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Fuzzy Differential Equations via Interactive Arithmetic: Applications in Biomathematics

Equações Diferenciais Fuzzy via Aritmética Interativa: Aplicações em biomatemática

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In memory of José Francisco Filho and Santina Boscolo Wasques

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"Não há ensino sem pesquisa e pesquisa sem ensino [...] Enquanto ensino continuo buscando, reprocurando. [...] Pesquiso para conhecer o que ainda não conheço e comunicar ou anunciar a novidade."

Paulo Freire

"Get up, stand up: stand up for your rights! Get up, stand up: don't give up the fight!"

Bob Marley

Resumo

Apresentamos um estudo sobre equações diferenciais fuzzy (EDFs), tanto do ponto de vista analítico quanto numérico. Para ambas abordagens faz-se necessário estabelecer uma aritmética entre números fuzzy. Focamos em aritméticas que consideram uma relação entre números fuzzy chamada interatividade, que é atrelada ao conceito de distribuição de possibilidade conjunta. Através do princípio de extensão sup-J, propomos diferentes aritméticas entre números fuzzy interativos, podendo assim tratar de EDFs numericamente através dos métodos de Euler e Runge-Kutta, adaptando as operações aritméticas para números fuzzy interativos. Fornecemos soluções analíticas para problemas de valores iniciais fuzzy utilizando o princípio de extensão sup-J e estabelecemos conexões dessa solução com outras na literatura, por exemplo, as soluções obtidas através da derivada de Fréchet. Provamos que a derivada de Hukuhara e suas generalizações são casos particulares de derivadas interativas. Construímos, através de uma nova família de distribuições de possibilidade conjunta, operações aritméticas que produzem números fuzzy com norma e largura mínimas, em comparação com qualquer outra aritmética obtidas via extensão sup-J. Caracterizamos essas operações aritméticas por α-níveis, tornando seu cálculo mais simples. Investigamos também soluções para equações fuzzy lineares que consideram aritméticas interativas, estabelecendo condições necessárias e suficientes para existência de tais soluções. Ilustramos os métodos propostos nessa tese, fornecendo soluções numéricas para modelos epidemiológicos e reações químicas e soluções analíticas para problemas físicos do tipo massa-mola, corroborando os resultados teóricos. Por fim, exploramos as propriedades de simetria de números fuzzy em sequências de Fibonacci e retardamento.

Palavras-chave: Equações diferenciais fuzzy. Derivadas fuzzy. Aritmética fuzzy.

Abstract

We present a study of fuzzy differential equations (FDEs) from the analytical and numerical point of view. Both approaches require an arithmetic on fuzzy numbers. We focus on arithmetics that consider the relationship between fuzzy numbers called interactivity, which is associated to concept of joint possibility distribution. By means of sup-J extension principle, we propose different arithmetics on interactive fuzzy numbers, allowing to deal with FDEs numerically using Euler's and Runge-Kutta methods, adapting the arithmetic operations for interactive fuzzy numbers. We provide analytical solutions to fuzzy initial value problems via sup-J extension principle and we establish connections with others approaches in the literature, for example, the solutions obtained from Fréchet derivative. We proved that Hukuhara derivative and its generalizations are particular cases of interactive derivatives. We construct, from a new family of joint possibility distributions, arithmetic operations that produce fuzzy numbers with minimal norm and width, in comparison with any other arithmetic derived from sup-J extension. We characterized these arithmetic operations by means of α-cuts, making the computation simpler. Also we investigate solution to linear fuzzy equations that consider interactive arithmetic, establishing necessary and sufficient conditions for the existence of these solutions. We illustrate the methods proposed in this thesis, providing numerical solutions to epidemiological and chemical models and analytical solutions to physical mass-springer problems, corroborating the theoretical results. Finally, we explore the symmetry properties of fuzzy numbers in Fibonacci and Delay sequences.

Keywords: Fuzzy differential equation. Fuzzy derivatives. Fuzzy arithmetic.

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List of abbreviations and acronyms

IVP	Initial Value Problem
BVP	Boundary Value Problem
FIVP	Fuzzy Initial Value Problem
FBVP	Fuzzy Boundary Value Problem
ODE	Ordinary Differential Equation
FDE	Fuzzy Differential Equation
JPD	Joint Possibility Distribution

List of symbols

A, B, U, F	(Capital letters) Classical sets, fuzzy sets and fuzzy functions
μ_A	Membership function of A
$core(A)$	Core of the fuzzy set A
$supp(A)$	Support of the fuzzy set A
a, b, u, f	(Small letters) Elements of classical sets, elements of fuzzy sets and classical functions
χ	Characteristic function
$I(\mathbb{R})$	Set of real intervals
$\mathcal{F}(U)$	Fuzzy subsets class of U
$\mathcal{F}(\mathbb{R}^n)$	Fuzzy subsets class of $\mathbb{R}^n$
$\mathbb{R}_\mathcal{F}$	Class of fuzzy numbers
$\mathbb{R}_{\mathcal{F}_\mathcal{C}}$	Class of fuzzy numbers with continuous endpoints
$(a; b; c)$	Triangular fuzzy number
$\mathcal{F}_{Tr}$	Class of triangular fuzzy numbers
$(a; b; c; d)$	Trapezoidal fuzzy number
$\mathcal{F}_{Tp}$	Class of trapezoidal fuzzy numbers
$(\mu; \sigma; \delta)$	Gaussian fuzzy number
$\mathcal{F}_G$	Class of Gaussian fuzzy numbers
$[A]^\alpha$	The α-cut of fuzzy number A
a^-_α, a^+_α	The lower and upper endpoints of the α-cut of A, respectively
$\triangle, \triangledown$	Triangular norm and conorm, respectively
$\vee,\ \wedge$	Maximum and minimum operators, respectively
$\bigvee, \bigwedge$	Supremum and infimum operators, respectively
Π	Projection operator

d_∞	Pompeiu-Hausdorff distance for fuzzy subsets spaces
$\|\|.\|\|_{\mathcal{F}}$	Pompeiu-Hausdorff norm
$-_H$	Hukuhara difference
$-_{gH}$	Generalized Hukuhara difference
$-_{CIA}$	CIA difference
$-_g$	Generalized difference
F'_H	Hukuhara derivative
F'_{gH}	Generalized Hukuhara derivative
F'_g	Generalized derivative
F'_I	Interactive derivative
$conv$	Convex hull
cl	Closure
$\widehat{f}$	Zadeh's extension principle of the function f

Contents

Introduction

Over time several researchers have found difficulty to incorporate in their models some properties, such as uncertainty and subjectivity. In 1965, Lofti Asker Zadeh introduced a new theory, called Fuzzy Set Theory [156]. By this theory it was possible to incorporate subjectivity in mathematical models.

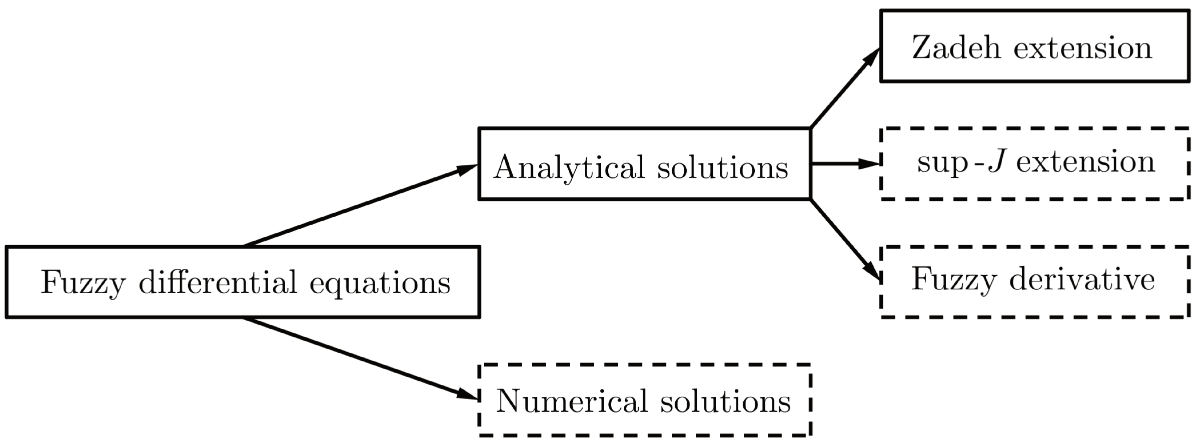
Zadeh established that elements may belong to a set with different degrees, which allows us to describe linguistic variables mathematically [111]. For instance, the authors of [123] used fuzzy-rule-based systems to describe linguistic variables such as small, medium or big sizes of tumor, in order to predict the risk of kidney cancer.

Fuzzy set theory can be used in many fields of science, such as biology [9, 107, 131], medicine [122, 128, 123], geology [151, 152] and chemistry [139]. A well-known methodology used to describe problems in these fields is fuzzy differential equation (FDE) [9, 16].

A FDE consists of a mathematical equation that relates some fuzzy function with its fuzzy derivative. In applications, the fuzzy functions represent the behaviour of quantities, biological or chemical for example, embracing uncertain and/or imprecise information. Fuzzy derivatives represent the variation of these quantities, and fuzzy differential equation establishes the relationship between them.

There are several ways to deal with FDEs. One can study these problems analytically and/or numerically, as depicted in Figure 1.

Figure 1 – Diagram of approaches for fuzzy differential equations



The solid squares are the most common approaches used in the literature. The dashed squares are the ones studied in this thesis. Source: Author

Many authors interpreted FDEs using the notion of derivatives for fuzzy-number-valued functions. Puri and Ralescu [118] introduced the Hukuhara derivative (H-derivative) and Kaleva [80] provided a detailed study of this type of fuzzy derivative. Nevertheless this approach produces only solutions with increasing width of the support, which means that in fuzzy dynamical systems the future state is always more uncertain than the present state [15].

Over the years many generalizations of the Hukuhara derivative were proposed. Bede and Gal [17] proposed the strongly generalized differentiability of fuzzy functions. Later, Stefanini and Bede [133] introduced the generalized Hukuhara derivative (gH-derivative). The gH-derivative is more general than the H-derivative, and they do not require the monotonicity width of the support for its existence.

Thereafter Bede and Stefanini [16] developed the generalized derivative (g-derivative). This fuzzy derivative arises from the g-difference. Gomes and Barros [68] proved that the existence of the generalized difference is only guaranteed, if one applies the convex hull to the definition proposed by Bede and Stefanini.

Another approach to deal with FDEs is through the relationship between fuzzy numbers called interactivity [161]. Barros and Santo Pedro [15] proposed a solution to FDEs using the concept of fuzzy interactive derivative. This paper reveals interesting properties in the study of fuzzy processes [109, 110]. Esmi *et al.* [52] used the notion of interactivity to introduce the Fréchet derivative for fuzzy functions. Salgado, Barros and Esmi [124] provided solutions of FDEs using the notion of interactivity via fuzzy Laplace transform.

Several models describe biological phenomena considering that the initial population is constant [46]. In the case where the states and/or parameters are given by fuzzy numbers, the only FDE that incorporates this hypothesis is the one that uses interactivity [15, 149]. This fact can be also observed in chemical and physical problems, which one may consider the conservation law. In chemistry, the conservation law of mass states that the mass of the final product must be equal to the mass of the reagents [106, 139]. In physics, the conservation law states that a particular measurable property of a physical system does not change as the system evolves over time [21]. This thesis deals with these models and other ones without this hypothesis.

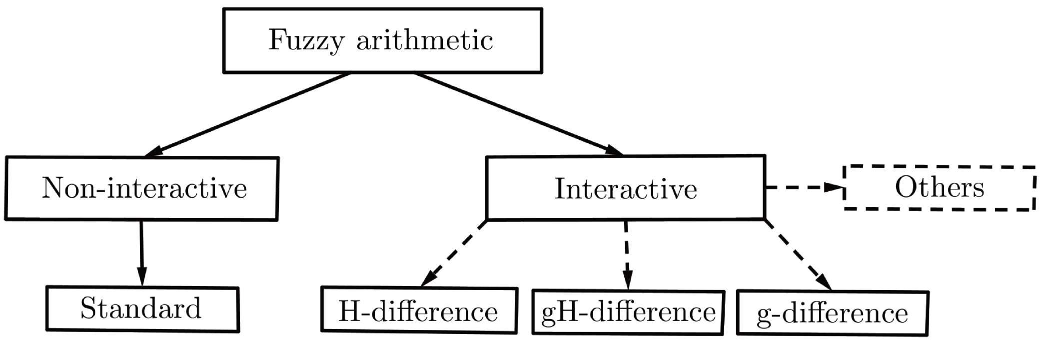
It is possible to provide analytical solutions to FDEs without using the notion of fuzzy derivative. In this case, one can use the Zadeh's extension principle [42, 25, 97, 68], which consists in extending the classical solution of the corresponding ordinary differential equation.

Also one can provide solutions to FDEs embracing the notion of interactivity via sup-J extension principle [62], which is a generalization of the Zadeh's extension principle. Cabral and Barros [29] provided a study of FDEs with parameters and initial conditions given by interactive fuzzy numbers. To this end, the authors consider the sup-J extension principle and a family of differential inclusions [39, 10]. Ibáñez *et al.* [77] proposed solutions to FDEs with interactive fuzzy boundary conditions, using sup-J extension.

The aforementioned approaches deal with FDEs analytically. This thesis also studies FDEs from the numerical point of view. Many researchers proposed numerical methods to produce fuzzy solutions to these problems [91, 3, 105, 64, 79, 1]. These methods do not take into account the concept of interactivity. In contrast of the methods provided in the literature, here numerical methods embracing the notion of interactivity will be investigated.

The study of solutions (analytical and/or numerical) to FDEs require an arithmetic on fuzzy numbers. The most common are the ones obtained by the standard arithmetic [85] and Hukuhara difference and its generalizations [118, 133, 16, 68]. New arithmetics will be proposed. Moreover they will be compared with the ones mentioned before, concluding that our approach embraces H-, gH- and g- differences (see diagram given in Figure 2).

Figure 2 – Diagram of arithmetics on fuzzy numbers



The solid squares are the arithmetic operations provided in the literature. The dashed square is the one provided in this thesis. The dashed arrows represents the associations established here. Source: Author

This thesis is divided as follows. Chapter 1 presents the necessary background for the understanding of this thesis. Chapter 2 provides a discussion about joint possibility distributions and the concept of interactivity. Moreover, it presents arithmetic of fuzzy numbers from two point of views, the interactive and non-interactive. Also it compares these two types of arithmetics, which is the first contribution. Chapter 3 investigates some natural questions that arise from fuzzy linear equations, such as the existence and uniqueness of solution for these equations when the arithmetic operations embraces the notion of interactivity. This is the second contribution. Chapter 4 presents the third contribution of this work, which is a generalization of one type of interactivity called linear interactivity (or also called completely correlation). From this type of relationship, a fuzzy solution for higher order fuzzy differential equations is provided. Moreover this chapter establishes some connections with others approaches given in the literature, such as Fréchet derivative. Chapter 5 deals with FDEs from two points of view: fuzzy derivatives and fuzzy numerical solutions. These approaches are connected from a family of fuzzy derivatives that take into account the relationship of interactivity. As the forth contribution, a fuzzy numerical solution for an n-dimensional fuzzy differential equations, with n initial conditions given by interactive fuzzy numbers, is proposed. As the fifth contribution, this chapter shows that the Hukuhara derivative is a particular case of this family of fuzzy derivatives, which makes the Hukuhara derivative be a type of interactive derivative. Chapter 6 proposes a family of joint possibility distributions, which gives rise to a new type of interactivity. From this family, arithmetic operations between interactive fuzzy numbers can be computed in a simpler way. The symmetry properties of fuzzy numbers and the consequences in addition and subtraction between fuzzy numbers are also explored, which is the sixth contribution. Finally, the last contribution is given in Chapter 7. Some applications are presented using different types of interactivity, such as in physical, biological and chemical models. Also the Fibonacci and Delay sequences composed by interactive fuzzy numbers are explored, in order to illustrate the properties of symmetry of fuzzy numbers.

1 Some Notions of Fuzzy Set Theory

This chapter presents the mathematical background for this thesis, which consists of the main definitions and theorems about fuzzy set theory. Fuzzy sets are characterized by a mapping called membership function. In order to interpret this concept and give a meaningful use to it, one can classify the fuzzy sets in two classes, *Ontic* and *Epistemic* fuzzy sets [38].

Ontic fuzzy sets represent objects originally constructed as sets, but to express the notion of subjectivity, for instance, to describe linguistic labels such as low, medium, high, etc [37]. On the other hand, epistemic fuzzy sets represent the idea of partial or imprecise information [20], for example, to determine the exact number of individuals infected with the HIV virus. This thesis deals with fuzzy sets without distinguishing between these two point of view.

This chapter is divided in six sections and it is based on the references [85, 83, 111, 70, 9, 140, 108]. Section 1.1 presents introductory definitions of fuzzy set theory. Section 1.2 focuses on a special class of fuzzy sets called fuzzy numbers. Section 1.3 recalls the extension principle, which is used in Section 1.4 to provide an arithmetic on fuzzy numbers. Finally, Sections 1.5 and 1.6 respectively establish the types of fuzzy functions and fuzzy derivatives most used in the literature.

1.1 Fuzzy Set Theory

Fuzzy set theory was introduced in 1965 by Zadeh [156]. A fuzzy set is defined as follows.

Definition 1.1. *[156] A fuzzy (sub)set A of a universe X is characterized by a function*

$$\mu_A : X \to [0, 1], \tag{1.1}$$

called the membership function *of A.*

The function μ_A applied at the element $x \in X$ represents the membership degree of x in the set A. This means that the greater value of $\mu_A(x)$, the greater association of the element x to the set A, where $\mu_A(x) = 1$ and $\mu_A(x) = 0$ represent the full and non-association of x to the set A, respectively.

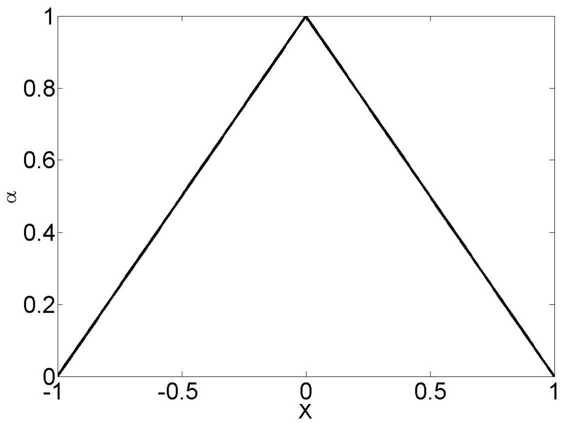
Example 1.1. *Let A be the set given by $A = \{x \in \mathbb{R} : x \text{ is close to } 0\}$. The elements that belong to A depend on the expression "close to", which is a subjective term. This implies* *that the set A is defined in an uncertain way. The membership function $\mu_A : \mathbb{R} \to [0,1]$ in order to characterize the fuzzy subset A is given by*

$$\mu_A(x) = \begin{cases} 1+x & \text{if} \quad -1 \leqslant x \leqslant 0 \\ 1-x & \text{if} \quad 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases} . \tag{1.2}$$

The image of the membership function μ_A is depicted in Figure 3. In this case, the elements outside the interval $I = [-1, 1]$ are not close to 0. For example, the number 1000 is not close to 0, since the membership degree of 1000 in A is 0, that is, $\mu_A(1000) = 0$.

Note that $\mu_A(0.25) = 0.75$ and $\mu_A(-0.1) = 0.9$. This means that both elements have an association with the set A (based on definition (1.2)) but -0.1 is closer to 0 than 0.25. Observe that the element 0 has full association with the set A, since $\mu_A(0) = 1$.

Figure 3 – Graphical representation of the fuzzy set A of Example 1.1.



Source: Author.

The choice of the interval I, in the above example, is subjective. One may choose an interval with larger or smaller size, this depends on the problem to be modeled.

The class of fuzzy subsets of X is denoted by $\mathcal{F}(X)$. Note that each classical subset U of X is identified by its characteristic function

$$\chi_U : X \to \{0, 1\}, \tag{1.3}$$

where $\chi_U(x) = 1$ if $x \in U$, and $\chi_U(x) = 0$ if $x \notin U$. This means that classical subsets are particular cases of fuzzy subsets. These subsets are also called *crisp subsets* in fuzzy set theory.

The *universe* (X) and the *empty* $(\varnothing)$ sets are fuzzy sets whose membership functions are given by $\mu_X(x) = 1,\ \forall x \in X$ and $\mu_\varnothing(x) = 0,\ \forall x \in X$, respectively. For notational convenience, the symbol $A(x)$ is used instead of $\mu_A(x)$.

The fuzzy set A is contained in B ($A \subseteq B$) if, and only if, $A(x) \leqslant B(x)$, for all $x \in X$. Moreover, A and B are equal if, and only if, $A(x) = B(x)$, $\forall x \in X$. The *intersection* ($\cap$) and *union* ($\cup$) between A and B are given by

$$(A \cap B)(x) = A(x) \wedge B(x) \quad \text{and} \quad (A \cup B)(x) = A(x) \vee B(x),$$

where the symbols $\wedge$ and $\vee$ stand for the maximum and minimum, respectively. The *complement* of the fuzzy set A in X, denoted by A^C, is defined by the membership function

$$A^C(x) = 1 - A(x), \ \forall x \in X.$$

Remark 1.1. *In general, the intersection between the fuzzy subsets $A \subseteq X$ and $A^C \subseteq X$ is not equal to the empty set, that is, $A \cap A^C \neq \varnothing$. A similar observation can be made for the union, that is, $A \cup A^C \neq X$.*

Example 1.2. *Let A be the set given in Example 1.1. The complement A^C has membership function given by*

$$A^C(x) = \begin{cases} -x & \textit{if} \quad -1 \leqslant x \leqslant 0 \\ x & \textit{if} \quad 0 \leqslant x \leqslant 1 \\ 1 & \quad \textit{otherwise} \end{cases}.$$

The union and intersection between A and A^C have the following membership function

$$(A \cup A^C)(x) = \begin{cases} -x & \textit{if} \quad -1 \leqslant x \leqslant -0.5 \\ 1+x & \textit{if} \quad -0.5 \leqslant x \leqslant 0 \\ 1-x & \textit{if} \quad 0 \leqslant x \leqslant 0.5 \\ x & \textit{if} \quad 0.5 \leqslant x \leqslant 1 \\ 1 & \quad \textit{otherwise} \end{cases}$$

and

$$(A \cap A^C)(x) = \begin{cases} 1+x & \textit{if} \quad -1 \leqslant x \leqslant -0.5 \\ -x & \textit{if} \quad -0.5 \leqslant x \leqslant 0 \\ x & \textit{if} \quad 0 \leqslant x \leqslant 0.5 \\ 1-x & \textit{if} \quad 0.5 \leqslant x \leqslant 1 \\ 0 & \quad \textit{otherwise} \end{cases}.$$

Therefore for this example, $A \cup A^C \neq \mathbb{R}$ and $A \cap A^C \neq \varnothing$.

Definition 1.2. *A fuzzy subset A of X is said to be normal if there exists $x \in X$ such that $A(x) = 1$.*

Every crisp set is a normal fuzzy subset (see (1.3)). Another example of normal fuzzy set is the fuzzy set A given in Example 1.1.

Definition 1.3. *The core of a fuzzy subset A of X is defined by the crisp set:*

$$core(A) = \{x \in X \ : \ A(x) = 1\}.$$

The $core(A)$ represents the set of all elements of X that have full association to the subset A. In the case of Example 1.1 it follows that $core(A) = \{0\}$, which means that 0 is the only element that is certainly *"close to 0"*.

Definition 1.4. *The* support *of the fuzzy set A is given by*

$$supp(A) = \{x \in X \ : \ A(x) > 0\}.$$

The crisp set $supp(A)$ consists of all elements that have "some" association to the set A. In Example 1.1 the support of A is equal to the interval $(-1, 1)$, which means that all elements between -1 and 1 are considered as close to some (non-zero) degree, based on membership function given as in (1.1).

The class of fuzzy subsets $\mathcal{F}(X)$ together with the partial ordering of fuzzy set inclusion ($\subseteq$) is a complete lattice that is isomorphic to the complete lattice given by $[0,1]^X = \{f : X \to [0,1]\}$ together with the usual partial ordering of functions ($f \leqslant g \Leftrightarrow f(x) \leqslant g(x), \ \forall x \in X$). The definitions of partial ordering and complete lattice are given as follows.

Definition 1.5. *[22] A set $\mathbb{L}$ is a partial ordered set if a binary relation ($\leqslant$) is defined, and satisfies*

1. *For all $x \in \mathbb{L}$, $x \leqslant x$ (Reflexive);*
2. *If $x \leqslant y$ and $y \leqslant x$, then $x = y$ $\forall x, y \in \mathbb{L}$ (Antisymmetric);*
3. *If $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$ $\forall x, y, z \in \mathbb{L}$ (Transitive);*

Definition 1.6. *[22] A partially ordered set $(\mathbb{L}, \leqslant)$ is a complete lattice if every limited $X \subseteq \mathbb{L}$ has an infimum, denoted $\bigwedge X$, and a supremum, denoted $\bigvee X$, in $\mathbb{L}$.*

The case where $X = \{x_i : i \in I\}$, for an arbitrary index set I, the infimum and supremum are written as $\bigwedge\limits_{i \in I} x_i$ and $\bigvee\limits_{i \in I} x_i$, respectively, instead of $\bigwedge X$ and $\bigvee X$. In the special case where $X = \{x, y\}$, one can simply write $x \wedge y$ and $x \vee y$.

Next, the definitions of algebraic erosions and dilations are presented.

Definition 1.7. *[72] A mapping $\Phi : \mathbb{L} \to \mathbb{M}$ is called an algebraic erosion if $\Phi(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} \Phi(x_i)$ for all $x_i \in \mathbb{L}$ and for all index sets I. Similarly, a mapping $\Phi : \mathbb{L} \to \mathbb{M}$ is called an algebraic dilation if $\Phi(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} \Phi(x_i)$ for all $x_i \in \mathbb{L}$ and for all index sets I.*

A mapping Φ that is both an algebraic erosion and an algebraic dilation is said to be a complete lattice homomorphism. If Φ is in addition bijective, then one speaks of a complete lattice isomorphism. Let us remark that a complete lattice isomorphism can also be characterized as a bijection $\Phi : \mathbb{L} \to \mathbb{M}$ that satisfies $x \leqslant_{\mathbb{L}} y$ if and only if $\Phi(x) \leqslant_M \Phi(y)$ $\forall x, y \in \mathbb{L}$ [22]. If $(\mathbb{L}, \leqslant)$ is a complete lattice and $\leqslant$ is a total order, that is, if $x \leqslant y$ or $y \leqslant x$ $\forall x, y \in \mathbb{L}$, then one speaks of a complete chain. Moreover, a chain $\mathbb{L}$ is said to be conditionally complete if the infimum and the supremum of any bounded subset of $\mathbb{L}$ exists in $\mathbb{L}$ [22]. If $\mathbb{L}$ is a complete lattice and X is an arbitrary non-empty set, then $\mathbb{L}^X = \{f : X \to \mathbb{L}\}$ is also a complete lattice with the partial ordering given by [22]

$$f \leqslant g \Leftrightarrow f(x) \leqslant g(x), \ \forall f, g \in \mathbb{L}^X. \tag{1.4}$$

Example 1.3. *The unit interval $[0, 1]$ and the extended $\mathbb{R}_{\pm\infty} = \mathbb{R} \cup \{+\infty, -\infty\}$ are examples of complete chains. The set $\mathbb{R}$ is known to be a conditionally complete lattice with the operation of addition.*

Example 1.4. *[22] The $\mathcal{P}(\mathbb{R})$ and $(\mathbb{R}_{\pm\infty})^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}_{\pm\infty}\}$ together with the partial order of set inclusion $(\subseteq)$ and the usual order of functions $(\leqslant)$, respectively, are examples of complete lattices.*

An example is given by $[0, 1]^X$ with the partial order given by (1.4) and this complete lattice is isomorphic to $\mathcal{F}(X)$. More precisely, $\phi : \mathcal{F}(X) \to [0, 1]^X$ given by $\phi(A) = \mu_A$ represents a complete lattice isomorphism because ϕ is bijective and because $A \subseteq B$ if and only if $\mu_A \leqslant \mu_B$.

Next the definition of triangular norms is presented. This concept started in 1942 by Menger [94], which was used to construct metric spaces based on probability distributions. Triangular norms arose as a generalization of the classical logical connectives [83].

Definition 1.8. *A triangular norm, t-norm for short, is a binary operation*

$$\begin{array}{rcl} \triangle : [0,1] \times [0,1] & \to & [0,1] \\ (x, y) & \longmapsto & x \triangle y \end{array}$$

that satisfies the following properties:

(a) Commutativity: $x \triangle y = y \triangle x$, $\forall x, y \in [0, 1]$;

(b) *Associativity:* $x \triangle (y \triangle z) = (x \triangle y) \triangle z$, $\forall x, y, z \in [0,1]$;

(c) *Monotonicity: if* $y \leqslant z$, *then* $x \triangle y \leqslant x \triangle z$, $\forall x \in [0,1]$;

(d) *Boundary conditions:* $x \triangle 1 = x$ *and* $x \triangle 0 = 0$, $\forall x \in [0,1]$.

Example 1.5. *The following operators are examples of t-norms*

(a) *Minimum:* $x \wedge y = \min\{x, y\}$;

(b) *Product:* $x \triangle_p y = xy$;

(c) *Łukasiewicz:* $x \triangle_Ł y = \max\{x + y - 1, 0\}$;

(d) *Drastic product:* $x \triangle_d y = \begin{cases} \min\{x,y\}, & \text{if } \max\{x,y\} = 1 \\ 0, & \text{otherwise} \end{cases}$.

Definition 1.9. *A triangular conorm, also called as s-norm, is a binary operation*

$$\begin{array}{rcl} \triangledown : [0,1] \times [0,1] & \rightarrow & [0,1] \\ (x,y) & \longmapsto & x \triangledown y \end{array}$$

that satisfies the following properties:

(a) *Commutativity:* $x \triangledown y = y \triangledown x$, $\forall x, y \in [0,1]$;

(b) *Associativity:* $x \triangledown (y \triangledown z) = (x \triangledown y) \triangledown z$, $\forall x, y, z \in [0,1]$;

(c) *Monotonicity: if* $y \leqslant z$, *then* $x \triangledown y \leqslant x \triangledown z$, $\forall x \in [0,1]$;

(d) *Boundary conditions:* $x \triangledown 1 = 1$ *and* $x \triangledown 0 = x$, $\forall x \in [0,1]$.

Example 1.6. *The following operators are examples of s-norms*

(a) *Maximum:* $x \vee y = \max\{x, y\}$;

(b) *Probabilistic sum:* $x \triangledown_p y = x + y - xy$;

(c) *Łukasiewicz:* $x \triangledown_Ł y = \min\{x + y, 1\}$;

(d) *Drastic sum:* $x \triangledown_d y = \begin{cases} \max\{x,y\}, & \text{if } \min\{x,y\} = 0 \\ 1, & \text{otherwise} \end{cases}$.

The t-norm and s-norm generalize the *and* and *or* connectives of classical logic, respectively. Moreover, s-norms can be viewed as dual operators of the t-norms. One can prove that $\triangledown$ is a s-norm if and only if there exists a t-norm $\triangle$ such that $x \triangledown y = 1 - (1 - x \triangle 1 - y)$, for every $x, y \in [0,1]$ [83, 111]. In general, both t-norms and s-norms can not be ordered. However, there exist the largest and the smallest t-norm and s-norm [111]. One can prove that the *minimum* operator is the largest t-norm, whereas the *drastic product* is the smallest one [140], that is, for all t-norm $\triangle$ it follows that

$$x \triangle_d y \leqslant x \triangle y \leqslant x \wedge y, \quad \forall x, y \in [0, 1].$$

In addition, the *maximum* operator is the smallest s-norm, whereas the *drastic sum* is the largest one [83], that is, for all s-norm $\bigtriangledown$ it follows that

$$x \vee y \leqslant x \bigtriangledown y \leqslant x \bigtriangledown_d y, \quad \forall x, y \in [0, 1].$$

Next the definition of Cartesian product via t-norm is presented [44].

Definition 1.10. *Let $A_i \in \mathcal{F}(X_i)$, for $i = 1, \ldots, n$. The* Cartesian product $A_1 \times_\triangle \ldots \times_\triangle A_n$*, via t-norm, is defined by the following membership function:*

$$(A_1 \times_\triangle \ldots \times_\triangle A_n)(x_1, \ldots, x_n) = A_1(x_1) \triangle \ldots \triangle A_n(x_n), \ \forall (x_1, \ldots, x_n) \in X_1, \ldots, X_n,$$

for some t-norm $\triangle$.

Consider the Cartesian product given by the minimum t-norm in Definition 1.10, which is called *usual Cartesian product* and it is denoted simply by $A_1 \times \ldots \times A_n$. The usual Cartesian product gives rise to fuzzy relations among fuzzy sets. A relation between classical sets indicates if there is (or not) an association among their elements, whereas the fuzzy relations not only indicates this relationship but also the degree of this association.

Definition 1.11. *An n-ary* fuzzy relation *R among the classical universes $X_1, \ldots, X_n$ is a fuzzy subset of $X_1 \times ... \times X_n$, whose membership function is given by*

$$R : X_1 \times \ldots \times X_n \to [0, 1].$$

The symbol $R(x_1, \ldots, x_n) \in [0, 1]$ represents the degree of relationship among the elements $x_1, \ldots, x_n$ according to the relation R. A fuzzy relation R between two classical universes X_1 and X_2 is called *binary fuzzy relation.*

The cylindrical extension of a fuzzy set is an example of fuzzy relation [70].

Definition 1.12. *[111] The* cylindrical extension *of a fuzzy set $A \subseteq X_i$, for $i = 1, 2$, is the fuzzy relation $cyl(A) \in \mathcal{F}(X_1 \times X_2)$ whose membership function is given by*

$$cyl(A)(x_1, x_2) = A(x_i), \ \forall x_i \in X_i.$$

The *fuzzy projection* provides fuzzy relations on some subspaces of the original space, in contrast to the cylindrical extension, which increases the number of coordinates of the Cartesian product over which the fuzzy relation is defined. That is, the projection reduces the dimensionality of the original fuzzy relation.

Definition 1.13. *[111] The projection of $R \in \mathcal{F}(X_1 \times ... \times X_n)$ onto X_i, where $1 \leqslant i \leqslant n$, is the fuzzy subset $\prod_R^i$ of X_i, whose membership function is given by*

$$\prod_R^i(y) = \bigvee_{\{x \in X \ : \ x_i = y\}} R(x_1, \ldots, x_n), \ \forall y \in X_i,$$

where $X = X_1 \times ... \times X_n$.

The α-cut plays a key role in the relationship between fuzzy and classical sets. The α-cut of a fuzzy set is defined as follows.

Definition 1.14. *[9] Let A be a fuzzy subset of $X \neq \emptyset$. The α-cuts of A are defined by the classical sets*

$$[A]^\alpha = \{x \in X \ : \ A(x) \geqslant 0\}, \quad \forall \alpha \in (0, 1].$$

In addition, if X is a topological space then the 0-cut of A is given by

$$[A]^0 = cl \ supp(A),$$

where $cl \ Y \subseteq X$ represents the closure of the classical Y.

It is easy to see, from the above definition, that the following statement holds true [85, 9]

$$A = B \Leftrightarrow [A]^\alpha = [B]^\alpha, \ \forall A, B \in \mathcal{F}(X), \ \forall \alpha \in [0, 1].$$

This fact implies that there is a relationship between the membership function of a fuzzy subset and the characteristic function of its α-cuts, as can be seen in the next corollary [9].

Corollary 1.1. *The membership function of $A \in \mathcal{F}(X)$ can be expressed in terms of the characteristic function of its α-cuts, as follows:*

$$A(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \chi_{[A]^\alpha}(x)), \quad \forall A \in \mathcal{F}(X)$$

where $\chi_{[A]^\alpha}(x) = \begin{cases} 1, & \text{if } x \in [A]^\alpha \\ 0, & \text{if } x \notin [A]^\alpha \end{cases}.$

The following theorem is known as *Negoita and Ralescu's Theorem of Representation* [102]. It shows a sufficient condition for a family of classical subsets of X to be formed as α-cuts of a fuzzy subset.

Theorem 1.1. *Given a family of classical subsets $\{M_\alpha \subseteq X : \alpha \in [0, 1]\}$ that satisfies the following conditions*

(a) $\bigcup_{0 \leqslant \alpha \leqslant 1} M_\alpha \subset M_0$;

(b) *If* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$, *then* $M_{\alpha_2} \subseteq M_{\alpha_1}$;

(c) *For any sequence* α_n *which converges from below to* $\alpha \in (0, 1]$, *it follows*

$$\bigcap_{n=1}^{\infty} M_{\alpha_n} = M_\alpha;$$

Then there exists a unique fuzzy set M, *such that* $[M]^\alpha = M_\alpha$, *for any* $\alpha \in [0, 1]$.

Proof. See [102, 16]. □

The next section discusses an important subclass of fuzzy sets, namely fuzzy numbers. They are connected to the definition of α-cuts.

1.2 Fuzzy Numbers

Fuzzy numbers form a special class of fuzzy sets, which are widely used in applications of fuzzy logic [9, 85, 104, 140], fuzzy analysis [102, 40, 16] and fuzzy differential equations [42, 80, 67, 70, 108]. Fuzzy numbers are fuzzy subsets of $\mathbb{R}$ (called *fuzzy quantities*) with some additional properties.

Definition 1.15. *[9] The fuzzy set* $A \in \mathcal{F}(\mathbb{R})$ *is said to be a fuzzy number if the following properties are satisfied:*

(a) A *is normal, that is,* $[A]^1 \neq \varnothing$;

(b) The α*-cuts of* A *are closed intervals of* $\mathbb{R}$, *for all* $\alpha \in [0, 1]$;

(c) $supp(A)$ *is bounded.*

The class of fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. Since the α-cuts of a fuzzy number A are given by closed intervals, $[A]^\alpha$ is denoted by

$$[A]^\alpha = [a_\alpha^-, a_\alpha^+], \ \forall A \in \mathbb{R}_{\mathcal{F}}, \ \forall \alpha \in [0, 1].$$

Note that real numbers are particular cases of fuzzy numbers. The most common fuzzy numbers used in the literature are *triangular*, *trapezoidal* and *Gaussian* fuzzy numbers [9].

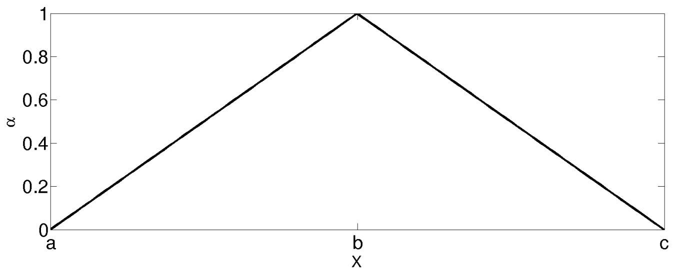
Definition 1.16. *A fuzzy number A is said to be triangular if its membership function is given by*

$$A(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a < x \leqslant b \\ \dfrac{c-x}{c-b}, & \text{if } b < x \leqslant c \\ 0, & \text{otherwise} \end{cases},$$

where $a \leqslant b \leqslant c$ are real numbers.

From Definition 1.16, the triangular fuzzy number A is determined by $a, b, c \in \mathbb{R}$. Hence, A is denoted by the triple $(a; b; c)$. In this thesis the class of triangular fuzzy numbers is represented by the symbol $\mathcal{F}_{Tr}$. The graphical representation of a triangular fuzzy number is depicted as in Figure 4.

Figure 4 – Graphical representation of a triangular fuzzy number $A = (a; b; c)$.



Source: Author

The α-cuts of the triangular fuzzy number $A = (a; b; c)$ are given by [9]

$$[A]^\alpha = [a + \alpha(b-a), c + \alpha(b-c)], \ \forall\alpha \in [0, 1].$$

Definition 1.17. *A fuzzy number A is said to be trapezoidal if its membership function is given by*

$$A(x) = \begin{cases} \dfrac{x-a}{b-a} & , \text{if } a < x \leqslant b \\ 1 & , \text{if } b < x \leqslant c \\ \dfrac{d-x}{d-c} & , \text{if } c < x \leqslant d \\ 0 & , \text{otherwise} \end{cases},$$

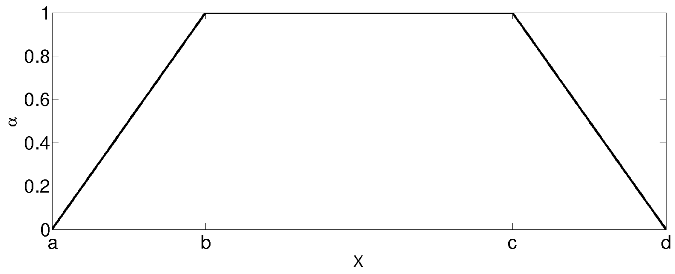
where $a \leqslant b \leqslant c \leqslant d$ are real numbers.

The trapezoidal fuzzy number A is denoted by the quadruple $A = (a; b; c; d)$. The class of trapezoidal fuzzy numbers is represented by the symbol $\mathcal{F}_{Tp}$. The graphical representation of a trapezoidal fuzzy number is given as in Figure 5.

The α-cuts of the trapezoidal fuzzy number $A = (a; b; c; d)$ are given by

$$[A]^\alpha = [a + \alpha(b-a), d + \alpha(c-d)], \ \forall\alpha \in [0, 1].$$

Figure 5 – Graphical representation of a trapezoidal fuzzy number $A = (a; b; c; d)$.



Source: Author

In the case where $b = c$ the trapezoidal fuzzy number $A = (a; b; c; d)$ becomes the triangular fuzzy number $A = (a; b; d)$. This implies that $\mathcal{F}_{Tr}$ is contained in $\mathcal{F}_{Tp}$.

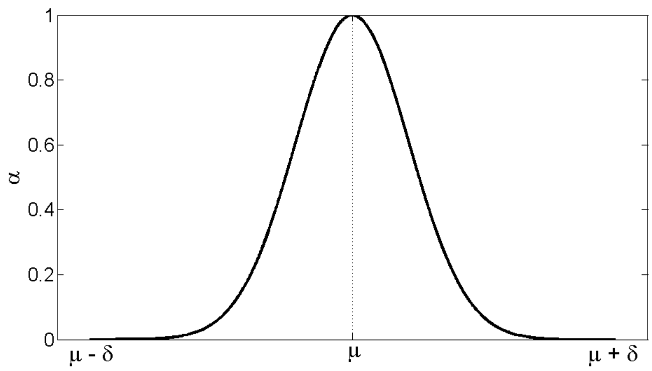
Definition 1.18. *A fuzzy number A is said to be Gaussian if its membership function is given by*

$$A(x) = \begin{cases} e^{-\frac{(\mu-x)^2}{\sigma}} & , \ if \ \ \mu-\delta \leqslant x \leqslant \mu+\delta \\ 0 & , \quad otherwise \end{cases},$$

where μ, σ and δ are real numbers, with $\delta > 0$.

The Gaussian fuzzy number A is denoted by $A = (\mu, \sigma, \delta)$ and its graphical representation is given as in Figure 6. The class of Gaussian fuzzy numbers is represented by the symbol $\mathcal{F}_G$.

Figure 6 – Graphical representation of a Gaussian fuzzy number $A = (\mu, \sigma, \delta)$.



Source: Author

The $\alpha-$cuts of the Gaussian fuzzy number $A = (\mu, \sigma, \delta)$ are given by

$$[A]^\alpha = \begin{cases} \left[\mu - \sigma\sqrt{ln\left(\frac{1}{\alpha}\right)}, \mu + \sigma\sqrt{ln\left(\frac{1}{\alpha}\right)}\right], & \text{if} \quad \alpha \geqslant \overline{\alpha} = e^{-\frac{\delta^2}{\sigma}} \\ [\mu-\delta, \mu+\delta], & \text{if} \quad \alpha < \overline{\alpha} = e^{-\frac{\delta^2}{\sigma}} \end{cases}.$$

The following results are consequences of Definition 1.15.

Proposition 1.1. *[8, 104] The following statements hold true for all $A \in \mathbb{R}_{\mathcal{F}}$:*

(a) The fuzzy numbers are convex fuzzy functions, i.e., *$A(y) \geqslant A(x) \wedge A(z)$ whenever $x \leqslant y \leqslant z$;*

(b) The fuzzy numbers are upper semi-continuous, i.e., *for all $x \in \mathbb{R}$ and $\epsilon > 0$ exists $\delta > 0$ such that $|x - y| < \delta$ implies that $A(y) < A(x) + \epsilon$;*

(c) A is non-decreasing on $(-\infty, a_1^-]$ and non-increasing on $[a_1^+, \infty)$.

Theorem 1.2. *[66] Consider the* endpoints *functions given by*

$$\begin{array}{rcl} a^- : [0,1] \to \mathbb{R} & \text{and} & a^- : [0,1] \to \mathbb{R} \\ a^-(\alpha) \longmapsto a_\alpha^- & & a^+(\alpha) \longmapsto a_\alpha^+ \end{array}$$

Thus

(a) $a^- \in \mathbb{R}$ is a bounded, non-decreasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.

(b) $a^+ \in \mathbb{R}$ is a bounded, non-increasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.

(c) $a^-(1) \leqslant a^+(1)$.

Conversely, given two functions $a^-, a^+ : [0,1] \to \mathbb{R}$ that satisfy the conditions (a)-(c) there exists a fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ such that $[A]^\alpha = [a_\alpha^-, a_\alpha^+]$, where $a_\alpha^- = a^-(\alpha)$ and $a_\alpha^+ = a^+(\alpha)$, $\forall \alpha \in [0,1]$.

Lemma 1.1. *[129] Let $I(\mathbb{R}) = \{[a_\alpha^-, a_\alpha^+] \subseteq \mathbb{R} \ : \ \alpha \in (0,1]\}$ be a family of non-empty closed intervals of $\mathbb{R}$. If the following properties are satisfied*

(a) $[b_\alpha^-, b_\alpha^+] \subset [a_\alpha^-, a_\alpha^+]$, for $0 < \alpha \leqslant \beta$;

(b) $\left[\lim_{k\to\infty} a_{\alpha_k}^-, \lim_{k\to\infty} a_{\alpha_k}^+\right] = [a_\alpha^-, a_\alpha^+]$, where (α_k) is a non-decreasing sequence which converges to $\alpha \in (0,1]$,

then the family $[a_\alpha^-, a_\alpha^+]$ represents the α-cuts of the fuzzy number A. On the other hand if $[a_\alpha^-, a_\alpha^+]$ are the α-cuts of the fuzzy number A, then the properties (a) and (b) hold true.

The set of fuzzy numbers such that the corresponding endpoint functions are continuous is denoted by the symbol $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$, that is, $\mathbb{R}_{\mathcal{F}_\mathcal{C}} = \{A \in \mathbb{R}_{\mathcal{F}} \ : \ a^- \text{ and } a^+ \text{ are continuous}\}$. It follows that $\mathcal{F}_{Tr} \subset \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ and $\mathcal{F}_{Tp} \subset \mathbb{R}_{\mathcal{F}_\mathcal{C}}$.

Next the concept of distance for fuzzy numbers is presented. Moreover, this thesis uses the *Pompeiu-Hausdorff* norm, which is obtained by the Pompeiu-Hausdorff metric.

Definition 1.19. *[71] Let A and B be two non-empty compact subsets of a metric space X. The pseudometric $\tilde{d}$, given by*

$$\tilde{d}(A, B) = \bigvee_{a \in A} d(a, B),$$

where

$$d(a, B) = \bigwedge_{b \in B} ||a - b||,$$

is called the Hausdorff separation. *The symbol* $\bigwedge$ *stands for the infimum operator.*

Definition 1.20. *[71] Let A and B be two non-empty compact subsets of a metric space X. The* Pompeiu-Hausdorff *metric d_H is given by*

$$d_H(A, B) = \max\{\tilde{d}(A, B), \tilde{d}(B, A)\}.$$

Particularly, if $X = \mathcal{F}(\mathbb{R})$ then the Pompeiu-Hausdorff metric for fuzzy sets is given as follows.

Definition 1.21. *[40] Let A and B be fuzzy sets. The* Pompeiu-Hausdorff *distance $d_\infty : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0, +\infty)$ is given by*

$$d_\infty(A, B) = \bigvee_{0 \leqslant \alpha \leqslant 1} d_H([A]^\alpha, [B]^\alpha), \ \forall A, B \in \mathcal{F}(\mathbb{R}).$$

In the case where A and B are given by fuzzy numbers, Definition 1.5 boils down to the next definition.

Definition 1.22. *[40] The Pompeiu-Hausdorff distance $d_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to [0, +\infty)$, between fuzzy numbers, is given by*

$$d_\infty(A, B) = \bigvee_{\alpha \in [0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}, \ \forall A, B \in \mathbb{R}_\mathcal{F}. \tag{1.5}$$

Moreover, the Pompeiu-Hausdorff norm of a fuzzy number $A \in \mathbb{R}_\mathcal{F}$ is defined by

$$||A||_\mathcal{F} = d_\infty(A, 0), \tag{1.6}$$

where the symbol 0 stands for the characteristic function of the real number 0.

One can observe that $\mathbb{R}_\mathcal{F}$ is not a vector space, thus the operator $|| \cdot ||_\mathcal{F}$ does not characterize a norm. In the literature the operator $|| \cdot ||_\mathcal{F}$ has been called by quasi-norm [121]. Here a language abuse is committed and this operator is called as the norm.

Theorem 1.3. *[14] The following statements are equivalent:*

(a) $f : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous in the usual metric of* $\mathbb{R}^n$*;*

(b) $F : (\mathcal{F}(\mathbb{R}^n), d_\infty) \to (\mathcal{F}(\mathbb{R}^n), d_\infty)$ *is continuous.*

Recall that $(\mathbb{R}_{\mathcal{F}_C}, d_\infty)$ is a complete and separable metric space, whereas $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is only a complete metric space [119].

Definition 1.23. *The* width *(or* diameter*) of a fuzzy number* $A \in \mathbb{R}_{\mathcal{F}}$ *is defined by*

$$width(A) = a_0^+ - a_0^- \geqslant 0. \tag{1.7}$$

The width of a fuzzy number is associated with the uncertainty that it models, which means that larger width is tied with greater uncertainty. Observe that every real number has width equals to 0.

The next section introduces the Zadeh extension principle which will be used in order to generate an arithmetic for fuzzy numbers, and to define what is meant by a fuzzy function and real-value analyses to fuzzy analyses.

1.3 Extension Principle

The Zadeh extension principle is a method that extends classical operations to operations with fuzzy sets as arguments. Given a function $f : X \to Z$, the Zadeh extension of f maps a fuzzy subset A of X in some element of $\mathcal{F}(Z)$ [157].

Definition 1.24. *[157, 103] Let* $f : X \to Z$ *be a classical function and* $A \in \mathcal{F}(X)$*. The Zadeh extension of* f *at* A*, denoted by* $\hat{f}(A)$*, is given by*

$$\widehat{f}(A)(z) = \begin{cases} \bigvee\limits_{f^{-1}(z)} A(x) & , \text{ if } f^{-1}(z) \neq \varnothing \\ 0 & , \text{ if } f^{-1}(z) = \varnothing \end{cases},$$

where $f^{-1}(z) = \{x \ : \ f(x) = z\}$*.*

For multiple variables the Zadeh extension principle is defined as follows.

Definition 1.25. *[59, 16] Let* $f : X_1 \times \ldots \times X_n \to Z$ *and* $A_i \in \mathcal{F}(X_i)$*, for* $i = 1, \ldots, n$*. The extension principle of* f *at* $(A_1, \ldots, A_n)$ *is given by*

$$\widehat{f}(A_1, \ldots, A_n)(z) = \begin{cases} \bigvee\limits_{f^{-1}(z)} A_1(x_1) \wedge \ldots \wedge A_n(x_n) & , \text{ if } f^{-1}(z) \neq \varnothing \\ 0 & , \text{ if } f^{-1}(z) = \varnothing \end{cases},$$

where $f^{-1}(z) = \{(x_1, \ldots, x_n) \ : \ f(x_1, \ldots, x_n) = z\}$*.*

The Zadeh extension principle can be generalize via the t-norm. The method known as sup-t extension principle [30, 41] is given by

$$f_{\triangle}(A_1, \ldots, A_n)(z) = \bigvee_{f^{-1}(z)} A_1(x_1) \triangle \ldots \triangle A_n(x_n). \tag{1.8}$$

Note that in the case where $\triangle = \wedge$, the equation given as in Definition 1.25 and Equation (1.8) are equivalent, that is, $\widehat{f} = f_{\wedge}$. The next theorem yields the α-cuts of the fuzzy number $\widehat{f}(A) \in \mathcal{F}(\mathbb{R})$ obtained by the extension principle.

Theorem 1.4. *[103, 8, 34] Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Thus the following statements hold true*

(a) If f is surjective, then

$$[\widehat{f}(A)]^{\alpha} = f([A]^{\alpha}),$$

if and only if each element of supp$(\widehat{f}(A))$ is attained, for all $z \in \mathbb{R}^m$.

(b) If f is continuous, then $\widehat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R})$ is well-defined and

$$[\widehat{f}(A)]^{\alpha} = f([A]^{\alpha}), \;\; \forall \alpha \in [0, 1].$$

An immediate consequence of Theorem 1.4 is that if f is continuous, then $A \subseteq B \Rightarrow \widehat{f}(A) \subseteq \widehat{f}(B)$ [8, 28].

The next two theorems provide the α-cuts of the Cartesian product when the extension principle is applied. These facts are used in the study of fuzzy initial value problems (FIVP).

Theorem 1.5. *[60] Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$ be a continuous function, $A \in \mathcal{F}(\mathbb{R}^m)$ and $B \in \mathcal{F}(\mathbb{R}^n)$. If $\triangle$ is an upper semi-continuous t-norm, then*

$$[\widehat{f}(A \times_{\triangle} B)]^{\alpha} = f([A \times_{\triangle} B]^{\alpha}),$$

where $f([A \times_{\triangle} B]^{\alpha}) = \{f(x, y) \;:\; (x, y) \in [A \times_{\triangle} B]^{\alpha}\}$.

For the usual Cartesian product the following theorem, known as Nguyen's Theorem, holds true.

Theorem 1.6. *[103, 8] Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$ be a continuous function, $A \in \mathcal{F}(\mathbb{R}^m)$ and $B \in \mathcal{F}(\mathbb{R}^n)$. Thus*

$$[\widehat{f}(A \times B)]^{\alpha} = [\widehat{f}(A, B)]^{\alpha} = f([A]^{\alpha}, [B]^{\alpha}),$$

where $f([A]^{\alpha}, [B]^{\alpha}) = \{f(x, y) \;:\; x \in [A]^{\alpha}$ and $y \in [B]^{\alpha}\}$.

The next section provides an arithmetic on fuzzy numbers. This arithmetic arises from Definition 1.25, that is, from the Zadeh extension principle for multiple variables.

1.4 Arithmetic on Fuzzy Numbers

The Zadeh extension principle allows us to generate an arithmetic on fuzzy numbers. To this end, one can consider an arithmetic operator as the function f and use the extension principle to provide the corresponding arithmetic operation between fuzzy numbers. Before providing the formalization of the fuzzy arithmetic, recall interval arithmetic. Henceforth open intervals of the real line will be denoted by $I \subseteq \mathbb{R}$. Also $K = \{[a, b] \,:\, a, b \in \mathbb{R} \text{ such that } a \leqslant b\}$ stands for the set of all closed intervals of $\mathbb{R}$.

Definition 1.26. *[100] Let $\lambda \in \mathbb{R}$ and let $A, B \in K$ given by $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two closed intervals of $\mathbb{R}$. The arithmetic operations between intervals are given by:*

(a) The sum $(+)$ *between A and B is the interval*

$$A + B = [a_1 + b_1, a_2 + b_2];$$

(b) The difference $(-)$ *between A and B is the interval*

$$A - B = [a_1 - b_2, a_2 - b_1];$$

(c) The product $(\cdot)$ *of A by B is the interval*

$$A \cdot B = [\min U, \max U],$$

where $U = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$;

(d) The quotient $(\div)$ *of A by B, if $0 \notin B$, is the interval*

$$A \div B = [a_1, a_2] \cdot \left[\frac{1}{b_2}, \frac{1}{b_1}\right];$$

(e) The product *of A by a scalar λ is the interval*

$$\lambda A = \begin{cases} [\lambda a_1, \lambda a_2], & \text{if } \lambda \geqslant 0 \\ [\lambda a_2, \lambda a_1], & \text{if } \lambda < 0 \end{cases}.$$

Interval arithmetic generalizes the arithmetic for real numbers, since every real number can be seen as a closed interval with equal endpoints. In addition, the next theorem provides interval arithmetic in terms of the extension principle.

Theorem 1.7. *[9] Let A and B be two closed intervals of $\mathbb{R}$. Then*

$$\chi_{A \otimes B}(z) = \bigvee_{(x,y):x \otimes y = z} \chi_A(x) \wedge \chi_B(y) = \begin{cases} 1, & \text{if } z \in A \otimes B \\ 0, & \text{if } z \notin A \otimes B \end{cases},$$

where $\otimes$ is some basic interval arithmetic operator $+, -, \cdot$ or $\div$.

The following corollary is an immediate consequence of Theorem 1.7.

Corollary 1.2. *[9] Let the interval $A \otimes B$. The α-cuts of the crisp set $A \otimes B$, with respective membership function $\chi_{A \otimes B}$, are given by*

$$[A \otimes B]^{\alpha} = A \otimes B,$$

$\forall \alpha \in [0, 1]$.

Next, the usual arithmetic on fuzzy numbers is defined.

1.4.1 Standard Arithmetic

Consider the following arithmetic operator given by

$$\begin{aligned} \otimes : \mathbb{R} \times \mathbb{R} &\quad \rightarrow \quad \mathbb{R} \\ (x, y) &\quad \longmapsto \quad x \otimes y, \end{aligned}$$

where $\otimes \in \{+, -, \cdot, \div\}$. Now let $A, B \in \mathbb{R}_{\mathcal{F}}$. From Zadeh's extension principle, it follows that

$$\begin{aligned} \widehat{\otimes} : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} &\quad \rightarrow \quad \mathbb{R}_{\mathcal{F}} \\ (A, B) &\quad \longmapsto \quad \widehat{\otimes}(A, B) = A \widehat{\otimes} B, \end{aligned}$$

where

$$(A \widehat{\otimes} B)(z) = \bigvee_{(x,y)\ :\ x \otimes y = z} A(x) \wedge B(y). \tag{1.9}$$

The arithmetic that arises from Equation (1.9) is called by *standard* fuzzy arithmetic or simply *usual* fuzzy arithmetic. For simplicity of notation, $A \widehat{\otimes} B$ will be denoted by $A \otimes B$.

Definition 1.27. *Let $A, B \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$. The standard arithmetic operations are defined as follows*

(a) The standard sum *between A and B is the fuzzy number $A + B$ whose membership function is given by*

$$(A + B)(z) = \bigvee_{(x,y)\ :\ x + y = z} A(x) \wedge B(y);$$

(b) The standard difference *between A and B is the fuzzy number $A - B$ whose membership function is given by*

$$(A - B)(z) = \bigvee_{(x,y)\ :\ x - y = z} A(x) \wedge B(y);$$

(c) *The* standard product *of A by B is the fuzzy number $A \cdot B$ whose membership function is given by*

$$(A \cdot B)(z) = \bigvee_{(x,y)\ :\ xy=z} A(x) \wedge B(y);$$

(d) *The* standard quotient *of A by B, if $0 \notin supp(B)$, is the fuzzy number $A \div B$ whose membership function is given by*

$$(A \div B)(z) = \bigvee_{(x,y)\ :\ x/y=z} A(x) \wedge B(y);$$

(e) *The* product *of A by scalar λ is the fuzzy number λA whose membership function is given by*

$$(\lambda A)(z) = \bigvee_{x\ :\ \lambda x=z} A(x).$$

The standard fuzzy arithmetic can be characterized by α-cuts, which is connected with the interval arithmetic. Theorem 1.8 provides the α-cuts of the standard arithmetic operations.

Theorem 1.8. *[9] Let A and B be fuzzy numbers with the respective α-cuts $[A]^\alpha = [a_\alpha^-, a_\alpha^+]$ and $[B]^\alpha = [b_\alpha^-, b_\alpha^+]$. Then the following properties hold true*

(a) *The α-cuts of the* standard sum *between A and B are given by*

$$[A+B]^\alpha = [A]^\alpha + [B]^\alpha = [a_\alpha^- + b_\alpha^-, a_\alpha^+ + b_\alpha^+],\ \forall \alpha \in [0,1];$$

(b) *The α-cuts of the* standard difference *between A and B are given by*

$$[A-B]^\alpha = [A]^\alpha - [B]^\alpha = [a_\alpha^- - b_\alpha^+, a_\alpha^+ - b_\alpha^-],\ \forall \alpha \in [0,1];$$

(c) *The α-cuts of the* standard product *of A by B are given by*

$$[A \cdot B]^\alpha = [A]^\alpha \cdot [B]^\alpha = [\min U^\alpha, \max U^\alpha],\ \forall \alpha \in [0,1];$$

where $U^\alpha = \{a_\alpha^- b_\alpha^-, a_\alpha^- b_\alpha^+, a_\alpha^+ b_\alpha^-, a_\alpha^+ b_\alpha^+\}$;

(d) *The α-cuts of the* standard quotient *of A by B, if $0 \notin supp(B)$, are given by*

$$[A \div B]^\alpha = [A]^\alpha \div [B]^\alpha = [a_\alpha^-, a_\alpha^+] \cdot \left[\frac{1}{b_\alpha^+}, \frac{1}{b_\alpha^-}\right],\ \forall \alpha \in [0,1];$$

(e) *The α-cuts of the product of A by a scalar λ are given by*

$$[\lambda A]^\alpha = \lambda [A]^\alpha = \begin{cases} [\lambda a_\alpha^-, \lambda a_\alpha^+] & ,\ \text{if } \lambda \geqslant 0 \\ [\lambda a_\alpha^+, \lambda a_\alpha^-] & ,\ \text{if } \lambda < 0 \end{cases},\ \forall \alpha \in [0,1];$$

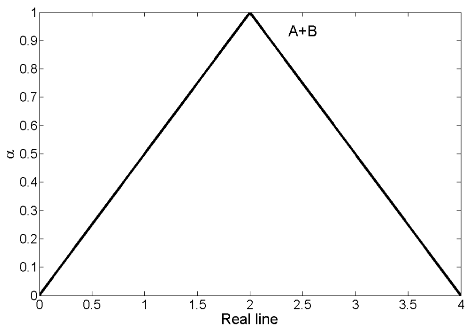
Proof. See [85, 60, 104]. □

Example 1.7. *Let $A = (-1; 0; 1)$ and $B = (1; 2; 3)$, whose α-cuts are given by $[A]^\alpha = [-1+\alpha, 1-\alpha]$ and $[B]^\alpha = [1+\alpha, 3-\alpha]$. The fuzzy numbers obtained from the result of the arithmetic operations are depicted in Figure 7 and given as follows*

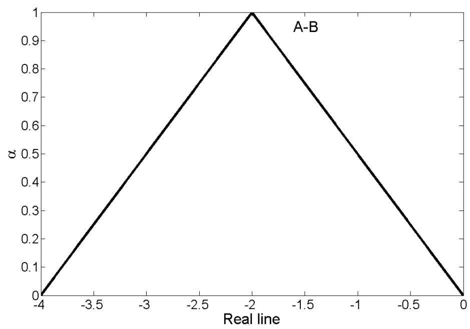
1. $[A+B]^\alpha = [2\alpha, 4-2\alpha]$;
2. $[A-B]^\alpha = [-4+2\alpha, -2\alpha]$;
3. $[A \cdot B]^\alpha = [-\alpha^2+4\alpha-3, \alpha^2-4\alpha+3]$;
4. $[A \div B]^\alpha = \left[\dfrac{-1+\alpha}{1+\alpha}, \dfrac{1-\alpha}{1+\alpha}\right]$.

Figure 7 – Graphical representation of the standard arithmetic operations of Example 1.7.

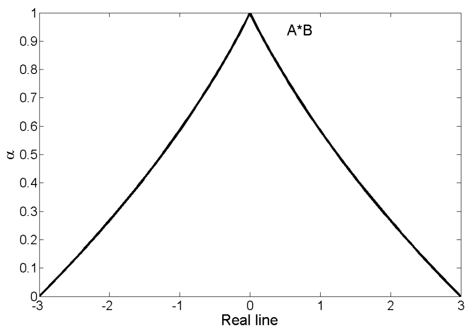
(a) Standard sum $A+B$.



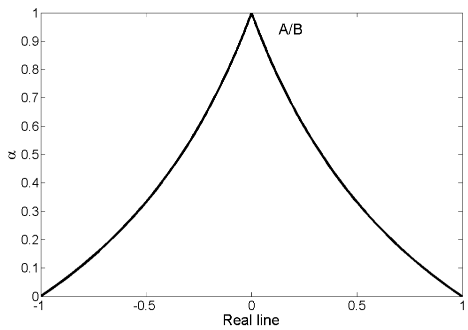
(b) Standard difference $A-B$.



(c) Standard product $A \cdot B$.



(d) Standard quocient $A \div B$.



Source: Author

The standard sum $A+B$, in Example 1.7, is equal to the triangular fuzzy number $(0; 2; 4)$. Moreover, the standard difference $A-B$ is equal to triangular fuzzy number $(-4; -2; 0)$. It is easy to verify that the standard sum (difference) between two triangular fuzzy numbers is a triangular fuzzy number [9]. In contrast to the latter operations, the standard product and quotient between triangular fuzzy numbers are not triangular fuzzy numbers, as Example 1.7 illustrates.

The standard arithmetic operations between fuzzy numbers can be easily computed. Moreover, the standard arithmetic is commutative, associative and sub-distributive

[45, 111]. However, this arithmetic do not satisfy some intuitive properties, such as

(i) $A - A = \chi_{\{0\}}$;

(ii) $(A + B) - B = A$;

(iii) $A \div A = \chi_{\{1\}}$,

where $A, B \in \mathbb{R}_{\mathcal{F}}$.

Examples 1.8, 1.9 and 1.10 exhibit the counterexamples for the statements (i), (ii) and (iii). From now on, $\chi_{\{u\}}$ is simply denoted by $u \in \mathbb{R}$.

Example 1.8. *Let $A = (-1; 0; 1)$, whose α-cuts are given by $[A]^\alpha = [-1 + \alpha, 1 - \alpha]$. For all $\alpha \in [0, 1]$, it follows that*

$$[A - A]^\alpha = [A]^\alpha - [A]^\alpha = [-2 + 2\alpha, 2 - 2\alpha] \neq [0, 0].$$

Therefore, $A - A \neq 0$. In fact $A - A = 0$ if, and only if A is a real number [9].

Example 1.9. *Let $A = (-1; 0; 1)$ and $B = (-2; 0; 2)$. Thus*

$$(A+B)-B = ((-1; 0; 1)+(-2; 0; 2))-(-2; 0; 2) = (-3; 0; 3)-(-2; 0; 2) = (-5; 0; 5) \neq A.$$

Therefore, $(A + B) - B \neq A$.

Example 1.10. *Let $A = (1; 2; 3)$, whose α-cuts are given by $[A]^\alpha = [1 + \alpha, 3 - \alpha]$. For all $\alpha \in [0, 1]$, it follows*

$$[A \div A]^\alpha = [A]^\alpha \div [A]^\alpha = \left[\frac{1 + \alpha}{3 - \alpha}, \frac{3 - \alpha}{1 + \alpha}\right] \neq [1, 1].$$

Therefore, $A \div A \neq 1$. Indeed $A \div A = 1$ if, and only if, A is a real number [9].

This thesis focuses on statements (i) and (ii). In order to avoid these mishaps, the next subsection provides the Hukuhara difference for fuzzy numbers.

1.4.2 Hukuhara Difference and its Generalizations

Several researchers have introduced other differences between fuzzy numbers such that $A - A = 0$ is verified. In 1967, Hukuhara [76] defined the Hukuhara difference to deal with integration of measurable functions and later Banks and Jacobs [6] generalized this concept and investigated its properties. Puri and Ralescu [118] used the Hukuhara difference in the context of fuzzy set theory for the first time in order to study differentials of fuzzy functions.

Definition 1.28. *[76, 118] The* Hukuhara difference *(or H-difference for short) between two fuzzy numbers A and B is defined by*

$$A -_H B = C \Longleftrightarrow A = B + C,$$

where $+$ is the standard sum.

Note that $A -_H A = 0$ holds true for all $A \in \mathbb{R}_{\mathcal{F}}$, since

$$A -_H A = 0 \Longleftrightarrow A = A + 0.$$

Moreover, $(A + B) -_H B = A$, since the standard sum is a commutative operation and by definition of the Hukuhara difference, it follows that

$$A + B = B + A \Longleftrightarrow (A + B) -_H B = A.$$

However, the difference $-_H$ is not defined for every pair of fuzzy numbers. A necessary condition for the existence of $A -_H B$ is given by

$$\begin{aligned} [A]^\alpha = [B + C]^\alpha \quad &\Leftrightarrow \quad [A]^\alpha = [B]^\alpha + [C]^\alpha, \quad \forall \alpha \in [0, 1] \\ &\Leftrightarrow \quad [a_\alpha^-, a_\alpha^+] = [b_\alpha^- + c_\alpha^-, b_\alpha^+ + c_\alpha^+] \\ &\Leftrightarrow \quad \begin{cases} c_\alpha^- = a_\alpha^- - b_\alpha^- \\ c_\alpha^+ = a_\alpha^+ - b_\alpha^+ \end{cases}. \end{aligned}$$

Since $c_\alpha^- \leqslant c_\alpha^+$ for all $\alpha \in [0, 1]$, it follows

$$width([A]^\alpha) = a_\alpha^+ - a_\alpha^- \geqslant b_\alpha^+ - b_\alpha^- = width([B]^\alpha). \tag{1.10}$$

Note that a sufficient condition for the existence of $A -_H B$ is that $B \subseteq A$.

The α-cuts of the Hukuhara difference between A and B, when it exists, is provided as follows:

Lemma 1.2. *[118] Let A and B be fuzzy numbers. If there exists the H-difference of A and B, then*

$$[A -_H B]^\alpha = [a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+], \ \forall \alpha \in [0, 1]. \tag{1.11}$$

The generalized Hukuhara difference or, for short, gH-difference, proposed by Bede, Gal and Stefanini [17, 18, 133, 132, 19], extends the H-difference, that is, if the difference $-_H$ exists then the difference $-_{gH}$ also exists and the equality $A -_H B = A -_{gH} B$ holds true. More precisely, the gH-difference is defined as follows.

Definition 1.29. *[16] Let $A, B \in \mathbb{R}_{\mathcal{F}}$. If there exists a fuzzy number C such that $A = B + C$ or $B = A - C$, where $+$ and $-$ are respectively the standard sum and difference, then $C = A -_{gH} B$ is the* generalized Hukuhara difference.

If the fuzzy numbers A and B satisfies the first condition of Definition 1.29, then there exists C such that $A = B+C$, which is exactly the case where the Hukuhara difference also exists. Therefore, a necessary condition to exist $A -_{gH} B$ is $width([A]^\alpha) \geqslant width([B]^\alpha)$, for all $\alpha \in [0,1]$. On the other hand, if the fuzzy numbers A and B satisfies the second condition of Definition 1.29 then there exists C such that $B = A - C$. Thus

$$\begin{aligned} [B]^\alpha = [A-C]^\alpha \quad &\Leftrightarrow \quad [B]^\alpha = [A]^\alpha - [C]^\alpha \\ &\Leftrightarrow \quad [b_\alpha^-, b_\alpha^+] = [a_\alpha^- - c_\alpha^+, a_\alpha^+ - c_\alpha^-] \\ &\Leftrightarrow \quad \begin{cases} c_\alpha^- = a_\alpha^+ - b_\alpha^+ \\ c_\alpha^+ = a_\alpha^- - b_\alpha^- \end{cases} \end{aligned}$$

Hence

$$width([A]^\alpha) \leqslant width([B]^\alpha), \tag{1.12}$$

since $c_\alpha^- \leqslant c_\alpha^+$ for all $\alpha \in [0,1]$.

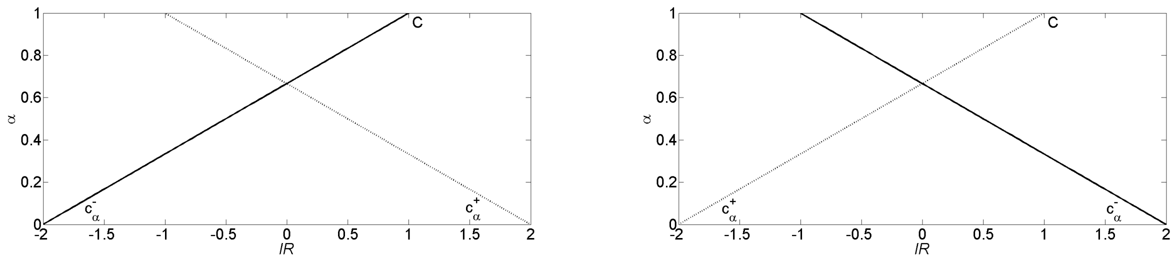
Note that the gH-difference extends the H-difference. However the gH-difference is not defined for every pair of fuzzy numbers. Example 1.11 illustrates this fact.

Example 1.11. *Let $A = (0;4;8)$ and $B = (2;3;5;6)$, whose α-cuts are given by $[A]^\alpha = [4\alpha, 8-4\alpha]$ and $[B]^\alpha = [2+\alpha, 6-\alpha]$. Suppose that $A -_{gH} B$ exists. Then A and B satisfies the condition i) $A = B+C$ or ii) $B = A - C$, for some $C \in \mathbb{R}_{\mathcal{F}}$.*

If the condition i) is satisfied then $[C]^\alpha = [-2+3\alpha, 2-3\alpha]$, however this is not an interval for values of α greater than $\dfrac{2}{3}$. Therefore, A and B do not satisfy the first condition. If the condition ii) is satisfied then $[C]^\alpha = [2-3\alpha, -2+3\alpha]$, which is not an interval for $\alpha < \dfrac{2}{3}$. In these two cases the α-cuts of C are not intervals for all $\alpha \in [0,1]$ (see Figure 8), thus C is not a fuzzy number. This means that the gH-difference of A and B do not exist.

Figure 8 – Graphical representation of C from Example 1.11.

(a) Fuzzy set C obtained from the case i). (b) Fuzzy set C obtained from the case ii).



The fuzzy set C obtained from the gH-difference between $A = (0;4;8)$ and $B = (2;3;5;6)$. The dotted and solid lines represent the left and right endpoints of $[C]^\alpha$, respectively.
Source: Author

The non-existence of the gH-difference between the fuzzy numbers provided in Example 1.11 is attached with the fact that the inequalities (1.10) or (1.12) are not satisfied for all $\alpha \in [0,1]$.

The gH-difference between fuzzy numbers, when it exists, can be given by means of α-cuts (see Lemma 1.3). Moreover, the properties given by Proposition 1.2 holds true.

Lemma 1.3. *[16] Let $A, B \in \mathbb{R}_{\mathcal{F}}$. If there exists the gH-difference between A and B, then the α-cuts of $A -_{gH} B$ are given by*

$$[A -_{gH} B]^{\alpha} = [\min\{a_{\alpha}^{-} - b_{\alpha}^{-}, a_{\alpha}^{+} - b_{\alpha}^{+}\}, \max\{a_{\alpha}^{-} - b_{\alpha}^{-}, a_{\alpha}^{+} - b_{\alpha}^{+}\}],\ \forall \alpha \in [0,1]. \tag{1.13}$$

Proposition 1.2. *[132] Let $A, B \in \mathbb{R}_{\mathcal{F}}$. If $A -_{gH} B$ exists, then the following properties are satisfied:*

- *(i) $A -_{gH} B$ is unique;*
- *(ii) $A -_{gH} A = 0$;*
- *(iii) $(A + B) -_{gH} B = A$;*
- *(iv) $A -_{gH} (A - B) = B$;*
- *(v) $A -_{gH} B = B -_{gH} A = C$ if, and only if, $C = -C$.*

Note that the gH-difference does not satisfy the property $(C -_{gH} B) + B = C$. Indeed, let $C = (0;1;2)$ and $B = (0;1;3)$. Then $(0;1;2) -_{gH} (0;1;3) = (-1;0;0)$, but $(-1;0;0) + (0;1;3) \neq (0;1;2)$.

Later Bede and Stefanini [19] proposed another difference, called generalized difference (or g-difference for short), which is defined in Definition 1.30. In contrast to the generalized Hukuhara difference, the difference $-_g$ is well-defined for any pair of fuzzy numbers [16].

Definition 1.30. *[16, 19] The* g-difference *of two fuzzy numbers $A, B \in \mathbb{R}_{\mathcal{F}}$ is defined by*

$$[A -_g B]^{\alpha} = cl\left\{conv\left(\bigcup_{\beta \geqslant \alpha}([A]^{\beta} -_{gH} [B]^{\beta})\right)\right\},\ \forall \alpha \in [0,1] \tag{1.14}$$

where the $-_{gH}$ denotes the gH-difference between the intervals $[A]^{\beta}$ and $[B]^{\beta}$, $\forall \beta \in [0,1]$, conv(Y) *denotes the convex hull of Y.*

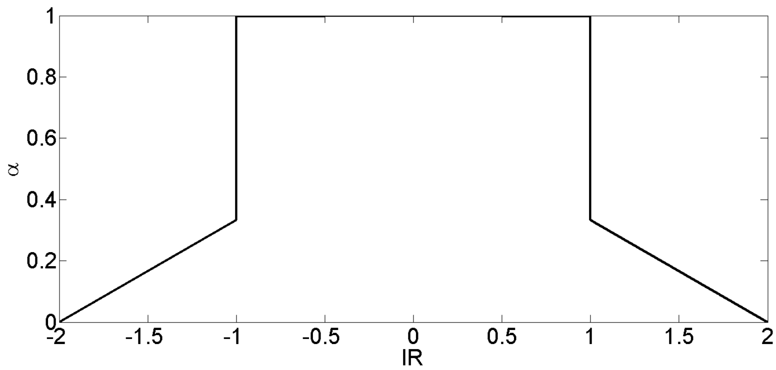
Remark 1.2. *In [16] the authors do not consider the convex hull as in Definition 1.30. More recently, Gomes and Barros [69] showed that the* convexification *(convex hull) is needed to guarantee that g-difference of any pair of fuzzy numbers is a fuzzy number. Considering the convexification as given by Definition 1.30, one can obtain the statement given as in Theorem 1.9.*

Theorem 1.9. *[16] Let $A, B \in \mathbb{R}_{\mathcal{F}}$. The α-cuts of $A -_g B$ are given by*

$$[A -_g B]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} \min\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}, \bigvee_{\beta \geqslant \alpha} \max\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}\right], \ \forall \alpha \in [0,1].$$

The g-difference between the fuzzy numbers A and B, given as in Example 1.11, is depicted in Figure 9. This difference extends the gH-difference in the following sense: if the gH-difference exists, then the g-difference also exists, and $A -_{gH} B = A -_g B$. Consequently, the generalized difference also extends the Hukuhara difference.

Figure 9 – Graphical representation of the fuzzy number $A -_g B$, where A and B are given as in Example 1.11.



The fuzzy number C obtained from de g-difference between $A = (0; 4; 8)$ and $B = (2; 3; 5; 6)$. Source: Author.

Note that the property $(C -_g B) + B = C$ may not hold true for the g-difference. For instance consider $C = (0; 1; 2; 3)$ and $B = (0; 1; 3)$, hence $(0; 1; 2; 3) -_g (0; 1; 3) = [0, 1]$. On the other hand, $((0; 1; 2; 3) -_g (0; 1; 3)) + (0; 1; 3) = (0; 1; 3; 4) \neq (0; 1; 2; 3)$.

Alternatively, one can define other arithmetic on fuzzy numbers which also satisfies the property $A - A = 0$, for all $A \in \mathbb{R}_{\mathcal{F}}$. This arithmetic is called CIA arithmetic and it is defined in the next subsection.

1.4.3 CIA Arithmetic

The Constraint Interval Arithmetic, or CIA for short, arises from an extension of a special type of arithmetic between intervals [86]. These interval arithmetic operations can be similarly defined as the arithmetic operations between sets. Moore [99] established an arithmetic obtained by endpoints of intervals. This arithmetic can be extend as [100]

$$X \odot Y = \{x \odot y \ : \ x \in X \ , \ y \in Y\},$$

for all intervals X and Y, where $\odot \in \{+, -, \cdot, \div\}$.

The CIA arithmetic is obtained by rewriting those intervals as real-valued functions, that is, each element x of the interval $A = [a_1, a_2]$ is associated by a combination of the endpoints of $A = [a_1, a_2]$ in the form of $x := (1-\lambda_A)a_1 + \lambda_A a_2$, for some $0 \leqslant \lambda_A \leqslant 1$. Hence, the CIA arithmetic for intervals is presented as follows.

Definition 1.31. *[86] Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$. The CIA arithmetic is defined by*

$$\begin{aligned} A \odot_{CIA} B &= [\min\{(1-\lambda_A)a_1 + \lambda_A a_2] \odot [(1-\lambda_B)b_1 + \lambda_B b_2],\ 0 \leqslant \lambda_A, \lambda_B \leqslant 1\}, \\ &\quad \max\{(1-\lambda_A)a_1 + \lambda_A a_2] \odot [(1-\lambda_B)b_1 + \lambda_B b_2],\ 0 \leqslant \lambda_A, \lambda_B \leqslant 1\}] \end{aligned}$$

where $\odot$ is an arithmetic operation.

Lodwick and Untiedt [89] extend the CIA arithmetic for fuzzy numbers by using the operations, given as in Definition 1.31, in each α-cut of the corresponding fuzzy arithmetic operation.

Definition 1.32. *[89] Let $A, B \in \mathbb{R}_{\mathcal{F}}$. For all $\alpha \in [0,1]$, the CIA arithmetic between fuzzy numbers is defined by*

$$\begin{aligned} [A \odot_{CIA} B]^\alpha &= [\min(1-\lambda_A)a_\alpha^- + \lambda_A a_\alpha^+] \odot [(1-\lambda_B)b_\alpha^- + \lambda_B b_\alpha^+],\ 0 \leqslant \lambda_A, \lambda_B \leqslant 1\}, \\ &\quad \max(1-\lambda_A)a_\alpha^- + \lambda_A a_\alpha^+] \odot [(1-\lambda_B)b_\alpha^- + \lambda_B b_\alpha^+],\ 0 \leqslant \lambda_A, \lambda_B \leqslant 1\}] \end{aligned}$$

where $\odot$ is an arithmetic operation.

One can define, in particular, the CIA difference by taking $\odot = -$. Note that in this case $A -_{CIA} A = 0$ for all $A \in \mathbb{R}_{\mathcal{F}}$, since

$$\begin{aligned} [A -_{CIA} A]^\alpha &= [\min(1-\lambda_A)a_\alpha^- + \lambda_A a_\alpha^+] - [(1-\lambda_A)a_\alpha^- + \lambda_A a_\alpha^+],\ 0 \leqslant \lambda_A \leqslant 1\}, \\ &\quad \max(1-\lambda_A)a_\alpha^- + \lambda_A a_\alpha^+] - [(1-\lambda_A)a_\alpha^- + \lambda_A a_\alpha^+],\ 0 \leqslant \lambda_A \leqslant 1\}] \\ &= [0,0], \quad \forall \alpha \in [0,1]. \end{aligned}$$

There are other types of arithmetic on fuzzy numbers that are based on interval arithmetic [84, 96, 87, 108, 113]. This thesis focuses on the *interactive arithmetic*, whose definition is provided in Chapter 2. For instance, the CIA arithmetic takes into account the concept of interactivity [108]. The various differences are used to deal with fuzzy derivatives, which are based on differences between fuzzy numbers, but first, the concept of fuzzy functions is established in the next section.

1.5 Fuzzy Functions

The definition of fuzzy function is required to study various problems such as in fuzzy differential equations [9, 70], fuzzy integrals [135, 120] and fuzzy optimization

[112, 154, 88]. The approach of this thesis is to develop results and connections mainly on the first subject, that is, in fuzzy differential equations. The definitions and examples provided in this section are based on the references [63, 67, 108].

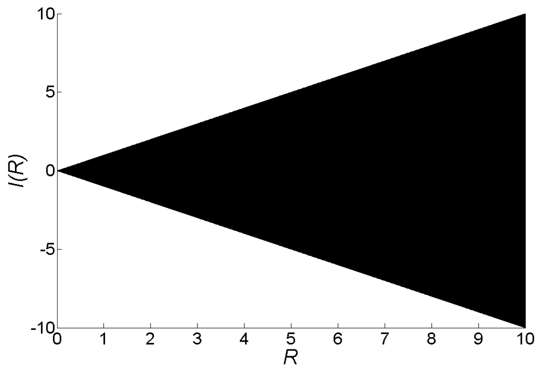
There are several types of fuzzy functions [45] in contrast to the classical definition functions (real-valued functions), which are defined in the space $E(I, \mathbb{R}) = \{f : I \to \mathbb{R},\ I \subseteq \mathbb{R}\}$. One of them is defined by mappings from crisp space to fuzzy space, that is, $F : X \to \mathcal{F}(Y)$, where X and Y are a crisp sets. These fuzzy functions are called *fuzzy-set-valued* functions, which are generalizations of set-valued functions [5].

Example 1.12. *[70] Let $f : \mathbb{R}_+ \to K$ be a function given by*

$$f(x) = [-1, 1]x. \tag{1.15}$$

For each $x \in \mathbb{R}_+$ the function f associates the interval $f(x) \in K$, which is graphically represented in Figure 10.

Figure 10 – Graphical representation of the fuzzy-set-valued function f given in Example 1.12.



The black region represents the function $f(x) = [-1, 1]x$, whose images are given by closed intervals. Source: Author

Functions that associate real values to fuzzy number values, in particular, are also functions of the form $F : X \to \mathcal{F}(Y)$. These functions are called *fuzzy-number-valued* functions.

Definition 1.33. *[45] The functions of the following form*

$$F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}},$$

are called fuzzy-number-valued functions.

Example 1.13. *Let $F : \mathbb{R}_+ \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function given by*

$$F(x) = Ax,$$

where $A = (-1; 0; 1) \in \mathbb{R}_{\mathcal{F}}$.

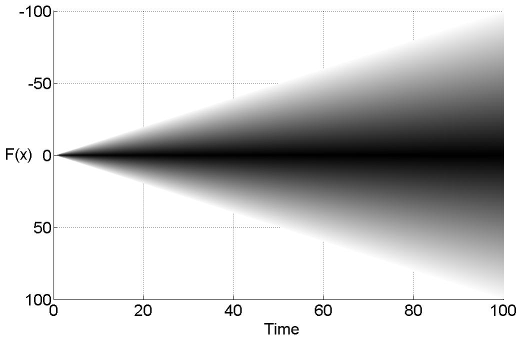
The fuzzy function F *maps each element* $x \in \mathbb{R}_+$ *to the triangular fuzzy number* $(-x; 0; x)$. *Since* $F(x)$ *is a fuzzy number for all* $x \in \mathbb{R}_+$, *it is possible to describe the image of* F *in terms of its* α*-cuts*

$$\begin{aligned} [F(x)]^\alpha &= [A]^\alpha x \\ &= [-1+\alpha, 1-\alpha]x \\ &= [(-1+\alpha)x, (1-\alpha)x] \end{aligned}$$

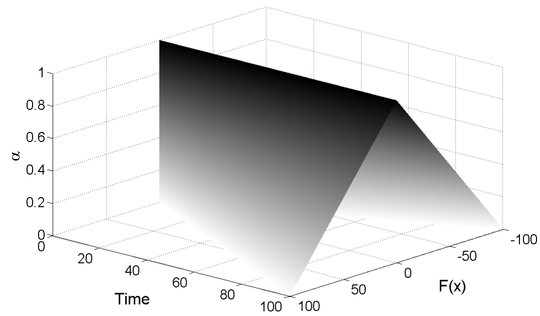
The fuzzy function F *is depicted in Figure 11*

Figure 11 – Graphical representation of the fuzzy function F from Example 1.13.

(a) The top view of the fuzzy function F.



(b) The tri-dimensional view of the fuzzy function F.



The left and right subfigures represent the top and tri-dimensional view of the fuzzy number-valued-function F, respectively. The gray lines represent the α-cuts of the fuzzy function F, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author

The above example presents a fuzzy-number-valued function and the characterization of its image. In general, the α-cuts of $F(x)$ are denoted by

$$[F(x)]^\alpha = [F^-_\alpha(x), F^+_\alpha(x)], \ \forall \alpha \in [0,1].$$

where $F^-_\alpha, F^+_\alpha : I \to \mathbb{R}$.

Another type of fuzzy functions are given by maps that simply associates fuzzy sets from one universe to the corresponding fuzzy sets from the other universe. These functions have the following form $F : \mathcal{F}(X) \to \mathcal{F}(Y)$, where X and Y are crisp sets. The Zadeh extension of a classical function $f : X \to Y$, denoted by $\widehat{f}$ (see Definition 1.24), is an example of fuzzy function of this form.

Although all these functions are defined in different ways, all of them are denoted by a capital letter F and, in each problem, it will be clarified which one is being used.

Next, Section 1.6 presents the derivatives of fuzzy-number-valued functions that are most used in the literature.

1.6 Fuzzy Derivatives

This section focuses on derivatives obtained from differences between fuzzy numbers. These derivatives are used in order to develop the theory of fuzzy differential equations, which may describe biological (chemical, physical,...) phenomena considering uncertainty parameters/states.

Chang and Zadeh [36] introduced the notion of fuzzy derivative. Later, several authors proposed different approaches for fuzzy derivatives [42, 118, 66, 80, 129, 57], which are all equivalent, as Buckley and Feuring showed [25]. In 2004, Bede and Gal [17, 18] proposed a derivative based on difference, which is called SGH-derivative. In 2013, Stefanini and Bede [133] presented the gH- and g-derivatives, which are based on the differences $-_{gH}$ and $-_g$, respectively. In the same year, Barros *et al.* [12, 68] introduced the fuzzy derivative for fuzzy bunches of functions[70], which is based on the Zadeh's extension of the classical derivative operator. Recently, Esmi *et al.* [52] introduced the notion of Fréchet derivative for fuzzy-number-valued functions.

The derivatives of fuzzy-number-valued functions F, that are based on difference between fuzzy numbers, arise from

$$\lim_{h\to 0} \frac{F(t+h) - F(t)}{h}. \tag{1.16}$$

This expression depends on the choice of a difference operation " $-$ " for fuzzy numbers. Consequently, different derivatives arise from this limit.

Note that, if F is a constant fuzzy-number-valued function, that is, $F(x) = A \in \mathbb{R}_{\mathcal{F}}$, $\forall x \in \mathbb{R}$, then the limit (1.16) based on the standard difference is not equal to 0, since $A - A \neq 0$ for all $A \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$. Therefore, the derivative via standard difference does not satisfy classical results for fuzzy functions. However, the H-, gH-, and g-derivatives satisfy this property.

The Hukuhara derivative (or H-derivative for short) is defined as follows.

Definition 1.34. *[118] The function $F : I \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, where I is an open set of $\mathbb{R}$, is said to be* Hukuhara differentiable *(H-differentiable for short) at $t_0 \in I$, if the limits*

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} \qquad \textit{and} \qquad \lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h} \tag{1.17}$$

exist and they are equal to some element $F'_H(t_0) \in \mathbb{R}_{\mathcal{F}}$, in d_∞ metric. The fuzzy number $F'_H(t_0)$ is called H-derivative of F at t_0.

The H-derivative satisfies a similar property of classical derivatives, that is, if a fuzzy-number-valued function F is Hukuhara differentiable then F is continuous [80]. In addition, the α-cuts of F'_H are given as follows.

Lemma 1.4. *[80] Let $F : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function. If F is H-differentiable, then the real-valued functions F^-_α and F^+_α are also differentiable and*

$$[F'_H(t)]^\alpha = [(F')^-_\alpha(t), (F')^+_\alpha(t)], \quad \forall t \in \mathbb{R}. \tag{1.18}$$

Example 1.14. *Let $F : (0, \infty) \to \mathbb{R}_{\mathcal{F}}$ be the fuzzy function given by $F(t) = (-t^2; 0; t^2)$ for all $t \in (0, \infty)$. Then*

$$[F(t)]^\alpha = [-1 + \alpha, 1 - \alpha]t^2 = [(-1 + \alpha)t^2, (1 - \alpha)t^2].$$

On the one hand,

$$\begin{aligned}\lim_{h\to 0^+} \frac{[F(t_0 + h) -_H F(t_0)]^\alpha}{h} &= \lim_{h\to 0^+} \frac{[(-1 + \alpha)(2t_0h + h^2), (1 - \alpha)(2t_0h + h^2)]}{h} \\ &= [(-1 + \alpha)2t_0, (1 - \alpha)2t_0]. \end{aligned} \tag{1.19}$$

On the other hand,

$$\begin{aligned}\lim_{h\to 0^+} \frac{[F(t_0) -_H F(t_0 - h)]^\alpha}{h} &= \lim_{h\to 0^+} \frac{[(1 - \alpha)(-2t_0h + h^2), (-1 + \alpha)(-2t_0h + h^2)]}{h} \\ &= [(-1 + \alpha)2t_0, (1 - \alpha)2t_0]. \end{aligned} \tag{1.20}$$

Since (1.19) and (1.20) are equal, it follows that F is H-differentiable. Moreover, $[F'_H(t)]^\alpha = [(-1+\alpha)2t, (1-\alpha)2t]$, for all $t \in \mathbb{R}_+$. Figure 12 depicts the H-derivative of F.

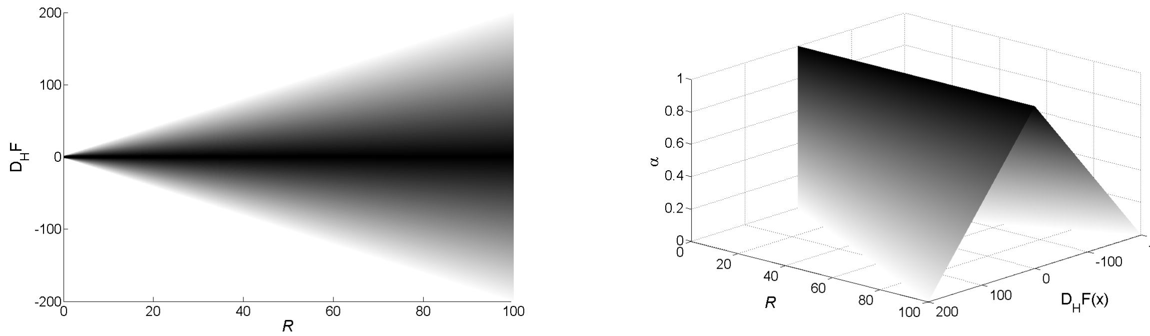
A necessary condition for the existence of the H-derivative of a fuzzy-number-valued function F at a given point t is that the width of F be an increasing function in the neighborhood of t. This means that $width(F(t_2)) > width(F(t_1))$, for all $t_1, t_2 \in (t_0 - h, t_0 + h)$ with $h > 0$ and $t_1 < t_2$.

The function F provided in Example 1.14 satisfies this condition for every $t \in \mathbb{R}_+$. However, if the domain of the function F is given by $\mathbb{R}$, then F is not H-differentiable in every point t ($t < 0$). In this case, the α-cut $[(-1 + \alpha)2t_0, (1 - \alpha)2t_0]$, obtained by the limits (1.19) and (1.20), is not an interval for all $\alpha \in [0, 1)$. Consequently, F is not H-differentiable at $t_0 < 0$. Figure 13 portrays the changing endpoints of $[F'_H]^\alpha$ at $t = 0$.

Next, the definition of the gH-derivative is presented, which arises from the gH-difference. The generalized Hukuhara differentiability were first considered for set-valued functions [92]. Several authors used this concept to deal with fuzzy-number-valued functions [18, 133].

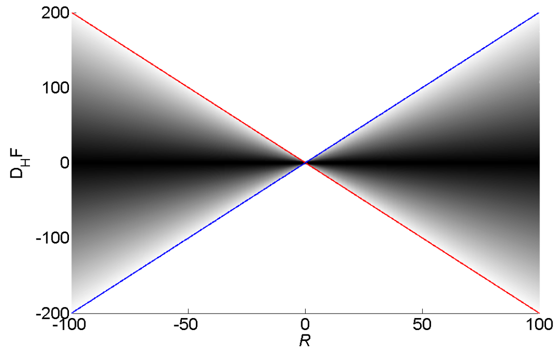
Figure 12 – Graphical representation of the H-derivative of the fuzzy function F given in Example 1.14.

(a) The top view of F'_H. (b) The tri-dimensional view of F'_H.



The left and right subfigure represent the top and tri-dimensional view of the H-derivative of F given by $F(t) = (-t^2; 0; t^2)$, respectively. The gray lines represent the α-cuts of F'_H, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author

Figure 13 – Graphical representation of the changing endpoints of $[F'_H]^\alpha$ given in Example 1.14.



The top view of the H-derivative of F. The red and blue lines represent the lower and upper endpoint of $[F'_H]^0$, respectively. The gray lines represent the α-cuts of F'_H, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author

Definition 1.35. *[16] The function $F : I \subseteq \mathbb{R} \to \mathbb{R}_\mathcal{F}$ is said to be* generalized Hukuhara *differentiable (gH-differentiable) at $t_0 \in I$, if the limit*

$$\lim_{h\to 0} \frac{F(t_0 + h) -_{gH} F(t_0)}{h} \tag{1.21}$$

exists and it is equal to some $F'_{gH}(t_0) \in \mathbb{R}_\mathcal{F}$, in d_∞ metric. The fuzzy number $F'_{gH}(t_0)$ is said to be the gH-derivative of F at t_0.

The implications of gH-differentiability are given in Theorem 1.10.

Theorem 1.10. *[16] Let $F : (a,b) \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable at t_0. Then the endpoints F^-_α and F^+_α of $[F'_{gH}(t_0)]^\alpha$ are continuous at t_0.*

The next theorem provides a necessary and sufficient condition for the existence of the generalized Hukuhara derivative.

Theorem 1.11. *[19] Let $F : (a,b) \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function such that F^-_α and F^+_α are real-valued functions, differentiable with respect to t and uniformly continuous with respect to $\alpha \in [0,1]$. The fuzzy function F is gH-differentiable at $t \in (a,b)$, if and only if, one of the following cases hold:*

(i) $(F')^-_\alpha(t)$ is an increasing function and $(F')^+_\alpha(t)$ is a decreasing function with respect to α, and $(F')^-_1(t) \leqslant (F')^+_1(t)$, for all $t \in (a,b)$;

(ii) $(F')^-_\alpha(t)$ is a decreasing function and $(F')^+_\alpha(t)$ is an increasing function with respect to α, and $(F')^+_1(t) \leqslant (F')^-_1(t)$, for all $t \in (a,b)$.

The α-cuts of the gH-derivative of F are given as follows.

Theorem 1.12. *Let $F : (a,b) \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable function. The α-cuts of F'_{gH} are given by*

$$[F'_{gH}(t)]^\alpha = [\min\{(F')^-_\alpha(t), (F')^+_\alpha(t)\}, \max\{(F')^-_\alpha(t), (F')^+_\alpha(t)\}], \tag{1.22}$$

for all $t \in [a,b]$ and $\alpha \in [0,1]$.

Example 1.15. *Let $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be the fuzzy function given by $F(t) = (-t^2; 0; t^2)$ for all $t \in \mathbb{R}$. The gH-derivative of F is given by*

$$\lim_{h\to 0} \frac{[F(t_0+h) -_{gH} F(t_0)]^\alpha}{h} = \lim_{h\to 0} \frac{[\min\{u(h), v(h)\}, \max\{u(h), v(h)\}]}{h},$$

where $u(h) = (-1+\alpha)(2t_0 h + h^2)$ and $v(h) = (1-\alpha)(2t_0 h + h^2)$.

Therefore,

$$\lim_{h\to 0} \frac{[F(t_0+h) -_{gH} F(t_0)]^\alpha}{h} = [\min\{u, v\}, \max\{u, v\}]. \tag{1.23}$$

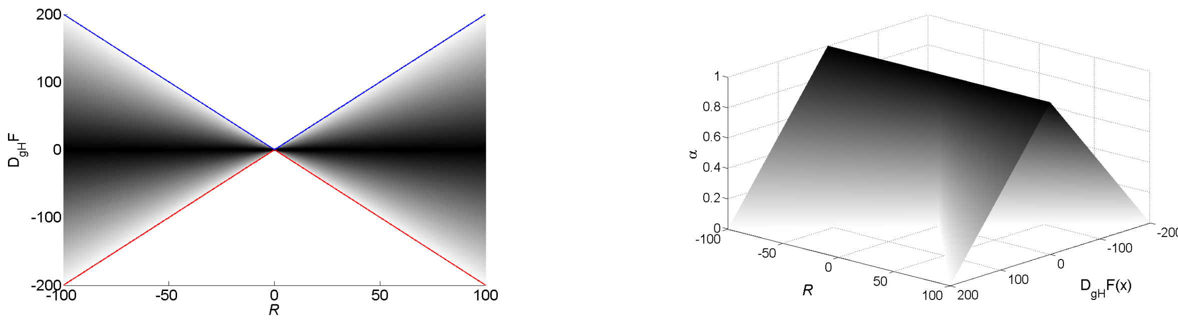
where $u = (-1+\alpha)2t_0$ and $v = (1-\alpha)2t_0$.

Note that the expression, given by (1.23), is an interval for all $\alpha \in [0,1]$. Thus, the gH-derivative of F exists and is equal to (1.23). Figure 14 graphically represents F'_{gH}.

One can observe that in Figure 14 (a) the endpoints of $[F'_{gH}]^\alpha$ do not switch, which corroborates the fact that (1.23) is always an interval.

Figure 14 – Graphical representation of the gH-derivative of the fuzzy function F given in Example 1.15.

(a) The top view of F'_{gH}. (b) The tri-dimensional view of F'_{gH}.



Sugfigure (a) represents the top view of the gH-derivative of F. The red and blue lines represent the lower and upper endpoints of $[F'_{gH}]^0$, respectively. Subfigure (b) represents the tri-dimensional view of the gH-derivative of F. The gray lines represent the α-cuts of F'_{gH}, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author.

Recall that the gH-derivative is not a linear operator with respect to the sum. Indeed, consider the functions $F, G : (-1, 1) \to \mathbb{R}_{\mathcal{F}}$, whose α-cuts are given by $[F(t)]^\alpha = [-1, 1]t$ and $[G(t)]^\alpha = [0, 1]e^{-t}$. The functions F and G are gH-differentiable at $t_0 = 0$, but $F + G$ is not [35].

The g-difference generates another derivative, which is called g-derivative. This fuzzy derivative extends the gH-derivative and it was introduced by Bede and Stefanini [19].

Definition 1.36. *The function $F : I \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is said to be* generalized differentiable *(or g-*differentiable *for short) at $t_0 \in I$, if the limit*

$$\lim_{h \to 0} \frac{F(t_0 + h) -_g F(t_0)}{h}, \tag{1.24}$$

exists and it is equal to some $F'_g(t_0) \in \mathbb{R}_{\mathcal{F}}$, in d_∞ metric. The fuzzy number $F'_g(t_0)$ is said to be the g-derivative of F at t_0.

Note that the g-derivative extends the gH-derivative, since $F(t_0+h) -_g F(t_0) = F(t_0 + h) -_{gH} F(t_0)$ whenever the gH-difference on the right side exists. The following theorem provides a characterization for the g-derivative of a fuzzy-number-valued function.

Theorem 1.13. *[19] Let $F : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be a g-differentiable function. The α-cuts of F'_g are given by*

$$[F'_g(t)]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} \min\{(F')^-_\beta(t), (F')^+_\beta(t)\}, \bigvee_{\beta \geqslant \alpha} \max\{(F')^-_\beta(t), (F')^+_\beta(t)\} \right] \tag{1.25}$$

The gH- and g-derivatives extend the Hukuhara derivative. In fact, if a fuzzy function is H-differentiable, then it is also gH- and g-differentiable, and all these derivatives are equal.

Next, an example is presented. This example illustrates a case where a fuzzy function is g-differentiable but, it is not gH-differentiable in all domain [16].

Example 1.16. *Let $F : [0, 1] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number-valued function given by*

$$[F(x)]^{\alpha} = [F_{\alpha}^{-}(x), F_{\alpha}^{+}(x)], \quad \forall \alpha \in [0, 1],$$

where

$$F_{\alpha}^{-}(x) = xe^{-x} + \alpha^2(e^{-x^2} + x - xe^{-x})$$

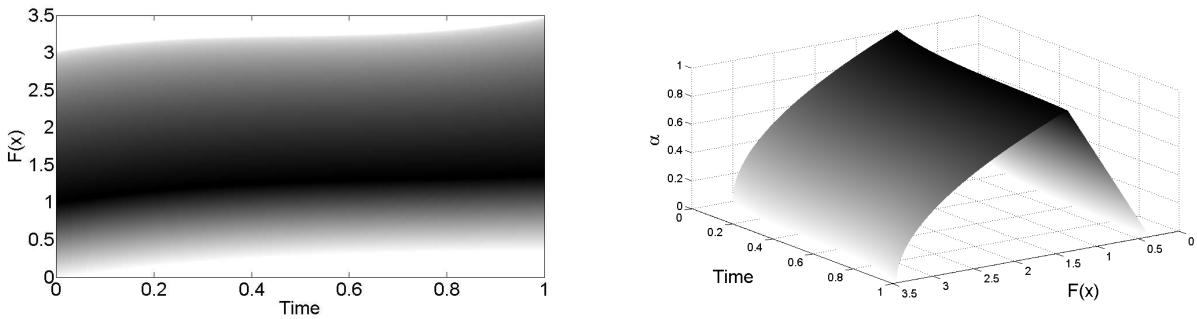
and

$$F_{\alpha}^{+}(x) = e^{-x^2} + x + (1 - \alpha^2)(e^{x} - x + e^{-x^2}).$$

One can show that the fuzzy function F, depicted in Figure 15, is gH-differentiable in the intervals $[0, x_1]$ and $(x_2, 1]$, where $x_1 \approx 0.61$ and $x_2 \approx 0.71$. However, in the interval $[x_1, x_2]$ this function is only g-differentiable [16]. The g-derivative of F is depicted in Figure 16.

Figure 15 – Graphical representation of the fuzzy function F given in Example 1.16.

(a) The top view of F. (b) The tri-dimensional view of F.



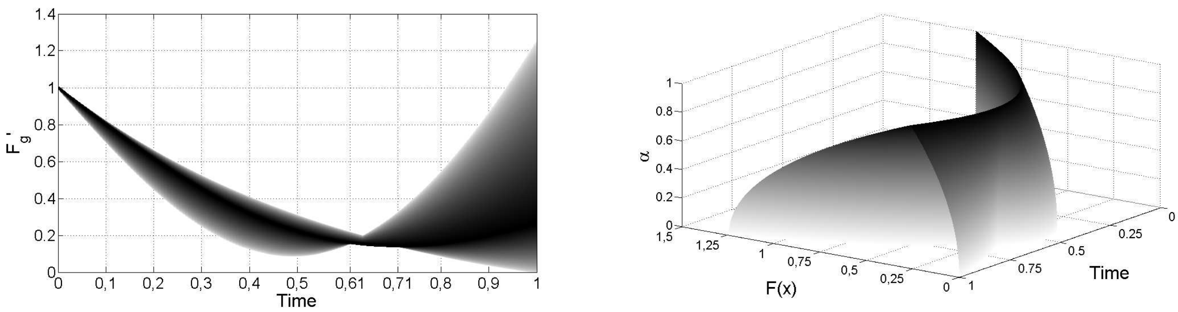
The left and right subfigures represent the top and tri-dimensional view of F. The gray lines represent the α-cuts of F, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author

The existence of the g-derivative depends on specific conditions of the functions F_{α}^{-} and F_{α}^{+}. For example, if F_{α}^{-} and F_{α}^{+} are differentiable functions with respect to t and uniformly continuous with respect to $\alpha \in [0, 1]$, then F is g-differentiable [19]. For more details about the gH- and g-derivatives the reader can refer to [16].

Other derivative studied here is the Fréchet derivative for fuzzy functions. Let us recall the Fréchet derivative for classical functions, before presenting this type of derivative.

Figure 16 – Graphical representation of the g-derivative of the fuzzy function F given in Example 1.16.

(a) The top view of F'_g. (b) The tri-dimensional view of F'_g.



The left and right subfigures represent the top and tri-dimensional view of the g-derivative of F, respectively. The gray lines represent the α-cuts of F'_g, varying from 0 to 1 which are represented respectively from the gray-scale lines varying from white to black. Source: Author

The function $f : M \to N$, where M and N are Banach spaces, is *Fréchet differentiable* at t_0 if there exists a linear continuous function $\kappa : M \to N$ and $o : M \to N$ that satisfies $\dfrac{||o(h)||}{|h|} \to 0$ when $h \to 0$, such that the expression (1.26) holds true [162]:

$$f(t_0 + h) = f(t_0) + \kappa(h) + o(h). \tag{1.26}$$

Esmi *et al.* [52] proposed a derivative for fuzzy-number-valued functions based on the Fréchet derivative for classical functions. To this end, they considered the following fuzzy functions: For a fixed fuzzy number $A \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$, let $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be given by

$$F(t) = q(t)A + r(t), \tag{1.27}$$

where $q, r : \mathbb{R} \to \mathbb{R}$. If q and r are differentiable functions, then F is Fréchet differentiable and it obeys the following expression [52]

$$F'(t) = q'(t)A + r'(t). \tag{1.28}$$

Moreover,

$$[F'(t)]^\alpha = q'(t)[A]^\alpha + r'(t), \quad \forall \alpha \in [0, 1]. \tag{1.29}$$

Here the notation from [52] is used to denote the Fréchet derivative by $F'_{\mathcal{F}}$. This type of derivative is associated with the arithmetic on fuzzy numbers that takes into account a relationship called interactivity, as will be proved in Chapter 4. For more details about the Fréchet derivative of fuzzy functions the reader can refer to [52].

The next chapter presents the concept of interactivity, which can be used to extend the Hukuhara derivative (and its generalizations).

1.7 Conclusion

This chapter reviewed the basic concepts about the fuzzy set theory required in the chapters that ensue. The definition of fuzzy numbers and the usual arithmetic for them were presented. In particular, the focus was on the operation of difference. Different types of fuzzy differences proposed in the literature were delineated. This chapter ended by presenting the concept of fuzzy derivatives based on the various differences and the Fréchet derivative.

2 Interactive Fuzzy Numbers

A discussion about interactive fuzzy numbers will be the focus of this chapter. The relationship of *interactivity* was introduced by Zadeh in 1975. According to him, "fuzzy variables are interactive if the assignment of a value to one affects the fuzzy restrictions placed on the others" [157, 158, 159]. The interactivity can be described in terms of a possibility distribution.

The possibility theory is based on the fuzzy set theory [160]. This approach consists in identifying the membership functions of fuzzy sets with possibility distributions. In this case, the possibility distribution is interpreted as a function that measures the possibility of an element belonging to a certain set [45, 48, 85]. In [85] possibility is defined axiomatically.

Later, in 2004, Fullér, Carlsson, and Majlender [62, 32] associated the concept of interactivity with the definition of joint possibility distribution. This thesis follows the same approach proposed by them.

This chapter is divided in four sections and it is based on the references [157, 158, 159, 161, 31, 88, 62, 33, 61, 95, 32, 78]. Section 2.1 provides the definition of joint possibility distribution. Section 2.2 presents the notion of interactivity and some examples to clarify this concept. Finally, Section 2.3, uses the relationship of interactivity to construct an arithmetic on interactive fuzzy numbers.

2.1 Joint Possibility Distribution

The definition of a *possibility distribution* is provided as follows.

Definition 2.1. *[41]A possibility distribution over $A \subset X \neq \emptyset$, $Pos(A) : X \to [0,1]$, is defined by*

$$Pos(A) = \bigvee_{x \in A} A(x)$$

Note that every normal fuzzy set A satisfies $Pos(A) = 1$, in particular for fuzzy numbers. Also, it is important to observe that a fuzzy set can not be viewed as a possibility distribution.

The definition of *joint possibility distribution* is presented as follows [62].

Definition 2.2. *[31] Let $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$. The possibility distribution J, defined in $\mathbb{R}^n_{\mathcal{F}}$, is said to be a* joint possibility distribution *(JPD) of fuzzy numbers $A_1, \ldots, A_n$, if it satisfies* *the following condition*

$$A_i(y) = \bigvee_{x \in \mathbb{R}^n : y = x_i} J(x_1, \ldots, x_n), \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n \tag{2.1}$$

for all $i = 1, \ldots, n$. *Furthermore, the fuzzy number* A_i *is called as the* i*th* marginal possibility distribution *of* J*.*

The fuzzy relation J, from the mathematical point of view, is a joint possibility distribution among the fuzzy numbers $A_1, \ldots, A_n$ if each fuzzy number A_i can be given as the projection of J in the i direction, for $i = \{1, \ldots, n\}$. From the practical point of view, this fact ensures that all the aggregated information in the joint possibility distribution may be transferred to their marginal possibilities.

Example 2.1. *The fuzzy relation* $J_\wedge \in \mathcal{F}(\mathbb{R}^n)$ *obtained from the usual cartesian product,* i.e.*,*

$$J_\wedge(x_1, \ldots, x_n) = A_1(x_1) \wedge \ldots \wedge A_n(x_n), \ \forall (x_1, \ldots, x_n) \in \mathbb{R}^n \tag{2.2}$$

is an example of JPD [62].

The next remark presents some properties that hold for all joint possibility distribution J.

Remark 2.1. *(a) Let* J *be a bivariate joint possibility distribution of the fuzzy numbers* A_1 *and* A_2*. Hence* J *satisfies*

$$\bigvee_{y \in \mathbb{R}} J(x_1, y) = A_1(x_1) \quad \textit{and} \quad \bigvee_{y \in \mathbb{R}} J(y, x_2) = A_2(x_2). \tag{2.3}$$

(b) Let J *be an arbitrary multivariate joint possibility distribution of* $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$*. Hence* J *satisfies*

$$J(x_1, \ldots, x_n) \leqslant A_1(x_1) \wedge \ldots \wedge A_n(x_n) = J_\wedge(x_1, \ldots, x_n), \ \forall (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{2.4}$$

Consequently, J *is contained in the usual Cartesian product, that is,*

$$[J]^\alpha \subseteq [A_1]^\alpha \times \ldots \times [A_n]^\alpha, \tag{2.5}$$

for all $\alpha \in [0, 1]$*.*

The distribution $J_\wedge$, given by (2.2), is a particular case of *joint possibility distribution based on the t-norm*, whose definition is given as follows. Let $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$ and some t-norm $\triangle$. The fuzzy relation $J_\triangle$ given by

$$J_\triangle(x_1, \ldots, x_n) = A_1(x_1) \triangle \ldots \triangle A_n(x_n), \quad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{2.6}$$

is said to be a *joint possibility distribution based on t-norm* [41].

Recall from Chapter 1, the *sup-J extension principle* is a mathematical tool to extend functions with two or more real arguments to a fuzzy domain. In contrast to the extension principle (see Definition 1.25) which is based on t-norm, the sup-J extension principle is based on a joint possibility distribution.

Definition 2.3. *[62] Let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $(A_1, ..., A_n) \in \mathbb{R}^n_{\mathcal{F}}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The* sup-$J$ extension *of f at $(A_1, ..., A_n) \in \mathbb{R}^n_{\mathcal{F}}$, denoted by $f_J(A_1, ..., A_n)$, is the fuzzy set whose membership function is given as follows:*

$$f_J(A_1, ..., A_n)(y) = \bigvee_{(x_1,...,x_n) \in f^{-1}(y)} J(x_1, ..., x_n), \tag{2.7}$$

where $f^{-1}(y) = \{(x_1, ..., x_n) \in \mathbb{R}^n : f(x_1, ..., x_n) = y\}$.

Equation (2.7) boils down to Equation (1.8), in the case where $J = J_{\triangle}$. Moreover if $J = J_{\wedge}$, then Definitions 1.25 and 2.3 are equivalent. This means that the Zadeh's extension principle for multiple variables is a particular case of sup-J extension principle. Clearly, if there is only one fuzzy number A in Equation (2.7), then this only makes sense, if Definition 1.24 is used.

The next theorem presents a practical method to obtain the α-cuts of the sup-J extension principle of a continuous function.

Theorem 2.1. *[103, 11] Let J be a joint possibility distribution of $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. For every $\alpha \in [0, 1]$, the α-cut of the sup-J extension principle f_J is given by*

$$[f_J(A_1, \ldots, A_n)]^{\alpha} = f([J]^{\alpha}). \tag{2.8}$$

Recall that the condition of continuity over the function f is sufficient to guarantee (2.8). However, it is not a necessary condition [11]. In the case where the sup-J extension of f at $(A_1, ..., A_n)$ is a fuzzy number, the α-cuts of $f_J(A_1, ..., A_n)$ can be given in terms of the infimum and supremum of the classical function f restricted by the α-cuts of J, that is,

$$[f_J(A_1, ..., A_n)]^{\alpha} = \left[\bigwedge_{(x_1,...,x_n) \in [J]^{\alpha}} f(x_1, ..., x_n) \, , \bigvee_{(x_1,...,x_n) \in [J]^{\alpha}} f(x_1, ..., x_n) \right]. \tag{2.9}$$

Remark 2.2. *When the function f is continuous and $[J]^{\alpha}$ is closed and connected set for all $\alpha \in [0, 1]$, then $f_J(A_1, ..., A_n)$ is a fuzzy number [90]. Consequently the characterization by means of α-cuts, provided in (2.9), holds. Recall that, these are sufficient but not necessary conditions.*

The next section presents a special concept called *interactivity*.

2.2 Interactive Fuzzy Numbers

Let us begin this section by defining non-interactive fuzzy numbers.

Definition 2.4. *The fuzzy numbers $A_1, \ldots, A_n$ are said to be* non-interactive *if their joint possibility distribution J is given by*

$$J(x_1, \ldots, x_n) = J_\wedge(x_1, \ldots, x_n) = A_1(x_1) \wedge \ldots \wedge A_n(x_n), \tag{2.10}$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Moreover, for all $\alpha \in [0,1]$ it follows

$$[J]^\alpha = [J_\wedge]^\alpha = [A_1]^\alpha \times \ldots \times [A_n]^\alpha. \tag{2.11}$$

This means that in the case where $A_1, \ldots, A_n$ are non-interactive, there are no restrictions in the joint possibility distribution $J_\wedge$, *i.e.*, the elements $x_1 \in A_1, \ldots, x_n \in A_n$ that are related with respect to $J_\wedge$ may be chosen independently from each other. In order to clarify this idea, let us consider the following example.

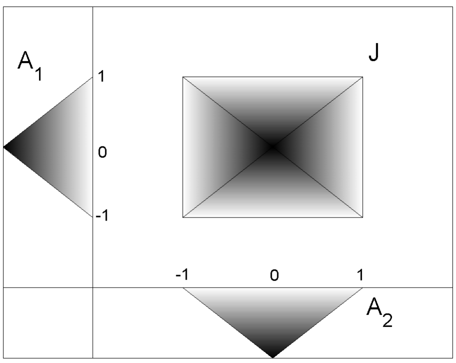
Example 2.2. *Let $A_1, A_2 \in \mathbb{R}_\mathcal{F}$ given by $A_1 = A_2 = (-1; 0; 1)$. The joint possibility distribution $J_\wedge$ between A_1 and A_2 is given by*

$$[J_\wedge]^\alpha = [A_1]^\alpha \times [A_2]^\alpha = [-1+\alpha, 1-\alpha] \times [-1+\alpha, 1-\alpha].$$

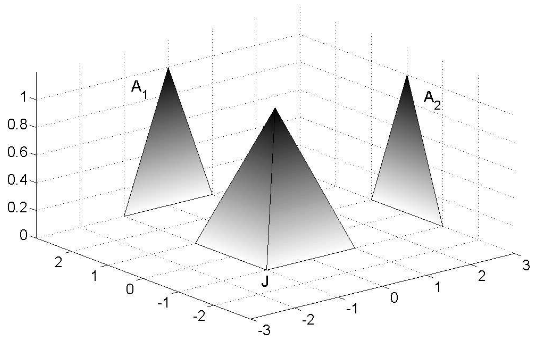
The JPD $J_\wedge$ is graphically represented in Figure 17.

Figure 17 – Graphical representation of the joint possibility distribution $J_\wedge$ between two non-interactive fuzzy numbers.

(a) The top view of the JPD $J_\wedge$.



(b) The tri-dimensional view of the JPD $J_\wedge$.



The fuzzy numbers $A_1 = (-1; 0; 1)$ and $A_2 = (-1; 0; 1)$ are depicted by the shaded triangles. The joint possibility distribution $J = J_\wedge = A_1 \times A_2$ is depicted by the shaded pyramid. The gray lines represent the α-cuts of J, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author.

Note that for each element of A_1 the joint possibility distribution establishes which elements of A_2 are associated with it. In this case, for each $x_1 \in [A_1]^\alpha$ the entire interval $[A_2]^\alpha = [(a_2)^-_\alpha, (a_2)^+_\alpha]$ is linked to x_1.

The concept of non-interactivity is similar (but not equivalent) to the concept of independence of random variable in probability theory [157]. Non-interactive fuzzy variables are defined by the JPD given by (2.10), whereas independent random variables are defined by the joint *probability* distribution given by the product of all the probabilities ($P(u_1, \ldots, u_n) = P(u_1) \ldots P(u_n)$) [27]. In both cases, the membership function (probability) of one marginal does not interfere the membership (probability) of the others. Hisdal [74] provided a study between the difference of conditional independence and non-interactivity. For more details on this topic the reader can refer to [7].

Next, the definition of interactive fuzzy numbers is presented.

Definition 2.5. *Let $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ and J be their joint possibility distribution. The fuzzy numbers $A_1, \ldots, A_n$ are called J-*interactive*, or simply* interactive*, if $J \neq J_\wedge$.*

In contrast to non-interactivity, the above definition ensures that interactive fuzzy numbers may have a dependence relation, which means that if $A_1, \ldots, A_n$ are interactive with respect to some joint possibility distribution J, then for each element $x_i \in [A_i]^\alpha$ there are specific elements $x_j \in [A_j]^\alpha$, with $i \neq j$, related to it. In this sense, the concept of interactivity is similar to dependence in the case of random variables.

Example 2.3. *Let $A_1 = A_2 = (-1; 0; 1) \in \mathbb{R}_{\mathcal{F}}$. Consider the fuzzy relation (depicted in Figure 18) defined by the following membership function*

$$J(x_1, x_2) = \begin{cases} A_1(x_1) \wedge A_2(x_2) & , \text{ if } (x_1, x_2) \in [-1, 0] \times [-1, 0] \\ A_1(x_1) \wedge A_2(x_2) & , \text{ if } (x_1, x_2) \in [0, 1] \times [0, 1] \\ 0 & , \text{ otherwise} \end{cases} \tag{2.12}$$

whose α-cuts are given by

$$[J]^\alpha = [-1 + \alpha, 0] \times [-1 + \alpha, 0] \cup [0, 1 - \alpha] \times [0, 1 - \alpha]. \tag{2.13}$$

Let us prove that J is a joint possibility distribution for A_1 and A_2. To this end, two cases can be considered: a) $-1 \leqslant x_1 \leqslant 0$ *and* b) $0 \leqslant x_1 \leqslant 1$.

a) *For $-1 \leqslant x_1 \leqslant 0$, it follows*

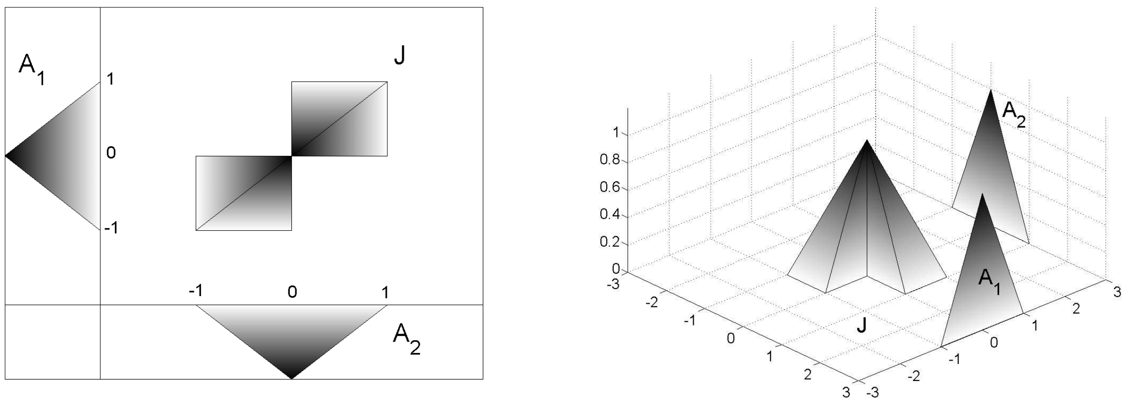
$$\bigvee_{x_2 \in [-1,1]} J(x_1, x_2) = \left(\bigvee_{x_2 \in [-1,0]} A_1(x_1) \wedge A_2(x_2) \right) \vee \left(\bigvee_{x_2 \in [0,1]} 0 \right)$$

The left endpoint of $[A_2]^\alpha$ is increasing w.r.t α, therefore for $x_2 \in [-1, 0]$ the supremum of $A_1(x_1) \wedge A_2(x_2)$ is attached in $x_2 = 0$. Thus,

$$\bigvee_{x_2 \in [-1,1]} J(x_1, x_2) = (A_1(x_1) \wedge 1) \vee 0 = A_1(x_1).$$

Figure 18 – Graphical representation of the joint possibility distribution J between two interactive fuzzy numbers of Example 2.3.

(a) The top view of the JPD. (b) The tri-dimensional view of the JPD.



The fuzzy numbers $A_1 = (-1;0;1)$ and $A_2 = (-1;0;1)$ are depicted by the shaded triangles, whereas the joint possibility distribution J, given in (2.12), is depicted by the shaded pyramid. The gray lines represent the α-cuts of J, varying from 0 to 1, which are represented respectively from the gray-scale lines varying from white to black. Source: Author.

b) *On the other hand, for* $0 \leqslant x_1 \leqslant 1$ *it follows*

$$\begin{aligned} \bigvee_{x_2 \in [-1,1]} J(x_1,x_2) &= \left(\bigvee_{x_2\in[-1,0]} 0\right) \vee \left(\bigvee_{x_2\in[0,1]} A_1(x_1) \wedge A_2(x_2)\right) \\ &= 0 \vee (A_1(x_1) \wedge 1) \\ &= A_1(x_1), \end{aligned}$$

since the right endpoint of $[A_2]^\alpha$ *is decreasing w.r.t* α *and consequently for* $x_2 \in [0,1]$ *the supremum of* $A_1(x_1) \wedge A_2(x_2)$ *is attached in* $x_2 = 0$.

Similarly, it can be shown that $\bigvee_{x_1\in[-1,1]} J(x_1,x_2) = A_2(x_2)$, *for all* $x_2 \in [-1,1]$. *Therefore* J *is a joint possibility distribution for* A_1 *and* A_2, *which means that* A_1 *and* A_2 *are interactive with respect to* J, *or simply* J*-interactive.*

In the above example, the elements $x_2 \in [A_2]^\alpha$ that are linked to $x_1 \in [A_1]^\alpha$ are given by

$$x_2 \in \begin{cases} [-1+\alpha, 0] & , \text{ if } \; -1+\alpha \leqslant x_1 \leqslant 0 \\ [0, 1-\alpha] & , \text{ if } \; 0 \leqslant x_1 \leqslant 1-\alpha \end{cases}. \tag{2.14}$$

Therefore, the set of elements x_2 that are connected to each x_1 is more specific than in the case of non-interactivity (see Figure 17). This means that the elements $x_1 \in [A_1]^\alpha$ and $x_2 \in [A_2]^\alpha$ may not be chosen independently, they have to satisfy the restrictions placed by the joint possibility distribution J.

Note that Examples 2.2 and 2.3 illustrate that the same fuzzy numbers can be interactive or not, all depends on the JPD under consideration. The next example ensures that fuzzy numbers may be interactive with respect to different joint possibility distributions.

Example 2.4. *Let $A_1 = A_2 = (-1; 0; 1) \in \mathbb{R}_{\mathcal{F}}$ and the fuzzy relation J (see Figure 19) given by*

$$J(x_1, x_2) = \begin{cases} A_1(x_1) \wedge A_2(x_2) & , \text{ if } (x_1, x_2) \in [-1, 0] \times [0, 1] \\ A_1(x_1) \wedge A_2(x_2) & , \text{ if } (x_1, x_2) \in [0, 1] \times [-1, 0] \\ 0 & , \text{ otherwise} \end{cases} \tag{2.15}$$

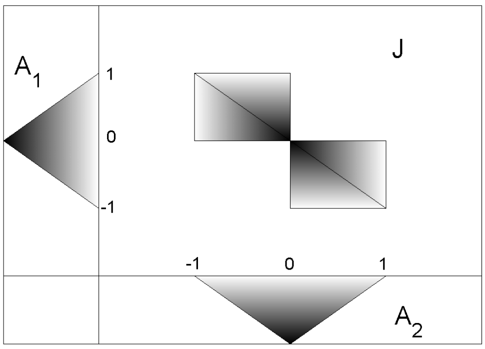
whose α-cuts are given as follows

$$[J]^\alpha = [-1 + \alpha, 0] \times [0, 1 - \alpha] \cup [0, 1 - \alpha] \times [-1 + \alpha, 0]. \tag{2.16}$$

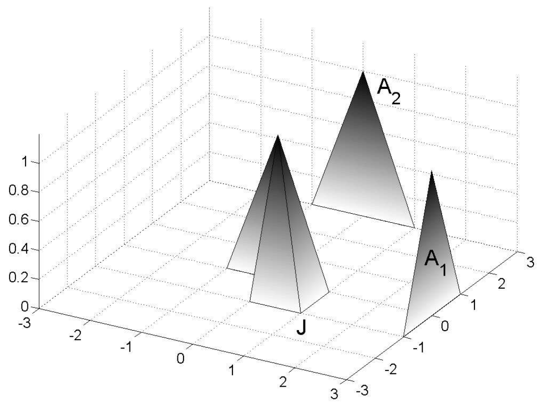
One can prove that J is a joint possibility distribution for A_1 and A_2 (cf. Example 2.3).

Figure 19 – Graphical representation of the joint possibility distribution J between two interactive fuzzy numbers of Example 2.4.

(a) The top view of the JPD.



(b) The 3 dimensional view of the JPD.



The fuzzy numbers $A_1 = (-1; 0; 1)$ and $A_2 = (-1; 0; 1)$ are depicted by the shaded triangles, whereas the joint possibility distribution J, given in (2.15), is depicted by the shaded pyramid. The gray lines represent the α-cuts of J, varying from 0 to 1 which are represented respectively from the gray-scale lines varying from white to black. Source: Author.

In this case, the elements $x_2 \in [A_2]^\alpha$ that are linked to $x_1 \in [A_1]^\alpha$ are given by

$$x_2 \in \begin{cases} [-1 + \alpha, 0] & , \text{ if } 0 \leqslant x_1 \leqslant 1 - \alpha \\ [0, 1 - \alpha] & , \text{ if } -1 + \alpha \leqslant x_1 \leqslant 0 \end{cases}. \tag{2.17}$$

Note that the joint possibility distribution J, given by (2.15), establishes a different dependence between the fuzzy numbers A_1 and A_2, than the JPD given by (2.12) (see Figure 18). This means that fuzzy numbers may be interactive in different ways, and this relation is described by the joint possibility distribution under consideration.

Another type of interactivity is the one based on t-norms [107, 41]. The fuzzy numbers are said to be $J_{\triangle}$-interactive, if their joint possibility distribution is given by $J_{\triangle} \neq J_{\wedge}$ (see (2.6)). For more details about this type of interactivity the reader can refer to [78].

The next section provides the *interactive arithmetic* on fuzzy numbers, which is obtained by the sup-J extension principle.

2.3 Arithmetic on Interactive Fuzzy Numbers

The arithmetic for interactive fuzzy numbers was introduced by Dubois and Prade [41]. More precisely, they provided the notion of addition between interactive fuzzy numbers considering the joint possibility distribution $J_{\triangle}$ (cf. Equation (2.6)). This subject has been the topic of interest of many researchers [24, 61, 82]. Later different types of interactive arithmetics, which are not based on t-norms, were proposed [62, 32, 50, 78, 55]. The focus of this thesis is on the interactive arithmetic that does not come from t-norms.

It is important to observe that, motivated by the properties of the standard arithmetic that do not hold, such as $A - A = 0$ and $A \div A = 1$ (see Examples 1.8 and 1.10), Klir [84] proposed in 1997 a fuzzy arithmetic with *requisite constraints*. The main idea is to constrain the arithmetic operations based on the extension principle. This approach resembles the interactive arithmetic, which takes into account the interactivity between the operands.

The interactive arithmetic is defined as follows: consider a function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \ldots, x_n) = x_1 \otimes \ldots \otimes x_n, \; \forall (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{2.18}$$

where $\otimes \in \{+, -, \cdot, \div\}$. Definition 2.3 gives raise to the following definition of interactive arithmetic.

Definition 2.6. *Let $A_1, A_2 \in \mathbb{R}_{\mathcal{F}}$ and J be their joint possibility distribution. The interactive arithmetic operations are defined by:*

(a) The interactive sum *between A_1 and A_2 is the fuzzy number $A_1 +_J A_2$, whose membership function is given by*

$$(A_1 +_J A_2)(z) = \bigvee_{x_1 + x_2 = z} J(x_1, x_2);$$

(b) *The* interactive difference *between A_1 and A_2 is the fuzzy number $A_1 -_J A_2$, whose membership function is given by*

$$(A_1 -_J A_2)(z) = \bigvee_{x_1 - x_2 = z} J(x_1, x_2);$$

(c) *The* interactive product *of A_1 by A_2 is the fuzzy number $A_1 \cdot_J A_2$, whose membership function is given by*

$$(A_1 \cdot_J A_2)(z) = \bigvee_{x_1 \cdot x_2 = z} J(x_1, x_2);$$

(d) *The* interactive quotient *of A_1 by A_2, if $0 \notin supp(A_2)$, is the fuzzy number $A_1 \div_J A_2$, whose membership function is given by*

$$(A_1 \div_J A_2)(z) = \bigvee_{x_1 / x_2 = z} J(x_1, x_2).$$

(e) *The* product *of $A \in \mathbb{R}_{\mathcal{F}}$ by a scalar λ is the fuzzy number λA, whose membership function is given by*

$$(\lambda A)(z) = \bigvee_{\lambda x = z} A(x).$$

Remark 2.3. *(I) Since fuzzy numbers can be interactive with respect to different JPDs (see Example 2.4), the interactive arithmetic operations are denoted by $\otimes_J$, in order to clarify which JPD is being used.*

(II) When J is given by $J_\wedge$, that is, A_1 and A_2 are non-interactive, the arithmetic provided by the sup-J extension principle boils down to the standard arithmetic.

(III) The product given by item (e) of Definition 2.6 is the same scalar product as the one given by standard arithmetic (see item (e) of Definition 1.27).

The observation *(II)* of Remark 2.3 ensures that the arithmetic obtained from Zadeh's extension principle is a particular case of the arithmetic based on sup-J extension principle. Henceforth, the arithmetic for non-interactive fuzzy numbers is called standard arithmetic. The case $J \neq J_\wedge$, the arithmetic is called the interactive arithmetic.

The next theorem is the first contribution of this thesis, which consists of some connections between these two arithmetics.

Theorem 2.2. *[146] Let $A, B, C, D \in \mathbb{R}_{\mathcal{F}}$ and $J_\wedge$ be the joint possibility distribution given by (2.10). Then for all joint possibility distribution J and $\otimes \in \{+, -, \div, \cdot\}$ it follows*

a) $A \otimes_J B \subseteq A \otimes_\wedge B$;

b) $\lambda(A \otimes_J B) \subseteq \lambda(A \otimes_\wedge B)$, for all $\lambda \in \mathbb{R}$;

c) If $A \subseteq B$ then $A \otimes_J C \subseteq B \otimes_\wedge C$;

d) If $A \subseteq B$ and $C \subseteq D$ then $A \otimes_J C \subseteq B \otimes_\wedge D$;

e) $(A \otimes_J (B \otimes_J C)) \subseteq (A \otimes_\wedge (B \otimes_\wedge C))$.

Proof.

a) Theorem 2.1 and item *b*) of Remark 2.1 ensures that

$$[A \otimes_J B]^\alpha = \{a \otimes b : (a,b) \in [J]^\alpha\} \subseteq \{a \otimes b : (a,b) \in [J_\wedge]^\alpha\} = [A \otimes_\wedge B]^\alpha, \quad \forall \alpha \in [0,1].$$

Therefore $A \otimes_J B \subseteq A \otimes_\wedge B$.

b) Since $\lambda[A \otimes_J B]^\alpha = [\lambda(A \otimes_J B)]^\alpha$ and $\lambda[A \otimes_{J_\wedge} B]^\alpha = [\lambda(A \otimes_{J_\wedge} B)]^\alpha$, for all $\alpha \in [0,1]$, from item *a*) it follows that $\lambda(A \otimes_J B) \subseteq \lambda(A \otimes_\wedge B)$.

c) If $A \subseteq B$ then a combination of Theorem 2.1, Remark 2.1 and Equation (2.11) gives rise to the following

$$\begin{aligned} [A \otimes_J C]^\alpha = \{a \otimes c : (a,c) \in [J]^\alpha\} &\subseteq \{a \otimes c : (a,c) \in [J_\wedge]^\alpha\} \\ &= \{a \otimes c : (a,c) \in [A]^\alpha \times [C]^\alpha\} \\ &\subseteq \{b \otimes c : (b,c) \in [B]^\alpha \times [C]^\alpha\} \\ &= [B \otimes_\wedge C]^\alpha, \quad \forall \alpha \in [0,1]. \end{aligned}$$

Therefore $A \otimes_J C \subseteq B \otimes_\wedge D$.

d) The proof is similar to item c).

e) A combination of items a) and c) proves this statement. □

Theorem 2.2 establishes that all the arithmetics that take into account the concept of interactivity is contained in the standard arithmetic.

This thesis develops different types of interactive arithmetics and it compares with the standard arithmetic. The above theorem will be used specially in the comparison of numerical methods using different arithmetics on fuzzy numbers.

From the point of view of dynamic population, the use of standard arithmetic results in the spread of uncertainty. For example, let us describe the growth of population of rabbits by the Fibonacci sequence [116].

For this problem consider:

1. A pair of rabbits that are well-fed;

2. The population of rabbits does not escape;

3. There are no predators for them.

Under these assumptions, how many rabbits do we have at time t?

In order to describe this behaviour mathematically, assume that the original pair of rabbits at the end of each instant of time (months) give birth to another pair of rabbits, then immediately mate again and give birth to another pair of rabbits, and so on. Moreover, consider that each newly-born pair of rabbits take one instant of time to be able to reproduce. Also, for simplification, suppose that rabbits do not die.

With these hypothesis at hand, the number of rabbits at time t is given by

$$x_t = x_{t-1} + x_{t-2}, \tag{2.19}$$

where the initial values of populations x_0 and x_1 are given. If $x_0 = 1$ and $x_1 = 1$, then the sequence (2.19) is given by

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}.$$

Particularly, if the initial numbers of rabbits are uncertain, then one can model the initial values by fuzzy numbers X_0 and X_1. This means that one can consider a number "around" one, instead considering $x_0 = x_1 = 1$. For example $X_0 = (0; 1; 2)$ and $X_1 = (0; 1; 2)$.

Since $X_0, X_1 \in \mathbb{R}_{\mathcal{F}}$, the usual sum between real numbers, given as in Equation (2.19), needs to be extend to a sum between fuzzy numbers. This extension is given by means of sup-J extension principle. Hence, the following sequence of fuzzy numbers is obtained

$$X_t = X_{t-1} \oplus X_{t-2}, \tag{2.20}$$

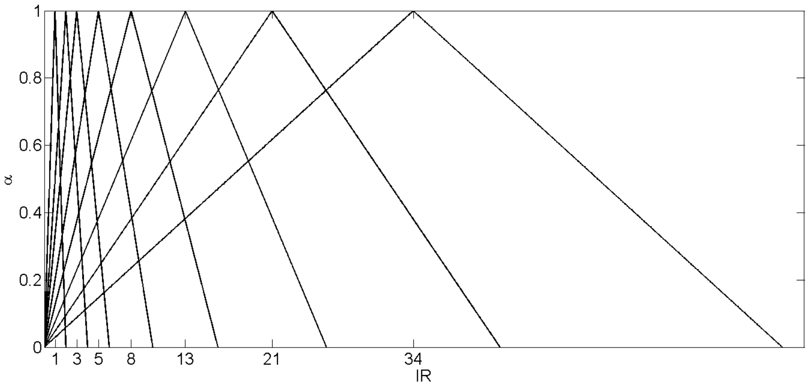
where $X_t \in \mathbb{R}_{\mathcal{F}}$ for all $t \in \mathbb{R}_+$ and the symbol $\oplus$ stands for some (interactive or non-interactive) sum between fuzzy numbers.

One can use the standard sum, that is, $\oplus = +_{\wedge}$. This implies that $width(X_{t+1}) \geqslant width(X_t)$, for all $t \in \mathbb{R}_+$, since

$$width(X_{t+1}) = width(X_t +_{\wedge} X_{t-1}) \geqslant width(X_t). \tag{2.21}$$

Figure 20 corroborates the statement given by Equation (2.21).

Figure 20 – Fibonacci sequence for standard sum



Fibonacci sequence using the standard sum. The initial conditions are given by the triangular fuzzy numbers $X_0 = X_1 = (0; 1; 2)$. Source: Author

The Fibonacci sequence for fuzzy numbers, depicted as in Figure 20, can be interpreted by the following sequence

$$\{\text{“around” one}, \text{“around” one}, \text{“around” two}, \text{“around” three}, \text{“around” five}, \ldots\},$$

where $X_{t_0} =$ “around” u means that at the instant time t_0, each number of rabbits in the interval containing u, has a membership degree in the fuzzy set X_{t_0}.

In this case, $[X_t]^1 = \{x_t\}$ for all $t \in \mathbb{R}_+$, as one can observe in Figure 20. However the uncertainty, which is measured by the value of width, increases over time. Consequently, the standard sum may not be a good approach to describe this phenomena.

Theorem 2.2 reveals that interactive arithmetic operations may provide a better approach to the Fibonacci model, since the width of X_t using the interactive arithmetic is always less or equal than the width of X_t using the standard arithmetic (see item a) of Theorem 2.2). Chapter 7 provides this comparison.

For each $t_0 \in \mathbb{R}_+$ the Fibonacci sequence gives rise to a fuzzy equation $X_{t_0-1} \oplus X_{t_0-2} = X_{t_0}$. In general, the main goal of a dynamical systems is to obtain an information about the future, knowing the present. Also one can focus on the past, that is, knowing the present X_t what can be established about the past X_{t-1} or X_{t-2}? This question is equivalent to a linear fuzzy equation of the form $X \oplus A = B$ or $A \oplus X = B$, where $X \in \mathbb{R}_{\mathcal{F}}$ is the free variable.

Consider the case where $\oplus = +_{\wedge}$. For simplification, denote $+_{\wedge} = +$. If there exists $X \in \mathbb{R}_{\mathcal{F}}$ such that $A + X = B$, then the fuzzy number X is the Hukuhara difference between B and A, that is, $X = B -_H A$ (see Definition 1.34). Moreover, the Hukuhara difference $X = B -_H A$ is the only solution for $A + X = B$.

However, in general the equation $A + X = B$ may not have a solution. One of the reasons is because of the standard sum. Basically, three fundamental questions about linear fuzzy equations can be made:

1) Given fuzzy numbers B and C, is there a fuzzy number X and an arithmetic sum such that $X \oplus B = C$?
2) Given fuzzy numbers A, B, and C, is there an arithmetic sum such that $A \oplus B = C$?
3) Given an arithmetic sum and the fuzzy numbers A and B, is there an arithmetic difference such that $(A \oplus B) \ominus B = A$?

As previously observed, the standard arithmetic do not provide a positive answer to questions 1) and 2). One can observe that the pair $(+, -)$ consisting of standard sum and difference is not a solution for the third problem. In contrast, the pair $(+, -_H)$ consisting of standard sum and Hukuhara difference is a solution for the problem.

The next chapter considers a relaxation of the hypothesis of A and X be non-interactive. This means that the above questions, in the case where the operands are interactive fuzzy numbers, will be investigated.

2.4 Conclusion

This chapter presented the notion of interactivity and the joint possibility distribution. The definition of sup-J extension principle, which gives rise to different arithmetic for interactive fuzzy numbers, was provided. This chapter ended by comparing the interactive arithmetic operations with the standard ones.

3 Interactive Fuzzy Equations

This chapter contains the second contribution of this thesis. Some fundamental questions about addition of interactive fuzzy numbers are investigated. These questions involve fuzzy linear equations, such as:

1) Given fuzzy numbers B and C, is there a fuzzy number X and a joint possibility distribution J of X and B such that $X +_J B = C$?
2) Given fuzzy numbers A, B, and C, is there a joint possibility distribution J of A and B such that $A +_J B = C$?
3) Given a joint possibility distribution J of fuzzy numbers A and B, is there a joint possibility distribution N of $(A +_J B)$ and B such that $(A +_J B) -_N B = A$?

This chapter intends to answer these questions, providing the maximal solution for each of these problems, if it exists. The concept of maximal solution will be given as well. This chapter is based on reference [51].

3.1 Fuzzy Equations

Fuzzy linear equations were studied by several authors. Dubois and Prade [43] started to investigate fuzzy equations of the form (3.1)

$$A \otimes X = C, \tag{3.1}$$

where $\otimes \in \{+, -, \cdot, \div\}$ stands for arithmetic operations between fuzzy numbers and A, X and C are the fuzzy variables.

The equality of (3.1) means that both inclusions $A \otimes X \subseteq C$ and $C \subseteq A \otimes X$ holds.

Mizumoto and Tanaka [98] and Yager [155] discussed the non-existence of inverse fuzzy numbers for the standard arithmetic operations, in particular, for the standard sum. This means that given $A, C \in \mathbb{R}_{\mathcal{F}}$, the fuzzy equation $A + X = C$ in general can not be solved by $X = C - A$, where $+$ and $-$ are the standard sum and difference, respectively.

Sanchez [127] provided necessary and sufficient conditions for Equation (3.1) has solution X. Buckley [26] in addition to study the existence of these solutions, also studied a way of calculating them. Many authors proposed different methods to solve this problem, for instance, Kawaguchi [81] solved this problem using fuzzy t-norm and Mazarbhuiya [93] used the method of superimposition.

The aforementioned papers consider the usual arithmetic operations between fuzzy numbers. This thesis focuses in providing a solution to (3.1), where not only A, X and C are the variables but also the joint possibility distribution J that gives rise to the arithmetic operation. In the case where the arithmetic under consideration is given by an interactive arithmetic operation, the equation

$$A +_J X = C, \tag{3.2}$$

is called by an *interactive fuzzy equation*.

An interactive fuzzy equation embraces the interactivity between A and X. In order to solve (3.2), it is necessary to establish which $X \in \mathbb{R}_{\mathcal{F}}$ and the joint possibility distribution J, between A and X, that satisfies $A +_J X = C$. This means that the solutions for (3.2) are always pairs (X, J). Note that the fuzzy equation given by (3.1) is a particular case of an interactive fuzzy equation given by (3.2), considering the joint possibility distribution $J = J_{\wedge}$.

The next sections provide the maximal solution, if it exists, to the problems 1), 2) and 3). In this case, the concept of the maximal solution is given by the following definition.

Definition 3.1. *Let $A, C \in \mathbb{R}_{\mathcal{F}}$. Consider the following interactive fuzzy equation*

$$A +_J X = C, \tag{3.3}$$

where J is a JPD between A and X. The fuzzy solution (X, J) is said to be the maximal solution of (3.3) if all solutions $(\hat{X}, \hat{J})$ of (3.3) satisfy $\hat{X} \subseteq X$ and $\hat{J} \subseteq J$.

The following sections investigate solutions of Equation (3.2), where $X \in \mathbb{R}_{\mathcal{F}}$ and J is a JPD of A and X. Section 3.2 provides the maximal solution for (3.2). In this case, A and X are marginal distributions of the joint possibility distribution J, which means that the free variable is the marginal distribution X. Section 3.3 considers X as a fixed marginal distribution, say $X = B$, and the joint possibility distribution J as free variable. More precisely, for $A, B, C \in \mathbb{R}_{\mathcal{F}}$, the following equation is considered

$$A +_J B = C, \tag{3.4}$$

where J is a JPD of A and B, and it is investigated when a fuzzy number C can be written as a J-interactive sum of A and B. Section 3.4 studies the existence of a joint possibility distribution N of $(A +_J B)$ and B such that Equation (3.5) holds true

$$(A +_J B) -_N B = A. \tag{3.5}$$

3.2 Given fuzzy numbers B and C, is there a fuzzy number X and a joint possibility distribution J of X and B such that $X +_J B = C$?

This section provides a solution of

$$X +_J B = C \tag{3.6}$$

where B and C are given fuzzy numbers and the fuzzy number X and the joint possibility distribution J of X and B are the free variables.

Moreover, this section shows that $X +_J B = B +_J X$, as the following theorem guarantees.

Theorem 3.1. *[51] Let J be a joint possibility distribution of fuzzy numbers A and B and let $\bar{J}$ and $J^\star$ be the fuzzy relations given by*

$$\bar{J}(x,y) = J(y,x) \tag{3.7}$$

and

$$J^\star(x,y) = J(x,-y) \tag{3.8}$$

for all $(x,y) \in \mathbb{R}^2$. The fuzzy set $\bar{J}$ is a joint possibility distribution of B and A and $A +_J B = B +_{\bar{J}} A$. Moreover, the fuzzy set $J^\star$ is a joint possibility distribution of A and $(-B)$ and $A -_J B = A +_{J^\star} (-B)$.

Proof. For all $z \in \mathbb{R}$, it follows that

$$(A +_J B)(z) = \bigvee_{z=x+y} J(x,y) = \bigvee_{z=x+y} \bar{J}(y,x) = (B +_{\bar{J}} A)(z)$$

and

$$(A -_J B)(z) = \bigvee_{z=x-y} J(x,y) = \bigvee_{z=x+(-y)} J^\star(x,-y) = (A +_{J^\star} (-B))(z).$$

$\square$

Theorem 3.2 reveals a maximal solution of Equation (3.6).

Theorem 3.2. *[51] Let $B, C \in \mathbb{R}_{\mathcal{F}}$ and S be the fuzzy relation whose membership function is defined by*

$$S(x,y) = B(y) \wedge C(x+y), \quad \forall (x,y) \in \mathbb{R}^2. \tag{3.9}$$

Thus

1. *The fuzzy relation S is a joint possibility distribution with marginals distributions $X, B \in \mathbb{R}_{\mathcal{F}}$, where $X(x) = \bigvee_{y \in \mathbb{R}} B(y) \wedge C(x+y), \quad \forall x \in \mathbb{R}$.*

2. *$X +_S B = C$ and $[X]^\alpha = [c_\alpha^- - b_\alpha^+, c_\alpha^+ - b_\alpha^-]$, $\forall \alpha \in [0,1]$;*

3. *X and S are the maximal solution of (3.2), that is, if there are a fuzzy number $\tilde{X}$ and a joint possibility distribution $\tilde{J}$ of $\tilde{X}$ and B where $\tilde{X} +_{\tilde{J}} B = C$, then $\tilde{J} \subseteq S$ and $\tilde{X} \subseteq X$.*

Proof. 1. Let X be the fuzzy set whose membership function is given by

$$X(x) = \bigvee_{y \in \mathbb{R}} B(y) \wedge C(x+y), \quad \forall x \in \mathbb{R}.$$

From definition of X, it follows $X(x) = \bigvee_{y \in \mathbb{R}} S(x,y)$. Now, let $\bar{z} \in [C]^1$. From definition of the fuzzy relation S, for every $y, z \in \mathbb{R}$, one obtains

$$B(y) \geqslant \bigvee_{x \in \mathbb{R}} S(x,y) \geqslant S(\bar{z} - y, y) = C(\bar{z}) \wedge B(y) = B(y),$$

which implies that $B(y) = \bigvee_{x \in \mathbb{R}} S(x,y)$. Therefore, S is a joint possibility distribution for X and B.

2. Let us prove that $X +_S B = C$ and $[X]^\alpha = [c_\alpha^- - b_\alpha^+, c_\alpha^+ - b_\alpha^-]$. To this end, consider $0 < \alpha \leqslant 1$. On the one hand, if $y \in [B]^\alpha$ and $z \in [C]^\alpha$ then $S(z-y, y) \geqslant \alpha$ which implies that $z - y \in [X]^\alpha$. Thus,

$$[c_\alpha^- - b_\alpha^+, c_\alpha^+ - b_\alpha^-] = \{z - y \mid z \in [C]^\alpha, y \in [B]^\alpha\} \subseteq [X]^\alpha.$$

On the other hand, if $x \in [X]^\alpha$ then there exists (y_n) such that $\lim_{n \to \infty} B(y_n) \wedge C(x + y_n) \geqslant \alpha$. For n sufficient large, one obtains $B(y_n) \geqslant B(y_n) \wedge C(x + y_n) > \dfrac{\alpha}{2}$. This implies that (y_n) is bounded and, therefore, there exists a convergent subsequence $(y_{n_k}) \in [B]^{\frac{\alpha}{2}}$, say $\lim_{k \to \infty} y_{n_k} = \hat{y} \in [B]^{\frac{\alpha}{2}}$. If $\beta := B(\hat{y}) < \alpha$ then for $\beta < \rho < \alpha$ and for k sufficient large, it follows that $y_{n_k} \in [B]^\rho$, because $B(y_{n_k}) \geqslant B(y_{n_k}) \wedge C(x + y_{n_k}) \geqslant \rho$. Since $[B]^\rho$ is a bounded closed interval, one concludes that $\hat{y} \in [B]^\rho$ which produces the following contradiction: $B(\hat{y}) < \rho \leqslant B(\hat{y})$. Therefore, $B(\hat{y}) \geqslant \alpha$. Similarly, it can be shown that $x + \hat{y} \in [C]^\alpha$. These last observations imply that

$$[X]^\alpha \subseteq [c_\alpha^- - y_\alpha^+, c_\alpha^+ - y_\alpha^-].$$

Hence, $[X]^\alpha = [c_\alpha^- - b_\alpha^+, c_\alpha^+ - b_\alpha^-]$, for all $\alpha \in (0,1]$. For $\alpha = 0$ it follows

$$[X]^0 = \overline{supp(X)} \;=\; \overline{\bigcup_{\alpha \in (0,1]} [X]^\alpha} = \left[\bigwedge_{\alpha \in (0,1]} c_\alpha^- - b_\alpha^+, \bigvee_{\alpha \in (0,1]} c_\alpha^+ - b_\alpha^- \right] = \left[c_0^- - b_0^+, c_0^+ - b_0^-\right].$$

Therefore the equality holds for all $\alpha \in [0,1]$. Finally, let $\bar{y} \in [B]^1$. Since $(X +_S B)(z) = \bigvee_{z=x+y} S(x,y)$, one obtains

$$C(z) \geqslant \bigvee_{z=x+y} S(x,y) \geqslant S(z-\bar{y},\bar{y}) = C(z) \wedge B(\bar{y}) = C(z).$$

Thus $X +_S B = C$.

3. Let $\tilde{J}$ be a joint possibility distribution of $\tilde{X} \in \mathbb{R}_{\mathcal{F}}$ and B such that $\tilde{X} +_{\tilde{J}} B = C$. For $x, y \in \mathbb{R}$ and $z = x + y$, it follows $B(y) = \bigvee_{w\in\mathbb{R}} \tilde{J}(w,y) \geqslant \tilde{J}(x,y)$ and $C(z) = \bigvee_{u+v=z} \tilde{J}(u,v) \geqslant \tilde{J}(x,y)$. Thus, $S(x,y) = B(y) \wedge C(x+y) \geqslant \tilde{J}(x,y)$ for all $(x,y) \in \mathbb{R}^2$, which implies that $X(x) = \bigvee_{y\in\mathbb{R}} S(x,y) \geqslant \bigvee_{y\in\mathbb{R}} \tilde{J}(x,y) = \tilde{X}(x)$ for all $x \in \mathbb{R}$. Hence $\tilde{J} \subseteq S$ and $\tilde{X} \subseteq X$.

□

Theorem 3.2 ensures that the set of all fuzzy numbers X and a joint possibility distribution J of X and B, such that $X +_J B = C$ is not empty and the maximal solution is given by the marginal distribution X of the joint possibility distribution S, given as in Equation (3.9).

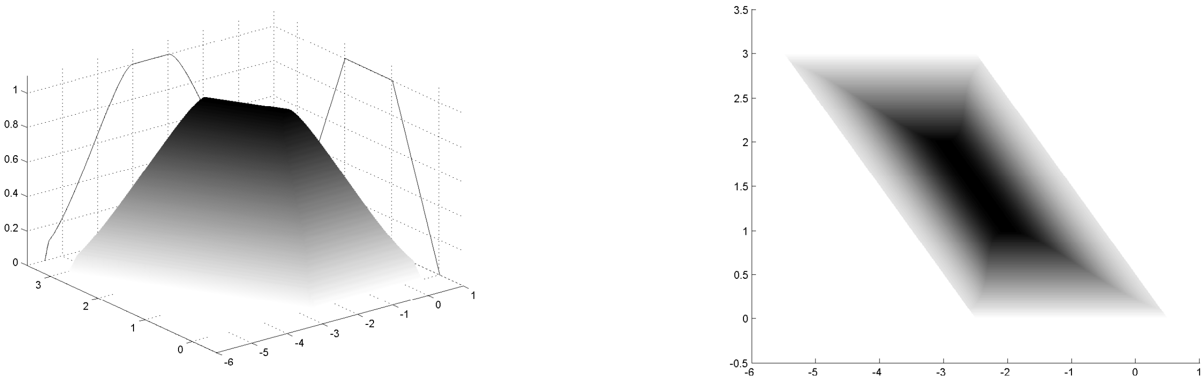
Observe that X and J are free variables of Equation (3.6). However, the JPD S given by (3.9) determines uniquely the fuzzy number X. This means that the truly free variable of the problem is the JPD J.

The next example illustrates this result.

Example 3.1. *Let $B = (0;1;2;3)$ and $C = (-1;1;1.5)_G$. Figure 21 exhibits the joint possibility distribution S of X and B, given as in Theorem 3.2. Figure 22 presents a visual representation of $X +_S B = C$.*

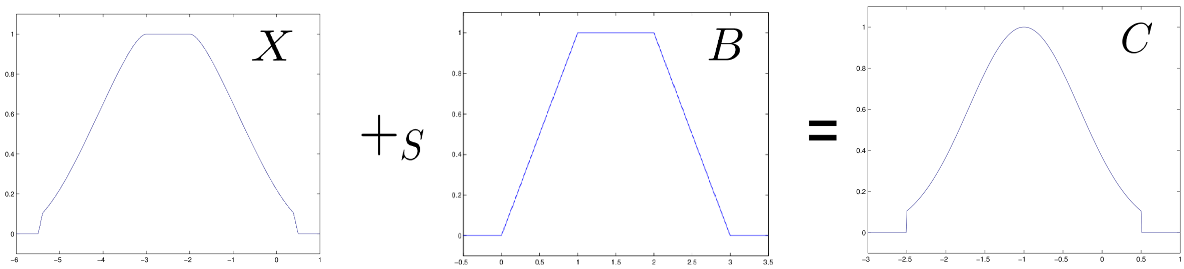
Figure 21 – Graphical representation of the joint possibility distribution S given as in Example 3.1

(a) Tri-dimensional view of the JPD S. (b) Top view of the JPD S.



The joint possibility distribution S given as in Equation (3.9), with $B = (0; 1; 2; 3)$ and $C = (-1; 1; 1.5)_G$, and its marginal distributions X and B. The gray region represent the α-cuts of S, varying from 0 to 1 which are represented respectively from the gray-scale varying from white to black. Source: [51].

Figure 22 – Graphical representation of S-interactive sum between X and B given as in Example 3.1.



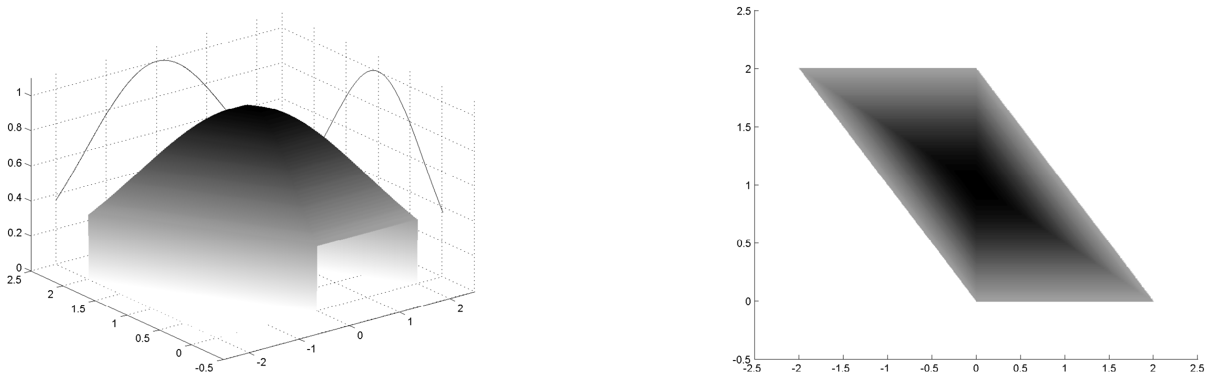
The S-interactive sum between X and B, where $B = (0; 1; 2; 3)$, $C = (-1; 1; 1.5)_G$, and X and S are given as in Figure 21. Source: [51].

Note that in the case where $C = B$, that is, when the interactive fuzzy equation is given by $X +_J B = B$, the fuzzy number $X = 0$ is not a unique solution for it, as one can see in the next example.

Example 3.2. *Consider the Gaussian fuzzy numbers $B = C = (1; 1; 1)_G$. Figure 23 exhibits the greatest fuzzy number X and joint possibility distribution S of X and B, given as in Theorem 3.2. A visual representation of $X +_S B = B$ is illustrated in Figure 24.*

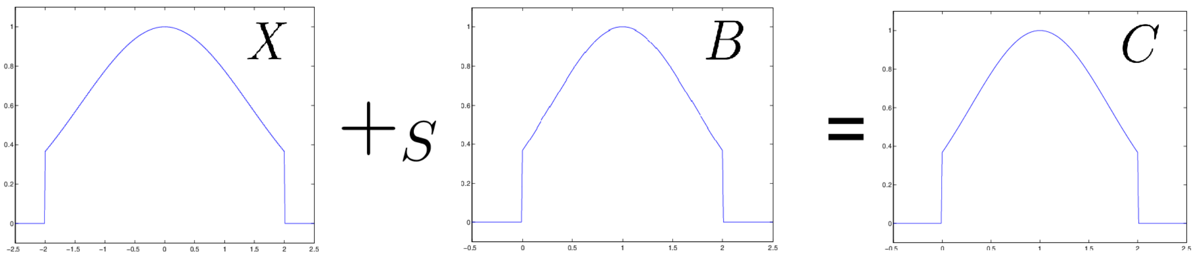
Figure 23 – Graphical representation of the JPD S given as in Example 3.2.

(a) Tri-dimensional view of JPD S (b) Top view of JPD S



The joint possibility distribution S given as in Equation (3.9), with $B = C = (1; 1; 1)_G$, and its marginal distributions X and B. The gray region represent the α-cuts of S, varying from 0 to 1 which are represented respectively from the gray-scale varying from white to black. Source: [51].

Figure 24 – Graphical representation of S-interactive sum between X and B given as in Example 3.2.



The S-interactive sum between X and B, where $B = C = (1; 1; 1)_G$, X and S are given as in Figure 23. Source: [51].

The next corollary is an immediate consequence of Theorems 3.1 and 3.2.

Corollary 3.1. *[51] Let $A, B, C \in \mathbb{R}_\mathcal{F}$ and let R, T and V be fuzzy relations of $\mathbb{R}^2$ given by*

$$R(x, y) = A(x) \wedge C(x + y), \quad T(x, y) = B(y) \wedge C(x - y), \quad V(x, y) = A(x) \wedge C(x - y) \tag{3.10}$$

for all $(x, y) \in \mathbb{R}^2$. it follows that

1. *R is a joint possibility distribution with marginals distributions A and $X \in \mathbb{R}_\mathcal{F}$ such that $A +_R X = C$;*

2. *T is a joint possibility distribution with marginals distributions B and $X \in \mathbb{R}_{\mathcal{F}}$ such that $X -_T B = C$;*

3. *V is a joint possibility distribution with marginals distributions A and $X \in \mathbb{R}_{\mathcal{F}}$ such that $A -_V X = C$.*

Proof. 1. From Theorem 3.2, one obtains

$$X +_S A = C$$

where $S(y, x) = A(x) \wedge C(y + x),\ \forall (y, x) \in \mathbb{R}^2$.

Theorem 3.1 implies that

$$X +_S A = C = A +_{\bar{S}} X,$$

and $\bar{S}(x, y) = S(y, x) = R(x, y),\ \forall (x, y) \in \mathbb{R}^2$.

2. Theorem 3.2 ensures that

$$X +_S (-B) = C,$$

where $S(x, y) = B(-y) \wedge C(x + y),\ \forall (x, y) \in \mathbb{R}^2$.

Theorem 3.1 implies that

$$X +_S (-B) = C = X -_{S^\star} B,$$

and $S^\star(x, y) = S(x, -y) = B(y) \wedge C(x - y) = T(x, y),\ \forall (x, y) \in \mathbb{R}^2$.

3. Theorem 3.2 ensures that

$$X +_S (-A) = C,$$

where $S(x, y) = A(-y) \wedge C(x + y),\ \forall (x, y) \in \mathbb{R}^2$.

Theorem 3.1 implies that

$$X +_S (-A) = C = X -_{S^\star} A$$

and $S^\star(x, y) = S(x, -y) = A(y) \wedge C(x - y) = V(x, y),\ \forall (x, y) \in \mathbb{R}^2$.

□

Item 1. of Corollary 3.1 reveals that equation $A +_R X = C$ has maximal solution (X, R). Item 2. guarantees that the equation $X -_T B = C$ has maximal solution (X, T). Item 3. ensures that the equation $A -_V X = C$ has maximal solution (X, V).

The next section investigates the second question.

3.3 Given fuzzy numbers A, B, and C, is there a joint possibility distribution J of A and B such that $A +_J B = C$?

This section focuses on studying the equation given by

$$A +_J B = C, \tag{3.11}$$

where A, B and C are given fuzzy numbers and the JPD J is the free variable.

Necessary and sufficient conditions are established for a J-interactive sum, between A and B, results in the fuzzy number C. For instance, one can see that there is no joint possibility distribution J of the real numbers 1 and 2 such that $1 +_J 2 = (2; 3; 3.5)$. However, it is not obvious if there exists or not a joint possibility distribution J of $(1; 1; 1)_G$ and $(1; 1; 1)_G$ such that $(1; 1; 1)_G +_J (1; 1; 1)_G = (1; 2; 3)$.

The next theorem yields a sufficient condition for the existence of a maximum solution of Equation (3.11).

Theorem 3.3. *[49] Let A, B, and C be fuzzy numbers. If there exists a joint possibility distribution J of A and B such that $A +_J B = C$, then the fuzzy relation M of $\mathbb{R}^2$ given by*

$$M(x, y) = A(x) \wedge B(y) \wedge C(x + y) \tag{3.12}$$

is a joint possibility distribution of A and B such that $J \subseteq M$ and $A +_M B = C$.

Proof. Since J is a joint possibility distribution of A and B, it follows that $J(x, y) \leqslant A(x) \wedge B(y)$. Moreover, since $A +_J B = C$ one obtains

$$C(x + y) = \bigvee_{f(u,v)=x+y} J(u, v) \geqslant J(x, y), \ \forall (x, y) \in \mathbb{R}^2,$$

where f is the sum operator.

Thus $J \subseteq M$. Now let us prove that M is a joint possibility distribution of A and B. The aforementioned comments imply that

$$A(x) = \bigvee_{y \in \mathbb{R}} J(x, y) \leqslant \bigvee_{y \in \mathbb{R}} M(x, y) = \bigvee_{y \in \mathbb{R}} A(x) \wedge B(y) \wedge C(x + y) \leqslant A(x).$$

Similarly one can show that $B(y) = \bigvee_{x \in \mathbb{R}} M(x, y)$, for all $y \in \mathbb{R}$.

Finally, since $A +_J B = C$ it follows that

$$C(z) = \bigvee_{x+y=z} J(x, y) \leqslant \bigvee_{x+y=z} M(x, y) = \bigvee_{x+y=z} A(x) \wedge B(y) \wedge C(z) \leqslant C(z), \ \forall z \in \mathbb{R}.$$

Therefore $A +_M B = C$.

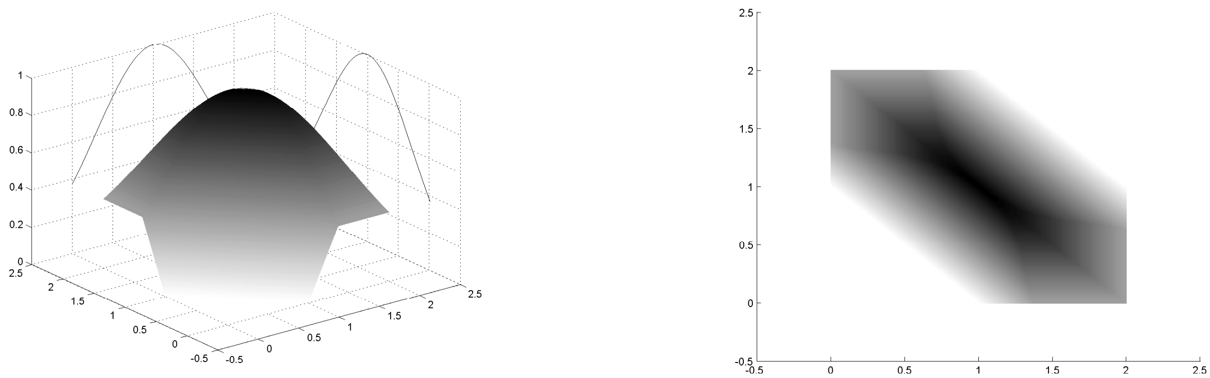
□

The next example illustrate an interactive fuzzy equation of the form $A +_J B = C$ that has solution.

Example 3.3. *Let $A = B = (1; 1; 1)_G$ and $C = (1; 2; 3)$. The joint possibility distribution M given by Equation (3.12) and its marginal distributions are depicted in Figure 25. One can observe in Figure 26 that A and B are the marginal distributions of M. Moreover, the M-interactive sum between A and B produces the fuzzy number C.*

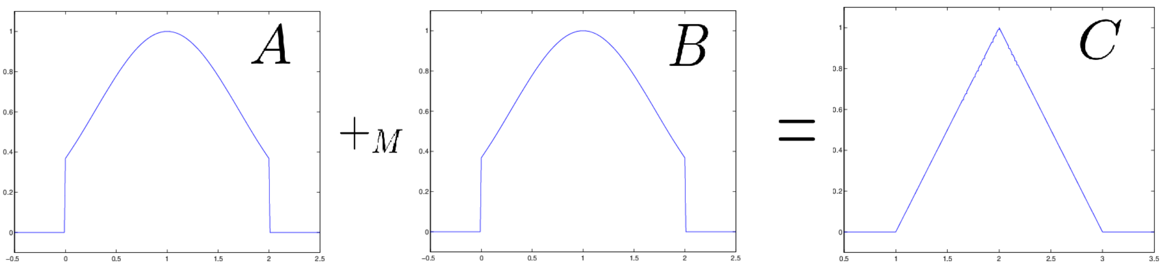
Figure 25 – Graphical representation of the JPD M given as in Example 3.3.

(a) Tri-dimensional view of JPD M (b) Top view of JPD M



The joint possibility distribution M given as in Equation (3.12), with $A = B = (1; 1; 1)_G$ and $C = (1; 2; 3)$, and its marginal distributions A and B. The gray region represent the α-cuts of S, varying from 0 to 1 which are represented respectively from the gray-scale varying from white to black. Source: [51].

Figure 26 – Graphical representation of M-interactive sum between A and B given as in Example 3.3.



The M-interactive sum between A and B, where $A = B = (1; 1; 1)_G$, A and S are given as in Figure 25. Source: [51].

The next corollary is an immediate consequence of Theorem 3.3 and it provides necessary and sufficient conditions for the existence of a solution of $A +_J B = C$.

Corollary 3.2. *[49] Let A, B, and C be fuzzy numbers. There exists a joint possibility distribution J of A and B such that $A +_J B = C$ if, and only if, the fuzzy relation M given as in Equation (3.12) is a joint possibility distribution of A and B and $A +_M B = C$.*

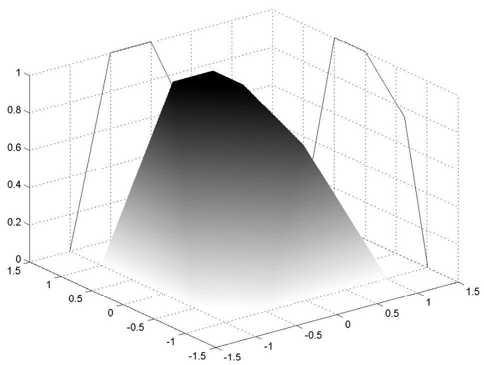
Proof. On the on hand, if M is a joint possibility distribution of A and B such that $A +_M B = C$, then $J = M$ is the required joint probability distribution. On the other hand, if there is a joint possibility distribution J of A and B such that $A +_J B = C$, then Theorem 3.3 yields that M is a joint possibility distribution of A and B, with $A +_M B = C$. □

Note that from Corollary 3.2, if the joint possibility distribution M given by (3.12) does not satisfy $A +_M B = C$ or it is not a JPD between A and B, then there is no solution of $A +_J B = C$. The next example illustrates a case where M does not satisfy $A +_M B = C$.

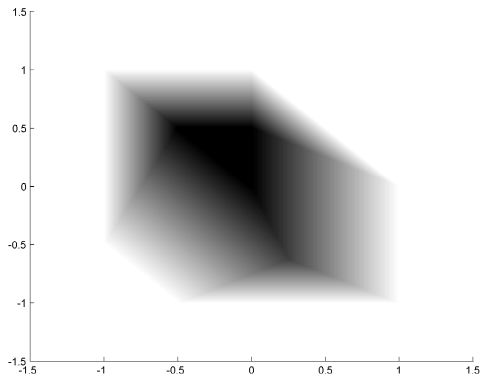
Example 3.4. *Let $A = (-1; -0.5; 0; 1)$, $B = (-1; -0.5; 0.5; 1)$ and $C = (-1.5; 0; 0.5; 1)$. The joint possibility distribution M given by Equation (3.12) and its marginal distributions are depicted in Figure 27. One can observe in Figure 27 that B is not a marginal distribution of M. From Corollary 3.2, there is no joint possibility distribution J of A and B such that $(-1; -0.5; 0; 1) +_J (-1; -0.5; 0.5; 1) = (-1.5; 0; 0.5; 1)$.*

Figure 27 – Graphical representation of the fuzzy relation M given as in Example 3.3.

(a) Tri-dimensional view of M



(b) Top view of M



The fuzzy relation M given as in Equation (3.12), and the fuzzy numbers $A = (-1; -0.5; 0; 1)$, $B = (-1; -0.5; 0.5; 1)$ and $C = (-1.5; 0; 0.5; 1)$, where the fuzzy number B is not a marginal distribution of M. Source: [51].

The above example reveals that not always an interactive fuzzy equation of the form (3.4) has a solution for J. Hence the equation $A +_J B = C$ does not always work, but $X +_J B = C$ has a solution. The next section studies the last question pointed out in this section.

3.4 Given a joint possibility distribution J of fuzzy numbers A and B, is there a joint possibility distribution N of $(A +_J B)$ and B such that $(A +_J B) -_N B = A$?

This section focuses on equation given by

$$(A +_J B) -_N B = A, \tag{3.13}$$

where J is a JPD between the fuzzy numbers A and B and the JPD N are the free variables.

Let C be the J-interactive sum between the fuzzy numbers A and B. A natural question that arises here is if there is a joint possibility distribution N of C and B, such that $A = C -_N B$. Theorem 3.4 provides an affirmative answer to this question, establishing a formula for the maximal solution.

Theorem 3.4. *[49] Let J be a joint possibility distribution of two fuzzy numbers A and B such that $C := A +_J B \in \mathbb{R}_{\mathcal{F}}$. The fuzzy relation N of $\mathbb{R}^2$, whose membership function is given by*

$$N(z, y) = A(z - y) \wedge B(y) \wedge C(z), \quad \forall (z, y) \in \mathbb{R}^2, \tag{3.14}$$

is a joint possibility distribution of C and B and $A = C -_N B$. Moreover, if there exists a joint possibility distribution $\tilde{J}$ of C and B such that $A = C -_{\tilde{J}} B$, then $\tilde{J} \subseteq N$.

Proof. Let $z \in \mathbb{R}$. If $C(z) = 0$, then $N(z, y) = 0$, for all $y \in \mathbb{R}$, which implies that $\bigvee_{y \in \mathbb{R}} N(z, y) = 0$. If $C(z) > 0$, then there exists (y_n) such that $\lim_{n\to\infty} J(z - y_n, y_n) = C(z)$, since $C = A +_J B$. Using the fact that $J(z - y_n, y_n) \leqslant A(z - y_n) \wedge B(y_n)$ and $J(z - y_n, y_n) \leqslant C(z)$, one concludes that

$$C(z) \geqslant \bigvee_{y \in \mathbb{R}} A(z - y) \wedge B(y) \wedge C(z) \geqslant \lim_{n\to\infty} A(z - y_n) \wedge B(y_n) \wedge C(z) \geqslant \lim_{n\to\infty} J(z - y_n, y_n) = C(z).$$

Therefore C is a marginal distribution of N.

Now let us prove that B is also a marginal distribution of N. To this end, let $y \in \mathbb{R}$. If $B(y) = 0$ then $N(z, y) = 0$, for all $z \in \mathbb{R}$, which implies that $\bigvee_{y \in \mathbb{R}} N(z, y) = 0$. If $B(y) > 0$, then there is a sequence (x_n) such that $B(y) = \lim_{n\to\infty} J(x_n, y)$, since J is a joint possibility distribution of A and B. Consider $z_n = x_n + y$, thus

$$C(z_n) = \bigvee_{z = u + v} J(u, v) \geqslant J(x_n, y)$$

and

$$J(z_n - y, y) \leqslant A(z_n - y) \wedge B(y).$$

A combination of the above observations gives rise to the following:

$$B(y) \geqslant \bigvee_{z\in\mathbb{R}} A(z-y)\wedge B(y)\wedge C(z) \geqslant \lim_{n\to\infty} A(z_n-y)\wedge B(y)\wedge C(z_n) \geqslant \lim_{n\to\infty} J(z_n-y,y) = B(y).$$

Therefore, B is a marginal distribution of N.

Let us prove that $A = C -_N B$. To this end, let $x \in \mathbb{R}$. On the one hand if $A(x) = 0$, then

$$\bigvee_{x=z-y} A(x) \wedge B(y) \wedge C(z) = 0.$$

On the other hand, if $A(x) > 0$ then there exists (y_n) such that $A(x) = \lim_{n\to\infty} J(x, y_n)$, since J is a joint possibility distribution of A and B. For every $z_n = x + y_n$, it follows

$$C(z_n) = \bigvee_{z_n=u+v} J(u,v) \geqslant J(x,y_n)$$

and

$$A(z_n - y_n) \wedge B(y_n) \geqslant J(x, y_n).$$

From these last observations,

$$A(x) \geqslant \bigvee_{x=z-y} A(x)\wedge B(y)\wedge C(z) \geqslant \lim_{n\to\infty} A(z_n-y_n)\wedge B(y_n)\wedge C(z_n) \geqslant \lim_{n\to\infty} J(x,y_n) = A(x).$$

Therefore, $A = C -_N B$.

Finally, let us prove that N is the maximal solution for (3.5). To this end, suppose that $\tilde{J}$ is a joint possibility distribution of C and B such that $A = C -_{\tilde{J}} B$. This hypothesis implies that $C(z) \geqslant \tilde{J}(z,y)$, $B(y) \geqslant \tilde{J}(z,y)$, and $A(z-y) \geqslant \tilde{J}(z,y)$ for every $z, y \in \mathbb{R}$. Thus

$$\tilde{J}(z,y) \leqslant A(z-y) \wedge B(y) \wedge C(z) = N(z,y), \ \forall (z,y) \in \mathbb{R}^2.$$

Therefore, $\tilde{J} \subseteq N$.

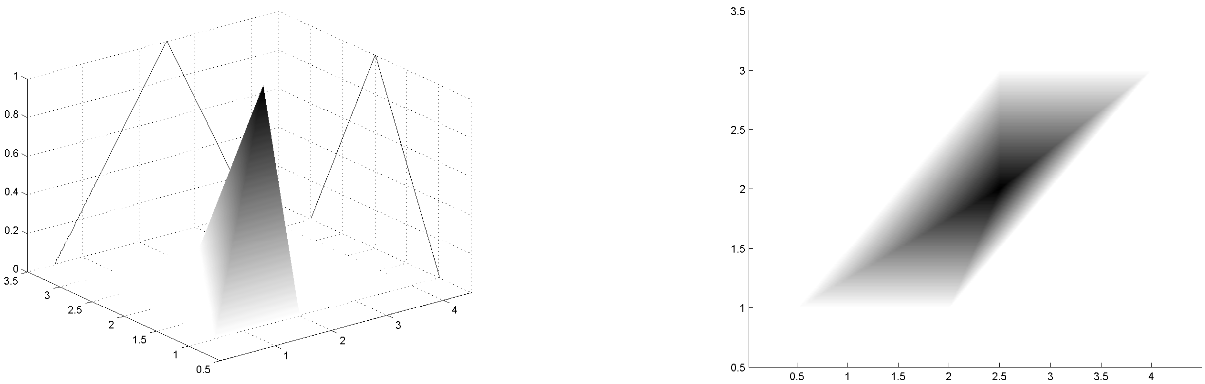
□

Recall that the standard addition of two fuzzy numbers A and B coincides with the non-interactive addition. From Theorem 3.4 one obtains $C -_H B = A = C -_N B$ where $C = A + B$ and N is given by Equation (3.14). Example 3.5 illustrates this last observation.

Example 3.5. *Let $A = (-0.5; 0.5; 1)$, $B = (1; 2; 3)$, and $C = A + B = (0.5; 2.5; 4)$. The joint possibility distribution N given by Equation (3.14) and its marginal distributions C and B are depicted in Figure 28. As expected from Theorem 3.4, the equality $C -_N B = A$ holds true.*

Figure 28 – Graphical representation of the JPD N given as in Example 3.5.

(a) Tri-dimensional view of JPD N (b) Top view of JPD N



The joint possibility distribution N given as in Equation (3.14), with $A = (-0.5; 0.5; 1)$, $B = (1; 2; 3)$, and $C = A + B = (0.5; 2.5; 4)$. The fuzzy numbers C and B are the marginal distributions of N. The gray region represent the α-cuts of S, varying from 0 to 1 which are represented respectively from the gray-scale varying from white to black. Source: [51].

This chapter is ended by connecting the joint possibility distributions M and N. The next corollary is a consequence of Theorem 3.4 and Corollary 3.1, the proof is provided in [49].

Corollary 3.3. *[49] Let $A, B, C \in \mathbb{R}_{\mathcal{F}}$. Thus*

$$A +_M B = C \Leftrightarrow A = C -_N B.$$

where M and N are the fuzzy sets of $\mathbb{R}^2$ given respectively by (3.12) and (3.14).

The next table presents a brief summary of questions brought in this chapter.

Table 1 – Solutions for interactive fuzzy equations.

Question	Answer	Maximal Solution
$\exists X \in \mathbb{R}_{\mathcal{F}}, J \in \mathcal{F}(\mathbb{R}^2)\ :\ X +_J B = C?$	Yes	$X = \bigvee_{y \in \mathbb{R}} B(y) \wedge C(x+y)$
$\exists J \in \mathcal{F}(\mathbb{R}^2)\ :\ A +_J B = C?$	Not always	$M(x,y) = A(x) \wedge B(y) \wedge C(x+y)$
$\exists N \in \mathcal{F}(\mathbb{R}^2)\ :\ (A +_J B) -_N B = A?$	Yes	$N(z,y) = A(z-y) \wedge B(y) \wedge C(z)$

Solutions of the questions raised in Chapter 3. Source: Author

This chapter shows the usefulness of interactive arithmetic in the context of fuzzy linear equations. In particularly, the equation $A -_V X = C$ (see Corollary 3.1) gives raise to the following.

Let us consider a fuzzy function F. The expression

$$F(t_0 + h) -_J F(t_0) = hC \tag{3.15}$$

is an equation in the form of $A -_J X = C$ and it is connected with the definition of fuzzy derivative that is obtained from a limit of difference between fuzzy numbers. Hence, Equation (3.15) allows us to define a fuzzy derivative that embraces the concept of interactivity.

Moreover, the equation

$$X(t) +_J hf(t, X(t)) = X(t + h) \tag{3.16}$$

can be interpreted by the Euler's numerical method for a fuzzy initial value problem.

The expression given by (3.16) is an equation in the form of $A +_J X = C$. This means that linear interactive fuzzy equations can be associated with numerical solutions for fuzzy differential equations. So the aforementioned questions can be associated with fuzzy differential equations from both numerical and analytical points of view.

Motivated by these connections, the next chapters will study fuzzy differential equations, using different types of interactivity (consequently, different types of JPDs).

3.5 Conclusion

This chapter asked fundamental questions regarding additions and subtractions involving fuzzy linear equations. More precisely, fuzzy equations that take into account the relationship of interactivity were investigated. Section 3.2 stated that for every $B, C \in \mathbb{R}_{\mathcal{F}}$ there exist a fuzzy number X and a joint possibility distribution S of X and B such that $X +_S B = C$. Section 3.3 established necessary and sufficient conditions for the equation $A +_J B = C$ has a solution in the free variable J. In this case, in contrast to standard addition, Example 3.3 showed that a triangular fuzzy number can be written as an interactive addition of two Gaussian fuzzy numbers. Section 3.4 ensured that if a fuzzy number C is given by means of a J-interactive sum of the fuzzy numbers A and B, then the fuzzy number A corresponds to the N-interactive difference of C and B (see Theorem 3.4). This chapter ended by establishing that linear interactive fuzzy equations can be associated with fuzzy differential equations from both numerical and analytical points of view.

4 Linearly Interactive Fuzzy Numbers

This chapter presents the concept of *linear interactive correlation*. This type of interactivity takes into account a relationship between fuzzy numbers that associates their membership functions via a linear correlation. The linear interactive fuzzy numbers are used in various problems such as arithmetic on fuzzy numbers [32, 77], fuzzy dynamic systems [15, 148, 138], fuzzy regression problems [115, 114], etc.

This relationship, which is also called completely correlation, was first introduced by Fullér, Carlsson and Majlender [62, 32]. Carlsson et al. [32] established the sum between two completely correlated fuzzy numbers. Barros and Pedro [15] characterized the arithmetic operations between completely correlated fuzzy numbers by means of α-cuts. Also, they introduced the notion of derivative for autocorrelated fuzzy process via complete correlation.

The third contribution of this thesis is to provide a generalization of this concept. In this case n completely correlated fuzzy numbers are called linear interactive fuzzy numbers [53, 148, 77]. Moreover, solutions of fuzzy differential equations are provided, using linear interactive correlation. More precisely, this chapter focuses on fuzzy initial value problems, where the initial conditions are given by n linear interactive fuzzy numbers. These solutions are compared with the one obtained from the Zadeh extension principle and it is shown that this approach is connected with the Fréchet derivative.

This chapter is based on references [32, 62, 15, 53, 148, 77, 149].

4.1 Completely Correlated Fuzzy Numbers

The completely correlated fuzzy numbers are defined by the following joint possibility distribution.

Definition 4.1. *[32] The fuzzy numbers A and B are said to be completely correlated if there exist $q, r \in \mathbb{R}$, where $q \neq 0$, such that their joint possibility distribution J_C is given by*

$$J_C(x,y) = A(x)\chi_{\{v=qu+r\}}(x,y) = B(y)\chi_{\{v=qu+r\}}(x,y), \tag{4.1}$$

where

$$\chi_{\{v=qu+r\}}(x,y) = \begin{cases} 1 & \text{if } y = qx + r \\ 0 & \text{if } y \neq qx + r \end{cases} \tag{4.2}$$

is the characteristic function of the line $C = \{(u,v) \in \mathbb{R}^2 \, : \, qu + r = v\}$.

This type of interactivity allows to write the α-cuts of the fuzzy number B in terms of the α-cuts of A [62]

$$[B]^\alpha = q[A]^\alpha + r, \quad \forall \alpha \in [0,1]. \tag{4.3}$$

It is important to observe that if A and B are completely correlated, then there exist q and r that satisfies the expression (4.3), for all $\alpha \in [0,1]$, which means that the values of q and r do not change, once the JPD J_C is chosen.

Also, the α-cuts of the joint possibility distribution J_C can be written in terms of A and the parameters q and r, as follows

$$[J_C]^\alpha = \{(x, qx + r) \ : \ x \in [A]^\alpha\}, \quad \forall \alpha \in [0,1]. \tag{4.4}$$

Equation (4.4) means that in the case where the fuzzy numbers A and B are completely correlated, the first marginal distribution determines completely the second one, and vice versa. The joint possibility distribution J_C may also be denoted by $J_{q,r}$, in order to make clear which parameters q and r are been used. Example 4.1 illustrates this concept.

Example 4.1. *Let $A = B = (-1;0;1) \in \mathbb{R}_\mathcal{F}$. Note that A and B are completely correlated with respect to the joint possibility distribution $J_C = J_{1,0}$ (see Figure 29), whose membership function is given by*

$$J_{1,0}(x,y) = A(x)\chi_{\{v=u\}}(x,y).$$

The α-cuts of $J_{1,0}$ are given by

$$[J_{1,0}]^\alpha = \{(x,x) \ : \ x \in [A]^\alpha\}, \quad \forall \alpha \in [0,1].$$

The fuzzy numbers A and B can also be completely correlated with respect to the joint possibility distribution $J_C = J_{-1,0}$ (see Figure 30), whose membership function is given by

$$J_{-1,0}(x,y) = A(x)\chi_{\{v=-u\}}(x,y).$$

The α-cuts of $J_{-1,0}$ are given by

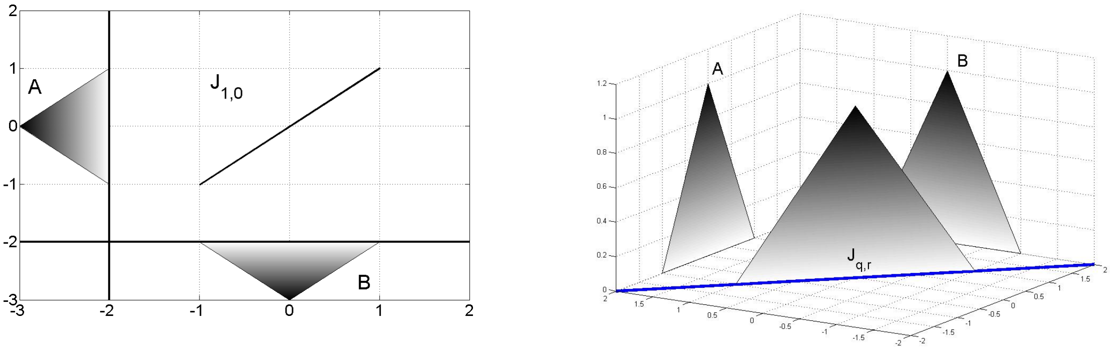
$$[J_{-1,0}]^\alpha = \{(x,-x) \ : \ x \in [A]^\alpha\}, \quad \forall \alpha \in [0,1].$$

Observe that the pairs $(1,0)$ and $(-1,0)$ are the only possible values for (q,r) which make A and B be completely correlated.

Example 4.1 suggests that two fuzzy numbers may be completely correlated in two different forms, when $q > 0$ or $q < 0$. In fact, if two fuzzy numbers are completely correlated then there are at most two joint possibility distribution for them [108].

Figure 29 – Graphical representation of the joint possibility distribution $J_{1,0}$ given by Example 4.1.

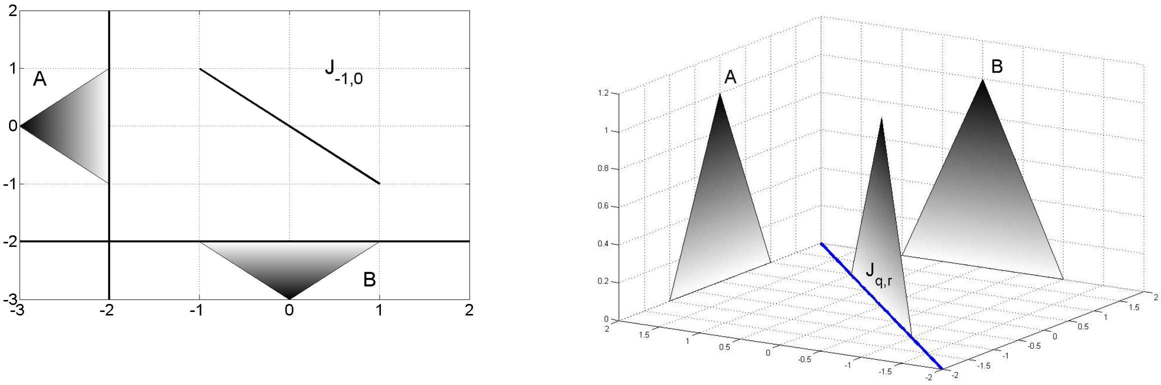
(a) The top view of $J_{1,0}$. (b) The tri-dimensional view of $J_{1,0}$.



The blue line represents the set $C = \{(u, v) : u = v\}$ and the shaded triangle on it represents the joint possibility distribution $J_{1,0}$. The gray lines represent the α-cuts of J, varying from 0 to 1 which are represented respectively from the gray-scale lines varying from white to black. Source: Author.

Figure 30 – Graphical representation of the joint possibility distribution $J_{-1,0}$ given by Example 4.1.

(a) The top view of $J_{-1,0}$. (b) The 3 dimensional view of $J_{-1,0}$.



The blue line represents the set $C = \{(u, v) : u = -v\}$ and the shaded triangle on it represents the joint possibility distribution $J_{-1,0}$. The gray lines represent the α-cuts of J, varying from 0 to 1 which are represented respectively from the gray-scale lines varying from white to black. Source: Author.

Definition 4.2. *Two fuzzy numbers are said to be positively (negatively) completely correlated if the parameter q of $J_{q,r}$ is positive (negative).*

The definition of completely correlation requires that membership functions of the involved fuzzy numbers have the same shape. For instance, triangular and trapezoidal fuzzy numbers can never be completely correlated, since their membership functions do not have the same form.

Next, the arithmetic for completely correlated fuzzy numbers are presented.

4.1.1 Arithmetic on completely correlated fuzzy numbers

The arithmetic operations between completely correlated fuzzy numbers are defined by

$$(A \otimes B)(z) = \bigvee_{z = x \otimes y} J_{q,r}(x, y), \tag{4.5}$$

where $\otimes$ represents some arithmetic operation.

The α-cuts of the arithmetic operations, via completely correlation, are provided in Theorem 4.1

Theorem 4.1. *[15] Let A and B be completely correlated fuzzy numbers. Thus*

(*a*) $[A +_C B]^\alpha = (1 + q)[A]^\alpha + r, \ \ \forall \alpha \in [0, 1]$;

(*b*) $[A -_C B]^\alpha = (1 - q)[A]^\alpha - r, \ \ \forall \alpha \in [0, 1]$;

(*c*) $[A \cdot_C B]^\alpha = \{qx^2 + rx \in \mathbb{R} \ : \ A(x) \geqslant \alpha\}$;

(*d*) $[A \div_C B]^\alpha = \left\{\dfrac{x}{qx + r} \in \mathbb{R} \ : \ A(x) \geqslant \alpha\right\}$.

Note that every fuzzy number A is completely correlated with itself, since A can be written as $A = 1.A + 0$. In this case, from item (b) of Theorem 4.1 it follows that

$$[A -_C A]^\alpha = (1 - 1)[A]^\alpha + 0 = [0, 0], \ \forall \alpha \in [0, 1]. \tag{4.6}$$

Therefore, the difference based on the joint possibility distribution J_C also satisfies the property of the generalized (Hukuhara) difference, that is, $A -_g A = 0$. Moreover, if $q < 0$ then $A +_C B = A + B$, where $+$ is the standart sum. Furthermore, if $q > 0$ then $A -_C B = A - B$, where $-$ is the standart difference [15].

For more details about this type of interactivity the reader can refer to [108, 15].

The next section provides the generalization of the concept of completely correlation.

4.2 Linearly Interactive Fuzzy Numbers

The fuzzy numbers $A_1, ..., A_n \in \mathbb{R}_\mathcal{F}$ are said to be linear interactive if their joint possibility distribution J_L is given by [148, 77, 53]

$$\begin{aligned} J_L(x_1, ..., x_n) &= A_1(x_1)\chi_L(x_1, ..., x_n) \\ &= A_2(x_2)\chi_L(x_1, ..., x_n) \\ &\vdots \\ &= A_n(x_n)\chi_L(x_1, ..., x_n) \end{aligned} \tag{4.7}$$

where $L = \{(u, q_2u + r_2, ..., q_nu + r_n) : u \in \mathbb{R};\ q_i, r_i \in \mathbb{R},\ \text{with } q_1q_2 \ldots q_n \neq 0\}$.

Note that J_L extends J_C, since from (4.7) one can see that A_1 and A_i, with $i > 1$, are completely correlated. In this case $[A_i]^\alpha = q_i[A_1]^\alpha + \{r_i\}$, for all $i = 2, ..., n$. For each $\alpha \in [0, 1]$, the $\alpha-$cuts of J_L can be given by:

$$[J_L]^\alpha = \{(x, q_2x + r_2, \ldots, q_nx + r_n)\ :\ x \in [A_1]^\alpha\}. \tag{4.8}$$

Equation (4.8) reveals that the $\alpha-$cuts of the joint possibility distribution J_L can be obtained only in terms of α-cuts of A_1 and the parameters q_i and r_i, for all $i = 2, ..., n$.

Since each $x \in [A]^\alpha = [a^-_\alpha, a^+_\alpha]$ can be written as $x = (1 - \lambda)a^-_\alpha + \lambda a^+_\alpha$ for $\lambda \in [0, 1]$, the expression given by (4.8) becomes

$$[J_L]^\alpha = \left\{(1 - \lambda)(a^-_\alpha, q_2a^-_\alpha + r_2, \ldots, q_na^-_\alpha + r_n) + \lambda(a^+_\alpha, q_2a^+_\alpha + r_2, \ldots, q_na^+_\alpha + r_n) : \lambda \in [0, 1]\right\}. \tag{4.9}$$

The total possible number of joint possibility distributions that makes the fuzzy numbers $A_1, \ldots, A_n$ be linear interactive depends on the symmetry of the first fuzzy number A_1.

Definition 4.3 (Symmetry). *[52] A fuzzy number A is said to be symmetric, with respect to $x \in \mathbb{R}$, if $A(x - y) = A(x + y)$, $\forall y \in \mathbb{R}$. If there is no $x \in \mathbb{R}$ such that this property is satisfied, A is said to be non-symmetric.*

For example the fuzzy number $A = (1; 2; 3)$ is symmetric with respect to $x = 2$, since $A(2 - y) = A(2 + y)$ for all $y \in \mathbb{R}$.

The follow proposition is obtained from the definition of symmetry.

Proposition 4.1. *[52] Let $A \in \mathbb{R}_\mathcal{F}$ and $q_1, q_2, r_1, r_2 \in \mathbb{R}$, where $q_1q_2 \neq 0$. If A satisfies*

$$q_1A + r_1 = q_2A + r_2, \tag{4.10}$$

then

$$\begin{cases} q_1 = q_2 \quad \text{and} \quad r_1 = r_2 \quad \rightarrow \quad \text{if A is non-symmetric} \\ \left.\begin{array}{l} q_1 = q_2 \quad \text{and} \quad r_1 = r_2 \quad \text{or} \\ q_1 = -q_2 \quad \text{and} \quad r_1 = 2q_2\bar{x} + r_2 \end{array}\right\} \text{if A is symmetric (w.r.t. } \bar{x}) \end{cases} \tag{4.11}$$

where $\bar{x}$ is the midpoint of $[A]^1$.

Proposition 4.1 implies that if A_1 is non-symmetric then $A_1, \ldots, A_n$ are linear interactive with respect to only one joint possibility distribution. However, if A_1 is symmetric then $A_1, \ldots, A_n$ are linear interactive with respect to 2^{n-1} joint possibility distributions [53].

Example 4.2. *Let $A_1 = (1;2;4), A_2 = (3;4;6), A_3 = (0;1;3) \in \mathbb{R}_{\mathcal{F}}$. Note that A_1, A_2 and A_3 are linear interactive with respect to $J_{q,r}$, where $q = (1,1,1)$ and $r = (0,2,-1)$. Since A_1 is non-symmetric, the joint possibility distribution $J_{q,r}$ is the only one that makes A_1, A_2 and A_3 be linear interactive, as Proposition 4.1 ensures.*

Example 4.3. *Let $A_1 = (1;2;3), A_2 = (2;3;4), A_3 = (3;4;5) \in \mathbb{R}_{\mathcal{F}}$. Note that A_1, A_2 and A_3 are linear interactive with respect to four different joint possibility distributions:*

$$[J_1]^\alpha = \left\{ \begin{pmatrix} x \\ x+1 \\ x+2 \end{pmatrix} : x \in [A_1]_\alpha \right\} \quad , \quad [J_2]^\alpha = \left\{ \begin{pmatrix} x \\ x+1 \\ -x+6 \end{pmatrix} : x \in [A_1]_\alpha \right\},$$

$$[J_3]^\alpha = \left\{ \begin{pmatrix} x \\ -x+5 \\ x+2 \end{pmatrix} : x \in [A_1]_\alpha \right\} \quad \text{and} \quad [J_4]^\alpha = \left\{ \begin{pmatrix} x \\ -x+5 \\ -x+6 \end{pmatrix} : x \in [A_1]_\alpha \right\},$$

as Proposition 4.1 establishes.

The above examples clarify that if the fuzzy number A_1 is non-symmetric and $A_1, \ldots, A_n$ are linear interactive, then $A_2, \ldots, A_n$ are necessarily non-symmetric as well. On the other hand, if A_1 is symmetric and $A_1, \ldots, A_n$ are linear interactive, then $A_2, \ldots, A_n$ are necessarily symmetric [53].

Remark 4.1. *If $A_1, \ldots, A_n$ are linear interactive symmetric fuzzy numbers, then is possible to consider 2^{n-1} JPDs.*

The representation (4.9) is used in order to provide fuzzy solutions to fuzzy initial value problems (FIVPs) and/or fuzzy boundary value problems (FBVPs), where the additional conditions (initial or boundary) may have an intrinsic linear correlation, which is described by linear interactive fuzzy numbers. This approach is presented in the next section.

4.2.1 Initial Value Problems with Linearly Interactive Fuzzy Conditions

Let us first present a brief review on the theory of ordinary differential equations (ODEs). Let $I \subseteq \mathbb{R}$ be an open set. A linear n-th order ordinary differential equation with constant coefficients is given by an equation, for all $t \in I$, of the form

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = g(t), \tag{4.12}$$

where $y^{(i)}$ denotes the $i-$ary derivative of y, $a_i \in \mathbb{R}$, for all $i = 1, ..., n$, and $g : I \to \mathbb{R}$ is a continuous function. If $g(t) = 0$, for all $t \in I$, then the corresponding ODE is said to be homogeneous. Otherwise, the ODE is called non-homogeneous.

Since Equation (4.12) involves the n-th derivative of y with respect to t, in order to obtain a unique solution it is necessary to specify n additional conditions. Here the focus is on fuzzy initial value problems (FIVPs), that is, the n initial conditions condition are given by

$$\begin{cases} y(t_0) = y_0 \\ y^{(1)}(t_0) = y_0^{(1)} \\ \quad\vdots \\ y^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}, \tag{4.13}$$

where $t_0 \in I$.

The general solution y for Equation (4.12) is given by the sum of a particular solution y_p and a solution y_h of the corresponding homogeneous ODE

$$y(t) = y_p(t) + y_h(t), \ \forall t \in I. \tag{4.14}$$

Recall that the function y_h, given in Equation (4.14), can be written as

$$y_h(t) = c_1 w_1(t) + ... + c_n w_n(t), \tag{4.15}$$

where $c_1, ..., c_n$ are constants values that can be determined by the n initial conditions and $w_1, ..., w_n$ are the fundamental set of solutions of the corresponding homogeneous ODE [23].

For $t = t_0$ one can represent (4.14), in matrix form, by

$$Mc = b - y_P, \tag{4.16}$$

where

$$M = \begin{bmatrix} w_1(t_0) & \ldots & w_n(t_0) \\ \vdots & \ddots & \vdots \\ w_1^{(n-1)}(t_0) & \ldots & w_n^{(n-1)}(t_0) \end{bmatrix}, \ c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \ b = \begin{bmatrix} y_0 \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}, \quad \text{and} \quad y_P = \begin{bmatrix} y_p(t_0) \\ \vdots \\ y_p^{(n-1)}(t_0) \end{bmatrix}.$$

Since the linear combination (4.15) of the functions $w_1, ..., w_n$ is a solution for the corresponding homogeneous equation, then M is a non-singular matrix [153]. Therefore, the vector c is obtained as follows

$$c = M^{-1}b - M^{-1}y_P. \tag{4.17}$$

Thus, for each $t \in \mathbb{R}$, the solution $y(t)$ is given by

$$y(t) = y_p(t) + W^T(M^{-1}b - M^{-1}y_P), \tag{4.18}$$

where $W^T = [w_1(t) \dots w_n(t)]$.

Let us consider the case where the initial conditions are given by interactive fuzzy numbers, that is, the vector b is formed by n linear interactive fuzzy numbers. Hence, consider the following FIVP

$$\begin{cases} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = g(t) \\ y(t_0) = Y_0 \\ \vdots \\ y^{(n-1)}(t_0) = Y_0^{(n-1)} \end{cases} \tag{4.19}$$

where $a_i \in \mathbb{R}$ for all $i = 1, \dots, n$ and the initial conditions $Y_0, Y_0^{(1)}, \ldots, Y_0^{(n-1)} \in \mathbb{R}_{\mathcal{F}}$ are linear interactive.

Let $y(\cdot, b)$ be the deterministic solution of the associated IVP given by Equation (4.18). A fuzzy solution for (4.19) is a fuzzy function y_L given by

$$y_L(\cdot) = y_{J_L}(\cdot, B), \tag{4.20}$$

where $B = (Y_0, \dots, Y_0^{(n-1)})$. In view of Theorem 2.1, the α-cut of the fuzzy solution y_L, for each $t \in \mathbb{R}$, can be written as

$$[y_L(t)]^\alpha = \{y_p(t) + W^T(M^{-1}b - M^{-1}y_P) : b \in [J_L]^\alpha \subseteq \mathbb{R}^n\}, \tag{4.21}$$

where $[J_L]^\alpha$ is given by (4.8).

From (4.17) and (4.9), for each $\lambda \in [0,1]$ and $\alpha \in [0,1]$, the vector c_λ can be obtained by:

$$c_\lambda = (1-\lambda)M^{-1}b_\alpha^- + \lambda M^{-1}b_\alpha^+ - M^{-1}y_P, \tag{4.22}$$

where

$$b_\alpha^- = \begin{bmatrix} y_{0_\alpha}^- \\ q_2 y_{0_\alpha}^- + r_2 \\ \vdots \\ q_n y_{0_\alpha}^- + r_n \end{bmatrix} \quad \text{and} \quad b_\alpha^+ = \begin{bmatrix} y_{0_\alpha}^+ \\ q_2 y_{0_\alpha}^+ + r_2 \\ \vdots \\ q_n y_{0_\alpha}^+ + r_n \end{bmatrix}.$$

with $[Y_0]^\alpha = [y_{0_\alpha}^-, y_{0_\alpha}^+]$, for all $\alpha \in [0,1]$.

This leads us to the following deterministic solution for the associated IVP:

$$\begin{aligned} h(t,\alpha,\lambda) &= y_p(t) + W^T c_\lambda \\ &= \underbrace{y_p(t) - W^T M^{-1} y_P}_{Y_1(t)} + (1-\lambda)\underbrace{W^T M^{-1} b_\alpha^-}_{Y_2(t,\alpha)} + \lambda \underbrace{W^T M^{-1} b_\alpha^+}_{Y_3(t,\alpha)} \\ &= Y_1(t) + (1-\lambda)Y_2(t,\alpha) + \lambda Y_3(t,\alpha). \end{aligned} \tag{4.23}$$

The above comments and Equations (4.21) and (2.9) imply that [148]

$$\begin{aligned}[y_L(t)]^\alpha &= \{y_p(t) + W^T c_\lambda \,:\, \lambda \in [0,1]\} \\ &= \{h(t,\alpha,\lambda) \,:\, \lambda \in [0,1]\} \\ &= \left[\bigwedge_{\lambda\in[0,1]} h(t,\alpha,\lambda), \bigvee_{\lambda\in[0,1]} h(t,\alpha,\lambda)\right]. \end{aligned} \tag{4.24}$$

Equation (4.24) reveals that the task of determining the endpoints of α-cuts of $y_L(t)$ boils down to calculate the minimum and maximum values of $h(t,\alpha,\lambda)$ with respect to $\lambda \in [0,1]$. Since Y_1 does not depend on the parameters λ or α, it is only necessary to analyze Y_2 and Y_3 in (4.23). On the one hand, if $Y_2(t,\alpha) \leqslant Y_3(t,\alpha)$, then

$$\underbrace{Y_1(t) + Y_2(t,\alpha)}_{h(t,\alpha,0)} \leqslant \underbrace{Y_1(t) + Y_2(t,\alpha) + \lambda(Y_3(t,\alpha) - Y_2(t,\alpha))}_{h(t,\alpha,\lambda)} \leqslant \underbrace{Y_1(t) + Y_3(t,\alpha)}_{h(t,\alpha,1)},$$

for all $\lambda \in [0,1]$.

Hence, the minimizer and maximizer of the function $h(t,\alpha,\cdot)$ are given by $\lambda = 0$ and $\lambda = 1$, respectively. Similarly, if $Y_3(t,\alpha) < Y_2(t,\alpha)$, then the minimum and maximum values of $h(t,\alpha,\cdot)$ are achieved at $\lambda = 1$ and $\lambda = 0$, respectively. In other words, the global minimizer and maximizer of $h(t,\alpha,\lambda)$ with respect to $\lambda \in [0,1]$ are given at $\lambda = 0$ or $\lambda = 1$. Therefore, for each $t \in \mathbb{R}$, the α-cuts of the fuzzy solution $y_L(t)$ are given by

$$[y_L(t)]_\alpha = [\min\{h(t,\alpha,0), h(t,\alpha,1)\}, \max\{h(t,\alpha,0), h(t,\alpha,1)\}]\,, \tag{4.25}$$

where

$$h(t,\alpha,0) = Y_1(t) + Y_2(t,\alpha) \quad \text{and} \quad h(t,\alpha,1) = Y_1(t) + Y_3(t,\alpha).$$

One can observe that in the case where the initial condition is symmetric, there are 2^{n-1} joint possibility distributions. But once the JPD is chosen there is only one solution for (4.19). Therefore, the problem involving the fuzzy differential equations (FDEs) with the additional conditions given by interactive fuzzy numbers should be written by the differential equation, the additional conditions given by fuzzy numbers and the joint possibility distribution that turns the fuzzy numbers interactive, that is,

$$\begin{cases} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = g(t) \\ y^{(i-1)}(t_0) = Y_0^{(i-1)} \in \mathbb{R}_{\mathcal{F}}, \quad i = 1,\ldots,n \\ J_L \end{cases}. \tag{4.26}$$

Other approach to solve FDEs is given by the Zadeh's extension principle, which is based on extending the classical solution y of the corresponding ODE, according to Definition 1.25. In this case, the fuzzy conditions are considered as non-interactive and this solution is denoted by $\widehat{y}$.

Recall from Subsection 2.1 that if $J = J_\wedge$, then the sup-J and Zadeh's extension principles coincide. Consequently, the fuzzy solution $\widehat{y}$ is given by $y_{J_\wedge}$. In view of Theorem 2.1, the endpoints of $[\widehat{y}(t)]^\alpha$ are obtained by

$$[\widehat{y}(t)]^\alpha = \left[\min_{[J_\wedge]^\alpha} y(t,\cdot)\ ,\ \max_{[J_\wedge]^\alpha} y(t,\cdot)\right]. \tag{4.27}$$

The next theorem establishes the connection between the fuzzy solutions $\widehat{y}$ and y_L, in the case where the initial fuzzy condition B_0 is symmetric.

Theorem 4.2. *Consider the FDE, given by (4.19), with n initial fuzzy conditions $B_0, \ldots, B_{n-1}$, where B_0 is symmetric. Let $\widehat{y}$ be the solution via Zadeh's extension and y_{J_i} be the solution via sup-J extension, where $i = \{1, \ldots, 2^{n-1}\}$. Then,*

$$[J_\wedge]^\alpha = conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right),\quad \forall \alpha \in [0,1], \tag{4.28}$$

where $conv\,(W)$ represents the convex hull of the set W. Moreover,

$$[\widehat{y}(t)]_\alpha = \left[\min_{i=1,\ldots,2^{n-1}} (y_{J_i}(t))^-_\alpha,\ \max_{i=1,\ldots,2^{n-1}} (y_{J_i}(t))^+_\alpha\right] \tag{4.29}$$

Proof. Let us first prove the equality (4.28). To this end, the following inclusion will be proved

$$conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right) \subseteq [J_\wedge]^\alpha.$$

Note that $[J_i]^\alpha \subseteq [J_\wedge]^\alpha$, for all $i \in \{1, \ldots, 2^{n-1}\}$ and for all $\alpha \in [0,1]$, which implies that $\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha \subseteq [J_\wedge]^\alpha$. Therefore,

$$conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right) \subseteq conv([J_\wedge]^\alpha) = [J_\wedge]^\alpha,$$

since $[J_\wedge]^\alpha$ is convex.

Now, let us prove the other inclusion. For a fixed $\alpha \in [0,1]$, let $x \in [J_\wedge]^\alpha \subseteq \mathbb{R}^n$. If $x \in U_1 = [J_1]^\alpha$ then exists $\lambda \in [0,1]$ such that $x = (1-\lambda)b^-_\alpha + \lambda b^+_\alpha$, where $b^-_\alpha, b^+_\alpha \in [J_1]^\alpha$. Thus, $x \in conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right)$.

If $x \notin U_1$, then let J_2 such that $[J_2]^\alpha \nsubseteq \text{span}(U_1)$, where $\text{span}(S)$ is defined as the set of all finite linear combinations of elements of S. Define U_2 as follows

$$U_2 = \{u : u = (1-\lambda)a + \lambda b, 0 \leqslant \lambda \leqslant 1,\ a \in [J_2]^\alpha \text{ and, } b \in U_1\}.$$

If $x \in U_2$, then $x \in conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right)$, otherwise, let J_3 such that $J_3 \nsubseteq span(U_2)$. Define U_3 as follows

$$U_3 = \{u : u = (1-\lambda)a + \lambda b, 0 \leqslant \lambda \leqslant 1,\ a \in [J_3]^\alpha \text{ and},\ b \in U_2\}.$$

If $x \in U_3$, then $x \in conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right)$, otherwise, repeat this process successively. Since there are 2^{n-1} joint possibility distributions and $2^{n-1} \geqslant n = dim(\mathbb{R}^n)$, $\forall n \geqslant 2$, then exist $\{J_1, \ldots, J_n\}$, such that $x \in U_i$, where

$$U_i = \{u : u = (1-\lambda)a + \lambda b, 0 \leqslant \lambda \leqslant 1,\ a \in [J_i]^\alpha \text{ and},\ b \in U_{i-1}\},$$

for $i \in \{2, \ldots, n\}$ and $U_1 = [J_1]^\alpha$.

Therefore, $[J_\wedge]^\alpha \subseteq conv\left(\bigcup_{i=1}^{2^{n-1}} [J_i]^\alpha\right)$ and the equality (4.28) holds. Consequently, the fuzzy solution $\widehat{y}$ is given as follows

$$\begin{aligned}
[\widehat{y}(t)]_\alpha &= \left[\min_{(x_1,\ldots,x_n)\in[J_\wedge]^\alpha} y(t, B_0, \ldots, B_{n-1}), \max_{(x_1,\ldots,x_n)\in[J_\wedge]^\alpha} y(t, B_0, \ldots, B_{n-1})\right] \\
&= \left[\min_{(x_1,\ldots,x_n)\in conv\left(\bigcup_{i=1}^{2^{n-1}}[J_i]^\alpha\right)} y(t, B_0, \ldots, B_{n-1}),\right. \\
&\qquad \left.\max_{(x_1,\ldots,x_n)\in conv\left(\bigcup_{i=1}^{2^{n-1}}[J_i]^\alpha\right)} y(t, B_0, \ldots, B_{n-1})\right] \\
&= \left[\min_{i=1,\ldots,2^{n-1}} (y_{J_i}(t))^-_\alpha, \max_{i=1,\ldots,2^{n-1}} (y_{J_i}(t))^+_\alpha\right],
\end{aligned}$$

where $[y_{J_i}(t)]^\alpha = [y_{J_i}(t))^-_\alpha, y_{J_i}(t))^+_\alpha]$, for all $i = 1, \ldots, 2^{n-1}$. □

Theorem 4.2 ensures that if the initial condition B_0 is symmetric then the fuzzy solution given by Zadeh's extension is equal to the convex hull of the fuzzy solutions produced by the sup-J extension.

The next theorem shows that the fuzzy solution via sup-J extensions is contained in the solution via Zadeh's extension, in the case where the initial condition B_0 is non-symmetric.

Theorem 4.3. *Consider the FDE, given by (4.19), with n initial fuzzy conditions $B_0, \ldots, B_{n-1}$, where B_0 is non-symmetric. Let $\widehat{y}$ be the solution via Zadeh's extension and y_J be the solution via sup-J extension. Then,*

$$[y_J(t)]^\alpha \subseteq [\widehat{y}(t)]^\alpha, \tag{4.30}$$

for all $\alpha \in [0, 1]$.

Proof. Since $[J]^\alpha \subseteq [J_\wedge]^\alpha$, $\forall \alpha \in [0,1]$, it follows that

$$\min y|_{[J_\wedge]^\alpha} \leqslant \min y|_{[J]^\alpha} \quad \text{and} \quad \max y|_{[J_\wedge]^\alpha} \geqslant \max y|_{[J]^\alpha},$$

where $y|_A$ represents the restriction of the function y to the set A.

Therefore,

$$\begin{aligned}
[y_J(t)]_\alpha &= \left[\min_{(x_1,\ldots,x_n)\in[J]^\alpha} y(t, B_0, \ldots, B_{n-1}), \max_{(x_1,\ldots,x_n)\in[J]^\alpha} y(t, B_0, \ldots, B_{n-1}) \right] \\
&\subseteq \left[\min_{(x_1,\ldots,x_n)\in[J_\wedge]^\alpha} y(t, B_0, \ldots, B_{n-1}), \max_{(x_1,\ldots,x_n)\in[J_\wedge]^\alpha} y(t, B_0, \ldots, B_{n-1}) \right] \\
&= [\widehat{y}(t)]^\alpha.
\end{aligned}$$

□

Theorem 4.3 yields that the fuzzy solution via sup-J extension is more specific than the solution given by Zadeh's extension in the case that the initial condition B_0 is non-symmetric.

The next subsection presents a method to produce a solution for fuzzy boundary value problems (FBVs), where the boundary conditions are given by linearly interactive fuzzy numbers. This method was provided in [77].

4.2.2 Boundary Value Problem with Linearly Interactive Fuzzy Conditions

Ibáñez *et al.* [77] provided a solution for fuzzy boundary conditions [125] for higher order differential equations. To this end, they used the arithmetic for linear interactive fuzzy numbers. Before presenting the similarities of this approach with ours, let us present the method given by them.

Consider the following FBVP

$$\begin{cases} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = g(t) \\ y(t_{i-1}) = B_{(i-1)} \in \mathbb{R}_\mathcal{F} \\ J_L \end{cases} . \tag{4.31}$$

The fuzzy solution of (4.31) is given by the fuzzy function $(S_t)_L$ given by

$$(S_t)_L(B) \quad = \quad y_p(t) + w^T(t)B - w^T(t)p, \tag{4.32}$$

with $B^T = \begin{bmatrix} B_0 & B_1 & \cdots & B_{n-1} \end{bmatrix}$, for all $t \in [t_0, T]$, w^T is a vector of a fundamental solution and p is a particular solution of the ODE. From the arithmetic on interactive fuzzy numbers (see Theorem 4.1), one obtains

$$(S_t)_L(B) \quad = \quad y_p(t) - w^T(t)p + w^T(t)qB_0 + w^T(t)r, \tag{4.33}$$

where $q^T = \begin{bmatrix} 1 & q_2 & \cdots & q_n \end{bmatrix}$ and $r^T = \begin{bmatrix} 0 & r_2 & \cdots & r_n \end{bmatrix}$.

The α-cuts of $(S_t)_L(B)$, for all $\alpha \in [0,1]$, are given by [77]

$$[(S_t)_L(B)]^\alpha = y_p(t) - w^T(t)p + w^T(t)r + \gamma_t[B_0]^\alpha, \tag{4.34}$$

where $\gamma_t = w^T(t)q = \sum_{i=1}^{n} w_i(t)q_i \in \mathbb{R}$ and $\gamma_t[B_0]^\alpha$ is given as follows

$$\gamma_t[B_0]^\alpha = \begin{cases} [\gamma_t(b_0)^-_\alpha, \gamma_t(b_0)^+_\alpha], & \text{if} \quad \gamma_t \geqslant 0, \\ [\gamma_t(b_0)^+_\alpha, \gamma_t(b_0)^-_\alpha], & \text{if} \quad \gamma_t < 0. \end{cases}$$

The next theorem establishes that y_L and $(S_t)_L$, given by (4.25) and (4.35), are the same solution to the higher order fuzzy differential equation, where the additional conditions are given by linear interactive fuzzy numbers.

Theorem 4.4. *Consider a higher order differential equation with constant coefficients of the form (4.12). If $B_0, \ldots, B_{n-1}$ are the n additional conditions (initial or boundary) of (4.12), then the fuzzy solutions y_L and $(S_t)_L$ given by (4.25) and (4.35), respectively, are equal.*

Proof. Let $t \in [t_0, T]$. Then,

$$\begin{aligned}
\Big[y_L(t)\Big]^\alpha &= \Big[\min\big\{h(t,\alpha,0), h(t,\alpha,1)\big\}, \max\big\{h(t,\alpha,0), h(t,\alpha,1)\big\}\Big] \\
&= \Big[\min\big\{Y_1(t) + Y_2(t,\alpha), Y_1(t) + Y_3(t,\alpha)\big\}, \\
&\qquad \max\big\{Y_1(t) + Y_2(t,\alpha), Y_1(t) + Y_3(t,\alpha)\big\}\Big] \\
&= Y_1(t) + \Big[\min\big\{Y_2(t,\alpha), Y_3(t,\alpha)\big\}, \max\big\{Y_2(t,\alpha), Y_3(t,\alpha)\big\}\Big] \\
&= y_p(t) - y^T(t)M^{-1}p + \Big[\min\big\{y^T(t)M^{-1}b^-_\alpha, y^T(t)M^{-1}b^+_\alpha\big\}, \\
&\qquad \max\big\{y^T(t)M^{-1}b^-_\alpha, y^T(t)M^{-1}b^+_\alpha\big\}\Big] \\
&= y_p(t) - w^T(t)p + \Big[\min\big\{w^T(t)(q(b_0)^-_\alpha + r), w^T(t)(q(b_0)^+_\alpha + r)\big\}, \\
&\qquad \max\big\{w^T(t)(q(b_0)^-_\alpha + r), w^T(t)(q(b_0)^+_\alpha + r)\big\}\Big] \\
&= y_p(t) - w^T(t)p + w^T(t)r + \Big[\min\big\{\gamma_t(b_0)^-_\alpha, \gamma_t(b_0)^+_\alpha\big\}, \\
&\qquad \max\big\{\gamma_t(b_0)^-_\alpha, \gamma_t(b_0)^+_\alpha\big\}\Big] \\
&= y_p(t) - w^T(t)p + w^T(t)r + \gamma_t\big[B_0\big]^\alpha \\
&= \Big[(S_t)_L(B)\Big]^\alpha, \quad \forall \alpha \in [0,1].
\end{aligned} \tag{4.35}$$

□

Theorem 4.5 reveals that the solution via sup-J extension is associated with an arithmetic for interactive fuzzy numbers. The next section associates the fuzzy solution y_L with the one produced by Fréchet derivative.

4.2.3 Fréchet Derivative for Fuzzy Functions

The Fréchet derivative for fuzzy functions is one type of derivative that considers the concept of interactivity [52]. The focus here is to compare the solution via Fréchet derivative and the solution proposed in this chapter, since both of them embraces the relationship of interactivity.

Esmi *et al.* [52] proposed a method to produce a solution of fuzzy differential equations by means of Fréchet derivative. Before present the similarities of this approach with ours, let us present the method given by them.

Let $y_{\mathcal{F}}$ be a fuzzy function given by

$$y_{\mathcal{F}}(t) = q(t)B_0 + r(t), \tag{4.36}$$

where $q, r : \mathbb{R} \to \mathbb{R}$. The n-th order Fréchet derivative of (4.36) at t is given as follows

$$y_{\mathcal{F}}^{(n)}(t) = q^{(n)}(t)B_0 + r^{(n)}(t). \tag{4.37}$$

Thus the fuzzy function $y_{\mathcal{F}}$ is a solution of FDE given by (4.19), if $y_{\mathcal{F}}$ satisfies

$$\begin{aligned} &(I) \quad (y_{\mathcal{F}})^{(n)}(t) + \sum_{i=0}^{n-1} k_i (y_{\mathcal{F}})^{(i)}(t) = f(t) \quad \text{and} \\ &(II) \quad y_{\mathcal{F}}^{(i_k)}(t_i) = B_{\{i,k\}}. \end{aligned}$$

The condition (II) and Equation (4.37) imply $B_{\{i,k\}} = q^{(i_k)}(t_i)B_0 + r^{(i_k)}(t_i)$. Since $B_{\{i,k\}} = q_{\{i,k\}}B_0 + r_{\{i,k\}}$, the following conditions are obtained

$$q^{(i_k)}(t_i) = q_{\{i,k\}} \quad \text{and} \quad r^{(i_k)}(t_i) = r_{\{i,k\}}.$$

A combination of condition (II) and Equation (4.37) result in the following FDE

$$(q^{(n)}(t) + \sum_{i=0}^{n-1} k_i q^{(i)}(t))B_0 + r^{(n)}(t) + \sum_{i=0}^{n-1} k_i r^{(i)}(t) = f(t). \tag{4.38}$$

Since f is a classical function, Equation (4.38) implies that

$$q^{(n)}(t) + \sum_{i=0}^{n-1} k_i q^{(i)}(t) = 0 \quad \text{and} \quad r^{(n)}(t) + \sum_{i=0}^{n-1} k_i r^{(i)}(t) = f(t),$$

which lead us to the following deterministic problems

$$\begin{cases} q^{(n)}(t) + \sum_{i=0}^{n-1} k_i q^{(i)}(t) = 0 \\ q^{(i_k)}(t_i) = q_{\{i,k\}} \in \mathbb{R} \end{cases} \tag{4.39}$$

and

$$\begin{cases} r^{(n)}(t) + \sum_{i=0}^{n-1} k_i r^{(i)}(t) = f(t) \\ r^{(i_k)}(t_i) = r_{\{i,k\}} \in \mathbb{R} \end{cases} . \tag{4.40}$$

Thus the classical solutions of (4.39) and (4.40) are given by

$$q(t) = y^T(t)M^{-1}q, \quad \text{and} \quad r(t) = y_p(t) - y^T(t)M^{-1}p + y^T(t)M^{-1}r, \tag{4.41}$$

where $y^T(t) = (y_i(t))$, $q = (q_{\{i,k\}})$, $r = (r_{\{i,k\}})$ and the vector p is the particular solution and the matrix M is composed by the fundamental solution of the ODE.

Therefore, Equations (4.36) and (4.41) ensure that the fuzzy solution of (4.19), via Fréchet derivative, is the fuzzy function $y_{\mathcal{F}}$ given by

$$\begin{aligned} y_{\mathcal{F}} &= q(t)B_0 + r(t) \\ &= y^T(t)M^{-1}qB_0 + y_p(t) - y^T(t)M^{-1}p + y^T(t)M^{-1}r, \end{aligned} \tag{4.42}$$

where q and r are solutions of (4.39) and (4.40), respectively.

The next theorem shows that the Fréchet derivative produces a fuzzy solution that is equal to the solution $(S_t)_L$.

Theorem 4.5. *Consider a higher order differential equation with constant coefficients of the form (4.12). If $B_0, \ldots, B_{n-1}$ are the n additional conditions (initial or boundary) of (4.12), then the fuzzy solutions $y_{\mathcal{F}}$ and $(S_t)_L$ given by (4.42) and (4.35), respectively, are equal.*

Proof. Let $t \in [t_0, T]$.

$$\begin{aligned} \left[y_{\mathcal{F}}(t)\right]^\alpha &= q(t)\left[B_0\right]^\alpha + r(t) \\ &= y^T(t)M^{-1}q\left[B_0\right]^\alpha + y_p(t) - y^T(t)M^{-1}p + y^T(t)M^{-1}r \\ &= y_p(t) - w^T(t)p + w^T(t)r + w^T(t)q\left[B_0\right]^\alpha \\ &= y_p(t) - w^T(t)p + w^T(t)r + \gamma_t\left[B_0\right]^\alpha \\ &= \left[(S_t)_L(B)\right]^\alpha, \quad \forall \alpha \in [0,1]. \end{aligned} \tag{4.43}$$

□

A combination of Equations (4.35) and (4.43) reveals the following

$$y_L(t) = (S_t)_L(B) = y_{\mathcal{F}}(t), \quad \forall t \in [t_0, T]. \tag{4.44}$$

This means that in the case where the additional conditions of the FDE (4.19) are given by linearly interactive fuzzy numbers, the solution produced by the sup-J extension is in fact associated with the notion of fuzzy derivative.

One can observe that in the case where $B_0, \ldots, B_{n-1}$ are symmetric, even though there is more than one JPD that makes these fuzzy numbers interactive, the result provided by Theorem 4.5 does not dependent on the choice of J_L. Indeed, the values of q_i and r_i for both J_L and the additional conditions of systems (4.39) and (4.40) are fixed, once the JPD is chosen.

Table 2 presents a brief summary of these fuzzy solutions. Table 3 presents the equivalence between these methodologies in the case where the additional condition B_0 is symmetric. Table 4 presents these equivalences when B_0 is non-symmetric.

Table 2 – Solutions for higher order fuzzy differential equations.

Solution	Notation	Equation
Sup-J extension	y_L	(4.21)
Zadeh's extension	$\widehat{y}$	(4.27)
Interactive arithmetic	$(S_t)_L$	(4.34)
Fréchet derivative	$y_{\mathcal{F}}$	(4.42)

Solutions via sup-J extension, Zadeh's extension, Interactive arithmetic and Fréchet derivative. Source: [53].

Table 3 – Fuzzy solutions when the additional condition is symmetric.

Solutions	Equivalences
Sup-J/Interactive Arithmetic/Fréchet	$y_L = (S_t)_L = y_{\mathcal{F}}$
Sup-J/Zadeh	$\widehat{y} = conv(\cup y_J)$

Comparison of the methods in the case where the additional condition is symmetric, where $\cup y_J$ stands for $\bigcup_{i=1}^{n} y_{J_i}$. Source: [53].

Table 4 – Fuzzy solutions when the additional condition is non-symmetric.

Solutions	Equivalences
Sup-J/Interactive Arithmetic/Fréchet	$y_L = (S_t)_L = y_{\mathcal{F}}$
Sup-J/Zadeh	$y_L \subseteq \widehat{y}$

Comparison of the methods in the case where the additional condition is non-symmetric. Source: [53].

The proposed method solves problems that can be given by linear systems as (4.16). In particular this thesis deals with initial value problem, where the initial conditions are given by linear interactive fuzzy numbers. For boundary conditions a similar method can be made, as one can see in [53]. Recall that this approach can be used in optimization problems as well, as Pinto *et al.* proposed [115, 114]. Chapter 7 presents an application in a physical problem that involves linear correlations among the position, velocity and acceleration of a particle, in order to illustrate this method.

The completely correlated fuzzy numbers have interesting properties. Several authors use this type of interactivity in problems that take this notion into account. However, this concept requires that the membership function of these interactive fuzzy numbers have the same shape. The next chapter produces a joint possibility distribution that does not require any restrictions of the shape of the involved fuzzy numbers.

4.3 Conclusion

This chapter presented one type of interactivity, which is widely used in the literature, called completely correlation. The arithmetic for this subclass of fuzzy numbers were established and a generalization of this concept was provided. Solutions for FIVPs were yielded and they were associated with the ones given by Fréchet derivative and the interactive arithmetic. This chapter ended by presenting a comparison among the solutions via linear interactivity, Zadeh's extension, interactive arithmetic and Fréchet derivative.

5 The g-difference as a Particular Case of an Interactive Difference

This chapter presents a family of joint possibility distributions denoted by J_γ, which was proposed by Esmi *et al.* [50, 54]. This family can be applied to every pair of fuzzy numbers. Moreover, they showed that one can have a certain control of the Pompeiu-Hausdorff norm of the interactive sum via this family of JPD's.

Later, Sussner *et al.* [136] proposed the concept of translated fuzzy numbers in order to "control" the width of this interactive sum as well. The width of a fuzzy number is associated with the uncertainty that it models. Consequently, controlling the width of the arithmetic operations via J_γ means controlling uncertainties in different ways, using the same fuzzy numbers.

This thesis used J_γ to provide a numerical solution for fuzzy differential equations, which is our fourth contribution. This numerical solution is based on extending the classic arithmetic operations of the Euler and Runge Kutta methods via sup-J extension principle. For each element of this family, there will be a numerical solution for the FDE. The proposed numerical method can be used in every n-dimensional initial value problem, in contrast to the other methods based on JPDs, J_L for example.

This chapter shows that this numerical solution is more specific than the solution using Zadeh's extension principle [146]. In order to illustrate this fact and the aforementioned ones, several applications of this method are presented, such as in epidemiology [142, 149, 145, 143, 144, 141, 146] and chemistry [147, 139] problems (see Chapter 7). In these cases, the numerical solutions will simulate different behaviours, with respect to the uncertainty along time, of the disease of a population and chemical reactions.

In addition, as the fifth contribution, this chapter proves that the generalized difference ($-_g$) is a particular case of interactive difference [150]. This means that the g-derivative (as well as gH- and H-derivatives) is interactive. This chapter is based on references [78, 50, 136, 150, 149, 146, 142, 143, 141, 144, 147, 139].

5.1 Joint Possibility Distribution J_γ

Esmi *et al.* [50, 54, 78] defined a parametrized family of joint possibility distributions J_γ, $\gamma \in [0, 1]$. In contrast to the joint possibility distribution J_L (cf. Equation (4.7)), the family of JPD's J_γ can be applied to every pair of fuzzy numbers. These joint possibility distributions are defined as follows. Given $A_1, A_2 \in \mathbb{R}_{\mathcal{F}}$, considers the auxiliary functions $g^i_\wedge$, $g^i_\vee$ and v^i defined by

$$g^i_\wedge(z,\alpha) = \bigwedge_{w \in [A_{3-i}]^\alpha} |w+z|, \tag{5.1}$$

$$g^i_\vee(z,\alpha) = \bigvee_{w \in [A_{3-i}]^\alpha} |w+z| \tag{5.2}$$

and

$$v^i(z,\alpha,\gamma) = (1-\gamma)g^i_\wedge(z,\alpha) + \gamma g^i_\vee(z,\alpha), \tag{5.3}$$

for all $z \in \mathbb{R}$, $\alpha \in [0,1]$, $\gamma \in [0,1]$ and $i \in \{1,2\}$.

Note that the functions $g^i_\wedge$ and $g^i_\vee$ calculate, respectively, the minimum and maximum absolute values of the sum between the elements in $[A_1]^\alpha$ and $[A_2]^\alpha$. Also note that the function $v^i(z,\alpha,\gamma)$, with respect to z, assumes greater values as long as γ increases in $[0,1]$. Also, consider the sets R^i_α and $L^i(z,\alpha,\gamma)$, given by

$$R^i_\alpha = \begin{cases} \{a^-_{i_\alpha}, a^+_{i_\alpha}\} & \text{if } \alpha \in [0,1) \\ [A_i]^1 & \text{if } \alpha = 1 \end{cases} \tag{5.4}$$

and

$$L^i(z,\alpha,\gamma) = [A_{3-i}]^\alpha \cap [-v^i(z,\alpha,\gamma) - z, v^i(z,\alpha,\gamma) - z]. \tag{5.5}$$

Finally, J_γ is given by

$$J_\gamma(x_1,x_2) = \begin{cases} A_1(x_1) \wedge A_2(x_2), & \text{if } (x_1,x_2) \in P(\gamma) \\ 0 \quad , & \text{otherwise} \end{cases} \tag{5.6}$$

where

$$P(\gamma) = \bigcup_{i=1}^{2} \bigcup_{\alpha \in [0,1]} P^i(\gamma,\alpha) \tag{5.7}$$

and for all $i \in \{1,2\}$, $\gamma \in [0,1]$ e $\alpha \in [0,1]$, $P^i(\gamma,\alpha)$ is defined by

$$P^i(\gamma,\alpha) = \{(x_1,x_2) : x_i \in R^i_\alpha \text{ and } x_{3-i} \in L^i(x_i,\alpha,\gamma)\}. \tag{5.8}$$

Since the function v^i is increasing with respect to γ, the set L^i is a larger interval for greater values of γ. This implies that for greater values of $\gamma \in [0,1]$, the greater the number of pairs (x_1,x_2) such as $J_\gamma(x_1,x_2) > 0$. In this case the "level" of interactivity is measure by the parameter γ, in the following sense: the smaller the value of $\gamma \in [0,1]$, the greater the interactivity.

One can prove that for $\gamma = 1$, $J_1 = J_\wedge$ [50], that is, A_1 and A_2 are non-interactive. In this case the sup-J extension principle boils down to the Zadeh extension principle.

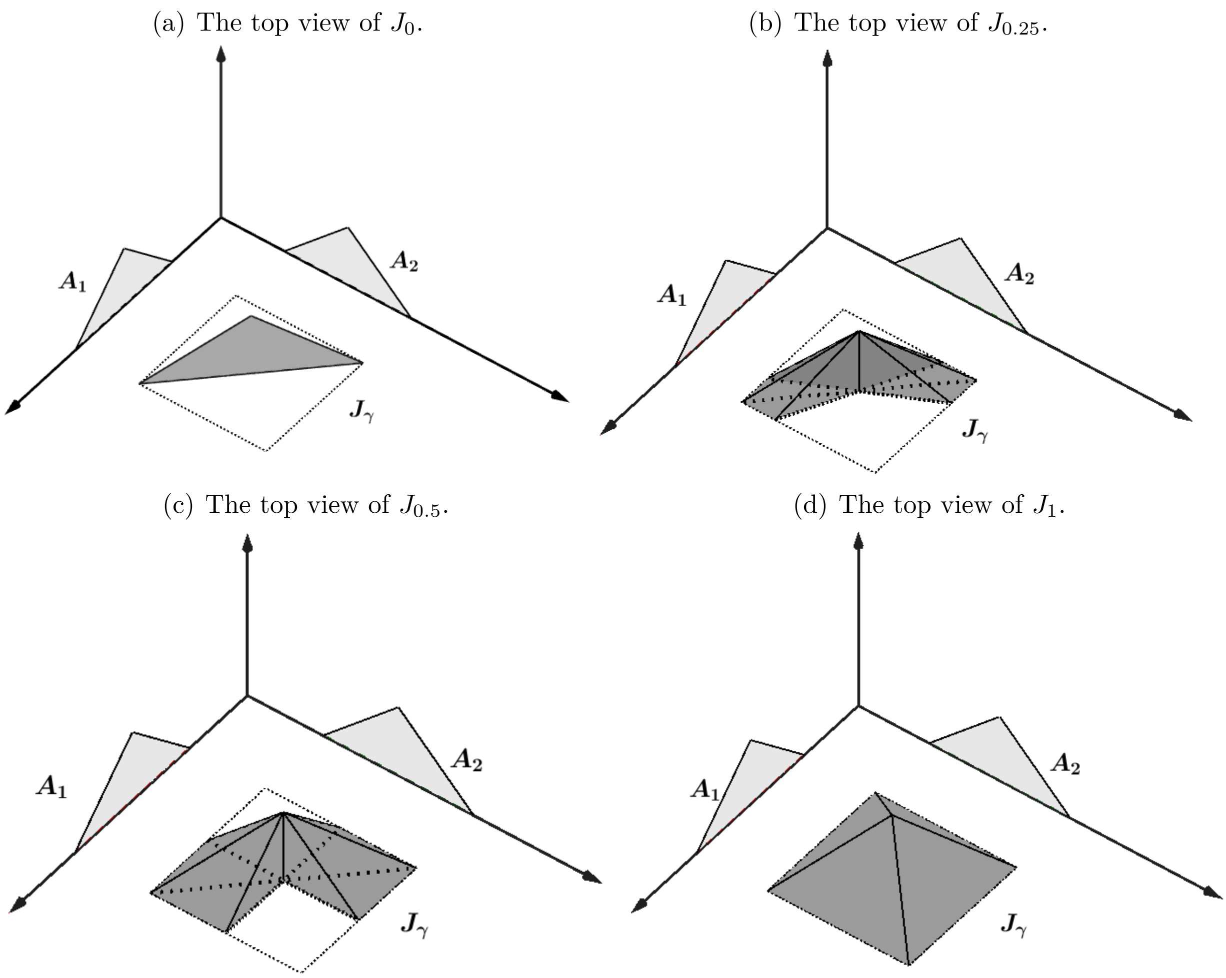
Theorem 5.1. *[50] Let $A_1, A_2 \in \mathbb{R}_{\mathcal{F}}$. If $0 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant 1$, then*

1. $J_0 \subseteq J_{\gamma_1} \subseteq J_{\gamma_2} \subseteq J_1$;
2. $||A_1 \otimes_{J_0} A_2||_{\mathcal{F}} \leqslant ||A_1 \otimes_{J_{\gamma_1}} A_2||_{\mathcal{F}} \leqslant ||A_1 \otimes_{J_{\gamma_2}} A_2||_{\mathcal{F}} \leqslant ||A_1 \otimes_{J_1} A_2||_{\mathcal{F}}$, *where* $\otimes \in \{+, -, \cdot, \div\}$

Geometrically, the larger the value of γ, the bigger the fuzzy set J_γ, as illustrated in the following example.

Example 5.1. *Let $A_1 = A_2 = (-1; 0; 1) \in \mathbb{R}_{\mathcal{F}}$. The joint possibility distribution J_γ, where $\gamma \in \{0, 0.25, 0.5, 1\}$, between A_1 and A_2 are graphically represent in Figure 31.*

Figure 31 – Graphical representation of the joint possibility distribution J_γ given as in Example (5.1).



The light gray triangles represent the triangular fuzzy numbers $A_1 = (-1; 0; 1)$ and $A_2 = (-1; 0; 1)$. The dark grey region represents the joint possibility distribution J_γ. Source: [142].

Note that $J_0 \subseteq J_{0.25} \subseteq J_{0.5} \subseteq J_1$. Moreover, the joint possibility distribution obtained by $\gamma = 1$ is the $J_{\wedge}$. Also, the JPD J_0 resembles one of the cases of J_L (see Figure 29).

The next example illustrates that J_γ can be applied to every pair of fuzzy numbers, in contrast to the joint possibility distribution J_L.

Example 5.2. *Let $A_1 = (-2;-1;1;2) \in \mathcal{F}_{Tp}$ and $A_1 = (-1;0;1) \in \mathcal{F}_{Tr}$. The joint possibility distribution J_γ, where $\gamma \in \{0, 0.25, 0.5, 0.75, 1\}$, between A_1 and A_2 are graphically represent in Figure 32.*

Note that in Example 5.2 the fuzzy numbers $A_1 = (-2;-1;1;2)$ and $A_2 = (-1;0;1)$ are not interactive with respect to the joint possibility distribution J_L, since their membership function can not be associated linearly. However, $A_1 = (-2;-1;1;2)$ and $A_2 = (-1;0;1)$ are interactive with respect to J_γ, for all $0 \leqslant \gamma < 1$.

One can observe that J_0 is a more general JPD than the distribution J_L, since it can be applied to every pair of fuzzy numbers. Even though J_0 resembles J_L, in the case where A_1 and A_2 are triangular fuzzy numbers (see Figures 31 and 30), the JPD J_L may produce two types of interactivity, the positive and the negative, and the JPD J_0 can only reproduce one of them. Moreover, the JPD J_0 can not be used to describe the α-cuts of A_1 in terms of the α-cuts of A_2, as it is possible for JPD J_L.

Although the width of a fuzzy number is limited by two times its norm [136], the approach proposed by [54] did not produce results about the width of the numerical solutions for FIVPs. In order to have a better control of the width, Sussner *et al.* [136] introduced the concept of translated fuzzy numbers.

Definition 5.1. *The translation of $A \in \mathbb{R}_{\mathcal{F}}$ by $k \in \mathbb{R}$ is defined as the fuzzy number $A^{(k)}$, whose membership function is given by*

$$A^{(k)}(x) = A(x+k), \ \forall x \in \mathbb{R}. \tag{5.9}$$

By the above definition it is possible to construct a family of JPDs, according to Theorem 5.2.

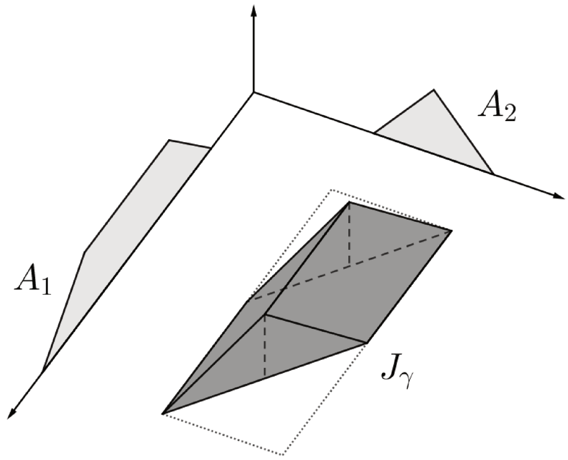
Theorem 5.2. *[136] Given $A_1, A_2 \in \mathbb{R}_{\mathcal{F}}$ and $c = (c_1, c_2) \in \mathbb{R}^2$, consider the translated fuzzy numbers $A_1^{(c_1)}, A_2^{(c_2)} \in \mathbb{R}_{\mathcal{F}}$. Let $\gamma \in [0,1]$ and $\tilde{J}_\gamma$ be a joint possibility distribution between $A_1^{(c_1)}$ and $A_2^{(c_2)}$. If J_γ^c is the following fuzzy relation given by*

$$J_\gamma^c(x_1, x_2) = \tilde{J}_\gamma(x_1 - c_1, x_2 - c_2), \ \forall (x_1, x_2) \in \mathbb{R}^2 \tag{5.10}$$

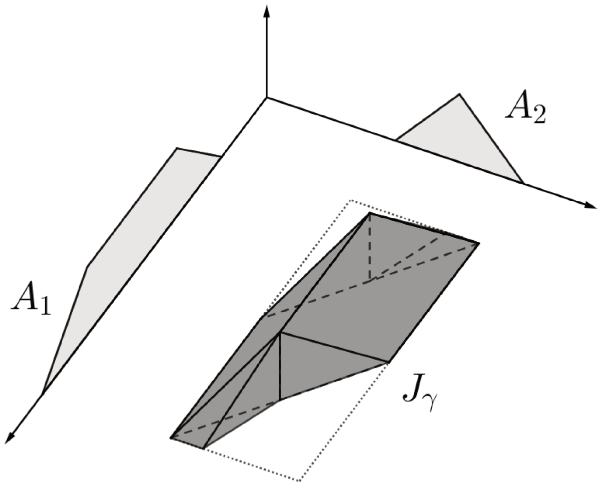
then J_γ^c is a JPD for A_1 and A_2, furthermore, $(A_1 +_\gamma^c A_2) \in \mathbb{R}_{\mathcal{F}}$, where $+_\gamma^c$ represents the sum via sup-J extension principle.

Figure 32 – Graphical representation of the joint possibility distribution J_γ given as in Example (5.2).

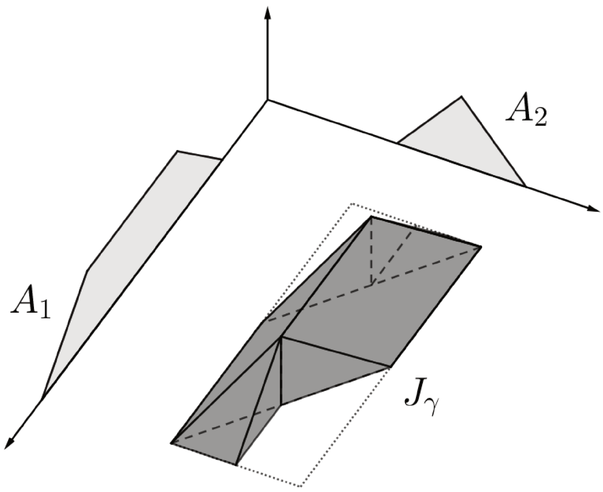
(a) The top view of J_0.



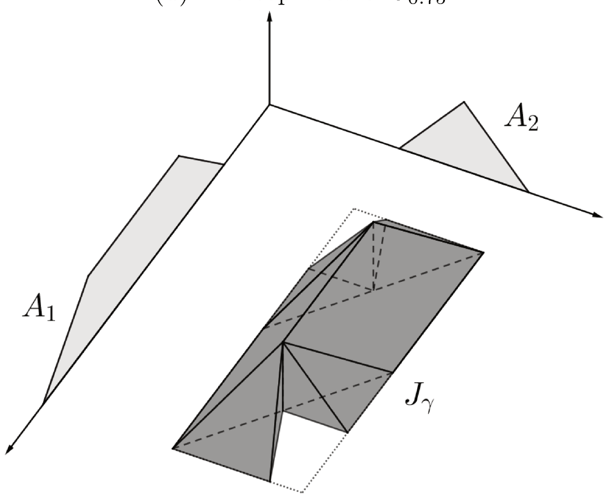
(b) The top view of $J_{0.25}$.



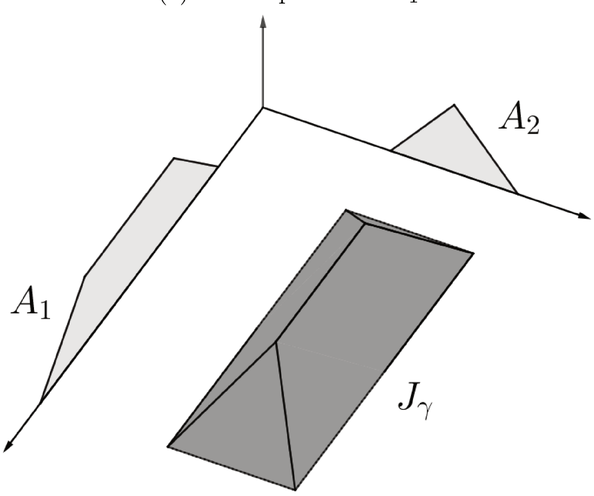
(c) The top view of $J_{0.5}$.



(d) The top view of $J_{0.75}$.



(e) The top view of J_1.



The light gray triangles represent the fuzzy numbers $A_1 = (-2; -1; 1; 2)$ and $A_2 = (-1; 0; 1)$. The dark grey region represents the joint possibility distribution J_γ. Source: Author.

From the JPD, given by (5.10), the following statements are equivalent:

Proposition 5.1. *[136, 50] Let J_γ^c be the joint possibility distribution of the fuzzy numbers A_1 and A_2. The following statements are equivalent:*

(a) $\gamma \leqslant \beta$;

(b) $J_\gamma^c \subseteq J_\beta^c$;

(c) $A_1 +_\gamma^c A_2 \subseteq A_1 +_\beta^c A_2$; *and*

(d) $width(A_1 +_\gamma^c A_2) \leqslant width(A_1 +_\beta^c A_2)$.

It is important to observe that $A_1 +_\gamma^c A_2$ and $A_1 +_\gamma A_2$ produce different results. However, one can show that $[A_1^{(c_1)} +_\gamma A_2^{(c_2)}]^\alpha - \{c_1 + c_2\} = [A_1 +_\gamma^c A_2]^\alpha$ for all $\alpha \in [0, 1]$ [136]. These equivalences ensure that the width of the sum via sup-J extension principle, with $J = J_\gamma^c$, is connected with the parameter γ in the following sense, the width of the fuzzy number obtained from the interactive sum decreases as the value of γ goes to 0.

The next section uses this family of JPDs to produce an arithmetic on interactive fuzzy numbers. Moreover, from this arithmetic a numerical solution for fuzzy differential equations will be provided.

5.2 Numerical Solution for Initial Value Problems with Interactive Fuzzy Conditions

This section focuses on fuzzy initial values problems where the initial conditions are uncertain and given by interactive fuzzy numbers via joint possibility distribution J_γ^c. From now on, in order to simplify the notation, J_γ^c will be denoted by J_γ and the arithmetic operations $\otimes_\gamma^c$ will be denoted by $\otimes_\gamma$.

There are several methods in the literature that provide numerical solutions of FDEs [3, 79, 2]. Most of them use arithmetic operators that are obtained from different joint possibility distributions. For instance, the authors of [2] used the generalized difference and standard sum in their numerical solutions. This chapter proves that the generalized difference is one type of interactive arithmetic but, as it was point out before, the standard sum is not. Consequently it is necessary to exhibit the JPD that has this property.

In contrast of these methods, our approach provides a numerical solution whose arithmetic operations are derived from the same JPD. Moreover, in the most numerical solutions [105, 91, 64, 1, 3, 79, 2] the authors study the problem via α-cuts, treating the FDEs in a classical way. Our proposed method do not require the study of α-cuts.

The numerical solution proposed in this thesis is based on the classical numerical solutions of the Euler's and Runge-Kutta methods, with the arithmetic operations involved in the method adapted for interactive fuzzy numbers. Before presenting this methodology, let us briefly review the Euler's and Runge-Kutta method.

Let $y_i : \mathbb{R} \to \mathbb{R}^n$, with $i = 1, ..., n$, functions that depend on time t. Consider the following IVP composed of ODEs and initial condition

$$\begin{cases} \dfrac{dy_i}{dt} = f_i(t, y_1, y_2, ..., y_n) \\ y(t_0) = y_0 \in \mathbb{R}^n \end{cases}, \tag{5.11}$$

where f_i is a function that depends on $y_1, y_2, ..., y_n$ and t, for each $i = 1, ..., n$.

Euler's and Runge-Kutta methods consist in determining numerical solutions for (5.11). The steps of algorithms are given by

$$y_i^{(k+1)} = y_i^{(k)} + hf_i(t^{(k)}, y_1^{(k)}, ..., y_n^{(k)}), \qquad \text{(Euler)} \tag{5.12}$$

and

$$y_i^{(k+1)} = y_i^{(k)} + \frac{h}{6}(k_1^{(k)} + 2k_2^{(k)} + 2k_3^{(k)} + k_4^{(k)}), \qquad \text{(Runge-Kutta)} \tag{5.13}$$

where

$$\begin{aligned} k_1^{(k)} &= f_i(t^{(k)}, y_1^{(k)}, ..., y_n^{(k)}), \\ k_2^{(k)} &= f_i\left(t^{(k)} + \frac{h}{2}, y_1^{(k)} + \frac{h}{2}k_1^{(k)}, ..., y_n^{(k)} + \frac{h}{2}k_1^{(k)}\right), \\ k_3^{(k)} &= f_i\left(t^{(k)} + \frac{h}{2}, y_1^{(k)} + \frac{h}{2}k_2^{(k)}, ..., y_n^{(k)} + \frac{h}{2}k_2^{(k)}\right), \\ k_4^{(k)} &= f_i\left(t^{(k)} + h, y_1^{(k)} + hk_3^{(k)}, ..., y_n^{(k)} + hk_3^{(k)}\right), \end{aligned} \tag{5.14}$$

with $0 \leqslant k \leqslant N - 1$, where N is the number of partitions of the interval time divided in equally spaced intervals $[t^{(k)}, t^{(k+1)}]$ with size h and initial condition $(t^{(0)}, y_1^{(0)}, \ldots, y_n^{(0)})$.

Since the initial conditions are given by fuzzy numbers, the arithmetic operations in each iteration must be extended to operations between fuzzy numbers, for all $k > 0$. Therefore, the Euler's algorithm becomes

$$Y_i^{k+1} = Y_i^k +_\gamma hf_i(t^k, Y_1^k, ..., Y_n^k), \tag{5.15}$$

and the Runge-Kutta algorithm is given by

$$Y_i^{(k+1)} = Y_i^{(k)} +_\gamma \frac{h}{6}(K_1^{(k)} +_\gamma 2K_2^{(k)} +_\gamma 2K_3^{(k)} +_\gamma K_4^{(k)}), \tag{5.16}$$

where

$$\begin{array}{rcl} K_1^{(k)} & = & f_i(t^{(k)}, Y_1^{(k)}, ..., Y_n^{(k)}), \\ K_2^{(k)} & = & f_i\left(t^{(k)} + \dfrac{h}{2}, Y_1^{(k)} +_\gamma \dfrac{h}{2}K_1^{(k)}, ..., Y_n^{(k)} +_\gamma \dfrac{h}{2}K_1^{(k)}\right), \\ K_3^{(k)} & = & f_i\left(t^{(k)} + \dfrac{h}{2}, Y_1^{(k)} +_\gamma \dfrac{h}{2}K_2^{(k)}, ..., Y_n^{(k)} +_\gamma \dfrac{h}{2}K_2^{(k)}\right), \\ K_4^{(k)} & = & f_i\left(t^{(k)} + h, Y_1^{(k)} +_\gamma hK_3^{(k)}, ..., Y_n^{(k)} +_\gamma hK_3^{(k)}\right), \end{array} \tag{5.17}$$

where Y_i is a fuzzy number for all $i = 1, \ldots, n$.

A combination of Proposition 5.1 and Theorem 2.2 ensures that both numerical solutions (5.15) and (5.16) are more specific when there is interactivity than the solutions using the standard arithmetic, since for all $\gamma \in [0, 1]$ it follows that $J_0 \subseteq J_\gamma \subseteq J_1 = J_\wedge$. This fact is illustrated in some applications in epidemiological problems, given in Chapter 7.

The next section focuses on a particular joint possibility distribution of this family, namely J_0. The interactive sum, obtained from J_0, produces a fuzzy number with smaller Pompeiu-Hausddorf norm. This is not the only interesting property that J_0 has. The next section provides the fifth result of this thesis, which is, the difference based on J_0 extends the g-difference, which means that the g-difference is interactive [150].

5.3 The joint possibility distribution J_0

This section provides new results about the joint possibility distribution J_0. Let us first present some preliminary results. The next proposition describes some properties of the joint possibility distribution J_0 [50].

Proposition 5.2. *Let J_0 be the joint possibility distribution given by (5.6), with $\gamma = 0$. Then*

(a) $A_1 +_0 A_2 \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$*;*

(b) $||A_1 +_0 A_2||_\mathcal{F} \leqslant ||A_1 +_J A_2||_\mathcal{F}$ *for every joint possibility distribution J of A_1 and A_2 such that $A_1 +_J A_2$ is a fuzzy number.*

Using the idea of translated fuzzy numbers, a specific joint possibility distribution I_{k_1,k_2} is defined for a given pair of fuzzy numbers in $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$.

Theorem 5.3. *[150] Let $A_1, A_2 \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ and let $k_i \in \mathbb{R}$ for $i = 1, 2$. Consider the joint possibility distribution J_0 of $A_1^{(k_1)}$ and $-A_2^{(k_2)}$ given by (5.9). The fuzzy relation I defined by*

$$I_{k_1,k_2}(x_1, x_2) = J_0(x_1 - k_1, k_2 - x_2), \ \forall (x_1, x_2) \in \mathbb{R}^2 \tag{5.18}$$

has the following properties:

(a) I_{k_1,k_2} *is a joint possibility distribution of* A_1 *and* A_2*;*

(b) $A_1 -_{I_{k_1,k_2}} A_2 = \left(A_1^{(k_1)} +_0 (-A_2^{(k_2)})\right) + k_1 - k_2 \in \mathbb{R}_{\mathcal{F}_C}$.

Proof. Let us prove that A_1 and A_2 are marginal distributions of I_{k_1,k_2} given by (5.18).

One the one hand,

$$A_1(x) = A_1^{(k_1)}(x - k_1) = \bigvee_{y \in \mathbb{R}} J_0(x - k_1, y) = \bigvee_{y \in \mathbb{R}} J_0(x - k_1, k_2 - y) = \bigvee_{y \in \mathbb{R}} I_{k_1,k_2}(x, y),$$

for all $x \in \mathbb{R}$.

On the other hand, the following equations are satisfied for all $y \in \mathbb{R}$:

$$\begin{aligned} A_2(y) &= A_2^{(k_2)}(y - k_2) = -A_2^{(k_2)}(k_2 - y) = \bigvee_{x \in \mathbb{R}} J_0(x, k_2 - y) \\ &= \bigvee_{x \in \mathbb{R}} J_0(x - k_1, k_2 - y) = \bigvee_{x \in \mathbb{R}} I_{k_1,k_2}(x, y). \end{aligned}$$

Therefore, I_{k_1,k_2} is a joint possibility distribution for A_1 and A_2.

(b) Let $i \in \{1, 2\}$ and let $a_i^+, a_i^-, (a_i^{(k_i)})^+, (a_i^{(k_i)})^- : [0, 1] \to \mathbb{R}$ be the functions that map $\alpha \in [0, 1]$ to $(a_i)_\alpha^+$, $(a_i)_\alpha^-$, $(a_i^{(k_i)})_\alpha^+$, and $(a_i^{(k_i)})_\alpha^-$, respectively. Since a_i^+ and a_i^- are continuous and since $(a_i^{(k_i)})^+ = (a_i)^+ - k_i$ and $(a_i^{(k_i)})^- = (a_i)^- - k_i$, the functions $(a_i^{(k_i)})^+$ and $(a_i^{(k_i)})^-$ are continuous as well. Thus, $A_i^{(k_i)} \in \mathbb{R}_{\mathcal{F}_C}$, for $i = 1, 2$. Proposition 5.2 implies that $\left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right) \in \mathbb{R}_{\mathcal{F}_C}$ and, consequently, $D := \left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right) + k_1 - k_2 \in \mathbb{R}_{\mathcal{F}_C}$.

Also,

$$\begin{aligned} D(y - (k_1 - k_2)) &= \left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right)(y) = \bigvee_{x_1 + x_2 = y} J_0(x_1, x_2) \\ &= \bigvee_{x_1 + x_2 = y} I_{k_1,k_2}(x_1 + k_1, k_2 - x_2), \end{aligned}$$

for every $y \in \mathbb{R}$.

By taking $z_1 = x_1 + k_1$ and $z_2 = k_2 - x_2$, it follows

$$D(y - (k_1 - k_2)) = \bigvee_{z_1 - z_2 = y - (k_1 - k_2)} I(z_1, z_2) = (A_1 -_I A_2)(y - (k_1 - k_2)).$$

Therefore, $D = A_1 -_{I_{k_1,k_2}} A_2$.

$\square$

Remark 5.1. *If $k_1 = k_2$, then $A_1 -_{I_{k_1,k_2}} A_2 = A_1 +_0 (-A_2)$. This expression resembles the definition of the difference of real numbers in terms of the additive inverse.*

The next corollary is a consequence of Theorem 5.3 and of Proposition 5.2.

Corollary 5.1. *For every $A \in \mathbb{R}_{\mathcal{F}_C}$, it follows that*

$$A -_{I_{k_1,k_2}} A = 0.$$

Proof. Theorem 5.3 implies that

$$A -_{I_{k_1,k_2}} A = (A +_0 (-A)) + (k - k) = A +_0 (-A).$$

Thus

$$A +_{J_{\{-1,0\}}} (-A) = 0,$$

since A and $-A$ are completely correlated with respect to $J_{\{-1,0\}}$ [32].

Item (c) of Proposition 5.2 ensures that

$$0 \leqslant ||A +_0 (-A)||_{\mathcal{F}} \leqslant ||A +_{J_{\{-1,0\}}} (-A)||_{\mathcal{F}} = 0$$

Therefore, $A +_0 (-A) = A -_{I_{k_1,k_2}} A = 0$. □

The next example illustrates the interactive difference, given by (5.18), and it compares I_{k_1,k_2}-difference with the gH-difference.

Example 5.3. *Let $A_1 = (-1; 0; 1)$ and $A_2 = (-2; 0; 2)$. Since $width([A_1]^\alpha) = 2 - 2\alpha < 4 - 4\alpha = width([A_2]^\alpha)$, for all $\alpha \in [0, 1]$, one obtains $A_1 -_{gH} A_2 = X = (-1; 0; 1)$.*

On the other hand, let $k_1 = 0$ and $k_2 = 0$. From Theorem 5.3, it follows

$$A_1 -_{I_{k_1,k_2}} A_2 = \left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right) + \{k_1 - k_2\}.$$

Hence

$$\left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right)(z) = \bigvee_{x_1+x_2=z} A_1(x_1) \wedge (-A_2)(x_2),$$

where $(x_1, x_2) \in P = \bigcup_{\alpha\in[0,1]} P^1(\alpha) \cup P^2(\alpha)$, with

$$P^1(\alpha) = \{(x_1, x_2) \in \{-1 + \alpha\} \times \{1 - \alpha\}\} \cup \{(x_1, x_2) \in \{1 - \alpha\} \times \{-1 + \alpha\}\}$$

and

$$P^2(\alpha) = \{(x_1, x_2) \in \{-1 + \alpha\} \times \{2 - 2\alpha\}\} \cup \{(x_1, x_2) \in \{1 - \alpha\} \times \{-2 + 2\alpha\}\}.$$

Therefore, $A_1 +_0 (-A_2) = (-1;0;1)$. *Hence*

$$A_1 -_{I_{k_1,k_2}} A_2 = (-1;0;1) + \{0\} = (-1;0;1) = A_1 -_{gH} A_2.$$

Now, let $A_1 = (-3;0;3)$ *and* $A_2 = (-1;0;1)$. *Since* $width([A_1]^\alpha) = 3 - 3\alpha > 1 - \alpha = width([A_2]^\alpha)$, *for all* $\alpha \in [0,1]$, *one obtains* $A_1 -_{gH} A_2 = X = (-2;0;2)$.

On the other hand, considering $k_1 = 0$ *and* $k_2 = 0$, *from Theorem 5.3 it follows*

$$(A_1 -_{I_{k_1,k_2}} A_2)(z) = \left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right)(z) = \bigvee_{x_1+x_2=z} A_1(x_1) \wedge (-A_2)(x_2),$$

where $(x_1,x_2) \in P = \bigcup_{\alpha\in[0,1]} P^1(\alpha) \cup P^2(\alpha)$, *with*

$$P^1(\alpha) = \{(x_1,x_2) \in \{-1+\alpha\} \times \{3-3\alpha\}\} \cup \{(x_1,x_2) \in \{1-\alpha\} \times \{-3+3\alpha\}\}$$

and

$$P^2(\alpha) = \{(x_1,x_2) \in \{-1+\alpha\} \times \{1-\alpha\}\} \cup \{(x_1,x_2) \in \{1-\alpha\} \times \{-1+\alpha\}\}.$$

Therefore, $A_1 +_0 (-A_2) = (-2;0;2)$. *Hence*

$$A_1 -_{I_{k_1,k_2}} A_2 = (-2;0;2) + \{0\} = (-2;0;2) = A_1 -_{gH} A_2.$$

Example 5.3 suggests that if the translations k_i are given by $k_i = 0.5((a_i)_1^- + (a_i)_1^+)$ for $i = 1,2$, then the gH- and I_{k_1,k_2}-differences lead to the same results. The next section shows that this statement holds true. More precisely, it proves that if the gH-difference exists, then $A -_{gH} B = A -_{I_{k_1,k_2}} B$, where $k_i = 0.5((a_i)_1^- + (a_i)_1^+)$ for $i = 1,2$ and for all $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$. Henceforth, the joint possibility distribution I used here is given by

$$I = I_{k_1,k_2}, \tag{5.19}$$

where $k_1 = 0.5((a_1)_1^- + (a_1)_1^+)$ and $k_2 = 0.5((a_2)_1^- + (a_2)_1^+)$.

The following example compares the I-interactive difference, given by (5.19), with the standard and the g-difference.

Example 5.4. *Let* $A_1 = (1;2;3)$ *and* $A_2 = (1;3;4)$. *From Theorem 5.3, it follows that* $A_1 -_I A_2 = \left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right) + \{k_1 - k_2\}$, *where* $k_1 = 2$ *and* $k_2 = 3$. *Hence*

$$\left(A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right)\right)(z) = \bigvee_{x_1+x_2=z} A_1^{(k_1)}(x_1) \wedge \left(-A_2^{(k_2)}\right)(x_2),$$

where $(x_1,x_2) \in P = \bigcup_{\alpha\in[0,1]} P^1(\alpha) \cup P^2(\alpha)$, *with*

$$P^1(\alpha) = \{(x_1,x_2) \in \{-1+\alpha\} \times \{1-\alpha\}\} \cup \{(x_1,x_2) \in \{1-\alpha\} \times [-2+2\alpha, 0]\}$$

and

$$P^2(\alpha) = \{(x_1, x_2) \in \{1-\alpha\} \times \{-2+2\alpha\}\} \cup \{(x_1, x_2) \in \{1-\alpha\} \times \{-1+\alpha\}\}.$$

Therefore, $A_1^{(k_1)} +_0 \left(-A_2^{(k_2)}\right) = (0;0;1)$. *Hence*

$$A_1 -_I A_2 = (0;0;1) + \{2-3\} = (-1;-1;0).$$

Note that

$$[A_1 -_g A_2]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} \min\{-\beta, -1\}, \bigvee_{\beta \geqslant \alpha} \max\{-\beta, -1\}\right] = [-1, -\alpha].$$

The standard, generalized, and I-interactive differences between A_1 and A_2 satisfy

$$A_1 -_g A_2 = A_1 -_I A_2 = (-1;-1;0) \subset (-3;-1;2) = A_1 - A_2$$

which implies that

$$width(A_1 -_g A_2) = width(A_1 -_I A_2) = 1 < 5 = width(A_1 - A_2),$$

and

$$||(A_1 -_g A_2)||_{\mathcal{F}} = ||(A_1 -_I A_2)||_{\mathcal{F}} = 1 < 3 = ||(A_1 - A_2)||_{\mathcal{F}}.$$

Example 5.4 suggests that the g-difference and I-interactive difference coincide. In fact this statement always holds true. The next three lemmas are used to prove Theorem 5.5. The main idea is to characterize the set $[J_0]^\alpha$ and, thereby, $[I]^\alpha$. From now on, for simplicity of notation, the sets $P(0)$ and $P^i(0,\alpha)$, given by (5.6), will be denoted respectively by P and $P^i(\alpha)$. Moreover, since for $\gamma = 0$ the function v^i (see (5.3)) is equal to $g^i_\wedge$, then the function $g^i_\wedge$ will be denoted by g_i.

Lemma 5.1. *Let $A_\beta \subseteq \mathbb{R}$, $A_\beta \neq \varnothing$, for all $\beta \in \mathbb{I}$, where $\mathbb{I}$ is an arbitrary non-empty index set. The following equalities hold true:*

$$\bigvee \left(\bigcup_{\beta \in \mathbb{I}} A_\beta\right) = \bigvee_{\beta \in \mathbb{I}} \left(\bigvee A_\beta\right) \quad \textit{and}$$
$$\bigwedge \left(\bigcup_{\beta \in \mathbb{I}} A_\beta\right) = \bigwedge_{\beta \in \mathbb{I}} \left(\bigwedge A_\beta\right)$$

Proof. Let us prove this lemma by using a few concepts of lattice theory and mathematical morphology [72]. Consider the following mappings F, G:

$$\begin{array}{rcl} F : \mathcal{P}(\mathbb{R}) & \to & (\mathbb{R}_{\pm\infty})^{\mathbb{R}} \\ A & \mapsto & f_A \end{array} \text{ and } \begin{array}{rcl} G : \mathcal{P}(\mathbb{R}) & \to & (\mathbb{R}_{\pm\infty})^{\mathbb{R}} \\ A & \mapsto & g_A \end{array}$$

where

$$f_A(x) = \begin{cases} x & , \text{ if } x \in A \\ -\infty & , \text{ otherwise} \end{cases} \quad \text{and}$$

$$g_A(x) = \begin{cases} x & , \text{ if } x \in A \\ +\infty & , \text{ otherwise} \end{cases}$$

For simplicity, f_{A_β} and g_{A_β} are written by f_β and g_β, respectively. Since $\mathcal{P}(\mathbb{R})$ and $(\mathbb{R}_{\pm\infty})^{\mathbb{R}}$ yield complete lattices, the following equations hold true for all $A_\beta \subseteq \mathbb{R}$ and all $\mathbb{I}$:

$$f_{\bigcup_{i\in\mathbb{I}} A_\beta} = \bigvee_{i\in\mathbb{I}} f_\beta \;\; \text{and} \;\; g_{\bigcup_{i\in\mathbb{I}} A_\beta} = \bigwedge_{i\in\mathbb{I}} g_\beta. \tag{5.20}$$

Let $\delta_{\mathbb{R}}, \varepsilon_{\mathbb{R}} : (\mathbb{R}_{\pm\infty})^{\mathbb{R}} \to (\mathbb{R}_{\pm\infty})^{\mathbb{R}}$ denote respectively the dilation and the erosion by the structuring element $\mathbb{R}$ based on the threshold or flat approach [72, 130, 137], that is:

$$\delta_{\mathbb{R}}(f) = \bigvee_{x\in\mathbb{R}} f(x), \;\; \varepsilon_{\mathbb{R}}(f) = \bigwedge_{x\in\mathbb{R}} f(x). \tag{5.21}$$

Thus

$$\delta_{\mathbb{R}}(f_A) = \bigvee_{x\in\mathbb{R}} f_A(x) = \bigvee \{x \mid x \in A\} = \bigvee A, \tag{5.22}$$

$$\varepsilon_{\mathbb{R}}(g_A) = \bigwedge_{x\in\mathbb{R}} g_A(x) = \bigwedge \{x \mid x \in A\} = \bigwedge A, \tag{5.23}$$

for every $A \neq \emptyset$.

A combination of (5.20), (5.21), (5.22), and (5.23) leads to the following equations that are valid for non-empty A_β such that $\beta \in \mathbb{I} \neq \emptyset$:

$$\bigvee \left(\bigcup_{\beta\in\mathbb{I}} A_\beta \right) = \bigvee_{x\in\mathbb{R}} f_{\bigcup A_\beta}(x) = \delta_{\mathbb{R}}(f_{\bigcup A_\beta}) = \delta_{\mathbb{R}}(\bigvee_{\beta\in\mathbb{I}} f_\beta),$$

$$\bigwedge \left(\bigcup_{\beta\in\mathbb{I}} A_\beta \right) = \bigwedge_{x\in\mathbb{R}} g_{\bigcup A_\beta}(x) = \varepsilon_{\mathbb{R}}(g_{\bigcup A_\beta}) = \varepsilon_{\mathbb{R}}(\bigwedge_{\beta\in\mathbb{I}} g_\beta).$$

Since $\delta_{\mathbb{R}}$ and ε_R are, respectively, an algebraic dilation and an algebraic erosion [137], the proof of Lemma 1 is concluded as follows:

$$\delta_{\mathbb{R}} \left(\bigvee_{\beta\in\mathbb{I}} f_\beta \right) = \bigvee_{\beta\in\mathbb{I}} \delta_{\mathbb{R}}(f_\beta) = \bigvee_{\beta\in\mathbb{I}} \bigvee A_\beta,$$

$$\varepsilon_{\mathbb{R}} \left(\bigwedge_{\beta\in\mathbb{I}} g_\beta \right) = \bigwedge_{\beta\in\mathbb{I}} \varepsilon_{\mathbb{R}}(g_\beta) = \bigwedge_{\beta\in\mathbb{I}} \bigwedge A_\beta.$$

$\square$

Lemma 5.2. *[150] Let $A_1, A_2 \in \mathbb{R}_{\mathcal{F}}$. If J_0 is the joint possibility distribution of A_1 and $A_2 \in \mathbb{R}_{\mathcal{F}}$ that is defined in (5.24), then*

$$[J_0]^\alpha = \bigcup_{i=1}^{2}\left(\bigcup_{\beta\in[\alpha,1]} P^i(\beta)\right), \quad \forall\alpha \in [0,1], \tag{5.24}$$

where $P^i(\beta) = \{(x_1, x_2) : x_i \in R^i_\beta$ and $x_{3-i} \in L^i(x_i, \beta)\}$ for $i = 1, 2$ and for all $\beta \in [\alpha, 1]$.

Proof. Let us first prove that the set on the right side of (5.24) is contained in the set on the left side. Recall that the sets R^i_β and $L^i(x_i, \beta)$ are given by (5.4) and (5.5). Note that every element (x_1, x_2) of $\bigcup_{i=1}^{2}\left(\bigcup_{\beta\in[\alpha,1]} P^i(\beta)\right)$ satisfies $(x_1, x_2) \in P^j(\overline{\beta})$ for some $\overline{\beta} \geqslant \alpha$ and some $j \in \{1, 2\}$. Therefore,

$$\begin{aligned} x_j \in R^j_{\overline{\beta}} \subseteq [A_j]^{\overline{\beta}} \quad &\text{and} \quad x_{3-j} \in L(x_j, \overline{\beta}) \subseteq [A_{3-j}]^{\overline{\beta}} \\ \Rightarrow A_j(x_j) \geqslant \overline{\beta} \quad &\text{and} \quad A_{3-j}(x_{3-j}) \geqslant \overline{\beta}, \end{aligned}$$

which implies that $A_1(x_1) \wedge A_2(x_2) \geqslant \overline{\beta} \;\Rightarrow\; (x_1, x_2) \in [J_0]^{\overline{\beta}} \subseteq [J_0]^\alpha$.

These considerations reveal that

$$\bigcup_{i=1}^{2}\left(\bigcup_{\beta\in[\alpha,1]} P^i(\beta)\right) \subseteq [J_0]^\alpha.$$

Let us proceed by showing

$$[J_0]^\alpha \subseteq \bigcup_{i=1}^{2}\left(\bigcup_{\beta\in[\alpha,1]} P^i(\beta)\right).$$

First, note that $J_0(x_1, x_2) = A_1(x_1) \wedge A_2(x_2) \geqslant \alpha$ for all $(x_1, x_2) \in [J_0]^\alpha$. By the definition of J_0 (see (5.6)), it follows

$$(x_1, x_2) \in P = \bigcup_{i=1}^{2}\left(\bigcup_{\alpha\in[0,1]} P^i(\alpha)\right).$$

Thus, there exists $\overline{\alpha} \in [0, 1]$ for some $j \in \{1, 2\}$ such that $(x_1, x_2) \in P^j(\overline{\alpha})$, that is, $x_j \in R^j_{\overline{\alpha}} \subseteq [A_i]^{\overline{\alpha}}$ and $x_{3-j} \in L^j(x_i, \overline{\alpha}) \subseteq [A_{3-j}]^{\overline{\alpha}}$. This implies that $A_i(x) \geqslant \overline{\alpha}$ for $i = 1, 2$ and, therefore, $A_1(x_1) \wedge A_2(x_2) \geqslant \overline{\alpha}$. If $\overline{\alpha} \geqslant \alpha$ then $(x_1, x_2) \in P^i(\overline{\alpha})$ which implies that

$$[J_0]^\alpha \subseteq \bigcup_{i=1}^{2}\left(\bigcup_{\beta\in[\alpha,1]} P^i(\beta)\right).$$

On the other hand, suppose that $\overline{\alpha} < \alpha$. Thus,

$$\overline{\alpha} < \alpha \leqslant \beta := A_1(x_1) \wedge A_2(x_2) \leqslant A_i(x_i) \leqslant 1.$$

From (5.4), it follows $x_i \in R^i_{\overline{\alpha}} = \{a^-_{i_{\overline{\alpha}}}, a^+_{i_{\overline{\alpha}}}\}$, because $\overline{\alpha} < 1$. Let us suppose without loss of generality that $x_i = a^-_{i_{\overline{\alpha}}}$. Note that $A_i(a^-_{i_\gamma}) \geqslant \gamma$, $\forall \gamma \in [0,1]$, because $a^-_{i_\gamma} \in [A_i]^\gamma$. The last observation and the fact the function $a^-_i : [0,1] \to \mathbb{R}$ given by $a^-_i(\gamma) = a^-_{i_\gamma}$ is increasing by Theorem 4.9 of [16], then

$$x_i = a^-_i(\overline{\alpha}) \leqslant a^-_i(\beta) \leqslant a^-_i(A_i(x_i)) \leqslant x_i \in [a^-_{i_{A_i(x_i)}}, a^+_{i_{A_i(x_i)}}],$$

Thus, $x_i = a^-_i(\beta) \in R^i_\beta$. In addition, the sequence of inequalities

$$|x_1 + x_2| = g_i(x_i, \overline{\alpha}) = \bigwedge_{w \in [A_{3_i}]^{\overline{\alpha}}} |w + x_i| \leqslant \bigwedge_{w \in [A_{3_i}]^{\beta}} |w + x_i| = g_i(x_i, \beta) \leqslant |x_1 + x_2|$$

holds true since $x_{3-i} \in [A_{3-i}]^\beta \subseteq [A_{3-i}]^{\overline{\alpha}}$. These last observations imply that $x_{3-i} \in L^i(x_i, \beta)$ because $x_{3-i} \in [A_{3-i}]^\beta$ and $x_{3-i} \in [-g_i(x_i,\beta) - x_i, g_i(x_i,\beta) - x_i]$. Therefore,

$$[J_0]^\alpha \subseteq \bigcup_{i=1}^{2} \left(\bigcup_{\beta \in [\alpha,1]} P^i(\beta) \right)$$

since $(x_i, x_2) \in P^i(\beta) \subseteq \bigcup_{i=1}^{2} \left(\bigcup_{\gamma \in [\alpha,1]} P^i(\gamma) \right)$.

Therefore, the equality $[J_0]^\alpha = \bigcup_{i=1}^{2} \left(\bigcup_{\beta \in [\alpha,1]} P^i(\beta) \right)$ holds. □

Lemma 5.2 shows that the set $[J_0]^\alpha$ equals $\bigcup_{\beta \in [\alpha,1]} P(\beta)$ where $P(\beta) := P^1(\beta) \cup P^2(\beta)$. Note that the following function f is continuous:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_1 + x_2 \end{aligned}$$

A combination of Theorem 2.1, Lemmas 5.1 and 5.2 reveals that, for every $\alpha \in [0,1]$, the α-cut of $[A_1 +_0 A_2]$ is given by

$$\begin{aligned} [A_1 +_0 A_2]^\alpha &= [f_{J_0}(A_1, A_2)]^\alpha \\ &= f([J_0]^\alpha) \\ &= \{x_1 + x_2 \mid (x_1, x_2) \in [J_0]^\alpha\} \\ &= \left\{ x_1 + x_2 \mid (x_1, x_2) \in \bigcup_{\beta \in [\alpha,1]} P(\beta) \right\} \\ &= \bigcup_{\beta \in [\alpha,1]} \{x_1 + x_2 \mid (x_1, x_2) \in P(\beta)\} \\ &= \bigcup_{\beta \in [\alpha,1]} S(\beta), \end{aligned}$$

where

$$S(\beta) = \{x_1 + x_2 \mid (x_1, x_2) \in P(\beta)\}. \tag{5.25}$$

The next lemma presents a characterization of the set $S(\beta)$ defined in (5.25) for certain pairs of fuzzy numbers.

Lemma 5.3. *Let $A_1, A_2 \in \mathbb{R}_{\mathcal{F}_C}$ such that $(a_i)_1^- + (a_i)_1^+ = 0$, $i = 1, 2$. For every $\beta \in [0, 1]$, the set $S(\beta)$ in (5.25) is given by*

$$S(\beta) = \begin{cases} \{u_\beta,\ 0,\ v_\beta\} & , \text{ if } \beta \in [0,1) \\ [u_\beta \wedge v_\beta,\ u_\beta \vee v_\beta] & , \text{ if } \beta = 1 \end{cases}, \tag{5.26}$$

where

$$u_\beta = (a_1)_\beta^- + (a_2)_\beta^+ \text{ and } v_\beta = (a_1)_\beta^+ + (a_2)_\beta^-.$$

Proof. Let $\beta \in [0, 1]$ be arbitrary. First, consider an arbitrary index $j \in \{1, 2\}$. If $(x_1, x_2) \in P^j(\beta)$, then $x_j \in R_\beta^i$ and $x_{3-j} \in L^j(x_j, \beta)$. Note that $L^j(x_j, \beta)$ corresponds to the set of minimizers of the function $h_j(w) = |x_j + w|$ subject to $w \in [A_{3-j}]^\beta$, that is,

$$x_{3-j} \in L^j(x_j, \beta) \Leftrightarrow x_{3-j} = \text{argmin}_{w \in [A_{3-j}]^\beta} |x_j + w|.$$

Since $(a_1)_1^- + (a_1)_1^+ = 0 = (a_2)_1^- + (a_2)_1^+$, it follows that $(a_j)_\beta^- \leqslant 0 \leqslant (a_j)_\beta^+$, where $(a_j)_\beta^+ \geqslant 0$, $\forall \beta \in [0, 1]$. Therefore, for $i = 1, 2$:

$$\begin{aligned} x_{3-i} &= \text{argmin}_{w \in [A_{3-i}]^\beta} |x_i + w| \\ &= \begin{cases} -x_i & , \text{ if } 0 \leqslant x_i \text{ and } (a_{3-i})_\beta^- \leqslant -x_i \\ (a_{3-i})_\beta^- & , \text{ if } 0 \leqslant x_i \text{ and } (a_{3-i})_\beta^- \geqslant -x_i \\ -x_i & , \text{ if } 0 \geqslant x_i \text{ and } (a_{3-i})_\beta^+ \geqslant -x_i \\ (a_{3-i})_\beta^+ & , \text{ if } 0 \geqslant x_i \text{ and } (a_{3-i})_\beta^+ \leqslant -x_i \end{cases}. \end{aligned} \tag{5.27}$$

Using (5.27), let us divide the rest of the proof into four cases.

(i) Suppose that $(a_1)_\beta^- \leqslant -(a_2)_\beta^+$ and $(a_1)_\beta^+ \geqslant -(a_2)_\beta^-$.

If $\beta < 1$, then $x_i \in \{(a_i)_\beta^-, (a_i)_\beta^+\}$ for $i = 1, 2$. Thus, (5.27) yields

$$P^1(\beta) = \left\{ \left((a_1)_\beta^-, (a_2)_\beta^+\right), \left((a_1)_\beta^+, (a_2)_\beta^-\right) \right\}$$

and

$$P^2(\beta) = \left\{ \left(-(a_2)_\beta^-, (a_2)_\beta^-\right), \left(-(a_2)_\beta^+, (a_2)_\beta^+\right) \right\}$$

which implies that

$$\begin{aligned} S(\beta) &= \{x_1 + x_2 \mid (x_1, x_2) \in P(\beta) = \bigcup_{i=1}^{2} P^i(\beta)\} \\ &= \{u_\beta,\ 0,\ v_\beta\}, \end{aligned}$$

where $u_\beta = (a_1)^-_\beta + (a_2)^+_\beta$ and $v_\beta = (a_1)^+_\beta + (a_2)^-_\beta$.

If $\beta = 1$, then $x_i \in [A_i]^1$ for $i = 1, 2$. From (5.27), it follows that

$$\begin{aligned} P^1(1) &= \left\{\left(x_1, (a_2)^+_1\right) \mid x_1 \in [(a_1)^-_1, -(a_2)^+_1]\right\} \\ &= \bigcup \left\{(x_1, -x_1) \mid x_1 \in [-(a_2)^+_1, -(a_2)^-_1]\right\} \\ &= \bigcup \left\{\left(x_1, (a_2)^-_1\right) \mid x_1 \in [-(a_2)^-_1, (a_1)^+_1]\right\} \end{aligned}$$

and

$$P^2(1) = \left\{(-x_2, x_2) \mid x_2 \in [(a_2)^-_1, (a_2)^+_1]\right\}.$$

These observations reveal that

$$\begin{aligned} S(1) &= \{x_1 + x_2 \mid (x_1, x_2) \in P(1) = \bigcup_{i=1}^{2} P^i(1)\} \\ &= [u_1 \wedge v_1,\ u_1 \vee v_1], \end{aligned}$$

where $u_1 = (a_1)^-_1 + (a_2)^+_1$ and $v_1 = (a_1)^+_1 + (a_2)^-_1$. Note that $u_1 \leqslant 0 \leqslant v_1$ or $v_1 \leqslant 0 \leqslant u_1$ since $(a_i)^-_1 + (a_i)^+_1 = 0$ for $i = 1, 2$.

(ii) Suppose that $(a_1)^-_\beta \leqslant -(a_2)^+_\beta$ and $(a_1)^+_\beta \leqslant -(a_2)^-_\beta$.

If $\beta < 1$, then $x_i \in R^i_\beta = \{(a_i)^-_\beta, (a_i)^+_\beta\}$ for $i = 1, 2$. Using (5.27), one obtains

$$P^1(\beta) = \left\{\left((a_1)^-_\beta, (a_2)^+_\beta\right), \left((a_1)^+_\beta, -(a_1)^+_\beta\right)\right\}$$

and

$$P^2(\beta) = \left\{\left((a_1)^+_\beta, (a_2)^-_\beta\right), \left(-(a_2)^+_\beta, (a_2)^+_\beta\right),\right\}$$

which implies that

$$\begin{aligned} S(\beta) = &\ \{x_1 + x_2 \mid (x_1, x_2) \in P(\beta) = \bigcup_{i=1}^{2} P^i(\beta)\} \\ = &\ \{u_\beta,\ 0,\ v_\beta\}, \end{aligned}$$

where $u_\beta = (a_1)^-_\beta + (a_2)^+_\beta$ and $v_\beta = (a_1)^+_\beta + (a_2)^-_\beta$.

If $\beta = 1$, then $x_i \in R^i_1 = [A_i]^1$ for $i = 1, 2$. Thus, (5.27) reveals that

$$\begin{aligned} P^1(1) &= \left\{\left(x_1, (a_2)^+_1\right) \mid x_1 \in [(a_1)^-_1, -(a_2)^+_1]\right\} \\ &= \bigcup \left\{(x_1, -x_1) \mid x_1 \in [-(a_2)^+_1, (a_1)^+_1]\right\} \end{aligned}$$

and

$$\begin{aligned} P^2(1) &= \left\{\left((a_1)^+_1, x_2\right) \mid x_2 \in [(a_2)^-_1, -(a_1)^+_1]\right\} \\ &= \bigcup \left\{(-x_2, x_2) \mid x_1 \in [-(a_1)^+_1, (a_2)^+_1]\right\}. \end{aligned}$$

These equations imply that

$$S(\beta) = \{x_1 + x_2 \mid (x_1, x_2) \in P(1) = P^1(1) \cup P^2(1)\} = [u_1 \wedge v_1,\ u_1 \vee v_1],$$

where $u_1 = (a_1)^-_1 + (a_2)^+_1$ and $v_1 = (a_1)^+_1 + (a_2)^-_1$. In addition, note that $u_1 \leqslant 0 \leqslant v_1$ or $v_1 \leqslant 0 \leqslant u_1$ since $(a_i)^-_1 + (a_i)^+_1 = 0$ for $i = 1, 2$.

(iii) The proof of (5.26) for the case where $(a_1)^-_\beta \geqslant -(a_2)^+_\beta$ and $(a_1)^+_\beta \leqslant -(a_2)^-_\beta$ is similar to the one of (i).

(iv) The proof of (5.26) for the case where $(a_1)^-_\beta \geqslant -(a_2)^+_\beta$ and $(a_1)^+_\beta \geqslant -(a_2)^-_\beta$ is similar to the one of (ii).

□

Note that in the above lemma, the condition $(a_i)^-_1 + (a_i)^+_1 = 0$, for $i = 1, 2$, stands only for fuzzy numbers that are centered in the origin, which means that the translation of A_1 and A_2 are given by the midpoint of the *core* of the respective fuzzy number.

Let us now employ Lemmas 1, 2, and 3 to prove the following two theorems that characterize the α-cuts of the difference $A_1 -_I A_2$.

Theorem 5.4. *Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. The α-cuts of $[A -_I B]^\alpha$ are given by*

$$\left[\bigwedge_{\beta\geqslant\alpha}\left((a^-_\beta - b^-_\beta)\wedge(a^+_\beta - b^+_\beta)\right), \bigvee_{\beta\geqslant\alpha}\left((a^-_\beta - b^-_\beta)\vee(a^+_\beta - b^+_\beta)\right)\right],$$

for every $\alpha \in [0, 1]$.

Proof. Let $\alpha \in [0, 1]$ be arbitrary. By Theorem 5.3,

$$[A -_I B]^\alpha = [A^{(k_A)} +_0 (-B^{(k_B)})]^\alpha + (k_A - k_B)$$

where $k_A = 0.5(a^-_1 + a^+_1)$ and $k_B = 0.5(b^-_1 + b^+_1)$.

Note that $(a^{(k_A)})^\pm_\alpha = a^\pm_\alpha - k_A$ and $(b^{(k_B)})^\pm_\alpha = b^\pm_\alpha - k_B$ for all $\alpha \in [0, 1]$. Defining $C = -B^{(k_B)}$, one obtains $c^-_\alpha = -(b^{(k_B)})^+_\alpha = k_B - b^+_\alpha$ and $c^+_\alpha = -(b^{(k_B)})^-_\alpha = k_B - b^-_\alpha$ for all $\alpha \in [0, 1]$. Moreover,

$$(a^{(k_A)})^-_1 + (a^{(k_A)})^+_1 = (a^-_1 + a^+_1) - 2k_A = 0$$

and

$$c^-_1 + c^+_1 = (k_B - b^+_1) + (k_B - b^-_1) = 2k_B - (b^+_1 + b^-_1) = 0.$$

A brief glance at Lemmas 5.2 and 5.3 reveals that

$$[A^{(k_A)} +_0 (-B^{(k_B)})]^\alpha = [A^{(k_A)} +_0 C]^\alpha = \bigcup_{\beta\in[\alpha,1]} S(\beta). \tag{5.28}$$

Recall that $S(\beta)$ is given by

$$S(\beta) = \begin{cases} \{u_\beta,\ 0,\ v_\beta\} & , \text{ if } \beta \in [0, 1) \\ [u_\beta \wedge v_\beta,\ u_\beta \vee v_\beta] & , \text{ if } \beta = 1 \end{cases},$$

where

$$u_\beta = (a^{(k_A)})^-_\beta + c^+_\beta = (a^-_\beta - b^-_\alpha) + (k_B - k_A)$$

and

$$v_\beta = (a^{(k_A)})^+_\beta + c^-_\beta = (a^+_\beta - b^+_\alpha) + (k_B - k_A).$$

Thus $u_1 \leqslant 0 \leqslant v_1$ or $v_1 \leqslant 0 \leqslant u_1$, since $(a^{(k_A)})^-_1 + (a^{(k_A)})^+_1 = c^-_1 + c^+_1 = 0$. This implies that

$$0 \in [u_\beta \wedge v_\beta,\ u_\beta \vee v_\beta]$$

and, consequently,

$$\bigcup_{\beta\in[\alpha,1]} S(\beta) = \bigcup_{\beta\in[\alpha,1]} \bar{S}(\beta), \tag{5.29}$$

where

$$\bar{S}(\beta) = \begin{cases} S(\beta)\backslash\{0\} & , \text{ if } \beta \in [0,1) \\ S(1) & , \text{ if } \beta = 1 \end{cases}.$$

One obtains $[A^{(k_A)} +_0 (-B^{(k_B)})]^\alpha = \bigcup_{\beta\in[\alpha,1]} \bar{S}(\beta)$ by merging (5.28) and (5.29). Since $\bar{S}(\beta) \neq \varnothing$ for all $\beta \in [\alpha,1]$, Lemma 5.1 can be applied to $\bigcup_{\beta\in[\alpha,1]} \bar{S}(\beta)$, yielding

$$\bigcup_{\beta\in[\alpha,1]} \bar{S}(\beta) = \left[\bigwedge_{\beta\geqslant\alpha}(u_\beta \wedge v_\beta), \bigvee_{\beta\geqslant\alpha}(u_\beta \vee v_\beta)\right].$$

Moreover, note that $A^{(k_A)} +_0 (-B^{(k_B)})$ is a fuzzy number by Theorem 5.2 which implies that the sets $\{u_\beta \wedge v_\beta \mid \beta \geqslant \alpha\}$ and $\{u_\beta \vee v_\beta \mid \beta \geqslant \alpha\}$ are bounded for every $\alpha \in [0,1]$. Since $\mathbb{R}$ is conditionally complete lattice (see Example 1.3), every group translation is an order-automorphism [22]. As an immediate consequence, it follows that

$$\begin{aligned} \bigwedge_{\beta\geqslant\alpha}(u_\beta \wedge v_\beta) &= \bigwedge_{\beta\geqslant\alpha}([(a^-_\beta - b^-_\alpha) + (k_B - k_A)] \wedge [(a^+_\beta - b^+_\alpha) + (k_B - k_A)]) \\ &= \bigwedge_{\beta\geqslant\alpha}([(a^-_\beta - b^-_\alpha) \wedge (a^+_\beta - b^+_\alpha)] + (k_B - k_A)) \\ &= \bigwedge_{\beta\geqslant\alpha}[(a^-_\beta - b^-_\alpha) \wedge (a^+_\beta - b^+_\alpha)] + (k_B - k_A). \end{aligned}$$

Similarly,

$$\bigvee_{\beta\geqslant\alpha}(u_\beta \vee v_\beta) = \bigvee_{\beta\geqslant\alpha}[(a^-_\beta - b^-_\alpha) \vee (a^+_\beta - b^+_\alpha)] + (k_B - k_A).$$

Since $[a+c, b+c] = [a,b] + c$, for all $a,b,c \in \mathbb{R}$, thus

$$[A^{(k_A)} +_0 (-B^{(k_B)})]^\alpha = \left[\bigwedge_{\beta\geqslant\alpha}((a^-_\beta - b^-_\alpha) \wedge (a^+_\beta - b^+_\alpha)),\ \bigvee_{\beta\geqslant\alpha}((a^-_\beta - b^-_\alpha) \vee (a^+_\beta - b^+_\alpha))\right] + (k_B - k_A)$$

At last, the proof finishes as follows:

$$\begin{aligned}
[A -_I B]^\alpha &= [A^{(k_A)} +_0 (-B^{(k_B)})]^\alpha + (k_A - k_B) \\
&= \left[\bigwedge_{\beta \geqslant \alpha} ((a_\beta^- - b_\beta^-) \wedge (a_\beta^+ - b_\beta^+)),\ \bigvee_{\beta \geqslant \alpha} ((a_\beta^- - b_\beta^-) \vee (a_\beta^+ - b_\beta^+)) \right].
\end{aligned}$$

□

Theorems 1.9 and 5.4 reveal that α-cuts of the g-difference and of the I-interactive difference of two fuzzy numbers in $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$ coincide. Consequently, the g-difference and the I-interactive difference of A and B are equal if $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ [102].

Theorem 5.5. *If $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$, then*

$$A -_I B = A -_g B.$$

The g-difference extends the $gH-$ as well as the $H-$difference, so all well-known differences between fuzzy numbers in $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$ can be derived using the sup-J extension principle for particular choices of J, and the following corollary is obtained.

Corollary 5.2. *Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$. If the Hukuhara difference between A and B exists, then the gH-, g-, and I-differences also exist and they satisfy*

$$A -_H B = A -_{gH} B = A -_g B = A -_I B.$$

There are two types of fuzzy arithmetic based on joint possibility distributions of fuzzy numbers: the interactive and non-interactive arithmetics [51]. Theorem 5.5 shows that the joint possibility distribution I gives rise to the generalized difference. Since $I \neq J_\wedge$, the g-difference is an interactive arithmetic operation.

These differences give raise to the H-, gH-, and g-derivatives, as it was pointed out before. Both concepts of gH- and g-derivatives, given by Definitions 1.35 and 1.36 extend the Hukuhara derivative (see (1.34)) [118]. This is also true for the interactive derivative that is defined as follows:

Definition 5.2. *Let $F : \mathbb{R} \to \mathbb{R}_\mathcal{F}$, $x_0 \in \mathbb{R}$, and $\delta > 0$. For every $h \in (-\delta, \delta) \backslash \{0\}$, let J_h be a joint possibility distribution of $F(x_0 + h)$ and $F(x_0)$ and let $\mathcal{J} = \{J_h \,|\, 0 < |h| < \delta\}$. The* interactive derivative *of a fuzzy-number-valued function $F : \mathbb{R} \to \mathbb{R}_\mathcal{F}$ at x_0 with respect to $\mathcal{J}$ is given by*

$$\lim_{h \to 0} \frac{F(x_0 + h) -_{J_h} F(x_0)}{h}, \tag{5.30}$$

if this limit exists. In this case, the limit in (5.30) is called the interactive derivative of F at x_0 and is denoted $F'_\mathcal{J}(x_0)$.

This definition yields a concept of interactive derivative for a fuzzy-number-valued function. From the point of view of fuzzy differential equations, a fuzzy initial value problem can be written by

$$\begin{cases} F'(x) = f(F(x), x) \\ F(x_0) = F_0 \end{cases} \tag{5.31}$$

for some fuzzy derivative F'.

Considering interactive fuzzy derivative J_h, one obtains

$$\begin{cases} \lim\limits_{h \to 0} \dfrac{F(t+h) -_{J_h} F(t)}{h} = f(F(t), t) \\ F(t_0) = F_0 \end{cases} \tag{5.32}$$

where the expression in the left side of equation depends on $F(t)$ and J_h. This means that in order to study a solution for this problem, it is necessary to establish which joint possibility distribution is at hand. In particular, if the family of JPDs under consideration is $\mathcal{I} = \{I_h \,|\, 0 < |h| < \delta\}$ where I_h are the joint possibility distributions of $F(x_0 + h)$ and $F(x_0)$ given by (5.19), then $F'_{\mathcal{I}}$ is called the $\mathcal{I}$-interactive derivative of F.

Observe that for a given fuzzy-number-valued function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}_C}$, the joint possibility distribution I_h connects the fuzzy numbers $F(t)$ and $F(t+h)$, for all $h \in \mathbb{R}$. From the dynamical point of view, one intends to yield an information about the future $F(t+h)$ knowing the present $F(t)$, where $h > 0$. Here this information is provided from the joint possibility distribution given by I. For instance, if one establishes that a fuzzy function has decreasing width, then the formula (5.6) with $\gamma = 0$ produces the joint possibility distribution I with this property.

The next theorem, which is an immediate consequence of Theorem 5.5, shows that the g-derivative of a fuzzy-number-valued function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}_C}$ coincides with the $\mathcal{I}$–interactive derivative of F.

Theorem 5.6. *Let $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}_C}$. Thus F is g-differentiable if and only if F is $\mathcal{I}$-differentiable.*

Hence, the g-derivative of a fuzzy-number-valued function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}_C}$ is a particular case of the interactive derivative. In other words, the g-derivative of $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}_C}$ is interactive. Since the g-derivative extends the notions of H- and gH-derivative, these are interactive as well. In fact, Barros and Pedro suggested this statement as a hypothesis in [15]. This thesis proves this result.

In conclusion, the family of JPDs defined as in (5.6) has the following properties:

1. Produces an interactive sum with smaller Pomepeiu-Hausdorff norm;

2. Extend classical numerical solutions of FDEs;

3. Produces an interactive difference that extends the Hukuhara difference and its generalizations;

4. Produces an interactive derivative that embraces the g-derivative;

5. Associates numerical solutions with the notion of interactive fuzzy derivatives.

However from the computational point of view, the interactive arithmetic operations based on family J_γ may be difficult to successfully implement. The next chapter proposes a new family of JPDs denoted by $\mathcal{J}_\gamma$. This new family has similar properties with the one defined as in (5.18). But in advantage to J_γ, the interactive sum and difference derived from the family $\mathcal{J}_\gamma$ can be characterized by means of α-cuts, which makes the computation more simpler.

5.4 Conclusion

This chapter presented a parametrized family of joint possibility distributions. The joint possibility distribution J_γ can be applied to every pair of fuzzy numbers, in contrast to J_L. From the concept of translated fuzzy numbers, it was exhibited another joint possibility distribution in order to control the Pompeiu-Hausdorff norm and width of the arithmetic operations via sup-J extension. From these family a numerical solution for fuzzy initial value problems were proposed, where the initial conditions are given by J_γ-interactive fuzzy numbers. In particular this chapter focused on J_0. It was demonstrated that, under some weak conditions, the generalized difference represents a particular case of the interactive difference. This chapter ended by showing that the generalized derivative of a fuzzy-number-valued function $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}_C}$ is a particular case of the interactive derivative.

6 Family of joint possibility distributions $\mathcal{J}_\gamma$

This chapter proposes a new family of joint possibility distributions and defines arithmetic operations based on it. More precisely, this chapter focuses on the interactive sum and difference. The Pompeiu-Hausdorff norm and width of these operations can be controled as well as the arithmetic operations obtained via J_γ, given in the previously chapter.

The family J_γ provided very interesting results. However, the arithmetic operations may be difficult to compute. From this new family proposed here, it is possible to characterize the interactive sum and difference by means of α-cuts, which makes the computation simpler. This new family also extends the g-difference, that is, the generalized difference can be obtained from one joint possibility distributions of this family.

This chapter is also connected with Chapter 3, since it deals with fuzzy discrete models involving only interactive sum and difference. This chapter also discusses the consequences of symmetric fuzzy numbers in the context of interactive fuzzy equations. The applications of this approach are provided in Chapter 7, where the Fibonacci and delay discrete sequence are studied. This chapter is based on the reference [55].

6.1 Family of joint possibility distributions $\mathcal{J}_\gamma$

This section constructs a new parametrized family of joint possibility distributions $\{\mathcal{J}_\gamma : \gamma \in [0,1]\}$. Each element of this family is associated with the parameter γ, which considers the notion of interactivity as well as the family of joint possibility distributions provided in Chapter 5.

Before providing the construction of this family, consider the fuzzy numbers $A, B \in \mathbb{R}_{\mathcal{F}_C}$. In view of Definition 5.1 the α-cuts of the translated fuzzy numbers $A^{(\overline{a})}$ and $B^{(\bar{b})}$ are denoted by

$$[A^{(\overline{a})}]^\alpha = [(a^{(\overline{a})})^-_\alpha, (a^{(\overline{a})})^+_\alpha] \quad \text{and} \quad [B^{(\bar{b})}]^\alpha = [(b^{(\bar{b})})^-_\alpha, (b^{(\bar{b})})^+_\alpha],$$

where $\overline{a} = 0.5(a_1^- + a_1^+)$ and $\bar{b} = 0.5(b_1^- + b_1^+)$. Recalling that $[A]^\alpha - \overline{a} = [A^{(\overline{a})}]^\alpha$ and $[B]^\alpha - \bar{b} = [B^{(\bar{b})}]^\alpha$, for all $\alpha \in [0,1]$.

For the construction of $\mathcal{J}_\gamma$, let us consider for each $\gamma \in [0,1]$ and for each $x \in \mathbb{R}$, the following intervals

$$I_A(x,\alpha,\gamma) = [\bar{b} + f_A^\alpha(x) + \gamma((b^{(\bar{b})})^-_\alpha - f_A^\alpha(x)), \bar{b} + f_A^\alpha(x) + \gamma((b^{(\bar{b})})^+_\alpha - f_A^\alpha(x))],$$

and

$$I_B(x,\alpha,\gamma) = [\overline{a} + f_B^\alpha(x) + \gamma((a^{(\overline{a})})_\alpha^- - f_B^\alpha(x)), \overline{a} + f_B^\alpha(x) + \gamma((a^{(\overline{a})})_\alpha^+ - f_B^\alpha(x))],$$

where the functions f_A^α and f_B^α are respectively given by

$$f_A^\alpha(x) = (-(x-\overline{a}) \vee ((b^{(\overline{b})})_\alpha^-)) \wedge ((b^{(\overline{b})})_\alpha^+)$$

and

$$f_B^\alpha(x) = (-(x-\overline{b}) \vee ((a^{(\overline{a})})_\alpha^-)) \wedge ((a^{(\overline{a})})_\alpha^+).$$

Note that the intervals I_A and I_B are well-defined, since $(b^{(\overline{b})})_\alpha^- \leqslant (b^{(\overline{b})})_\alpha^+$ and $(a^{(\overline{a})})_\alpha^- \leqslant (a^{(\overline{a})})_\alpha^+$, for all $\alpha \in [0,1]$. Also, the functions f_A^α and f_B^α are continuous and decreasing, for each $\alpha \in [0,1]$, hence by Weierstrass Theorem it follows that $f_A^\alpha(a_\alpha^+) \leqslant f_A^\alpha(x) \leqslant f_A^\alpha(a_\alpha^-)$ and $f_B^\alpha(b_\alpha^+) \leqslant f_B^\alpha(y) \leqslant f_B^\alpha(b_\alpha^-)$, $\forall x \in [A]^\alpha$ and $\forall y \in [B]^\alpha$.

Therefore, for a fixed $\gamma \in [0,1]$ it follows

$$f_A^\alpha(a_\alpha^+)(1-\gamma) + \gamma b_\alpha^- \leqslant f_A^\alpha(a_\alpha^-)(1-\gamma) + \gamma b_\alpha^-,\ \forall\alpha \in [0,1]$$

and

$$f_A^\alpha(a_\alpha^+)(1-\gamma) + \gamma b_\alpha^+ \leqslant f_A^\alpha(a_\alpha^-)(1-\gamma) + \gamma b_\alpha^+,\ \forall\alpha \in [0,1].$$

This means that for each $\alpha \in [0,1]$ the function f_A^α applied at a_α^+ produces the smallest left endpoint of the interval I_A and f_A^α applied at a_α^- produces the biggest right endpoint of the interval I_A. Analogously, f_B^α applied at b_α^+ and b_α^- produce the smallest left endpoint and the biggest right endpoint of the interval I_B, respectively.

Now, for each $\gamma \in [0,1]$ consider the sets $P_A(\gamma), P_B(\gamma) \subseteq \mathbb{R}^2$ given by

$$P_A(\gamma) = \left(\bigcup_{\alpha\in[0,1)}\left(\bigcup_{x\in\{a_\alpha^-,a_\alpha^+\}}\{x\}\times I_A(x,\alpha,\gamma)\right)\right) \cup \left(\bigcup_{x\in[A]^1}\{x\}\times I_A(x,1,\gamma)\right) \tag{6.1}$$

and

$$P_B(\gamma) = \left(\bigcup_{\alpha\in[0,1)}\left(\bigcup_{x\in\{b_\alpha^-,b_\alpha^+\}} I_B(x,\alpha,\gamma)\times\{x\}\right)\right) \cup \left(\bigcup_{x\in[B]^1} I_B(x,1,\gamma)\times\{x\}\right). \tag{6.2}$$

Define the fuzzy relation $\mathcal{J}_\gamma$ by the following membership function

$$\mathcal{J}_\gamma(x_1,x_2) = \begin{cases} A(x_1) \wedge B(x_2) & \text{, if } (x_1,x_2) \in P(\gamma) \\ 0 & \text{, otherwise} \end{cases}, \tag{6.3}$$

where the set

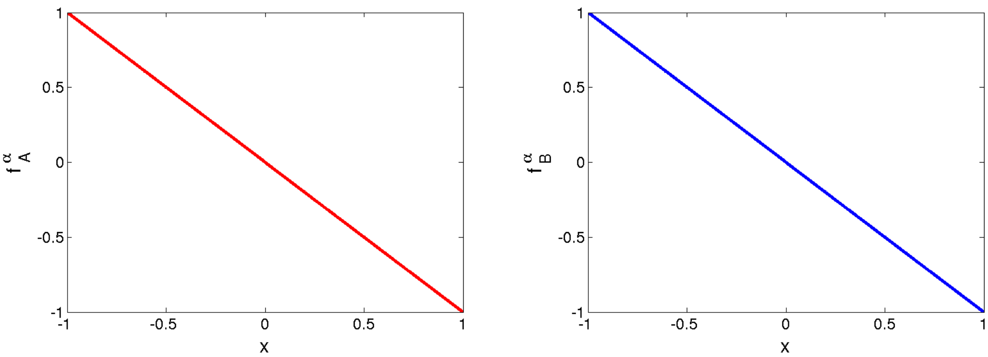
$$P(\gamma) := P_A(\gamma) \cup P_B(\gamma) \tag{6.4}$$

is defined as the region such that $J_\gamma(u,v) > 0$, $\forall (u,v) \in \mathbb{R}^2$.

In order to clarify this construction, the following example presents the set $P(\gamma)$ for specific values of γ.

Example 6.1. *Let $A = B = (-1;0;1) \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$, whose α-cuts are given by $[A]^\alpha = [B]^\alpha = [-1+\alpha, 1-\alpha]$. For these fuzzy numbers one obtains $f_A^\alpha(a_\alpha^-) = 1-\alpha$, $f_A^\alpha(a_\alpha^+) = -1+\alpha$, $f_B^\alpha(b_\alpha^-) = 1-\alpha$ and $f_B^\alpha(b_\alpha^+) = -1+\alpha$. The functions $f_A^\alpha(\cdot)$ and $f_B^\alpha(\cdot)$ are depicted in Figure 33.*

Figure 33 – Graphical representation of f_A^α and f_B^α of Example 6.1.



The red and blue lines represent the functions f_A^α and f_B^α, respectively, where $A = B = (-1;0;1)$. Source: Author.

The intervals I_A and I_B are given by

$$\begin{aligned}
I_A(a_\alpha^-, \alpha, \gamma) &= [1-\alpha+\gamma(-2+2\alpha), 1-\alpha], \\
I_A(a_\alpha^+, \alpha, \gamma) &= [-1+\alpha, -1+\alpha+\gamma(2-2\alpha)], \\
I_B(b_\alpha^-, \alpha, \gamma) &= [1-\alpha, 1-\alpha+\gamma(-2+2\alpha)], \\
I_B(b_\alpha^+, \alpha, \gamma) &= [-1+\alpha+\gamma(2-2\alpha), -1+\alpha].
\end{aligned}$$

For $\gamma = 0$, the intervals

$$I_A(a_\alpha^-, \alpha, 0) = [1-\alpha, 1-\alpha] \quad \textit{and} \quad I_A(a_\alpha^+, \alpha, 0) = [-1+\alpha, -1+\alpha]$$

are depicted in Figure 34 for different values of α.

For $\gamma = 0.25$, the intervals

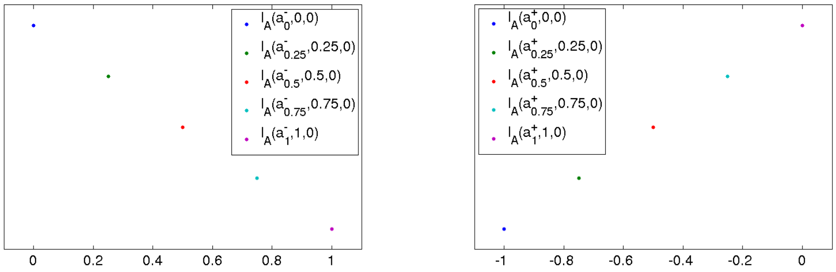
$$I_A(a_\alpha^-, \alpha, 0.25) = [0.5-0.5\alpha, 1-\alpha] \quad \textit{and} \quad I_A(a_\alpha^+, \alpha, 0.25) = [-1+\alpha, -0.5+0.5\alpha]$$

are depicted in Figure 35 for different values of α.

For $\gamma = 0.5$, the intervals

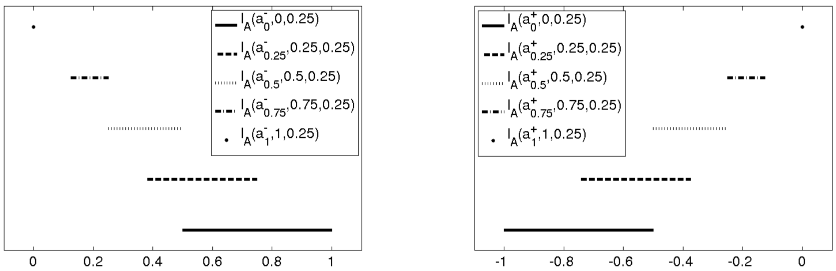
$$I_A(a_\alpha^-, \alpha, 0.5) = [0, 1-\alpha] \quad \textit{and} \quad I_A(a_\alpha^+, \alpha, 0.5) = [-1+\alpha, 0]$$

Figure 34 – Graphical representation of the interval I_A for $\gamma = 0$ given as in Example 6.1.



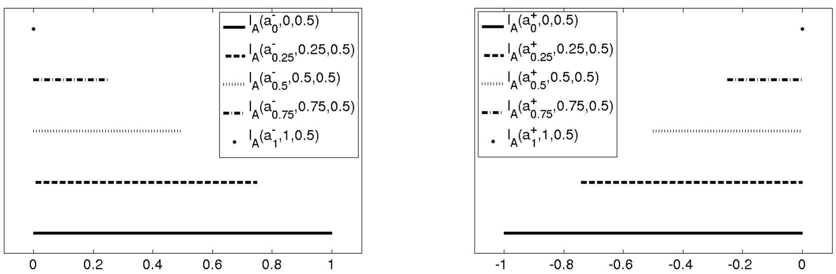
The left subfigure contains the interval I_A evaluated at a_α^- for $\gamma = 0$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The right subfigure contains the interval I_A evaluated at a_α^+ for $\gamma = 0$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. Source: Author

Figure 35 – Graphical representation of the interval I_A for $\gamma = 0.25$ given as in Example 6.1.



The left subfigure contains the interval I_A evaluated at a_α^- for $\gamma = 0.25$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The right subfigure contains the interval I_A evaluated at a_α^+ for $\gamma = 0.25$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. Source: Author

Figure 36 – Graphical representation of the interval I_A for $\gamma = 0.5$ given as in Example 6.1.



The left subfigure contains the interval I_A evaluated at a_α^- for $\gamma = 0.5$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The right subfigure contains the interval I_A evaluated at a_α^+ for $\gamma = 0.5$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. Source: Author

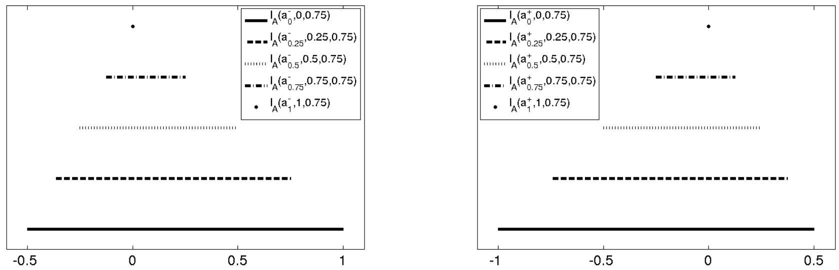
are depicted in Figure 36 for different values of α.

For $\gamma = 0.75$, the intervals

$$I_A(a_\alpha^-, \alpha, 0.75) = [-0.5 + 0.5\alpha, 1 - \alpha] \quad \textit{and} \quad I_A(a_\alpha^+, \alpha, 0.75) = [-1 + \alpha, 0.5 - 0.5\alpha]$$

are depicted in Figure 37 for different values of α.

Figure 37 – Graphical representation of the interval I_A for $\gamma = 0.75$ given as in Example 6.1.



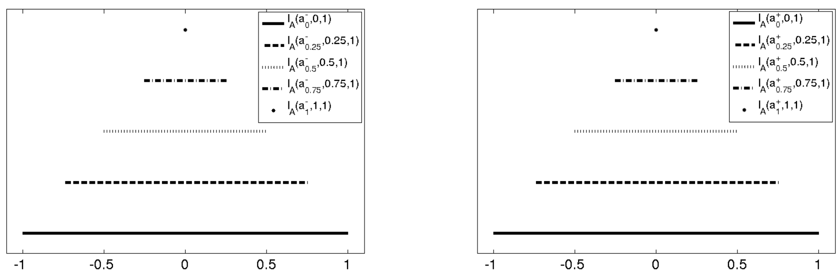
The left subfigure contains the interval I_A evaluated at a_α^- for $\gamma = 0.75$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The right subfigure contains the interval I_A evaluated at a_α^+ for $\gamma = 0.75$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. Source: Author

Finally for $\gamma = 1$, the intervals

$$I_A(a_\alpha^-, \alpha, 1) = I_A(a_\alpha^+, \alpha, 1) = [-1 + \alpha, 1 + \alpha]$$

are depicted in Figure 38 for different values of α.

Figure 38 – Graphical representation of the interval I_A for $\gamma = 1$ given as in Example 6.1.



The left subfigure contains the interval I_A evaluated at a_α^- for $\gamma = 1$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The right subfigure contains the interval I_A evaluated at a_α^+ for $\gamma = 1$ and for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. Source: Author

From Equations 6.1 and 6.2 the sets $P_A(\gamma)$ and $P_B(\gamma)$ are given by

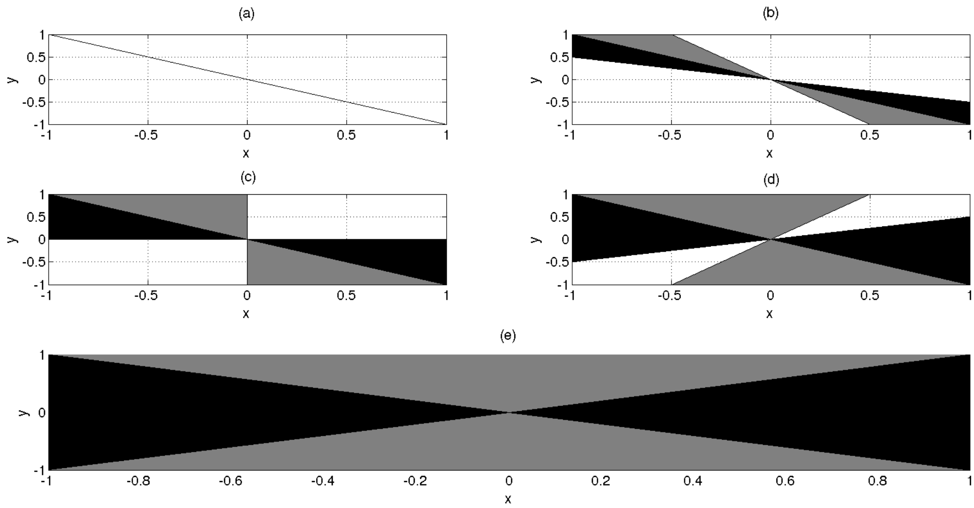
$$\begin{aligned} P_A(\gamma) &= \{-1 + \alpha\} \times [1 - \alpha + \gamma(-2 + 2\alpha), 1 - \alpha] \cup \\ & \quad \{1 - \alpha\} \times [-1 + \alpha, -1 + \alpha + \gamma(2 - 2\alpha)] \cup \\ & \quad [-1 + \alpha, 1 - \alpha] \times \{0\}. \end{aligned}$$

and

$$
\begin{aligned}
P_B(\gamma) \;=\; & [1-\alpha+\gamma(-2+2\alpha), 1-\alpha] \times \{-1+\alpha\} \cup \\
& [-1+\alpha, -1+\alpha+\gamma(2-2\alpha)] \times \{1-\alpha\} \cup \\
& \{0\} \times [-1+\alpha, 1-\alpha].
\end{aligned}
$$

Figure 39 depicts the set $P(\gamma)$ for different values of γ. The black region represents the set $P_A(\gamma)$, whereas the gray region represents the set $P_B(\gamma)$. Hence, $P(\gamma)$ is given by the union of these two regions.

Figure 39 – Graphical representation of the set P_γ for different values of γ given as in Example 6.1.



The subfigures $(a),(b),(c),(d),(e)$ exhibit the top view of $P_A(\gamma)$ (represented by black region) and $P_B(\gamma)$ (represented by gray region) for $\gamma = 0, 0.25, 0.5, 0.75, 1$, respectively.
Source: Author

Observe that for $\gamma = 0$, $P_A(\gamma) = P_B(\gamma)$. This statement holds since the functions f_A^α and f_B^α are equal, in this example. Note that for small values of γ, one obtains more constrains for the region $P(\gamma)$, where $\mathcal{J}_\gamma(u,v) > 0$. Those constrains are associated with interactivity in the following sense: the larger the constrain of the region P the greater the interactivity between the fuzzy numbers.

One can observe that in the previous example the chosen fuzzy numbers can be also interactive with respect to the joint possibility distribution J_L (see (4.7)), for $q = 1, r = 0$ and $q = -1, r = 0$. However the joint possibility distribution J_L can not be applied when the involved fuzzy numbers have different shapes, as already observed in Chapter 5. The next example illustrates that the proposed joint possibility distribution $\mathcal{J}_\gamma$ does not have this type of restriction nor does the joint possibility distribution J_γ given by (5.6).

Example 6.2. *Let $A = (-2; -1; 1; 2)$ and $B = (-1; 0; 1)$ be fuzzy numbers whose $\alpha-$cuts are given by $[A]^\alpha = [-2+\alpha, 2-\alpha]$ and $[B]^\alpha = [-1+\alpha, 1-\alpha]$. Thus*

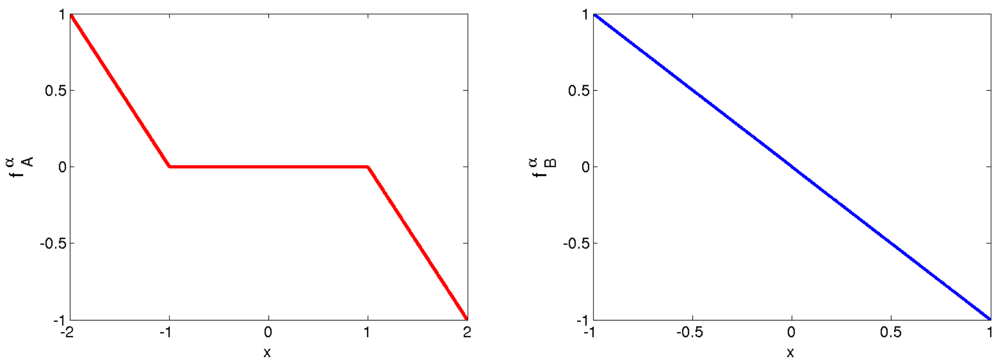
$$f_A^\alpha(a_\alpha^-) = \begin{cases} 1-\alpha & , \textit{if} \quad 0 \leqslant \alpha < 1 \\ 0 & , \textit{if} \quad \alpha = 1 \end{cases} \quad \textit{and} \quad f_A^\alpha(a_\alpha^+) = \begin{cases} -1+\alpha & , \textit{if} \quad 0 \leqslant \alpha < 1 \\ 0 & , \textit{if} \quad \alpha = 1 \end{cases}$$

and

$$f_B^\alpha(b_\alpha^-) = 1-\alpha \quad \textit{and} \quad f_B^\alpha(b_\alpha^+) = -1+\alpha,$$

which are depicted in Figure 40.

Figure 40 – Graphical representation of the functions f_A^α and f_B^α given as in Example 6.2.



The red line at left subfigure represents the function f_A^α and the blue line at right subfigure represents the function f_B^α, where $A = (-2; -1; 1; 2)$ and $B = (-1; 0; 1)$. Source: Author

For each $\gamma \in [0, 1]$, it follows

$$\begin{aligned} I_A(a_\alpha^-, \alpha, \gamma) &= [1-\alpha+\gamma(-2+2\alpha), 1-\alpha], \ 0 \leqslant \alpha < 1 \\ I_A(a_\alpha^+, \alpha, \gamma) &= [-1+\alpha, -1+\alpha+\gamma(2-2\alpha)], \ 0 \leqslant \alpha < 1 \\ I_A(x, 1, \gamma) &= \{0\}, \ \forall x \in [A]^1, \end{aligned}$$

and

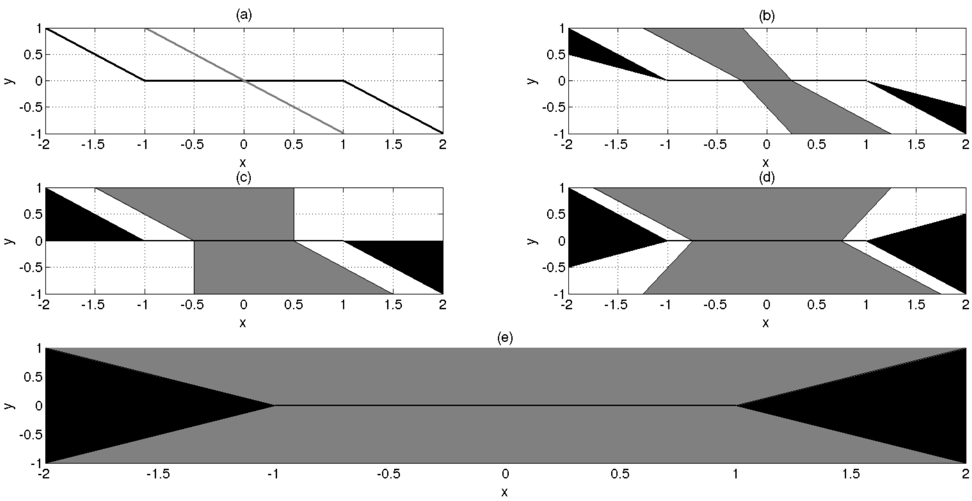
$$\begin{aligned} I_B(b_\alpha^-, \alpha, \gamma) &= [1-\alpha+\gamma, 1-\alpha+\gamma(-3+2\alpha)], \ 0 \leqslant \alpha \leqslant 1 \\ I_B(b_\alpha^+, \alpha, \gamma) &= [-1+\alpha+\gamma(3-2\alpha), -1+\alpha-\gamma], \ 0 \leqslant \alpha \leqslant 1. \end{aligned}$$

Figure 41 depicts the sets $P_A(\gamma)$ (black region) and $P_B(\gamma)$ (gray region) for $\gamma = 0, 0.25, 0.5, 0.75, 1$. Note that independently of the choice of the fuzzy numbers A and B, the larger the parameter γ the greater the region $P(\gamma)$.

Figure 42 graphically represents the tri-dimensional view of the joint possibility distribution $\mathcal{J}_\gamma$ for some values of γ.

Figure 43 compares the regions $P(\gamma)$ of the joint possibility distributions J_γ and $\mathcal{J}_\gamma$, for different values of γ. One can observe that the region where $\mathcal{J}_\gamma(u, v) > 0$ is smaller than the region where $J_\gamma(u, v) > 0$, for all values of $\gamma \in \{0, 0.25, 0.5, 0.75, 1\}$.

Figure 41 – Graphical representation of the sets $P_A(\gamma)$ and $P_B(\gamma)$ given as in Example 6.2.



The subfigures $(a), (b), (c), (d), (e)$ exhibit a top view of $P_A(\gamma)$ (represented by black region) and $P_B(\gamma)$ (represented by gray region) for $\gamma = 0, 0.25, 0.5, 0.75, 1$, respectively, where $A = (-2; -1; 1; 2)$ and $B = (-1; 0; 1)$. Source: Author

Theorem 6.1 shows that $\mathcal{J} = \{\mathcal{J}_\gamma : \gamma \in [0,1]\}$ is a family of joint possibility distributions of A and B. Moreover, the interactive sum between A and B, obtained via sup-J extension principle and denoted by $A +_\gamma B$, is a fuzzy number in $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$.

Theorem 6.1. *Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ and $\mathcal{J}_\gamma$ be the fuzzy relation given by (6.3). For each $\gamma \in [0,1]$, it follows that:*

- (a) *$\mathcal{J}_\gamma$ is a joint possibility distribution between A and B;*
- (b) *$\mathcal{J}_\gamma(x_1, x_2) \leqslant \mathcal{J}_\lambda(x_1, x_2)$, for all $0 \leqslant \gamma \leqslant \lambda \leqslant 1$ and for all $(x_1, x_2) \in \mathbb{R}^2$;*
- (c) *$A +_\gamma B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$.*

Proof. (a) Let us first prove that $I_A(x, \alpha, \gamma) \subseteq [B]^\alpha$, $\forall x \in [A]^\alpha$ and $I_B(y, \alpha, \gamma) \subseteq [A]^\alpha$, $\forall y \in [B]^\alpha$ for all $\alpha \in [0,1]$ and for all $\gamma \in [0,1]$. Suppose that $\alpha < 1$. Let $z \in I_A(x, \alpha, \gamma)$, where $x \in \{a^-_\alpha, a^+_\alpha\}$. Let us divide this proof into two cases.

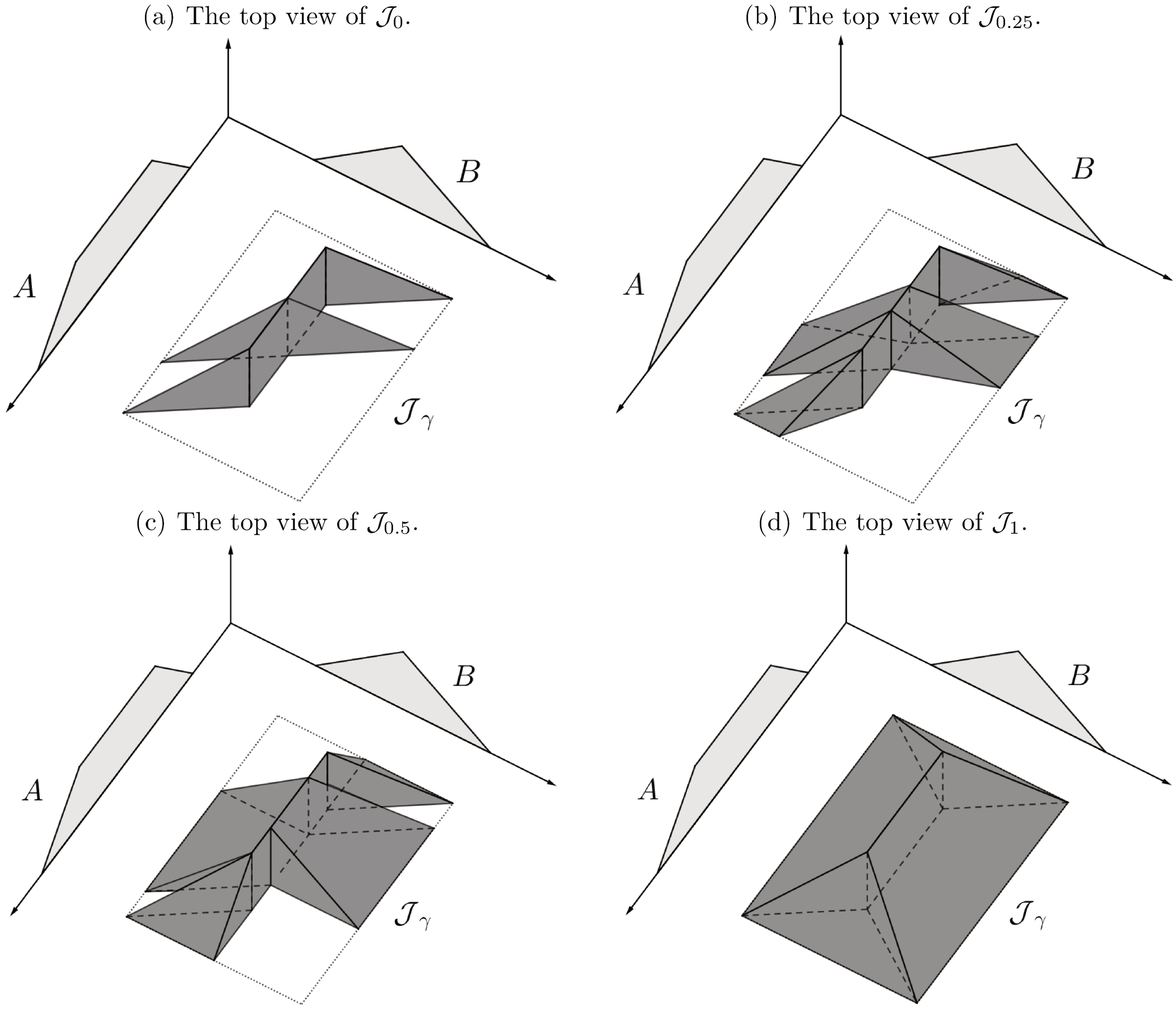
(i) Suppose that $x = a^-_\alpha$.

If $-(a^{(\bar{a})})^-_\alpha \leqslant (b^{(\bar{b})})^+_\alpha$ then

$$b^-_\alpha \leqslant \bar{b} - (a^{(\bar{a})})^-_\alpha + \gamma((b^{(\bar{b})})^-_\alpha + (a^{(\bar{a})})^-_\alpha) \leqslant z \leqslant \bar{b} - (a^{(\bar{a})})^-_\alpha + \gamma((b^{(\bar{b})})^+_\alpha + (a^{(\bar{a})})^-_\alpha) \leqslant b^+_\alpha,$$

which implies that $z \in [B]^\alpha$.

Figure 42 – Graphical representation of the joint possibility distribution $\mathcal{J}_\gamma$ given as in Example 6.2.



The light gray trapezoid represents the trapezoidal fuzzy number $A = (-2; -1; 1; 2)$ and the light gray triangle represents the triangular fuzzy number $B = (-1; 0; 1)$. The dark grey region represents the joint possibility distribution $\mathcal{J}_\gamma$. Source: Author.

On the other hand, if $-(a^{(\overline{a})})^-_\alpha \geqslant (b^{(\overline{b})})^+_\alpha$ then

$$b^-_\alpha \leqslant \overline{b} + (b^{(\overline{b})})^+_\alpha + \gamma((b^{(\overline{b})})^-_\alpha - (b^{(\overline{b})})^+_\alpha) \leqslant z \leqslant \overline{b} + (b^{(\overline{b})})^+_\alpha = b^+_\alpha,$$

which implies that $z \in [B]^\alpha$.

Therefore, $I_A(a^-_\alpha, \alpha, \gamma) \subseteq [B]^\alpha$, $\forall \alpha \in [0, 1)$ and $\forall \gamma \in [0, 1]$.

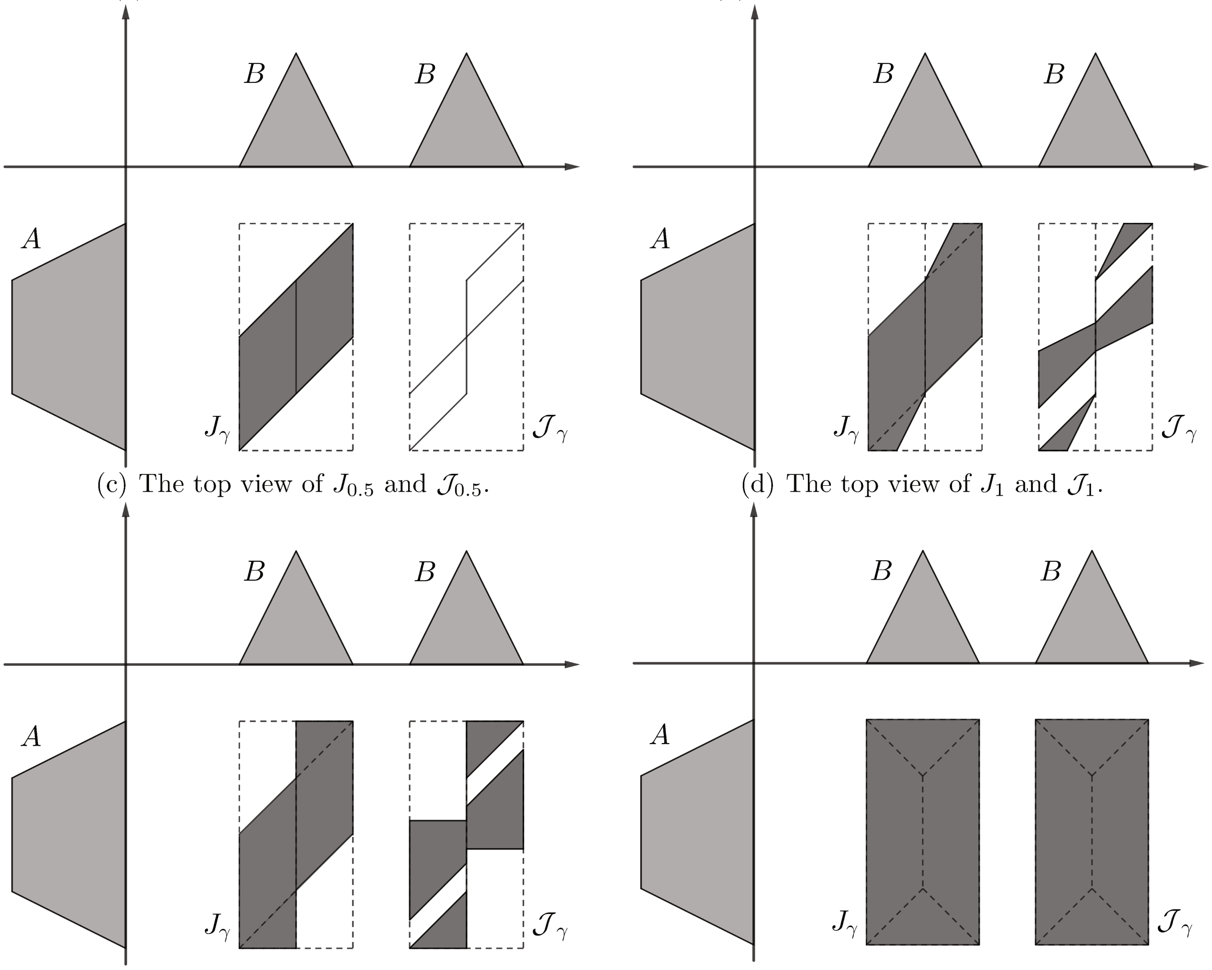
(ii) Suppose $x = a^+_\alpha$.

If $-(a^{(\overline{a})})^+_\alpha \leqslant (b^{(\overline{b})})^-_\alpha$ then

$$b^-_\alpha = \overline{b} + (b^{(\overline{b})})^-_\alpha \leqslant z \leqslant \overline{b} + (b^{(\overline{b})})^-_\alpha + \gamma((b^{(\overline{b})})^+_\alpha - (b^{(\overline{b})})^-_\alpha) \leqslant b^+_\alpha,$$

which implies that $z \in [B]^\alpha$.

Figure 43 – Graphical representation of the joint possibility distribution $\mathcal{J}_\gamma$ given as in Example 6.2.



(a) The top view of J_0 and $\mathcal{J}_0$.

(b) The top view of $J_{0.25}$ and $\mathcal{J}_{0.25}$.

(c) The top view of $J_{0.5}$ and $\mathcal{J}_{0.5}$.

(d) The top view of J_1 and $\mathcal{J}_1$.

The light gray trapezoids represent the trapezoidal fuzzy numbers $A = (-2; -1; 1; 2)$ and the light gray triangles represent the triangular fuzzy numbers $B = (-1; 0; 1)$. The left dark grey regions represent the joint possibility distribution J_γ. The right dark grey regions represent the joint possibility distribution $\mathcal{J}_\gamma$. Source: Author.

If $-(a^{(\overline{a})})^+_\alpha \geqslant (b^{(\bar{b})})^-_\alpha$ then

$$b^-_\alpha \leqslant \bar{b} - (a^{(\overline{a})})^+_\alpha + \gamma((b^{(\bar{b})})^-_\alpha + (a^{(\overline{a})})^+_\alpha) \leqslant z \leqslant \bar{b} - (a^{(\overline{a})})^+_\alpha + \gamma((b^{(\bar{b})})^+_\alpha + (a^{(\overline{a})})^+_\alpha) \leqslant b^+_\alpha,$$

which implies that $z \in [B]^\alpha$.

Therefore, $I_A(a^+_\alpha, \alpha, \gamma) \subseteq [B]^\alpha, \ \forall \alpha \in [0, 1)$ and $\forall \gamma \in [0, 1]$.

Let us prove the above statement in the case where $\alpha = 1$. Let $z \in I_A(x, 1, \gamma)$, where $x \in [A]^1$. Also, let us divide this proof into two cases.

(i) Suppose $x \in [a^-_1, 0]$.

If $-(x-\bar{a}) \leqslant (b^{(\bar{b})})_1^+$ then

$$b_1^- \leqslant \bar{b} - (x-\bar{a}) + \gamma((b^{(\bar{b})})_1^- + (x-\bar{a})) \leqslant z \leqslant \bar{b} - (x-\bar{a}) + \gamma((b^{(\bar{b})})_1^+ + (x-\bar{a})) \leqslant b_1^+,$$

which implies that $z \in [B]^1$. If $-(x-\bar{a}) \geqslant (b^{(\bar{b})})_1^+$ then

$$b_1^- \leqslant \bar{b} + (b^{(\bar{b})})_1^+ + \gamma((b^{(\bar{b})})_1^- - (b^{(\bar{b})})_1^+) \leqslant z \leqslant \bar{b} + (b^{(\bar{b})})_1^+ = b_1^+,$$

which implies that $z \in [B]^1$.

(ii) Suppose $x \in [0, a_1^+]$.
If $-(x-\bar{a}) \leqslant (b^{(\bar{b})})_1^-$ then

$$b_1^- = \bar{b} + (b^{(\bar{b})})_1^- \leqslant z \leqslant \bar{b} + (b^{(\bar{b})})_1^- + \gamma((b^{(\bar{b})})_1^+ - (b^{(\bar{b})})_1^-) \leqslant b_1^+,$$

which implies that $z \in [B]^1$. If $-(x-\bar{a}) \geqslant (b^{(\bar{b})})_1^-$ then

$$b_1^- \leqslant \bar{b} - (x-\bar{a}) + \gamma((b^{(\bar{b})})_1^- + (x-\bar{a})) \leqslant z \leqslant \bar{b} - (x-\bar{a}) + \gamma((b^{(\bar{b})})_1^+ + (x-\bar{a})) \leqslant b_1^+,$$

which implies that $z \in [B]^1$.

Therefore, $I_A(x,1,\gamma) \subseteq [B]^1$, concluding that the inclusion holds for all $\alpha \in [0,1]$. Similarly, one can prove that $I_B(y,\alpha,\gamma) \subseteq [A]^\alpha,\ \forall \alpha,\gamma \in [0,1]$.

Now, let us prove that $\mathcal{J}_\gamma$ is a joint possibility distribution for A and B for all $\gamma \in [0,1]$. To this end, let $\gamma \in [0,1]$. On the one hand, $\mathcal{J}_\gamma(x_1,x_2) = A(x_1) \wedge B(x_2) \leqslant A(x_1),\ \forall (x_1,x_2) \in P(\gamma)$. Hence,

$$\prod_{\mathcal{J}_\gamma}^{A}(x_1) = \bigvee_{x_2 \in \mathbb{R}} \mathcal{J}_\gamma(x_1,x_2) \leqslant A(x_1).$$

On the other hand, for each $\alpha \in [0,1]$ let $\overline{x} \in [A]^\alpha$ such that $A(\overline{x}) = \alpha$.

If $\alpha < 1$, then $a^-_{A(\overline{x})} = \alpha$ or $a^+_{A(\overline{x})} = \alpha$, since $a^\pm_\alpha$ are continuous. From definition of $P(\gamma)$, exists $w \in I_A(\overline{x}, A(\overline{x}), \gamma)$ such that $(\overline{x}, w) \in P_A(\gamma)$, which implies that $\varnothing \neq P_A(\gamma) \subseteq P(\gamma)$. Since $w \in I_A(\overline{x}, A(\overline{x}), \gamma) \subseteq [B]^{A(\overline{x})}$, it follows that $B(w) \geqslant A(\overline{x}) = \alpha$. Therefore,

$$A(\overline{x}) = A(\overline{x}) \wedge B(w) = \mathcal{J}_\gamma(\overline{x}, w) \leqslant \bigvee_{u \in \mathbb{R}} \mathcal{J}_\gamma(\overline{x}, u) = \Pi^A_{\mathcal{J}_\gamma}(\overline{x}).$$

If $\alpha = 1$, then $\overline{x} \in [a_1^-, a_1^+]$. By definition of $P(\gamma)$ exists $w \in I_A(\overline{x}, 1, \gamma)$ such that $(\overline{x}, w) \in P_A(\gamma)$, which implies that $\varnothing \neq P_A(\gamma) \subseteq P(\gamma)$. Since $w \in I_A(\overline{x}, 1, \gamma) \subseteq [B]^1$, it follows that $B(w) \geqslant 1 = A(\overline{x})$. Therefore,

$$A(\overline{x}) = A(\overline{x}) \wedge B(w) = \mathcal{J}_\gamma(\overline{x}, w) \leqslant \bigvee_{u \in \mathbb{R}} \mathcal{J}_\gamma(\overline{x}, u) = \Pi^A_{\mathcal{J}_\gamma}(\overline{x}).$$

Consequently, $\Pi^A_{\mathcal{J}_\gamma} = A$. Similarly, it can be prove that $\Pi^B_{\mathcal{J}_\gamma} = B$. Hence, $\mathcal{J}_\gamma$ is a joint possibility distribution for A and B, for all $\gamma \in [0,1]$.

(b) Let us prove that $\mathcal{J}_\gamma(x_1,x_2) \leqslant \mathcal{J}_\lambda(x_1,x_2)$, for $0 \leqslant \gamma \leqslant \lambda \leqslant 1$. To this end, let $(x,y) \in [\mathcal{J}_\gamma]^\alpha$, for some $\alpha \in [0,1]$. Then $(x,y) \in P_A(\gamma)$ or $(x,y) \in P_B(\gamma)$. Without loss of generality, suppose that $(x,y) \in P_A(\gamma)$.

If $\alpha < 1$, then $(x,y) \in \{a^-_\alpha, a^+_\alpha\} \times I_A(x,\alpha,\gamma)$. Let us divide this proof in two cases.

(i) Suppose $x = a^-_\alpha$.
If $(a^{(\overline{a})})^-_\alpha + (b^{(\overline{b})})^+_\alpha \geqslant 0$ then

$$\overline{b} - (a^{(\overline{a})})^-_\alpha + \lambda((b^{(\overline{b})})^-_\alpha + (a^{(\overline{a})})^-_\alpha) \leqslant \overline{b} - (a^{(\overline{a})})^-_\alpha + \gamma((b^{(\overline{b})})^-_\alpha + (a^{(\overline{a})})^-_\alpha) \leqslant y$$

and

$$y \leqslant \overline{b} - (a^{(\overline{a})})^-_\alpha + \gamma((b^{(\overline{b})})^+_\alpha + (a^{(\overline{a})})^-_\alpha) \leqslant \overline{b} - (a^{(\overline{a})})^-_\alpha + \lambda((b^{(\overline{b})})^+_\alpha + (a^{(\overline{a})})^-_\alpha).$$

This implies that $y \in I_A(a^-_\alpha, \alpha, \lambda)$.
If $(a^{(\overline{a})})^-_\alpha + (b^{(\overline{b})})^+_\alpha \leqslant 0$ then

$$\overline{b} + (b^{(\overline{b})})^+_\alpha + \lambda((b^{(\overline{b})})^-_\alpha - (b^{(\overline{b})})^+_\alpha) \leqslant \overline{b} + (b^{(\overline{b})})^+_\alpha + \gamma((b^{(\overline{b})})^-_\alpha - (b^{(\overline{b})})^+_\alpha) \leqslant y,$$

and

$$y \leqslant \overline{b} + (b^{(\overline{b})})^+_\alpha = \overline{b} + (b^{(\overline{b})})^+_\alpha + \gamma((b^{(\overline{b})})^+_\alpha - (b^{(\overline{b})})^+_\alpha) = \overline{b} + (b^{(\overline{b})})^+_\alpha + \lambda((b^{(\overline{b})})^+_\alpha - (b^{(\overline{b})})^+_\alpha),$$

which also implies that $y \in I_A(a^-_\alpha, \alpha, \lambda)$.

(ii) Suppose $x = a^+_\alpha$.
If $(a^{(\overline{a})})^+_\alpha + (b^{(\overline{b})})^-_\alpha \geqslant 0$ then

$$\overline{b} + (b^{(\overline{b})})^-_\alpha + \lambda((b^{(\overline{b})})^-_\alpha - (b^{(\overline{b})})^-_\alpha) = \overline{b} + (b^{(\overline{b})})^-_\alpha + \gamma((b^{(\overline{b})})^-_\alpha - (b^{(\overline{b})})^-_\alpha) = \overline{b} + (b^{(\overline{b})})^-_\alpha \leqslant y,$$

and

$$y \leqslant \overline{b} + (b^{(\overline{b})})^-_\alpha + \gamma((b^{(\overline{b})})^+_\alpha - (b^{(\overline{b})})^-_\alpha) \leqslant \overline{b} + (b^{(\overline{b})})^-_\alpha + \lambda((b^{(\overline{b})})^+_\alpha - (b^{(\overline{b})})^-_\alpha).$$

This implies that $y \in I_A(a^+_\alpha, \alpha, \lambda)$.
If $(a^{(\overline{a})})^+_\alpha + (b^{(\overline{b})})^-_\alpha \leqslant 0$ then

$$\overline{b} - (a^{(\overline{a})})^+_\alpha + \lambda((b^{(\overline{b})})^-_\alpha + (a^{(\overline{a})})^+_\alpha) \leqslant \overline{b} - (a^{(\overline{a})})^+_\alpha + \gamma((b^{(\overline{b})})^-_\alpha + (a^{(\overline{a})})^+_\alpha) \leqslant y,$$

and

$$y \leqslant \overline{b} - (a^{(\overline{a})})^+_\alpha + \gamma((b^{(\overline{b})})^+_\alpha + (a^{(\overline{a})})^+_\alpha) \leqslant \overline{b} - (a^{(\overline{a})})^+_\alpha + \lambda((b^{(\overline{b})})^+_\alpha + (a^{(\overline{a})})^+_\alpha),$$

this also implies that $y \in I_A(a^+_\alpha, \alpha, \lambda)$.

If $\alpha = 1$ then suppose $x \in [a_1^-, 0]$. If $-(x - \bar{a}) \leqslant (b^{(\bar{b})})_1^+$ then

$$\bar{b} - (x - \bar{a}) + \lambda((b^{(\bar{b})})_1^- + (x - \bar{a})) \leqslant \bar{b} - (x - \bar{a}) + \gamma((b^{(\bar{b})})_1^- + (x - \bar{a})) \leqslant y,$$

and

$$y \leqslant \bar{b} - (x - \bar{a}) + \gamma((b^{(\bar{b})})_1^+ + (x - \bar{a})) \leqslant \bar{b} - (x - \bar{a}) + \lambda((b^{(\bar{b})})_1^+ + (x - \bar{a})),$$

this means that $y \in I_A(x, 1, \lambda)$.

If $-(x - \bar{a}) \geqslant (b^{(\bar{b})})_1^+$ then

$$\bar{b} + (b^{(\bar{b})})_1^+ + \lambda((b^{(\bar{b})})_1^- - (b^{(\bar{b})})_1^+) \leqslant \bar{b} - (x - \bar{a}) + \gamma((b^{(\bar{b})})_1^- - (b^{(\bar{b})})_1^+) \leqslant y,$$

and

$$y \leqslant \bar{b} + (b^{(\bar{b})})_1^+ + \gamma((b^{(\bar{b})})_1^+ - (b^{(\bar{b})})_1^+) = \bar{b} + (b^{(\bar{b})})_1^+ + \lambda((b^{(\bar{b})})_1^+ - (b^{(\bar{b})})_1^+),$$

which implies that $y \in I_A(x, 1, \lambda)$.

The case where $x \in [0, a_1^+]$ is analogous.

Therefore, $[J_\gamma]^\alpha \subseteq [J_\lambda]^\alpha$, for all $0 \leqslant \gamma \leqslant \lambda \leqslant 1$. Consequently, $J_\gamma(x_1, x_2) \leqslant J_\lambda(x_1, x_2)$, $\forall (x_1, x_2) \in \mathbb{R}^2$.

(c) Let us first prove that $A +_\gamma B$ is a fuzzy number by showing that $[A +_\gamma B]^\alpha$ is a non-empty, compact and connected set of $\mathbb{R}$, for all $\alpha \in [0, 1]$. These conditions imply that $[A +_\gamma B]^\alpha$ is a closed and boundary interval, which by definition yields $A +_\gamma B \in \mathbb{R}_\mathcal{F}$.

By Theorem 2.1, $[A +_\gamma B]^\alpha = +[\mathcal{J}_\gamma]^\alpha$. Thus, it is sufficient to show that $[\mathcal{J}_\gamma]^\alpha$ satisfies the above properties for all $\alpha \in [0, 1]$. Since $[\mathcal{J}_\gamma]^1 \neq \varnothing$, it follows that $[A +_\gamma B]^\alpha \neq \varnothing$, $\forall \alpha \in [0, 1]$. Hence $[A +_\gamma B]^\alpha$ is a non-empty set.

Let us prove that $[\mathcal{J}_\gamma]^\alpha$ is a compact set, that is, all sequences in $[\mathcal{J}_\gamma]^\alpha$ has a subsequence that converges to a point in $[\mathcal{J}_\gamma]^\alpha$. First, note that $P(\gamma)$ is bounded for all $\gamma \in [0, 1]$, since $P(\gamma) \subseteq [A]^0 \times [B]^0$. Let (x_n, y_n) be a sequence in $[\mathcal{J}_\gamma]^\alpha$, so exists (α_n) and (i_n) such that $(x_n, y_n) \in P_{i_n}(\gamma)$, where $i_n \in \{A, B\}$ and $\alpha \leqslant \alpha_n \leqslant A(x_n), B(y_n)$, $\forall n \in \mathbb{N}$. Since $\{A, B\}$ is a finite set, there is a convergent subsequence (i_{n_k}) of (i_n), that is, for every $\epsilon > 0$ there exists n_{k_0} such that $i_{n_k} = A$ or $i_{n_k} = B$, for all $n_k \geqslant n_{k_0}$.

Suppose, without loss of generality, that $A_{n_k} = A$, $\forall n \in \mathbb{N}$. By definition of $P_A(\gamma)$, the subsequence (x_{n_k}) of (x_n) satisfies $x_{n_k} = a^-_{\alpha_{n_k}}$, $x_{n_k} = a^+_{\alpha_{n_k}}$ or $x_{n_k} \in [a_1^-, a_1^+]$, $\forall n_k \geqslant n_{k_0}$. Let us divide this proof into three cases.

(i) If $x_{n_k} = a^-_{\alpha_{n_k}}$, then $y_{n_k} \in I_A(a^-_{\alpha_{n_k}}, \alpha_{n_k}, \gamma) \subseteq [B]^{\alpha_{n_k}}$.

Since $\alpha_{n_k} \in [0, 1]$, $y_{n_k} \in [B]^{\alpha_{n_k}}$ and B is upper semi-continuous (see Proposition 1.1), there are convergent subsequences $(\alpha_{n_{k_i}})$ and $(y_{n_{k_i}})$ such that

$$(\alpha_{n_{k_i}}) \to \overline{\alpha} \geqslant \alpha \text{ and } (y_{n_{k_i}}) \to \overline{y} \in I_A(a^-_{\overline{\alpha}}, \overline{\alpha}, \gamma) \subseteq [B]^{\overline{\alpha}}.$$

Consequently, $B(\overline{y}) \geqslant \overline{\alpha}$. Since $A \in \mathbb{R}_{\mathcal{F}_C}$, one obtains

$$\overline{x} = \lim_{i\to\infty} x_{n_{k_i}} = \lim_{i\to\infty} a^-_{\alpha_{n_{k_i}}} = a^-_{\overline{\alpha}},$$

thus $A(\overline{x}) = A(a^-_{\overline{\alpha}}) \geqslant \overline{\alpha}$.

Therefore, $(\overline{x}, \overline{y}) \in \{a^-_{\overline{\alpha}}\} \times \{I_A(a^-_{\overline{\alpha}}, \overline{\alpha}, \gamma)\} \subseteq P_A(\gamma)$. Finally,

$$\mathcal{J}_\gamma(\overline{x}, \overline{y}) = A(\overline{x}) \wedge B(\overline{y}) \geqslant \overline{\alpha} \Rightarrow (\overline{x}, \overline{y}) \in [\mathcal{J}_\gamma]^{\overline{\alpha}} \subseteq [\mathcal{J}_\gamma]^\alpha.$$

(*ii*) If $x_{n_k} = a^+_{\alpha_{n_k}}$, then the proof is analogous to the item (*i*).

(*iii*) If $x_{n_k} \in [a^-_1, a^+_1]$, then $y_{n_k} \in I_A(x_{n_k}, 1, \gamma) \subseteq [B]^1$. Let $(x_{n_{k_i}}, y_{n_{k_i}})$ be a convergent subsequence such that $x_{n_{k_i}} \to \overline{x} \in [A]^1$ and $y_{n_{k_i}} \to \overline{y}$. From the continuity of the endpoints of $I_A(\overline{x}, 1, \gamma)$, it follows $\overline{y} \in I_A(\overline{x}, 1, \gamma)$. Thus $(\overline{x}, \overline{y}) \in P(\gamma)$ and

$$\mathcal{J}_\gamma(\overline{x}, \overline{y}) = A(\overline{x}) \wedge B(\overline{y}) = 1 \geqslant \alpha \Rightarrow (\overline{x}, \overline{y}) \in [\mathcal{J}_\gamma]^\alpha$$

Therefore, by items (*i*), (*ii*), and (*iii*) one concludes that $[\mathcal{J}_\gamma]^\alpha$ is compact.

Now, let us show that $[\mathcal{J}_\gamma]^\alpha$ is a (path) connected set.

Let $(x_1, x_2), (y_1, y_2) \in [\mathcal{J}_\gamma]^\alpha$. Thus exist $(i_x), (i_y), (\alpha_x)$ and (α_y) such that $(x_1, x_2) \in P_{i_x}(\gamma)$, $(y_1, y_2) \in P_{i_y}(\gamma)$, where $(i_x), (i_y) \in \{A, B\}$. Thus $\alpha \leqslant \alpha_x \leqslant A(x_1), B(x_2)$ and $\alpha \leqslant \alpha_y \leqslant A(y_1), B(y_2)$.

Let us consider the case where $i_x = i_y$ and w.l.o.g suppose that $i_x = i_y = A$.

Suppose $A(x_1) \leqslant A(y_1)$ (the other case is similar). Let us divide this proof into three cases.

(*i*) If $x_1 = a^-_{\alpha_x}$ and $y_1 = a^-_{\alpha_y}$ then the path

$$\begin{aligned}
&\{x_1\} \times [\bar{b} + f_A^{\alpha_x}(x_1) + \gamma((b^{\bar{b}})^-_{\alpha_x} - f_A^{\alpha_x}(x_1)), x_2] \cup \\
&\{(a^-_\beta, \bar{b} + f_A^\beta(a^-_\beta) + \gamma((b^{\bar{b}})^-_\beta - f_A^\beta(a^-_\beta))) : \beta \in (\alpha_x, \alpha_y)\} \cup \\
&\{y_1\} \times [\bar{b} + f_A^{\alpha_y}(y_1) + \gamma((b^{\bar{b}})^-_{\alpha_y} - f_A^{\alpha_y}(y_1)), y_2],
\end{aligned}$$

connects (x_1, x_2) and (y_1, y_2) in $[\mathcal{J}_\gamma]^\alpha$, since the above sets are contained in $P_A(\gamma)$.

(*ii*) If $x_1 = a^+_{\alpha_x}$ and $y_1 = a^+_{\alpha_y}$ then the path can be constructed in a similar way of item (*i*).

(*iii*) If $x_1, y_1 \in [a^-_1, a^+_1]$, i.e, $\alpha_x = \alpha_y = 1$. Suppose w.l.o.g that $x_1 \leqslant y_1$. Thus, the points (x_1, x_2) and (y_1, y_2) are connected by the following path:

$$\begin{aligned}
&\{x_1\} \times [\bar{b} + f_A^1(x_1) + \gamma((b^{(\bar{b})})^-_1 - f_A^1(x_1)), x_2] \cup \\
&\{(z, \bar{b} + f_A^1(z) + \gamma((b^{(\bar{b})})^-_1 - f_A^1(z))) : z \in (x_1, y_1)\} \cup \\
&\{y_1\} \times [\bar{b} + f_A^1(y_1) + \gamma((b^{(\bar{b})})^-_1 - f_A^1(y_1)), y_2],
\end{aligned}$$

since the above sets are contained in $P_A(\gamma)$.

On the other hand, let us consider the case where $i_x \neq i_y$, that is, $(x_1, x_2) \in P_A(\gamma)$ and $(y_1, y_2) \in P_B(\gamma)$. Since $[\mathcal{J}_\gamma]^1 \neq \varnothing$, there exists $(z_1, z_2) \in I_B(z_2, 1, \gamma) \times I_A(z_1, 1, \gamma) \subseteq [A]^1 \times [B]^1$. A combination of the previously construction ensures that there are paths ρ_1 and ρ_2 which connect (x_1, x_2) to (z_1, z_2) and (y_1, y_2) to (z_1, z_2). Thus $\rho_1 \cup \rho_2$ is a path connecting (x_1, x_2) to (y_1, y_2), since $(z_1, z_2) \in P_A(\gamma) \cap P_B(\gamma)$. Therefore, $[\mathcal{J}_\gamma]^\alpha$ is a (path) connected set, which implies that $+([\mathcal{J}_\gamma]^\alpha)$ is a closed interval of $\mathbb{R}$, since $[\mathcal{J}_\gamma]^\alpha$ is also compact. By Theorem 1.1, $A +_\gamma B$ is a fuzzy number, for all $\gamma \in [0, 1]$.

Finally, it will be shown that $A +_\gamma B \in \mathbb{R}_{\mathcal{F}_C}$. In view of Proposition 1.2, it remains to prove that c^-_α and c^+_α are right-continuous functions in $(0, 1]$, where $[A +_\gamma B]^\alpha = [c^-_\alpha, c^+_\alpha]$. The fact that $[A +_\gamma B]^\alpha$ is a non-empty, compact and connected set, for all $\alpha \in [0, 1]$, implies that for every $z \in [A +_\gamma B]^\alpha$ there exists $(x_1, x_2) \in [\mathcal{J}_\gamma]^\alpha$ such that $z = x_1 + x_2$. Let (α_n) be a sequence that converges to $\alpha \in (0, 1)$ such that $\alpha \leqslant \alpha_n$, $\forall n \in \mathbb{N}$. If $z_n = c^-_{\alpha_n} \in [A +_\gamma B]^\alpha$ then $z_n \geqslant z := c^-_\alpha \in [A +_\gamma B]^\alpha$, since c^-_α is increasing. This observation reveals that $z_n \in [z, c^-_1]$, for all $n \in \mathbb{N}$. Thus, one can extract a decreasing convergent subsequence $(z_{n_k}) \subseteq (z_n)$ such that $\lim_{k \to \infty} z_{n_k} = \overline{z}$ and $z_{n_{k+1}} \leqslant z_{n_k}$. Obviously, $z \leqslant \overline{z}$. Let us prove that $z = \overline{z}$. To this end, suppose $z < \overline{z}$. Thus $(A +_\gamma B)(w) = \alpha$, $\forall w \in [z, \overline{z})$, since $[A +_\gamma B]^\alpha$ is close interval of $\mathbb{R}$.

If $\mathcal{J}_\gamma(x_1, x_2) = \alpha \in (0, 1)$, then by Equation (6.3), one obtains $A(x_1) = \alpha$ or $B(x_2) = \alpha$. Due to the continuity of the functions a^-_α and a^+_α the set $\{x : A(x) = \alpha\}$ is equal to $\{a^-_\alpha : A(a^-_\alpha) = \alpha\} \cup \{a^+_\alpha : A(a^+_\alpha) = \alpha\}$ for $\alpha \in (0, 1)$, consequently, this set has at most two elements. Consider the following elements $z < w_1 < w_2 < w_3 < w_4 < \overline{z}$ and $(x^j_1, x^j_2) \in [\mathcal{J}_\gamma]^\alpha$ such that $w_j = x^j_1 + x^j_2$ for $j = 1, \ldots, 4$. Thus, it is possible to determine w_{p_1} and w_{p_2} with $p_1 < p_2$ such that $\overline{x}_i := x^{p_1}_i = x^{p_2}_i$, $A(x^{p_k}_1) = \alpha$ and $B(x^{p_k}_2) > \alpha$, for $k = 1, 2$, which implies that $a^-_\alpha = \overline{x}_i$ or $a^+_\alpha = \overline{x}_i$, where $i \in \{1, 2\}$. The definition of $P_A(\gamma)$ ensures that $x^{p_k}_2 \in I_A(x^{p_k}_1, \alpha, \gamma)$, for $k = 1, 2$. Since $x^{p_1}_2 = w_{p_1} - \overline{x}_1 < w_{p_2} - \overline{x}_1 = x^{p_2}_2$, for $x^{p_1}_2 < \overline{x}_2 < x^{p_2}_2$ it follows that $(\overline{x}_1, \overline{x}_2) \in P_A(\gamma) \subseteq [\mathcal{J}_\gamma]^\alpha$ and $B(\overline{x}_2) > \alpha$. Suppose that $a^-_\alpha = \overline{x}_1$. For $\overline{z} - w_{p_2} > \epsilon > 0$ exists $\delta > 0$, such that $\alpha < \beta < \alpha + \delta$ implies $0 \leqslant a^-_\beta - \overline{x}_1 < \epsilon$ and $Y := I_A(a^-_\beta, \beta, \gamma) \cap (x^{p_1}_2, x^{p_2}_2) \neq \varnothing$, since a^-_α and the endpoints of $I_A(a^-_\alpha, \alpha, \gamma)$ are continuous. Thus $(x_1, x_2) \in P_A(\gamma)$, where $x_1 = a^-_\beta$ and $x_2 \in Y \subseteq [B]^\beta$. Consequently, $w_{p_1} = \overline{x}_1 + x^{p_1}_2 \leqslant x_1 + x_2 < a^-_\beta + x^{p_2}_2 \leqslant \overline{x}_1 + x^{p_2}_2 + \epsilon \leqslant w_{p_2} + \epsilon < \overline{z}$. Thus, one obtains the contradicting statement

$$\alpha = (A +_\gamma B)(x_1 + x_2) \geqslant \mathcal{J}_\gamma(x_1, x_2) \geqslant \beta > \alpha.$$

A similar analysis can be made in the case where $a^+_\alpha = \overline{x}_1$. Therefore, $z = \overline{z}$. Thus c^-_α is right-continuous in the interval $(0, 1)$. Similarly, it can be prove that c^+_α is right-continuous in the interval $(0, 1)$. Hence, $A +_\gamma B \in \mathbb{R}_{\mathcal{F}_C}$.

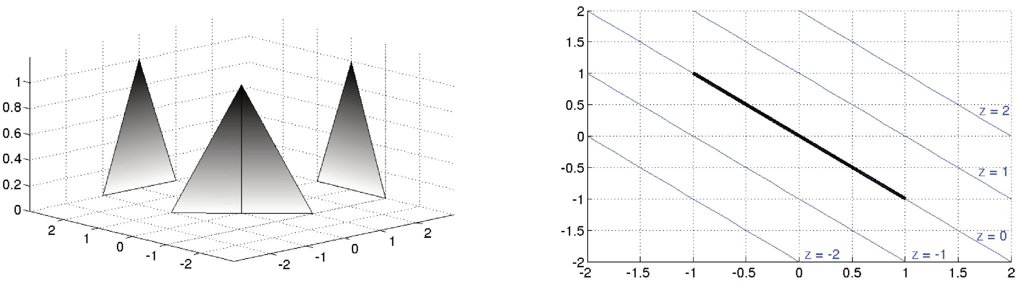
□

An immediately consequence of item (b) of Theorem 6.1 is that $\mathcal{J}_1$ has greater region where $\mathcal{J}_\gamma(u,v) \neq 0$, than $\mathcal{J}_\gamma$ for all $\gamma \in [0,1)$. One can prove that $\mathcal{J}_1 = \mathcal{J}_\wedge$, by following the similar steps given as in [54], which means that if $\mathcal{J}_1$ is the joint possibility distribution under consideration, then non-interactive fuzzy numbers are being considered. Hence one obtains $J_1 = \mathcal{J}_1 = J_\wedge$, as expected from Figure 43 - (d).

The next example illustrates the properties given by Theorem 6.1.

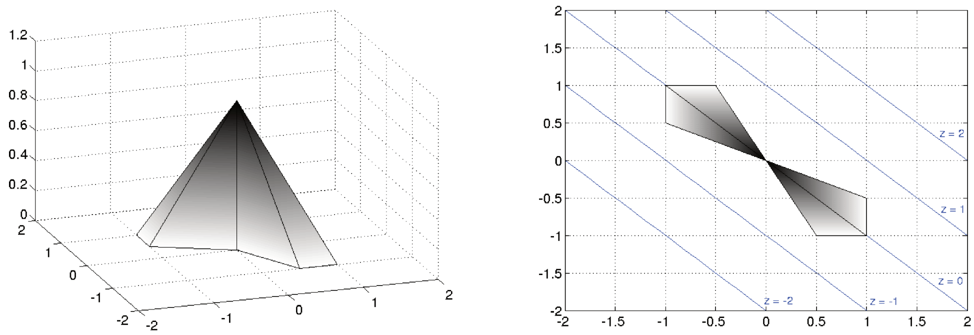
Example 6.3. *Let $A = B = (-1;0;1) \in \mathbb{R}_{\mathcal{F}_C}$ and $\mathcal{J}_\gamma$ be the joint possibility distribution of A and B given by (6.3). Figures 44, 45, 46, 47 and 48 depict $\mathcal{J}_\gamma$ for different values of γ. Note that these figures corroborate the property given by item (b) of Theorem 6.1, that is, $\mathcal{J}_0 \subseteq \mathcal{J}_\gamma \subseteq \mathcal{J}_1$, for all $\gamma \in [0,1]$.*

Figure 44 – Graphical representation of $\mathcal{J}_0$ given as in Example 6.3.



The left subfigure represents the joint possibility distribution $\mathcal{J}_\gamma$, for $\gamma = 0$, between fuzzy numbers $A = B = (-1;0;1)$. The right subfigure represents the top view of $\mathcal{J}_\gamma$, the region where $\mathcal{J}_\gamma(u,v) \neq 0$ is depicted by the black line and the blue lines represent the level sets of the function $z = x + y$ for $z \in \{-2,-1,0,1,2\}$. Source: Author

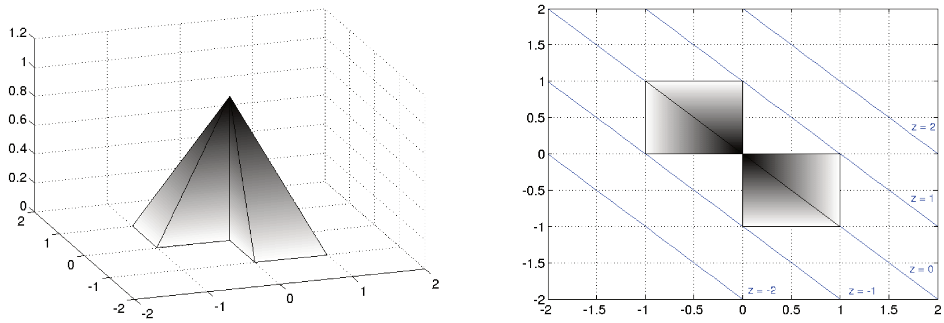
Figure 45 – Graphical representation of $\mathcal{J}_{0.25}$ given as in Example 6.3.



The left subfigure represents the joint possibility distribution $\mathcal{J}_\gamma$, for $\gamma = 0.25$, between fuzzy numbers $A = B = (-1;0;1)$. The right subfigure represents the top view of $\mathcal{J}_\gamma$, the region where $\mathcal{J}_\gamma(u,v) \neq 0$ is depicted by the gray shadow region and the blue lines represent the level sets of the function $z = x + y$ for $z \in \{-2,-1,0,1,2\}$. Source: Author

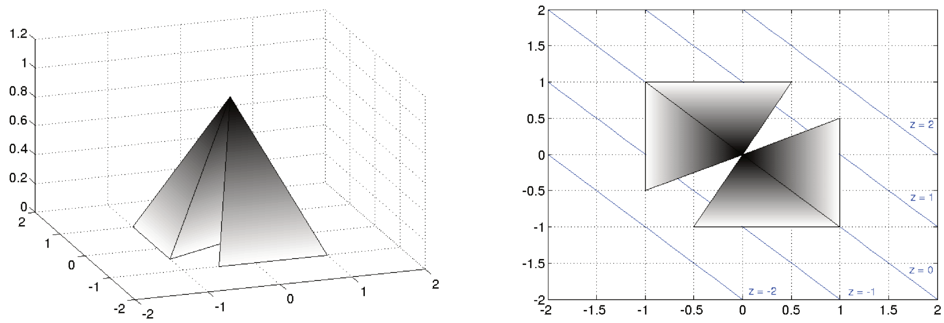
Theorem 6.1 ensures that the interactive sum via sup-$\mathcal{J}_\gamma$ extension principle is a fuzzy number in $\mathbb{R}_{\mathcal{F}_C}$. Here the focus is on the characterization of this interactive

Figure 46 – Graphical representation of $\mathcal{J}_{0.5}$ given as in Example 6.3.



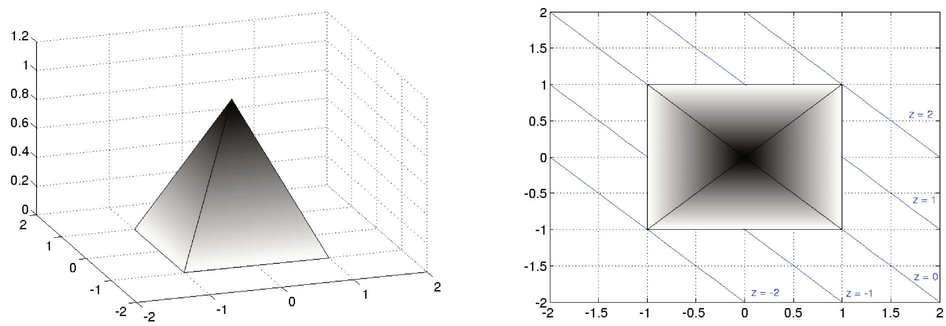
The left subfigure represents the joint possibility distribution $\mathcal{J}_\gamma$, for $\gamma = 0.5$, between fuzzy numbers $A = B = (-1; 0; 1)$. The right subfigure represents the top view of $\mathcal{J}_\gamma$, the region where $\mathcal{J}_\gamma(u, v) \neq 0$ is depicted by the gray shadow region and the blue lines represent the level sets of the function $z = x + y$ for $z \in \{-2, -1, 0, 1, 2\}$. Source: Author

Figure 47 – Graphical representation of $\mathcal{J}_{0.75}$ given as in Example 6.3.



The left subfigure represents the joint possibility distribution $\mathcal{J}_\gamma$, for $\gamma = 0.75$, between fuzzy numbers $A = B = (-1; 0; 1)$. The right subfigure represents the top view of $\mathcal{J}_\gamma$, the region where $\mathcal{J}_\gamma(u, v) \neq 0$ is depicted by the gray shadow region and the blue lines represent the level sets of the function $z = x + y$ for $z \in \{-2, -1, 0, 1, 2\}$. Source: Author

Figure 48 – Graphical representation of $\mathcal{J}_1$ given as in Example 6.3.



The left subfigure represents the joint possibility distribution $\mathcal{J}_\gamma$, for $\gamma = 1$, between fuzzy numbers $A = B = (-1; 0; 1)$. The right subfigure represents the top view of $\mathcal{J}_\gamma$, the region where $\mathcal{J}_\gamma(u, v) \neq 0$ is depicted by the gray shadow region and the blue lines represent the level sets of the function $z = x + y$ for $z \in \{-2, -1, 0, 1, 2\}$. Source: Author arithmetic operation. In contrast to the joint possibility distribution J_γ, it is possible to describe the sum via $\mathcal{J}_\gamma$ by means of α-cuts.

Note that in the right subfigures depicted in Example 6.3 (see Figures 44, 45, 46, 47 and 48), the level curves of the sum operator $f(x, y) = x + y$ are presented. The goal is to illustrate which elements $(u, v) \in \mathbb{R}^2$, where $\mathcal{J}_\gamma(u, v) \neq 0$, are associated with respect to $f(x, y) = x + y$.

The next section presents the interactive sum via joint possibility distribution given as in (6.3).

6.2 Interactive Sum via $\mathcal{J}_\gamma$

This section provides a characterization of the interactive sum between fuzzy numbers by means of α-cuts. To this end, the next lemma is used as an auxiliary result.

Lemma 6.1. *Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ and $\mathcal{J}_\gamma$ be a joint possibility distribution for A and B given by (6.3). Thus*

$$[\mathcal{J}_\gamma]^\alpha = \bigcup_{\beta \in [\alpha,1]} (P_A(\beta, \gamma) \cup P_B(\beta, \gamma)), \quad \forall \gamma \in [0, 1], \tag{6.5}$$

where

$$P_A(\beta, \gamma) = \begin{cases} \left(\{a^-_\beta\} \times I_A(a^-_\beta, \beta, \gamma)\right) \cup \left(\{a^+_\beta\} \times I_A(a^+_\beta, \beta, \gamma)\right), & \text{if } \beta \in [0, 1) \\ \bigcup\limits_{x \in [A]^1} \{x\} \times I_A(x, 1, \gamma), & \text{if } \beta = 1 \end{cases},$$

and

$$P_B(\beta, \gamma) = \begin{cases} \left(I_B(b^-_\beta, \beta, \gamma) \times \{b^-_\beta\}\right) \cup \left(I_B(b^+_\beta, \beta, \gamma) \times \{b^+_\beta\}\right), & \text{if } \beta \in [0, 1) \\ \bigcup\limits_{x \in [B]^1} I_B(x, 1, \gamma) \times \{x\}, & \text{if } \beta = 1 \end{cases}.$$

Proof. Let us prove Equation (6.5) by first showing the inclusion $\bigcup\limits_{\beta \in [\alpha,1]} (P_A(\beta, \gamma) \cup P_B(\beta, \gamma)) \subseteq [\mathcal{J}_\gamma]^\alpha$. To this end, let $(x, y) \in \bigcup\limits_{\beta \in [\alpha,1]} (P_A(\beta, \gamma) \cup P_B(\beta, \gamma))$. Thus, there exists $\overline{\beta} \geqslant \alpha$ such that $(x, y) \in P_A(\overline{\beta}, \gamma)$ or $(x, y) \in P_B(\overline{\beta}, \gamma)$. Suppose w.l.o.g $(x, y) \in P_A(\overline{\beta}, \gamma)$.

If $\overline{\beta} = 1 \geqslant \alpha$, then $x \in [A]^1$ and consequently $A(x) = 1$. By definition of the set I_A, one obtains $y \in I_A(x, 1, \gamma) \subseteq [B]^1$, which implies that $B(y) = 1$. Thus, $\mathcal{J}_\gamma(x, y) = A(x) \wedge B(y) = 1$. Hence $(x, y) \in [\mathcal{J}_\gamma]^1 \subseteq [\mathcal{J}_\gamma]^\alpha$.

If $\overline{\beta} < 1$, then $x \in \{a^-_{\overline{\beta}}, a^+_{\overline{\beta}}\} \subseteq [A]^{\overline{\beta}}$ and $y \in I_A(x, \overline{\beta}, \gamma) \subseteq [B]^{\overline{\beta}}$. Thus, $\mathcal{J}_\gamma(x, y) = A(x) \wedge B(y) \geqslant \overline{\beta}$ which implies that $(x, y) \in [\mathcal{J}_\gamma]^{\overline{\beta}} \subseteq [\mathcal{J}_\gamma]^\alpha$. Therefore, $\bigcup\limits_{\beta \in [\alpha,1]} (P_A(\beta, \gamma) \cup P_B(\beta, \gamma)) \subseteq [\mathcal{J}_\gamma]^\alpha$.

Let us prove the other inclusion. To this end, consider $(x, y) \in [\mathcal{J}_\gamma]^\alpha$. Hence $\mathcal{J}_\gamma(x, y) \geqslant \alpha$. Suppose that $(x, y) \notin \bigcup_{\beta \in [\alpha, 1]} (P_A(\beta, \gamma) \cup P_B(\beta, \gamma))$. Thus, exists $\lambda < \alpha$ such that

$$(x, y) \in \left(\{a_\lambda^-\} \times I_A(a_\lambda^-, \lambda, \gamma)\right) \cup \left(\{a_\lambda^+\} \times I_A(a_\lambda^+, \lambda, \gamma)\right) \cup \left(\bigcup_{x \in [A]^1} \{x\} \times I_A(x, 1, \gamma)\right),$$

or

$$(x, y) \in \left(I_B(b_\lambda^-, \lambda, \gamma) \times \{b_\lambda^-\}\right) \cup \left(I_B(b_\lambda^+, \lambda, \gamma) \times \{b_\lambda^+\}\right) \cup \left(\bigcup_{x \in [B]^1} I_B(x, 1, \gamma) \times \{x\}\right).$$

Since $\lambda < \alpha \leqslant 1$, it follows that

$$(x, y) \notin \left(\bigcup_{x \in [A]^1} \{x\} \times I_A(x, 1, \gamma)\right) \quad \text{and} \quad (x, y) \notin \left(\bigcup_{x \in [B]^1} I_B(x, 1, \gamma) \times \{x\}\right).$$

Without loss of generality suppose that

$$(x, y) \in \left(\{a_\lambda^-\} \times I_A(a_\lambda^-, \lambda, \gamma)\right) \cup \left(\{a_\lambda^+\} \times I_A(a_\lambda^+, \lambda, \gamma)\right).$$

Thus, $x \in \{a_\lambda^-, a_\lambda^+\}$ and $y \in I_A(x, \lambda, \gamma) \subseteq [B]^\lambda$. Since $A \in \mathbb{R}_{\mathcal{F}_C}$, one obtains $A(x) = \lambda \leqslant B(y)$, for $x \in \{a_\lambda^-, a_\lambda^+\}$. Hence, $\mathcal{J}_\gamma(x, y) = A(x) \wedge B(y) = A(x) = \lambda < \alpha$, contradicting the fact that $\mathcal{J}_\gamma(x, y) \geqslant \alpha$. Therefore, $[\mathcal{J}_\gamma]^\alpha \subseteq \bigcup_{\beta \in [\alpha, 1]} (P_A(\beta, \gamma) \cup P_B(\beta, \gamma))$, concluding that the equality holds. □

The next theorem reveals the α-cuts of the interactive sum between fuzzy numbers.

Theorem 6.2. *Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$ whose α-cuts are given by $[A]^\alpha = [a_\alpha^-, a_\alpha^+]$ and $[B]^\alpha = [b_\alpha^-, b_\alpha^+]$. Then, for all $\alpha \in [0, 1]$*

$$[A +_\gamma B]^\alpha = [c_\alpha^-, c_\alpha^+] + \{\overline{a} + \overline{b}\} \tag{6.6}$$

where,

$$c_\alpha^- = \bigwedge_{\beta \geqslant \alpha} h_{(A+B)}^-(\beta, \gamma) \quad \text{and} \quad c_\alpha^+ = \bigvee_{\beta \geqslant \alpha} h_{(A+B)}^+(\beta, \gamma). \tag{6.7}$$

with

$$\begin{aligned} h_{(A+B)}^-(\beta, \gamma) = \min\{ & (a^{(\overline{a})})_\beta^- + (b^{(\overline{b})})_\beta^+ + \gamma((b^{(\overline{b})})_\beta^- - (b^{(\overline{b})})_\beta^+), \\ & (a^{(\overline{a})})_\beta^+ + (b^{(\overline{b})})_\beta^- + \gamma((a^{(\overline{a})})_\beta^- - (a^{(\overline{a})})_\beta^+), \\ & \gamma((a^{(\overline{a})})_\beta^- + (b^{(\overline{b})})_\beta^-)\} \end{aligned}$$

and

$$\begin{aligned} h_{(A+B)}^+(\beta, \gamma) = \max\{ & (a^{(\overline{a})})_\beta^- + (b^{(\overline{b})})_\beta^+ + \gamma((a^{(\overline{a})})_\beta^+ - (a^{(\overline{a})})_\beta^-), \\ & (a^{(\overline{a})})_\beta^+ + (b^{(\overline{b})})_\beta^- + \gamma((b^{(\overline{b})})_\beta^+ - (b^{(\overline{b})})_\beta^-), \\ & \gamma((a^{(\overline{a})})_\beta^+ + (b^{(\overline{b})})_\beta^+)\} \end{aligned}$$

Proof. First note that

$$I_A(x,\alpha,\gamma) = [f_A^\alpha(x) + \gamma((b^{(\bar{b})})_\alpha^- - f_A^\alpha(x)), f_A^\alpha(x) + \gamma((b^{(\bar{b})})_\alpha^+ - f_A^\alpha(x))] + \{\bar{b}\}$$

and

$$I_B(x,\alpha,\gamma) = [f_B^\alpha(x) + \gamma((a^{(\bar{a})})_\alpha^- - f_B^\alpha(x)), f_B^\alpha(x) + \gamma((a^{(\bar{a})})_\alpha^+ - f_B^\alpha(x))] + \{\bar{a}\}.$$

In order to prove this theorem, let us analyse the intervals

$$[f_A^\alpha(x) + \gamma((b^{(\bar{b})})_\alpha^- - f_A^\alpha(x)), f_A^\alpha(x) + \gamma((b^{(\bar{b})})_\alpha^+ - f_A^\alpha(x))]$$

and

$$[f_B^\alpha(x) + \gamma((a^{(\bar{a})})_\alpha^- - f_B^\alpha(x)), f_B^\alpha(x) + \gamma((a^{(\bar{a})})_\alpha^+ - f_B^\alpha(x))].$$

To this end, consider for each $\beta \in [0,1]$ the following cases:

(i) $(a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ \geqslant 0$ and $(b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ \geqslant 0$;

(ii) $(a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ \geqslant 0$ and $(b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ \leqslant 0$;

(iii) $(a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ \leqslant 0$ and $(b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ \geqslant 0$;

(iv) $(a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ \leqslant 0$ and $(b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ \leqslant 0$.

Case (i):

If $(a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ \geqslant 0$ and $(b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ \geqslant 0$, then

$$\gamma((b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^-) \leqslant y + (a^{(\bar{a})})_\beta^- \leqslant \gamma((b^{(\bar{b})})_\beta^+ + (a^{(\bar{a})})_\beta^-)$$

and

$$(b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ \leqslant y + (a^{(\bar{a})})_\beta^+ \leqslant (b^{(\bar{b})})_\beta^- + (a^{(\bar{a})})_\beta^+ + \gamma((b^{(\bar{b})})_\beta^+ - (b^{(\bar{b})})_\beta^-), \tag{6.8}$$

for all $y \in [B]^\beta$.

Also, for all $x \in [A]^\beta$,

$$\gamma((a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^-) \leqslant x + (b^{(\bar{b})})_\beta^- \leqslant \gamma((a^{(\bar{a})})_\beta^+ + (b^{(\bar{b})})_\beta^-)$$

and

$$(a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ \leqslant x + (b^{(\bar{b})})_\beta^+ \leqslant (a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ + \gamma((a^{(\bar{a})})_\beta^+ - (a^{(\bar{a})})_\beta^-). \tag{6.9}$$

Let c_β^- and c_β^+ be the endpoints of $[A +_\gamma B]^\beta$, that is, $[A +_\gamma B]^\beta = [c_\beta^-, c_\beta^+]$. Since $\gamma((a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^-)$ is less or equal than the left side of inequalities (6.8) and (6.9), one obtains

$$c_\beta^- \leqslant \gamma((a^{(\bar{a})})_\beta^- + (b^{(\bar{b})})_\beta^-).$$

On the other hand, since

$$\gamma((b^{(\bar{b})})^+_\beta + (a^{(\bar{a})})^-_\beta) \leqslant (a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((a^{(\bar{a})})^+_\beta - (a^{(\bar{a})})^-_\beta)$$

and

$$\gamma((a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta) \leqslant (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((b^{(\bar{b})})^+_\beta - (b^{(\bar{b})})^-_\beta),$$

then

$$c^+_\beta \geqslant \max\{(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((a^{(\bar{a})})^+_\beta - (a^{(\bar{a})})^-_\beta), (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((b^{(\bar{b})})^+_\beta - (b^{(\bar{b})})^-_\beta)\}.$$

Case (ii):

If $(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta \geqslant 0$ and $(b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta \leqslant 0$, then

$$\gamma((b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^-_\beta) \leqslant y + (a^{(\bar{a})})^-_\beta \leqslant \gamma((b^{(\bar{b})})^+_\beta + (a^{(\bar{a})})^-_\beta) \tag{6.10}$$

and

$$\gamma((b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta) \leqslant y + (a^{(\bar{a})})^+_\beta \leqslant \gamma((b^{(\bar{b})})^+_\beta + (a^{(\bar{a})})^+_\beta), \tag{6.11}$$

for all $y \in [B]^\beta$.

Also, for all $x \in [A]^\beta$

$$(a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((a^{(\bar{a})})^-_\beta - (a^{(\bar{a})})^+_\beta) \leqslant x + (b^{(\bar{b})})^-_\beta \leqslant (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta \tag{6.12}$$

and

$$(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta \leqslant x + (b^{(\bar{b})})^+_\beta \leqslant (a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((a^{(\bar{a})})^+_\beta - (a^{(\bar{a})})^-_\beta). \tag{6.13}$$

Since $(a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((a^{(\bar{a})})^-_\beta - (a^{(\bar{a})})^+_\beta)$ is less or equal than the left side of inequalities (6.10), (6.11), and, (6.13),

$$c^-_\beta \leqslant (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((a^{(\bar{a})})^-_\beta - (a^{(\bar{a})})^+_\beta).$$

On the other hand, since $(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((a^{(\bar{a})})^+_\beta - (a^{(\bar{a})})^-_\beta)$ is greater or equal than the right side of inequalities (6.10), (6.11), and, (6.12), one obtains

$$c^+_\beta \geqslant (a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((a^{(\bar{a})})^+_\beta - (a^{(\bar{a})})^-_\beta).$$

Case (iii):

If $(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta \leqslant 0$ and $(b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta \geqslant 0$, then

$$(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((b^{(\bar{b})})^-_\beta - (b^{(\bar{b})})^+_\beta) \leqslant y + (a^{(\bar{a})})^-_\beta \leqslant (b^{(\bar{b})})^+_\beta + (a^{(\bar{a})})^-_\beta \tag{6.14}$$

and

$$(b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta \leqslant y + (a^{(\bar{a})})^+_\beta \leqslant (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((b^{(\bar{b})})^+_\beta - (b^{(\bar{b})})^-_\beta), \tag{6.15}$$

for all $y \in [B]^\beta$.

Also, for all $x \in [A]^\beta$,

$$\gamma((a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^-_\beta) \leqslant x + (b^{(\bar{b})})^-_\beta \leqslant \gamma((a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta) \tag{6.16}$$

and

$$\gamma((a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta) \leqslant x + (b^{(\bar{b})})^+_\beta \leqslant \gamma((a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^+_\beta). \tag{6.17}$$

Since $(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((b^{(\bar{b})})^-_\beta - (b^{(\bar{b})})^+_\beta)$ is less than or equal to the left side of inequalities (6.15), (6.16), and, (6.17) then

$$c^-_\beta \leqslant (a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((b^{(\bar{b})})^-_\beta - (b^{(\bar{b})})^+_\beta).$$

On the other hand, since $(a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((b^{(\bar{b})})^+_\beta - (b^{(\bar{b})})^-_\beta)$ is greater than or equal to the right side of inequalities (6.14), (6.16), and, (6.17) then

$$c^+_\beta \geqslant (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((b^{(\bar{b})})^+_\beta - (b^{(\bar{b})})^-_\beta).$$

Case (iv):

If $(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta \leqslant 0$ and $(b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta \leqslant 0$ then

$$(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((b^{(\bar{b})})^-_\beta - (b^{(\bar{b})})^+_\beta) \leqslant y + (a^{(\bar{a})})^-_\beta \leqslant (b^{(\bar{b})})^+_\beta + (a^{(\bar{a})})^-_\beta \tag{6.18}$$

and

$$\gamma((b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta) \leqslant y + (a^{(\bar{a})})^+_\beta \leqslant \gamma((a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^+_\beta),$$

for all $y \in [B]^\beta$.

Also, for all $x \in [A]^\beta$

$$(a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((a^{(\bar{a})})^-_\beta - (a^{(\bar{a})})^+_\beta) \leqslant x + (b^{(\bar{b})})^-_\beta \leqslant (a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta \tag{6.19}$$

and

$$\gamma((a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta) \leqslant x + (b^{(\bar{b})})^+_\beta \leqslant \gamma((a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^+_\beta).$$

Since

$$(a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta + \gamma((b^{(\bar{b})})^-_\beta - (b^{(\bar{b})})^+_\beta) \leqslant \gamma((a^{(\bar{a})})^-_\beta + (b^{(\bar{b})})^+_\beta)$$

and

$$(a^{(\bar{a})})^+_\beta + (b^{(\bar{b})})^-_\beta + \gamma((a^{(\bar{a})})^-_\beta - (a^{(\bar{a})})^+_\beta) \leqslant \gamma((b^{(\bar{b})})^-_\beta + (a^{(\bar{a})})^+_\beta),$$

then

$$c_\beta^- \leqslant \min\{(a^{(\overline{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ + \gamma((b^{(\bar{b})})_\beta^- - (b^{(\bar{b})})_\beta^+), (a^{(\overline{a})})_\beta^+ + (b^{(\bar{b})})_\beta^- + \gamma((a^{(\overline{a})})_\beta^- - (a^{(\overline{a})})_\beta^+)\}.$$

On the other hand, since $\gamma((a^{(\overline{a})})_\beta^+ + (b^{(\bar{b})})_\beta^+)$ is greater or equal than the right side of inequalities (6.18) and (6.19), then

$$c_\beta^+ \geqslant \gamma((a^{(\overline{a})})_\beta^+ + (b^{(\bar{b})})_\beta^+)$$

Next, let $h_{(A+B)}^-, h_{(A+B)}^+ : [0,1]^2 \to \mathbb{R}$ be the functions defined by

$$\begin{aligned} h_{(A+B)}^-(\beta,\gamma) = \min\{ & (a^{(\overline{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ + \gamma((b^{(\bar{b})})_\beta^- - (b^{(\bar{b})})_\beta^+), \\ & (a^{(\overline{a})})_\beta^+ + (b^{(\bar{b})})_\beta^- + \gamma((a^{(\overline{a})})_\beta^- - (a^{(\overline{a})})_\beta^+), \\ & \gamma((a^{(\overline{a})})_\beta^- + (b^{(\bar{b})})_\beta^-)\}, \end{aligned}$$

and

$$\begin{aligned} h_{(A+B)}^+(\beta,\gamma) = \max\{ & (a^{(\overline{a})})_\beta^- + (b^{(\bar{b})})_\beta^+ + \gamma((a^{(\overline{a})})_\beta^+ - (a^{(\overline{a})})_\beta^-), \\ & (a^{(\overline{a})})_\beta^+ + (b^{(\bar{b})})_\beta^- + \gamma((b^{(\bar{b})})_\beta^+ - (b^{(\bar{b})})_\beta^-), \\ & \gamma((a^{(\overline{a})})_\beta^+ + (b^{(\bar{b})})_\beta^+)\}. \end{aligned}$$

A combination of the Cases $(i), (ii), (iii)$, and, (iv) reveals that

$$h_{(A+B)}^-(\beta,\gamma) \leqslant \overline{x} + \overline{y} \leqslant h_{(A+B)}^+(\beta,\gamma),$$

for all $\overline{x} \in [A^{(\overline{a})}]^\beta$ and $\overline{y} \in [B^{(\bar{b})}]^\beta$. Therefore, one obtains

$$h_{(A+B)}^-(\beta,\gamma) + \{\overline{a} + \bar{b}\} \leqslant x + y \leqslant h_{(A+B)}^+(\beta,\gamma) + \{\overline{a} + \bar{b}\},$$

for all $x \in [A]^\beta$ and $y \in [B]^\beta$, which leads us to the following equivalence

$$(x,y) \in P_A(\beta,\gamma) \cup P_B(\beta,\gamma) \Leftrightarrow x + y \in [h_{(A+B)}^-(\beta,\gamma), h_{(A+B)}^+(\beta,\gamma)] + \{\overline{a} + \bar{b}\}. \qquad (6.20)$$

Since the sum operator is continuous, by Theorem 2.1 it follows that

$$[A +_\gamma B]^\alpha = +[\mathcal{J}_\gamma]^\alpha = \{x + y \ : \ (x,y) \in [\mathcal{J}_\gamma]^\alpha\}, \ \forall \alpha \in [0,1],$$

which is given by $[\mathcal{J}_\gamma]^\alpha = [c_\alpha^-, c_\alpha^+]$, where

$$c_\alpha^- = \bigwedge_{(x,y)\in[\mathcal{J}_\gamma]^\alpha} (x+y) \quad \text{and} \quad c_\alpha^+ = \bigvee_{(x,y)\in[\mathcal{J}_\gamma]^\alpha} (x+y).$$

Theorem 6.1 and Lemma 5.1 lead us to

$$\begin{aligned} c_\alpha^- &= \bigwedge (x + y \ : \ (x,y) \in [\mathcal{J}_\gamma]^\alpha) \\ &= \bigwedge \left(x + y \ : \ (x,y) \in \bigcup_{\beta \geqslant \alpha} (P_A(\beta,\gamma) \cup P_B(\beta,\gamma)) \right) \\ &= \bigwedge_{\beta\geqslant\alpha} \left(\bigwedge (x + y \ : \ (x,y) \in (P_A(\beta,\gamma) \cup P_B(\beta,\gamma)) \right). \end{aligned}$$

By (6.20), it follows that

$$c_\alpha^- = \bigwedge_{\beta \geqslant \alpha} h_{(A+B)}^-(\beta, \gamma) + \{\overline{a} + \overline{b}\}.$$

Analogously,

$$c_\alpha^+ = \bigvee_{\beta \geqslant \alpha} h_{(A+B)}^+(\beta, \gamma) + \{\overline{a} + \overline{b}\}.$$

Therefore, for all $\alpha \in [0, 1]$

$$[A +_\gamma B]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} h_{(A+B)}^-(\beta, \gamma), \bigvee_{\beta \geqslant \alpha} h_{(A+B)}^+(\beta, \gamma) \right] + \{\overline{a} + \overline{b}\}.$$

$\square$

Next, an example of this interactive sum is provided.

Example 6.4. *Let $A = (1; 2; 3)$, $B = (1; 2; 3; 4)$ and $C = (0.5; 1.5; 4)_G$ be fuzzy numbers. In view of Theorem 6.2 one obtains:*

$$[A +_\gamma A]^\alpha = [\gamma(-2 + 2\alpha), \gamma(2 - 2\alpha)] + 4.$$

$$[A +_\gamma B]^\alpha = [-0.5 + \gamma(-2 + 2\alpha), 0.5 + \gamma(2 - 2\alpha)] + 4.5.$$

For all $\alpha \in [0, 1]$ such that $\alpha \geqslant \overline{\alpha}$, where $\overline{\alpha} = e^{-32}$, it follows that

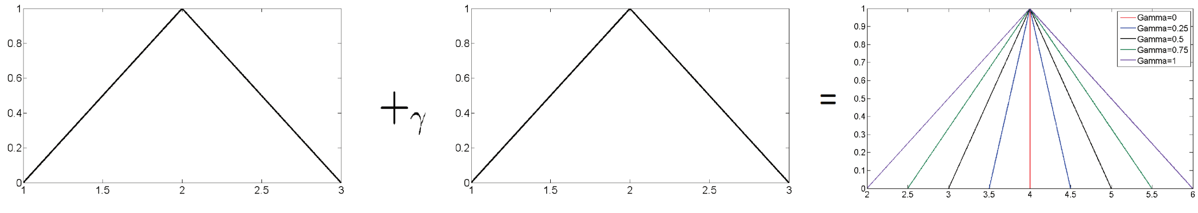
$$[A +_\gamma C]^\alpha = \left[1 - \alpha - \sqrt{-1.5ln(\alpha)} + \gamma(-2 + 2\alpha), -1 + \alpha + \sqrt{-1.5ln(\alpha)} + \gamma(2 - 2\alpha)\right] + 2.5.$$

For $\alpha < \overline{\alpha}$,

$$[A +_\gamma C]^\alpha = [-3 - \alpha + \gamma(-2 + 2\alpha), 3 + \alpha + \gamma(2 - 2\alpha)] + 2.5.$$

Figure 49 depicts the interactive sum between the triangular fuzzy numbers A and A for the following values of $\gamma \in \{0, 0.25, 0.5, 0.75, 1\}$.

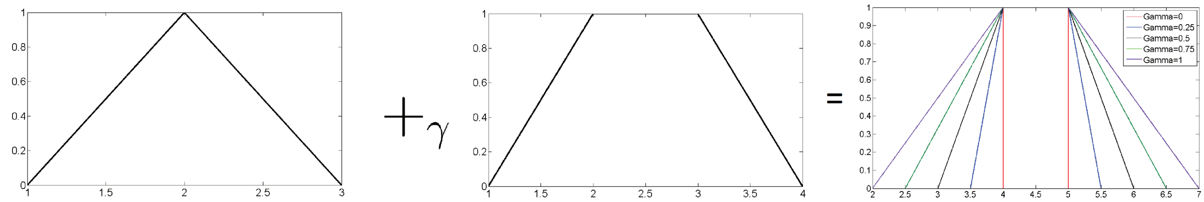
Figure 49 – Graphical representation of the interactive sum $A +_\gamma A$ given as in Example 6.4.



The interactive sum $A +_\gamma A$, where $A = (1; 2; 3)$. The red, blue, black, green and purple lines represent the fuzzy numbers obtained by the interactive sum for $\gamma = 0$, $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, respectively. Source: Author.

Figure 50 graphically represents the interactive sum between the triangular fuzzy number A and the trapezoidal fuzzy number B for the same values of γ.

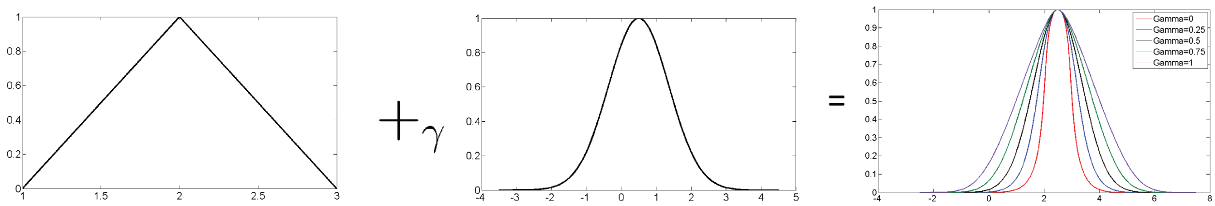
Figure 50 – Graphical representation of the interactive sum $A +_\gamma B$ given as in Example 6.4.



The interactive sum $A +_\gamma B$, where $A = (1; 2; 3)$ and $B = (1; 2; 3; 4)$. The red, blue, black, green and purple lines represent the fuzzy numbers obtained by the interactive sum for $\gamma = 0$, $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, respectively. Source: Author.

Finally, Figure 51 graphically portrays the interactive sum between the triangular fuzzy number A and the Gaussian fuzzy number C, for $\gamma \in \{0, 0.25, 0.5, 0.75, 1\}$.

Figure 51 – Graphical representation of the interactive sum $A +_\gamma C$ given as in Example 6.4.



The interactive sum $A +_\gamma B$, where $A = (1; 2; 3)$ and $C = (0.5; 1.5; 4)_G$. The red, blue, black, green and purple lines represent the fuzzy numbers obtained by the interactive sum for $\gamma = 0$, $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, respectively. Source: Author.

Note that in all these cases the sum for $\gamma = 1$ boils down to the standard one.

Tables 5, 6 and 7 exhibit the values of the Pompeiu-Hausdorff norm and width of $A +_\gamma A$, $A +_\gamma B$ and $A +_\gamma C$, respectively, for $\gamma = 0, 0.25, 0.5, 0.75, 1$.

One can observe, in Example 6.4, that the width and the Pompeiu-Hausdorff norm of the interactive sums based on this family of joint possibility distributions are increasing with respect to the parameter γ. In fact Theorems 6.3 and 6.4 show that this result holds true for every pair of fuzzy numbers.

Table 5 – Pompeiu-Hausdorff norm and width of $A +_\gamma A$.

γ	Pompeiu-Hausdorff Norm	Width
0	4	0
0.25	4.5	1
0.5	5	2
0.75	5.5	3
1	6	4

Table 6 – Pompeiu-Hausdorff norm and width of $A +_\gamma B$.

γ	Pompeiu-Hausdorff Norm	Width
0	5	1
0.25	5.5	2
0.5	6	3
0.75	6.5	4
1	7	5

Table 7 – Pompeiu-Hausdorff norm and width of $A +_\gamma C$.

γ	Pompeiu-Hausdorff Norm	Width
0	5.5	6
0.25	6	7
0.5	6.5	8
0.75	7	9
1	7.5	10

Theorem 6.3. *Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$. Consider the function $\phi_{A+B} : [0,1] \to \mathbb{R}$ given by $\phi_{A+B}(\gamma) = ||A +_\gamma B||$. Thus, the following properties hold.*

(*a*) *ϕ_{A+B} is a continuous and increasing function;*

(*b*) *$\phi_{A+B}(0) \leqslant ||A +_\gamma B|| \leqslant \phi_{A+B}(1), \ \forall\ 0 \leqslant \gamma \leqslant 1$.*

Proof. (*a*) Consider the function $\phi : [0,1] \to \mathbb{R}$ given by $\phi(\gamma) = ||A +_\gamma B||$, where A and B are fixed arbitrary fuzzy numbers in $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$. First, it will be shown that ϕ is an increasing function. To this end, let $\gamma, \lambda \in [0,1]$ such that $\gamma \leqslant \lambda$. Let us call $A +_\gamma B = C_\gamma$ and $A +_\lambda B = C_\lambda$, where $[C_\gamma]^\alpha = [c^-_{\gamma_\alpha}, c^+_{\gamma_\alpha}]$ and $[C_\lambda]^\alpha = [c^-_{\lambda_\alpha}, c^+_{\lambda_\alpha}]$. From Definition 1.22, one obtains that $||C|| = \max\{|c^-_0|, |c^+_0|\}$, for all $C \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$. By item (*c*) of Theorem (6.1) and the definition of joint possibility distribution, it follows that $c^-_{\lambda_0} \leqslant c^-_{\gamma_0} \leqslant 0$ and $0 \leqslant c^+_{\gamma_0} \leqslant c^+_{\lambda_0}$ which implies that $|c^-_{\gamma_0}| \leqslant |c^-_{\lambda_0}|$ and $|c^+_{\gamma_0}| \leqslant |c^+_{\lambda_0}|$. Therefore,

$$\phi(\gamma) = ||C_\gamma|| = \max\{|c^-_{\gamma_0}|, |c^+_{\gamma_0}|\} \leqslant \max\{|c^-_{\lambda_0}|, |c^+_{\lambda_0}|\} = ||C_\lambda|| = \phi(\lambda),$$

concluding that ϕ is an increasing function.

Now, it will be shown that ϕ is a continuous function. To this end, let $(\gamma_n) \subseteq [0,1]$ be a sequence that converges to $\overline{\gamma} \in [0,1]$. Let us call $C_n = A +_{\gamma_n} B$ and $C = A +_{\overline{\gamma}} B$. Since $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$, it follows that $h^-_{C_n}(\beta, \gamma_n)$ and $h^+_{C_n}(\beta, \gamma_n)$ converges to $h^-_C(\beta, \overline{\gamma})$ and $h^+_C(\beta, \overline{\gamma})$, respectively. Since $\beta \in [\alpha, 1]$, which is a compact set, then

$$c^-_{n_\alpha} = \bigwedge_{\beta \geqslant \alpha} h^-_{C_n}(\beta, \gamma_n) \to \bigwedge_{\beta \geqslant \alpha} h^-_C(\beta, \overline{\gamma}) = c^-_\alpha$$

and

$$c^+_{n_\alpha} = \bigwedge_{\beta \geqslant \alpha} h^+_{C_n}(\beta, \gamma_n) \to \bigwedge_{\beta \geqslant \alpha} h^+_C(\beta, \overline{\gamma}) = c^+_\alpha.$$

Consequently,

$$\phi(\gamma_n) = ||C_n|| = \max\{|c^-_{n_0}|, |c^+_{n_0}|\} \longrightarrow \max\{|c^-_0|, |c^+_0|\} = ||C|| = \phi(\overline{\gamma}),$$

which implies that ϕ is continuous.

(b) Since, ϕ is continuous and $[0,1]$ is compact, Weierstrass Theorem concludes that there exist γ_1 and γ_2 such that $\phi(\gamma_1) \leqslant \phi(\gamma) \leqslant \phi(\gamma_2),\ \forall \gamma \in [0,1]$. Since ϕ is increasing, it follows that $\phi(0) \leqslant \phi(\gamma) \leqslant \phi(1)$. Hence,

$$||A +_0 B|| \leqslant ||A +_\gamma B|| \leqslant ||A +_1 B||,\ \forall \gamma \in [0,1].$$

$\square$

Theorem 6.3 reveals that the norm of the interactive sum obtained from $\mathcal{J}_\gamma$ is increasing with respect to γ. Moreover, the minimum and maximum norm are achieved at $\gamma = 0$ and $\gamma = 1$, respectively.

Theorem 6.4. *Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$. Consider the function $\varphi_{A+B} : [0,1] \to \mathbb{R}$ given by $\varphi_{A+B}(\gamma) = width(A +_\gamma B)$. Thus, the following properties hold.*

(a) *φ_{A+B} is a continuous and increasing function;*

(b) $\varphi_{A+B}(0) \leqslant width(A +_\gamma B) \leqslant \varphi_{A+B}(1),\ \forall\ 0 \leqslant \gamma \leqslant 1.$

Proof. (a) Consider the function $\varphi : [0,1] \to \mathbb{R}$ given by $\varphi(\gamma) = width(A +_\gamma B)$, where A and B are fixed arbitrary fuzzy numbers in $\mathbb{R}_{\mathcal{F}_\mathcal{C}}$. In a similar way to the item (a) of Theorem (6.3), it can be prove that φ is a continuous function. Let us call $A +_\gamma B = C_\gamma$ and $A +_\lambda B = C_\lambda$, where $[C_\gamma]^\alpha = [c^-_{\gamma_\alpha}, c^+_{\gamma_\alpha}]$ and $[C_\lambda]^\alpha = [c^-_{\lambda_\alpha}, c^+_{\lambda_\alpha}]$ for $\gamma \leqslant \lambda$. Item (c) of Theorem (6.1) concludes that $0 \leqslant c^+_{\gamma_0} - c^-_{\gamma_0} \leqslant c^+_{\lambda_0} - c^-_{\lambda_0}$. Therefore,

$$\varphi(\gamma) = width(C_\gamma) = |c^+_{\gamma_0} - c^-_{\gamma_0}| \leqslant |c^+_{\lambda_0} - c^-_{\lambda_0}| = width(C_\lambda) = \varphi(\lambda),$$

which implies that φ is increasing.

(*b*) An immediately consequence of Weierstrass Theorem applied to the function φ, ensures that

$$width(A +_0 B) \leqslant width(A +_\gamma B) \leqslant width(A +_1 B), \ \forall \gamma \in [0, 1].$$

$\square$

Theorem 6.4 reveals that the width of the interactive sum obtained from $\mathcal{J}_\gamma$ is increasing with respect to γ. Moreover, the minimum and maximum width are achieved at $\gamma = 0$ and $\gamma = 1$, respectively.

The next proposition reveals an interesting property of the interactive sum via $\mathcal{J}_0$ for symmetric triangular fuzzy numbers (see Definition 4.3).

Proposition 6.1. *[55] Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ be symmetric triangular fuzzy numbers and $\mathcal{J}_0$ be their joint possibility distribution given as in (6.3), for $\gamma = 0$. Hence*

$$A +_0 B = (\delta - \mu)(-1; 0; 1) + \chi_{\{\overline{a}+\overline{b}\}}, \tag{6.21}$$

where $+$ stands for the standard sum and $A = (\overline{a} - \delta; \overline{a}; \overline{a} + \delta)$ and $B = (\overline{b} - \mu; \overline{b}; \overline{b} + \mu)$.

Proof. Let A and B be triangular fuzzy numbers given by $A = (\overline{a} - \delta; \overline{a}; \overline{a} + \delta)$ and $B = (\overline{b} - \mu; \overline{b}; \overline{b} + \mu)$. From Theorem 6.2

$$[A +_0 B]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} h^-_{A+B}(\beta, 0), \bigvee_{\beta \geqslant \alpha} h^+_{A+B}(\beta, 0) \right],$$

where

$$\begin{aligned} h^-_{A+B}(\beta, 0) &= \min\{-\delta + \delta\beta + \mu - \mu\beta, \delta - \delta\beta - \mu + \mu\beta, 0\} + \{\overline{a} + \overline{b}\} \\ &= (\delta - \mu) \min\{-1 + \beta, 1 - \beta\} + \{\overline{a} + \overline{b}\} \end{aligned}$$

and

$$\begin{aligned} h^+_{A+B}(\beta, 0) &= \max\{-\delta + \delta\beta + \mu - \mu\beta, \delta - \delta\beta - \mu + \mu\beta, 0\} + \{\overline{a} + \overline{b}\} \\ &= (\delta - \mu) \max\{-1 + \beta, 1 - \beta\} + \{\overline{a} + \overline{b}\} \end{aligned}$$

Hence,

$$\begin{aligned} [A +_0 B]^\alpha &= \left[\bigwedge_{\beta \geqslant \alpha} (\delta - \mu) \min\{-1 + \beta, 1 - \beta\}, \bigvee_{\beta \geqslant \alpha} (\delta - \mu) \max\{-1 + \beta, 1 - \beta\} \right] + \{\overline{a} + \overline{b}\} \\ &= (\delta - \mu) \left[\bigwedge_{\beta \geqslant \alpha} \min\{-1 + \beta, 1 - \beta\}, \bigvee_{\beta \geqslant \alpha} \max\{-1 + \beta, 1 - \beta\} \right] + \{\overline{a} + \overline{b}\} \\ &= (\delta - \mu) \left[\bigwedge_{\beta \geqslant \alpha} (-1 + \beta), \bigvee_{\beta \geqslant \alpha} (1 - \beta) \right] + \{\overline{a} + \overline{b}\} \\ &= (\delta - \mu) [-1 + \alpha, 1 - \alpha] + \{\overline{a} + \overline{b}\} \quad \forall \alpha \in [0, 1]. \end{aligned}$$

Therefore,

$$A +_0 B = (\delta - \mu)(-1; 0; 1) + \chi_{\{\bar{a}+\bar{b}\}}.$$

□

Proposition 6.1 ensures that if $\delta = \mu$, then the interactive sum via $\mathcal{J}_0$ between A and B produces a real number.

Corollary 6.1. *Let A and B be symmetric triangular fuzzy numbers and $\mathcal{J}_0$ be their joint possibility distribution. If $width(A) = width(B)$, then*

$$A +_0 B = \chi_{\{\bar{a}+\bar{b}\}}.$$

Chapter 7 provides an application in the Fibonacci sequence in order to illustrate the results given by Proposition 6.1 and Corollary 6.1. One can prove that the same results provided by the above statements hold for trapezoidal and Gaussian fuzzy numbers as well.

The next section defines the interactive difference between fuzzy numbers based on joint possibility distribution given by (6.3).

6.3 Interactive Difference via $\mathcal{J}_\gamma$

This section focuses on defining a difference between fuzzy numbers that is obtained as the inverse operation of the sum, that is, $A -_{J_1} B = A +_{J_2} (-B)$. To this end, consider the following fuzzy relation

$$\widehat{\mathcal{J}}_\gamma(x, y) = \mathcal{J}_\gamma(x, -y), \quad \forall x, y \in \mathbb{R}, \tag{6.22}$$

where $\mathcal{J}_\gamma$ is given as in (6.3).

Hence, the following theorem is obtained.

Theorem 6.5. *[55] Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ and $\mathcal{J}_\gamma$ be the joint possibility distribution of A and B given by (6.3). The fuzzy relation $\widehat{\mathcal{J}}_\gamma$ defined by*

$$\widehat{\mathcal{J}}_\gamma(x, y) = \mathcal{J}_\gamma(x, -y) \tag{6.23}$$

satisfies the following properties for all $\gamma \in [0, 1]$

(a) *$\widehat{\mathcal{J}}_\gamma$ is a joint possibility distribution of A and B;*

(b) *$\widehat{\mathcal{J}}_\gamma(x_1, x_2) \leqslant \widehat{\mathcal{J}}_\lambda(x_1, x_2)$, for all $\gamma \leqslant \lambda$ and $\forall (x_1, x_2) \in \mathbb{R}^2$;*

(c) *$A -_{\widehat{\mathcal{J}}_\gamma} B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$.*

Proof. The proof is similar to the proof of Theorem 6.1. □

Theorem 6.5 reveals that the fuzzy relation given by (6.22) is a joint possibility distribution between A and B. Moreover, the properties (b) and (c) of Theorem 6.1 also hold for $\widehat{\mathcal{J}}_\gamma$ and the following equation is satisfied

$$A -_{\widehat{\mathcal{J}}_\gamma} B = A +_{\mathcal{J}_\gamma} (-B). \tag{6.24}$$

For simplicity of notation, $A -_{\widehat{\mathcal{J}}_\gamma} B$ is denoted by $A -_\gamma B$.

The characterization of the interactive difference via $\widehat{\mathcal{J}}_\gamma$ can be provided by means of α-cuts. Theorem 6.6 establishes these α-cuts.

Theorem 6.6. *Let $A, B \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}$ whose α-cuts are given by $[A]^\alpha = [a^-_\alpha, a^+_\alpha]$ and $[B]^\alpha = [b^-_\alpha, b^+_\alpha]$. Then, for all $\alpha \in [0,1]$*

$$[A -_\gamma B]^\alpha = [d^-_\alpha, d^+_\alpha] + \{\overline{a} - \overline{b}\}, \tag{6.25}$$

where

$$d^-_\alpha = \bigwedge_{\beta \geqslant \alpha} h^-_{(A-B)}(\beta, \gamma) \quad \textit{and} \quad d^+_\alpha = \bigvee_{\beta \geqslant \alpha} h^+_{(A-B)}(\beta, \gamma). \tag{6.26}$$

with

$$\begin{aligned} h^-_{(A-B)}(\beta,\gamma) = \min\{ & (a^{(\overline{a})})^-_\beta - (b^{(\overline{b})})^-_\beta + \gamma((b^{(\overline{b})})^-_\beta - (b^{(\overline{b})})^+_\beta), \\ & (a^{(\overline{a})})^+_\beta - (b^{(\overline{b})})^+_\beta + \gamma((a^{(\overline{a})})^-_\beta - (a^{(\overline{a})})^+_\beta), \\ & \gamma((a^{(\overline{a})})^-_\beta - (b^{(\overline{b})})^+_\beta)\} \end{aligned}$$

and

$$\begin{aligned} h^+_{(A-B)}(\beta,\gamma) = \max\{ & (a^{(\overline{a})})^-_\beta - (b^{(\overline{b})})^-_\beta + \gamma((a^{(\overline{a})})^+_\beta - (a^{(\overline{a})})^-_\beta), \\ & (a^{(\overline{a})})^+_\beta - (b^{(\overline{b})})^+_\beta + \gamma((b^{(\overline{b})})^+_\beta - (b^{(\overline{b})})^-_\beta), \\ & \gamma((a^{(\overline{a})})^+_\beta - (b^{(\overline{b})})^-_\beta)\} \end{aligned} .$$

Proof. The proof is similar to the proof of Theorem 6.2, recalling that $[-B]^\alpha = [-b^+_\alpha, -b^-_\alpha]$. □

Remark 6.1. *Theorem 6.6 guarantees that the interactive difference via $\widehat{\mathcal{J}}_0$ also satisfies*

$$A -_0 A = A -_L A = A -_I A = A -_g A = 0, \quad \forall A \in \mathbb{R}_{\mathcal{F}_\mathcal{C}}. \tag{6.27}$$

The next example illustrates the interactive difference between triangular, trapezoidal and Gaussian fuzzy numbers for different values of γ.

Example 6.5. *Let $A = (2;3;5;6)$, $B = (0;4;8)$ and $C = (0.5;1.5;4)_G$ be fuzzy numbers. In view of Theorem 6.6, for all $\gamma \in [0,1]$, it follows that:*

$$[A -_\gamma A]^\alpha = [\gamma(-4+2\alpha), \gamma(4-2\alpha)] + 0.$$

$$[A -_\gamma B]^\alpha \;=\; \Bigg[\bigwedge_{\beta \geqslant \alpha} \min\{2 - 3\beta + \gamma(-8 + 8\beta), -2 + 3\beta + \gamma(-4 + 2\beta), \gamma(-6 + 5\beta)\},$$
$$\bigvee_{\beta \geqslant \alpha} \max\{2 - 3\beta + \gamma(4 - 2\beta), -2 + 3\beta + \gamma(8 - 8\beta), \gamma(6 - 5\beta)\}\Bigg] + 0.$$

For all $\alpha \in [0,1]$ such that $\alpha \geqslant \overline{\alpha}$, where $\overline{\alpha} = e^{-32}$, one obtains

$$[A -_\gamma C]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} h^-_{A-B}(\beta,\gamma), \bigvee_{\beta \geqslant \alpha} h^+_{A-B}(\beta,\gamma)\right] + 3.5,$$

where

$$h^-_{A-B}(\beta,\gamma) = \min\Big\{-2 + \beta + (1 - 2\gamma)\sqrt{-1.5ln(\beta)}, 2 - \beta - \sqrt{-1.5ln(\beta)} + \gamma(-4 + 2\beta),$$
$$\gamma\left(-2 + \beta - \sqrt{-1.5ln(\beta)}\right)\Big\}$$

and

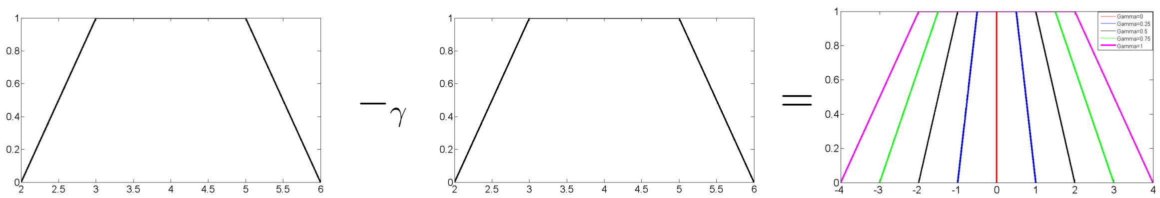
$$h^+_{A-B}(\beta,\gamma) = \max\Big\{-2 + \beta + \sqrt{-1.5ln(\beta)} + \gamma(4 - 2\beta), 2 - \beta + (2\gamma - 1)\sqrt{-1.5ln(\beta)},$$
$$\gamma\left(2 - \beta - \sqrt{-1.5ln(\beta)}\right)\Big\}$$

For $\alpha < \overline{\alpha}$

$$[A -_\gamma C]^\alpha = \Bigg[\bigwedge_{\beta \geqslant \alpha} \min\{2 + \beta - 8\gamma, -2 - \beta + \gamma(-4 + 2\beta), \gamma(-6 + \beta)\},$$
$$\bigvee_{\beta \geqslant \alpha} \max\{2 + \beta + \gamma(4 - 2\beta), -2 - \beta + 8\gamma, \gamma(6 - \beta)\}\Bigg] + 3.5.$$

Figure 52 depicts the interactive difference between the trapezoidal fuzzy numbers A and A for the following values of $\gamma \in \{0, 0.25, 0.5, 0.75, 1\}$.

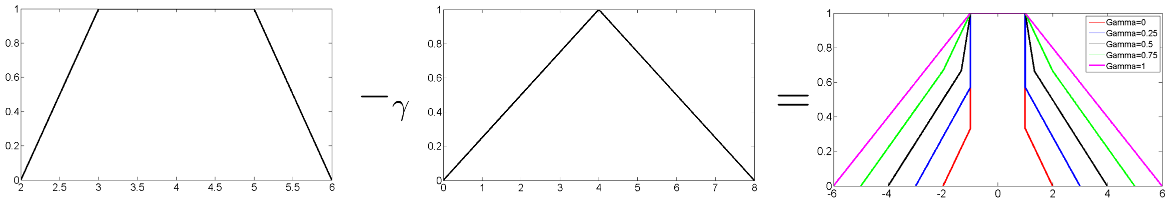
Figure 52 – Graphical representation of the interactive difference $A -_\gamma A$ given as in Example 6.5.



The interactive sum $A -_\gamma A$, where $A = (2; 3; 5; 6)$. The red, blue, black, green and purple lines represent the fuzzy numbers obtained by the interactive difference for $\gamma = 0$, $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, respectively. Source: Author.

Figure 53 graphically represents the interactive difference between the trapezoidal fuzzy number A and the triangular fuzzy number B for the same values of γ.

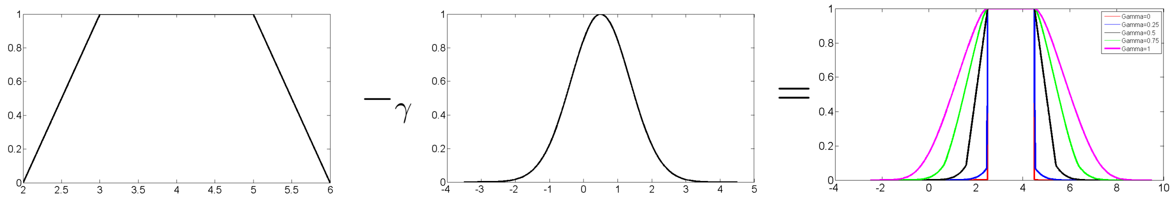
Figure 53 – Graphical representation of the interactive difference $A -_\gamma B$ given as in Example 6.5.



The interactive sum $A -_\gamma B$, where $A = (2; 3; 5; 6)$ and $B = (0; 4; 8)$. The red, blue, black, green and purple lines represent the fuzzy numbers obtained by the interactive difference for $\gamma = 0$, $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, respectively. Source: Author.

Finally, Figure 54 graphically portrays the interactive difference between the trapezoidal fuzzy number A and the Gaussian fuzzy number B, for $\gamma \in \{0, 0.25, 0.5, 0.75, 1\}$.

Figure 54 – Graphical representation of the interactive difference $A -_\gamma C$ given as in Example 6.5.



The interactive sum $A -_\gamma C$, where $A = (2; 3; 5; 6)$ and $C = (0.5; 1.5; 4)_G$. The red, blue, black, green and purple lines represent the fuzzy numbers obtained by the interactive difference for $\gamma = 0$, $\gamma = 0.25$, $\gamma = 0.5$, $\gamma = 0.75$ and $\gamma = 1$, respectively. Source: Author.

Figure 52 corroborates the fact that $A -_0 A = 0$, for all $A \in \mathbb{R}_{\mathcal{F}_C}$. Moreover, one can observe in Figures 52, 53 and 54 that the proposed interactive difference has increasing Pompeiu-Hausdorff norm and width with respect to γ.

Theorem 6.7. *Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. Consider the function $\widehat{\phi}_{A-B} : [0, 1] \to \mathbb{R}$ given by $\widehat{\phi}_{A-B}(\gamma) = ||A -_\gamma B||$. Hence the following properties hold.*

(a) *$\widehat{\phi}_{A-B}$ is a continuous and an increasing function;*

(b) *$\widehat{\phi}_{A-B}(0) \leqslant ||A -_\gamma B|| \leqslant \widehat{\phi}_{A-B}(1), \ \forall\ 0 \leqslant \gamma \leqslant 1.$*

Proof. (a) The proof of the continuity of the function $\widehat{\phi}$ is similar to the proof of item (a) of Theorem (6.3) considering $h^{-}_{A-B}(\beta,\gamma_n)$ and $h^{+}_{A-B}(\beta,\gamma_n)$. The item (b) of Theorem 6.5 implies that $\widehat{\phi}$ is increasing.

(b) Since, $A -_\gamma B = A -_{\hat{\mathcal{J}}_\gamma} B = A +_{\mathcal{J}_\gamma} (-B)$ then

$$\begin{aligned} ||A -_0 B|| &= ||A +_0 (-B)|| \leqslant ||A +_{\mathcal{J}_\gamma} (-B)|| = ||A -_{\hat{\mathcal{J}}} B|| \\ &\leqslant ||A +_1 (-B)|| = ||A -_1 B||. \end{aligned}$$

□

Theorem 6.7 reveals that the width of the interactive difference obtained from $\widehat{\mathcal{J}}_\gamma$ is increasing with respect to γ. Moreover, the minimum and maximum width are achieved at $\gamma = 0$ and $\gamma = 1$, respectively.

Theorem 6.8. *Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$. Consider the function $\widehat{\varphi}_{A-B} : [0,1] \to \mathbb{R}$ given by $\widehat{\varphi}_{A-B}(\gamma) = width(A -_\gamma B)$. Hence the following properties hold.*

(a) *$\widehat{\varphi}_{A-B}$ is a continuous and an increasing function;*

(b) *$\widehat{\varphi}_{A-B}(0) \leqslant width(A -_\gamma B) \leqslant \widehat{\varphi}_{A-B}(1),\ \forall\ 0 \leqslant \gamma \leqslant 1$.*

Proof. The proof is similar to the proof of Theorem 6.4. □

Theorem 6.8 reveals that the norm of the interactive difference obtained from $\widehat{\mathcal{J}}_\gamma$ is increasing with respect to γ. Moreover, the minimum and maximum norm are achieved at $\gamma = 0$ and $\gamma = 1$, respectively.

One can observe, in Figure 53, that the interactive difference (for $\gamma = 0$) between the fuzzy numbers $A = (2;3;5;6)$ and $B = (0;4;8)$ resembles the g-difference between them (see Figure 9). Chapter 5 already proved that the g-difference is an interactive difference. The next theorem shows that $-_g$ also coincides with $-_\gamma$.

Theorem 6.9. *Let $\widehat{\mathcal{J}}_0$ be the joint possibility distribution of A and B given as in (6.24), for $\gamma = 0$. Hence*

$$A -_0 B = A -_g B.$$

Proof. By Theorem 6.6, it follows that

$$[A -_0 B]^\alpha = \left[\bigwedge_{\beta \geqslant \alpha} h^{-}_{(A-B)}(\beta,0), \bigvee_{\beta \geqslant \alpha} h^{+}_{(A-B)}(\beta,0)\right] + (\overline{a} - \overline{b}),$$

where

$$h^{-}_{(A-B)}(\beta,0) = \min\{(a^{(\overline{a})})^{-}_{\beta} - (b^{(\overline{b})})^{-}_{\beta}, (a^{(\overline{a})})^{+}_{\beta} - (b^{(\overline{b})})^{+}_{\beta}\}$$

and

$$h^+_{(A-B)}(\beta, 0) = \max\{(a^{(\overline{a})})^-_\beta - (b^{(\overline{b})})^-_\beta, (a^{(\overline{a})})^+_\beta - (b^{(\overline{b})})^+_\beta\}.$$

Therefore,

$$\begin{aligned}
[A -_0 B]^\alpha &= \Bigg[\bigwedge_{\beta\geqslant\alpha} \min\{(a^{(\overline{a})})^-_\beta - (b^{(\overline{b})})^-_\beta, (a^{(\overline{a})})^+_\beta - (b^{(\overline{b})})^+_\beta\}\ , \\
&\quad \bigvee_{\beta\geqslant\alpha} \max\{(a^{(\overline{a})})^-_\beta - (b^{(\overline{b})})^-_\beta, (a^{(\overline{a})})^+_\beta - (b^{(\overline{b})})^+_\beta\}\Bigg] + (\overline{a} - \overline{b}) \\
&= \Bigg[\bigwedge_{\beta\geqslant\alpha} \min\{a^-_\beta - b^-_\beta, a^+_\beta - b^+_\beta\}, \bigvee_{\beta\geqslant\alpha} \max\{a^-_\beta - b^-_\beta, a^+_\beta - b^+_\beta\}\Bigg] \\
&= [A -_g B]^\alpha, \quad \forall \alpha \in [0,1]
\end{aligned}$$

□

Theorem 6.9 reveals that the interactive difference coincides with the g-difference, consequently, from Theorem 5.5 the next Corollary is obtained

Corollary 6.2. *Let $A \in \mathbb{R}_{\mathcal{F}_C}$. Thus*

$$A -_0 B = A -_I B = A -_g B.$$

The next proposition shows a particular property of the interactive difference via $-_{\hat{\mathcal{J}}_0}$ between symmetric triangular fuzzy numbers.

Proposition 6.2. *Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$ be symmetric triangular fuzzy numbers and $\hat{\mathcal{J}}_0$ be their joint possibility distribution given as in (6.3), for $\gamma = 0$. Hence*

$$A -_0 B = (\delta - \mu)(-1; 0; 1) + \chi_{\{\overline{a}-\overline{b}\}}, \tag{6.28}$$

where + stands for the standard sum and $A = (\overline{a} - \delta; \overline{a}; \overline{a} + \delta)$ and $B = (\overline{b} - \mu; \overline{b}; \overline{b} + \mu)$.

Proof. Let $A = (\overline{a} - \delta; \overline{a}; \overline{a} + \delta)$ and $B = (\overline{b} - \mu; \overline{b}; \overline{b} + \mu)$ be symmetric triangular fuzzy numbers. Since $A -_0 B = A +_0 (-B)$, Proposition 6.1 guarantees that

$$A -_0 B = A +_0 (-B) = (\overline{a} - \delta; \overline{a}; \overline{a} + \delta) +_0 (-\overline{b} - \mu; -\overline{b}; -\overline{b} + \mu) = (\delta - \mu)(-1; 0; 1) + \chi_{\{\overline{a}-\overline{b}\}}.$$

□

Proposition 6.2 implies that if $\delta = \mu$, then the proposed interactive difference between A and B results in a real number.

Corollary 6.3. *Let A and B be symmetric triangular fuzzy numbers and $\widehat{\mathcal{J}}_0$ be their joint possibility distribution. If $width(A) = width(B)$, then*

$$A -_0 B = \chi_{\{\overline{a}-\bar{b}\}}.$$

Chapter 7 provides an application in discrete delay models for fuzzy numbers in order to illustrate the results given by Proposition 6.2 and Corollary 6.3. One can prove that the same results provided by the above statements hold for trapezoidal and Gaussian fuzzy numbers as well.

6.4 Conclusion

This chapter presented a new family of joint possibility distribution $\mathcal{J}_\gamma$, where $\gamma \in [0,1]$. This family has similar properties as the family (5.6) presented in Chapter 5, that is, the interactive sum and difference via sup-J extension produce fuzzy numbers in $\mathbb{R}_{\mathcal{F}_C}$ with smaller Pompeiu-Hausdorff norm and width. More precisely, the interactive sum and difference via $\mathcal{J}_0$ produce fuzzy numbers with minimum norm and width, than any other sum and difference obtained from joint possibility distributions. Some examples were presented to illustrate the proposed arithmetic operations between different types of fuzzy numbers (triangular, trapezoidal and Gaussian). This chapter proved that the g-difference is a particular case of the interactive difference based on this family as well. The interactive sum and difference were characterized by means of $\alpha-$cuts, which allowed to compute these operations in a simpler way. Some properties involving the symmetries of fuzzy numbers were explored. This chapter ended presenting the conditions so that the sum and difference between interactive fuzzy numbers result in a real number.

7 Applications in Biomathematics

This chapter presents some applications of the methods proposed in Chapters 4, 5 and 6. Briefly comments about ontic and epistemic fuzzy sets were made in Chapter 1. In that context there was no distinction between these two points of view. However, in this chapter, the fuzzy numbers in the initial and/or boundary conditions will be chosen in order to describe the imprecise/uncertain information. This means that the approach considered in this chapter is the epistemic one.

Section 7.1 presents examples in fuzzy differential equations with initial conditions given by linearly interactive fuzzy numbers. For this section the reader can refer to [77, 138, 53, 148, 126]. Section 7.2 provides several examples in epidemiology and chemical reactions in order to illustrate the numerical fuzzy solutions of FDE's, where the initial conditions are given by J_γ-interactive fuzzy numbers. For this section the reader can refer to [141, 142, 143, 144, 145, 146, 147, 149, 139]. Finally, Section 7.3 uses the proposed family of joint possibility distribution $\mathcal{J}_\gamma$, defined in Chapter 6, in order to describe the Fibonacci and discrete delay sequences in the case where the first and second conditions are given by $\mathcal{J}_\gamma$-interactive fuzzy numbers. For this section the reader can refer to [55].

7.1 Application on FIVPs with linearly interactive fuzzy conditions

This section provides solutions of fuzzy differential equations, where the additional conditions are given by linearly interactive fuzzy numbers. First it is presented some examples in order to illustrate the method given in Chapter 4, which are based on reference [148]. Next it is provided applications in physical problems such as: to describe the trajectory of a particle according to a Hypocycloid curve [138] and mechanical vibrations [126].

7.1.1 Example 1

Let us consider a 3rd order differential equation given by

$$\begin{cases} y'''(t) - 5y''(t) + 6y'(t) = 2t + 4 \\ y(0) = 1, \ y'(0) = 2, \ y''(0) = 3 \end{cases} . \tag{7.1}$$

Here the initial position, velocity and acceleration are supposed to be uncertain and modeled by linearly interactive fuzzy numbers. In classical physics theory, this consideration is justifiable in problems such as determining the position of an aircraft far from earth. Ashby *et al.* [4] demonstrate that the position cannot be accurately determined unless either the velocity of the detector is known or determined from the same data used to determine the position. In other words, they show that position and velocity are correlated.

The particular (y_p) and homogeneous (y_h) solutions of IVP (7.1) are given by

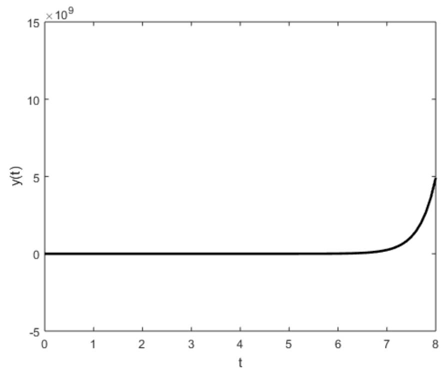
$$y_p(t) = \frac{17}{18}t + \frac{t^2}{6} \quad \text{and} \quad y_h(t) = w_1(t)k_1 + w_2(t)k_2 + w_3(t)k_3,$$

where $w_1(t) = \frac{1}{2}e^{2t}$, $w_2(t) = \frac{1}{3}e^{3t}$, $w_3(t) = 1$, k_1, k_2 and k_3 are constants. Therefore, under these initial conditions the deterministic solution is given by

$$y(t) = \frac{1}{108}\left(61 + 27e^{2t} + 20e^{3t} + 102t + 18t^2\right). \tag{7.2}$$

Figure 55 exhibits the deterministic solution $y(t)$.

Figure 55 – Graphical representation of the deterministic solution $y(t)$ given by (7.2).



Solution of Equation (7.1), where the initial conditions are given by real numbers. Source: [148]

Now suppose that the initial conditions are uncertainties and modelled by the fuzzy numbers

$$\begin{cases} y(0) = (0;1;2) = Y_0 \\ y'(0) = (1;2;3) = Y_0' \\ y''(0) = (2;3;4) = Y_0'' \end{cases}, \tag{7.3}$$

where Y_0, Y_0' and Y_0'' are linearly interactive fuzzy numbers with respect to the joint possibility distribution J_L, whose membership function is given by

$$J_L(x,y,z) = \chi_{\left\{\begin{pmatrix} u \\ u+1 \\ u+2 \end{pmatrix} : \, u \in \mathbb{R}\right\}}(x,y,z)Y_0(x), \tag{7.4}$$

for each $(x, y, z) \in \mathbb{R}^3$. From Equation (4.9), the $\alpha-$cuts of (7.4) are given as follows

$$[J_L]^\alpha = \left\{ (1-\lambda) \begin{pmatrix} \alpha \\ \alpha+1 \\ \alpha+2 \end{pmatrix} + \lambda \begin{pmatrix} 2-\alpha \\ 3-\alpha \\ 4-\alpha \end{pmatrix} : \lambda \in [0,1] \right\}.$$

By Equation (4.25) the fuzzy solution $y_L(t)$, for each $\alpha \in [0,1]$ and $t \in \mathbb{R}$, is given by

$$[y_L(t)]^\alpha = [\min\{h(t,\alpha,0), h(t,\alpha,1)\}, \max\{h(t,\alpha,0), h(t,\alpha,1)\}],$$

where

$$h(t,\alpha,0) = \frac{17}{18}t + \frac{1}{6}t^2 - \frac{3}{4}e^{2t} + \frac{14}{27}e^{3t} + \frac{25}{108} + \alpha\left(e^{2t} - \frac{1}{3}e^{3t} + \frac{1}{3}\right)$$

and

$$h(t,\alpha,1) = \frac{17}{18}t + \frac{1}{6}t^2 + \frac{5}{4}e^{2t} - \frac{4}{27}e^{3t} + \frac{97}{108} + \alpha\left(-e^{2t} + \frac{1}{3}e^{3t} - \frac{1}{3}\right).$$

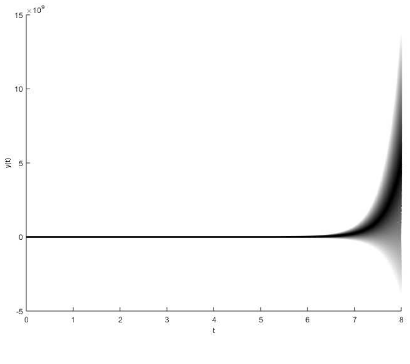
Note that the $\alpha-$cut of the fuzzy solution can also be written as

$$\begin{aligned} [y_L(t)]^\alpha &= \left\{\frac{17}{18}t + \frac{t^2}{6} + \frac{79}{108} + \frac{1}{2}e^{2t}\left((1-\lambda)(1+2\alpha) + \lambda(5-2\alpha) - \frac{5}{2}\right)\right. \\ &+ \frac{1}{3}e^{3t}\left((1-\lambda)(-\alpha) + \lambda(-2+\alpha) + \frac{14}{9}\right) \\ &+ \left.(1-\lambda)\left(-\frac{1}{2} + \frac{1}{3}\alpha\right) + \lambda\left(\frac{1}{6} - \frac{1}{3}\alpha\right) : \lambda \in [0,1]\right\} \end{aligned} \tag{7.5}$$

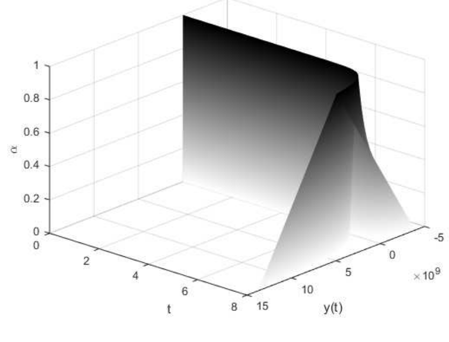
Figure 56 depicts the top and tri-dimensional views of the fuzzy solution (7.5).

Figure 56 – Graphical representation of the fuzzy solution based on the joint possibility distribution J_L given by (7.4).

(a) The top view of the fuzzy solution.



(b) The tri-dimensional view of the fuzzy solution.



The gray-scale lines represent the $\alpha-$cuts of the fuzzy solution given as in (7.5), where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [148].

Note that the joint possibility distribution $J = J_L$, given by (7.4), suggests that Y_0 and Y_0' are positively linear interactive, as well as Y_0 and Y_0'', since the diameter of the fuzzy solution always increases. One can observe that Y_0, Y_0', and Y_0'' may be linearly interactive with respect to others joint possibility distributions. The next subsection illustrates this fact.

7.1.2 Example 2

Consider the following 3rd order differential equation with the initial conditions:

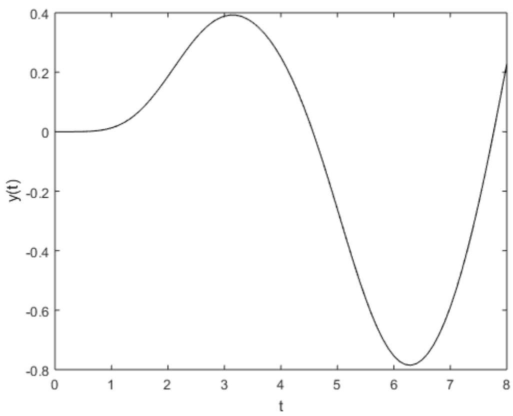
$$\begin{cases} y'''(t) + 9y'(t) = tsint \\ y(0) = 0, y'(0) = 0, y''(0) = 0 \end{cases} . \tag{7.6}$$

The deterministic solution for this IVP is given by

$$y(t) = \frac{1}{96}\left(-12t\cos t + 9sint + sin3t\right).$$

The solution y is depicted in Figure 57.

Figure 57 – Graphical representation of the deterministic solution $y(t)$ of the system (7.6).



Solution of Equation (7.6), where the initial conditions are given by real numbers. Source: [148]

Suppose that the initial conditions are modelled by the fuzzy numbers

$$\begin{cases} y(0) = (-1; 0; 1) = Y_0 \\ y'(0) = (-1; 0; 1) = Y_0' \\ y''(0) = (-1; 0; 1) = Y_0'' \end{cases} .$$

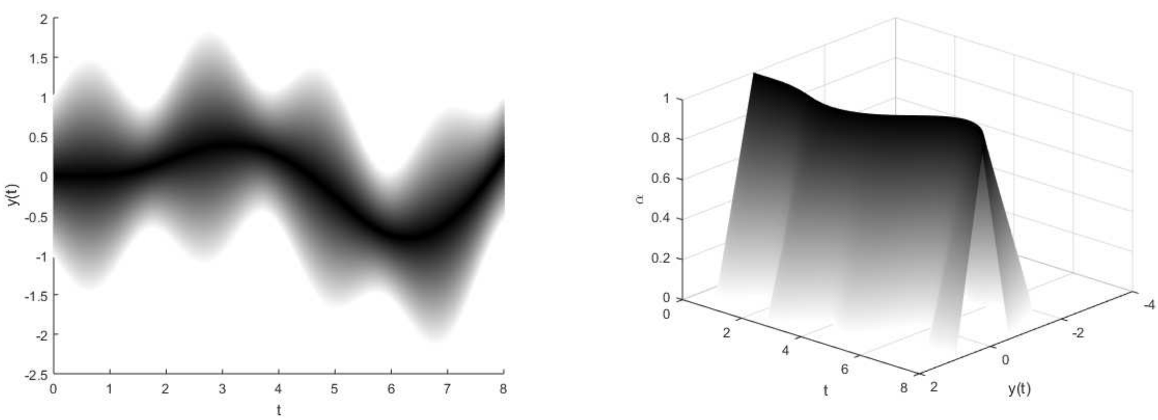
Thus Y_0, Y_0' and Y_0'' may be linearly interactive with respect to the following joint possibility distributions:

$$J_1(x,y,z) = \chi_{\left\{\begin{pmatrix} u \\ u \\ u \end{pmatrix} : u \in \mathbb{R}\right\}}(x,y,z) Y_0(x). \tag{7.7}$$

From J_1, it follows that Y_0 and Y_0' are positively linearly interactive, as well as Y_0 and Y_0''. The fuzzy solution y_L obtained by our approach is depicted in Figure 58.

Figure 58 – Graphical representation of the fuzzy solution based on joint possibility distribution J_1 given by (7.7).

(a) The top view of the fuzzy solution. (b) The tri-dimensional view of the fuzzy solution.



The top and tri-dimensional view of the fuzzy solution obtained using the joint possibility distribution J_1 as in (7.7). The gray lines represent the $\alpha-$cuts of the fuzzy solution $y_L(t)$. The endpoints of the $\alpha-$cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [148].

Now, suppose that Y_0, Y_0' and Y_0'' are linearly interactive with respect to joint possibility distribution J_2, given by

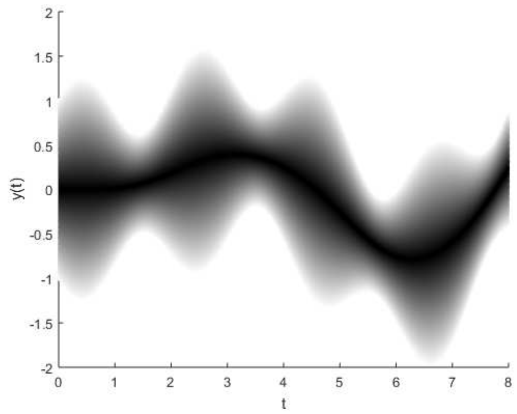
$$J_2(x,y,z) = \chi_{\left\{\begin{pmatrix} u \\ u \\ -u \end{pmatrix} : u \in \mathbb{R}\right\}}(x,y,z) Y_0(x). \tag{7.8}$$

The fuzzy solution y_L based on J_2 is exhibited in Figure 59. Note that, from J_2, it follows that Y_0 and Y_0' are positively linearly interactive and Y_0 and Y_0'' are negatively linearly interactive.

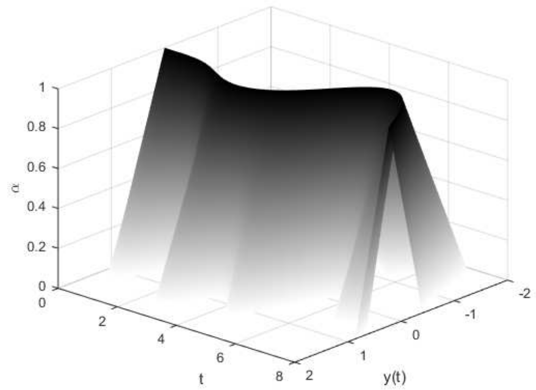
If Y_0, Y_0' and Y_0'' are linearly interactive with respect to joint possibility distri-

Figure 59 – Graphical representation of the fuzzy solution based on joint possibility distribution J_2 given by (7.8).

(a) The top view of the fuzzy solution.



(b) The tri-dimensional view of the fuzzy solution.



The top and tri-dimensional view of the fuzzy solution obtained using the joint possibility distribution J_2 as in (7.8). The gray lines represent the $\alpha-$cuts of the fuzzy solution $y_L(t)$. The endpoints of the $\alpha-$cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [148].

bution J_3, given by

$$J_3(x,y,z) = \chi_{\left\{\begin{pmatrix} u \\ -u \\ u \end{pmatrix} \,:\, u \in \mathbb{R}\right\}}(x,y,z)Y_0(x), \tag{7.9}$$

then Y_0 and Y_0' are negatively linearly interactive whereas Y_0 and Y_0'' are positively linearly interactive. The fuzzy solution y_L via the proposed approach is depicted in Figure 60.

Finally, suppose that Y_0, Y_0' and Y_0'' are linearly interactive with respect to joint possibility distribution J_4, given by

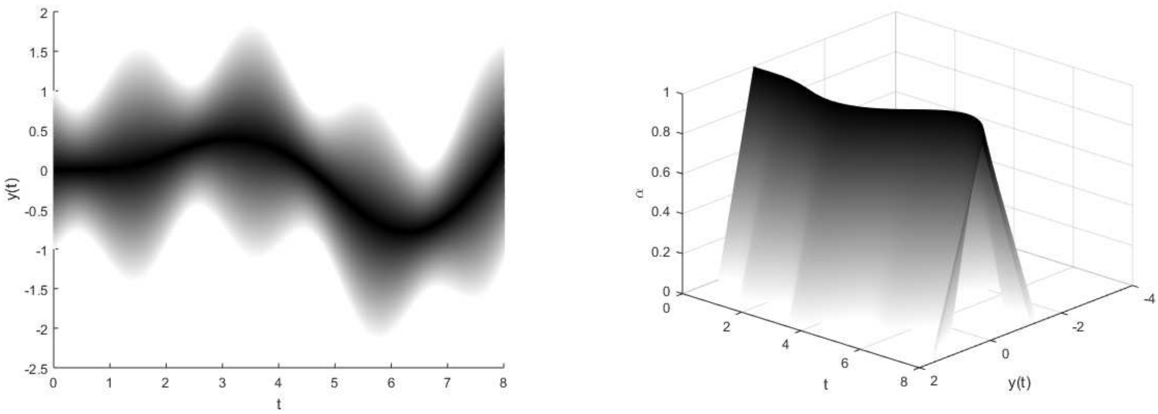
$$J_4(x,y,z) = \chi_{\left\{\begin{pmatrix} u \\ -u \\ -u \end{pmatrix} \,:\, u \in \mathbb{R}\right\}}(x,y,z)Y_0(x), \tag{7.10}$$

The fuzzy solution y_L, obtained by J_4, is exhibited in Figure 61. By Equation (7.10), one can conclude that Y_0 and Y_0' as well as Y_0 and Y_0'' are negatively linearly interactive.

Note that the joint possibility distribution J_1 indicates that initial positive positions $(y(t_0))$ is associated with initial positives velocity $(y'(t_0))$ and acceleration $(y''(t_0))$, since Y_0 and Y_0' as well as Y_0 and Y_0'' are positively linear interactive. This means that $y(t)$ increases initially with increasing rate if the initial position $y(t_0)$ is positive. On the other hand, if $y(t_0)$ is negative, then the initial velocity $y'(t_0)$ and acceleration $y''(t_0)$

Figure 60 – Graphical representation of the fuzzy solution based on joint possibility distribution J_3 given by (7.9).

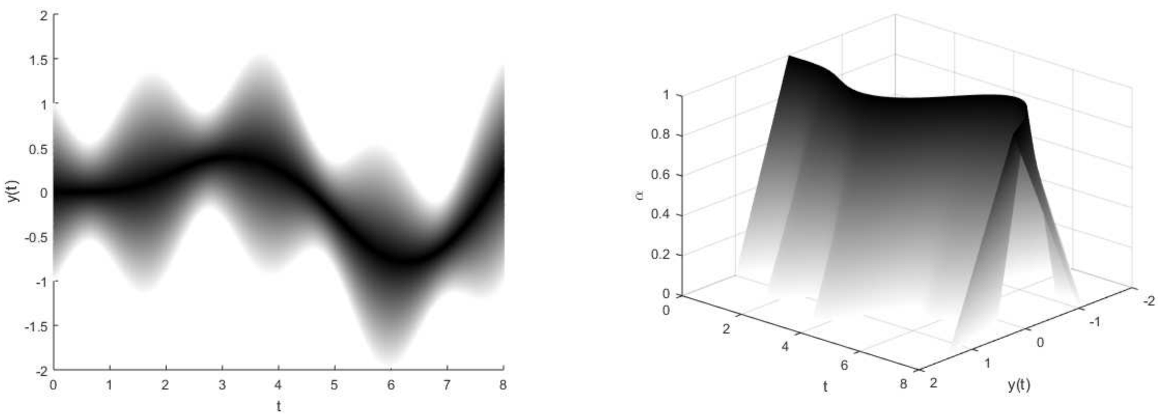
(a) The top view of the fuzzy solution. (b) The tri-dimensional view of the fuzzy solution.



The top and tri-dimensional view of the fuzzy solution obtained using the joint possibility distribution J_3 as in (7.9). The gray lines represent the $\alpha-$cuts of the fuzzy solution $y_L(t)$. The endpoints of the $\alpha-$cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [148].

Figure 61 – Graphical representation of the fuzzy solution based on joint possibility distribution J_4 given by (7.10).

(a) The top view of the fuzzy solution. (b) The tri-dimensional view of the fuzzy solution.



The top and tri-dimensional view of the fuzzy solution obtained using the joint possibility distribution J_4 as in (7.10). The gray lines represent the $\alpha-$cuts of the fuzzy solution $y_L(t)$. The endpoints of the $\alpha-$cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [148].

are also negative values. In this case $y(t)$ decreases with decreasing rate initially, causes the expansion of the diameter of the fuzzy solution $y_L(t)$ initially, as one can observe in Figure 58.

The fuzzy solution based on J_2 has the same behaviour as the solution produced by J_1. However, the acceleration at t_0 is negatively linear interactive with the initial position $y(t_0)$. This means that the expansion of the solution based on J_2 is smaller than the expansion of the solution given by J_1 (see Figures 58 and 59), as one can observe in Figure 62.

In contrast to J_1 and J_2, the joint possibility distribution J_3 indicates that the $y(t)$ decreases initially with increasing rate if the initial position $y(t_0)$ is positive since Y_0 and Y_0' are negatively linear interactive and Y_0 and Y_0'' are positively linear interactive. Thus, the solution $y_L(t)$ obtained by J_3 is initially contractive (see Figure 60).

Finally, the JPD J_4 also produces a fuzzy solution that is initially contractive. However, the contraction is smaller than the contraction of the solution given by J_3, since the acceleration is negatively linear interactive with the initial position $y(t_0)$.

A briefly study of the diameter of the fuzzy solution is given as follows. For J_1, it follows

$$diam(y_L(t)) = \frac{2}{3}sin(3t) - \frac{2}{9}cos(3t) + \frac{20}{9}$$

Note that the diameter of the fuzzy solution increases on the intervals

$$\left[\frac{2}{3}\pi n, \frac{\pi}{6} + \frac{2}{3}\pi n\right] \quad \text{and} \quad \left[\frac{\pi}{2} + \frac{2}{3}\pi n, \frac{2\pi}{3} + \frac{2}{3}\pi n\right],$$

for all $n \in \mathbb{N}$, since the derivative of $diam(y_L(t))$ with respect to t is positive in these intervals. Moreover, it decreases on intervals

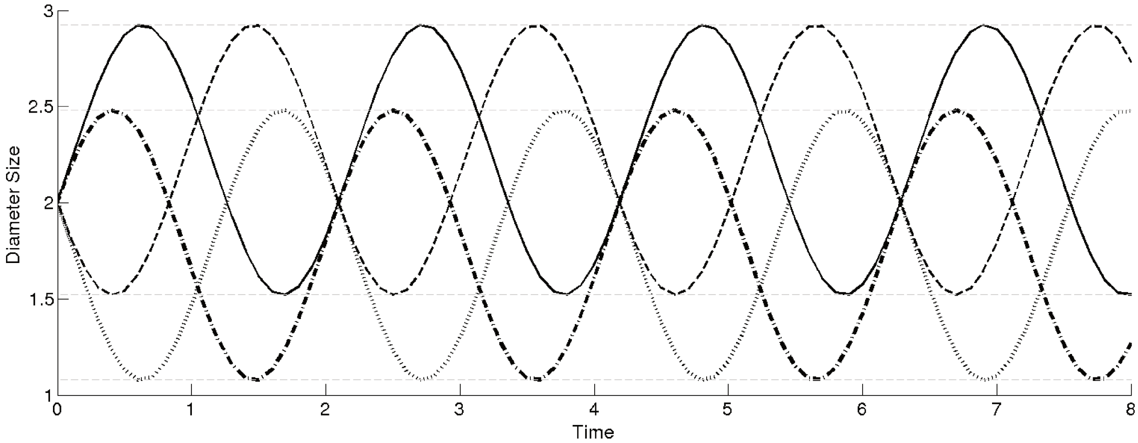
$$\left[\frac{\pi}{6} + \frac{2}{3}\pi n, \frac{\pi}{3} + \frac{2}{3}\pi n\right] \quad \text{and} \quad \left[\frac{\pi}{3} + \frac{2}{3}\pi n, \frac{\pi}{2} + \frac{2}{3}\pi n\right],$$

for all $n \in \mathbb{N}$ since the derivative of $diam(y_L(t))$ is negative in these intervals.

A similar analysis can be made for the diameter of the solution obtained by using J_2, J_3 and J_4. Figure 62 exhibits the diameters of the solutions $y_L(t)$ with respect to the joint possibility distributions J_1, J_2, J_3 and J_4. Figure 62 also corroborates the above comments about the initially expansion and contraction of the fuzzy solutions.

The next subsection presents the last example involving this type of interactivity, in order to clarify the properties of the fuzzy solution that arises from the JPD J_L.

Figure 62 – Graphical representation of the diameter of the solutions produced by J_1, J_2, J_3, and J_4.



The solid, dash-dotted, dotted, and dashed lines represent the diameter of the fuzzy solution y_L obtained using J_1, J_2, J_3 and J_4, respectively. Source: [148].

7.1.3 Example 3

For this example, consider the following FBVP, where the additional conditions are given by linearly interactive fuzzy numbers:

$$\begin{cases} y'''(t) + 5y'(t) = t\cos^2 t \\ y(0) = A,\ y'(0) = B,\ y(10) = C \end{cases}, \tag{7.11}$$

where $t \in [0, 10]$ and $A, B, C \in \mathbb{R}_{\mathcal{F}}$.

The classical solution of the associated ODE, where the boundary conditions are $y(0) = 0,\ y''(0) = 0$ and $y(10) = 5$, is given by

$$\begin{aligned} y(t) \quad &= \frac{1}{40\left(\cos\left(10\sqrt{5}\right) - 1\right)}\left(-2t^2 + 2t^2\cos\left(10\sqrt{5}\right) - 10tsin(2t)\right. \\ &- 35\cos\left(10\sqrt{5}\right)\cos(2t) + 35\cos(2t) + 35\cos(20)\cos\left(\sqrt{5}t\right) \\ &- 35\cos\left(\sqrt{5}t\right) + 10t\cos\left(10\sqrt{5}\right) sin(2t) - 35\cos(20) \\ &\left. + 100sin(20) - 100sin(20)\cos\left(\sqrt{5}t\right) + 35\cos\left(10\sqrt{5}\right)\right) \end{aligned} \tag{7.12}$$

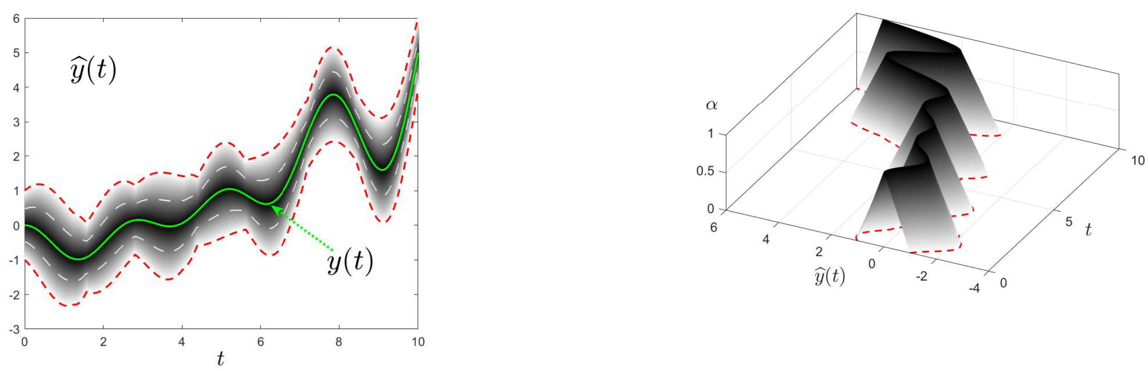
Suppose that the boundary conditions are given by the symmetric fuzzy numbers $y(0) = (-1; 0; 1) = A$, $y'(0) = (-1; 0; 1) = B$, and, $y(10) = (4; 5; 6) = C$. The fuzzy solution y_L of (7.11), via Zadeh's extension, is depicted in Figure 63.

The fuzzy numbers A, B and C may be linearly interactive with respect to four different joint possibility distributions given by

$$[J_1]_\alpha = \left\{ \begin{pmatrix} x \\ x \\ x+5 \end{pmatrix} : x \in [A]_\alpha \right\} \quad , \quad [J_2]_\alpha = \left\{ \begin{pmatrix} x \\ x \\ -x+5 \end{pmatrix} : x \in [A]_\alpha \right\},$$

Figure 63 – Graphical representation of the fuzzy solution via Zadeh's extension principle of the FBVP given as in (7.11).

(a) The top view of $\widehat{y}$. (b) The tri-dimensional view of $\widehat{y}$.



The green line represents the deterministic solution y, given by (7.12). The red and white dashed-lines illustrate respectively the 0-cut and the 0.5-cut of y_L. The gray-scale lines varying from white to black represent the α-cuts of $\widehat{y}$, where α varying from 0 to 1. Source: [138].

$$[J_3]_\alpha = \left\{ \begin{pmatrix} x \\ -x \\ x+5 \end{pmatrix} : x \in [A]_\alpha \right\} \quad \text{and} \quad [J_4]_\alpha = \left\{ \begin{pmatrix} x \\ -x \\ -x+5 \end{pmatrix} : x \in [A]_\alpha \right\}.$$

The fuzzy solutions of (7.11) via sup-J extension are depicted in Figure 64.

It is important to observe in Figures 63 and 64 that the convex hull of the solutions $y_{J_1}, y_{J_2}, y_{J_3}$ and y_{J_4} is equal to the solution given by Zadeh's extension, which corroborates the statement of Theorem 4.2.

Now, suppose that the boundary conditions are given by the non-symmetric fuzzy numbers $y(0) = (-1;0;2) = A$, $y'(0) = (-1;0;2) = B$, and, $y(10) = (4;5;7) = C$. The fuzzy numbers A, B, and, C are linearly interactive with respect to the joint possibility distribution given by

$$[J]_\alpha = \left\{ \begin{pmatrix} x \\ x \\ x+5 \end{pmatrix} : x \in [A]_\alpha \right\}. \tag{7.13}$$

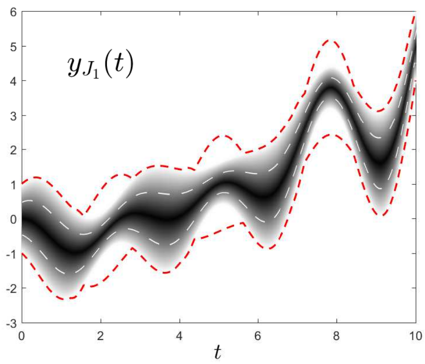
The fuzzy solution y_J of (7.11) via sup-J extension is depicted in Figure 65.

Observe that in this case there is only one joint possibility distribution of A, B, and, C. Also, note that Figure 65 (a) verifies the result of Theorem 4.3, that is, the solution y_J is contained in $\widehat{y}$. Moreover, Equation (4.38) ensures that $\tilde{y}_F$, given by (4.36), is also contained in $\widehat{y}$.

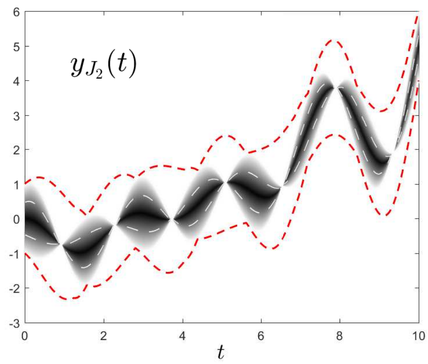
The next subsection presents an example to describe the trajectory of a particle, according to a hypocycloid curve.

Figure 64 – Graphical representation of the fuzzy solution based on joint possibility distribution J_L given by (7.10).

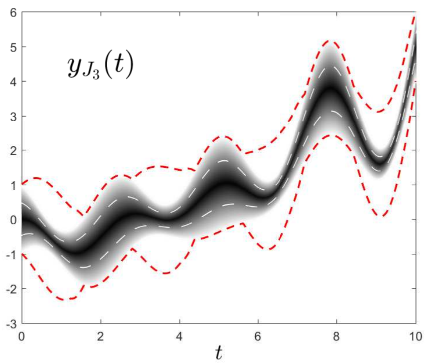
(a) The top view of y_{J_1}.



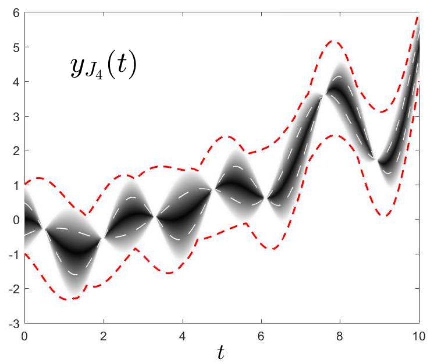
(b) The top view of y_{J_2}.



(c) The top view of y_{J_3}.



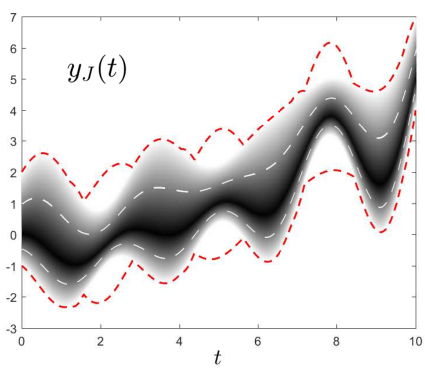
(d) The top view of y_{J_4}.



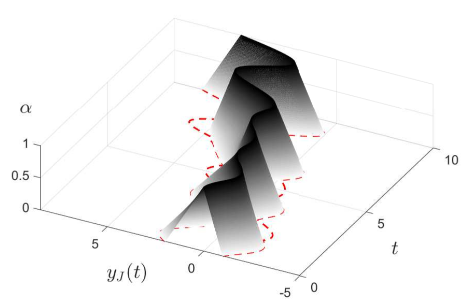
The red and white dashed-lines illustrate respectively the 0-cut of $\widehat{y}$ and the 0.5-cut of y_{J_i}, where $i = 1, \ldots, 4$. The gray-scale lines varying from white to black represent the α-cuts of y_{J_i}, where α varying from 0 to 1. Source: [138].

Figure 65 – Graphical representation of the fuzzy solution y_J given as in (7.11).

(a) The top view of y_J.



(b) The tri-dimensional view of y_J.

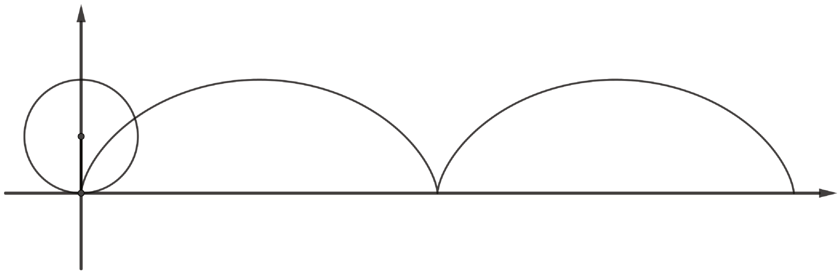


The red and white dashed-lines illustrate respectively the 0-cut of $\widehat{y}$ and the 0.5-cut of y_J. The gray-scale lines varying from white to black represent the α-cuts of y_J, where α varying from 0 to 1. Source: [138].

7.1.4 Fuzzy Initial Value Problems for Fuzzy Hypocycloid Curves

This subsection focuses on a system of differential equations that describes the trajectory of a particle that moves in the plane according to a hypocycloid curve (See Figure 66).

Figure 66 – Graphical representation of the hypocycloid curve.



Hypocycloid curve generated by the trace of a fixed point on a circle that rolls along a line. Source: Author

To this end, consider a coupled system composed of two ordinary second order differential equations with initial conditions given by interactive fuzzy numbers. In this case, the same approach given in Chapter 4 can be used, as it will be illustrated.

Hypocycloid curves are solutions of IVPs, which are given by a homogeneous system of two ODEs of second order. This solution represents a dynamic response of two states variables that graphically represent the movement of particles on a phase plane [47]. Consider that the initial conditions are given by linearly interactive fuzzy numbers.

Before presenting this solution, let us provide the method for a system of two ODEs. Consider the second order linear and homogeneous system given by

$$\begin{cases} x''(t) + k_1\, y'(t) + k_2\, x(t) = 0 \\ y''(t) + k_3\, x'(t) + k_4\, y(t) = 0 \end{cases}, \tag{7.14}$$

for all $t \in [t_0, T]$, where the parameters $k_1, k_2, k_3, k_4 \in \mathbb{R}$ are constants and x, y are the state variables.

The linear system (7.14) can be rewritten in the matrix form using the substitution $x_1 = x(t)$, $x_2 = x'(t)$, $x_3 = y(t)$ and $x_4 = y'(t)$ given by

$$x' = K\,x, \tag{7.15}$$

where $x' = \begin{bmatrix} x_1' & x_2' & x_3' & x_4' \end{bmatrix}^T$, $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ and

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_2 & 0 & 0 & -k1 \\ 0 & 0 & 0 & 1 \\ 0 & -k_3 & -k_4 & 0 \end{bmatrix}.$$

Then in order to provide a fuzzy solution for this problem, one can use the same method, provided in Chapter 4, for system (7.15) by taking the particular and homogeneous solution of the problem and calculating the vector of constants in terms of the fuzzy initial condition.

So let us illustrate the above comments by considering a linear second order and homogeneous FIVP given by

$$\begin{cases} x''(t) - 2\,y'(t) + 3\,x(t) = 0, \\ y''(t) + 2\,x'(t) + 3\,y(t) = 0, \\ x(0) = (3.5; 4; 5), \quad x'(0) = y(0) = y'(0) = (-1; 0; 0.5), \end{cases} \tag{7.16}$$

for $t \in [0, 2\pi]$, where the initial conditions are given by linearly interactive fuzzy numbers.

The classical solution of the associated FIVP (7.16), where the initial conditions are real numbers $x(0) = 4$ and $x'(0) = y(0) = y'(0) = 0$, is given by

$$\begin{aligned} x(t) &= 3\cos(t) + \cos(3t) \\ y(t) &= 3\mathrm{sen}(t) - \mathrm{sen}(3t) \end{aligned} \;. \tag{7.17}$$

The α-cuts of the fuzzy solution of (7.16) are given by

$$\begin{aligned} [x_J(t)]^\alpha &= u_1(t) + v_1(t)[x(0)]^\alpha \\ [y_J(t)]^\alpha &= u_3(t) + v_3(t)[x(0)]^\alpha \end{aligned} \tag{7.18}$$

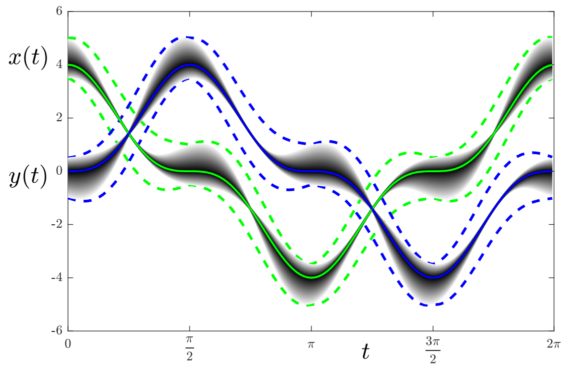
where $[x(0)]^\alpha = [0.5\alpha + 3.5\,, 5 - \alpha]$ for all $\alpha \in [0,1]$ and $u_1(t), u_3(t), v_1(t), v_3(t)$ are determined as in (4.34) with $q = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}^T$ and $r = \begin{bmatrix} 0 & 4 & 4 & 4 \end{bmatrix}^T$.

Note that the fuzzy solutions given by (7.18) depend only on the initial condition $x(0) \in \mathbb{R}_{\mathcal{F}}$ and the parameters q_i and r_i, for $i = 1, \ldots, 4$, as well as in the method provided in Section 4.2. Also observe that the initial conditions are given by non-symmetric fuzzy numbers, which means that there is only one joint possibility distribution J_L among $x(0), x'(0), y(0)$ and $y'(0)$, as Proposition 4.1 ensures.

Figure 67 depicts the fuzzy solution given by (7.18). In this figure the fuzzy solutions were superimposed on the same graph to observe the behaviour of solutions at the same points on the domain. One can observe that the fuzzy solution via sup-J extension is contained in the one given by the Zadeh's extension. From the physical point of view the fuzzy numbers can represent the uncertainty of the initial points $x(0)$ and $y(0)$ or stability points $x'(0)$ and $y'(0)$ [148].

The equations in (7.17) describe the hypocycloid curve traced by a fixed point $P(x, y)$ on the circumference of a circle of radio 1, which rolls internally around from a circle of radius 4. A fuzzy hypocycloid curve, depicted in Figure 68, can be considered as

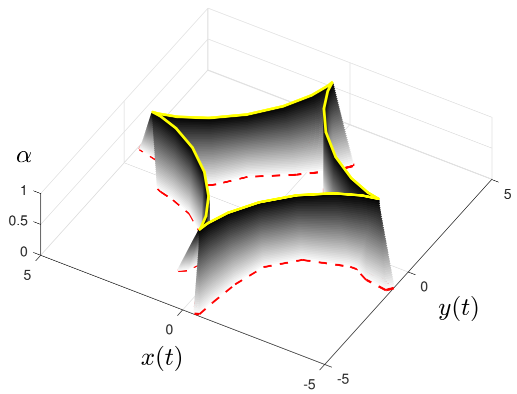
Figure 67 – Graphical representation of the fuzzy solution of hyocycloid curve problem.



The green and blue solid lines represent the deterministic solution $x(t)$ and $y(t)$, given by (7.17). The green and blue dashed-lines represent the 0-cut of the solution via Zadeh's extension. The gray lines represent the α-cuts of the fuzzy solution via sup-J extension given by (7.18). The endpoints of the α-cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [138].

the diagram phase of the fuzzy solution via sup-J extension principle, given by (7.18), of the states variables $x(t)$ and $y(t)$.

Figure 68 – Graphical representation of the fuzzy hyocycloid curve.



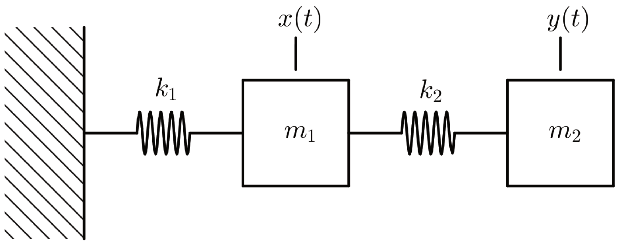
Tri-dimensional view of the fuzzy hypocycloid curve as diagram phase. The yellow solid line represents the deterministic solutions $x(t)$ and $y(t)$, given by (7.17). The red dashed-line represents the 0-cut of the solution via Zadeh's extension. The gray lines represent the α-cuts of the fuzzy solution via sup-J extension given by (7.18). The endpoints of the α-cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [138].

The next subsection presents an application involving mechanical vibrations.

7.1.5 Fuzzy Initial Value Problems for Fuzzy Mechanical Vibrations

The field of mechanical vibrations plays a fundamental role in the study and design of physical phenomena and structures. For instance, computer designs of engines, planes or cars help to lower costs and visualize possible problems related to noise or attrition of parts due to vibrations. In particular, mechanical vibration describes the dynamics (position and/or velocity) of one or more particles subjected to free or forced external conditions in a simple or coupled mass-spring model (see Figure 69).

Figure 69 – Graphical representation of the mass-spring system.



The spring k_1 is attached to the wall and mass m_1. Mass m_1 is also attached to mass m_2 through spring k_2. The trajectories of particles of mass m_1 and m_2 are described by x and y, respectively. Source: Author

These physical models are represented by two ODEs for any mass coupled and empirical initial conditions values. Particularly, these ODEs can be transform in a linear first order system, in according to the classical differential equation theory [101].

This subsection presents an example of mass-spring system with initial conditions given by linearly interactive fuzzy numbers. This example considers the trajectory of two particles without damper and external force.

So consider the mass-springer system given by

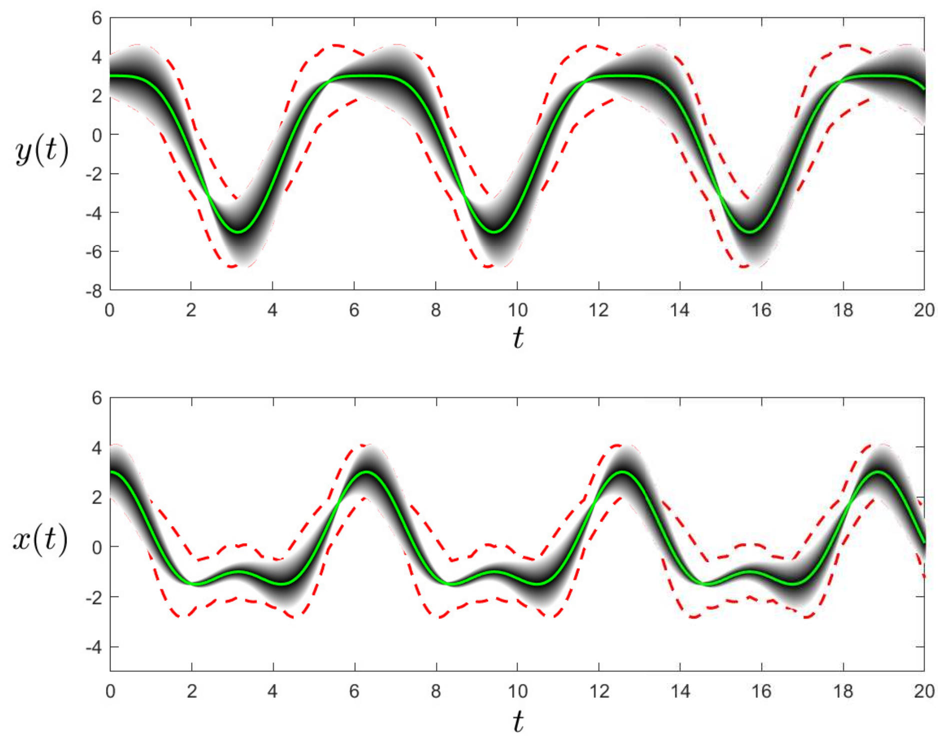
$$\begin{cases} 2\,x''(t) + 6\,x(t) - 2\,y(t) = 0\,, \\ y''(t) + 2\,y(t) - 2\,x(t) = 0\,, \\ x(t_0) = (2\,;\,3\,;\,4)\,,\ x'(t_0) = (-1\,;\,0\,;\,1)\,, y(t_0) = (2\,;\,3\,;\,4)\,,\ y'(t_0) = (-1\,;\,0\,;\,1)\,. \end{cases} \tag{7.19}$$

where $t \in [0, 20]$ and the initial conditions are given by linearly interactive fuzzy numbers.

From the method described in last subsection and Chapter 4, the two ODEs given by (7.19) can be written in a matrix form using the substitution $x_1 = x(t)$, $x_2 = x'(t)$, $x_3 = y(t)$ and $x_4 = y'(t)$. Hence, the fuzzy solution of (7.19) can be obtained in the same way as before (see Subsection 7.1.4), and it is depicted in Figure 70.

Figure 70 represents two dynamic trajectories $y(t)$ and $x(t)$ with masses m_2 and m_1, respectively. Observe that the interactive fuzzy solution is contained in the one given by the Zadeh's extension, as it was expected. Note that the curve given by the 1-cut of the fuzzy solution $x_J(t)$ coincides with the deterministic solution.

Figure 70 – Graphical representation of the solution of a mass-spring system given as in (7.19).



The green lines represent the deterministic solutions $x(t)$ and $y(t)$ for the coupled mechanical vibration problem (7.19), for $x(0) = y(0) = 3$ and $x'(0) = y'(0) = 0$. In both subfigures, the dashed red lines represent the 0-cut of the solution via Zadeh's extension. The gray lines represent the α-cuts of the fuzzy solution via sup-J extension. The endpoints of the α-cuts for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [126]

Since B_0 is symmetric, there exist 2^3 JPDs among the initial fuzzy numbers. Figure 70 portrays the JPD J_L in the case where $q = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $r = \begin{bmatrix} 0 & -3 & 0 & -3 \end{bmatrix}^T$.

The next section presents applications in epidemiology and chemical reaction problems.

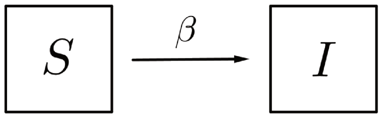
7.2 Numerical solutions for IVP's with initial conditions given by J_γ-interactive fuzzy numbers

In the epidemiology field, the mathematical models are fundamental tools for understanding the dynamic and spread of the disease. These mathematical models are usually given by ordinary or partial differential equations [46]. Here, the focus is on models that are widely studied in the literature, such as SI, SIS, and SIR models, where S,

I, and R stands for susceptible, infected and recovered populations, respectively. Each individual is classified in one of these states. The interactions among the populations are given by the law of mass action [13] and it is assumed that the infected individual transmits the disease with the same infection rate to the susceptible ones. In other words, the population is homogeneous.

1. For the SI model the population is divided in two states, susceptible and infected. The susceptible individual becomes infected with rate β and in this state remains (see Figure 71). One example of this disease is the AIDS [144].

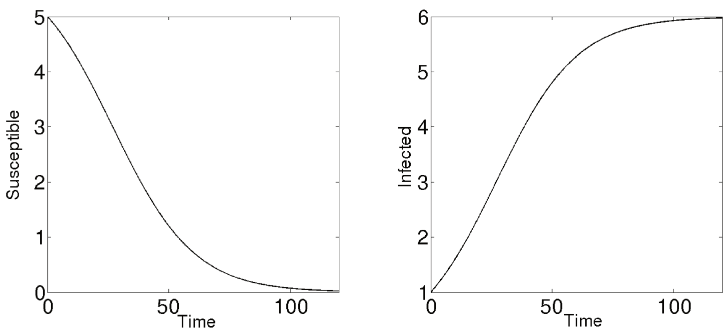
Figure 71 – Diagram of the SI epidemiological model.



The susceptible and infected individuals are represented by S and I, respectively. The infection rate is represented by β. Source: Author

The behaviour of populations S and I in the SI model is depicted in Figure 72

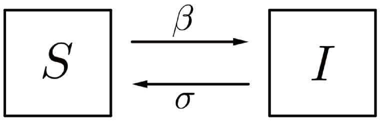
Figure 72 – Graphical representation of the populations S and I in the SI epidemiological model.



The left and right subfigures represent respectively the susceptible and infected populations, where $S(0) = 5$ and $I(0) = 1$. Source: [145]

2. In the SIS model the infected individual may return to the susceptible state. However, those who recover do not have permanent immunity to infection, returning to susceptible population with rate σ (see Figure 73). One example of this disease is the Chagas disease [145].

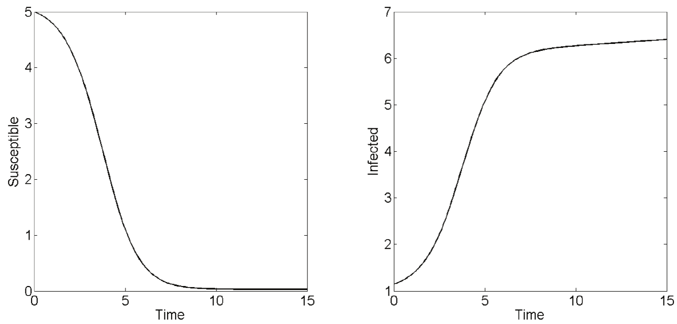
Figure 73 – Diagram of the SIS epidemiological model.



The susceptible and infected individuals are represented by S and I, respectively. The infection rate is represented by β and the recovery is represented by σ. Source: Author

The behaviour of populations S and I in the SIS model is depicted in Figure 74

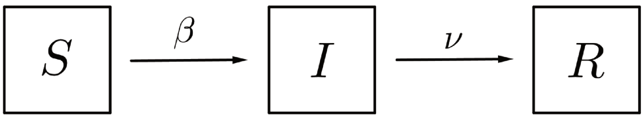
Figure 74 – Graphical representation of the populations S and I in the SIS epidemiological model.



The left and right subfigures represent respectively the susceptible and infected populations, where $S(0) = 5$ and $I(0) = 1$. Source: [145]

3. In the SIR model there are three states, susceptible, infected and recovered. The infected individual after recovering the disease is moved to the recovered state with rate ν and in this state remains (see Figure 75). One example of this disease is Caxumba [142].

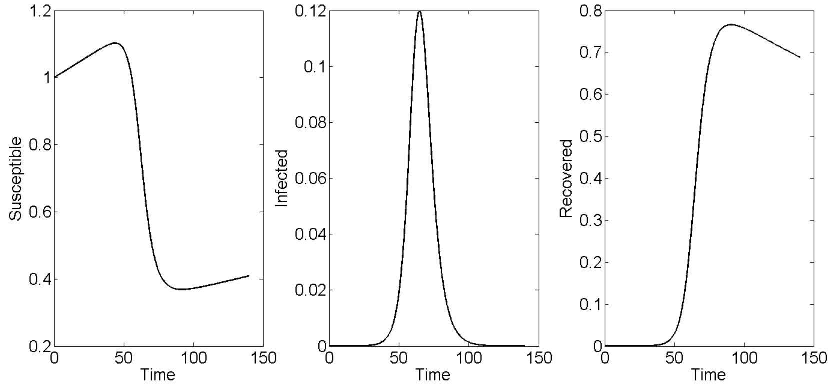
Figure 75 – Diagram of the SIR epidemiological model.



The susceptible, infected and recovered individuals are represented by S, I and R, respectively. The infection rate is represented by β and the recovery is represented by ν. Source: Author

The behaviour of populations S, I and R in the SIR model is depicted in Figure 76

Figure 76 – Graphical representation of the populations S, I and R in the SIR epidemiological model.



The left to right subfigures represent respectively the susceptible, infected and recovered populations, where $S(0) = 1$, $I(0) = 0$ and $R(0) = 0$. Source: [145]

In all these models, the number of individuals S, I, and R are uncertain, since the immunological system of each individual reacts in different ways [13, 144]. The classical models do not take into account this fact.

The FIVPs consider initial conditions and/or parameters uncertain in their differential equations. However, the most FIVPs do not take into account, explicitly, the interactivity relationship. In the epidemiological models the interactivity relationship may arise in any restriction of the biological phenomena, such as the relations of mutualism, protocooperation, commensalism, etc [142]. Mathematically, this interactivity is described by the concept of interactive fuzzy numbers.

Next, numerical solutions of FDEs, which are based on the method proposed in Chapter 5, are presented. These FDEs describe the epidemiological models SI, SIS, and SIR.

7.2.1 Numerical solution of SI model

The SI model is described by diagram in Figure 71, and it is written mathematically as follows [46]

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI, & S(0) = S_0 \\ \dfrac{dI}{dt} = \beta SI, & I(0) = I_0 \end{cases}. \tag{7.20}$$

where β is the rate of the infection of disease and S_0 and I_0 are given by interactive fuzzy numbers.

Therefore, the numerical solution based on Euler's method adapted (see (5.15)) is given by

$$\begin{cases} S^{k+1} = S^k +_\gamma h(-\beta S^k *_\gamma I^k) \\ I^{k+1} = I^k +_\gamma h(\beta S^k *_\gamma I^k) \end{cases}. \tag{7.21}$$

One can observe that, a fuzzy solution for (7.20) using the joint possibility distribution J_L can also be given. Indeed, suppose that the initial values S_0 and I_0 are given by interactive fuzzy numbers with respect the joint possibility distribution $J_L = J_{\{q,r\}}$, given by (4.1). Note that $S *_L I$ and S are not necessarily interactive, even though S and I are interactive. The same observation can be made for $S *_L I$ and I. Thus, the arithmetic operations in (5.12) can not be applied. First it is necessary to make some appropriated changes in Equation (5.12). Since a vital dynamics is not considered in this model, it follows that $S(t) + I(t) = p$, with $p \in \mathbb{R}$ and $\forall t \in [0, \infty)$. Thus the following equations are equivalent

$$S^{k+1} = S^k + h(-\beta S^k I^k) \Leftrightarrow S^{k+1} = S^k(1 - ph\beta + h\beta S^k). \tag{7.22}$$

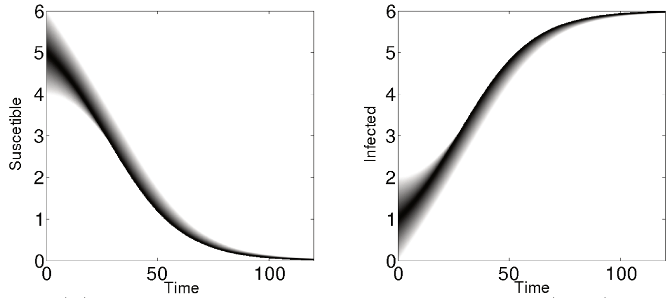
In this case, S^k and $(1 - ph\beta + h\beta S^k)$ are also linearly interactive and then one can use the sup-J extension principle for $J_L = J_{\{q,r\}}$, where $q = h\beta$ and $r = 1 - ph\beta$. The same holds for I^k. Thus, the fuzzy numerical solution for (7.20), with respect $J_L = J_{\{q,r\}}$, is given by

$$\begin{cases} S^{k+1} = S^k *_L (1 - ph\beta + h\beta S^k) \\ I^{k+1} = I^k *_L (1 + ph\beta - h\beta I^k) \end{cases}. \tag{7.23}$$

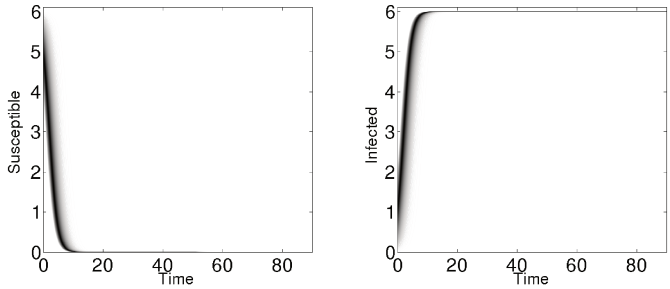
In order to illustrate this method, let us consider $S_0 = (4; 5; 6)$ and $I_0 = (0; 1; 2)$. Note that S_0 and I_0 are interactive with respect to J_L and J_γ. Chapter 5 observed that the JPDs J_L and J_0 are similar (but not equal), in the case where the involved fuzzy numbers are triangular. Hence, in order to compare these JPDs, it is depicted in Figure 77 the fuzzy solutions provided by (7.23) and (7.21), for $\gamma = 0$.

Figure 77 – Graphical representation of the fuzzy solutions of the SI model (7.21), via J_0 and J_L.

(a) Fuzzy solution, via JPD J_0, given by (7.21).



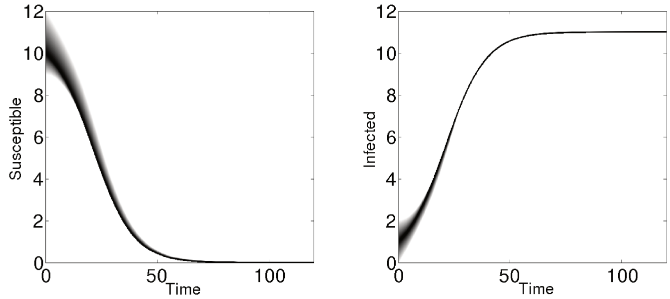
(b) Fuzzy solution, via JPD J_L, given by (7.23).



The fuzzy solutions provided by (7.21) and (7.23), where $S_0 = (4; 5; 6)$ and $I_0 = (0; 1; 2)$. The left and right subfigures present the susceptible and infected populations, respectively. The parameters used were $h = 0.125$ and $\beta = 0.01$. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [149].

Now, let us consider other fuzzy initial conditions. Let $S_0 = (9; 10; 12)$ and $I_0 = (0; 1; 2)$. Even though S_0 and I_0 are triangular fuzzy numbers, there is no linear correlation between their membership functions. Therefore the fuzzy solution provided by (7.23) can not be applied in this case. However, the fuzzy solution provided by (7.21) has no restriction. Figure 78 presents the fuzzy solution given by (7.21).

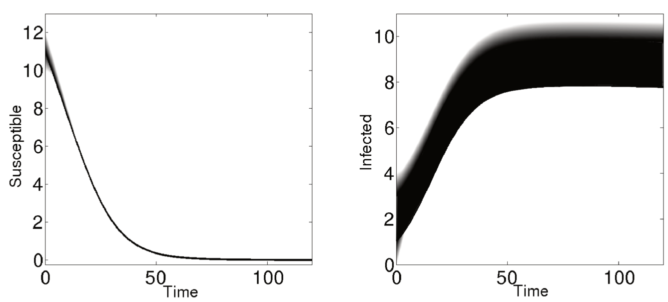
Figure 78 – Graphical representation of the fuzzy solution of the SI model (7.21) via J_0.



The fuzzy solution provided by (7.21), where $S_0 = (9; 10; 12)$ and $I_0 = (0; 1; 2)$. The left and right subfigures present the susceptible and infected populations, respectively. The parameters used were $h = 0.125$ and $\beta = 0.01$. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [149].

Chapter 5 states that the joint possibility distribution J_γ has no restrictions with respect to the shape of the fuzzy numbers. In order to corroborate this statement, let us consider the following case: suppose that the initial number of susceptible population is *around* 11 and the initial number of infected population is *around* the interval $[1, 3]$. Thus, the population S_0 and I_0 are modeled by the interactive fuzzy numbers $S_0 = (10; 11; 12)$ and $I_0 = (0; 1; 3; 4)$. Since, S_0 is a triangular fuzzy number and I_0 is a trapezoidal fuzzy number, obviously S_0 and I_0 are not linearly interactive. Hence, between the two approaches only the joint possibility distribution J_0 can be used in this case. The fuzzy solution is depicted in Figure 79.

Figure 79 – Graphical representation of the fuzzy solution of the SI model (7.21) via J_0.



The fuzzy solution provided by (7.21), where $S_0 = (10; 11; 12)$ and $I_0 = (0; 1; 3; 4)$. The left and right subfigures present the susceptible and infected populations, respectively. The parameters used were $h = 0.125$ and $\beta = 0.01$. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [149].

Note that in all the cases that was considered, the width of the fuzzy solutions given by J_0 decreases over time. This means that it is possible to obtain a certain control of the uncertainty in the temporal evolution of both populations.

The next subsection presents a numerical solution for the SI model considering vital dynamics.

7.2.2 Numerical solution for SI model with vital dynamics

This subsection provides a numerical solution for the SI model with vital dynamics. This solution is based on the Runge-Kutta adapted method (see (5.16)). Recall that one may use the Euler adapted method as well [143]. The bidimensional SI model with vital dynamics is given by

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI + (\eta - \mu)S, \\ \dfrac{dI}{dt} = \beta SI - \mu I, \\ S(0) = S^0, \quad I(0) = I^0 \end{cases}, \tag{7.24}$$

where η, μ, and, β are the rate of birth, death and infection of disease, respectively, and the initial conditions S^0 and I^0 are given by fuzzy numbers.

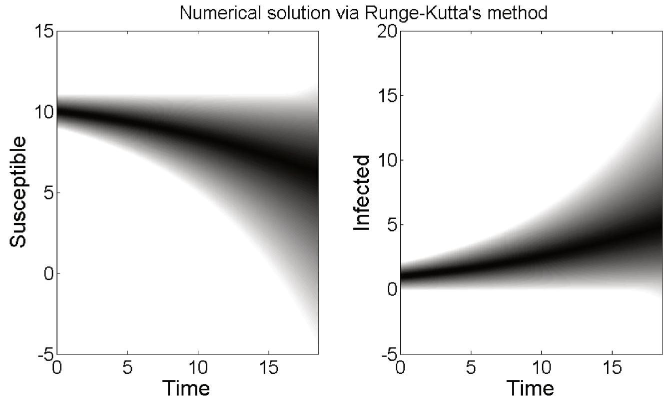
In the case where the initial conditions are considered as non-interactive, the fuzzy numerical solution have to be based on the standard arithmetic. Therefore the solution via Runge-Kutta method of forth order is given by (7.25)

$$\begin{cases} S^{k+1} = S^k + \dfrac{h}{6}(K_1^S + 2K_2^S + 2k_3^S + K_4^S) \\ I^{k+1} = I^k + \dfrac{h}{6}(K_1^I + 2K_2^I + 2k_3^I + K_4^I) \end{cases}, \tag{7.25}$$

where the values of K_i^S and K_i^I are given by

$$\begin{aligned}
K_1^S &= f_S(t^k, S^k, I^k) = -\beta S^k \cdot I^k + (\eta - \mu)S^k, \\
K_1^I &= f_I(t^k, S^k, I^k) = \beta S^k \cdot I^k - \mu S^k, \\
K_2^S &= f_S\left(t^k + \frac{h}{2}, S^k + K_1^S\frac{h}{2}, I^k + K_1^S\frac{h}{2}\right), \\
K_2^I &= f_I\left(t^k + \frac{h}{2}, S^k + K_1^S\frac{h}{2}, I^k + K_1^S\frac{h}{2}\right), \\
K_3^S &= f_S\left(t^k + \frac{h}{2}, S^k + K_2^S\frac{h}{2}, I^k + K_2^S\frac{h}{2}\right), \\
K_3^I &= f_I\left(t^k + \frac{h}{2}, S^k + K_2^S\frac{h}{2}, I^k + K_2^S\frac{h}{2}\right), \\
K_4^S &= f_S\left(t^k + h, S^k + hK_3^S, S^k + hK_3^S\right), \\
K_4^I &= f_I\left(t^k + h, S^k + hK_3^S, S^k + hK_3^S\right).
\end{aligned}$$

Figure 80 – Graphical representation of the fuzzy solution of the SI model with vital dynamics (7.24) via standard arithmetic.



The fuzzy solution given by Runge-Kutta method via standard arithmetic. The left and right subfigures present the susceptible and infected populations, respectively. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [146].

The parameters used in all simulations are $h = 0.125$, $\eta = 0.2 \times 10^{-4}$, $\mu = 0.1 \times 10^{-4}$, $\beta = 0.01$, $S^0 = (9; 10; 11)$ and $I^0 = (0; 1; 2)$.

One can observe in Figure 80 the numerical method given by (7.25) yields a fuzzy solution which width increases over time. This fact holds due to the standard arithmetic for non-interactive fuzzy numbers. More precisely, $width(S^k + A) \geqslant width(S^k)$, $\forall A \in \mathbb{R}_{\mathcal{F}}$, which implies that $width(S^{k+1}) \geqslant width(S^k)$ for all iteration k.

From the epidemiological point of view the numerical method based on the standard arithmetic is not a good approach to model the dynamic of the disease, since the uncertainty of the temporal evolution increases.

On the other hand, it seems reasonable to consider that the future number of the susceptible and infected individuals depends on the current or past number of the population. This dependence can be describe by the concept of interactivity. For this approach the numerical solution has to be based on the interactive arithmetic.

In this case, the JPD J_L can not be used, since the equality $S(t) + I(t) = p$ do not hold true for all $t \in [0, \infty)$ (see (7.24)). Hence, only the JPD J_γ is used to provide the solution for this model.

Thus the following numerical solution for (7.20) via Runge-Kutta method is proposed

$$\begin{cases} S^{k+1} = S^k +_0 \dfrac{h}{6}(K_1^S +_0 2K_2^S +_0 2k_3^S +_0 K_4^S) \\ I^{k+1} = I^k +_0 \dfrac{h}{6}(K_1^I +_0 2K_2^I +_0 2k_3^I +_0 K_4^I) \end{cases} \tag{7.26}$$

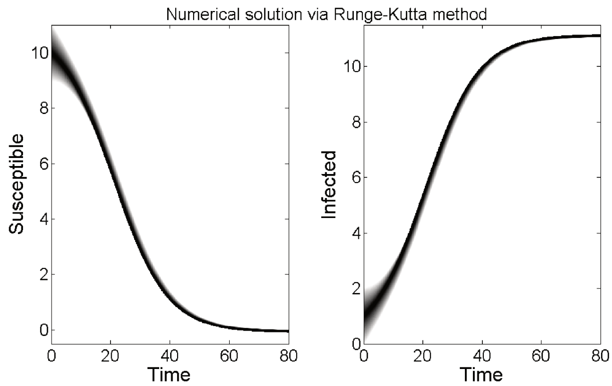
where,

$$
\begin{aligned}
K_1^S &= f_S(t^k, S^k, I^k) = -\beta S^k \cdot_0 I^k +_0 (\eta - \mu) S^k, \\
K_1^I &= f_I(t^k, S^k, I^k) = \beta S^k \cdot_0 I^k -_0 \mu S^k, \\
K_2^S &= f_S\left(t^k + \frac{h}{2}, S^k +_0 K_1^S \frac{h}{2}, I^k +_0 K_1^S \frac{h}{2}\right), \\
K_2^I &= f_I\left(t^k + \frac{h}{2}, S^k +_0 K_1^S \frac{h}{2}, I^k +_0 K_1^S \frac{h}{2}\right), \\
K_3^S &= f_S\left(t^k + \frac{h}{2}, S^k +_0 K_2^S \frac{h}{2}, I^k +_0 K_2^S \frac{h}{2}\right), \\
K_3^I &= f_I\left(t^k + \frac{h}{2}, S^k +_0 K_2^S \frac{h}{2}, I^k +_0 K_2^S \frac{h}{2}\right), \\
K_4^S &= f_S\left(t^k + h, S^k +_0 h K_3^S, S^k +_0 h K_3^S\right), \\
K_4^I &= f_I\left(t^k + h, S^k +_0 h K_3^S, S^k +_0 h K_3^S\right).
\end{aligned}
$$

and the arithmetic operations are obtained from J_0.

The fuzzy solution presented in (7.26) is depicted in Figure 81.

Figure 81 – Graphical representation of the fuzzy solution of the SI model with vital dynamics (7.24) via J_0.



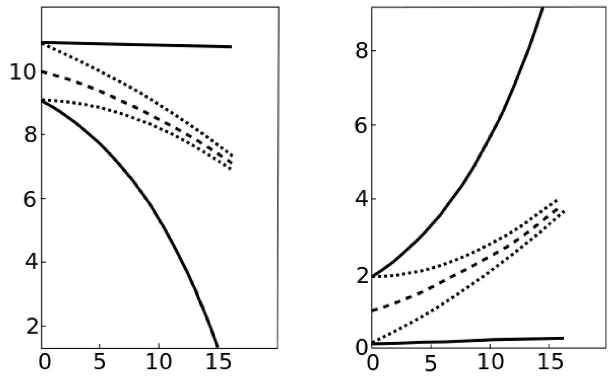
The fuzzy solution given by Runge-Kutta method via interactive arithmetic based on J_0. The left and right subfigures present the susceptible and infected populations, respectively. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [146].

The fuzzy solution produced by (7.26) has non-increasing width over time, as one can observe in Figure 81. This fact is associated with the interactive arithmetic based on J_0, whose operations have the smallest norm and width than any other operation based on joint possibility distributions J_γ [54]. In contrast to the standard arithmetic, one may have $width(S^k +_0 A) < width(S^k)$ where $A \in \mathbb{R}_{\mathcal{F}}$ and $+_0$ is the interactive sum. For example, let be $S^k = (-2; 0; 2)$ and $A = (-1; 0; 1)$ then

$$S^k +_0 A = (-1; 0; 1) \quad \Rightarrow \quad width(S^k +_0 A) = 2 < 4 = width(S^k).$$

As an immediately consequence of a combination of Proposition 5.1 and Theorem 2.2 ensures that the numerical solution given by (7.26) is contained in the numerical solution given by (7.25), as one can observe in Figure 82.

Figure 82 – Comparison of the fuzzy numerical solutions of (7.24) produced by the standard and interactive arithmetic via J_0.



The left and right Figures present the susceptible and infected populations. The dotted and solid lines represent the 0-cut of the fuzzy solutions provided by the Runge-Kutta method based on interactive and standard arithmetic, respectively. The dashed line represents the classical solution of (7.20) considering the initial conditions given by $S^0 = 10$ and $I^0 = 1$. Source: [146].

From the epidemiological point of view the fuzzy solution produced by the interactive arithmetic describes the dynamic of the disease more specifically than the solution given by (7.25). Note that over time the fuzzy solution (7.26) behaves similarly to the classical solution (see Figure 82).

The next subsection presents a numerical solution for the SIS model. In this case different values of γ are used, in order to illustrate the "control" of the width of the numerical fuzzy solutions.

7.2.3 Numerical solution for SIS model

Let us consider now the SIS model. In this case consider a vital dynamic, which is described by the following system

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI + \sigma I + (\eta - \mu)S, & S(0) = S_0 \\ \dfrac{dI}{dt} = \beta SI + (-\mu - \sigma)I, & I(0) = I_0 \end{cases}, \tag{7.27}$$

where η, μ, β, $\sigma \in \mathbb{R}$ are the birth, mortality, infection, and recovery rates, respectively.

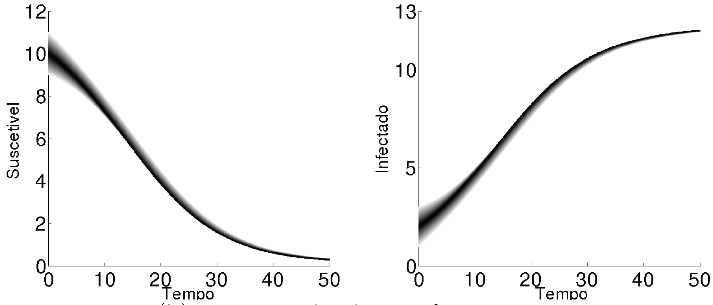
Therefore the fuzzy numerical solution, via Euler's method, is given by

$$\begin{cases} S^{k+1} = S^k +_\gamma h(-\beta S^k *_\gamma I^k +_\gamma \sigma I^k +_\gamma (\eta - \mu)S^k) \\ I^{k+1} = I^k +_\gamma h(\beta S^k *_\gamma I^k +_\gamma (-\mu - \sigma)I^k) \end{cases}. \tag{7.28}$$

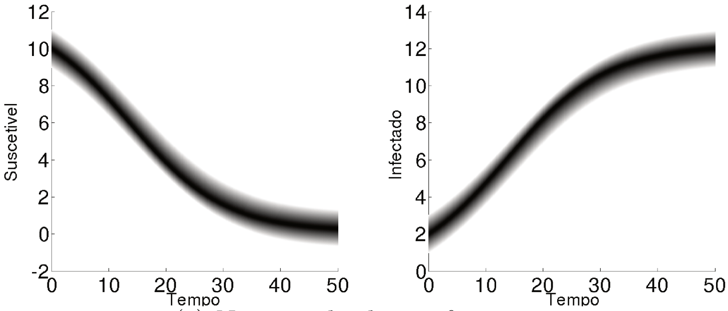
In order to illustrate the "control" of the width of the solution, consider three values of γ, which are $\gamma = 0$, $\gamma = 0.5$, and $\gamma = 1$. Recalling that for $\gamma = 1$ one obtains that the involved fuzzy numbers are non-interactive. Figure 83 depicts the numerical solution for these values of γ, where the initial conditions are given by $S_0 = (9; 10; 11) \in \mathcal{F}_{Tr}$ and $I_0 = (1; 2; 3) \in \mathcal{F}_{Tr}$.

Figure 83 – Graphical representation of the fuzzy solution of the SIS model (7.27).

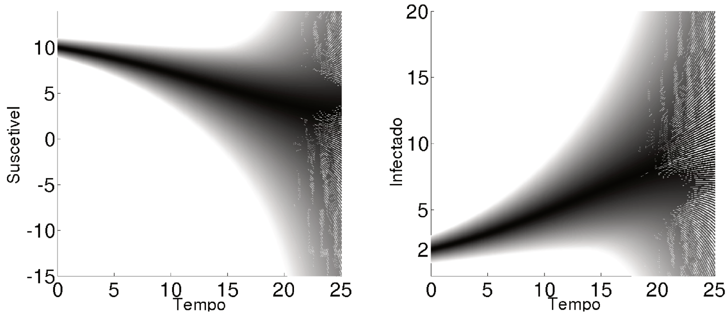
(a) Numerical solution for $\gamma = 0$.



(b) Numerical solution for $\gamma = 0.5$.



(c) Numerical solution for $\gamma = 1$.



The fuzzy solution provided by (7.28). The left and right curves represent the susceptible and infected populations, respectively. The parameters used were $h = 0,125$, $\beta = 0,01$, $\sigma = 7.10^{-4}$, $\eta = 2.10^{-5}$, and $\mu = 1.10^{-5}$. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [145]. Source: [145].

One can observe that in Figures 83 (a) and (b) the susceptible population decreases whereas the infected ones increase. The same behaviour occurs in the classical models. Moreover, the width of the fuzzy numerical solution increases when the values of γ increases, where the smallest and largest width is attached in $\gamma = 0$ and $\gamma = 1$, respectively, corroborating the theoretical results.

7.2.4 Numerical solution for SIR model

Let us provide the last example in the biological field. In this case consider the SIR model with vital dynamic. The SIR model is described by the following system

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI + (\eta - \mu)S, & S(0) = S_0 \\ \dfrac{dI}{dt} = \beta SI - (\mu + \nu)I, & I(0) = I_0 \\ \dfrac{dR}{dt} = \nu I - \mu R, & R(0) = R_0 \end{cases}, \tag{7.29}$$

where η, μ, ν, $\beta \in \mathbb{R}$ are the birth, mortality, recovery, and infection rates, respectively.

The fuzzy numerical solution, via Euler's method, for (7.29) is given by

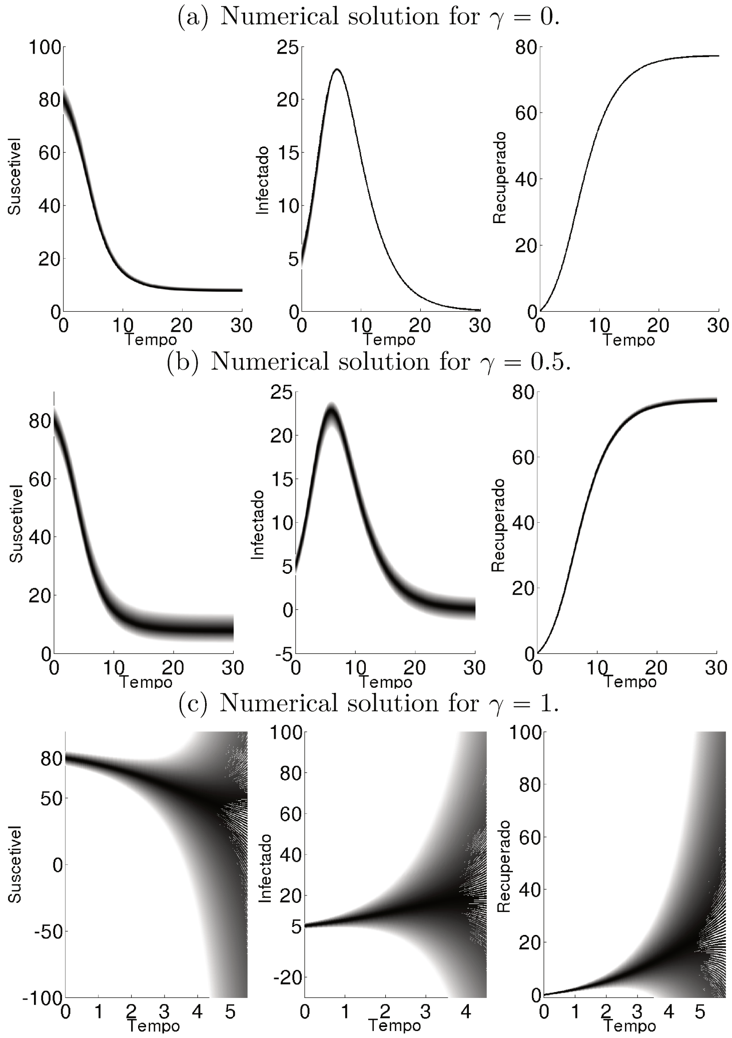
$$\begin{cases} S^{k+1} = S^k +_\gamma h(-\beta S^k *_\gamma I^k +_\gamma (\eta - \mu)S^k) \\ I^{k+1} = I^k +_\gamma h(\beta S^k *_\gamma I^k -_\gamma (\mu + \nu)I^k) \\ R^{k+1} = R^k +_\gamma h(\nu I^k -_\gamma \mu R^k) \end{cases}. \tag{7.30}$$

Figure 84 depicts the numerical solutions considering the values $\gamma = 0$, $\gamma = 0.5$, and $\gamma = 1$. The initial conditions are given by $S_0 = (75; 80; 85)$, $I_0 = (4; 5; 6)$, and $R_0 = (0; 0; 0)$. Recall that, at the beginning of the temporal evolution the recovered population begins with 0 individuals, since there is no infection yet.

One can observe that in Figures 84 (a) and (b) the susceptible population decreases, whereas the infected population increases. The recovery population increases and stabilizes.

It is important to observe that it was considered the case where only the initial conditions are given by interactive fuzzy numbers. In fact, the same methodology can be applied in the case where the parameters are given by interactive fuzzy numbers [144].

Figure 84 – Graphical representation of the fuzzy solution of the SIR model (7.29).



The fuzzy solution provided by (7.28). The left and right curves represent the susceptible and infected populations, respectively. The parameters used were $h = 0,125$, $\nu = \frac{1}{3}$, $\beta = 0,01$, $\eta = 2.10^{-5}$, and $\mu = 1.10^{-5}$. The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [142].

The next subsection presents the last application of the Euler's adapted method in a chemical reactions problem.

7.2.5 Numerical solution for Lotka-Volterra Model of Oscillating Chemical Reactions

Chemical Kinetic deals with chemistry experiments and interprets them in terms of mathematical models. In particular, Chemical Kinetic studies the chemical reactions, as well as the factors that influence the final result [58]. This subsection focuses on chemical reactions of the type

$$U + V \to cW, \tag{7.31}$$

where U and V are the consumed reagents and W is the final product of this reaction, with proportion c.

Some factors may influence the velocity of these reactions, for instance, concentration, activation energy, temperature, pressure, etc. The velocity (v) of a reaction can be determined from $v = k[U]^m[V]^n$, where k is the reaction rate, $[U]$ and $[V]$ are the concentration of the reagents and m and n are the orders of the reactions, which are determined experimentally. Thus, there may be imprecision (or uncertainty) in the process of obtaining such parameters. The classic models do not consider this fact [65]. On the other hand, fuzzy sets theory can be used to describe these uncertainties.

This thesis focuses on Lotka-Volterra model of oscillating chemical reactions, which is based on a molecular mechanism where at each step the reagent molecules combine to produce intermediate reagents or final products. Fundamentally [117]:

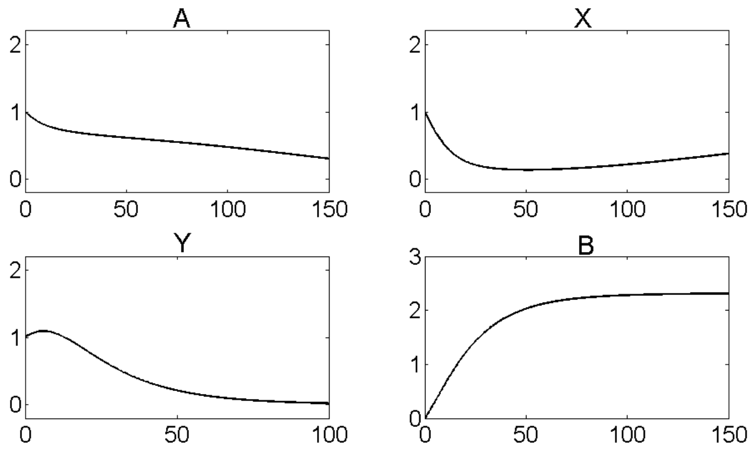
$$\begin{array}{rcll} A + X & \to & 2X & \text{with reaction rate } k_1 \\ X + Y & \to & 2Y & \text{with reaction rate } k_2 \\ Y & \to & B & \text{with reaction rate } k_3 \end{array} \tag{7.32}$$

The effective rate laws for the reagent A, the product B, and the intermediates reagents X and Y are described by the initial value problem (IVP) [73]:

$$\begin{cases} \dfrac{d[A]}{dt} = -k_1[A][X], & [A(0)] = [A_0] \\ \dfrac{d[X]}{dt} = k_1[A][X] - k_2[X][Y], & [X(0)] = [X_0] \\ \dfrac{d[Y]}{dt} = k_2[X][Y] - k_3[Y], & [Y(0)] = [Y_0] \\ \dfrac{d[B]}{dt} = k_3[Y], & [B(0)] = [B_0] \end{cases}. \tag{7.33}$$

The classical solution of (7.33) is depicted in Figure 85.

Figure 85 – Classical solution of Lotka-Volterra model of oscillating chemical reactions given as in (7.33)



The subfigures represent the chemical reagents A, X, Y and B, with initial conditions given by $A(0) = X(0) = Y(0) = 1$ and $B(0) = 0$. Source: [139].

Law of conservation of mass guarantees that a mass is neither created nor destroyed in chemical reactions. This means that the mass of any element at the beginning of a reaction will be equal to the mass of that element at the end of the reaction. Moreover, for all reagents and products in a chemical reaction, one obtains that the total mass will be the same at any point in time in any closed system [134]. Therefore, Equation (7.34) holds true.

$$\frac{d[A]}{dt} + \frac{d[X]}{dt} + \frac{d[Y]}{dt} + \frac{d[B]}{dt} = 0. \tag{7.34}$$

Consequently,

$$[A(t)] + [X(t)] + [Y(t)] + [B(t)] = k, \quad \forall t \in \mathbb{R} \tag{7.35}$$

for some $k \in \mathbb{R}$.

In particular, for initial quantities

$$[A_0] + [X_0] + [Y_0] + [B_0] = k. \tag{7.36}$$

The initial conditions and/or parameters may be uncertain [65]. In case where the initial conditions $[A_0]$, $[X_0]$, $[Y_0]$ and $[B_0]$ are uncertain and modeled by fuzzy numbers, it follows that $[A_0], [X_0], [Y_0]$ and $[B_0]$ need to be interactive [62] in order to guarantee that the total quantity (k), given in (7.36), be a real number [15].

The concept of interactivity is requiried to ensure that Equation (7.36) holds. Moreover, interactivity is used to model the intrinsic dependence of the reagents/products with its concentration [56]. The sum is obtained via sup-J extension principle, with $J = J_\gamma$.

Consequently the sum operation depends on the values of $\gamma \in [0,1]$. Hence Equations (7.35) and (7.36) become

$$[A(t)] +_\gamma [X(t)] +_\gamma [Y(t)] +_\gamma [B(t)] = k, \tag{7.37}$$

and

$$[A_0] +_\gamma [X_0] +_\gamma [Y_0] +_\gamma [B_0] = k. \tag{7.38}$$

Since $[B]$ represents the concentration of the final product B, then $[B_0] = 0$. The combination of (7.37) and (7.38) leads us to the following

$$[B] = [A_0] -_\gamma [A(t)] +_\gamma [X_0] -_\gamma [X(t)] +_\gamma [Y_0] -_\gamma [Y(t)]. \tag{7.39}$$

Therefore it is only necessary to solve the first three equations of (7.33). The numerical solution based on the Euler's adapted method for this problem is given by

$$\begin{cases} [A]^{k+1} = [A]^k -_\gamma hk_1([A]^k \cdot_\gamma [X]^k) \\ [X]^{k+1} = [X]^k +_\gamma h(k_1[A]^k \cdot_\gamma [X]^k -_\gamma k_2[X]^k \cdot_\gamma [Y]^k) \\ [Y]^{k+1} = [Y]^k +_\gamma h((k_2[X]^k \cdot_\gamma [Y]^k) -_\gamma (k_3[Y]^k)) \\ [B]^{k+1} = [A_0] -_\gamma [A]^k +_\gamma [X_0] -_\gamma [X]^k +_\gamma [Y_0] -_\gamma [Y]^k \end{cases}, \tag{7.40}$$

with initial conditions $[A_0], [X_0], [Y_0] \in \mathbb{R}_\mathcal{F}$.

Figures 86, 87 and 88 depict the simulations for three different "levels" of interactivity, that is, $\gamma = 0$, $\gamma = 0.5$ and $\gamma = 0.75$. The parameters used were $h = 0.125$, $k_1 = 0.03$, $k_2 = 0.09$, $k_3 = 0.06$ and $[A_0] = [X_0] = [Y_0] = (0; 1; 2)$.

Note that for different values of γ one obtains different final products. This fact is associated with the interactive arithmetic that is based on the family of the joint possibility distribution J_γ [149].

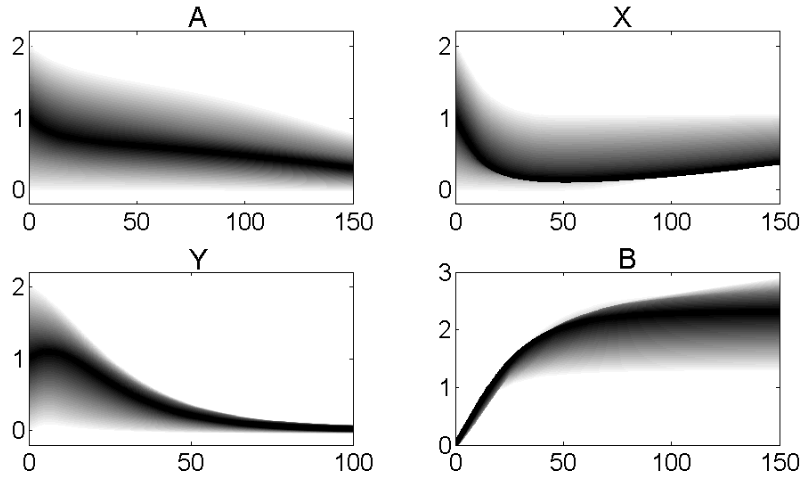
Figure 86 reveals that for the highest level of interactivity ($\gamma = 0$) one obtains decreasing width for the reagents A, X and Y over time. However the width of the final product increases initially and thereafter has few variations.

Figure 87 reveals that for $\gamma = 0.5$ (medium level of interactivity) the width of A, X and Y has few variations. The width of the product also has few variations but always with width smaller than width of the fuzzy solution provided by $\gamma = 0$.

Even though for $\gamma = 0.5$ the reagents have a greater uncertainty than for $\gamma = 0$, the uncertainty in the final product is smaller. Thus, in this sense, the solution via $J_{0.5}$ may describe this final product in a more precisely way.

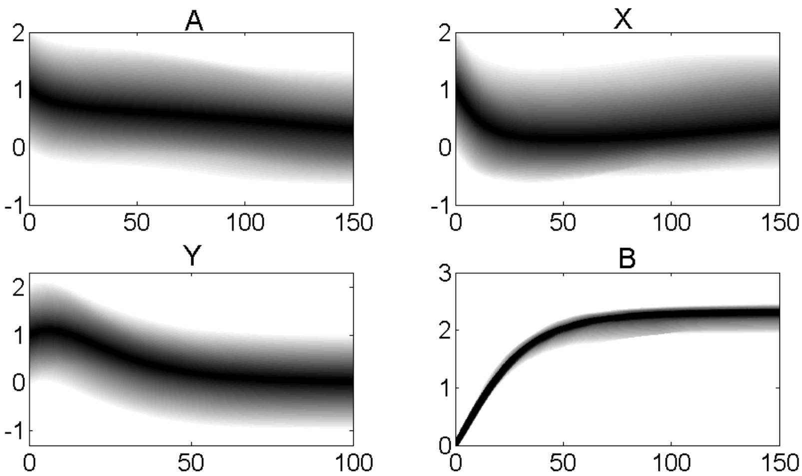
For $\gamma = 0.75$ the uncertainty increases over time as it was expected, since the value of γ is closer to 1 [149]. This fact is corroborated in Figure 88.

Figure 86 – Numerical solution for (7.33) produced by the Euler's adapted method for $\gamma = 0$.



The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [139].

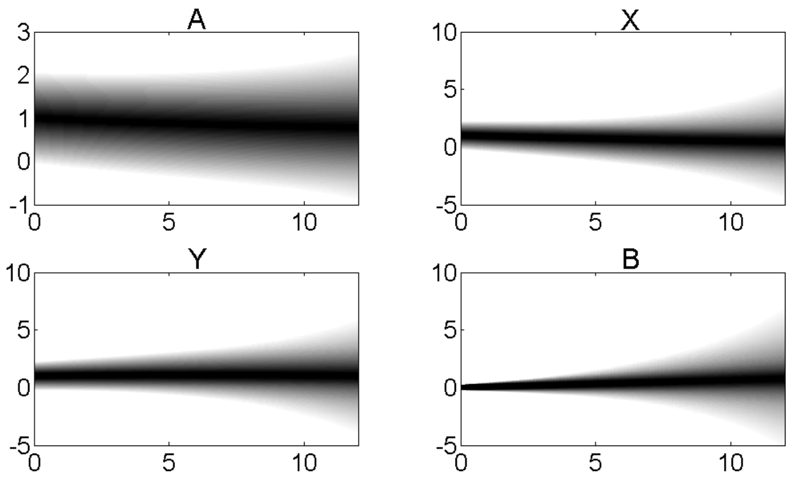
Figure 87 – Numerical solution for (7.33) produced by the Euler's adapted method for $\gamma = 0.5$.



The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [139].

From the chemical point of view, the joint possibility distributions J_0 and $J_{0.5}$ produce solutions which are qualitatively similar to the deterministic case (see Figures 86, 87 and 85). On the other hand, the joint possibility distribution $J_{0.75}$ produces a numerical solution with uncertainty so high that the final result does not resemble (qualitatively) the deterministic case.

Figure 88 – Numerical solution for (7.33) produced by the Euler's adapted method for $\gamma = 0.75$.



The gray lines represent the α-cuts of the fuzzy solutions, where their endpoints for α varying from 0 to 1 are represented respectively from the gray-scale lines varying from white to black. Source: [139].

Hence, in the context of fuzzy sets theory, the relationship of interactivity (as well as the level of interactivity given by γ) influences in the width of the final product. This means that from the chemical point of view, different quantities and/or concentration of the reagents produce products with different uncertainties.

The next section provides the last application of this thesis.

7.3 Fuzzy numerical sequences for Fibonacci and Discret Delay models with initial conditions given by $\mathcal{J}_\gamma$-interactive fuzzy numbers

This section provides examples on Fibonacci and Delay discrete sequences, using the interactive sum and difference obtained from $\mathcal{J}_\gamma$ (see Chapter 6).

7.3.1 Fuzzy Fibonacci Sequence

The Fibonacci sequence is a well-known sequence in Mathematics. This sequence is defined by

$$x_{n+2} = x_{n+1} + x_n, \tag{7.41}$$

where x_0 and x_1 are the initial conditions.

For example, if the initial conditions are given by $x_0 = x_1 = 1$, then the Fibonacci sequence is given by

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}.$$

This sequence has several applications as one can find in [75].

This subsection focuses on the Fibonacci sequence where the initial conditions X_0 and X_1 are $\mathcal{J}_\gamma$-interactive fuzzy numbers, in order to illustrate the properties presented by Chapter 6.

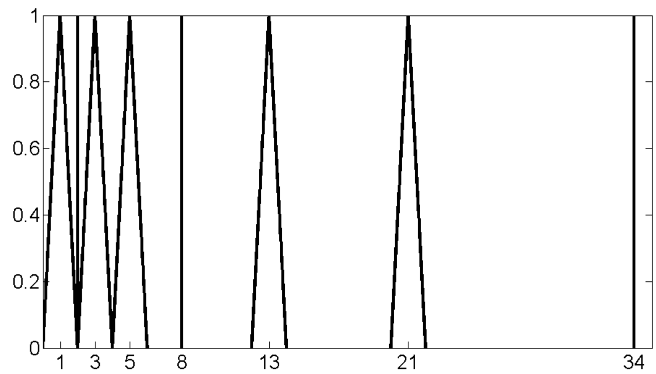
Hence, considering the initial conditions given by $\mathcal{J}_\gamma$-interactive fuzzy numbers, the Fibonacci sequence is given by

$$X_{n+2} = X_{n+1} +_\gamma X_n. \tag{7.42}$$

where X_{n+2} is a fuzzy number for all $n \in \mathbb{N}$.

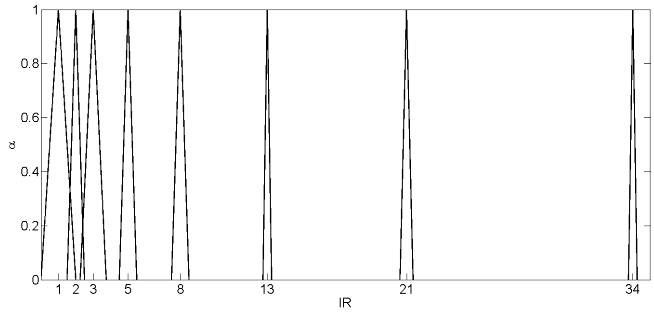
Figures 89, 90, 91 and 92 depict the numerical simulations for the Fibonacci sequence given by (7.42), where the initial conditions are given by $X_1 = X_0 = (0; 1; 2)$.

Figure 89 – Graphical representation of the Fibonacci sequence for $\gamma = 0$.



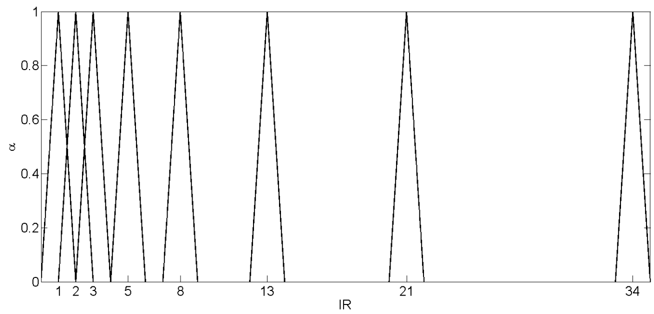
Fibonacci sequence for $\gamma = 0$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$. Source: Author.

Figure 90 – Graphical representation of the Fibonacci sequence for $\gamma = 0.25$.



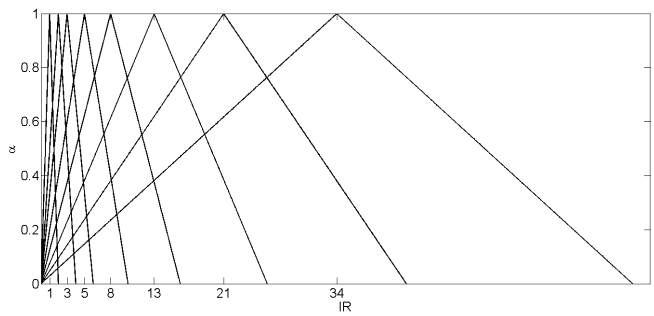
Fibonacci sequence for $\gamma = 0.25$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$. Source: Author.

Figure 91 – Graphical representation of the Fibonacci sequence for $\gamma = 0.5$.



Fibonacci sequence for $\gamma = 0.5$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$. Source: Author.

Figure 92 – Graphical representation of the Fibonacci sequence for $\gamma = 1$.



Fibonacci sequence for $\gamma = 1$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$. Source: Author.

One can observe that, for $\gamma = 1$ the Fibonacci sequence obtained is the same as the one given by the standard sum, that is, $X_{n+2} = X_{n+1} +_{\wedge} X_n$.

Note that as long as the values of γ decreases, one obtains elements X_k with smaller width. In fact, for $\gamma = 0$, it follows that X_2, X_5, X_8... are real numbers. This fact is associated with the symmetry of the fuzzy numbers X_0 and X_1, as it had been prove in Theorem 6.1.

The next subsection presents the fuzzy delay discrete sequence.

7.3.2 Delay Discrete Sequence

The discrete delay sequence is defined by

$$y_{n+2} = y_{n+1} - r y_n, \tag{7.43}$$

where $r \in (0, 1)$ is a constant, y_0 and y_1 are the initial conditions.

For example, if the initial conditions are given by $x_0 = x_1 = 1$ and $r = 0.25$, then the delay discrete sequence is given by

$$\{1, 1, 0.75, 0.5, 0.3125, 0.1875, \ldots\}.$$

This subsection focuses on the delay discrete sequence where the initial conditions Y_0 and Y_1 are $\mathcal{J}_\gamma$-interactive fuzzy numbers, in order to illustrate the properties presented by Chapter 6.

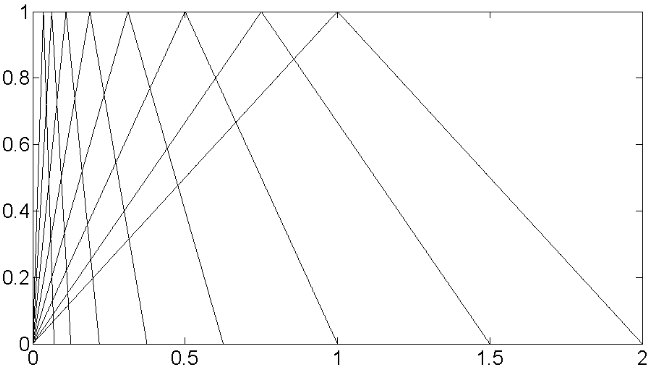
Hence, considering the initial conditions given by $\mathcal{J}_\gamma$-interactive fuzzy numbers, the delay discrete sequence is given by

$$Y_{n+2} = Y_{n+1} -_\gamma r Y_n, \tag{7.44}$$

where Y_{n+2} is a fuzzy number for all $n \in \mathbb{N}$.

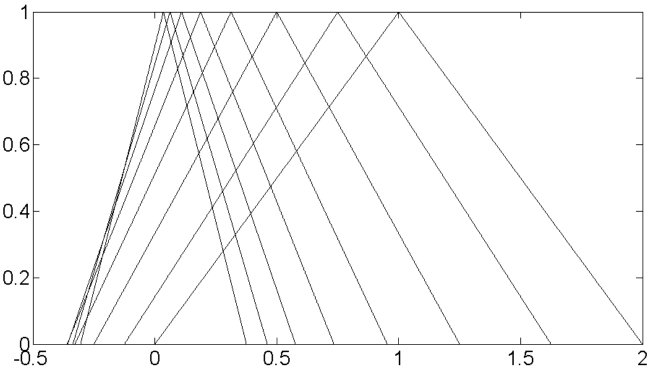
Figures 93, 94, 95 and 96 depict the numerical simulations for the delay discrete sequence given by (7.44), where the initial conditions are given by $X_1 = X_0 = (0; 1; 2)$ and $r = 0.25$.

Figure 93 – Graphical representation of the discrete delay sequence for $\gamma = 0$.



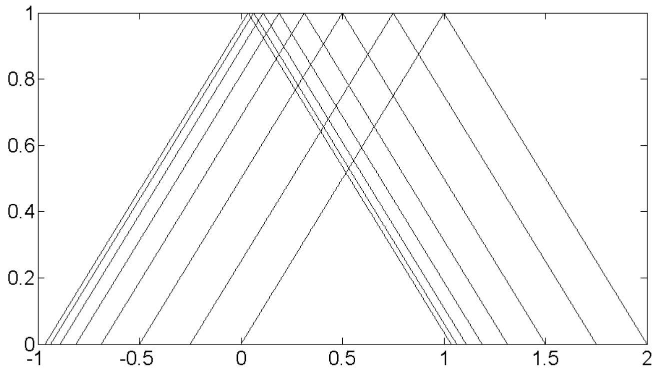
Discrete delay sequence for $\gamma = 0$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$ and $r = 0.25$. Source: Author.

Figure 94 – Graphical representation of the discrete delay sequence for $\gamma = 0.25$.



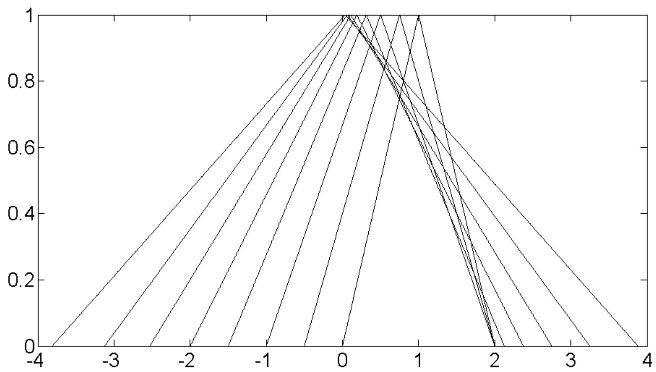
Discrete delay sequence for $\gamma = 0.25$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$ and $r = 0.25$. Source: Author.

Figure 95 – Graphical representation of the discrete delay sequence for $\gamma = 0.5$.



Fibonacci sequence for $\gamma = 0.5$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$ and $r = 0.25$. Source: Author.

Figure 96 – Graphical representation of the discrete delay sequence for $\gamma = 1$.



Discrete delay sequence for $\gamma = 1$, where the initial conditions are given by $X_0 = X_1 = (0; 1; 2)$ and $r = 0.25$. Source: Author.

One can observe that for $\gamma = 1$ the delay sequence obtained is the same as the one given by the standard difference. For $\gamma = 0$ one obtains that the sequence approximates to the fuzzy number with width closer to 0.

In case where the delay coefficient r is given by $r = 1$, it follows that the elements Y_2, Y_5, Y_8... are real numbers, as Theorem 6.2 reveals.

Also note that for this particular sequence the joint possibility distribution $\mathcal{J}_0$ is the only one that produced a sequence where all values in the support of Y_k, with $k \in \mathbb{N}$, are positive.

7.4 Conclusion

This chapter provided several examples where the concept of interactivity can be used. First, the method proposed in Chapter 4 was applied in a physical problem, where the position, velocity and acceleration were linear connected. Second, the family of joint possibility distributions J_γ, defined in Chapter 5, was used in order to provide numerical solutions for epidemiological and chemical models, where the populations/reagents were considered as J_γ-interactive. This chapter ended applying the family of joint possibility distribution $\mathcal{J}_\gamma$, given in Chapter 6, in the Fibonacci and delay sequences, in order to explore the symmetry of these interactive fuzzy numbers.

Conclusion

This thesis presented a study about fuzzy differential equations (FDEs) considering the relationship of interactivity. It was proposed some methods to solve fuzzy initial value problems (FIVPs), where the initial conditions are given by fuzzy numbers with different types of interactivity. To this end several types of interactive arithmetics were provided, which give raise to methods that can be used in the solution of FDEs analytically as well as numerically.

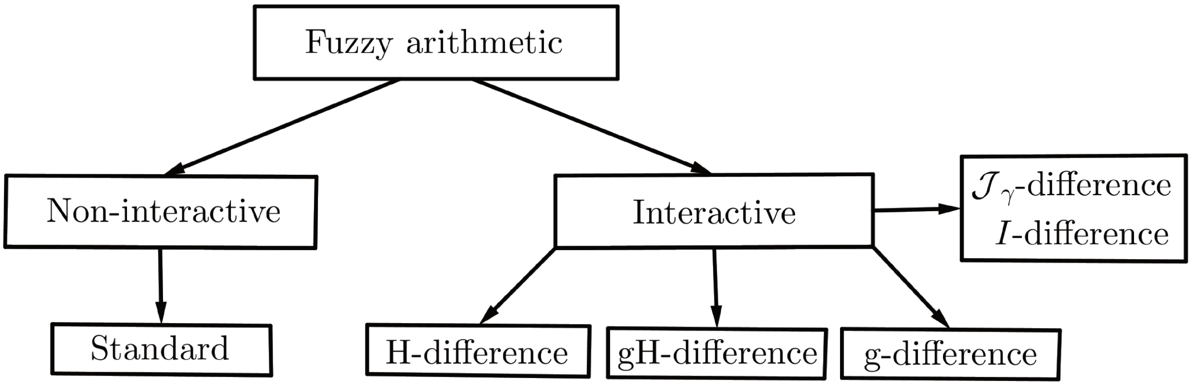
Before presenting the study of FDEs, it was investigated some fundamental questions about fuzzy equations of the form $A \oplus_J B = C$, where the operation $\oplus_J$ is given by an interactive sum or difference. In the case where A and J are free variables, it was showed that there is always a solution for it and this solution is the maximal solution (see Definition (3.1)). However, in the case where only J is the free variable, the equation $A \oplus_J B = C$ not always has solution; in the affirmative case, necessary and sufficient conditions to exist the solution were established.

The study of FDEs began by developing a theory using the linear interactivity, which arises from the joint possibility distribution (JPD) J_L (see Equation (4.7)). By means of J_L analytical solutions of FIVPs were provided, where the initial conditions are given by linearly interactive fuzzy numbers. These solutions were obtained by the sup-J extension principle. It was showed that the solution provided by J_L is associated with Fréchet derivative approach, in the sense that both methods produce the same solution to FIVPs.

The linear interactivity requires that the membership functions of the fuzzy numbers have a linear correlation. Hence, a more general JPD, denoted by J_γ (see Equation (5.6)) was studied. This family can be applied to every pair of fuzzy numbers. From J_γ numerical solutions of FIVPs were provided, where the fuzzy initial conditions are J_γ-interactive. These numerical solutions were obtained by extending the classical arithmetic operations in the Euler's and Runge-Kutta methods. The proposed approach is not computed via α-cuts, in contrast to the methods provided in the literature.

In addition it was proved that the difference via J_γ coincides with the g-difference (see Equation (1.30)). This means that the g-difference is one type of interactive difference, and consequently, embraces the notion of interactivity. Since g-difference generalizes the gH- and H-differences, it was possible to conclude that these differences are interactive as well. Hence the connection between interactive differences and the Hukuhara difference (and its generalizations), investigated in Figure 2, was answered as Figure 97 depicts.

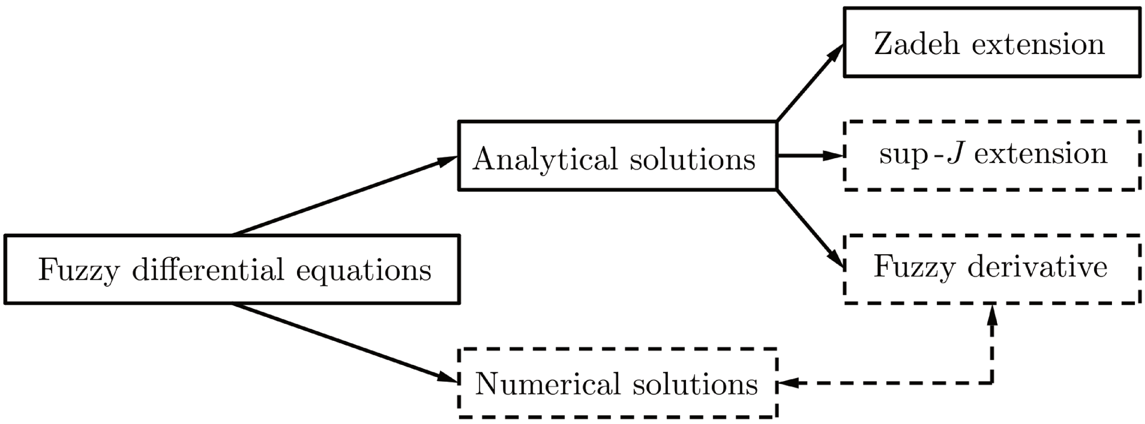
Figure 97 – Diagram of arithmetics on fuzzy numbers and its connections



The solid squares are the arithmetics operations investigated in this thesis. The solid arrows stands for the connections established here. Source: Author

The proposed approach allowed to treat of FDEs numerically and analytically. Moreover, the numerical solution is connected with the analytical solutions, since both of them arose from the JPD J_0. Figure 98 depicts this new association.

Figure 98 – Diagram of approaches for fuzzy differential equations



The solid squares are the most common approaches used in the literature. The dashed squares are the ones studied in this thesis. The dashed arrow establishes the connection provided here. Source: Author

The interactive arithmetic operations, via J_γ, may be difficult to manually compute, demanding a computational effort. In this sense another JPD, denoted by $\mathcal{J}_\gamma$ (see Equation (6.3)), was proposed. This JPD produces arithmetic operations that can be characterized by α-cuts, making the computation of the operations simpler.

The $\mathcal{J}_\gamma$-interactive sum (difference), with $\gamma = 0$, produce fuzzy numbers with smaller Pompeiu-Hausdorff norm and width than any other sum (difference) derived from the sup-J extension. In addition, sufficient conditions for the $\mathcal{J}_\gamma$-interactive sum and difference between fuzzy numbers resulting in real numbers were established.

The proposed approaches were used in applications involving Physics, Biology and Chemistry fields. From J_L, analytical fuzzy solutions of FIVPs were provided. These FIVPs describe problems involving hypocycloid curves and a mass-spring system, where the initial conditions are given by linearly interactive fuzzy numbers, describing the correlation among the position, velocity and acceleration.

From J_γ numerical fuzzy solutions of FIVPs were provided. These FIVPs describe the epidemiological models SI, SIS and SIR. Since the immunological system of each individual reacts in different ways, different values of γ were used in order to study these fuzzy solutions taking into account different interactivities.

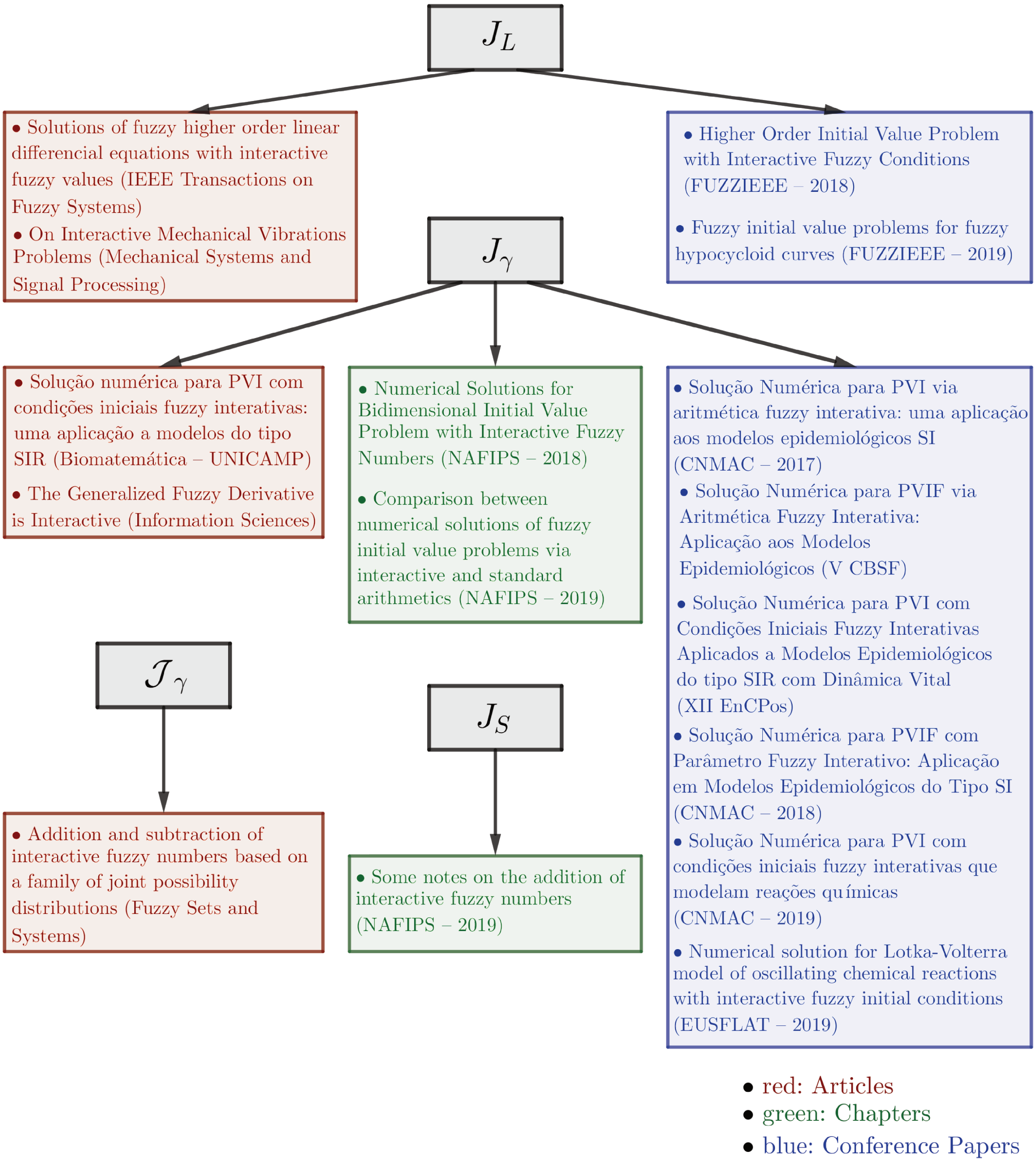
Chemical reactions were studied numerically by providing numerical solutions, via J_γ, to the FIVPs that describe the Lotka-Volterra model of oscillating chemical reactions. The values of γ model different interactions of reagents that influence the final result of the reaction.

Finally the JPD $\mathcal{J}_\gamma$ was used to introduce the Fibonacci and Delay discrete sequences that incorporate the concept of interactivity. The properties of symmetry of fuzzy numbers in these sequences were exhibited.

For further works, we intend to continue the study of FDEs by providing an analysis of the stability of fuzzy solutions. Also we aim to establish the conditions for the existence of the family of $\mathcal{J}$-interactive derivatives. Our other goal is provide a characterization of the family of $\mathcal{I}$-interactive derivatives by means of α-cuts. In addition, we intend to study fundamental properties of arithmetic on interactive fuzzy numbers, such as associativity and distributivity. Finally, we aim to provide a more deeply study of the Fibonacci and Delay sequences, using the notion of interactivity.

This thesis provides a study in the field of FDEs, using different types of interactive arithmetic on fuzzy numbers. Some important and significant connections with the approaches proposed in the literature were established, proving that some of them embrace the concept of interactivity. The importance and richness of interactivity in the context of FDE is showed in the diagram given as in Figure 99, which contains all the contributions from this thesis.

Figure 99 – Contributions of the thesis.



The contributions written in red, green and blue represent the manuscripts submitted/published in form of articles, chapters of books and conference papers, respectively.

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