mathbf{u}_t^n\}$ é seqüência de Cauchy em $L^\infty(\gamma, T_2; H)$ e em $L^2(\gamma, T_2; V)$; por sua vez, $\{\rho^n\}$ é seqüência de Cauchy em $L^\infty(\gamma, T_2; H^3_{_N})$ e em $L^2(\gamma, T_2; H^4_{_N})$ e $\{\rho_t^n\}$ é seqüência de Cauchy em $L^\infty(\gamma, T_2; H^1)$ e em $L^2(\gamma, T_2; H^2_{_N})$. Usando a completude de tais espaços, teremos as seguintes convergências fortes:

$$\lim_{n\to+\infty} \mathbf{u}^n = \mathbf{u} \text{ em } L^\infty(\gamma, T_2; D(A)),$$

$$\lim_{n\to+\infty} \mathbf{u}_t^n = \mathbf{u}_t \text{ em } L^\infty(\gamma, T_2; H) \cap L^2(\gamma, T_2; V),$$

$$\lim_{n\to+\infty} \rho^n = \rho \text{ em } L^\infty(\gamma, T_2; H^3_{_N}) \cap L^2(\gamma, T_2; H^4_{_N}),$$

$$\lim_{n\to+\infty} \rho_t^n = \rho_t \text{ em } L^\infty(\gamma, T_2; H^1) \cap L^2(\gamma, T_2; H^2_{_N})$$

Fazendo o limite quando $s \to +\infty$ em (4.90), (4.91), (4.92) e (4.93), obteremos as taxas dadas por (4.82), (4.83), (4.84), (4.85), (4.86) e (4.87). ■

4.6 Resultados sobre a Pressão

Notamos que na formulação do problema aproximado (4.42)-(4.45), aplicou-se o operador projeção P à equação de momento em (4.41), eliminando-se assim, o termo ∇p e evitando-se que na equação (4.42) tivéssemos um termo do tipo ∇p^n que seria uma aproximação de ∇p.

Nosso objetivo neste momento, é recuperar a pressão aproximada p^n de modo que a seqüência (p^n) convirja para uma pressão p tal que $(\mathbf{u}, \rho, p)$ seja solução do problema simplificado. Nossa metodologia será a de, em primeiro lugar, obter a existência de p^n, para cada n; em seguida, estabelecer estimativas a priori (independentes de n) para p^n e finalmente, mostrar que a seqüência (p^n) é de Cauchy em algum espaço de Banach conveniente. Ao final, obteremos os resultados dados na proposição abaixo.

Proposição 4.5 *Nas condições da Proposição 4.3, temos que existe uma única pressão* $p \in L^\infty(\gamma, T_2; H^1(\Omega)/I\!R) \cap L^2(0, T_2; H^1(\Omega)/I\!R$, *para* $\forall \gamma > 0$, *tal que* $(\mathbf{u}, \rho, p)$ *é a única solução do problema* (4.41). *Além disso, temos as seguintes estimativas para a seqüência* (p^n) *de aproximações da pressão* p:

$$|p^n(t)|^2_{_{H^1(\Omega)/I\!R}} \leq C, \tag{4.94}$$

$$\sigma(t)|p^n(t) - p(t)|^2_{_{H^1(\Omega)/I\!R}} \leq K_6\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}, \tag{4.95}$$

$$\int_0^t |p^n(\tau) - p(\tau)|^2_{_{H^1(\Omega)/I\!R}} d\tau \leq K_7\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}, \tag{4.96}$$

para todo $n \geq 1$ *e todo* $t \in [0, T_2]$.

Demonstração. Vemos, pela equação (4.42), que

$$\begin{array}{rcl} \mu A\mathbf{u}^{n+1} & = & -P[\rho^n \mathbf{u}_t^{n+1} - (\rho^n \mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \rho^n F \\ & & +\lambda(\mathbf{u}^n \cdot \nabla)\nabla\rho^n + \lambda(\nabla\rho^n \cdot \nabla)\mathbf{u}^n] \\ & = & F^n \end{array}$$

Observamos inicialmente que como $\mathbf{u}^n \in L^\infty(0, T_2; D(A))$, uniformemente em n, então deve ocorrer que $F^n \in L^\infty(0, T_2; L^2(\Omega))$, uniformemente em n. Assim, para cada n temos que existe $p^n \in L^2(0, T_2; H^1(\Omega)/\mathbb{R})$ tal que $(\mathbf{u}^n, p^n)$ é solução de um problema de Stokes:

$$\left\{ \begin{array}{l} -\mu\Delta\mathbf{u}^n + \nabla p^n = F^n \\ \text{div } \mathbf{u}^n = 0 \\ \mathbf{u}^n|_\Gamma = 0 \end{array} \right.$$

com $|\mathbf{u}^n|^2_{H^2(\Omega)} + |p^n|^2_{H^1(\Omega)/\mathbb{R}} \leq C|F^n|^2$. Como $|F^n| \leq C$, para todo n, temos que $|p^n|^2_{H^1(\Omega)/\mathbb{R}} \leq C$, verificando então (4.94). Assim, concluímos que

$$p^n \in L^\infty(0, T_2; H^1(\Omega)/\mathbb{R})$$

uniformemente em n.

Mostraremos a seguir, que a seqüência $\{p^n\}$ é de Cauchy em $L^\infty(\gamma, T_2; H^1(\Omega)/\mathbb{R})$. Para tal, definamos $p^{n,s} = p^{n+s} - p^n$, para todos n, $s \geq 1$. Vemos então, que $p^{n,s}$ satisfaz

$$\begin{array}{rcl} -\mu\Delta\mathbf{u}^{n,s} + \nabla p^{n,s} & = & -\rho^{n-1}\mathbf{u}_t^{n,s} - \rho^{n-1,s}\mathbf{u}_t^{n+s} + (\rho^{n-1+s}\mathbf{u}^{n-1,s} \cdot \nabla)\mathbf{u}^{n+s} \\ & + & (\rho^{n-1+s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n,s} + (\rho^{n-1,s}\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^n - \rho^{n-1,s}F \\ & - & \lambda(\mathbf{u}^{n-1,s} \cdot \nabla)\nabla\rho^{n-1+s} - \lambda(\mathbf{u}^{n-1} \cdot \nabla)\nabla\rho^{n-1,s} \\ & - & \lambda(\nabla\rho^{n-1,s} \cdot \nabla)\mathbf{u}^{n-1+s} - \lambda(\nabla\rho^{n-1} \cdot \nabla)\mathbf{u}^{n-1,s} \\ & = & F^{n,s}, \end{array}$$

ou seja, para cada n e s fixados, $(\mathbf{u}^{n,s}, p^{n,s})$ é solução de um problema de Stokes pois

$$\left\{ \begin{array}{l} -\mu\Delta\mathbf{u}^{n,s} + \nabla p^{n,s} = F^{n,s} \\ \text{div } \mathbf{u}^{n,s} = 0 \\ \mathbf{u}^{n,s}|_\Gamma = 0, \end{array} \right.$$

de onde podemos concluir que $|p^{n,s}|^2_{H^1(\Omega)/\mathbb{R}} \leq |F^{n,s}|^2$. Daí, teremos:

$$|p^{n,s}|^2_{H^1(\Omega)/\mathbb{R}} \leq C\Big(|\mathbf{u}_t^{n,s}|^2 + |\rho^{n-1,s}|^2_\infty|\mathbf{u}_t^{n+s}|^2 + |\mathbf{u}^{n-1,s}|^2_4|\nabla\mathbf{u}^{n+s}|^2_4 + |\mathbf{u}^{n-1}|^2_\infty|\nabla\mathbf{u}^{n,s}|^2$$

$$+|\rho^{n-1,s}|^2_4|\mathbf{u}^{n-1}|^2_\infty|\nabla\mathbf{u}^n|^2_4 + |\rho^{n-1,s}|^2_6|F|^2_3 + |\mathbf{u}^{n-1,s}|^2_4|\nabla^2\rho^{n-1+s}|^2_4$$

$$+|\mathbf{u}^{n-1}|^2_\infty|\Delta\rho^{n-1,s}|^2 + |\nabla\rho^{n-1,s}|^2_4|\nabla\mathbf{u}^{n-1+s}|^2_4 + |\nabla\rho^{n-1}|^2_\infty|\nabla\mathbf{u}^{n-1,s}|^2\Big)$$

$$\leq C\Big(|\mathbf{u}_t^{n,s}|^2 + |\Delta\rho^{n-1,s}|^2|\mathbf{u}_t^{n+s}|^2 + |\nabla\mathbf{u}^{n-1,s}|^2|A\mathbf{u}^{n+s}|^2 + |A\mathbf{u}^{n-1}|^2|\nabla\mathbf{u}^{n,s}|^2$$

$$+|\rho^{n-1,s}|_{H^1}^2|A\mathbf{u}^{n-1}|^2|A\mathbf{u}^n|^2 + |\rho^{n-1,s}|_{H^1}^2|F|_3^2 + |\nabla\mathbf{u}^{n-1,s}|^2|\nabla\Delta\rho^{n-1+s}|^2$$

$$+|A\mathbf{u}^{n-1}|^2|\Delta\rho^{n-1,s}|^2 + |\Delta\rho^{n-1,s}|^2|A\mathbf{u}^{n-1+s}|^2 + |\nabla\Delta\rho^{n-1}|^2|\nabla\mathbf{u}^{n-1,s}|^2\Big)$$

$$\leq C\Big(|\mathbf{u}_t^{n,s}|^2 + |\nabla\mathbf{u}^{n,s}|^2 + |\nabla\mathbf{u}^{n-1,s}|^2 + |\rho^{n-1,s}|_{H^1}^2 + |\Delta\rho^{n-1,s}|^2\Big). \tag{4.97}$$

Em seguida, multiplicando esta última desigualdade pela função $\sigma(t) = \min\{t, 1\}$ teremos:

$$\sigma(t)|p^{n,s}|_{H^1(\Omega)/I\!R}^2 \leq C\sigma(t)|\mathbf{u}_t^{n,s}|^2 + C|\nabla\mathbf{u}^{n,s}|^2 + C|\nabla\mathbf{u}^{n-1,s}|^2 + C|\rho^{n-1,s}|_{H^1}^2 + C|\Delta\rho^{n-1,s}|^2$$

$$\leq CC^*\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} + CM\Big[\frac{(CT_2)^{n}}{n!}\Big]^{\frac{1}{2}} + CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}} + CK_1\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{4}},$$

em virtude de (4.75), (4.79) e (4.90). Portanto,

$$\sigma(t)|p^{n,s}|_{H^1(\Omega)/I\!R}^2 \leq K_6\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}}. \tag{4.98}$$

Por outro lado, usando (4.75), (4.76) e (4.78) e integrando (4.97) entre 0 e t, teremos:

$$\int_0^t |p^{n,s}(\tau)|_{H^1(\Omega)/I\!R}^2 d\tau \leq C\Big(\int_0^t |\mathbf{u}_t^{n,s}(\tau)|^2 d\tau + \int_0^t |\nabla\mathbf{u}^{n,s}(\tau)|^2 d\tau$$

$$+\int_0^t |\nabla\mathbf{u}^{n-1,s}(\tau)|^2 d\tau + \int_0^t |\rho^{n-1,s}(\tau)|_{H^1}^2 d\tau + \int_0^t |\Delta\rho^{n-1,s}(\tau)|^2 d\tau\Big)$$

$$\leq CM\Big[\frac{(CT_2)^{n}}{n!}\Big]^{\frac{1}{2}} + CM\Big[\frac{(CT_2)^{n-1}}{(n-1)!}\Big]^{\frac{1}{2}} + CK\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} \leq K_7\Big[\frac{(CT_2)^{n-2}}{(n-2)!}\Big]^{\frac{1}{2}} \tag{4.99}$$

Desta forma, as desigualdades (4.98) e (4.99) nos dizem que $\{p^n\}$ é seqüência de Cauchy em $L^\infty(\gamma, T_2; H^1(\Omega)/I\!R)$, para todo $\gamma > 0$, e em $L^2(0, T_2; H^1(\Omega)/I\!R)$.

Portanto, existe $p \in L^\infty(\gamma, T_2; H^1(\Omega)/I\!R) \cap L^2(0, T_2; H^1(\Omega)/I\!R)$ tal que

$$\lim_{n\to+\infty} p^n = p \quad \text{em} \quad L^\infty(\gamma, T_2; H^1(\Omega)/I\!R) \cap L^2(0, T_2; H^1(\Omega)/I\!R);$$

fazendo o limite quando $s \to +\infty$ em (4.98) e (4.99), obtemos (4.95) e (4.96). Procedendo exatamente como no capítulo 2, mostramos que a tripla $(\mathbf{u}, \rho, p)$ é a única solução do problema simplificado. ∎

Bibliography

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] C. Amrouche, V. Girault, *On the existence and regularity of the solution of Stokes problem in arbitrary dimensions,* Proc. Japan Acad., 67, Ser. A, 1991, 171-175.

[3] A. Avez, *Calcul Differéntiel,* Masson, 1983.

[4] H. Beirão da Veiga, *Diffusion on viscous fluids, existence and asymptotic properties of solutions*, Ann. Sc. Norm. Sup. Pisa, 10 (1983), 341-355.

[5] H. Beirão da Veiga, R. Serapioni, A. Valli, *On the motion of non-homogeneous fluids in the presence of diffusion*, Journ. Math. Anal. Applic., 85, 1982, 179-191.

[6] H. Beirão da Veiga, *Long time behaviour of the solutions to the Navier-Stokes equations with diffusion*, Nonl. Anal.- Theory, Meth. and Applic., vol. 27, no. 11, 1996, 1229–1239.

[7] J.L. Boldrini, M.A. Rojas-Medar, *On the convergence rate of spectral approximations for the equations for nonhomogeneous asymmetric fluids*, Math. Modelling and Num. Anal, 1994.

[8] J.L. Boldrini, M.A. Rojas-Medar, *Global strong solutions of the equations for the motion of nonhomogeneous incompressible fluids*, Numerical Meth. in Mech., Pitman Res. Notes in Math. Series, 371, 1997, 35-45.

[9] H. Brezis, *Analyse fonctienelle, théorie et applications,* Masson, 1983.

[10] R. M. Brown, Z. Shen, *Estimates for the Stokes operator in Lipschiz domains*, Indiana Univ. Math. Journal, 44, 4, 1995, 1183-1206.

[11] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazoni di Stokes*, Rend. Sem. Univ. Padova, 31, 1961, 308-340.

[12] E. Coddington, N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

[13] P. Constantin, C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, 1988.

[14] P. D. Damázio, M.A. Rojas-Medar, *On the convergence rate of semi-Galerkin approximations for the equations of viscous fluids in the presence of diffusion,* Matemática Contemporânea, vol. 15, 1998, 105-125.

[15] D.A. Frank and V.I. Kamenetskii, *Diffusion and Heat Transfer in Chemical Kinetics*, Plenum Press, 1969.

[16] V. Girault, P. A. Raviart, *Finite element approximation of the Navier-Stokes equatios*, Lect. Notes Math., Springer-Verlag, 1981.

[17] F. M. Guillén Gonzales, *Nuevos resultados sobre el problema de Navier-Stokes con densidad variable y algunas variantes: existência, unicidad, otros resultados teóricos y aproximación numérica,* Tese de Doutorado, Universidad de Sevilha, 1992.

[18] J.G. Heywood, *An error estimate uniform in time for spectral Galerkin approximations of the Navier-Stokes problem*, P.J. Math., 96 (1982), 333-345.

[19] J.G. Heywood, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. 29 (1980), 639-681.

[20] J.G. Heywood, R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem I: regularity of solutions and second order error estimates for spatial discretization*, SIAM J. Num. Anal., 19 (1982), 275-311.

[21] A.V. Kazhikov, Sh. Smagulov, *The correctness of boundary-value problems in a diffusion model of inhomogeneous fluid*. Dokl. Akad. Nauk. SSSR, 234 (1977), 330-332.

[22] J.V. Kim, *Weak solutions of an initial boundary-value problem for an incompressible viscous fluid with non-negative density*, SIAM J. Math. Anal. 18 (1987), 89-96.

[23] O.A. Ladyzhenkaya, *The Mathematical Theory of Viscous Incompressible Fluid*, Gordon and Breach, New York, 1969.

[24] O.A. Ladyzhenkaya, V.A. Solonnikov, *Unique solvability of an initial and boundary value problem for viscous incompressible fluids*, Zap. Naučn Sem. Leningrado Otdel Math. Inst. Steklov, 52, 1975, 52-109, English Transl., J. Soviet Math., 9, 1978, 697-749.

[25] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, Paris, 1969.

[26] G. Lukaszewicz, *Micropolar Fluids, Theory abd Application,* Birkhäuser-Boston, 1999.

[27] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Norm. Sup. Pisa, 13, 1959, 116-162.

[28] G. Prouse, *Modelli matematici in inquinamento dei fluidi*, Bolettino U.M.I., Analisi Funzionale e Applicazioni, Serie VI, II (1984), 1-13.

[29] G. Prouse, A. Zaretti, *On the inequalities associated to a model of Graffi for the motion of a mixture of two viscous incompressible fluids*, Rend. Accad. Nazion. Scienze, Mem. Matematica, 105o., vol. XI, fasc. 17, 1987, 253-275.

[30] R. Rautmann, *On the convergence rate of nonstationary Navier-Stokes approximations*, Proc. IUTAM Symp. 1979,

[31] M.A. Rojas-Medar and J.L. Boldrini, *Spectral Galerkin approximations for the Navier-Stokes equations: uniform in time errror estimates*, Rev. Mat. Apl. 14 (1993), 63-74.

[32] R. Salvi, *On the existence of weak solutions of boundary-value problems in a diffusion model of an inhomogeneous liquid in regions with moving boundaries*, Portugaliae Math. 43 (1986), 213-233.

[33] R. Salvi, *Disequazioni variazionali per fluidi viscoisi incomprimibili non omogenei in presenza di diffusione*, Riv. Mat. Univ. Parma. 4, 10, 1984, 43-55.

[34] R. Salvi, *The equations of viscous incompressible nonhomogeneous fluid: on the existence and regularity*, J. Australian Math. Soc. Series B, Applied Math., 33 (1991), 94-110.

[35] P. Secchi,*On the motion of viscous fluids in the presence of diffusion*, SIAM J. Math. Anal. 19 (1988), 22-31.

[36] P. Secchi, *On the inicial value problem for the equations of motion of viscous incompressible fluids in the presence of diffusion*, Bollettino U.M.I., 6 1-B, 1982, 1117–1130.

[37] J. Simon, *Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure*, SIAM J. Math. Anal., 21 (1990), 1093-1117.

[38] J. Simon, *Compacts sets in the space $L^p(0,T;B)$*, Annali di Matematica Pura ed Applicata, CXLVI (1987), 65-96.

[39] R. Temam, *Navier-Stokes equations, theory and numerical analysis*, North-Holland, Amsterdam, 1979.

[40] A. G. Zarubin, *On an iterational method for the approximate solution of an initial and boundary value problem for the heat-convection equations*, Comput. Maths Math. Phys., 33, no. 8, 1993, 1077-1085.