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Orthogonal Instanton Bundles on $\mathbb{P}^n$ and their splitting type

Fibrados instanton ortogonais sobre $\mathbb{P}^n$ e seus tipos de splitting

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Forget it all for an instanton!
(José Figueroa-O'Farrill)

Resumo

Fibrados instanton são um importante link entre a física matemática e a geometria algébrica. Desde a década de 1970 o estudo dessa família de fibrados e de seus espaços de moduli despertam um grande interesse na comunidade matemática. Mas o estudo desses espaços de moduli pode se tornar muito complicado se considerarmos fibrados de posto mais alto ou variedades projetivas de dimensão alta. Por este motivo, muitos autores estudam fibrados instanton com uma estrutura adicional. O foco deste trabalho serão os fibrados instanton ortogonais.

A fim de obter critérios de existência para fibrados instanton ortogonais em $\mathbb{P}^n$, construímos uma bijeção entre classes de equivalência de fibrados instanton ortogonais sem seções globais e formas simétricas. Usando esta correspondência fomos capazes de construir exemplos explícitos de fibrados instanton ortogonais sem seções globais em $\mathbb{P}^n$ e provar que todo fibrado instanton ortogonal sem seções globais em $\mathbb{P}^n$ com carga c tem posto $(n-1)c$, para $n, c \geqslant 3$. Também provamos que $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$, o espaço de moduli de fibrados instanton ortogonais sem seções globais em $\mathbb{P}^n$, com carga c e posto $(n-1)c$, para $n, c \geqslant 3$ é afim. Por último, construímos módulos de Kronecker para determinar o tipo de splitting dos fibrados em $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$.

Palavras-chave: Fibrados instanton ortogonais. Formas simétricas. Espaços de moduli. Teoria geométrica dos invariantes. Tipo de splitting. Módulos de Kronecker.

Abstract

Instanton bundles are an important link between mathematical physics and algebraic geometry. Since the 1970's the study of this family of bundles and its moduli space awakes great interest in the mathematical community. But the study of its moduli space can be very complicated if we consider bundles of higher rank or higher dimensional projective spaces. Because of this, several authors study instanton bundles with an additional structure. In this work, we will focus on orthogonal instanton bundles.

In order to obtain existence criteria for orthogonal instanton bundles on $\mathbb{P}^n$, we provide a bijection between equivalence classes of orthogonal instanton bundles with no global sections and symmetric forms. Using such correspondence we are able to provide explicit examples of orthogonal instanton bundles with no global sections on $\mathbb{P}^n$ and prove that every orthogonal instanton bundle with no global sections on $\mathbb{P}^n$ and charge c has rank $(n-1)c$, for $n, c \geqslant 3$. We also prove that $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$ the coarse moduli space of orthogonal instanton bundles with no global sections on $\mathbb{P}^n$, charge c and rank $(n-1)c$, for $n, c \geqslant 3$ is affine. Last, we construct Kronecker modules to determine the splitting type of the bundles of $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$.

Keywords: Orthogonal intanton bundles. Symmetric forms. Moduli spaces. Geometric invariant theory. Splitting type. Kronecker modules.

Contents

Introduction

The study of vector bundles in projective varieties has been a topic of great interest in algebraic geometry, see for instance Hartshorne's problems list in [25]. Specifically, in the case of instanton bundles the main interest in the past has been the link that they provide between algebraic geometry and mathematical physics. Since the 1970's the "instantons" or pseudo-particle solutions of the classical Yang-Mills equations in the Euclidean 4-space has awaken great interest in the physical and mathematical communities (see [5] and [6]). In [5] Atiyah, Hitchin and Singer proved that instantons correspond to certain real algebraic bundles on $\mathbb{CP}^3$ and also proved that the complete set of solutions depends on $8c-3$ parameters, where c is the quantum number (or charge) of the instanton. Atiyah and Ward in [6] used the Penrose's program (see [43]) to construct explicit solutions. In [4], Atiyah, Drinfield, Hitchin and Manin, using tools of linear algebra, provided the classical "ADHM construction of instantons". The moduli space $\mathcal{M}_{\mathbb{P}^3}(c)$ of the c-instanton bundles on $\mathbb{P}^3$, i.e. of stable 2–bundles $\mathcal{E}$ with Chern classes $(c_1, c_2) = (0, c)$ and $\mathrm{H}^1(\mathcal{E}(-2)) = 0$ is expected to be a smooth and irreducible variety with dimension $8c-3$ for $c \geqslant 1$. For $c = 1$, $\mathcal{M}_{\mathbb{P}^3}(1)$ is isomorphic to the complement in $\mathbb{P}^5$ of the Grassmann manifold of lines in $\mathbb{P}^3$ (see [40] - Theorem 4.3.4). In [24], Hartshorne described $\mathcal{M}_{\mathbb{P}^3}(2)$ as a smooth and irreducible fibration. In [16] Ellingsrud and Strømme proved that $\mathcal{M}_{\mathbb{P}^3}(3)$ is smooth irreducible of dimension 21. LePotier proved in [35] that $\mathcal{M}_{\mathbb{P}^3}(4)$ is smooth and in [7] Barth proved that $\mathcal{M}_{\mathbb{P}^3}(4)$ is irreducible. Katsylo and Ottavianni proved in [34] that $\mathcal{M}_{\mathbb{P}^3}(5)$ is smooth. In [12] Coandă, Tikhomirov and Trautmann proved that $\mathcal{M}_{\mathbb{P}^3}(5)$ is irreducible and smooth, unifying the proof of this previous results for $n \leqslant 4$. In [45] and [46] Tikhomirov proved the irreducibility of $\mathcal{M}_{\mathbb{P}^3}(c)$ for arbitrary $c \geqslant 1$. The question about the smoothness was solved on $\mathbb{CP}^3$ by Jardim and Verbitsky (see [33]), but it is important to highlight that the smoothness on $\mathbb{P}^3$, that means for any 3-dimensional projective space over an algebraically closed field of characteristic 0 is still open.

In 1986 Okonek and Spindler in [41] used a Salamon's generalization of the Penrose transformation to extend the definition of instanton bundles. They defined the so called mathematical instanton bundles on $\mathbb{P}^{2n+1}$ $(n \geqslant 1)$, i.e. holomorphic $2n$-rank bundles $\mathcal{E}$ on $\mathbb{P}^{2n+1}$, with Chern polynomial $c_t(\mathcal{E}) = \left(\dfrac{1}{1-t^2}\right)^c$ and natural cohomology $\mathrm{H}^q(\mathcal{E}(l))$ in the range $-2n-1 \leqslant l \leqslant 0$. In addition, these bundles are simple, trivial on generic lines and have a symplectic structure. For $c = 1$ and $n \geqslant 1$ we have the so called Nullcorrelation bundles (see [40]), while for $n = 1$ and $c \geqslant 1$ we have all the mathematical instanton bundles, which came from physics (see [47]). In $\mathbb{CP}^{2n+1}$ the mathematical instanton bundles are generalizations of special instanton bundles over $\mathbb{CP}^3$, also called special 't Hooft bundles. Spindler and Trautmann proved in [44] that any special instanton bundle $\mathcal{E}$ of charge c on $\mathbb{P}^{2n+1}$ can be defined by an exact sequence

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus c} \longrightarrow S^{\vee} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where S is a Schwarzenberger bundle of rank $2n + c$.

Ancona and Ottaviani in [2] defined mathematical instanton bundles without assuming that they are all symplectic and proved that every special symplectic instanton bundles is stable.

Let $\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ denote the moduli space of stable instanton bundles on $\mathbb{P}^{2n+1}$, with charge c. In [3] Ancona and Ottaviani proved that $\mathcal{M}_{\mathbb{P}^{2n+1}}(2)$ is smooth and irreducible, while $\mathcal{M}_{\mathbb{P}^{2n+1}}(3)$ and $\mathcal{M}_{\mathbb{P}^{2n+1}}(4)$ are singular. Costa and Ottaviani in [14] proved that $\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ is affine and introduced an invariant which allowed Farnik, Frapporti and Marchesi to prove in [17] that there are no orthogonal instanton bundles with rank $2n$ on $\mathbb{P}^{2n+1}$. In [37] Miró-Roig and Orus-Lacort proved that $\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ is singular for $n \geqslant 2$ and $c \geqslant 3$. Hoffmann proved in [28] that $\mathcal{M}_{\mathbb{P}^{2n+1}}(c)$ is rational for $n \geqslant 1$ and $c \geqslant 2$. Later, in [13] Costa, Hoffmann, Miró-Roig and Schmitt proved that the moduli space all symplectic instanton bundles on $\mathbb{P}^{2n+1}$ with $n \geqslant 2$ is reducible.

In order to understand moduli spaces of stable vector bundles over a projective variety, in [31] Jardim extended the definition of instantons to even-dimensional projective spaces and allowed non-locally-free sheaves of arbitrary rank. Jardim defined an instanton sheaf on $\mathbb{P}^n$ ($n \geqslant 2$) as a torsion-free coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ with first Chern class $c_1(\mathcal{E}) = 0$ satisfying some cohomological conditions (see Definition 1.2.4 for details). If $\mathcal{E}$ is locally-free, $\mathcal{E}$ is called an instanton bundle. In addition, if $\mathcal{E}$ is a $2n$-rank bundle on $\mathbb{P}^{2n+1}$ with trivial splitting type, $\mathcal{E}$ is a mathematical instanton bundle as defined by Okonek and Spindler. In that paper Jardim showed that every instanton sheaf is the cohomology of a linear monad, and that rank r instanton sheaves on $\mathbb{P}^n$ exist if and only if $r \geqslant n - 1$. But to study the moduli space of instanton bundles becomes more complicated for higher dimensional projective spaces or higher rank, because of this many authors have considered instanton bundles with some additional structure: special, symplectic, orthogonal for instance. Using the ADHM construction introduced by Henni, Jardim and Martins in [26], Jardim, Marchesi and Wißdorf in [32] consider autodual instantons of arbitrary rank on projective spaces, with focus on symplectic and orthogonal instantons. They describe the moduli space of framed autodual instanton bundles and showed that there are no orthogonal instanton bundles of trivial splitting type, arbitrary rank r and charge 2 or odd on $\mathbb{P}^n$.

While in [1] Abauf and Boralevi proved that the moduli space of rank r stable orthogonal bundles on $\mathbb{P}^2$, with Chern classes $(c_1, c_2) = (0, c)$ and trivial splitting type on the general line, is smooth and irreducible for $r = c$ and $c \geqslant 4$, and $r = c - 1$ and $c \geqslant 8$, the results of Farnik, Frapport and Marchesi in [17] and Jardim, Marchesi and Wißdorf in [32], already mentioned, show us that orthogonal instanton bundles on $\mathbb{P}^n$, $n \geqslant 3$ are for some reason hard to find and that it is interesting to establish existence criteria for orthogonal instanton bundles with higher rank on projective spaces.

The main goal of this work is to provide existence criteria for orthogonal instanton bundles with higher rank on $\mathbb{P}^n$, for $n \geqslant 3$ and then to study their moduli space and splitting type.

Overview. We will now give a short overview of the contents of this thesis.

Chapter 1: we introduce some preliminaries necessary through the text. In the first section we give a brief summary of some of the algebraic geometry concepts and tools that we use: from the definition of torsion-free sheaf, passing by splitting type and monads till hyperdeterminants. In the second section we recall the definition of instanton bundles and some important results and properties that we will need. In the third section we present the definition of moduli space and a collection of results in geometric invariant theory necessary to construct moduli spaces.

Chapter 2: in the first section, in order to establish existence criteria for orthogonal intanton bundles on $\mathbb{P}^n$, for $n \geqslant 3$ we define certain equivalence classes of orthogonal instanton bundles and provide a bijection between those classes and symmetric forms which give us our first main result (see Theorem 2.1.3). Using such correspondence we prove the following result.

Theorem 2.1.4 Let c be an integer, with $c \geqslant 3$. Every orthogonal instanton bundle with no global sections on $\mathbb{P}^n$ and charge c has rank $(n-1)c$. Moreover, there are no orthogonal instanton bundles with no global sections, and charge c equal 1 or 2 on $\mathbb{P}^n$.

These results translate our existence problem in to a linear problem: to find invertible symmetric matrices and give us a very useful tool to construct explicit examples of orthogonal instanton bundles with no global sections, and charge $c \geqslant 3$ on $\mathbb{P}^n$, for $n \geqslant 3$.

In the second section, we first define an algebraic action on our parameter space in order to determine an orbit space (see Theorem 2.2.2). After this we prove that this orbit space is in fact an affine coarse moduli space for our problem (see Theorem 2.2.3).

Chapter 3: given an orthogonal instanton bundle $\mathcal{E}$ on $\mathbb{P}^n$ with charge c, rank $(n-1)c$ and no global sections, for $c, n \geqslant 3$, we construct a Kronecker module to determine if $\mathcal{E}$ has trivial splitting type. In fact, we have the following result.

Theorem 3.0.3 Let $\mathcal{E}$ be an orthogonal instanton bundle on $\mathbb{P}^n = \mathbb{P}(V)$, with charge c, rank $(n-1)c$ and no global sections, for $n, c \geqslant 3$. Let γ be its Kronecker module associated. If $L \subset \mathbb{P}^n$ is the line defined by $v_1, v_2 \in V$, $v_1 \wedge v_2 \neq 0$, the the restriction $\mathcal{E}|_L$ is trivial if and only if

$\gamma(v_1 \wedge v_2)$ is an isomorphism.

1 Preliminaries

The purpose of this chapter is to introduce the tools and fix the notation that we will use throughout this work. In the first section we summarize some fundamental results in algebraic geometry necessary to understand this text. In the second section we introduce our main object of study: the instanton bundles. Finally, in the last section we introduce some results about moduli spaces and geometric invariant theory that will be necessary in the construction of the moduli space of orthogonal instanton bundles on Chapter 2. For more details see [23], [38], [39] and [40].

1.1 Basic concepts

Through this work, $\mathbb{K}$ is an algebraically closed field of characteristic zero, V is a $\mathbb{K}$-vector space with dimension $n+1$ and $\mathbb{P}^n = \mathbb{P}(V)$ the associated projective space of lines in V.

Definition 1.1.1. Let X be a scheme over $\mathbb{K}$ with structure sheaf $\mathcal{O}_X$. Let $\mathcal{F}$ be a coherent sheaf on X and denote by $\mathcal{F}_x$ the **stalk** of $\mathcal{F}$ at a point $x \in X$. We define the **dual** of $\mathcal{F}$ as $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ and say that

(a) $\mathcal{F}$ is **torsion-free** if and only if the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is a monomorphism, or equivalently if for all $x \in X$ and all sections $s \in \mathcal{O}_{X,x} \backslash \{0\}$, the multiplication map $\times s : \mathcal{F}_x \to \mathcal{F}_x$ is injective.

(b) $\mathcal{F}$ is **locally-free** of rank r if $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$ for all $x \in X$.

(c) $\mathcal{F}$ is **reflexive** if the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

(d) $\mathcal{F}$ is **simple** if $\mathrm{End}(\mathcal{F}) = \mathcal{H}om(\mathcal{F}, \mathcal{F}) \cong \mathbb{K}$.

Concerning to the properties of coherent sheaves defined above, we have the following relations (see [40] - Chapter 2)

$$\text{locally-free} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

But the converse is not always true (see Example 1.1.28 and Example 1.1.29).

Let $\mathcal{E}$ and $\mathcal{F}$ be two sheaves on a scheme X. A **morphism of sheaves** $\phi : \mathcal{E} \to \mathcal{F}$ consists of a morphism $\phi_U : \mathcal{E}(U) \to \mathcal{F}(U)$ for each open set U of X, such that for all inclusion $V \subset U$ we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{E}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho^{\mathcal{E}}_{UV}"'] & \mathcal{F}(U) \arrow[d, "\rho^{\mathcal{F}}_{UV}"] \\ \mathcal{E}(U) \arrow[r, "\phi(V)"'] & \mathcal{F}(U), \end{tikzcd}$$

where $\rho^{\mathcal{E}}$ and $\rho^{\mathcal{F}}$ are the restriction maps in $\mathcal{E}$ and $\mathcal{F}$, respectively.

Definition 1.1.2. A **vector bundle of rank r** (or **line bundle** if $r = 1$) over an algebraic variety X is an algebraic variety $\mathcal{F}$ equipped with a surjective morphism $\pi : \mathcal{F} \to X$ such that there exists a covering $X = \bigcup_{i \in I} U_i$ by (Zariski) open subsets such that:

(i) For each $i \in I$ there is an isomorphism $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^r$ satisfying that the composition $\pi \circ \psi_i^{-1} : U_i \times \mathbb{K}^r \to U_i$ is the first projection.

(ii) For each $i, j \in I$ there is an invertible $(r \times r)$-matrix A_{ij} (called **transition matrix**, or **transition function** if $r = 1$) whose entries are regular functions in $U_i \cap U_j$ satisfying that the composition

$$\varphi_{ij} = \psi_j \circ \psi^{-1}_{i|U_i \cap U_j} : (U_i \cap U_j) \times \mathbb{K}^r \to (U_i \cap U_j) \to (U_i \cap U_j) \times \mathbb{K}^r$$

takes the form

$$\varphi_{ij}(x, v) = (x, A_{ij}(x)v).$$

Let $\pi_{\mathcal{E}} : \mathcal{E} \to X$ and $\pi_{\mathcal{F}} : \mathcal{F} \to X$ be two vector bundles over an algebraic variety X. A **morphism of vector bundles** $\phi : \mathcal{E} \to \mathcal{F}$ is given by a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\phi"] \arrow[dr, "\pi_{\mathcal{E}}"'] & & \mathcal{F}, \arrow[dl, "\pi_{\mathcal{F}}"] \\ & X & \end{tikzcd}$$

where for any $x \in X$ the map $\phi_x : \mathcal{E}_x \to \mathcal{F}_x$ is a linear map.

For any vector bundle of rank r over a scheme X we have a locally free sheaf of rank r, which is nothing but the sheaf of sections. So in what follows we will not distinguish locally free shaves and vector bundles on a scheme, and this is justified by the following theorem.

Theorem 1.1.3 ([23] - Chapter 2 - Exercise 5.18)**.** There exists an equivalence of the category of locally free sheaves of rank r and the category of vector bundles of rank r over a scheme X.

Definition 1.1.4. We call a vector bundle $\mathcal{E}$ **autodual** if it is isomorphic to its dual, i.e. there exists an isomorphism $\phi : \mathcal{E} \to \mathcal{E}^\vee$. If the isomorphism ϕ satisfies $\phi^\vee = -\phi$ the vector bundle is called **symplectic**. If the isomorphism ϕ satisfies $\phi^\vee = \phi$ the vector bundle is called **orthogonal**.

Example 1.1.5. Recall the well know **Euler sequence**

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0 \,,$$

where $\alpha = \begin{pmatrix} x_0 & x_1 & \cdots & x_n \end{pmatrix}$ and $(x_0, \cdots, x_n)$ is a basis of $V^\vee$ with $\mathbb{P}^n = \mathbb{P}(V)$.
The kernel of $\alpha(-1)$ is the **cotangent bundle** of $\mathbb{P}^n$. With its dual $\mathcal{T}_{\mathbb{P}^n}$ the **tangent bundle** of $\mathbb{P}^n$ we can construct an example of symplectic vector bundle. Indeed, for odd n the kernel of a bundle epimorphism

$$\mathcal{T}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}(1),$$

also known as the **null correlation bundle** is a symplectic bundle (For more details see [40] - Chapter 1 - Section 4.2).

1.1.1 Serre duality

For a coherent sheaf $\mathcal{F}$ over a scheme X we will denote by $\mathrm{H}^i(X, \mathcal{F})$ the **i-th cohomology module** and by $h^i(X, \mathcal{F}) := \dim_{\mathbb{K}} \mathrm{H}^i(X, \mathcal{F})$ its dimension. When the scheme X is clear in the context we will omit it.

Definition 1.1.6. A coherent sheaf $\mathcal{E}$ has **natural cohomology in the range** $a \leqslant k \leqslant b$ if for the values of k in this range, at most one of the cohomology modules $\mathrm{H}^q(\mathcal{E}(k))$ is nontrivial, for any q.

Theorem 1.1.7. For each short exact sequence $0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$ of coherent sheaves over a scheme X there exists a natural morphism $\delta^i : \mathrm{H}^i(\mathcal{A}'') \to \mathrm{H}^{i+1}(\mathcal{A}')$, such that we obtain a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(A') \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(A'') \xrightarrow{\delta^0} \cdots \longrightarrow \mathrm{H}^i(A'') \xrightarrow{\delta^i} \mathrm{H}^{i+1}(A') \longrightarrow \cdots$$

Proof. See [23] - Chapter 3 - Theorem 1.1A. □

Theorem 1.1.8 (Serre)**.** Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over Spec A. Let $\mathcal{F}$ be a coherent sheaf on X. Then:

(a) for each $i \geqslant 0$, $\mathrm{H}^i(\mathcal{F})$ is finitely generated A-module;

(b) there is an integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $\mathrm{H}^i(\mathcal{F}(n)) = 0$.

Proof. See [23] - Chapter 3 - Theorem 5.2. □

Let X be a projective scheme of dimension n over a field $\mathbb{K}$, and let $\mathcal{F}$ be a coherent sheaf on X. We have $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > n$ and by the condition (a) of the previous theorem $\mathrm{H}^i(\mathcal{F})$ is a $\mathbb{K}$-vector space of finite dimension. Therefore we define the **Euler characteristic** of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{K}} \mathrm{H}^i(X, \mathcal{F}).$$

An important tool in the computation of the dimension of cohomology modules of coherent sheaves is the Serre duality which now we present a brief overview.

Definition 1.1.9. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the functors $\mathrm{Ext}^i(\mathcal{F}, \cdot)$ as the right derived functors of $\mathrm{Hom}(\mathcal{F}, \cdot)$.

According to [23] - Chapter 3 - Section 6, let $\mathcal{L}$ be a locally-free sheaf of finite rank. For any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ we have

$$\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong \mathrm{H}^i(X, \mathcal{F}),$$

and

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}),$$

for all $i \geqslant 0$.

Theorem 1.1.10 (Serre Duality)**.** Let X be a projective scheme of dimension n over an algebraically closed field $\mathbb{K}$ and let $\mathcal{F}$ be a coherent sheaf over X. Let ω_X be the dualizing sheaf on X, and let $\mathcal{O}_X(1)$ be a very ample sheaf on X, then for all $i \geqslant 0$ there is natural functorial isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \to \mathrm{H}^{n-i}(X, \mathcal{F})^\vee.$$

Proof. See [23] - Chapter 3 - Theorem 7.6. □

Example 1.1.11. Let $\mathcal{F}$ be a locally-free sheaf on $\mathbb{P}^n$, we have $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. By the results above, for all $i \geqslant 0$ we have

$$\begin{aligned}
\mathrm{H}^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee &\cong \mathrm{Ext}^i(\mathcal{F}, \omega_{\mathbb{P}^n}) \\
&\cong \mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^n}, \mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1)) \\
&\cong \mathrm{H}^i(\mathbb{P}^n, \mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1)) \\
&\cong \mathrm{H}^i(\mathbb{P}^n, \mathcal{F}^\vee(-n-1))
\end{aligned}$$

the **Serre duality** on $\mathbb{P}^n$.

1.1.2 Chern Classes

Let X be a projective variety over $\mathbb{K}$. A **cycle of codimension r on X** is an element of the free abelian group $Z(X)$ generated by the closed irreducible subvarieties of X of codimension r. We say that two cycles $C_1, C_2 \in Z(X)$ are **rationally equivalent** if exists a cycle over $\mathbb{P}^1 \times X$ such that the restriction in two fibres $\{t_1\} \times X$ and $\{t_2\} \times X$ are C_1 and C_2, respectively.

For each integer r let $A^r(X)$ be the group of cycles of codimension r on X modulo rational equivalence. The **Chow group** $A(X) = \bigoplus_i A^i(X)$ is the group of cycles modulo rational equivalence. If X is a smooth projective variety $A(X)$ is a commutative associative graded ring called **Chow ring**, whose product is given by the intersection of equivalence class of cycles.

Definition 1.1.12. Let $\mathcal{E}$ be a locally-free sheaf of rank r on a nonsingular quasi-projective variety X, let $\mathbb{P}(\mathcal{E})$ be the associated projective space bundle, and let $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ be the class of divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. For each $i = 0, 1, \cdots, r$ we define the **ith Chern class** to be the element $c_i(E) \in A^i(X)$, with the requirement $c_0(E) = 1$, and

$$\sum_{i=0}^{r}(-1)^i c_i(\mathcal{E})\xi^{r-1} = 0$$

in $A^r(\mathbb{P}(\mathcal{E}))$.

For convenience we define the **total Chern class**

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E})$$

and the **Chern polynomial**

$$c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r.$$

Let $\mathcal{E}$ be a rank r vector bundle on a nonsingular projective variety X. We have the following properties.

(C1) [Normalization] Let $\mathcal{E} \cong \mathcal{O}_X(\mathcal{D})$ be the line bundle associated to a divisor $\mathcal{D}$. Then $c_t(\mathcal{E}) = 1 + \mathcal{D}t$.

(C2) [Pullback] If $f : X' \to X$ is a morphism, then for each i

$$c_i(f^*\mathcal{E}) = f^* c_i(\mathcal{E}),$$

where f^* is the pullback of f.

(C3) [Cartan formula] If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves on X, then

$$c_t(\mathcal{E}) = c_t(\mathcal{E}') \cdot c_t(\mathcal{E}'').$$

(C4) If $\mathcal{E}$ has a filtration $\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots \supseteq \mathcal{E}_s = 0$ whose successive quotients $\mathcal{L}_1, \cdots, \mathcal{L}_s$ are all invertible sheaves, then

$$c_t(\mathcal{E}) = \prod_{i=1}^{s} c_t(\mathcal{L}_i).$$

(C5) Let $\mathcal{E}, \mathcal{F}$ be locally free sheaves of rank r and s, respectively. Write

$$c_t(\mathcal{E}) = \prod_{i=1}^{r} (1 + a_i t)$$

and

$$c_t(\mathcal{F}) = \prod_{i=1}^{s} (1 + b_i t),$$

then we have

$$c_t(\mathcal{E} \otimes \mathcal{F}) = \prod_{i,j} (1 + (a_i + b_j)t)$$

$$c_t(\bigwedge^p \mathcal{E}) = \prod_{1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant r} (1 + (a_{i_1} + \cdots + a_{i_p})t)$$

$$c_t(\mathcal{E}^\vee) = c_{-t}(\mathcal{E}).$$

The properties (C1)-(C3) and the requirement $c_0(\mathcal{E}) = 1$ define the Chern Classes uniquely. For more details see [23] - Appendix A.

Example 1.1.13. If we tensor the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n}(-1) \to 0$$

with $\mathcal{O}_{\mathbb{P}^n}(1)$, one gets the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}(1) \to \mathcal{T}_{\mathbb{P}^n} \to 0,$$

thus by $(C1)$ and $(C3)$

$$c_t(\mathcal{T}_{\mathbb{P}^n}) = (1 + t)^{n+1}.$$

Therefore $c_i(\mathcal{T}_{\mathbb{P}^n}) = \binom{n+1}{i} t^i$ and we often write $c_i = \binom{n+1}{i}$.

Let

$$c_t(\mathcal{E}) = \prod_{i=1}^{r} (1 + a_i t)$$

as above, where the a_i are formal symbols. Then we define the **exponential Chern character**

$$\operatorname{ch}(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i},$$

and the **Todd class** of $\mathcal{E}$,

$$\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{r} \frac{a_i}{1-e^{-a_i}},$$

where $\dfrac{x}{1-e^{-x}} = 1 + \dfrac{1}{2}x + \dfrac{1}{12}x^2 - \dfrac{1}{720}x^4 + \cdots$.

Theorem 1.1.14 (Hirzebruch-Riemann-Roch)**.** For a locally free sheaf $\mathcal{E}$ of rank r on a nonsingular projective variety X of dimension n,

$$\chi(\mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X))_n,$$

where $()_n$ denotes the component of degree $n \in A(X) \otimes \mathbb{Q}$.

This theorem was proved in [27] by Hirzebruch over $\mathbb{C}$ and by Grothendieck in a generalized form over any algebraically closed field [10].

Example 1.1.15. Let X be a curve of genus g and $\mathcal{D}$ a divisor on X. If $\mathcal{E} = \mathcal{O}_X(\mathcal{D})$, we have $\mathrm{ch}(\mathcal{E}) = 1 + \mathcal{D}$. The tangent sheaf is given by $\mathcal{T}_X \cong \mathcal{O}_X(-\mathcal{K})$, where $\mathcal{K}$ is the canonical divisor, thus $\mathrm{td}(\mathcal{T}_X) = 1 - \dfrac{1}{2}\mathcal{K}$. Therefore, by the Hirzebruch-Riemann-Roch theorem, we have

$$\begin{aligned} \chi(\mathcal{O}_X(\mathcal{D})) &= \deg((1+\mathcal{D})(1-\frac{1}{2}\mathcal{K}))_1 \\ &= \deg(\mathcal{D} - \frac{1}{2}\mathcal{K}). \end{aligned}$$

For $\mathcal{D} = 0$, we have $1 - g = \dfrac{1}{2}\mathcal{K}$, thus we can write

$$\chi(\mathcal{O}_X(\mathcal{D})) = \deg\mathcal{D} + 1 - g$$

which is the Riemann-Roch theorem for curves.

1.1.3 Splitting type

We shall say that a rank r vector bundle **splits** when it can be represented as a direct sum of r line bundles. On the projective line we have the following.

Theorem 1.1.16 (Grothendieck)**.** Every rank r vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ has the form

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

with uniquely determined numbers $a_1, \cdots, a_r \in \mathbb{Z}$ and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$.

Proof. See [40] - Theorem 2.1.1. □

By the Cartan formula it follows that for a vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ over $\mathbb{P}^1$ the first Chern class is

$$c_1(\mathcal{E}) = \sum_{i=1}^{r} a_i.$$

Let $\mathcal{E}$ be a rank r vector bundle over $\mathbb{P}^n$. Let G_n be the Grassmann manifold of lines in $\mathbb{P}^n$. Denote by l the point of G_n which corresponds to a projective line $L \subset \mathbb{P}^n$. By Grothendieck's theorem for every $l \in G_n$ there is a r-tuple $(a_1, \cdots, a_r) \in \mathbb{Z}^r$, with $a_1 \geqslant \cdots \geqslant a_r$ such that $\mathcal{E}|_L \cong \oplus_{i=1}^{r} \mathcal{O}_L(a_i)$. The r-tuple $(a_1, \cdots, a_r) \in \mathbb{Z}^r$ is called the **splitting type** of $\mathcal{E}$ on L.

Definition 1.1.17. Let $\mathcal{E}$ be a rank r vector bundle over $\mathbb{P}^n$. Let $L \subset \mathbb{P}^n$ be a line. We say that $\mathcal{E}$ has **trivial splitting type on** L if $\mathcal{E}|_L \cong \mathcal{O}_L^{\bigoplus r}$. Moreover, if $\mathcal{E}$ has trivial splitting type on L, an isomorphism $\Phi : \mathcal{E}|_L \to \mathcal{O}_L^{\oplus r}$ is called a **framing**.

1.1.4 Monads

Monads were first introduced by Horrocks in [29] which has shown that every vector bundle on $\mathbb{P}^2$ and $\mathbb{P}^3$ is the cohomology bundle of a monad. In [8] Barth and Hulek generalized this result on $\mathbb{P}^n$.

Definition 1.1.18. A **monad** over X a projective variety is a complex,

$$\mathcal{M} := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0 \tag{1.1}$$

of coherent sheaves over X which is exact at $\mathcal{A}$ and at $\mathcal{C}$, that means $ba = 0$, a is injective and b is surjective. The coherent sheaf $\mathcal{E} := \ker(b)/\mathrm{Im}(a)$ will be called **cohomology** of $\mathcal{M}$ and also denoted by $\mathrm{H}^\bullet(\mathcal{M}) = \mathcal{E}$.

A cohomology sheaf $\mathcal{E}$ need not be locally free, even if the sheaves in the monad (1.1) are locally free. The **degeneration locus** of the monad (1.1) is defined as

$$\Sigma(\mathcal{M}) := \{x \in X \mid \text{the stalk } \mathcal{E}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\},$$

and $\mathcal{E}$ is locally free if and only if $\Sigma(\mathcal{M})$ is empty. In this case $\mathcal{E}$ is said a **cohomology bundle**.

Next we will give an example of a cohomology sheaf which is not a bundle. First we will need the following lemma.

Lemma 1.1.19. Let

$$\mathcal{M} := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

be a monad of locally free sheaves on a projective variety X. Then

$$\Sigma(\mathcal{M}) = \{x \in X \ \mid a(x) \text{ is not injective}\}.$$

Proof. See [19] - Lemma 4. □

Example 1.1.20. Let $(x_0, \cdots, x_3)$ be a basis of $V^\vee$ and $\mathbb{P}^3 = \mathbb{P}(V)$. Consider the monad

$$\mathcal{M}_1 := 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{b} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0,$$

where $a = \begin{pmatrix} x_0 \\ x_1 \\ 0 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} -x_1 & x_0 & x_2 & x_3 \end{pmatrix}$.

Note that $b \circ a = 0$ and b is surjective. Moreover a is injective if and only if $x_0 \neq 0$ or $x_1 \neq 0$. By Lemma 1.1.19 we have

$$\Sigma(\mathcal{M}_1) = \{\{x_0 = 0\} \cap \{x_1 = 0\}\}.$$

Thus $\Sigma(\mathcal{M}_1) \neq \varnothing$, therefore the cohomology sheaf $\mathcal{E}_1$ of $\mathcal{M}_1$ is not locally-free.

Now let us see an example of a monad whose cohomology sheaf is a bundle.

Example 1.1.21. Let $(x_0, \cdots, x_3)$ be a basis of $V^\vee$ and $\mathbb{P}^3 = \mathbb{P}(V)$. Consider the monad

$$\mathcal{M}_2 := 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{b} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0,$$

where $a = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} -x_1 & x_0 & -x_3 & x_2 \end{pmatrix}$.

Note that $b \circ a = 0$, b is surjective and a is injective. Then by Lemma 1.1.19, $\Sigma(\mathcal{M}_2) = \varnothing$ and therefore the cohomology sheaf $\mathcal{E}_2$ of $\mathcal{M}_2$ is locally-free.

A monad $\mathcal{M}$ has a so called display defined bellow.

Definition 1.1.22. The **display** of a monad $\mathcal{M}$ is the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r, "a"] & \mathcal{B} \arrow[r] \arrow[d, "b"'] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{C} \arrow[r, equal] \arrow[d] & \mathcal{C} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $\mathcal{K} = \ker b$ and $\mathcal{Q} = \operatorname{coker} a$.

From the display one deduces the following.

Lemma 1.1.23. If $\mathcal{E}$ is the cohomology of the monad $\mathcal{M}$, then the rank and Chern polynomial of $\mathcal{E}$ are given by

$$\mathrm{rk}\mathcal{E} = \mathrm{rk}\mathcal{B} - \mathrm{rk}\mathcal{A} - \mathrm{rk}\mathcal{C}$$

$$c_t(\mathcal{E}) = c_t(\mathcal{B})c_t(\mathcal{A})^{-1}c_t(\mathcal{C})^{-1}.$$

Definition 1.1.24. Let $\mathcal{M}$ be a monad of locally-free shaves over a projective variety X, whose cohomology sheaf is locally-free. The complex

$$\mathcal{M}^\vee := 0 \longrightarrow \mathcal{C}^\vee \xrightarrow{b^\vee} \mathcal{B}^\vee \xrightarrow{a^\vee} \mathcal{A}^\vee \longrightarrow 0$$

is called the **dual monad** associated at $\mathcal{M}$.

The condition on the cohomology of the monad being locally-free is crucial. As we can see in the next example.

Example 1.1.25. Consider the monad $\mathcal{M}_1$ of the Example 1.1.20. Dualizing this complex we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{b^\vee} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{a^\vee} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \cdots .$$

Since $a^\vee$ is not surjective in the set $\{\{x_0 = 0\} \cap \{x_1 = 0\}\}$, therefore this complex is not a monad.

Given two monads

$$\mathcal{M} := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$
$$\mathcal{M}' := 0 \longrightarrow \mathcal{A}' \xrightarrow{a'} \mathcal{B}' \xrightarrow{b'} \mathcal{C}' \longrightarrow 0\ ,$$

a **morphism of monads** is a morphism of complexes, i.e. is a triple of morphisms (f, g, h) such that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{M} := 0 \arrow[r] & \mathcal{A} \arrow[r, "a"] \arrow[d, "f"'] & \mathcal{B} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{C} \arrow[r] \arrow[d, "h"'] & 0\ . \\
\mathcal{M}' := 0 \arrow[r] & \mathcal{A}' \arrow[r, "a'"] & \mathcal{B}' \arrow[r, "b'"] & \mathcal{C}' \arrow[r] & 0
\end{tikzcd}$$

In addition, if f, g and h are isomorphisms we say that the monads are **isomorphic**.

Definition 1.1.26. Let X be a projective scheme and let $\mathcal{O}_X$ be a very ample line bundle on X. A monad over X is called **linear** if it is of the form

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus a} \longrightarrow \mathcal{O}_X^{\oplus b} \longrightarrow \mathcal{O}_X(1)^{\oplus c} \longrightarrow 0\ .$$

The cohomology sheaf of a linear monad is called **linear sheaf**.

We have the following useful result.

Proposition 1.1.27. Let $\mathcal{E}$ be a linear sheaf.

(i) $\mathcal{E}$ is reflexive if and only if its degeneration locus is a subvariety of codimension at least 3;

(ii) $\mathcal{E}$ is torsion-free if and only if its degeneration locus is a subvariety of codimension at least 2.

Proof. See [31] - Proposition 4. □

By this proposition we have the following.

Example 1.1.28. The sheaf $\mathcal{E}_1$ given in the Example 1.1.20 is torsion-free, but is not reflexive.

Next we will give an example of a sheaf that is reflexive but is not locally-free.

Example 1.1.29. Let $(x_0, \cdots, x_3)$ be a basis of $V^\vee$ and $\mathbb{P}^3 = \mathbb{P}(V)$. Consider the monad

$$\mathcal{M}_1 := 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{b} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0,$$

where $a = \begin{pmatrix} x_0 \\ x_1 \\ 0 \\ 0 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} -x_1 & x_0 & x_2 & x_3 & 0 \end{pmatrix}$.

Note that $b \circ a = 0$, b is surjective and moreover a is injective if and only if $x_0 \neq 0$, $x_1 \neq 0$ and $x_2 \neq 0$. By Lemma 1.1.19 we have

$$\Sigma(\mathcal{M}_3) = \{\{x_0 = 0\} \cap \{x_1 = 0\} \cap \{x_2 = 0\}\}\,.$$

Thus $\Sigma(\mathcal{M}_3) = \{[0 : 0 : 0 : 1]\}$, therefore by Proposition 1.1.27 the cohomology sheaf $\mathcal{E}_3$ of $\mathcal{M}_3$ is reflexive and is not locally-free.

The existence of linear monads on $\mathbb{P}^n$ was completely characterized by Fløystad in [18] and generalize to projective varieties for Marchesi, Marques and Soares in [36].

Lemma 1.1.30. Let $\mathcal{E} = \mathrm{H}^{\bullet}(\mathcal{M})$, $\mathcal{E}' = \mathrm{H}^{\bullet}(\mathcal{M}')$ be the cohomology bundles of the two monads

$$\mathcal{M} := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$
$$\mathcal{M}' := 0 \longrightarrow \mathcal{A}' \xrightarrow{a'} \mathcal{B}' \xrightarrow{b'} \mathcal{C}' \longrightarrow 0$$

of locally-free sheaves over a projective variety X. The mapping

$$h : \mathrm{Hom}(\mathcal{M}, \mathcal{M}') \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$$

which associates to each homomorphism of monads the induced homomorphism of cohomology bundles is bijective if the following hypotheses are satisfied:

$$\mathrm{Hom}(\mathcal{B}, \mathcal{A}') = \mathrm{Hom}(\mathcal{C}, \mathcal{B}') = 0$$

$$\mathrm{H}^1(X, \mathcal{C}^\vee \otimes \mathcal{A}') = \mathrm{H}^1(X, \mathcal{B}^\vee \otimes \mathcal{A}') = \mathrm{H}^1(X, \mathcal{C}^\vee \otimes \mathcal{B}') = \mathrm{H}^2(X, \mathcal{C}^\vee \otimes \mathcal{A}') = 0.$$

Proof. See [41] - Lemma 4.1.3. □

From this lemma we draw the following two conclusions.

Corollary 1.1.31. If the hypotheses of the Lemma 1.1.30 are satisfied for the pairs of monads $(\mathcal{M}, \mathcal{M})$, $(\mathcal{M}, \mathcal{M}')$, $(\mathcal{M}', \mathcal{M})$ and $(\mathcal{M}', \mathcal{M}')$, then the isomorphisms of the monads $\mathcal{M}$, $\mathcal{M}'$ correspond bijectively (under h) to the isomorphisms of the cohomology bundles $\mathcal{E}$, $\mathcal{E}'$.

Corollary 1.1.32. Let

$$\mathcal{M} := 0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0$$

be a monad of locally-free sheaves with cohomology bundle $\mathcal{E} = \mathrm{H}^\bullet(\mathcal{M})$, and

$$\mathcal{M}^\vee := 0 \longrightarrow \mathcal{C}^\vee \xrightarrow{b^\vee} \mathcal{B}^\vee \xrightarrow{a^\vee} \mathcal{A}^\vee \longrightarrow 0$$

be the dual monad. Furthermore suppose that the hypotheses of the lemma are satisfied for the pairs of monads $(\mathcal{M}, \mathcal{M})$, $(\mathcal{M}, \mathcal{M}^\vee)$, $(\mathcal{M}^\vee, \mathcal{M})$ and $(\mathcal{M}^\vee, \mathcal{M}^\vee)$.

If $f : \mathcal{E} \to \mathcal{E}^\vee$ is a symplectic (respectively an orthogonal) isomorphism of the cohomology bundles, then there are isomorphisms

$$h : \mathcal{C} \to \mathcal{A}^\vee \text{ and } q : \mathcal{B} \to \mathcal{B}^\vee,$$

such that $q^\vee = -q$ (respectively $q^\vee = q$) and $h \circ b = a^\vee \circ q$.

Proof. See [8] - Proposition 5. □

Now we present the theorems of Beilinson about spectral sequences, for the proof and more details see [40] - Chapter 2 - Section 3.

Theorem 1.1.33 (Beilinson, Theorem I)**.** Let $\mathcal{E}$ be a vector bundle over $\mathbb{P}^n$ of rank r. There is a spectral sequence E_r^{pq} with E_1-term

$$E_1^{pq} = \mathrm{H}^q(\mathbb{P}^n, \mathcal{E}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p),$$

which converges to

$$E^i = \begin{cases} \mathcal{E} & \text{for } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $E_\infty^{pq} = 0$, for $p + q = 0$ and $\bigoplus_{p=0}^{n} E_\infty^{-p,p}$ is the associated graded sheaf of a filtration of $\mathcal{E}$.

Theorem 1.1.34 (Beilinson, Theorem II)**.** Let $\mathcal{E}$ be a vector bundle over $\mathbb{P}^n$ of rank r. There is a spectral sequence E_r^{pq} with E_1-term

$$E_1^{pq} = \mathrm{H}^q(\mathbb{P}^n, \mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p),$$

which converges to

$$E^i = \begin{cases} \mathcal{E} & \text{for } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $E_\infty^{pq} = 0$, for $p + q = 0$ and $\bigoplus_{p=0}^{n} E_\infty^{-p,p}$ is the associated graded sheaf of a filtration of $\mathcal{E}$.

1.1.5 Hyperdeterminants of matrices

In [20] Gelfand, Zelevinsky and Kapranov provide a higher-dimensional generalization of the classical notion of the determinant of a square matrix: the hyperdeterminant. The goal of this subsection is to present the relation between hyperdeterminants and degeneracy of bilinear maps. For more details see [42] and [[21] - Chapter 14].

Let $r \geqslant 2$ be an integer. By a **r-dimensional matrix** we shall mean an array $A = (a_{i_1,\cdots,i_r})$ with entries on $\mathbb{K}$, where each index ranges over some finite set, i.e. $0 \leqslant i_j \leqslant k_j$.

The definition of the hyperderminant of A can be stated in geometric, analytic or algebraic terms. Let us give all three formulations.

Geometrically: consider the product $X = \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$ of several projective spaces in the Segre embedding into the projective space $\mathbb{P}^{(k_1+1)\cdots(k_r+1)-1}$ (if $\mathbb{P}^{k_j}$ is the projectivization of a $\mathbb{K}$-vector space $V_j^\vee = \mathbb{K}^{k_j+1}$ then the ambient projective space is $\mathbb{P}(V_1^\vee \otimes \cdots \otimes V_r^\vee)$). Let $X^\vee$ be the projective dual variety of X consisting of all hyperplanes in $\mathbb{P}^{(k_1+1)\cdots(k_r+1)-1}$ tangent at X at some point. The **hyperdeterminant of format** $(k_1+1) \times \cdots \times (k_r+1)$ is the X-discriminant, i.e. a homogeneous polynomial function on $V_1 \otimes \cdots \otimes V_r$ which is a defining equation of $X^\vee$. We denote the hyperdeterminant by Det. If $X^\vee$ is not a hypersurface, we set Det equal to 1, and refer to this case as **trivial**. If each V_j is equipped with a basis then an element $f \in V_1 \otimes \cdots \otimes V_r$ is represented by a matrix $A = (a_{i_1,\cdots,i_r})$, $0 \leqslant i_j \leqslant k_j$ as above, and so $\mathrm{Det}(A)$ is a polynomial function of matrix entries.

Analytically: the hyperplane $\{f = 0\}$ belongs to $X^\vee$ if and only if f vanishes at some point of X with all its first derivatives. If we choose a coordinate system $x^{(j)} = (x_0^{(j)}, x_1^{(j)}, \cdots, x_{k_j}^{(j)})$ on each $V_j^\vee$ then $f \in V_1 \otimes \cdots \otimes V_r$ is represented after restriction on X by a multilinear form

$$f(x^{(1)}, \cdots, x^{(r)}) = \sum_{i_1,\cdots,i_r} a_{i_1,\cdots,i_r}, x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}.$$

Therefore, the condition $\mathrm{Det}(A) = 0$ means that for i, j the system of equations

$$f(x) = \frac{\partial f(x)}{\partial x_i^{(j)}} = 0,$$

has solution $x = (x_{i_1}^{(1)}, \cdots, x_{i_r}^{(r)})$ with all $x^{(j)} \neq 0$. We say that a multilinear form f (or a matrix A) satisfying this condition is **degenerate**.

Algebraically: the degeneracy of a form f can be characterized as follows. We denote by $\mathcal{K}(f)$ (or $\mathcal{K}(A)$) the set of points

$$x = (x^{(1)}, \cdots, x^{(r)}) \in \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$$

such that

$$f(x^{(1)}, \cdots, x^{(j-1)}, y, x^{(j+1)}, \cdots, x^{(r)}) = 0$$

for every $j = 1, \cdots, r$ and every $y \in V_j^\vee$. We shall sometimes call $\mathcal{K}(A)$ the **kernel** of A. For a bilinear form $f(x, y)$, there is a notion of left and right kernels

$$\mathcal{K}_l(f) = \{x | f(x, y) = 0, \text{ for all } y\},$$
$$\mathcal{K}_r(f) = \{y | f(x, y) = 0, \text{ for all } x\}$$

and $\mathcal{K}(f) = \mathcal{K}_l(f) \times \mathcal{K}_r(f)$.

Proposition 1.1.35. A form f is degenerate if and only if $\mathcal{K}(f)$ is non-empty.

Proof. See [21] - Chapter 14 - Proposition 1.1. □

Remark 1.1.36. In particular, for $r = 2$ when f is a bilinear form with a matrix A, the degeneracy of f just defined coincides with the usual notion of degeneracy and means that A is not of maximal rank. Obviously, this condition is of codimension one if and only if A is a square matrix, and in this case $\mathrm{Det}(A)$ coincides with the ordinary determinant $\det(A)$.

1.2 Instanton bundles

In [4] Atiyah, Drinfeld, Hitchin and Manin showed that instanton bundles on $\mathbb{P}^3$ are holomorphic vector bundles on $\mathbb{P}^3$ in correspondence to self-dual solutions of the $SU(2)$ Yang Mills equations over the 4-dimensional sphere S^4 via the Penrose-Ward correspondence.

In [41] Okonek and Spindler used the Salamon's generalization of the Penrose-Ward correspondence to extend the definition of instanton bundles to odd dimensional projective spaces as follows.

Definition 1.2.1. An algebraic rank-$2n$ bundle $\mathcal{E}$ on $\mathbb{P}^{2n+1}$ is a **mathematical instanton bundle** with quantum number (or charge) $c \geqslant 1$ if it has the following properties:

(i) the Chern polynomial of $\mathcal{E}$ is $c_t(E) = \left(\frac{1}{1-t^2}\right)^c$;

(ii) $\mathcal{E}$ has natural cohomology in the range $-2n - 1 \leqslant l \leqslant 0$;

(iii) $\mathcal{E}$ has trivial splitting type;

(iv) $\mathcal{E}$ is simple;

(v) $\mathcal{E}$ has a symplectic structure.

The authors also proved in [41] that a mathematical instanton bundle is the cohomology of a linear monad.

Theorem 1.2.2. Any mathematical instanton bundle $\mathcal{E}$ can be represented as the cohomology of a monad of the form

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E} \otimes \Omega^2(1)) \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(-1) \longrightarrow \mathrm{H}^1(\mathcal{E} \otimes \Omega^1)) \otimes \mathcal{O}_{\mathbb{P}^{2n+1}} \longrightarrow$$

$$\longrightarrow \mathrm{H}^1(\mathcal{E}(-1)) \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(1) \longrightarrow 0.$$

Proof. See [41] - Corollary 1.4. □

In [2] Ancona and Ottaviani proved that every $2n$-bundle satisfying (i) and (ii) is simple (Proposition 2.11), thus the condition (iv) is superfluous. In this paper the authors also defined the mathematical instanton bundles excluding the condition (v), which on $\mathbb{P}^3$ is superfluous. The condition (iii) is also superfluous on $\mathbb{P}^3$, since simple rank 2 bundles $\mathcal{E}$ with first Chern class equals to zero has trivial splitting type by the Grauert-Mülich theorem (See [40]).

In order to extend the study to even-dimensional projective spaces, in [31] Jardim considers a more general class of objects.

Definition 1.2.3. A coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ is called an **instanton sheaf** of charge c and rank r if it is defined as the cohomology of a linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus c}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(r+2c)} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus c}(1) \to 0.$$

Moreover, if $\mathcal{E}$ is locally free, we call it an **instanton bundle**.

The Chern polynomial of an instanton sheaf is then $c_t(\mathcal{E}) = \left(\dfrac{1}{1-t^2}\right)^c$. In particular, we have $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = c > 0$.

In [31] Jardim gave the following alternative cohomological characterization.

Definition 1.2.4. An **instanton sheaf on** $\mathbb{P}^n$ $(n \geqslant 2)$ is a torsion-free coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ with $c_1 = 0$ satisfying the following cohomological conditions:

(1) $\mathrm{H}^0(\mathcal{E}(-1)) = \mathrm{H}^n(\mathcal{E}(-n)) = 0$;

(2) $\mathrm{H}^1(\mathcal{E}(-2)) = \mathrm{H}^{n-1}(\mathcal{E}(1-n)) = 0$, if $n \geqslant 3$;

(3) $\mathrm{H}^i(\mathcal{E}(-k)) = 0$ for $2 \leqslant p \leqslant n-2$ and all k, if $n \geqslant 4$.

The integer $c = -\chi(\mathcal{E}(-1))$ is called charge of $\mathcal{E}$.

In addition, if $\mathcal{E}$ is a rank $2n$ bundle on $\mathbb{P}^{2n+1}$ with trivial splitting type, then $\mathcal{E}$ is a mathematical instanton bundle as defined above. In [31] Jardim generalized the Theorem 1.2.2 for any torsion-free sheaf on $\mathbb{P}^n$ satisfying the cohomological conditions (1)-(3) of Definition 1.2.4.

Theorem 1.2.5. If $\mathcal{E}$ is a torsion-free sheaf on $\mathbb{P}^n$ satisfying:

(1) $\mathrm{H}^0(\mathcal{E}(-1)) = \mathrm{H}^n(\mathcal{E}(-n)) = 0$;

(2) $\mathrm{H}^1(\mathcal{E}(-2)) = \mathrm{H}^{n-1}(\mathcal{E}(1-n)) = 0$, if $n \geqslant 3$;

(3) $\mathrm{H}^i(\mathcal{E}(-k)) = 0$ for $2 \leqslant p \leqslant n-2$ and all k, if $n \geqslant 4$.

then $\mathcal{E}$ is linear, and can be represented as the cohomology of the monad:

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E} \otimes \Omega^2(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathrm{H}^1(\mathcal{E} \otimes \Omega^1)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow$$

$$\longrightarrow \mathrm{H}^1(\mathcal{E}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0.$$

Proof. See [31] - Theorem 3.

□

As we can see in the next result, the dual of an instanton bundle $\mathcal{E}$ is a instanton bundle and cohomology of the dual monad of $\mathcal{E}$.

Lemma 1.2.6. If $\mathcal{E}$ is an instanton bundle, then its dual bundle $\mathcal{E}^\vee$ is the cohomology of the monad which is dual to the monad that defines $\mathcal{E}$. In particular, $\mathcal{E}^\vee$ is again instanton.

Proof. See [32] - Lemma 2.2. □

Let $\mathcal{E}$ be an orthogonal instanton bundle over $\mathbb{P}^n$ ($n \geqslant 3$), i. e., an instanton vector bundle on $\mathbb{P}^n$ which is orthogonal, with charge c, rank r and no global sections ($\mathrm{H}^0(\mathcal{E}) = 0$). Let us compute some of the cohomology groups that will be needed in the next chapter.

By the instantons cohomological conditions in Definition 1.2.4 one has

(1) $\mathrm{H}^0(\mathcal{E}(-1)) = \mathrm{H}^n(\mathcal{E}(-n)) = 0$;

(2) $\mathrm{H}^1(\mathcal{E}(-2)) = \mathrm{H}^{n-1}(\mathcal{E}(1-n)) = 0$, if $n \geqslant 3$;

(3) $\mathrm{H}^i(\mathcal{E}(-k)) = 0$ for $2 \leqslant p \leqslant n-2$ and all k, if $n \geqslant 4$.

Consider the following exact sequence

$$0 \longrightarrow \mathcal{E}(-i-1) \longrightarrow \mathcal{E}(-i) \longrightarrow \mathcal{E}(-i)\mid_{\mathbb{P}^{n-1}} \longrightarrow 0\,.$$

For $i = 0$, one has $\mathrm{H}^0(\mathcal{E}(-1)) = 0$.

For $i = 1$, one has $\mathrm{H}^0(\mathcal{E}(-2)) = \mathrm{H}^0(E(-1)\mid_{\mathbb{P}^{n-1}}) = 0$.

For $i = 2$, one has $\mathrm{H}^1(\mathcal{E}(-3)) = \mathrm{H}^0(\mathcal{E}(-3)) = \mathrm{H}^0(\mathcal{E}(-2)\mid_{\mathbb{P}^{n-1}}) = 0$.

For $i = 3$, one has $\mathrm{H}^1(\mathcal{E}(-4)) = \mathrm{H}^0(\mathcal{E}(-4)) = \mathrm{H}^0(\mathcal{E}(-3)\mid_{\mathbb{P}^{n-1}}) = 0$.

For $i = 4$, one has $\mathrm{H}^1(\mathcal{E}(-5)) = \mathrm{H}^0(\mathcal{E}(-5)) = \mathrm{H}^0(\mathcal{E}(-4)\mid_{\mathbb{P}^{n-1}}) = 0$.

Continuing this process, one has $\mathrm{H}^0(\mathcal{E}(-i)\mid_{\mathbb{P}^{n-1}}) = 0$ for all $i \geqslant 1$, $\mathrm{H}^0(\mathcal{E}(-i)) = 0$ for all $i \geqslant 0$ and $\mathrm{H}^1(\mathcal{E}(-i)) = 0$ for all $i \geqslant 2$. So by Serre duality one has

$\mathbb{P}^n$, $n \geqslant 3$	$\mathcal{E}(-n-1)$	$\mathcal{E}(-n)$	$\mathcal{E}(1-n)$	$\dots$	$\mathcal{E}(-2)$	$\mathcal{E}(-1)$	$\mathcal{E}$
h^0	0	0	0	0	0	0	0
h^1	0	0	0	0	0	$*$	$*$
h^2	0	0	0	0	0	0	0
$\vdots$	0	0	0	0	0	0	0
h^{n-2}	0	0	0	0	0	0	0
h^{n-1}	$*$	$*$	0	0	0	0	0
h^n	0	0	0	0	0	0	0

Recall that the charge $c = -\chi(\mathcal{E}(-1))$, thus by Serre duality we have $h^{n-1}(\mathcal{E}(-n)) = h^1(\mathcal{E}(-1)) = c$. Now we can use Hirzebruch-Riemann-Roch theorem to complete the table.

As we saw in Example 1.1.13 the Chern classes of tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ on $\mathbb{P}^n$ are given by $c_i := c_i(\mathcal{T}_{\mathbb{P}^n}) = \binom{n+1}{i}$, and therefore the Todd class of $\mathcal{T}_{\mathbb{P}^n}$ is given by

$$td(T_{\mathbb{P}^n}) \quad = \quad 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \ldots$$

On the other hand, the Chern character of $\mathcal{E}$ is given by

$$ch(\mathcal{E}) \quad = \quad r + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2 + 2c_2(\mathcal{E})}{2} + \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})}{6} + \ldots$$

So by Hirzebruch-Riemann-Roch theorem one has

$$\begin{aligned} \chi(\mathcal{E}) &= \deg\,(ch(\mathcal{E}).td(\mathcal{T}_{\mathbb{P}^n}))_n \\ &= (n-1)c - r. \end{aligned}$$

Since, $\mathrm{H}^0(\mathcal{E}) = \mathrm{H}^2(\mathcal{E}) = \cdots = \mathrm{H}^n(\mathcal{E}) = 0$ and

$$\chi(\mathcal{E}) = \sum_{i=0}^{n} (-1)^i h^i(\mathcal{E})$$

we have

$$h^1(\mathcal{E}) = (n-1)c - r,$$

and by Serre Duality

$$h^1(\mathcal{E}) = h^{n-1}(\mathcal{E}(-n-1)) = (n-1)c - r.$$

Therefore, for $n \geqslant 3$ and $-n-1 \leqslant k \leqslant 0$, we have

$$h^i(\mathcal{E}(k)) = \begin{cases} c, & \text{if } (i,k) \in \{(1,-1),(n-1,-n)\}; \\ (n-1)c - r, & \text{if } (i,k) \in \{(1,0),(n-1,-n-1)\}; \\ 0, & \text{otherwise.} \end{cases}$$

1.3 Moduli space and Geometric invariant theory

A moduli problem is essentially a classification problem. Given a collection of algebraic objects $\mathcal{A}$ and an equivalence relation $\sim$ on $\mathcal{A}$ our goal is to describe the set of equivalence classes $\mathcal{A}/\sim$. Moreover, we would like to endow the set $\mathcal{A}/\sim$ with an algebraic structure which reflects the algebraic structure it is on $\mathcal{A}$.

Definition 1.3.1. A **naive moduli problem** (in algebraic geometry) is a collection $\mathcal{A}$ of algebraic objects and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 1.3.2. Let $\mathcal{A}$ be the collection of vector bundles on a fixed scheme X and $\sim$ be the relation given by isomorphisms of vector bundles.

Definition 1.3.3. Let $(\mathcal{A}, \sim)$ be a naive moduli problem. Then an **(extended) moduli problem** is given by

(1) sets $\mathcal{A}_X$ of families over X and an equivalence relation $\sim_X$ on $\mathcal{A}_X$, for all schemes X;

(2) pullback maps $f^* : \mathcal{A}_Y \to \mathcal{A}_X$, for every morphism of schemes $X \to Y$, satisfying the following properties:

   (i) $(\mathcal{A}_{\mathrm{Spec}\,\mathbb{K}}, \sim_{\mathrm{Spec}\,\mathbb{K}}) = (\mathcal{A}, \sim)$;

   (ii) for the identity $\mathrm{Id} : X \to X$ and any family $\mathcal{F}$ over X, we have $\mathrm{Id}^*\mathcal{F} = \mathcal{F}$;

   (iii) for a morphism $f : Y \to X$ and equivalent families $\mathcal{F} \sim_X \mathcal{G}$ over X, we have $f^*\mathcal{F} \sim_Y f^*\mathcal{G}$;

   (iv) for morphisms $f : X \to Y$ and $g : Y \to Z$, and a family $\mathcal{F}$ over Z, we have an equivalence $(g \circ f)^*\mathcal{F} \sim_X f^*g^*\mathcal{F}$;

For a family $\mathcal{F}$ over X and a point $s : \mathrm{Spec}\,\mathbb{K} \to X$, we write $\mathcal{F}_s := s^*\mathcal{F}$ to denote the corresponding family over $\mathrm{Spec}\,\mathbb{K}$.

Definition 1.3.4. Given a moduli problem $\mathcal{M}$, we say that a family $\mathcal{F}$ over a scheme X has the **local universal property** if for any family $\mathcal{G}$ over a scheme Y and for any point $y \in Y$, there exists a neighbourhood U of y in Y and a morphism $f : U \to X$ such that $\mathcal{G}|_U \sim_U f^*\mathcal{F}$.

A moduli problem defines a contravariant functor $\mathcal{M} : \mathrm{Sch} \to \mathrm{Set}$ from the category of schemes to the category of sets, given by

$$\mathcal{M}(X) := \{\text{families over } X\}/ \sim_S$$

$$\mathcal{M}(f : Y \to X) := f^* : \mathcal{M}(X) \to \mathcal{M}(Y).$$

The contravariant functor defined above is called **moduli functor**. We will often refer to a moduli problem simply by its moduli functor.

Definition 1.3.5. We say that a moduli functor $\mathcal{M} : \mathrm{Sch} \to \mathrm{Set}$ is represented by an object $M \in Ob(\mathrm{Sch})$ if it is isomorphic to h_M the functor of points of M, defined by $h_M(X) = \mathrm{Hom}_{\mathcal{C}}(X, M)$. The object M is called a **fine moduli space** for the moduli functor $\mathcal{M}$.

If a fine moduli space exists, then it is unique up to isomorphism. Unfortunately there are many moduli problems which not admit a fine moduli space. Therefore it is necessary to find some weaker condition which nevertheless determines a unique algebraic structure on $\mathcal{A}/ \sim$.

Definition 1.3.6. We say that a moduli functor $\mathcal{M} : \mathrm{Sch} \to \mathrm{Set}$ is corepresented by an object $M \in Ob(\mathrm{Sch})$ if there is a natural transformation $\eta : \mathcal{M} \to h_M$ such that $\eta(\{pt\})$ is bijective and for any object $N \in Ob(\mathrm{Sch})$ and for any natural transformation $\beta : \mathcal{M} \to h_N$ there exists a unique morphism $\gamma : M \to N$ such that $\beta = h_\gamma \eta$. The object M is called a **coarse moduli space** for the contravariant moduli functor $\mathcal{M}$.

If a coarse moduli space exists, then it is unique up to isomorphism. A fine moduli space for a given contravariant moduli functor $\mathcal{M}$ is always a coarse moduli space for this moduli functor but the reciprocal is not true in general.

Proposition 1.3.7. . Let (M, η) be a coarse moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if

(1) there exists a family U over M such that $\eta_M(U) = \mathrm{Id}_M$;

(2) for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G} \Leftrightarrow \eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

Proof. See [39] - Proposition 1.8. □

Unfortunately, sometimes is not even possible to obtain a coarse moduli space, as we can see in the following result.

Lemma 1.3.8. Let $\mathcal{M}$ be a moduli problem and suppose there exists a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_1$ for all $s \neq 0$ and $\mathcal{F}_0 \nsim \mathcal{F}_1$. Then for any scheme M and natural transformation $\eta : M \to h_M$, we have that $\eta_{\mathbb{A}^1}(\mathcal{F}) : \mathbb{A}^1 \to M$ is constant. In particular, there is no coarse moduli space for this moduli problem.

Proof. See [30] - Lemma 2.27. □

1.3.1 Actions

In this subsection we introduce some concepts about actions of algebraic groups.

Definition 1.3.9. An **algebraic group** over $\mathbb{K}$ is a scheme G over $\mathbb{K}$ with morphisms $e : \mathrm{Spec}\,\mathbb{K} \to G$ (identity element), $m : G \times G \to G$ (group law) and $i : G \to G$ (group inversion) such that we have commutative diagrams

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{Id}\times m"] \arrow[d, "m\times \mathrm{Id}"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}\mathbb{K} \times G \arrow[r, "e\times \mathrm{Id}"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times \mathrm{Spec}\mathbb{K} \arrow[l, "\mathrm{Id}\times e"'] \arrow[dl, "\cong"] \\
& G &
\end{tikzcd}
$$

$$\begin{tikzcd} G \arrow[r, "(i,\mathrm{Id})"] \arrow[d, "a"'] & G \times G \arrow[d, "m"'] & G \arrow[l, "(\mathrm{Id},i)"'] \arrow[d, "a"] \\ \mathrm{Spec}\mathbb{K} \arrow[r, "e"'] & G & \mathrm{Spec}\mathbb{K}, \arrow[l, "e"] \end{tikzcd}$$

where $a : G \to \mathrm{Spec}\mathbb{K}$ sends all elements $g \in G$ into the single point $\{\mathrm{pt}\} \in \mathrm{Spec}\ \mathbb{K}$.

We say G is an **affine algebraic group** if the underlying scheme G is affine. A **homomorphism** of algebraic groups G and H is a morphism of schemes $f : G \to H$ such that the following diagram commutes

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "m_H"'] & H. \end{tikzcd}$$

An **algebraic subgroup** of G is a closed subscheme H such that the immersion $H \hookrightarrow G$ is a homomorphism of algebraic groups. We say that an algebraic group G' is an **algebraic quotient** of G if there is a homomorphism of algebraic groups $f : G \to G'$ which is flat and surjective.

Many of the groups that we are already familiar with are affine algebraic groups.

Example 1.3.10. The general linear group GL_n over $\mathbb{K}$ is an affine algebraic group.

Definition 1.3.11. A **linear algebraic group** is a subgroup of GL_n which is defined by polynomial equations.

In particular, any linear algebraic group is an affine algebraic group. In fact, the converse statement is also true: any affine algebraic group is a linear algebraic group (See [30] - Theorem 3.9). Moreover, any linear algebraic group G possesses an unique maximal connected normal solvable subgroup, called **radical**.

Definition 1.3.12. An algebraic group G is called **reductive** if the radical of G is a torus.

Example 1.3.13. The algebraic groups GL_n, SL_n and PGL_n are all reductive.

Definition 1.3.14. An algebraic **action** of an affine algebraic group G on a $\mathbb{K}-$scheme X is a morphism of schemes $\sigma : G \times X \to X$ such that the following diagrams commute:

$$\begin{tikzcd} \mathrm{Spec}\mathbb{K} \times X \arrow[r, "e\times\mathrm{Id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{Id}_G\times\sigma"] \arrow[d, "m_G\times\mathrm{Id}_X"'] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

Definition 1.3.15. Let G be an affine algebraic group acting on a scheme X by $\sigma : G \times X \to X$ and let $x \in X$ be a point.

(i) The **stabiliser** $G \cdot x$ of x is the subset

$$G_x = \{g \in G | \sigma(g, x) = x\}.$$

(ii) The **orbit** G_x of x is the closed subgroup of G

$$O(x) = \{\sigma(g, x) | \text{ for all } g \in G\}.$$

If all the orbits are closed subsets of X, we say that the action of G is **closed**.

A point x (subset W) of X is said to be **invariant** under G if $\sigma(g, x) = x$ $(\sigma(g, W) = W)$ for every $g \in G$. Given actions of G on two schemes X and Y, we say that the morphism $\phi : X \to Y$ is a G**-morphism** if $\phi(\sigma_X(g, x)) = \sigma_Y(g, \phi(x))$, for all $g \in G$, $x \in X$. In particular, when G acts trivially on Y (i.e. $\sigma_Y(g, y) = y$), ϕ is said a G**-invariant morphism**.

If only the Id_G has fixed points (i.e. $\sigma(g, x) = x$ for some $x \in X$ implies $g = e$) then we say the action is **free**. Moreover, if the action is free, then all the orbits are closed (See [9] - Chapter 1 - Section 1.8).

1.3.2 Quotients

In this subsection we introduce the definition of categorical quotient and its relation with moduli spaces.

Definition 1.3.16. A **categorical quotient** for the action of G on X is a pair (Y, π), where $\pi : X \to Y$ is a G-invariant morphism of schemes which is universal; that is, every other G-invariant morphism $f : X \to Z$ factors uniquely through π so that there exists a unique morphism $h : Y \to Z$ such that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & Y \arrow[dl, dashed, "\exists! h"] \\ & Z & \end{tikzcd}$$

Furthermore, if the preimage of each point in Y is a unique orbit, then we say π is an **orbit space**.

We shall often speak of the scheme as a categorical quotient or orbit space without mention of the morphism φ and use the following notation $Y = X//G$. A categorical quotient is determined up to isomorphism.

Definition 1.3.17. A categorical quotient (Y, π) for the action of G on X is a **universal categorical quotient** if the following holds: Let Y' be an algebraic variety and $Y' \to Y$ a morphism. Consider the action of G on $X \times_Y Y'$ given by the fiber product. If p_2 denotes the projection $X \times_Y Y' \to Y'$, then (Y', p_2) is a categorical quotient of $X \times_Y Y'$ by G. In this case we say also that Y is a universal categorical quotient of X by G, without mention the morphism.

Let G be an affine algebraic group acting on a scheme X over $\mathbb{K}$. The group G acts on the $\mathbb{K}$-algebra $\mathcal{O}(X)$ of regular functions on X by $g \cdot f(x) = f(g^{-1} \cdot x)$. We denote the **subalgebra of invariant functions** by

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) | g \cdot f = f \text{ for all } g \in G\}.$$

Similarly if $U \subset X$ is a subset which is G-invariant, then G acts on $\mathcal{O}_X(U)$ and we denote by $\mathcal{O}_X(U)^G$ the subalgebra of invariant functions.

Definition 1.3.18. A **good quotient** for the action of G on X is a pair (Y, π), where

(i) π is G-invariant.

(ii) π is surjective.

(iii) If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\pi^{-1}(U))^G$.

(iv) If $W \subset X$ is a G-invariant closed subset of X, its image $\pi(W)$ is closed in Y.

(v) If W_1 and W_2 are disjoint G-invariant closed subsets, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint.

(vi) π is affine.

Moreover, if the preimage of each point is an unique orbit then we say π is a **geometric quotient**.

In particular, any good quotient is a categorical quotient.

Proposition 1.3.19. Let G be an affine algebraic group acting on a scheme X and suppose we have a morphism $\pi : X \to Y$ satisfying properties (i), (iii), (iv) and (v) of Definition 1.3.18. Then π is a categorical quotient.

Proof. See [30] - Proposition 3.30. □

In some cases, a good quotient $X//G$ shares the same properties of X.

Proposition 1.3.20. Let G be an algebraic group acting on an scheme X, and suppose that a good quotient $(X//G, \pi)$ of X by G exists. Then

(i) If X is reduced, then $(X//G)$ is reduced.

(ii) If X is connected, then $(X//G)$ is connected.

(iii) If X is irreducible, then $(X//G)$ is irreducible.

(iv) If X is normal, then $(X//G)$ is normal.

Proof. See [15] - Proposition 2.15. □

Our interest in the existence of categorical quotients and orbit spaces lies in the fact that they can be related with moduli spaces as show the following results.

Proposition 1.3.21. For a moduli problem $\mathcal{M}$, let $\mathcal{F}$ be a family with the local universal property over a scheme X. Furthermore, suppose that there is an algebraic group G acting on X such that two points x, y lie in the same G-orbit if and only if $\mathcal{F}_x \sim \mathcal{F}_y$. Then

(i) any coarse moduli space is a categorical quotient of the G-action on X;

(ii) a categorical quotient of the G-action on X is a coarse moduli space if and only if it is an orbit space.

Proof. See [30] - Proposition 3.35. □

Theorem 1.3.22. Let G be a reductive group acting on an affine scheme X. Then a good quotient of X by G exists, and $X//G$ is affine scheme. Moreover it is an universal categorical quotient.

Proof. See [15] - Theorem 2.16. □

2 Orthogonal instanton bundles and symmetric forms

The focus of this chapter is the study of orthogonal instanton bundles on $\mathbb{P}^n$, $n \geqslant 3$. In the first section we provide a bijection between equivalence classes of orthogonal instanton bundles with no global sections and symmetric forms, this construction was inspired by the work of Bruzzo, Markushevich and Tikhomirov which in [11] relate symplectic instanton bundles on $\mathbb{P}^3$ and the space of hyperwebs of quadrics. Using such correspondence we are able to construct explicit examples of orthogonal instanton bundles on $\mathbb{P}^n$ with no global sections and charge $c \geqslant 3$. In the second section we use the techniques of geometric invariant theory described on Chapter 1 to construct $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}$ the moduli space of orthogonal instanton bundles on $\mathbb{P}^n$ with no global sections and charge $c \geqslant 3$.

2.1 The equivalence

From now on, let us consider $\mathbb{P}^n = \mathbb{P}(V)$, where V is a $(n+1)$-dimensional $\mathbb{K}$-vector space, for $n \geqslant 3$. Let H_c be a c-dimensional $\mathbb{K}$-vector space, with $c \geqslant 1$.

Let $\mathcal{E}$ be an orthogonal instanton bundle on $\mathbb{P}^n$ with charge c, rank r and no global sections. We saw in the previous chapter that $\dim \mathrm{H}^{n-1}(\mathcal{E}(-n)) = c$. Thus consider a triple $(\mathcal{E}, \phi, f)$, where $f : H_c \xrightarrow{\cong} \mathrm{H}^{n-1}(\mathcal{E}(-n))$ and ϕ is an orthogonal structure of $\mathcal{E}$, i.e. an isomorphism $\phi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}^\vee$, such that $\phi^\vee = \phi$. Consider the set of all triples $(\mathcal{E}, \phi, f)$, next we will define an equivalence relation between triples which makes the orthogonal structures ϕ and the isomorphisms f compatible.

Definition 2.1.1. Two triples $(\mathcal{E}_1, \phi_1, f_1)$ and $(\mathcal{E}_2, \phi_2, f_2)$ are called **equivalent** if there is an isomorphism $g : \mathcal{E}_1 \xrightarrow{\cong} \mathcal{E}_2$ such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\phi_1"] \arrow[d, "g"'] & \mathcal{E}_1^\vee \\
\mathcal{E}_2 \arrow[r, "\phi_2"'] & \mathcal{E}_2^\vee \arrow[u, "g^\vee"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H_c \arrow[r, "f_1"] \arrow[d, "\lambda \mathrm{Id}_c"'] & \mathrm{H}^{n-1}(\mathcal{E}_1(-n)) \arrow[d, "g_*"] \\
H_c \arrow[r, "f_2"'] & \mathrm{H}^{n-1}(\mathcal{E}_2(-n)),
\end{tikzcd}$$

where $g_* : \mathrm{H}^{n-1}(\mathcal{E}_1(-n)) \xrightarrow{\cong} \mathrm{H}^{n-1}(\mathcal{E}_2(-n))$ is the induced isomorphism in cohomology and $\lambda \in \{-1, 1\}$.

We denote by $[\mathcal{E},\phi,f]$ the **equivalence class** of a triple $(\mathcal{E},\phi,f)$.

Fixing the integers c and r, we will denote by $\mathbb{E}[c,r]$ the set of all equivalence classes $[\mathcal{E},\phi,f]$ of orthogonal instanton bundles, with charge c, rank r and no global sections over $\mathbb{P}^n$. Consider the Euler exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \xrightarrow{i_1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0 \tag{2.1}$$

and the sequences

$$0 \longrightarrow \Omega^{i+1}_{\mathbb{P}^n} \longrightarrow \wedge^{i+1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-i-1) \longrightarrow \Omega^i_{\mathbb{P}^n} \longrightarrow 0\,, \tag{2.2}$$

with $1 \leqslant i \leqslant n-2$, and

$$0 \longrightarrow \bigwedge^{n+1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1) \longrightarrow \bigwedge^{n} V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \xrightarrow{i_2} \Omega^{n-1}_{\mathbb{P}^n} \longrightarrow 0 \tag{2.3}$$

induced by the Koszul complex of $V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^n}$, where ev denotes the canonical evaluation map.

Tensoring these sequences with $\mathcal{E}$:

(i) from (2.1) we obtain $\mathrm{H}^i(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}) = 0$, for $i = 0, 3, \ldots, n$ and the exact sequence

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}) \xrightarrow{i_1} \mathrm{H}^1(V^\vee \otimes \mathcal{E}(-1)) \longrightarrow \mathrm{H}^1(\mathcal{E}) \longrightarrow H^2(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}) \longrightarrow 0; \tag{2.4}$$

(ii) from (2.2) we obtain

$$\mathrm{H}^j(\mathcal{E} \otimes \Omega^i_{\mathbb{P}^n}) \cong \mathrm{H}^{j+1}(\mathcal{E} \otimes \Omega^{i+1}_{\mathbb{P}^n}) \tag{2.5}$$

and $\mathrm{H}^0(\mathcal{E} \otimes \Omega^i_{\mathbb{P}^n}) = \mathrm{H}^{n-1}(\mathcal{E} \otimes \Omega^{i+1}_{\mathbb{P}^n}) = 0$, for $1 \leqslant i \leqslant n-2$ and $0 \leqslant j \leqslant n-1$;

(iii) from (2.3) we obtain $\mathrm{H}^i(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}) = 0$, for $i = 0, \ldots, n-3, n$ and the exact sequence

$$0 \longrightarrow \mathrm{H}^{n-2}(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}) \longrightarrow \mathrm{H}^{n-1}(\bigwedge^{n+1} V^\vee \otimes \mathcal{E}(-n-1)) \longrightarrow \tag{2.6}$$

$$\longrightarrow \mathrm{H}^{n-1}(\bigwedge^{n} V^\vee \otimes \mathcal{E}(-n)) \xrightarrow{i_2} \mathrm{H}^{n-1}(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}) \longrightarrow 0.$$

Therefore we have

$$\mathrm{H}^2(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}) = \mathrm{H}^3(\mathcal{E} \otimes \Omega^2_{\mathbb{P}^n}) = \cdots = \mathrm{H}^n(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}) = 0,$$

$$\mathrm{H}^{n-2}(\mathcal{E}\otimes\Omega^{n-1}_{\mathbb{P}^n}) = \mathrm{H}^{n-3}(\mathcal{E}\otimes\Omega^{n-2}_{\mathbb{P}^n}) = \cdots = \mathrm{H}^{0}(\mathcal{E}\otimes\Omega^{1}_{\mathbb{P}^n}) = 0,\ \text{and}$$

$$h^{n-1}(\mathcal{E}\otimes\Omega^{n-1}_{\mathbb{P}^n}) = h^1(\mathcal{E}\otimes\Omega^{1}_{\mathbb{P}^n}) = h^1(V^\vee\otimes\mathcal{E}(-1)) - h^1(\mathcal{E}) = 2c + r.$$

and one has the exact sequences

$$0 \longrightarrow \mathrm{H}^{n-1}(\mathcal{E}(-n-1))\otimes \bigwedge^{n+1} V^\vee \xrightarrow{a} \mathrm{H}^{n-1}(\mathcal{E}(-n))\otimes \bigwedge^{n} V^\vee \xrightarrow{i_2} \mathrm{H}^{n-1}(\mathcal{E}\otimes\Omega^{n-1}_{\mathbb{P}^n}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E}\otimes\Omega^1_{\mathbb{P}^n}) \xrightarrow{i_1} \mathrm{H}^1(\mathcal{E}(-1))\otimes V^\vee \xrightarrow{b} \mathrm{H}^1(\mathcal{E}) \longrightarrow 0. \tag{2.7}$$

Now observe that by the functoriality of the Serre duality we have $i_1 = i_2^\vee$. And moreover, since $\mathcal{E}$ is orthogonal by Theorem 1.2.5 and Corollary 1.1.32, there exists a symmetric isomorphism $\partial\otimes\mathrm{Id} : \mathrm{H}^1(\mathcal{E}\otimes\Omega^1))\otimes\mathcal{O}_{\mathbb{P}^n} \to \mathrm{H}^{n-1}(\mathcal{E}\otimes\Omega^{n-1})\otimes\mathcal{O}_{\mathbb{P}^n}$. Thus we have the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{n-1}(\mathcal{E}(-n-1))\otimes \bigwedge^{n+1} V^\vee \arrow[r, "a"] & \mathrm{H}^{n-1}(\mathcal{E}(-n))\otimes \bigwedge^{n} V^\vee \arrow[r, "i_2"] \arrow[d, "A'"] & \mathrm{H}^{n-1}(\mathcal{E}\otimes\Omega^{n-1}_{\mathbb{P}^n}) \arrow[r] & 0 \\
0 & \mathrm{H}^1(\mathcal{E}) \arrow[l] & \mathrm{H}^1(\mathcal{E}(-1))\otimes V^\vee \arrow[l, "b"'] & \mathrm{H}^1(\mathcal{E}\otimes\Omega^1_{\mathbb{P}^n}) \arrow[l, "i_2^\vee"'] \arrow[u, "\partial", "\cong"'] & 0 \arrow[l]
\end{tikzcd} \tag{2.8}$$

where $A' = i_2^\vee \circ \partial^{-1}\circ i_2$.
The Euler exact sequence (2.1) yields the canonical isomorphism $\omega_{\mathbb{P}^n} \xrightarrow{\cong} \bigwedge^{n+1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1)$.
So fixing an isomorphism $\tau : \mathbb{K} \xrightarrow{\cong} \bigwedge^{n+1} V^\vee$ we have the isomorphisms

$$\tau_1 : V \xrightarrow{\cong} \bigwedge^{n} V^\vee \ \text{ and } \ \tau_2 : \omega_{\mathbb{P}^n} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^n}(-n-1)\ . \tag{2.9}$$

Thus each class $[\mathcal{E}, f, \phi] \in \mathbb{E}$ defines a morphism $A : \mathrm{H}_c\otimes V \longrightarrow \mathrm{H}_c^\vee \otimes V^\vee$ by the following composition

$$A : H_c\otimes V \xrightarrow{\mathrm{Id}\otimes\tau_1} H_c\otimes\bigwedge^n V^\vee \xrightarrow{f\otimes\mathrm{Id}} \mathrm{H}^{n-1}(\mathcal{E}(-n))\otimes\bigwedge^n V^\vee \xrightarrow{A'} \mathrm{H}^1(\mathcal{E}(-1))\otimes V^\vee \xrightarrow{\phi\otimes\mathrm{Id}} \mathrm{H}^1(\mathcal{E}^\vee(-1))\otimes V^\vee$$

$$\xrightarrow{\mathrm{SD}\otimes\mathrm{Id}} \mathrm{H}^{n-1}(\mathcal{E}(1)\otimes\omega_{\mathbb{P}^n})^\vee\otimes V^\vee \xrightarrow{\tau_2\otimes\mathrm{Id}} \mathrm{H}^{n-1}(\mathcal{E}(-n))^\vee\otimes V^\vee \xrightarrow{f^\vee\otimes\mathrm{Id}} H_c^\vee\otimes V^\vee,$$

where SD is the Serre duality isomorphism.

Therefore we can write

$$A = ((f^\vee \circ \tau_2 \circ \mathrm{SD} \circ \phi) \otimes \mathrm{Id}) \circ A' \circ (f \otimes \tau_1). \tag{2.10}$$

Note that, since τ is a multiplication by scalar, A does not depend on the choice of τ. And moreover, note that A is symmetric, indeed

(i) first note that $A' = A'^\vee$, this follows from

$$A'^\vee = (i_2^\vee \circ \partial^{-1} \circ i_2)^\vee = i_2^\vee \circ (\partial^{-1})^\vee \circ i_2 = i_2^\vee \circ \partial^{-1} \circ i_2 = A'.$$

(ii) SD is functorial and auto-dual, so $SD^\vee = SD$ and it commutes with every morphism.

(iii) the maps ϕ and A' commute, because the cohomology groups of $\mathcal{E}$ and $\mathcal{E}^\vee$ are isomorphic, since $\phi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}^\vee$.

(iv) As $\tau : \mathbb{K} : \xrightarrow{\simeq} \bigwedge^{n+1} V^\vee$, one has that τ is a multiplication by a scalar λ. So $\tau_1 = \lambda \mathrm{Id}$ and τ_2 is also a multiplication by λ. It follows that τ_1 and τ_2 are both auto-dual and commute with every morphism.

Then

$$\begin{aligned} A^\vee &= (f^\vee \otimes \tau_1^\vee) \circ A'^\vee \circ ((\phi^\vee \circ \mathrm{SD}^\vee \circ \tau_2^\vee \circ f) \otimes \mathrm{Id}) \\ &= (f^\vee \otimes \tau_1) \circ A'^\vee \circ ((\phi \circ \mathrm{SD} \circ \tau_2 \circ f) \otimes \mathrm{Id}) \\ &= ((f^\vee \circ \tau_2 \circ \mathrm{SD} \circ \phi) \otimes \mathrm{Id}) \circ A' \circ (f \otimes \tau_1) \\ &= A, \end{aligned}$$

so

$$A \in (S^2 H_c^\vee \otimes S^2 V^\vee) \oplus (\bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee). \tag{2.11}$$

Now we will show that $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$.

Denote $W = \dfrac{H_c \otimes V}{\text{Ker } A}$. By (2.8) and (2.10) we have the commutative diagram of exact rows

$$\begin{tikzcd}
 & & H_c \otimes V \arrow[d, "f\otimes\tau_1"] \arrow[ddd, dashed, bend right=40, "A"'] & & \\
0 \arrow[r] & H^{n-1}(\mathcal{E}(-n-1)) \otimes \bigwedge^{n+1} V^\vee \arrow[r, "a"] & H^{n-1}(\mathcal{E}(-n)) \otimes \bigwedge^{n} V^\vee \arrow[r, "i_2"] \arrow[d, "A'"] & H^{n-1}(\mathcal{E} \otimes \Omega^{n-1}_{\mathbb{P}^n}) \arrow[r] & 0 \\
0 & H^1(\mathcal{E}) \arrow[l] & H^1(\mathcal{E}(-1)) \otimes V^\vee \arrow[l, "b"'] \arrow[d, "(f^\vee \circ\tau_2\circ SD\circ\phi)\otimes \mathrm{Id}"] & H^1(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}) \arrow[l, "i_2^\vee"'] \arrow[u, "\partial", "\cong"'] & 0 \arrow[l] \\
 & & H_c^\vee \otimes V^\vee & &
\end{tikzcd} \tag{2.12}$$

Thus

$$\begin{array}{rcl}
\text{Ker } (A) & \cong & \text{Ker } (A') \\
 & \cong & \text{Ker } (i_2^\vee \circ \partial^{-1} \circ i_2) \\
 & \cong & \text{Ker } (i_2) \\
 & \cong & \text{Im } (a)
\end{array}
\quad \text{and} \quad
\begin{array}{rcl}
\text{Im } (A) & \cong & \text{Im } (A') \\
 & \cong & \text{Im } (i_2^\vee \circ \partial^{-1} \circ i_2) \\
 & \cong & \text{Im } (i_2^\vee) \\
 & \cong & \text{Ker } (b)
\end{array}$$

and

$$\begin{array}{rcl}
\dim W & = & \dim(H_c \otimes V) - \dim\, (\ker A) \\
 & = & \dim\, (H_c \otimes V) - \dim\, (\text{Im } a) \\
 & = & \dim\, (H_c \otimes V) - \dim\, (\text{H }^{n-1}(E(-n-1) \otimes \bigwedge^{n+1} V^\vee) \\
 & = & (n+1)c - ((n-1)c - r) \\
 & = & 2c + r.
\end{array}$$

Hence, we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \text{Ker } A \arrow[r] & H_c \otimes V \arrow[r, "p"] \arrow[d, "A"] & W \arrow[r] \arrow[d, "q_A", "\cong"'] & 0 \\
0 & \text{Ker } A^\vee \arrow[l] & H_c^\vee \otimes V^\vee \arrow[l] & W^\vee \arrow[l, "p^\vee"'] & 0. \arrow[l]
\end{tikzcd} \tag{2.13}$$

where p is the canonical projection.

Combining $A = p^\vee \circ q_A \circ p$ and $A = A^\vee$ we have $p^\vee \circ q_A \circ p = p^\vee \circ q_A^\vee \circ p$. But the projection is an epimorphism and its dual a monomorphism, hence $q_A^\vee = q_A$ and $q_A : W \xrightarrow{\cong} W^\vee$ is a symmetric isomorphism.

So we can define the induced morphism of sheaves

$$a_A^\vee : W^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{p^\vee \otimes \mathrm{Id}} H_c^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\mathrm{Id} \otimes ev} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \tag{2.14}$$

which is surjective, therefore $a^\vee$ is injective and the composition

$$\psi : H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{q_A \otimes \mathrm{Id}} W^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{a_A^\vee} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

is zero. Indeed, by the diagram (2.12) we have

$$\begin{aligned}
\psi &= a_A^\vee \circ (q_A \otimes \mathrm{Id}) \circ a_A \\
&= (\mathrm{Id} \circ ev) \circ (A \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes ev^\vee) \\
&= (\mathrm{Id} \circ ev) \circ ((((f^\vee \circ \tau_2 \circ \mathrm{SD} \circ \phi) \otimes \mathrm{Id}) \circ A' \circ (f \otimes \tau_1)) \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes ev^\vee) \\
&= ((f^\vee \circ \tau_2 \circ \mathrm{SD} \circ \phi) \otimes ev) \circ (A' \otimes \mathrm{Id}) \circ ((f \otimes \tau_1) \otimes ev^\vee) \\
&= ((f^\vee \circ \tau_2 \circ \mathrm{SD} \circ \phi) \otimes ev) \circ ((c \circ \partial^{-1} \circ b) \otimes \mathrm{id}) \circ ((f \otimes \tau_1) \otimes ev^\vee) \\
&= 0,
\end{aligned}$$

since by (2.1) Im $i_2^\vee \subset \ker ev$.

Now let us prove that $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$. Since A is symmetric, we can write $A = A_1 + A_2$, where $A_1 \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$ and $A_2 \in S^2 H_c^\vee \otimes S^2 V^\vee$. By the Euler sequence (2.1) we have

$$0 \longrightarrow \textstyle\bigwedge^2(\Omega(1)) \longrightarrow \bigwedge^2 V^\vee \otimes \mathcal{O} \xrightarrow{(\mathrm{Id} \otimes ev) \circ (i \otimes \mathrm{Id})} V^\vee \otimes \mathcal{O}(1) \xrightarrow{ev} \mathcal{O}(2) \longrightarrow 0, \tag{2.15}$$

where $i : \bigwedge^2 V^\vee \hookrightarrow V^\vee \otimes V^\vee$ is the inclusion.

Note that $\psi = (\mathrm{Id} \otimes ev) \circ A \circ (\mathrm{Id} \otimes ev^\vee)$, thus

$$\psi = (\mathrm{Id} \otimes ev) \circ A_1 \circ (\mathrm{Id} \otimes ev^\vee) + (\mathrm{Id} \otimes ev) \circ A_2 \circ (\mathrm{Id} \otimes ev^\vee).$$

By the sequence (2.15), we have Im$A_1 \subset \ker ev$ and therefore $\psi = (\mathrm{Id} \otimes ev) \circ A_2 \circ (\mathrm{Id} \otimes ev^\vee)$. Moreover, Im$A_2 \subset S^2 H_c^\vee \otimes S^2 V^\vee \not\subset \ker ev$, otherwise the evaluation map would be the zero map. Hence, $\psi = 0$ implies $A_2 = 0$ and therefore $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$.

For each $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$, by the discussion of the previous paragraph we have $(\mathrm{Id} \otimes ev) \circ A \circ (\mathrm{Id} \otimes ev^\vee) = 0$, therefore we can associate the monad

$$\mathcal{M}_A : H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{a_A^\vee \circ (q_A \otimes \mathrm{Id})} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1), \tag{2.16}$$

whose cohomology sheaf is defined by

$$\mathcal{E}_A := \frac{\text{Ker } (a_A^\vee \circ (q_A \otimes Id))}{\text{Im } a_A}. \tag{2.17}$$

Recall that A is called non-degenerate if $A(h \otimes v) \neq 0$ for any non-zero decomposable tensor $h \otimes v \in H_c \otimes V$. Hence similar to [12] the following are equivalent:

(i) $a_A^\vee \circ (q_A \otimes \text{Id})$ is surjective;

(ii) the image of a_A is a subbundle;

(iii) A is non-degenerate.

Indeed, the conditions (i) and (ii) are equivalent by definition. Now $a_A^\vee \circ (q_A \otimes \text{Id})$ is surjective if and only if for any $< v > \in \mathbb{P}(V)$ the induced homomorphism $W \to H_c^\vee \otimes < v >^\vee$ on the fibre is surjective, or equivalently, that $H_c \otimes < v > \xrightarrow{A} H_c^\vee \otimes V^\vee$ is injective.

From the previous observation, the map A defined by (2.10) has the following properties:

(A1) rank $(A : H_c \otimes V \to H_c^\vee \otimes V^\vee) = 2c + r$;

(A2) A is non degenerate;

(A3) there exists $q_A : W \xrightarrow{\cong} W^\vee$ symmetric, where $W = \dfrac{H_c \otimes V}{\text{Ker } A}$.

Consider the set

$$\mathcal{A}[c, r] := \left\{ A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee;\ \text{such that (A1)-(A3) holds} \right\},$$

our next step is to prove a bijection between sets $\mathcal{A}[c, r]$ and $\mathbb{E}[c, r]$. To do so, we will need the next lemma.

Lemma 2.1.2. For any $A \in \mathcal{A}[c, r]$, there are isomorphisms

$$\begin{array}{ccc} H_c \cong \mathrm{H}^{n-1}(\mathcal{E}_A \otimes \Omega^n(1)) & W \cong \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^1) & \ker A^\vee \cong \mathrm{H}^1(\mathcal{E}_A) \\ H_c^\vee \cong \mathrm{H}^1(\mathcal{E}_A(-1)) & W^\vee \cong \mathrm{H}^{n-1}(\mathcal{E}_A \otimes \Omega^{n-1}) & \end{array}$$

which are compatible with the Serre duality and the orthogonal structure $\mathcal{E}_A \cong \mathcal{E}_A^\vee$ and give the following commutative diagram

$$\begin{tikzcd}
H_c \otimes V \arrow[r] \arrow[d, "\cong"] & W^\vee \arrow[d, "\cong"] & W \arrow[l, "q_A"'] \arrow[r] \arrow[d, "\cong"] & H_c^\vee \otimes V^\vee \arrow[r] \arrow[d, "\cong"] & \ker A^\vee \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathrm{H}^{n-1}(\mathcal{E}_A \otimes \Omega^n(1)) \otimes V \arrow[r] & \mathrm{H}^{n-1}(\mathcal{E}_A \otimes \Omega^{n-1}) & \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^1) \arrow[l, "\cong"] \arrow[r] & \mathrm{H}^1(\mathcal{E}_A(-1)) \otimes V^\vee \arrow[r] & \mathrm{H}^1(\mathcal{E}_A) \arrow[r] & 0
\end{tikzcd}$$

Proof. Given $A \in \mathcal{A}[c,r]$ we have the monad

$$\mathcal{M}_A : H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{a_A^\vee \circ (q_A \otimes \mathrm{Id})} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1), \tag{2.18}$$

whose cohomology bundle is $\mathcal{E}_A$. On the other hand, applying the Beilinson spectral sequence (Theorem 1.1.34) to $\mathcal{E}_A(-1)$, one has

$$E_1^{p,q} = H^q(\mathcal{E}_A(-1) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

$$d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

$$E_2^{p,q} = \frac{\text{Ker } d_1^{p,q}}{\text{Im } d_1^{p-1,q}}$$

and

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

Thus $d_2^{p,q} = 0$, for all p,q and

$$E_2^{p,q} = E_\infty^{p,q}.$$

So we have $d_2^{-2,1} = \text{Ker } d_1^{p,q} = 0$ and $d_2^{0,1} = \dfrac{\text{Ker } d_1^{0,1}}{\text{Im } d_1^{-1,1}} = \dfrac{E_1^{0,1}}{\text{Im } d_1^{-1,1}} = \text{Coker } d_1^{-1,1} = 0$. Thus $d_1^{-2,1}$ is a monomorphism, $d_1^{-1,1}$ is an epimorphism, and

$$\mathcal{E}_A(-1) \cong \frac{\text{Ker } d_1^{-1,1}}{\text{Im } d_1^{-2,1}}.$$

Therefore we obtain the monad

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E}_A(1) \otimes \Omega^2_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \xrightarrow{d_1^{-2,1}} \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^1_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{d_1^{-1,1}} \mathrm{H}^1(\mathcal{E}_A(-1)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0\,,$$

and tensoring this monad by $\mathcal{O}_{\mathbb{P}^n}(1)$, we obtain the monad

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E}_A(1) \otimes \Omega^2_{\mathbb{P}^3}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{d_1^{-2,1}} \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^1_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{d_1^{-1,1}} \mathrm{H}^1(\mathcal{E}_A(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0\,, \tag{2.19}$$

whose cohomology is isomorphic to $\mathcal{E}_A$.

Obviously, $\mathcal{E}_A \cong \mathcal{E}_A$ thus by Corollary 1.1.31 we have the isomorphism of the monads (2.18) and (2.19), which gives us the isomorphisms $H_c \cong \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^2(1)) \cong \mathrm{H}^{n-1}(\mathcal{E}_A \otimes \Omega^n(1))$, $W \cong \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^1)$ and $H_c^\vee \cong \mathrm{H}^1(\mathcal{E}_A(-1))$. By Serre duality, we have $W^\vee \cong \mathrm{H}^1(\mathcal{E}_A \otimes \Omega^1)^\vee \cong \mathrm{H}^{n-1}(\mathcal{E}_A \otimes \Omega^{n-1})$. The last isomorphism follows by (2.13) and (2.7).

Finally the commutativity of the diagram follows by the functoriality of Serre-duality (compare the sequences (2.4) and (2.6) with the diagram (2.13)). □

Thanks to the previous lemma, we have the following result.

Theorem 2.1.3. There exists a bijection between the equivalence classes $[E, \phi, f] \in \mathbb{E}[c, r]$ of orthogonal instanton bundles of charge c, rank r, with no global sections on $\mathbb{P}^n$ and the set $A \in \mathcal{A}[c, r]$.

Proof. By the previous construction, given an equivalence class $[\mathcal{E}, \phi, f] \in \mathbb{E}[c, r]$, there exists $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$ which satisfies (A1)-(A3). Reciprocally, given $A \in \mathcal{A}[c, r]$, there exists a monad

$$\mathcal{M}_A : H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{a_A^\vee \circ (q_A \otimes \mathrm{Id})} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)\ . \tag{2.20}$$

Now, by Theorem 1.2.5, $\mathcal{E}$ is cohomology of the monad

$$0 \longrightarrow \mathrm{H}^1(\mathcal{E}(1) \otimes \Omega^2_{\mathbb{P}^3}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{d_1^{-2,1}} \mathrm{H}^1(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{d_1^{-1,1}} \mathrm{H}^1(\mathcal{E}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0\ . \tag{2.21}$$

By the Lemma (2.1.2) the monads (2.20) and (2.21) are isomorphic, thus by Corollary 1.1.31, A defines a monad whose cohomology sheaf $\mathcal{E}_A$ is isomorphic to $\mathcal{E}$.

Tensoring $\mathcal{M}_A$ by $\mathcal{O}_{\mathbb{P}^n}(-n)$ and using (2.17), we obtain $\mathrm{H}^{n-1}(\mathcal{E}_A(-n)) \cong \mathrm{H}^n(H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1))$. Note that $\dim_{\mathbb{K}} \mathrm{H}^n(H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1)) = c$, then there exists $f_A : H_c \xrightarrow{\cong} \mathrm{H}^{n-1}(\mathcal{E}_A(-n))$.

Furthermore, the symmetric map q_A induces a canonical isomorphism of monads

$$\begin{tikzcd}
\mathcal{M}_A : \arrow[d, "\Phi_A"'] & H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r, "a_A"] \arrow[d, "\mathrm{Id}"'] & W \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "a_A^\vee \circ (q_A \otimes \mathrm{Id})"] \arrow[d, "q_a \otimes \mathrm{Id}"] & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \arrow[d, "\mathrm{Id}"'] \\
\mathcal{M}_A^\vee : & H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r, "(q_A \otimes \mathrm{Id}) \circ a_A"'] & W^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "a_A^\vee"'] & H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1)
\end{tikzcd}$$

which by Corollary 1.1.31 induces a symmetric isomorphism of vector bundles $\phi_A : \mathcal{E}_A \xrightarrow{\cong} \mathcal{E}_A^\vee$. By Lemma 1.2.6 we have that $\mathcal{E}_A^\vee$ is the cohomology bundle of $\mathcal{M}_A^\vee$.

Thus, the data $[\mathcal{E}_A, \phi_A, f_A]$ can be recovered from A.

□

By Theorem 2.1.3 the existence of orthogonal instanton bundles, with charge c, rank r and no global sections on $\mathbb{P}^n$ is related with the existence of symmetric and non-degenerate linear forms. This approach will be extremely helpful in the proof of the next result.

Theorem 2.1.4. Let c be an integer, with $c \geqslant 3$. Every orthogonal instanton bundle with no global sections on $\mathbb{P}^n$ and charge c has rank $(n-1)c$. Moreover, there are no orthogonal instanton bundles with no global sections, and charge c equal 1 or 2 on $\mathbb{P}^n$.

Proof. First suppose that there exists an orthogonal instanton bundle $\mathcal{E}$ with no global sections, with charge c and rank r over $\mathbb{P}^n$ and consider its equivalence class $[\mathcal{E}, \phi, f]$. By Theorem 2.1.3 there exists $A \in \mathcal{A}[c, r]$ associated with $[\mathcal{E}, \phi, f]$.

Given $A \in \mathcal{A}[c, r]$, with some abuse of notation, let us also denote by A the matrix associated with the morphism $A : H_c \otimes V \longrightarrow H_c^\vee \otimes V^\vee$. By Proposition 1.1.35 and Remark 1.1.36, A is non degenerate if and only if $\det A \neq 0$. This means that A is invertible and we have

$$\operatorname{rank} A = 2c + r = (n+1)c, \tag{2.22}$$

therefore $\operatorname{rank} \mathcal{E} = r = (n+1)c - 2c = (n-1)c$.

Finally, if $c = 1$, then $A \in \bigwedge^2 H_1^\vee \otimes \bigwedge^2 V^\vee \cong 0$, but the zero map is degenerate.

If $c = 2$, then $A \in \bigwedge^2 H_2^\vee \otimes \bigwedge^2 V^\vee \cong \mathbb{K} \otimes \bigwedge^2 V^\vee$, so A is skew-symmetric. But A is also symmetric, hence A is the zero map.

Therefore there are no orthogonal instanton bundles, with no global sections and charge 1 or 2 on $\mathbb{P}^n$.

$\square$

Now, we will show how to use the equivalence of Theorem 2.1.3 to construct explicit examples of orthogonal instanton bundles on $\mathbb{P}^n$. To compute the next examples we use Macaulay2 [22].

By Theorem 2.1.4, we can rewrite the diagram (2.13) as

$$\begin{tikzcd}
0 \arrow[r] & \ker A \arrow[r] & H_c \otimes V \arrow[r, "\mathrm{Id}"] \arrow[d, "A"] & W \arrow[r] \arrow[d, "q_A", "\cong"'] & 0 \\
0 & \ker A^\vee \arrow[l] & H_c^\vee \otimes V^\vee \arrow[l] & W^\vee \arrow[l, "\mathrm{Id}"'] & 0, \arrow[l]
\end{tikzcd}$$

then $A \cong q_A$. Moreover, we have

$$a_A^\vee : W^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\mathrm{Id}} H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{ev} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1).$$

So if $\{x_0, x_1, \ldots, x_n\}$ is a basis of $V^\vee$, we have the monad

$$\mathcal{M}_A : H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{a_A^\vee \circ (A \otimes \mathrm{Id})} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \,, \tag{2.23}$$

where $a_A^\vee$ is given by

$$a_A^\vee = \begin{pmatrix} x_0 & x_1 & \ldots & x_n & 0 & 0 & \ldots & 0 & \cdots & & & & & & & & \\ 0 & 0 & \ldots & 0 & x_0 & x_1 & \ldots & x_n & \cdots & & & & & & & & \\ & & & & & & & & \ddots & & & & & & & & \\ & & & & & & & & \cdots & 0 & 0 & \ldots & 0 & x_0 & x_1 & \ldots & x_n \end{pmatrix}_{(n+1)\times(n+1)c.}$$

Theorem 2.1.3 combined with Proposition 1.1.35 simplifies the search for orthogonal instanton bundles, indeed it translates our existence problem in a linear algebra problem: we have to look for invertible matrices in $\bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$. Recall that every skew-symmetric matrix M can be written as a block diagonal matrix

$$\begin{pmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{matrix} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{matrix} 0 & \lambda_l \\ -\lambda_l & 0 \end{matrix} \end{pmatrix}_{2l\times 2l} . \tag{2.24}$$

where $\pm i\lambda_i$ are the non-zero eigenvalues of M. In order to build examples of orthogonal instanton bundles with charge c even on $\mathbb{P}^n$, with n odd, we can take two matrices B an C as in (2.24), where:

- B is a $c \times c$ skew-symmetric matrix;
- C is an $(n+1) \times (n+1)$ skew-symmetric matrix.

So if we consider $A = B \otimes C$, then $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$.

Example 2.1.5. Let us construct an example of orthogonal instanton bundle with no global sections and charge 6 on $\mathbb{P}^3$. Let $\{x_0, x_1, x_2, x_3\}$ be a basis for $V^\vee$. Consider

$$B = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Let $A = B \otimes C \in \bigwedge^2 H_6^\vee \otimes \bigwedge^2 V^\vee$. We have that rank $A = 24$, so A is invertible and therefore non degenarate by Theorem 1.1.35 and Remark 1.1.36. Thus by Theorem 2.1.3 and Theorem 2.1.4 we have the monad

$$\mathcal{O}_{\mathbb{P}^3}^6(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{24} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}^6(1)$$

where

$$\alpha = \begin{pmatrix}
x_0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 0 & 0 \\
0 & x_0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & 0 & 0 \\
0 & x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & x_0 & 0 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & x_2 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & x_0 & 0 & 0 \\
0 & 0 & 0 & x_1 & 0 & 0 \\
0 & 0 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & x_0 & 0 \\
0 & 0 & 0 & 0 & x_1 & 0 \\
0 & 0 & 0 & 0 & x_2 & 0 \\
0 & 0 & 0 & 0 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & x_0 \\
0 & 0 & 0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & 0 & x_3
\end{pmatrix}, \ \beta^t = \begin{pmatrix}
0 & 2x_1 & 0 & 0 & 0 & 0 \\
0 & -2x_0 & 0 & 0 & 0 & 0 \\
0 & -6x_3 & 0 & 0 & 0 & 0 \\
0 & 6x_2 & 0 & 0 & 0 & 0 \\
-2x_1 & 0 & 0 & 0 & 0 & 0 \\
2x_0 & 0 & 0 & 0 & 0 & 0 \\
6x_3 & 0 & 0 & 0 & 0 & 0 \\
-6x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & 0 & 0 \\
0 & 0 & 0 & x_0 & 0 & 0 \\
0 & 0 & 0 & -3x_3 & 0 & 0 \\
0 & 0 & 0 & 3x_2 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & x_0 & 0 & 0 & 0 \\
0 & 0 & -3x_3 & 0 & 0 & 0 \\
0 & 0 & 3x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & -x_0 \\
0 & 0 & 0 & 0 & 0 & -3x_3 \\
0 & 0 & 0 & 0 & 0 & 3x_2 \\
0 & 0 & 0 & 0 & -x_1 & 0 \\
0 & 0 & 0 & 0 & x_0 & 0 \\
0 & 0 & 0 & 0 & 3x_3 & 0 \\
0 & 0 & 0 & 0 & -3x_2 & 0
\end{pmatrix}$$

whose cohomology bundle is an orthogonal instanton bundle with no global sections, charge 6 and rank 12 on $\mathbb{P}^3$.

As the reader can see in the next example, when c is odd or n is even, we need to be a little more careful, because skew-symmetric matrices of odd order do not have complete rank.

Example 2.1.6. For $c = 5$ and $n = 3$, consider

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \end{pmatrix} B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} C_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

And let $A = B_1 \otimes C_1 + B_2 \otimes C_2 + B_3 \otimes C_3$. We have that rank $A = 20$, so A is invertible and therefore non degenarate by Theorem 1.1.35 and Remark 1.1.36. Thus by Theorem 2.1.3 and Theorem 2.1.4 we have the monad

$$\mathcal{O}_{\mathbb{P}^3}^5(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{20} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}^5(1)$$

where

$$\alpha = \begin{pmatrix} x_0 & 0 & 0 & 0 & \\ x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & x_0 \\ 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & x_3 \end{pmatrix} \text{ and } \beta^t = \begin{pmatrix} 0 & x_1 & x_3 & 0 & x_3 \\ 0 & -x_0 & x_2 & 0 & x_2 \\ 0 & x_3 & -x_1 & 0 & -x_1 \\ 0 & -x_2 & -x_0 & 0 & -x_0 \\ -x_1 & 0 & x_1 & x_3 & 0 \\ x_0 & 0 & -x_0 & 0 & 0 \\ -x_3 & 0 & x_3 & x_3 & 0 \\ x_2 & 0 & -x_2 & -x_0 - x_2 & 0 \\ -x_3 & -x_1 & 0 & 0 & x_3 \\ x_2 & x_0 & 0 & 0 & 0 \\ x_1 & -x_3 & 0 & 0 & x_3 \\ x_0 & x_2 & 0 & 0 & -x_0 - x_2 \\ 0 & -x_3 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & -x_3 & 0 & 0 & -x_1 \\ 0 & x_0 + x_2 & 0 & 0 & -x_0 \\ -x_3 & 0 & -x_3 & -x_3 & 0 \\ -x_2 & 0 & 0 & -x_2 & 0 \\ -x_3 & 0 & -x_3 & 0 & 0 \\ x_0 & 0 & x_0 + x_2 & x_0 & 0 \end{pmatrix}$$

as constructed before. The vector bundle cohomology of the monad (2.1.6) is an orthogonal instanton bundle with no global sections, charge 5 and rank 10.

2.2 Moduli space of Orthogonal instanton bundles on $\mathbb{P}^n$

Now that we ensured the existence of orthogonal instanton bundles of higher rank, with no global sections on $\mathbb{P}^n$, it is natural to ask: can we give to this family of bundles a structure of moduli space? In this section we will describe how we use techniques of geometric invariant theory (GIT) to construct $\mathcal{M}^{O}_{\mathbb{P}^n}(c)$, the moduli space of orthogonal instanton bundles with charge c, rank $(n-1)c$ and no global sections on $\mathbb{P}^n$, for $n, c \geqslant 3$.

First note that if $r = (n-1)c$, the conditions (A1) and (A3) are superfluous, and we have that

$$\mathcal{A}[c,(n-1)c] = \{A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee; A \text{ is non degenerate}\}.$$

Denote $\mathbb{E}_c = \mathbb{E}[c,(n-1)c]$, $\mathcal{A}_c = \mathcal{A}[c,(n-1)c]$, $G = Gl(H_c)$, and let $\widetilde{\mathbb{E}_c}$ be the set of isomorphism classes $[\mathcal{E}, \phi]$ such that $[\mathcal{E}, \phi, f] \in \mathbb{E}_c$. Consider the action

$$\begin{array}{rcl} \alpha : G \times \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee & \to & \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee \\ (h, A) & \mapsto & (h \otimes \mathrm{Id})A(h^\vee \otimes \mathrm{Id}). \end{array}$$

We will need the following lemmas in order to prove the main theorem of this section.

Lemma 2.2.1. The set $\mathcal{A}_c$ is a G-invariant subset of $\bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$.

Proof. Let $h \in G$, $A \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$ and $B = \alpha(h, A)$ the image of h and A by the previous action, that means

$$B = (h \otimes \mathrm{Id})A(h^\vee \otimes \mathrm{Id}).$$

We can write $A = \sum_i (C_i \otimes D_i)$, where $C_i \in \bigwedge^2 H_c^\vee$ and $D_i \in \bigwedge^2 V^\vee$ for all integer i. Thus

$$\begin{aligned} B &= (h \otimes \mathrm{Id})(\sum_i (C_i \otimes D_i))(h^\vee \otimes \mathrm{Id}) \\ &= \sum_i ((hC_i h^\vee) \otimes D_i). \end{aligned}$$

Since $hC_ih^\vee \in \bigwedge^2 H_c^\vee$ for all integer i, it follows that $B \in \bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$. $\square$

The bijection given in the next theorem is the key ingredient to construct the $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$ moduli space of orthogonal instanton bundles with charge c, rank $(n-1)c$ and no global sections on $\mathbb{P}^n$, for $n, c \geqslant 3$.

Theorem 2.2.2. There is a bijection from the set of isomorphism classes $\widetilde{\mathbb{E}_c}$ with the orbit space $\mathcal{A}_c/G$. The isotropy group in each point is $\{\pm \mathrm{Id}_{H_c}\}$.

Proof. Given $A \in \mathcal{A}_c$ by Theorem 2.1.3 and Theorem 2.1.4 there exists $[\mathcal{E}_A, \phi_A, f_A] \in \mathbb{E}_c$, then we can define

$$\begin{array}{rccc} \Psi: & \mathcal{A}_c & \rightarrow & \widetilde{\mathbb{E}_c} \\ & A & \mapsto & [\mathcal{E}_A, \phi_A]. \end{array}$$

We will prove that $\Psi/G : A_c/G \rightarrow \widetilde{\mathbb{E}_c}$ is a bijection.

First note that Ψ factors through $\mathcal{A}_c/G$. Indeed, consider $A, B \in \mathcal{A}_c$ such that there exists $h \in G$ with $\alpha(h, A) = B$. Thus we have the following commutative diagram.

$$\begin{tikzcd} H_c \otimes V \arrow[r, "A"] \arrow[d, "h\otimes Id"'] & H_c^\vee \otimes V^\vee \arrow[d, "(h^\vee)^{-1}\otimes Id"] \\ H_c \otimes V \arrow[r, "B"'] & H_c^\vee \otimes V^\vee. \end{tikzcd}$$

Since $A, B \in \mathcal{A}_c$, we have A and B invertible and by diagram (2.13), we have the following commutative diagram,

$$\begin{tikzcd} H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r, "\mathrm{Id}\otimes ev^\vee"] \arrow[d, "h\otimes\mathrm{Id}"'] & W_A \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "A\otimes\mathrm{Id}"] \arrow[d, "h\otimes\mathrm{Id}"'] & W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "\mathrm{Id}\otimes ev"] \arrow[d, "(h^\vee)^{-1}\otimes\mathrm{Id}"] & H_c^\vee \mathcal{O}_{\mathbb{P}^n}(1) \arrow[d, "(h^\vee)^{-1}\otimes\mathrm{Id}"] \\ H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r, "ev^\vee"'] & W_B \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "B\otimes\mathrm{Id}"'] & W_B^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "ev"'] & H_c^\vee \mathcal{O}_{\mathbb{P}^n}(1). \end{tikzcd}$$

Thus we have the following isomorphism of monads

$$\begin{tikzcd} \mathcal{M}_A: & 0 \arrow[r] & H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r, "a_A"] \arrow[d, "h\otimes\mathrm{Id}"'] & W_A \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "a_A^\vee\circ(A\otimes\mathrm{Id})"] \arrow[d, "h\otimes\mathrm{Id}"] & H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \arrow[r] \arrow[d, "(h^\vee)^{-1}\otimes\mathrm{Id}"] & 0 \\ \mathcal{M}_B: & 0 \arrow[r] & H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r, "a_B"'] & W_B \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "a_B^\vee\circ(B\otimes\mathrm{Id})"'] & H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \arrow[r] & 0. \end{tikzcd}$$

Considering the cohomology of the previous monads $\mathcal{M}_A$ and $\mathcal{M}_B$, by Corollary 1.1.31 and Corollary 1.1.32 we get $\Psi(A) = [\mathcal{E}_A, \phi_A] = [\mathcal{E}_B, \phi_B] = \Psi(B)$. Thus we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}_c \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & \widetilde{\mathbb{E}}_c \\
& \mathcal{A}_c/G \arrow[ur, "\Psi/G"'] &
\end{tikzcd}$$

The projection π is surjective by definition and we have by Theorem 2.1.3 and Theorem 2.1.4 that Ψ is surjective as well. This implies that Ψ/G is surjective.

We need now to prove that Ψ/G is injective. Indeed, let $A, B \in \mathcal{A}_c$ such that $\Psi(A) = [\mathcal{E}_A, \phi_A] = [\mathcal{E}_B, \phi_B] = \Psi(B)$, we will show that there exists $h \in G$ such that $A = \alpha(h, B)$. If $[\mathcal{E}_A, \phi_A] = [\mathcal{E}_B, \phi_B]$, then by definition there exists an isomorphism $g : \mathcal{E}_A \xrightarrow{\cong} \mathcal{E}_B$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{E}_A \arrow[r, "\phi_A"] \arrow[d, "g"'] & \mathcal{E}_A^\vee \\
\mathcal{E}_B \arrow[r, "\phi_B"'] & \mathcal{E}_B^\vee . \arrow[u, "g^\vee"']
\end{tikzcd}$$

Thus by Lemma 2.1.2 we have the following commutative diagram

$$\begin{tikzcd}
H_c \otimes V \arrow[r] \arrow[d, "\cong"] \arrow[rrr, bend left, dotted, "A"] & W_A \arrow[r, "q_A"] \arrow[d, "\cong"] & W_A^\vee \arrow[r] \arrow[d, "\cong"] & H_c^\vee \otimes V^\vee \arrow[r] \arrow[d, "\cong"] & \ker A^\vee \arrow[r] \arrow[d, "\cong"] & 0 \\
H^{n-1}(\mathcal{E}_A \otimes \Omega^n(1)) \otimes V \arrow[r] \arrow[d, "g^*"] & H^{n-1}(\mathcal{E}_A \otimes \Omega^{n-1}) \arrow[d, "g^*"] & H^1(\mathcal{E}_A \otimes \Omega^1) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "g^*"] & H^1(\mathcal{E}_A(-1)) \otimes V^\vee \arrow[r] \arrow[d, "g^*"] & H^1(\mathcal{E}_A) \arrow[r] \arrow[d, "g^*"] & 0 \\
H^{n-1}(\mathcal{E}_B \otimes \Omega^n(1)) \otimes V \arrow[r] & H^{n-1}(\mathcal{E}_B \otimes \Omega^{n-1}) & H^1(\mathcal{E}_B \otimes \Omega^1) \arrow[l, "\cong"'] \arrow[r] & H^1(\mathcal{E}_B(-1)) \otimes V^\vee \arrow[r] & H^1(\mathcal{E}_B) \arrow[r] & 0 \\
H_c \otimes V \arrow[r] \arrow[u, "\cong"] \arrow[rrr, bend right, dotted, "B"'] & W_B \arrow[r, "q_B"'] \arrow[u, "\cong"] & W_B^\vee \arrow[r] \arrow[u, "\cong"] & H_c^\vee \otimes V^\vee \arrow[r] \arrow[u, "\cong"] & \ker B^\vee \arrow[r] \arrow[u, "\cong"] & 0,
\end{tikzcd}$$

where g^* denotes the morphisms induced by g on the cohomology groups. Thus, the middle blocks are commutative and the commutativity of the top and bottom blocks follows by Lemma 2.1.2. Therefore, there exists $h \in G$ such that $B = \alpha(h, A)$.

Finally, we will prove that the isotropy group (i.e. the stabiliser of the action) is given by $\{\pm \mathrm{Id}_{H_c}\}$. Let $h \in G$ and $A \in \mathcal{A}_c$, such that $A = \alpha(h, A)$. Thus by Theorem 2.1.3 we have

$[\mathcal{E}_A, \phi_A, f_A] = [\mathcal{E}_{\alpha(h,A)}, \phi_{\alpha(h,A)}, f_{\alpha(h,A)}]$, since they came from the same symmetric map. Therefore, there exists an isomorphism $g : \mathcal{E}_A \xrightarrow{\cong} \mathcal{E}_{\alpha(h,A)}$ such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{E}_A \arrow[r, "\phi_A"] \arrow[d, "g"'] & \mathcal{E}_A^\vee \\
\mathcal{E}_{\alpha(h,A)} \arrow[r, "\phi_{\alpha(h,A)}"'] & \mathcal{E}_{\alpha(h,A)}^\vee \arrow[u, "g^\vee"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H_c \arrow[r, "f_A"] \arrow[d, "\lambda \mathrm{Id}_c"'] & \mathrm{H}^{n-1}(\mathcal{E}_A(-n)) \arrow[d, "g_*"] \\
H_c \arrow[r, "f_{\alpha(h,A)}"'] & \mathrm{H}^{n-1}(\mathcal{E}_{\alpha(h,A)}(-n)),
\end{tikzcd}$$

with $\lambda \in \{-1, 1\}$.

Thus

$$g^* \circ f_A = \pm f_{\alpha(h,A)}. \tag{2.25}$$

On the other hand, since $A = \alpha(h, A)$ by Lemma 2.1.2 we have the following commutative diagram

$$\begin{tikzcd}[column sep=small]
H_c \otimes V \arrow[r] \arrow[d, "f_A"] \arrow[rrr, dotted, bend left, "A"] \arrow[ddd, dotted, bend right, "h"'] & W_A \arrow[r, "q_A"] \arrow[d] & W_A^\vee \arrow[r] \arrow[d] & H_c^\vee \otimes V^\vee \arrow[d, "(f_A^\vee)^{-1}"'] \arrow[ddd, dotted, bend left, "(h^\vee)^{-1}"] \\
H^{n-1}(\mathcal{E}_A \otimes \Omega^n(1)) \otimes V \arrow[r] \arrow[d, "g^*"] & H^{n-1}(\mathcal{E}_A \otimes \Omega^{n-1}) \arrow[d, "g^*"] & H^1(\mathcal{E}_A \otimes \Omega^1) \arrow[l, "\cong"] \arrow[r] \arrow[d, "g^*"] & H^1(\mathcal{E}_A(-1)) \otimes V^\vee \arrow[d, "g^*"'] \\
H^{n-1}(\mathcal{E}_{\alpha(h,A)} \otimes \Omega^n(1)) \otimes V \arrow[r] & H^{n-1}(\mathcal{E}_{\alpha(h,A)} \otimes \Omega^{n-1}) & H^1(\mathcal{E}_{\alpha(h,A)} \otimes \Omega^1) \arrow[l, "\cong"] \arrow[r] & H^1(\mathcal{E}_{\alpha(h,A)}(-1)) \otimes V^\vee \\
H_c \otimes V \arrow[r] \arrow[u, "f_{\alpha(h,A)}"'] \arrow[rrr, dotted, bend right, "\alpha(h,A)"'] & W_B \arrow[r, "q_{\alpha(h,A)}"'] \arrow[u] & W_{\alpha(h,A)}^\vee \arrow[r] \arrow[u] & H_c^\vee \otimes V^\vee . \arrow[u, "(f_{\alpha(h,A)}^\vee)^{-1}"]
\end{tikzcd}$$

Therefore looking at the left column we have

$$\begin{aligned}
h &= (f_{\alpha(h,A)})^{-1} \circ g^* \circ f_A \\
&= (f_{\alpha(h,A)})^{-1} \circ (\pm f_{\alpha(h,A)}) && \text{(By (2.25))} \\
&= \pm \mathrm{Id}_{H_c},
\end{aligned}$$

hence the isotropy group is $\{\pm \mathrm{Id}_{H_c}\}$.

□

Since $G = Gl(H_c)$ is a reductive group, and once that its isotropy group $\pm\{\mathrm{Id}_{H_c}\}$ is a discreet subgroup, the quotient $G_0 = G/\{\pm \mathrm{Id}_{H_c}\}$ is also reductive. Moreover, the action of G_0 on $\mathcal{A}_c$ is free. Keeping this on mind we have all the ingredients to prove the next result, which gives the structure of moduli space that we are looking for.

Theorem 2.2.3. The geometric quotient $\mathcal{M}^{O}_{\mathbb{P}^n}(c) := \mathcal{A}_c//G_0$ is an affine coarse moduli space of dimension $\binom{c}{2}\binom{n+1}{2} - c^2$ for orthogonal instanton bundles with charge c, rank $(n-1)c$ and no global sections on $\mathbb{P}^n$, for $n, c \geqslant 3$.

Proof. First note that $\mathcal{A}_c$ is an open dense subset of $\bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee$ which is affine, thus $\mathcal{A}_c$ is also affine. Moreover note that G_0 is reductive, then by Theorem 1.3.22, $\mathcal{M}^{O}_{\mathbb{P}^n}(c) := \mathcal{A}_c//G_0$ is an affine good quotient. By Proposition 1.3.19 $\mathcal{M}^{O}_{\mathbb{P}^n}(c)$ is an affine categorical quotient, therefore by Proposition 1.3.21 is an affine coarse moduli space.

The dimension of $\mathcal{M}^{O}_{\mathbb{P}^n}(c)$ is

$$\begin{aligned} \dim \mathcal{A}_c - \dim G_0 &= \dim(\bigwedge^2 H_c^\vee \otimes \bigwedge^2 V^\vee) - \dim G \\ &= \binom{c}{2}\binom{n+1}{2} - c^2. \end{aligned}$$

□

3 Some properties of orthogonal instanton bundles

The goal of this chapter is to determine the splitting type of orthogonal instanton bundles on $\mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$ for $n, c \geqslant 3$. As described before the mathematical instanton bundles on $\mathbb{P}^{2n+1}$ have trivial splitting type by definition, but they are all symplectic instanton bundles. Jardim, Marchesi and Wißdorf proved in ([32] - Lemma 4.3 and Theorem 4.4) that there are no orthogonal instanton bundles of trivial splitting type, arbitrary rank r, and charge 2 or odd on $\mathbb{P}^n$. But what can be said about the splitting type of orthogonal instanton bundles with even charge? The main theorem of this chapter gives us an important tool to answer this question.

In order to determine the splitting type of the orthogonal instanton bundles $\mathcal{E} \in \mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$, with $n, c \geqslant 3$ we will associate these bundles to Kronecker modules.

Definition 3.0.1. A **Kronecker module** of rank r is a linear map

$$\gamma : \bigwedge^2 V \to \mathrm{Hom}(H_c, H_c^\vee),$$

such that for the associated linear map,

$$\hat{\gamma} : V \otimes H_c \to V^\vee \otimes H_c^\vee,$$

defined by $\hat{\gamma}(v_1 \otimes h_1)(v_2 \otimes h_2) = [\gamma(v_1 \wedge v_2)(h_1)](h_2)$, the followings hold

(K1) $\hat{\gamma}(v \otimes -) : H_c \to V^\vee \otimes H_c^\vee$ is injective for all $v \neq 0$.

(K2) If $v^{\vee\vee} : V^\vee \otimes H_c^\vee \to H_c^\vee$ is the evaluation map associated to $v \in V$, then $v^{\vee\vee} \circ \hat{\gamma} : V \otimes H_c \to H_c^\vee$ is surjective for all $v \neq 0$.

(K3) $\mathrm{rank}\, \hat{\gamma} = 2n + r$.

Additionally, a Kronecker module is said

(a) **symmetric** (respectively **skew-symmetric**): if the image of γ lies in the subspace $S^2 H_c^\vee \subset \mathrm{Hom}(H_c, H_c^\vee)$ (respectively $\bigwedge^2 H_c^\vee \subset \mathrm{Hom}(H_c, H_c^\vee)$), i.e. if $\hat{\gamma}$ is symplectic (respectively symmetric).

(b) **simple**: if for each pair $\varphi_1, \varphi_2 \in \mathrm{End}(H_c)$ with $\varphi_2^\vee \gamma = \gamma \varphi_1$ it follows that $\varphi_1 = \varphi_2 = \lambda \mathrm{Id}_H$, with $\lambda \in \mathbb{K}$.

(c) **non-degenerate**: if for almost all $v_1, v_2 \in V$ the bilinear form $\gamma(v_1 \wedge v_2)$ is non-degenerate.

(d) **irreducible**: if $U \in H_c$, $U' \in H_c^\vee$ are linear subspaces, such that $U' \neq 0$, $U' \neq H_c^\vee$ and $\gamma(v_1 \wedge v_2)(U) \subset U'$ for linearly independent $v_1, v_2 \in V$, then $\dim U < \dim U'$.

In Chapter 2 we saw that given $\mathcal{E} \in \mathcal{M}^{O}_{\mathbb{P}^n}(c)$, with $n, c \geqslant 3$, by Theorem 2.1.3 there exists $A \in \mathcal{A}_c$ and the monad below

$$\mathcal{M}_A : 0 \longrightarrow H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{a_A^\vee \circ (q_A \otimes \mathrm{Id})} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0\,, \tag{3.1}$$

whose cohomological bundle $\mathcal{E}_A$ is isomorphic to $\mathcal{E}$.

Now let us use the maps a_A and $b_A = a_A^\vee \circ (q_A \otimes \mathrm{Id})$ on the monad (3.1) to construct a Kronecker module associated to $\mathcal{E}$. We can associate to a_A and b_A the linear maps $\alpha \in \mathrm{Hom}(V, \mathrm{Hom}(H_c, W))$ and $\beta \in \mathrm{Hom}(V, \mathrm{Hom}(W, H_c^\vee))$ as follows

$$\begin{array}{rccl} \alpha : V \to & \mathrm{Hom}(H_c, W) & & \\ v \mapsto & \alpha(v) : H_c & \to & W \\ & h & \mapsto & a_A(x)(h \otimes v) \end{array}$$

and

$$\begin{array}{rccl} \beta : V \to & \mathrm{Hom}(W, H_c^\vee) & & \\ v \mapsto & \beta(v) : W & \to & H_c^\vee \\ & w & \mapsto & b_A(x)(w)(h \otimes v) \end{array}$$

where $x = \mathbb{P}(\mathbb{K}v)$.

First note that $b_A(x)(w)(h \otimes v) = q_A(w)(\alpha(v)(h))$, this is why β it is also known as the **transpose map of α (with respect to $q_A : W \to W^\vee$)**.

This pair of maps (α, β) has the following properties.

(P1) $\alpha(v) : H_c \to W$ is injective for all $v \neq 0$;

(P2) $\beta(v) \circ \alpha(v) : H_c \to H_c^\vee$ is the zero mapping for all $v \in V$;

(P3) the map $\hat{\alpha} : V \otimes H_c \to W$ is surjective, with $\hat{\alpha}(v \otimes h) = \alpha(v)(h)$.

The property (P1) happens if and only if a_A is injective in each fibre.

The property (P2) is equivalent to the composition $a_A^\vee \circ (q_A \otimes \mathrm{Id}) \circ a_A$ be the zero mapping in each fibre. Indeed, for each $x \in \mathbb{P}(\mathbb{K}v) \in \mathbb{P}^n$, we have

$$\begin{aligned}(b_A \circ a_A)(x)(h \otimes v) &= [a_A^\vee \circ (q_A \otimes \mathrm{Id}) \circ a_A](x)(h \otimes v)\\ &= q_A(\alpha(v)(h))(\alpha(v)(h))\\ &= \beta(v) \circ \alpha(v).\end{aligned}$$

Now, let us prove that (P3) happens if and only if the cohomology bundle of (3.1) has no global sections. Indeed, by the display of the monad (3.1), we have the following exact sequences.

$$0 \longrightarrow H_c \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{a_A} \ker(b_A) \longrightarrow \mathcal{E}_A \longrightarrow 0$$

$$0 \longrightarrow \ker(b_A) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{b_A} H_c^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0,$$

Thus,

$$H^0(\mathcal{E}_A) \cong H^0(\ker(b_A)) \cong \ker(W \to H_c^\vee \otimes V^\vee),$$

that means $H^0(\mathcal{E}_A) = 0$ if and only if $H^0(b_A) : W \to H_c^\vee \otimes V^\vee$ is injective, if and only if $\hat{\alpha} : V \otimes H_c \to W$ is surjective.

Now let us define

$$\begin{aligned}\gamma' : V \times V &\to \mathrm{Hom}(H_c, H_c^\vee)\\ (v_1, v_2)) &\mapsto \beta(v_2) \circ \alpha(v_1),\end{aligned} \tag{3.2}$$

which defines an element $\gamma \in \mathrm{Hom}(\bigwedge^2 V, \mathrm{Hom}(H_c, H_c^\vee))$. The map γ is a Kronecker module of rank $(n-1)c$ as we can see in the next result.

Lemma 3.0.2. The element $\gamma \in \mathrm{Hom}(\bigwedge^2 V, \mathrm{Hom}(H_c, H_c^\vee))$ constructed as above is a Kronecker module of rank $(n-1)c$.

Proof. Let $\hat{\alpha} : V \otimes H_c \to W$, $\hat{\beta} : W \to V^\vee \otimes H_c^\vee$ and $\hat{\gamma} : H_c \otimes V \to H_c^\vee \otimes V^\vee$ be the linear maps associated to α, β and γ, respectively. By the definition of γ' in (3.2) we have $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$. Now let us prove that γ satisfies the properties (K1)-(K3) of Definition 3.0.1.

For each $v \neq 0$ we have

$$\hat{\gamma}(v \otimes h)[\gamma(v \wedge v_1)(h)](h_1),$$

but

$$
\begin{aligned}
[\gamma'(v \wedge v_1)(h)](h_1) &= [\beta(v_1) \circ \alpha(v)](h)(h_1) \\
&= q(\alpha(v)(h))(\alpha(v_1)(h_1)).
\end{aligned}
$$

Thus (K1) follows by property (P1). Also observe that $v^{\vee\vee} \circ \hat{\gamma} = \hat{\gamma}(v \otimes -)^{\vee}$, therefore (K1) implies (K2).

By (P3) we have that $\hat{\alpha}$ is injective and $\hat{\beta}$ is surjective, thus it follows that rank $\hat{\gamma} = (n+1)c$. Moreover, since $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$, we have rank $\gamma = (n-1)c$ and the property (P3).

Therefore γ is a Kronecker module of rank $(n-1)c$

□

Let $\mathcal{K}$ be the set of all stable Kronecker modules of rank $(n-1)c$. The following result describes how we can obtain the splitting type of a bundle by the Kronecker module associated with it by (3.2).

Theorem 3.0.3. Let $\mathcal{E}$ be an orthogonal instanton bundle on $\mathbb{P}^n$, with charge c, rank $(n-1)c$ and no global sections, for $n, c \geqslant 3$. Let γ be its Kronecker module associated by the previous construction. If $L \subset \mathbb{P}^n$ is the line defined by $v_1, v_2 \in V$, $v_1 \wedge v_2 \neq 0$, the the restriction $E|_L$ is trivial if and only if $\gamma(v_1 \wedge v_2)$ is an isomorphism.

Proof. Given $\mathcal{E} \in \mathcal{M}^{\mathcal{O}}_{\mathbb{P}^n}(c)$ consider the maps (α, β) and the Kronecker module γ constructed above. Let $v_1, v_2 \in V$ such that $v_1 \wedge v_2 \neq 0$, and consider the $\mathbb{K}$-subspace $K = \mathbb{K}v_1 + \mathbb{K}v_2$. The restriction of the monad (3.1) to $L = \mathbb{P}(K)$ is the monad

$$
\mathcal{M}_A : 0 \longrightarrow H_c \otimes \mathcal{O}_L(-1) \xrightarrow{a_A|_L} W \otimes \mathcal{O}_L \xrightarrow{b_A|_L} H_c^{\vee} \otimes \mathcal{O}_L(1) \longrightarrow 0. \tag{3.3}
$$

The display of the monad (3.3) gives the exact sequences

$$
0 \longrightarrow H_c \otimes \mathcal{O}_L(-1) \xrightarrow{a_A|_L} \ker(b_A|_L) \longrightarrow \mathcal{E}|_L \longrightarrow 0,
$$

$$
0 \longrightarrow \ker(b_A|_L) \longrightarrow W \otimes \mathcal{O}_L \xrightarrow{b_A|_L} H_c^{\vee} \otimes \mathcal{O}_L(1) \longrightarrow 0.
$$

Thus

$$
H^0(L, \mathcal{E}|_L) \cong H^0(L, \ker(b_A|_L)) \cong \ker(W \to H_c^{\vee} \otimes K^{\vee}).
$$

Observe that $\mathcal{E}|_L$ has trivial splitting type if and only if no section $s \in H^0(L, \mathcal{E}|_L) \backslash \{0\}$ has zeros. Indeed, suppose that $\mathcal{E}|_L$ is trivial, then $\mathcal{E}|_L \cong \mathcal{O}_L^r$, and $H^0(L, \mathcal{E}|_L) \cong H^0(L, \mathcal{O}_L^r) \cong \mathbb{K}^r$. Thus if

$s \in H^0(L, \mathcal{E}|_L)\backslash\{0\}$, then $s = (k_1, \cdots, k_r) \in \mathbb{K}^r\backslash\{0\}$. Therefore s has no zeros. Reciprocally, if $s \in H^0(L, \mathcal{E}|_L)$ has no zeros, suppose that $\mathcal{E}|_L$ has no trivial splitting type, i.e. $\mathcal{E}|_L \cong \bigoplus_{i=1}^{r} \mathcal{O}_L(a_i)$, where at least on $a_i \geqslant 0$, because $c_i(\mathcal{E}) = 0$. But $\sum_{i=1}^{r} a_i = 0$, so without loss of generality we can assume $a_1 > 0$. Let $s_1 \in H^0(L, \mathcal{O}_L(a_1))$, then $s = (s_1, 0, \cdots, 0) \in H^0(L, \mathcal{E}|_L)$ and has zeros, a contradiction.

Now we will show that this happens if and only if $\gamma(v_1 \wedge v_2)$ is invertible. Consider the inclusions

$$H_c \otimes \mathcal{O}_L(-1) \overset{i}{\hookrightarrow} \ker(b_A|_L) \overset{j}{\hookrightarrow} W \otimes \mathcal{O}_L.$$

Let $s \in H^0(L, \ker(b_A|_L)) \cong H^0(L, \mathcal{E}|_L)$ be a section. Being $H^0(W \otimes \mathcal{O}_L) \cong W$, there exists $w \in W$ with $j \circ s(x) = w$ for all $x \in L$. So the section $s' \in H^0(L, \mathcal{E}|_L)$ defined by s has zeros at $x = \mathbb{P}(\mathbb{K}v) \in L$ if and only if $s(x)$ lies in the image of the inclusion $i(x) : H_c \otimes \mathcal{O}_L(-1) \hookrightarrow \ker(b_A|_L)(x)$, i.e. if and only if there exists $h \in H_c$ with $\alpha(v)(h) = w$. Because s is a section in $\ker(b_A|_L)$, for every $v' \in K$ we must have $\beta(v')(w) = 0$. Thus $\mathcal{E}|_L$ has no trivial section with zeros if and only if

$$\mathrm{Im}\alpha(v) \subset \bigcap_{v' \in K} \ker \beta(v'),$$

for at least one vector $v \in K\backslash\{0\}$. Which means that for any basis $v, v' \in K$ of K the map

$$\gamma(v \wedge v') = \beta(v') \circ \alpha(v)$$

is not an isomorphism. □

Now let us use the Theorem 3.0.3 to determine the type of splitting of the orthogonal instanton bundle with no global sections, charge 6 and rank 12 on $\mathbb{P}^3$ that we obtained in Chapter 2 - Example 2.1.5.

Example 3.0.4. Let $\{x_0, x_1, x_2, x_3\}$ be a basis for $V^\vee$. In Example 2.1.5 we saw that the cohomology bundle $\mathcal{E}$ of the monad

$$\mathcal{O}_{\mathbb{P}^3}^6(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{24} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}^6(1)$$

where

$$\alpha = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & x_3 \end{pmatrix}, \quad \beta^t = \begin{pmatrix} 0 & 2x_1 & 0 & 0 & 0 & 0 \\ 0 & -2x_0 & 0 & 0 & 0 & 0 \\ 0 & -6x_3 & 0 & 0 & 0 & 0 \\ 0 & 6x_2 & 0 & 0 & 0 & 0 \\ -2x_1 & 0 & 0 & 0 & 0 & 0 \\ 2x_0 & 0 & 0 & 0 & 0 & 0 \\ 6x_3 & 0 & 0 & 0 & 0 & 0 \\ -6x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & -3x_3 & 0 & 0 \\ 0 & 0 & 0 & 3x_2 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & -3x_3 & 0 & 0 & 0 \\ 0 & 0 & 3x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 0 & 0 & -3x_3 \\ 0 & 0 & 0 & 0 & 0 & 3x_2 \\ 0 & 0 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & 3x_3 & 0 \\ 0 & 0 & 0 & 0 & -3x_2 & 0 \end{pmatrix}$$

is an orthogonal instanton bundle with no global sections, charge 6 and rank 12 on $\mathbb{P}^3$.

Let $L \in \mathbb{P}^3$ be the line joining the general points $P = [a : b : c : d]$ and $Q = [e : f : g : h]$. By Theorem 3.0.3 $\mathcal{E}|_L$ is trivial if and only if $\beta(Q)\alpha(P)$ is invertible. Indeed,

$$\beta(Q)\alpha(P) = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 \end{pmatrix}$$

where $\lambda_1 = 2be - 2af - 6dg + 6ch$, $\lambda_2 = -be + af + 3dg - 3ch$ and $\lambda_3 = be - af - 3dg + 3ch$. Thus $\beta(Q)\alpha(P)$ is invertible and therefore $\mathcal{E}|_L$ is trivial. Since the points are general this is also true for every general line, therefore $\mathcal{E}$ has trivial splitting type.
On the other hand, let L_0 be the line joining the points $P = [1 : 0 : 0 : 0]$ and $Q = [0 : 0 : 0 : 1]$. By the previous construction $\beta(Q)\alpha(P) = 0$, therefore by Theorem 3.0.3 $\mathcal{E}|_{L_0}$ is not trivial, hence L_0 is a jumping line for $\mathcal{E}$.

Finally, we can see in the next example that the orthogonal instanton bundle of Example 2.1.6 has no trivial splitting type, as expected ([32] - Lemma 4.3).

Example 3.0.5. Again let $\{x_0, x_1, x_2, x_3\}$ be a basis for $V^\vee$. In Example 2.1.6 we saw that the cohomology bundle $\mathcal{E}$ of the monad

$$\mathcal{O}_{\mathbb{P}^3}^5(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{20} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}^5(1)$$

where

$$\alpha = \begin{pmatrix} x_0 & 0 & 0 & 0 & \\ x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 & x_0 \\ 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & x_3 \end{pmatrix} \text{ and } \beta^t = \begin{pmatrix} 0 & x_1 & x_3 & 0 & x_3 \\ 0 & -x_0 & x_2 & 0 & x_2 \\ 0 & x_3 & -x_1 & 0 & -x_1 \\ 0 & -x_2 & -x_0 & 0 & -x_0 \\ -x_1 & 0 & x_1 & x_3 & 0 \\ x_0 & 0 & -x_0 & 0 & 0 \\ -x_3 & 0 & x_3 & x_3 & 0 \\ x_2 & 0 & -x_2 & -x_0 - x_2 & 0 \\ -x_3 & -x_1 & 0 & 0 & x_3 \\ x_2 & x_0 & 0 & 0 & 0 \\ x_1 & -x_3 & 0 & 0 & x_3 \\ x_0 & x_2 & 0 & 0 & -x_0 - x_2 \\ 0 & -x_3 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & -x_3 & 0 & 0 & -x_1 \\ 0 & x_0 + x_2 & 0 & 0 & -x_0 \\ -x_3 & 0 & -x_3 & -x_3 & 0 \\ -x_2 & 0 & 0 & -x_2 & 0 \\ -x_3 & 0 & -x_3 & 0 & 0 \\ x_0 & 0 & x_0 + x_2 & x_0 & 0 \end{pmatrix}$$

is an orthogonal instanton bundle with no global sections, charge 5 and rank 10 on $\mathbb{P}^3$.

Let $L \in \mathbb{P}^3$ be the line joint the points $P = [a : b : c : d]$ and $Q = [e : f : g : h]$. We have

$$\beta(Q)\alpha(P) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & 0 & \lambda_2 \\ -\lambda_1 & 0 & \lambda_1 & \lambda_2 & 0 \\ -\lambda_2 & -\lambda_1 & 0 & 0 & \lambda_2 \\ 0 & -\lambda_2 & 0 & 0 & \lambda_2 \\ -\lambda_2 & 0 & -\lambda_2 & -\lambda_2 & 0 \end{pmatrix}$$

where $\lambda_1 = be - af + dg - ch$ and $\lambda_2 = de + cf - bg - ah$. Since $\beta(Q)\alpha(P)$ is a skew-symmetric matrix of odd order, $\beta(Q)\alpha(P)$ is not invertible for all P and Q. Thus by Theorem 3.0.3 $\mathcal{E}|_L$ is not trivial for every line $L \in \mathbb{P}^3$. Therefore $\mathcal{E}$ has no trivial splitting type.

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