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Variedades quiver ADHM aumentado

Enhanced ADHM quiver varieties

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Rodrigo Aguiar von Flach

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Orientador: Marcos Benevenuto Jardim

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"Here We Stand"

(The motto of the House Mormont, fictional noble house from Game of Thrones)

Resumo

Este trabalho apresenta os resultados obtidos acerca da investigação das propriedades dos espaços de moduli das representações referenciais estáveis do quiver ADHM aumentado, denominados variedades ADHM aumentadas. Dentre os resultados obtidos, destacamos que estas variedades possuem correspondência biunívoca com o espaço de moduli de bandeira de feixes no plano projetivo. Também foi obtido uma caracterização das variedades ADHM aumentadas suaves. Ademais as variedades ADHM aumentadas são quasi-projetivas e podem ser imersas em uma variedade hyperkähler. Devido a esta imersão, verifica-se que, nos casos em que esta variedade é suave, obtemos que ela é Kähler e possui uma 2-forma, fechada, degenerada herdada pela variedade hyperkähler. Em outras palavras, as variedades ADHM aumentadas podem ser dotadas de uma estrutura definida neste trabalho denominada estrutura holomorfa pré-simplética.

Palavras-chave: representações de quivers. equações ADHM. teoria de moduli.

Abstract

This work presents the results obtained about the investigation of the properties of the moduli spaces of framed stable representations of the enhanced ADHM quiver, called enhanced ADHM quiver varieties, such as the bijection between the enhanced ADHM quiver varieties and the moduli space of flag sheaves in the projective plane that we proved it exists. Also it was proved a characterization of the smooth enhanced ADHM quiver varieties. Moreover, these varieties are quasi-projective and they can be embedded in a hyperkähler manifold. Because of the immersion, it was proved that when the enhanced ADHM quiver variety is smooth, it is a Kähler manifold and has a closed, degenerated 2-form inherited by the hyperkähler manifold. In other words, smooth enhanced ADHM quiver varieties can be endowed with a structure defined in the work as holomorphic pre-symplectic structure.

Keywords: representations of quivers. ADHM equation. moduli theory.

Contents

Introduction

The concept of ADHM quiver was defined by Atiyah, Drinfeld Hitchin and Manin in 1978 in the paper [1]. There they described in terms of linear algebra in a unique way all self-dual connections on euclidean 4-dimensional space (instantons). In order to do this, they used the Ward's correspondence and a few algebro-geometric techniques (monads) introduced by Horrocks. Donaldson restarted in 1984 the ADHM description in terms of maps between complex vector spaces, see [7]. One important result proved by Donaldson in this paper is the fact that the regular solutions of the ADHM equation modulo a specific $\mathcal{G}$-action parametrizes the moduli space of holomorphic bundles on $\mathbb{P}^2$ that are framed at the line at infinity, where $\mathcal{G}$ is the general linear group over a fixed vector space.

In 1999, Nakajima in [13] changed the regularity condition to a weaker stability condition and obtained the moduli space of framed torsion free sheaves on $\mathbb{P}^2$. In particular, Nakajima obtained an algebraic description of the Hilbert schemes of points on $\mathbb{C}^2$ and that it admits a hyperkähler structure.

The concept of enhanced ADHM quiver was defined by Bruzzo, Diaconescu, Jardim, et al. in [4] and it appears as a tool to study the moduli space of supersymmetric flat directions of a quantum mechanical potential. They noticed that this moduli space, constructed using Geometric Invariant Theory techniques, is isomorphic to the moduli space of framed stable representations of the enhanced ADHM quiver. Moreover, they proved that it is a quasi-projective smooth irreducible variety.

In this thesis we define and study a quiver slightly different from the quiver defined in [4] that shares many proprieties with it. Because of this similarity, it is also called enhanced ADHM quiver. Here we proved that the moduli space of framed stable representations of the enhanced ADHM quiver is a quasi-projective variety that can be embedded in a hyperkähler manifold, $\mathcal{W}$, and it has a bijection with the moduli space of flags of sheaves. Moreover, for some special cases, this moduli space is smooth and irreducible. Also we present a study of the complex and holomorphic structures that can be inherited by $\mathcal{W}$. Defining new structures denominated holomorphic pre-symplectic structures.

This thesis is divided in four chapters. In Chapter 1, it is presented briefly the ADHM quiver and the moduli space of stable representations of this quiver. In Chapter 2, we defined the enhanced ADHM quiver and exposed the construction of the moduli space of framed stable representations of this quiver. Also it is proved that for a special case this moduli space is smooth. On Chapter 3 it is proved that this moduli space is a subvariety of a hyperkähler manifold. In this chapter it is also defined the holomorphic pre-symplectic structure and it is checked that this moduli space admits this structure. Finally, in Chapter 4 it is proved that the moduli space of framed stable representations of the enhanced ADHM quiver has a bijection with the moduli space of flags of sheaves.

1 Preliminaries

In this chapter it is presented briefly representations of quivers, ADHM equations and the ADHM varieties and a introduction to Kähler and hyperkähler verieties. Here, the reader can find the main definitions and results relating to subrepresentations of a quiver, morphisms between representations, smoothness of the set of stable solutions of the ADHM equations and a few equivalences of the definition of Kähler and hyperkähler manifolds.

In the first section, we will present the topics related to quivers and representation of quivers, in the second section, we will define the ADHM equations and the ADHM varieties. Finally, in the third and las section we will talk briefly about the complexes structures mentioned above.

1.1 Quivers and representation of quivers

It will be presented in this section the first definitions about quivers and representations of quivers. We will talk about this very briefly, but the reader can find more details about this topic in [12, Chapter 3].

Definition 1.1.1. A *quiver* is a quadruple $Q = (Q_0, Q_1, h, t)$ where Q_0, Q_1 are two finite sets and

$$h, t : Q_1 \longrightarrow Q_0$$

maps.

In general, Q_0 is a set of vertices and Q_1 is a set of arrows. The maps h and t are called *head* and *tail*, respectively. Given an arrow $s \in Q_1$, $t(s)$ and $h(s)$ represent the source and the target of s, respectively. In other words, a quiver can be view as a oriented graph.

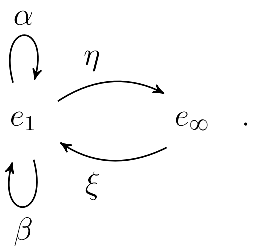
Definition 1.1.2. A *relation* in a quiver is a linear combination of concatenation of arrows that have the same source and target.

Consider the following example.

Example 1.1.3 (ADHM quiver)**.** Let $Q = (Q_0, Q_1, h, t)$ a quiver such that $Q_0 = \{e_1, e_\infty\}$, $Q_1 = \{\alpha, \beta, \xi, \eta\}$ and $h, t : Q_1 \longrightarrow Q_0$ are given by

$$t(s) = \begin{cases} e_1, & \text{se} \quad s \in \{\alpha,\, \beta,\, \eta\} \\ e_\infty, & \text{se} \quad s \in \{\xi\} \end{cases} \qquad h(s) = \begin{cases} e_1, & \text{se} \quad s \in \{\alpha,\, \beta,\, \xi\} \\ e_\infty, & \text{se} \quad s \in \{\eta\} \end{cases} .$$

Then Q is the oriented graph



Furthermore

$$\alpha\beta - \beta\alpha + \xi\eta \tag{1.1}$$

is an example of relation in Q. The quiver $Q = (e_1, e_\infty, h, t)$ above with relation (1.1) is called *ADHM quiver*.

Definition 1.1.4. Let $Q = (Q_0, Q_1, h, t)$ be a quiver. A *representation* $E = (E_i, \varphi_s)$ of Q is a colection of vector spaces over a filed K, $\{E_i : i \in Q_0\}$, and a colection of K-linear maps,

$$\{\varphi_s : E_{t(s)} \longrightarrow E_{h(s)}; s \in Q_1\}.$$

A **morphism** between representations $E = (E_i, \varphi_s)$ and $F = (F_i, \phi_s)$ of the quiver Q, is a set of linear maps

$$f = \{f_i : E_i \longrightarrow F_i : i \in Q_0\}$$

such that the diagram

$$\begin{array}{ccc} E_{t(s)} & \xrightarrow{\varphi_s} & E_{h(s)} \\ \downarrow{\scriptstyle f_{t(s)}} & & \downarrow{\scriptstyle f_{h(s)}} \\ F_{t(s)} & \xrightarrow{\phi_s} & F_{h(s)} \end{array}$$

commutes for all $s \in Q_1$. If these morphisms are isomorphisms, then (E_i, φ_s) and (F_i, φ_s) are *isomorphic* and f is an *isomorphism* between . If $E_i \subset F_i$, for all $i \in Q_0$ and the morphisms f_i are inclusions for all $i \in Q_0$, then (E_i, φ_s) is a *subrepresentation* of (F_i, φ_s).

Example 1.1.5 (Representetions od the ADHM quiver)**.** Let Q be the ADHM quiver, as Example 1.1.3. Let V, W be complex vector spaces. Let $A, B \in Hom(V, V)$, $I \in Hom(W, V)$ and $J \in Hom(V, W)$. Then, $X = (W, V, A, B, I, J)$ is a representation of Q and V, W, A, B, I, J, are identified with e_1, e_∞, α, β, ξ, η, respectively. Let $X' = (V', W', A', B', I', J')$ be another representation of Q. Thus, X and X' are isomorphic, if there existe linear maps

$$f: \ V \ \longrightarrow \ V', \qquad g: \ W \ \longrightarrow \ W'$$

such that

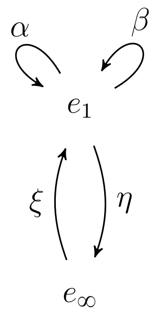
$$fA = A'f, \quad fI = I'g, \quad fB = B'f, \quad gJ = J'f.$$

Let $Q = (Q_0, Q_1, h, t)$ be a quiver. Let n be the the cardinality of Q_0. The *numerical type* or *dimension vector* of a representation $E = (E_i, \varphi_s)$ of Q is given by the vector $(\dim E_1, \ldots, \dim E_n) \in \mathbb{Z}^n$. If $X = (V, W, A, B, I, J)$ is a representation of the ADHM quiver, the numerical type of X is $(r, c) = (\dim W, \dim V)$.

1.2 ADHM equations and ADHM varieties

In this section we will present the stability conditions for representations of the ADHM quiver and talk briefly how the moduli space of the stable representations of the ADHM quiver can be found. Furthermore, we will present a few properties just for further references. More details can be found in [10].

As we saw on Example 1.1.3 the quiver



with the relation

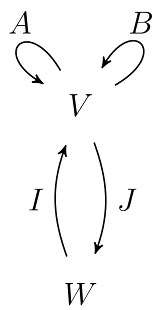
$$\alpha\beta - \beta\alpha + \xi\eta.$$

is called *ADHM quiver*. A representation of this quiver is given by $X = (W, V, A, B, I, J)$, such that W and V are complex vector spaces, A, $B \in End(V)$, $I \in Hom(W, V)$ and $J \in Hom(V, W)$ and X satisfies

$$[A, B] + IJ = 0.$$

This equation is called *ADHM equation*. The vector $(\dim(W), \dim(V))$ is called *dimension vector* or *numerical type* of the representation.

One can express a representation $X = (W, V, A, B, I, J)$ as



Definition 1.2.1. Let $X = (W, V, A, B, I, J)$ be a representation of the ADHM quiver. This representation X is called

(*i*) *stable*, if there is no subspace $0 \subsetneq S \subset V$ such that $A(S)$, $B(S)$, $I(W) \subset S$;

(*ii*) *costable*, if there is no $0 \neq S \subset V$ such that $A(S)$, $B(S) \subset S \subset \ker(J)$;

(*iii*) *regular*, if it is stable and costable.

Let V and W be complex vector spaces of dimension c and r, respectively. We define the *ADHM data* as the complex vector space given by

$$\mathbf{B} := End(V) \oplus End(V) \oplus Hom(W,V) \oplus Hom(V,W).$$

A point of $\mathbf{B}$ is called *ADHM datum*, if $X = (A, B, I, J)$ with

$$A, B \in End(V),\ I \in Hom(W,V) \text{ and } J \in Hom(V,W)$$

where V and W are fixed vector spaces.

Denote by $\mathbf{B}_0 := \{X \in \mathbf{B} :\ X \text{ satisfies the ADHM equation and } X \text{ is stable}\}$. Let $\mathcal{G} = GL(V)$ and consider the following $\mathcal{G}$-action

$$\mathcal{G} \times \mathbf{B} \longrightarrow \mathbf{B} \tag{1.2}$$

$$(h, X) \longmapsto (hAh^{-1}, hBh^{-1}, hI, Jh^{-1}) \tag{1.3}$$

Then, it is well known that the moduli space of the stable representations of the ADHM quiver, $\mathcal{M}(r,c) := \mathbf{B}_0/\mathcal{G}$, is a hyperkähler quotient with complex dimension $2rc$. A proof of this can be found in [2, Section 3.1.1]. Moreover, it was proved in [10] that $\mathcal{M}(r,c)$ is an irreducible quasi-affine variety. Consider the following result just for further references.

Proposition 1.2.2. Let $\mathcal{C}(\mathcal{A})$ be a deformation complex of a stable ADHM datum $X = (A, B, I, J)$, i.e.,

$$\mathcal{C}(\mathcal{A}) : \qquad End(V) \xrightarrow{d_0} \begin{matrix} End(V)^{\oplus 2} \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,W) \end{matrix} \xrightarrow{d_1} End(V)$$

with

$$d_0(a) = ([a, A], [a, B], aI, -Ja);$$
$$d_1(a, b, i, j) = [a, B] + [A, b] + Ij + iJ.$$

Then,

$$H^0(\mathcal{C}(\mathcal{A})) = H^2(\mathcal{C}(\mathcal{A})) = 0.$$

Proof. In order to prove this result, it is enough to check that d_0 is injective and d_1 is surjective. Suppose that

$$d_0(a) = ([a, A], [a, B], aI, -Ja) = 0. \tag{1.4}$$

We must to prove that

$$a = 0.$$

Define $S := \ker(a)$. It follows from equation (1.4) that

$$\begin{aligned} aA &= Aa; \\ aB &= Ba; \\ aI &= 0 \quad \Rightarrow imI \subset S. \end{aligned} \tag{1.5}$$

Let $v \in S$. Then, it follows from equation (1.5) that

$$aAv = Aav = 0 \quad \Rightarrow Av \in S, \text{ for all } v \in S.$$

Therefore, $A(S) \subset S$. Analogously one can prove that $B(S) \subset S$. Thus,

$$A(S), \quad B(S), \quad Im(I) \subset S$$

and it follows from the stability of (A, B, I, J) that $\ker(a) = V$. Therefore $a = 0$. Thus, $\ker(d_0) = 0$ and $H^0(\mathcal{C}(\mathcal{A})) = 0$. The proof of the surjectivity of d_1 can be found on [4, Proposition 2.3.1]. □

1.3 Kähler and hyperkähler structures

This section is a brief introduction to Kähler and hyperkähler structures. Those theories are important to comprehend and work into the subjects presented on the other chapters. The notions defined here will be useful to understand a tool to build hyperkähler manifolds, called hyperkähler reductions or hyperkähler quotients, that will be presented in Chapter 3 of this thesis. The hyperkähler quotient was develop by Hitchin, Karlhede, Lindström and Roček in 1987 on the paper "Hyperkähler Metrics and Supersymmetry", [9]. More details about hyperkähler quotients in portuguese can be find on [2].

Definition 1.3.1. Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and Γ a complex structure defined in the tangent bundle of M. The triple $(M, \langle\ ,\ \rangle, \Gamma)$ is a *Kähler manifold* if satisfies

1. the complex structure Γ is compatible with the Riemannian metric $\langle\ ,\ \rangle$, i.e., if $\langle Iu, Iv\rangle = \langle u, v\rangle$, para todo $u,\ v \in TM$;
2. the non-degenerate 2-form $\omega(\cdot, \cdot) := \langle I\cdot, \cdot\rangle$ is closed, i.e., $dw = 0$.

The 2-form ω above is, in particular, a symplectic form and is called *Kähler form*. Let (M, Γ) be a complex manifold. This manifold is called a *holomorphic symplectic* *manifold*, if it admits a symplectic form. A manifold is called *hypercomplex manifold*, if it admits three complex structures on TM, Γ_1, Γ_2 and Γ_3 that satisfy the relation, i.e,

$$I^2 = J^2 = K^2 = IJK = -1. \tag{1.6}$$

In 1965, Obata proved that there exists a unique torsion-free connection ∇ on a hypercomplex manifold $(M, \Gamma_1, \Gamma_2, \Gamma_3)$ that satisfies

$$\nabla\Gamma_1 = \nabla\Gamma_2 = \nabla\Gamma_3 = 0 \tag{1.7}$$

(see [14]).

Let $(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ be a hypercomplex manifold endowed with a Riemannian metric. This manifold is called a *hyperkähler manifold* if the Riemannian metric $\langle\ ,\ \rangle$ is compatible with the complex structures Γ_n, for $n \in \{1, 2, 3\}$ and if the Levi-Civita connection of $(M, \langle\ ,\ \rangle)$ satisfies (1.7). The next Proposition give to us equivalent definitions of hyperkähler manifolds.

Proposition 1.3.2. Let $(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ be a hypercomplex manifold endowed with a Riemannian metric. Let $\omega_n(\cdot, \cdot) = \langle\Gamma_n\cdot, \cdot\rangle$ for all $n \in \{1, 2, 3\}$. The following sentences are equivalents:

1. the Levi-Civita connection of $(M, \langle\ ,\ \rangle)$ satisfies the equation (1.7);
2. $\nabla\omega_n = 0$, for all $n \in \{1, 2, 3\}$;
3. $d\omega_n = 0$, for all $n \in \{1, 2, 3\}$.

The proof of this proposition can be found in [2, Proposition2.3.1]. It follows from this Proposition that $(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ is a hyperkähler manifold if and only if $(M, \langle\ ,\ \rangle, \Gamma_1)$, $(M, \langle\ ,\ \rangle, \Gamma_2)$ and $(M, \langle\ ,\ \rangle, \Gamma_3)$ are Kähler manifolds.

Let $(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ be a hyperkähler manifold. Let ω_2 e ω_3 be the Kähler forms associated with the Kähler varieties $(M, \langle\ ,\ \rangle, \Gamma_2)$ e $(M, \langle\ ,\ \rangle, \Gamma_3)$, respectivamente. É verdade que ω_J e ω_K satisfazem

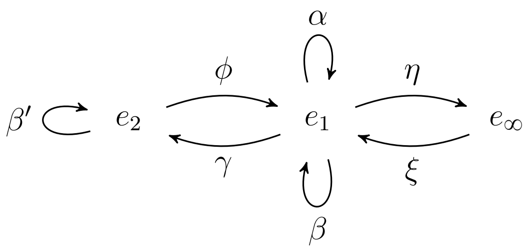
$$\omega_2(u, v) \;:=\; g(Ju, v) = g(IJu, Iv) = g(Ku, Iv) := \omega_3(u, Iv); \tag{1.8}$$
$$\omega_3(u, v) \;:=\; g(Ku, v) = g(IIJu, Iv) = -g(Ju, Iv) := \omega_2(u, Iv). \tag{1.9}$$

Define $\Omega := \omega_J + \sqrt{-1}\omega_K$. It is easy to check that Ω is a symplectic form. Therefore, (M, Γ, Ω) in a holomorphic symplectic manifold. On the other hand, if (M, Γ_1, Ω) is a irreducible compact holomorphic symplectic Kähler manifold, then there exists a unique hyperkähler manifold,$(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ such that $\Omega = \omega_2\sqrt{-1}\omega_3$. Ver [8, Theorem 5.11, p. 26; Theorem 23.5, p. 179].

In the Chapter 3, we will define a new structures called holomorphic pre-symplectic manifold and holomorphic pre-hyperkähler manifold and see that it satisfies one of the sides if a manifold has a holomorphic pre-hyperkähler structure, then we can construct a holomorphic pre-symplectic structure in this manifold. However, the reciprocate is still unknown. We will also proof in Chapter 3 that if we fix a numerical type $(1, 2, 1)$, the moduli space of the framed stable representations of the enhanced ADHM quiver admits a holomorphic pre-symplectic structure. The construction of this moduli space will be presented in the next chapter.

2 Enhanced ADHM varieties

In 2011, Bruzzo, Chuang, Diaconescu, Jardim, Pan and Zhang defined in [4, Chapter 3] the *enhanced ADHM quiver* as the quiver



(2.1)

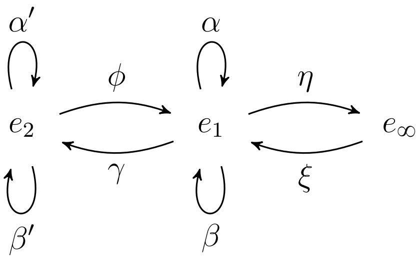
with ideal generated by the relations

$$\begin{matrix} \alpha\beta - \beta\alpha + \xi\eta, & \alpha\phi, & \beta\phi - \phi\beta', & \eta\phi, \\ \gamma\xi, & \phi\gamma, & \gamma\alpha, & \gamma\beta - \beta'\gamma. \end{matrix} \tag{2.2}$$

In [4, Chapter 3], they proved the moduli space of the framed stable representations of the quiver (2.1) is a smooth, quasi-projective variety of dimension $(2c - c')r$, [4, Theorem 3.2]. In this thesis, the main object studied is the moduli space of framed stable representations of another quiver quite similar to the quiver (2.1). The moduli space presented here and the one worked on [4] also share a few properties and some propositions are true for both of them. Because of those similarities the objects defined here will carry the same name as the aforementioned paper. The definition and properties of this quiver and its representations are presented in the following sections of this chapter.

2.1 Enhanced ADHM quiver and its representations

In this section we will present the enhanced ADHM quiver as the following. Consider the quiver



(2.3)

with ideal generated by the the relations

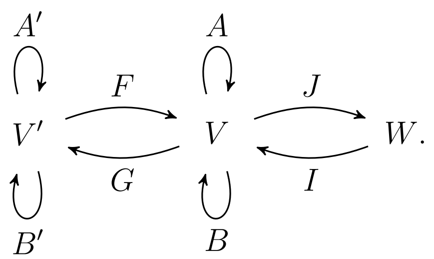
$$\alpha\beta - \beta\alpha + \xi\eta, \tag{2.4}$$
$$\alpha'\beta' - \beta'\alpha', \tag{2.5}$$
$$\alpha\phi - \phi\alpha', \tag{2.6}$$
$$\beta\phi - \phi\beta', \tag{2.7}$$
$$\eta\phi, \tag{2.8}$$
$$\gamma\xi, \tag{2.9}$$
$$\phi\gamma, \tag{2.10}$$
$$\gamma\alpha - \alpha'\gamma, \tag{2.11}$$
$$\gamma\beta - \beta'\gamma. \tag{2.12}$$

Note that the difference between the quiver above and the quiver (2.1) is the arrow α' and the relations (2.5), (2.6) and (2.11) that were adapted to this new arrow. This quiver will also be called *enhanced ADHM quiver*.

A *representation of the enhanced ADHM quiver of type* (r, c, c') in the category of complex vector spaces is given by $X = (W, V, V', A, B, I, J, A', B', F, G)$, where W, V, V' are vector spaces of complex dimension r, c and c', respectively, and A, $B \in End(V)$, $I \in Hom(W, V)$, $J \in Hom(V, W)$, A', $B' \in End(V')$, $F \in Hom(V', V)$ and $G \in Hom(V, V')$ that satisfy the following equations called *enhanced ADHM equations*

$$\begin{aligned} &[A, B] + IJ = 0, \quad [A', B'] = 0, \quad AF - FA' = 0, \quad BF - FB' = 0, \quad JF = 0, \\ &GI = 0, \quad FG = 0, \quad GA - A'G = 0, \quad GB - B'G = 0. \end{aligned} \tag{2.13}$$

A representation $X = (W, V, V', A, B, I, J, A', B', F, G)$ can be illustrated as the diagram below



Let $\varphi : W \longrightarrow \mathbb{C}^r$ be an isomorphism. Then, if X is a representation of the enhanced ADHM, (X, φ) is called a *framed representation* of the enhanced ADHM quiver. Two framed representations (X, φ) and $(\widetilde{X}, \widetilde{\varphi})$ are said to be isomorphic if there exists an isomorphism $(\xi_1, \xi_2, \xi_\infty) : X \longrightarrow \widetilde{X}$ such that $\widetilde{\varphi}\xi_\infty = \varphi$.

$$(\xi_1, \xi_2, \xi_\infty) : X \longrightarrow \widetilde{X},$$

such that $\widetilde{\varphi}\xi_\infty = \varphi$. Since $(\xi_1, \xi_2, \xi_\infty)$ is in particular a morphism between the representations X and $\widetilde{X}$, the following diagrams commute

$$\begin{array}{ccc} V & \xrightarrow{A,\ B} & V \\ \big\downarrow{\scriptstyle \xi_1} & & \big\downarrow{\scriptstyle \xi_1} \\ \widetilde{V} & \xrightarrow{\widetilde{A},\ \widetilde{B}} & \widetilde{V} \end{array}, \quad \begin{array}{ccc} V & \overset{J}{\underset{I}{\rightleftarrows}} & W \\ \big\downarrow{\scriptstyle \xi_1} & & \big\downarrow{\scriptstyle \xi_\infty} \\ \widetilde{V} & \overset{\widetilde{J}}{\underset{\widetilde{I}}{\rightleftarrows}} & \widetilde{W} \end{array}, \quad \begin{array}{ccc} V' & \xrightarrow{A',\ B'} & V' \\ \big\downarrow{\scriptstyle \xi_2} & & \big\downarrow{\scriptstyle \xi_2} \\ \widetilde{V'} & \xrightarrow{\widetilde{A'},\ \widetilde{B'}} & \widetilde{V'} \end{array}, \quad \begin{array}{ccc} V' & \overset{F}{\underset{G}{\rightleftarrows}} & V \\ \big\downarrow{\scriptstyle \xi_2} & & \big\downarrow{\scriptstyle \xi_1} \\ \widetilde{V'} & \overset{\widetilde{F}}{\underset{\widetilde{G}}{\rightleftarrows}} & \widetilde{V} \end{array}.$$

In other words, $(\xi_1, \xi_2, \xi_\infty) : X \longrightarrow \widetilde{X}$ is a morphism if and only if the following equations are satisfied

$$\begin{gathered} \xi_1 A = \widetilde{A}\xi_1, \quad \xi_1 B = \widetilde{B}\xi_1, \quad \xi_1 I = \widetilde{I}\xi_\infty, \quad \xi_\infty J = \widetilde{J}\xi_1, \\ \xi_2 A' = \widetilde{A}'\xi_2, \quad \xi_2 B' = \widetilde{B'}\xi_2, \quad \xi_1 F = \widetilde{F}\xi_2, \quad \xi_2 G = \widetilde{G}\xi_2. \end{gathered} \tag{2.14}$$

2.2 Stability conditions

In order to construct the moduli space, first it is needed to introduce a stability condition. In this sections two stability conditions will be presented and will be proved that these are equivalent in a suitable chamber. The first one, called Θ-stability condition, is inspired in the stability condition presented by King in 1994 in [12]. The Θ-stability condition is a good one because there exists techniques using Geometric Invariant Theory to construct the moduli space of Θ-stable framed representations of quivers, as we prove in the Section 2.3.

While the second stability condition is more resembling with the stability condition usually defined for representations of the ADHM quiver. This resemblance plays an important role to prove that the moduli space stable framed representation of the enhanced ADHM quiver of numerical type (r, c, c'), where $c' \geqslant 2$, is not smooth. Furthermore, this condition is very useful to prove Lemma 2.3.8 and Proposition 4.0.3.

Definition 2.2.1. Let $\Theta = (\theta, \theta', \theta_\infty) \in \mathbb{Q}^3$ a triple satisfying the relation

$$c\theta + c'\theta' + r\theta_\infty = 0. \tag{2.15}$$

A representation X of numerical type (r, c, c') is called $\Theta-$*stable* if X satisfies the following conditions:

(i) Any subrepresentation $0 \neq \widetilde{X} \subset X$ of numerical type $(0, \widetilde{c}, \widetilde{c'})$ satisfies

$$\theta\widetilde{c} + \theta'\widetilde{c'} < 0; \tag{2.16}$$

(ii) Any subrepresentation $0 \neq \widetilde{X} \subset X$ of numerical type $(\widetilde{r}, \widetilde{c}, \widetilde{c'})$ satisfies

$$\theta_\infty\widetilde{r} + \theta\widetilde{c} + \theta'\widetilde{c'} < 0. \tag{2.17}$$

A representation X of numerical type (r, c, c') is called $\Theta-$*semistable* if X satisfies

(iii) Any subrepresentation $0 \neq \widetilde{X} \subset X$ of numerical type $(0, \widetilde{c}, \widetilde{c'})$ satisfies

$$\theta\widetilde{c} + \theta'\widetilde{c'} \leqslant 0; \tag{2.18}$$

(iv) Any subrepresentation $0 \neq \widetilde{X} \subset X$ of numerical type $(\widetilde{r}, \widetilde{c}, \widetilde{c'})$ satisfies

$$\theta_\infty\widetilde{r} + \theta\widetilde{c} + \theta'\widetilde{c'} \leqslant 0. \tag{2.19}$$

Note that this stability condition is slightly different from the notion defined by King, since here only the subrepresentations of numerical type $(0, c, c')$ and (r, c, c'). However, this notion defined in [4] is enough to construct the moduli space of Θ-stable framed representations of the enhanced ADHM quiver, as we will see in Section 2.3. Let (r, c, c') be a fixed dimension vector. Then the space of stability parameters $\Theta = (\theta, \theta', \theta_\infty) \in \mathbb{Q}^3$ satisfying (2.15) can be identified with the $(\theta, \theta')-$plane in $\mathbb{Q}^2$, after solving for θ_∞. If the set of the representations X with numerical type (r, c, c') which is strictly $\Theta-$semistable is nonempty, the parameter Θ is called *critical of type* (r, c, c'). Otherwise, if this set is empty, the parameter Θ is called *generic*. The following lemma establishes the existence of generic stability parameters for any given dimension vector (r, c, c'). Moreover, this lemma is analogous to [4, Lemma 3.1] and a proof is written only for completeness.

Lemma 2.2.2. Fix a triple $(r, c, c') \in (\mathbb{Z}_{>0})^3$. Suppose $\theta' > 0$ and $\theta + c'\theta' < 0$. Let $X = (W, V, V', A, B, I, J, A', B', F, G)$ be a representation of numerical type (r, c, c'). Then the following are equivalent:

(i) X is $\Theta-$stable;

(ii) X is $\Theta-$semistable;

(iii) X satisfies the following conditions:

- $(S.1)$ $F \in Hom(V', V)$ is injective;
- $(S.2)$ The ADHM data $\mathcal{A} = (W, V, A, B, I, J)$ is stable, i.e., there is no proper subspace $0 \subset S \subsetneq V$ preserved by A, B and containing the image of I.

Proof. If X is $\Theta-$stable, then X is clearly $\Theta-$semistable.

Suppose that X is $\Theta-$semistable and F is not injective. Then

$$A'(\ker(F)) \subset \ker(F)$$
$$B'(\ker(F)) \subset \ker(F)$$

In fact, let $v \in \ker(F)$. Then it follows from the enhanced ADHM equations that

$$\begin{aligned} 0 &= \underbrace{(BF - FB')}_{=0} v \\ &= B \underbrace{Fv}_{=0} - FB'v \\ &\Rightarrow F(B'v) = 0, \text{ for all } v \in \ker(F) \\ &\Rightarrow B'(v) \in \ker(F), \text{ for all } v \in \ker(F) \\ &\Rightarrow B'(\ker(F)) \subset \ker(F) \end{aligned}$$

Analogously the same can be proved for the endomorphism A'.

Then, $\widetilde{X} = (0, 0, \ker(F), 0, 0, 0, 0, A'|_{\ker(F)}, B'|_{\ker(F)}, F|_{\ker(F)}, 0)$ is a subrepresentation of X with numerical type $(\widetilde{r}, \widetilde{c}, \widetilde{c'})$ in which $\widetilde{r} = \widetilde{c} = 0$ and $\widetilde{c'} = \dim(\ker(F))$. However,

$$\widetilde{c}\theta + \widetilde{c'}\theta' = \theta' \cdot \dim(\ker(F)) > 0$$

and this contradicts the inequation (2.18).

Now suppose that X is $\Theta-$semistable and the condition $(S.2)$ is false. Then, there is a proper subspace $0 \subset S \subsetneq V$ such that

$$A(S), B(S), Im(I) \subseteq S.$$

Therefore, $\widetilde{X} = (W, S, V', A|_S, B|_S, I, J|_S, 0, 0, 0, 0)$ is a subrepresentation with numerical type $(r, \dim(S), c')$. However, it follows from the equation (2.15) and from conditions $\theta' > 0$, $\theta + c'\theta' < 0$ that $\dim(S)\theta + c'\theta' + \underbrace{\theta_\infty r}_{=-c\theta - c'\theta'} > 0$. Indeed,

$$\theta' > 0,\ \theta + c'\theta' < 0 \Rightarrow \theta < 0,$$

moreover,

$$\dim(S)\theta + c'\theta' + \underbrace{\theta_\infty r}_{=-c\theta - c'\theta'} = \underbrace{(\dim(S) - c)}_{<0}\theta > 0 \tag{2.20}$$

and this contradicts the inequality (2.19). Thus, if X is $\Theta-$semistable, then X satisfies the conditions $(S.1)$ and $(S.2)$.

Now suppose that X satisfies the conditions $(S.1)$ and $(S.2)$, thus X is $\Theta-$stable. Indeed, let $\widetilde{X} = (\widetilde{W}, \widetilde{V}, \widetilde{V'}, \widetilde{A}, \widetilde{B}, \widetilde{I}, \widetilde{J}, \widetilde{A'}, \widetilde{B'}, \widetilde{F}, \widetilde{G})$ be a subrepresentation of X with numerical type $(\widetilde{r}, \widetilde{c}, \widetilde{c'})$. There are two cases to study: $\widetilde{r} = r$ and $\widetilde{r} = 0$.

First suppose $\widetilde{r} = r$. Since $\widetilde{W}$ is a subspace of W, by definition, and $\widetilde{r} = r$, $\widetilde{W} = W$. It follows from condition $(S.2)$ that $I \neq 0$. Indeed, otherwise, $0 \subset V$ would satisfy

$A(0), B(0), Im(0) = 0$ and then $\mathcal{A} = (W, V, A, B, I, J)$ is not stable. Since $\widetilde{X}$ is a subrepresentation of X, the diagram below commutes:

$$\begin{array}{ccc} W & \xrightarrow{I} & V \\ {\scriptstyle 1_W}\uparrow & & \uparrow{\scriptstyle \iota} \\ W & \xrightarrow{\widetilde{I}} & \widetilde{V} \end{array}$$

i.e.,

$$\iota \circ \widetilde{I} = I \circ 1_W. \tag{2.21}$$

Thus, if $\widetilde{c} = 0$, $I \equiv 0$, which is a contradiction. Therefore $\widetilde{c} > 0$.

If $\widetilde{c} < c$, then $0 \subset \widetilde{V} \subsetneq V$ is a proper subspace such that

$$A(\widetilde{V}), B(\widetilde{V}), Im(I) \subset \widetilde{V}.$$

Indeed, since $\widetilde{X}$ is a subrepresentation of X, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{A} & V \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\ \widetilde{V} & \xrightarrow{\widetilde{A}} & \widetilde{V} \end{array}$$

thus,

$$\iota \circ \widetilde{A} = A \circ \iota \Rightarrow A|_{\widetilde{V}} \subset \widetilde{V}.$$

Analogously, B preserves $\widetilde{V}$. Moreover, it follows from (2.21) that $Im(I) \subset \widetilde{V}$, and this contradicts the condition $(S.2)$. Therefore, $\widetilde{c} = c$. Since $\widetilde{X}$ is a proper subrepresentation, $\widetilde{c'} < c'$ and then

$$\begin{aligned} \theta \underbrace{\widetilde{c}}_{=c} + \theta'\widetilde{c'} + \theta_\infty \underbrace{\widetilde{r}}_{=r} &= \theta c + \theta'\widetilde{c'} + \theta_\infty r - \underbrace{(\theta c + \theta' c' + \theta_\infty r)}_{=0} \\ &= \theta'(\widetilde{c'} - c') < 0 \end{aligned}$$

Now suppose $\widetilde{r} = 0$. If $\widetilde{c} = 0$, since F is injective, $\widetilde{V'} \subset \ker(F) = 0$. Thus $\widetilde{V'} = 0$. However, only nontrivial subrepresentations are being considered. Then $\widetilde{c} > 0$. If $\widetilde{c} < c$, again $0 \subset \widetilde{V} \subsetneq V$ contradicts the condition $(S.2)$ and $\widetilde{c} = c$. Since $\widetilde{X}$ is a proper subrepresentation, $\widetilde{c'} < c$. Thus

$$\widetilde{c}\theta + \widetilde{c'}\theta' \leqslant \theta + c'\theta' < 0$$

Therefore, X is $\Theta-$stable. □

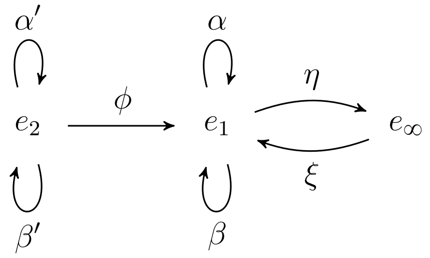
The following corollary is trivial.

Corollary 2.2.3. Let $X = (A, B, I, J, A', B', F, G)$ be a stable representation of the enhanced ADHM quiver. Then $G \equiv 0$.

Proof. Note that F is injective, see condition $(S.1)$, and $FG = 0$, see enhanced ADHM equations (2.13). Then $G = 0$. $\square$

Due to the Lemma 2.2.2, from now on a representation X of the enhanced ADHM quiver will be called *stable* if X satisfies $(S.1)$ and $(S.2)$. Due to Corollary 2.2.3, framed stable representations are quite simpler and easier to manipulate.

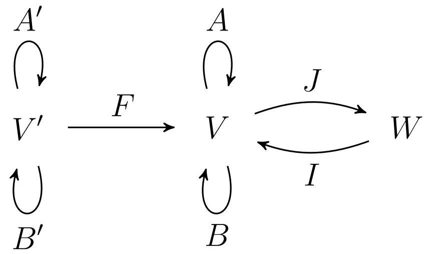
In this work almost always the representations of the enhanced ADHM quiver considered are stable. Thus, sometimes it will be considered the following quiver as the *enhanced ADHM quiver*



with ideal generated by relations

$$\alpha\beta - \beta\alpha + \xi\eta, \quad \alpha\phi - \phi\alpha', \quad \beta\phi - \phi\beta', \quad \eta\phi, \quad \alpha'\beta' - \beta'\alpha'.$$

Then a representation of the quiver above is given by $X = (A, B, I, J, A', B', F)$ such that $A, B \in End(V)$, $I \in Hom(W, V)$, $J \in Hom(V, W)$, $A', B' \in End(V')$ and $F \in Hom(V', V)$, see the diagram below,



satisfying the equations

$$[A, B] + IJ = 0, \quad AF - FA' = 0, \quad BF - FB' = 0, \quad JF = 0, \quad [A', B'] = 0, \tag{2.22}$$

which sometimes it will be also called *enhanced ADHM equations* in this work.

2.3 The Moduli Space

In this section the construction of the moduli spaces of framed Θ−semistable representation of the enhanced ADHM quiver is presented. In order to do this, Geometric Invariant Theory techniques are used by analogy of [12] and [4, Section 3.2]. This construction is presented in details just for completeness.

Let V, V', W be complex vector spaces such that $\dim V = c$, $\dim V' = c'$ and $\dim W = r$. We define the space of *enhanced ADHM data*, denoted by $\mathbb{X}$ or $\mathbb{X}(r, c, c')$, as the following complex vector space

$$\mathbb{X} = End(V)^{\oplus^2} \oplus Hom(W, V) \oplus Hom(V, W) \oplus End(V')^{\oplus^2} \oplus Hom(V', V) \oplus Hom(V, V').$$

A vector $X \in \mathbb{X}$, $X = (A, B, I, J, A', B', F, G)$, is called *enhanced ADHM datum*. Let

$$\mathcal{G} = GL(V) \times GL(V').$$

Consider the map

$$\begin{array}{rccl} \Psi : & \mathcal{G} \times \mathbb{X} & \longrightarrow & \mathbb{X} \\ & (h, h', X) & \longmapsto & (hAh^{-1}, hBh^{-1}, hI, Jh^{-1}, h'A'h'^{-1}, h'B'h'^{-1}, hFh'^{-1}, h'Gh^{-1}) \end{array} . \tag{2.23}$$

This map defines a $\mathcal{G}-$action on $\mathbb{X}$. Indeed, let (h_1, h'_1), $(h_2, h'_2) \in \mathcal{G}$ and $X = (A, B, I, J, A', B', F, G) \in \mathbb{X}$. Denote

$$(h, h') \cdot X := \Psi(h, h', X).$$

Then

$$\begin{aligned} (h_1, h'_1) \cdot ((h_2, h'_2) \cdot X) &= (h_1, h'_1) \cdot (h_2Ah_2^{-1}, h_2Bh_2^{-1}, h_2I, Jh_2^{-1}, h'_2A'h_2'^{-1}, h'_2B'h_2'^{-1}, \\ &= h_2Fh_2'^{-1}, h'_2Gh_2^{-1}) \\ &= (h_1h_2Ah_2^{-1}h_1^{-1}, h_1h_2Bh_2^{-1}h_1^{-1}, h_1h_2I, Jh_2^{-1}h_1^{-1}, h'_1h'_2A'h_2'^{-1}h_1'^{-1}, \\ &\quad h_1'^{-1}h'_2B'h_2'^{-1}h_1'^{-'}, h_1h_2Fh_2'^{-1}h_1'^{-1}) \\ &= (h_1h_2A(h_1h_2)^{-1}, h_1h_2B(h_1h_2)^{-1}, h_1h_2I, J(h_1h_2)^{-1}, \\ &\quad h'_1h'_2A(h'_1h'_2)^{-1}, h_1'^{-1}h'_2B'(h'_1h'_2)^{-1}, h_1h_2F(h'_1h'_2)^{-1}) \\ &= (h_1h_2, h'_1h_2, h'_2) \cdot X. \end{aligned}$$

Moreover, let 1_V and $1_{V'}$ be the identities of $GL(V)$ and $GL(V')$, respectively. Then

$$\begin{aligned} (1_V, 1_{V'}) \cdot X &= (1_VA1_V^{-1}, 1_VB1_V^{-1}, 1_VI, J1_V^{-1}, 1_{V'}A'1_{V'}, 1_{V'}B'1_{V'}, 1_VF1_{V'}, 1_{V'}G1_V) \\ &= X, \end{aligned}$$

i.e., the map (2.23) is a $\mathcal{G}-$action on $\mathbb{X}$.

Proposition 2.3.1. The $\mathcal{G}-$action (2.23) is free on stable points of the enhanced ADHM data $\mathbb{X}$.

Proof. Suppose that there exists $(h, h') \in \mathcal{G}$ such that

$$(h, h') \cdot X = X, \text{ for all } X \in \mathbb{X}.$$

Then, the following equations are satisfied

$$hAh^{-1} = A, \quad hA = Ah;$$
$$hBh^{-1} = B, \quad hB = Bh; \tag{2.24}$$
$$hI = I, \quad (h - 1_V)I = 0; \tag{2.25}$$
$$hA'h^{-1} = A', \quad hA' = A'h;$$
$$hB'h^{-1} = B', \quad hB' = B'h;$$
$$hFh'^{-1} = F \tag{2.26}$$

Let $S := \ker(h - 1_V)$. It follows from the equation (2.25) that $Im(I) \subset S$. We claim that $A(S)$, $B(S) \subset S$. Indeed, let $v \in S$. Then,

$$(h - 1_V)v = 0 \quad \Rightarrow \quad hv = v. \tag{2.27}$$

Therefore, it follows from the equations (2.24) e (2.27) that

$$\begin{aligned} (h - 1_V)Bv &= hBv - Bv \\ &= Bhv - Bv \\ &= Bv - Bv \\ &= 0. \end{aligned}$$

Hence $Bv \in S$ for all $v \in S$, i.e., $B(S) \subset S$. Analogously, one can prove that $A(S) \subset S$. Thus,

$$A(S), \quad B(S), \quad Im(I) \subset S.$$

It follows from the stability condition of the ADHM datum (A, B, I, J) that $S = V$. Then,

$$h = 1_V. \tag{2.28}$$

We claim that $h' = 1_{V'}$. Indeed, it follows from the equations (2.26) and (2.28) that $F(1_{V'} - h'^{-1}) = 0$. Since F is injective, it follows that $(1_{V'} - h'^{-1}) = 0$. Therefore, $h' = 1_{V'}$ and $(h, h') = (1_V, 1_{V'})$, which concludes the proof. □

The *stabilizer* of a given point $X \in \mathbb{X}$ is denoted by $\mathcal{G}_X \subset \mathcal{G}$. It is easy to check the following.

Lemma 2.3.2. Let $\mathbb{X}_0 = \mathbb{X}_0(r, c, c') \subset \mathbb{X}(r, c, c')$ be the subscheme defined by the equations (2.13). Then $\mathbb{X}_0$ is preserved by the $\mathcal{G}$-action (2.23).

Proof. If $X = (A, B, I, J, A', B', F, G) \in \mathbb{X}_0$ and $(h, h') \in \mathcal{G}$, then

$$(h, h') \cdot X = (hAh^{-1}, hBh^{-1}, hI, Jh^{-1}, h'A'h'^{-1}, h'B'h'^{-1}, hFh'^{-1}, h'Gh^{-1}).$$

Furthermore, it follows from the equations (2.13) that

$$\begin{aligned}[hAh^{-1}, hBh^{-1}] + hIJh^{-1} &= hAh^{-1}hBh^{-1} - hBh^{-1}hAh^{-1} + hIJh^{-1} \\ &= h(\underbrace{AB - BA + IJ}_{=0})h^{-1} \\ &= 0, \end{aligned} \tag{2.29}$$

$$\begin{aligned}[h'A'h'^{-1}, h'B'h'^{-1}] &= h'A'h'^{-1}h'B'h'^{-1} - h'B'h'^{-1}h'A'h'^{-1} \\ &= h' \underbrace{(AB - BA)}_{=0} h'^{-1} \\ &= 0, \end{aligned} \tag{2.30}$$

$$\begin{aligned} hAh^{-1}hFh'^{-1} - hFh'^{-1}hA'h'^{-1} &= h \underbrace{(AF - FA')}_{=0} h^{-1} \\ &= 0, \end{aligned} \tag{2.31}$$

$$\begin{aligned} Jh^{-1}hFh'^{-1} &= \underbrace{JF}_{=0} h'^{-1} \\ &= 0, \end{aligned} \tag{2.32}$$

$$\begin{aligned} h'Gh^{-1}hI &= h' \underbrace{GI}_{=0} \\ &= 0, \end{aligned} \tag{2.33}$$

$$\begin{aligned} hFh'^{-1}h'Gh^{-1} &= h \underbrace{FB}_{=0} h^{-1} \\ &= 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} h'Gh^{-1}hAh^{-1} - h'A'h'^{-1}h'Gh^{-1} &= h' \underbrace{(GA - A'G)}_{=0} h^{-1} \\ &= 0. \end{aligned} \tag{2.35}$$

Analogously to (2.31) and (2.35), one can obtain

$$hBh^{-1}hFh'^{-1} - hFh'^{-1}hB'h'^{-1} = h'Gh^{-1}hBh^{-1} - h'B'h'^{-1}h'Gh^{-1} = 0.$$

Therefore, the $\mathcal{G}$-action mentioned preserves the equations (2.13). □

Remark 2.3.3. Each representation $X = (W, V, V', A, B, I, J, A', B', F, G)$ corresponds to a datum vector $X \in \mathbb{X}_0$. Moreover, two framed representations are isomorphic if and only if the corresponding points in $\mathbb{X}_0$ are in the same orbit.

Indeed, let (X, φ) and $(\widetilde{X}, \widetilde{\varphi})$ be framed representations such that $X = (W, V, V', A, B, I, J, A', B', F, G)$ and $\widetilde{X} = (W, V, V', \widetilde{A}, \widetilde{B}, \widetilde{I}, \widetilde{J}, \widetilde{A'}, \widetilde{B'}, \widetilde{F}, \widetilde{G})$. Suppose that (X, φ) and $(\widetilde{X}, \widetilde{\varphi})$ are isomorphic. Then, the corresponding points in $\mathbb{X}_0$ are in the same $\mathcal{G}-$orbit. In fact, there is an isomorphism $(\xi_1, \xi_2, \xi_\infty)$ such that the equations (2.14) are satisfied and $\widetilde{\varphi}\xi_\infty = \varphi$. However, one can choose a basis for W and for W' such that, $\xi_\infty = 1_W$. Then, the equations (2.14) reduce to

$$\begin{gathered} \xi_1 A = \widetilde{A}\xi_1, \quad \xi_1 B = \widetilde{B}\xi_1, \quad \xi_1 I = \widetilde{I}1_W, \quad 1_W J = \widetilde{J}\xi_1, \\ \xi_2 A' = \widetilde{A}'\xi_2, \quad \xi_2 B = \widetilde{B'}\xi_2, \quad \xi_1 F = \widetilde{F}\xi_2, \quad \xi_2 G = \widetilde{G}\xi_2. \end{gathered}$$

This means that $X = (\xi_1, \xi_2) \cdot \widetilde{X}$ where $(\xi_1, \xi_2) \in \mathcal{G}$. On the other hand, if they are in the same $\mathcal{G}-$orbit, there is $(h, h') \in \mathcal{G}$ such that $X = (h, h') \cdot \widetilde{X}$. In other words,

$$\begin{gathered} hAh^{-1} = \widetilde{A}, \quad hBh^{-1} = \widetilde{B}, \quad hI = \widetilde{I}, \quad Jh^{-1} = \widetilde{J}, \\ h'A'h'^{-1} = \widetilde{A}', \quad h'B'h'^{-1} = \widetilde{B'}, \quad hFh'^{-1} = \widetilde{F}, \quad h'Gh^{-1} = \widetilde{G}, \end{gathered}$$

i.e.,

$$\begin{gathered} hA = \widetilde{A}h, \quad hB = \widetilde{B}h, \quad hI = \widetilde{I}1_W, \quad 1_W J = \widetilde{J}h, \\ h'A' = \widetilde{A}'h', \quad h'B = \widetilde{B'}h', \quad hF = \widetilde{F}h', \quad h'G = \widetilde{G}h \end{gathered}$$

and $(h, h', 1_W) : X \longrightarrow \widetilde{X}$ is an isomorphism. Moreover, one can take a basis such that, $\varphi 1_W = \widetilde{\varphi}$ and (X, φ) and $(\widetilde{X}, \widetilde{\varphi})$ are isomorphic.

Now a recall of a few results about Geometric Invariant Theory for representations of quivers. More details can be found in [12]. First, the notion of χ-(semi)stability for a given character $\chi : \mathcal{G} \longrightarrow \mathbb{C}^*$ will be defined. Then it will be proved that the notion of Θ-(semi)stability is equivalent to the notion of χ-(semi)stability for a specific character that will be defined below.

Definition 2.3.4. Let $\mathcal{G}$ be a reductive algebraic group acting on a vector space $\mathbb{X}$. Given an algebraic character

$$\chi : \mathcal{G} \longrightarrow \mathbb{C}^*,$$

a point $X_0 \in \mathbb{X}$ is called:

(i) χ-*semistable* , if there exists a polynomial function $p(X)$ on $\mathbb{X}(r, c, c')$ satisfying:

$$p((h, h') \cdot X_0) = \chi(h, h')^l p(X_0), \tag{2.36}$$

for some $l \in \mathbb{Z}_{\geqslant 1}$, such that $p(X_0) \neq 0$;

(ii) χ-*stable*, if there exists a polynomial function $p(X)$ on $\mathbb{X}(r, c, c')$ satisfying (2.36) for some $l \in \mathbb{Z}_{\geqslant 1}$, such that $p(X_0) \neq 0$ and such that

$$\dim(\mathcal{G} \cdot X_0) = \dim(\mathcal{G}/\Delta),$$

where $\Delta \subset \mathcal{G}$ is the subgroup which acts trivially on $\mathbb{X}$, and the action of $\mathcal{G}$ on $\{x \in \mathbb{X} : p(x) \neq 0\}$ is closed.

The next Lemma gives to us an equivalent definition of χ-(semi)stability.

Lemma 2.3.5. Let $\mathcal{G}$ act on the direct product $\mathbb{X}_0(r, c, c') \times \mathbb{C}$ by

$$(h, h') \times (X, z) \longmapsto ((h, h') \cdot X, \chi(h, h')^{-1} z).$$

A point $X \in \mathbb{X}$ is

(i) χ-semistable if and only if the closure of the orbit $\mathcal{G} \cdot (X, z)$ is disjoint from the zero section $\mathbb{X}(r, c, c') \times \{0\}$, for all $z \neq 0$;

(ii) χ-stable if and only if the orbit is closed in the complement of the zero section, and the stabilizer $G_{(X,z)}$ is a finite index subgroup of Δ.

The proof of this Lemma can be found in [12, Lemma 2.2]. One can form the quasi-projective scheme:

$$\mathcal{N}_{\chi}^{ss}(r, c, c') = \mathbb{X}_0(r, c, c') /\!/_{\chi} \mathcal{G} := \mathrm{Proj}(\oplus_{n \geqslant 0} A(\mathbb{X}_0(r, c, c'))^{\mathcal{G}, \chi^n}), \tag{2.37}$$

in which

$$A(\mathbb{X}_0(r, c, c'))^{\mathcal{G}, \chi^n} := \{f \in A(\mathbb{X}_0(r, c, c')); f((h, h') \cdot X) = \chi(h, h')^n f(X), \text{ for all } (h, h') \in \mathcal{G}\}.$$

Remark 2.3.6. It is well known that $\mathcal{N}_{\chi}^{ss}(r, c, c')$ is projective over $\mathrm{Spec}(\mathbb{X}_0(r, c, c')^{\mathcal{G}})$, and it is quasi-projective over $\mathbb{C}$. Geometric Invariant Theory says that $\mathcal{N}_{\chi}^{ss}(r, c, c')$ is the space of χ-semistable orbits; moreover it contains an open subscheme $\mathcal{N}_{\theta}^{s}(r, c, c') \subseteq \mathcal{N}_{\theta}^{ss}(r, c, c')$ consisting of χ-stable orbits.

The following proposition holds by analogy with [12, Proposition 3.1, Theorem 4.1] and the proof is analogous to [4, Proposition 3.1]. The proof is given in details just for completeness.

Proposition 2.3.7. Suppose that $\Theta = (\theta, \theta') \in \mathbb{Z}^2$ and let $\chi_{\Theta} : \mathcal{G} \longrightarrow \mathbb{C}^*$ be the character

$$\chi_{\Theta}(h, h') = \det(h)^{-\theta} \det(h')^{-\theta'}.$$

Let $X = (W, V, V', A, B, I, J, A', B', F, G)$ be a representation of the enhanced ADHM quiver and X the corresponding point in $\mathbb{X}_0$. Thus, X is Θ-(semi)stable if and only if X is χ_{Θ}-(semi)stable.

Proof. Suppose that X is χ_Θ-semistable. Let $\theta_\infty \in \mathbb{Z}$ such that it satisfies (2.15). Suppose that there exists a nontrivial proper subrepresentation $\widetilde{X} = (\widetilde{W}, \widetilde{V}, \widetilde{V'}, \widetilde{A}, \widetilde{B}, \widetilde{I}, \widetilde{J}, \widetilde{A'}, \widetilde{B'}, \widetilde{F}, \widetilde{G})$ of numerical type (r, c, c') of the representation X such that $\widetilde{r} = \dim(\widetilde{W}) \in \{0,\ r\}$ satisfies

$$\widetilde{c}\theta + \widetilde{c'}\theta' + \widetilde{r}\theta_\infty > 0.$$

First take $\widetilde{r} = 0$. Then $\widetilde{W} = \{0\}$. Since $\widetilde{X}$ is a subrepresentation of X, $\widetilde{V}$ and $\widetilde{V'}$ are subspaces of V and V', respectively, and it follows that

$$F(\widetilde{V'}) \subseteq \widetilde{V}, \quad G(\widetilde{V}) \subseteq \widetilde{V'}, \quad A(\widetilde{V}),\ B(\widetilde{V}) \subseteq \widetilde{V}, \quad A'(\widetilde{V'}),\ B'(\widetilde{V'}) \subseteq \widetilde{V'}, \quad J(\widetilde{V}) = 0.$$

Thus, there exist direct sum decompositions

$$\begin{cases} V \cong \widetilde{V} \oplus \widehat{V} \\ V' \cong \widetilde{V'} \oplus \widehat{V'} \end{cases} \tag{2.38}$$

such that the linear maps A, B, A', B', F, G have block decomposition of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \tag{2.39}$$

while the linear maps I and J have block decomposition of the form

$$I = \begin{bmatrix} * \\ * \end{bmatrix}, \quad J = \begin{bmatrix} 0 & * \end{bmatrix}.$$

Consider a one-parameter subgroup of $\mathcal{G}$ of the form

$$h(t) = \begin{bmatrix} t1_{\widetilde{V}} & 0 \\ 0 & 1_{\widehat{V}} \end{bmatrix}, \quad h'(t) = \begin{bmatrix} t1_{\widetilde{V}} & 0 \\ 0 & 1_{\widehat{V'}} \end{bmatrix}. \tag{2.40}$$

It follows that the linear maps

$$(A(t), B(t), I(t), J(t), A'(t), B'(t), F(t), G(t)) = (h(t), h'(t)) \cdot X$$

have block decomposition of the form

$$\begin{bmatrix} * & t* \\ 0 & * \end{bmatrix} \tag{2.41}$$

and

$$I(t) = \begin{bmatrix} t* \\ * \end{bmatrix}, \quad J(t) = \begin{bmatrix} 0 & * \end{bmatrix}. \tag{2.42}$$

However,

$$\begin{aligned} \chi_\Theta(h(t), h'(t)) \cdot z &= \left(\det \begin{bmatrix} t1_{\widetilde{V}} & 0 \\ 0 & 1_{\widehat{V}} \end{bmatrix}^{-\theta} \det \begin{bmatrix} t1_{\widetilde{V}} & 0 \\ 0 & 1_{\widehat{V'}} \end{bmatrix}^{-\theta'} \right)^{-1} \cdot z \\ &= (t^{-\theta\widetilde{c} - \theta'\widetilde{c'}})^{-1} \cdot z \\ &= t^{\theta\widetilde{c} + \theta'\widetilde{c'}} \cdot z \end{aligned}$$

with $\theta\widetilde{c} + \theta'\widetilde{c'} > 0$. Therefore,

$$\lim_{t\to 0}(h(t), h'(t)\cdot(X,z)) \in \mathbb{X}\times\{0\},$$

which contradicts the χ_Θ-semistability condition.

Now suppose $\widetilde{r} = r$. Thus, analogously to the case $r = 0$, one can obtain

$$F(\widetilde{V'}) \subseteq \widetilde{V}, G(\widetilde{V}) \subseteq \widetilde{V'}, \quad A(\widetilde{V}),\ B(\widetilde{V}) \subseteq \widetilde{V}, \quad A'(\widetilde{V'}),\ B'(\widetilde{V'}), \quad I(\widetilde{W}) \subseteq \widetilde{V'}.$$

Therefore, there exist direct sum decompositions like in (2.38) such that the maps A, B, A', B', F and G have block form decomposition of the form (2.39), while I, J have block form decompositions of the form

$$I = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad J = \begin{bmatrix} * & * \end{bmatrix}. \tag{2.43}$$

Now consider a one-parameter subgroup of $\mathcal{G}$ of the form

$$h(t) = \begin{bmatrix} 1_{\widetilde{V}} & 0 \\ 0 & t^{-1}1_{\widehat{V}} \end{bmatrix}, \quad h'(t) = \begin{bmatrix} t1_{\widetilde{V}} & 0 \\ 0 & t^{-1}1_{\widehat{V'}} \end{bmatrix}. \tag{2.44}$$

It follows that the linear maps

$$(A(t), B(t), I(t), J(t), A'(t), B'(t), F(t), G(t)) = (h(t), h'(t))\cdot X$$

have block decomposition of the form (2.41) and

$$I(t) = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad J(t) = \begin{bmatrix} * & t* \end{bmatrix}. \tag{2.45}$$

However,

$$\begin{aligned}\chi_\Theta(h(t), h'(t))\cdot z &= \left(\det\begin{bmatrix} 1_{\widetilde{V}} & 0 \\ 0 & t^{-1}1_{\widehat{V}} \end{bmatrix}^{-\theta} \det\begin{bmatrix} t1_{\widetilde{V}} & 0 \\ 0 & t^{-1}1_{\widehat{V'}} \end{bmatrix}^{-\theta'}\right)^{-1}\cdot z \\ &= (t^{(\widetilde{c}-c)\theta + (\widetilde{c'}-c')\theta'})\cdot z\end{aligned}$$

in which $((\widetilde{c}-c)\theta + (\widetilde{c'}-c')\theta') > 0$. Indeed

$$\begin{aligned}(\widetilde{c}-c)\theta + (\widetilde{c'}-c')\theta' &= \underbrace{(\widetilde{c}\theta + \widetilde{c'}\theta')}_{>-\widetilde{r}\theta_\infty} - \underbrace{(c\theta + c'\theta')}_{=-r\theta_\infty} \\ &> -\underbrace{\widetilde{r}}_{=r}\theta_\infty + r\theta_\infty \\ &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{t\to 0}(h(t), h'(t)\cdot(X,z)) &= \lim_{t\to 0} t^{((\widetilde{c}-c)\theta + (\widetilde{c'}-c')\theta')}\cdot Z \\ &= 0 \in \mathbb{X}\times\{0\},\end{aligned}$$

which contradicts the χ_Θ-semistability condition since the closure of the orbit intersects $\mathbb{X}(r,c,c') \times \{0\}$ for some $z \neq 0$. Then, in both cases, χ_Θ-semistability implies Θ-semistability.

Now suppose that X is χ_Θ-stable but it is not Θ-stable. In particular if X is χ_Θ-stable, X is χ_Θ-semistable and thus Θ-semistable in consequence. Therefore, there exists a proper subrepresentation $\widetilde{X}$ of X with numerical type $(\widetilde{r}, \widetilde{c}, \widetilde{c'})$ such that $\widetilde{r} \in \{0, r\}$ and

$$\widetilde{c}\theta + \widetilde{c'}\theta' + \widetilde{r}\theta_\infty = 0.$$

There are two cases to consider, $\widetilde{r} = 0$ and $\widetilde{r} = r$. In both cases, it will be proved that X has a nontrivial stabilizer, which contradicts the χ_Θ-stability condition.

First, consider $\widetilde{r} = 0$. As above, A, B, F, G, A' and B' have block decomposition of the form (2.39) the direct sum decomposition of V and V' like (2.38). Consider a one-parameter subgroup, $(h(t), h'(t))$, of $\mathcal{G}$ of the form (2.40). The linear maps $(A(t), B(t), I(t), J(t), A'(t), B'(t), F(t), G(t)) = (h(t), h'(t)) \cdot (X, z)$ have block form decomposition of the form (2.41) and (2.42). Therefore, the limit of $(h(t), h'(t)) \cdot (X, z)$ as $t \to 0$ has block decomposition of the form

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \text{for } A(t),\ B(t),\ A'(t),\ B'(t),\ F(t) \text{ and } G(t), \quad I(t) = \begin{bmatrix} 0 \\ * \end{bmatrix}, \quad J(t) = \begin{bmatrix} 0 & * \end{bmatrix}$$

On the other hand, since $\mathcal{G} \cdot (X, z)$ is closed for $z \neq 0$, the linear maps A, B, A', B', F, G must have block decomposition of the form

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

while I, J have block decomposition of the form

$$I = \begin{bmatrix} 0 \\ * \end{bmatrix}, \quad J = \begin{bmatrix} 0 & * \end{bmatrix}.$$

Thus, the subgroup $(h(t), h'(t))$ stabilizes (X, z) which contradicts the χ_Θ-stability condition.

Now consider $\widetilde{r} = r$. One can repeat the step above obtaining the block decomposition in (2.39) for the linear maps A, B, A', B', F, and G, while I and J have the block decomposition in (2.43). Thus, let $(h(t), h'(t))$ be a one-parameter subgroup of $\mathcal{G}$ of the form (2.44). Then, the linear maps

$$(A(t), B(t), I(t), J(t), A'(t), B'(t), F(t), G(t)) = (h(t), h'(t)) \cdot X$$

are such that $A(t)$, $B(t)$, $A'(t)$, $B'(t)$, $F(t)$ and $G(t)$ have block decomposition of the form (2.41) while $I(t)$ and $J(t)$ have block decomposition of the form (2.45). Therefore, the limit of $(h(t), h'(t)) \cdot (X, z)$ as $t \to 0$ have block decomposition of the form

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \text{for } A(t),\ B(t),\ A'(t),\ B'(t),\ F(t) \text{ and } G(t), \quad I(t) = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad J(t) = \begin{bmatrix} * & 0 \end{bmatrix}.$$

Again, this implies that (X, z) have a nontrivial stabilizer leading to a contradiction.

The other side of this proof is analogous. □

Therefore, it follows from Lemma 2.2.2 and Proposition 2.3.7 that there exists a chamber in $\mathbb{Q}^2$ given by $\theta' > 0$ and $\theta + c'\theta' < 0$ such that all the stability conditions defined until now are the same. Thus, given a representation of the enhanced ADHM quiver X with numerical type (r, c, c') and $\Theta = (\theta, \theta', \theta_\infty)$ satisfying $\theta' > 0$ and $\theta + c'\theta' < 0$ from now on X will be called *stable* if it satisfies one of the conditions below:

(i) X satisfies the conditions $(S.1)$ and $(S.2)$ of the Lemma 2.2.2;

(ii) X is Θ-stable;

(iii) X is Θ-semistable;

(iv) X is χ_Θ-stable;

(v) X is χ_Θ-semistable.

Then, in a suitable chamber, the moduli space of framed stable representations of numerical type (r, c, c') of the enhanced ADHM quiver denoted, $\mathcal{N}^{st}(r, c, c') = \mathcal{N}^{ss}_\chi(r, c, c')$ is given by the equation (2.37).

For further reference, let $X = (W, V, V', A, B, I, J, A', B', F)$ be a framed stable representation of numerical type (r, c, c') of the enhanced ADHM quiver. One can consider the stable representation of the ADHM quiver $X'' = (W, V'', A'', B'', I'', J'')$ of numerical type $(r, c - c')$, where $V'' := V/Im(F)$ and the maps A'', $B'' \in End(V'')$, $I \in Hom(W, V'')$ and $J \in Hom(V'', W)$ are inherited by the quotient $V/Im(F)$. Moreover, X'' is indeed stable and satisfies the ADHM equation $[A'', B''] + I''J'' = 0$. See the proof below.

Consider a basis in V such that

$$F = \begin{bmatrix} 1_{V'} \\ 0 \end{bmatrix}.$$

Let

$$\begin{array}{cccc} \pi': & V' \oplus V'' & \longrightarrow & V' \\ & (v', v'') & \longmapsto & v' \end{array}, \qquad \begin{array}{cccc} \pi'': & V' \oplus V'' & \longrightarrow & V'' \\ & (v', v'') & \longmapsto & v'' \end{array}.$$

Then V can be decomposed as $V = V' \oplus V/Im(F) = V' \oplus V''$ and A'', B'', I'', J'' are given by

$$A'' = A|_{V''}, \quad B'' = B|_{V''}, \quad I'' = \pi'' \circ I, \quad J'' = J|_{V''}.$$

Therefore,

$$\begin{aligned}[A'', B''] + I''J'' &= [A|_{V''}, B|_{V''}] + (\pi'' \circ I) \circ (J|_{V''}) \\ &= [A, B]|_{V''} + (IJ)|_{V''} \\ &= (\underbrace{[A, B] + IJ}_{=0})|_{V''} \\ &= 0 \end{aligned} \tag{2.46}$$

and $X'' = (W, V'', A'', B'', I'', J'')$ is stable. Indeed, suppose that there exists $0 \subset S'' \subsetneq V''$ a subspace of V'' such that

$$A''(S''), \quad B''(S''), \quad I''(W) \subset S''.$$

Then $0 \subset V' \oplus S'' \subsetneq V$ is a subspace such that

$$A(V' \oplus S''), \quad B(V' \oplus S''), \quad I(W) \subset V' \oplus S''.$$

In fact, fix $(v', s'') \in V' \oplus S''$. Thus

$$\begin{aligned} A(v', s'') &= (A|_{V'}(v'), A|_{V''}(s'')) \\ &= (\underbrace{A|_{V'}(v')}_{\in V'}, \underbrace{A''(s)}_{\in S''}) \\ &\in V' \oplus S'', \end{aligned}$$

which means that $A(v', s'') \in V' \oplus S''$ for all $(v', s'') \in V' \oplus S''$, i.e., $A(V' \oplus S'') \subset V' \oplus S''$. Analogously one can obtain that $B(V' \oplus S'') \subset V' \oplus S''$. Moreover,

$$\begin{aligned} I(W) &= I(W) \cap V' \oplus I(W) \cap V'' \\ &= \pi' \circ I(W) \oplus \pi'' \circ I(W) \\ &= \underbrace{\pi' \circ I(W)}_{\subset V'} \oplus \underbrace{I''(W)}_{\subset S''} \\ &\subset V' \oplus S'' \end{aligned} \tag{2.47}$$

which contradicts the condition $(S.2)$ of Lemma 2.2.2.

Therefore, if $X = (W, V, V', A, B, I, J, A', B', F)$ is a framed stable representation of the enhanced ADHM quiver with numerical type (r, c, c'), $X'' = (W, X'', A'', B'', I'', J'')$ is a stable representation of the ADHM quiver of numerical type $(r, c - c')$.

The following Lemma is analogous to [4, Lemma 3.2]. Again the proof can be found below only for completeness.

Lemma 2.3.8. Let $\mathcal{M}(r, c - c')$ be the moduli space of the stable representations of the ADHM quiver of numerical type $(r, c - c')$. There exists a surjective morphism

$$\begin{array}{cccc} \mathfrak{q}: & \mathcal{N}^{st}(r, c, c') & \longrightarrow & \mathcal{M}(r, c - c') \\ & [(W, V, V', A, B, I, J, A', B', F)] & \longmapsto & [(W, V'', A'', B'', I'', J'')] \end{array}$$

where $[(W, V, V', A, B, I, J, A', B', F)]$ and $[(W, V'', A'', B'', I'', J'')]$ denote the isomorphism class of the framed stable representation $(W, V, V', A, B, I, J, A', B', F)$ of the enhanced ADHM quiver and the isomorphism class of the stable representation $(W, V'', A'', B'', I'', J'')$ of the ADHM quiver constructed above, respectively.

Proof. The construction above shows the existence of the morphism $\mathfrak{q}$. It is enough to prove that this morphism is surjective. So, fix an ADHM data (A'', B'', I'', J'') of numerical type $(r, c-c')$ and the morphisms A', $B' \in End(V')$. Set $V = V' \oplus V''$ and

$$F = \begin{bmatrix} 1_{V'} \\ 0 \end{bmatrix}.$$

Now let A, $B \in End(V)$, $I \in Hom(W, V)$ and $J \in Hom(V, W)$ be of the following form

$$A = \begin{bmatrix} A' & \widetilde{A} \\ 0 & A'' \end{bmatrix}, \quad B = \begin{bmatrix} B' & \widetilde{B} \\ 0 & B'' \end{bmatrix}, \quad I = \begin{bmatrix} \widetilde{I} \\ I'' \end{bmatrix}, \quad J = \begin{bmatrix} 0 & J'' \end{bmatrix},$$

according to the decomposition $V = V' \oplus V''$. This means that

$$\widetilde{A},\ \widetilde{B} \in Hom(V'', V') \quad \text{and} \quad \widetilde{I} \in Hom(W, V').$$

It is easy to check that

$$AF - FA' = BF - FB' = JF = 0 \tag{2.48}$$

and

$$[A, B] + IJ = 0 \Leftrightarrow \begin{cases} [A', B'] & = 0 \\ A'\widetilde{B} + \widetilde{A}B'' - B'\widetilde{A} - \widetilde{B}A'' + \widetilde{I}J'' & = 0 \end{cases}. \tag{2.49}$$

Indeed,

$$AF = \begin{bmatrix} A' & \widetilde{A} \\ 0 & A'' \end{bmatrix} \cdot \begin{bmatrix} 1_{V'} \\ 0 \end{bmatrix} = \begin{bmatrix} A' \\ 0 \end{bmatrix} = \begin{bmatrix} 1_{V'} \\ 0 \end{bmatrix} \cdot A' = FA'.$$

Analogously, $BF = FB'$. Moreover,

$$JF = \begin{bmatrix} 0 & J'' \end{bmatrix} \cdot \begin{bmatrix} 1_{V'} \\ 0 \end{bmatrix} = 0$$

and

$$\begin{aligned} [A, B] + IJ &= \begin{bmatrix} A' & \widetilde{A} \\ 0 & A'' \end{bmatrix} \cdot \begin{bmatrix} B' & \widetilde{B} \\ 0 & B'' \end{bmatrix} - \begin{bmatrix} B' & \widetilde{B} \\ 0 & B'' \end{bmatrix} \cdot \begin{bmatrix} A' & \widetilde{A} \\ 0 & A'' \end{bmatrix} + \\ &\quad \begin{bmatrix} \widetilde{I} \\ I'' \end{bmatrix} \cdot \begin{bmatrix} 0 & J'' \end{bmatrix} \\ &= \begin{bmatrix} A'B' - B'A' & A'\widetilde{B} + \widetilde{A}B'' - B'\widetilde{A} - \widetilde{B}A'' + \widetilde{I}J'' \\ 0 & A''B'' - B''A'' + I''J'' \end{bmatrix} \\ &= \begin{bmatrix} [A, B] & A'\widetilde{B} + \widetilde{A}B'' - B'\widetilde{A} - \widetilde{B}A'' + \widetilde{I}J'' \\ 0 & \underbrace{[A'', B''] + I''J''}_{=0} \end{bmatrix} \end{aligned}.$$

Thus the equations (2.48) and (2.49) are obtained. The map F above is injective. In order to conclude the proof, the following has to be proved. The ADHM data (A, B, I, J) is stable if and only if it satisfies:

(i) at least on of the maps $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{I}$ is nontrivial;

(ii) there is no proper subspace $S' \subsetneq V'$ such that

$$\widetilde{A}(V''), \quad \widetilde{B}(V''), \quad \widetilde{I}(W) \subset S' \quad \text{and} \quad A'(S), B'(S) \subset S'. \tag{2.50}$$

In fact, first suppose that (A, B, I, J) is stable and $\widetilde{A} = \widetilde{B} = \widetilde{I} = 0$. Then

$$A = \begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix}, \quad B = \begin{bmatrix} B' & 0 \\ 0 & B'' \end{bmatrix}, \quad I = \begin{bmatrix} 0 \\ I'' \end{bmatrix}, J = \begin{bmatrix} 0 & J'' \end{bmatrix}.$$

Fix $(0, v'') \in 0 \oplus V''$. Thus,

$$A(0, v'') = \begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix} \begin{bmatrix} 0 \\ v'' \end{bmatrix} = \begin{bmatrix} 0 & A''(v'') \end{bmatrix} \in 0 \oplus V''$$

for all $(0, v'') \in 0 \oplus V''$, which means $A(0 \oplus V'') \subset 0 \oplus V''$. Analogously $B(0 \oplus V'') \subset 0 \oplus V''$. Moreover, fixing $w \in W$

$$I(W) = \begin{bmatrix} 0 \\ I'' \end{bmatrix} [w] = \begin{bmatrix} 0 \\ I''(w) \end{bmatrix} \in 0 \oplus V''$$

for all $w \in W$. Therefore

$$A(0 \oplus V''), \quad B(0 \oplus V''), \quad I(W) \subset 0 \oplus V'',$$

which is a contradiction.

Now suppose that there exists a proper subspace $S' \subsetneq V'$ such that the conditions (2.50) are satisfied. Thus, $S = S' \oplus V'' \subsetneq V$ is a subspace such that $A(S)$, $B(S)$, $I(W) \subset S$. Indeed, let $(s', v'') \in S' \oplus V''$. Then

$$A(s', v'') = \begin{bmatrix} A' & \widetilde{A} \\ 0 & A'' \end{bmatrix} \begin{bmatrix} s' \\ v'' \end{bmatrix} = \begin{bmatrix} A'(s') + \widetilde{A}(v'') & A''(v'') \end{bmatrix} \in S' \oplus V''$$

for all $(s', v'') \in S' \oplus V''$, i.e., $A(S' \oplus V'') \subset S' \oplus V''$. Analogously, $B(S' \oplus V'') \subset S' \oplus V''$. Moreover, fixing $w \in W$

$$I(W) = \begin{bmatrix} \widetilde{I} \\ I'' \end{bmatrix} [w] = \begin{bmatrix} \widetilde{I}(w) \\ I''(w) \end{bmatrix} \in S' \oplus V''$$

for all $w \in W$. Therefore

$$A(S' \oplus V''), B(S' \oplus V''), I(W) \subset S' \oplus V'',$$

which contradicts the stability condition. Therefore, if (A, B, I, J) is stable, it satisfies the conditions (i) and (ii) above.

Now suppose that (A, B, I, J) satisfies the conditions (i) and (ii). One can check that (A, B, I, J) is a stable data. Indeed, let $S = S' \oplus S'' \subset V$ such that $A(S)$, $B(S)$, $I(W) \subset S$ and $(s', s'') \in S$. Thus,

$$A(s', s'') = \begin{bmatrix} A' & \widetilde{A} \\ 0 & A'' \end{bmatrix} \begin{bmatrix} s' \\ s'' \end{bmatrix} = \begin{bmatrix} A'(s') + \widetilde{A}(s'') & A''(s'') \end{bmatrix} \in S' \oplus S''$$

for all $(s', s'') \in S' \oplus S''$, which means that $A'(S') + \widetilde{A}(S'') \subset S'$ and $A''(S'') \subset S''$. Analogously, $B'(S') + B''(S'') \subset S'$ and $B''(S'') \subset S''$. Moreover, given $w \in W$

$$I(w) = \begin{bmatrix} \widetilde{I} \\ I'' \end{bmatrix} [w] = \begin{bmatrix} \widetilde{I}(w) \\ I''(w) \end{bmatrix} \in S' \oplus S''$$

However, since the ADHM data (A'', B'', I'', J'') is stable, $S'' = V''$. Thus, S' is a subspace which satisfies the conditions in (2.50). It follows from (ii) that $S' = 0$ or $S' = V'$. If $S' = 0$, $\widetilde{A}(V'')$, $\widetilde{B}(V'')$, $\widetilde{I}(W) \subset \{0\}$ and $\widetilde{A} = \widetilde{B} = \widetilde{I} = 0$, which contradicts the condition (i). Therefore, $S' = V'$ and $S = V$, i.e., the ADHM data (A, B, I, J) is in fact stable if and only if (A, B, I, J) satisfies the conditions (i) and (ii) above.

In order to finish the proof, it is enough to show that there exists a nontrivial solution for the equation (2.49) which satisfies the conditions (i) and (ii). First choose a basis $\{v_1, \ldots, v_{c'}\}$ for V' and let A' and B' be two diagonal matrix,

$$A' = \operatorname{diag}(\alpha_1, \ldots, \alpha_{c'}), \quad B' = \operatorname{diag}(\beta_1, \ldots, \beta_{c'}),$$

such that

$$\begin{cases} \alpha_i \neq \alpha_j, \text{ for} i \neq j \\ \beta_i \neq \beta_j, \text{ for } i \neq j \end{cases}.$$

Let $\widetilde{I} : W \longrightarrow V'$ be a linear map of rank is 1 and $Im(\widetilde{I})$ is generated by the vector

$$v = \sum_{i=1}^{c'} v_i.$$

Therefore, $\{v, B'v, \ldots, B'^{c'-1}v\}$ is a basis for V', otherwise, there would exist a nontrivial linear relation of the form

$$\sum_{i=0}^{c'-1} x_i B'^i v = 0.$$

Thus, for $B' = \operatorname{diag}(\beta_1, \ldots, \beta_{c'})$, $x_i's$ are a solution for the linear system

$$\sum_{i=1}^{c'} \beta_j^i x_i = 0, \text{ for } j \in \{1, \ldots, c'\},$$

where $B'^0 = 1_{V'}$. However, the discriminant of the linear system is the Vandermond determinant

$$\Delta(\beta_1, \ldots, \beta_{c'}) = \prod_{i<j}^{c'} (\beta_j - \beta_i) \neq 0,$$

since $\beta_i \neq \beta_j$ for all $i \neq j$. Thus, $x_i = 0$, for all $i \in \{1, \ldots, c'\}$, leading to a contradiction. In conclusion, $\{v, B'v, \ldots, B'^{c-1}v\}$ is a basis for V'. In particular, there is no subspace $0 \subset S' \subset V'$ preserved by B' and contained in the image of $\widetilde{I}$. Analogously, there is no subspace $0 \subset S' \subset V'$ preserved by A' and contained in the image of $\widetilde{I}$ as well.

Fixing A', B' and $\widetilde{I}$ as above, the equation (2.49) is a linear system with $c'(c-c')$ equations in the $2c'(c-c')$ variables $\widetilde{A}$, $\widetilde{B}$. Such system a has a $c'(c-c')$-dimensional space of solutions. Any nontrivial solution determines a stable ADHM datum (A, B, I, J). □

2.4 Deformation complex

Consider the following complex

$$\mathcal{C}(X): \quad \begin{matrix} End(V) \\ \oplus \\ End(V') \end{matrix} \xrightarrow{d_0} \begin{matrix} End(V)^{\oplus 2} \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,W) \\ \oplus \\ End(V')^{\oplus 2} \\ \oplus \\ Hom(V',V) \end{matrix} \xrightarrow{d_1} \begin{matrix} End(V) \\ \oplus \\ Hom(V',V)^{\oplus 2} \\ \oplus \\ Hom(V',W) \\ \oplus \\ End(V') \end{matrix} \xrightarrow{d_2} Hom(V',V) \tag{2.51}$$

with

$$\begin{aligned} d_0(h,h') &= ([h,A],[h,B],hI,-Jh,[h',A'],[h',B'],hF-Fh') \\ d_1(a,b,i,j,a',b',f) &= ([a,B]+[A,b]+Ij+iJ, Af+aF-Fa'-fA', \\ &\qquad Bf+bF-Fb'-fB', jF+Jf, [a',B']+[A',b']) \\ d_2(c_1,c_2,c_3,c_4,c_5) &= c_1F+Bc_2-c_2B'+c_3A'-Ac_3-Ic_4-Fc_5. \end{aligned}$$

The differentials of the complex $\mathcal{C}(R)$ were obtained as follows, d_0 is the linearization of the free action (2.23) and d_1 is the linearization of the equations in (2.2). Moreover, $Im(d_1) \subset \ker(d_2)$, i.e.,

$$d_2(d_1(a,b,i,j,a',b',f)) = 0, \text{ for all } (a,b,i,j,a',b',f) \in \mathcal{C}(R)^1$$

Indeed,

$$\begin{aligned}
d_2(d_1(a,b,i,j,a',b',f)) &= d_2([a,B]+[A,b]+Ij+iJ, Af+aF-Fa'-fA',\\
&\quad Bf+bF-Fb'-fB', jF+Jf, [a',B']+[A',b'])\\
&= [a,B]F+[A,b]F+IjF+i\underbrace{JF}_{=0}+BAf+BaF-\\
&\quad -\underbrace{BF}_{=FB'}a'-BfA'-AfB'-a\underbrace{FB'}_{=BF}+Fa'B'+\\
&\quad +fA'B'+BfA'+b\underbrace{FA'}_{=AF}-Fb'A'-fB'A'-ABf-AbF+\\
&\quad +\underbrace{AF}_{FA'}b'+AfB'-IjF-IJf-F([a',B']+[A',b'])\\
&= (\underbrace{[a,B]+Ba-aB}_{=0}+\underbrace{[A,b]+bA-Ab}_{=0}+\underbrace{BA-AB-IJ}_{=-([A,B]+IJ)=0})F\\
&\quad F(\underbrace{-B'a'+a'B'-[a',B']}_{=0}\underbrace{-b'A'+A'b'-[A',b']}_{=0})+f\underbrace{[A',B']}_{=0}\\
&= 0.
\end{aligned}$$

Therefore, d_1 is not a surjective map.

Theorem 2.4.1. Let $X=(V,V',A,B,I,J,A',B',F)$ be a enhanced ADHM datum. Let $\dim(W)=r$, $\dim(V)=c$ and $\dim(V')=c'$. Then

$$H^0(\mathcal{C}(X))=H^3(\mathcal{C}(X))=0,$$

where $\mathcal{C}(X)$ is the complex (2.51).

Proof. The complex $\mathcal{C}(X)[1]^i := \mathcal{C}(X)^{i+1}$ with $(d_i)_{\mathcal{C}(X)[1]} := (-1)d_{i+1}$ it is given by:

$$\mathcal{C}(X)[1]: \quad \begin{array}{c} End(V)^{\oplus 2}\\ \oplus\\ Hom(W,V)\\ \oplus\\ Hom(V,W)\\ \oplus\\ End(V')^{\oplus 2}\\ \oplus\\ Hom(V',V)\end{array} \xrightarrow{d_0} \begin{array}{c} End(V)\\ \oplus\\ Hom(V',V)^{\oplus 2}\\ \oplus\\ Hom(V,W)\\ \oplus\\ End(V')\end{array} \xrightarrow{d_1} Hom(V',V)$$

with

$$\begin{aligned}
d_0(a,b,i,j,a',b',f) &= -([a,B]+[A,b]+Ij+iJ, Af+aF-Fa'-fA',\\
&\quad Bf+bF-Fb'-fB', jF+Jf, [a',B']+[A',b'])\\
d_1(c_1,c_2,c_3,c_4,c_5) &= -c_1F-Bc_2+c_2B'-c_3A'+Ac_3+Ic_4+Fc_5.
\end{aligned}$$

Consider the following complexes:

$$\mathcal{C}(\mathcal{A}): \quad End(V) \xrightarrow{d_0} \begin{array}{c} End(V)^{\oplus 2}\\ \oplus\\ Hom(W,V)\\ \oplus\\ Hom(V,W)\end{array} \xrightarrow{d_1} End(V)$$

where

$$\begin{aligned} d_0(a) &= ([h,A],[h,B],hI,-Jh) \\ d_1(a,b,i,j) &= [a,B]+[A,b]+Ij+iJ; \end{aligned}$$

$$\mathcal{C}(\mathcal{B}): \qquad End(V') \xrightarrow{d_0} End(V') \tag{2.52}$$

where

$$d_0(h') = [h',B];$$

and

$$\mathcal{C}(\mathcal{A},\mathcal{B}): \qquad \begin{array}{c} Hom(V',V) \\ \oplus \\ End(V') \end{array} \xrightarrow{d_0} \begin{array}{c} Hom(V',V)^{\oplus 2} \\ \oplus \\ Hom(V',W) \\ \oplus \\ End(V') \end{array} \xrightarrow{d_1} Hom(V',V)$$

where

$$\begin{aligned} d_0(f,a') &= (-Af+fA'-Fa',-Bf+fB',-Jf,[a',B']) \\ d_1(c_2,c_3,c_4,c_5) &= -Bc_2+c_2B'-c_3A'+Ac_3+Ic_4+Fc_5. \end{aligned}$$

Define the map

$$\rho: \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A},\mathcal{B})$$

by

$$\begin{aligned} \rho_0(h,h') &= (hF-Fh',[h',A']) \\ \rho_1(a,b,i,j,b') &= (aF,bF-Fb',jF,[A',b']) \\ \rho_2(c_1) &= c_1F \end{aligned}$$

We claim that the map ρ is a morphim. Indeed, the following diagram comutes

$$\begin{array}{ccccc} \begin{array}{c} End(V) \\ \oplus \\ End(V') \end{array} & \xrightarrow{d_0} & \begin{array}{c} End(V)^{\oplus 2} \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,W) \\ \oplus \\ End(V') \end{array} & \xrightarrow{d_1} & End(V) \\ \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ \begin{array}{c} Hom(V',V) \\ \oplus \\ End(V) \end{array} & \xrightarrow{d_0} & \begin{array}{c} Hom(V',V)^{\oplus 2} \\ \oplus \\ Hom(V',W) \\ \oplus \\ End(V') \end{array} & \xrightarrow{d_1} & Hom(V',V) \end{array}$$

Indeed,

$$\begin{aligned} \rho_2(d_1(a,b,i,j,b')) &= \rho_2([a,B]+[A,b]+Ij+iJ,0) \\ &= ([a,B]F+[A,b]B+IjB+i\underbrace{JF}_{=0},0) \\ &= ([a,B]F+[A,b]B+IjB,0), \end{aligned}$$

$$\begin{aligned} d_1(\rho_1(a,b,i,j,b')) &= d_1((aF, bF - Fb', jF, [A', b'])) \\ &= -BaF + a\underbrace{FB'}_{=BF} - b\underbrace{FA'}_{=AF} + Fb'A' + AbF - \underbrace{AF}_{=FA'} b' \\ &\quad + IjF + F[A', b'] \\ &= (aB - Ba)F + (Ab - bA)F + IjF + \underbrace{F(b'A' - A'b') + F[A', b']}_{=0} \\ &= [a, B]F + [A, b]B + IjB, \end{aligned}$$

$$\begin{aligned} d_0(\rho_0(h,h')) &= d_0(hF - Fh', [h', A']) \\ &= (-A(hF - Fh') + (hF - Fh')A' + F[h', A], \\ &\quad -B(hF - Fh') + (hF - Fh')B', J(hF - Fh'), [[h', A'], B']) \\ &= (-AhF + \underbrace{AF}_{=FA'} h' + h\underbrace{FA'}_{=AF} - Fh'A' + F[h', A'], -BhF + \underbrace{BF}_{=FB'} h' + \\ &\quad h\underbrace{FB'}_{=BF} - Fh'B', -JhF + \underbrace{JF}_{=0} h', [\underbrace{[A', B']}_{=0}, h'] + [A', [h', B]]) \\ &= ([h, A]F + \underbrace{F[A', b'] + F[h', A']}_{=0}, [h, B]F - F[h, B'], -JhF, [A', [h', B']]) \\ &= ([h, A]F, [h, B]F - F[h, B'], -JhF, [A, [h, B']]) \end{aligned}$$

and

$$\begin{aligned} \rho_1(d_0(a,b')) &= \rho_1([a, A], [a, B], aI, -Ja, [b', B']) \\ &= ([a, A]F, [a, B]F - F[b, B'], -JaF, [A, [b, B']]). \end{aligned}$$

We assert that the cone of the map ρ is equivalent to $\mathcal{C}(X)[1]$. In fact, denote the cone by (C, d_C), it follows that

$$\begin{cases} C^i = \mathcal{C}(\mathcal{A}, \mathcal{B})^i \oplus (\mathcal{C}(\mathcal{A})^{i+1} \oplus \mathcal{C}(\mathcal{B})^{i+1}) \\ (d_i)_C = ((d_i)_{\mathcal{C}(\mathcal{A},\mathcal{B})^i} - \rho_{i+1}, -(d_{i+1})_{(\mathcal{C}(\mathcal{A})^{i+1} \oplus \mathcal{C}(\mathcal{B})^{i+1})}) \end{cases}.$$

Therefore,

$$\begin{aligned} C^0 &= Hom(V', V) \oplus End(V') \oplus End(V)^{\oplus^2} \oplus Hom(W, V) \oplus Hom(V, W) \oplus End(V') \\ C^1 &= Hom(V', V)^{\oplus^2} \oplus Hom(V, W) \oplus End(V') \oplus End(V) \\ C^2 &= Hom(V', V) \end{aligned}$$

$$\begin{aligned} (d_0)_C(f, a', a, b, i, j, b') &= (d_0(f, a') - \rho_1(a, b, i, j, b'), -d_1(a, b, i, j, b')) \\ &= ((-Af + fA' + Fa', -Bf + fB', -Jf, -[a', B']) - \\ &= (aF, bF - Fb', jF, [A', b']), -[a, B] - [A, b] - Ij - iJ) \\ &= (-Af + fA' + Fa' - aF, -Bf + fB' - bF + Fb', \\ &\quad -Jf - jF, -[A', b'] - [a, B'], -[a, B] - [A, b] - Ij - iJ) \end{aligned}$$

$$\begin{aligned} (d_1)_C(c_2, c_3, c_4, c_5) &= (d_1(c_2, c_3, c_4, c_5) - \rho_1(c_1), -d_2(c_1)) \\ &= (-c_1F - Bc_2 + c_2B' - c_3A' + Ac_3 + Ic_4 + Fc_5, 0). \end{aligned}$$

Hence the cone of the map ρ is equivalent to $\mathcal{C}(X[1])$. So, one can obtain the following exact triangle

$$\mathcal{C}(X) \longrightarrow \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \xrightarrow{\rho} \mathcal{C}(\mathcal{A}, \mathcal{B}) \tag{2.53}$$

We obtain from Proposition 1.2.2 that

$$H^0(\mathcal{C}(\mathcal{A})) = H^2(\mathcal{C}(\mathcal{A})) = 0. \tag{2.54}$$

Let us prove that $H^2(\mathcal{C}(\mathcal{A}, \mathcal{B})) = 0$. In fact, the dual of the differential

$$d_1 : \mathcal{C}(\mathcal{A}, \mathcal{B})^1 \longrightarrow \mathcal{C}(\mathcal{A}, \mathcal{B})^2$$

is given by

$$\begin{array}{rcl} d_1^\vee : \; Hom(V, V') & \longrightarrow & Hom(V, V')^{\oplus 2} \oplus Hom(W, V') \oplus End(V') \\ \varphi & \longmapsto & (B'\varphi - \varphi B, \varphi A - A'\varphi, \varphi I, \varphi F) \end{array}.$$

Suppose that $d_1^\vee(\varphi) = 0$. Thus,

$$B'\varphi - \varphi B = A'\varphi - \varphi A = \varphi I = 0.$$

Therefore,

$$Im(I), A(\ker(\varphi)), B(\ker(\varphi)) \subseteq \ker(\varphi).$$

The fact that $Im(I) \subseteq \ker(\varphi)$ is trivial. If $v \in B(\ker(\varphi))$, then there exists $w \in \ker(\varphi)$ such that $v = Bw$. Then

$$\varphi(v) = \varphi(B(w)) = B'(\varphi(w)) = B'(0) = 0.$$

Therefore $v \in \ker(\varphi)$. Hence, $B(\ker(\varphi)) \subseteq \ker(\varphi)$. One can prove that $A(\ker(\varphi)) \subseteq \ker(\varphi)$ analogously. It follows from the stability of x that $\ker(\varphi) = 0$ or $\ker(\varphi) = V'$. If $\ker(\varphi) = 0$, then $I \equiv 0$. This lead us to a contradiction. Therefore, $\varphi = 0$, i.e., $d_1^\vee$ it is injective and hence d_1 it is surjective. So,

$$\begin{array}{rcl} H^2(\mathcal{C}(\mathcal{A}, \mathcal{B})) & = & coker(d_1) \\ & = & Hom(V', V)/Im(d_1) \\ & = & 0 \end{array}$$

Let us prove that $H^0(\rho)$ is injective. Since $H^0(\mathcal{C}(\mathcal{A})) = 0$,

$$H^0(\mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B})) = 0 \oplus H^0(\mathcal{C}(\mathcal{B})).$$

Therefore,

$$\begin{array}{rcl} H^0(\rho) : \; 0 \oplus H^0(\mathcal{C}(\mathcal{B})) & \longrightarrow & H^0(\mathcal{C}(\mathcal{A}, \mathcal{B})) \\ (\overline{0}, \overline{h'}) & \longmapsto & (\overline{-Fh'}, \overline{[h', A]}) \end{array}$$

where $\overline{x} \in H^0(\mathcal{C})$ denotes the equivalence class of $x \in \mathcal{C}$, with $\mathcal{C} \in \{\mathcal{C}(\mathcal{A}), \mathcal{C}(\mathcal{B}), \mathcal{C}(\mathcal{A}, \mathcal{B})\}$. Suppose that $H^0(\rho)(\overline{0}, \overline{h'}) = 0$. Then, $\overline{-Fh'} = 0$. Since F it is injective, it is true that

$\overline{h'} = \overline{0}$. Hence $H^0\rho$ it is injective.

It follows from equation (2.54) that the exact sequence of cohomologies of (2.53) is given by

$$0 \longrightarrow H^0(\mathcal{C}(R)) \xrightarrow{\delta} H^0(\mathcal{C}(B)) \xrightarrow{H^0(\rho)} H^0(\mathcal{C}(\mathcal{A},\mathcal{B})) \longrightarrow \dots \tag{2.55}$$

$$\dots \longrightarrow \underbrace{H^2(\mathcal{C}(B))}_{=0} \longrightarrow \underbrace{H^2(\mathcal{C}(\mathcal{A},\mathcal{B}))}_{=0} \xrightarrow{\gamma} H^3(\mathcal{C}(R)) \longrightarrow 0.$$

Thus, the map δ is injective.This proves the Theorem. □

Remark 2.4.2. Since $H^0(\rho)$ it is injective,

$$Im(\delta) = \ker(H^0(\rho)) = 0.$$

Hence $H^0(\mathcal{C}(R)) = 0$. It follows from the sequence (2.55) that the map γ is surjective. Since $H^2(\mathcal{C}(\mathcal{A},\mathcal{B})) = 0$, $H^3(\mathcal{C}(R)) = 0$. Moreover,

$$0 \longrightarrow H^0(\mathcal{C}(\mathcal{B})) \xrightarrow{H^0\rho} H^1(\mathcal{C}(R)) \longrightarrow \dots$$

$$\dots \longrightarrow H^1(\mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B})) \xrightarrow{H^1\rho} H^1(\mathcal{C}(\mathcal{A},\mathcal{B})) \longrightarrow H^2(\mathcal{C}(R)) \longrightarrow 0$$

it is the exact sequence of cohomologies (2.53).

Now, we are going to consider the particular case $c' = \dim(V') = 1$. Since

$$[A', B'] = 0, \text{ for all } A',\ B' \in End(V'),$$

we can omit this equation from the set of enhanced ADHM equation, obtaining the following enhanced ADHM equations

$$[A, B] + IJ = 0, \quad AF - FA' = 0, \quad BF - FB' = 0, \quad JF = 0.$$

Consider $\mathcal{C}(X)$ the complex below

$$\mathcal{C}(X) : \quad \begin{matrix} End(V) \\ \oplus \\ End(V') \end{matrix} \xrightarrow{d_0} \begin{matrix} End(V)^{\oplus 2} \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,W) \\ \oplus \\ End(V')^{\oplus 2} \\ \oplus \\ Hom(V',V) \end{matrix} \xrightarrow{d_1} \begin{matrix} End(V) \\ \oplus \\ Hom(V',V)^{\oplus 2} \\ \oplus \\ Hom(V',W) \end{matrix} \xrightarrow{d_2} Hom(V',V)$$

in which

$$\begin{aligned} d_0(h,h') &= ([h,A],[h,B],hI,-Jh,[h',A'],[h',B'],aF-Fa') \\ d_1(a,b,i,j,a',b',f) &= ([a,B]+[A,b]+Ij+iJ, Af+aF-Fa'-fA', \\ &\quad Bf+bF-Fb'-fB', jF+Jf) \\ d_2(c_1,c_2,c_3,c_4) &= c_1F+Bc_2-c_2B'+c_3A'-Ac_3-Ic_4. \end{aligned}$$

Moreover d_0 is the linearization of the action (2.23), d_1 is the linearization of the equations (2.4) and d_2 is a map such that $im(d_1) \subset \ker(d_2)$, in particular d_1 it is not a surjective map. In fact,

$$\begin{aligned} d_2(d_1(a,b,i,j,a',b',f)) &= d_2([a,B]+[A,b]+Ij+iJ, Af+aF-Fa'-fA', \\ &\quad Bf+bF-Fb'-fB', jF+Jf) \\ &= [a,B]F+[A,b]F+IjF+i\underbrace{JF}_{=0}+BAf+BaF- \\ &\quad -\underbrace{BF}_{=FB'}a'-BfA'-AfB'-a\underbrace{FB'}_{=BF}+Fa'B'+fA'B'+ \\ &\quad +BfA'+b\underbrace{FA'}_{=AF}-Fb'A'-fB'A'-ABf-AbF+ \\ &\quad +\underbrace{AF}_{FA'}b'+AfB'-IjF-IJf \\ &= (\underbrace{[a,B]+Ba-aB}_{=0}+\underbrace{[A,b]+bA-Ab}_{=0}+\underbrace{BA-AB-IJ}_{=-([A,B]+IJ)=0})F \\ &\quad F(a'B'-B'a'+A'b'-b'A')+f\underbrace{[A',B']}_{=0} \\ &= F([a',B']+[A',b']) \\ &= 0 \end{aligned}$$

Theorem 2.4.3. Let $\mathcal{R} = (A,B,I,J,A',B',F)$ be a stable enhanced ADHM datum which satisfies the enhanced ADHM equations. Let $\dim(W)=r$, $\dim(V)=c$ and $\dim(V')=1$. Then the moduli space of stable solutions of the enhanced ADHM equations, $\mathcal{N}^{st}(r,c,1)$, is non-singular, quasi-projective and its dimension is $2rc-r+1$.

Proof. Let $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ be the complex (2.4) and (2.52), respectively. Let $\mathcal{C}(\mathcal{A},\mathcal{B})$ be the complex given by

$$\mathcal{C}(\mathcal{A},\mathcal{B}): \quad \begin{matrix} Hom(V',V) \\ \oplus \\ End(V') \end{matrix} \xrightarrow{d_0} \begin{matrix} Hom(V',V)^{\oplus 2} \\ \oplus \\ Hom(V',W) \\ \oplus \\ 0 \end{matrix} \xrightarrow{d_1} Hom(V',V)$$

with

$$\begin{aligned} d_0(f,a') &= (-Af+fA'-Fa', -Bf+fB', -Jf, 0) \\ d_1(c_2,c_3,c_4) &= -Bc_2+c_2B'-c_3A'+Ac_3+Ic_4. \end{aligned}$$

Define a map

$$\rho: \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A},\mathcal{B})$$

given by

$$\begin{array}{lcl}\rho_0(a,a') & = & (aF - Fa', [a',A']) \\ \rho_1(a,b,i,j,b') & = & (aF, bF - Fb', jF, 0) \ . \\ \rho_2(c_1) & = & c_1 F\end{array}$$

We assert that ρ is a morphism and that the sequence

$$\mathcal{C}(R) \longrightarrow \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \xrightarrow{\ \rho\ } \mathcal{C}(\mathcal{A},\mathcal{B}) \tag{2.56}$$

is exact. Indeed, the following diagram commutes

$$\begin{array}{ccccc}
\begin{array}{c} End(V) \\ \oplus \\ End(V') \end{array} & \xrightarrow{d_0} & \begin{array}{c} End(V)^{\oplus 2} \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,W) \\ \oplus \\ End(V') \end{array} & \xrightarrow{d_1} & End(V) \quad . \\
\downarrow \rho_0 & & \downarrow \rho_1 & & \downarrow \rho_2 \\
\begin{array}{c} Hom(V',V) \\ \oplus \\ End(V') \end{array} & \xrightarrow{d_0} & \begin{array}{c} Hom(V',V)^{\oplus 2} \\ \oplus \\ Hom(V',W) \\ \oplus \\ 0 \end{array} & \xrightarrow{d_1} & Hom(V',V)
\end{array}$$

Indeed,

$$\begin{array}{rcl}
\rho_1(d_0(a,a')) & = & \rho_1([a,A],[a,B],aI,-Ja,0) \\
& = & ([a,A]F,[a,B]F,-JaF,0) \\
& = & (aAF - AaF + \underbrace{FA'a' - Fa'A' + F0}_{=0}, \\
& & aBF - BaF + \underbrace{FB'a' - Fa'B'}_{=0}, -JaF + \underbrace{JF}_{=0} a', 0) \\
& = & (-A(aF - Fa') + (aF - Fa')A' + F0, \\
& & -B(aF - Fa') + (aF - Fa')B', -J(aF - Fa'), 0) \\
& = & d_0(aF - Fa', 0) \\
& = & d_0(\rho_0(a,a')),
\end{array}$$

$$\begin{array}{rcl}
\rho_2(d_1(a,b,i,j,b')) & = & \rho_2([a,B] + [A,b] + Ij + iJ) \\
& = & [a,B]F + [A,b]F + IjF + i\underbrace{JF}_{=0} \\
& = & aBF - BaF - bAF + AbF - Fb'A' + FA'b' + IjF \\
& = & aFB' - BaF - (bF - Fb')A' + A(bF - Fb') + IjF \\
& = & d_1(aF, bF - Fb', jF, 0) \\
& = & d_1(\rho_1(a,b,i,j,b')).
\end{array}$$

Analogously to Theorem 2.4.1, one can prove that $\mathcal{C}(R)[1] = C(\rho)$, where $C(\rho)$ denotes the cone of ρ. Indeed,

$$C(\rho): \quad \begin{matrix} End(V)^{\oplus 2} \\ \oplus \\ Hom(W,V) \\ \oplus \\ Hom(V,W) \\ \oplus \\ End(V')^{\oplus 2} \\ \oplus \\ Hom(V',V) \end{matrix} \xrightarrow{d_0} \begin{matrix} End(V) \\ \oplus \\ Hom(V,V')^{\oplus 2} \\ \oplus \\ Hom(V,W) \\ \oplus \\ 0 \end{matrix} \xrightarrow{d_1} Hom(V,V')$$

with

$$\begin{aligned} d_0(a,b,i,j,a',b',f) &= (-[a,B]-[A,b]-Ij-iJ, \\ & \quad (-Af+fA'+Fa',-Bf+fB',-Jf,0)-(aF,bF-Fb',jF,0)) \\ &= -d_1(a,b,i,j,a',b',f) \end{aligned}$$

$$\begin{aligned} d_1(c_1,c_2,c_3,c_4,0) &= -c_1F-Bc_2+c_2B'-c_3A'+Ac_3+Ic_4 \\ &= -d_2(c_1,c_2,c_3,c_4,0). \end{aligned}$$

Hence, the sequence (2.56) is exact. Analogously to Theorem 2.4.1, one can prove that

$$H^2(\mathcal{C}(\mathcal{A},\mathcal{B})) = H^3(\mathcal{C}(R)) = 0,$$

therefore

$$0 \longrightarrow H^0(\mathcal{C}(\mathcal{B})) \xrightarrow{H^0\rho} H^1(\mathcal{C}(R)) \longrightarrow \dots$$

$$\dots \longrightarrow H^1(\mathcal{C}(\mathcal{A})\oplus\mathcal{C}(\mathcal{B})) \xrightarrow{H^1\rho} H^1(\mathcal{C}(\mathcal{A},\mathcal{B})) \longrightarrow H^2(\mathcal{C}(R)) \longrightarrow 0$$

is the exact sequence of cohomologies (2.56).

To complete the proof, it remains to show that the obstruction $H^2(\mathcal{C}(R)) = 0$. Since the sequence above it is exact, is enough to show that $H^1(\rho)$ is a surjective map. To prove this, we are going to show that the map

$$Z_1(\rho): Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B})) \longrightarrow Z^1(\mathcal{C}(\mathcal{A},\mathcal{B})), \tag{2.57}$$

in which $Z^1(\mathcal{C}(\mathcal{A})) := \ker(d_1)$, with $d_1 : \mathcal{C}(\mathcal{A})^1 \longrightarrow \mathcal{C}(\mathcal{A})^2$ and $Z_1(\rho)$ is the induced map is surjective. It is true that $Z^1(\mathcal{C}(\mathcal{B})) = 0$. In order to prove that $Z_1(\rho)$ is surjective, we are going to prove that:

1. the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^1(\mathcal{C}(\mathcal{A}))\oplus Z^1(\mathcal{C}(\mathcal{B})) & \xrightarrow{i} & \mathcal{C}(\mathcal{A})^1\oplus\mathcal{C}(\mathcal{B})^1 & \xrightarrow{d_1} & \overbrace{\mathcal{C}(\mathcal{A})^2}^{=End(V)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Z_1(\rho)} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \\ 0 & \longrightarrow & Z^1(\mathcal{C}(\mathcal{A},\mathcal{B})) & \xrightarrow{i} & \mathcal{C}(\mathcal{A},\mathcal{B})^1 & \xrightarrow{d_1} & \underbrace{\mathcal{C}(\mathcal{A},\mathcal{B})^2}_{=Hom(V',V)} & \longrightarrow & 0 \end{array}$$

   commutes;

2. the maps ρ_1 and ρ_2 are surjective;

3. for all $p \in Z^1(\mathcal{C}(\mathcal{A},\mathcal{B}))$, $\rho_1^{-1}(p) \cap (Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B}))) \neq 0$ in $\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1$.

Proof of item 1: The map $Z^1(\rho)$ it is well-defined. Indeed, let $(a, b, i, j, b') \in \ker((d_1)_{\mathcal{C}(\mathcal{A})\oplus\mathcal{C}(\mathcal{B})})$. Thus,

$$Ij = -[a, B] - [A, b] - iJ.$$

Therefore,

$$\begin{aligned} d_1(\rho_1)(a,b,i,j,b') &= d_1(aF, bF - Fb', jF, 0) \\ &= -BaF + a\underbrace{FB'}_{=BF} - b\underbrace{FA'}_{=AF} + Fb'A' + AbF - \underbrace{AF}_{=FA'} b' + IjF \\ &= [a,B]F + [A,b]F + i\underbrace{JF}_{=0} + IjF - F\underbrace{[A',b']}_{=0} \\ &= ([a,B] + [A,b] + iJ)F + IjF \\ &= IjF - IjF \\ &= 0. \end{aligned}$$

Hence, the map $Z_1(\rho)$ it is well-defined. Since ρ is a morphism, it follows that $d_1 \circ \rho_1 = \rho_2 \circ d_1$. Moreover, $i \circ Z^1(\rho) = i \circ \rho_1$. Proof of item 2: Let $p = (c_2, c_3, c_4, 0) \in \mathcal{C}(\mathcal{A},\mathcal{B})^1$. Let $E : V \longrightarrow V'$, such that $EF = Id_{V'}$. Since F is injective, there exists a surjective map E. Thus,

$$\begin{aligned} \rho_1(c_2E, c_3E, i, c_4E, 0) &= (c_2EF, c_3EF - F0, c_4EF, 0) \\ &= (c_2, c_3, c_4, 0) \end{aligned}.$$

Then, ρ_1 is surjective. In order to show that ρ_2 is surjective, consider $c_1 \in Hom(V', V)$. Therefore

$$\begin{aligned} \rho_2(c_1E) &= (c_1EF) \\ &= c_1 \end{aligned}.$$

Hence ρ_2 is surjective.

Proof of item 3: Since ρ_1 is surjective, $\rho_1^{-1}(p)$ is a fiber over the linear space $\ker(\rho_1)$ for all

$p \in \mathcal{C}(\mathcal{A},\mathcal{B})^1$. Since $Z^1(\mathcal{C}(\mathcal{A}))\oplus Z^1(\mathcal{C}(\mathcal{B}))$ it is a proper subspace $\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1$, it remains to show that

$$\Delta = \dim(\ker(\rho_1)) + \dim(Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B}))) - \dim(\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1) \geqslant 0.$$

Indeed,

$$\Delta = \dim(\ker(\rho_1)) - \dim((\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1)\backslash(Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B})))$$

and $\Delta \geqslant 0$ means that $\ker(\rho_1) \cap (Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B})) \neq 0$ and this conclude the proof, since the diagram above commutes. So, let us prove that $\Delta \geqslant 0$

$$\begin{aligned}
\Delta &= \dim(\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1) - \dim(Im(\rho_1)) + \dim(\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1) - \\
&\quad \dim(Im(d_1)) - \dim(\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1) \\
&= \dim(\mathcal{C}(\mathcal{A})^1 \oplus \mathcal{C}(\mathcal{B})^1) - \dim(Im(\rho_1)) - \dim(Im(d_1)) \\
&= 2c^2 + 2rc + 1 - 2c - r - 1 - c^2 \\
&= c^2 + 2c(r-1) - r \\
&\geqslant 0
\end{aligned}$$

At last, the dimension of the moduli space is equal to the dimension of $H^1(\mathcal{C}(R))$ which is

$$\begin{aligned}
\dim(h^1(\mathcal{C}(R))) &= -\dim(C(R))^0 + \dim(C(R))^1 - \dim(C(R))^2 + \dim(C(R))^3 \\
&= -(c^2 + c'^2) + (2c^2 + 2c'^2 + 2rc + cc') - (c^2 + 2c'c + c'r) + cc' \\
&= 2rc + c'^2 - c'r \\
&= 2rc - r + 1.
\end{aligned}$$

□

3 Geometric Structures

In this chapter we will prove that the moduli space of framed stable representations of the enhanced ADHM quiver can be embedded into a hyperkähler manifold of complex dimension $2(rc + cc')$. Then, one can define a complex structure on $\mathcal{N}$ and a closed degenerate 2-form such that it has a structure that it will be defined as *holomorphic pre-symplectic variety*. It is still unknown in which points this form is degenerated or non-degenerated. However, this study was done for the particular case $\mathcal{N}^{st}(1, 2, 1)$.

In the first section, one can find the proof of the fact that $\mathcal{N}^{st}(r, c, c')$ is a subvariety in a hyperkähler manifold os complex dimension $2(rc + cc')$. In the second section, one can find a study of the holomorphic structures on $\mathcal{N}^{st}(r, c, 1)$ and of the holomorphic pre-symplectic 2-form Ω for the particular case $\mathcal{N}^{st}(1, 2, 1)$.

3.1 Enhanced ADHM quiver varieties as a submanifold of a hyperkähler manifold

Let (M, ω) be a symplectic variety, $\mathcal{G}$ a Lie group and $\Psi : \mathcal{G} \longrightarrow Sympl(M, \omega)$ a symplectic action. The symplectic action Ψ is called hamiltonian if there exists a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

which satisfies

(i) $d\mu^\xi = iX_\xi\omega$, for all $\xi \in \mathfrak{g}$;

(ii) $Ad_g^* \circ \mu = \mu \circ \Psi_g$, for all $g \in \mathcal{G}$,

where $\mu^\xi \in C^\infty(M)$, $\mu^\xi(p) := \mu(p)(\xi)$. A map μ as above is called a *moment map*. The next Lemma is useful and gives us a tool to find moment maps.

Lemma 3.1.1. Let (M, ω) be a symplectic variety and $\mathcal{G}$ be a Lie group. Suppose that there is a symplectic $\mathcal{G}$-action on (M, ω). If the 2-form ω is exact, i.e. if $\omega = d\theta$, and this action preserves the 1-form θ, i.e. $\Psi_g^*\theta = \theta$, for all $g \in \mathcal{G}$, then the map $\mu : M \longrightarrow \mathfrak{g}^*$ given by

$$\mu(m)(\xi) = -\theta(m)(X_\xi)_m$$

is a moment map for this $\mathcal{G}$-action.

The proof of this Lemma can be found on [2, Lemma 2.1.1]. Moreover, a *hyperkähler moment map* is a map

$$\mu :\longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

such that

(i) $d\mu^\xi = ix_\xi\omega$, where $\mu^\xi : p \in M \longmapsto \mu(p)(\xi)$, for all $\xi \in \mathfrak{g}$;

(ii) $\mu \circ \Psi_h = Ad_h^* \circ \mu$, for all $f \in \mathcal{G}$.

Theorem 3.1.2. Let $\mathcal{G}$ be a compact Lie group acting on a hyperkähler variety

$$(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$$

. Suppose that $\zeta \in \mathfrak{g}^* \otimes \mathbb{R}^3$ is invariant by the coadjoint action. Suppose also that $\mathcal{G}$ acts freely on μ^ζ. Then $M_\zeta = \mu^{-1}(\zeta)/\mathcal{G}$ is a smooth manifold and inherits the hyperkähler structure of M. This quotient is called *hyperkähler quotient*.

The proof the this Theorem can be find at [2, Teorema 2.3.5].
Consider the following vector space

$$\mathbb{X} = End(V)^{\oplus 2} \oplus Hom(W, V) \oplus Hom(V, W) \oplus End(V')^{\oplus 2} \oplus Hom(V', V) \oplus Hom(V, V').$$

Define in $\mathbb{X}$ the following equations

$$[A, B] + IJ + FG = 0, \quad [A', B'] - GF = 0, \tag{3.1}$$

where $X = (A, B, I, J, A', B', F, G) \in \mathbb{X}$. A vector $X \in \mathbb{X}$ is called *stable* if it satisfies the conditions $(S.1)$ and $(S.2)$ of Lemma 2.2.2, i.e., if F is injective and the ADHM data given by (A, B, I, J) is stable. Let

$$\mathbb{W} = \mathbb{W}(r, c, c') := \{X \in \mathbb{X} : X \text{ satisfies (3.1) and X is stable}\}.$$

Note that the $\mathcal{G}$-action in (2.23) is free on $\mathbb{W}$ and that the equations in (3.1) are preserved by the $\mathcal{G}$-action in (2.23). Indeed the freeness of this action has been already proved in Proposition 2.3.1. Analogously to Lemma 2.3.2, one can prove that the equations in (3.1) are preserved by the $\mathcal{G}$-action (2.23). The same is true if the one consider the action of $\mathcal{U} := U(V) \times U(V')$ on $\mathbb{W}(r, c, c')$ given by (2.23), i.e.,

$$\begin{array}{ccc} \mathcal{U} \times \mathbb{X} & \longrightarrow & \mathbb{W} \\ (h, h', X) & \longmapsto & (hAh^{-1}, hBh^{-1}, hI, Jh^{-1}, h'A'h'^{-1}, h'B'h'^{-1}, hFh'^{-1}, h'Gh^{-1}) \end{array} \tag{3.2}$$

is a free action of $\mathcal{U}$ on $\mathbb{W}$ which preserves the equations (3.1). The moduli space of stable points of $\mathbb{W}$, $\mathcal{W}(r, c, c')$, can be constructed by using Geometric Invariant Theory techniques. Moreover, the moduli space of framed stable representations of the enhanced ADHM quiver, $\mathcal{N}^{st}(r,c,c')$ is embedded in $\mathcal{W}(r,c,c')$. In fact, if $X \in \mathbb{X}_0$ is stable, than X satisfies

$$[A,B] + IJ = 0, \quad [A',B'] = 0, \quad G \equiv 0.$$

Thus, X satisfies the equations (3.1) and $X \in \mathbb{W}$. In this section, it will be proved that the moduli space $\mathcal{W}$ can be obtained by a hyperkähler reduction. In other words, $\mathcal{N}^{st}(r,c,c')$ is embedded in the hyperkähler variety $\mathcal{W}(r,c,c')$. The hyperkähler reduction is presented in details below.

In order to prove that $\mathcal{W}(r,c,c')$ is a hyperkähler variety, consider the following equations

$$\begin{cases} [A,A^\dagger] + [B,B^\dagger] + II^\dagger - J^\dagger J + FF^\dagger - G^\dagger G & = & 0 \\ [A',A'^\dagger] + [B',B'^\dagger] - F^\dagger F + GG^\dagger & = & 0 \end{cases}, \tag{3.3}$$

where $A^\dagger$ denotes the hermitian adjoint of the map A. It will be proved that the equations (3.1) and (3.3) can be obtained as a hyperkähler moment map μ. Thus, one can viewed the hyperkähler variety $\widetilde{\mathcal{W}} := \mu^{-1}(0)/\mathcal{U}$. However, it follows from the Kempf-Ness Theorem that the moduli space $\mathcal{W}$ obtained above is isomorphic to the hyperkähler variety $\widetilde{\mathcal{W}}$.

The proof that one can find the hyperkähler variety $\widetilde{\mathcal{W}}$ is below. Define on $\mathbb{W}$ the hermitian metric

$$\langle\ ,\ \rangle : T\mathbb{W} \times T\mathbb{W} \longrightarrow \mathbb{C}$$

given by

$$\begin{aligned} \langle x_1, x_2 \rangle = \frac{1}{2} tr(& a_1 a_2^\dagger + a_2 a_1^\dagger + b_1 b_2^\dagger + b_2 b_1^\dagger + i_1 i_2^\dagger + i_2 i_1^\dagger + j_2^\dagger j_1 + j_1^\dagger j_2 + \\ & a_1' a_2'^\dagger + a_2' a_1'^\dagger + b_1' b_2'^\dagger + b_2' b_1'^\dagger + f_2^\dagger f_1 + f_1^\dagger f_2 + g_1 g_2'^\dagger + g_2 g_1'^\dagger), \end{aligned}$$

where $x_1 = (a_1, b_1, i_1, j_1, a_1', b_1', f_1, g_1)$ and $x_2 = (a_2, b_2, i_2, j_2, a_2', b_2', f_2, g_2)$. Define in $T\mathbb{W}$ the following complex structures. Let $x = (a,b,i,j,a',b',f,g) \in T\mathbb{W}$,

$$\begin{cases} \Gamma_1(x) & = & (\sqrt{-1}a, \sqrt{-1}b, \sqrt{-1}i, \sqrt{-1}j, \sqrt{-1}a', \sqrt{-1}b', \sqrt{-1}f, \sqrt{-1}g) \\ \Gamma_2(x) & = & (-b^\dagger, a^\dagger, -j^\dagger, i^\dagger, -b'^\dagger, a'^\dagger, -g^\dagger, f^\dagger) \\ \Gamma_3(x) & = & \Gamma_1 \circ \Gamma_2(x) \end{cases}.$$

It is easy to check that Γ_n, $n \in \{1,2,3\}$, satisfies the quaternion identities

$$\Gamma_1\Gamma_1 = \Gamma_2\Gamma_2 = \Gamma_3\Gamma_3 = \Gamma_1\Gamma_2\Gamma_3 = -\mathbf{1} \tag{3.4}$$

where $\mathbf{1} : T\mathbb{W} \longrightarrow T\mathbb{W}$ is the identity map.

In fact, it is trival that $\Gamma_1\Gamma_1 = -\mathbf{1}$. Moreover, let $u = (a,b,i,j,a',b',f,g) \in T\mathcal{W}$, then

$$\begin{aligned}\Gamma_2(\Gamma_2(u)) &= \Gamma_2(-b^\dagger, a^\dagger, -j^\dagger, i^\dagger, -b'^\dagger, a'^\dagger, -g^\dagger, f^\dagger)\\ &= (-(a^\dagger)^\dagger, (-b^\dagger)^\dagger, -(i^\dagger)^\dagger, (-j^\dagger)^\dagger, -(a'^\dagger)^\dagger, (-b'^\dagger)^\dagger - (f^\dagger)^\dagger, (-g^\dagger)^\dagger)\\ &= -(a, b, i, j, a', b', f, g)\\ &= -\mathbf{1}(a, b, i, j, a', b', f, g).\end{aligned}$$

In order to prove that

$$\Gamma_3\Gamma_3 = \Gamma_1\Gamma_2\Gamma_3 = -\mathbf{1},$$

it is enough to check that

$$\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1. \tag{3.5}$$

Indeed, suppose that $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$, thus

$$\begin{aligned}\Gamma_1\Gamma_2\Gamma_3 &= \Gamma_1 \underbrace{\Gamma_2\Gamma_1}_{-\Gamma_1\Gamma_2} \Gamma_2\\ &= -\Gamma_1\Gamma_1\Gamma_2\Gamma_2\\ &= -(-\mathbf{1})(-\mathbf{1})\\ &= -\mathbf{1}.\end{aligned}$$

Moreover, one can check that (3.5) is true. Indeed,

$$\begin{aligned}\Gamma_1\Gamma_2(u) =&\Gamma_1(-b^\dagger, a^\dagger, -j^\dagger, i^\dagger, -b'^\dagger, a'^\dagger, -g^\dagger, f^\dagger)\\ =&(-\sqrt{-1}b^\dagger, \sqrt{-1}a^\dagger, -\sqrt{-1}j^\dagger, \sqrt{-1}i^\dagger, -\sqrt{-1}b'^\dagger, \sqrt{-1}a'^\dagger, -\sqrt{-1}g^\dagger,\\ &\sqrt{-1}f^\dagger)\\ =&((\sqrt{-1}b)^\dagger, (-\sqrt{-1}a)^\dagger, (\sqrt{-1}j)^\dagger, (-\sqrt{-1}i)^\dagger, (\sqrt{-1}b)'^\dagger, (-\sqrt{-1}a)'^\dagger, (\sqrt{-1}g)^\dagger,\\ &-(\sqrt{-1}f)^\dagger)\\ =&-(-(\sqrt{-1}b)^\dagger, (\sqrt{-1}a)^\dagger, -(\sqrt{-1}j)^\dagger, (\sqrt{-1}i)^\dagger, -(\sqrt{-1}b)'^\dagger, (\sqrt{-1}a)'^\dagger, -(\sqrt{-1}g)^\dagger,\\ &(\sqrt{-1}f)^\dagger)\\ =&-\Gamma_2(\sqrt{-1}a, \sqrt{-1}b, \sqrt{-1}i, \sqrt{-1}j, \sqrt{-1}a', \sqrt{-1}b', \sqrt{-1}f, \sqrt{-1}g)\\ =&-\Gamma_2\Gamma_1(a, b, i, j, a', b', f, g)\end{aligned}$$

Therefore, Γ_n, $n \in \{1, 2, 3\}$ satisfies the quaternions identities (3.4). Thus, one has the following

Lemma 3.1.3. Consider $\mathbb{W}$, $\langle\ ,\ \rangle$, Γ_n, $n \in \{1, 2, 3\}$ as above. Then $(\mathbb{W}, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ is a hyperkähler manifold.

Proof. First, note that $\langle\ ,\ \rangle$ is compatible with Γ_n, $n \in \{1,2,3\}$. Indeed, let $x_1, x_2 \in T\mathbb{W}$. Then

$$\langle x_1, x_2\rangle = \langle\sqrt{-1}x_1, \sqrt{-1}x_2\rangle = \langle\Gamma_1(x_1), \Gamma_1(x_2)\rangle.$$

Moreover,

$$\begin{aligned}
\langle\Gamma_2(x_1), \Gamma_2(x_2)\rangle &= \langle(a_1, b_1, i_1, j_1, a_1', b_1', f_1, g_1), (a_2, b_2, i_2, j_2, a_2', b_2', f_2, g_2)\rangle \\
&= \langle(-b_1^\dagger, a_1^\dagger, -j_1^\dagger, i_1^\dagger, -b_1'^\dagger, a_1'^\dagger, -g_1^\dagger, f_1^\dagger), (-b_2^\dagger, a_2^\dagger, -j_2^\dagger, i_2^\dagger, -b_2'^\dagger\prime\dagger_2, -g_2^\dagger, f_2^\dagger)\rangle \\
&= \frac{1}{2}tr(b_1^\dagger(b_2^\dagger)^\dagger + b_2^\dagger(b_1^\dagger)^\dagger + a_1^\dagger(a_2^\dagger)^\dagger + a_2^\dagger(a_1^\dagger)^\dagger + (j_2^\dagger)^\dagger j_1^\dagger) + (j_1^\dagger)^\dagger j_2^\dagger) + i_1^\dagger(i_2^\dagger)^\dagger + \\
&i_2^\dagger(i_1^\dagger)^\dagger + b_1'^\dagger(b_2'^\dagger)^\dagger + b_2'^\dagger(b_1'^\dagger)^\dagger + a_1'^\dagger(a_2'^\dagger)^\dagger + a_2'^\dagger(a_1'^\dagger)^\dagger + (f_2^\dagger)^\dagger f_1^\dagger) + (f_1^\dagger)^\dagger f_2^\dagger) + \\
&g_1^\dagger(g_2^\dagger)^\dagger + g_2^\dagger(g_1^\dagger)^\dagger) \\
&= \frac{1}{2}tr(a_1a_2^\dagger + a_2a_1^\dagger + b_1b_2^\dagger + b_2b_1^\dagger + i_1i_2^\dagger + i_2i_1^\dagger + j_2^\dagger j_1 + j_1^\dagger j_2 + \\
&a_1'a_2'^\dagger + a_2'a_1'^\dagger + b_1'b_2'^\dagger + b_2'b_1'^\dagger + f_2^\dagger f_1 + f_1^\dagger f_2 + g_1g_2'^\dagger + g_2g_1'^\dagger) \\
&= \langle x_1, x_2\rangle.
\end{aligned}$$

and

$$\begin{aligned}
\langle\Gamma_3(x_1), \Gamma_3(x_2)\rangle &= \langle\Gamma_1\Gamma_2(x_1), \Gamma_1\Gamma_2(x_2)\rangle \\
&= \langle\Gamma_2(x_1), \Gamma_2(x_2)\rangle \\
&= \langle(x_1), (x_2)\rangle
\end{aligned}$$

Define the following 2-forms, $\omega_n(x_1, x_2) := \langle\Gamma_n(x_1), x_2\rangle$, for all $n \in \{1,2,3\}$. Since $\langle\ ,\ \rangle$ is non-degenerate, these 2-forms are non-degenerate too. Moreover, they are antisymmetric. In fact, since $\langle\ ,\ \rangle$ is compatible with the complex structure Γ_n, for all $n \in \{1,2,3\}$, one has from (3.4)

$$\begin{aligned}
\omega_n(x_1, x_2) &= \langle\Gamma_n(x_1), x_2\rangle \\
&= \langle\Gamma_n\Gamma_n(x_1), \Gamma_n x_2\rangle \\
&= \langle-(x_1), \Gamma_n x_2\rangle \\
&= -\langle\Gamma_n(x_2), x_1\rangle \\
&= -\omega_n(x_2, x_1).
\end{aligned}$$

Therefore, ω_n is a Kähler form for all $n \in \{1,2,3\}$ and $(\mathbb{W}, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ is in fact a hyperkähler variety. □

Moreover, the $\mathcal{U}$-action (3.2) satisfies

$$\langle(h,h')\cdot u, (h,h')\cdot v\rangle = \langle u, v\rangle$$

for all $(h,h') \in \mathcal{U}$ and preserves the complex structures Γ_n, $n \in \{1,2,3\}$, i.e., it satisfies

$$\Gamma_n((h,h')\cdot u) = (h,h')\cdot \Gamma_n(u)$$

for all $n \in \{1,2,3\}$ and for all $(h,h') \in \mathcal{U}$. Recall that

$$\operatorname{tr}(abc) = \operatorname{tr}(bca) = \operatorname{tr}(cab) \tag{3.6}$$

and

$$\begin{cases} h^{\dagger} &= h^{-1} \\ h'^{\dagger} &= h'^{-1} \end{cases}, \tag{3.7}$$

for all $h \in U(V)$ and $h' \in U(V')$. Thus, let $u = (a,b,i,j,a',b',f,g)$

$$\begin{aligned}
\Gamma_1((h,h')\cdot u) &= \Gamma_1((hah^{-1}, hbh^{-1}, hi, jh^{-1}, h'a'h'^{-1}, h'b'h'^{-1}, hfh'^{-1}, h'gh^{-1})) \\
&= (h(\sqrt{-1}a)h^{-1}, h(\sqrt{-1}b)h^{-1}, h(\sqrt{-1}i), (\sqrt{-1}j)h^{-1}, \\
&\quad h'(\sqrt{-1}a')h'^{-1}, h'(\sqrt{-1}b')h'^{-1}, h(\sqrt{-1}f)h'^{-1}, h'(\sqrt{-1}g)h^{-1}) \\
&= (h,h')\cdot(\sqrt{-1}a, \sqrt{-1}b, \sqrt{-1}i, \sqrt{-1}j, \sqrt{-1}a', \sqrt{-1}b', \sqrt{-1}f, \sqrt{-1}g) \\
&= (h,h')\cdot\Gamma_1(u),
\end{aligned}$$

$$\begin{aligned}
\Gamma_2((h,h')\cdot u) =&\Gamma_2((hah^{-1}, hbh^{-1}, hi, jh^{-1}, h'a'h'^{-1}, h'b'h'^{-1}, hfh'^{-1}, h'gh^{-1})) \\
=&\Gamma_2((hah^{\dagger}, hbh^{\dagger}, hi, jh^{\dagger}, h'a'h'^{\dagger}, h'b'h'^{\dagger}, hfh'^{\dagger}, h'gh^{\dagger})) \\
=&(-(hbh^{\dagger})^{\dagger}, (hah^{\dagger})^{\dagger}, -(jh^{\dagger})^{\dagger}, (hi)^{\dagger}, -(h'b'h'^{\dagger})^{\dagger}, (h'a'h'^{\dagger})^{\dagger}, -(h'gh^{\dagger})^{\dagger}, \\
&(hfh'^{\dagger})^{\dagger}) \\
=&(h(-b^{\dagger})h^{\dagger}, h(a^{\dagger})h^{\dagger}, h(-j^{\dagger}), (i^{\dagger})h^{\dagger}, h'(-b'^{\dagger})h'^{\dagger}, h'(a'^{\dagger})h'^{\dagger}, h(-g^{\dagger})h'^{\dagger}, \\
&h'(f^{\dagger})h^{\dagger}) \\
=&(h(-b^{\dagger})h^{-1}, h(a^{\dagger})h^{-1}, h(-j^{\dagger}), (i^{\dagger})h^{-1}, h'(-b'^{\dagger})h'^{-1}, h'(a'^{\dagger})h'^{-1}, \\
&h(-g^{\dagger})h'^{-1}, h'(f^{\dagger})h^{-1}) \\
&= (h,h')\cdot(-b^{\dagger}, a^{\dagger}, -j^{\dagger}, i^{\dagger}, -b'^{\dagger}, a'^{\dagger}, -g^{\dagger}, f^{\dagger}) \\
&= (h,h')\cdot\Gamma_2(u)
\end{aligned}$$

and finally,

$$\begin{aligned}
\Gamma_3((h,h')\cdot u) &= \Gamma_1(\Gamma_2((h,h')\cdot u)) \\
&= \Gamma_1((h,h')\cdot\Gamma_2(u)) \\
&= (h,h')\cdot\Gamma_1\Gamma_2(u) \\
&= (h,h')\cdot\Gamma_3(u)
\end{aligned}$$

Moreover, let $x_1 = (a_1, b_1, i_1, j_1, a_1', b_1', f_1, g_1)$ and $x_2 = (a_2, b_2, i_2, j_2, a_2', b_2', f_2, g_2)$. Then it follows from the equations (3.6) and (3.7) that

$$\begin{aligned}
\langle (h,h')\cdot x_1, (h,h')\cdot x_2\rangle =& \langle (ha_1h^{-1}, hb_1h^{-1}, hi_1, j_1h^{-1}, h'a_1'h'^{-1}, h'b_1'h'^{-1}, hf_1h'^{-1}, h'g_1h^{-1}),\\
& (ha_2h^{-1}, hb_2h^{-1}, hi_2, j_2h^{-1}, h'a_2'h'^{-1}, h'b_2'h'^{-1}, hf_2h'^{-1}, h'g_2h^{-1})\rangle\\
=& \frac{1}{2}tr(ha_1h^{-1}((ha_2h^{-1})^\dagger, ha_2h^{-1}(ha_1h^{-1})^\dagger +\\
& hb_1h^{-1}(hb_2h^{-1})^\dagger, hb_2h^{-1}(hb_1h^{-1})^\dagger + hi_1h^{-1}(hi_2h^{-1})^\dagger,\\
& hi_2h^{-1}(hi_1h^{-1})^\dagger + (hj_2h^{-1})^\dagger hj_1h^{-1}, (haJ_1h^{-1})^\dagger hj_2h^{-1} +\\
& ha_1'h^{-1}(ha_2'h^{-1})^\dagger, ha_2'h^{-1}(ha_1'h^{-1})^\dagger + hb_1'h^{-1}(hb_2'h^{-1})^\dagger,\\
& hb_2'h^{-1}(hb_1'h^{-1})^\dagger + (hf_2h'^{-1})^\dagger hf_1h'^{-1}, (hf_1h'^{-1})^\dagger hf_2h'^{-1} +\\
& h'g_1h^{-1}(h'g_2h^{-1})^\dagger, h'g_2h^{-1}(h'g_1h^{-1})^\dagger)\\
=& \frac{1}{2}tr(h(a_1a_2^\dagger)h^{-1}), h(a_2a_1^\dagger)h^{-1} + h(b_1b_2^\dagger)h^{-1}), h(b_2b_1^\dagger)h^{-1} +\\
& h(i_1i_2^\dagger)h^{-1}), h(i_2i_1^\dagger)h^{-1} + (h(j_2^\dagger j_1)h^{-1}), h(j_1^\dagger j_2)h^{-1} +\\
& h'(a_1'a_2'^\dagger)h'^{-1}), h'(a_2'a_1'^\dagger)h'^{-1} + h'(b_1'b_2'^\dagger)h'^{-1}), h'(b_2'b_1'^\dagger)h'^{-1} +\\
& h'(f_1f_2^\dagger)h'^{-1}), h'(f_2f_1^\dagger)h'^{-1} + (h(g_2^\dagger g_1)h^{-1}), h(g_1^\dagger g_2)h^{-1})\\
=& \frac{1}{2}tr(a_1a_2^\dagger + a_2a_1^\dagger + b_1b_2^\dagger + b_2b_1^\dagger + i_1i_2^\dagger + i_2i_1^\dagger + j_2^\dagger j_1 + j_1^\dagger j_2 +\\
& a_1'a_2'^\dagger + a_2'a_1'^\dagger + b_1'b_2'^\dagger + b_2'b_1'^\dagger + f_2^\dagger f_1 + f_1^\dagger f_2 + g_1g_2'^\dagger + g_2g_1'^\dagger)\\
=& \langle x_1, x_2\rangle.
\end{aligned}$$

Let

$$(\xi, \xi') \in \mathfrak{u} = \mathfrak{u}(V) \times \mathfrak{u}(V') := \{(\xi,\xi') \in GL(V) \times GL(V'); \xi + \xi^\dagger = \xi' + \xi'^\dagger = 0\}.$$

One can compute the fundamental vector field $(W_{(\xi,\xi')})$ as following.
Let $W \in \mathbb{W}$ and

$$(W_{(\xi,\xi')})_W = d\Psi_W(1_V, 1_{V'})(\xi,\xi')$$

where

$$\begin{array}{rlcl}\Psi_W: & \mathcal{U} & \longrightarrow & \mathbb{W}\\ & (h,h') & \longmapsto & (h,h')\cdot W\end{array}$$

and consider the smooth curve

$$\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{U}$$

given by the ODE

$$\begin{cases}\gamma(0) & = & (1_V, 1_{V'})\\ \dfrac{d}{dt}(\gamma)|_{t=0} & = & (\xi,\xi')\end{cases}.$$

Thus,

$$\begin{aligned}\frac{d}{dt}(\Psi_W \circ \gamma)|_{t=0} &= (d\Psi_w(\gamma(0)) \cdot \frac{d}{dt}(\gamma(0))) \\ &= d\Psi_W(1_V, 1_{V'})(\xi, \xi') \\ &= (W_{\xi,\xi'})_W\end{aligned}$$

Consider

$$\begin{array}{rlcl}\pi_1: & \mathcal{U} & \longrightarrow & U(V) \\ & (h,h') & \longmapsto & h\end{array}, \qquad \begin{array}{rlcl}\pi_2: & \mathcal{U} & \longrightarrow & U(V') \\ & (h,h') & \longmapsto & h'\end{array}.$$

Since $\gamma(t) \in \mathcal{U}$, for all $t \in (-\epsilon, \epsilon)$, one has

$$\gamma(t)^\dagger = \gamma(t)^{-1},$$

for all $t \in (-\epsilon, \epsilon)$. Moreover,

$$\frac{d}{dt}\gamma(t)^\dagger = -\frac{d}{dt}\gamma(t),$$

for all $t \in (-\epsilon, \epsilon)$. Hence,

$$\begin{aligned}\Psi_x(\gamma(t)) = (&\gamma(t)a\gamma(t)^{-1}, \gamma(t)b\gamma(t)^{-1}, \gamma(t)i, j\gamma(t)^{-1}, \\ &\gamma(t)'a'\gamma(t)'^{-1}, \gamma(t)'b'\gamma(t)'^{-1}, \gamma(t)f\gamma(t)'^{-1}, \gamma(t)'g\gamma(t)^{-1}),\end{aligned}$$

where on the right side of the equation above, $\gamma(t)$ and $\gamma(t)'$ denote $\pi_1(\gamma(t))$ and $\pi_2(\gamma(t))$, respectively. Therefore, using the same notation,

$$\begin{aligned}(W_{(\xi,\xi')})_W =& \frac{d}{dt}(\Psi_W \circ \gamma)|_{t=0} \\ =&(\frac{d}{dt}\gamma(t)a\gamma(t)^{-1} + \gamma(t)a\frac{d}{dt}\gamma(t)^{-1}, \frac{d}{dt}\gamma(t)b\gamma(t)^{-1} + \gamma(t)b\frac{d}{dt}\gamma(t)^{-1}, \\ &\frac{d}{dt}\gamma(t)i, j\frac{d}{dt}\gamma(t)^{-1}, \frac{d}{dt}\gamma(t)'a'\gamma(t)'^{-1} + \gamma(t)'a'\frac{d}{dt}\gamma(t)'^{-1}, \\ &\frac{d}{dt}\gamma(t)'b'\gamma(t)'^{-1} + \gamma(t)'b'\frac{d}{dt}\gamma(t)'^{-1}, \frac{d}{dt}\gamma(t)f\gamma(t)'^{-1} + \gamma(t)f\frac{d}{dt}\gamma(t)'^{-1}, \\ &\frac{d}{dt}\gamma(t)'g\gamma(t)^{-1} + \gamma(t)'g\frac{d}{dt}\gamma(t)^{-1})|t = 0 \\ &= (\xi a - a\xi, \xi b - b\text{ß}, \xi i, -j\xi, \xi a' - a'\xi, \xi b' - b'\xi, \xi f - f\xi', \xi' g - g\xi) \\ &= ([\xi, a], [\xi, b], \xi i, -j\xi, [\xi, a'], [\xi, b'], \xi f - f\xi', \xi' g - g\xi)\end{aligned}$$

Now, in order to construct a moment map, one can prove that the Kähler forms ω_n, $n \in \{1,2,3\}$ are exact, i.e. that there exists a 1-form θ_n in $\mathbb{W}$ such that $\omega_n = d\theta_n$, for all $n \in \{1,2,3\}$. First, note that for all $x_1,\, x_2 \in T\mathbb{W}$

$$
\begin{aligned}
\omega_1(x_1,x_2) :=& \langle \Gamma_1 x_1, x_2\rangle \\
=& \langle (\sqrt{-1}a_1, \sqrt{-1}b_1, \sqrt{-1}i_1, \sqrt{-1}j_1, \sqrt{-1}a'_1, \sqrt{-1}b'_1, \sqrt{-1}f_1, \sqrt{-1}g_1), \\
& (a_2, b_, 2i_2, j_2, a'_2, b'_2, f_2, g_2)\rangle \\
=& \frac{1}{2} tr(\sqrt{-1}a_1 a_2^\dagger + a_2(\sqrt{-1}a_1)^\dagger + \sqrt{-1}b_1 b_2^\dagger + b_2(\sqrt{-1}b_1)^\dagger + \sqrt{-1}i_1 i_2^\dagger + \\
& i_2(\sqrt{-1}i_1)^\dagger + \sqrt{-1}j_2^\dagger j_1 + (\sqrt{-1}j_1)^\dagger j_2 \sqrt{-1}a'_1 a_2'^\dagger + a'_2(\sqrt{-1}a'_1)^\dagger + \\
& + \sqrt{-1}b'_1 b_2'^\dagger + b'_2(\sqrt{-1}b'_1)^\dagger + \sqrt{-1}g_1 g_2^\dagger + g_2(\sqrt{-1}g_1)^\dagger + \sqrt{-1}f_2^\dagger f_1 + (\sqrt{-1}f_1)^\dagger f_2) \\
=& \frac{\sqrt{-1}}{2} tr(a_1 a_2^\dagger - a_2 a_1^\dagger + b_1 b_2^\dagger - b_2 b_1^\dagger + i_1 i_2^\dagger - i_2 i_1^\dagger + j_2^\dagger j_1 - j_1^\dagger j_2 + \\
& a'_1 a_2'^\dagger - a'_2 a_1'^\dagger + b'_1 b_2'^\dagger - b'_2 b_1'^\dagger + f_2^\dagger f_1 - f_1^\dagger f_2 + g_1 g_2'^\dagger - g_2 g_1'^\dagger)
\end{aligned}
$$

Let $\pi_1 : \mathbb{W} \longrightarrow \mathbb{W}$ given by $\pi_1(a,b,i,j,a',b',f,g) = (a,0,0,0,0,0,0,0)$, one can introduce the following 2-form

$$
d\pi_1 \wedge d\pi_1^\dagger((a_1,b_1,i_1,j_1,a'_1,b'_1,f_1,g_1),(a_1,b_2,i_2,j_2,a'_2,b'_2,f_2,g_2)) = a_1 a_2^\dagger - a_2 a_1^\dagger.
$$

Defining $\pi_i : \mathbb{W} \longrightarrow \mathbb{W}$ as the projection in the i-th coordinate, one can write ω_1 as

$$
\begin{aligned}
\omega_1 =& \frac{\sqrt{-1}}{2} tr(d\pi_1 \wedge d\pi_1^\dagger + d\pi_2 \wedge d\pi_2^\dagger + d\pi_3 \wedge d\pi_3^\dagger + d\pi_4 \wedge d\pi_4^\dagger \\
& + d\pi'_5 \wedge d\pi_5'^\dagger + d\pi'_6 \wedge d\pi_6'^\dagger + d\pi_7 \wedge d\pi_7^\dagger + d\pi_8 \wedge d\pi_8^\dagger).
\end{aligned}
$$

Therefore, θ_1 if you put

$$
\begin{aligned}
\theta_1 =& \frac{\sqrt{-1}}{2} tr(\pi_1 \wedge d\pi_1^\dagger + \pi_2 \wedge d\pi_2^\dagger + \pi_3 \wedge d\pi_3^\dagger + \pi_4 \wedge d\pi_4^\dagger \\
& + \pi'_5 \wedge d\pi_5'^\dagger + \pi'_6 \wedge d\pi_6'^\dagger + \pi_7 \wedge d\pi_7^\dagger + \pi_8 \wedge d\pi_8^\dagger),
\end{aligned}
$$

$\omega_1 = d\theta_1$. Thus, it follows from Lemma 3.1.1 that one can find the moment map $\mu_1(W)(\xi,\xi') := -\theta_1(W_{(\xi,xi')})_W$. Recall that since $(\xi \in \mathfrak{u}(V))$, $\xi^\dagger = -\xi$, and then $[\xi,a]^\dagger = (\xi a)^\dagger - (a\xi)^\dagger$. Analogously, $[\xi',a']^\dagger = (\xi' a')^\dagger - (a'\xi')^\dagger$. Moreover,

$$
\begin{aligned}
\mu_1(w)(\xi,\xi') =& -\theta_1(W_{(\xi,\xi')})_w \\
=& \frac{-\sqrt{-1}}{2} tr(a[\xi,a]^\dagger + b[\xi,b]^\dagger + i(\xi i)^\dagger + j(-j\xi)^\dagger \\
& a'[\xi',a']^\dagger + b'[\xi',b']^\dagger + f(\xi f - f\xi')^\dagger) + g(\xi' g - g\xi)^\dagger) \\
=& \frac{1}{2\sqrt{-1}} tr(a((\xi a)^\dagger - (a\xi)^\dagger) + b(\xi b)^\dagger - (b\xi)^\dagger + i i^\dagger \xi^\dagger - j\xi^\dagger j^\dagger \\
& a'(\xi' a')^\dagger - (a'\xi')^\dagger + b'(\xi' b')^\dagger - (b'\xi')^\dagger + f f^\dagger \xi^\dagger - f\xi'^\dagger f^\dagger + g g^\dagger \xi'^\dagger - g\xi' g\dagger \\
& \frac{1}{2\sqrt{-1}}[tr([a,a^\dagger] + [b,b^\dagger] + i i^\dagger - j^\dagger j + f f^\dagger - g^\dagger g)\xi + \\
& tr([a',a'^\dagger] + [b',b'^\dagger] - f^\dagger f + g g^\dagger)\xi'].
\end{aligned}
$$

Identifying $\mathfrak{u}(V)\times\mathfrak{u}(V')\cong\mathfrak{u}(V)^*\times\mathfrak{u}(V')^*$ via the inner product $(a,b)=tr(ab^\dagger)$, one obtains the moment map

$$\begin{aligned}\mu_1(x)=&((\frac{1}{2\sqrt{-1}}[a,a^\dagger]+[b,b^\dagger]+ii^\dagger-j^\dagger j+ff^\dagger-g^\dagger g)),\\&(\frac{1}{2\sqrt{-1}})([a',a'^\dagger]+[b',b'^\dagger]-f^\dagger f+gg^\dagger)).\end{aligned}$$

Repeating this procedure for ω_2 and ω_3, one can find

$$\begin{aligned}\omega_2(x_1,x_2)=&-\frac{1}{2}tr(a_2b_1-b_2a_1+b_1^\dagger a_2^\dagger-a_1^\dagger b_2^\dagger+i_2j_1-j_2i_1+j_1^\dagger i_2^\dagger-i_1^\dagger j_2^\dagger\\&+b_1'^\dagger a_2'^\dagger-a_1'^\dagger b_2'^\dagger+a_2'b_1'-b_2'a_1'+f_2g_1+g_1^\dagger f_2^\dagger-g_2f_1-f_1^\dagger g_2^\dagger),\end{aligned}$$

$$\begin{aligned}\omega_3(x_1,x_2)=&-\frac{-\sqrt{-1}}{2}tr(-a_2b_1+b_2a_1+b_1^\dagger a_2^\dagger-a_1^\dagger b_2^\dagger-i_2j_1+j_2i_1+j_1^\dagger i_2^\dagger-i_1^\dagger j_2^\dagger\\&+b_1'^\dagger a_2'^\dagger-a_1'^\dagger b_2'^\dagger-a_2'b_1'+b_2'a_1'+g_1^\dagger f_2^\dagger-f_1g_1-f_1^\dagger g_2^\dagger+g_2f_1)\end{aligned}$$

and the moment maps

$$\begin{aligned}\mu_2(x)=&\frac{-1}{2}(([a,b]+[a^\dagger,b^\dagger+ij-j^\dagger i^\dagger+fg-g^\dagger f^\dagger]),\\&([a',b']+[a^\dagger,b^\dagger]-gf+f^\dagger g^\dagger))\end{aligned}$$

and

$$\begin{aligned}\mu_3(x)=&\frac{-1}{2\sqrt{-1}}((([a,b]-[a^\dagger,b^\dagger+ij+j^\dagger i^\dagger+fg+g^\dagger f^\dagger]),\\&([a',b']-[a^\dagger,b^\dagger]-gf-f^\dagger g^\dagger)).\end{aligned}$$

Then, defining the moment map

$$\begin{aligned}\mu_{\mathbb{C}}(x)=&(\mu_2+\sqrt{-1}\mu_3)(x)\\=&-(([a,b]+ij+fg),[a',b']-gf).\end{aligned}$$

it follows from Theorem 3.1.2 that

$$\mathcal{W}=\frac{\mu_1^{-1}(0)\cap\mu_{\mathbb{C}}^{-1}(0)\cap\mathbb{W}}{\mathcal{U}}$$

is a hyperkähler variety, since the $\mathcal{U}$-action acts freely on the stable points of $\mathbb{W}$. Moreover, its real dimension is given by

$$\dim_{\mathbb{R}}(\mathcal{W})=4(c^2+rc+c'^2+cc')-4(c^2-c'^2)=4(rc+cc')$$

This concludes the proof that the moduli space $\mathcal{N}^{st}(r,c,c')$ is a subvariety of the hyperhähler manifold $(\mathcal{W},\langle\ ,\ \rangle,\Gamma_1,\Gamma_2,\Gamma_3)$.

3.2 Holomorphic structures on $\mathcal{N}^{st}(r, c, 1)$

Let M be a complex manifold. We define *holomorphic pre-symplectic variety* as the triple (M, Γ, Ω), where Ω is a holomorphic closed 2-form, non-degenerated in an open dense subset defined on TM and Γ is a complex structure on M. Let $(M, \langle\ ,\ \rangle, \Gamma_1)$ be a Kähler manifold. Let ω_2 and ω_3 be two closed, not necessarily non-degenerated, 2-forms defined on M satisfying

$$\omega_2(u,v) \ := \ g(Ju,v) = g(IJu,Iv) = g(Ku,Iv) := \omega_3(u,Iv); \tag{3.8}$$

$$\omega_3(u,v) \ := \ g(Ku,v) = g(IIJu,Iv) = -g(Ju,Iv) := \omega_2(u,Iv), \tag{3.9}$$

then $(M, \langle\ ,\ \rangle, \Gamma_1, \omega_2, \omega_3)$ is called *pre-hyperkähler variety*. It is easy to check that given a pre-hyperkähler manifold, $(M, \langle\ ,\ \rangle, \Gamma_1, \omega_2, \omega_3)$, one can obtain that (M, Γ_1, Ω), with $\Omega := \omega_2 + \sqrt{-1}\omega_3$, is a holomorphic pre-symplectic manifold.

The reciprocate is not proved, i.e., if (M, Γ_1, Ω) has a holomorphic pre-symplectic structure, is unknown if there exists unique pre-hyperkähler structure $(M, \langle\ ,\ \rangle, \Gamma_1, \omega_2, \omega_3)$ such that $\Omega = \omega_2 + \sqrt{-1}\omega_3$. However, if M is compact, ω_2 and ω_3 are symplectic forms, Ω is non-degenerate and $(M, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$ a hyperkähler manifold where Γ_n, for $n \in \{2, 3\}$ is such that $\omega_n(\cdot,\cdot) = \langle \Gamma_n \cdot, \cdot\rangle$, this is true and the reader can find a proof of this in [8, Theorem 5.11, p. 26; Theorem 23.5, p. 179].

In this section, one can find the consequences of the fact that the moduli space $\mathcal{N}(r, c, 1)$ is a subvariety of the hyperkähler manifold $\mathcal{W}(r, c, 1) = (\mathcal{W}, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3)$. It was proved in the last section that this is true for the general case $\mathcal{N}(r, c, c')$. However, here it is fixed the moduli space of framed stable representations of the ADHM quiver of numerical type $(r, c, 1)$, because this is the only case in which the variety is smooth. First, note that there exists the inclusion map

$$\mathcal{N}^{st}(r, c, 1) \xhookrightarrow{\ \iota\ } (\mathcal{W}, \langle\ ,\ \rangle, \Gamma_1, \Gamma_2, \Gamma_3).$$

Hence, associated with this inclusion, there exists a complex structure on $\mathcal{N}^{st}(r, c, 1)$ inherited by the pull-back, $\iota^*\Gamma_1$, and a closed degenerate 2-form $\Omega = \iota^*\omega_2 + \sqrt{-1}\iota^*\omega_3$. Indeed, let $(a, b, i, j, a', b', f, 0) \in \mathcal{N}^{st}(r, c, 1)$. Thus,

$$\begin{aligned}\iota^*\Gamma_1(a, b, i, j, a', b', f, 0) =& \Gamma_1(\iota_* a, \iota_* b, \iota_* i, \iota_* j, \iota_* a', \iota_* b', \iota_* f, 0)\\ =& (\sqrt{-1}a, \sqrt{-1}b, \sqrt{-1}i, \sqrt{-1}j, \sqrt{-1}a', \sqrt{-1}b', \sqrt{-1}f, 0)\end{aligned}$$

is clearly a complex structure on $\mathcal{N}^{st}$. Moreover, let $x_1 = (a_1, b_1, i_1, j_1, a_1', b_1', f_1, 0)$ and $x_2 = (a_2, b_2, i_2, j_2, a_2', b_2', f_2, 0)$ in $\mathcal{N}^{st}$. It is easy to check that $(\mathcal{N}^{st}, \iota^*\langle\ ,\ \rangle, \iota^*\Gamma_1)$ has a

Kähler structure. The 2-form Ω is given by,

$$\begin{aligned}\Omega(x_1,x_2) =&(\iota^*\omega_2+\sqrt{-1}\iota^*\omega_3)(x_1,x_2)\\ =&(\omega_2+\sqrt{-1}\omega_3)(\iota_*x_1,\iota_*x_2)\\ =&(\omega_2+\sqrt{-1}\omega_3)((a_1,b_1,i_1,j_1,a_1',b_1',f_1,0),(a_2,b_2,i_2,j_2,a_2',b_2',f_2,0))\\ =&-\frac{1}{2}tr(a_2b_1-b_2a_1+b_1^\dagger a_2^\dagger-a_1^\dagger b_2^\dagger+i_2j_1-j_2i_1+j_1^\dagger i_2^\dagger-i_1^\dagger j_2^\dagger\\ &+b_1'^\dagger a_2'^\dagger-a_1'^\dagger b_2'^\dagger+a_2'b_1'-b_2'a_1')-\\ &-\frac{(\sqrt{-1})^2}{2}tr(-a_2b_1+b_2a_1+b_1^\dagger a_2^\dagger-a_1^\dagger b_2^\dagger-i_2j_1+j_2i_1+j_1^\dagger i_2^\dagger-i_1^\dagger j_2^\dagger\\ &+b_1'^\dagger a_2'^\dagger-a_1'^\dagger b_2'^\dagger-a_2'b_1'+b_2'a_1')\\ =&\frac{1}{2}tr(-a_2b_1+b_2a_1-b_1^\dagger a_2^\dagger+a_1^\dagger b_2^\dagger-i_2j_1+j_2i_1-j_1^\dagger i_2^\dagger+i_1^\dagger j_2^\dagger\\ &-b_1'^\dagger a_2'^\dagger+a_1'^\dagger b_2'^\dagger-a_2'b_1'+b_2'a_1'+b_1'^\dagger a_2'^\dagger-a_1'^\dagger b_2'^\dagger-a_2'b_1'+b_2'a_1'-\\ &-a_2b_1+b_2a_1+b_1^\dagger a_2^\dagger-a_1^\dagger b_2^\dagger-i_2j_1+j_2i_1+j_1^\dagger i_2^\dagger-i_1^\dagger j_2^\dagger)\\ &=tr(-a_2b_1+b_2a_1-i_2j_1+i_1j_2-a_2'b_1'+b_2'a_1').\end{aligned}$$

Note that taking $u=(0,0,0,0,0,0,f,0)\in T\mathcal{N}^{st}$, $\Omega_X(u,v)\equiv 0$ for all $v\in T\mathcal{N}^{st}$, i.e., Ω is in fact a degenerate 2-form. Also, it is easy to check that the 2-forms $\iota^*\omega_2$ and $\iota^*\omega_3$ satisfy

$$\begin{cases}\iota^*\omega_2(u,v) &= \iota^*\omega_3(u,\Gamma_1 v)\\ \iota^*\omega_3(u,v) &= -\iota^*\omega_2(u,\Gamma_1 v)\end{cases}.$$

Given the 2-form

$$\Omega: T\mathcal{N}^{st}(r,c,1)\times T\mathcal{N}^{st}(r,c,1)\longrightarrow\mathbb{C}$$

as above, one can define the maps

$$\begin{array}{rccc}\Omega_X: & T_X\mathcal{N}^{st}(r,c,1)\times T_X\mathcal{N}^{st}(r,c,1) & \longrightarrow & \mathcal{C}\\ & (u,v) & \longmapsto & \Omega(u_X,v_X)\end{array}$$

and

$$\begin{array}{rccc}\Omega_X(u,\cdot): & T_X\mathcal{N}(r,c,1) & \longrightarrow & \mathcal{C}\\ & v & \longmapsto & \Omega_X(u,v)\end{array}$$

It is still unknown in which points Ω_X is non-degenerate in the general case. However, the investigation for the particular case for Ω_X defined in $T_X\mathcal{N}(1,2,1)\times T_X\mathcal{N}(1,2,1)$ is done as one can see below. First, we will proof an auxiliary Lemma.

Lemma 3.2.1. Let $X=(W,V,V',A,B,I,J,A',B',F,G)$ be a framed stable representation of the enhanced ADHM quiver of numerical type $(1,2,1)$. Thus, there exists a change of basis for V such that

(i) $A = \begin{bmatrix} A' & 0 \\ 0 & a_2 \end{bmatrix}$, $B = \begin{bmatrix} B' & 0 \\ 0 & b_2 \end{bmatrix}$, $F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, if A and B are diagonalizable;

(ii) $A = \begin{bmatrix} A' & 1 \\ 0 & A' \end{bmatrix}$, $B = \begin{bmatrix} B' & B_{12} \\ 0 & B' \end{bmatrix}$, $F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, if A and B are not diagonalizable;

(iii) $A = A' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} B' & 1 \\ 0 & B' \end{bmatrix}$, $F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, if A is diagonalizable and B is not diagonalizable.

Proof. First, note that

$$\begin{cases} AF(1) - F(1)A' = 0 \\ BF(1) - F(1)B' = 0 \end{cases},$$

i.e., $F(1) \in V$ is an eigenvector of A and B. Also, A' and B' are eigenvalues of A and B, respectively, associated with the vector $F(1)$. Suppose that both A and B are diagonalizable. Therefore, there exists a vector $w \in V$ that satisfies

$$Aw - a_2 \cdot 1_V w = 0,$$

where $A' \neq a_2 \in V'$. There exists a change of basis for V such that

$$F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, we obtain

$$0 = AF - FA' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} A' = \begin{bmatrix} a_{11} - A' \\ a_{21} \end{bmatrix}.$$

Hence

$$A = \begin{bmatrix} A' & a_{12} \\ 0 & a_{22} \end{bmatrix}.$$

Analogously, we obtain

$$B = \begin{bmatrix} B' & b_{12} \\ 0 & b_{22} \end{bmatrix}.$$

Since a_2 is an eigenvalue of A associated with the vector $w \in V$, we get

$$0 = Aw - a_2 \cdot 1_V w = \begin{bmatrix} A' & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} - a_2 \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} A' & 0 \\ 0 & a_2 \end{bmatrix}.$$

Since $r = 1$, we have $J = 0$. Hence

$$0 = [A, B] = \begin{bmatrix} A' & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} B' & b_{12} \\ 0 & b_{22} \end{bmatrix} - \begin{bmatrix} B' & b_{12} \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & a_2 \end{bmatrix}.$$

$$= \begin{bmatrix} A'B' - B'A' & A'b_{12} - b_{12}a_2 \\ 0 & a_2b_{22} - b_{22}a_2 \end{bmatrix} = \begin{bmatrix} 0 & b_{12}(A' - a_2) \\ 0 & 0 \end{bmatrix}$$

Thus $b_{12} = 0$ and

$$B = \begin{bmatrix} B' & 0 \\ 0 & b_{22} \end{bmatrix}.$$

If both A and B are not diagonalizable, there exists $w \in V$ such that $v := Aw - A' \cdot 1_V$ is an eigenvector of A.There exist a change of basis for V such that

$$F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$v = \begin{bmatrix} A' & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ A' \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} - A' \end{bmatrix}$$

Hence,

$$0 = Av - A' \cdot 1_V v = \begin{bmatrix} A' & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} - A' \end{bmatrix} - A' \begin{bmatrix} a_{12} \\ a_{22} - A' \end{bmatrix},$$

$$= \begin{bmatrix} a_{12}(a_{22} - A') \\ (a_{22} - A')^2 \end{bmatrix}$$

i.e., $a_{12} \neq 0$ and $a_{22} = A'$. Thus,

$$A = \begin{bmatrix} A' & a_{12} \\ 0 & A' \end{bmatrix}.$$

In the other hand,

$$0 = [A, B] = \begin{bmatrix} A' & a_{12} \\ 0 & A' \end{bmatrix} \begin{bmatrix} B' & b_{12} \\ 0 & b_{22} \end{bmatrix} - \begin{bmatrix} B' & b_{12} \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} A' & a_{12} \\ 0 & A' \end{bmatrix}$$

$$= \begin{bmatrix} A'B' - B'A' & A'b_{12} + b_{22} - B' - b_{12}A' \\ 0 & A'b_{22} - b_{22}A' \end{bmatrix} = \begin{bmatrix} 0 & b_{22} - B' \\ 0 & 0 \end{bmatrix}$$

Therefore

$$B = \begin{bmatrix} B' & b_{12} \\ 0 & B' \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix}.$$

Then

$$S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & a_{12} \end{bmatrix}$$

and we obtain

$$SAS^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix} \begin{bmatrix} A' & a_{12} \\ 0 & A' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_{12} \end{bmatrix} = \begin{bmatrix} A' & 1 \\ 0 & A' \end{bmatrix},$$

$$SBS^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix} \begin{bmatrix} B' & b_{12} \\ 0 & B' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_{12} \end{bmatrix} = \begin{bmatrix} B' & b_{12}a_{12} \\ 0 & B' \end{bmatrix},$$

$$SF = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Denoting $B_{12} = b_{12}a_{12}$, this concludes the proof of the case where A and B are both non-diagonalizable. The last case, A diagonalizable and B not diagonalizable, is entirely analogous. □

Now we can proof the next Proposition.

Proposition 3.2.2. Let $\mathcal{N}(1,2,1)$ be the moduli space of framed stable representations of the enhanced ADHM quiver of numerical type $(1,2,1)$. Fix a framed stable representation $X = (A, B, I, J, A', B', F)$. Then the 2-form Ω_X defined on $T_X\mathcal{N}(1,2,1)$ is non-degenerate if and only if the matrices associated with the endomorphisms A and B are diagonalizable.

Proof. Recall that if $r = 1$, then the map $J \in Hom(V, W)$ is null, since X is stable (see [13, Proposition 2.8]) and recall that if $c' = 1$, thus $[A', B'] = 0$, for all A', $B' \in V'$. Thus, the enhanced ADHM equations are reduced to

$$[A, B] = 0, \quad AF - FA' = 0, \quad BF - FB' = 0.$$

Suppose that A and B are not diagonalizable. Thus, it follows from Lemma 3.2.1 (ii) that there exists change of basis for V such that

$$A = \begin{bmatrix} A' & 1 \\ 0 & A' \end{bmatrix}, \quad B = \begin{bmatrix} B' & B_{12} \\ 0 & B' \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.10}$$

In order to $X = (A, B, I, J, A', B', F)$ is a stable representation of the enhanced ADHM quiver,

$$I = \begin{bmatrix} \mu \\ 1 \end{bmatrix}. \tag{3.11}$$

Indeed, F is clearly injective. Furthermore, Consider

$$I = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}.$$

Note that if $i_2 \neq 0$, then there exists a basis of W such that $i_2 = 1$ and

$$I = \begin{bmatrix} \mu \\ 1 \end{bmatrix} \tag{3.12}$$

for some $\mu \in \mathbb{C}$ Indeed, suppose that there exists $0 \subset S \subset V$ such that

$$A(S), B(S), I(W) \subset S.$$

Let $0 \neq v \in I(W)$, thus, there exists $w \in W$ such that

$$v = Iw = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \cdot w = \begin{bmatrix} i_1 w \\ i_2 w \end{bmatrix} \in S$$

Moreover, since $v \in S$, $A(v)$ $B(v) \in S$ by hypothesis. Hence

$$A(v) = \begin{bmatrix} A' & 1 \\ 0 & A' \end{bmatrix} \cdot \begin{bmatrix} i_1 w \\ i_2 w \end{bmatrix} = \begin{bmatrix} A' i_1 + i_2 \\ A' i_2 \end{bmatrix} \cdot w$$

Analogously,

$$B(v) = \begin{bmatrix} B' i_1 + B_{12} i_2 \\ B' i_2 \end{bmatrix} \cdot w$$

Then, if $i_2 = 0$, $S = \langle i_1 \rangle \subsetneq V$ is a subset such that $A(S), B(S), I(w) \subset S$. Indeed,

$$I(w) = \begin{bmatrix} i_1 w \\ 0 \end{bmatrix} \in S$$

for all $w \in W$, which means $I(W) \subset S$. Let $s \in S$ given by $s = \lambda i_1$. Thus,

$$A(s) = \begin{bmatrix} A' & 1 \\ 0 & A' \end{bmatrix} \cdot \begin{bmatrix} \lambda i_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A' \lambda i_1 \\ 0 \end{bmatrix} \in S$$

for all $s \in S$. And therefore, $A(S) \subset S$. Analogously, $B(S) \subset S$. Thus, in order to prove that X is stable, $i_2 \neq 0$ and then there exists a basis of W such that I is given by (3.12). This concludes that if X is a framed stable representation of the enhanced ADHM quiver, then A, B, I, A', B', F are of the form (3.10), (3.11).

Now, consider $v \in T_X\mathcal{N}$ given by $v = (a, b, i, j, a', b', f)$ such that X satisfies (3.10) and (3.11). Then, if follows from Theorem 2.4.3 and from the fact that $J = 0$ that

$$j = 0, \quad [a, B] + [A, b] = 0, \quad fA' + Fa' - aF - Af = 0, \quad fB' + Fb' - bF - Bf = 0.$$

Then, denoting

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad i = \begin{bmatrix} i_1 \\ i_1 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

one gets

$$\begin{aligned} fA' + Fa' - aF - Af &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \cdot A' + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot a' - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \\ &\quad - \begin{bmatrix} A' & 1 \\ 0 & A' \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} A'f_1 \\ A'f_2 \end{bmatrix} + \begin{bmatrix} a' \\ 0 \end{bmatrix} - \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} - \begin{bmatrix} A'f_1 + f_2 \\ f_2A' \end{bmatrix} \\ &= \begin{bmatrix} a' - a_{11} - f_2 \\ a_{21} \end{bmatrix} \end{aligned}.$$

Therefore,

$$fA' + Fa' - aF - Af = 0 \Leftrightarrow \begin{cases} a' &= a_{11} + f_2 \\ a_{21} &= 0 \end{cases}. \tag{3.13}$$

Analogously,

$$fB' + Fb' - bF - Bf = 0 \Leftrightarrow \begin{cases} b' &= b_{11} + B_{12}f_2 \\ b_{21} &= 0 \end{cases} \tag{3.14}$$

It follows from equations (3.13) and (3.14) that

$$\begin{aligned}
[a,B]+[A,b] &= \left[\begin{bmatrix} a_{11} & a_{12} \\ \underbrace{a_{21}}_{=0} & a_{22}\end{bmatrix}, \begin{bmatrix} B' & B_{12} \\ 0 & B'\end{bmatrix}\right] + \\
&+ \left[\begin{bmatrix} A' & 1 \\ 0 & A'\end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ \underbrace{b_{21}}_{=0} & b_{22}\end{bmatrix}\right] \\
&= \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{bmatrix}\cdot\begin{bmatrix} B' & B_{12} \\ 0 & B'\end{bmatrix} - \begin{bmatrix} B' & B_{12} \\ 0 & B'\end{bmatrix}\cdot\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{bmatrix} + \\
&\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22}\end{bmatrix}\cdot\begin{bmatrix} A' & 1 \\ 0 & A'\end{bmatrix} - \begin{bmatrix} A' & 1 \\ 0 & A'\end{bmatrix}\cdot\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22}\end{bmatrix} \\
&= \begin{bmatrix} a_{11}B' & B_{12}a_{11}+B'a_{12} \\ 0 & a_{22}B'\end{bmatrix} - \begin{bmatrix} B'a_{11} & B'a_{12}+B_{12}a_{22} \\ 0 & B'a_{22}\end{bmatrix} + \\
&\begin{bmatrix} A'b_{11} & A'b_{12}+b_{22} \\ 0 & A'b_{22}\end{bmatrix} - \begin{bmatrix} b_{11}A' & b_{11}+A'b_{12} \\ 0 & b_{22}A'\end{bmatrix} \\
&= \begin{bmatrix} 0 & B_{12}(a_{11}-a_{22})+b_{22}-b_{11} \\ 0 & 0\end{bmatrix}
\end{aligned}$$

Therefore,

$$[a,B]+[A,b]=0 \Leftrightarrow B_{12}(a_{11}-a_{22})+b_{22}-b_{11}=0. \tag{3.15}$$

Thus, $v=(a,b,i,j,a',b',f)\in T_X\mathcal{N}$ is such that

$$a=\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{bmatrix}, \qquad b=\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22}\end{bmatrix}, \qquad i=\begin{bmatrix} i_1 \\ i_1\end{bmatrix},$$
$$f=\begin{bmatrix} f_1 \\ f_2\end{bmatrix}, \qquad a'=f_2+a_{11}, \qquad b'=B_{12}f_2+b_{11}.$$

and satisfies (3.15).

Thus, for $u=(a,b,i,j,a',b',f)$ and $v=(\tilde{a},\tilde{b},\tilde{i},\tilde{a}',\tilde{b}',\tilde{f})\in T_X\mathcal{N}$,

$$\begin{aligned}
\Omega_X(u,v) &= tr\left(-\begin{bmatrix} \widetilde{a_{11}} & \widetilde{a_{12}} \\ 0 & \widetilde{a_{22}}\end{bmatrix}\cdot\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22}\end{bmatrix} + \begin{bmatrix} \widetilde{b_{11}} & \widetilde{b_{12}} \\ 0 & \widetilde{b_{22}}\end{bmatrix}\cdot\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{bmatrix}\right) - \\
&-(\widetilde{a_{11}}+\tilde{f}_2)(b_{11}+B_{12}f_2)+(\widetilde{b_{11}}+B_{12}\tilde{f}_2)(a_{11}+f_2) \\
&= tr\left(-\begin{bmatrix} \widetilde{a_{11}}b_{11} & \widetilde{a_{11}}b_{12}+\widetilde{a_{12}}b_{22} \\ 0 & \widetilde{a_{22}}b_{22}\end{bmatrix} + \begin{bmatrix} \widetilde{b_{11}}a_{11} & \widetilde{b_{11}}a_{12}+\widetilde{b_{12}}a_{22} \\ 0 & \widetilde{b_{22}}a_{22}\end{bmatrix}\right) - \\
&-\widetilde{a_{11}}b_{11}-\widetilde{a_{11}}B_{12}f_2-\tilde{f}_2b_{11}-\tilde{f}_2f_2B_{12}+\widetilde{b_{11}}a_{11}+\widetilde{b_{11}}f_2+\tilde{f}_2B_{12}a_{11} \\
&+\tilde{f}_2f_2B_{12} \\
&= (-\widetilde{a_{11}}b_{11}-\widetilde{a_{22}}b_{22}+\widetilde{b_{11}}a_{11}+\widetilde{b_{22}}a_{22}-\widetilde{a_{11}}b_{11}-\widetilde{a_{11}}B_{12}f_2-\tilde{f}_2b_{11}+ \\
&+\widetilde{b_{11}}a_{11}+\widetilde{b_{11}}f_2+\tilde{f}_2B_{12}a_{11}) \\
&= (-2\widetilde{a_{11}}b_{11}+2\widetilde{b_{11}}a_{11}-\widetilde{a_{22}}b_{22}+\widetilde{b_{22}}a_{22}+f_2(\widetilde{b_11}-\widetilde{a_11})+\tilde{f}_2(a_{11}-b_{11})) \\
&= (2(-\widetilde{a_{11}}b_{11}+\widetilde{b_{11}}a_{11})-\widetilde{a_{22}}b_{22}+\widetilde{b_{22}}a_{22}+f_2(\widetilde{b_11}-\widetilde{a_11})+\tilde{f}_2(a_{11}-b_{11})).
\end{aligned}$$

Thus, if $u \in T_X\mathcal{N}$ satisfies

$$\begin{cases} b_{11} &= B_{12}a_{11} \\ f_2 &= 0 \end{cases}, \tag{3.16}$$

then

$$\Omega_X(u,v) = (2(\widetilde{b_{11}}a_{11} - \widetilde{a_{11}}B_{12}a_{11}) - \widetilde{a_{22}}b_{22} + \widetilde{b_{22}}a_{22})$$

Note that it follows from equation (3.15) that if $B_{12}a_{11} = b_{11}$, then $B_{12}a_{22} = b_{22}$. Thus one obtains

$$\begin{aligned} \Omega_X(u,v) &= 2(-\widetilde{a_{11}}B_{12}a_{11} + \widetilde{b_{11}}a_{11}) - \widetilde{a_{22}}B_{12}a_{22} + \widetilde{b_{22}}a_{22} \\ &= 2a_{11}(\widetilde{b_{11}} - B_{12}\widetilde{a_{11}}) + a_{22}(\widetilde{b_{22}} - B_{12}\widetilde{a_{22}}) \end{aligned}.$$

Since $v \in T_X\mathcal{N}$,

$$B_{12}\widetilde{a_{11}} - B_{12}\widetilde{a_{22}} + \widetilde{b_{22}} - \widetilde{b_{11}} = 0 \quad \Rightarrow \quad \widetilde{b_{22}} - B_{12}\widetilde{a_{22}} = \widetilde{b_{11}} - B_{12}\widetilde{a_{11}}.$$

Therefore

$$\begin{aligned} \Omega_X(u,v) &= 2a_{11}(\widetilde{b_{11}} - B_{12}\widetilde{a_{11}}) + a_{22}(\widetilde{b_{11}} - B_{12}\widetilde{a_{11}}) \\ &= (2a_{11} + a_{22})(\widetilde{b_{11}} - B_{12}\widetilde{a_{11}}) \end{aligned}$$

Then, if $u \in T_X\mathcal{N}$ satisfies (3.16) and $a_{22} = -2a_{11}$,

$$\Omega_X(u,\cdot) \equiv 0$$

Denote by $[u]$ the equivalence class of u. It is easy to check that if $a_{11} \neq 0$, $[u] \neq [0]$. Indeed, it follows from Theorem 2.4.3 that $[u] = [0]$ if and only if $u \in Im(d_0)$, where

$$\begin{aligned} do: \quad End(V) \oplus End(V') &\longrightarrow \mathbb{X} \\ (h,h') &\longmapsto ([h,A],[h,B],hI,-Jh,[h',A'],[h',B'],hF-Fh') \end{aligned}.$$

Since $c' = 1$, $[h',A'] = 0$. Then, if $a_{11} \neq 0$, since $f_2 = 0$ and $a' = a_{11} + f_2$, $a' \neq 0$. Therefore, there is no $(h,h') \in End(V) \oplus End(V')$ such that $d_0(h,h') = u$. In other words,

$$[u] = \left[\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & -2a_{11} \end{bmatrix}, \begin{bmatrix} B_{12}a_{11} & b_{12} \\ 0 & -2B_{12}a_{11} \end{bmatrix}, \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, a_{11}, a_{11}, \begin{bmatrix} f_1 \\ 0 \end{bmatrix}\right)\right] \neq [0]$$

is a non-null vector in $T_X\mathcal{N}^{st}(1,2,1)$ such that $\Omega_X(u,v) = 0$, for all $v \in T_X\mathcal{N}^{st}(1,2,1)$. This concludes that if X satisfies (3.10), and (3.11), then Ω_X is degenerate.

Now suppose that A is dianagolizable and B is non-diagonalizable. Analogously to the previous case, one can check that $u \in T_X\mathcal{N}(1,2,1)$ is given by

$$u = \left(\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}, \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, 0, 0, b_{11} + f_2, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)$$

and

$$\Omega_X(u,v) = \widetilde{b_{22}} a_{22} - \widetilde{a_{22}} b_{22}$$

Hence, taking $a_{22} = b_{22} = 0$ e $b_{11} \neq f_2$, i.e.,

$$u = \left(\begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, 0, 0, b_{11} + f_2, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)$$

one gets $\Omega_x(u,v) = 0$ for all $v \in T_X\mathcal{N}(1,2,1)$ with $[u] \neq [0]$. Thus Ω_X is degenerate.

In order to conclude the proof, we have now to prove that if

$$X = (A, B, I, J, A', B', F)$$

is such that A and B are diagonalizable matrices, then it follows from Lemma 3.2.1 that there exists a change of basis for V such that

$$A = \begin{bmatrix} A' & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} B' & 0 \\ 0 & b_2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.17}$$

Moreover, analogously to the previous case, one can check that, in order for X to be stable,

$$I = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}, \quad a_1 \neq a_2, \quad b_1 \neq b_2$$

and a vector $u = (a, b, i, j, a', b', f) \in T_X\mathcal{N}$ is given by

$$u = \left(\begin{bmatrix} a_{11} & a_{12} \\ f_2(a_1 - a_2) & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & \delta a_{12} \\ \delta f_2(a_1 - a_2) & b_{22} \end{bmatrix}, \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, a_{11}, b_{11}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right)$$

Therefore, one can check that given $u = (a, b, i, j, a', b', f)$, $v = (\widetilde{a}, \widetilde{b}, \widetilde{i}, \widetilde{j}, \widetilde{a'}, \widetilde{b'}, \widetilde{f}) \in T_X\mathcal{N}$,

$$\Omega_X(u,v) = 2(a_{11}\widetilde{b_{11}} - \widetilde{a_{11}} b_{11}) + a_{22}\widetilde{b_{22}} - \widetilde{a_{22}} b_{22}.$$

Moreover, $\Omega_X(u,\cdot) \equiv 0$ if and only if $a_{11} = b_{11} = a_{22} = b_{22} = 0$. Indeed, if $a_{11} = b_{11} = a_{22} = b_{22} = 0$, it is trivial that $\Omega_X(u,\cdot) \equiv 0$. Now suppose that $\Omega_X(u,\cdot) = 0$ for all $v \in T_X\mathcal{N}$. In particular, by taking

$$v = \left(\begin{bmatrix} 0 & a_{12} \\ f_2(a_1 - a_2) & 0 \end{bmatrix}, \begin{bmatrix} \overline{a_{11}} & \delta a_{12} \\ \delta f_2(a_1 - a_2) & \overline{a_{22}} \end{bmatrix}, \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, 0, \overline{a_{11}}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right),$$

$$w = \left(\begin{bmatrix} -\overline{b_{11}} & a_{12} \\ f_2(a_1 - a_2) & -\overline{b_{22}} \end{bmatrix}, \begin{bmatrix} 0 & \delta a_{12} \\ \delta f_2(a_1 - a_2) & 0 \end{bmatrix}, \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, -\overline{b_{11}}, 0, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right).$$

$\Omega_X(u,v) = \Omega_X(u,w) = 0$, but

$$\Omega_X(u,v) = 2a_{11}\overline{a_{11}} + a_{22}\overline{a_{22}},$$

and this vanishes if and only if $a_{11} = a_{22} = 0$. Moreover,

$$\Omega_X(u,w) = 2b_{11}\overline{b_{11}} + b_{22}\overline{b_{22}}$$

and this vanishes if and only if $b_{11} = b_{22} = 0$. Therefore, $\Omega_x(u,\cdot) \equiv 0$ if and only if

$$u = \left(\begin{bmatrix} 0 & a_{12} \\ f_2(a_1 - a_2) & 0 \end{bmatrix}, \begin{bmatrix} 0 & \delta a_{12} \\ \delta f_2(a_1 - a_2) & 0 \end{bmatrix}, \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, 0, 0, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right).$$

However, $u = d_0(h,h')$, for

$$h = \begin{bmatrix} f_1 - \dfrac{a_{12} + i_1(i_1 - i_2) - \lambda(a_1 - a_2)f_1}{\lambda(a_1 - a_2)} & -\dfrac{a_{12}}{(a_1 - a_2)} \\ f_2 & i_2 - \lambda f_2 \end{bmatrix},$$

$$h' = \frac{a_{12} + i_1 - \lambda(a_1 - a_2)f_1}{\lambda(a_1 - a_2)}.$$

Indeed, one can check that

$$d_0(h,h') = \left((a_1 - a_2)H, (b_1 - b_2)H, \begin{bmatrix} \lambda h_{11} + h_{12} \\ \lambda h_{21} + h_{22} \end{bmatrix}, 0, 0, 0, \begin{bmatrix} h_{11} - h' \\ h_{21} \end{bmatrix} \right)$$

where

$$H = \begin{bmatrix} 0 & -h_{12} \\ h_{21} & 0 \end{bmatrix}$$

and then, $d(h,h') = u$. This means that $[u] = [0]$. In other words, if X satisfies (3.17), Ω_X is non-degenerate. □

4 Enhanced ADHM quiver varieties and flag of sheaves

In this chapter it is discussed the relation between the moduli space of framed stable representations of the enhanced ADHM quiver with numerical type (r, c, c') and the moduli space of flags of sheaves. We proved in details that there exists a bijection between the moduli space $\mathcal{N}^{st}(r, c, c')$ and the moduli space of flag of sheaves (E, F, φ), where (F, φ) is a framed torsion free sheaf in $\mathbb{P}^2$ with rank r and second Chern class $(c - c')$, (E, φ) is a subsheaf of (F, φ) of rank r and second Chern class c such that the support of the quotient F/E is given by c' points outside the line at infinity l_∞. In this chapter, $\mathcal{F}(E, F, \varphi)$ the moduli space of flags of sheaves as above and (E, F, φ) denotes a single flag of sheaves.

In literature one can find results which associate moduli space of stable representations of the ADHM quiver with numerical type (r, c) and the framed moduli space of torsion-free sheaves on $\mathbb{P}^2$ with rank r and second Chern class c. Most of them can be found on [13, Chapter 2]. In the first section one can find the main facts that will be used to prove the existence of the bijection above. In the second section it is proved this existence, i.e., the following

Proposition 4.0.3. Let $\mathcal{N}^{st}(r, c, c')$ be the moduli space of framed stable representations of the enhanced ADHM quiver of numerical type (r, c, c'). Let (E, F, φ) be the moduli space of the flag of sheave given by (F, φ) a framed torsion free sheaf in $\mathbb{P}^2$ with rank r and second Chern class c, (E, φ) is a subsheaf of (F, φ) of rank r and second Chern class $(c - c')$ such that the support of the quotient F/E is given by c' points outside the line in infinity l_∞. Then there exists a bijection between $\mathcal{N}^{st}(r, c, c')$ and (E, F, φ).

Moreover, it was proved by Patrícia Borges dos Santos in [16], that if $r = 1$, the moduli space $\mathcal{N}^{st}(1, c, c')(\mathbb{C})$ and the nested Hilbert Scheme $Hilb^{c-c',c}(\mathbb{C})$ are isomorphic.

4.1 Framed torsion free sheaves and stable representations of the ADHM quiver

In this section one can find a few results that can be found on [13, Chapter 2], [10, Chapter 5] and [15, Chapter II, Section 3]. These results will be presented briefly but the reader can check the references for more details.

Definition 4.1.1. Let $(x : y : z)$ be fixed homogeneous coordinates in $\mathbb{P}^2$. Let $X = (W, V, A, B, I, J)$ be a representation of the ADHM quiver. An *ADHM complex* is a complex on $\mathbb{P}^2$ of the form

$$E_X^\bullet : V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} (V \oplus V \oplus W) \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where

$$\alpha = \begin{bmatrix} zA + x1_V \\ zB + y1_V \\ zJ \end{bmatrix}, \quad \beta = \begin{bmatrix} -zB - y1_V & zA + x1_V & zI \end{bmatrix}.$$

Note that the ADHM equation is equivalent to the condition $\beta\alpha = 0$. Indeed,

$$\beta\alpha = \begin{bmatrix} -zB - y1_V & zA + x1_V & zI \end{bmatrix} \cdot \begin{bmatrix} zA + x1_V \\ zB + y1_V \\ zJ \end{bmatrix} = z^2([A, B] + IJ).$$

Moreover, given a morphism (ξ_1, ξ_2) between two representations X and $\widetilde{X}$, one has the following morphism $\xi^\bullet = (\xi_1 \oplus 1_V, (\xi_1 \oplus \xi_1 \oplus \xi_2) \otimes 1_V, \xi_1 \otimes 1_V)$ between the ADHM complexes $E_X^\bullet$ and $E_{\widetilde{X}}^\bullet$

$$\begin{array}{ccccc} V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha} & (V \oplus V \oplus W) \otimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta} & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow{\scriptstyle \xi_1 \oplus 1_V} & & \downarrow{\scriptstyle (\xi_1 \oplus \xi_1 \oplus \xi_2) \otimes 1_V} & & \downarrow{\scriptstyle \xi_1 \otimes 1_V} \\ V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\tilde{\alpha}} & (V \oplus V \oplus W) \otimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\tilde{\beta}} & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \end{array}$$

From now on we will denote by $\mathcal{A}$ and $\mathrm{Kom}(\mathbb{P}^2)$ the abelian categories of representations of the ADHM quiver and the category of complexes of sheaves on $\mathbb{P}^2$, respectively.

Proposition 4.1.2. The functor

$$\mathbb{F} : \mathcal{A} \longrightarrow \mathrm{Kom}(\mathbb{P}^2)$$

given by

$$\mathbb{F}(X) = E_X^\bullet, \qquad \mathbb{F}(\xi_1, \xi_2) = \xi^\bullet.$$

is exact, full and faithful.

This Proposition was proved by Patrícia Borges dos Santos and the proof can be found in [16].

Lemma 4.1.3. Let us fix a representation X and the corresponding ADHM complex $E_X^\bullet$. Then:

(*i*) the sheaf map α is injective. The fiber maps α_P are injective for every $P \in \mathbb{P}^2$ if and only if X is costable;

(*ii*) if X is stable, then $\mathcal{H}^1(E^\bullet_X) = 0$ and $\mathcal{H}^0(E^\bullet_X)$ is a torsion free sheaf whose restriction to l_∞ is trivial of rank $r = \dim W$ and second Chern class $c = \dim V$;

(*iii*) for any $X \in \mathcal{A}$, $H^0(\mathcal{H}^0(E^\bullet_X(-1)) = 0$.

The proof of this Lemma can be found in [10, Lemma 5.2 and Lemma 5.4] and [13, Section 2.1].

For further references, we state the following:

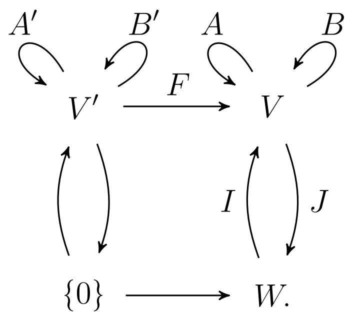
Theorem 4.1.4. There exists bijection between the moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank r and second Chern class c and the moduli space $\mathcal{M}(r, c)$ of stable representations of the ADHM quiver with numerical type (r, c).

The proof of this Theorem can be found on [13, Chapter 2] and [15, Chapter II, Section 3].

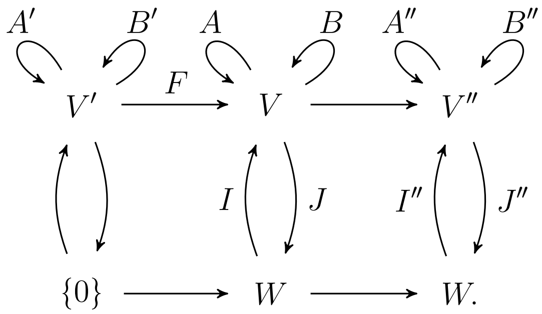
All results that are needed to prove the main Proposition of this Chapter were enunciated here. This proof can be found in the next section.

4.2 Proof of the Proposition 4.0.3

First let $X = (A, B, I, J, A', B', F)$ be a framed stable representation of the enhanced ADHM quiver. Then F is injective and the ADHM datum (A, B, I, J) is stable. Thus one has the diagram

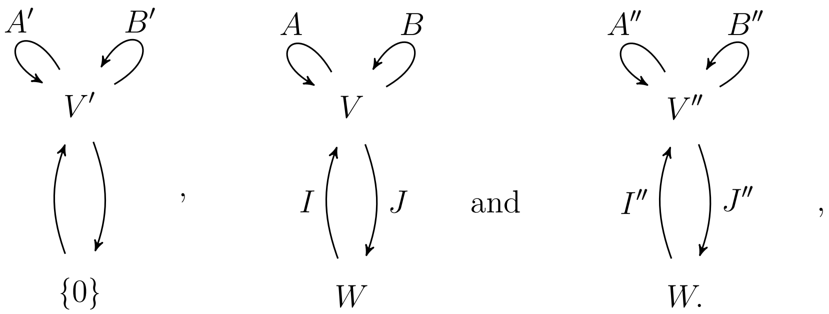


Since X is stable, one can obtain the stable representation of the ADHM quiver with numerical type $(r, c - c')$, $X'' = (A'', B'', I'', J'')$ (see Lemma 2.3.8). Then, incluing this representation, one can obtains the diagram



(4.1)

Denote by $\mathbf{Z}$, $\mathbf{S}$ and $\mathbf{Q}$ the stable representations of the ADHM quiver



respectively. Thus, since F is injective and there exists a surjective map between the moduli spaces $\mathcal{N}^{st(r,c,c')}$ and $\mathcal{M}(r, c-c')$, see Lemma 2.3.8, the diagram (4.1) can be expressed as the exact sequence of representations

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{S} \longrightarrow \mathbf{Q} \longrightarrow 0. \tag{4.2}$$

It follows from Proposition 4.1.2 the exactness of the sequence of monads

$$0 \longrightarrow E^{\bullet}_{\mathbf{Z}} \longrightarrow E^{\bullet}_{\mathbf{S}} \longrightarrow E^{\bullet}_{\mathbf{Q}} \longrightarrow 0,$$

where $E^{\bullet}_{\mathbf{Z}}$, $E^{\bullet}_{\mathbf{S}}$ and $E^{\bullet}_{\mathbf{Q}}$ are the ADHM complexes on $\mathbb{P}^2$ associated with the representations of the ADHM quiver $\mathbf{Z}$, $\mathbf{S}$ and $\mathbf{Q}$, respectively.

from the exact sequence 4.2, one can construct the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathcal{H}^{-1}(E^{\bullet}_{\mathbf{Z}}) \longrightarrow \mathcal{H}^{-1}(E^{\bullet}_{\mathbf{S}}) \longrightarrow \mathcal{H}^{-1}(E^{\bullet}_{\mathbf{Q}}) \longrightarrow \mathcal{H}^{0}(E^{\bullet}_{\mathbf{Z}}) \longrightarrow \mathcal{H}^{0}(E^{\bullet}_{\mathbf{S}}) \longrightarrow \\ \longrightarrow \mathcal{H}^{0}(E^{\bullet}_{\mathbf{Q}}) \longrightarrow \mathcal{H}^{1}(E^{\bullet}_{\mathbf{Z}}) \longrightarrow \mathcal{H}^{1}(E^{\bullet}_{\mathbf{S}}) \longrightarrow \mathcal{H}^{1}(E^{\bullet}_{\mathbf{Q}}) \longrightarrow 0 \end{array} \tag{4.3}$$

It follows from Lemma 4.1.3, that the exact sequence in (4.3) reduces to

$$0 \longrightarrow \mathcal{H}^{0}(E^{\bullet}_{\mathbf{Z}}) \longrightarrow \mathcal{H}^{0}(E^{\bullet}_{\mathbf{S}}) \longrightarrow \mathcal{H}^{0}(E^{\bullet}_{\mathbf{Q}}) \longrightarrow \mathcal{H}^{1}(E^{\bullet}_{\mathbf{Z}}) \longrightarrow 0. \tag{4.4}$$

Indeed, it follows from Lemma 4.1.3 (i) that α', α and α'' on the ADHM complexes $E^\bullet_{\mathbf{Z}}$, $E^\bullet_{\mathbf{S}}$ and $E^\bullet_{\mathbf{Q}}$, respectively, are injective maps. Thus,

$$\mathcal{H}^{-1}(E^\bullet_{\mathbf{Z}}) = \mathcal{H}^{-1}(E^\bullet_{\mathbf{S}}) = \mathcal{H}^{-1}(E^\bullet_{\mathbf{Q}}) = 0.$$

Moreover, since $E^\bullet_{\mathbf{S}}$ and $E^\bullet_{\mathbf{Q}}$ are stable, Lemma 4.1.3 (ii) leads to

$$\mathcal{H}^0(E^\bullet_{\mathbf{S}}) = \mathcal{H}^0(E^\bullet_{\mathbf{Q}}) = 0$$

and then one has the exact sequence (4.4). However, $\mathcal{H}^0(E^\bullet_{\mathbf{Z}}) = 0$. Indeed,

$$E^\bullet_{\mathbf{Z}} : V' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} (V' \oplus V') \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V' \otimes \mathcal{O}_{\mathbb{P}^2}(1) \ ,$$

and taking $P = (x : y : 0) \in l_\infty$, one obtains

$$\alpha' = \begin{bmatrix} x1_{V'} \\ y1_{V'} \end{bmatrix}, \quad \beta' = \begin{bmatrix} y1_{V'} & x1_{V'} \end{bmatrix}.$$

Thus,

$$\ker(\beta') = 0$$

for all $P \in l_\infty$. Since

$$\mathcal{H}^0(E^\bullet_{\mathbf{Z}}) = \ker(\beta')/Im(\alpha'),$$

the stalks of this sheaf vanishes at P. Therefore, the support of this sheaf is a 0-dimensional scheme, since it does not intersect the line at infinity l_∞. However,

$$H^0(\mathcal{H}^0(E^\bullet_{\mathbf{Z}})) = H^0(\mathcal{H}^0(E^\bullet_{\mathbf{Z}}(-1)))$$

and according with Lemma 4.1.3 (iii),

$$H^0(\mathcal{H}^0(E^\bullet_{\mathbf{Z}}(-1))) = 0.$$

Therefore, since $\mathcal{H}^0(E^\bullet_{\mathbf{Z}})$ is supported at finitely many points,

$$\mathcal{H}^0(E^\bullet_{\mathbf{Z}}) = 0.$$

Hence, finally one obtains the exact sequence

$$0 \longrightarrow \mathcal{H}^0(E^\bullet_{\mathbf{S}}) \xrightarrow{\varphi} \mathcal{H}^0(E^\bullet_{\mathbf{Q}}) \longrightarrow \mathcal{H}^1(E^\bullet_{\mathbf{Z}}) \longrightarrow 0. \tag{4.5}$$

Therefore, $(\mathcal{H}^0(E^\bullet_{\mathbf{S}}), \mathcal{H}^0(E^\bullet_{\mathbf{Q}}), \varphi)$ is a flag of sheaves, $(\mathcal{H}^0(E^\bullet_{\mathbf{Q}}), \varphi)$ is a framed torsion free sheaf with rank r and second Chern class $(c - c')$ and $(\mathcal{H}^0(E^\bullet_{\mathbf{S}}), \varphi)$ a subsheaf of $(\mathcal{H}^0(E^\bullet_{\mathbf{Q}})\varphi)$ with rank r and second Chern class c, furthermore,

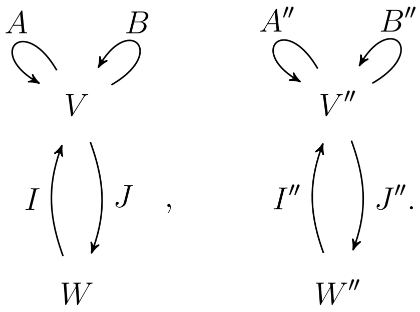
$$\mathcal{H}^1(E^\bullet_{\mathbf{Z}}) \cong \mathcal{H}^0(E^\bullet_{\mathbf{Q}})/\mathcal{H}^0(E^\bullet_{\mathbf{S}})$$

has rank 0 and consists of $(c - (c - c')) = c'$ points outside the line at infinity l_∞.

Now suppose that (E, F, φ) is a flag of sheaves with ranks and second Chern classes as above. One can find a framed stable representation of the enhanced ADHM quiver

$$X = (W, V, V', A, B, I, J, A', B', F)$$

such that its numerical type is (r, c, c'). In fact, since (E, F, φ) is a flag of sheaves, (F, φ) and (E, φ) are framed torsion free sheaves of rank r and second Chern class $(c - c')$ and c, respectively. It follows from Theorem 4.1.4 that there are stable representations of the ADHM quiver $\mathbf{Q} = (W'', V'', A'', B'', I'', J'')$ with numerical type $(r, c - c')$ and $\mathbf{S} = (W, V, A, B, I, J)$ with numerical type (r, c) associated with the torsion free sheaves (F, φ) and (E, φ), respectively. Thus one has the diagrama

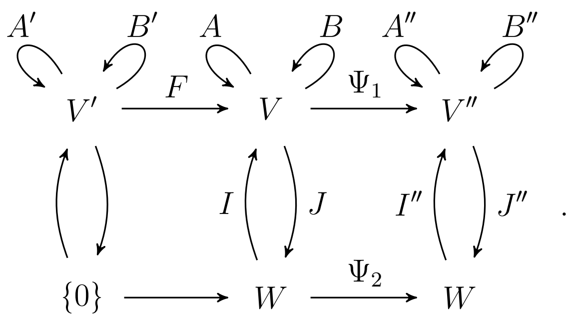


(4.6)

Suppose that there exists a surjective map

$$\Psi : \mathbf{S} \longrightarrow \mathbf{Q},$$

then there are surjective maps $\Psi_1 \in Hom(V, V'')$ and $\Psi_2 \in Hom(W, W'')$. By putting $V' = \ker(\Psi_1)$, one can define the representation of the ADHM quiver $\mathbf{Z} = (0, \ker(\Psi_1), A|_{\ker(\Psi_1)}, B|_{\ker(\Psi_1)}, 0, 0)$. Denoting $V' = \ker(\Psi_1)$, $A' = A|_{\ker(\Psi_1)}$ and $B'|_{\ker(\Psi_1)}$, and defining by $F : V' \longrightarrow V$ as the inclusion map, one gets the following diagram



(4.7)

This means that the framed stable representation of the enhanced ADHM quiver

$$X = (W, V, V', A, B, I, J, A', B', F)$$

is such that the cohomology groups of the monads $E^\bullet_{\mathbf{Z}}$, $E^\bullet_{\mathbf{S}}$ and $E^\bullet_{\mathbf{Q}}$ associated with the representations of the ADHM quiver $\mathbf{Z}$, $\mathbf{S}$ and $\mathbf{Q}$, respectively, lead to the exact sequence in (4.5). Therefore, in order to conclude the proof, it is enough to show that the surjection Ψ does exist.

First of all, it follows from the proof of Theorem 4.1.4 on [13, Chapter 2] that there is a unique description in terms of monads of framed torsion-free sheaves on $\mathbb{P}^2$. According to this description, the sheaf (E,φ) leads to the monad

$$E^\bullet : V\otimes\mathcal{O}_{\mathbb{P}^2}(-1)\xrightarrow{\ \alpha\ }(V\oplus V\oplus W)\otimes\mathcal{O}_{\mathbb{P}^2}\xrightarrow{\ \beta\ }V\otimes\mathcal{O}_{\mathbb{P}^2}(1)$$

where

$$\alpha=\begin{bmatrix} zA+x1_V\\ zB+y1_V\\ zJ\end{bmatrix},\quad \beta=\begin{bmatrix} -zB-y1_V & zA+x1_V & zI\end{bmatrix}$$

and

$$V=H^1(F(-1)),\quad W=H^0(F|_{l_\infty})$$

while (F,φ) leads to the monad

$$F^\bullet : V''\otimes\mathcal{O}_{\mathbb{P}^2}(-1)\xrightarrow{\ \alpha''\ }(V''\oplus V''\oplus W'')\otimes\mathcal{O}_{\mathbb{P}^2}\xrightarrow{\ \beta''\ }V''\otimes\mathcal{O}_{\mathbb{P}^2}(1)$$

where

$$\alpha=\begin{bmatrix} zA''+x1_{V''}\\ zB''+y1_{V''}\\ zJ''\end{bmatrix},\quad \beta=\begin{bmatrix} -zB''-y1_{V''} & zA''+x1_{V''} & zI''\end{bmatrix}$$

and

$$V''=H^1(E(-1)),\quad W''=H^0(E|_{l_\infty}).$$

Since (E,φ) is a subsheaf of (F,φ) there exists the exact sequence

$$0\longrightarrow E\hookrightarrow F\longrightarrow Q\longrightarrow 0$$

where $Q=F/E$. Therefore, since Q is supported in finitely many points,

$$0\longrightarrow\underbrace{H^1(E(-1))}_{=V}\overset{\Psi_1}{\hookrightarrow}\underbrace{H^1(F(-1))}_{=V''}\longrightarrow\underbrace{H^1(Q(-1))}_{=0}\longrightarrow 0.$$

In other words, there exists a surjection $\Psi_1 : V\longrightarrow V''$. Moreover, since (F,φ) and (E,φ) have the same framing, $E|_\infty\cong F|_\infty$. Then

$$W=H^0(F|_{l_\infty})\cong H^0(E|_{l_\infty})=W''.$$

In other words, there exists an isomorphims $\Psi_2 : W \longrightarrow W''$. Then,

$\Psi = (\Psi_1 \otimes \mathbf{1}, (\Psi_1 \oplus \Psi_1 \oplus \Psi_2) \otimes \mathbf{1}, \Psi_1 \otimes \mathbf{1})$

$$\begin{array}{cccccccc}
E^{\bullet}: & V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha} & (V \oplus V \oplus W) \otimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta} & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \\
\downarrow{\scriptstyle \Psi} & \downarrow{\scriptstyle \Psi_1 \otimes \mathbf{1}} & & \downarrow{\scriptstyle (\Psi_1 \oplus \Psi_1 \oplus \Psi_2) \otimes \mathbf{1}} & & \downarrow{\scriptstyle \Psi_1 \otimes \mathbf{1}} \\
F^{\bullet}: & V'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha''} & (V'' \oplus V'' \oplus W'') \otimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta''} & V'' \otimes \mathcal{O}_{\mathbb{P}^2}(1)
\end{array}$$

is a surjective maps between monads. It follows from Proposition 4.1.2 that $\mathbb{F}(\Psi) = (\xi_1, \xi_2)$ is a surjective map between $\mathbb{F}(E^{\bullet}) = \mathbf{S}$ and $\mathbb{F}(F^{\bullet}) = \mathbf{Q}$. This concludes the proof.

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