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Combinatorics on Schubert varieties

Combinatória em variedades de Schubert

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Combinatória em variedades de Schubert

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Resumo

Esta tese explora aspectos combinatórios relacionados com a topologia/geometria das variedades de Schubert.

O primeiro problema consiste em obter uma fórmula explícita para o cálculo dos coeficientes do operador fronteira da homologia inteira das Grassmannianas isotrópicas e ortogonais ímpares reais. Apesar da natureza geométrica deste problema, este cálculo depende apenas da combinatória das permutações associadas às variedades de Schubert da decomposição celular das Grassmannianas isotrópicas.

Também consideramos um estudo combinatório de permutações que se associam a uma classe mais geral de variedades de Schubert, chamadas de permutações theta-vexillary com sinal. O principal resultado é o desenvolvimento de descrições equivalentes para as permutações theta-vexillary dadas em termos de *pattern avoidance* e do conjunto de cantos do diagrama da permutação.

Palavras-chave: Variedades de Schubert, Topologia algébrica, Permutações, Permutações evitando padrões.

Abstract

This thesis presents combinatorial aspects related to topology/geometry of Schubert varieties.

The first problem consists to obtain an explicit formula to compute the coefficients of the boundary operator of the integral homology of real isotropic and odd orthogonal Grassmannians. Despite the geometric nature of this problem, this computation only depends on the combinatorics of permutations associated to Schubert varieties of a cellular decomposition of an isotropic Grassmannians.

We also consider a combinatorial study of permutations that are associated to an even more general class of Schubert varieties called theta-vexillary signed permutations. The main result is the development of equivalent descriptions of theta-vexillary permutations in terms of pattern avoidance, and the set of corners of the permutation's diagram.

Keywords: Schubert varieties, Algebraic topology, Permutations, Pattern-avoiding permutations.

Contents

Introduction

A real or complex flag manifold is a homogeneous space $\mathbb{F} = G/P$ where G is a semisimple Lie group and P is a parabolic subgroup of G. The Bruhat decomposition of a flag manifold is a disjoint union of P-orbits that can be parametrized by certain elements of the Weyl group $\mathcal{W}$. A Schubert variety is the closure of some P-orbit associated to an element w of the Weyl group. When G is a classical Lie group, points in a flag manifold are flags $L_\bullet = (L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_r} \subset V)$ in a finite-dimensional vector space V over the real or complex field, where each subspace L_{i_j} has dimension i_j and the sequence $(i_1 < \cdots < i_r)$ is associated to the choice of parabolic subgroup P. Fixing a flag $V_\bullet$, a Schubert variety is also the loci of flags satisfying certain incidence conditions with $V_\bullet$.

The main results provided by this thesis are essentially about some combinatorial problems that comes from Schubert varieties. The first problem is to obtain an explicit formula for integral homology of real isotropic and odd orthogonal Grassmannians. The second one is to find equivalent ways of describing a specific kind of permutation in the Weyl group $\mathcal{W}$ of type B (or C), called theta-vexillary signed permutations. Such permutations are interesting because they contain all permutations that parametrize the Schubert varieties of the isotropic and odd orthogonal Grassmannians. However, it is important to say that both problems are independent of each other. In what follows a short introduction of the main results of each one of these parts is given.

Homology of isotropic and odd orthogonal Grassmannians

Let G be one of the following classical split real Lie groups: the indefinite special orthogonal group

$$\mathrm{SO}(2n+1) := \{g \in \mathrm{Sl}(2n+1, \mathbb{R}) \mid gg^T = 1\},$$

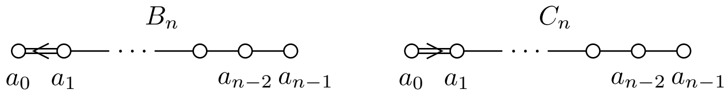
or the symplectic group

$$\mathrm{Sp}(n, \mathbb{R}) := \{g \in \mathrm{Sl}(2n, \mathbb{R}) \mid gJ_ng^T = 1\},$$

where,

$$J_n = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

Their respective Lie algebra $\mathfrak{g}$ are $\mathfrak{so}(n, n+1)$, the Lie algebra of type B, or $\mathfrak{sp}(n, \mathbb{R})$, the Lie algebra of type C. The simple root system Σ is the set of simple roots $a_0,\ a_1, \ldots,\ a_{n-1}$ such that: for type B, a_0 is the unique short root; for type C, a_0 is the unique large root. In both situations, we denote as a_0 the extreme root in the Dynkin with the double bound, i.e., the roots are ordered as below



Given some $k = 0, \ldots, n-1$, *odd orthogonal real Grassmannians* and *isotropic real Grassmannians* are, respectively, the minimal flag manifold $\mathrm{SO}(n, n+1)/P_{(k)}$ and $\mathrm{Sp}(n, \mathbb{R})/P_{(k)}$, where $(k) = \Sigma \backslash \{a_k\}$ is the maximal proper subset of simple roots that does not contain a_k, and $P_{(k)}$ is a maximal parabolic subgroup associated to (k).

Both Grassmannians can be geometric realized as a set whose points are flags. The orthogonal Grassmannian of type B is the set of $(n-k)$-dimensional isotropic subspaces in the vector space $V = \mathbb{R}^{2n+1}$ equipped with a inner product. It will be denoted by $\mathrm{OG}(n-k, 2n+1)$. The isotropic Grassmannian of type C is the set of $(n-k)$-dimensional isotropic subspaces in the symplectic vector space $V = \mathbb{R}^{2n}$. This set will be denoted by $\mathrm{IG}(n-k, 2n)$. They have the same Weyl group, namely, the hyperoctahedral group: the elemens are permutations in the symmetric group S_n with a sign, positive or negative, attached to each entry. This group were introduced by Young [45] in 1930s.

In the general context of flag manifolds, the topology of the complex flag manifolds is well known and its first results date back to Ehresmann in the 1930s (cf. [18]). Yet the first results about the topology of real flag manifolds dates to the 1970s (cf. Burghelea–Hangan–Moscovici–Verona [10]) and 1980s (cf. Duistermaat-Kolk-Varadarajan [16]). Only in the 1990s a complete description of its integral homology was done by Kocherlakota [30]. A more sophisticated point of view is developed by Casian-Stanton [15]. From the point of view of cellular decomposition, the main difficulty met in the context of real flag manifolds is the existence of cells in all dimensions whereas the complex one has only even dimensional cells.

In both the real and complex cases, the Schubert varieties form a cellular decomposition for the flag manifold, hence the integral homology can be determined if the boundary operator is known. For a general real flag manifold, a formula for the boundary map was obtained by Kocherlakota [30] via Morse homology. The same result was also developed by Rabelo and San Martin [36] in the 2010s, where they computed directly the boundary map using the Bruhat decomposition as a CW complex. According to Kocherlakota, the data required to compute the boundary operators depends exclusively on the permutations, and they are extracted from the set Π_w of positive roots sent to negative ones by w^{-1} for a Weyl group element w in $\mathcal{W}$. However, it is not easy to compute the formula even for classical Lie groups, including the odd orthogonal and isotropic groups, and the task of computing homology of such Grassmannians becomes hard even for a computer.

Every element w in the Weyl group of type B and C can be represented as a signed permutation, and thus Π_w can be associated with the set of inversions of w. Pragacz and Ratajski in [33] have shown that there is a bijection between the permutations that parametrize Schubert varieties for the Grassmannians and a specific set constituted of pairs of partitions (α, λ), where $\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_k > 0)$ is a partition, and $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$ is a strict partition such that $\alpha_k \geqslant r$. Such pair can be represented as a diagram of boxes that we call half-shifted Young diagrams (HSYD). Namely, a half-shifted Young diagram is composed by two parts: the top part is the Young diagram of α in a rectangle; and the bottom part is the shifted Young diagram of λ in a staircase shape.

In this thesis, we show that there is a straight relationship between the set of inversions Π_w and the half-shifted Young diagrams associated with the permutation w. Namely, the boxes of a HSYD can be filled in with a unique inversion in Π_w. In the context of the maximal Grassmannians $G/P_{(0)}$, this was done by Ikeda-Naruse [28] and Graham-Kreiman [23].

The half-shifted Young diagrams described here are slight modifications of the diagrams defined by Pragacz and Ratajski, and they are also related to the k-strict partitions defined by Buch, Kresch and Tamvakis in [39]. The purpose of such change was to get an easy row-reading of the permutation as well as to insert the inversions in Π_w inside the diagrams similarly to what is done in [28] and [23]. Another consequence is the occurrence of removable boxes in these diagrams. Given a Schubert variety, some boxes can be removed from its diagram to get another Schubert variety that drops the dimension by one. Such pairs of Schubert varieties appear as the possible cases of non-trivial coefficients of the boundary map.

One of the main results of the present work is to apply the above construction to get an explicit formula for the coefficients of the boundary map of the integral homology groups of $G/P_{(k)}$ (cf. Theorem 2.7). This formula is not as elegant to state as the general theorem of Kocherlakota, but it is clearly easier to compute since it is described in terms of removable boxes of the half-shifted Young diagrams of permutations. The formula is also a generalization of the work of Rabelo [35] in which these coefficients are obtained when $k = 0$, the Lagrangian and maximal orthogonal Grassmannians. The method to compute integral homology of the usual Grassmannians of type A may be obtained as a particular case.

This study is developed in the first two chapters of this thesis. Chapter 1 describes all the basic tools required to deal with the half-shifted Young diagrams. The first section is devoted to the preliminaries on Lie theory, which includes the formal definition of the Grassmannians subject to our study. After we define a half-shifted Young diagram and establish its relationship to some permutations in the Weyl group, we can introduce the idea of H-relation and V-relation in the half-shifted Young diagram. Then the removable boxes in a HSYD can be classified according to their positions in the diagram. Finally, we present a correlation between the set of inversions Π_w and the HSYD associated to the permutation w.

In Chapter 2 we state and prove the main result describing the boundary maps of the cellular homology of Grassmannians of type B and C. The homology of the isotropic Grassmannian $\mathrm{IG}(2, 8)$ is computed as an example. We also have some results about orientability, 1- and 2-homology, and a brief explanation about the cohomology of such Grassmannians.

Theta-vexillary signed permutations

For classical complex Lie groups, a Schubert variety of some permutation w is a subvariety of some flag manifold $\mathbb{F}$, where it is the loci of flags satisfying certain incidence conditions. This idea of set of flags that characterizes a Schubert variety can be extended to flags of subbundles over some arbitrary variety. Consider, in a first moment, the type A case where the complex Lie group is special linear group $\mathrm{Sl}(n, \mathbb{C})$ and the Weyl group is S_n. Let V be an vector bundle of rank n over some arbitrary variety X, and flags of subbundles $E_1 \subset E_2 \subset \cdots \subset E_n \subset V$ and $F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset V$ where E_i has rank i and F_i has rank $n - i$. Given a permutation $w \in S_n$, the degeneracy locus is a subvariety $\Omega_w \subset X$ defined by

$$\Omega_w := \{x \in X \mid \dim(E_p(x) \cap F_q(x)) \geqslant r_w^A(q, p), \text{ for every } 1 \leqslant p, q \leqslant n\},$$

where $r_w^A(q, p)$ is the cardinality of the set $\{w(1), w(2), \ldots, w(p)\} \cap \{q + 1, q + 2, \ldots, n\}$. The Schubert varieties are the particular case where X is chosen as the flag manifold $\mathbb{F} = \mathrm{Sl}(n, \mathbb{C})/P$, and $E_1 \subset E_2 \subset \cdots \subset E_n \subset V$ is some fixed flag bundle. This generalization is better than considering only the Schubert varieties because, for certain sets of permutations, it is possible to determine a formula for the cohomology class $[\Omega_w]$ of the degeneracy locus as a polynomial in the Chern classes of the vectors bundles involved.

In order to understand the theta-vexillary signed permutations, it is helpful to discuss about two classes of permutations that inspired them: the vexillary permutations, and its particular case, the vexillary signed permutations.

A first class of permutation that we will consider are the vexillary permutations in S_n, in which the easiest definition is given in terms of pattern avoidance as it follows: a permutation w is called vexillary if and only if it avoids the patters $[2\ 1\ 4\ 3]$, i.e., there are no indices $a < b < c < d$ such that $w(b) < w(a) < w(d) < w(c)$. The vexillary permutations were found by Lascoux and Schützenberger [31] in the 1980s. Fulton [20] in the 1990s obtained other equivalent characterizations for the vexillary permutations: in addition to the pattern avoidance criterion, he figured out one in terms of the essential set of a permutation, among others. Since S_n is the Weyl group of type A, the vexillary permutations represent Schubert varieties in some flag manifold where the Lie group is $G = \mathrm{Sl}(n, \mathbb{C})$.

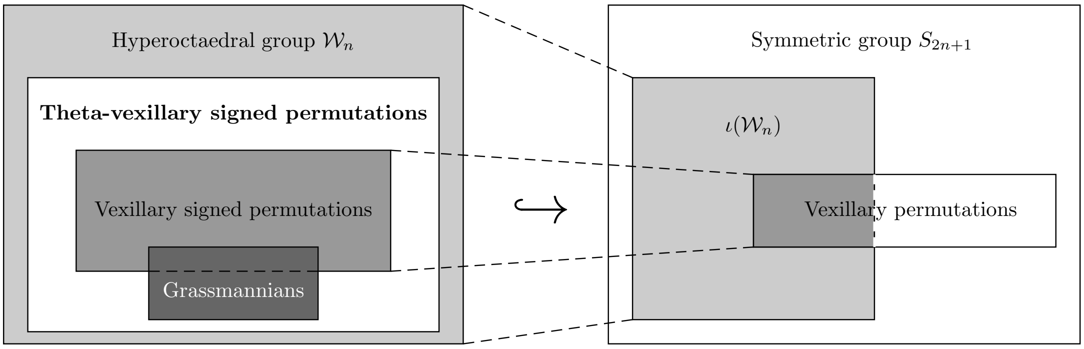
A few years later, the notion of vexillary signed permutations were introduced by Billy and Lam [6] in the hyperoctaedral group, i.e., in the Weyl group of type B or C. Recently, Anderson and Fulton [1, 2] provided a diffent characterization for the vexillary signed permutations. They defined them by using the notion of associated triple of integers: given three s-tuple of positive integers $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, where $\mathbf{k} = (0 < k_1 < \cdots < k_s)$, $\mathbf{p} = (p_1 \geqslant \cdots \geqslant p_s > 0)$, and $\mathbf{q} = (q_1 \geqslant \cdots \geqslant q_s > 0)$, satisfying $p_i - p_{i+1} + q_i - q_{i+1} > k_{i+1} - k_i$ for $1 \leqslant i \leqslant s-1$, one constructs a signed permutation $w = w(\boldsymbol{\tau})$. Since the Weyl group of type B can be included in the group S_{2n+1}, a signed permutation w in $\mathcal{W}_n$ is vexillary if and only if its inclusion $\iota(w)$ in S_{2n+1} is a vexillary permutation as mentioned above. Anderson and Fulton in [1] provided two other characterizations of vexillary signed permutations: one in terms of essential sets and other by pattern avoidance of a signed permutation.

In this thesis, we obtained alternative characterizations for an even more general class of signed permutations called *theta-vexillary signed permutations*. As well as vexillary signed permutations, they are defined using a triple of integers $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, where we allow negative values for $\mathbf{q}$, which requires adding many other conditions to the triple. The precise definition of a theta-vexillary permutation $w = w(\boldsymbol{\tau})$ in terms of a triple $\boldsymbol{\tau}$ can be found in Chapter 5. These triples also have a geometric interpretation in terms of degeneracy loci. For our purpose, it is easier to denote the hyperoctahedral group as the Weyl group of type B. Consider a vector bundle V of rank $2n+1$ over X, equipped with a nondegenerate form and two flags of bundles $E_\bullet = (E_{p_1} \subset E_{p_2} \subset \cdots \subset E_{p_s} \subset V)$ and $F_\bullet = (F_{q_1} \subset F_{q_2} \subset \cdots \subset F_{q_s} \subset V)$ such that: for $q > 0$, the subbundles F_q are isotropic, of rank $n+1-q$; for $q < 0$, F_q is coisotropic, of corank $n+q$; and all the subbundles E_p are isotropic, of rank $n+1-p$. The degeneracy locus of $w = w(\boldsymbol{\tau})$ is

$$\Omega_w = \Omega_{\boldsymbol{\tau}} := \{x \in X \mid \dim(E_{p_i} \cap F_{q_i}) \geqslant k_i, \text{ for } 1 \leqslant i \leqslant s\}.$$

If a triple is chosen such that all values p_i are constant and equal to p, then the permutation $w = w(\boldsymbol{\tau})$ are the ones associated to the Grassmannian Schubert varieties. The following diagram helps to understand how these three different classes of permutations are related to each other.

The conditions for the triples associated to theta-vexillary permutations were



introduced by Anderson and Fulton in [3]. They figured out that the cohomology class $[\Omega_{\boldsymbol{\tau}}]$ is given in terms of the Chern classes of the vector bundles E_{p_i} and F_{q_i}, applied to multi-theta-polynomials Θ_λ, which derives from the theta-polynomials defined via raising operators by Bush, Kresch, and Tamvakis [9].

If a permutation w in the Weyl group $\mathcal{W}_n$ of type B is represented as a matrix of dots in a $(2n+1) \times n$ array of boxes, the (Rothe) extended diagram is the subset of boxes that remains after striking out the boxes weakly south or east of each dot. The southeast (SE) corners in the extended diagram form the set of corners $\mathscr{C}(w)$. One characterization of theta-vexillary signed permutations is the set of corners $\mathscr{C}(w)$ is the disjoint union of the set $\mathscr{N}e(w)$ which is composed by all corners that form a piecewise path that goes to the northeast direction, and the set $\mathscr{U}(w)$ of unessential corners. We also have a characterization via pattern avoidance.

Theorem. *Let w be a signed permutation. The following are equivalent:*

1. *w is theta-vexillary, i.e., there is a triple $\boldsymbol{\tau}$ such that $w = w(\boldsymbol{\tau})$;*
2. *the set of corner $\mathscr{C}(w)$ is the disjoint union*

$$\mathscr{C}(w) = \mathscr{N}e(w)\dot{\cup}\mathscr{U}(w),$$

3. *w avoids the follow thirteen signed patterns* $[\overline{1}\ 3\ 2]$, $[\overline{2}\ 3\ 1]$, $[\overline{3}\ 2\ 1]$, $[\overline{3}\ 2\ \overline{1}]$, $[2\ 1\ 4\ 3]$, $[2\ \overline{3}\ 4\ \overline{1}]$, $[\overline{2}\ \overline{3}\ 4\ \overline{1}]$, $[3\ \overline{4}\ 1\ \overline{2}]$, $[3\ \overline{4}\ \overline{1}\ \overline{2}]$, $[\overline{3}\ \overline{4}\ 1\ \overline{2}]$, $[\overline{3}\ \overline{4}\ \overline{1}\ \overline{2}]$, $[\overline{4}\ 1\ \overline{2}\ 3]$, *and* $[\overline{4}\ \overline{1}\ \overline{2}\ 3]$.

This theorem is consequence of Propositions 4.8 and 4.14 and it is similar to the vexillary signed permutation's version. It is interesting to notice that, comparing to the vexillary case, we admit some SE corners in the diagram that are not in an ordered northeast path, which we call the unessential corners. Besides, the characterization via signed pattern avoidance for the Theta-vexillary permutations has eight patterns in common with those for the vexillary case and [2 1] is the unique not present in this list.

It worth to notice that if we consider the pattern avoidance criterion, the theta-vexillary signed permutations are not listed yet in the "Database of Permutation

Pattern Avoidance" maintained by Tenner [42]. Hence, they form a whole new class of permutations among other 51 classes that already belongs to the database.

In Chapter 3 we define some basic conceptions of the combinatorics of permutations in S_n and $\mathcal{W}_n$ the Weyl group of type B, and their associated diagrams, and we describe the vexillary and vexillary signed permutations. The definition of vexillary permutations (signed or not) are not require to define the theta-vexillary signed permutations but it would certainly facilitate the understanding about constructions and properties related to theta-vexillary permutations. Chapter 4 proves the theorem stated above by making use of several combinatorics tools. Finally, in Appendix A we give the geometric interpretation of such types of permutations, their association to the Schubert varieties and degeneracy loci, as well as the formulas for the cohomology class of such associated Schubert varieties.

Chapter 1

Young diagrams of isotropic and odd orthogonal Grassmannians

In this chapter, we introduce the basic definitions about Lie theory and isotropic Grassmannians required to deal with the problem in the next chapter. We also establish all the concepts around the half-shifted Young diagrams, such as the row-reading, H-related and V-related columns, removable boxes, and the set of inversions.

1.1 Preliminaries

The main facts about semi-simple Lie Groups and flag manifolds may be found in Helgason [24], Knapp [29], Warner [43], and San Martin [37]. We also refer Björner and Brenti [7] for some results about the Weyl group. Flag manifolds are defined as homogeneous spaces G/P where G is a non-compact semi-simple Lie group and P is a parabolic subgroup of G. Let $\mathfrak{g}$ be a non-compact real semi-simple Lie algebra.

Let G be a non-compact semi-simple Lie group. Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and let $\mathfrak{a}$ be a maximal abelian sub-algebra contained in $\mathfrak{s}$. We denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and fix a simple system of roots $\Sigma \subset \Pi$. Denote by $\Pi^{\pm}$ the set of positive and negative roots, respectively, and by $\mathfrak{a}^+$ the Weyl chamber $\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma\}$. Let $\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ be the direct sum of root spaces corresponding to the positive roots. The Iwasawa decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The notations K and N are used to indicate the connected subgroups whose Lie algebras are $\mathfrak{k}$ and $\mathfrak{n}$ respectively. A sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be a Cartan sub-algebra if $\mathfrak{h}_{\mathbb{C}}$ is a Cartan sub-algebra of $\mathfrak{g}_{\mathbb{C}}$.

A minimal parabolic sub-algebra of $\mathfrak{g}$ is given by $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let P be the minimal parabolic subgroup with Lie algebra $\mathfrak{p}$. Note that P is the normalizer of $\mathfrak{p}$ in G. We call $\mathbb{F} = G/P$ the maximal flag manifold of G and denote by b_0 the base point $1 \cdot P$ in G/P.

Associated to a subset of simple roots $\Theta \subset \Sigma$ there are several Lie algebras and groups. We write $\mathfrak{g}(\Theta)$ for the semi-simple Lie algebra generated by $\mathfrak{g}_{\pm\alpha}$, $\alpha \in \Theta$. Let $G(\Theta)$ be the connected group with Lie algebra $\mathfrak{g}(\Theta)$. Moreover, let $\mathfrak{n}_\Theta$ be the sub-algebra generated by the roots spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \Theta$ and put $\mathfrak{p}_\Theta = \mathfrak{n}_\Theta \oplus \mathfrak{p}$.

The normalizer P_Θ of $\mathfrak{p}_\Theta$ in G is a standard parabolic subgroup which contains P. The corresponding flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ is called a partial flag manifold of G or flag manifold of type Θ. We denote by b_Θ the base point $1 \cdot P_\Theta$ in G/P_Θ.

A central role in our context will be played by the Weyl group $\mathcal{W}$ associated to $\mathfrak{a}$. This is the finite group generated by the reflections over the root hyperplanes $\alpha = 0$ contained in $\mathfrak{a}$, $\alpha \in \Sigma$. Alternatively, it may be given as the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ where $N_K(\mathfrak{a})$ and $M = Z_K(\mathfrak{a})$ are respectively the normalizer and the centralizer of $\mathfrak{a}$ in K (the Lie algebra of M is $\mathfrak{m}$).

The elements in the Weyl group $\mathcal{W}$ of G also can be described as product of simple reflections $s_i = s_{\alpha_i}$ through simple roots $\alpha_i \in \Sigma$. The length $\ell(w)$ of $w \in \mathcal{W}$ is the number of simple reflections in any reduced decomposition of w. There is a partial order in the Weyl group called the Bruhat-Chevalley order: we say that $w_1 \leqslant w_2$ if given a reduced decomposition $w_2 = s_{j_1} \cdots s_{j_r}$ then $w_1 = s_{j_{i_1}} \cdots s_{j_{i_k}}$ for some $1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant r$.

For a subset $\Theta \subset \Sigma$, the subgroup $\mathcal{W}_\Theta$ is defined to be the stabilizer of $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0, \alpha \in \Theta\}$. Alternatively, $\mathcal{W}_\Theta$ may be seen as the subgroup of the Weyl group generated by the reflections with respect to the roots $\alpha \in \Theta$.

We also define the subset $\mathcal{W}^\Theta$ of $\mathcal{W}$ by

$$\mathcal{W}^\Theta = \{w \in \mathcal{W} : \ell(ws_\alpha) = \ell(w) + 1 \ , \ \alpha \in \Theta\}.$$

There exists a unique element $w^\Theta \in \mathcal{W}^\Theta$ of minimal length in each coset $w\mathcal{W}_\Theta$. The set $\mathcal{W}^\Theta$ is called the subset of minimal representatives of the cosets of $\mathcal{W}_\Theta$ in $\mathcal{W}$.

The Bruhat decomposition presents the flag manifolds as union of disjoint N-orbits, namely,

$$\mathbb{F}_\Theta = \coprod_{w \in \mathcal{W}/\mathcal{W}_\Theta} N \cdot wb_\Theta$$

where $N \cdot w_1 b_\Theta = N \cdot w_2 b_\Theta$ if $w_1 \mathcal{W}_\Theta = w_2 \mathcal{W}_\Theta$.

Each N-orbit through w is diffeomorphic to an euclidean space. Such an orbit $N \cdot wb_\Theta$ is called a Bruhat cell. Its dimension is given by the formula

$$\dim (N \cdot wb_\Theta) = \sum_{\alpha \in \Pi_w \setminus \langle\Theta\rangle} m_\alpha$$

where $m_\alpha = \dim(\mathfrak{g}_\alpha)$ is the multiplicity of the root space $\mathfrak{g}_\alpha$ and $\langle\Theta\rangle$ denotes the roots in Π^+ generated by Θ.

Given any $w \in \mathcal{W}/\mathcal{W}_\Theta$, in order to establish a relationship between the dimension $\dim(N \cdot wb_\Theta)$ and the length $\ell(w)$, we must choose the minimal representative for w in $\mathcal{W}^\Theta$. In this case, for $w = s_{j_1} \cdots s_{j_r} \in \mathcal{W}^\Theta$, $\dim(N \cdot wb_\Theta) = \sum_{i=1}^{r} m_{\alpha_{j_i}} + m_{2\alpha_{j_i}}$ (see [44], Corollary 2.6).

A Schubert variety is the closure of a Bruhat cell, i.e., $\mathcal{S}_w^\Theta = \mathrm{cl}(N \cdot wb_\Theta)$. The Bruhat-Chevalley order defines an order between the Schubert varieties by $\mathcal{S}_{w_1}^\Theta \subset \mathcal{S}_{w_2}^\Theta$ if, and only if, $w_1 \leqslant w_2$.

A particular case of Lie algebra is the split real form. If $\mathfrak{h} = \mathfrak{a}$ is a Cartan sub-algebra of $\mathfrak{g}$ we say that $\mathfrak{g}$ is a split real form of $\mathfrak{g}_\mathbb{C}$. In this case, we have that $\mathfrak{m} = 0$, $m_\alpha = 1$, and $m_{2\alpha} = 0$ for $\alpha \in \Pi$. Clearly, we conclude that $\dim(N \cdot wb_\Theta) = \ell(w)$.

1.1.1 Isotropic and odd orthogonal Grassmannians

Let us briefly describe the two different types of Grassmannians studied here: the isotropic $\mathrm{IG}(n-k, 2n)$ and the odd orthogonal $\mathrm{OG}(n-k, 2n+1)$.

First of all, we require some basic algebraic concepts. Suppose that V is vector space of dimension $2n$ and equipped with symplectic bilinear form ω. Given a subspace W of V, the perpendicular of W is the subspace $W^\perp = \{v \in W \mid \omega(v, w) = 0 \text{ for every } w \in W\}$. A subspace W of V is called

- isotropic: if the symplectic form vanishes in W, i.e., $W \subset W^\perp$;
- coisotropic: if the symplectic form vanishes in $W^\perp$, i.e., $W^\perp \subset W$.

The same definition of (co)isotropic space applies to a vector space V of dimension $2n+1$ and equipped with a inner product.

Consider the isotropic Grassmannian $\mathrm{IG}(n-k, 2n)$ which parametrizes $(n-k)$-dimensional isotropic subspaces of a real $2n$-dimensional symplectic vector space considered as a minimal flag manifold of the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ with the Lie algebra $\mathfrak{sp}(n, \mathbb{R})$.

The root system of type C is realized as a set of vectors

$$\Pi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm 2\varepsilon_i \mid 1 \leqslant i \leqslant n\}$$

in the Euclidean space $\mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$. Denote the simple roots[1] by $a_0 = 2\varepsilon_1$ and $a_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leqslant i < n$. The isotropic Grassmannians are minimal flag manifolds

$$\mathrm{IG}(n-k, 2n) = \mathrm{Sp}(n, \mathbb{R})/P_{(k)}$$

[1] In contrast to the usual definition of simple root given in the previous section, from now on we are going to denote a simple root by a_k because the alpha symbol "α" will be used to denote a partition.

where $(k) = \Sigma \backslash \{a_k\}$ is the maximal proper subset of the simple set of roots that does not contain the simple root a_k.

Now, consider the Grassmannian $\mathrm{OG}(n-k, 2n+1)$ which parametrizes $(n-k)$-dimensional isotropic subspaces of a real $(2n+1)$-dimensional vector space equipped with a inner product considered as a minimal flag manifold of the indefinite special orthogonal group $\mathrm{SO}(n, n+1)$ with Lie algebra $\mathfrak{so}(n, n+1)$ of type B.

The root system of type B is realized as a set of vectors

$$\Pi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm\varepsilon_i \mid 1 \leqslant i \leqslant n\}$$

in the Euclidean space $\mathbb{R}^n = \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$. Denote the simple roots by $a_0 = \varepsilon_1$ and $a_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leqslant i < n$. The odd orthogonal Grassmannians are minimal flag manifolds

$$\mathrm{OG}(n-k, 2n+1) = \mathrm{SO}(2n+1, \mathbb{R})/P_{(k)}$$

where $(k) = \Sigma \backslash \{a_k\}$ is the maximal proper subset of the simple set of roots that does not contain the simple root a_k.

The Weyl group $\mathcal{W}_n$ for the root system B_n and C_n are equal and it is isomorphic to the semidirect product $S_n \ltimes \mathbb{Z}_2^n$, i.e., they are given by permutations in S_n with a sign attached to each entry. We will write these elements as barred permutations of the form

$$\overline{n}, \ldots, \overline{2}, \overline{1}, 1, 2, \ldots, n$$

using the bar to denote a negative sign, and we take the natural order on them, as above. The hyperoctahedral group $\mathcal{W}_n$ has simple reflections $s_0, \ldots, s_{n-1}$ which acts on the right of the semidirect product $S_n \ltimes \mathbb{Z}_2^n$ by

$$(x_1, x_2, \ldots, x_n)s_0 = (\overline{x}_1, x_2, \ldots, x_n);$$
$$(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n)s_i = (x_1, \ldots, x_{i+1}, x_i, \ldots, x_n), \text{ for } 1 \leqslant i \leqslant n-1,$$

where each x_i is an integer value between $\overline{n}$ and n.

If $\mathcal{W}_k$ is the parabolic subgroup generated by $\{s_i : i \neq k\}$ then the set $\mathcal{W}^{(k)} \subset \mathcal{W}_n$ of minimal length coset representatives of $\mathcal{W}_k$ parametrizes the Schubert varieties in $\mathrm{IG}(n-k, 2n)$ and $\mathrm{OG}(n-k, 2n+1)$. This indexing set $\mathcal{W}^{(k)}$ can be identified as the set of barred permutations of the form

$$w = w_{u,\lambda} = (u_k, \ldots, u_1, \overline{\lambda}_1, \ldots, \overline{\lambda}_r, v_{n-k-r}, \ldots, v_1) \tag{1.1.1}$$

where $r \leqslant n-k$ and the sets $\lambda = (\lambda_1, \ldots, \lambda_r)$, $u = (u_1, \ldots, u_k)$, and $v = (v_1, \ldots, v_{n-k-r})$ satisfy

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0;$$
$$0 < u_k < \cdots < u_1;$$
$$0 < v_{n-k-r} < \cdots < v_1.$$

This description follows the notation given by Tamvakis [39]. A general version, for any choice of Θ, is also described by Stanley [38].

1.2 Half-shifted Young diagrams

The goal of this section is to provide the main tool for dealing with the combinatorics of isotropic and odd orthogonal Grassmannians.

A (standard) integer partition is a set of non-increasing integers $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m > 0)$. Partitions can be graphically represented with a finite collection of boxes, arranged in left-justified rows, with rows lengths equal to α_i. This representation is called a Young diagram. Let $\mathcal{R}(m,n)$ denote the set of integer partitions $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m > 0)$ with $\alpha_1 \leqslant n$ so that the Young diagram of each α fits inside an $m \times n$ rectangle. For instance, $\alpha = (5,5,4)$ is represented as the Young diagram in Figure 1.

A strict integer partition is a set of decreasing integers $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$. Strict partitions also can be graphically represented with a finite collection of boxes, arranged in a staircase shape, with rows lengths equal to λ_i. This representation is called a strict Young diagram. Define $\mathcal{D}_n$ as the set of all strict partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$ with $\lambda_1 \leqslant n$. For instance, $\lambda = (8,7,4,1)$ is represented as the strict Young diagram in Figure 1.

In both cases, the number of nonzero entries in the partition α and in the strict partition λ is denoted by $\ell(\alpha)$ and $\ell(\lambda)$, respectively.

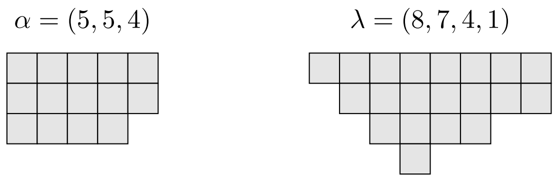


Figure 1 – Young diagram of $\alpha = (5,5,4)$ and strict Young diagram of $\lambda = (8,7,4,1)$. In this case, $\ell(\alpha) = 3$ and $\ell(\lambda) = 4$.

Pragacz and Ratajski in [33] describe the elements of $\mathcal{W}^{(k)}$ by means of Young diagrams, which combines one standard and one strict partition. Each element $w_{u,\lambda} \in \mathcal{W}^{(k)}$ corresponds to *a double partition* $\Lambda = \alpha|\lambda$ where $\alpha = \alpha(u,\lambda)$ is defined by

$$\alpha_i = u_i + i - k - 1 + d_i \tag{1.2.1}$$

for $1 \leqslant i \leqslant k$ and $d_i = \#\{j \mid \lambda_j > u_i\}$. We will call α *the top partition of* Λ and λ *the bottom partition of* Λ. The length of w is $\ell = \ell(w) = |\alpha| + |\lambda|$, where $|\alpha| := \sum_{j=1}^{k} \alpha_j$ and $|\lambda| := \sum_{j=1}^{r} \lambda_j$.

For example, consider the permutation $w = (2, 5, 6, \overline{8}, \overline{7}, \overline{4}, \overline{1}, 3)$ in IG(5, 16). From Equation (1.1.1), the strict partition λ correspond to the negtive entries of w, i.e., $\lambda = (8, 7, 4, 1)$. Using Equation (1.2.1), the partition α is $\alpha = (5, 5, 4)$. Both partitions are the ones given in Figure 1.

Define $\mathcal{P}(k, n)$ as the set of pairs $\Lambda = \alpha|\lambda$ with $\alpha \in \mathcal{R}(k, n-k)$ and $\lambda \in \mathcal{D}_n$ such that $\alpha_k \geqslant \ell(\lambda)$, i.e., Λ satisfies

$$\begin{aligned} & n - k \geqslant \alpha_1 \geqslant \cdots \alpha_k \geqslant 0, \\ & n \geqslant \lambda_1 > \cdots > \lambda_r > 0, \\ & \alpha_k \geqslant \ell(\lambda). \end{aligned} \tag{1.2.2}$$

Lemma 1.1 ([33], Lemma 1.2)**.** *There is a bijection between $\mathcal{W}^{(k)}$ and $\mathcal{P}(k, n)$.*

The bijection allows us to consider the Schubert varieties $\mathcal{S}_\Lambda$, parametrized by the set $\mathcal{P}(k, n)$ of double partitions. When we want to emphasize that a permutation w is associated to some double partition Λ, we denote the permutation by w_Λ.

Previous works already represented a partition Λ as a diagram of arranged boxes, but they do not fit in our purpose. We may arrange the top and the bottom partition in boxes in a Young diagram's style. This diagrams will be called *half-shifted Young diagrams* (HSYD), which corresponds to a half-shifted diagram in the sense presented by Tamvakis [39].

Consider the top diagram α left justified in the $k \times (n-k)$ rectangle and denote by D_α the set of square boxes with coordinates (i, j) in this rectangle arranged such that $(1, 1)$ is the upper left box. Also, consider, for each $1 \leqslant i \leqslant r$, that the i-th row of λ is shifted to the right $(i-1)$ units. With that shift, the bottom diagram may be seen inside a staircase partition with n rows and define SD_λ the set of square boxes with coordinates (i, j) of the bottom diagram arranged such that $(1, 1)$ is the upper left box.

The diagram D_Λ of $\Lambda = \alpha|\lambda$ is the juxtaposition of D_α and SD_λ. Boxes, rows, or columns that are contained in the top part D_α (resp. bottom part SD_λ) will be called *top* (resp. *bottom*) boxes, rows or columns. Note that the condition $\alpha_k \geqslant \ell(\lambda)$ implies that the number of rows in the bottom diagram does not exceed the number of boxes in the last row of the top diagram. For instance, Figure 2 presents the half-shifted diagram of the partition $\Lambda = 5, 5, 4|8, 7, 4, 1$ of a Schubert variety in IG(5, 16) and how to read an (i, j) box in each diagram. It is very important to keep in mind this example of partition because it will be used throughout Chapters 1 and 2.

The advantage of arranging the boxes of a double partition Λ as the half-shifted Young diagram is because there is a easy way get the permutation w_Λ in terms of simple reflections s_i. This is known as the *row-reading expression* of $w_\Lambda \in \mathcal{W}^{(k)}$, where $\Lambda = \alpha|\lambda \in \mathcal{P}(k, n)$. With respect to the top partition α, let $s^T : D_\alpha \to \{s_1, \ldots, s_{n-1}\}$ be

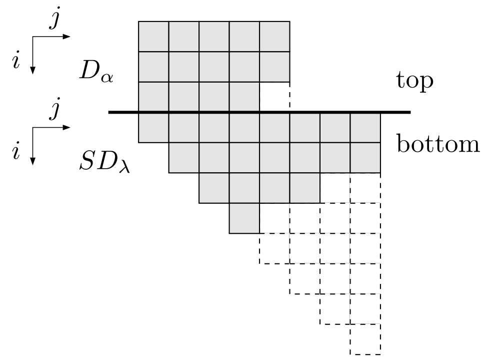


Figure 2 – Half-shifted Young diagram of $\Lambda = 5, 5, 4|8, 7, 4, 1$ in $\mathrm{IG}(5, 16)$. It shows the direction to read the boxes of D_α and SD_λ.

defined, for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n - k$, by

$$s^T(i, j) = s_{j-i+k}. \tag{1.2.3}$$

For a given partition $\alpha \in \mathcal{R}(k, n-k)$ associated with $\Lambda = \alpha|\lambda \in \mathcal{P}(k, n)$, the row-reading map is a bijection $\eta^T : D_\alpha \to \{1, 2, \ldots, |\alpha|\}$ defined by assigning the numbers increasingly to the boxes of D_α from right to left starting from the bottom row to the top row. Then, we can form a word

$$w_\alpha = s_{i_1} \cdots s_{i_{|\alpha|}} \tag{1.2.4}$$

where $s_{i_l} = s^T((\eta^T)^{-1}(l))$, for all $1 \leqslant l \leqslant |\alpha|$.

With respect to the bottom partition λ, let $s^B : SD_\lambda \to \{s_0, \ldots, s_{n-1}\}$, for $1 \leqslant i \leqslant n - k$, $1 \leqslant j \leqslant n$ and $i \leqslant j$, by

$$s^B(i, j) = s_{j-i}. \tag{1.2.5}$$

As before, for a given partition $\lambda \in \mathcal{D}_n$ associated with $\Lambda = \alpha|\lambda \in \mathcal{P}(k, n)$, the row-reading map is a bijection $\eta^B : SD_\lambda \to \{1, 2, \ldots, |\lambda|\}$ defined by assign the numbers increasingly to the boxes of SD_λ from right to left starting from the bottom row to the top row. Then, we can form a word

$$w_\lambda = s_{j_1} \cdots s_{j_{|\lambda|}} \tag{1.2.6}$$

where $s_{j_l} = s^B((\eta^B)^{-1}(l))$, for all $1 \leqslant l \leqslant |\lambda|$. The concatenation of expressions (1.2.6) and (1.2.4) gives

$$w = w_\Lambda = w_\lambda w_\alpha. \tag{1.2.7}$$

The maps s^T, s^B, η^T, and η^B for the Schubert variety given by the partition $5, 5, 4|8, 7, 4, 1$ of $\mathrm{IG}(5, 16)$ are illustrated in Figure 3. Then, $w_\lambda = s_0 \cdot s_3 s_2 s_1 s_0 \cdot s_6 s_5 s_4 s_3 s_2 s_1 s_0 \cdot s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_0$ and $w_\alpha = s_4 s_3 s_2 s_1 \cdot s_6 s_5 s_4 s_3 s_2 \cdot s_7 s_6 s_5 s_4 s_3$.

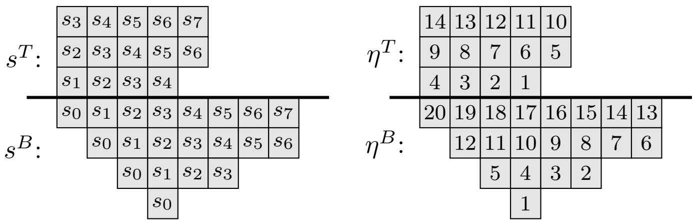


Figure 3 – Row-reading of $5,5,4|8,7,4,1$ in $\mathrm{IG}(5,16)$.

1.2.1 Related bottom columns

The HSYD gives us an easy way to get the permutation w when we know the double partition $\Lambda = \alpha|\lambda$, due to Tamvakis [39].

First of all, we need to establish a relationship between top rows and top columns with bottom columns of D_Λ.

Given any top row $1 \leqslant i \leqslant k$, the $(\alpha_i + k - i + 1)$-th bottom column in the $n \times n$ staircase shape will be called *horizontal-related*, or simply *H-related*. This definition has a geometrical explanation in D_Λ: a bottom column is H-related if we can draw a 45-degree line from the center of the first box in this column to the center of rightmost box of some top row. For example, the H-related columns of $\Lambda = 5,5,4|8,7,4,1$ are the bottom columns with gray lines in the left diagram of Figure 4.

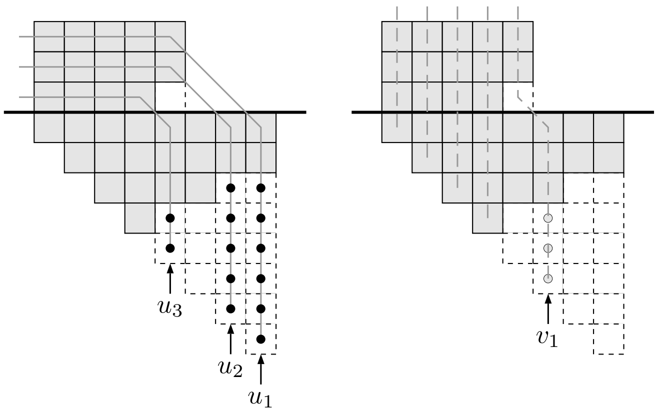


Figure 4 – The left figure presents the H-related columns (marked by the gray lines) and the right one presents the V-related columns (marked by the dashed gray lines) for $\Lambda = 5,5,4|8,7,4,1$. The number of dots of each bottom column are their vacant length.

On the other hand, given any top column $1 \leqslant j \leqslant n-k$, it also can be associated with some bottom column. The *conjugate partition* α^* of α is a partition in $\mathcal{R}(n-k,k)$ defined by $\alpha^*_i = \#\{j \mid \alpha_j \geqslant i\}$, for all $1 \leqslant i \leqslant n-k$, i.e., α^* is the partition given by the columns of α. For instance, if $\alpha = (5,5,4)$ then the respective conjugate partition is $\alpha^* = (3,3,3,2)$.

For any top column $1 \leqslant j \leqslant n-k$, the $(k+j-\alpha_j^*)$-th bottom column in the $n \times n$ staircase shape will be called *vertical-related*, or simply *V-related*. Geometrically in D_Λ, suppose that a t-th bottom column is chosen: (a) if $t \leqslant \alpha_k$ then such bottom column is V-related; (b) else, if we can draw a 45-degree line from the center of the first box in this column to the center of the blank box immediately below a box of D_α, then this bottom column is V-related. For example, the V-related columns of $\Lambda = 5,5,4|8,7,4,1$ are the bottom columns with dash gray lines in the right diagram of Figure 4.

In short:

H-relation: i-th top row $\longleftrightarrow$ $(\alpha_i + k - i + 1)$-th bottom column;

V-relation: j-th top column $\longleftrightarrow$ $(k + j - \alpha_j^*)$-th bottom column.

Lemma 1.2. *We have the following properties:*

1. *Any bottom column is either H-related or V-related;*
2. *Let $\Lambda = \alpha|\lambda$ be a double partition, and suppose that we admit the extreme values $\alpha_0 = n-k$ and $\alpha_{k+1} = 0$. For any $i \in \{0, \ldots, k\}$, then all $(\alpha_i - \alpha_{i+1})$ bottom columns between the columns H-related to the i-th and $(i+1)$-th top rows are V-related, as indicated in Figure 5.*

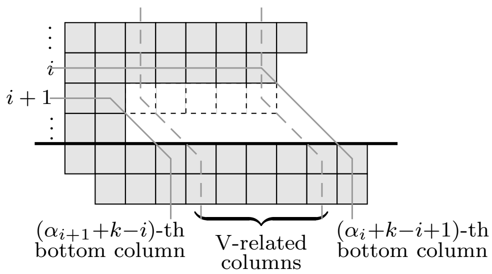


Figure 5 – Interpretation of Lemma 1.2 in the HSYD.

Proof. Suppose that the t-th bottom column is, simultaneously, H-related and V-related. Then, there are i and j such that $t = \alpha_i + k - i + 1 = k + l - \alpha_j^*$, i.e., $\alpha_i - j + 1 = i - \alpha_j^*$. If $\alpha_i \geqslant j$ then $i - \alpha_j^* \geqslant 1$ and $\alpha_j^* = \#\{l \mid \alpha_l \geqslant j\} \geqslant i$, a contradiction. On the other hand, if $\alpha_i < j$ then $i - \alpha_j^* \leqslant 0$ and $\alpha_j^* = \#\{l \mid \alpha_l \geqslant j\} < i$, also a contradiction. Hence, no bottom column can be, simultaneously, H-related and V-related. Since there are exactly k H-related bottom columns and $(n-k)$ V-related bottom columns, a bottom column is either H-related or V-related.

For the second statement, observe that if $t < i$ then $\alpha_t + k - t + 1 > \alpha_i + k - i + 1$ or if $t > i+1$ then $\alpha_t + k - t + 1 < \alpha_{i+1} + k - i$. So, all the H-related bottom columns occur either to the left of the $(\alpha_{i+1} + k - i)$-th column or to the right of the $(\alpha_i + k - i + 1)$-th column. Hence, there is no H-related column between them. $\square$

Notice that the previous lemma still works even if $k = 0$. In this situation, all bottom columns are V-related.

The *vacant length of a bottom column* is the number of empty boxes below the boxes of λ in the staircase $n \times n$ shape. Explicitly, the vacant length of the j-th bottom column is the number $j - \#\{i \mid \lambda_i + i > j\}$.

We may recover the permutation associated with such diagram by taking the vacant length of the H-related and V-related bottom columns. Namely, the permutation element for $\Lambda = \alpha|\lambda$ is defined by $w_{u,\lambda}$ in the Equation (1.1.1), where $0 < u_k < \ldots < u_1$ are the vacant length of the H-related columns, and $0 < v_{n-k-\ell(\lambda)} < \cdots < v_1$ are the vacant length of the V-related columns. For example, the partition $\Lambda = 5, 5, 4|8, 7, 4, 1$ (cf. Figure 4) corresponds to the element $w = (2, 5, 6, \overline{8}, \overline{7}, \overline{4}, \overline{1}, 3)$ since $u_1 = 6$, $u_2 = 5$, $u_3 = 2$ and $v_1 = 3$.

Remark 1.3. The notion of (H-)related boxes is defined in [9] using the model of k-strict partitions. Our definition is an adaptation of this in the context of HSYD in terms of columns, instead of diagonals.

Recall that the definition of α depends mainly on u (see Equation (1.2.1)). On the other hand, the next proposition says that the conjugate α^* is essentially given in terms of v.

Proposition 1.4. *Let $\Lambda = \alpha|\lambda$ be a partition associated with w. The conjugate partition $\alpha^* \in \mathcal{R}(n-k, k)$ can be written as*

$$\alpha^*_j = \begin{cases} k & , \text{ if } 1 \leqslant j \leqslant \ell(\lambda); \\ -v_{n-k-j+1} + j + k - \widetilde{d}_j & , \text{ if } \ell(\lambda) < j \leqslant n-k; \end{cases}$$

where $\widetilde{d}_j = \#\{l \mid \lambda_l > v_{n-k-j+1}\}$.

Proof. By definition of Λ, we know that $\alpha_k \geqslant \ell(\lambda)$. If $1 \leqslant j \leqslant \ell(\lambda)$ then $\alpha^*_j = k$, the number of rows of α.

If $\ell(\lambda) < j \leqslant n-k$ then the $(k + j - \alpha^*_j)$-th bottom column is V-related. Denoting $t := k + j - \alpha^*_j$, such t-th column has vacant length v_m, for some m. Let us figure out m.

Since $v_1 > v_2 > \cdots > v_{n-k-\ell(\lambda)} > 0$, observe that v_1 is the vacant length of the rightmost V-related column, which is associated with the $(n-k)$-th top column; v_2 is the vacant length of the second V-related column from right to left, which is associated with the $(n-k-1)$-th top column. Inductively, v_m is the vacant length of a V-related column associated with the $(n-k-m+1)$-th top column, which is the j-th column. Hence, $m = n - k - j + 1$.

The t-th bottom column has t rows in the staircase shape $n \times n$, which implies that t is equal to the vacant length of such column plus $\widetilde{d}_j$, which represents the number of boxes filled by λ in such column. In short, $t = v_{n-k-j+1} + \widetilde{d}_j$ and, therefore, $\alpha_j^* = -v_{n-k-j+1} + k + j - \widetilde{d}_j$. □

1.3 Removable boxes of a double partition

Given any permutation $w \in \mathcal{W}^{(k)}$, it is important to ask what are all the possible permutations $w' \leqslant w$ such that $\ell(w') = \ell(w) - 1$. To answer this question we should look at the HSYD.

1.3.1 Corners of a double partition

Let $\Lambda = \alpha|\lambda$ be a double partition related to a permutation w_Λ. A *corner* of Λ is a box of the HSYD such that if we delete it from D_Λ then the resulting diagram $D_{\Lambda'}$ is the HSYD of a double partition Λ', which is associated to a permutation w' that satisfies the Equations (1.2.2). Therefore, a corner of a diagram is the rightmost box of a row where there is no box just below it. Additionally, we also consider the rightmost corner of the last row of α being a corner if the strict inequality $\alpha_k > \ell(\lambda)$ holds. In other words, there is a corner in the i-th top row if and only if $\alpha_i > \alpha_{i+1}$, for $i < k = \ell(\alpha)$, or $\alpha_k > \ell(\lambda)$, and there is a corner in the i-th bottom row if and only if $\lambda_i + 1 > \lambda_{i+1}$. The last row of λ always contains a corner.

Corners that are contained in the top part D_α (resp. bottom part SD_λ) will be called *top* (resp. *bottom*) corners. We will sort the corners in different classes based on their location in the diagram. We call:

- *T-corner* any top corner;
- *D-corner* any bottom corner that lies in a diagonal box $(i,i) \in SD_\lambda$;
- *H-corner* any bottom corner that lies in an H-related bottom column;
- *V-corner* any bottom corner that lies in a V-related bottom column and it is not diagonal.

Figure 6 illustrates the corners of the double partition $5,5,4|8,7,4,1$ in $\mathrm{IG}(5,16)$. Observe that it contains all four types of corners and the last top row does not contain a corner since $\alpha_3 = \ell(\lambda)$.

The following lemma gives a different way to determine the bottom corners by using u, v, and λ as written in Equation (1.1.1).

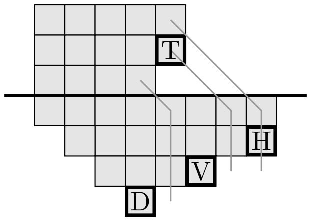


Figure 6 – Corners of $5, 5, 4|8, 7, 4, 1$ in $\mathrm{IG}(5, 16)$. They are labeled according to their type.

Lemma 1.5. *Let w be a permutation in $\mathcal{W}^{(k)}$ and $\Lambda = \alpha|\lambda$ be the associated double partition. Suppose that the t-th bottom row contains a bottom corner C. Then C is*

1. *a D-corner if, and only if, $\lambda_t = 1$ and $t = \ell(\lambda)$;*
2. *an H-corner if, and only if, there is some $1 \leqslant p_C \leqslant k$ such that $u_{p_C} = \lambda_t - 1$;*
3. *a V-corner if, and only if, there is some $1 \leqslant q_C \leqslant n-k-\ell(\lambda)$ such that $v_{q_C} = \lambda_t - 1$.*

Proof. A corner only lies in the diagonal iff it is unique in the row, proving the first statement. For the others, observe that the vacant length of the column of carrying such corner is, by symmetry, $\lambda_t - 1$. Being this an H-related or V-related column, either some u_p or some v_q is equal to $\lambda_t - 1$. □

The integer p_C of some H-corner (resp. q_C of some V-corner) C of Λ is the p_C-th H-related bottom column (resp. q_C-th V-related bottom column) counted from right to left (see Equation (1.1.1)). In Figure 6, for instance, $p_C = 1$ for the H-corner because it lays in the 1st H-related bottom column, and $q_C = 1$ for the V-corner since it belongs to the 1st V-related column.

1.3.2 Middle bottom boxes

Let $\Lambda = \alpha|\lambda$ be a double partition related to a permutation w_Λ. Given $1 \leqslant t \leqslant \ell(\lambda)$ and $1 \leqslant x \leqslant \lambda_t$, suppose that C is a bottom box that lies in the t-th bottom row and the $(t + x - 1)$-th bottom column. Then C is called a *middle bottom box* or an *M-box* of Λ if it satisfies the following conditions:

- C is neither a corner nor a diagonal box, i.e., $1 < x < \lambda_t$;
- C belongs to an H-related column;
- $x > \lambda_{t+1} + 1$;
- $\lambda_t - x + \alpha_{p_C} \leqslant \alpha_{p_C - 1}$, where p_C is the index $1 \leqslant p_C \leqslant k$ such that $\alpha_{p_C} + k - p_C + 1 = t + x - 1$.

For instance, the highlighted box in Figure 7 (left) is the unique middle bottom box of $5, 5, 4|8, 7, 4, 1$.

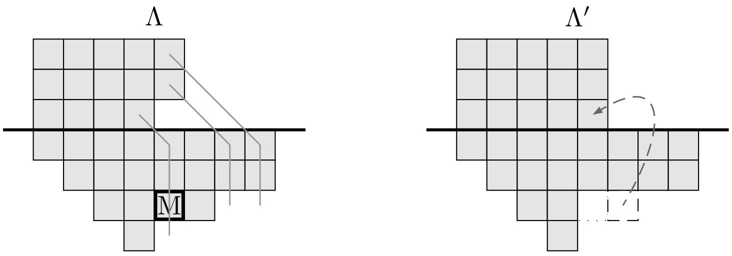


Figure 7 – The unique M-box of $5, 5, 4|8, 7, 4, 1$ in IG(5, 16). To remove this box, we need to move the last box in this row to the last row of α (the H-related top row of the M-box) to get a valid partition such that $w_{\Lambda'} \leqslant w_\Lambda$.

Notice that if we just remove such M-box, we do not get a Young diagram. But, if we remove C and move the last $(\lambda_t - x)$ boxes from the t-th bottom row to the p_C-th top row, we get a partition. In fact, this partition $\Lambda' = \alpha'|\lambda'$ is given as following:

$$\alpha'_i = \begin{cases} \alpha_i + \lambda_t - x & \text{, if } i = p_C; \\ \alpha_i & \text{, otherwise;} \end{cases} \qquad \lambda'_j = \begin{cases} x - 1 & \text{, if } j = t; \\ \lambda_j & \text{, otherwise.} \end{cases} \tag{1.3.1}$$

Hence, the permutation $w_{\Lambda'}$ associated to Λ' satisfies that $w_{\Lambda'} \leqslant w_\Lambda$ and $\ell(\Lambda) = \ell(\Lambda') + 1$. We call Λ' the *rearrangement of* Λ *with respect to the M-box* C.

The conditions of an M-box C implies that all boxes to the right of C lie in a V-related bottom column. Indeed, Lemma 1.2 says that there are $(\alpha_{p_C - 1} - \alpha_{p_C})$ V-related columns after to the right of C and all boxes to the right fit in such range.

We also have a formula of u_{p_C} for M-boxes likely for corners in Lemma 1.5. Given an M-box C, the index p_C also satisfies that

$$u_{p_C} = x - 1. \tag{1.3.2}$$

1.3.3 Bruhat order and Young's graph

The *Young's graph* is defined by the graph whose vertices are all the HSYD of $\Lambda \in \mathcal{P}(k, n)$ and the edges are defined by $\Lambda \to \Lambda'$ where $w_{\Lambda'} \leqslant w_\Lambda$ and $\ell(\Lambda) = \ell(\Lambda') + 1$. In other word, $\Lambda \to \Lambda'$ if, and only if, one of the following happens

- Λ' is obtained by deleting for some corner C from Λ; or
- Λ' is a rearrangement of Λ with respect to the M-box C.

In the Grassmannian IG$(n - k, 2n)$ or OG$(n - k, 2n + 1)$, the top cell $\mathcal{S}_{w_\circ}$ corresponding to the longest element $w_\circ \in \mathcal{W}^{(k)}$ has associated double diagram $\Lambda_\circ =$

$n-k,\dots,n-k|n,n-1,\dots,k+1$, which represents the $k\times(n-k)$ rectangle for α and all the $n-k$ rows in the staircase shape for λ. Then, we can construct the corresponding Young's graph from the larger diagram $\Lambda_\circ$ and get all the vertices and edges by deleting corners and M-boxes. Figure 8 illustrates this process and presents the Young's graph for IG$(2,8)$ and OG$(2,9)$, where $n=4$ and $k=2$.

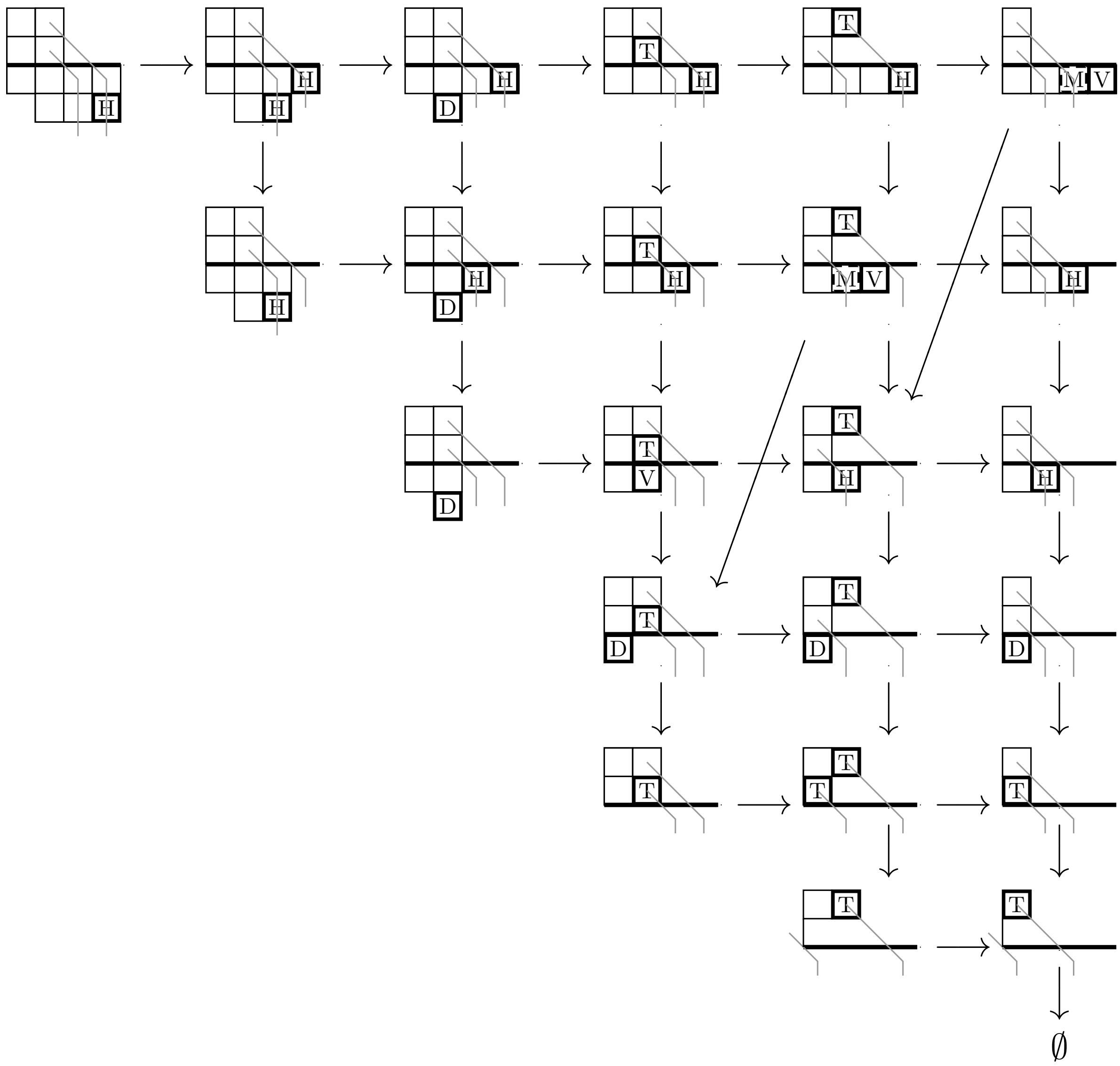


Figure 8 – Young's graph for IG$(2,8)$ and OG$(2,9)$. The highlighted boxes are the corners and M-boxes, and the letters inside them tell us their type.

1.3.4 Lagrangian and maximal odd orthogonal Grassmannians

As a particular case of what was described above, the minimal flag manifolds $\mathbb{F}_{(0)}$ yields the Lagrangian Grassmannian L$(n,2n)$ = IG$(n,2n)$ for type C and the maximal odd orthogonal Grassmannian OG$(n,2n+1)$ for type B.

The double permutation is $\Lambda=\varnothing|\lambda$, i.e., $\Lambda=\lambda$ and there is no top diagram.

Hence, $\mathcal{W}^{(0)}$ corresponds to all strict partitions in $\mathcal{D}_\lambda$. Moreover, there is no H-related columns and only two types of corners occur, namely, D-corners and V-corners.

By the row-reading expression (1.2.7), recall that any Weyl group element is written as $w = w_\lambda w_\alpha$. In particular, if $k = 0$, then $w = w_\lambda$ and we may consider w_λ as an element of $\mathcal{W}^{(0)}$. The remark below explains how we can obtain the permutation w_λ (the bottom row-reading) once the permutation w is known.

Remark 1.6. If $k = 0$, there is no top partition and the permutation Λ is only given by a strict partition λ. Hence, $w_\lambda = w$ is a permutation in $\mathcal{W}^{(0)}$ and Equation (1.1.1) for w_λ is

$$w_\lambda = (\overline{\lambda}_1, \ldots, \overline{\lambda}_r, \widetilde{v}_{n-k}, \ldots, \widetilde{v}_1) \tag{1.3.3}$$

where $0 < \widetilde{v}_{n-k} < \cdots < \widetilde{v}_1$. Then, for any double permutation $\Lambda = \alpha|\lambda$ chosen from an isotropic Grassmannian, w_λ is written as in Equation (1.3.3). But recall from Equation (1.1.1) that w_α does not contain the simple reflection s_0. Since $w = w_\lambda w_\alpha$ and the simple reflections act on the right, it follows that w_α simply permutes the entries of the permutation w_λ. Therefore, w_λ can be obtained from w by reordering the entries increasingly.

For example, considering $\Lambda = 5, 5, 4|8, 7, 4, 1$ in IG(5, 16), we can reorder the entries of $w_\Lambda = (2, 5, 6, \overline{8}, \overline{7}, \overline{4}, \overline{1}, 3)$ in order to get $w_\lambda = (\overline{8}, \overline{7}, \overline{4}, \overline{1}, 2, 3, 5, 6)$.

1.4 Inversions and Young diagrams

This section is devoted to show how to fill in the half-shifted Young diagrams with roots of the set of inversions. As far as we know, such description has not been done elsewhere. In particular, this will be very useful to compute boundary maps of cellular homology.

Let $w = s_{j_1} \cdots s_{j_r} \in \mathcal{W}^{(k)}$ be a reduced decomposition of w, where $r = \ell(w)$. Define the *β-sequences of* w as the roots

$$\beta_t = s_{j_1} \cdots s_{j_{t-1}}(a_{j_t}) \ , \text{ for } 1 \leqslant t \leqslant r. \tag{1.4.1}$$

Consider the set $\Pi_w = \Pi^+ \cap w\Pi^-$ of positive roots sent to negative roots by w^{-1}, which is also called *set of inversions of* w. Using a reduced decomposition of w, it is known that

$$\Pi_w = \{\beta_1, \ldots, \beta_r\}.$$

Although the definition of β-sequences of w depends on the choice of a reduced decomposition, different reduced decompositions can be obtained by permuting such roots.

Since $|\Lambda|$ is equal to the cardinality of Π_w where Λ is the double diagram of w, we aim to describe each β-sequence of $w \in \mathcal{W}^{(k)}$ using the position (i, j) of a box in D_Λ of w.

Let $\mathcal{S}_\Lambda$ be the Schubert variety parametrized by $\Lambda = \alpha|\lambda$ in $\mathcal{P}(k,n)$. Denote by β_t^α and β_t^λ the β-sequences of w_α and w_λ, respectively.

Given a bottom box (i,j) in SD_λ set $\beta_{i,j}^B := \beta_{\eta^B(i,j)}^\lambda$ and given a top box (i,j) in D_α set $\beta_{i,j}^T := w_\lambda \cdot \beta_{\eta^T(i,j)}^\alpha$, where η^T and η_B are the row-reading maps of D_Λ.

Proposition 1.7. *Every β-sequence of Λ can be obtained from boxes in D_λ as follows:*

1. *For top boxes: $\beta_{i,j}^T = \varepsilon_{w(k-i+1)} - \varepsilon_{w(k+j)}$, for $(i,j) \in D_\alpha$;*
2. *For bottom boxes, it depends on the type of G:*
   a) *Type C: $\beta_{i,j}^B = \varepsilon_{w_\lambda(2n+1-i)} - \varepsilon_{w_\lambda(j)}$, for $(i,j) \in SD_\lambda$;*
   b) *Type B: $\beta_{i,j}^B = 2^{-\delta_{ij}}(\varepsilon_{w_\lambda(2n+1-i)} - \varepsilon_{w_\lambda(j)})$, for $(i,j) \in SD_\lambda$.*

Proof. By definition, β_t^B is the β-sequence for w_λ and this formula is done in [28]. Lemma 3 of [28] says that $\beta_{\eta^T(i,j)}^\alpha = \varepsilon_{w_\alpha(k-i+1)} - \varepsilon_{w_\alpha(k+j)}$. Therefore,

$$\beta_{i,j}^T = w_\lambda(\varepsilon_{w_\alpha(k-i+1)} - \varepsilon_{w_\alpha(k+j)}) = \varepsilon_{w(k-i+1)} - \varepsilon_{w(k+j)}. \qquad \square$$

Clearly, this proposition describes all β-sequences of w. Notice that $\beta_{i,j}^B$, in both cases, is obtained from w_λ instead of w.

We can use this proposition to compute the set Π_w of all β-sequences by placing such roots into each box of the HSYD of w. Let us start with the top part: label all k rows of the top diagram with $w(1), \ldots, w(k)$ from the bottom upwards, and label all the $n-k$ columns with $w(k+1), \ldots, w(n)$ from left to right. Then, the β-sequence associated with each box $(i,j) \in D_\alpha$ is the root $\varepsilon_a - \varepsilon_b$, where a and b are the labels of i-th row and j-th column, respectively. For example, considering $\Lambda = 5,5,4|8,7,4,1$ in $\mathrm{IG}(5,16)$, we label it as in Figure 9. Notice that the top box $(1,1) \in D_\alpha$ has labels 6 and $\overline{8}$ for row and column, respectively, implying that $\beta_{(1,1)}^T = \varepsilon_6 - \varepsilon_{\overline{8}} = \varepsilon_6 + \varepsilon_8$.

Now, for the bottom part, start labeling the n columns of the bottom diagram with $w_\lambda(1), \ldots, w_\lambda(n)$ from left to right, and label the n rows with $\overline{w_\lambda(1)}, \ldots, \overline{w_\lambda(n)}$ from top to bottom. The β-sequence associated with each box $(i,j) \in SD_\lambda$ is the root $\varepsilon_a - \varepsilon_b$, where a and b are the labels of i-th row and j-th column, respectively. Figure 9 gives an example of how to label the bottom part and what are the β-sequences of each box.

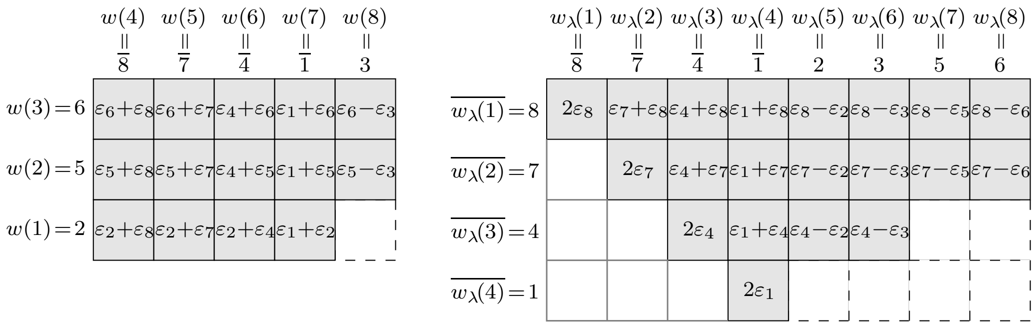


Figure 9 – β-sequences associated with the partition $5, 5, 4|8, 7, 4, 1$.

Chapter 2

Homology of isotropic and odd orthogonal Grassmannians

In this Chapter, we state and prove the theorem about the coefficients of the boundary map of isotropic and odd orthogonal Grassmannians.

2.1 Boundary map and integral homology of real flag manifolds

A general formula for the boundary map of real flags manifolds was first obtained by Kocherlakota [30] by a Morse homology approach. The same result was developed by Rabelo and San Martin [36], where they computed directly the boundary map using the Bruhat decomposition as a CW complex. We choose to summarize these results using [36] since it follows the same notation we presented in the previous chapter.

Consider first a maximal flag manifold $\mathbb{F}$. Its cellular homology is defined from a cellular decomposition provided by the Schubert varieties. Given a Schubert variety $\mathcal{S}_w$, we fix once and for all reduced decompositions

$$w = s_{j_1} \cdots s_{j_r}$$

as a product of simple reflections, for each $w \in \mathcal{W}$, and $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$. Let $\mathcal{C}$ be the $\mathbb{Z}$-module freely generated by $\mathcal{S}_w$, for every element w of the Weyl group $\mathcal{W}$. The boundary maps $\partial : \mathcal{C} \to \mathcal{C}$ are defined by

$$\partial \mathcal{S}_w = \sum_{w'} c(w, w') \mathcal{S}_{w'} \tag{2.1.1}$$

for some coefficients $c(w, w') \in \mathbb{Z}$. First of all, if $\ell(w) - \ell(w') \neq 1$ or if w' and w are not comparable by the Bruhat-Chevalley order "$\leqslant$" then set $c(w, w') = 0$. Now, it is only missing to define $c(w, w')$ when $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$, i.e., when $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$.

Proposition 2.1 ([36], Proposition 4.1)**.** *Let $w, w' \in \mathcal{W}$. The following statements are equivalent.*

1. *$\mathcal{S}_{w'} \subset \mathcal{S}_w$ and $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$.*

2. *Given a the reduced decomposition $w = s_{j_1} \cdots s_{j_r}$ then a reduce decomposition of w' can be obtained from w by removing some simple reflection s_{j_i}, i.e., $w' = s_{j_1} \cdots \widehat{s}_{j_i} \cdots s_{j_r}$.*

The first main result is that the coefficient $c(w, w')$ is the sum of the degree of two sphere homeomorphisms which have degree one.

Theorem 2.2 ([36], Theorem 4.3)**.** *For any $w, w' \in \mathcal{W}$ such that $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$, the coefficient $c(w, w') = 0, \pm 2$.*

Suppose that $\mathfrak{g}$ is a split real form. It is possible to get a more accurate expression for the coefficients $c(w, w')$ in terms of roots. For $w \in \mathcal{W}$, define

$$\sigma(w) = \sum_{\beta \in \Pi_w} \beta \tag{2.1.2}$$

the sum of β-sequences $\Pi_w = \Pi^+ \cap w\Pi^-$.

Given w and w' of $\mathcal{W}^{(k)}$ such that $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$, denote by $w = s_{j_1} \cdots s_{j_r}$ and $w' = s_{j_1} \cdots \widehat{s}_{j_i} \cdots s_{j_r}$ a reduced decomposition for them. Let γ be the unique root (not necessarily simple) satisfying $w = s_\gamma w'$, that is, $\gamma = s_{j_1} \cdots s_{j_{i-1}}(a_{j_i})$.

Theorem 2.3 ([30], Thm. 1.1.4 and [36], Theorem 4.9)**.** *Suppose that $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$, i.e., there is a root γ such that $w = s_\gamma w'$. Then*

$$\sigma(w) - \sigma(w') = \kappa \cdot \gamma \tag{2.1.3}$$

for some integer $\kappa = \kappa(w, w')$. Moreover, no nontrivial multiple of γ is a root, so that κ is a well-defined integer. Then the coefficient $c(w, w')$ is given as follows:

$$c(w, w') = \pm(1 + (-1)^\kappa) = \begin{cases} 0 & , \text{ if } \kappa \text{ is odd,} \\ \pm 2 & , \text{ if } \kappa \text{ is even.} \end{cases} \tag{2.1.4}$$

Thus, the signs on $c(w, w')$ can be chosen so that $\partial^2 = 0$ and the homology of $(\mathcal{C}, \partial)$ is the integral homology of $\mathbb{F}$.

The sign of $c(w, w') = \pm 2$ depends on the choice of a reduced decomposition of the elements in the Weyl group, but observe that the sign is not relevant in order to compute the homology. Besides, in our context, we have a fixed choice of reduced decompositions.

Remark 2.4. The formula (2.1.4) as stated in [36] includes another ± 1 factor which is omitted here because the reduced decompositions for the Weyl group elements are previously defined.

In the context of the partial flag manifolds $\mathbb{F}_\Theta$, recall that the Schubert varieties are $\mathcal{S}_w^\Theta$ are the closure of the Bruhat cells $N \cdot wb_\Theta$, for $w \in \mathcal{W}^\Theta$. Let $\mathcal{C}^\Theta$ be the $\mathbb{Z}$-module freely generated by $\mathcal{S}_w^\Theta$, for every element w of $\mathcal{W}^\Theta$. The boundary maps $\partial^\Theta : \mathcal{C}^\Theta \to \mathcal{C}^\Theta$ are defined by

$$\partial^\Theta \mathcal{S}_w^\Theta = \sum_{w' \in \mathcal{W}^\Theta} c^\Theta(w, w') \mathcal{S}_{w'}^\Theta$$

for some coefficients $c^\Theta(w, w') \in \mathbb{Z}$.

Theorem 2.5 ([36], Theorem 5.4)**.** *The integral homology of the flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ is isomorphic to the homology of $(\mathcal{C}^\Theta, \partial^\Theta)$, where ∂^Θ obtained by restricting ∂ and projecting it onto $\mathcal{C}^\Theta$.*

Hence the coefficients $c^\Theta(w, w')$ for the boundary map ∂^Θ of the cellular homology of the partial flag manifolds $\mathbb{F}_\Theta$ is

$$c^\Theta(w, w') = c(w, w').$$

and the computation of $c^\Theta(w, w')$ reduces to a computation of $c(w, w')$ on $\mathbb{F}$.

In short, to compute the boundary map of any flag manifold, we need to be able to compute κ for any given $w, w' \in \mathcal{W}^\Theta$ such that $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$, as in Equation (2.1.3).

2.2 Integral homology of isotropic and odd orthogonal Grassmannians

Let G be either an indefinite odd orthogonal or a symplectic group. Then the flag manifold $\mathbb{F}_{(k)}$ with respect to the set $(k) = \Sigma - \{a_k\}$ is, respectively, the isotropic Grassmannian $\mathrm{IG}(n-k, 2n)$ or odd orthogonal Grassmannians $\mathrm{OG}(n-k, 2n+1)$.

The integral homology $H_*(\mathbb{F}_{(k)}, \mathbb{Z})$ can be computed after we determine the boundary map as given in Theorem 2.3 which gives us a neat general formula for the boundary coefficients $c(w, w')$. In the sequel, we will state our main theorem as a consequence of the next proposition that provides an explicit combinatorial expression of κ in terms of the half-shifted Young diagrams used to index the Schubert varieties.

Remember that given $w, w' \in \mathcal{W}^{(k)}$ and their respective double diagrams Λ, Λ', then $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$ if, and only if, one of the following happens

- Λ' is obtained by deleting for some corner C from Λ; or
- Λ' is a rearrangement of Λ with respect to the M-box C.

Also recall from Lemma 1.5 that if C is an H-corner (resp. a V-corner) and t is the t-th bottom row containing C then there is a positive integer $p_C \leqslant k$ such that $u_{p_C} = \lambda_t - 1$ (resp. there is a positive integer $q_C \leqslant n-k-\ell(\lambda)$ such that $v_{q_C} = \lambda_t - 1$).

Proposition 2.6. *Let $\Lambda = \alpha|\lambda$ be the double diagram associated with a permutation w and C either a corner or an M-box of D_Λ. Denote by w' the permutation associated with the diagram Λ' obtained by deleting C (and rearranging it with respect to an M-box). Suppose that t is either the t-th top row of D_α or the t-th bottom row of SD_λ containing C. Then,*

$$\sigma(w) - \sigma(w') = \kappa \cdot \gamma,$$

where γ is a root such that $w = s_\gamma w'$ and κ is an integer that depends on the type of the group G as following:

1. *If $\mathbb{F}_{(k)}$ is an isotropic Grassmannian (i.e., G is of type C), then*

$$\kappa = \begin{cases} t + \alpha_t - 1 & \text{, if } C \text{ is a T-corner;} \\ t + k & \text{, if } C \text{ is a D-corner;} \\ t + 2k - p_C + 1 & \text{, if } C \text{ is an H-corner or an M-box;} \\ t + n + k - q_C + 1 & \text{, if } C \text{ is a V-corner.} \end{cases}$$

2. *If $\mathbb{F}_{(k)}$ is an odd orthogonal Grassmannian (i.e., G is of type B), then*

$$\kappa = \begin{cases} t + \alpha_t - 1 & \text{, if } C \text{ is a T-corner;} \\ 2t + 2k - 1 & \text{, if } C \text{ is a D-corner;} \\ t + 2k - p_C & \text{, if } C \text{ is an H-corner or an M-box;} \\ t + n + k - q_C & \text{, if } C \text{ is a V-corner.} \end{cases}$$

This proposition will be proved in section 2.4. Now, we can rewrite Theorem 2.3 for isotropic and odd orthogonal Grassmannians as follows:

Theorem 2.7. *Consider the hypothesis of Proposition 2.6. The coefficients $c(w, w')$ are non-zero for the cases described below.*

1. *Suppose that $\mathbb{F}_{(k)}$ is an isotropic Grassmannian. Then $c(w, w') = \pm 2$ if, and only if, the box C satisfies one of the following statements:*

    a) C is a T-corner and $t + \alpha_t \equiv 1 \mod 2$,

    b) C is a D-corner and $t + k \equiv 0 \mod 2$,

    c) C is an H-corner or an M-box and $t + p_C \equiv 1 \mod 2$,

    d) C is a V-corner and $t + n + k + q_C \equiv 1 \mod 2$,

2. *Suppose that $\mathbb{F}_{(k)}$ is an odd orthogonal Grassmannian. Then $c(w, w') = \pm 2$ if, and only if, the box C satisfies one of the following statements:*

a) C is a T-corner and $t + \alpha_t \equiv 1 \bmod 2$*,*

b) C is an H-corner or an M-box and $t + p_C \equiv 0 \bmod 2$*,*

c) C is a V-corner and $t + n + k + q_C \equiv 0 \bmod 2$*.*

Proof. Clearly, this theorem takes the formulas of κ in Proposition 2.6 and states when it should be even. □

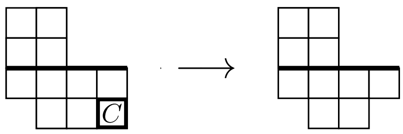
In order to compute the boundary map, we are going to modify the Young's graph so that the graph can carry the information of $c(w, w')$. If $c(w, w') = \pm 2$, replace the edge $\to$ between the respective vertices by $\Rightarrow$ in the Young's graph. Casian and Kodama in [14] call such graph the *incidence graph* of a Grassmannian. Notice that the incidence graph encodes the information about the boundary map.

Let us compute an explicit example of homology for some Grassmannian. Consider the isotropic Grassmannian IG(2, 8), which is a case where $n = 4$ and $k = 2$. First of all, we need to get the incidence graph in this situation. We will compute four examples of $c(w, w')$, one for each type of corner.

- Suppose that

$$w = (1, 2, \overline{4}, \overline{3}) = (u_2, u_1, \overline{\lambda_1}, \overline{\lambda_2});$$
$$w' = (1, 3, \overline{4}, \overline{2}) = (u_2, u_1 + 1, \overline{\lambda_1}, \overline{\lambda_2 - 1})$$

are the permutations which corresponds to the partitions $\Lambda = 2, 2|4, 3$ and $\Lambda' = 2, 2|4, 2$, respectively, as below:

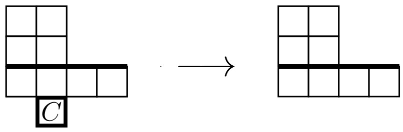


Notice that C is an H-corner because it is H-related to the first row, and $t = 2$ since C is the corner at the 2nd bottom row of D_Λ. On the other hand, remember that there is some integer p_C such that $u_{p_C} = \lambda_t - 1$ when C is an H-corner. Since we have the relation $u_1 = \lambda_2 - 1$, then $p_C = 1$. Therefore, applying 1.c of Theorem 2.7, we have that $t + p_C = 3 \equiv 1 \bmod 2$ and $c(w, w') = \pm 2$.

- Suppose that

$$w = (2, 3, \overline{4}, \overline{1}) = (u_2, u_1, \overline{\lambda_1}, \overline{\lambda_2});$$
$$w' = (2, 3, \overline{4}, 1) = (u_2, u_1, \overline{\lambda_1}, \lambda_2)$$

are the permutations which corresponds to the partitions $\Lambda = 2, 2|4, 1$ and $\Lambda' = 2, 2|4$, respectively, as below:

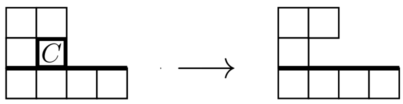


Notice that C is a D-corner, and $t = 2$. Therefore, applying 1.b of Theorem 2.7, we have that $t + k = 4 \equiv 0 \bmod 2$ and $c(w, w') = \pm 2$.

- Suppose that

$$w = (2, 3, \overline{4}, 1) = (u_2, u_1, \overline{\lambda_1}, v_1);$$
$$w' = (1, 3, \overline{4}, 2) = (v_1, u_1, \overline{\lambda_1}, u_2)$$

are the permutations which corresponds to the partitions $\Lambda = 2, 2|4$ and $\Lambda' = 2, 1|4$, respectively, as below:

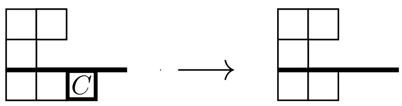


Notice that C is a T-corner, and $t = 2$ since C is the corner at the 2nd top row of D_Λ. Therefore, applying 1.a of Theorem 2.7, we have that $t + \alpha_t = 4 \equiv 0 \bmod 2$ and $c(w, w') = 0$.

- Suppose that

$$w = (1, 4, \overline{3}, 2) = (u_2, u_1, \overline{\lambda_1}, v_1);$$
$$w' = (1, 4, \overline{2}, 3) = (u_2, u_1, \overline{\lambda_1 - 1}, v_1 + 1)$$

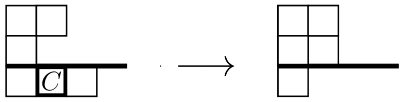
are the permutations which corresponds to the partitions $\Lambda = 2, 1|3$ and $\Lambda' = 2, 1|2$, respectively, as below:



Notice that C is a V-corner because it is neither D-related nor H-related to any top row, and $t = 1$ since C is the corner at the 1st bottom row of D_Λ. On the other hand, remember that there is some integer q_C such that $v_{q_C} = \lambda_t - 1$ when C is a V-corner. Since we have the relation $v_1 = \lambda_1 - 1$, then $q_C = 1$. Therefore, applying 1.d of Theorem 2.7, we have that $t + n + k + q_C = 8 \equiv 0 \bmod 2$ and $c(w, w') = 0$.

- Suppose that

$$w = (1, 4, \overline{3}, 2) = (u_2, u_1, \overline{\lambda_1}, v_1);$$
$$w' = (3, 4, \overline{1}, 2) = (u_2 + 2, u_1, \overline{\lambda_1 - 2}, v_1)$$



are the permutations which corresponds to the partitions $\Lambda = 2, 1|3$ and $\Lambda' = 2, 2|1$, respectively, as below:

Notice that for $x = 2$ and $t = 1$, we have C is an M-box because: it is neither a corner nor a diagonal box; it is H-related to the second row, i.e., $p_C = 2$; $x > \lambda_2 = 0$; and $\lambda_1 - x + \alpha_2 \leqslant \alpha_1$. Therefore, applying 1.e of Theorem 2.7, we have that $t + p_C = 3 \equiv 1 \bmod 2$ and $c(w, w') = \pm 2$.

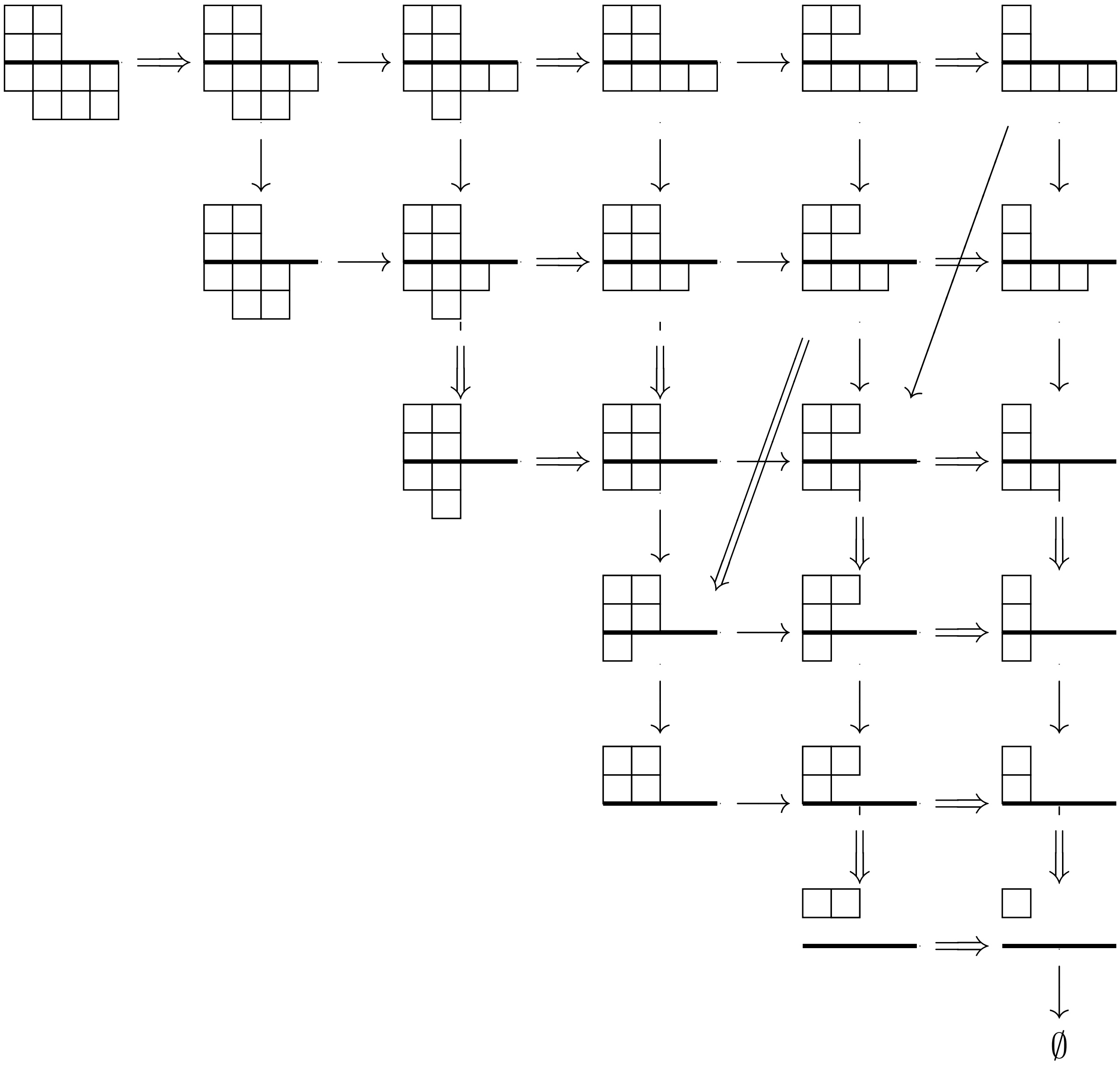


Figure 10 – Incidence graph of IG(2, 8).

Proceeding as above, we can get all the coefficients $c(w, w')$ for any pair w, w'. The incidence graph of the isotropic Grassmannian IG(2, 8) is given in Figure 10. The boundary map is given below, where they are presented considering each cell dimension.

11-dim: $\partial\mathcal{S}_{2,2|4,3} = \pm 2\mathcal{S}_{2,2|4,2}$;

10-dim: $\partial\mathcal{S}_{2,2|4,2} = 0$;

9-dim: $\partial\mathcal{S}_{2,2|3,2} = 0$ and $\partial\mathcal{S}_{2,2|4,1} = \pm 2\mathcal{S}_{2,2|4}$;

8-dim: $\partial\mathcal{S}_{2,2|3,1} = \pm 2\mathcal{S}_{2,2|2,1} \pm 2\mathcal{S}_{2,2|3}$ and $\partial\mathcal{S}_{2,2|4} = 0$;

7-dim: $\partial\mathcal{S}_{2,2|2,1} = \pm 2\mathcal{S}_{2,2|2}$ and $\partial\mathcal{S}_{2,2|3} = \pm 2\mathcal{S}_{2,2|2}$ and $\partial\mathcal{S}_{2,1|4} = \pm 2\mathcal{S}_{1,1|4}$;

6-dim: $\partial\mathcal{S}_{2,2|2} = 0$ and $\partial\mathcal{S}_{2,1|3} = \pm 2\mathcal{S}_{1,1|3} \pm 2\mathcal{S}_{2,2|1}$ and $\partial\mathcal{S}_{1,1|4} = 0$;

5-dim: $\partial\mathcal{S}_{2,2|1} = 0$ and $\partial\mathcal{S}_{2,1|2} = \pm 2\mathcal{S}_{2,1|1} \pm 2\mathcal{S}_{1,1|2}$ and $\partial\mathcal{S}_{1,1|3} = 0$;

4-dim: $\partial\mathcal{S}_{2,2|\varnothing} = 0$ and $\partial\mathcal{S}_{2,1|1} = \pm 2\partial\mathcal{S}_{1,1|1}$ and $\partial\mathcal{S}_{1,1|2} = \pm 2\partial\mathcal{S}_{1,1|1}$;

3-dim: $\partial\mathcal{S}_{2,1|\varnothing} = \pm 2\mathcal{S}_{2,0|\varnothing} \pm 2\mathcal{S}_{1,1|\varnothing}$ and $\partial\mathcal{S}_{1,1|1} = 0$;

2-dim: $\partial\mathcal{S}_{2,0|\varnothing} = \pm 2\mathcal{S}_{1,0|\varnothing}$ and $\partial\mathcal{S}_{1,1|\varnothing} = \pm 2\mathcal{S}_{1,0|\varnothing}$;

1-dim: $\partial\mathcal{S}_{1,0|\varnothing} = 0$.

Therefore, the integral homology of $\mathrm{IG}(2,8)$ is

$$\begin{aligned}
H_{11}(\mathrm{IG}(2,8),\mathbb{Z}) &= 0; & H_5(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}\oplus\mathbb{Z}_2;\\
H_{10}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2; & H_4(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}\oplus\mathbb{Z}_2;\\
H_9(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}; & H_3(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2;\\
H_8(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2; & H_2(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2;\\
H_7(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2; & H_1(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2;\\
H_6(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2\oplus\mathbb{Z}_2; & H_0(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}.
\end{aligned}$$

Notice that $\mathrm{IG}(2,8)$ is not orientable. The following corollary states when a Grassmannian is orientable.

Corollary 2.8. *About the orientability of isotropic or odd orthogonal Grassmannians:*

1. $\mathrm{IG}(n-k,2n)$ *is orientable if, and only if,* $n-k$ *is odd.*

2. *For* $k>0$, $\mathrm{OG}(n-k,2n+1)$ *is orientable if, and only if,* $n-k$ *is even. If* $k=0$, $\mathrm{OG}(n,2n+1)$ *is orientable for every* n.

Proof. The top cell $\mathcal{S}_{w_\circ}$ corresponding to the longest element $w_\circ \in \mathcal{W}^{(k)}$ has associated double diagram $\Lambda_\circ = n-k,\dots,n-k|n,n-1,\dots,k+1$. Hence, the associated double diagram associated with the unique codimension one cell $\mathcal{S}_{w'_\circ}$ is $\Lambda'_\circ = n-k,\dots,n-k|n,n-1,\dots,k+2,k$ and there is a corner C in the diagram $D_{\Lambda_\circ}$ of $w_\circ$ such that $D_{\Lambda'_\circ} = D_{\Lambda_\circ} - \{C\}$.

Being orientable is equivalent to boundary map in the top cell equals zero. The analysis is divided in two cases. If $k > 0$ then the corner C is an H-corner in the $(n-k)$-th bottom row with $p_C = 1$, by Lemma 1.5. Hence, by Theorem 2.7,

- $\mathrm{IG}(n-k, 2n)$ is orientable iff $t + p_C = (n-k) + 1 \equiv 0 \bmod 2$ iff $n - k$ is odd.
- $\mathrm{OG}(n-k, 2n+1)$ is orientable iff $t + p_C = (n-k) + 1 \equiv 1 \bmod 2$ iff $n - k$ is even.

Now if $k = 0$, the corner C is a D-corner in n-th bottom row. By Theorem 2.7, the odd orthogonal Grassmannian is orientable for every n and the Lagrangian Grassmannian is orientable iff $n + k \equiv 1 \bmod 2$ iff n is odd. □

Corollary 2.9. *Let G be an isotropic or odd orthogonal Grassmannians. The 1-homology and 2-homology of G are:*

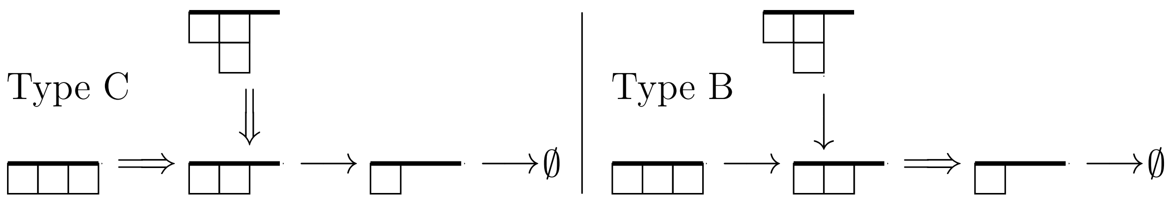
$$H_1(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & , \text{ if } k = 0 \text{ and } G \text{ is of type } C, \\ \mathbb{Z} & , \text{ if } k = 0,\ n = 1, \text{ and } G \text{ is of type } B, \\ \mathbb{Z} & , \text{ if } k = 1,\ n = 2, \text{ and } G \text{ is of type } B, \\ \mathbb{Z}_2 & , \text{ otherwise.} \end{cases}$$

$$H_2(G,\mathbb{Z}) = \begin{cases} 0 & , \text{ if } k = 0, \text{ and } G \text{ is of type } B, \\ 0 & , \text{ if } k = 1,\ n = 2, \text{ and } G \text{ is of type } C, \\ \mathbb{Z}_2 & , \text{ otherwise.} \end{cases}$$

Proof. We will proof using the incidence graph related to Schubert varieties of dimension 1, 2, and 3. It requires to split in some cases.

- $k = 0$:

  In this case, there is no top part and the incidence graph is

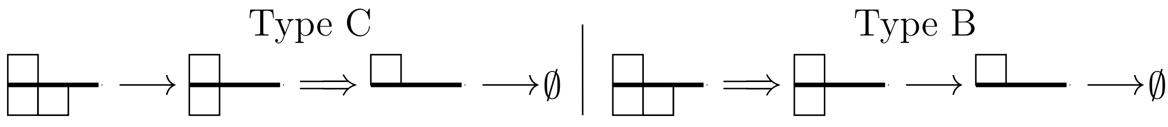


  Then, for $n \geqslant 1$, we have that

$$H_1(\mathrm{IG}(n,2n)) = \mathbb{Z}, \qquad H_1(\mathrm{OG}(1,2)) = \mathbb{Z},$$
$$H_2(\mathrm{IG}(n,2n)) = \mathbb{Z}_2, \qquad H_1(\mathrm{OG}(n,2n)) = \mathbb{Z}_2\ (n > 1),$$
$$H_2(\mathrm{OG}(n,2n)) = 0.$$

- $k = 1$ and $n = 2$:

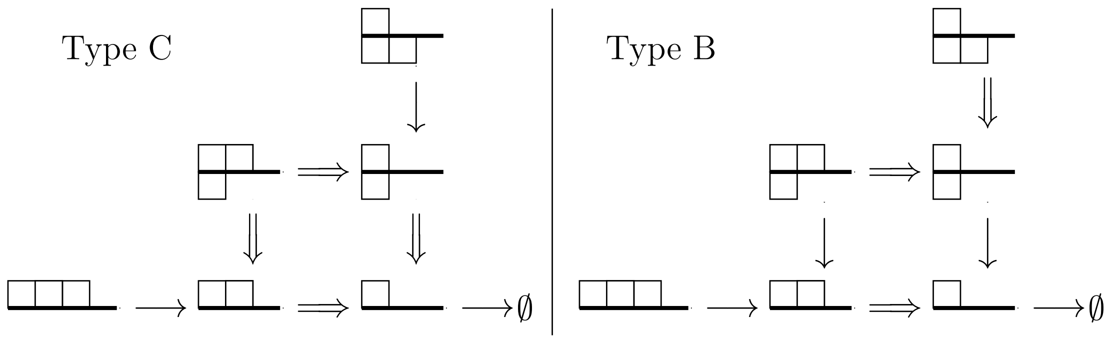
  In this case, the incidence graph is



  Then, we have that

$$H_1(\mathrm{IG}(1,4)) = \mathbb{Z}_2, \qquad H_1(\mathrm{OG}(1,4)) = \mathbb{Z},$$
$$H_2(\mathrm{IG}(1,4)) = 0, \qquad H_2(\mathrm{OG}(1,4)) = \mathbb{Z}_2.$$

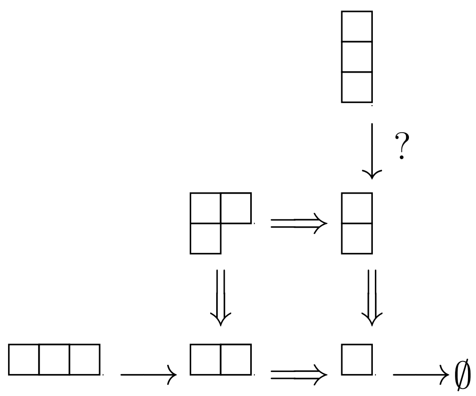
- $k = 1$ and $n > 2$: In this case, the incidence graph is



  Then, for $n > 2$, we have that

$$H_1(\mathrm{IG}(n-1,2n)) = \mathbb{Z}_2, \qquad H_1(\mathrm{OG}(n-1,2n)) = \mathbb{Z}_2,$$
$$H_2(\mathrm{IG}(n-1,2n)) = \mathbb{Z}_2, \qquad H_2(\mathrm{OG}(n-1,2n)) = \mathbb{Z}_2.$$

- $k \geqslant 2$: In this case, the incidence graph is



  The coefficient in the question mark depends on the choice of G. But this won't affect the result since it image through ∂_2 does not belongs to the kernel of ∂_1.

  Then, for $n > k \geqslant 2$, we have that

$$H_1(\mathrm{IG}(n-k,2n)) = \mathbb{Z}_2, \qquad H_1(\mathrm{OG}(n-k,2n)) = \mathbb{Z}_2,$$
$$H_2(\mathrm{IG}(n-k,2n)) = \mathbb{Z}_2, \qquad H_2(\mathrm{OG}(n-k,2n)) = \mathbb{Z}_2. \qquad \square$$

2.3 Cohomology groups of isotropic and odd orthogonal Grassmannians

Before we prove Proposition 2.6, we will provide a brief discussion about the cohomology of isotropic and odd orthogonal Grassmannians.

The problem of compute the integral cohomology of some real flag manifold was developed by Casian and Kodama in [11]. They found a way to determine the (cohomological) incidence graph for the coboundary map for the Bruhat decomposition of a flag manifold G/P_Θ. Indeed, they managed to relate the value of the coefficient $c(w, w')$, i.e., determine if an edge in the incidence graph is either $\longrightarrow$ (simple) or $\Longrightarrow$ (double) in term of the set of singularities (blow-up points) of the trajectories of the Toda lattice.

In [14], Casian and Kodama worked on the special case of a standard real Grassmannian $\mathrm{Gr}(k, n) = \mathrm{Sl}(n)/P_{(k)}$, where they found out how to get the incidence graph by the standard Young diagrams associated to each Schubert cell in the cellular decomposition of $\mathrm{Gr}(k, n)$. In fact, their process to compute the coefficients $c(w, w')$ for the real Grassmannians $\mathrm{Gr}(k, n)$ are pretty similar to what we show in case of a T-corner in the proof of Proposition 2.6.

We do not plan to show this construction in this text but we can compute the cohomology using the universal coefficient theorem for cohomology once we know are able to get the integral homology of the isotropic and odd orthogonal Grassmannians.

In general, let $(\mathcal{C}, \partial)$ be a chain complex of free abelian groups, where $\partial_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ is the boundary map, and G be any group. Denote the cochain groups by the homomorphism groups $\mathcal{C}_i^* = \mathrm{Hom}(\mathcal{C}_i, G)$ and the coboundary map $\delta_i = \partial_i^* : \mathcal{C}_{i-1}^* \to \mathcal{C}_i^*$ the dual map of ∂. The cohomology group $H^i(\mathcal{C}; G)$ is defined by the quotient $\mathrm{Ker}\ \delta / \mathrm{Im}\ \delta$. The universal coefficient theorem says that the cohomology groups $H^i(\mathcal{C}; G)$ are determined solely by G and the homology groups $H_i(\mathcal{C})$.

Theorem 2.10 (Universal coefficient theorem)**.** *If a chain complex $\mathcal{C}$ of free abelian groups has homology groups $H_i(\mathcal{C})$, then the cohomology groups $H^i(\mathcal{C}; G)$ of the cochain complex* $\mathrm{Hom}(\mathcal{C}, G)$ *are determined by split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{i-1}(\mathcal{C}), G) \longrightarrow H^i(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_i(\mathcal{C}), G) \longrightarrow 0.$$

Consider the special case of computing the integral cohomology groups of a CW complex X. In this situation, $H_i(X, \mathbb{Z})$ is finitely generated and

$$H_i(X; \mathbb{Z}) \cong \mathbb{Z}^{\beta_i(X)} \oplus T_i$$

where $\beta_i(X)$ are the Betti numbers of X and T_i is the torsion part of $H_i(X, \mathbb{Z})$. Using some properties of Hom and Ext, we have that

$$\begin{aligned}\mathrm{Hom}(H_i(X,\mathbb{Z}),\mathbb{Z}) &\cong \mathrm{Hom}(\mathbb{Z}^{\beta_i(X)},\mathbb{Z})\oplus \mathrm{Hom}(T_i,\mathbb{Z}) \cong \mathbb{Z}^{\beta_i(X)},\\ \mathrm{Ext}(H_i(X,\mathbb{Z}),\mathbb{Z}) &\cong \mathrm{Ext}(\mathbb{Z}^{\beta_i(X)},\mathbb{Z})\oplus \mathrm{Ext}(T_i,\mathbb{Z}) \cong T_i\end{aligned}$$

Hence, the universal coefficient theorem says that the integral cohomology groups $H^i(X,\mathbb{Z})$ of X are

$$H^i(X,\mathbb{Z}) \cong \mathbb{Z}^{\beta_i(X)} \oplus T_{i-1} = Free(H_k(X,\mathbb{Z})) \oplus Torsion(H_{k-1}(X,\mathbb{Z})).$$

This theorem allows us to easily compute the integral cohomology groups of isotropic and odd orthogonal Grassmannians since we can determine their homology groups using Theorem 2.7.

For instance, the integral cohomology of the isotropic Grassmannian $\mathrm{IG}(2,8)$ is

$$\begin{aligned}
H^{11}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2; & H^{5}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}\oplus\mathbb{Z}_2;\\
H^{10}(\mathrm{IG}(2,8),\mathbb{Z}) &= 0; & H^{4}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}\oplus\mathbb{Z}_2;\\
H^{9}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}\oplus\mathbb{Z}_2; & H^{3}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2;\\
H^{8}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2; & H^{2}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2;\\
H^{7}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2\oplus\mathbb{Z}_2; & H^{1}(\mathrm{IG}(2,8),\mathbb{Z}) &= 0;\\
H^{6}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}_2; & H^{0}(\mathrm{IG}(2,8),\mathbb{Z}) &= \mathbb{Z}.
\end{aligned}$$

Notice that this process only allow us to compute the integral cohomology groups of such Grassmannians and the problem of compute the cohomology ring remains unsolved. In fact, even for the standard real Grassmannians $\mathrm{Gr}(k,n)$, this problem was not solved yet, although Casian and Kodama in [14, Conjecture 6.1] conjectured a solution.

2.4 Proof of Proposition 2.6

To get the boundary operator, we must compute $\sigma(w)-\sigma(w')$, for $w' < w$ and $\ell(w') = \ell(w)-1$, where $\sigma(w) = \sum_{\alpha\in\Pi_w}\alpha$ is the sum of the elements in the β-sequence of w. The analysis for each possible kind of corners for C must be computed separately. Furthermore, the top and the bottom part, for each type of corners, will be worked out one by one. Denote by β' the β-sequences associated with w'. From now on, it will be useful to write $\sigma(w)-\sigma(w') = S^T(w,w') + S^B(w,w')$, where

$$S^T(w,w') = \sum_{(i,j)\in D_\alpha}\beta^T_{i,j} - \sum_{(i,j)\in D_{\alpha'}}(\beta')^T_{i,j}, \tag{2.4.1}$$

$$S^B(w,w') = \sum_{(i,j)\in SD_\lambda}\beta^B_{i,j} - \sum_{(i,j)\in SD_{\lambda'}}(\beta')^B_{i,j}. \tag{2.4.2}$$

The proof will be concentrated in the computation of the top part $S^T(w, w')$ whereas the bottom part $S^B(w, w')$ will be described in the next proposition using the result of [35].

Proposition 2.11. *Consider the hypothesis of Proposition 2.6. If C is a corner then*

$$S^B(w, w) = \sigma(w_\lambda) - \sigma(w_{\lambda'}) = \kappa(w_\lambda, w_{\lambda'}) \cdot \gamma \tag{2.4.3}$$

where γ is the given as in Equation (2.1.3) and $\kappa(w_\lambda, w_{\lambda'})$ is described below:

1. *If $\mathbb{F}_{(k)}$ is an isotropic Grassmannian (i.e., of type C), then*

$$\kappa(w_\lambda, w_{\lambda'}) = \begin{cases} 0 & \text{, if } C \text{ is a T-corner;} \\ t & \text{, if } C \text{ is a D-corner } (\lambda_t = 1); \\ \lambda_t + 2t - 1 & \text{, if } C \text{ is an H-corner or a V-corner } (\lambda_t > 1).. \end{cases}$$

2. *If $\mathbb{F}_{(k)}$ is a maximal odd orthogonal Grassmannian (i.e., of type B), then*

$$\kappa(w_\lambda, w_{\lambda'}) = \begin{cases} 0 & \text{, if } C \text{ is a T-corner;} \\ \lambda_t + 2(t-1) & \text{, if } C \text{ is any other kind of corner.} \end{cases}$$

Proof. It follows directly from definition of $\beta^B_{i,j}$ that $S^B(w, w') = \sigma(w_\lambda) - \sigma(w_{\lambda'}) = \kappa(w_\lambda, w_{\lambda'}) \cdot \delta$, for some root δ. We know that $w = w_\lambda w_\alpha$ and $w = s_\gamma w'$. Since $w' = w_{\lambda'} w_\alpha$ it follows that $w_\lambda = s_\gamma w_{\lambda'}$ and, hence, $\delta = \gamma$.

If C is a T-corner then $w_\lambda = w_{\lambda'}$ and the bottom β-sequences β^B of w and w' agree, implying that $S^B(w, w')$ is zero.

If C is a D-corner, H-corner or V-corner then, by Proposition 1.7, the β-sequences and β'-sequences of the bottom part depends only on w_λ instead of w. This corresponds to the case $k = 0$ as observed in Remark 1.6. Hence, $\kappa(w_\lambda, w_{\lambda'})$ is given by Proposition 6.4 of [35]. □

Proposition 2.12. *Consider the hypothesis of Proposition 2.6. If C is an M-box in the $(t + x - 1)$-th column then*

$$S^B(w, w) = \sigma(w_\lambda) - \sigma(w_{\lambda'}) = \kappa(w_\lambda, w_{\lambda'}) \cdot \gamma + \sum_{j=1}^{\lambda_t - x} \left(\varepsilon_{\lambda_t} - \varepsilon_{w(k+\alpha_{p_C}+j)}\right) \tag{2.4.4}$$

where γ is the given as in Equation (2.1.3) and $\kappa(w_\lambda, w_{\lambda'})$ is described below:

1. *If $\mathbb{F}_{(k)}$ is an isotropic Grassmannian (i.e., of type C), then*

$$\kappa(w_\lambda, w_{\lambda'}) = x + 2t - 1$$

2. If $\mathbb{F}_{(k)}$ is a maximal odd orthogonal Grassmannian (i.e., of type B), then

$$\kappa(w_\lambda, w_{\lambda'}) = x + 2(t-1)$$

Proof. The formula of $\kappa(w_\lambda, w_{\lambda'})$ is proved as it was for H-related corners. It only differs by the fact that we also remove the last $\lambda_t - x$ boxes from the t-th row of λ to get λ', then the sum $\sum_{i=1}^{\lambda_t - x} \beta^B_{t,x+j}$ cannot be canceled. Then, we have

$$S^B(w,w) = \sigma(w_\lambda) - \sigma(w_{\lambda'}) = \kappa(w_\lambda, w_{\lambda'}) \cdot \gamma + \sum_{j=1}^{\lambda_t - x} \beta^B_{t,x+j}. \tag{2.4.5}$$

Notice that $\beta^B_{t,x+j} = \varepsilon_{\lambda_t} - \varepsilon_{w_\lambda(t-1+x+j)}$. Hence, we just need to show that $w_\lambda(t-1+x+j) = w(k+\alpha_{p_C}+j)$. By the definition of p_C for an M-box, $w_\lambda(t-1+x+j) = w_\lambda(\alpha_{p_C} + k - p_C + 1 + j)$.

Equation (1.3.2) says that the index $1 \leqslant p_C \leqslant k$ satisfies $u_{p_C} = x - 1$ and the permutation is

$$w = (u_k, \ldots, x-1, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, v_1).$$

Denote $y_j = \alpha_{p_C} + k + j$ for each $1 \leqslant j \leqslant \lambda_t - x$. We know that the all the $(\lambda_t - x)$ boxes we remove from λ_t are in V-related columns and w is written as following

$$w = (u_k, \ldots, x-1, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, w(y_1), \ldots, w(y_{\lambda_t - x}), \ldots, v_1).$$

Recall that w_λ is obtained from w by reordering its entries in an increasing order. Then, we need to count how many entries to the left of y_j are bigger than $w(y_j)$, because after reordering it they will appear to the right of $w(y_j)$. In other words, we need to find a value z_j such that $w(y_j) = w_\lambda(y_j - z_j)$.

Now, the entries of w can be comparable as following:

$$\begin{gathered} u_k < \cdots < u_{p_C+1} < x-1 < w(y_j), \\ w(y_j) < u_{p_C-1} < \cdots < u_1 \\ \overline{\lambda_1} < \ldots < \overline{\lambda_{\ell(\lambda)}} < w(y_j), \\ v_{n-k-\ell(\lambda)} < \cdots < w(y_1) < w(y_2) < \cdots < w(y_{\lambda_y - x}), \end{gathered}$$

for all $1 \leqslant j \leqslant \lambda_t - x$. Hence, we see that $z_j = p_C - 1$ for all j, since only $u_{p_C-1}, \cdots, u_1$ are greater than $w(y_j)$. This shows that $w_\lambda(t-1+x+j) = w(k+\alpha_{p_C}+j)$. □

Remember that the permutation of w associated with $\Lambda = \alpha|\lambda$ can be written as

$$w_{u,\lambda} = (u_k, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_r}, v_{n-k-r}, \ldots, v_1)$$

and these values are obtained by counting the vacant length of H-related and V-related bottom columns.

C **is a T-corner:** Consider C in the t-th top row of D_α, which is H-related to the $(\alpha_t + k - t + 1)$-th bottom column in D_λ.

Since C is a T-corner, $\alpha_{t+1} < \alpha_t$ and the $(\alpha_t + k - t)$-th bottom column must be V-related, by Lemma 1.2.

Now, we need to find out the vacant length of the $(\alpha_t + k - t + 1)$-th and $(\alpha_t + k - t)$-th bottom columns. There are $(t - 1)$ H-related columns to the right of the $(\alpha_t + k - t + 1)$-th which have vacant length $u_1 > \cdots > u_{t-1}$. Then the vacant length of the $(\alpha_t + k - t + 1)$-th column is u_t. On the other hand, by Lemma 1.2, there are $\alpha_0 - \alpha_t = (\alpha_0 - \alpha_1) + \cdots + (\alpha_{t-1} - \alpha_t) = n - k - \alpha_t$ V-related columns to the right of the $(\alpha_t + k - t)$-th column which have vacant length $v_1 > \cdots > v_{n-k-\alpha_t}$. Then the vacant length of the $(\alpha_t + k - t)$-th column is $v_{n-k-\alpha_t+1}$. Define $q := n - k - \alpha_t + 1$.

After remove the corner C, the H-related $(\alpha_t + k - t + 1)$-th and V-related $(\alpha_t + k - t)$-th columns of w switch their role in the permutation and they become, respectively, V-related and H-related columns of w'. Then, the permutation of w' is obtained from w by swapping u_t and v_q:

$$w' = (u_k, \ldots, v_q, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_r}, v_{n-k-r}, \ldots, u_t, \ldots, v_1).$$

That is,

$$\begin{aligned} w(k - t + 1) &= u_t, & w'(k - t + 1) &= v_q, \\ w(k + \alpha_t) &= v_q, & w'(k + \alpha_t) &= u_t. \end{aligned}$$

Using Proposition 1.7, the top β-sequences β^T of w and w' are

$$\beta^T_{t,\alpha_t} = \varepsilon_{w(k-t+1)} - \varepsilon_{w(k+\alpha_t)} = \varepsilon_{u_t} - \varepsilon_{v_q};$$
$$\beta^T_{i,\alpha_t} = \varepsilon_{w(k-i+1)} - \varepsilon_{v_q} \quad \text{and} \quad (\beta')^T_{i,\alpha_t} = \varepsilon_{w(k-i+1)} - \varepsilon_{u_t}, \quad \text{for } 1 \leqslant i < t;$$
$$\beta^T_{t,j} = \varepsilon_{u_t} - \varepsilon_{w(k+j)} \quad \text{and} \quad (\beta')^T_{t,j} = \varepsilon_{v_q} - \varepsilon_{w(k+j)}, \quad \text{for } 1 \leqslant j < \alpha_t;$$
$$(\beta')^T_{i,j} = \beta^T_{i,j}, \quad \text{for } (i,j) \in D_{\alpha'}, i \neq t \text{ and } j \neq \alpha_t.$$

Hence, reordering the terms of $S^T(w, w')$, we get

$$\begin{aligned} S^T(w, w') &= \beta^T_{t,\alpha_t} + \sum_{(i,j)\in D_{\alpha'}} \left(\beta^T_{i,j} - (\beta')^T_{i,j}\right) \\ &= \beta^T_{t,\alpha_t} + \sum_{1\leqslant i<t} \left(\beta^T_{i,\alpha_t} - (\beta')^T_{i,\alpha_t}\right) + \sum_{1\leqslant j<\alpha_t} \left(\beta^T_{t,j} - (\beta')^T_{t,j}\right) \\ &= (\varepsilon_{u_t} - \varepsilon_{v_q}) + (t-1)(\varepsilon_{u_t} - \varepsilon_{v_q}) + (\alpha_t - 1)(\varepsilon_{u_t} - \varepsilon_{v_q}) \\ &= (t + \alpha_t - 1)(\varepsilon_{u_t} - \varepsilon_{v_q}). \end{aligned}$$

By Proposition 2.11, the sum $S^B(w, w')$ is zero and, therefore,

$$\sigma(w) - \sigma(w') = (t + \alpha_t - 1)(\varepsilon_{u_t} - \varepsilon_{v_{n-k-\alpha_t+1}}).$$

Notice that the positive root $\varepsilon_{u_t} - \varepsilon_{v_{n-k-\alpha_t+1}}$ is not necessarily a simple root.

We may visualize the above computations directly in the diagram D_Λ. In this case where C is a T-corner, besides the box C, we fill in the remaining boxes in the t-th row and in the α_t-th column with 1's. It follows that κ is the sum of such 1's in D_Λ as shown in Figure 11.

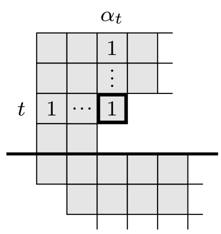


Figure 11 – For a T-corner, fill D_Λ as shown. The sum of these numbers is κ.

Remark 2.13. We should notice that the Schubert varieties of usual Grassmannian (of type A) are parametrized by the standard Young diagrams which correspond to a particular case of the half-shifted Young diagrams when the bottom part $\lambda = \varnothing$. Hence, we only have T-corners and the homology coefficients for the usual Grassmannians is computed exactly as done above.

***C* is a D-corner:** Consider C in the t-th bottom row of SD_λ. Lemma 1.5 says that $t = \ell(\lambda)$ and $\lambda_t = 1$ and the permutation of w is

$$w = (u_k, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_{t-1}}, \overline{1}, v_{n-k-t}, \ldots, v_1).$$

The double partition of w' is $\Lambda' = \alpha|\lambda'$, where $\lambda' = (\lambda_1, \ldots, \lambda_{t-1})$. Observe that C belongs to a V-related bottom column of vacant length zero in D_Λ, but when C is removed it adds 1 to the vacant length of such column in $D_{\Lambda'}$. Hence, the permutation w' is gotten from w by deleting the sign of $\overline{1}$, i.e.,

$$w' = (u_k, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_{t-1}}, 1, v_{n-k-t}, \ldots, v_1).$$

That is, $w(k+t) = \overline{1}$ and $w'(k+t) = 1$. By Proposition 1.7, the top β-sequences β^T of w and w' are

$$\beta^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_1 \quad \text{and} \quad (\beta')^T_{i,t} = \varepsilon_{w(k-i+1)} - \varepsilon_1, \quad \text{for } 1 \leqslant i \leqslant k;$$
$$(\beta')^T_{i,j} = \beta^T_{i,j}, \quad \text{for } (i,j) \in D_\alpha, j \neq t.$$

Consequently we have that

$$S^T(w,w') = \sum_{(i,j)\in D_\alpha} \left(\beta^T_{i,j} - (\beta')^T_{i,j}\right) = \sum_{1\leqslant i\leqslant k} \left(\beta^T_{i,t} - (\beta')^T_{i,t}\right) = 2k\varepsilon_1.$$

For type C, $a_1 = 2\varepsilon$ which gives that $S^T(w,w') = ka_1$. By Proposition 2.11, $S^B(w,w') = ta_1$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (k+t)a_1.$$

For type B, $a_1 = \varepsilon$ which gives that $S^T(w,w') = (2k)a_1$. By Proposition 2.11, $S^B(w,w') = (2t-1)a_1$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (2k+2t-1)a_1.$$

We can also visualize the above computations only using the diagram D_Λ. In this case where C is a D-corner, we simply fill in the t-column of D_Λ as shown in Figure 12. It follows that κ is the sum of such numbers in D_Λ.

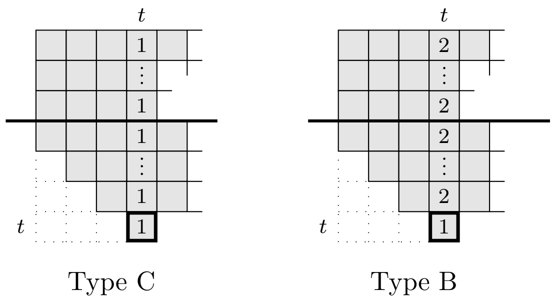


Figure 12 – For a D-corner, fill D_Λ as shown and according to the type of the group. The sum of these numbers is κ.

C is an H-corner: Consider C in the t-th bottom row of SD_λ. Lemma 1.5 says that there is $1 \leqslant p_C \leqslant k$ such that $u_{p_C} = \lambda_t - 1$ and the permutation is

$$w = (u_k, \ldots, \lambda_t - 1, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, v_1).$$

The double partition of w' is $\Lambda' = \alpha|\lambda'$, where $\lambda' = (\lambda_1, \ldots, \lambda_t - 1, \ldots, \lambda_{\ell(\lambda)})$. When C is removed it adds 1 to the vacant length of such H-related column in $D_{\Lambda'}$, which is $p_C + 1 = \lambda_t$. Hence, the permutation w' is

$$w' = (u_k, \ldots, \lambda_t, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t - 1}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, v_1).$$

That is,

$$w(k - p_C + 1) = \lambda_t - 1, \qquad w'(k - p_C + 1) = \lambda_t,$$
$$w(k+t) = \overline{\lambda_t}, \qquad w'(k+t) = \overline{\lambda_t - 1}.$$

By Proposition 1.7, the top β-sequences β^T of w and w' are

$$\beta^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t} \quad \text{and} \quad (\beta')^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t - 1}, \quad \text{for } 1 \leqslant i \leqslant k;$$
$$\beta^T_{p_C,j} = \varepsilon_{\lambda_t - 1} - \varepsilon_{w(k+j)} \quad \text{and} \quad (\beta')^T_{p_C,j} = \varepsilon_{\lambda_t} - \varepsilon_{w(k+j)}, \quad \text{for } 1 \leqslant j \leqslant \alpha_{p_C};$$
$$(\beta')^T_{i,j} = \beta^T_{i,j}, \quad \text{for } (i,j) \in D_\alpha, i \neq \alpha_{p_C} \text{ and } j \neq t.$$

Then,

$$\begin{aligned} S^T(w,w') &= \sum_{(i,j)\in D_\alpha} \left(\beta^T_{i,j} - (\beta')^T_{i,j}\right) \\ &= \beta^T_{p_C,t} - (\beta')^T_{p_C,t} + \sum_{\substack{1\leqslant i\leqslant k \\ i\neq p_C}} \left(\beta^T_{i,t} - (\beta')^T_{i,t}\right) + \sum_{\substack{1\leqslant j\leqslant \alpha_{p_C} \\ j\neq t}} \left(\beta^T_{p_C,j} - (\beta')^T_{p_C,j}\right) \\ &= 0 + (k-1)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t-1}) + (\alpha_{p_C} - 1)(-\varepsilon_{\lambda_t} + \varepsilon_{\lambda_t - 1}) \\ &= (k - \alpha_{p_C})(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t - 1}). \end{aligned}$$

The definition of α in Equation (1.2.1) implies that $\alpha_{p_C} = \lambda_t + p_C - k - 2 + d_{p_C}$, where $d_{p_C} = \#\{j \mid \lambda_j > u_{p_C}\} = t$. Hence, $S^T(w,w') = (2k - \lambda_t - p_C - t + 2)a_{\lambda_t}$.

For type C, by Proposition 2.11, $S^B(w,w') = (\lambda_t + 2t - 1)a_{\lambda_t}$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (2k - p_C + t + 1)a_{\lambda_t}.$$

For type B, by Proposition 2.11, $S^B(w,w') = (\lambda_t + 2t - 2)a_{\lambda_t}$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (2k - p_C + t)a_{\lambda_t}.$$

We also can compute κ only using the diagram D_Λ. In this case where C is an H-corner, the p_C-th top row is filled in with -1's while the t-th column is filled in with 1's except the box (p_C, t) which is filled in with zero. Besides, the t-th bottom row is filled in with 1's (except for type C where the diagonal box is filled in with 2) and both the t-th and the H-related bottom columns are filled in with 1's. So κ is the sum of such numbers as shown in Figure 13.

C is a V-corner: Consider C in the t-th bottom row of SD_λ. Lemma 1.5 says that there is some $1 \leqslant q_C \leqslant n - k - \ell(\lambda)$ such that $v_{q_C} = \lambda_t - 1$ and the permutation is

$$w = (u_k, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, \lambda_t - 1, \ldots, v_1).$$

The double partition of w' is $\Lambda' = \alpha|\lambda'$, where $\lambda' = (\lambda_1, \ldots, \lambda_t - 1, \ldots, \lambda_{\ell(\lambda)})$. When C is removed it adds 1 to the vacant length of such V-related column in $D_{\Lambda'}$, which is $q_C + 1 = \lambda_t$. Hence, the permutation w' is

$$w' = (u_k, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t - 1}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, \lambda_t, \ldots, v_1).$$

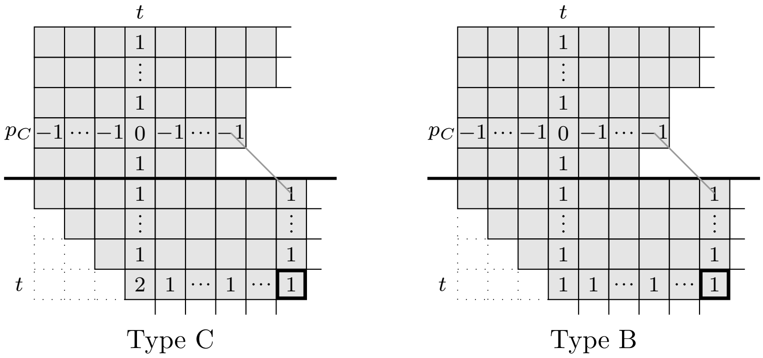


Figure 13 – For an H-corner, fill D_Λ as shown and according to the type of the group. The sum of these numbers is κ.

Observe that the entries $\lambda_t - 1$ and λ_t of w and w', resp., are at position $n - q_C + 1$. Defining the value $m := n - k - q_C + 1$, we have that $n - q_C + 1 = k + m$ and $1 \leqslant t \leqslant \ell(\lambda) < m \leqslant n - k$. That means,

$$w(k+t) = \overline{\lambda_t}, \qquad w'(k+t) = \overline{\lambda_t - 1},$$
$$w(k+m) = \lambda_t - 1, \qquad w'(k+m) = \lambda_t.$$

Remember that the *conjugate partition* α^* of α is a partition in $\mathcal{R}(n-k,k)$ defined by $\alpha^*_i = \#\{j \mid \alpha_j \geqslant i\}$, for all $1 \leqslant i \leqslant n-k$. By Proposition 1.7, the top β-sequences β^T of w and w' are

$$\beta^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t} \quad \text{and} \quad (\beta')^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t - 1}, \quad \text{for } 1 \leqslant i \leqslant k;$$
$$\beta^T_{i,m} = \varepsilon_{w(k-i+1)} - \varepsilon_{\lambda_t - 1} \quad \text{and} \quad (\beta')^T_{i,m} = \varepsilon_{w(k-i+1)} - \varepsilon_{\lambda_t}, \quad \text{for } 1 \leqslant i \leqslant \alpha^*_m;$$
$$(\beta')^T_{i,j} = \beta^T_{i,j}, \quad \text{for } (i,j) \in D_\alpha, j \neq t \text{ and } j \neq m.$$

Then,

$$\begin{aligned} S^T(w,w') &= \sum_{(i,j)\in D_\alpha} \left(\beta^T_{i,j} - (\beta')^T_{i,j}\right) \\ &= \sum_{1\leqslant i\leqslant k} \left(\beta^T_{i,t} - (\beta')^T_{i,t}\right) + \sum_{1\leqslant i\leqslant \alpha^*_m} \left(\beta^T_{i,m} - (\beta')^T_{i,m}\right) \\ &= k(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t-1}) + \alpha^*_m(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t-1}) \\ &= (k + \alpha^*_m)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t-1}). \end{aligned}$$

Since $m > \ell(\lambda)$, it follows from Proposition 1.4 that $\alpha^*_m = -v_{n-k-m+1} + m + k - \widetilde{d}_m$, where $\widetilde{d}_m = \#\{l \mid \lambda_l > v_{n-k-m+1}\} = \#\{l \mid \lambda_l > v_{q_C}\} = t$. Thus, $\alpha^*_m = -v_{q_C} + n - q_C + 1 - t = -\lambda_t + n - q_C - t + 2$ and

$$S^T(w,w') = (k - \lambda_t + n - q_C - t + 2)a_{\lambda_t}.$$

For type C, by Proposition 2.11, $S^B(w,w) = (\lambda_t + 2t - 1)a_{\lambda_t}$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (k + n - q_C + t + 1)a_{\lambda_t}.$$

For type B, by Proposition 2.11, $S^B(w,w') = (\lambda_t + 2t - 2)a_{\lambda_t}$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (k + n - q_C + t)a_{\lambda_t}.$$

We can also illustrate the computation of κ in the diagram D_Λ. In this case where C is a V-corner, we fill in with 1's both the t-th and $(n-k-q_C+1)$-th top columns. Besides, the t-th bottom row is filled in with 1's (except for type C where the diagonal box is filled in with 2) and both the t-th and the V-related bottom columns are filled in with 1's. Hence, κ is the sum of such numbers as shown in Figure 14.

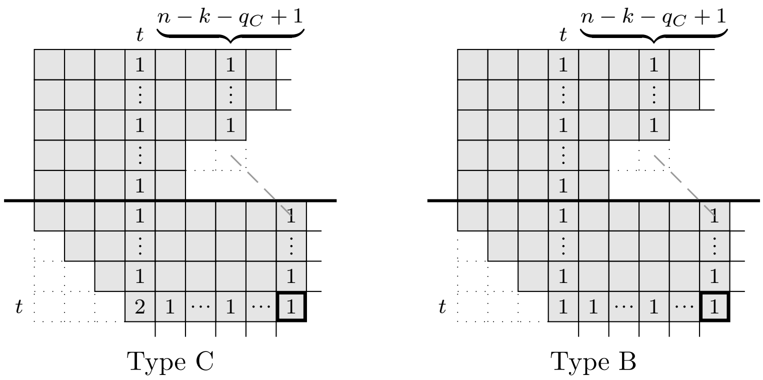


Figure 14 – For a V-corner, fill D_Λ as shown and according to the type of the group. The sum of these numbers is κ.

***C* is an M-box:** Consider C in the t-th bottom row and in the $(t+x-1)$-th bottom column of SD_λ. Equation (1.3.2) says that the index $1 \leqslant p_C \leqslant k$ satisfies $u_{p_C} = x - 1$ and the permutation is

$$w = (u_k, \ldots, x-1, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{\lambda_t}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, v_1).$$

The double partition of w' is $\Lambda' = \alpha'|\lambda'$, where $\alpha' = (\alpha_1, \ldots, \alpha_{p_C} + \lambda_t - x, \ldots, \alpha_k)$ and $\lambda' = (\lambda_1, \ldots, x-1, \ldots, \lambda_{\ell(\lambda)})$. Hence, the permutation w' is

$$w' = (u_k, \ldots, \lambda_t, \ldots, u_1, \overline{\lambda_1}, \ldots, \overline{x-1}, \ldots, \overline{\lambda_{\ell(\lambda)}}, v_{n-k-\ell(\lambda)}, \ldots, v_1).$$

That is,

$$w(k - p_C + 1) = x - 1, \qquad w'(k - p_C + 1) = \lambda_t,$$
$$w(k+t) = \overline{\lambda_t}, \qquad w'(k+t) = \overline{x-1}.$$

By Proposition 1.7, the top β-sequences β^T of w and w' are

$$\beta^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t} \quad \text{and} \quad (\beta')^T_{i,t} = \varepsilon_{w(k-i+1)} + \varepsilon_{x-1}, \quad \text{for } 1 \leqslant i \leqslant k;$$
$$\beta^T_{p_C,j} = \varepsilon_{x-1} - \varepsilon_{w(k+j)} \quad \text{and} \quad (\beta')^T_{p_C,j} = \varepsilon_{\lambda_t} - \varepsilon_{w(k+j)}, \quad \text{for } 1 \leqslant j \leqslant \alpha_{p_C};$$
$$(\beta')^T_{i,j} = \beta^T_{i,j}, \quad \text{for } (i,j) \in D_\alpha, i \neq \alpha_{p_C} \text{ and } j \neq t.$$

But, remember that we added $(\lambda_t - x)$ boxes to the p_C-th row of α', which implies that

$$(\beta')^T_{p_C,\alpha_{p_C}+j} = \varepsilon_{\lambda_t} - \varepsilon_{w(k+\alpha_{p_C}+j)}, \quad \text{for } 1 \leqslant j \leqslant \lambda_t - x.$$

Then,

$$\begin{aligned}
S^T(w,w') &= \sum_{(i,j)\in D_\alpha} \beta^T_{i,j} - \sum_{(i,j)\in D_{\alpha'}} (\beta')^T_{i,j} \\
&= \beta^T_{p_C,t} - (\beta')^T_{p_C,t} + \sum_{\substack{1\leqslant i\leqslant k \\ i\neq p_C}} \left(\beta^T_{i,t} - (\beta')^T_{i,t}\right) + \sum_{\substack{1\leqslant j\leqslant \alpha_{p_C}+\lambda_t-x \\ j\neq t}} \left(\beta^T_{p_C,j} - (\beta')^T_{p_C,j}\right) \\
&= 0 + (k-1)(\varepsilon_{\lambda_t} - \varepsilon_{x-1}) + (\alpha_{p_C} - 1)(-\varepsilon_{\lambda_t} + \varepsilon_{x-1}) - \sum_{j=1}^{\lambda_t - x} (\beta^T)'_{p_C,\alpha_{p_C}+j} \\
&= (k - \alpha_{p_C})(\varepsilon_{\lambda_t} - \varepsilon_{x-1}) - \sum_{j=1}^{\lambda_t-x} \left(\varepsilon_{\lambda_t} - \varepsilon_{w(k+\alpha_{p_C}+j)}\right).
\end{aligned}$$

The definition of α in Equation (1.2.1) implies that $\alpha_{p_C} = x + p_C - k - 2 + d_{p_C}$, where $d_{p_C} = \#\{j \mid \lambda_j > u_{p_C}\} = t$. Hence, $S^T(w,w') = (2k - x - p_C - t + 2)a_{\lambda_t} - \sum_{j=1}^{\lambda_t-x} \left(\varepsilon_{\lambda_t} - \varepsilon_{w(k+\alpha_{p_C}+j)}\right)$.

For type C, by Proposition 2.12, $S^B(w,w') = \sum_{j=1}^{\lambda_t-x} \left(\varepsilon_{\lambda_t} - \varepsilon_{w(k+\alpha_{p_C}+j)}\right) + (x + 2t - 1)a_{\lambda_t}$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (2k - p_C + t + 1)a_{\lambda_t}.$$

For type B, by Proposition 2.12, $S^B(w,w') = \sum_{j=1}^{\lambda_t-x} \left(\varepsilon_{\lambda_t} - \varepsilon_{w(k+\alpha_{p_C}+j)}\right) + (x + 2t - 2)a_{\lambda_t}$. Therefore,

$$\sigma(w) - \sigma(w') = S^T(w,w') + S^B(w,w') = (2k - p_C + t)a_{\lambda_t}.$$

It is exactly the same formula of deleting an H-corner and we can compute it as shown in Figure 13.

This concludes the proof of Proposition 2.6.

Remark 2.14. Notice that Theorem 2.7 can be easily used in a computational algorithm to obtain the boundary map of any isotropic or odd orthogonal Grassmannian. On the human viewpoint, applying such theorem to compute the coefficient for some bigger diagram becomes a harder task, since it requires us to compute p_C or q_C when C is, respectively, an H-corner or a V-corner. So, instead of using Theorem 2.7, we can get κ as in Figures 11, 12, 13 and 14 according to the type of the corner C, and then we apply Theorem 2.3 which says that $c(w,w') = 0$ if κ is odd and $c(w,w') = \pm 2$ is κ is even.

For instance, consider $\Lambda = 5,5,4|8,7,4,1$ in the isotropic Grassmannian IG$(5,16)$ and the four types of corners according to the Figure 6. Let us denote by

- $\Lambda_1 = 5,4,4|8,7,4,1$ the partition obtained when the T-corner is removed;
- $\Lambda_2 = 5,5,4|8,7,4$ the partition obtained when the D-corner is removed;
- $\Lambda_3 = 5,5,4|8,6,4,1$ the partition obtained when the H-corner is removed;
- $\Lambda_4 = 5,5,4|8,7,3,1$ the partition obtained when the V-corner is removed.
- $\Lambda_5 = 5,5,5|8,7,2,1$ the partition obtained when the M-box is removed and rearranged.

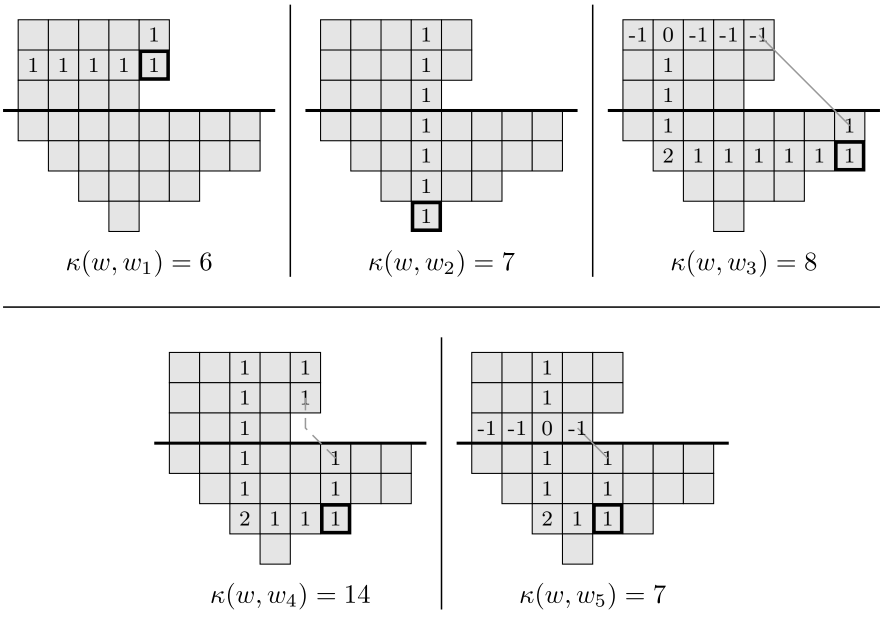


Figure 15 – Computation of κ for $\Lambda = 5,5,4|8,7,4,1$ and each possible removable box, as described in Figures 11, 12, 13 and 14.

Let w, w_1, w_2, w_3, w_4, and w_5 be the respective Weyl group elements. In order to compute $c(w,w_i)$ for $i = 1,2,3,4,5$, instead of using Theorem 2.7, we will compute κ and apply it to Theorem 2.3.

Figure 15 shows us how to fill the Young diagram for each pair (w, w_i). The value $\kappa(w, w_i)$ is the sum of values in the each diagram. Then, we have that

$$\sigma(w) - \sigma(w_1) = 6(\varepsilon_5 - \varepsilon_3); \qquad \sigma(w) - \sigma(w_2) = 7(2\varepsilon_1);$$
$$\sigma(w) - \sigma(w_3) = 8(\varepsilon_7 - \varepsilon_6); \qquad \sigma(w) - \sigma(w_4) = 14(\varepsilon_4 - \varepsilon_3);$$
$$\sigma(w) - \sigma(w_5) = 7(\varepsilon_4 - \varepsilon_2).$$

Therefore, the coefficients $c(w, w_i)$ are

$$c(w, w_1) = \pm 2; \qquad c(w, w_2) = 0;$$
$$c(w, w_3) = \pm 2; \qquad c(w, w_4) = \pm 2;$$
$$c(w, w_5) = 0.$$

Chapter 3

Permutations and diagrams

This chapter starts the second part of this dissertation. Here, we develop all the basic combinatorial concepts required to state and prove the main result about theta-vexillary signed permutations.

Understanding the idea behind the construction of a vexillary signed permutation will facilitate the comprehension of the theta-vexillary signed permutation that we will present in the next chapter. In short, a vexillary signed permutation is a special case of the vexillary permutations. This chapter compiles results and notations from Anderson and Fulton [1, 2, 3, 4], and Fulton [20].

All the geometric construction that associates the vexillary and vexillary signed permutations to the Schubert varieties is described in Appendix A.

3.1 Permutations in S_n

First of all, we are going to study the usual permutations S_n. We will see that such permutations are required to understand the vexillary permutations. The notation presented in this section is slightly different from [20], in order to match the notation we will use afterwards.

Permutations are chosen to be finite, that means $w(m) = m$ whenever $|m|$ is sufficiently large. Usually, the permutations will be written in S_n using the one-line notation $w(1)\ w(2) \cdots w(n)$.

Given a permutation w, it has a *descent* at position i if $w(i) > w(i+1)$ for some integer i.

A permutation belongs to S_n if $w(m) = m$ for all $m > n$; remember that the permutation group S_n is the Weyl group of type A_n. The group S_n is generated by the *simple transpositions* $s_1, \ldots, s_n$, where for $i > 0$, right-multiplication by s_i exchanges entries in positions i. Every permutation w can be written as $w = s_{i_1} \cdots s_{i_\ell}$ such that ℓ

is minimal. This number $\ell = \ell(w)$ is called the *length* of w and it can be determined by counting the number of inversions of w, i.e.,

$$\ell(w) = \#\{1 \leqslant i < j \leqslant n \mid w(i) > w(j)\}.$$

3.1.1 Diagram of a permutation in S_n

Consider an $n \times n$ array of boxes with rows and columns indexed by integers $[1, n] := \{1, \ldots, n\}$ in matrix style. The *permutation matrix* associated to a permutation $w \in S_n$ is obtained by placing dots in positions $(w(i), i)$, for all $1 \leqslant i \leqslant n$, in the array. The *diagram* of w is the collection of boxes that remain after removing those which are south and east of a dot in the permutation matrix. In other words, the diagram $D(w)$ is defined by

$$D(w) = \{(i, j) \in [1, n] \times [1, n] \mid w(j) > i \text{ and } w^{-1}(i) > j\}.$$

For example, take $w = 4\ 8\ 6\ 2\ 7\ 3\ 1\ 5$. Figure 16 presents the permutation matrix of w. The dots indicate the points $(w(i), i)$, the shaded boxes are south or east of a dot, and the white boxes make up the diagram $D(w)$.

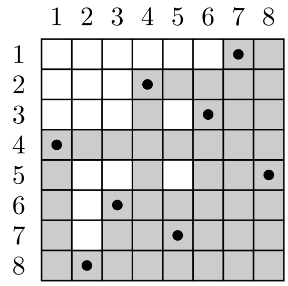


Figure 16 – Diagram for $w = 4\ 8\ 6\ 2\ 7\ 3\ 1\ 5$.

The number of boxes in the diagram is equal to the length of the permutation. In fact, if $l = w^{-1}(i)$ then

$$\begin{aligned} \#D(w) &= \#\{(l, j) \in [1, n] \times [1, n] \mid w(j) > w(l) \text{ and } l > j\} \\ &= \#\{j < l \mid w(j) > w(l)\} = \ell(w) \end{aligned}$$

For any permutation w in S_n and any $1 \leqslant p, q \leqslant n$, the *type A rank function* of a permutation w for a pair (q, p) counts the number of dots strictly south and weakly west of the box (q, p) in the permutation matrix of w, i.e., this function is defined by

$$\begin{aligned} r_w^A(q, p) &= \#\{i \leqslant p \mid w(i) > q\} \\ &= \#(\{w(1), w(2), \ldots, w(p)\} \cap \{q + 1, q + 2, \ldots, n\}). \end{aligned} \tag{3.1.1}$$

A permutation is clearly determined by its rank functions. However, in many cases, we want to figure out a smaller set of rank functions that also determine such permutation.

Remark 3.1. We use a different definition of r_w^A in comparison with [20]. Instead of defining the rank function as the number of dots to the left and above the box (q, p) in the diagram, we define it as the number of dots to the left and below (q, p). In other words, $\tilde{r}_w(q, p) = p - r_w^A(q, p)$ is the rank function as defined by Fulton. We changed this definition in order to give a better intuition for the type B case.

We say that a box (a, b) is a southeast (SE) corner of the diagram of w if w has a descent at b, with a lying in the interval of the jump, and w^{-1} has a descent at a, with b lying in the interval of the jump. This can be written as

$$\begin{aligned} w(b) > a \geqslant w(b+1) \quad &\text{and} \\ w^{-1}(a) > b \geqslant w^{-1}(a+1). \end{aligned} \tag{3.1.2}$$

A *essential position* of w is a pair (q, p) such that the box (q, p) is a southeast (SE) corner of the diagram $D(w)$ of w. Define the *essential set* $\mathscr{E}ss(w)$ of w to be the set of triples (k, p, q) such that (q, p) is a SE corner and $k = r_w^A(q, p)$ is the value of the rank function. Although k is given by the pair (q, p), it will be important to preserve the value of the rank function.

A *basic triple (of type A)* is a triple (k, q, p) such that $k > \max\{0, 1 - p - q\}$. Observe that the elements of $\mathscr{E}ss(w)$ are basic triples since (k, p, q) in $\mathscr{E}ss(w)$ implies that $k > 0$.

Using the example above of $w = 4\ 8\ 6\ 2\ 7\ 3\ 1\ 5$, Figure 17 outlines the boxes labelling essential positions with the corresponding values of the rank inside. The essential set in this example is

$$\mathscr{E}ss(w) = \{(1,2,7), (2,3,5), (3,3,3), (3,5,5), (4,5,3), (6,6,1)\}.$$

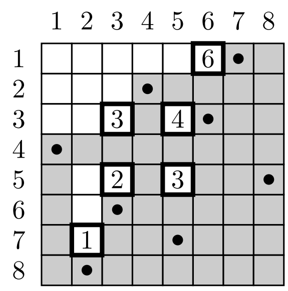


Figure 17 – SE corners and rank values for $w = 4\ 8\ 6\ 2\ 7\ 3\ 1\ 5$.

3.2 Vexillary permutations on S_n

We say that a permutation $w \in S_n$ is *vexillary* if there is no subpermutation isomorphic to the permutation [2 1 4 3]; in other words, there do not exist four numbers

$a < b < c < d$ such that $w(b) < w(a) < w(d) < w(c)$. A vexillary permutation also called a 2143-avoiding permutation.

Now, we want to describe vexillary permutations in a different way, and for that we define a *triple of type A*. We say that a three s-tuples $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of non-negative integer is a *triple of type A* if they satisfy

$$\mathbf{k} = (0 < k_1 < k_2 < \cdots < k_s),$$
$$\mathbf{p} = (0 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_s),$$
$$\mathbf{q} = (q_1 \geqslant q_2 \geqslant \cdots \geqslant q_s > 0),$$

and, setting $l_i = q_i - p_i + k_i$, we also require that

$$l_1 > l_2 > \cdots > l_s > 0. \tag{3.2.1}$$

Notice that Equation (3.2.1) is equivalent to the following

$$p_{i+1} - p_i + q_i - q_{i+1} > k_{i+1} - k_i \tag{3.2.2}$$

for every $1 \leqslant i \leqslant s$, and we have the extreme value $(k_{s+1}, p_{s+1}, q_{s+1}) = (n, n, 0)$ for some integer $n > \max\{k_s, p_s, q_1\}$.

Related to a triple of type A there is a partition $\lambda = \lambda(\boldsymbol{\tau})$, which is defined by taking $\lambda_{k_i} = l_i$ for each $1 \leqslant i \leqslant s$, and filling the remaining parts minimally so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{k_s} > 0$. Since $l_i > l_{i+1}$, then all corners of the Young diagram associated to the (standard) partition of λ are in the rows k_i and they carry information about the triple $\boldsymbol{\tau}$.

The next theorem, due to Fulton, show us how to associate a vexillary permutation with a triple of type A.

Theorem 3.2 ([20] Proposition 9.6)**.** *Let w be a permutation of S_n. Then, w is a vexillary permutation if and only if there is a unique triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of type A such that the essential set $\mathscr{E}ss(w)$ is given by*

$$\mathscr{E}ss(w) = \{(k_1, q_1, p_1), (k_2, q_2, p_2), \ldots, (k_s, q_s, p_s)\},$$

where

$$k_i = r_w^A(q_i, p_i) \quad , \textit{for } 1 \leqslant i \leqslant s.$$

This theorem also gives another equivalence for such permutations. A permutation w is vexillary if and only if the essential corners of w strung to the northeast direction, i.e., there are no essential positions (q, p) and (q', p') such that $q < q'$ and $p < p'$.

Given a triple $\boldsymbol{\tau}$ of type A, we can construct the vexillary permutation $w = w(\boldsymbol{\tau})$ using the following steps:

(1) Ending in the p_1 position, place k_1 consecutive entries to the left, in increasing order, starting with $q_1 + 1$. Mark these numbers as "used";

(i) For $1 < i \leqslant s$, ending in the p_i position, or the next available position to the left, fill the next available $k_i - k_{i-1}$ positions with entries chosen consecutively from the unused integers, in increasing order, stating with $q_i + 1$ or, if it is not available, the smallest unused integer above $q_i + 1$. Again, mark these numbers as "used";

($s + 1$) Fill the remaining available positions with the unused positive integers, in increasing order.

For instance, consider the triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ such that

$$\begin{aligned} \mathbf{k} &= (1, 3, 4), \\ \mathbf{p} &= (2, 4, 7), \\ \mathbf{q} &= (9, 7, 5). \end{aligned}$$

In this case, notice that $l = (8, 6, 2)$ satisfies Equation (3.2.1) and, then, $\boldsymbol{\tau}$ is triple of type A. We can obtain w using the steps above as follows: put 10 in position 2; then, put 8 and 9 in positions 3 and 4; then put 6 in position 7; finally, fill the remaining positions with unused integers from 1 to 10. In short, we can represent these four steps as follows:

$$\begin{array}{rcccccccccc} & \cdot & \mathbf{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & 10 & \mathbf{8} & \mathbf{9} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & 10 & 8 & 9 & \cdot & \cdot & \mathbf{6} & \cdot & \cdot & \cdot \\ w = & \mathbf{1} & 10 & 8 & 9 & \mathbf{2} & \mathbf{3} & 6 & \mathbf{4} & \mathbf{5} & \mathbf{7}. \end{array}$$

Since $\boldsymbol{\tau}$ is a triple of type A, then the permutation $w = 1\ 10\ 8\ 9\ 2\ 3\ 6\ 4\ 5\ 7$ is vexillary. If we draw the diagram $D(w)$ then, by Theorem 3.2, we can verify that the essential positions lie in a northeast path, as in Figure 18.

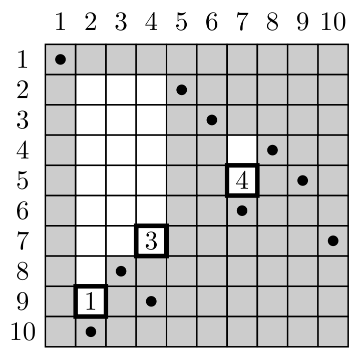


Figure 18 – Diagram for the vexillary permutation $w = 1\ 10\ 8\ 9\ 2\ 3\ 6\ 4\ 5\ 7$.

Also notice that the permutation given in Figure 17 is not a vexillary permutation since the essential positions does not satisfy the conditions in Theorem 3.2.

The geometric setup of vexillary permutations and their association to the Schubert varieties is developed in the first section of Appendix A.

3.3 Signed permutations in $\mathcal{W}_n$

The notation present here is the same used in [4]. We also refer [7, §8.1] for further details.

Consider the permutation of positive and negative integers, where the bar over the number denotes the negative sign, and consider the natural order of them

$$\ldots, \overline{n}, \ldots, \overline{2}, \overline{1}, 0, 1, \ldots, n, \ldots$$

A *signed permutation* is a permutation w satisfying that $w(\bar{\imath}) = \overline{w(i)}$, for each i. A signed permutation belongs to $\mathcal{W}_n$ if $w(m) = m$ for all $m > n$; this is a group isomorphic to the hyperoctahedral group, the Weyl group of types B_n and C_n. Since $w(\bar{\imath}) = \overline{w(i)}$, we just need the positive positions when writing signed permutation in one-line notation, i.e., a permutation $w \in \mathcal{W}_n$ is represented by $w(1)\ w(2)\ \cdots\ w(n)$. For example, the full form of the signed permutation $w = \overline{2}\ 1\ \overline{3}$ in $\mathcal{W}_3$ is $3\ \overline{1}\ 2\ 0\ \overline{2}\ 1\ \overline{3}$, but we can omit the values at the position $\overline{3}, \overline{2}, \overline{1}$ and 0. The group $\mathcal{W}_n$ is generated by the *simple transpositions* $s_0, \ldots, s_n$, where for $i > 0$, right-multiplication by s_i exchanges entries in positions i and $i+1$, and right-multiplication by s_0 replaces $w(1)$ with $\overline{w(1)}$. Every signed permutation w can be written as $w = s_{i_1} \cdots s_{i_\ell}$ such that ℓ is minimal; call the number $\ell = \ell(w)$ the *length* of w. This value counts the number of inversions of $w \in \mathcal{W}_n$, and it is given by the formula

$$\ell(w) = \#\{1 \leqslant i < j \leqslant n \mid w(i) > w(j)\} + \#\{1 \leqslant i \leqslant j \leqslant n \mid w(-i) > w(j)\}. \quad (3.3.1)$$

The element $w_\circ^{(n)} = \overline{1}\ \overline{2} \cdots \overline{n}$ is the longest element in $\mathcal{W}_n$ and it is called the involution of $\mathcal{W}_n$. Notice that the involution $w_\circ^{(n)}$ has length n^2.

The group of permutations $\mathcal{W}_n$ can be embedded in the symmetric group S_{2n+1}, considering S_{2n+1} the permutations of $\overline{n}, \ldots, 0, \ldots, n$. Indeed, define the *odd* embedding by $\iota : \mathcal{W}_n \hookrightarrow S_{2n+1}$ where it sends $w = w(1)\ w(2)\ \cdots\ w(n)$ to the permutation

$$\overline{w(n)}\ \cdots\ \overline{w(2)}\ \overline{w(1)}\ 0\ w(1)\ w(2)\ \cdots\ w(n)$$

in S_{2n+1}. The embedding ι will be used when it is necessary to highlight that we need the full permutation of w.

There is also a *even* embedding $\iota' : \mathcal{W}_n \hookrightarrow S_{2n}$ defined by omitting the value $w(0) = 0$.

Considering the natural inclusions $\mathcal{W}_n \subset \mathcal{W}_{n+1} \subset \cdots$, we get the infinite Weyl group $\mathcal{W}_\infty = \cup \mathcal{W}_n$. When the value n is understood or irrelevant, we can consider w as an element of $\mathcal{W}_\infty$. The odd embeddings are compatible with the corresponding inclusions $S_{2n+1} \subset S_{2n+3} \subset \cdots$.

3.3.1 Diagram of a permutation in S_{2n+1}

Let us consider the specific case where the permutation group is S_{2n+1}. It is important to consider this case because we need to do some modification in the notation that will be useful for us.

Consider a $(2n+1) \times (2n+1)$ arrays of boxes with rows and columns indexed by integers $[\overline{n}, n] = \{\overline{n}, \ldots, \overline{1}, 0, 1, \ldots, n\}$ in matrix style. The *permutation matrix* associated to a permutation $w \in S_{2n+1}$ is obtained by placing dots in positions $(w(i), i)$, for all $\overline{n} \leqslant i \leqslant n$, in the array. Again the *diagram* of w is the collection of boxes that remain after removing those which are (weakly) south and east of a dot in the permutation matrix. Observe that the number of boxes in the diagram is equal to the length of the permutation.

The *rank function* of a permutation $w \in S_{2n+1}$ for a pair (p, q), where $\overline{n} \leqslant p, q \leqslant n$, is the number of dots strictly south and weakly west of the box $(q-1, \overline{p})$ in the permutation matrix of w. In other words, it will be defined by

$$r_w(p, q) := \#\{i \leqslant \overline{p} \mid w(i) \geqslant q\}, \tag{3.3.2}$$

for $\overline{n} \leqslant p, q \leqslant n$. Notice that we defined a different rank function for $w \in S_{2n+1}$ compared to S_n: the rank function r_w of a pair (p, q) is, in some sense, the type A rank function r_w^A associated to the box $(q-1, \overline{p})$. This change is going to make easier to deal with signed permutations.

A *corner position* of w is a pair (p, q) such that the box $(q-1, \overline{p})$ is a southeast (SE) corner of the diagram of w. The *set of corners* of w is the set $\mathscr{C}(w)$ of triples (k, p, q) such that (p, q) is a corner position and $k = r_w(p, q)$.

Remark 3.3. Observe that the definition of $\mathscr{C}(w)$ is exactly the same definition of essential set $\mathscr{E}ss(w)$ that we gave in the previous section. The reason we are using here $\mathscr{C}(w)$ instead of $\mathscr{E}ss(w)$ is because the essential set of a signed permutation defined by Anderson and Fulton (Definition 1.2 of [4]) is properly contained in the set of corner for some signed permutations, since their definition removes some "redundant" corners positions. In the present work, we won't disconsider such corners in our constructions and, from now on, we will use the expression "corner" in place of "essential" to avoid some later misuse of the definition of essential set of a signed permutation.

For example, consider $w = \iota(\overline{2}\ 3\ 1) = \overline{1}\ \overline{3}\ 2\ 0\ \overline{2}\ 3\ 1$. Figure 19 shows the diagram of w. The SE corners $(q-1, \overline{p})$ are highlighted and they are filled with the rank function values $r_w(p,q)$. In this case, the set of corners is

$$\mathscr{C}(w) = \{(1,3,\overline{1}),(1,1,2),(3,0,\overline{1}),(2,\overline{2},2)\}.$$

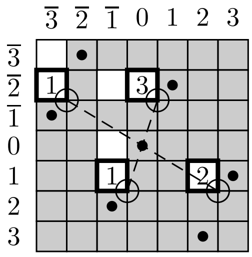


Figure 19 – Diagram for $w = \iota(\overline{2}\ 3\ 1) = \overline{1}\ \overline{3}\ 2\ 0\ \overline{2}\ 3\ 1$. The circle corners connected with dashad lines illustrate the symmetry of Lemma 3.4

Notice that if a box $(q-1,\overline{p})$ is a SE corner that satisfies (3.1.2), then (p,q) is a corner position and $k = r_w(p,q)$.

3.3.2 Extended diagram of a signed permutation in $\mathcal{W}_n$

We know that signed permutations must satisfy the relation $w(\overline{\imath}) = \overline{w(i)}$, then the negative positions can be obtained from the positive ones. Hence, a signed permutation $w \in \mathcal{W}_n$ corresponds to a $(2n+1)\times n$ array of boxes, with rows indexed by $\{\overline{n},\dots,n\}$ and the columns indexed by $\{\overline{n},\dots,\overline{1}\}$, where the dots are placed in the boxes $(w(i),i)$ for $\overline{n} \leqslant i \leqslant \overline{1}$.

For each dot, we place an "$\times$" in those boxes (a,b) such that $a = \overline{w(i)}$ and $i \leqslant b$, in other words, an $\times$ is placed in the same column and opposite along with the boxes to the right of this $\times$.

The *extended diagram* $D^+(w)$ of a signed permutation w is the collection of boxes in the $(2n+1)\times n$ rectangle that remain after removing those which are south or east of a dot. The *diagram* $D(w) \subseteq D^+(w)$ is obtained from extended diagram $D^+(w)$ by removing the ones marked with $\times$. Namely, $D(w)$ is defined by

$$D(w) = \{(i,j) \in [\overline{n},\overline{1}] \times [\overline{n},n] \mid w(i) > j, w^{-1}(j) > i, \text{ and } w^{-1}(-j) > i\}.$$

The number of boxes of $D(w)$ is equal to the length of w. Indeed, if define $J = \{j \in [\overline{n},n] \mid w^{-1}(j) < 0\}$ then define the following set

$$D_1(w) = \{(i,j) \in D(w) \mid j \in J\}.$$

Notice that $w^{-1}(-j) > i$ is trivially satisfied because $w^{-1}(-j) > 0$. If we denote $l = -w^{-1}(j)$ and $m = -i$ then

$$\begin{aligned}\#D_1(w) &= \#\{(i,j) \in [\overline{n}, \overline{1}] \times J \mid w(i) > j \text{ and } w^{-1}(j) > i\} \\ &= \#\{(l,m) \in [1,n] \times [1,n] \mid w(l) > w(m) \text{ and } l < m\} \\ &= \#\{1 \leqslant l < m \leqslant n \mid w(l) > w(m)\}.\end{aligned}$$

On the other hand, if we define the set

$$D_2(w) = \{(i,j) \in D(w) \mid j \notin J\}$$

then $w^{-1}(j) > i$ is trivially satisfied because $w^{-1}(j) \geqslant 0$. So, denoting $l = w^{-1}(j)$ and $m = -i$

$$\begin{aligned}\#D_2(w) &= \#\{(i,j) \in [\overline{n}, \overline{1}] \times ([\overline{n}, n] - J) \mid w(i) > j \text{ and } w^{-1}(-j) > i\} \\ &= \#\{(l,m) \in [0,n] \times [1,n] \mid w(-l) > w(m) \text{ and } l < m\} \\ &= \#\{(l,m) \in [1,n] \times [1,n] \mid w(-l) > w(m) \text{ and } l \leqslant m\} \\ &= \#\{1 \leqslant l \leqslant m \leqslant n \mid w(-l) > w(m)\}.\end{aligned}$$

Hence, $D(w)$ is the disjoint union $D_1(w) \cup D_2(w)$ and, by Equation (3.3.1), $\#D_w = \ell(w)$.

Observe that if we use the embedding $\iota : \mathcal{W}_n \hookrightarrow S_{2n+1}$, the matrix and extended diagram of $w \in \mathcal{W}_n$ corresponds, respectively, to the first n columns of the matrix and diagram of $\iota(w)$. The notation $\iota(D^+(w))$ will be used when we need to use the respective $(2n+1) \times (2n+1)$ diagram of $\iota(w)$.

The rank function of a permutation w in $\mathcal{W}_n$ is defined by

$$r_w(p,q) = \#\{i \geqslant p \mid w(i) \leqslant \overline{q}\}, \tag{3.3.3}$$

for $1 \leqslant p \leqslant n$, and $\overline{n} \leqslant q \leqslant n$.

Since $w(\overline{\imath}) = \overline{w(i)}$, then the rank function $r_w(p,q)$ is also equal to $\#\{i \leqslant \overline{p} \mid w(i) \geqslant q\}$, so the rank functions r_w coincides to $r_{\iota(w)}$.

We say that integers (k,p,q) form a *basic triple (of type B)* if it a basic triple of type A, i.e., it satisfies $k > \max\{0, 1-p-q\}$ along with the following conditions: $p > 0$, $q \neq 0$ and if $p = 1$ then $q > 0$.

Given $w \in \mathcal{W}_n$, the next lemma states that there is a symmetry about the origin of the corner positions corresponding to $\iota(w)$. In order to simplify the notation, given (k,p,q) a basic triple of type A, define the *reflected* basic triple $(k,p,q)^\perp$ by $(k+p+q-1, \overline{p}+1, \overline{q}+1)$.

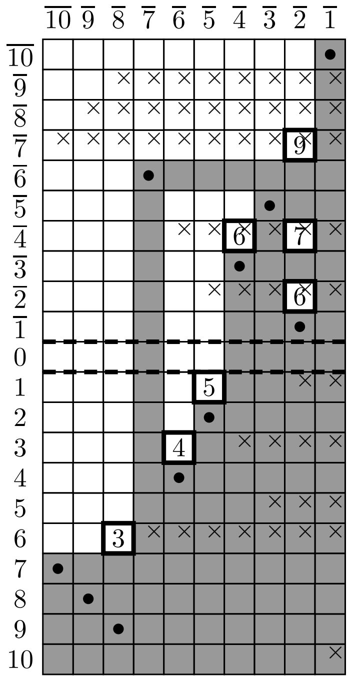
Lemma 3.4 ([4], Lemma 1.1)**.** *For $w \in \mathcal{W}_n$, the set of corners of $\iota(w) \in S_{2n+1}$ has the following symmetry: (k,p,q) is in $\mathscr{C}(\iota(w))$ if and only if $(k,p,q)^{\perp}$ is in $\mathscr{C}(\iota(w))$.*

We can see in Figure 19 that both corners in the left half of the diagram are symmetric by the origin to other two corners in the right side. This behavior will happen for every signed permutation w, implying that half of $\mathscr{C}(\iota(w))$ suffices to determine the signed permutation w; we will consider those corners appearing in the first n columns.

A *corner position* of signed permutation w is a pair (p,q) such that the box $(q-1,\overline{p})$ is a southeast (SE) corner of the extended diagram of w. The *set of corners* of a signed permutation w is the set (k,p,q) such that $(q-1,\overline{p})$ is a SE corner of the extended diagram $D^{+}(w)$ and $k = r_w(p,q)$, except for corner positions (p,q) where $p = 1$ and $q < 0$. This exception comes from the fact that $(1,q)$, for $q < 0$, is not a corner position in $\iota(w)$ because the respective box $(q-1,\overline{1})$ cannot be a SE corner since $w(0) = 0$.

Observe that every element (k,p,q) in the set of corners $\mathscr{C}(w)$ is also a basic triple of type B. Since the integer k is the rank of w in (p,q), sometimes we can simply say that the corner position $(p,q) \in \mathscr{C}(w)$, instead of the basic triple (k,p,q).

The Figure 20 illustrates the extended diagram and the set of cornet of the signed permutation $w = 10\ 1\ 5\ 3\ \overline{2}\ \overline{4}\ 6\ \overline{9}\ \overline{8}\ \overline{7}$.



$w = 10\ 1\ 5\ 3\ \overline{2}\ \overline{4}\ 6\ \overline{9}\ \overline{8}\ \overline{7}$

$$\mathscr{C}(w) = \{(3,8,7),(4,6,4),(5,5,2),(6,4,\overline{3}),(6,2,\overline{1}),(7,2,\overline{3}),(9,2,\overline{6})\}$$

$\ell(w) = 65$

Figure 20 – Diagram and set of corners of a signed permutation.

To make the diagrams look cleaner, from now on we won't denote $\times$ in the extended diagrams $D^{+}(w)$. During the text, it can happen that we omit the word "extended" since we are only interested in studying the extended diagram of a signed permutation w so that the diagram $D(w)$ won't be useful for us.

3.4 Vexillary signed permutations on $\mathcal{W}_n$

In this section, we will figure out what is a vexillary permutation in $\mathcal{W}_n$. Recall that there is an inclusion $\iota : \mathcal{W}_n \to S_{2n+1}$, so that makes sense to think about permutations $w \in \mathcal{W}_n$ such that its inclusion $\iota(w)$ is a vexillary permutation in S_{2n+1}. Indeed, the symmetry in the diagram of $\iota(w)$ allows us to get a description of a vexillary signed permutation intrinsically in $\mathcal{W}_n$.

A *triple of type B* is a three s-tuples $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of non-negative integer such that

$$\begin{aligned} \mathbf{k} &= (0 < k_1 < k_2 < \cdots < k_s), \\ \mathbf{p} &= (p_1 \geqslant p_2 \geqslant \cdots \geqslant p_s > 0), \\ \mathbf{q} &= (q_1 \geqslant q_2 \geqslant \cdots \geqslant q_s > 0), \end{aligned}$$

satisfying

$$p_i - p_{i+1} + q_i - q_{i+1} > k_{i+1} - k_i$$

for every $1 \leqslant i \leqslant s$, and we have the extreme value $(k_{s+1}, p_{s+1}, q_{s+1}) = (n, 0, -n)$ for some integer $n > \max\{k_s, p_s, q_1\}$ (when $i = s$, we have that $p_s + q_s + k_s > 0$). Notice that this definition is similar to the definition of a triple of type A.

Given a triple of type B, we can construct a permutation $w(\boldsymbol{\tau})$ as follows:

(1) Starting in the p_1 position, place k_1 consecutive entries, in increasing order, ending with $\overline{q_1}$. Mark these numbers as "used";

(i) For $1 < i \leqslant s$, starting in the p_i position, or the next available position to the right, fill the next available $k_i - k_{i-1}$ positions with entries chosen consecutively from the unused absolute numbers, in increasing order, ending with $\overline{q_i}$ or, if it is not available, the biggest unused number below $\overline{q_i}$. Again, mark these numbers as "used";

($s + 1$) Fill the remaining available positions with the unused positive numbers, in increasing order.

Let us consider the following example. Suppose that $\boldsymbol{\tau} = (1\ 3\ 4\ 5\ 8,\ 9\ 9\ 6\ 4\ 3,\ 12\ 9\ 8\ 8\ 5)$. The six steps to obtain $w(\boldsymbol{\tau})$ are:

(1): $(k_1, p_1, q_1) = (1, 9, 12)$. From position $p_1 = 9$ we are going to place $k_1 = 1$ consecutive numbers, ending with $\overline{q_1} = \overline{12}$. Then,

$$w(9) = \overline{12}.$$

We also need to keep tracking what positions and values where have already been used. **Used positions:** 9; **Used values:** $\overline{12}$.

(2): $(k_2, p_2, q_2) = (3, 9, 9)$. From position $p_2 = 9$ we are going to place $k_2 - k_1 = 2$ consecutive numbers, ending with $\overline{q_2} = \overline{9}$. Since we already set a value in position 9, then we go to the next available position. Then,

$$w(10) = \overline{10}, \qquad w(11) = \overline{9}.$$

Used positions: $9, 10, 11$; **Used values:** $\overline{12}, \overline{10}, \overline{9}$.

(3): $(k_3, p_3, q_3) = (4, 6, 8)$. From position $p_3 = 6$ we are going to place $k_3 - k_2 = 1$ consecutive numbers, ending with $\overline{q_3} = \overline{8}$. Then,

$$w(6) = \overline{8}.$$

Used positions: $6, 9, 10, 11$; **Used values:** $\overline{12}, \overline{10}, \overline{9}, \overline{8}$.

(4): $(k_4, p_4, q_4) = (5, 4, 8)$. From position $p_3 = 4$ we are going to place $k_4 - k_3 = 1$ consecutive numbers, ending with $\overline{q_4} = \overline{8}$. We have already used $\overline{8}$, so we are going to use the biggest available value smaller than $\overline{8}$. Then,

$$w(4) = \overline{11}.$$

Used positions: $4, 6, 9, 10, 11$; **Used values:** $\overline{12}, \overline{11}, \overline{10}, \overline{9}, \overline{8}$.

(5): $(k_5, p_5, q_5) = (8, 3, 5)$. From position $p_3 = 3$ we are going to place $k_5 - k_4 = 3$ consecutive numbers, ending with $\overline{q_5} = \overline{5}$. We cannot place values in positions 4 and 6, so we need to skip them. Then,

$$w(3) = \overline{7}, \qquad w(5) = \overline{6}, \qquad w(7) = \overline{5}.$$

Used positions: $3, 4, 5, 6, 7, 9, 10, 11$; **Used values:** $\overline{12}, \overline{11}, \overline{10}, \overline{9}, \overline{8}, \overline{7}, \overline{6}, \overline{5}$.

(6:) The final step fills the vacant positions with unused positive values. Then,

$$w(1) = 1, \qquad w(2) = 2, \qquad w(8) = 3, \qquad w(12) = 4.$$

In short, we can represent these six steps as follows:

$$
\begin{array}{ccccccccccccc}
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\mathbf{12}} & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{12} & \overline{\mathbf{10}} & \overline{\mathbf{9}} & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \overline{\mathbf{8}} & \cdot & \cdot & \overline{12} & \overline{10} & \overline{9} & \cdot & \cdot \\
 & \cdot & \cdot & \overline{\mathbf{11}} & \cdot & \overline{8} & \cdot & \cdot & \overline{12} & \overline{10} & \overline{9} & \cdot & \cdot \\
 & \cdot & \overline{\mathbf{7}} & \overline{11} & \overline{\mathbf{6}} & \overline{8} & \overline{\mathbf{5}} & \cdot & \overline{12} & \overline{10} & \overline{9} & \cdot & \cdot \\
w = & \mathbf{1} & \mathbf{2} & \overline{7} & \overline{11} & \overline{6} & \overline{8} & \overline{5} & \mathbf{3} & \overline{12} & \overline{10} & \overline{9} & \mathbf{4}.
\end{array}
$$

A signed permutation $w \in \mathcal{W}_n$ is called a *vexillary signed permutation* if there is some triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of type B such that $w = w(\boldsymbol{\tau})$.

We can determine the corner positions of a vexillary signed permutation as in Theorem 3.2 for vexillary permutation. Remember that a corner position is a pair (p, q) such that the box $(q-1, \overline{p})$ in the extended diagram $D^+(w)$ is a SE corner. The set $\mathscr{C}(w)$ is the set of all triples (k, p, q) such that (p, q) is a corner position and $k = r_w(p, q)$, except for corner positions (p, q) where $p = 1$ and $q < 0$.

Proposition 3.5 ([1], Lemma 2.4)**.** *Let $w = w(\boldsymbol{\tau})$ be a vexillary signed permutation, for a triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$. Then the set of corners $\mathscr{C}(w)$ is*

$$\mathscr{C}(w) = \{(k_1, q_1, p_1), (k_2, q_2, p_2), \ldots, (k_s, q_s, p_s)\},$$

In other words, all SE corners of the diagram of w are the boxes $(q_i - 1, \overline{p_i})$, and k_i is the number of dots strictly south and weakly west of the i-th SE corner in the diagram.

In order to give all possible ways to determine a vexillary signed permutation, we need to define the concept of *pattern avoidance*. Given a signed permutation $w \in \mathcal{W}_n$ and a signed pattern $\pi = [\pi(1)\ \pi(2) \cdots \pi(m)]$ in $\mathcal{W}_m$, $m < n$, we say that w *contains* π if there is a subsequence $w(i_1) \cdots w(i_m)$ such that the signs of $w(i_j)$ and $\pi(j)$ are the same for all j, and also the absolute values of the subsequence are in the same relative order as the absolute values of π. Otherwise w *avoids* π.

For example, $\overline{5}\ 1\ 3\ \overline{2}\ 4$ contains the pattern $[\overline{3}\ 2\ \overline{1}]$, as the subsequence $\overline{5}\ 3\ \overline{2}$, but $\overline{5}\ 1\ 2\ \overline{3}\ \overline{4}$ avoids $[\overline{3}\ 2\ \overline{1}]$.

The following theorem gives four characterizations of vexillary signed permutations.

Theorem 3.6 ([1])**.** *Let w be a signed permutation in $\mathcal{W}_n$. The following are equivalent:*

1. *w is vexillary, i.e., there is a triple $\boldsymbol{\tau}$ of type B such that $w = w(\boldsymbol{\tau})$;*
2. *$\iota(w)$ is vexillary, as a permutation in S_{2n+1};*

3. *The corner positions of w can be ordered $(p_1, q_1), \dots, (p_s, q_s)$, so that $p_1 \geqslant \cdots \geqslant p_s > 0$ and $q_1 \geqslant \cdots \geqslant q_s > 0$;*

4. *w avoids the nine signed patterns* $[2\ 1]$, $[\overline{3}\ 2\ \overline{1}]$, $[2\ \overline{3}\ 4\ \overline{1}]$, $[\overline{2}\ \overline{3}\ 4\ \overline{1}]$, $[3\ \overline{4}\ \overline{1}\ \overline{2}]$, $[\overline{3}\ \overline{4}\ 1\ \overline{2}]$, $[\overline{3}\ \overline{4}\ \overline{1}\ \overline{2}]$, $[\overline{4}\ 1\ \overline{2}\ 3]$ *and* $[\overline{4}\ \overline{1}\ \overline{2}\ 3]$.

The geometric setup of vexillary signed permutations and their association to the Schubert varieties is developed in the second section of Appendix A.

3.5 NE path and unessential corners

Suppose that $w \in \mathcal{W}_n$ is any signed permutation. There are two notable classes of SE corner in the set $\mathscr{C}(w)$ that we will be important to our main theorem stated in the next chapter. They are the corners in the northeast path and the unessential corners.

Given any signed permutation w, consider a *(strict) partial order* for the set of corners $\mathscr{C}(w)$ by $(p, q) < (p', q')$ if and only if $p > p'$ and $q < q'$, for corner positions $(p, q), (p', q') \in \mathscr{C}(w)$. For example, in Figure 20, the unique possible relation is $(4, \overline{3}) < (2, \overline{2})$, the two boxes filled in with the value 6.

Define the *northeast (NE) path* as the set $\mathscr{N\!e}(w)$ of minimal elements of $\mathscr{C}(w)$ relative to the poset "<". Using the same example in Figure 20, we have that $\mathscr{N\!e}(w) = \mathscr{C}(w) - \{(6, 2, \overline{1})\}$, since all the corners are minimal under this poset except the basic triple $(6, 2, \overline{1})$.

The positions (p_i, q_i) of the NE path $\mathscr{N\!e}(w)$ can be ordered so that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_r > 0$ and $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_r$. In fact, suppose that we order $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_r > 0$ but there is i such that and $q_i < q_{i+1}$. If $p_i = p_{i+1}$ then we can exchange i and $i + 1$. Otherwise, if $p_i > p_{i+1}$ then $(p_i, q_i) < (p_{i+1}, q_{i+1})$ and (p_{i+1}, q_{i+1}) does not belongs to the NE path.

Given a signed permutation w, we say that a corner position (p, q) of $\mathscr{C}(w)$ is *unessential* if there are corners (p_1, q_1), (p_2, q_2) and (p_3, q_3) in the NE path $\mathscr{N\!e}(w)$ satisfying the following conditions:

$$p_1 = p \text{ and } q_1 < q < 0;$$
$$p_2 > 0 \text{ and } q_2 = \overline{q} + 1;$$
$$(p_3, q_3) < (p, q).$$

In other words, (p, q) is not a minimal corner in the poset in the upper half of the diagram, the box $(q_1 - 1, \overline{p_1})$ lays above and in the same column of the box $(q - 1, \overline{p})$, and the box $(\overline{q_2}, p_2 - 1)$ reflected from $(q_2 - 1, \overline{p_2})$ lays to the right and in the same row of $(q - 1, \overline{p})$, as shown in Figure 21. Define by $\mathscr{U}(w)$ the set of all unessential corners of w.

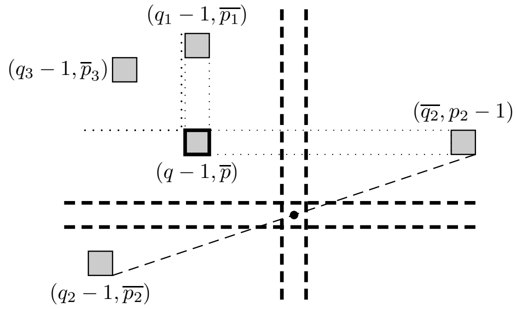


Figure 21 – Configuration of an unessential corner (p,q). The highlighted box $(q-1,\overline{p})$ satisfies all the conditions.

It is important to emphasize that we require all three corners (p_1,q_1), (p_2,q_2) and (p_3,q_3) must belong to the NE path $\mathscr{Ne}(w)$.

Considering the example above for a signed permutation $w = 10\,1\,5\,3\,\overline{2}\,\overline{4}\,6\,\overline{9}\,\overline{8}\,\overline{7}$, the set of unessential corners $\mathscr{U}(w)$ only contains the triple $(6,2,\overline{1})$. In fact, Figure 22 shows that the corner $(6,\overline{2},\overline{2})$ is an unessential corner and there are no other one since this is the unique non-minimal corner for this case.

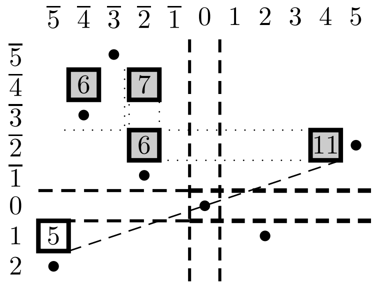


Figure 22 – Part of the diagram $\mathscr{C}(\iota(w))$ showing that $(6,\overline{2},\overline{2})$ is an unessential corner for $w = 10\,1\,5\,3\,\overline{2}\,\overline{4}\,6\,\overline{9}\,\overline{8}\,\overline{7}$.

Chapter 4

Theta-vexillary signed permutations

In this chapter, we will define a class of degeneracy loci that generalize the ones got from a vexillary signed permutation. Such permutations are the ones such that the cohomolgy classes $[\Omega_w]$ are represented by a polynomial Θ, that generalize the raising operators associated to the vexillary and vexillary signed permutations.

4.1 Theta-triples and theta-vexillary permutations

A *theta-triple* is three s-tuples $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ with

$$\begin{aligned} \mathbf{k} &= (0 < k_1 < k_2 < \cdots < k_s), \\ \mathbf{p} &= (p_1 \geqslant p_2 \geqslant \cdots \geqslant p_s > 0), \\ \mathbf{q} &= (q_1 \geqslant q_2 \geqslant \cdots \geqslant q_s), \end{aligned} \tag{4.1.1}$$

where q_i is allowed to be negative and satisfying eight conditions. The first three are

A1. $q_i \neq 0$ for all i;

A2. $q_i \neq -q_j$, for any $i \neq j$.

A3. If $q_s < 0$ then $p_s > 1$;

Now, let $a = a(\boldsymbol{\tau})$ be the integer such that $q_{a-1} > 0 > q_a$, allowing $a = 1$ and $a = s+1$ for the cases where all q's are negative or all q's are positive, respectively. For all $i \geqslant a$, denote by $R(i)$ (or $R(i)_{\boldsymbol{\tau}}$ to specify the triple) the unique integer such that $q_{R(i)} > -q_i > q_{R(i)+1}$; is necessary consider $k_0 = 0$, $p_0 = +\infty$, $q_0 = +\infty$, and $R(a-1) = a-1$. The next three conditions are

B1. $(p_i - p_{i+1}) + (q_i - q_{i+1}) > k_{i+1} - k_i$, for $1 \leqslant i < a-1$;

B2. $(p_i - p_{i+1}) + (q_i - q_{i+1}) > (k_{i+1} - k_i) + (k_{R(i)} - k_{R(i+1)})$, for $a \leqslant i < s$;

B3. $p_s + q_s + k_s > k_{R(s)} + 1$, if $a \leqslant s$.

It is important to observe that none of the above conditions compare indexes $a-1$ and a. Finally, consider $a \leqslant i \leqslant s$ and let $L(i) = L_{\boldsymbol{\tau}}(i)$ be the biggest integer j in $\{R(i)+1, \ldots, a-1\}$ satisfying $k_j - k_{R(i)+1} \geqslant q_{R(i)+1} - q_j$, i.e., $L(i) = \max\{R(i)+1 \leqslant j \leqslant a-1 \mid k_j - k_{R(i)+1} \geqslant q_{R(i)+1} - q_j\}$. The last two conditions are

C1. $-q_i \geqslant k_i - k_{R(i)}$ for all $a \leqslant i \leqslant s$;

C2. $-q_i \geqslant q_{L(i)} + k_{L(i)} - k_{R(i)}$ for all $a \leqslant i \leqslant s$.

Notice that we intentionally divided the conditions in three blocks. We are going to see that conditions in the same block share common characteristics.

Given a theta-triple $\boldsymbol{\tau}$, we can construct a permutation $w(\boldsymbol{\tau})$ using the same step-by-step construction given for a triple of type B as follows:

(1) Starting in the p_1 position, place k_1 consecutive entries, in increasing order, ending with $-q_1$. Mark the *absolute* value of these numbers as "used";

(i) For $1 < i \leqslant s$, starting in the p_i position, or the next available position to the right, fill the next available $k_i - k_{i-1}$ positions with entries chosen consecutively from the unused *absolute* numbers, in increasing order, ending with $-q_i$ or, if it is not available, the biggest unused number below $-q_i$. Again, mark the absolute value of these numbers as "used";

($s+1$) Fill the remaining available positions with the unused positive numbers, in increasing order.

Notice that we should mark as used the absolute of the placed values because we allow negative q_i for a theta-triple.

A signed permutation $w \in \mathcal{W}_n$ is called *theta-vexillary* if $w = w(\boldsymbol{\tau})$ comes from some theta-triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$.

Example 1. The permutation w given in Figure 20 can be obtained from the triple $\boldsymbol{\tau} = (3\,4\,5\,6\,9,\ 8\,6\,5\,4\,2,\ 7\,4\,2\,\overline{3}\,\overline{6})$ using the steps as follows:

$$\begin{array}{rcccccccccc}
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\mathbf{9}} & \overline{\mathbf{8}} & \overline{\mathbf{7}} \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\mathbf{4}} & \cdot & \overline{9} & \overline{8} & \overline{7} \\
 & \cdot & \cdot & \cdot & \cdot & \overline{\mathbf{2}} & \overline{4} & \cdot & \overline{9} & \overline{8} & \overline{7} \\
 & \cdot & \cdot & \cdot & \mathbf{3} & \overline{2} & \overline{4} & \cdot & \overline{9} & \overline{8} & \overline{7} \\
 & \cdot & \mathbf{1} & \mathbf{5} & 3 & \overline{2} & \overline{4} & \mathbf{6} & \overline{9} & \overline{8} & \overline{7} \\
w = & \mathbf{10} & 1 & 5 & 3 & \overline{2} & \overline{4} & 6 & \overline{9} & \overline{8} & \overline{7}
\end{array}$$

We can check that $\boldsymbol{\tau}$ is a theta-triple: clearly $\boldsymbol{\tau}$ satisfies A1, A2, A3. Then, the smaller index where $q_a < 0$ is $a = 4$. We also have that the inequalities

$$q_2 > -q_4 > q_3, \qquad q_1 > -q_5 > q_2,$$

implies that $R(4) = 2$ and $R(5) = 1$, respectively. Then, the inequalities

$$(p_1 - p_2) + (q_1 - q_2) = 5 > 1 = k_2 - k_1$$
$$(p_2 - p_3) + (q_2 - q_3) = 3 > 1 = k_3 - k_2$$

prove that B1 is satisfied,

$$(p_4 - p_5) + (q_4 - q_5) = 5 > 4 = (k_5 - k_4) + (k_{R(4)} - k_{R(5)})$$

prove that B2 is satisfied,

$$p_5 + q_5 + k_5 = 5 > 4 = k_{R(5)} + 1$$

prove that B3 is satisfied, and

$$-q_4 = 3 \geqslant 2 = k_4 - k_{R(4)}$$
$$-q_5 = 6 = k_5 - k_{R(5)}$$

prove that C1 is satisfied. Finally, remember that $L(i) = \max\{R(i) + 1 \leqslant j \leqslant a - 1 \mid k_j - k_{R(i)+1} \geqslant q_{R(i)+1} - q_j\}$, then $L(4) = 3$ and $L(5) = 2$. So, C2 is satisfied because

$$-q_4 = 3 = q_{L(4)} + k_{L(4)} - k_{R(4)}$$
$$-q_5 = 6 \geqslant 5 = q_{L(5)} + k_{L(5)} - k_{R(5)}$$

Hence, w is theta-vexillary signed permutation. Observe that every (k, p, q) in this triple is also a corner position in the diagram. In fact, this is not a coincidence, and we will show that every theta-triple are corner positions on the permutation.

Notice in this construction that it does not create a descent inside a step, i.e., if $a < b$ are positions placed by a Step (i) then $w(a) < w(b)$.

The geometric setup of theta-vexillary signed permutations and their association to the Schubert varieties is developed in the second section of Appendix A.

4.2 Descents of theta-vexillary permutations

A theta-triple $\boldsymbol{\tau}$ also can be seen as a set of basic triples of type B (k_i, p_i, q_i), satisfying the eight conditions. We aim to prove that $\boldsymbol{\tau} \subset \mathscr{C}(w)$. In other words, every basic triple (k_i, p_i, q_i) of a theta-triple $\boldsymbol{\tau}$ is a corner position in the extended diagram of

$w(\boldsymbol{\tau})$. This requires us to study descents in a theta-vexillary signed permutation, which proofs are extentions of the one's given by Anderson and Fulton for Lemmas 2.2, 2.3 and 2.4 of [1].

Before we study descents in a theta-vexillary signed permutation, we need some properties related to the conditions and the step-by-step construction of a permutation from a triple.

First of all, consider the following interpretation of the eight conditions of a theta-triple.

Conditions A1, A2 and A3 are required in order to guarantee that $w(\boldsymbol{\tau})$ is a signed permutation, i.e., the elements (k_i, p_i, q_i) are basic triples of type B.

Conditions B1, B2 and B3, in some sense, characterize a theta-vexillary permutations as well as the condition $(p_i - p_{i+1}) + (q_i - q_{i+1}) > k_{i+1} - k_i$ does for the vexillary permutations. For B2, an extra $k_{R(i)} - k_{R(i+1)}$ is added to the right side because Step (i) skips an equal number of entries, since they have already been used from Step $(R(i+1)+1)$ to $(R(i))$. Moreover, condition B3 is equivalent to apply $i = s$ in condition B2, where we consider the extreme cases $(k_0, p_0, q_0) = (0, n, n)$ and $(k_{s+1}, p_{s+1}, q_{s+1}) = (n, 1, -n)$.

Finally, for conditions C1 and C2, we have the following lemma:

Lemma 4.1. *The conditions* C1 *and* C2 *are equivalent, respectively, to*

C1′. *Given any $a \leqslant i \leqslant s$, all entries placed by Steps (a) to (i) are positive;*

C2′. *Given any $a \leqslant i \leqslant s$, all entries placed by Steps $(R(i)+1)$ to $(a-1)$ are strictly bigger than q_i.*

Proof. For the first statement, observe that all Steps from (a) to (i) must skip at most $k_{a-1} - k_{R(i)}$ values because they were already used in Steps $(R(i)+1)$ to $(a-1)$ and denote by $\alpha := -q_i - (k_{a-1} - k_{R(i)})$ the number of available positive entries from 1 to $\overline{q_i}$ that can be used by Steps (a) to (i). Then, condition C1 is equivalent to say that $\alpha \geqslant k_i - k_{a-1}$, i.e., there is enough positive values available to be placed by Steps (a) to (i).

For the second assertion, remember that the definition of $L(i)$ says that it is the biggest integer in $\{R(i)+1, \ldots, a-1\}$ where $k_{L(i)} - k_{R(i)+1} \geqslant q_{R(i)+1} - q_{L(i)}$. The smallest possible entry placed by Steps $(R(i)+1)$ to $(L(i))$ is limited below by $\overline{q_{L(i)} + k_{L(i)} - k_{R(i)}} + 1$. Since for any Step (j) after $L(i)$, we have that $k_j - k_{R(i)+1} < q_{R(i)+1} - q_j$, then no entry placed by such step cannot be smaller than $\overline{q_{L(i)}}$. So, every entry placed by Steps $(R(i)+1)$ to $(a-1)$ is limited below by $\overline{q_{L(i)} + k_{L(i)} - k_{R(i)}} + 1$, and we conclude that both conditions C2 and C2′ imply that $q_i < \overline{q_{L(i)} + k_{L(i)} - k_{R(i)}} + 1$. □

In other words, conditions C1 and C2 guarantee that given $i \geqslant a$, then all values placed by Steps $(R(i)+1)$ to (i) ranges from q_i to $\overline{q_i}$.

Lemma 4.2. *Let $w = w(\boldsymbol{\tau})$ be a theta-vexillary permutation and $\boldsymbol{\tau}$ be a triple. Then, for each $a \leqslant i \leqslant s$ the Step (i) places only positive integers in the construction of the permutation w.*

Proof. Observe that Step (i) must skip at most $k_{i-1} - k_{R(i)}$ values because they were already used in Steps $(R(i)+1)$ to $(i-1)$. Then, $\alpha = -q_i - (k_{i-1} - k_{R(i)})$ represent the number of available positive entries from 1 to $\overline{q_i}$ that can be placed by Step (i). Using the condition (iv), $\alpha \geqslant (k_i - k_{R(i)}) - (k_{i-1} - k_{R(i)}) \geqslant k_i - k_{i-1}$, which means that all the $k_i - k_{i-1}$ positive entries to be placed in Step (i) fit in the positive entries 1 to $\overline{q_i}$. □

Proposition 4.3. *Let $w = w(\boldsymbol{\tau})$ be a theta-vexillary signed permutation and $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ be a theta-triple. Then all the descents of w are at positions $p_i - 1$, i.e, for each i, we have $w(p_i - 1) > \overline{q_i} \geqslant w(p_i)$ and there are no other descents.*

Proof. In Step (1), no descents are created, unless $p_1 = 1$, in which case the permutation has a single descent at 0. For $1 < i < a$, this is proved in Lemma 2.2 of [1]. Now, supposing that $a \leqslant i \leqslant s$ and $i \geqslant 2$, assume inductively that for $j < i$, there is a descent at position $p_j - 1$ whenever this positions has been filled, satisfying $w(p_j - 1) > \overline{q_j} \geqslant w(p_j)$, and there are no other descents. By Lemma 4.1, only positive entries are placed in consecutive vacant positions of Step (i), from left to right, at position p_i (or the next vacant position to the right, if $p_{i-1} = p_i$). We consider "sub-steps" of Step (i), where we are placing an entry at position $p \geqslant p_i$, and distinguish three cases. First, suppose we are at position p, with $p < p_{i-1} - 1$. In this case, the previous entry placed in Step (i) (if any) was placed at position $p - 1$, so we did not create a descent at $p - 1$. Position $p + 1$ is still vacant, so no new descents are created.

To clarify this proof, let $\boldsymbol{\tau} = (3\,4\,5\,6\,9,\ 8\,6\,5\,4\,2,\ 7\,4\,2\,\overline{3}\,\overline{6})$ as in Example 1. In Step (5), the first entry placed is 1 and it does not create a descent:

$$w = \cdot\, \mathbf{1} \,\cdot\, 3\,\overline{2}\,\overline{4} \,\cdot\, \overline{9}\,\overline{8}\,\overline{7}$$

Next, suppose we are at position $p = p_{i-1} - 1$. This means that $p_{i-1} - p_i \leqslant k_i - k_{i-1}$, so let $\beta = (k_i - k_{i-1}) - (p_{i-1} - p_i)$ be the number of entries remaining to be placed in Step (i), after placing the current at position p. Condition (v′) tell us that $q_i \leqslant q_i + \beta < q_{i-1}$, then considering the integer interval $\mathcal{I}_i = \{\overline{q_{i-1}} + 1, \ldots, \overline{q_i}\}$, it must be non-empty. We claim that the entry $w(p) = w(p_{i-1} - 1)$ lies in $\mathcal{I}_i$ and therefore $w(p_{i-1} - 1) > \overline{q_{i-1}} \geqslant w(p_{i-1})$, proving this situation. Remember that the step-by-step construction must skip those entries that its absolute value have already been used, and then this claim is equivalent to say that even removing from $\mathcal{I}_i$ those repetitions, there still is some value to be picked by $w(p)$ in $\mathcal{I}_i$.

To prove that claim, lets count how many values in $\mathcal{I}_i$ were used in previous steps. For $a \leqslant j < i$, any entry x of Steps (j) satisfies $x \leqslant \overline{q_j} \leqslant \overline{q_{i-1}}$, that means $x \notin \mathcal{I}_i$. If $1 \leqslant j \leqslant R(i)$ then any entry x placed in Steps (j) satisfies $x \leqslant \overline{q_j} \leqslant \overline{q_{R(i)}} < q_i$, implying that $\overline{x} \notin \mathcal{I}_i$. If $R(i-1) < j < a$ then by condition (viii′), any entry x placed in Steps (j) satisfies $x > q_{i-1}$, implying that $\overline{x} \notin \mathcal{I}_i$. Finally, if $R(i) < j \leqslant R(i-1)$ then by condition (viii′), any entry x placed in Step (j) satisfies $q_i < x \leqslant \overline{q_j} \leqslant \overline{q_{R(i-1)}} < q_{i-1}$, hence, $\overline{x} \in \mathcal{I}_i$. We conclude that the only absolute values placed in previous steps that belongs to the interval $\mathcal{I}_i$ are all the ones from Steps $(R(i)+1)$ to $(R(i-1))$. So there are $\alpha := k_{R(i-1)} - k_{R(i)}$ values in $\mathcal{I}_i$ that cannot be used in the Step (i) in position p. In order to place the correct value for position p of Step (i), we need to consider that the values which are going to be placed after position p also must belong to $\mathcal{I}_i$ and are bigger than $w(p)$, i.e., it also is required to skip the β biggest values in $\mathcal{I}_i$. Since the number of elements of $\mathcal{I}_i$ is $\overline{q_i} - \overline{q_{i-1}}$, follows from condition (vi′) that $\#(\mathcal{I}_i) > \alpha + \beta$ and, therefore, there is some value in $\mathcal{I}_i$ to pick for $w(p)$.

Continuing the example, in Step (5), the second entry placed is 5, creating an descent at position 3:

$$w = \cdot\, 1\, \mathbf{5}\, 3\, \overline{2}\, \overline{4}\, \cdot\, \overline{9}\, \overline{8}\, \overline{7}$$

Finally, suppose we are at position $p \geqslant p_{i-1}$. Using the previous case, the entry to be placed is some $x \in \mathcal{I}_i$. When an entry is placed in a vacant position to the right of a filled position, it does not create a descent since either all entries already placed the previous steps are smaller than $\overline{q_{i-1}} < x$ or the entries placed in this step is smaller than x. When it is placed to the left of a filled position, which can only happen at positions $p_j - 1$ for some $j < i-1$, and it does create a descent at the position $p_j - 1$ satisfying $w(p_j - 1) > \overline{q_{i-1}} \geqslant \overline{q_j} \geqslant w(p_j)$

In Step (5) of our example, it remains to place the 3rd value 6 in the next vacant position, which occurs at position 7. Observe that we do not create a descent at the filled position to its left, but we do create a descent at position 7, since the position 8 is already filled:

$$w = \cdot\, 1\, 5\, 3\, \overline{2}\, \overline{4}\, \mathbf{6}\, \overline{9}\, \overline{8}\, \overline{7}$$

At the Step $(s+1)$, we can apply the previous case for $i = s+1$, adding the values $k_{s+1} = n$, $p_{s+1} = 0, q_{s+1} = -n+1$ to $\boldsymbol{\tau}$. This procedure will create descents only at those $p_j - 1$ which are still vacant. □

Given a triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, the dual triple is defined by $\boldsymbol{\tau}^* = (\mathbf{k}, \mathbf{q}, \mathbf{p})$, where p and q were switched. Clearly, a dual triple could not be a theta-vexillary permutation, but the dual triple is useful to compute the inverse of $w(\boldsymbol{\tau})$.

The dual triple $\boldsymbol{\tau}^* = (k, q, p)$ determines a signed permutation $\iota(w(\boldsymbol{\tau}^*))$ in S_{2n+1} using the follow steps:

(0) Put a zero at the position 0;

(1) Starting in the q_1 position, place k_1 consecutive entries, in increasing order, ending with $-p_1$. Mark the absolute value of these numbers as "used" and fill the reflection through 0 with the respective reflection $w(\overline{a}) = \overline{w(a)}$;

(i) For $1 < i \leqslant s$, starting in the q_i position (if $q_i < 0$ then use a position before zero), or the next available position to the right, fill the next available $k_i - k_{i-1}$ positions with entries chosen consecutively from the unused absolute numbers, in increasing order, ending with $-p_i$ or, if it is not available, the biggest unused number below $-p_i$. Again, mark the absolute value of these numbers as "used" and fill the reflection through 0 with the respective reflection $w(\overline{a}) = \overline{w(a)}$;

(s+1) Fill the remaining available positions after 0 with the unused positive numbers, in increasing order.

The difference here compared to the construction using the theta-vexillary permutation is that we allow to have negative positions, so we need the full form of the permutation. The signed permutation $w(\boldsymbol{\tau}^*)$ is obtained from $\iota(w(\boldsymbol{\tau}^*))$ by restricting it to the positions $\{1, \ldots, n\}$.

Lemma 4.4. *We have* $w(\boldsymbol{\tau}^*) = w(\boldsymbol{\tau})^{-1}$.

Proof. We can prove in the same way as Lemma 2.3 of [1], adding the fact that for $a \leqslant i \leqslant s$, the permutation $\iota(w)$ maps the set $a(i)$ to $b(i)$ and, hence, the inverse $\iota(w)^{-1}$ maps $b(i)$ to $a(i)$. □

Example 2. Consider the dual triple $\boldsymbol{\tau}^* = (3\,4\,5\,6\,9,\ 7\,4\,2\,\overline{3}\,\overline{6},\ 8\,6\,5\,4\,2)$ of the one gave in Example 1. The permutation $\iota(w(\boldsymbol{\tau}^*))$ is constructed as follows:

$$
\begin{array}{cccccccccc|c|cccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \mathit{8} & \mathit{9} & \mathit{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\mathbf{10}} & \overline{\mathbf{9}} & \overline{\mathbf{8}} & \cdot \\
\cdot & 8 & 9 & 10 & \cdot & \cdot & \mathit{6} & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \overline{\mathbf{6}} & \cdot & \cdot & \overline{10} & \overline{9} & \overline{8} & \cdot \\
\cdot & 8 & 9 & 10 & \cdot & \cdot & 6 & \cdot & \mathit{5} & \cdot & 0 & \cdot & \overline{\mathbf{5}} & \cdot & \overline{6} & \cdot & \cdot & \overline{10} & \overline{9} & \overline{8} & \cdot \\
\cdot & 8 & 9 & 10 & \cdot & \cdot & 6 & \overline{\mathbf{4}} & 5 & \cdot & 0 & \cdot & \overline{5} & \mathit{4} & \overline{6} & \cdot & \cdot & \overline{10} & \overline{9} & \overline{8} & \cdot \\
\cdot & 8 & 9 & 10 & \overline{\mathbf{7}} & \overline{\mathbf{3}} & 6 & \overline{4} & 5 & \overline{\mathbf{2}} & 0 & \mathit{2} & \overline{5} & 4 & \overline{6} & \mathit{3} & \mathit{7} & \overline{10} & \overline{9} & \overline{8} & \cdot \\
\overline{\mathit{1}} & 8 & 9 & 10 & \overline{7} & \overline{3} & 6 & \overline{4} & 5 & \overline{2} & 0 & 2 & \overline{5} & 4 & \overline{6} & 3 & 7 & \overline{10} & \overline{9} & \overline{8} & \mathbf{1}
\end{array}
$$

For each step, the bold numbers represent the values placed for such step, and the italic ones are their reflection through zero.

So, $w(\boldsymbol{\tau}^*) = 2\,\overline{5}\,4\,\overline{6}\,3\,7\,\overline{10}\,\overline{9}\,\overline{8}\,1$ and we can easily verify that this permutation is the inverse of $w = 10\ 1\ 5\ 3\ \overline{2}\ \overline{4}\ 6\ \overline{9}\ \overline{8}\ \overline{7}$.

Although $w(\boldsymbol{\tau}^*)$ is not theta-vexillary, a similar version of Proposition 4.3 holds for this case and the proof follows that same idea.

Proposition 4.5. *Let $w = w(\boldsymbol{\tau})$ be a theta-vexillary signed permutation, for a theta-triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$. Then all the descents of w^{-1} are at positions $q_i - 1$, when $i < a$, and $\overline{q_i}$, when $i \geqslant a$. In fact, we have*

$$w^{-1}(q_i - 1) > \overline{p_i} \geqslant w^{-1}(q_i)\ ,\ \text{for } i < a;$$
$$w^{-1}(\overline{q_i}) > p_i - 1 \geqslant w^{-1}(\overline{q_i} + 1)\ ,\ \text{for } i \geqslant a;$$

and there are no other descents.

4.3 Extended diagrams for theta-vexillary permutations

In this section, we aim to understand how a theta-vexillary permutation looks like in the extended diagram.

Given a position (p, q) in the extended diagram $D^+(w)$, define the *left lower region* of (p, q) by the set boxes in the extended diagram strictly south and weakly west of the SE corner $(q - 1, \overline{p})$. In other words, denoting it by $\Lambda(p, q)$, this set is

$$\Lambda(p, q) := \{(a, b) \in D^+(w) \mid a \geqslant q, b \leqslant \overline{p}\}.$$

Notice that the step-by-step construction of a theta-vexillary permutation can also be seen as a process of placing dots in the extended diagram, since each pair $(w(i), i)$ corresponds to a dot in the diagram. We can say that a Step (i) places dots in the diagram using the following rule: if an entry x is placed at a position z in the permutation, i.e., $w(z) = x$, then it produces a dot at the box $(\overline{x}, \overline{z})$ in the diagram. For instance, Figure 23 shows how each step places dots in the diagram of the triple $\boldsymbol{\tau} = (34569,\ 86542,\ 742\overline{3}\overline{6})$ (c.f. Example 1). The first step places the entries $\overline{9}$, $\overline{8}$ and $\overline{7}$, respectively, at position 8, 9 and 10. This means that the dots for Step (1) are placed in boxes $(9, \overline{8})$, $(8, \overline{9})$ and $(7, \overline{10})$. The same idea happens for all steps. It is important to notice in this example that each step "creates" a SE corner, but not all of then. In fact, two SE corners must appeared in the diagram after the last step. This is the moment in which we can establish a relationship between theta-triples and corners in the extended diagram. The next proposition states how are some SE corners since we know the theta-triple.

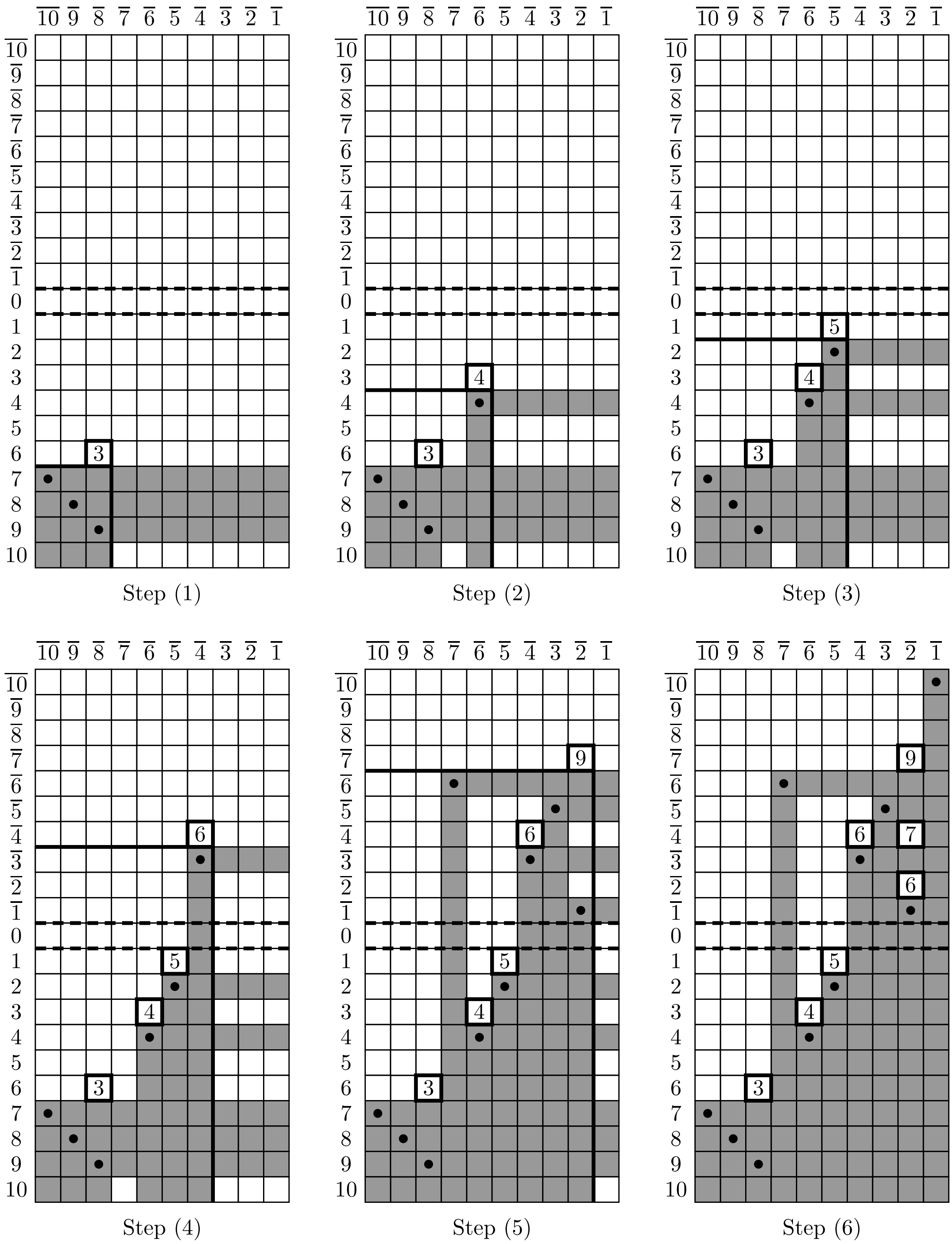


Figure 23 – Placing dots during a step-by-step construction. The thicker line encloses the region $\Lambda(p_i, q_i)$.

Proposition 4.6. *Let $w = w(\boldsymbol{\tau})$ be a theta-vexillary signed permutation and $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ be a theta-triple. Then we have the following:*

1. *The boxes $(q_i - 1, \overline{p_i})$ and their reflection $(\overline{q_i}, p_i - 1)$ are SE corners of the diagram of $\iota(w)$ (not necessary all of them);*

2. *For any $1 \leqslant i \leqslant s+1$, all the dots placed by Step (i) in the diagram are inside region $\Lambda(p_i, q_i)$ and outside $\Lambda(p_{i-1}, q_{i-1})$;*

3. *k_i is the number of dots inside the region $\Lambda(p_i, q_i)$.*

Proof. Lemma 3.4 says that there is a symmetry between boxes $(q_i - 1, \overline{p_i})$ and their reflection $(\overline{q_i}, p_i - 1)$. Then, it suffices to prove that every $(q_i - 1, \overline{p_i})$ is a SE corner. If $p > 0$, then a signed permutation w has a descent at position $p-1$ if and only if $i(w)$ has descents at position $p-1$ and $\overline{p}$. By proposition 4.3, $\iota(w)$ satisfies $\iota(w)(p_i - 1) > \overline{q_i} \geqslant \iota(w)(p_i)$, and it implies that

$$\iota(w)(\overline{p_i}) > q_i - 1 \geqslant \iota(w)(\overline{p_i} + 1).$$

On the other hand, by Proposition 4.5, $\iota(w)^{-1}$ satisfies

$$\iota(w)^{-1}(q_i - 1) > \overline{p_i} \geqslant \iota(w)^{-1}(q_i),$$

for any i. This proves that $(q_i - 1, \overline{p_i})$ satisfies Equation 3.1.2, which proves item (1).

For item (2), first of all, observe that every entry x placed at position z in Step (i) satisfies $p_i \leqslant z$ and $x \leqslant \overline{q_i}$, implying that the correspondent dot at box $(\overline{x}, \overline{z})$ in the diagram belongs to $\Lambda(p_i, q_i)$.

Now, we need to check that all dots placed by Step (i) are outside $\Lambda(p_{i-1}, q_{i-1})$. It is enough to verify that whenever in Step (i) we are placing an entry x at a position $z \geqslant p_{i-1}$ in the permutation, then $x > \overline{q_{i-1}}$. Set $\beta = (k_i - k_{i-1}) - (p_{i-1} - p_i)$ the number of entries to be placed after the position p_{i-1} during the Step (i). If $1 \leqslant i < a$ then condition B1$'$ implies that $\beta < q_{i-1} - q_i$. The entries that will be placed are $\overline{q_i + \beta} + 1, \ldots, \overline{q_i}$ and they are all strictly greater than $\overline{q_{i-1}}$ (in the diagram, it is equivalent to say that we have $q_{i-1} - q_i$ available rows to place the dots above q_{i-1} but we only need β rows). If $i = a$ then by Lemma 4.1, $x > 0 > \overline{q_{i-1}}$.

If $a < i \leqslant s+1$ then condition B2$'$ implies that $\beta < (q_{i-1} - q_i) - (k_{R(i-i)} - k_{R(i)})$, which means that have $(q_{i-1} - q_i) - (k_{R(i-1)} - k_{R(i)})$ available rows in the diagram to place the dots above q_{i-1} but we only need β rows. Observe that we must skip $k_{R(i-1)} - k_{R(i)}$ rows in the diagram since their reflection have already been used between Steps $(R(i) + 1)$ to $(R(i-1))$. This proves item (2).

Finally for (3), k_i is the total of dots placed until Step (i) and they are all placed inside the region $\Lambda(p_i, q_i)$. Any other dot placed after this step is placed outside $\Lambda(p_i, q_i)$. □

Portraying $\boldsymbol{\tau}$ as a set of corner positions (p_i, q_i), item (1) of Proposition 4.6 simply says that $\boldsymbol{\tau} \subset \mathscr{C}(w)$.

Remember that there is a poset "<" in the set of corners $\mathscr{C}(w)$ where two corners positions satisfy $(p, q) < (p', q')$ if and only if $p > p'$ and $q < q'$. Also remember that the NE path $\mathscr{N}\!e(w) \subset \mathscr{C}(w)$ is the set of minimal elements of this poset.

Lemma 4.7. *Let $w = w(\boldsymbol{\tau})$ be a theta-vexillary signed permutation, and $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ be a theta-triple. Then every corner position (p_i, q_i) of $\boldsymbol{\tau}$ is minimal in the poset "<", i.e., $\boldsymbol{\tau} \subset \mathscr{N}\!e(w)$.*

Proof. Suppose that there is a pair (p_i, q_i) of $\boldsymbol{\tau}$ and a corner position $(p, q) \in \mathscr{C}(w)$ such that $(p, q) < (p_i, q_i)$, i.e., $p > p_i$ and $q < q_i$. The pair (p, q) is not in $\boldsymbol{\tau}$ because $\mathbf{p}$ and $\mathbf{q}$ are strictly decreasing s-tuples. Since the box $(q - 1, \overline{p})$ is a SE corner, Equation (3.1.2) implies that

$$\overline{q} < x \quad \text{and} \quad p \leqslant y, \tag{4.3.1}$$

where $x := w(p - 1)$ and $y := w^{-1}(\overline{q})$.

When we use the step-by-step construction to produce the permutation w, observe that the position $p - 1$ must be filled by some step and the entry $\overline{q}$ must be placed in some step. So, there must be integers $1 \leqslant m, l \leqslant s + 1$ such that:

a) The entry x is placed in the position $p - 1$ during some Step (m). This places a dot at the box $(\overline{x}, \overline{p} + 1) \in \Lambda(p_m, q_m)$;

b) The entry $\overline{q}$ is placed in the position y during some Step (l). This places a dot at the box $(q, \overline{y}) \in \Lambda(p_l, q_l)$.

Although there exist such integers m and i, we are going to show that they cannot be either equal, smaller or greater than each other. Hence, this contradicts the assumption that (p_i, q_i) is not minimal in the poset.

If $m = l$ then, using Equation 4.3.1, $p - 1 < y$ are positions in Step $(m = l)$ and the entry in such positions are $w(p - 1) = x > \overline{q} = w(y)$, i.e., there is a descent in it. This contradicts the fact that there are no descents in a step.

If $m < l$ then, using Equation 4.3.1, we got that $\overline{y} \leqslant \overline{p}$ and $\overline{q} \leqslant x \leqslant \overline{q_m}$ (the former relation comes from the fact that every entry placed by Step (m) is weakly smaller than $\overline{q_m}$). This implies that the box $(q, \overline{y})$ also belongs to the region $\Lambda(p_m, q_m)$, a contradiction of item 2 of Proposition 4.6.

If $m > l$ then observe that Step (l) must fill all positions from p_l to y in the step-by-step construction of the permutation w. Since $y > p - 1 \geqslant p_i \geqslant p_l$ (because $i < l$), we have that the position $p - 1$ is also filled by Step (l), which contradicts the fact that it is filled during Step (m). $\square$

Recall that a corner position (p, q) of $\mathscr{C}(w)$ is unessential if there are corners (p_1, q_1), (p_2, q_2) and (p_3, q_3) in the NE path $\mathscr{N}e(w)$ such that (p, q) is not a minimal corner in the poset in the upper half of the diagram, the box $(q_1 - 1, \overline{p_1})$ lays above and in the same column of the box $(q - 1, \overline{p})$, and the box $(\overline{q_2}, p_2 - 1)$ reflected from $(q_2 - 1, \overline{p_2})$ lays to the right and in the same row of $(q - 1, \overline{p})$, as we can see in figure Figure 21.

Proposition 4.8. *Given $w \in \mathcal{W}_n$, suppose that the set of corner $\mathscr{C}(w)$ is the disjoint union*

$$\mathscr{C}(w) = \mathscr{N}e(w)\dot{\cup}\mathscr{U}(w).$$

Then w is a theta-vexillary.

Proof. Suppose that the set of corners $\mathscr{C}(w)$ of a permutation w is given by the disjoint union of the NE path $\mathscr{N}e(w)$ and the set of unessential corners $\mathscr{U}(w)$. Since all corner positions (p_i, q_i) of $\mathscr{N}e(w)$ can be ordered so that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_r > 0$ and $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_r$, set k_i as the rank $r_w(p_i, q_i)$ and define the triple $\boldsymbol{\tau}' = (\mathbf{k}, \mathbf{p}, \mathbf{q})$. We will prove that $\boldsymbol{\tau}$ is almost a theta-vexillary triple, i.e., it satisfies A1, A2, A3, C1, C2, and B1. In order to get B2 and B3, occasionally some elements (k_i, p_i, q_i) should be removed from $\boldsymbol{\tau}'$.

Conditions A1, A2 and A3 are true because w is a signed permutation in $\mathcal{W}_n$. In fact, A1 and A3 come direct from the fact that there is no SE corner at row -1 or above the middle in column -1 since $w(0) = 0$, and A2 is satisfied just because we cannot have dots laying in opposite rows.

Now, a and $R(i)$, for $a \leqslant i \leqslant s$, can be defined. Let us prove that $\boldsymbol{\tau}$ satisfies conditions $C1$ and $C2$. Consider the diagrams sketched in Figure 24.

For condition C1, let $a \leqslant i \leqslant s$ and consider the regions A and B as in Figure 24. Denote by $d(A)$ and $d(B)$ the number of dots in each one of them. The definition of $R(i)$ can be translated to the diagram as follows: $R(i)$ is the unique index smaller than a such that there is no other corner of $\boldsymbol{\tau}$ laying to the right of it and in the rows $q_{R(i)} - 1, \ldots, \overline{q_i}$. Suppose that there is a dot in the darker region A of Figure 24. This dot must be placed by some Step (j), for $j > R(i)$, which implies that the corner position (p_j, q_j) is located above the row $\overline{q_i}$ and it also places a dot above $\overline{q_i}$. However, the construction of a step says that we must fill all entries between them, including q_i. So, we should have a dot at row q_i and another in the row $\overline{q_i}$, a contradiction of condition A2. Hence, $d(A) = 0$. On the other hand, $d(B) \leqslant -q_i$ because condition A2 says that we cannot have dots in opposite rows. Thus, $-q_i \geqslant d(A) + d(B) = k_i - k_{R(i)}$, since $d(A) + d(B)$ is the amount of dots to be placed from Step $(R(i) + 1)$ to (i).

By Lemma 4.1, we may show that $\boldsymbol{\tau}$ satisfies condition C2$'$ instead of C2. In the previous case, we proved that region A contains no dots. It means that no step from $(R(i) + 1)$ to $(a - 1)$ place dots in A, which is equivalent to say that all entries placed by Steps $(R(i) + 1)$ to $(a - 1)$ are strictly bigger than q_i, proving C2$'$.

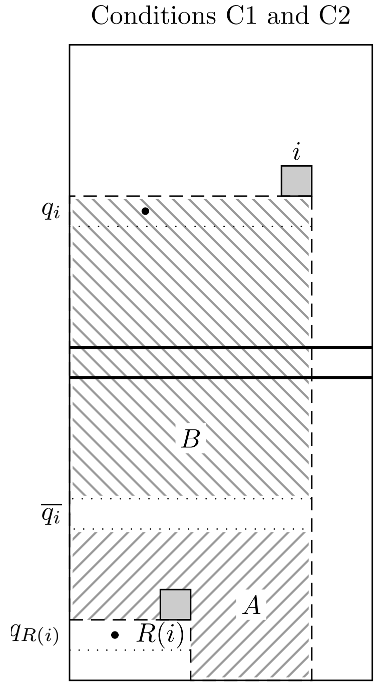


Figure 24 – Configuration required to get conditions C1 and C2.

In order to show that $\boldsymbol{\tau}$ satisfies conditions B1, B2, and B3, consider the diagrams sketched in Figure 25.

For condition B1, let $1 \leqslant i < a-1$ and consider the rectangular regions A and B as in Figure 25 (left). Denote by $d(A)$ and $d(B)$ the number of dots in each one of them. Notice that the number of dots in each rectangle is limited by the length of their sides, and $d(A)+d(B)$ is the number of dots in Step (i). If $p_i = p_{i+1}$ then $d(B) = 0$ and $d(A) < q_i - q_{i+1}$, since we cannot place a dot in row $q_i - 1$. So, $(p_i - p_{i+1}) + (q_i - q_{i+1}) > d(B) + d(A) = k_{i+1} - k_i$. If $p_i > p_{i+1}$ then, we cannot have the dot in column $\overline{p_i} + 1$ inside B because it would not create a SE corner $(q_i - 1, \overline{p_i})$. Hence, $(p_i - p_{i+1}) + (q_i - q_{i+1}) > d(B) + d(A) = k_{i+1} - k_i$.

For conditions B2 and B3, let $a \leqslant i \leqslant s$ and consider the rectangular regions A, B and C of Figure 25 (right). Suppose that $p_i > p_{i+1}$. Using the same argument of condition C1, all dots between the rows $\overline{q_i}$ and $\overline{q_{i+1}}$ are in rectangle C and the number of dots in this region is $d(C) = k_{R(i)} - k_{R(i+1)}$. As well as the previous case, $d(B) < p_i - p_{i+1}$ and the number of dots in region A is $d(A) \leqslant (q_i - q_{i+1}) - d(C)$, since we cannot have dots in opposite rows. Therefore, $(p_i - p_{i+1}) + (q_i - q_{i+1}) > d(B) + d(A) + d(C) = (k_{i+1} - k_i) + (k_{R(i)} - k_{R(i+1)})$.

The difficulty appears when $p_i = p_{i+1}$. In this case, $d(B) = 0$ and $d(A) \leqslant (q_i - q_{i+1}) - d(C)$. Then, $(p_i - p_{i+1}) + (q_i - q_{i+1}) \geqslant d(B) + d(A) + d(C) = (k_{i+1} - k_i) + (k_{R(i)} - k_{R(i+1)})$, which means that the equality can happen. So, we need to remove these elements from $\boldsymbol{\tau'}$ where the equality holds. Denote the set of index $I = I_{\boldsymbol{\tau'}} \subset [1, s]$ by

$$I = \{i \geqslant a \mid (p_i - p_{i+1}) + (q_i - q_{i+1}) = (k_{i+1} - k_i) + (k_{R(i)} - k_{R(i+1)})\}$$

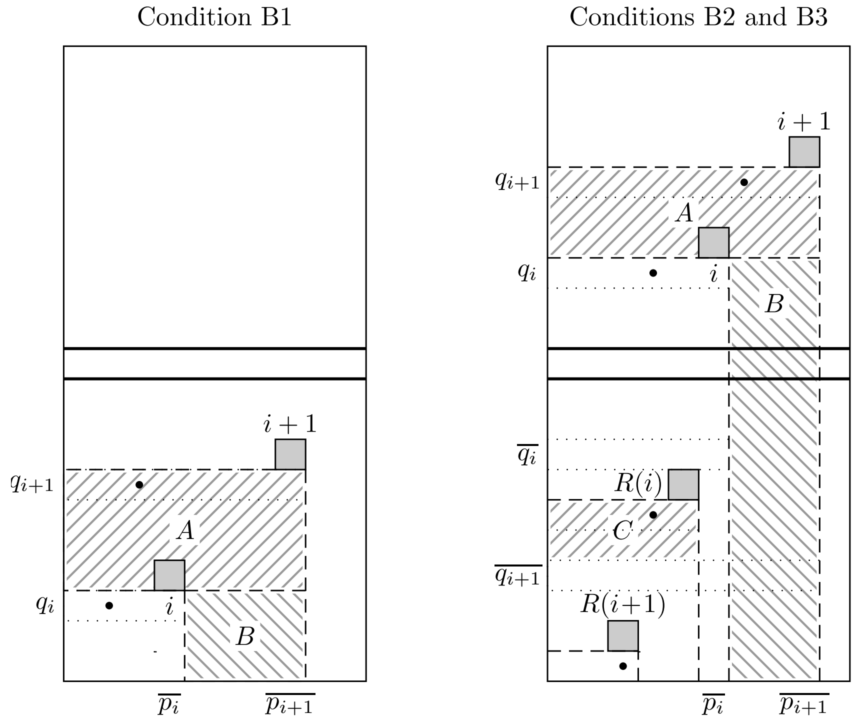


Figure 25 – Configuration required to get conditions B1, B2, and B3.

Define $\boldsymbol{\tau}$ the triple

$$\boldsymbol{\tau} := \{(k_i, p_i, q_i) \in \boldsymbol{\tau}' \mid i \notin I\}$$

Clearly, $\boldsymbol{\tau}$ satisfies A1, A2, A3, C1, C2, and B1. Suppose that $a \leqslant i < j$ are integers such that $i, j \notin I$ and $i+1, i+2, \ldots, j-1 \in I$, i.e., i and j are consecutive indexes in $\boldsymbol{\tau}$. Then they satisfy

$$\begin{aligned}
(p_i - p_{i+1}) + (q_i - q_{i+1}) &> (k_{i+1} - k_i) + (k_{R(i)} - k_{R(i+1)}),\\
(p_{i+1} - p_{i+2}) + (q_{i+1} - q_{i+2}) &= (k_{i+2} - k_{i+1}) + (k_{R(i+1)} - k_{R(i+2)}),\\
(p_{i+2} - p_{i+3}) + (q_{i+2} - q_{i+3}) &= (k_{i+3} - k_{i+2}) + (k_{R(i+2)} - k_{R(i+3)}),\\
&\vdots\\
(p_{j-1} - p_j) + (q_{j-1} - q_j) &= (k_j - k_{j-1}) + (k_{R(j-1)} - k_{R(j)}).
\end{aligned}$$

Therefore,

$$(p_i - p_j) + (q_i - q_j) > (k_j - k_i) + (k_{R(i)} - k_{R(j)}),$$

and $\boldsymbol{\tau}$ also satisfies B2 and B3.

Finally, observe that the extended diagram of $w(\boldsymbol{\tau})$ is exactly the extended diagram of w, which means that $w(\boldsymbol{\tau}) = w$. □

Now, we aim to proof the converse of this proposition.

Let w be a theta-vexillary permutation and $\boldsymbol{\tau}$ a theta-triple. Denote by $\iota(\boldsymbol{\tau}) \subset \mathscr{C}(\iota(w))$ the set of all corner positions of $\boldsymbol{\tau}$ and their reflections, i.e.,

$$\iota(\boldsymbol{\tau}) = \bigcup_{i=1}^{s} \{(k_i, p_i, q_i) \cup (k_i, p_i, q_i)^{\perp}\}.$$

Remember that Propositions 4.3 and 4.5 state that all descents of w and w^{-1} are exclusively determined by elements in $\boldsymbol{\tau}$. More over, this assertion can be extended to the diagram $D(\iota(w)))$ of $\iota(w)$: all descents of $\iota(w)$ are at position $p_i - 1$ and $\overline{p_i}$, and all descents of $\iota(w)^{-1}$ are at position $q_i - 1$ and $\overline{q_i}$, where i ranges from 1 to s. Thus, if there is other SE corner, it should not create descents, but it must match existing descents. For instance, considering $\boldsymbol{\tau}$ of Example 1 and its diagram in Figure 20, observe that the corner position $(2, \overline{3}) \notin \iota(\boldsymbol{\tau})$ but it is in the same row of $(4, \overline{3}) \in \iota(\boldsymbol{\tau})$ and same column of $(2, \overline{6}) \in \iota(\boldsymbol{\tau})$.

We conclude that if there exists a corner position $T \notin \iota(\boldsymbol{\tau})$ then there are corner positions $T', T'' \in \iota(\boldsymbol{\tau})$ such that T' is in the same column of T, and T'' is in the same row of T.

Now, consider the following situation: suppose that $T' = (p_i, q_i)$ and $T'' = (p_j, q_j)$, for some $i < j$ such that $q_i > q_j > 0$. Is there possible, for instance, that the descent of $\iota(w)$ at position $(p_i - 1)$ and the decent of $\iota(w)^{-1}$ at position $\overline{q_j}$ create a corner $T = (p_i, q_j)$, at the box $(q_j - 1, \overline{p_i})$? The answer depends on the arrangement of the dots in the diagram, as shown in Figure 26.

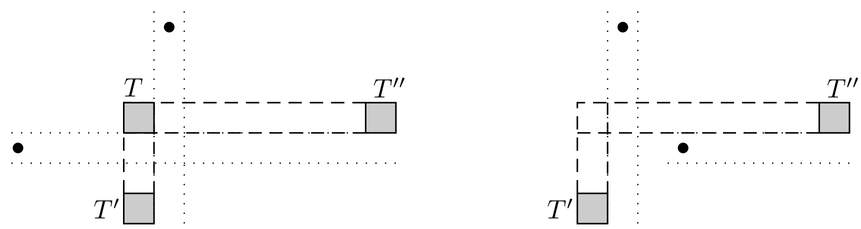


Figure 26 – Example of combination of two descents. On the left, we can observe that the dots are placed in a such way that there is a corner T. On the other hand, the figure to the right does not have a corner T because the dots are not arranged properly to create it.

This question allow us to figure out where we can find all the corner of w by combination of descents of corner T' and T'' in $\iota(\boldsymbol{\tau})$.

Next lemma states a first situation where a SE corner cannot occur.

Lemma 4.9. *Let $w = w(\boldsymbol{\tau})$ be a theta-vexillary signed permutation, and $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ be a theta-triple. Then for any $1 \leqslant i \leqslant s$ such that $p_i > p_{i+1}$, there is no corner position (p, q)*

different of (p_i, q_i) *satisfying* $p > p_{i+1}$ *and* $q_i \geqslant q$. *In other words,* (p_i, q_i) *is the unique SE corner in the region highlighted in Figure 27.*

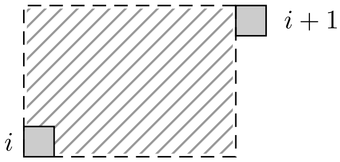


Figure 27 – Region in the extended diagram where we cannot have a SE corner.

Proof. Suppose that there is (p, q) for some i such that $p > p_{i+1}$. If $p_i > p > p_{i+1}$ then the position $p - 1$ is a descent of w, which is impossible since all descents of w are at positions $p_i - 1$ and no one matches to $p - 1$.

If $p_i = p$ and $q > q_i$ then the box $(q - 1, \overline{p_i})$ is a SE corner, and the dots in row $\overline{q}$ and column $\overline{p_i + 1}$ are placed as in Figure 28. The dot placed in row $\overline{q}$ lies inside the region $\Lambda(p_{i+1}, q_{i+1})$, and outside $\Lambda(p_i, q_i)$, implying that such dot is placed during Step $(i + 1)$.

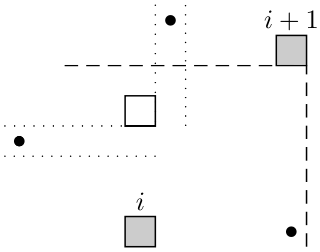


Figure 28 – Sketch for proof of Lemma 4.9.

Notice that the dot at column $\overline{p_i} + 1$ cannot be placed during Step $(i + 1)$ because it would create a descent in Step $(i + 1)$. Then, there is $j > i + 1$ such that Step (j) placed such dot. In this case, Step $(i + 1)$ should skip column $\overline{p_i} + 1$, which is impossible (by construction, this step places dots in all available columns between $w^{-1}(\overline{q}$ and $p_{i+1})$. □

The NE path also can contain another kind of SE corner defined as follows: given theta-vexillary signed permutation w and theta-triple $\boldsymbol{\tau}$, a corner position $(p, q) \notin \boldsymbol{\tau}$ is called *optional* if there are $a \leqslant i \leqslant s$ and $1 \leqslant j < a$ such that $p = p_i$, $q_i < q = \overline{q_j} + 1$ and $q_{i-1} \geqslant q > q_i$. In other words, (p, q) belongs to the NE path just between the corners (p_{i-1}, q_{i-1}) and (p_i, q_i), and the box $(q_i - 1, \overline{p_i})$ lays above and in the same column of $(q - 1, \overline{p})$, as shown in Figure 29. Denote by $\mathcal{O}p_{\boldsymbol{\tau}}(w)$ the set of all optional corners and observe that $\mathcal{O}p_{\boldsymbol{\tau}}(w) \subset \mathcal{N}e(w)$.

Observe that such box only occurs if the number of available rows between q_i and q is smaller than the number of dots to be placed by Step (i), which is $k_i - k_{i-1}$. In

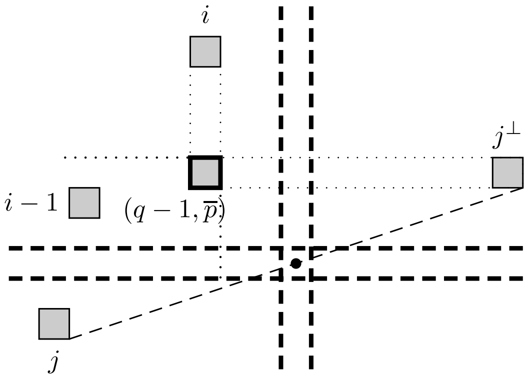


Figure 29 – Configuration of an optional corner (p, q).

other words, we need to have enough dots to place during Step (i) such that some of them are placed below the corner (p, q). This implies that the equation

$$q - q_i = k_i - k + k_j - k_{R(i)} \tag{4.3.2}$$

is satisfied. Thus, a triple $\boldsymbol{\tau}'$ obtained by adding (k, p, q) to $\boldsymbol{\tau}$ also gives the same permutation but it is not a theta-triple anymore.

Lemma 4.10. *Let w be a theta-vexillary and $\boldsymbol{\tau}$ be a theta-triple. Then, the set of corners is the disjoint union*

$$\mathscr{C}(w) = \boldsymbol{\tau} \ \dot\cup \ \mathscr{O}p_{\boldsymbol{\tau}}(w) \ \dot\cup \ \mathscr{U}(w).$$

Proof. Consider the diagram $D(\iota(w))$ divided in quadrants as in Figure 30. As we can see, a box $(a, b) \in D(\iota(w))$ belongs to

- Quadrant **A** if $a \geqslant 0$ and $b < 0$;
- Quadrant **B** if $a < 0$ and $b < 0$;
- Quadrant **C** if $a < 0$ and $b \geqslant 0$;
- Quadrant **D** if $a \geqslant 0$ and $b \geqslant 0$.

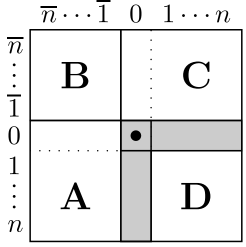


Figure 30 – Quadrants of the diagram.

Then, given $(p, q) \in \iota(\boldsymbol{\tau})$, we say that the SE corner $(q-1, \overline{p})$ belongs to:

- Quadrant **A** if $(p, q) = (p_i, q_i)$ for some $i < a$;
- Quadrant **B** if $(p, q) = (p_i, q_i)$ for some $i \geqslant a$;
- Quadrant **C** if $(p, q) = (p_i, q_i)^{\perp}$ for some $i < a$;
- Quadrant **D** if $(p, q) = (p_i, q_i)^{\perp}$ for some $i \geqslant a$.

Considering $T' = (p', q')$ and $T'' = (p'', q'')$ in $\iota(\boldsymbol{\tau})$ such that $p' > p''$ and $q' \neq q''$ (T' and T'' are in different rows and columns), we say that T' and T'' has a *cross descent box* (a, b) *of type*

- α if $q' > q''$ and $(a, b) = (q'' - 1, \overline{p'})$;
- β if $q' > q''$ and $(a, b) = (q' - 1, \overline{p''})$;
- γ if $q' < q''$ and $(a, b) = (q'' - 1, \overline{p'})$;
- δ if $q' < q''$ and $(a, b) = (q' - 1, \overline{p''})$.

Figure 31 shows how the cross descents boxes are arranged in the diagram for given T' and T''.

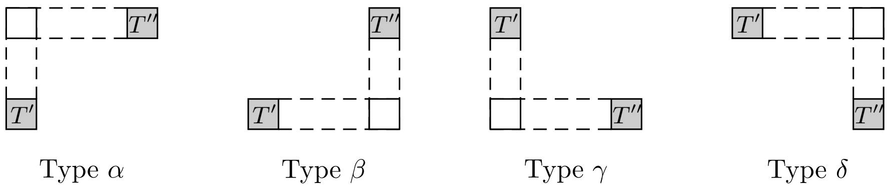


Figure 31 – Possible cross descents boxes.

Suppose that $T' = (p', q')$ and $T'' = (p'', q'')$ are two corners of $\iota(\tau)$. Consider that T' lays in some quadrant $\mathbf{X}$, T'' lays in some quadrant $\mathbf{Y}$, and cross descent box (a, b) has type ξ, where $\mathbf{X}, \mathbf{Y} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ and $\xi \in \{\alpha, \beta, \gamma, \delta\}$. We say that this configuration has *shape* $\mathbf{X}\xi\mathbf{Y}$. Also denote by $c_\xi(T', T'') = (a, b)$ the respective cross descent box. For instance, Figure 31 shows a shape $\mathbf{A}\alpha\mathbf{A}$.

First of all, we need to figure out all possible shapes and, then, verify if such shapes can create a SE corner from the cross descent box.

There are 64 different combination of shapes $\mathbf{X}\xi\mathbf{Y}$, where $\mathbf{X}, \mathbf{Y} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ and $\xi \in \{\alpha, \beta, \gamma, \delta\}$. However, not every shape is possible because $\boldsymbol{\tau}$ is a theta-triple and T', T'' are chosen in $\iota(\boldsymbol{\tau})$. An example of impossible shape is $\mathbf{A}\delta\mathbf{A}$ since, by definition, there is no $i < j$ where $T' = (p_i, q_i)$ and $T'' = (p_j, q_j)$ such that $q_i < q_j$. Thus, it remains only 24 possible shapes. We listed them in Table 1, divided in two categories: the shapes

Table 1 – Possible shapes

Shapes $\mathbf{X}\xi\mathbf{Y}$ where $c_\xi(T',T'')$ belongs to $\mathbf{A}$ or $\mathbf{B}$:	Shapes $\mathbf{X}\xi\mathbf{Y}$ where $c_\xi(T',T'')$ belongs to $\mathbf{C}$ or $\mathbf{D}$:
$\mathbf{A}\alpha\mathbf{A}$, $\mathbf{A}\alpha\mathbf{B}$, $\mathbf{A}\alpha\mathbf{C}$, $\mathbf{A}\alpha\mathbf{D}$, $\mathbf{B}\alpha\mathbf{B}$, $\mathbf{B}\alpha\mathbf{C}$, $\mathbf{A}\beta\mathbf{A}$, $\mathbf{A}\beta\mathbf{B}$, $\mathbf{B}\beta\mathbf{B}$, $\mathbf{A}\gamma\mathbf{D}$, $\mathbf{B}\gamma\mathbf{C}$, $\mathbf{B}\gamma\mathbf{D}$.	$\mathbf{C}\beta\mathbf{C}$, $\mathbf{D}\beta\mathbf{C}$, $\mathbf{A}\beta\mathbf{C}$, $\mathbf{B}\beta\mathbf{C}$, $\mathbf{D}\beta\mathbf{D}$, $\mathbf{A}\beta\mathbf{D}$, $\mathbf{C}\alpha\mathbf{C}$, $\mathbf{D}\alpha\mathbf{C}$, $\mathbf{D}\alpha\mathbf{D}$, $\mathbf{B}\delta\mathbf{C}$, $\mathbf{A}\delta\mathbf{D}$, $\mathbf{B}\delta\mathbf{D}$.

$\mathbf{X}\xi\mathbf{Y}$ where the cross descent box $c_\xi(T',T'')$ belongs to the quadrants $\mathbf{A}$ or $\mathbf{B}$, and the shapes $\mathbf{X}\xi\mathbf{Y}$ where $c_\xi(T',T'')$ belongs to the quadrants $\mathbf{C}$ of $\mathbf{D}$.

However, observe that if $c_\xi(T',T'')$ belongs to quadrants $\mathbf{C}$ or $\mathbf{D}$ then its reflection $c_\xi(T',T'')^\perp$ belongs to quadrant $\mathbf{A}$ or $\mathbf{B}$ and corresponds to the cross descent box of corners $(T'')^\perp = (\overline{p''}+1, \overline{q''}+1)$ and $(T')^\perp = (\overline{p'}+1, \overline{q'}+1)$. In other words, each shape in the left column of Table 1 is equivalent to another one to the right column. Hence, let us consider only the 12 shapes where $c_\xi(T',T'')$ belongs to quadrants $\mathbf{A}$ or $\mathbf{B}$.

It follows from Lemma 4.7 that $\boldsymbol{\tau} \cap \mathscr{U}(w) = \varnothing$ because no unessential corner is minimal in the poset. By definition of optional corner, we also have that $\boldsymbol{\tau} \cap \mathcal{O}p_{\boldsymbol{\tau}}(w) = \varnothing$ and $\mathcal{O}p_{\boldsymbol{\tau}}(w) \cap \mathscr{U}(w) = \varnothing$. Then, all the sets are disjoint.

Suppose that $T' = (p',q')$ and $T'' = (p'',q'')$ of $\iota(\tau)$ has some shape $\mathbf{X}\xi\mathbf{Y}$, where $\mathbf{X},\mathbf{Y} \in \{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$ and $\xi \in \{\alpha,\beta,\gamma,\delta\}$, such that the cross descent box $c_\xi(T',T'')$ is a SE corner in quadrant $\mathbf{A}$ or $\mathbf{B}$ which *does not* belongs to $\boldsymbol{\tau}$. Then, analyzing each situation in the first column of Table 1, we must show that either $c_\xi(T',T'') \in \mathcal{O}p_{\boldsymbol{\tau}}(w) \,\dot\cup\, \mathscr{U}(w)$ or it leads us to a contradiction.

Consider $\xi = \alpha$, where $p' > p''$, $q' > q''$, and $T = (p',q'')$ is a SE corner $(q''-1, \overline{p'})$ not in $\boldsymbol{\tau}$ and satisfying the following conditions

$$\begin{aligned} \iota(w)(p'-1) > \overline{q''} \geqslant \iota(w)(p') \\ \iota(w)^{-1}(\overline{q''}) > p'-1 \geqslant \iota(w)^{-1}(\overline{q''}+1). \end{aligned} \tag{4.3.3}$$

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{A}\alpha\mathbf{A}$, $\mathbf{A}\alpha\mathbf{B}$ or $\mathbf{B}\alpha\mathbf{B}$, then $T' = (p_i,q_i)$, $T'' = (p_j,q_j)$, where $1 \leqslant i < j \leqslant s$, and $T = (p_i,q_j)$ is a SE corner $(q_j-1, \overline{p_i})$. But Lemma 4.9 says that T cannot be a corner.

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{A}\alpha\mathbf{C}$ or $\mathbf{B}\alpha\mathbf{C}$, then $T' = (p_i,q_i)$, for some i, $T'' = (p_j,q_j)^\perp = (\overline{p_j}+1, \overline{q_j}+1)$, for some $j < a$, and $q_i > \overline{q_j}+1$. We can assume that i is chosen such that there is no $l > i$ satisfying $p_i = p_l$ and $q_i > q_l > \overline{q_j}+1$, i.e., there is no corner of $\boldsymbol{\tau}$ in the same column and between the SE corners T' and T. If $p_i > p_{i+1}$ then Lemma 4.9 is contradicted. Thus, we have that $p_i = p_{i+1}$ and $q_i > \overline{q_j}+1 > q_{i+1}$, implying that T is an optional SE corner.

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{A}\alpha\mathbf{D}$, then $T' = (p_i, q_i)$, for some $i < a$, $T'' = (p_j, q_j)^\perp = (\overline{p_j} + 1, \overline{q_j} + 1)$, for some $a \leqslant j \leqslant s$, and $q_i > \overline{q_j} + 1$. As in the previous case, we can assume that i is chosen such that there is no $l > i$ satisfying $p_i = p_l$ and $q_i > q_l > \overline{q_j} + 1$, i.e., there is no corner of $\boldsymbol{\tau}$ in the same column and between the SE corners T' and T. If $p_i > p_{i+1}$, then the corner T contradict Lemma 4.9. Thus, $p_i = p_{i+1}$, $q_i > \overline{q_j} + 1 > q_{i+1}$ and $i = R(j)$. Notice that the dot in the row $\overline{q_j} + 1$ is between rows q_i and $\overline{q_j}$ since T is a corner, which is impossible as shown in proof of Proposition 4.8 (see Figure 32).

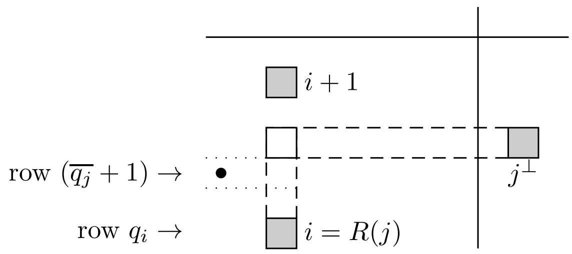


Figure 32 – Sketch for the shape $\mathbf{A}\alpha\mathbf{D}$.

Consider $\xi = \beta$, where $p' > p''$, $q' > q''$, and $T = (p'', q')$ is a SE corner $(q' - 1, \overline{p''})$ is a SE corner not in $\boldsymbol{\tau}$ and satisfying the following conditions

$$\begin{aligned} \iota(w)(p'' - 1) &> \overline{q'} \geqslant \iota(w)(p'') \\ \iota(w)^{-1}(\overline{q'}) &> p'' - 1 \geqslant \iota(w)^{-1}(\overline{q'} + 1). \end{aligned} \tag{4.3.4}$$

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{A}\beta\mathbf{A}$ or $\mathbf{A}\beta\mathbf{B}$, then $T' = (p_i, q_i)$ and $T'' = (p_j, q_j)$, for some $i < a$ and $i < j$. Observe that the dots at column $\overline{p_j}$ and row q_j are placed by Step (j) (or some previous one). Then, by construction, the dot at row $q_i - 1$ must be placed by some Step (l) for $l \leqslant j$. Thus, $\iota(w)^{-1}(q_i - 1) \geqslant \overline{p_j}$, contradicting Equation (4.3.4).

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{B}\beta\mathbf{B}$, then $T' = (p_i, q_i)$ and $T'' = (p_j, q_j)$, for some $a \leqslant i < j \leqslant s$. If $\iota(w)^{-1}(q_i - 1) < 0$, i.e., the dot in the row $q_i - 1$ is in quadrant $\mathbf{B}$ then we can proceed as the previous case. If $\iota(w)^{-1}(q_i - 1) > 0$ then belongs to the quadrant $\mathbf{C}$ is a reflection of a dot placed during some Step (l) for $l < a$. Since $\iota(w)(q_i) < \overline{p_j} + 1 \leqslant 0$, then $q_l = \overline{q_i} + 1$ and the corner (p_l, q_l) lays in row $\overline{q_i} + 1$. Therefore, the reflection $(p_l, q_l)^\perp$ is in the row $q_i - 1$, and the corner T is optional or unessential (see Figure 33).

Consider $\xi = \gamma$, where $p' > p''$, $q' < q''$, and $T = (p', q'')$ is a SE corner $(q'' - 1, \overline{p'})$ is a SE corner not in $\boldsymbol{\tau}$ and satisfying the following conditions

$$\begin{aligned} \iota(w)(p' - 1) &> \overline{q''} \geqslant \iota(w)(p') \\ \iota(w)^{-1}(\overline{q''}) &> p' - 1 \geqslant \iota(w)^{-1}(\overline{q''} + 1). \end{aligned} \tag{4.3.5}$$

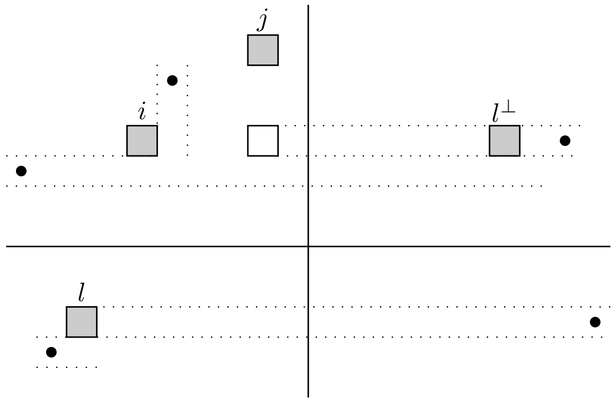


Figure 33 – Sketch for the shape $\mathbf{B}\beta\mathbf{B}$.

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{A}\gamma\mathbf{D}$ or $\mathbf{B}\gamma\mathbf{D}$, then $T' = (p_i, q_i)$, for any i, $T'' = (p_j, q_j)^\perp = (\overline{p_j} + 1, \overline{q_j} + 1)$, for some $a \leqslant j \leqslant s$, and $q_i < \overline{q_j} + 1$. By Equation (4.3.5), $q_j - 1 \geqslant \iota(w)(p_i)$ and $\iota(w)^{-1}(q_j - 1) > p_i - 1 \geqslant 0 > \iota(w)^{-1}(q_j)$, implying that no Step (l), for $l < a$, can place the dot at row $\overline{q_j}$. Hence, $q_{R(j)} = \overline{q_j} + 1$ and T is exactly the corner $(p_{R(j)}, q_{R(j)})$ of $\boldsymbol{\tau}$ (see Figure 34).

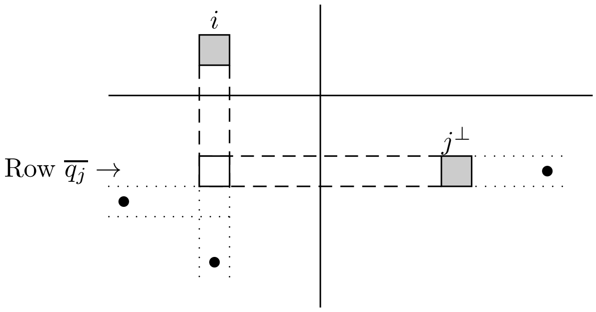


Figure 34 – Sketch for the shape $\mathbf{A}\gamma\mathbf{D}$ or $\mathbf{B}\gamma\mathbf{D}$.

- If $\mathbf{X}\xi\mathbf{Y}$ is a shape $\mathbf{B}\gamma\mathbf{C}$, then we clearly have that T is an unessential or optional corner. □

Proposition 4.11. *For $w \in \mathcal{W}_n$, w is theta-vexillary if and only if the set of corner $\mathscr{C}(w)$ is the disjoint union*

$$\mathscr{C}(w) = \mathscr{N}\!e(w) \dot\cup \mathscr{U}(w).$$

Proof. Suppose that $w(\boldsymbol{\tau})$ is theta-vexillary. From Lemma 4.7, $\boldsymbol{\tau} \cup \mathcal{O}p_{\boldsymbol{\tau}}(w) \subset \mathscr{N}\!e(w)$. On the other hand, Lemma 4.10 implies that $\mathscr{N}\!e(w) \subset \boldsymbol{\tau} \cup \mathcal{O}p_{\boldsymbol{\tau}}(w)$ since $\mathscr{N}\!e(w) \cap \mathscr{U}(w) = \varnothing$. □

Remark 4.12. If w is a theta-vexillary signed permutation but we don't know a theta-triple such that $w = w(\boldsymbol{\tau})$, we can use the process in the proof of Proposition 4.8 to get $\boldsymbol{\tau}$.

Basically, set $\boldsymbol{\tau}$ with all the corners in the NE path $\mathscr{N}e(w)$. Then, withdraw all the optional corners from it, which results in a valid theta-triple $\boldsymbol{\tau}$ of w.

Proposition 4.13. *The theta-triple is unique for each theta-vexillary signed permutation.*

Proof. Suppose that $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ are two different theta-triples such that $w = w(\boldsymbol{\tau}) = w(\tilde{\boldsymbol{\tau}})$. Then, $\boldsymbol{\tau}\dot{\cup}\mathcal{O}p_{\boldsymbol{\tau}}(w) = \mathscr{N}e(w) = \tilde{\boldsymbol{\tau}}\dot{\cup}\mathcal{O}p_{\tilde{\boldsymbol{\tau}}}$. If there is a corner position $(p, q_1) \in \mathcal{O}p_{\boldsymbol{\tau}} \cap \tilde{\boldsymbol{\tau}}$ then there is $q_2 > q_1$ such that $(p, q_2) \in \boldsymbol{\tau}$ is a corner position immediately above it. Notice that (p, q_2) does not belong to $\tilde{\boldsymbol{\tau}}$, otherwise condition B2 of $\tilde{\boldsymbol{\tau}}$ for both corners would contradict Equation (4.3.2) for the optional corner (p, q_1). Then, $(p, q_2) \in \mathcal{O}p_{\tilde{\boldsymbol{\tau}}} \cap \boldsymbol{\tau}$. For the same reason, there is $q_3 > q_2$ such that $(p, q_3) \in \mathcal{O}p_{\boldsymbol{\tau}} \cap \tilde{\boldsymbol{\tau}}$, and keep going. Hence, this process should be repeated forever, which is impossible since the sets are finite. Therefore, $\mathcal{O}p_{\boldsymbol{\tau}} \cap \tilde{\boldsymbol{\tau}} = \varnothing$, and by the same reason $\mathcal{O}p_{\tilde{\boldsymbol{\tau}}} \cap \boldsymbol{\tau} = \varnothing$, which implies that $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}}$. □

4.4 Pattern avoidance

Recall that given a signed pattern $\pi = \pi(1)\ \pi(2) \cdots \pi(m)$ in $\mathcal{W}_m$, a signed permutation w *contains* π if there is a subsequence $w(i_1) \cdots w(i_m)$ such that the signs of $w(i_j)$ and $\pi(j)$ are the same for all j, and also the absolute values of the subsequence are in the same relative order as the absolute values of π. Otherwise w *avoids* π.

Proposition 4.14. *A signed permutation w is theta-vexillary if and only if w avoids the follow thirteen signed patterns* $[\overline{1}\ 3\ 2]$, $[\overline{2}\ 3\ 1]$, $[\overline{3}\ 2\ 1]$, $[\overline{3}\ 2\ \overline{1}]$, $[2\ 1\ 4\ 3]$, $[2\ \overline{3}\ 4\ \overline{1}]$, $[\overline{2}\ \overline{3}\ 4\ \overline{1}]$, $[3\ \overline{4}\ 1\ \overline{2}]$, $[3\ \overline{4}\ \overline{1}\ \overline{2}]$, $[\overline{3}\ \overline{4}\ 1\ \overline{2}]$, $[\overline{3}\ \overline{4}\ \overline{1}\ \overline{2}]$, $[\overline{4}\ 1\ \overline{2}\ 3]$, *and* $[\overline{4}\ \overline{1}\ \overline{2}\ 3]$.

Proof. We know, by Proposition 4.8, how to describe a theta-vexillary permutation by the SE corners of the extended diagram.

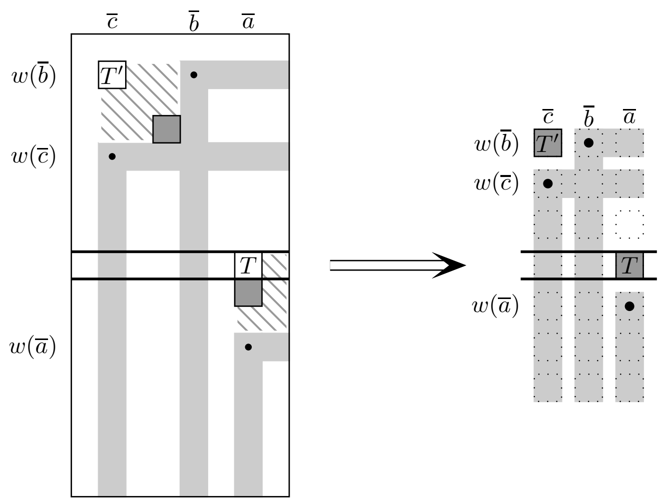


Figure 35 – Suppose that w contains $[\overline{1}\ 3\ 2]$. We can restrict the diagram of w to the pattern $[\overline{1}\ 3\ 2]$.

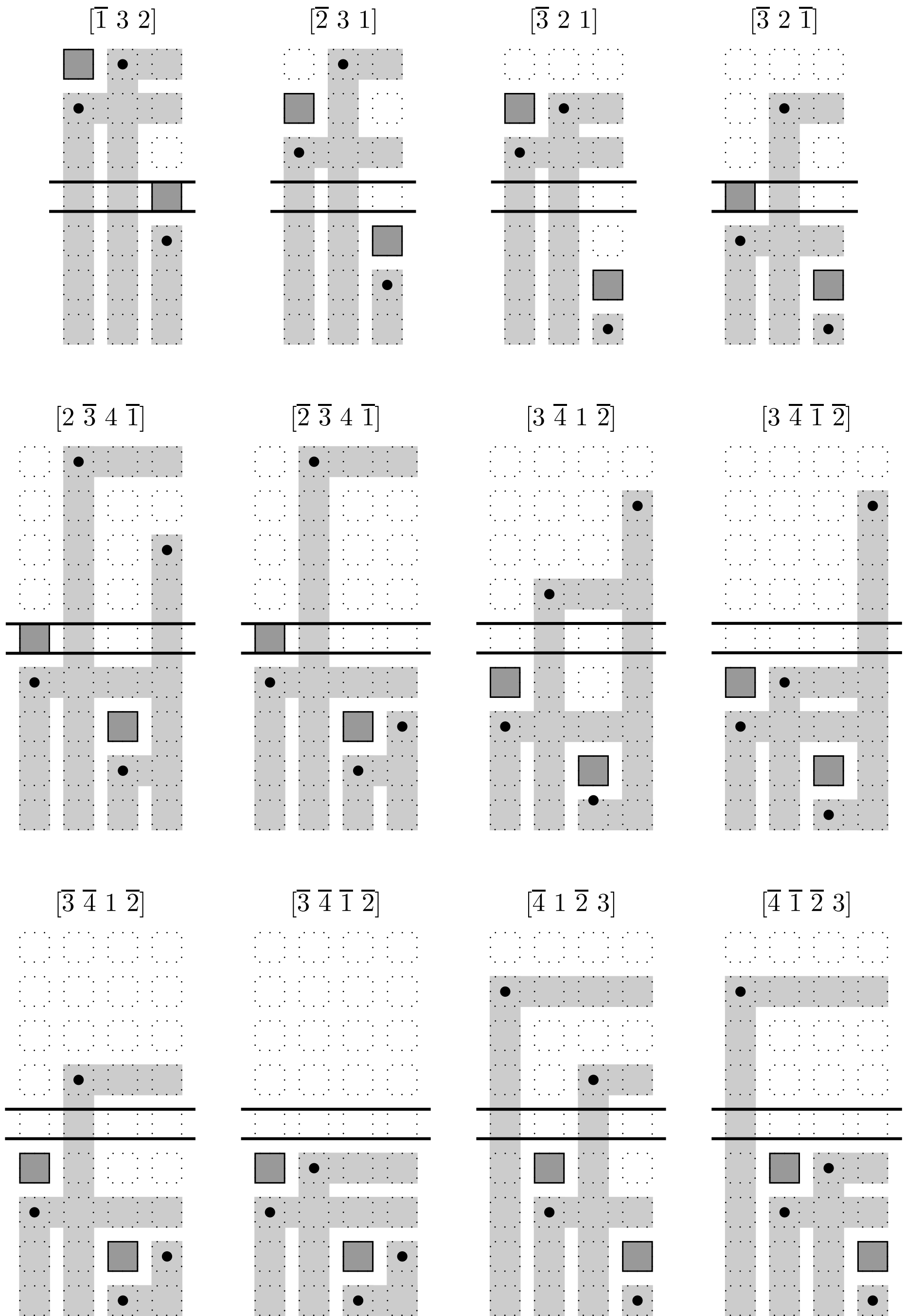


Figure 36 – Diagram of w restricted to 12 different patterns.

Suppose that w is a theta-vexillary signed permutation. To prove that it avoids all these 13 patterns, we will assume if one of these patterns is contained in w, then show that there is a SE corner T such that $T \notin \mathscr{Ne}(w) \cup \mathscr{U}(w)$.

Assume that w contains $[\overline{1}\ 3\ 2]$ as a subsequence $(w(a)\ w(b)\ w(c))$ satisfying $\overline{w(a)} < w(c) < w(b)$ for some $a < b < c$. Figure 35 shows the diagram of w and its restriction to the columns $\overline{a}, \overline{b}, \overline{c}$ and rows $0, \pm w(\overline{a}), \pm w(\overline{b}), \pm w(\overline{c})$. Notice that the box T (resp. T') is not necessarily a SE corner in the diagram of w, but certainly there is a SE corner in the shaded area. So, we can assume that T and T' are SE corner when restricting it. Clearly, T is neither an unessential corner nor minimal under the poset.

We can use the same idea to prove that other 11 patterns should be avoided, as shown in Figure 36. Only the pattern [2 1 4 3] requires more arguments to show that it also is avoided.

Assume that w contains [2 1 4 3] as a subsequence $(w(a)\ w(b)\ w(c)\ w(d))$ satisfying $w(b) < w(a) < w(d) < w(c)$ for some $a < b < c < d$ (see Figure 37). Suppose that T and T' are SE corner in the respective shaded area. Clearly $T \notin \mathscr{N}e(w)$, but it could be an unessential corner. If $\boldsymbol{\tau}$ is a theta-triple of w then notice that there are i and j such that Step (i) and (j) place a dot in the column $\bar{b}$ and $\bar{a}$, respectively. It follows from the step-by-step construction that $a < i < j$, however it also lead us to a contradiction because we cannot place a dot in the row $w(\bar{b})$ during Step (i) and skip the row $w(\bar{a})$ since it will be place further.

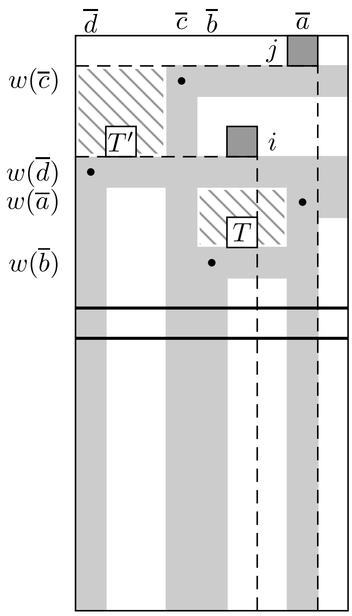


Figure 37 – Suppose that w contains [2 1 4 3].

Now, let us assume that w is permutation that avoids all the thirteen patterns listed above. We are going to prove that $\mathscr{C}(w) = \mathscr{N}e(w) \cup \mathscr{U}(w)$, and hence, w is a theta-vexillary permutation.

Suppose that there are corners $T = (p, q)$ and $T' = (p', q')$ such that $q > 0 > q'$ and $p' > p > 0$, i.e., T is in quadrant **A**, T' is in quadrant **B**, and $T' < T$. If we denote $a := p$, $b := p' - 1$ and $c := w^{-1}(\overline{q'})$, then they satisfy $0 < a < b < c$ and $w(a) < 0 < w(c) < w(b)$. Observe that $\bar{a}, \bar{b}, \bar{c}$ are the columns of the dots in Figure 38.

In order to relate the subsequence $(w(a)\ w(b)\ w(c))$ of w to some 3-pattern π, we need describe all possible orderings of $\overline{w(a)}, w(b)$ and $w(c)$.

- If $\overline{w(a)} < w(c) < w(b)$ then $\pi = [\bar{1}\ 3\ 2]$;

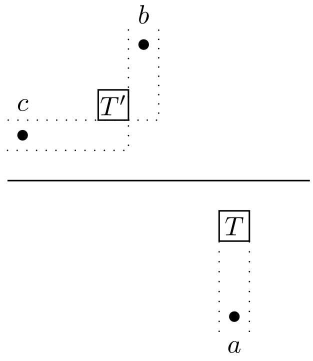


Figure 38 – Sketch for the case where T is in quadrant **A**, T' is in quadrant **B**, and $T' < T$.

- If $w(c) < \overline{w(a)} < w(b)$ then $\pi = [\overline{2}\ 3\ 1]$;
- If $w(c) < w(b) < \overline{w(a)}$ then $\pi = [\overline{3}\ 2\ 1]$.

By hypothesis, the pattern in each case should be avoid. Hence the configuration in Figure 38 is impossible.

Now, suppose that there are corners $T = (p, q)$ and $T' = (p', q')$ such that $q > q' > 0$ and $p' > p > 0$, i.e., both T and T' are in quadrant **A** and $T' < T$. Denote $i := w^{-1}(\overline{q} + 1)$, $a = p$, $b = p' - 1$ and $c = w^{-1}(\overline{q'})$.

If $i > 0$, then they satisfy $0 < i < a < b < c$ and $w(a) < w(i) < w(c) < w(b)$. Observe that $\bar{\imath}, \bar{a}, \bar{b}, \bar{c}$ are the columns of the dots in Figure 39 (left).

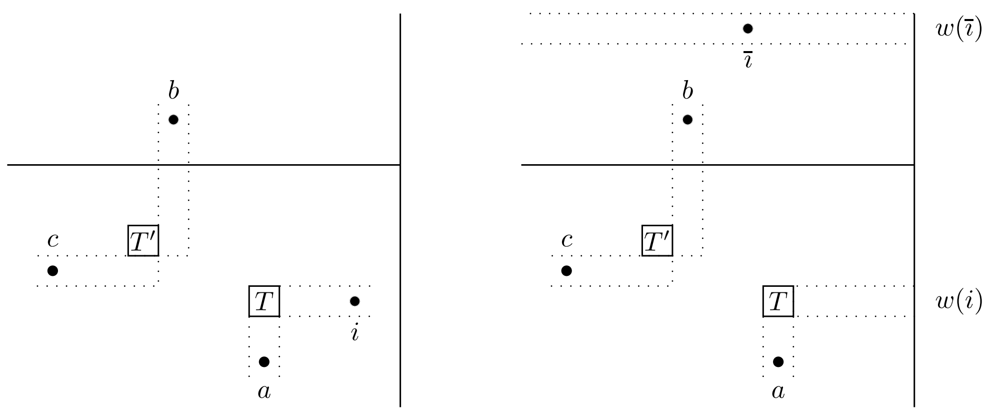


Figure 39 – Sketch for the case where both T and T' are in quadrant **A**, and $T' < T$.

In order to relate the subsequence $(w(i)\ w(a)\ w(b)\ w(c))$ of w to some 4-pattern π, we need to describe all possible orderings of $\overline{w(i)}, \overline{w(a)}, \overline{w(c)}$ and $\pm w(b)$.

- If $\overline{w(b)} < \overline{w(c)} < \overline{w(i)} < \overline{w(a)}$ then $\pi = [\overline{3}\ \overline{4}\ \overline{1}\ \overline{2}]$;
- If $w(b) < \overline{w(c)} < \overline{w(i)} < \overline{w(a)}$ then $\pi = [\overline{3}\ \overline{4}\ 1\ \overline{2}]$;
- If $\overline{w(c)} < w(b) < \overline{w(i)} < \overline{w(a)}$ then $\pi = [\overline{3}\ \overline{\mathbf{4}}\ \mathbf{2}\ \overline{\mathbf{1}}]$;

- If $\overline{w(c)} < \overline{w(i)} < w(b) < \overline{w(a)}$ then $\pi = [\bar{2}\ \bar{\mathbf{4}}\ \mathbf{3}\ \bar{\mathbf{1}}]$;
- If $\overline{w(c)} < \overline{w(i)} < \overline{w(a)} < w(b)$ then $\pi = [\bar{2}\ \bar{3}\ 4\ \bar{1}]$.

By hypothesis, the pattern in each case should be avoided (in some cases, the highlighted parts are avoid by $[\bar{3}\ 2\ \bar{1}]$). Hence this configuration is impossible.

If $i < 0$ then we have four possibilities to place $\bar{\imath} > 0$ in the sequence $0 < a < b < c$. Observe that $i, \bar{a}, \bar{b}, \bar{c}$ are the columns of the dots in Figure 39 (right). Table 2 combines all this possibilities along with all possible orderings of $w(\bar{\imath}), \overline{w(a)}, \overline{w(c)}$ and $\pm w(b)$, in order to get the respective 4-pattern π relative to the correspondent subsequence of w.

Table 2 – Combinations to get the respective 4-pattern of the subsequence of w.

	$\bar{\imath}<a<b<c$	$a<\bar{\imath}<b<c$	$a<b<\bar{\imath}<c$	$a<b<c<\bar{\imath}$
$\overline{w(b)}<\overline{w(c)}<w(\bar{\imath})<\overline{w(a)}$	$[3\ \bar{4}\ \bar{1}\ \bar{2}]$	$[\bar{\mathbf{4}}\ \mathbf{3}\ \bar{\mathbf{1}}\ \bar{2}]$	$[\bar{\mathbf{4}}\ \bar{1}\ \mathbf{3}\ \bar{\mathbf{2}}]$	$[\bar{4}\ \bar{1}\ \bar{2}\ 3]$
$w(b)<\overline{w(c)}<w(\bar{\imath})<\overline{w(a)}$	$[3\ \bar{4}\ 1\ \bar{2}]$	$[\bar{\mathbf{4}}\ \mathbf{3}\ 1\ \bar{\mathbf{2}}]$	$[\bar{\mathbf{4}}\ 1\ \mathbf{3}\ \bar{\mathbf{2}}]$	$[\bar{4}\ 1\ \bar{2}\ 3]$
$\overline{w(c)}<w(b)<w(\bar{\imath})<\overline{w(a)}$	$[3\ \bar{\mathbf{4}}\ \mathbf{2}\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{4}}\ \mathbf{3}\ 2\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{4}}\ \mathbf{2}\ 3\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{4}}\ \mathbf{2}\ \bar{\mathbf{1}}\ 3]$
$\overline{w(c)}<w(\bar{\imath})<w(b)<\overline{w(a)}$	$[2\ \bar{\mathbf{4}}\ \mathbf{3}\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{4}}\ \mathbf{2}\ 3\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{4}}\ \mathbf{3}\ 2\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{4}}\ \mathbf{3}\ \bar{\mathbf{1}}\ 2]$
$\overline{w(c)}<w(\bar{\imath})<\overline{w(a)}<w(b)$	$[2\ \bar{3}\ 4\ \bar{1}]$	$[\bar{\mathbf{3}}\ \mathbf{2}\ 4\ \bar{\mathbf{1}}]$	$[3\ \bar{\mathbf{4}}\ \mathbf{2}\ \bar{\mathbf{1}}]$	$[\bar{\mathbf{3}}\ \mathbf{4}\ \bar{1}\ \mathbf{2}]$

By hypothesis, the pattern in each case should be avoid and, hence, this case is impossible.

Finally, suppose that there are corners $T = (p, q)$ and $T' = (p', q')$ such that $0 > q > q'$ and $p' > p > 0$, i.e., both T and T' are in quadrant **B**, and $T' < T$. If we denote $i = w^{-1}(\bar{q} + 1)$, $a = p - 1$, $b = p$, $c = p' - 1$ and $d = w^{-1}(\overline{q'})$, then they satisfy $i < a < b < c < d$, $w(b) < w(a)$ and $w(b) < w(i) < w(d) < w(c)$. Observe that $\bar{\imath}, \bar{a}, \bar{b}, \bar{c}$ are the columns of the dots in Figure 40 (left).

Consider the following situations:

- If $w(b) < 0$ then the subsequence $(w(b)\ w(c)\ w(d))$ of w is a 3-pattern π equal to $[\bar{1}\ 3\ 2]$, $[\bar{2}\ 3\ 1]$ or $[\bar{3}\ 2\ 1]$, which is impossible;
- If $0 < w(b) < w(a) < w(d) < w(c)$ then the subsequence $(w(a)\ w(b)\ w(c)\ w(d))$ is a 4-pattern $\pi = [2\ 1\ 4\ 3]$ and also should be avoided;
- If $i > 0$ and $w(b) > 0$ then the subsequence $(w(i)\ w(b)\ w(c)\ w(d))$ is a 4-pattern $\pi = [2\ 1\ 4\ 3]$ and also should be avoided;
- If $0 > i > \bar{c}$ and $w(b) > 0$ then the subsequence $(w(\bar{\imath})\ w(c)\ w(d))$ of w is a 3-pattern π equal to $[\bar{1}\ 3\ 2]$, $[\bar{2}\ 3\ 1]$ or $[\bar{3}\ 2\ 1]$, which is impossible;

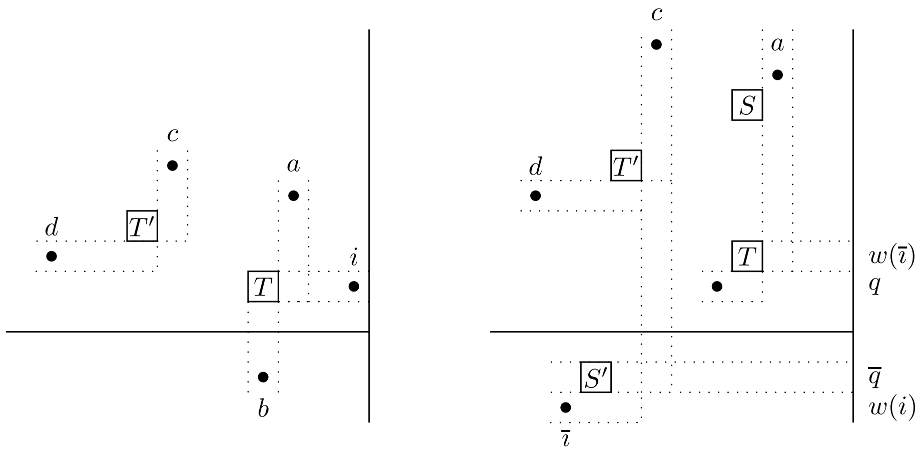


Figure 40 – Sketch for the case where both T and T' are in quadrant $\mathbf{B}$, and $T' < T$.

- If $i < \bar{c}$ and $0 < w(b) < w(d) < w(a)$ then clearly there are SE corners S and S' as in Figure 40 (right). Therefore, such construction implies that T is an unessential box. □

Therefore, Propositions 4.8, 4.11, and 4.14 prove the main theorem.

Theorem 4.15. *Let w be a signed permutation. The following are equivalent:*

1. *w is theta-vexillary, i.e., there is a triple $\boldsymbol{\tau}$ such that $w = w(\boldsymbol{\tau})$;*

2. *the set of corner $\mathscr{C}(w)$ is the disjoint union*

$$\mathscr{C}(w) = \mathscr{N}e(w)\dot{\cup}\mathscr{U}(w),$$

3. *w avoids the follow thirteen signed patterns* $[\bar{1}\ 3\ 2]$, $[\bar{2}\ 3\ 1]$, $[\bar{3}\ 2\ 1]$, $[\bar{3}\ 2\ \bar{1}]$, $[2\ 1\ 4\ 3]$, $[2\ \bar{3}\ 4\ \bar{1}]$, $[\bar{2}\ \bar{3}\ 4\ \bar{1}]$, $[3\ \bar{4}\ 1\ \bar{2}]$, $[3\ \bar{4}\ \bar{1}\ \bar{2}]$, $[\bar{3}\ \bar{4}\ 1\ \bar{2}]$, $[\bar{3}\ \bar{4}\ \bar{1}\ \bar{2}]$, $[\bar{4}\ 1\ \bar{2}\ 3]$, *and* $[\bar{4}\ \bar{1}\ \bar{2}\ 3]$.

4.5 Particular case: Grassmannian permutations

Recall that all Schubert varieties of a odd orthogonal Grassmannian $\mathrm{OG}(n-k, 2n+1)$ are parametrized by elements in set $\mathcal{W}^{(k)}$ of minimal representatives of the coset of $\mathcal{W}_{(k)}$ in the Weyl group $\mathcal{W}_n$. This permutations can be denoted, by Equation (1.1.1), as follows

$$w = w_{u,\lambda} = (u_k, \ldots, u_1, \bar{\lambda}_1, \ldots, \bar{\lambda}_r, v_{n-k-r}, \ldots, v_1)$$

where $r \leqslant n-k$, $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$, $u = (u_1 > u_2 > \cdots > u_k > 0)$, and $v = (v_1 > v_{2>} \cdots > v_{n-k-r} > 0)$.

Observe that each $w \in \mathcal{W}^{(k)}$ avoids all the thirteen signed patterns in Theorem 4.15 by construction of w. Hence, every w in $\mathcal{W}^{(k)}$ is a theta-vexillary signed permutation.

The unique descent of $w \in \mathcal{W}^{(k)}$ is in position k since $w(k) = u_1 > \overline{\lambda_1} = w(k+1)$, and $w(i) < w(i+1)$ for $i \neq k$. If $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ is the theta-triple of w then, by Proposition 4.3, all decents of w are in position $p_i - 1$, so we conclude that $p_1 = p_2 = \cdots = p_s = k + 1$. Moreover, all the SE corners $\mathscr{C}(w)$ of the extended diagram of w lie in the column $\overline{k+1}$.

For instance, consider the permutation $w = (2, 5, 6, \overline{8}, \overline{7}, \overline{4}, \overline{1}, 3)$ given by the double partition of Figure 2. The extended diagram of this permutation is given in Figure 41.

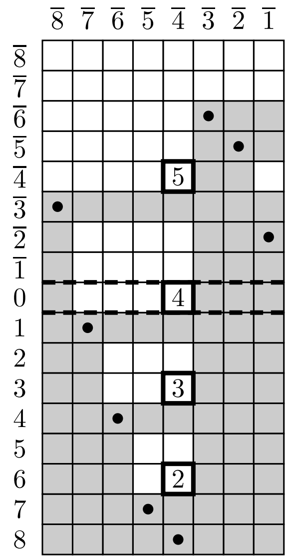


Figure 41 – Extended diagram of corners of a Grassmannian permutation.

As expected, all SE corners lies in column $\overline{4}$ (remember that, by definition, the white boxes in the top half of column $\overline{1}$ are not considered SE corners). Furthermore, there is neither unessential nor optional corners. Hence, we can obtain the theta-triple of w as in the proof of Proposition 4.8, where $\boldsymbol{\tau}$ is given by the corners in the NE path, i.e., $\boldsymbol{\tau} = (2\ 3\ 4\ 5, 4\ 4\ 4\ 4, 7\ 4\ 1\ \overline{3})$.

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Appendix

APPENDIX A

Degeneraci loci and Chern class formulas

In this appendix, we give the geometric construction that explains the vexillary permutations (signed or not) and the theta-vexillary signed permutations. We recommend the reading of this appendix after a complete understanding of definition of each type of permutation.

This chapter is a compilation of Anderson and Fulton [1, 2, 3, 4], Fulton [20], and Manivel [32].

A.1 Degeneraci loci of type A

Let X be an arbitrary nonsingular complex variety (or scheme). Consider $h : E \to \widetilde{F}$ a map of vector bundles over the variety X, and

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_n = E \xrightarrow{h} \widetilde{F} = \widetilde{F}_n \twoheadrightarrow \widetilde{F}_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{F}_1,$$

are flags of subbundles, where each E_i and $\widetilde{F}_i$ are vector bundles over X of rank i. Notice that for any $1 \leqslant p, q \leqslant n$, there is an induced map $E_p \to \widetilde{F}_q$. The *degeneracy locus* corresponding to a permutation $w \in S_n$ is a subvariety $\Omega_w \subseteq X$ defined by

$$\Omega_w = \{x \in X \mid \operatorname{rank}(E_p(x) \to \widetilde{F}_q(x)) \leqslant p - r_w^A(q,p), \text{ for every } 1 \leqslant p, q \leqslant n\}. \quad \text{(A.1.1)}$$

Notice that the degeneracy loci depend on h, $E_\bullet$, and $\widetilde{F}_\bullet$. If we consider $\tilde{r}_w(p,q) = p - r_w^A(q,p)$ as given in Remark 3.1, the conditions are written as $\operatorname{rank}(E_p(x) \to F_q(x)) \leqslant \tilde{r}_w(q,p)$.

If the map $h : E \to \widetilde{F}$ is sufficiently generic, the degeneracy locus Ω_v is a subvariety of X of codimension $\ell(v)$. In other word, the sufficiently generic condition means that Ω_v is a subvariety of X of codimension $\ell(v)$ for "almost all" maps $h : E \to \widetilde{F}$.

Now, we are going to consider a particular construction. Let V be a vector bundle on X of rank n, and two flags of vector bundles over X,

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset V,$$
$$F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset V$$

where E_i has rank i and F_i has rank $n-i$. In this case, set $\widetilde{F}_i := V/F_i$ with rank i, and observe that

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset V \twoheadrightarrow \widetilde{F}_n \twoheadrightarrow \widetilde{F}_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{F}_1.$$

Let us rewrite the conditions for the degeneracy locus Ω_w. If $f_{p,q,x} : E_p(x) \to \widetilde{F}_q(x) = V/F_q(x)$ is the induced map, then it follows, by the rank-nullity theorem, that

$$\operatorname{rank}(f_{p,q,x}) = \dim E_p(x) - \dim \operatorname{Ker}(f_{p,q,x}) = p - \dim(E_p(x) \cap F_q(x)).$$

Hence, the degeneracy locus of w is

$$\Omega_w = \{x \in X \mid \dim(E_p(x) \cap F_q(x)) \geqslant r_w^A(q,p), \text{ for every } 1 \leqslant p, q \leqslant n\}. \tag{A.1.2}$$

For our purposes, it is enough to consider the degeneracy locus for this particular situation.

The Schubert varieties are the special cases for X being a complex flag manifold $\mathbb{F}_\Theta = \mathrm{Sl}(n,\mathbb{C})/P_\Theta$. Let $V = \mathbb{C}^n$ be the complex vector of dimension n. It is known that a flag manifold $\mathbb{F}_\Theta$ is homeomorphic to the set $F\ell(V) = F\ell_{i_1,i_2,\cdots,i_r}(V)$ of all flags

$$L_\bullet = (L_{i_1} \subset L_{i_2} \subset \cdots L_{i_r} \subset V)$$

where $\dim L_{i_j} = i_j$ for any j. In order to simplify our notation, suppose that $F\ell(V)$ is the complete flag manifold where any element of the flag manifold is $L_\bullet = (L_1 \subset L_2 \subset \cdots \subset L_n = V)$.

Fix a complete flag $V_\bullet$ of vector spaces in V. For each w in S_n, we are going to construct the degeneracy locus Ω_w in the variety $X = F\ell(V)$ associated to some choice of vector bundles $E_1 \subset \cdots \subset E_n$ and $F_n \subset \cdots \subset F_1$.

Let us define $E_1, \ldots, E_n$ as follows: the vector bundle E_i on the flag manifold $F\ell(V)$ consists of the vector bundle whose total space is

$$T_i = \{(L_\bullet, v) \in F\ell(V) \times V \mid v \in L_i\}$$

so that it projects the pair $(L_\bullet, u)$ on $L_\bullet$. Notice that the fiber on $L_\bullet$ is $E_i(L_\bullet) = L_i$. This also can be constructed using the tautological bundles over $\mathbb{CP}(V)$. Define F_i as the trivial bundle of V_{n-i} on $F\ell(V)$, where, for every $L_\bullet$, the fiber is $F_i(L_\bullet) = V_{n-i}$. Clearly, we have that $F_n \subset \cdots \subset F_1 \subset V$ and $\operatorname{rank}(\widetilde{F}_i) = \operatorname{rank}(V/F_i) = i$. Hence, the degeneracy locus for a permutation w in S_n is as in Equation (A.1.2) becomes

$$\Omega_w(V_\bullet) = \{L_\bullet \in F\ell(V) \mid \dim(L_p \cap V_{n-q}) \geqslant r_w^A(q,p), \text{ for every } 1 \leqslant p, q \leqslant n\}.$$

Recall from Chapter 1 that we defined a Schubert variety by $\mathcal{S}_w = \mathrm{cl}(N \cdot wb_\Theta)$, and it has dimension $\ell(w)$. Instead of defining the Schubert variety as a subset of the homogeneous space G/P_Θ, it can be written as a subset of $F\ell(V)$ as follows

$$\mathcal{S}_w = \mathcal{S}_w(V_\bullet) = \{L_\bullet \in F\ell(V) \mid \dim(L_p \cap V_q) \geqslant p - r_w^A(q,p), \text{ for every } 1 \leqslant p, q \leqslant n\}.$$

Observe that the choice of $V_\bullet$ is associated to the point $b_\Theta = 1 \cdot P_\Theta$ in the flag $\mathbb{F}_\Theta$. Clearly, $\mathcal{S}_w(V_\bullet)$ and $\Omega_w(V_\bullet)$ are not equal since they have, respectively, dimension and codimension $\ell(w)$. However, we can find a relation between them.

Let $w_\circ \in S_n$ be the involution, i.e., $w_\circ(i) = n - i + 1$. For all p, q, the condition $\dim(L_p \cap V_{n-q}) \geqslant r_w^A(q,p)$ is equivalent to

$$\begin{aligned}
\dim(L_p \cap V_q) &\geqslant r_w^A(n-q,p) \\
&= \#\{i \leqslant p \mid w(i) > n-q\} \\
&= \#\{i \leqslant p \mid w_\circ w(i) \leqslant q\} \\
&= p - \#\{i \leqslant p \mid w_\circ w(i) > q\} \\
&= p - r_{w_\circ \cdot w}^A(q,p).
\end{aligned}$$

Therefore, we have that

$$\Omega_w = \mathcal{S}_{w_\circ \cdot w}.$$

A.1.1 Chern class formula of vexillary permutations

Observe that the definition of the degeneracy locus Ω_w on a variety X takes into account every (q,p) in the diagram $D(w)$, i.e., there are n^2 conditions to satisfy. But, there exists a subset of these n^2 conditions which defines the same locus more efficiently, given by the essential set of w, as if follows:

Proposition A.1 ([20] Proposition 4.2)**.** *The degeneracy locus Ω_w is also defined by*

$$\Omega_w = \{x \in X \mid \dim(E_p(x) \cap F_q(x)) \geqslant r_w^A(q,p), \text{ for every } (q,p) \in \mathscr{E}ss(w)\}.$$

This proposition tell us that we require only the essential set of a permutation to determine its degeneracy locus. Thus, if w is a vexillary permutation then we can use Theorem 3.2 and define the degeneracy locus of w by its associated triple of type A $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$. Given two flags vector bundles

$$\begin{aligned}
E_{p_1} \subset E_{p_2} \subset \cdots \subset E_{p_s} \subset V, \\
F_{q_1} \subset F_{q_2} \subset \cdots \subset F_{q_s} \subset V,
\end{aligned}$$

the degeneracy locus of the vexillary permutation $\boldsymbol{\tau}$ is

$$\Omega_{\boldsymbol{\tau}} = \Omega_w = \{x \in X \mid \dim(E_{p_i}(x) \cap F_{q_i}(x)) \geqslant k_i, \text{ for every } 1 \leqslant i \leqslant s\}.$$

Now, consider the integral cohomology ring $H^\bullet(X) = H^\bullet(X, \mathbb{Z})$. The cohomology class $[\Omega_w]$ belongs to the cohomology ring $H^{2\ell(w)}(X)$. In fact, since X is a nonsingular complex variety of dimension m, it is an oriented real $2m$-manifold, and the group $H_{2m}(X)$ has a canonical generator $[X]$; then, the Poincaré duality map $H^i(X) \to H_{2m-i}(X)$ is an isomorphism. The (closed) subvariety Ω_w determines a class denoted $[\Omega_w]$ in $H_{2d}(X)$, where d is the dimension of Ω_w. By Poincaré duality, we have $[\Omega_w] \in H_{2d}(X) = H^{2\ell(w)}(X)$ since $\ell(w)$ is the codimension of Ω_w in X.

It is reasonable to ask if there is any formula that describes the cohomology class $[\Omega_w]$. Indeed, the class $[\Omega_w]$ is a polynomial that depends on the total Chern classes of E_{p_i} and F_{q_i}.

Recall that, given any vector bundle E on a variety X of rank r, cohomology classes $c_j(V) = H^{2j}(X)$, for $j = 0, 1, 2, \ldots$, are called Chern classes for the bundle V if they are invariant under vector bundle isomorphisms and satisfy certain axioms. The total Chern class of E is the sum $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$.

Back to the vexillary permutations, denote

$$c(k_i) := c(F_{q_i} - E_{p_i}) = c(F_{q_i})/c(E_{p_i}) \quad , \text{ for every } 1 \leqslant i \leqslant s, \text{ and}$$
$$c(k) := c(k_i) \quad , \text{ whenever } k_{i-1} < k \leqslant k_i.$$

where, by convention, we set $k_0 = 0$. Since $H^\bullet(X)$ is a graded ring, we can decompose $c(k)$, for every k, as follows

$$c(k) = 1 + c_1(k) + c_2(k) + \cdots ,$$

where $c_i(k) \in H^{2i}(X)$. The next theorem gives the explicit formula for class $[\Omega_w]$ for a vexillary permutation w.

Theorem A.2 ([20] Proposition 9.6)**.** *Let $w \in S_n$ be a vexillary permutation such that $\boldsymbol{\tau}$ is the respective triple of type A and $\lambda(\boldsymbol{\tau}) = (\lambda_1, \ldots, \lambda_r)$ is the partition as defined above, where $r = k_s$. Then we have*

$$[\Omega_{\boldsymbol{\tau}}] = \Delta_{\lambda(\boldsymbol{\tau})}(c(1), \ldots, c(r)) := \det(c_{\lambda_i + j - i}(i))_{1 \leqslant i,j \leqslant r}.$$

The polynomial given by this determinant is also known as *Schur determinant.* The polynomial Δ_λ has a variation where it is described as a raising operator as follows: given $i < j$, we define $R_{ij}(\lambda)$ the function that adds 1 to λ_i, and subtracts 1 from λ_j, i.e.,

$$R_{ij}(\lambda) = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_r).$$

A raising operator R is any monomial in these R_{ij}'s. Set $m_\lambda = \prod_i c_{\lambda_i}(i)$ and $R \cdot m_\lambda = m_{R\lambda}$ where R acts on the monomial m_λ. If $r > 0$ is any integer and $R^{(r)}$ is the raising operator defined by

$$R^{(r)} = \prod_{1 \leqslant i < j \leqslant r} (1 - R_{ij})$$

then an application of the Vandermonde identity shows that the cohomology class $[\Omega_\tau]$ is given by

$$[\Omega_\tau] = \Delta_\lambda(c(1), \ldots, c(r)) = R^{(r)} \cdot m_\lambda = R^{(r)} \cdot (c_{\lambda_1}(1) \cdots c_{\lambda_r}(r)).$$

Such description of $[\Omega_\tau]$ in terms of the raising operator $R^{(r)}$ will inspire an equivalent result for the theta-vexillary signed permutation, where the raising operator will be slighted modified.

For the general case where $w \in S_n$ is any permutation we consider the double Schubert polynomial, which can be found in [20, 32].

A.2 Degeneraci loci of type B

Let V be a vector bundle of rank $2n+1$ on a variety X, equipped with a nondegenerate quadratic form. Consider two flags of vector isotropic subbundles on X,

$$E_n \subset E_{n-1} \subset \cdots \subset E_1 \subset V,$$
$$F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset V$$

where E_i and F_i have rank $n+1-i$. We can extend F_i to complete the flag in V, by setting $F_{\overline{q}} := F_{q+1}^{\perp}$ when $q > 0$. Then,

$$F_n \subset \cdots \subset F_1 \subset F_0 \subset F_{\overline{1}} \subset \cdots \subset F_{\overline{n}} = V$$

We can define the degeneracy locus $\Omega_w \subset X$ of a signed permutation $w \in \mathcal{W}_n$, as in Equation (A.1.2), by

$$\Omega_w = \{x \in X \mid \dim(E_p(x) \cap F_q(x)) \geqslant r_w(p,q), \text{ for every } 1 \leqslant p \leqslant n \text{ and } \overline{n} \leqslant q \leqslant n\}.$$

As for S_n, we also have an interpretation of Schubert varieties as degeneracy loci for a signed permutation in $\mathcal{W}_n$. We will give a brief explaination of such relation.

The Schubert varieties are the special cases for X being any complex flag manifold $\mathbb{F}_\Theta = \mathrm{SO}(2n+1, \mathbb{C})/P_\Theta$. Indeed, let $V = \mathbb{C}^{2n+1}$ be the complex vector space with nondegenerate quadratic form. Fix a complete isotropic flag $V_\bullet = (0 \subset V_n \subset V_{n-1} \subset \cdots \subset$

$V_1 \subset V$) of isotropic vector spaces in V, where V_q has dimension $n+1-q$. Set $V_{\overline{q}} := V_{q+1}^{\perp}$ when $q > 0$. Then, the Schubert variety $\Omega_w(V_\bullet)$ of a signed permutation is

$$\Omega_w(V_\bullet) = \{L_\bullet \in \mathbb{F}_\Theta \mid \dim(L_p \cap V_q) \geqslant r_w(p,q), \text{ for every } 1 \leqslant p \leqslant n \text{ and } \overline{n} \leqslant q \leqslant n\}.$$

This degeneracy locus is a Schubert variety of codimension $\ell(w)$, and it is related to the Schubert variety S_w as follows:

$$\Omega_w = \mathcal{S}_{w_\circ \cdot w},$$

where $w_\circ$ is the involution in $\mathcal{W}_n$.

Notice that the definition of the degeneracy locus Ω_w on a variety X takes into account every (p,q) in the extended diagram $D^+(w)$, i.e., there are $(2n^2+n)$ conditions to satisfy. A version of Proposition A.1 for signed permutations was given by Anderson and Fulton.

Proposition A.3 ([4] Corollary 2.6)**.** *The degeneracy locus Ω_w is also defined by*

$$\Omega_w = \{x \in X \mid \dim(E_p(x) \cap F_q(x)) \geqslant r_w(p,q), \textit{ for every } (p,q) \in \mathscr{C}(w)\}.$$

A.2.1 Chern class formula of vexillary signed permutations

If $w = w(\boldsymbol{\tau})$ is a vexillary signed permutation then, by Proposition A.3, the degeneracy locus of the triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ of type B is

$$\Omega_{\boldsymbol{\tau}} = \Omega_w = \{x \in X \mid \dim(E_{p_i}(x) \cap F_{q_i}(x)) \geqslant k_i, \text{ for every } 1 \leqslant i \leqslant s\}.$$

Given $\boldsymbol{\tau}$, we define a *strict* partition $\lambda = \lambda(\boldsymbol{\tau}) = (\lambda_1 > \lambda_2 > \cdots > \lambda_{k_s} > 0)$ such that

$$\lambda_{k_i} = p_i + q_i - 1, \quad \text{for every } 1 \leqslant i \leqslant s,$$

and we set the remaining parts $\lambda_k = \lambda_{k_i} - (k_i - k)$ for $k_{i-1} < k < k_i$.

The cohomology class $[\Omega_w]$ is belongs to the cohomology ring $H^{2\ell(w)}(X)$. Denote the total Chern classes

$$c(k_i) := c(V - E_{q_i} - F_{p_i}) \quad , \text{ for every } 1 \leqslant i \leqslant s, \text{ and}$$
$$c(k) := c(k_i) \quad , \text{ whenever } k_{i-1} < k \leqslant k_i.$$

where, by convention, we set $k_0 = 0$. Then, we get the explicit formula for the cohomology class $[\Omega_w]$.

Theorem A.4 ([3])**.** *Let $w \in \mathcal{W}_n$ be a vexillary signed permutation such that $\boldsymbol{\tau}$ is the respective triple of type B and $\lambda(\boldsymbol{\tau}) = (\lambda_1, \ldots, \lambda_r)$ is the strict partition as defined above, where $r = k_s$. Then,*

$$2^r \cdot [\Omega_{\boldsymbol{\tau}}] = \mathrm{Pf}_{\lambda(\boldsymbol{\tau})}(c(1), \ldots, c(r)) := \left(\prod_{1 \leqslant i < j \leqslant r} \frac{1 - R_{ij}}{1 + R_{ij}} \right) \cdot (c_{\lambda_1}(1) \cdots c_{\lambda_r}(r))$$

Remark A.5. This construction is based on the inclusion of $\mathcal{W}_n$ in the odd permutations S_{2n+1}. We could also consider a signed permutation as an even permutation in S_{2n}. In this case, a triple $\boldsymbol{\tau}$ of type C is defined as a triple of type B, and the geometry starts with a vector bundle V on X of rank $2n$ equipped with a symplectic form. Given two flag isotropic subbundles $E_n \subset E_{n-1} \subset \cdots \subset E_1 \subset V$ and $F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset V$, the degeneracy locus $\Omega_{\boldsymbol{\tau}}$ of a triple $\boldsymbol{\tau}$ of type C is

$$\Omega_{\boldsymbol{\tau}} = \Omega_w = \{x \in X \mid \dim(E_{p_i}(x) \cap F_{q_i}(x)) \geqslant k_i, \text{ for every } 1 \leqslant i \leqslant s\},$$

and the cohomology class of $\Omega_{\boldsymbol{\tau}}$ is

$$[\Omega_{\boldsymbol{\tau}}] = \mathrm{Pf}_{\lambda(\boldsymbol{\tau})}(c(1), \ldots, c(r)) = \left(\prod_{1 \leqslant i < j \leqslant r} \frac{1 - R_{ij}}{1 + R_{ij}}\right) \cdot (c_{\lambda_1}(1) \cdots c_{\lambda_r}(r))$$

It follows that the cohomology class $[\Omega_{\boldsymbol{\tau}}]$ for a triple of type C differs by the term 2^r when compared to a triple of type B.

A.2.2 Chern class formula of theta-vexillary signed permutations

If $w = w(\boldsymbol{\tau})$ is a theta-vexillary signed permutation then the degeneracy locus of a theta-triple $\boldsymbol{\tau} = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ is

$$\Omega_{\boldsymbol{\tau}} = \Omega_w = \{x \in X \mid \dim(E_{p_i}(x) \cap F_{q_i}(x)) \geqslant k_i, \text{ for every } 1 \leqslant i \leqslant s\}.$$

Consider the sequence $\rho(\boldsymbol{\tau}) = (\rho_1, \rho_2, \ldots, \rho_{k_s})$ defined by

$$\rho_k = \begin{cases} k - 1 & \text{, if } k \leqslant k_{a-1}, \\ k_{R(i)} & \text{, if } k > k_{a-1} \text{ and } k_{i-1} < k \leqslant k_i. \end{cases}$$

Now, we can define a partition $\lambda(\boldsymbol{\tau}) = (\lambda_1, \lambda_2, \ldots, \lambda_{k_s})$ as follows: first of all, set the values of λ for k_i by

$$\lambda_{k_i} = \begin{cases} p_i + q_i - 1 & \text{, if } i < a, \\ p_i + q_i + k_i - 1 - \rho_{k_i} & \text{, if } i \geqslant a. \end{cases}$$

Since $\boldsymbol{\tau}$ is a theta-triple, we got the following inequalities:

- Condition B1 implies that $\lambda_{k_i} \geqslant \lambda_{k_{i+1}} + (k_{i+1} - k_i)$ for $i < a - 1$;
- Condition B2 implies that $\lambda_{k_i} \geqslant \lambda_{k_{i+1}}$ for $a \leqslant i < s$;
- Condition B3 implies that $\lambda_{k_s} \geqslant 0$;
- Condition C1 implies that $\lambda_{k_{a-1}} > \lambda_{k_a}$.

Then, we can fill the other parts of λ minimally, subject to the requirement

$$\lambda_1 > \cdots > \lambda_{k_{a-1}} > \lambda_{k_a} \geqslant \cdots \geqslant \lambda_{k_s} \geqslant 0. \tag{A.2.1}$$

In general, a sequence of nonnegative integers $\rho = (\rho_1, \ldots, \rho_r)$, a *$\rho$-strict partition* is a partition $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$ such that $\lambda + \rho = (\lambda_1 + \rho_1, \ldots, \lambda_r + \rho_r)$ is also a partition. In particular, the partition $\lambda(\boldsymbol{\tau})$ is a $\rho(\boldsymbol{\tau})$-strict partition.

For instance, given the triple $\boldsymbol{\tau} = (3\,4\,5\,6\,9,\ 8\,6\,5\,4\,2,\ 7\,4\,2\,\overline{3}\,\overline{6})$ from Example 1, we have that $\rho_1 = 0$, $\rho_2 = 1$, $\rho_3 = 2$, $\rho_4 = 3$, $\rho_5 = 4$, $\rho_6 = \rho_{k_4} = k_{R(4)} = k_2 = 4$, and $\rho_9 = \rho_{k_5} = k_{R(5)} = k_1 = 3$. Then,

$$\rho(\boldsymbol{\tau}) = (0, 1, 2, 3, 4, 4, 3, 3, 3).$$

We also have that $\lambda_3 = \lambda_{k_1} = 14$, $\lambda_4 = \lambda_{k_2} = 9$, $\lambda_5 = \lambda_{k_3} = 6$, $\lambda_6 = \lambda_{k_4} = 2$, and $\lambda_9 = \lambda_{k_5} = 1$. Filling λ minimally to satisfy Equation (A.2.1), then

$$\lambda(\boldsymbol{\tau}) = (16, 15, 14, 9, 6, 2, 1, 1, 1)$$

Since $\rho + \lambda = (16, 16, 16, 12, 10, 6, 4, 4, 4)$ is a partition, then λ is a ρ-strict partition. This partition can be illustrated in a skew shape Young diagram as Figure 42.

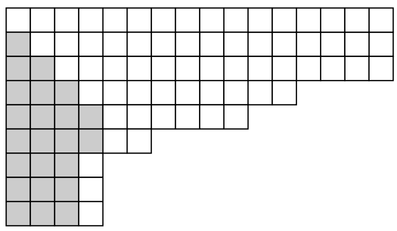

Figure 42 – The skew shape for $\lambda(\boldsymbol{\tau})$ (white boxes) and $\rho(\boldsymbol{\tau})$ (shaded boxes).

Given an integer $r > 0$, and a sequence of nonnegative integers $\rho = (\rho_1, \ldots, \rho_r)$ with $\rho_i < i$, define the raising operator

$$R^{(\rho,r)} = \left(\prod_{1 \leqslant i < j \leqslant r} (1 - R_{ij})\right)\left(\prod_{1 \leqslant i \leqslant \rho_j < j \leqslant r} (1 + R_{ij})^{-1}\right).$$

The theta-polynomial for a ρ-strict partition λ is defined as

$$\Theta_\lambda^{(\rho)}(c(1), \ldots, c(r)) = R^{(\rho,r)} \cdot (c_{\lambda_1}(1) \cdots c_{\lambda_r}(r)),$$

where $c(i) = 1 + c_1(i) + c_2(i) + \cdots$ are indexed variables. Observe that when $\rho = \varnothing$, Θ_λ is a Schur determinant Δ_λ, and when $\rho_j = j - 1$, $\Theta_\lambda^{(\rho)}$ is a Pfaffian Pf_λ.

Denote the total Chern classes $c(k_i) := c(V - E_{q_i} - F_{p_i})$ for every $1 \leqslant i \leqslant s$ and $c(k) := c(k_i)$ whenever $k_{i-1} < k \leqslant k_i$. Then, we get the explicit formula for the cohomology class $[\Omega_w]$.

Theorem A.6 ([3]). *Let $w \in \mathcal{W}_n$ be a vexillary signed permutation with theta-triple $\boldsymbol{\tau}$. Then,*

$$2^r \cdot [\Omega_{\boldsymbol{\tau}}] = \Theta_\lambda^{(\rho)}(c(1), \ldots, c(r))$$

Remark A.7. This dissertation also contributes with a formal definition of theta-vexillary signed permutations. Indeed, Anderson and Fulton in [3] managed to describe the conditions of a (theta-)triple using the sequence $\rho(\boldsymbol{\tau})$ and partition $\lambda(\boldsymbol{\tau})$, which is equivalent to our eight conditions. But they are not worried about the permutation w associated to such triple. Worth mentioning that the name "theta-vexillary" comes from Theorem A.6 by suggestion of David Anderson and Sara Billey.