UNIVERSIDADE ESTADUAL DE
CAMPINAS

Instituto de Matemática, Estatística e
Computação Científica

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One-Dimensional Random Interlacements

Entrelaçamentos aleatórios unidimensionais

Campinas

2017

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Tese de doutorado apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Estatística.

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Doctorate Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Statistics.

Advisor: Serguei Popov

Este exemplar corresponde à versão final da Tese de doutorado defendida pelo aluno Darcy Gabriel Augusto de Camargo Cunha e orientada pelo Prof. Dr. Serguei Popov.

Campinas

2017

Agência(s) de fomento e nº(s) de processo(s): FAPESP, 2013/23081-6
ORCID: https://orcid.org/0000-0002-5812-8424

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Silvania Renata de Jesus Ribeiro - CRB 8/6592

Cunha, Darcy Gabriel Augusto de Camargo, 1991-
C914o One-dimensional random interlacements / Darcy Gabriel Augusto de Camargo Cunha. – Campinas, SP : [s.n.], 2017.

Orientador: Serguei Popov.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Entrelaçamentos aleatórios. 2. Tempos locais (Processo estocástico). 3. Passeios aleatórios (Matemática). I. Popov, Serguei, 1972-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Entrelaçamentos aleatórios unidimensionais
Palavras-chave em inglês:
Random interlacements
Local times (Stochastic processes)
Random walk (Mathematics)
Área de concentração: Estatística
Titulação: Doutor em Estatística
Banca examinadora:
Serguei Popov [Orientador]
Christophe Frédéric Gallesco
Élcio Lebensztayn
Augusto Quadros Teixeira
Anatoli Iambartsev
Data de defesa: 21-11-2017
Programa de Pós-Graduação: Estatística

Tese de Doutorado defendida em 21 de novembro de 2017 e aprovada pela banca examinadora composta pelos Profs. Drs.

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Ao meu irmão Igor, que mesmo não podendo ver esse trabalho concluído sempre foi meu maior apoiador.

Agradecimentos

Agradeço a minha mãe Vilma que sempre batalhou mais que tudo para que eu chagasse até aqui, e que mesmo nas horas mais difíceis me apoiava e era compreensiva em relação a tudo. Agradeço a minha irmã Lilian que compartilhando do mesmo caminho, mesmo que em áreas opostas, sempre serviu de exemplo e inspiração para seguir em frente e buscar novos horizontes.

Aos meus amigos Charles, Gabriel, Victor, Marcelo, Vanessa, Thaís, Giuliana e Luísa, que tornaram prazerosos dias que seriam insuportáveis e me permitiram nisso seguir em frente.

Ao meu orientador Serguei Popov não tenho palavras pra agradecer, pois ele que despertou em mim o fascínio pela pesquisa e o gosto pelo descobrir, e sempre me deu a segurança necessária para poder seguir em frente, não importa os males que me atingissem. Se eu um dia posso me considerar um pesquisador é por completo mérito dele.

Aos membros da Banca Examinadora Anatoli Iambartsev, Augusto Quadros Teixeira, Christophe Frédéric Gallesco e Elcio Lebensztayn pelo seu auxilio na versão final desse trabalho.

À FAPESP, o qual financiamento do projeto proporcionou esse trabalho possível, bem como todo meu crescimento como pesquisador nesses quase quatro anos.

Por fim um agradecimento especial ao meu irmão Igor que não pôde ver o fim de meu trabalho, mas viu o fim da pessoa que ele ajudou a moldar: eu. Espero que de onde ele esteja eu esteja vivendo honrando toda confiança e apoio que sempre depositou em mim.

Resumo

Baseado na construção dos entrelaçamentos aleatórios bidimensionais (Comets, F.; Popov, S.; Vachkovskaia, M., 2016), definimos a versão unidimensional do processo. Para isso, consideramos passeios aleatórios condicionados a não entrar na origem. Nós comparamos esse processo com o passeio aleatório condicionado no grafo anel. Nossos resultados são a convergência do conjunto vacante do passeio no grafo anel em lei para o conjunto vacante dos entrelaçamentos, um teorema central do limite para os tempos locais dos entrelaçamentos e a convergência em lei dos tempos locais do passeio no grafo anel para os tempos locais dos entrelaçamentos.

Palavras-chave: entrelaçamentos aleatórios. tempos locais. passeio aleatório em uma faixa. passeio aleatório condicional. transformada h de Doob.

Abstract

Based on the construction of the two-dimensional random interlacements (Comets, F.; Popov, S.; Vachkovskaia, M., 2016), we define the one-dimensional version of the process. For this, we consider simple random walks conditioned on never hitting the origin. We compare this process to the conditional random walk on the ring graph. Our results are the convergence of the vacant set on the ring graph to the vacant set of one-dimensional random interlacements, a central limit theorem for the interlacements' local time and the convergence in law of the local times of the conditional walk on the ring graph to the interlacements' local times.

Keywords: random interlacements. local times. random walk on a stripe. conditional random walk. Doob's h-transform.

List of symbols

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ Ring graph or one-dimensional torus.

$\mathbb{Z}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ d-dimensional torus.

$X_{[a,b]}$ Range of stochastic process X_t on the time interval $[a, b]$.

$\mathcal{B}^+$ Borel sigma-algebra of $\mathbb{R}^+$.

$\mathbb{Q}^u$ Probability measure for the original random interlacements model at level u with $d \geqslant 3$.

$\mathcal{I}_d^\alpha$ Interlacement set for the random interlacements in dimension d, including the cases $d = 1, 2$.

$\mathcal{V}_d^\alpha$ Vacant set for the random interlacements in dimension d, including the cases $d = 1, 2$.

$\ell(x)$ Local time for the one-dimensional random interlacements model at site $x \in \mathbb{Z}$.

$L_n(x)$ Local time for the conditional random walk on $\mathbb{Z}_n$ started on $\lfloor n/2 \rfloor$ and running up to time $\lfloor \alpha n^3/(2\pi^2) \rfloor$.

$a(x)$ Potential kernel for the simple random walk.

$\mathrm{Cap}(A)$ Capacity of set A.

$\widetilde{\mathbb{P}}$ Probability measure of the conditional walk on $\mathbb{Z}^+$.

$\widehat{\mathbb{P}}^t$ Probability measure for the conditional random walk on $\mathbb{Z}_n$, running up to time t.

$\mathbb{P}_x$ Probability measure for the simple random walk on $\mathbb{Z}$ starting on x.

$h_n(x, t)$ Probability that a simple random walk started on x avoids 0 and n up to time t

Contents

1 Introduction

1.1 The original random interlacements process

The process of random interlacements was initially introduced by Alain Sol Sznitman in (Sznitman, A. S., 2010). The original problem that motivates its definition comes from the fragmentation of a d-dimensional torus (with $d \geqslant 3$) by a random walk. Let $\mathbb{Z}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ be the d dimensional torus of size n and let X_t be the simple symmetric random walk on $\mathbb{Z}_n^d$. If we denote the range of the random walk at time t by $X_{[0,t]} = \{X_0, X_1, \ldots, X_t\}$, then the vacant set of the torus by time t is defined by $V_t = \mathbb{Z}_n^d \backslash X_{[0,t]}$. It turns out that one can find non-trivial fractal properties on V_t considering times of the form $t_u = un^d$, which includes a phase transition on u about the fragmentation of the torus.

Roughly speaking, the random interlacements process can be viewed as a Poissonian soup of (transient) doubly infinite trajectories of simple random walks in $\mathbb{Z}^d$ for $d \geqslant 3$. To be defined rigorously, consider the following space of doubly-infinite trajectories

$$W = \{\omega : \mathbb{Z} \to \mathbb{Z}^d \, : \, \|\omega(k+1) - \omega(k)\| = 1 \text{ and } |\omega^{-1}(\{x\})| < \infty \text{ for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{Z}^d\}.$$

This space has an issue, where different elements represent the same trajectory in $\mathbb{Z}^d$, to solve this we consider an equivalence relation $\sim$, where

$$\omega \sim \omega' \Leftrightarrow \omega(\cdot) \equiv \omega'(\cdot + k) \text{ for some } k \in \mathbb{Z}.$$

We denote the space of trajectories modulo the equivalence relations by $W^* = W \backslash \sim$ and denote the σ-algebra generated by the canonical projections of this space by $\mathcal{W}^*$. There is a positive parameter u entering the intensity measure of the corresponding Poisson process on the space of such trajectories. The process is then defined as a Poisson point process on the space $(W^* \times \mathbb{R}^+, \mathcal{W}^* \times \mathcal{B}^+)$. The use of the space $\mathbb{R}^+$ is a coupling to construct the process for all parameters at once. If we denote each point of the Poisson point process by (ω, v), then the process at level u is the restriction of the points where $v \leqslant u$.

In this definition we are intentionally omitting the intensity measure of the Poisson point process due to the complexity of its definition. We refer to equation (3.9) of (Teixeira, A.; Černỳ, J., 2012) to the definition of the measure and for Theorem 3.1 of (Teixeira, A.; Černỳ, J., 2012) to a proof of existence and uniqueness of such measure.

The process is then almost-surely characterized by the law of its vacant set by

$$\mathbb{P}[A \subset \mathcal{V}^u] = \exp\big\{ -u \, \mathrm{Cap}(A) \big\},$$

where Cap stands for the classical capacity for the simple random walk.

The main initial questions about the process were about percolative properties of the vacant set (i.e. the set of unvisited sites), where Theorem 3.4 of (Sznitman, A. S.; Sidoravicius, V., 2009) says that for $d \geqslant 3$ we have a non-trivial phase transition for the vacant set percolation, i.e. there exist a positive finite u^* such that

$$\begin{aligned}
&\text{If } u < u^*, \text{ then } \mathbb{Q}^u[\text{there is a infinite connected component inside } \mathcal{V}^u] = 1;\\
&\text{If } u > u^*, \text{ then } \mathbb{Q}^u[\text{there is a infinite connected component inside } \mathcal{V}^u] < 1,
\end{aligned}$$

where $\mathbb{Q}^u$ stands for the probability measure of the random interlacements at level u.

We give the full definition of the process in Section 2.1.

1.2 The two-dimensional random interlacements process

In (Comets, F.; Popov, S.; Vachkovskaia, M., 2016; Comets, F.; Popov, S., 2017) the model of random interlacements in two dimensions was introduced and studied. This process could not be defined using the classical approach, as the simple random walk in two dimensions is recurrent and so just one trajectory would cover the entire discrete plane $\mathbb{Z}^2$, leaving nothing to be seen. Therefore, in order to construct the process, one uses simple random walks conditioned to never hitting the origin. This conditioning makes the walk transient and the construction of the process becomes possible, at cost of losing the stationarity (there is, however, a so-called *conditional stationarity*, see Theorem 2.3 (i) of (Comets, F.; Popov, S.; Vachkovskaia, M., 2016)). In its construction a parameter change was made to make the formulas cleaner, so the law of the vacant set is characterized by

$$\mathbb{P}[A \subset \mathcal{V}^\alpha] = e^{-\alpha\pi \operatorname{Cap}(A \cup \{0\})}. \tag{1.1}$$

Later we will introduce this model properties with more details.

1.3 Introduction to the model and objectives

Here we base ourselves on the approach of (Comets, F.; Popov, S.; Vachkovskaia, M., 2016) to construct the one-dimensional random interlacements process. Analogously to the two-dimensional case, we define the process making use of conditional random walks in this construction. It turns out that for any $A \subset \mathbb{Z}$ an analogous to (2.3) formula holds:

$$\mathbb{P}[A \subset \mathcal{V}^\alpha] = e^{-\alpha \operatorname{Cap}(A \cup \{0\})} = e^{-\alpha \operatorname{Diam}(A \cup \{0\})/2}, \tag{1.2}$$

where Diam(A) stands for the diameter of the set.

As usual in dimension 1, percolation questions are not of interest, since our vacant set is an interval containing the origin; so, we focus on other problems, mainly about the relation to random walks on the ring graph (the "one-dimensional torus") and the local times of the process.

A well studied problem about random interlacements is how it represents the local picture of a random walk on a torus, when it is left to run for a certain fixed time. Consider the d-dimensional torus $\mathbb{Z}^d/n\mathbb{Z}^d$ and denote the trace left for a random walk until time t for $X_{[0,t]}$. In Theorem 1.1 of (Teixeira, A.; Windisch, D., 2011) it was shown that for any $u > 0$, $\delta > 0$ and $\varepsilon \in (0,1)$ one can construct a coupling between the random interlacements and the random walk on this torus in such a way that for a constant c depending on u, δ, ε we have

$$\mathbb{P}[\mathcal{I}^{u(1-\varepsilon)} \cap A \subseteq X_{[0,\lfloor un^d \rfloor]} \cap A \subseteq \mathcal{I}^{u(1+\varepsilon)} \cap A] \geqslant 1 - cn^{-\delta}, \tag{1.3}$$

where A is a "mesoscopic" d-dimensional box of size $n^{(1-\varepsilon)}$ (i.e., of volume $n^{(1-\varepsilon)d}$). More recently in Theorem 4.1 of (Černý, J.; Teixeira, A., 2016) this result was improved for a box of size $(1-\delta)n$ and success probability $1 - C_1 \exp\{-C_2 n^{C_3}\}$, where C_1, C_2 and C_3 are constants. From now on we will work with weak convergence between random subsets of the torus and $\mathbb{Z}^d$. To be precise, let $\pi : \mathbb{Z}_n^d \to \mathbb{Z}^d$ be the canonical projection between the torus and $\mathbb{Z}^d$. We will say that a random set $A_n \subset \mathbb{Z}_n^d$ converges weakly to $\mathcal{A} \subset \mathbb{Z}^d$ and denote this by $A_n \overset{\text{law}}{\Rightarrow} \mathcal{A}$ iff for every fixed $B \subset \mathbb{Z}_n^d$ (with n large enough to contain B) it holds

$$\lim_{n\to\infty} \mathbb{P}[B \subset A_n] = \mathbb{P}[\pi(B) \subset \mathcal{A}].$$

Denoting the vacant set left by the random walk on the torus by $V_t^d = (\mathbb{Z}/n\mathbb{Z})^d \backslash X_{[0,t]}$ for $d \geqslant 3$, it holds that

$$V^d_{\lfloor un^d \rfloor} \overset{\text{law}}{\Rightarrow} \mathcal{V}^u_d,$$

where $\mathcal{V}^u_d$ stands for the vacant set for the random interlacements in $\mathbb{Z}^d$.

In Theorem 2.6 of (Comets, F.; Popov, S.; Vachkovskaia, M., 2016) this result was extended to the random interlacements in two dimensions and a simple random walk on the torus conditioned to never hitting the origin. For $d = 2$ we use an analogous notation for the multidimensional version of vacant set, only with conditional random walks. So, let $\{\hat{X}_t\}_{t\geqslant 0}$ be the random walk on the two-dimensional torus conditioned to never hitting the origin and $\hat{X}_{[0,t]}$ its trajectory until time t. Denoting $V_t = \mathbb{Z}^2 \backslash \hat{X}_{[0,t]}$, Theorem 2.6 of (Comets, F.; Popov, S.; Vachkovskaia, M., 2016) states that

$$V^2_{\lfloor \frac{4\alpha}{\pi} n^2 \ln^2 n \rfloor} \overset{\text{law}}{\Rightarrow} \mathcal{V}^\alpha_2. \tag{1.4}$$

1.3.1 Main Theorems

The first result we prove is the one-dimensional random interlacements version of this theorem. The one-dimensional discrete torus $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ with n sites is in fact a "ring" graph with n vertices; we usually identify $\mathbb{Z}_n$ with $\{0, \dots, n-1\}$, remembering that the sites 0 and $n-1$ are neighbors as well. In the following, we will consider a simple random walk conditioned to never hitting the origin (i.e., the site $0 \in \mathbb{Z}_n$).

Theorem 1.3.1. *Let X_t be the conditional random walk on $\mathbb{Z}_n$ started at $\lfloor n/2 \rfloor$ and $V_t = \{x \in \mathbb{Z}_n \mid X_k \neq x \text{ for all } k \leqslant t\}$. Then*

$$V_{\left\lfloor \frac{\alpha n^3}{2\pi^2} \right\rfloor} \overset{law}{\Rightarrow} \mathcal{V}_1^{\alpha}, \qquad \text{as } n \to \infty.$$

Our next results are about the local times (sometimes called occupation times) of the random walk. For random interlacements, the local time in x is the total number of visits to x of all particles. Some previous results regarding local times of random interlacements, such as a Ray-Knight-type theorems and large deviations, can be found in (Sznitman, A.S., 2012) and (Li, X; Sznitman, A.S., 2015).

Theorem 1.3.2. *Let $\ell(x)$ be the local time of the one-dimensional random interlacements for $x \in \mathbb{Z}^+$, and let Z be a Standard Normal random variable. Then, as $x \to \infty$*

$$\frac{\ell(x) - \alpha x^2}{x\sqrt{\alpha(4x-1)}} \overset{law}{\Rightarrow} Z.$$

Our last result is about the local times convergence for the random walk on the ring graph. Our approach in the proof is to construct a coupling of local times with independent trajectories of the conditional random walk in $\mathbb{Z}^+$.

Theorem 1.3.3. *Let $\ell(x)$ be the local time in x of the one-dimensional random interlacements, and $L_n(x)$ the local time in x for the random walk in $\mathbb{Z}_n$ started at $\lfloor n/2 \rfloor$ up to time $\lfloor \alpha n^3/(2\pi^2) \rfloor$ conditioned on not hitting the origin. Then as $n \to \infty$*

$$L_n(x) \overset{law}{\Rightarrow} \ell(x).$$

Observe that Theorem 1.3.1 is, in fact, a corollary of Theorem 1.3.3. We opted to state the former one separately because the proof of Theorem 1.3.1 is much more simple and straightforward than that of Theorem 1.3.3.

2 Preliminaries

2.1 Random Interlacements

Now we will give the rigorous definition of the original random interlacements process. As before, we first consider the space of doubly-infinite trajectories

$$W = \{\omega : \mathbb{Z} \to \mathbb{Z}^d : \|\omega(k+1) - \omega(k)\|_1 = 1 \text{ for every } k \in \mathbb{Z} \\ \text{and } |\omega^{-1}(\{x\})| < \infty \text{ for every } x \in \mathbb{Z}^d\}.$$

Where $\omega^{-1}(\cdot)$ stands for the inverse image of the function ω. We will also consider the space W^+ which is the space of trajectories with a initial point (therefore they are not doubly-infinite).

$$W^+ = \{\omega : \mathbb{N} \to \mathbb{Z}^d : \|\omega(k+1) - \omega(k)\|_1 = 1 \text{ for every } k \in \mathbb{N} \\ \text{and } |\omega^{-1}(\{x\})| < \infty \text{ for every } x \in \mathbb{Z}^d\}.$$

Observe that here we have a space of functions, but the graph of each of those functions represents a trajectory of a simple random walk in $\mathbb{Z}^d$, and it also also replicates the random walk behavior of only visiting finite subsets for a finite amount of time. Using these spaces we consider the σ-algebras generated by the canonical projections, $\mathcal{W}$ and $\mathcal{W}^+$ respectively.

For $A \subset \mathbb{Z}^d$ we define the space W_A of trajectories that visits A at some point, i.e.

$$W_A = \{\omega \in W : \omega(k) \in A \text{ for some } k \in \mathbb{Z}.$$

We also denote by θ_k the time-shift operator, i.e.

$$\theta_k(\omega)(\cdot) = \omega(\cdot + k), \text{ for } k \in \mathbb{Z}.$$

The shift operator points out to a representation problem in the space W, where multiple different functions represent the same trajectory, and to solve this problem we consider an equivalence relation $\sim$ defined as

$$\omega \sim \omega' \Leftrightarrow \omega(\cdot) \equiv \omega'(\cdot + k) \text{ for some } k \in \mathbb{Z}.$$

With this we can consider the space $W^* = W/\sim$ of trajectories modulo time shifts. In this space we consider elements in which we have the same trajectory and order of visits to be the same.

If π^* denotes the canonical projection from W to W^*, i.e. the mapping that takes each element $\omega \in W$ to its corresponding equivalence class in W^*. Using π^* we induce a sigma algebra on W^* by

$$\mathcal{W}^* = \{A \subset W^* : (\pi^*)^{-1}(A) \in \mathcal{W}\}.$$

Which is the largest σ-algebra such that the mapping π^* from $(W, \mathcal{W})$ to $(W^*, \mathcal{W}^*)$ is measurable. Analogously to the definition of W_A, for $A \subset \mathbb{Z}^d$ we can define the space W_A^* of trajectories that visit A modulo time-shifts

$$W_A^* = \pi^*(W_A).$$

The random interlacements process will then be a Poisson point process on the space $(W^* \times \mathbb{R}^+, \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}^+))$ with intensity measure $\tau \times \lambda$, where λ stands for the Lebesgue measure and η will be a suitable measure, which we will define now.

Consider the coordinate mapping $X_n = X_n(\omega) = \omega(n)$ for the elements of W. To define the measure η first we define the measure $\mathbb{Q}_A$ on $((W, \mathcal{W})$ by

$$\mathbb{Q}_A[(X_{-n})_{n \geqslant 0} \in F, X_0 = x, (X_n)_{n \geqslant 0} \in G] = \mathbb{P}_x[F | \overline{\tau}_A < \infty] e_A(x) \mathbb{P}_x[G].$$

The intensity measure of the Poisson point process defining the random interlacements, η, is then defined as the unique σ-finite measure on the space $(W^*, \mathcal{W}^*)$ satisfying, for every finite set $A \subset \mathbb{Z}^d$ the following relation

$$\mathbb{1}_{W_A^*} \cdot \eta = \pi^* \circ \mathbb{Q}_A.$$

Consider the space Ω of finite point measures on W

$$\Omega = \Big\{ \rho = \sum_{i \geqslant 1} \delta_{(\omega_i, u_i)} : \omega_i \in W^*, u_i \in \mathbb{R}^+$$
$$\text{and } \rho(W_G^* \times [0, u]) < \infty \text{ for all finite } G \subset \mathbb{Z}^d \text{ and } u \geqslant 0 \Big\}.$$

We endow this space with the σ-algebra $\mathcal{A}$ generated by the evaluation maps $\rho \to \rho(D)$ for $D \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}^+)$.

2.2 Capacity in one dimension

To work with capacity in lower dimensions we need to make use of the potential kernel of the random walk. The potential kernel $a(x)$ for any random walk X_t in $\mathbb{Z}^d$ is defined in section 4.4 of (Lawler, G.; Limic, V., 2010) by

$$a(x) = \sum_{k=0}^{\infty} \big(\mathbb{P}_0[X_k = 0] - \mathbb{P}_0[X_k = x] \big).$$

If the random walk is transient, then we can relate $a(x)$ to the Green function $G(x)$ by $a(x) = G(0) - G(x)$, but if the random walk is recurrent, the Green function does not exist. Theorem 4.4.8 of (Lawler, G.; Limic, V., 2010) states that for the simple random walk in one dimension the potential kernel is given by $a(x) = |x|$.

Now we define the capacity for one dimension, this definition is analogous to the one of section 6.6 of (Lawler, G.; Limic, V., 2010) for two dimensions and also used in (Comets, F.; Popov, S.; Vachkovskaia, M., 2016). The capacity of a set $A \subset \mathbb{Z}$ containing the origin is defined for any $z \in A$ as

$$\begin{aligned} \mathrm{Cap}(A) &:= \sum_{x \in A} \mathrm{hm}_A(x) a(x-z) \\ &= \frac{1}{2}\Big(a(\max A) + a(\min A)\Big). \end{aligned}$$

and for any other subset B the capacity is given by the capacity of any translation of B that contains the origin.

As the harmonic measure can only be non null on the extremal points of a set, we have that $\mathrm{Cap}(A) = \mathrm{Cap}([\min A; \max A])$. The explicit form of the harmonic measure and the translation invariance of capacity then imply that for any finite subset A of $\mathbb{Z}$ we have

$$\mathrm{Cap}(A) = \frac{\mathrm{Diam}(A)}{2}.$$

2.3 The conditional random walk

Here we construct random walks conditioned on never hitting 0; since such a walk never changes its sign, let us consider it only on $\mathbb{Z}^+$. Let $x \geqslant 1$ be a positive integer, and let us figure out the "right" way to define a simple random walk $(X_t)_{t \geqslant 0}$ started at x and conditioned on not hitting 0. To define the law of X_t, let us condition it on hitting N before 0 and take a limit in law. So for $x < N$ we have the following transition probabilities

$$\begin{aligned} p_{x,x+1} &= 1 - p_{x,x-1} \\ &= \mathbb{P}_x[X_1 = x+1 | \tau_0 > \tau_N] \\ &= \frac{\mathbb{P}_x[X_1 = x+1]\mathbb{P}_{x+1}[\tau_0 > \tau_N]}{\mathbb{P}_x[\tau_0 > \tau_N]} \\ &= \frac{x+1}{2x}. \end{aligned} \tag{2.1}$$

We then send N to infinity to take the restriction on x off, thus obtaining transition probabilities from any site $x \geqslant 1$. Now, the "canonical" way to define this kind of conditioned random walk is to apply the Doob's h-transform, using the potential kernel for one dimension $a(x) = |x|$. The Doob's h-transform with a non-negative function h defined on the state space S and a Markov chain with transitions $P(x, y)$, such that h is

P-harmonic outside the set $\{x : h(x) = 0\}$, is defined in the following way. For all sites x with $h(x) \neq 0$ the new transition probabilities $P^*(x, y)$ are

$$P^*(x, y) = \frac{P(x, y)h(y)}{h(x)}; \tag{2.2}$$

note that $P^*(\cdot, \cdot)$ are indeed transition probabilities due to the harmonicity condition $h(x) = \sum_y P(x, y)h(y)$.

So, when we apply the Doob's h-transform for the simple random walk S_t with constant transition probabilities $P(x, x + 1) = 1/2$ we get a new random walk X_t with transition probabilities given by

$$P^*(x, x + 1) = \frac{a(x + 1)P(x, x + 1)}{a(x)} = \frac{x + 1}{2x}$$

The same walk could also be seen as a random walk with conductances on $\mathbb{Z}^+$, where the conductances are given by $c_{x,x+1} = a(x)a(x + 1) = x(x + 1)$.

The following lemma is a standard fact, which is usually very useful for calculating exit probabilities and alike for the conditioned walk.

Lema 2.1. *Let $(X_t)_{t\geqslant 0}$ be a Markov chain in a countable state space S with transition probabilities $P(x, y)$. Let $(X^*_t)_{t\geqslant 0}$ be the Doob's h-transform of $(X_t)_{t\geqslant 0}$ with respect to a function h, which is non-negative and harmonic outside the set $\{x : h(x) = 0\}$. Let*

$$S^* = \big\{x \in S : \text{ there exists } y \text{ with } P(x, y) > 0 \text{ and } h(y) = 0\big\}.$$

Then the process $\big((h(X^*_{t\wedge\tau_{S^*}}))^{-1}\big)_{t\geqslant 0}$ with $h(X_0) \neq 0$ is a martingale.

Proof. Recall the definition (2.2). The following is just a straightforward calculation; being $\{\mathcal{F}_t\}_{t\geqslant 0}$ the associated filtration for the process $((h(X_{t\wedge\tau_{S^*}}))^{-1})_{t\geqslant 0}$, we have

$$\begin{aligned}
\mathbb{E}\big[(h(X^*_{(t+1)\wedge\tau_{S^*}}))^{-1} \mid \mathcal{F}_t\big] &= \sum_{y\in S:h(y)\neq 0} (h(y))^{-1}P^*(X^*_{t\wedge\tau_{S^*}}, y) \\
&= \sum_{y\in S:h(y)\neq 0} \frac{1}{h(y)} \frac{P(X^*_{t\wedge\tau_{S^*}}, y)h(y)}{h(X^*_{t\wedge\tau_{S^*}})} \\
&= \sum_{y\in S:h(y)\neq 0} \frac{P(X^*_{t\wedge\tau_{S^*}}, y)}{h(X^*_{t\wedge\tau_{S^*}})} \\
&= (h(X^*_{t\wedge\tau_{S^*}}))^{-1}\mathbb{P}\big[h(X^*_{(t+1)\wedge\tau_{S^*}}) \neq 0 \mid \mathcal{F}_t\big] \\
&= (h(X^*_{t\wedge\tau_{S^*}}))^{-1},
\end{aligned}$$

and this completes the proof. □

For the conditional random walk X_t, Lemma 2.1 implies that the process $\{(X_{t\wedge\tau_1})^{-1}\}_{t\geqslant 0}$ is a martingale. This fact will help us with calculations.

From now on $\widetilde{\mathbb{P}}$ will stand for the probability measure of the conditional walk on $\mathbb{Z}^+$. Next, we need

Lema 2.2. *Let X_t be the conditional random walk on $\mathbb{Z}^+$ started at y with $N > y > x > 1$. Then*

(i) $\widetilde{\mathbb{P}}_y[\tau_x < \tau_N] = \dfrac{x(N-y)}{y(N-x)}$,

(ii) $\widetilde{\mathbb{P}}_y[\tau_x < \infty] = \dfrac{x}{y}$,

(iii) $\widetilde{\mathbb{P}}_x[\overline{\tau}_x = \infty] = \dfrac{1}{2x}$.

Proof. These are also very straightforward calculations using the optional stopping theorem. The first result comes from using the martingale $(X_{t\wedge\tau_1})^{-1}$ with the stopping time $\tau(N) \wedge \tau(x)$:

$$\frac{1}{y} = \frac{1}{x}\widetilde{\mathbb{P}}_y[\tau_x < \tau_N] + \frac{1}{N}(1 - \widetilde{\mathbb{P}}_y[\tau_x < \tau_N]),$$

and then isolating the probability in the expression give us the desired result. For the second relation we just need to take limit in N, as clearly τ_N will diverge. For the last expression we observe that the first step should be to $x+1$ and then

$$\begin{aligned}\widetilde{\mathbb{P}}_x[\overline{\tau}_x = \infty] &= P(x, x+1)\widetilde{\mathbb{P}}_{x+1}[\overline{\tau}_x = \infty] \\ &= \frac{x+1}{2x}\Big(1 - \frac{x}{x+1}\Big) = \frac{1}{2x}.\end{aligned}$$

This concludes the proof. □

An important consequence of Lemma 2.2 (iii) is that the random walk conditioned on never hitting the origin is transient.

2.4 The two dimensional random interlacements

Most definition we use here, as potential kernel and capacity are analogous to the one-dimensional versions previously defined and therefore we will not define them again unless necessary.

Differently from the original version of the random interlacements, which is based on what is left of the torus when corroded by a random walk for some time such that this set have fractal properties, the two dimensional version of the process is defined based on another problem, which is: what does the last particle to be covered by a random walk in the two dimensional torus see around her? not only this model has a different construction than the original one, but the percolative properties and overall behavior are completely unexpected. This model is more interesting when defining the one dimensional version of the interlacements, since they share the problem keeping the original definition of working at all: recurrence.

We cannot define the random interlacements in dimension $d = 2$ directly because of the recurrence of the random walk, which means that each trajectory covers the entire plane and therefore the vacant set and interlacements set are trivial. One simple way to solve this would be to consider conditional random walks, which are basically simple random walks conditioned on never hitting the origin. Although it is intuitive what we mean when we say "define the random interlacements using conditional random walks", to rigorously define it we use (Teixeira, A., 2009), which defines the interlacements for any weighted graph in which the random walk is transient.

This definition is possible because the conditional random walk is the same as the simple random walk transformed by Doob's h-transform using as the h function the two dimensional potential kernel $a(x)$, later we will see the same holds for the one dimensional process. Observing the transition probabilities we can tell that the conditional random walk is then the same as a random walk on the weighted lattice $\mathbb{Z}^2$, where each edge e_{xy} has weight $a(x)a(y)$.

Using this random walk with the construction of (Teixeira, A., 2009) we have the two dimensional version of the process, which analogously to the original process is characterized by its vacant set law by

$$\mathbb{P}[A \subset \mathcal{V}_2^\alpha] = e^{-\alpha\pi \operatorname{Cap}(A\cup\{0\})}. \tag{2.3}$$

As stated before, the constant π was introduced to simplify all formulas and expressions, and being just a scaling of the parameter α, it makes no difference for the process behavior, but only to the parameter related to such behaviors.

The first meaningful difference between the behavior of the two dimensional version of the process and the original one is the fact that the two dimensional version has two phase-transitions, while in the original process both phase transitions coincide. The following theorem of (Comets, F.; Popov, S.; Vachkovskaia, M., 2016) shows these phase-transitions.

Theorem 2.4.1.

- *Let $B(r)$ be the two dimensional ball with radius r, then it holds*

$$\mathbb{E}|\mathcal{V}_2^\alpha \cap B(r)| \sim \begin{cases} \dfrac{2\pi C^\alpha}{2-\alpha} \times r^{2-\alpha}, & \text{for } \alpha < 2, \\ 2\pi C^\alpha \times \ln r, & \text{for } \alpha = 2, \\ \text{Constant}, & \text{for } \alpha > 2. \end{cases}$$

- *For $\alpha > 1$ it holds that $\mathcal{V}_2^\alpha$ is finite almost surely. Moreover $\mathbb{P}[\mathcal{V}_2^\alpha = \{0\}] > 0$ and $\mathbb{P}[\mathcal{V}_2^\alpha = \{0\}] \to 0$ as $\alpha \to \infty$.*

- *For* $\alpha \in (0,1)$, *we have* $|\mathcal{V}_2^\alpha| = \infty$ *almost surely. Moreover,*

$$\mathbb{P}[\mathcal{V}_2^\alpha \cap \big(B(r)\backslash B(r/2)\big)] \leqslant r^{-2(1-\sqrt{\alpha})+o(1)}.$$

This shows that we have an interval for the parameter α where the process's vacant set is almost surely finite, but the expectation of its size is infinite. More recently it was shown in (Comets, F.; Popov, S., 2017) that for $\alpha = 1$ the vacant set of the two dimensional random interlacements is infinite.

Another result worth mentioning is part (iii) of Theorem 2.3 from (Comets, F.; Popov, S.; Vachkovskaia, M., 2016), which states that for A such that $0 \in A \subset B(r)$ and $x \in \mathbb{Z}^2$ such hat $\|x\| \geqslant 2r$ it holds

$$\mathbb{P}[A \subset \mathcal{V}_2^\alpha \mid x \in \mathcal{V}_2^\alpha] = \exp\left(-\frac{\pi\alpha}{4}\operatorname{Cap}(a)\frac{1+\mathcal{O}\big(\frac{r\ln r \ln\|x\|}{\|x\|}\big)}{1-\frac{\operatorname{Cap}(A)}{2a(x)}+\mathcal{O}\big(\frac{r\ln r}{\|x\|}\big)}\right).$$

With this, if we consider any finite A and take the limit with$\|x\| \to \infty$ we get

$$\lim_{\|x\|\to\infty} \mathbb{P}[A \subset \mathcal{V}_2^\alpha \mid x \in \mathcal{V}_2^\alpha] = \exp\Big(-\frac{\pi\alpha}{4}\operatorname{Cap}(a)\Big).$$

So conditioned on a infinitely far point being vacant, the rate of the process in some fixed set A get reduced to $1/4$ of the original process, this is not only interesting but it also shows that we do not have asymptotic independence between sites.

2.5 Definition of the process

Even though there is a rigorous way to define the process that uses the Poisson process of trajectories, we first will present a constructive approach which also works in higher dimensions (substituting the conditioned random walks by its unconditioned version) and heuristically make it easier to understand the process behavior.

Recall that α is a parameter that rules over the number of trajectories in our process. Consider the following procedure depending on N:

- Consider $p(\alpha, N) \sim \text{Poisson}(\alpha N)$ independent particles.
- Each particle choose a starting point at random from N and $-N$.
- Each particle realize an independent conditional random walk, which is transient.
- Taking limit in N and then we have the random interlacements process.

Now to define rigorously the process, we will use the construction of (Teixeira, A., 2009), where the process of random interlacements is constructed for any weighted transient graph (i.e., a graph on which the random walk is transient). The graph considered here is $\mathbb{Z}$, so our weights (or conductances) $c_{x,y}$ are only positive if $|x-y|=1$. The conductances that generate the conditional random walk defined in (2.1) are

$$c_{x,x+1} = c_{x+1,x} = \frac{x(x+1)}{2}.$$

Then, the random walk on the graph with conductances is reversible with reversible measure $\mu_x := c_{x,x+1} + c_{x,x-1} = x^2$, and its transition probabilities are

$$P(x,x+1) = \frac{c_{x,x+1}}{\mu_x} = \frac{x+1}{2x},$$

as it should be. In accordance to (Teixeira, A., 2009), the capacity (denoted here by $\overline{\mathrm{Cap}}(A)$) with respect to the conditional walk is defined by

$$\overline{\mathrm{Cap}}(A) = \sum_{x\in A} e_A(x), \tag{2.4}$$

where $e_A(x)$ is the *equilibrium measure* defined by

$$e_A(x) = \mathbb{1}[x\in A]\widetilde{\mathbb{P}}_x[\overline{\tau}_A = \infty]\mu_x.$$

This definition uses the equilibrium measure of the conditional random walk, so its straightforward to see that for any finite $A \subset \mathbb{Z}$ we have $\overline{\mathrm{Cap}}(A) = \overline{\mathrm{Cap}}(A\cup\{0\})$.

We now show that for any set A containing the origin we have

$$\overline{\mathrm{Cap}}(A) = \mathrm{Cap}(A).$$

Since the capacity of any finite set A is the same as the capacity of the shortest interval containing it, we can assume without loss of generality that $A=[a,b]$. In the definition of $\mathrm{Cap}(A)$ we consider a set containing the origin, and for any other set we consider a translation of it containing the origin (this capacity is translation invariant). So we consider here $a<0$ and $b>0$, and then by Lemma 2.2 we have

$$\begin{aligned}
\overline{\mathrm{Cap}}([a,b]) &= \sum_{x=a}^{b} e_{[a,b]}(x) \\
&= e_{[a,b]}(a) + e_{[a,b]}(b) \\
&= \widetilde{\mathbb{P}}_a[\overline{\tau}_a = \infty]a^2 + \widetilde{\mathbb{P}}_b[\overline{\tau}_b = \infty]b^2 \\
&= \frac{b-a}{2} = \mathrm{Cap}([a,b]).
\end{aligned}$$

With this we can relate both capacities by

$$\overline{\mathrm{Cap}}(A) = \mathrm{Cap}(A\cup\{0\}).$$

Let W^* be the space of doubly infinite trajectories that spend a finite time in each finite set, the random interlacements process is defined as a Poisson point process on the space

$W^* \times \mathbb{R}^+$ with intensity by a measure $\nu \cdot \lambda$, where λ is the Lebesgue measure on $\mathbb{R}^+$ and ν is a measure on W^* characterized by

$$\nu(\{\omega^* \in W^* : \omega^*(\mathbb{Z}) \cap A \neq \varnothing\}) = \overline{\mathrm{Cap}}(A).$$

With this we can characterize the law of the process as

$$\begin{aligned}\mathbb{P}[A \subset \mathcal{V}^\alpha] &= \exp\{-\nu(\{\omega^* \in W^* : \omega(\mathbb{Z}) \cap A \neq \varnothing\}) \times \lambda([0,\alpha])\} \\ &= \exp\{-\alpha\,\overline{\mathrm{Cap}}(A)\}.\end{aligned}$$

For a complete description of the construction see (Teixeira, A., 2009).

One property of the construction that will be useful is that the number of trajectories that hit a set A (we will denote it by N_A) in the random interlacements process at level α is such that

$$N_A \sim \mathrm{Poisson}(\alpha\,\mathrm{Cap}(A \cup \{0\})) \sim \mathrm{Poisson}\Big(\alpha \frac{\mathrm{Diam}(A \cup \{0\})}{2}\Big).$$

2.6 Local times

The local time or occupation time of a transient random walk in site x can be defined as the time the random walk spends at site x. By the symmetry of the process around 0, we will consider $x > 0$ in this section. The local time of the random interlacements process is the sum of the local times of each trajectory. We know we have $\mathrm{Poisson}\big(\alpha x/2\big)$ trajectories that hit x. For each of those trajectories, at each visit to x a particle has a constant probability of escaping, making the local time of each particle a geometric random variable with success probability $(2x)^{-1}$ (see Lemma 2.2). Therefore, the random interlacements local time at x is a compound Poisson variable

$$\ell(x) = \sum_{k=1}^{N_{\{x\}}} V_k,$$

where V_k are i.i.d. $\mathrm{Geometric}((2x)^{-1})$, i.e., $\mathbb{P}[V_k = j] = \dfrac{1}{2x}\big(1 - \dfrac{1}{2x}\big)^{j-1}$.

Lema 2.3. *For any $x > 0$ the characteristic function of $\ell(x)$ is*

$$\varphi_{\ell(x)}(t) = \exp\left\{\alpha x^2 \frac{(e^{it} - 1)}{2x - (2x-1)e^{it}}\right\}.$$

Proof. This follows from a straightforward calculation of the characteristic function of a compound Poisson of geometric random variables with $N_{\{x\}}$ being a Poisson variable with parameter $\alpha\,\mathrm{Cap}(\{0, x\}) = \alpha x/2$. The characteristic function of geometric variables is also well-known, so we have

$$\varphi_{\ell(x)}(t) = \exp\left\{\frac{\alpha x}{2}\left(\frac{\frac{1}{2x}e^{it}}{1-\left(1-\frac{1}{2x}\right)e^{it}} - 1\right)\right\}$$
$$= \exp\left\{\alpha x^2 \frac{(e^{it}-1)}{2x-(2x-1)e^{it}}\right\}. \tag{2.5}$$

□

2.7 Random walk on the ring graph

In higher dimensions it is known that we can approximate the trace left by the random walk in a torus by the random interlacements process, main results about this can be found in (Teixeira, A.; Windisch, D., 2011) and more recently for two-dimensional random interlacements in (Comets, F.; Popov, S.; Vachkovskaia, M., 2016). Here we wish to establish the same fact in dimension one.

For the simple random walk (on the ring) conditioned on not hitting the origin until time t, we denote its law by $\widehat{\mathbb{P}}^t$ and its respective vacant set by V_t. This random walk can be seen as a random walk on $\mathbb{Z}$ conditioned to not hitting 0 and n. Let us define the quantity

$$h_n(x,t) = \mathbb{P}_x[\tau_{\{0,n\}} > t].$$

Then the law for the walk conditioned on not hitting $0 \in \mathbb{Z}_n$ until time s is given by

$$\begin{aligned}\widehat{\mathbb{P}}^s_x[X_1 = x+1] &= 1 - \widehat{\mathbb{P}}^s_x[X_1 = x-1]\\ &= \mathbb{P}_x[X_1 = x+1 | \tau_{\{0,n\}} > s]\\ &= \frac{\mathbb{P}_x[X_1 = x+1, \tau_{\{0,n\}} > s]}{\mathbb{P}_x[\tau_{\{0,n\}} > s]}\\ &= \frac{h_n(x+1, s-1)}{2h_n(x,s)},\end{aligned}$$

and then the probability of a path γ of size m starting at x when the remaining time is t is given by (note that γ_m is the last site of that path)

$$\widehat{\mathbb{P}}^t_x[\gamma] = \frac{h_n(\gamma_m, t-m)}{2^m h_n(x,t)}. \tag{2.6}$$

Observe also that the conditional random walk law for the same path (as it is always a valid path in $\mathbb{Z}^+$) is

$$\widetilde{\mathbb{P}}_x[\gamma] = \frac{\gamma_m}{2^m x}. \tag{2.7}$$

Now we need to understand better the asymptotic behavior of $h_n(x,t)$, this will be crucial in all the results we will prove.

We now analyze $h_n(x,t)$. First we present an application of result 5.7 from chapter XIV of (Feller, W., 1968).

Lema 2.4. *Consider the simple random walk in $\mathbb{Z}$, if $0 < x < n$ it holds that*

$$\mathbb{P}_x[\tau_0 < \tau_n,\, \tau_0 = k] = \frac{1}{n}\sum_{j=1}^{n-1} \cos^{k-1}\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi x j}{n}\right).$$

The utility of this comes from the fact that it includes the symmetric case $\mathbb{P}_x[\tau_0 > \tau_n,\, \tau_n = k] = \mathbb{P}_{n-x}[\tau_0 < \tau_n,\, \tau_0 = k]$ and so we can write

$$\mathbb{P}_x[\tau_{\{0,n\}} = k] = \mathbb{P}_x[\tau_0 < \tau_n,\, \tau_0 = k] + \mathbb{P}_x[\tau_0 > \tau_n,\, \tau_n = k]. \tag{2.8}$$

With this we can obtain an expression for $h_n(x,t)$:

Lema 2.5. *For any integer $x \in [0,n]$ it holds*

$$h_n(x,t) = \frac{2}{n}\sum_{j=1}^{\lfloor n/2 \rfloor} \cos^t\left(\frac{\pi(2j-1)}{n}\right) \cot\left(\frac{\pi(2j-1)}{2n}\right) \sin\left(\frac{\pi x(2j-1)}{n}\right). \tag{2.9}$$

Proof. Observe that

$$\begin{aligned}
\mathbb{P}_x[\tau_0 > \tau_n,\, \tau_n = k] &= \mathbb{P}_{n-x}[\tau_0 < \tau_n,\, \tau_0 = k] \\
&= \frac{1}{n}\sum_{j=1}^{n-1} \cos^{k-1}\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi (n-x) j}{n}\right) \\
&= \frac{1}{n}\sum_{j=1}^{n-1} (-1)^{j+1} \cos^{k-1}\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi x j}{n}\right).
\end{aligned}$$

So when we sum the probabilities in (2.8) and use Lemma 2.4, all terms with even j's disappear and we get

$$\begin{aligned}
\mathbb{P}_x[\tau_{\{0,n\}} = k] &= \mathbb{P}_x[\tau_0 < \tau_n,\, \tau_0 = k] + \mathbb{P}_x[\tau_0 > \tau_n,\, \tau_n = k] \\
&= \frac{2}{n}\sum_{j=1}^{\lfloor n/2 \rfloor} \cos^{k-1}\left(\frac{\pi(2j-1)}{n}\right) \sin\left(\frac{\pi(2j-1)}{n}\right) \sin\left(\frac{\pi x(2j-1)}{n}\right),
\end{aligned}$$

and then finally

$$\begin{aligned}
h_n(x,t) &= \frac{2}{n}\sum_{k=t+1}^{\infty}\sum_{j=1}^{\lfloor n/2 \rfloor} \cos^{k-1}\left(\frac{\pi(2j-1)}{n}\right) \sin\left(\frac{\pi(2j-1)}{n}\right) \sin\left(\frac{\pi x(2j-1)}{n}\right) \\
&= \frac{2}{n}\sum_{j=1}^{\lfloor n/2 \rfloor} \frac{\sin\left(\frac{\pi(2j-1)}{n}\right)}{1-\cos\left(\frac{\pi(2j-1)}{n}\right)} \sin\left(\frac{\pi x(2j-1)}{n}\right) \cos^t\left(\frac{\pi(2j-1)}{n}\right) \\
&= \frac{2}{n}\sum_{j=1}^{\lfloor n/2 \rfloor} \cos^t\left(\frac{\pi(2j-1)}{n}\right) \cot\left(\frac{\pi(2j-1)}{2n}\right) \sin\left(\frac{\pi x(2j-1)}{n}\right).
\end{aligned}$$

This concludes the proof. □

Our main concern now is to turn the expression in Lemma 2.5 into something tractable. In order to do this we first need to show that the only term asymptotically relevant in the sum is the first one.

Lema 2.6. *If $t = t(n)$ satisfies*

$$\liminf_{n\to\infty} \frac{t}{n^2 \ln n} \geqslant \frac{4}{\pi^2}, \tag{2.10}$$

then the asymptotic behavior of $h_n(x,t)$ as $n \to \infty$ is

$$\begin{aligned} h_n(x,t) &= \left(1 + \mathcal{O}(n^{-2})\right) \frac{4}{\pi} \cos^t\left(\frac{\pi}{n}\right) \sin\left(\frac{\pi x}{n}\right) \\ &\sim \frac{4}{\pi} \cos^t\left(\frac{\pi}{n}\right) \sin\left(\frac{\pi x}{n}\right). \end{aligned}$$

Proof. First we get an upper bound for (2.9) without the first term. Let us denote the first term of (2.9) by T_1; using the fact that cos and cot are decreasing functions on $[0, \pi/2]$ we have

$$\begin{aligned} \left|h_n(x,t) - T_1\right| &= \left| \frac{2}{n} \sum_{j=2}^{\lfloor \frac{n}{2} \rfloor} \cos^t\left(\frac{\pi(2j-1)}{n}\right) \cot\left(\frac{\pi(2j-1)}{2n}\right) \sin\left(\frac{\pi x(2j-1)}{n}\right) \right| \\ &\leqslant \frac{2}{n} \sum_{j=2}^{\lfloor \frac{n}{2} \rfloor} \left|\cos^t\left(\frac{\pi(2j-1)}{n}\right)\right| \left|\cot\left(\frac{\pi(2j-1)}{2n}\right)\right| \\ &\leqslant \frac{2}{n}\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \left|\cos^t\left(\frac{2\pi}{n}\right)\right| \left|\cot\left(\frac{\pi}{n}\right)\right| \\ &\leqslant \left|\cos^t\left(\frac{2\pi}{n}\right)\right| \left|\cot\left(\frac{\pi}{n}\right)\right|. \end{aligned}$$

Dividing both sides by T_1 we get

$$\left|\frac{h_n(x,t)}{T_1} - 1\right| \leqslant \left|\frac{\cos^t\left(\frac{2\pi}{n}\right)}{\cos^t\left(\frac{\pi}{n}\right)}\right| \left|\frac{\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)}\right| \frac{1}{\left|\sin\left(\frac{\pi x}{n}\right)\right|}. \tag{2.11}$$

Let us study the asymptotic behavior of the right-hand side of (2.11). We have that $\cos(x) = e^{-x^2/2}(1 + \mathcal{O}(x^4))$ and $\cot(x) = x^{-1}(1 + \mathcal{O}(x^2))$ as $x \to 0$. Using this in the above display

$$\frac{\cos^t\left(\frac{2\pi}{n}\right)}{\cos^t\left(\frac{\pi}{n}\right)} = (1 + \mathcal{O}(tn^{-4}))e^{-3t\pi^2/(2n^2)} \quad \text{and} \quad \frac{\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)} = \frac{1}{2}(1 + \mathcal{O}(n^{-2})).$$

So, we obtain

$$\begin{aligned} \left|\frac{h_n(x,t)}{T_1} - 1\right| &\leqslant \left|\frac{\cos^t\left(\frac{2\pi}{n}\right)}{\cos^t\left(\frac{\pi}{n}\right)}\right| \left|\frac{\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)}\right| \frac{1}{\left|\sin\left(\frac{\pi x}{n}\right)\right|} \\ &= (1 + \mathcal{O}(tn^{-4}) + \mathcal{O}(x^2 n^{-2})) \frac{1}{2\sin\left(\frac{\pi x}{n}\right)} e^{-3t\pi^2/(2n^2)}. \end{aligned}$$

If $x \sim cn$, then our bound is $\mathcal{O}(e^{-3t\pi^2/(2n^2)})$ and if $x = o(n)$ then our bound is $\mathcal{O}(nx^{-1}e^{-3t\pi^2/(2n^2)})$. As we do not need really sharp estimates at this point, we will just work with the worst case bound, so

$$\left|\frac{h_n(x,t)}{T_1} - 1\right| = \mathcal{O}(ne^{-3t\pi^2/(2n^2)}).$$

Since for sufficiently large n we have $\dfrac{3t\pi^2}{2n^2} \geqslant 3\ln n$, then $ne^{-3t\pi^2/(2n^2)} \leqslant n^{-2}$. Therefore

$$\begin{aligned} h_n(x,t) &= (1 + \mathcal{O}(ne^{-3t\pi^2/(2n^2)}))T_1 \\ &= (1 + \mathcal{O}(n^{-2}))\frac{2}{n}\cos^t\left(\frac{\pi}{n}\right)\cot\left(\frac{\pi}{2n}\right)\sin\left(\frac{\pi x}{n}\right), \end{aligned} \tag{2.12}$$

and again using the asymptotic expression of $\cot x$, we obtain

$$\frac{2}{n}\cot\left(\frac{\pi}{2n}\right) = \frac{2}{n}\left(\frac{2n}{\pi} + \mathcal{O}(n^{-1})\right) = \frac{4}{\pi}(1 + \mathcal{O}(n^{-2})).$$

and this gives us the asymptotic relation for $h_n(x,t)$. □

This concludes the preliminaries we need in order to prove our main results.

3 Convergence of vacant set law

We want to show that the random walk on the torus conditioned on not hitting the origin for a fixed time has as a limit the random interlacements process when we look at a fixed subset around the origin. The main question here is about how much time the conditional walk on the ring graph (the one-dimensional discrete torus) needs in order to match the random interlacements behavior. It turns out that here the time for the random interlacements convergence will be

$$t_{\alpha,n} = \frac{\alpha n^3}{2\pi^2}. \tag{3.1}$$

Now let us begin the proof.

Proof of Theorem 1.3.1. Although we already specified the value of $t_{n,\alpha}$, here we will work with a generic t and then find the "right" value. The only assumption we have here is that t should satisfy the condition of Lemma 2.6.

Our aim here is to find a time t such that for any fixed interval $[-a, b]$ with $a, b > 0$ we have for a starting point $x = \lfloor n/2 \rfloor$ outside the interval

$$\widehat{\mathbb{P}}_x^t\big[[-a,b] \subset V_t\big] \to \exp\left\{-\frac{\alpha(a+b)}{2}\right\}.$$

The choice of $x = \lfloor n/2 \rfloor$ is to keep the walk starting sufficiently away from the limit points of the interval so that the initial points do not affect the law of the vacant set. For this consider the conditional random walk on the ring graph as a random walk on $\mathbb{Z}$ conditioned on not hitting 0 or n for time t, so the site $-a$ will be equivalent of point $n-a$ in this walk.

With this we have

$$\begin{aligned}\widehat{\mathbb{P}}_x^t\big[[-a,b] \subset V_t\big] &= \mathbb{P}_x\big[\tau_{\{b,n-a\}} > t \mid \tau_{\{0,n\}} > t\big] \\ &= \frac{\mathbb{P}_x\big[\tau_{\{b,n-a\}} > t\big]}{\mathbb{P}_x\big[\tau_{\{0,n\}} > t\big]};\end{aligned} \tag{3.2}$$

here we used that the simple random walk necessarily needs to hit $\{b, n-a\}$ in order to hit $\{0, n\}$. Now that we are working with the simple random walk, we can use its symmetry to rewrite the probability in the numerator of (3.2):

$$\mathbb{P}_x\big[\tau_{\{b,n-a\}} > t\big] = \mathbb{P}_{x-b}\big[\tau_{\{0,n-a-b\}} > t\big].$$

Using this in (3.2), we obtain

$$\widehat{\mathbb{P}}_x^t\big[[-a,b] \subset V_t\big] = \frac{h_{n-a-b}(x-b,t)}{h_n(x,t)}.$$

Now using Lemma 2.6 we have

$$\widehat{\mathbb{P}}_x^t\big[[-a,b]\subset V_t\big] \sim \frac{\cos^t\left(\frac{\pi}{n-a-b}\right)\sin\left(\frac{\pi(x-b)}{n-a-b}\right)}{\cos^t\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi x}{n}\right)}.$$

As $x = \lfloor n/2 \rfloor$, both sines in the above expression are asymptotic to 1. Then, using again the asymptotic relation $\cos x \sim e^{-x^2/2}$ as $x \to 0$

$$\begin{aligned}\widehat{\mathbb{P}}_x^t\big[[-a,b]\subset V_t\big] &\sim \frac{\exp\left\{\frac{-t\pi^2}{2(n-a-b)^2}\right\}}{\exp\left\{\frac{-t\pi^2}{2n^2}\right\}}\\ &\sim \exp\left\{-\frac{t\pi^2(a+b)}{n^3}\right\}.\end{aligned}$$

So, by continuity of the exponential function we need this exponent to be asymptotic to $\alpha(a+b)/2$, this means

$$t \sim \frac{\alpha n^3}{a\pi^2}. \tag{3.3}$$

Now, we just need to observe that this value of t of (3.3) satisfies the condition on Lemma 2.6:

$$\liminf_{n\to\infty} \frac{t}{n^2 \ln n} = \infty.$$

This concludes the proof of Theorem 1.3.1, as for any fixed interval $[-a,b]$

$$\widehat{\mathbb{P}}_z^{\alpha n^3/(a\pi^2)}\big[[-a,b]\subset V_{\lfloor \alpha n^3/(a\pi^2)\rfloor}\big] \sim \exp\left\{-\frac{\alpha(a+b)}{2}\right\}.$$

Therefore,

$$V_{\lfloor \alpha n^3/(a\pi^2)\rfloor} \overset{\text{law}}{\Rightarrow} \mathcal{V}^\alpha,$$

as desired. ∎

4 Central Limit Theorem

Now we prove the central limit theorem for the local times. This will be done using the characteristic function of the local time together with Lévy's continuity theorem.

Proof of Theorem 1.3.2. By Lemma 2.3, the characteristic function of the local time is

$$\varphi_{\ell(x)}(t) = \exp\left\{\alpha x^2 \frac{(e^{it}-1)}{2x-(2x-1)e^{it}}\right\}.$$

We need to study the asymptotic behavior of the exponent when $t \to 0$ and $x \to \infty$. Here we need not only the main term, but also the error to see under which conditions the convergence holds. Write

$$\begin{aligned}\varphi_{\ell(x)-\alpha x^2}(t) &= \exp\left\{\alpha x^2 \frac{(e^{it}-1)}{2x-(2x-1)e^{it}} - \alpha x^2 it\right\} \\ &= \exp\left\{\alpha x^2 \left(\frac{(e^{it}-1)}{2x-(2x-1)e^{it}} - it\right)\right\}. \end{aligned} \tag{4.1}$$

Next, we obtain

$$\begin{aligned}\frac{(e^{it}-1)}{2x-(2x-1)e^{it}} - it &= \frac{(e^{it}-1) - it(2x-(2x-1)e^{it})}{2x-(2x-1)e^{it}} \\ &= \frac{(it - \frac{t^2}{2} + \mathcal{O}(t^3)) - it(2x-(2x-1)(1+it-\frac{t^2}{2}+\mathcal{O}(t^3)))}{1-\mathcal{O}(xt)} \\ &= -\frac{t^2}{2}(4x-1) + \mathcal{O}(x^2t^3),\end{aligned}$$

and using the asymptotic expansion

$$e^{it} - 1 = it - \frac{t^2}{2} + \mathcal{O}(t^3),$$

together with

$$\begin{aligned}2x-(2x-1)e^{it} &= 1 + (2x-1)(1-e^{it}) \\ &= 1 - \mathcal{O}(xt),\end{aligned}$$

therefore

$$\begin{aligned}\frac{(e^{it}-1) - it(2x-(2x-1)e^{it})}{2x-(2x-1)e^{it}} &= \frac{(it - \frac{t^2}{2} + \mathcal{O}(t^3)) - it(2x-(2x-1)(1+it-\frac{t^2}{2}+\mathcal{O}(t^3)))}{1-\mathcal{O}(xt)} \\ &= -\frac{t^2}{2}(4x-1) + \mathcal{O}(x^2t^3),\end{aligned}$$

and then, coming back to (4.1),

$$\begin{aligned}\varphi_{\ell(x)-\alpha x^2}(t) &= \exp\left\{\alpha x^2\left(-\frac{t^2}{2}(4x-1)+\mathcal{O}(x^2t^3)\right)\right\} \\ &= \exp\left\{-\frac{\alpha(4x-1)x^2t^2}{2}+\mathcal{O}(x^4t^3)\right\}. \qquad (4.2)\end{aligned}$$

So,

$$\varphi_{\ell^*(x)}(t) = \exp\left\{-\frac{t^2}{2}+\mathcal{O}\left(\frac{t^3}{\sqrt{x}}\right)\right\}, \text{ where } \ell^*(x) = \frac{\ell(x)-\alpha x^2}{\sqrt{\alpha(4x-1)x}}.$$

Then, as $x \to \infty$ this characteristic function converges to the one of the standard normal distribution, and, by the continuity theorem (see e.g. Theorem 9.5.2 of (Resnick, S., 2013)), we conclude our proof. □

5 Convergence of Local Times

5.1 Lemmas

In order to prove Theorem 1.3.3 we need more preliminaries. Again we will represent the conditional random walk in the ring graph as the simple random walk in $\mathbb{Z}$ conditioned to the event $\tau_{0,n} > t^*$, where $t^* = \alpha n^3/(2\pi^2)$ is the random interlacements convergence time.

Lema 5.1. *Consider the conditional random walk on the ring graph with n sites. Let $t = t(n)$ and $\Delta = \Delta(n)$ be such that t, Δ and $t - \Delta$ satisfy condition (2.10). For any $1 < x < \lfloor n/2 \rfloor$ the time until the conditional random walk hits the site $\lfloor n/2 \rfloor$ satisfies*

$$\widehat{\mathbb{P}}_x^t[\tau_{\lfloor n/2 \rfloor} > \Delta] \leqslant (1 + \mathcal{O}(\Delta n^{-3}))\frac{8}{\pi}\cos\left(\frac{\pi x}{n}\right)\exp\left\{-\frac{3\pi^2\Delta}{2n^2}\right\}.$$

Proof. Splitting the above probability into the sum of the probabilities of each path, we consider the set Γ of paths which does not include the sites 0 or $\lfloor n/2 \rfloor$ and have length Δ. Then by (2.6) we have

$$\begin{aligned}\widehat{\mathbb{P}}_x^t[\tau_{\lfloor n/2 \rfloor} > \Delta] &= \sum_{\gamma \in \Gamma} \widehat{\mathbb{P}}_x^t[\gamma] \\ &= \sum_{\gamma \in \Gamma} \frac{h_n(\gamma_\Delta, t - \Delta)}{2^\Delta h_n(x,t)}.\end{aligned}$$

As t and $t - \Delta$ satisfy condition (2.10), we can use Lemma 2.6 to obtain that

$$\begin{aligned}\widehat{\mathbb{P}}_x^t[\tau_{\lfloor n/2 \rfloor} > \Delta] &= (1 + \mathcal{O}(n^{-2}))\sum_{\gamma \in \Gamma}\cos^{-\Delta}\left(\frac{\pi}{n}\right)\frac{\sin\left(\frac{\pi\gamma_\Delta}{n}\right)}{2^\Delta \sin\left(\frac{\pi x}{n}\right)} \\ &= (1 + \mathcal{O}(n^{-2}))\frac{\cos^{-\Delta}\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)}\mathbb{E}_x\Big[\sin\left(\frac{\pi X_\Delta}{n}\right)\mathbb{1}[\tau_{\{0,\lfloor n/2 \rfloor\}} > \Delta]\Big] \\ &\leqslant (1 + \mathcal{O}(n^{-2}))\frac{\cos^{-\Delta}\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)}h_{\lfloor n/2 \rfloor}(x,\Delta).\end{aligned}$$

Now, as Δ also satisfies condition (2.10), we again use Lemma 2.6 and get

$$\begin{aligned}\widehat{\mathbb{P}}_x^t[\tau_n > \Delta] &\leqslant (1 + \mathcal{O}(n^{-2}))\frac{4\sin\left(\frac{\pi x}{\lfloor n/2 \rfloor}\right)}{\pi\sin\left(\frac{\pi x}{n}\right)}\frac{\cos^\Delta\left(\frac{\pi}{\lfloor n/2 \rfloor}\right)}{\cos^\Delta\left(\frac{\pi}{n}\right)} \\ &= (1 + \mathcal{O}(n^{-1}))\frac{8}{\pi}\cos\left(\frac{\pi x}{n}\right)\frac{\cos^\Delta\left(\frac{\pi}{\lfloor n/2 \rfloor}\right)}{\cos^\Delta\left(\frac{\pi}{n}\right)}.\end{aligned}$$

Using the asymptotic expansion of cosine $\cos(x) = e^{-x^2/2}(1 + \mathcal{O}(x^4))$ we have

$$\begin{aligned}\widehat{\mathbb{P}}_x^t[\tau_n > \Delta] &\leqslant (1+\mathcal{O}(n^{-1}))(1+\mathcal{O}(\Delta n^{-4}))\frac{8}{\pi}\cos\Big(\frac{\pi x}{n}\Big)\exp\Big\{-\frac{\pi^2\Delta}{2}\Big(\frac{1}{\lfloor n/2\rfloor^2}-\frac{1}{n^2}\Big)\Big\} \\ &= (1+\mathcal{O}(\Delta n^{-3}))\frac{8}{\pi}\cos\Big(\frac{\pi x}{n}\Big)\exp\Big\{-\frac{3\pi^2\Delta}{2n^2}\Big\}.\end{aligned}$$

This concludes the proof. □

Lema 5.2. *Let $\{X_t\}_{t\in\mathbb{Z}^+}$ be a simple random walk on $\mathbb{Z}$. Consider $\Delta = \Delta(n)$ satisfying condition (2.10). Then for any $a \in \{1, \ldots, n-1\}$ we have*

$$\mathbb{E}_a\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n\}} > \Delta\Big] = (1+\mathcal{O}(n^{-2}))\frac{\pi}{4}.$$

Proof. Consider a quantity $t > \Delta$ such that $t - \Delta$ satisfies (2.10). Then we have

$$\begin{aligned}h_n(a,t) &= \mathbb{P}_a[\tau_{\{0,n\}} > t] \\ &= \mathbb{P}_a[\tau_{\{0,n\}} > \Delta]\mathbb{P}_a[\tau_{\{0,n\}} > t \mid \tau_{\{0,n\}} > \Delta].\end{aligned}$$

Using the Markov property we obtain

$$\begin{aligned}\mathbb{P}_a[\tau_{\{0,n\}} > t \mid \tau_{\{0,n\}} > \Delta] &= \mathbb{E}_a[\mathbb{P}_{X_\Delta}[\tau_{\{0,n\}} > t-\Delta] \mid \tau_{\{0,n\}} > \Delta] \\ &= \mathbb{E}_a[h_n(X_\Delta, t-\Delta) \mid \tau_{\{0,n\}} > \Delta].\end{aligned}$$

That gives the relation

$$h_n(a,t) = h_n(a,\Delta)\cdot\mathbb{E}_a[h_n(X_\Delta, t-\Delta) \mid \tau_{\{0,n\}} > \Delta]$$

or, equivalently

$$\mathbb{E}_a[h_n(X_\Delta, t-\Delta) \mid \tau_{\{0,n\}} > \Delta] = \frac{h_n(a,t)}{h_n(a,\Delta)}.$$

Here, as t, Δ and $t-\Delta$ satisfy (2.10), we can use Lemma 2.6 and obtain

$$\mathbb{E}_a\Big[\frac{4}{\pi}\cos^{t-\Delta}\Big(\frac{\pi}{n}\Big)\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n\}} > \Delta\Big] = (1+\mathcal{O}(n^{-2}))\frac{\cos^t\big(\frac{\pi}{n}\big)\sin\big(\frac{\pi a}{n}\big)}{\cos^\Delta\big(\frac{\pi}{n}\big)\sin\big(\frac{\pi a}{n}\big)}.$$

Rearranging the terms, we obtain

$$\mathbb{E}_a\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n\}} > \Delta\Big] = (1+\mathcal{O}(n^{-2}))\frac{\pi}{4},$$

which concludes the proof of Lemma 5.2. □

Lema 5.3. *Consider the conditional random walk on the ring graph with n sites, $\mathbb{Z}_n$. Assume $x \in \mathbb{Z}^+$ is fixed, $t = t(n)$ and $\Delta = \Delta(n)$ are such that both Δ and $t-\Delta$ satisfy (2.10). Then as $n \to \infty$ we have*

$$\widehat{\mathbb{P}}_{\lfloor n/2\rfloor}^t[\tau_x > \Delta] = (1+\mathcal{O}(n^{-1})+\mathcal{O}(n^{-4}\Delta))\exp\Big\{-\frac{\Delta x\pi^2}{n^3}\Big\}.$$

Proof. First we calculate the probability that there will be no visits to a fixed site x in this interval by the random walk with initial site n:

$$\begin{aligned}
\widehat{\mathbb{P}}^t_{\lfloor n/2\rfloor}[\tau_x > \Delta] &= \mathbb{P}_{\lfloor n/2\rfloor}[\tau_x > \Delta \mid \tau_{\{0,n\}} > t] \\
&= \frac{\mathbb{P}_{\lfloor n/2\rfloor}[\tau_{\{x,n\}} > \Delta]\mathbb{P}_{\lfloor n/2\rfloor}[\tau_{\{0,n\}} > t \mid \tau_{\{x,n\}} > \Delta]}{\mathbb{P}_{\lfloor n/2\rfloor}[\tau_{\{0,n\}} > t]} \\
&= \frac{\mathbb{P}_{\lfloor n/2\rfloor - x}[\tau_{\{0,n-x\}} > \Delta]\mathbb{P}_{\lfloor n/2\rfloor}[\tau_{\{0,n\}} > t \mid \tau_{\{x,n\}} > \Delta]}{\mathbb{P}_{\lfloor n/2\rfloor}[\tau_{\{0,n\}} > t]} \\
&= \frac{h_{n-x}(\lfloor n/2\rfloor - x, \Delta)}{h_n(\lfloor n/2\rfloor, t)}\mathbb{P}_{\lfloor n/2\rfloor}[\tau_{\{0,n\}} > t \mid \tau_{\{x,n\}} > \Delta] \\
&= \frac{h_{n-x}(\lfloor n/2\rfloor - x, \Delta)}{h_n(\lfloor n/2\rfloor, t)}\mathbb{E}_{\lfloor n/2\rfloor}[h_n(X_\Delta, t-\Delta) \mid \tau_{\{x,n\}} > \Delta].
\end{aligned}$$

We can use Lemma 2.6 to obtain

$$\begin{aligned}
&\frac{h_{n-x}(\lfloor n/2\rfloor - x, \Delta)}{h_n(\lfloor n/2\rfloor, t)}\mathbb{E}_{\lfloor n/2\rfloor}[h_n(X_\Delta, t-\Delta) \mid \tau_{\{x,n\}} > \Delta] \\
&= (1+\mathcal{O}(n^{-2}))\frac{\frac{4}{\pi}\cos^\Delta\left(\frac{\pi}{n-x}\right)\sin\left(\frac{\pi(\lfloor n/2\rfloor - x)}{n-x}\right)}{\frac{4}{\pi}\cos^t\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi\lfloor n/2\rfloor}{n}\right)}\mathbb{E}_{\lfloor n/2\rfloor}\left[\frac{4}{\pi}\cos^{t-\Delta}\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi X_\Delta}{n}\right) \mid \tau_{\{x,n\}} > \Delta\right] \\
&= (1+\mathcal{O}(n^{-2}))\frac{4}{\pi}\left(\frac{\cos\left(\frac{\pi}{n-x}\right)}{\cos\left(\frac{\pi}{n}\right)}\right)^\Delta \mathbb{E}_{\lfloor n/2\rfloor}\left[\sin\left(\frac{\pi X_\Delta}{n}\right) \mid \tau_{\{x,n\}} > \Delta\right].
\end{aligned}$$

Again using $\cos x = e^{-\frac{x^2}{2}}(1+\mathcal{O}(x^4))$ we have

$$\begin{aligned}
(1+&\mathcal{O}(n^{-2})))\frac{4}{\pi}\left(\frac{\cos\left(\frac{\pi}{n-x}\right)}{\cos\left(\frac{\pi}{n}\right)}\right)^\Delta \mathbb{E}_{\lfloor n/2\rfloor}\left[\sin\left(\frac{\pi X_\Delta}{n}\right) \mid \tau_{\{x,n\}} > \Delta\right] \\
&= (1+\mathcal{O}(\Delta n^{-4}))\frac{4}{\pi}e^{-\Delta\pi^2((n-x)^{-2}-n^{-2})/2}\mathbb{E}_{\lfloor n/2\rfloor}\left[\sin\left(\frac{\pi X_\Delta}{n}\right) \mid \tau_{\{x,n\}} > \Delta\right] \\
&= (1+\mathcal{O}(\Delta n^{-4}))\frac{4}{\pi}e^{-\Delta x\pi^2/(n^3)}\mathbb{E}_{\lfloor n/2\rfloor}\left[\sin\left(\frac{\pi X_\Delta}{n}\right) \mid \tau_{\{x,n\}} > \Delta\right]. \qquad (5.1)
\end{aligned}$$

An important point here is that this probability asymptotically does not depend on t, just on n, Δ and x. Now, to work with the expectation, we will show that its value is asymptotically equal to $\pi/4$, using Lemma 5.2 for this. Consider the set $\Gamma_{n,\Delta}$ of all paths started in $\lfloor n/2\rfloor$ and of length Δ, so we can write the following expectation in terms of a sum of probabilities of paths and use translation invariance of simple random walk:

$$\begin{aligned}
\mathbb{E}_{\lfloor n/2 \rfloor}\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{x,n\}} > \Delta\Big] &= \sum_{\gamma \in \Gamma_{n,\Delta}} \sin\Big(\frac{\pi \gamma_\Delta}{n}\Big)\mathbb{P}_{\lfloor n/2 \rfloor}[\gamma \mid \tau_{\{x,n\}} > \Delta] \\
&= \sum_{\gamma \in \Gamma_{n,\Delta}} \sin\Big(\frac{\pi \gamma_\Delta}{n}\Big)\frac{\mathbb{P}_{\lfloor n/2 \rfloor}[\gamma, \tau_{\{x,n\}} > \Delta]}{\mathbb{P}_{\lfloor n/2 \rfloor}[\tau_{\{x,n\}} > \Delta]} \\
&= \sum_{\gamma \in \Gamma_{\lfloor n/2 \rfloor - x,\Delta}} \sin\Big(\frac{\pi (x+\gamma_\Delta)}{n}\Big)\frac{\mathbb{P}_{\lfloor n/2 \rfloor - x}[\gamma, \tau_{\{0,n-x\}} > \Delta]}{\mathbb{P}_{\lfloor n/2 \rfloor - x}[\tau_{\{0,n-x\}} > \Delta]} \\
&= \mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi (x+X_\Delta)}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big]. \qquad (5.2)
\end{aligned}$$

We can write

$$\begin{aligned}
&\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi (x+X_\Delta)}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \\
&\quad = \sin\Big(\frac{\pi x}{n}\Big)\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\cos\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \\
&\quad\quad + \cos\Big(\frac{\pi x}{n}\Big)\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \qquad (5.3) \\
&\quad = \mathcal{O}(n^{-1}) + (1-\mathcal{O}(n^{-2}))\mathbb{E}_{\lfloor n/2 \rfloor - x}[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta]. \qquad (5.4)
\end{aligned}$$

Now working with the sine in the expectation

$$\begin{aligned}
&\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \\
&= \mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n-x}\Big(1-\frac{x}{n}\Big)\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \\
&= \mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n-x}\Big)\cos\Big(\frac{\pi x X_\Delta}{n(n-x)}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \\
&\quad - \mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\cos\Big(\frac{\pi X_\Delta}{n-x}\Big)\sin\Big(\frac{\pi x X_\Delta}{n(n-x)}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big].
\end{aligned}$$

As $0 < X_\Delta < n-x$ we have the following asymptotic behavior,

$$\begin{aligned}
\cos\Big(\frac{\pi x X_\Delta}{n(n-x)}\Big) &= 1 - \mathcal{O}\Big(\frac{(\pi x X_\Delta)^2}{2(n(n-x))^2}\Big) \\
&= 1 - \mathcal{O}(n^{-2}),
\end{aligned}$$

and

$$\begin{aligned}
\sin\Big(\frac{\pi x X_\Delta}{n(n-x)}\Big) &= \mathcal{O}\Big(\frac{\pi x X_\Delta}{n(n-x)}\Big) \\
&= \mathcal{O}(n^{-1}).
\end{aligned}$$

So with this we get

$$\begin{aligned}
&\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] \\
&\quad = (1-\mathcal{O}(n^{-2}))\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n-x}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] + \mathcal{O}(n^{-1}). \qquad (5.5)
\end{aligned}$$

By Lemma 5.2 we have

$$\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n-x}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] = \frac{\pi}{4}(1 + \mathcal{O}(n^{-2})).$$

Then using this in (5.5) we get

$$\mathbb{E}_{\lfloor n/2 \rfloor - x}\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{0,n-x\}} > \Delta\Big] = \frac{\pi}{4}(1 + \mathcal{O}(n^{-1})).$$

Finally, using this in (5.4) and then in (5.2) we get

$$\mathbb{E}_{\lfloor n/2 \rfloor}\Big[\sin\Big(\frac{\pi X_\Delta}{n}\Big) \mid \tau_{\{x,n\}} > \Delta\Big] = \frac{\pi}{4}(1 + \mathcal{O}(n^{-1})). \tag{5.6}$$

Using this in (5.1), we conclude the proof of Lemma 5.3. □

Lema 5.4. *Suppose $x > 1$ is fixed and $y^2 = o(\Delta)$. It holds that*

$$\widetilde{\mathbb{P}}_x[X_\Delta \leqslant y] = \mathbb{P}_x[X_\Delta \leqslant y \mid \tau_0 = \infty] = \sqrt{\frac{2}{\pi}}\frac{y^3}{3\Delta^{3/2}}(1 + \mathcal{O}(y^2\Delta^{-1})).$$

Proof. Let us calculate this probability by splitting it according to the endpoints of the paths:

$$\widetilde{\mathbb{P}}_x[X_\Delta \leqslant y] = \sum_{k=1}^{y} \frac{k}{x2^\Delta}|\{\gamma : \gamma_0 = x,\, \gamma_i \neq 0,\, \gamma_\Delta = y\}|, \tag{5.7}$$

where $|A|$ stands for the cardinality of the set A. We need to estimate $N_k := |\{\gamma : \gamma_0 = x,\, \gamma_i \neq 0,\, \gamma_\Delta = k\}|$. In this sum we have some problems with parity. If Δ is even, then both x and γ_Δ need to have the same parity. Otherwise they need to have opposite parity. As we are mostly interested in asymptotic results, let us assume that Δ is even and so x and k are of the same parity.

Let us consider paths from $(0, x)$ to (Δ, y) that do not touch the line $y = 0$. The value of N_k can be explicitly calculated using the reflection principle (see section 1 of chapter III of (Feller, W., 1968)), so we have:

$$\begin{aligned} N_k &= \binom{\Delta}{\frac{\Delta+k-x}{2}} - \binom{\Delta}{\frac{\Delta-x-k}{2}} \\ &= \binom{\Delta}{\frac{\Delta+k-x}{2}}\left(1 - \frac{\left(\frac{\Delta+k-x}{2}\right)!\left(\frac{\Delta-k+x}{2}\right)!}{\left(\frac{\Delta+k+x}{2}\right)!\left(\frac{\Delta-k-x}{2}\right)!}\right). \end{aligned}$$

We have two fractions to work with, so as $\Delta + k$ goes to infinity, we use the asymptotic expansions valid for any real constant a:

$$\frac{\Gamma(n+a+1)}{\Gamma(n-a+1)} = n^{2a}\Big(1 + \frac{a}{n} + \mathcal{O}(n^{-2})\Big).$$

Then

$$
\begin{aligned}
N_k &= \binom{\Delta}{\frac{\Delta+k-x}{2}}\left(1-\left(\frac{\Delta-k}{\Delta+k}\right)^x\left(1-\frac{x}{\Delta+k}+\mathcal{O}(\Delta^{-2})\right)\right)\\
&= \binom{\Delta}{\frac{\Delta+k-x}{2}}\left(1-\left(1-\frac{2kx}{\Delta+k}+\mathcal{O}(k^2\Delta^{-2})\right)\left(1-\frac{x}{\Delta+k}+\mathcal{O}(\Delta^{-2})\right)\right)\\
&= \binom{\Delta}{\frac{\Delta+k-x}{2}}\left(\frac{(2k+1)x}{\Delta}+\mathcal{O}(k^2\Delta^{-2})\right). \qquad (5.8)
\end{aligned}
$$

Using the Stirling approximation $n! = \sqrt{2\pi n}(n/e)^n(1+\mathcal{O}(n^{-1}))$ we can work with the asymptotic expression of the remaining binomial term:

$$
\begin{aligned}
\binom{\Delta}{\frac{\Delta+k-x}{2}} &= \frac{\Delta!}{\frac{\Delta+k-x}{2}!\,\frac{\Delta-k+x}{2}!}\\
&= \frac{\sqrt{2\pi\Delta}\left(\frac{\Delta}{e}\right)^{\Delta}}{\pi\sqrt{\Delta^2-(k-x)^2}\left(\frac{\Delta+k-x}{2e}\right)^{(\Delta+k-x)/2}\left(\frac{\Delta-k+x}{2e}\right)^{(\Delta-k+x)/2}}(1+\mathcal{O}(\Delta^{-1}))\\
&= 2^{\Delta}\sqrt{\frac{2}{\pi\Delta}}(1+\mathcal{O}(k^2\Delta^{-1})).
\end{aligned}
$$

Now using this in (5.8) we get

$$N_k = 2^{\Delta}\sqrt{\frac{2}{\pi}}\frac{(2k+1)x}{\Delta^{3/2}}(1+\mathcal{O}(k^2\Delta^{-1})). \qquad (5.9)$$

We want to use (5.9) in (5.7), but we need to worry about the parity before. First, if x is even we get

$$
\begin{aligned}
\widetilde{\mathbb{P}}_x[X_\Delta \leqslant y] &= \sqrt{\frac{2}{\pi}}\frac{x}{x\Delta^{3/2}}\sum_{k=1}^{\lfloor y/2\rfloor} 2k(4k+1)(1+\mathcal{O}(k^2\Delta^{-1}))\\
&= \sqrt{\frac{2}{\pi}}\frac{y^3}{3\Delta^{3/2}}(1+\mathcal{O}(y^2\Delta^{-1})). \qquad (5.10)
\end{aligned}
$$

It is straightforward to see that $\widetilde{\mathbb{P}}_x[X_\Delta \leqslant y]$ is decreasing in x. This monotonicity property allow us to extend (5.10) to any value of x. The same argument can be used for Δ, as the positive drift makes $\widetilde{\mathbb{P}}_x[X_\Delta \leqslant y]$ also decreasing in Δ. This makes this asymptotic expression valid for all x, y, and Δ satisfying the condition in the hypothesis, which concludes the proof of Lemma 5.4. □

Corollary 5.1.1. *Consider the conditional random walk on the ring graph of size n. For a fixed $x > 0$, consider quantities Δ and y possibly depending on n such that $\Delta = o(n^2)$, $y^2 = o(\Delta)$, $y = o(n)$ and t satisfy Condition (2.10). Then*

$$\widehat{\mathbb{P}}_x^t[X_\Delta \leqslant y] = (1+\mathcal{O}(\Delta n^{-2})+\mathcal{O}(y^2\Delta^{-1}))\sqrt{\frac{2}{\pi}}\frac{y^3}{3\Delta^{3/2}}.$$

Proof. Consider the set Γ of paths γ of length Δ with the property: $\gamma_0 = x$, $\gamma_i \notin \{0, n\}$ for all i and $\gamma_\Delta \leqslant y$. Then, using Lemma 5.4

$$\begin{aligned}
\widehat{\mathbb{P}}^t_x[X_\Delta \leqslant y] &= \sum_{\gamma \in \Gamma} \widehat{\mathbb{P}}^t_x[\gamma] \\
&= \sum_{\gamma \in \Gamma} \frac{h_n(\gamma_\Delta, t-\Delta)}{2^\Delta h_n(x,t)} \\
&= (1 + \mathcal{O}(n^{-2})) \frac{\cos^{-\Delta}\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)} \sum_{\gamma \in \Gamma} \frac{\sin\left(\frac{\pi \gamma_\Delta}{n}\right)}{2^\Delta} \\
&= (1 + \mathcal{O}(y^2 n^{-2}))\cos^{-\Delta}\left(\frac{\pi}{n}\right) \sum_{\gamma \in \Gamma} \frac{\gamma_\Delta}{x 2^\Delta} \\
&= (1 + \mathcal{O}(\Delta n^{-2}) + \mathcal{O}(y^2 n^{-2}))\widetilde{\mathbb{P}}_x[X_\Delta \leqslant y] \\
&= (1 + \mathcal{O}(\Delta n^{-2}) + \mathcal{O}(y^2 \Delta^{-1}))\sqrt{\frac{2}{\pi}} \frac{y^3}{3\Delta^{3/2}}.
\end{aligned}$$

This concludes the proof of Corollary 5.1.1. □

5.2 Proof

Now we prove the convergence of the local time for a fixed x. First we present a sketch of the proof: Let X_t be our conditional random walk on the ring. We will define a second walk Y_t such that with high probability $X_t = Y_t$ for all t and for Y_t we can prove that its local times converge to these of the random interlacements.

- The particle Y_t will follow X_t until the time X_t hits x.
- When Y_t hits x, Y_t will follow the law of the conditional random walk on $\mathbb{Z}^+$ for a fixed time T (that will depend on n and will be specified later).
- During this time T we will consider this pieces of trajectory as random elements and couple the walks X_t and Y_t using the maximal coupling.
- After T, if X_t and Y_t are on the same site (i.e. the maximal coupling worked), they will continue moving together afterwards. Otherwise we say our coupling procedure failed.

To make our calculations work, we will also impose a condition on the number of pieces of trajectories of length T. For this we split our time interval (from 0 to $t^* = \alpha n^3/(2\pi^2)$) in $m = \lfloor \ln n \rfloor$ intervals, and only allow at most one "initial" visit to x (and therefore at most one alteration of Y_t for each interval). If this does not hold, we will also say our procedure failed.

As we are only interested in the situation where the procedure worked (which we will prove that have a high probability), we will refer to any particle as X_t during the proof.

Now in the proof of Theorem 1.3.3 we will rigorously define the terms used in the sketch.

Proof of Theorem 1.3.3. We begin by considering the ring graph of size n with the conditional random walk started at site $\lfloor n/2 \rfloor$. The vacant set convergence time is $t^* = \alpha n^3/(2\pi^2)$ by Theorem 1.3.1. We will split this interval in $m = \lfloor \ln n \rfloor$ random intervals. For this purpose let us define the sequence of points A_j in the following way: Let $\eta = \lfloor t^*/\ln n \rfloor$ and with it define

$$\begin{aligned} A_0 &= 0; \\ A_{j+1} &= \inf\{t \geqslant A_j + \eta : X_t = \lfloor n/2 \rfloor\}, \qquad \text{if } j < m; \\ A_{m+1} &= t^*. \end{aligned}$$

Then, the interval I_j for $j \leqslant m$ is defined as

$$I_j := [A_{j-1}, A_j), \tag{5.11}$$

and the remaining time interval R is defined as

$$R := [A_m, A_{m+1}]. \tag{5.12}$$

Now we want to show that, almost surely, the lengths of all above intervals are asymptotic to η; by definition of I_j we can say that its length can be represented as $I_j = \eta + T_j$, where T_j satisfies

$$\mathbb{P}[T_j = a] = \mathbb{E}_{\lfloor n/2 \rfloor}\big[\widehat{\mathbb{P}}^{t^*-A_{j-1}-\eta}_{X_\eta}[\tau_{\lfloor n/2 \rfloor} = a] \mid \tau_{\{0,n\}} > t^* - A_{j-1}\big]. \tag{5.13}$$

This is because from the moment A_{j-1} on, the process still has $t^* - A_{j-1}$ steps to run without hitting the origin. Also, we want the first time where $X_t = \lfloor n/2 \rfloor$ after η, so the starting point is X_η, and from there we are considering the hitting time of $\lfloor n/2 \rfloor$, which justifies the expression for the probability inside the expectation.

Consider $j < m$, then, for any $\varepsilon > 0$ we have

$$\begin{aligned} \mathbb{P}[|I_j| > (1+\varepsilon)\eta] &= \mathbb{P}[T_j > \varepsilon\eta] \\ &= \mathbb{E}_{\lfloor n/2 \rfloor}[\widehat{\mathbb{P}}^{t^*-A_{j-1}-\eta}_{X_\eta}[\tau_n > \varepsilon\eta] \mid \tau_{\{0,n\}} > t^* - A_{j-1}]. \end{aligned} \tag{5.14}$$

Observe also that $\varepsilon\eta$, $t^* - A_{j-1} - \eta$, and the difference $t^* - A_{j-1} - (1+\varepsilon)\eta$ all satisfy condition (2.10), so we use Lemma 5.1 and get

$$\begin{aligned}\mathbb{P}[|I_j| > (1+\varepsilon)\eta] &\leqslant (1+o(1))\frac{8}{\pi}\exp\Big(-\frac{3\pi^2\varepsilon\eta}{2n^2}\Big)\mathbb{E}_{\lfloor n/2\rfloor}\Big[\cos\Big(\frac{\pi X_\eta}{n}\Big) \mid \tau_{\{0,n\}} > t^* - A_{j-1}\Big]\\ &\leqslant (1+o(1))\frac{8}{\pi}\exp\Big(-\frac{3\pi^2\varepsilon\eta}{2n^2}\Big)\\ &= (1+o(1))\frac{8}{\pi}\exp\Big(-\frac{3\alpha\varepsilon n}{4\ln n}\Big).\end{aligned}$$

So, we have a summable bound (in n) for the tail probability, and therefore this shows that $I_j \sim \eta$ a.s..

Before constructing the coupling, let us discuss the probability of a successful coupling between trajectories of conditional random walks on the ring graph up to time t and on $\mathbb{Z}^+$. Assume that t is of order n^3 and let x be a fixed value, we are interested in coupling the paths of both processes up to a time $T = n^\mu$ (where $\mu < 1$), using the maximal coupling. Then the coupling event probability $\mathbb{P}[C]$ can be estimated using the expressions for the laws (2.6) and (2.7). Let Γ be the set of all paths started in x and with length T.

We have the following expression for the coupling event probability

$$\mathbb{P}[C^\complement] = \frac{1}{2}\sum_{\gamma\in\Gamma}\big|\widetilde{\mathbb{P}}_x[\gamma] - \widehat{\mathbb{P}}^t_x[\gamma]\big|.$$

We use Lemma 2.6, but we have the stronger condition that t is of order n^3; so, instead we use expression (2.12) inside the proof and for each term in the sum we have

$$\begin{aligned}\big|\widetilde{\mathbb{P}}_x[\gamma] - \widehat{\mathbb{P}}^t_x[\gamma]\big| &= \left|\frac{\gamma_T}{2^T x} - \frac{h_n(\gamma_T, t-T)}{2^T h_n(x,t)}\right|\\ &= \frac{1}{2^T}\left|\frac{\gamma_T}{x} - (1+\mathcal{O}(ne^{-3t\pi^2/(2n^2)}))\cos^{-T}\Big(\frac{\pi}{n}\Big)\frac{\sin\big(\frac{\pi\gamma_T}{n}\big)}{\sin\big(\frac{\pi x}{n}\big)}\right|.\end{aligned}$$

As $\cos^{-T}\Big(\dfrac{\pi}{n}\Big) = 1 - \mathcal{O}(n^{-2}T)$, the error term in the cosine asymptotic approximation dominates the error term in expression (2.12) and then

$$(1+\mathcal{O}(ne^{-3t\pi^2/(2n^2)}))\cos^{-T}\Big(\frac{\pi}{n}\Big) = 1 - \mathcal{O}(n^{-2}T).$$

So, we have

$$\begin{aligned}\big|\widetilde{\mathbb{P}}_x[\gamma] - \widehat{\mathbb{P}}^t_x[\gamma]\big| &= \frac{1}{2^T}\left|\frac{\gamma_T}{x} - (1-\mathcal{O}(n^{-2}T))\frac{\sin\big(\frac{\pi\gamma_T}{n}\big)}{\sin\big(\frac{\pi x}{n}\big)}\right|\\ &\leqslant \frac{1}{2^T}\left|\frac{\gamma_T}{x} - \frac{\sin\big(\frac{\pi\gamma_T}{n}\big)}{\sin\big(\frac{\pi x}{n}\big)}\right| + \mathcal{O}(n^{-2}T)\frac{\sin\big(\frac{\pi\gamma_T}{n}\big)}{2^T\sin\big(\frac{\pi x}{n}\big)}.\end{aligned} \tag{5.15}$$

Next, we sum the second terms in (5.15) in $\gamma \in \Gamma$ to obtain

$$\sum_{\gamma\in\Gamma} \frac{\sin\left(\frac{\pi\gamma_T}{n}\right)}{2^T \sin\left(\frac{\pi x}{n}\right)} = \frac{\mathbb{E}_x\Big[\sin\left(\frac{\pi X_T}{n}\right)\mathbb{1}[\tau_0 > T]\Big]}{\sin\left(\frac{\pi x}{n}\right)} \leqslant \frac{\mathbb{E}_x \sin\left(\frac{\pi X_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)}$$

As the sine is concave on the interval $[0, \pi]$ we can use Jensen's inequality and get

$$\frac{\mathbb{E}_x \sin\left(\frac{\pi X_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)} \leqslant \frac{\mathbb{E}_x \sin\left(\frac{\pi X_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)} \leqslant \frac{\sin\left(\frac{\pi \mathbb{E}_x X_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)} = 1. \tag{5.16}$$

Now we sum the first term of (5.15) in Γ to get

$$\sum_{\gamma\in\Gamma} \frac{1}{2^T}\left|\frac{\gamma_T}{x} - \frac{\sin\left(\frac{\pi\gamma_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)}\right| \leqslant \mathbb{E}_x\left|\frac{X_T}{x} - \frac{\sin\left(\frac{\pi X_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)}\right|.$$

As the maximum value that X_T can achieve starting at x is $x+T$, we have that $X_T = o(n)$ and then we can use the asymptotic expression for the sine and get

$$\begin{aligned}
\mathbb{E}_x\left|\frac{X_T}{x} - \frac{\sin\left(\frac{\pi X_T}{n}\right)}{\sin\left(\frac{\pi x}{n}\right)}\right| &= \mathbb{E}_x\left|\frac{X_T}{x} - \frac{X_T - \mathcal{O}(X_T^3 n^{-2})}{x - \mathcal{O}(n^{-2})}\right| \\
&= \mathbb{E}_x\left|\frac{X_T}{x}\mathcal{O}(X_T^2 n^{-2})\right| \\
&= \mathcal{O}(n^{-2})\mathbb{E}_x\left|X_T\right|^3.
\end{aligned} \tag{5.17}$$

By the fact that X_T is a simple random walk, we can represent the steps as a sequence of i.i.d. random variables Y_i, where Y_i take values $+1$ and -1 with probability $1/2$, so $X_t = x + S_t$, where $S_t = Y_1 + Y_2 + \ldots + Y_t$. We will use this to bound $\mathbb{E}_x\left|X_T\right|^3$.

$$\begin{aligned}
\mathbb{E}_x\left|X_T\right|^3 &= \mathbb{E}\left|x + S_t\right|^3 \\
&\leqslant \mathbb{E}\left|S_T\right|^3 + 3x\mathbb{E}_x\left|S_T\right|^2 + \mathcal{O}(T^{1/2}) \\
&= \mathbb{E}\left|S_T\right|^3 + \mathcal{O}(T)
\end{aligned}$$

Theorem 7.1.1 of (Matoušek, J.; Vondrák, J., 2001) shows that for the simple random walk S_t started at 0 and positive x it holds

$$\mathbb{P}[S_t \geqslant x] < \exp\left\{-\frac{x^2}{2t}\right\}.$$

For any constant $\beta > 0$ we have

$$\begin{aligned}\mathbb{P}[|S_T|^3 \geqslant \beta T^2] &= 2\mathbb{P}[S_T \geqslant \beta^{\frac{1}{3}} T^{\frac{2}{3}}] \\ &\leqslant 2\exp\Big\{-\frac{\beta^{\frac{2}{3}} T^{\frac{1}{3}}}{2}\Big\},\end{aligned}$$

a stretched exponential bound in T.

With this we can finally estimate $\mathbb{E}_x |S_T|^3$. Let D be the event $\{|S_T|^3 \geqslant \beta T^2\}$, then, since $|S_T| \leqslant T$, we have

$$\begin{aligned}\mathbb{E}|S_T|^3 &= \mathbb{P}[D]\mathbb{E}[|S_T|^3 \mid D] + \mathbb{P}[D^C]\mathbb{E}[|S_T|^3 \mid D^C] \\ &\leqslant 2\exp\Big\{-\frac{\beta^{2/3} T^{1/3}}{2}\Big\} T^3 + \beta T^2 = \mathcal{O}(T^2).\end{aligned}$$

Finally,

$$\mathbb{E}_x \left|X_T{}^3\right| \leqslant \mathbb{E}|S_T|^3 + \mathcal{O}(T) = \mathcal{O}(T^2). \tag{5.18}$$

Then we can bound the coupling event probability. Using (5.16), (5.17) and (5.18) together we have

$$\begin{aligned}2\mathbb{P}[C^{\complement}] &= \sum_{\gamma\in\Gamma} \left|\widetilde{\mathbb{P}}_x[\gamma] - \widehat{\mathbb{P}}^t_x[\gamma]\right| \\ &= \mathcal{O}(n^{-2})\mathcal{O}(T^2) + \mathcal{O}(n^{-2}T) = \mathcal{O}(n^{-2}T^2).\end{aligned} \tag{5.19}$$

We still have to estimate the probability that our procedure fails because the random walk has at least two excursions in at least one of the intervals.

Let us consider excursions of length T starting at x. The initial times of the first and (possibly) second excursion starting in x are denoted by

$$\begin{aligned}\tau^1_x(k) &= \inf\{t \in I_k : X_t = x\}, \\ \tau^2_x(k) &= \inf\{t > \tau^1_x(k) + T : X_t = x\}.\end{aligned}$$

Observe that we do not necessarily have $\tau^2_x(k) \in I_k$, so the event that two or more excursions happen during I_k is $\{\tau^2_x(k) \in I_k\}$. We want to calculate the probability of this event. The initial point of interval I_k is A_{k-1}, so the process still has time $t^* - A_{k-1}$ to run. So, using the Markov property, we have

$$\begin{aligned}&\widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^2_x(k) \in I_k] \\ &= \widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^1_x(k) \in I_k] \cdot \widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^2_x(k) \in I_k \mid \tau^1_x(k) \in I_k].\end{aligned} \tag{5.20}$$

Now let us work with each term of (5.20) separately. As both $|I_k|$ and $t^* - A_{k-1} - |I_k|$ satisfy almost surely condition (2.10), we can use Lemma 5.3 and the fact that $|I_k|$ is of order $n^3(\ln n)^{-1}$, then we get

$$\begin{aligned}\widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^1_x(k)\in I_k] &= \mathbb{P}_{\lfloor n/2\rfloor}[\tau_x \leqslant |I_k| \mid \tau_{\{0,n\}} > t^* - A_{k-1}]\\ &= \mathbb{E}\Big[\mathbb{E}\big[\mathbb{P}_{\lfloor n/2\rfloor}[\tau_x \leqslant |I_k| \mid \tau_{\{0,n\}} > t^* - A_{k-1}] \mid \sigma(I_k)\big]\Big]\\ &= 1 - \mathbb{E}\big[\mathbb{E}[(1+\mathcal{O}(n^{-1}) + \mathcal{O}(|I_k|n^{-4}))e^{-|I_k|\pi^2x/n^3} \mid \sigma(I_k)]\big]\\ &= 1-(1+\mathcal{O}(n^{-1}))\mathbb{E}e^{-|I_k|\pi^2x/n^3}\\ &= 1-(1+\mathcal{O}(n^{-1}))e^{-\eta\pi^2x/n^3}\mathbb{E}e^{-T_k\pi^2x/n^3}.\end{aligned}$$

Using Lemma 5.1 in the same way as we did in (5.14), we have

$$\begin{aligned}\mathbb{E}e^{-T_k\pi^2x/n^3} &= \mathbb{P}[T_k > n^2\ln n/\pi^2]\mathbb{E}[e^{-T_k\pi^2x/n^3} \mid T_k > n^2\ln n/\pi^2]\\ &\quad + \mathbb{P}[T_k \leqslant n^2\ln n/\pi^2]\mathbb{E}[e^{-T_k\pi^2x/n^3} \mid T_k \leqslant n^2\ln n/\pi^2]\\ &= \mathcal{O}(n^{-\frac{3}{2}}) + (1-\mathcal{O}(n^{-\frac{3}{2}}))]\mathbb{E}[e^{-T_k\pi^2x/n^3} \mid T_k \leqslant n^2\ln n/\pi^2]\\ &= \mathcal{O}(n^{-\frac{3}{2}}) + (1-\mathcal{O}(n^{-\frac{3}{2}}))(1-\mathcal{O}(n^{-1}\ln n))\\ &= 1-\mathcal{O}(n^{-1}\ln n),\end{aligned}$$

and therefore

$$\begin{aligned}\widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^1_x(k)\in I_k] &= 1-(1+\mathcal{O}(n^{-1}\ln n))e^{-\eta\pi^2x/n^3}\\ &= 1-(1+\mathcal{O}(n^{-1}\ln n))e^{-\alpha x/(2\ln n)}.\end{aligned}$$

As $e^{-\alpha x/(2\ln n)} = 1 - \dfrac{\alpha x}{2\ln n} + \mathcal{O}((\ln n)^{-2})$, we obtain

$$\begin{aligned}\widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^1_x(k)\in I_k] &= 1-(1+\mathcal{O}(n^{-1}\ln n))\Big(1-\frac{\alpha x}{2\ln n} + \mathcal{O}((\ln n)^{-2})\Big)\\ &= \frac{\alpha x}{2\ln n} + \mathcal{O}(n^{-1}\ln n).\end{aligned} \tag{5.21}$$

Now we work with the second probability of (5.20):

$$\begin{aligned}&\widehat{\mathbb{P}}^{t^*-A_{k-1}}_{\lfloor n/2\rfloor}[\tau^2_x(k)\in I_k \mid \tau^1_x(k)\in I_k]\\ &\qquad = \widehat{\mathbb{E}}^{t^*-A_{k-1}}_x\mathbb{P}_{X_T}[\tau_x \leqslant |I_k| - \tau^1_x(k) - T \mid \tau_{\{0,n\}} > t^* - P_{k-1} - \tau^1_x(k)].\end{aligned}$$

Let us abbreviate $t^{**} = t^* - A_{k-1}$. Again we use Corollary 5.1.1 and Lemma 5.3; then, as

$T = n^{\mu}$ we get

$$
\begin{aligned}
&\widehat{\mathbb{E}}_x^{t^{**}} \mathbb{P}_{X_T}[\tau_x \leqslant |I_k| - \tau_x^1(k) - T \mid \tau_{\{0,2n\}} > t^* - A_{k-1} - \tau_x^1(k)] \\
&\quad = \widehat{\mathbb{P}}_x^{t^{**}}[X_T \leqslant n^{\frac{\mu}{3}}] \widehat{\mathbb{E}}_x^{t^{**}} \big(\mathbb{P}_{X_T}[\tau_x \leqslant |I_k| - \tau_x^1(k) - T \mid \tau_{\{0,n\}} > t^* - P_{k-1} - \tau_x^1(k)] \mid X_T \leqslant n^{\frac{\mu}{3}} \big) \\
&\quad\quad + \widehat{\mathbb{P}}_x^{t^{**}}[X_T > n^{\frac{\mu}{3}}] \widehat{\mathbb{E}}_x^{t^{**}} \big(\mathbb{P}_{X_T}[\tau_x \leqslant |I_k| - \tau_x^1(k) - T \mid \tau_{\{0,n\}} > t^* - P_{k-1} - \tau_x^1(k)] \mid X_T > n^{\frac{\mu}{3}} \big) \\
&\quad = \mathcal{O}(n^{-\frac{\mu}{2}}) + (1 - \mathcal{O}(n^{-\frac{\mu}{2}})) \Big(1 - (1 + \mathcal{O}(n^{-1})) \widehat{\mathbb{E}}_x^{t^{**}} e^{-(|I_k| - \tau_x^1(k) - T) x \pi^2 / n^3} \Big) \\
&\quad \leqslant \mathcal{O}(n^{-\frac{\mu}{2}}) + (1 - \mathcal{O}(n^{-\frac{\mu}{2}})) \Big(1 - (1 + \mathcal{O}(n^{-1} \ln n)) \widehat{\mathbb{E}}_x^{t^{**}} e^{-|I_k| x \pi^2 / n^3} \Big) \\
&\quad = \mathcal{O}(n^{-\frac{\mu}{2}}) + (1 - \mathcal{O}(n^{-\frac{\mu}{2}})) \Big(1 - (1 + \mathcal{O}(n^{-1} \ln n)) e^{-\alpha x / (2 \ln n)} \Big) \\
&\quad = \mathcal{O}(n^{-\frac{\mu}{2}}) + (1 - \mathcal{O}(n^{-\frac{\mu}{2}})) \Big(1 - (1 + \mathcal{O}(n^{-1} \ln n)) \Big(1 - \frac{\alpha x}{2 \ln n} \Big) \Big) \\
&\quad = \frac{\alpha x}{2 \ln n} (1 + \mathcal{O}(n^{-\frac{\mu}{2}} \ln n)).
\end{aligned}
$$

So, we can bound the probability that a specific interval I_j contains at least two excursions to x:

$$
\begin{aligned}
\widehat{\mathbb{P}}_{\lfloor n/2 \rfloor}^{t^* - A_{k-1}}[\tau_x^2(k) \in I_k] &\leqslant \Big(\frac{\alpha x}{2 \ln n} + \mathcal{O}(n^{-1} \ln n) \Big) \Big(\frac{\alpha x}{2 \ln n} + \mathcal{O}(n^{-\frac{\mu}{2}}) \Big) \\
&= \frac{\alpha^2 x^2}{4 \ln^2 n} + \mathcal{O}(n^{-\frac{\mu}{2}} (\ln n)^{-1}).
\end{aligned}
$$

Finally we bound the probability that at least one interval contain at least two excursions:

$$
\begin{aligned}
\widehat{\mathbb{P}}_{\lfloor n/2 \rfloor}^{t^*} \Big[\bigcup_{k=1}^{m} \{ \tau_x^2(k) \in I_k \} \Big] &\leqslant \sum_{k=1}^{m} \widehat{\mathbb{P}}_{\lfloor n/2 \rfloor}^{t^* - A_{k-1}} [\tau_x^2(k) \in I_k] \\
&\leqslant m \Big(\frac{\alpha^2 x^2}{4 \ln^2 n} + \mathcal{O}(n^{-\frac{\mu}{2}} (\ln n)^{-1}) \Big) \\
&= \frac{\alpha^2 x^2}{4 \ln n} + \mathcal{O}((\ln n)^{-2}) \to 0 \quad \text{as } n \to \infty. \qquad (5.22)
\end{aligned}
$$

After this we only need to worry about the remaining time R. We surely have that it is smaller than any of the intervals, as for any $j \leqslant m$ we have $|I_j| \geqslant \eta$, but $(m+1)\eta > t^*$. Let us bound the probability that the process visits x in the remaining time.

We want to estimate $\widehat{\mathbb{P}}_{\lfloor n/2 \rfloor}^{R}[\tau_x > R]$; consider the function $f : \mathbb{Z}^+ \to [0, 1]$ defined by

$$
f(t) = \widehat{\mathbb{P}}_{\lfloor n/2 \rfloor}^{t}[\tau_x > t]
$$

By definition we have

$$
\begin{aligned}
\widehat{\mathbb{P}}_{\lfloor n/2 \rfloor}^{t}[\tau_x > t] &= \frac{\mathbb{P}_{\lfloor n/2 \rfloor}[\tau_{\{x,n\}} > t]}{\mathbb{P}_{\lfloor n/2 \rfloor}[\tau_{\{0,n\}} > t} \\
&= \frac{h_{n-x}(\lfloor n/2 \rfloor - x, t)}{h_n(\lfloor n/2 \rfloor, t)}.
\end{aligned}
$$

If $t = t(n)$ satisfies condition (2.10) we have an asymptotic expression in n for $f(t)$

$$\begin{aligned} f(t) &= (1+\mathcal{O}(n^{-2}))\frac{\sin\left(\frac{\pi(\lfloor n/2\rfloor - x)}{n-x}\right)}{\sin\left(\frac{\pi\lfloor n/2\rfloor}{n}\right)}\frac{\cos^t\left(\frac{\pi}{n-x}\right)}{\cos^t\left(\frac{\pi}{n}\right)} \\ &= (1+\mathcal{O}(tn^{-4})))\exp\left\{-\frac{t\pi^2 x}{n^3}\right\}. \end{aligned}$$

So, asymptotically the function is decreasing. We now show that R satisfies condition (2.10). Fix a constant $\beta > 4/\pi^2$, then write

$$\begin{aligned} \mathbb{P}[R > \beta n^2 \ln n] &= \mathbb{P}\Big[t^* - m\eta - \sum_{k=1}^{m} T_k > \beta n^2 \ln n\Big] \\ &= \mathbb{P}\Big[\sum_{k=1}^{m} T_k < t^* - m\eta - \beta n^2 \ln n\Big] \\ &\geqslant 1 - \frac{\sum_{k=1}^{m}\mathbb{E}T_k}{t^* - m\beta\eta n^2 \ln n}. \end{aligned} \tag{5.23}$$

To bound the expectations in (5.23) we use (5.13) and Lemma 5.1:

$$\begin{aligned} \mathbb{P}\Big[T_k > \frac{4n^2\ln n}{\pi^2}\Big] &= \widehat{\mathbb{E}}_{\lfloor n/2\rfloor}\Big[\widehat{\mathbb{P}}^{t^*-A_{k-1}-\eta}_{X_\eta}\Big[\tau_{\lfloor n/2\rfloor} > \frac{4n^2\ln n}{\pi^2}\Big] \mid \tau_{\{0,n\}} > t^* - A_{k-1}\Big] \\ &\leqslant (1+\mathcal{O}(n^{-2}\ln n))\frac{8}{\pi}n^{-6}. \end{aligned}$$

Also, as each T_k is bounded by t^*, we have

$$\begin{aligned} \mathbb{E}T_k &= \mathbb{E}\Big[T_k \mid T_k > \frac{4n^2\ln n}{\pi^2}\Big]\mathbb{P}\Big[T_k > \frac{4n^2\ln n}{\pi^2}\Big] \\ &\quad + \mathbb{E}\Big[T_k \mid T_k \leqslant \frac{4n^2\ln n}{\pi^2}\Big]\mathbb{P}\Big[T_k \leqslant \frac{4n^2\ln n}{\pi^2}\Big] \\ &\leqslant \mathcal{O}(t^* n^{-6}) + \frac{4n^2\ln n}{\pi^2}(1-\mathcal{O}(n^{-6})) \\ &\leqslant \frac{16n^2\ln n}{\pi^2} + \mathcal{O}(n^{-3}). \end{aligned}$$

Now observe that

$$\begin{aligned} t^* - m\eta &= t^* - \lfloor \ln n\rfloor \left\lfloor \frac{t^*}{\ln n}\right\rfloor \\ &= \mathcal{O}(t^*(\ln n)^{-1}). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{P}[R > \beta n^2 \ln n] &\geqslant 1 - \frac{\sum_{k=1}^{m}\mathbb{E}T_k}{t^* - m\eta - \beta n^2\ln n} \\ &\geqslant 1 - \frac{m\Big(16n^2\ln n/\pi^2 + \mathcal{O}(n^{-3})\Big)}{\mathcal{O}(t^*(\ln n)^{-1})} \\ &= 1 - \mathcal{O}((\ln n)^3 n^{-1}). \end{aligned}$$

Then with high probability R satisfies condition (2.10), consequently

$$\begin{aligned}\widehat{\mathbb{P}}^R_{[n/2]}[\tau_x > R] &= \mathbb{E}f(R) \\ &\geqslant \mathbb{E}[f(R) \mid R > \beta n^2 \ln n]\mathbb{P}[R > \beta n^2 \ln n] \\ &= \mathbb{E}[(1+\mathcal{O}(Rn^{-4}))e^{-R\pi^2 x/n^3} \mid R > \beta n^2 \ln n](1-\mathcal{O}((\ln n)^3 n^{-1})) \\ &\geqslant e^{-\beta \ln n \pi^2 x/n}(1-\mathcal{O}((\ln n)^3 n^{-1})) \\ &= 1-\mathcal{O}((\ln n)^3 n^{-1}). \end{aligned} \tag{5.24}$$

So, we have that $\widehat{\mathbb{P}}^R_n[\tau_x \leqslant R] = \mathcal{O}((\ln n)^3 n^{-1})$, which is an upper bound for the probability of a visit in the remaining time.

Now, we construct the coupling. The motivation behind the procedure is that all visits to x usually happen in "batches", so when a initial visit to x happens, the walk visits x again at some moments during a small time interval and then goes away again. When trying to couple the entire process we get an error of large order that turns the coupling almost impossible to happen, so, as we are only interested in the visits to x, we just need to couple then for these small time intervals of the excursions.

As x is fixed and we are working with asymptotic behaviors, these "batches" of visits are rare and, since we splitted our time in the intervals defined in (5.11) such that in each of them probability of hitting x is almost the same; this makes our calculations possible. So, again let us recall our definitions and the coupling procedure:

- Consider a conditional random walk on the ring graph of size n that will run for a time $t^* = \alpha n^3/(2\pi^2)$. We split the time into m intervals as defined in (5.11) and the remaining time.
- In each interval we have a small chance of visiting x. When the visit happens, from that moment on we couple the walk on the ring with the conditional random walk on $\mathbb{Z}^+$ for a time $T = n^\mu$.

Our procedure can fail if and only if any of the following happens:

- At least one of the m intervals has 2 or more excursions.
- There is a visit to x in the remaining time.
- The maximal coupling fails for at least one excursion.

Let us denote these three events by F_1, F_2 and F_3 respectively, and the event that the coupling fails by F, so $F = F_1 \cup F_2 \cup F_3$. Also, denote the local time of this walk in site x by $L_n(x)$.

The probability of F_1 was already bounded in (5.22) and we bounded the probability of F_3 in (5.24). As for F_2, the probability that the coupling of one excursion fails was dealt with in (5.19); since we have at most m excursions, it holds that

$$\mathbb{P}[F_2] \leqslant m\mathcal{O}(n^{-2}T^2) = \mathcal{O}(n^{-2(1-\mu)} \ln n).$$

Consequently,

$$\begin{aligned}\mathbb{P}[F] &\leqslant \mathbb{P}[F_1] + \mathbb{P}[F_2] + \mathbb{P}[F_3] \\ &\leqslant \frac{\alpha^2 x^2}{4 \ln n} + \mathcal{O}((\ln n)^{-2}) + \mathcal{O}(n^{-2(1-\mu)} \ln n) + \mathcal{O}((\ln n)^3 n^{-1}) \\ &= \frac{\alpha^2 x^2}{4 \ln n} + \mathcal{O}((\ln n)^{-2}).\end{aligned}$$

Now we have an estimate on the probability that the procedure fails; we need then to see which is the distribution of the number of visits to x of an excursion. We have a conditional random walk on $\mathbb{Z}^+$ running up to time n^μ. Let us consider each excursion as a part of a random walk in $\mathbb{Z}^+$ started in x and denote by V_i the number of visits of ith random walk to x if we let it run indefinitely; also denote by $T_{i,k}$ the time between the kth and $(k+1)$th visits. With this the number of visits of ith excursion W_i can be defined as the number of visits of the walk up to time n^μ and be represented as

$$W_i = \sum_{j=1}^{V_i} \mathbb{1}\left[\sum_{k=1}^{j-1} T_{i,k} \leqslant n^\mu\right].$$

Since $\sum_{k=1}^{V_i-1} T_{i,k}$ is the time of the last visit; by Lemma 2.2 it is finite as the number of visits is finite. This means that $W_i \to V_i$ almost surely, and since $|e^{itW_i}| = 1$, by the dominated convergence theorem

$$\varphi_{W_i}(t) = \mathbb{E}e^{itW_i} \to \varphi_{V_i}(t).$$

Each coupled excursion is independent, so if we denote by B_i the Bernoulli random variable indicating if the visit in x at interval I_i happened. Then $L_n(x)$ can be written as

$$L_n(x) = \sum_{i=1}^{m} B_i W_i,$$

where the variables W_i and B_i are independent. Therefore

$$\begin{aligned}\varphi_{L_n(x)}(t) &= \mathbb{E}e^{itL_n(x)} \\ &= \prod_{i=1}^{m} [\mathbb{P}[B_i = 0] + \mathbb{P}[B_i = 1]\varphi_W(t)].\end{aligned}$$

The probability that the interval I_i has a visit (and then an excursion) was calculated in (5.21), so we write

$$\begin{aligned}\varphi_{L_n(x)}(t) &= \prod_{i=1}^{m}\Big(1-\frac{\alpha x}{2\ln n}+\mathcal{O}(n^{-1})+\Big(\frac{\alpha x}{2\ln n}+\mathcal{O}(n^{-1})\Big)\varphi_W(t)\Big)\\ &= \Big(1+\frac{\alpha x}{2\ln n}(\varphi_W(t)-1)+\mathcal{O}(n^{-1})\Big)^m\\ &= \exp\Big\{m\ln\Big(1+\frac{\alpha x}{2\ln n}(\varphi_W(t)-1)+\mathcal{O}(n^{-1})\Big)\Big\}.\end{aligned}$$

As $|\varphi_W(t)-1|\leqslant 2$, we can use the asymptotic expansion $\ln(1+x)=x+\mathcal{O}(x^2)$ and

$$\varphi_{L_n(x)}(t)=\exp\Big\{\frac{mx\alpha}{2\ln n}(\varphi_W(t)-1)+\mathcal{O}(m(\ln n)^{-2})\Big\}$$

and, since $m=\lfloor \ln n\rfloor$, this becomes

$$\begin{aligned}\varphi_{L_n(x)}(t) &= \exp\Big\{\frac{\alpha x}{2}(\varphi_W(t)-1)+\mathcal{O}((\ln n)^{-1})\Big\}\\ &\to \exp\Big\{\frac{\alpha x}{2}(\varphi_V(t)-1)\Big\}.\end{aligned}$$

Now, using Lemma 2.2, we have that V is Geometric with $\mathbb{P}[V=k]=\frac{1}{2x}(1-\frac{1}{2x})^{k-1}$ and, finally,

$$\begin{aligned}\varphi_{L_n(x)}(t) &\to \exp\Big\{\alpha x^2\frac{(e^{it}-1)}{2x-(2x-1)e^{it}}\Big\}\\ &= \varphi_{\ell(x)}(t).\end{aligned}$$

Then, since $\varphi_{\ell(x)}(t)$ is continuous at 0, one can use the continuity theorem (cf. e.g. Theorem 9.5.2 of (Resnick, S., 2013)) and obtain that

$$L_n(x)\overset{\text{law}}{\Rightarrow}\ell(x),$$

as desired. This concludes the proof of Theorem 1.3.3. □

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